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ON THE CONSTRUCTION OF RIEMANNIAN METRICS FOR BERWALD SPACES BY AVERAGING

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Abstract. The construction of Riemannian metrics on the base manifold of any given Finsler space by averaging suitable objects over indicatrices, such that the Levi-Civita connection of the metric coincides with the canonical Berwald connection of the Finsler space when the Finsler space is a Berwald space, is discussed. Some examples of such metrics are already known, but several new ones, all in principle different, are defined and analysed.

1. Introduction

Szabó's celebrated analysis of Berwald spaces starts from the existence of a Riemannian metric whose Levi-Civita connection coincides with the symmetric linear connection given by the Berwald structure. Known methods of constructing such metrics apply to any Finsler space. Consider a Finsler space over a manifold M with Finsler function F: we require a way of constructing a Riemannian metric on M out of F. One approach, due to Vincze [8], is to average the fundamental tensor g of F over each indicatrix $\{y \in T_xM : F(x,y) = 1\}$ with respect to the volume form induced from g. In [2], and more recently in [4, 5, 7], further methods of associating Riemannian metrics with Finsler functions by averaging have been introduced. The metrics obtained in [2, 4, 5, 7, 8], of which there are actually three distinct ones, satisfy the required property when the space is Berwald. The question arises, are these the only ways of constructing a Riemannian metric for a Finsler function by averaging, such that when the Finsler space is a Berwald space

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its connection is the Levi-Civita connection of the metric? As a further desideratum, when the Finsler function is actually the length function of a Riemannian metric the construction should actually return that metric.

It turns out that there are surprisingly many ways of producing such a metric by averaging. In this note I shall discuss the general procedures involved, and give eight specific conceptually distinct examples; these include the three mentioned above but the rest are new. I do not claim that my list is exhaustive, only that it contains an interesting and instructive sample. The proof I give of the Berwald property is systematic, in the sense that it reduces to showing that the functions to be averaged all satisfy the same simple condition.

A Finsler structure is defined on the slit tangent bundle $T^\circ M$ of a manifold M. The averaging process takes place fibre by fibre, so that in the first place one may concentrate one's attention on an individual fibre $T^\circ_x M$, $x \in M$, of $T^\circ M \to M$. This is a Minkowski space, with Minkowski norm F_x determined by restriction of the Finsler function F. I begin therefore, in Section 2, by considering averaging in Minkowski spaces. The eight specific examples mentioned above are all described initially in this section. In Section 3 I extend the constructions to Finsler spaces, and prove the main result, that each of the metrics obtained satisfies the required condition when the space is a Berwald space. Section 4 contains some observations about the relation between the volume form induced on M by any of these Riemannian metrics and the canonically defined Busemann-Hausdorff form. There are some concluding remarks in Section 5.

References [4] and [5] mentioned above are concerned in part with the possiblity of generalizing the concept of a Finsler function by relaxing the strict convexity requirement and allowing the function to be only partially smooth in a certain sense. These generalizations are taken up in the specific case of a Berwald space in [7], where the notion of a Berwald-Matveev space is introduced. These developments are clearly interesting and important, but I have not attempted to pursue them here: so I emphasise that this paper is conerned entirely with smooth strongly convex y-global Finsler functions.

2. Averaging on Minkowski spaces

Consider a Minkowski norm F on $\mathbf{R}^n$. Denote the natural coordinates by (y^i), and set $d^n y = dy^1 \wedge dy^2 \wedge \cdots \wedge dy^n$. Set $\mathbf{R}^n_\circ = \mathbf{R}^n - \{0\}$. With $r > 0$, denote by $B(r)$ the closed ball of radius r determined by F: $B(r) = \{y \in \mathbf{R}^n_\circ : F(y) \leq r\}$; and by $S(r)$ the sphere: $S(r) = \{y \in \mathbf{R}^n_\circ : F(y) = r\}$. Set $B(1) = B$, $S(1) = S$.

