Unique characterization of the Fourier transform in the framework of representation theory

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Abstract. In this paper we elaborate upon the investigation initiated in [3] of typical and distinctive properties of the Fourier transform (FT), in particular the crucial role played by the Howe dual pair $(O(m), \mathfrak{sl}_2)$. We prove in detail a result on the unique characterization of the FT making extensive use of a representation of the Lie algebra $\mathfrak{sl}_2$. As an example, we consider the case $m = 1$. We refer to [3] for a detailed study involving the derivation of a class of operators portraying FT symmetry properties.

1. Introduction

Due to its useful properties the classical Fourier transform (FT) has many applications in a whole range of areas such as harmonic analysis and signal processing. Over $\mathbb{R}^m$, the FT is given by the integral transform

$$\mathcal{F}[f](y) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \mathrm{e}^{i\langle x,y\rangle} f(x)\,\mathrm{d}x,$$

where $\langle x, y\rangle = \sum_{j=1}^{m} x_j y_j$ denotes the standard inner product of two m-dimensional vectors $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$.

In [3], we initiated the investigation of typical and distinctive properties of the FT. This led to the question as to what suffices to uniquely characterize the Fourier transform. Hereto, we consider in particular invariance and symmetry properties governed by the dual pair $O(m)$ and $\mathfrak{sl}_2$. This is motivated by the fact that the FT can be expressed as an operator exponential containing orthogonally invariant operators that generate a realization of $\mathfrak{sl}_2$. Other means of characterizing the FT include for instance the interaction of the FT with the convolution product, which is used in [1] to obtain a uniqueness result. Our approach allows us to make explicit use of the representation theory of the Lie algebra $\mathfrak{sl}_2$.

Now, a natural course of actions is the following: We start from a select set of FT properties determined by the underlying symmetry and we set out to determine the class of all operators also satisfying these. This problem necessarily has at least one solution, namely the FT. Two possible scenarios then arise depending on the imposed properties. If the FT turns out to be the sole solution, this gives us a unique characterization the FT. To obtain other transforms we can then relax the prescribed properties to less strict ones. At the other end of the spectrum is the case where we find an overabundance of solutions. We can then investigate how we can reduce this class of solutions by imposing additional properties. In this way we arrive at a (preferably finite) set of interesting transforms. On top of that, by continuing this process it ultimately gives us a list of properties, the combination of which is exclusively satisfied by the FT.

The explicit derivation of operators having Fourier-like symmetry properties is the subject of [3]. Now we focus on a specific result which uniquely characterizes the FT. Moreover, we elaborate upon the case $m = 1$ where the approach of [3] does not yield any new operators.

The paper is organized as follows. In Section 2, we discuss the properties and symmetries of the FT relevant to our cause. In Section 3, we prove a uniqueness result for the FT. In Section 4, we consider in detail the one-dimensional case.

2. Properties

We will start by elaborating upon what makes the FT so useful. Our goal is then to determine a class of operators T having these specific properties.

A major property of the Fourier transform consists of its interaction with differential operators. In particular, we have for $j = 1, \ldots, m$:

$$\begin{aligned} \mathcal{F} \circ \partial_{x_j} &= -i\, y_j \circ \mathcal{F} \\ \mathcal{F} \circ x_j &= -i\, \partial_{y_j} \circ \mathcal{F} \end{aligned} \tag{1}$$

We want to relax these interactions as prescribing them uniquely characterizes the FT up to a multiplicative factor. They give rise to the following symmetries for $j = 1, \ldots, m$:

$$\mathcal{F} \circ \left(x_j^2 - \partial_{x_j}^2\right) = \left(y_j^2 - \partial_{y_j}^2\right) \circ \mathcal{F} \tag{2}$$

where one recognises the Hamiltonian of the quantum harmonic oscillator.

