UNIVERSITEIT GENT

Hogere spin operatoren in cliffordanalyse

Higher spin operators in Clifford analysis

Tim Raeymaekers

Promotoren:
Prof. Dr. David Eelbode (UA)
Prof. Dr. Hennie De Schepper (UGent)

Proefschrift voorgelegd aan de Faculteit Wetenschappen van de Universiteit Gent tot het behalen van de graad van Doctor in de Wetenschappen: Wiskunde.

Universiteit Gent
Faculteit Ingenieurswetenschappen en Architectuur
Vakgroep Wiskundige Analyse - Onderzoekseenheid Cliffordanalyse
Academiejaar 2014-2015

I can no other answer make, but, thanks, and thanks.
William Shakespeare

Dankwoord

Vooraleer ik begin met dit doctoraatsproefschrift wil ik eerst een aantal mensen bedanken zonder wie dit werk nooit tot stand was gekomen.

In de eerste plaats ben ik mijn promotor David Eelbode bijzonder dankbaar. Onder de begeleiding van David mocht ik tijdens het schrijven van mijn masterthesis al tip van de enorme sluier lichten waarachter de wereld van cliffordanalyse verborgen zat. Voornamelijk onder zijn deskundige begeleiding is dit werk geworden tot wat het nu is. Ondanks zijn drukke schema was er altijd wel eens tijd voor een bezoekje aan Antwerpen of een skypegesprek.

Ook zonder Hennie De Schepper, mijn andere promotor, was het niet gelukt. Zij was altijd beschikbaar voor een motiverend babbeltje. Ook op gebied van onderwijs heb ik enorm veel van haar geleerd. Bedankt voor het eindeloze vertrouwen bij de steeds groeiende stapel met onderwijstaken.

Verder bedank ik graag Fred Brackx. Dankzij zijn fantastische lessen Analyse V en transformatieanalyse leerde ik het plezier van analyse (ihb. cliffordanalyse) kennen, en maakte ik voor het eerst kennis met een aantal leden van de vakgroep. Dankzij zijn aanstekelijke lessen heb ik ervoor gekozen om me verder in deze materie te verdiepen.

Ik wil ook graag alle andere collega's van de vakgroep bedanken voor de aangename werksfeer, in het bijzonder de leden van de onderzoeksgroep Hendrik De Bie, Frank Sommen, Hilde, Lander, Michael en Sigiswald voor de wekelijkse cake en koffie. Bedankt ook, Matthias, voor de interessante discussies over Yangians.

Bedankt ook aan Wim en Claudia, Karel en Marijke, Hilde en Bart, en Timo. Onze 2-maandelijkse afspraakjes waren altijd fantastisch!

Ook Serena, Kenneth en Arne mag ik niet vergeten voor de leuke 'World of Warcraft'-spelletjes die me 's avonds toelieten om te ontspannen na een hele dag wiskunde. Verder moet ik ook Bert, Stijn, Michiel en Jos vermelden voor de wekelijkse lan-parties waarop ik me altijd kapot lachte.

Last, but not least bedank ik graag mijn ouders om er steeds voor me te zijn, de onuitputtelijke aanmoedigingen op alle gebied, en om me de

vrijheid te laten om wiskunde te gaan studeren, zodat ik jullie nu dit werk kan presenteren.

Tim
Gent, 30 januari 2015

Contents

Namely, we have no right to believe a thing true because everybody says so unless there are good grounds for believing that some one person at least has the means of knowing what is true, and is speaking the truth so far as he knows it.

William Kingdon Clifford

1 Introduction

Within the theory of Riemannian spin manifolds, there exists an entire system of conformally invariant elliptic first-order differential operators, see [16, 45, 73, 79]. The aim of this thesis is to study these operators. One of these operators is the (massless) Dirac operator, an operator that maps spinor-valued functions to the same space. When it comes to this operator, one often focusses on the rotational invariance with respect to the spin group or its orthogonal Lie algebra $\mathfrak{so}(m,\mathbb{C})$. Interesting results considering the Dirac operator, studied from a function theoretical point of view (i.e. studying polynomial solutions, integral representations, special functions etc.) can be found in standard references such as [12, 30, 48].

In recent years, Clifford analysis has shown to offer an elegant framework to study the aforementioned function theoretical problems not only for the Dirac operator, but also for far-reaching generalisations of it, acting on functions which take their values in arbitrary irreducible Spin(m)-representations $\mathbb{V}^{\pm}_{\lambda}$, with highest weight $\lambda = \left(l_1 + \frac{1}{2}, \ldots, l_{n-1} + \frac{1}{2}, \pm\frac{1}{2}\right)$, where $n = \left\lfloor \frac{m}{2} \right\rfloor$. For convenience, we will restrict ourselves to the case of an odd dimension, implying that we will be able to omit the $\pm$-signature. However, all results in this thesis can be generalised to the case of an even dimension as well.