Denote by Δ the Liouville vector field:

$$\Delta = y^i \frac{\partial}{\partial y^i}.$$

There is a Riemannian metric g on $\mathbf{R}^n_\circ$ canonically associated with F, given in terms of the natural coordinates by

$$g_{ij} = F\frac{\partial^2 F}{\partial y^i \partial y^j} + \frac{\partial F}{\partial y^i}\frac{\partial F}{\partial y^j} = \frac{\partial^2 E}{\partial y^i \partial y^j}$$

where $E = \frac{1}{2}F^2$ is the energy of F. If we allow linear transformations of coordinates then the g_{ij} transform tensorially in the manner indicated by the position of the indices; the formula defines what may be called a linear tensor. Let $\Omega = \sqrt{\det g}\, d^n y$ be the volume form on $\mathbf{R}^n_\circ$ defined by g. Then Δ is the unit outward normal field on S, and $\omega = \Delta \lrcorner\, \Omega$, pulled back to S, is the volume form on S induced by g. (Actually one should be a little careful here to take account of orientation. I mean that Ω is the unique n-form whose expression in terms of natural coordinates is that just given. If $\tilde{y}^i$ are new linear coordinates then $\Omega = \pm\sqrt{\det \tilde{g}}\, d^n \tilde{y}$ where $\tilde{g}$ is the transformed tensor, and one takes the plus sign if the transformation is orientation-preserving, the minus sign if it is orientation-reversing. However, this nicety will turn out to have no real significance.) We have $\omega = \sqrt{\det g}\, \lambda$ where $\lambda = \Delta \lrcorner\, d^n y$. In [1] volumes calculated with respect to $d^n y$ or λ are referred to as Euclidean, those calculated with respect to Ω or ω as Riemannian: I mention this as a way of drawing attention to the distinction, though I shan't use this terminology here.

For any function f on S define averages of f over S, with respect to the volume forms ω and λ, by

$$\langle f \rangle_\omega = \frac{\int_S f\omega}{\int_S \omega}, \qquad \langle f \rangle_\lambda = \frac{\int_S f\lambda}{\int_S \lambda}.$$

Observe that $\langle f \rangle_\omega$ is coordinate-independent, that is, unchanged under a linear change of coordinates: this is evidently the case if the transformation is orientation-preserving, while both terms change sign if the transformation is orientation-reversing. The same is true of $\langle f \rangle_\lambda$: the numerator and denominator both change, but by the same constant factor (the determinant of the coordinate transformation). One can easily express either average in terms of the other: for example

$$\langle f \rangle_\lambda = \frac{\langle f/\sqrt{\det g}\rangle_\omega}{\langle 1/\sqrt{\det g}\rangle_\omega}$$

In the literature averages are often defined instead in terms of integrals over the unit ball B. If f is actually the restriction to S of a function on $\mathbf{R}^n_\circ$ (also denoted by f) which is homogeneous then the two definitons are related in a simple way, as I now show. I first require a lemma.

Lemma 2.1. *Let f be a C^∞ function on $\mathbf{R}^n_\circ$ which is homogeneous of degree $k \geq 0$. Then $\int_B f d^n y$ is well-defined, and*

$$\int_S f\lambda = (n+k)\int_B f d^n y.$$

(This is a slight generalization of a result which appears in [7].)

Proof. I show first that $d(f\lambda) = (n+k) f d^n y$. We have $\Delta f = kf$, and so $\mathcal{L}_\Delta(f d^n y) = (n+k) d^n y$; the result follows immediately from the homotopy formula for the Lie derivative. By Stokes's Theorem, for small positive ϵ

$$\int_S f\lambda - \int_{S(\epsilon)} f\lambda = (n+k)\int_{B-B(\epsilon)} f d^n y.$$

But

$$\lim_{\epsilon\to 0}\int_{S(\epsilon)} f\lambda = \lim_{\epsilon\to 0}\int_{S(\epsilon)} f\Delta \lrcorner\, d^n y = \lim_{\epsilon\to 0}\epsilon^{n+k}\int_S f\Delta\lrcorner\, d^n y = 0.$$