Now, the FT also exhibits a symmetry with respect to the orthogonal group $O(m)$. The relations (2) combined with the invariance under $O(m)$ yield

$$\mathcal{F} \circ (-\Delta_x + |x|^2) = (-\Delta_y + |y|^2) \circ \mathcal{F} \tag{3}$$

which is compatible with $O(m)$. Here the Laplace operator Δ_x and the norm squared $|x|^2$ are defined as

$$\Delta_x := \sum_{j=1}^{m} \partial_{x_j}^2, \qquad |x|^2 := \sum_{j=1}^{m} x_j^2.$$

Also

$$\mathcal{F} \circ (\Delta_x + |x|^2) = -(\Delta_y + |y|^2) \circ \mathcal{F}. \tag{4}$$

This is related to an alternative formulation of the FT as an operator exponential:

$$\mathcal{F} = \mathrm{e}^{i\frac{\pi}{4}(-\Delta_x + |x|^2 - m)}. \tag{5}$$

The operators that occur here highlight a connection with the Lie algebra $\mathfrak{sl}_2$ as they satisfy

$$\left[\Delta_x/2, |x|^2/2\right] = \mathbb{E}_x + \tfrac{m}{2} \qquad \left[\mathbb{E}_x + \tfrac{m}{2}, |x|^2/2\right] = |x|^2 \qquad \left[\mathbb{E}_x + \tfrac{m}{2}, \Delta_x/2\right] = -\Delta_x. \tag{6}$$

Here we denoted by $\mathbb{E}_x$ the Euler operator, defined as

$$\mathbb{E}_x := \sum_{j=1}^{m} x_j \partial_{x_i}.$$

An operator T will have the same symmetries (3) and (4) if it satisfies

$$\begin{aligned} T \circ \Delta_x &= -|y|^2 \circ T \\ T \circ |x|^2 &= -\Delta_y \circ T, \end{aligned} \tag{7}$$

which are reminiscent of (1). These are called the Helmholtz relations because if (7) holds for an integral transform

$$Tf(y) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} K(x,y) f(x)\,\mathrm{d}x,$$

then its kernel will satisfy

$$\frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \big(\Delta_y K(x,y) + |x|^2 K(x,y)\big) f(x)\,\mathrm{d}x = 0 \qquad \big(\forall f \in \mathcal{S}(\mathbb{R}^m)\big). \tag{8}$$

Here $\mathcal{S}(\mathbb{R}^m)$ denotes the space of smooth and rapidly decreasing functions on $\mathbb{R}^m$, also called the Schwartz space, which is dense in $L^2(\mathbb{R}^m)$. Note that T satisfying the Helmholtz relations, together with the commutation relations (6), implies that one also has

$$T \circ (\mathbb{E}_x + \tfrac{m}{2}) = -(\mathbb{E}_x + \tfrac{m}{2}) \circ T.$$

Another property of the FT is that it is an automorphism on $\mathcal{S}(\mathbb{R}^m)$. A decomposition of $\mathcal{S}(\mathbb{R}^m)$ with respect to both the orthogonal $O(m)$ symmetry and the algebraic $\mathfrak{sl}_2$ symmetry is given by the generalized Hermite functions [2]

$$\phi_{j,k,\ell} := 2^j j!\, L_j^{\frac{m}{2}+k-1}(|x|^2)\, H_k^{(\ell)}\, \mathrm{e}^{-|x|^2/2}. \tag{9}$$

Here $j,k \in \mathbb{Z}_{\geq 0}$, $L_j^{\frac{m}{2}+k-1}$ is the Laguerre polynomial and $\{\, H_k^{(\ell)} \mid \ell = 1,...,\dim(\mathcal{H}_k) \,\}$ is a basis for the space of spherical harmonics of degree k, denoted by $\mathcal{H}_k := \ker \Delta_x \cap \mathcal{P}_k$, with $\mathcal{P}_k$ the space of homogeneous polynomials of degree k. The basis (9) forms eigenfunctions of the FT with action given by

$$\mathcal{F}(\phi_{j,k,\ell}) = \mathrm{e}^{i\frac{\pi}{2}(2j+k)} \phi_{j,k,\ell}.$$

Note that this action is independent of the index ℓ of the Hermite functions, which is precisely due to the orthogonal symmetry of the FT.