For each suitable choice of integers $l_1, \ldots, l_{n-1}$, there corresponds a higher spin Dirac operator $\mathcal{Q}_\lambda$ acting as

$$\mathcal{Q}_\lambda : \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{V}_\lambda) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{V}_\lambda).$$

The easiest of these operators is the classical Dirac operator, where the choice of the integers is $l_1 = \cdots = l_{n-1} = 0$. The corresponding irreducible representation $\mathbb{V}_\lambda$ is then modelled by the spinor space $\mathbb{S}$. In this way, higher spin Clifford analysis can be seen as a generalisation of standard Clifford analysis. The Dirac operator originally was used in a paper by P.A.M. Dirac [33] within the context of particle physics, describing the behaviour of an electron, an elementary particle with spin number $\frac{1}{2}$.

An important breakthrough was made in [48, 86], where it was shown that all finite-dimensional irreducible representations $\mathbb{V}_\lambda$ of Spin(m) can be modelled by means of certain spaces of polynomials (in several vector variables). This means that higher spin Clifford analysis can be handled as a function theory in combination with useful results coming from representation theory.

The first generalisation of the Dirac operator is the case where $l_1 = 1$ and $l_2 = \cdots = l_{n-1} = 0$. This results in an operator which is well-known in theoretical physics, and was first discovered by Rarita and Schwinger in [69], describing the behaviour of fermionic particles with spin number $\frac{3}{2}$. Within Clifford analysis, further generalisations have been made to the case where $\lambda = \left(l_1 + \frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}\right)$, under the name 'Rarita-Schwinger operators', after the writers of the original article. The earliest results were established by Bureš, Sommen, Souček and Van Lancker (see [20, 21]). In these articles, several properties of the Rarita-Schwinger operators are discussed, starting with an explicit expression for these operators as first-order differential operators. Moreover, a fundamental solution was obtained, a full decomposition of the polynomial kernel of these operators in terms of irreducible representations of Spin(m) was established, and the conformal invariance of the Rarita-Schwinger operators was proven. In more recent years, extensions of Rarita-Schwinger operators have been made to the sphere in e.g. [83, 52, 88, 17].

An aim of this thesis is to generalise some of these results to the case of general higher spin Dirac operators, hence for general choices of $l_1, \ldots, l_{n-1}$. In the remainder of this introductory chapter, we will give an overview of the contents of this thesis.

We start with the basics in Chapter 2. Here, we introduce Clifford algebras or geometric algebras, together with definitions, properties and important results that come with these algebras. Standard references are e.g. [12, 30, 48]. Two major groups are discussed, namely the spin and pin group, as double covers of SO(m) and O(m) respectively. Moreover, it will be shown that these groups can be realised within a Clifford algebra. As a lot of results in this thesis will use arguments coming from representation theory, the definition of a group representation is introduced and a classification of the finite-dimensional irreducible representations of the spin group is made.

In Chapter 3, we dig deeper into representation theory. More specifically, we discuss finite-dimensional representations of some classical Lie algebras, as they will be of utmost importance in this thesis. Standard references here are e.g. [47, 52]. In the first part of this chapter, Lie algebras and their properties are introduced. One class of Lie algebras is put in the spotlights, the simple Lie algebras. Of all simple Lie algebras the special linear Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ is the most basic one, hence the perfect candidate to start with when discussing representations. Also representations of symplectic and orthogonal Lie algebras are classified to some extent.

In Chapter 4, we finally start dealing with higher spin Clifford analysis. A construction method for higher spin operators is developed. This method uses an algebraic concept called a transvector algebra. First, the general theory is explained, based on the work of Zhelobenko, Mickelsson and Molev [90, 63, 66]. In order to get a set of generators of such a transvector algebra, an operator is constructed called an extremal projection operator. Next, the abstract theory is translated to Clifford analysis. Exploiting the properties of the extremal projector will enable us to construct higher spin operators. We focus on two types of higher spin operators in particular, namely the higher spin Dirac operators mentioned above, and the higher spin twistor operators. As the Dirac operator is conformally invariant, also the conformal invariance of these higher spin operators is proven in the last part of this chapter.

A transvector algebra is a concept that is quite difficult to grasp. To this end, in Chapter 5, an explicit example is discussed, clarifying the general construction made in Chapter 4. The main aim of this chapter is to explicitly give the generators of the transvector algebra in this example.

When discussing differential operators, another significant property is the existence of a fundamental solution. In Chapter 6, the fundamental solution of the higher spin Dirac operator is determined, using Riesz potentials and distribution theory. This fundamental solution then is used to prove three basic integral formulae for the higher spin Dirac operator: Stokes' theorem, Cauchy-Pompeiu theorem and the Cauchy integral formula.