So we may set

$$\int_B f d^n y = \lim_{\epsilon\to 0}\int_{B-B(\epsilon)} f d^n y$$

and the result follows. □

Corollary 2.2. *With f as before,*

$$\int_S f\omega = (n+k)\int_B f\Omega.$$

Proof. The function $\sqrt{\det g}$ is homogeneous of degree 0. □

Proposition 2.3. *With f as before,*

$$\langle f\rangle_\omega = \frac{n+k}{n}\frac{\int_B f\Omega}{\int_B \Omega}, \qquad \langle f\rangle_\lambda = \frac{n+k}{n}\frac{\int_B f d^n y}{\int_B d^n y}.$$

Proof. These results follow directly from the lemma above and its corollary. □

These averaging processes may be extended to (smooth) linear-tensor fields on $\mathbf{R}^n_\circ$. Consider, for example, a tensor field a of type $(0,2)$. Let a_{ij} be its components with respect to natural coordinates, and set $\bar{a}_{ij} = \langle a_{ij}\rangle_\omega$. Then $\bar{a}$ transforms tensorially under a linear change of coordinates, and therefore defines an element of $\mathbf{R}^{n*}\otimes\mathbf{R}^{n*}$, or in other words a bilinear form on $\mathbf{R}^n$, such that for any $u, v \in \mathbf{R}^n$, $\bar{a}(u,v) = \langle a(u,v)\rangle_\omega$. For convenience I shall write (in this case) $\bar{a} = \langle a\rangle_\omega$.

(Strictly speaking it is necessary only that a be defined on S, that is, that it be a tensor field along the injection $S \to \mathbf{R}^n_\circ$. However, in all cases of interest a will be a linear-tensor field on $\mathbf{R}^n_\circ$, which is homogeneous in the sense that its components with respect to natural coordinates are homogeneous functions of some common degree k, so that their values are determined everywhere from their values on S.)

Proposition 2.4. *Let a be a smooth type $(0,2)$ tensor field on $\mathbf{R}^n_\circ$, and set $\bar{a} = \langle a\rangle_\omega$ or $\langle a\rangle_\lambda$. Then*

(1) *If a is symmetric so is $\bar{a}$.*
(2) *If a is itself constant (that is, if it is an element of $\mathbf{R}^{n*} \otimes \mathbf{R}^{n*}$ considered as a linear-tensor field on $\mathbf{R}^n_\circ$) then $\bar{a} = a$.*
(3) *Suppose that for every non-zero $u \in \mathbf{R}^n$, $a(u,u)$ is non-negative on S, and non-vanishing at some point of S: then $\bar{a}$ is positive definite.*

If a is instead a type $(2,0)$ tensor field then the same results hold, mutatis mutandis.

Proof. Only item 3 is not immediately obvious. For any $u \in \mathbf{R}^n$ we have $\langle a(u,u)\rangle_\omega \geq 0$. But since for $u \neq 0$, $a(u,u) > 0$ at some point of S, and therefore in some neighbourhood of that point, in fact $\langle a(u,u)\rangle_\omega > 0$. The same argument works with λ in place of ω. □

This proposition, applied fibrewise, with different choices of a, provides the foundation for numerous methods of constructing a Riemannian metric on the base manifold of a Finsler space by averaging over the indicatrix in each fibre. I give below a list of possibilities, some of which have appeared in the literature, others of which are variations on methods that have appeared in the literature, and yet others are to the best of my knowledge completely new. Here we are concerned just with the construction in a Minkowski space, which will be a typical fibre in the application to Finsler spaces in the next section. The construction leads to a bilinear form on $\mathbf{R}^n$ which is symmetric and positive definite. In each case I denote this bilinear form by $\bar{g}$ and its components (with respect to the standard basis) by $\bar{g}_{ij}$. In certain cases apparently extraneous numerical factors appear: these have been introduced for later convenience. Note that in each case condition 3 of Proposition 2.4 is satisfied, so that $\bar{g}$ is indeed positive definite.

(1)
$$\bar{g}_{ij} = \langle g_{ij}\rangle_\lambda.$$

This construction, apart from some minor details, was introduced by Matveev et al. in [4], and appears also in the recent paper [7] by Szilasi et al. In both of these papers the term $\int_S \lambda$ is omitted from the definition of the average, but instead the construction begins with a volume n-form $c\,d^n y$, with constant c, normalized so that $\int_B c\,d^n y = 1$. As a result the bilinear form defined in these papers differs from mine by a numerical factor.