Another way to write the Hermite functions (see [2]) is

$$\phi_{j,k,\ell} = \left(-\frac{\Delta_x}{2} - \frac{|x|^2}{2} + \mathbb{E}_x + \frac{m}{2} \right)^j H_k^{(\ell)}\, \mathrm{e}^{-|x|^2/2}. \tag{10}$$

The operator that acts on the Hermite functions gives rise to another realization of the Lie algebra $\mathfrak{sl}_2$ by means of

$$h = -\frac{\Delta_x}{2} + \frac{|x|^2}{2}, \quad e = -\frac{\Delta_x}{4} - \frac{|x|^2}{4} + \frac{1}{2}(\mathbb{E}_x + \tfrac{m}{2}), \quad f = \frac{\Delta_x}{4} + \frac{|x|^2}{4} + \frac{1}{2}(\mathbb{E}_x + \tfrac{m}{2}), \tag{11}$$

which satisfy

$$[h,e] = 2e, \qquad [h,f] = -2f, \qquad [e,f] = h.$$

For every $k \in \mathbb{Z}_{\geq 0}$ and $\ell \in \{1, \ldots, \dim(\mathcal{H}_k)\}$, the set of Hermite functions $\{\, \phi_{j,k,\ell} \mid j \in \mathbb{Z}_{\geq 0} \,\}$ then forms a basis for the positive discrete series representation of $\mathfrak{sl}_2$ with lowest weight $k + m/2$. This can be seen from their action on the basis (9) which is given by

$$h\, \phi_{j,k,\ell} = \big(2j + k + \tfrac{m}{2}\big)\, \phi_{j,k,\ell}, \tag{12}$$

and

$$e\,\phi_{j,k,\ell} = \frac{1}{2}\,\phi_{j+1,k,\ell}, \qquad f\,\phi_{j,k,\ell} = -j\,(2j-2+m+2k)\,\phi_{j-1,k,\ell}.$$

Returning to the operator formulation (5) of the Fourier transform, this can now be rewritten in terms of h as

$$\mathcal{F} = \mathrm{e}^{i\frac{\pi}{4}(-\Delta_x+|x|^2-m)} = \mathrm{e}^{i\frac{\pi}{2}(h-\frac{m}{2})}.$$

We see that, up to a complex phase factor, the FT is an element of $\exp\big(\mathfrak{sl}_2\big)$. Now, we want to find out if any other elements of $\exp\big(\mathfrak{sl}_2\big)$ give rise to an operator that satisfies the properties we considered. The operators (11) allow us to prove a unique characterization of the Fourier transform. This is the subject of the following section.

3. Uniqueness result

Theorem 1. *Let T be an operator that satisfies the properties:*

(i) the Helmholtz relations

$$\begin{aligned} T\circ\Delta_x &= -|y|^2\circ T \\ T\circ|x|^2 &= -\Delta_y\circ T \end{aligned}$$

(ii) $T\,\phi_{j,k,\ell} = \mu_{j,k}\,\phi_{j,k,\ell}$ *with* $\mu_{j,k}\in\mathbb{C}$

(iii) $T^4 = \mathrm{id}$

(iv) $T = A\exp(z)$ *with* $A\in\mathbb{C}$ *and* $z\in\mathfrak{sl}_2 = \mathrm{span}\big\{\Delta_x, |x|^2, \big[\Delta_x, |x|^2\big]\big\}$.

Then T equals, up to multiplication by an integer power of i, the classical Fourier transform or its inverse.

Proof. To prove this result we will use the realization of $\mathfrak{sl}_2$ by means of the operators (11) and their relation with the basis (9) to our advantage. A general element of $\mathfrak{sl}_2$ can then be written as

$$z = B\,h + C\,e + D\,f$$

with complex numbers $B, C, D\in\mathbb{C}$.