The classical Cauchy-Kovalevskaya extension theorem (e.g. [30]) essentially tells us that there is an isomorphism between the space of polynomials in the kernel of the Dirac operator, homogeneous of degree k, and the space of homogeneous polynomials in one variable less. In Chapter 7, an analogue of this theorem is proven for the higher spin Dirac operators, enabling us to calculate the dimension of the h-homogeneous polynomial kernel of the higher spin Dirac operator.

The main aim of this thesis is to find the decomposition of the h-homogeneous polynomial kernel of the higher spin Dirac operator in irreducible spin-modules. Chapters 8, 9 and 10 are devoted to just that. We will discover an inductive structure in this kernel.

In Chapter 8, we introduce twisted operators, which are essentially operators that act on functions taking values in the 'wrong' space. Higher spin Dirac operators have a built-in inductive structure, which is revealed in this chapter. First of all the twisted Dirac operator is introduced, which has the property that it can be written as the sum of a higher spin Dirac operator and at most k twistor operators, where k is the order of the higher spin Dirac operator (the number of non-trivial entries in λ). This relation between a twisted operator and a normal higher spin Dirac operator will be used, as we will prove that a similar relation exists between a higher spin Dirac operator of order k and a twisted higher spin Dirac operator of order $k-1$. The proofs in this chapter heavily rely on representation theoretical arguments, which are collected in the final section of this chapter. Important is that the link between higher spin Dirac operators of different order suggests that there is a relation between the polynomial kernel spaces of those higher spin Dirac operators as well.

In Chapter 9, we show that the space of solutions of the higher spin Dirac operators can be classified into two subsets: so-called type A and type B solutions. It will be proven that the type A solutions are represented by a special polynomial space which we will refer to as skew simplicial monogenic polynomials. This space is, however, not an irreducible representation of $\mathrm{Spin}(m)$, but we will again make use of transvector algebras to decompose this space into irreducible modules. Even more, we will show how these modules can be embedded (as polynomial spaces) in the space of skew simplicial monogenic polynomials.

In Chapter 10, the results from the previous two chapters are combined. The main result in this chapter is that the set of h-homogeneous polynomials in the kernel of a higher spin Dirac operator is contained in a direct sum of type A solutions (as representations) of different higher spin Dirac operators. When calculating examples, it seems that this inclusion is in fact an equality. In order to prove this, we managed to reduce the problem at hand to a combinatorial problem, using a dimension analysis. Unfortunately, the (symbolic) formulas become so extensive that we did not manage to solve this combinatorial problem in full generality.

The inductive approach is not the only approach one can take to describe the polynomial kernel of the higher spin Dirac operators. Making use of the generalised CK-extension and branching rules in Chapter 11, the h-homogeneous polynomial kernel of a higher spin Dirac operator can be rewritten as a sum of tensor products. This does not give a full decomposition of the kernel as the tensor products have not yet been resolved, but nonetheless, this is an approach which might prove useful in future research.

In the twelfth and final chapter of this thesis, we build further on the fact that the twisted Dirac operator can be written as a sum of a higher spin Dirac operator and at most k twistor operators, up to a non-trivial embedding operator. In this chapter, this decomposition is explicitly determined, or in other words, the embedding operators are determined, again relying on a suitable extremal projection operator.

As far as the laws of mathematics refer to reality, they are not certain; as far as they are certain, they do not refer to reality.

Albert Einstein

2

Basic notions of Clifford analysis

Euclidean Clifford analysis is, in its simplest definition, the function theoretic and functional analytic study of hypercomplex functions defined in Euclidean space and taking values in a so-called Clifford algebra, which will be introduced in the first section of this chapter. General references for this theory are for instance [12, 30, 29, 48]. The cornerstone of this theory is the Dirac operator, a first-order differential operator which is elliptic and rotationally invariant (even conformally invariant, see Chapter 4 or [45]). This operator plays the same role in classical Clifford analysis as the Cauchy-Riemann operator does in the theory of holomorphic functions in the complex plane, so Clifford analysis is often seen as a generalisation of this function theory to higher dimensions. The Dirac operator factorises the Laplace operator, whence Clifford analysis can also be seen as a refinement of harmonic analysis.

In this chapter, we will introduce the basic notions that will be used throughout this thesis. The first section is devoted to real and complex Clifford algebras, and we introduce a group whose representation theory plays a crucial role in what follows, the group $\mathrm{Spin}(m)$. In the second section, a fundamental representation of $\mathrm{Spin}(m)$ is discussed, the so-called spinor space. The third section introduces the necessary operators in order to be able to discuss more general $\mathrm{Spin}(m)$-representations in the fourth and final one.