(2)

$$\bar{g}_{ij} = \langle g_{ij} \rangle_\omega.$$

This is Vincze's original specification, in [8], of a metric associated with a Finsler space obtained by averaging over the indicatrix. It has also been discussed in [3].

(3)

$$\bar{g}^{ij} = n \langle y^i y^j \rangle_\lambda.$$

Note that in this case the average leads to the contravariant form, so that, as a matrix, $\bar{g}$ is the inverse of the matrix whose components are given above. To the best of my knowledge this definition was first given by Centore in [2], albeit in terms of integrals over B; for exact comparison with his definition it is necessary to observe that in this case the argument is homogeneous of degree 2. Centore calls $\bar{g}$ the osculating metric. An equivalent definition appears in a recent paper by Matveev et al., [5], where the ellipsoid defined by the averaged metric is named the Binet-Legendre ellipsoid.

(4)

$$\bar{g}^{ij} = n \langle y^i y^j \rangle_\omega.$$

This is an obvious alternative to item 3. This example has not already appeared in the literature so far as I know, nor have any of those following.

(5)

$$\bar{g}_{ij} = n \langle y_i y_j \rangle_\lambda,$$

where

$$y_i = g_{ij} y^j = F \frac{\partial F}{\partial y^i} = \frac{\partial E}{\partial y^i}.$$

(6)

$$\bar{g}_{ij} = n \langle y_i y_j \rangle_\omega.$$

This example is dual to that of item 4, in the following sense. The map $\mathfrak{L} : \mathbf{R}^n_\circ \to \mathbf{R}^{n*}_\circ$ where $\mathfrak{L}(y^i) = (g_{ij} y^j) = (y_i)$ is the Legendre transformation associated with the Lagrangian E. It is a diffeomorphism, with $\partial \mathfrak{L}_i / \partial y^j =$

g_{ij}; its inverse is $\mathfrak{L}^{-1}(p_i) = (g^{ij}p_j)$. The function $F^* = F \circ \mathfrak{L}^{-1}$ is a Minkowski norm on $\mathbf{R}^{n*}_\circ$, whose associated metric is (g^{ij}) and associated volume form is $\Omega^* = (1/\sqrt{\det g})d^n p$. (For the details see [1] Section 14.8.) The construction of item 4 applied to these data on $\mathbf{R}^{n*}_\circ$ leads to an average $\bar{g}$ on $\mathbf{R}^{n*}_\circ$ where $\bar{g}_{ij} \circ \mathfrak{L} = n\langle y_i y_j \rangle_\omega$.

(7)
$$\bar{g}_{ij} = \frac{n}{n-1}\langle g_{ij} - y_i y_j \rangle_\lambda.$$

The argument here is the restriction to S of the component h_{ij} of the angular metric h. In order to express $\bar{g}$ as an average over B one needs to remember that the general formula for h_{ij} is $h_{ij} = g_{ij} - y_i y_j / F^2$ (both terms of which are homogeneous of degree zero).

(8)
$$\bar{g}_{ij} = \frac{n}{n-1}\langle h_{ij} \rangle_\omega.$$

There is a certain, probably spurious, elegance in this formula coming from the fact that ω is the volume form on S induced by h considered as a metric on S.

I now show that for each of these examples, if g is actually constant, so that $F(y) = \sqrt{g_{ij}y^i y^j}$ is effectively Euclidean, then $\bar{g} = g$.

Proposition 2.5. *If g is constant then $\bar{g} = g$.*

Proof. This is obvious for examples 1 and 2. We may assume without loss of generality that F is Euclidean with respect to the coordinates (y^i), that is, that $g_{ij} = \delta_{ij}$. Then $\lambda = \omega$, so it is enough to consider just one of each of the remaining pairs. For example 4 we take advantage of the fact that both S (which is just the Euclidean unit sphere) and ω are invariant under proper orthogonal transformations. Using this observation for the transformation, for any i, j with $i < j$,

$$(y^1, y^2, \dots, y^n) \mapsto (y^1 \dots, -y^j, \dots, y^i, \dots, y^n)$$

we see that

$$\int_S (y^i)^2 \omega = \int_S (y^j)^2 \omega = \frac{1}{n}\int_S \sum_{i=1}^n (y^i)^2 \omega = \frac{1}{n}\int_S \omega.$$