In terms of the operators h, e, f, the Helmholtz property translates to the (anti-)commutation relations

$$T\circ h = h\circ T, \qquad T\circ e = -e\circ T, \qquad T\circ f = -f\circ T.$$

Hence, in order for $T = A\exp(z)$ to satisfy the Helmholtz relations, we must have $z\in\mathfrak{sl}_2$ such that

$$\exp(z)\,h\,\exp(-z) = h, \qquad \exp(z)\,e\,\exp(-z) = -e, \qquad \exp(z)\,f\,\exp(-z) = -f.$$

Using $\exp(X)\,Y\exp(-X) = \exp(\mathrm{ad}_X)\,Y$ with $\mathrm{ad}_X Y = [X,Y]$ (Lemma 5.3 of [8]) and

$$[z,h] = -C\,h + 2D\,e, \qquad [z,e] = B\,h - 2D\,f, \qquad [z,f] = -2B\,e + 2C\,f$$

to work out the left-hand sides, these conditions give rise to

$$(1+4CDX)\,h + (-4BCX-2CY)\,e + (-4BDX+2DY)\,f = h \tag{13}$$

$$(-2BDX-DY)\,h + \big(1+(4B^2+2CD)X+2BY\big)\,e + (-2D^2X)\,f = -e \tag{14}$$

$$(-2BCX+CY)\,h + (-2C^2X)\,e + \big(1+(4B^2+2CD)X-2BY\big)\,f = -f \tag{15}$$

with

$$X=\sum_{n=1}^{\infty}\frac{(4B^2+4CD)^{n-1}}{(2n)!}\quad,\qquad Y=\sum_{n=0}^{\infty}\frac{(4B^2+4CD)^{n}}{(2n+1)!}.$$

We find requirements for B, C, D by equating the coefficients of the operators h, e, f in the left-hand sides with those in the right-hand sides.

From the coefficients of f in (14) and e in (15) we conclude that when $X \neq 0$, we must have $C = 0$ and $D = 0$. Similarly, if $X = 0$, we deduce from the coefficient of e in (14) that $Y \neq 0$. This in turn implies again that $C = 0$ and $D = 0$ (from the coefficients of e and f in (13)). Hence, we necessarily have $C = 0$ and $D = 0$. Note that this immediately covers $T = \exp(z)$ upholding property (ii), as for $z = B\,h$ we have using (12)

$$T\,\phi_{j,k,\ell} = A\exp\big(B(2j+k+\tfrac{m}{2})\big)\,\phi_{j,k,\ell}. \tag{16}$$

Plugging $C = 0$ and $D = 0$ into (13)–(15), the only non-trivial coefficient equations yield

$$-1 = 1 + 4B^2\sum_{n=1}^{\infty}\frac{(4B^2)^{n-1}}{(2n)!} + 2B\sum_{n=0}^{\infty}\frac{(4B^2)^{n}}{(2n+1)!} = \exp(2B),$$

and

$$-1 = \exp(-2B).$$

Both of which lead to the condition

$$2B = i(\pi + 2\pi n) \iff B = i\frac{\pi}{2}(2n+1),$$

for some integer n. This completely fixes all viable elements $z \in \exp(\mathfrak{sl}_2)$.

Now, in order for property (iii) to hold, the eigenvalues of T^4 must all equal 1. Together with the expression (16) for the eigenvalues of $T = A\exp(z)$ this leads to $A^4\exp(i\pi m) = 1$, which gives $A = \exp\big(i\frac{\pi}{2}(-\frac{m}{2}+N)\big)$ for some integer N.