2.1 Clifford Algebras

In 1878, the English mathematician and philosopher William Kingdon Clifford introduced *geometric algebras* in [24]. The importance of these algebras lies in the fact that they incorporate the classical inner product and the wedge product in a single structure. These algebras are nowadays called, after him, *Clifford algebras.*

2.1.1 Real Clifford algebras

Clifford analysis in its full generality can be seen as a generalisation of the theory of holomorphic functions in the complex plane to the case of several variables. This means that throughout the theory, we will work in a general dimension denoted by $m \in \mathbb{N}$. The real Euclidean m-dimensional space will be denoted as $\mathbb{R}^m$, and the standard orthogonal basis for $\mathbb{R}^m$ will be $(e_1, \ldots, e_m)$. A real Clifford algebra is an algebraic structure defined on $\mathbb{R}^m$ as follows.

Definition 2.1. *The real universal Clifford algebra $\mathbb{R}_m$ is the unital associative algebra over $\mathbb{R}^m$ ($m \in \mathbb{N}$) where the multiplication is governed by the relations*

$$e_i e_j + e_j e_i = -2\delta_{ij}, \text{ for all } 1 \leq i, j \leq m. \tag{2.1}$$

Remark 2.1. In standard literature (e.g. [12, 30]), the notation $\mathbb{R}_{0,m}$ is often used instead of $\mathbb{R}_m$, as Clifford algebras exist in general for different signatures.

The relations (2.1) signify that the basis vectors of the underlying vector space anticommute, and they square to -1. The real universal Clifford algebra has dimension 2^m, corresponding to the number of possibilities to take an ordered product of basis vectors e_i. A basis for the Clifford algebra $\mathbb{R}_m$ is given by

$$\mathbb{R}_m = \operatorname{span}_{\mathbb{R}} \{e_{i_1} e_{i_2} \cdots e_{i_h} : 1 \leq i_1 < i_2 < \cdots < i_h \leq m\}.$$

Alternatively, we will also denote these basis elements in a more compact way as

$$\mathbb{R}_m = \operatorname{span}_{\mathbb{R}} \{e_A : A \subseteq \{1, 2, \ldots, m\}\}, \tag{2.2}$$

where each considered set A is a subset of $\{1, 2, \ldots, m\}$. For $A = \varnothing$, we define $e_A = 1$, the identity element of $\mathbb{R}_m$. Every element a of the Clifford algebra $\mathbb{R}_m$ can then be written as a linear combination of these basis elements with real coefficients, i.e. a sum of the form

$$a = \sum_{A \subset \{1,\ldots,m\}} a_A e_A, \text{ with } a_A \in \mathbb{R}.$$

Elements of a Clifford algebra will be referred to as *Clifford numbers*. Let us give some examples of Clifford algebras.

Example 2.1. First of all, the simplest case where $m = 1$, gives us the Clifford algebra $\mathbb{R}_1$. This algebra is isomorphic to the space of complex numbers $\mathbb{R}_1 \cong \mathbb{C}$. Since the only generating basis vector e_1 squares to -1, the isomorphism is given by

$$\mathbb{R}_1 \to \mathbb{C} : a + be_1 \mapsto a + bi,$$

for all $a, b \in \mathbb{R}$, i denoting the imaginary unit.

Example 2.2. The Clifford algebra $\mathbb{R}_2$ is isomorphic to the space of quaternions $\mathbb{H}$, and the isomorphism is given by

$$\mathbb{R}_2 \to \mathbb{H} : a + be_1 + ce_2 + de_1e_2 \mapsto a + bi + cj + dk.$$

There exists a straightforward grading on the set of Clifford numbers, based on the cardinality $|A|$ of each subset A in the definition (2.2). Elements of the form

$$\sum_{|A|=k} a_A e_A,$$

where the number of elements in each of the considered sets A is equal to k, are called k-vectors. The corresponding basis vectors e_A generate the subspace $\mathbb{R}_m^{(k)}$:

$$\mathbb{R}_m^{(k)} = \text{span}_{\mathbb{R}} \{e_A : |A| = k\}$$

For $k = 0$, we have that $\mathbb{R}_m^{(0)} \cong \mathbb{R}$. When $k = 1$, the numbers are simply called *vectors* rather then 1-vectors. In the case where $k = 2$, 2-vectors are also called *bivectors*. Finally, the Clifford number $e_{1\ldots m}$ is called the *pseudoscalar* of the Clifford algebra as it commutes or anticommutes with each k-vector, depending on m and k. These observations reveal the following multivector structure on the Clifford algebra:

$$\mathbb{R}_m = \mathbb{R}_m^{(0)} \oplus \mathbb{R}_m^{(1)} \oplus \cdots \oplus \mathbb{R}_m^{(m)}.$$

Remark 2.2. The real space $\mathbb{R}^m$ is isomorphic to $\mathbb{R}_m^{(1)}$ by the identification between $(x_1, \ldots, x_m)$ and the vector $x = \sum_{j=1^m} x_j e_j$. This gives us an embedding of $\mathbb{R}^m$ into $\mathbb{R}_m$. We will use the same notation x in both cases, as its meaning will always be clear from the context.