On the other hand, using the transformation

$$(y^1, y^2, \dots, y^n) \mapsto \left(y^1 \dots, \frac{1}{\sqrt{2}}(y^i - y^j), \dots, \frac{1}{\sqrt{2}}(y^i + y^j), \dots, y^n\right)$$

we see that for $i \neq j$

$$\int_S (y^i y^j)\omega = 0.$$

It follows that $\bar{g}^{ij} = \delta^{ij}$, as required. Now

$$\int_S (y_i y_j)\omega = \delta_{ik}\delta_{jl}\int_S (y^k y^l)\omega = \delta_{ik}\delta_{jl}\delta^{kl} = \delta_{ij},$$

which deals with examples 6 and 5. Finally, we have

$$\langle \delta_{ij} - y_i y_j \rangle_\omega = \delta_{ij} - \frac{1}{n}\delta_{ij} = \frac{n-1}{n}\delta_{ij},$$

which proves the result for the final two examples. □

The proof above for item 4 is adapted from the one given in [5]. There is a quite different proof, which is perhaps more elegant, but less straightforward, in [2].

3. Constructing metrics for Finsler spaces by averaging

Now consider a Finsler space over a manifold M of dimension n; let F be its Finsler function, defined on $T^\circ M$, its slit tangent bundle. Given any coordinate system (x^i) about $x \in M$, we can identify $T_x M$ with $\mathbf{R}^n$ by means of the canonical fibre coordinates (y^i). Let $S_x \subset T^\circ_x M$ be the indicatrix at x, that is, the unit sphere of the Minkowski norm F_x defined on $T^\circ_x M$ by F; let $\mathcal{S}$ be the indicatrix bundle, the submanifold of $T^\circ M$ on which F takes the value 1, whose fibre over x is S_x. The quantities

$$g_{ij} = \frac{\partial^2 E_x}{\partial y^i \partial y^j}$$

are usually regarded as the components of a tensor along the projection $\tau : T^\circ M \to M$. But they also determine a Riemannian metric on each fibre, where the components of the metric g_x on $T^\circ_x M$ with respect to the fibre coordinates are $g_{ij}(x, y)$. Let ω_x be the volume form on S_x induced by g_x (that is, induced by the volume form on $T^\circ_x M$ determined by g_x). One can carry out the constructions of the previous section pointwise over M, to obtain eight possibly different Riemannian metrics $\bar{g}$ on M by averaging over indicatrices.

Recall that the canonical horizontal distribution of a Finsler space is spanned by local vector fields

$$H_i = \frac{\partial}{\partial x^i} - \Gamma^j_i \frac{\partial}{\partial y^j},$$

the horizontal lifts of the coordinate fields on M, where the coefficients Γ^i_j are given in terms of the connection coefficients Γ^i_{jk} of the canonical Berwald connection by

$$\Gamma^i_j = \Gamma^i_{jk} y^k, \quad \Gamma^i_{jk} = \frac{\partial \Gamma^i_j}{\partial y^k} = \Gamma^i_{kj}.$$

The Finsler function F is constant along horizontal curves, so the horizontal distribution is tangent to $\mathcal{S}$.

A Finsler space is a Landsberg space if g satisfies

$$H_k(g_{ij}) = g_{il}\Gamma^l_{jk} + g_{jl}\Gamma^l_{ik};$$

that is to say, g is covariant constant along horizontal curves with respect to the Berwald connection. In other words, in a Landsberg space, parallel transport (with respect to the Berwald connection) along horizontal curves is an isometry of fibres of $T^\circ M$, regarded as Riemannian manifolds. A Finsler space is a Berwald space if its canonical Berwald connection is actually a connection on M, that is, if the connection coefficients are local functions on M. It is known that every Berwald space is a Landsberg space (see for example [6]).

Proposition 3.1. *In a Landsberg space, for any function f on $T^\circ M$*

$$\frac{\partial}{\partial x^i}\langle f\rangle_\omega = \langle H_i f\rangle_\omega,$$

where $\langle f\rangle_\omega$ is the function on M whose value at x is $\langle f_x\rangle_{\omega_x}$.