In this way we arrive at the form

$$T=\exp\bigg(i\frac{\pi}{2}\Big((2n+1)\Big(h-\frac{m}{2}\Big)+nm+N\Big)\bigg)\qquad (n,N\in\mathbb{Z})$$

Now, the operator $h - m/2$ has integer eigenvalues and for integer k we have $i^k = i^{k \bmod 4}$. Hence, for n an even integer, T is precisely the operator exponential form of the Fourier transform, up to multiplication by an integer power of i, namely i^{nm+N}. Likewise, for n odd T equals the inverse Fourier transform

$$\exp\bigg(-i\frac{\pi}{2}\Big(h-\frac{m}{2}\Big)\bigg),$$

up to an integer power of i. □

Property (iv) in the previous theorem is rather restrictive. In order to see whether the other properties suffice to uniquely characterize the FT, we investigate what happens if we omit it. The objective is thus to determine all operators

$$T\colon \mathcal{S}(\mathbb{R}^m) \to \mathcal{S}(\mathbb{R}^m)$$

that satisfy

(i) the Helmholtz relations

$$\begin{aligned} T \circ \Delta_x &= -|y|^2 \circ T \\ T \circ |x|^2 &= -\Delta_y \circ T \end{aligned}$$

(ii) $T\phi_{j,k,\ell} = \mu_{j,k}\, \phi_{j,k,\ell}$ with $\mu_{j,k} \in \mathbb{C}$

(iii) $T^4 = \text{id}$

As opposed to the previous result, the solution to this problem will in general not be limited to the FT and its inverse. To determine these other solutions, we make explicit use of the fact that we already know of one specific solution, namely the FT, and use this to rewrite the requirements. As the FT is an automorphism on the space $\mathcal{S}(\mathbb{R}^m)$, we propose to look for $\widetilde{T}$ such that $\widetilde{T}\big[\mathcal{F}[f]\big]$ has the desired properties and then define the operator T by $T[f] = \widetilde{T}\big[\mathcal{F}[f]\big]$, or thus

$$T = \widetilde{T} \circ \mathcal{F}. \tag{17}$$

This leads to the following result:

Proposition 2. *Any operator* $\widetilde{T}$ *of the form*

$$\widetilde{T} = \exp\big(i\tfrac{\pi}{2}F\big), \tag{18}$$

with F *an operator that*

- *commutes with (the generators of)* $\mathfrak{sl}_2 = \operatorname{span}\big\{\Delta_x, |x|^2, \big[\Delta_x, |x|^2\big]\big\}$
- *has integer eigenvalues on the functions* $\{\phi_{j,k,\ell}\}$ *(independent of* ℓ*)*

will yield an operator T *by* (17) *that satisfies the properties (i)–(iii).*

The problem is thus reduced to finding suitable operators F, the requirements for which are more clear-cut than those for $\widetilde{T}$. Natural candidates for F are operators in the center of $\mathcal{U}\big(\mathfrak{sl}_2\big)$, where $\mathcal{U}\big(\mathfrak{sl}_2\big)$ denotes the universal enveloping algebra of $\mathfrak{sl}_2$. The center is the subspace consisting of those elements that commute with all of $\mathcal{U}\big(\mathfrak{sl}_2\big)$, or adequately with the elements of $\mathfrak{sl}_2$. This subspace is finitely generated by the Casimir element [4, 5, 6]. For a realization of $\mathfrak{sl}_2$ by means of a triplet h, e, f satisfying

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h,$$

the Casimir element has as explicit expression:

$$\Omega = h^2 + 2ef + 2fe = h^2 + 2h + 4fe. \tag{19}$$

Moreover, as the Casimir is a diagonal operator on the representation space, the condition of $\{\phi_{j,k,\ell}\}$ being eigenfunctions is immediately fulfilled for F an operator in the center of $\mathcal{U}\big(\mathfrak{sl}_2\big)$. Hence, the only remaining requirement on F is that its eigenvalues must be integers. To resolve this we resort to the notion of integer-valued polynomials, see [3].

Except for one particular case, this is a fruitful approach yielding a complete class of operator solutions. We elaborate on this special case in the next section. For the general study hereof we again refer to [3].