The sum of all even subspaces $\mathbb{R}_m^{(2j)}$ is a subalgebra of $\mathbb{R}_m$, called the *even subalgebra*:

$$\mathbb{R}_m^+ = \bigoplus_{j=0}^{\lfloor \frac{m}{2} \rfloor} \mathbb{R}_m^{(2j)}.$$

As was mentioned before, the universal Clifford algebras incorporate two types of products on vectors, the well-known Euclidean inner product and the so-called exterior or wedge product. These are defined as follows.

Definition 2.2. *For vectors x and y, explicitly given by $x = \sum_{j=1}^{m} x_j e_j$ and $y = \sum_{j=1}^{m} y_j e_j$, the Euclidean inner product $\langle \cdot, \cdot \rangle$ is defined as*

$$\langle x, y \rangle = \sum_{j=1}^{m} x_j y_j.$$

The Euclidean inner product always is scalar-valued, and the squared norm of a vector x directly is derived from the definition of the Euclidean inner product, $|x|^2 = \langle x, x \rangle$. The exterior or wedge product of two vectors generally is defined as follows.

Definition 2.3. *For vectors $x = \sum_{j=1}^{m} x_j e_j$ and $y = \sum_{j=1}^{m} y_j e_j$, we define the exterior product or wedge product as*

$$x \wedge y = \sum_{1 \leq i < j \leq m} e_i e_j (x_i y_j - x_j y_i).$$

This product of two vectors has the property that it always is bivector valued. When considering the Clifford algebra $\mathbb{R}_m$, it is interesting to compare it to the Grassmann algebra $\Lambda\mathbb{R}^m$, another algebra of dimension 2^m. This is the associative unital algebra generated by

$$\Lambda\mathbb{R}^m = \operatorname{span}_{\mathbb{R}} \left\{ e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_h} : 1 \leq i_1 < i_2 < \cdots < i_h \leq m \right\}.$$

Although the dimensions of the Clifford algebra $\mathbb{R}_m$ and the Grassmann algebra $\Lambda\mathbb{R}^m$ are equal, these algebras are not isomorphic as algebras. It suffices to look at the respective products of two vectors, for instance:

$$(ae_1 + be_2)e_2 = ae_1e_2 - b = -b + ae_1 \wedge e_2,$$

while on the other hand

$$(ae_1 + be_2) \wedge e_2 = ae_1 \wedge e_2.$$

Within the Clifford algebra $\mathbb{R}_m$, the product of two vectors x and y according to the multiplication rules (2.1) equals

$$xy = -\langle x, y \rangle + x \wedge y.$$

This means that the Clifford product of two vectors always splits in a scalar part given by the Euclidean inner product up to a minus sign, and a bivector part, given by the wedge product. This indeed justifies the saying that both the wedge product and the Euclidean inner product are incorporated in the Clifford algebra.

2.1.2 (Anti-)automorphisms on $\mathbb{R}_m$

On the real Clifford algebra $\mathbb{R}_m$, we define three (anti-)involutions. Let e_A be a basis element of $\mathbb{R}_m$, e.g. $e_A = e_{i_1} e_{i_2} \cdots e_{i_h}$, with $i_1 < i_2 < \cdots < i_h$.

(i) The inversion or main involution $a \mapsto \hat{a}$ is defined on the basis elements by means of

$$\hat{e}_A = (-1)^h e_{i_1} \cdots e_{i_h},$$

and then is linearly extended to the entire Clifford algebra $\mathbb{R}_m$:

$$(a_A e_A + a_B e_B)\hat{} = a_A \hat{e}_A + a_B \hat{e}_B,$$

for all $a_A, a_B \in \mathbb{R}$. This involution has the property that $\widehat{ab} = \hat{a}\hat{b}$ for all $a, b \in \mathbb{R}_m$. Note that the spaces of even (resp. odd) k-vectors are eigenspaces for the main involution with eigenvalues 1 (resp. -1).

(ii) The reversion or main anti-involution $a \mapsto a^*$ is defined on the basis elements by means of

$$e_A^* = e_{i_h} \cdots e_{i_1} = (-1)^{\frac{h(h-1)}{2}} e_A,$$

and then is linearly extended to the entire Clifford algebra $\mathbb{R}_m$:

$$(a_A e_A + a_B e_B)^* = a_A e_A^* + a_B e_B^*,$$

for all $a_A, a_B \in \mathbb{R}$. This anti-involution has the property that $(ab)^* = b^* a^*$ for all $a, b \in \mathbb{R}_m$.