Proof. Take a curve c in M with $c(0) = x$, $\dot c(0) = v \in T_xM$, and let $t \mapsto c^{\mathrm{H}}(t, y)$ be the horizontal lift of c through $y \in T^\circ_x M$. For t in the domain of c define a map $\rho(t) : T^\circ_x M \to T^o_{c(t)} M$ by $\rho(t)(y) = c^{\mathrm{H}}(t, y)$: then since the space is Landsberg, $\rho(t)$ maps S_x isometrically onto $S_{c(t)}$. Thus in particular $\rho(t)$ is volume preserving: $\rho(t)^*\omega_{c(t)} = \omega_x$. It follows that

$$\int_{S_{c(t)}} f\,\omega_{c(t)} = \int_{\rho(t)(S_x)} f\,\omega_{c(t)} = \int_{S_x} \rho(t)^*(f\,\omega_{c(t)}) = \int_{S_x} \rho(t)^*(f)\,\omega_x.$$

Moreover, $\int_{S_{c(t)}} \omega_{c(t)} = \int_{S_x} \omega_x$. Thus

$$\langle f\rangle_{\omega_{c(t)}} = \langle \rho(t)^*(f)\rangle_{\omega_x} = \langle f \circ c^{\mathrm{H}}(t, \cdot)\rangle_{\omega_x}.$$

On differentiating with respect to t at $t = 0$ we obtain

$$v\langle f\rangle_\omega = \langle v^{\mathrm{H}} f\rangle_{\omega_x},$$

which is equivalent to the stated result. □

(This proof is taken from [3].)

Corollary 3.2. *In a Berwald space*

$$\frac{\partial}{\partial x^i}\langle f\rangle_\lambda = \langle H_i f\rangle_\lambda.$$

Proof. I show first that in a Landsberg space

$$\frac{\partial}{\partial x^i}\langle f\rangle_\lambda = \langle H_i f\rangle_\lambda - \langle f\Gamma_i\rangle_\lambda + \langle f\rangle_\lambda\langle \Gamma_i\rangle_\lambda$$

where $\Gamma_i = \Gamma^j_{ji}$. Recall that

$$\langle f\rangle_\lambda = \frac{\langle f/\sqrt{\det g}\rangle_\omega}{\langle 1/\sqrt{\det g}\rangle_\omega}$$

We can obtain an expression for $\partial\langle f\rangle_\lambda/\partial x^i$ by differentiating the quotient on the right and applying the result of the proposition. For this we need to evaluate $H_i(1/\sqrt{\det g})$. In a Landsberg space $H_i(\det g) = 2\Gamma_i \det g$, by a calculation almost identical to the one which leads to the corresponding result in the Riemannian case. Thus $H_i(1/\sqrt{\det g}) = -\Gamma_i/\sqrt{\det g}$. One finds that

$$\begin{aligned}\frac{\partial}{\partial x^i}\langle f\rangle_\lambda &= \frac{\langle H_i(f/\sqrt{\det g})\rangle_\omega\langle 1/\sqrt{\det g}\rangle_\omega - \langle f/\sqrt{\det g}\rangle_\omega\langle H_i(1/\sqrt{\det g})\rangle_\omega}{\langle 1/\sqrt{\det g}\rangle_\omega^2}\\ &= \frac{\langle H_i f/\sqrt{\det g}\rangle_\omega}{\langle 1/\sqrt{\det g}\rangle_\omega} - \frac{\langle f\Gamma_i/\sqrt{\det g}\rangle_\omega}{\langle 1/\sqrt{\det g}\rangle_\omega} + \frac{\langle f/\sqrt{\det g}\rangle_\omega}{\langle 1/\sqrt{\det g}\rangle_\omega}\frac{\langle \Gamma_i/\sqrt{\det g})\rangle_\omega}{\langle 1/\sqrt{\det g}\rangle_\omega}\\ &= \langle H_i f\rangle_\lambda - \langle f\Gamma_i\rangle_\lambda + \langle f\rangle_\lambda\langle\Gamma_i\rangle_\lambda\end{aligned}$$

as claimed. But in a Berwald space

$$\langle f\Gamma_i\rangle_\lambda - \langle f\rangle_\lambda\langle\Gamma_i\rangle_\lambda = \langle f\rangle_\lambda\Gamma_i - \langle f\rangle_\lambda\Gamma_i = 0.$$

□

For any symmetric connection on M I denote the covariant derivative, in component form, with a semi-colon. For any Berwald connection on $T^\circ M$ I denote the covariant derivative (of a tensor along τ) with a solidus. When dealing with a Berwald space I assume of course that the two connections are the same, even if presented in different guises.