4. The one-dimensional case

We now consider the specific case where $m = 1$. Instead of dealing with vectors $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$, we now denote by x and y the single coordinate $x \equiv x_1$ and $y \equiv y_1$. In this case, the orthogonal group $O(1)$ becomes a finite group of order 2. The orthogonal symmetry of the FT then comes down to commuting with the sign inversion operator

$$\mathcal{F}\big[f(-x)\big](y) = \mathcal{F}\big[f(x)\big](-y).$$

Moreover for $m = 1$, the operators in our realization of $\mathfrak{sl}_2$ reduce to

$$\Delta_x := \frac{\mathrm{d}^2}{\mathrm{d}x^2}, \qquad |x|^2 := x^2, \qquad \mathbb{E}_x := x\frac{\mathrm{d}}{\mathrm{d}x}.$$

As the Laplace operator only has one term, the space of polynomials contained in its kernel is built up out of only two parts, namely the space $\mathcal{H}_0$ which is generated by 1, and $\mathcal{H}_1$ generated by x. The basis (9) then consists of

$$\phi_{j,0} = 2^j j!\, L_j^{-1/2}(x^2)\, \mathrm{e}^{-x^2/2}, \qquad \phi_{j,1} = 2^j j!\, L_j^{1/2}(x^2)\, x\, \mathrm{e}^{-x^2/2}, \tag{20}$$

for $j \in \mathbb{Z}_{\geq 0}$. Note that there is no index ℓ as the spaces $\mathcal{H}_0$ and $\mathcal{H}_1$ are one-dimensional. These are in fact the Hermite functions (up to a scalar factor), as the generalized Laguerre polynomials are related to the Hermite polynomials as follows

$$H_{2j}(x) = (-1)^j 2^{2j} j! L_j^{-1/2}(x^2), \quad H_{2j+1}(x) = (-1)^j 2^{2j+1} j! x L_j^{-1/2}(x^2).$$

Indeed, the basis (20) can be unified as

$$\psi_k(x) = \left(i\frac{\sqrt{2}}{2}\right)^k H_k(x)\, \mathrm{e}^{-x^2/2}, \tag{21}$$

where we have slightly adjusted the factor in front. The action of the FT on (20) is given by

$$\mathcal{F}(\phi_{j,0}) = (-1)^j \phi_{j,0}, \qquad \mathcal{F}(\phi_{j,1}) = (-1)^j i\, \phi_{j,1}.$$

On (21) this becomes

$$\mathcal{F}(\psi_k) = i^k \psi_k.$$

This in fact corresponds to unifying two representations of $\mathfrak{sl}_2$ as one representation of the Lie superalgebra $\mathfrak{osp}(1|2)$. Indeed, introducing the operators

$$b^+ = i\frac{\sqrt{2}}{2}\left(x - \frac{\mathrm{d}}{\mathrm{d}x}\right) \quad \text{and} \quad b^- = -i\frac{\sqrt{2}}{2}\left(x + \frac{\mathrm{d}}{\mathrm{d}x}\right), \tag{22}$$

they satisfy the relations

$$\big[\{b^-, b^+\}, b^\pm\big] = \pm 2 b^\pm,$$

where $\{a, b\} = ab + ba$ denotes the anti-commutator. Hence b^+ and b^- generate a realization of the Lie superalgebra $\mathfrak{osp}(1|2)$, [7]. The even (or "bosonic") elements of this algebra

$$h = \frac{1}{2}\{b^-, b^+\}, \qquad e = \frac{1}{4}\{b^+, b^+\}, \qquad f = -\frac{1}{4}\{b^-, b^-\}.$$

generate an even subalgebra isomorphic with $\mathfrak{sl}_2$. Working out the anti-commutators, these are precisely the operators (11) we considered earlier. For instance, using (22),

$$2h = \{b^-, b^+\} = \frac{1}{2}\left(x + \frac{\mathrm{d}}{\mathrm{d}x}\right)\left(x - \frac{\mathrm{d}}{\mathrm{d}x}\right) + \frac{1}{2}\left(x - \frac{\mathrm{d}}{\mathrm{d}x}\right)\left(x + \frac{\mathrm{d}}{\mathrm{d}x}\right) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2$$