(iii) The conjugation $a \mapsto \overline{a}$ is defined on the basis elements by means of

$$\overline{e}_A = (-1)^h e_A^* = (-1)^h e_{i_h} \cdots e_{i_1} = (-1)^{\frac{h(h+1)}{2}} e_A,$$

and then is linearly extended to the entire Clifford algebra $\mathbb{R}_m$:

$$\overline{(a_A e_A + a_B e_B)} = a_A \overline{e}_A + a_B \overline{e}_B,$$

for all $a_A, a_B \in \mathbb{R}$. This anti-involution has the property that $\overline{(ab)} = \overline{b}\,\overline{a}$.

2.1.3 The groups Spin(m) and Pin(m)

One of the interesting features of the Clifford algebra $\mathbb{R}_m$ is the existence of subgroups constituting double covers of the orthogonal group $\mathrm{O}(m)$ and the special orthogonal group $\mathrm{SO}(m)$ respectively. They are called $\mathrm{Pin}(m)$ and $\mathrm{Spin}(m)$. These two groups, and for this thesis in particular the latter one, are of crucial importance in Clifford analysis. Spin(m)-representations, as discussed below, will give rise to the entire theory of higher spin Clifford analysis.

First notice that each nonzero vector $x \in \mathbb{R}^m \backslash \{0\}$ is invertible within the Clifford algebra $\mathbb{R}_m$, with inverse given by

$$x^{-1} = -\frac{x}{|x|^2}, \tag{2.3}$$

For all invertible Clifford numbers $s \in \mathbb{R}_m$, such that $sx\hat{s}^{-1} \in \mathbb{R}^m$, we can introduce the linear transformation given by

$$\chi(s) : \mathbb{R}^m \to \mathbb{R}^m : x \mapsto \chi(s)x := sx\hat{s}^{-1}$$

Note that $\chi(s)$ indeed defines an element in $\mathrm{End}(\mathbb{R}^m)$ (see Lemma 2.2 below). This transformation has an interesting property.

Lemma 2.1. *For each Clifford number $s \in \mathbb{R}_m$, we have that $\chi(s)$ is an element of the orthogonal group:*

$$\chi(s) \in \mathrm{O}(m).$$

Proof. This directly follows from the fact that

$$(\chi(s)x)^2 = -\widehat{(\chi(s)x)}(\chi(s)x) = -\hat{s}\hat{x}s^{-1}sx\hat{s}^{-1} = x^2,$$

as $\mathrm{O}(m)$ can be seen as the group of linear transformations that leave the norm of a vector unchanged. □

We can now give the following definition.

Definition 2.4. *The Clifford group is defined as the group*

$$\Gamma(m) := \left\{ s \in \mathbb{R}_m : sx\hat{s}^{-1} \in \mathbb{R}_m^{(1)}, \forall x \in \mathbb{R}_m^{(1)} \right\}.$$

For a particular subset of the Clifford group, we have the following property.

Lemma 2.2. *Each nonzero vector $v \in \mathbb{R}_m^{(1)}$ is an element of $\Gamma(m)$; the corresponding transformation $\chi(s)$ defines a reflection.*

Proof. Take $x \in \mathbb{R}_m^{(1)}$ and $v \in \mathbb{R}_m^{(1)}$ with $v \neq 0$, whence it is invertible (see (2.3)). Since x can always be written as $x = \lambda v + v^\perp$, with $\lambda \in \mathbb{R}$ and $\langle v, v^\perp \rangle = 0$, we immediately find that

$$\chi(v)x = v(\lambda v + v^\perp)\hat{v}^{-1} = -\lambda v + v^\perp.$$

Thus, this means that the action of $\chi(v) \in \mathrm{O}(m)$ is nothing but a reflection with respect to the hyperplane H_v perpendicular to v. □

We can then use the theorem of Cartan-Dieudonné, stating that each orthogonal transformation $T \in \mathrm{O}(m)$ is a composition of at most m different reflections, i.e.

$$\forall T \in \mathrm{O}(m) : T = \prod_{i=1}^{p} \chi(v_i),$$

with $1 \leq p \leq m$, and $v_i \in \mathbb{R}_m^{(1)} \backslash \{0\}$. On top of that, we can use the fact that $\chi(s_1)\chi(s_2) = \chi(s_1 s_2)$, for all $s_1, s_2 \in \Gamma(m)$, such that

$$T = \chi\left(\prod_{i=1}^{p} v_i\right).$$

This proves that the map

$$\chi : \Gamma(m) \to \mathrm{O}(m)$$

in fact is a group morphism. The kernel of this mapping equals $\mathbb{R}\backslash\{0\} = \mathbb{R}_0$. This may be summarised as

$$\mathrm{O}(m) \cong \Gamma(m)/\mathbb{R}_0.$$

Note that we can also define the Clifford group as

$$\Gamma(m) := \left\{ v \in \mathbb{R}_m : \exists p \in \mathbb{N}, \exists v_1, \ldots, v_p \in \mathbb{R}_m^{(1)} \backslash \{0\}, v = \prod_{i=1}^{p} v_i \right\}.$$