Lemma 3.3. *Let a be any type $(0,2)$ tensor field along τ, $\bar{a}$ the type $(0,2)$ tensor field on M obtained by averaging over indicatrices with respect either to ω or to λ. If the space is a Berwald space then*

$$\bar{a}_{ij;k} = \langle a_{ij|k}\rangle$$

where the average on the right is taken with respect to ω or λ as appropriate. A similar result holds for any type $(2,0)$ tensor field along τ.

Proof. I give the proof for a type $(0,2)$ field using the ω average. The other results are obtained by similar means, taking into account Corollary 3.2. We have

$$a_{ij|k} = H_k(a_{ij}) + a_{lj}\Gamma^l_{ik} + a_{il}\Gamma^l_{jk}.$$

It follows, using the assumption that the space is Berwald, that

$$\begin{aligned}\langle a_{ij|k}\rangle_\omega &= \langle H_k(a_{ij})\rangle_\omega + \langle a_{lj}\rangle_\omega\Gamma^l_{ik} + \langle a_{il}\rangle_\omega\Gamma^l_{jk}\\ &= \frac{\partial \bar a_{ij}}{\partial x^k} + \bar a_{lj}\Gamma^l_{ik} + \bar a_{il}\Gamma^l_{jk}\\ &= \bar a_{ij;k}.\end{aligned}$$

□

For what it is worth, an analogous formula evidently holds for any tensor field along τ, whatever its type.

A symmetric connection on M is the Levi-Civita connection of a metric $\bar g$ if and only if $\bar g_{ij;k} = 0$, or equivalently $\bar g^{ij}{}_{;k} = 0$.

Theorem 3.4. *In a Berwald space, for each of the eight Riemannian metrics $\bar g$ defined in the previous section, the associated Levi-Civita connection coincides with the canonical Berwald connection.*

Proof. From the lemma, it is enough to show that for each of the three tensors

$$a_{ij} = \begin{cases} g_{ij} \\ y_i y_j \\ h_{ij}\end{cases}$$

$a_{ij|k} = 0$, while for

$$a^{ij} = y^i y^j,$$

$a^{ij}{}_{|k} = 0$. In the case $a_{ij} = g_{ij}$, this is true for a Landsberg space, and so a fortiori for a Berwald space. For the case $a^{ij} = y^i y^j$ we see first that $H_k(y^i) = -\Gamma^i_{kl} y^l$, so that

$$\begin{aligned}a^{ij}{}_{|k} &= H_k(a^{ij}) + a^{lj}\Gamma^i_{lk} + a^{il}\Gamma^j_{lk}\\ &= -\Gamma^i_{kl}y^l y^j - y^i\Gamma^j_{kl}y^l + y^l y^j\Gamma^i_{lk} + y^i y^l\Gamma^j_{lk}\\ &= 0.\end{aligned}$$

For $a_{ij} = y_i y_j$ we use the fact that $a_{ij} = g_{il}g_{jm}y^l y^m$: the result then follows from the previous case, given that $g_{il|k} = g_{jm|k} = 0$. The case $a_{ij} = h_{ij}$ then follows from the first and third cases. □

The case $a_{ij} = g_{ij}$ with volume ω gives Vincze's proof of Szabó's theorem [8], the case $a_{ij} = g_{ij}$ with volume λ is proved in [7], and the case $a^{ij} = y^i y^j$ with volume λ is proved in [5]. The remaining results are new.