The action on the eigenfunctions is as follows

$$b^+\,\psi_k = \psi_{k+1}, \qquad b^-\,\psi_k = k\,\psi_{k-1},$$

while

$$h\,\psi_k = (k + \tfrac{1}{2})\,\psi_k, \quad e\,\psi_k = \frac{1}{2}\,\psi_{k+2}, \quad f\,\psi_k = -\frac{1}{2}k\,(k-1)\,\psi_{k-2}.$$

In line with (10), we can write

$$\psi_k = (b^+)^k \psi_0$$

Now, for the basis functions (20) we have

$$h\,\phi_{j,0} = \left(2j + \tfrac{1}{2}\right)\phi_{j,0}, \qquad h\,\phi_{j,1} = \left(2j + \tfrac{3}{2}\right)\phi_{j,1}.$$

The sets $\{\, \phi_{j,0} \mid j \in \mathbb{Z}_{\geq 0} \,\}$ and $\{\, \phi_{j,1} \mid j \in \mathbb{Z}_{\geq 0} \,\}$ thus form a basis for the positive discrete series representation of $\mathfrak{sl}_2$ with lowest weights $1/2$ and $3/2$ respectively. Together they span the same space as $\{\, \psi_k \mid k \in \mathbb{Z}_{\geq 0} \,\}$. This last set forms a basis for the irreducible representation of the Lie superalgebra $\mathfrak{osp}(1|2)$ with lowest weight $1/2$.

Returning to the objective we had in mind we look at the Casimir element of $\mathfrak{sl}_2$ for $m = 1$. However, this element turns out to be a scalar constant as it is given by

$$\Omega_1 = h^2 + 2h + 4fe = -\frac{3}{4}.$$

Hence, looking back at Proposition 2, an operator F in the center of $\mathcal{U}(\mathfrak{sl}_2)$ will, by relation (18), yield the classical FT up to a multiplicative scalar factor.

In this case there is another set we can consider, as besides $\mathfrak{sl}_2$ we also have a realization of $\mathfrak{osp}(1|2)$. The center of $\mathcal{U}(\mathfrak{osp}(1|2))$ is finitely generated by another Casimir element:

$$C = \frac{1}{4} + \frac{1}{2}\left[b^-, b^+\right] + h^2 + 2h + 4fe.$$

We see that C differs from the Casimir element of $\mathfrak{sl}_2$, given by (19), by an additional term. This extra term is related to the Scasimir element (see [4])

$$S = \frac{1}{2}(b^-b^+ - b^+b^- - 1),$$

which is a square root of the Casimir operator. Using (22), for our realization of $\mathfrak{osp}(1|2)$ this element becomes

$$S = \frac{1}{4}\left(x + \frac{\mathrm{d}}{\mathrm{d}x}\right)\left(x - \frac{\mathrm{d}}{\mathrm{d}x}\right) - \frac{1}{4}\left(x - \frac{\mathrm{d}}{\mathrm{d}x}\right)\left(x + \frac{\mathrm{d}}{\mathrm{d}x}\right) - \frac{1}{2} = \frac{1}{4} + \frac{1}{4} - \frac{1}{2} = 0,$$

and hence $C = S^2 = 0$. In this way we again find no solutions other than the classical FT.

These results can also be explained in another fashion. In the one-dimensional case, the requirement for the kernel of an integral transform in order to satisfy the Helmholtz relations, (8), reduces to

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}K + x^2\,K = 0.$$

This differential equation has as general solution

$$K = C_1 \mathrm{e}^{ixy} + C_2 \mathrm{e}^{-ixy}, \qquad C_1, C_2 \in \mathbb{C}$$

where we recognize the kernel of the Fourier transform and that of its inverse. For $m = 1$, the Helmholtz relations thus suffice to uniquely characterize the FT or its inverse.

In dimension $m \geq 2$, the Casimir element Ω_m becomes

$$\Omega_m = h^2 + 2h + 4fe = \left(\mathbb{E}_x + \tfrac{m-2}{2}\right)^2 - |x|^2 \Delta_x - 1.$$

As in this case Ω_m does not reduce to a trivial operator, by Proposition 2 we do find solutions that differ from the FT or its inverse. We again refer to [3] for a detailed study.

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