Since each rotation (resp. improper rotation) is the product of an even (resp. odd) number of reflections in $\mathbb{R}$, we can conclude that $s \in \Gamma(m)$ gives rise to a *rotation* if $s \in \Gamma^+(m) := \Gamma(m) \cap \mathbb{R}_m^+$. Since we have for those elements that $s = \hat{s}$, the rotations are given by $\chi(s)x = sxs^{-1}$. $\Gamma^+(m)$ also is called the *even* Clifford group. For this subgroup, we have that

$$\chi : \Gamma^+(m) \to \mathrm{SO}(m),$$

with kernel $\mathbb{R}_0$. This results in the isomorphism

$$\mathrm{SO}(m) \cong \Gamma^+(m)/\mathbb{R}_0.$$

The Clifford group $\Gamma(m)$ is a normed space, where the *Clifford norm* is defined as follows.

Definition 2.5. *For each $s \in \Gamma(m)$, we define the Clifford norm $\mathcal{N}(s) = s\bar{s}$.*

It is easily seen that this always yields a positive scalar. This allows us to introduce the following subgroups of the Clifford group:

$$\begin{aligned} \mathrm{Pin}(m) &:= \{s \in \Gamma(m) : \mathcal{N}(s) = 1\} \\ \mathrm{Spin}(m) &:= \{s \in \Gamma^+(m) : \mathcal{N}(s) = 1\} \end{aligned}$$

called pin group and spin group. It holds that

$$\mathrm{Pin}(m) \cong \Gamma(m)/\mathbb{R}_0^+ \text{ and } \mathrm{Spin}(m) \cong \Gamma^+(m)/\mathbb{R}_0^+.$$

Clearly, we have that $\mathrm{Pin}(m)$ is generated by the set $\{e_1, \ldots, e_m\}$, while $\mathrm{Spin}(m)$ is generated by $\{e_{12}, e_{13}, \ldots, e_{(m-1)m}\}$. This leads us to the desired isomorphisms:

$$\mathrm{Pin}(m)/\mathbb{Z}_2 \cong \mathrm{O}(m) \text{ and } \mathrm{Spin}(m)/\mathbb{Z}_2 \cong \mathrm{SO}(m),$$

making the pin group and the spin group double covers of $\mathrm{O}(m)$ and $\mathrm{SO}(m)$ respectively. The groups themselves clearly are related in the following way:

$$\mathrm{Spin}(m) \subset \mathrm{Pin}(m) \subset \Gamma(m).$$

Since each nonzero vector in $\mathbb{R}_m$ is invertible, we can identify the set of vectors x, for which $\mathcal{N}(x) = 1$ with the unit sphere $S^{m-1} \subset \mathbb{R}^m$. In particular, we have that

$$\mathrm{Pin}(m) = \left\{ \prod_{j=1}^{p} \omega_j : \omega_j \in S^{m-1} \right\}$$

and

$$\mathrm{Spin}(m) = \left\{ \prod_{j=1}^{2p} \omega_j : \omega_j \in S^{m-1} \right\}. \tag{2.4}$$

Note that both groups are Lie groups. In this thesis, we will mainly work with $\mathrm{Spin}(m)$.

2.1.4 The Lie algebra of Spin(m)

We will not only work on the group level. As $\mathrm{Spin}(m)$ is a Lie group, we can also operate on the level of the corresponding Lie algebra. The next step thus is to construct the Lie algebra corresponding to $\mathrm{Spin}(m)$. First of all we will repeat the definition of a Lie algebra in general.

Definition 2.6. *A Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$ is a vector space, equipped with a binary operation*

$$[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

also called a Lie bracket, which satisfies the following conditions:

- *bilinearity: for all $x, y, z \in \mathfrak{g}$, and $\alpha, \beta \in \mathbb{K}$, it holds that*

$$[x, \alpha y + \beta z] = \alpha[x, y] + \beta[x, z],$$

- *anti-commutativity: for all* $x, y \in \mathfrak{g}$*,*

$$[x, y] = -[y, x].$$

- *Jacobi identity: for all* $x, y, z \in \mathfrak{g}$*,*

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$

The Lie algebra $\mathfrak{g}$ of a (matrix) Lie group G is the set of all matrices X such that e^{tX} is a smooth curve in G for all $t \in \mathbb{R}$. The *exponential map*, which gives a relations between a Lie group and its Lie algebra, is given by

$$\exp : \mathfrak{g} \to G : X \mapsto e^X.$$

Here, e^X is defined via its power series

$$e^X = \sum_{j=0}^{\infty} \frac{X^j}{j!}.$$

The inverse map is given by

$$X = \frac{d}{dt}\bigg|_{t=0} e^{tX}. \tag{2.5}$$

Then the following property holds.