4. Volume invariants

For any Finsler space there is a canonical volume n-form on M, called the Busemann-Hausdorff form: its expression in terms of coordinates (x^i) on M is

$$\left(\frac{\sigma_n}{\int_S \lambda}\right) d^n x,$$

where σ_n is a numerical factor, the volume of the Euclidean unit n-sphere, whose presence ensures that when F comes from a Riemannian metric the Busemann-Hausdorff form coincides with the induced volume form. It is easy to see that this is the coordinate expression of a well-defined nowhere-vanishing n-form. In fact the Busemann-Hausdorff form is usually defined in terms of unit balls rather than unit spheres (see [2, 5, 6]), but the formulation above is equivalent and more in keeping with the approach in the present paper.

On the other hand, any of the Riemannian metrics $\bar{g}$ defined earlier leads to a volume form in the usual way, namely $\sqrt{\det \bar{g}}\, d^n x$. In [2], Centore calls the non-vanishing function which is the ratio of the second of these volume forms to the first, in the case $\bar{g}^{ij} = n\langle y^i y^j \rangle_\lambda$ which he discusses, the volume invariant. He shows, by an argument involving the use of normal coordinates, that in the case of a Berwald space this volume invariant is a constant.

Evidently a volume invariant may be defined for each of the metrics $\bar{g}$. I show, by a direct calculation, that in a Berwald space each of them is constant, thus both extending Centore's result and simplifying its proof.

Theorem 4.1. *In a Berwald space, for each of the metrics $\bar{g}$ defined earlier the volume invariant is constant.*

Proof. Apart from the constant factor σ_n the volume invariant is

$$\sqrt{\det \bar{g}} \int_S \lambda.$$

Since $\bar{g}$ has the canonical Berwald connection as its Levi-Civita connection,

$$\frac{\partial \sqrt{\det \bar{g}}}{\partial x^i} = \Gamma_i \sqrt{\det \bar{g}}.$$

On the other hand,

$$\int_S \lambda = \int_S \frac{1}{\sqrt{\det g}}\, \omega,$$

and by slight modifications of the arguments in the proofs of Proposition 3.1 and Corollary 3.2 we obtain

$$\frac{\partial}{\partial x^i}\left(\int_S \lambda\right) = \int_S H_i\left(\frac{1}{\sqrt{\det g}}\right)\omega = -\int_S \Gamma_i\lambda = -\Gamma_i\int_S \lambda.$$

Thus

$$\frac{\partial}{\partial x^i}\left(\sqrt{\det \bar{g}}\int_S \lambda\right) = \Gamma_i\sqrt{\det \bar{g}}\int_S \lambda - \Gamma_i\sqrt{\det \bar{g}}\int_S \lambda = 0,$$

and the volume invariant is constant. □

Incidentally there is another volume form on M, called the Holmes-Thompson form ([6] Section 5.1), given by

$$\left(\frac{\int_S(\det g)\lambda}{\sigma_n}\right)d^n x;$$

the volume invariant between this and the Busemann-Hausdorff form, which can be written

$$\int_S \sqrt{\det g}\,\omega\int_S \frac{1}{\sqrt{\det g}}\,\omega,$$

is easily shown to be constant in a Berwald space by a similar argument to that used in the proof of the theorem above.

5. Concluding remarks

We appear to have an embarras de richesses, if all these metrics are distinct. Indeed, the situation is actually worse than this, because I have dealt only with metrics defined by averaging. There are yet other means of constructing metrics: for example, in each fibre there is a unique ellipsoid, centred at the origin, of least volume (calculated with respect to the volume form $c\,d^n y$ for which B has unit volume) which contains B, the so-called Loewner ellipsoid of B. The metric so defined is discussed in [7], where it is shown that, again, in a Berwald space its Levi-Civita connection is the canonical Berwald connection.

The freedom in choice of Riemannian metrics for which a given symmetric linear connection is the Levi-Civita connection (assuming there are any) is determined by the holonomy of the connection, via de Rham's decomposition theorem. In particular (as pointed out in [7]) if the holonomy group acts irreducibly then all such metrics are homothetic. In [7] this observation is used to show that under such circumstances the metric obtained by the Loewner ellipsoid construction is homothetic to the one obtained in item 1 above. But in fact it applies to all of the metrics discussed in the present paper, all of which must then be homothetic to each other.

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