Theorem 2.1. *The Lie algebra associated to the* $\mathrm{Spin}(m)$*-group is the Lie algebra* $\mathbb{R}_m^{(2)}$ *of the bivectors in* $\mathbb{R}_m$*, where the Lie bracket is given by the commutator.*

A proof of this theorem can be found in e.g. [61]. We will later show that the Lie algebra $\mathbb{R}_m^{(2)}$ is in fact isomorphic with the *(special) orthogonal Lie algebra* $\mathfrak{so}(m, \mathbb{R})$.

2.1.5 Complex Clifford algebras

Until now, we only considered *real* Clifford algebras. However, instead of working over the real space $\mathbb{R}^m$, we can also choose the m-dimensional complex space $\mathbb{C}^m$. Hence, we can define a complex Clifford algebra as follows.

Definition 2.7. *The complex Clifford algebra* $\mathbb{C}_m$ *is defined as the complexification of* $\mathbb{R}_m$*, i.e.* $\mathbb{C}_m = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}_m$*.*

As it is just a complexified version of the real Clifford algebra, most properties almost are identical. For the sake of completeness,we list them below:

- the dimension of $\mathbb{C}_m$ equals

$$\dim_{\mathbb{C}}(\mathbb{C}_m) = 2^m.$$

- if we take the standard orthonormal basis $(e_1, \ldots, e_m)$ of $\mathbb{C}^m$, then

$$\mathbb{C}_m = \operatorname{span}_{\mathbb{C}} \left\{ e_A : A \subseteq \{1, 2, \ldots, m\} \right\}.$$

- the complex spaces of k-vectors ($0 \leq k \leq m$) are defined as

$$\mathbb{C}_m^{(k)} = \operatorname{span}_{\mathbb{C}}\{e_A : |A| = k\}.$$

- the sum of all even (odd) spaces of k-vectors is a subalgebra (subspace) which we call the *even subalgebra* (odd subspace):

$$\mathbb{C}_m^+ = \bigoplus_{j=0}^{\lfloor \frac{m}{2} \rfloor} \mathbb{C}_m^{(2j)} \text{ and } \mathbb{C}_m^- = \bigoplus_{j=1}^{\lceil \frac{m}{2} \rceil} \mathbb{C}_m^{(2j-1)}.$$

In order to define a norm on $\mathbb{C}_m$, we define yet another anti-involution.

Definition 2.8. *The Hermitean conjugation $a \mapsto a^\dagger$ is defined on the basis elements e_A as*

$$e_A^\dagger = (-1)^h e_A^* = (-1)^h e_{i_h} \cdots e_{i_1},$$

and it is then extended anti-linearly:

$$(a_A e_A + a_B e_B)^\dagger = \overline{a}_A e_A^\dagger + \overline{a}_B e_B^\dagger,$$

for all $a_A, a_B \in \mathbb{C}$. Here, $\bar{\cdot}$ denotes the complex conjugation. We have that $(ab)^\dagger = b^\dagger a^\dagger$.

One can use the Hermitean conjugation to define an inner product on $\mathbb{C}_m$ (which is anti-linear in the second argument):

$$\langle a, b \rangle := [ab^\dagger]_0,$$

the subindex 0 meaning that we take the scalar part. A norm on $\mathbb{C}_m$ can then be defined as

$$|a| := \sqrt{[aa^\dagger]_0} = \sqrt{\sum_A |a_A|^2}.$$

We also mention a useful isomorphism, which we will use to define spinor spaces further on.

Theorem 2.2. *For complex Clifford algebras, we have*

$$\mathbb{C}_m^+ \cong \mathbb{C}_{m-1}.$$

Proof. This is straightforward, as there is an isomorphism between the generators $\{e_1 e_m, \ldots, e_{m-1} e_m\}$ of $\mathbb{C}_m^+$ and the generators $\{e_1, \ldots, e_{m-1}\}$ of $\mathbb{C}_{m-1}$. □

2.2 Spinor spaces

The entire theory of higher spin operators is connected to half-integer representations of $\mathrm{Spin}(m)$. Throughout this chapter, we will discuss all of the possible finite-dimensional irreducible representations, but we start with the most basic one, the so-called spinor space. This space plays a very important role in theoretical physics, in the context of the Dirac equation (e.g. [33]). Clifford analysis has abstracted the idea of the Dirac equation to a function theory, hence the spinor space will be of utmost importance there as well. First of all, let us introduce *representations* in general.

2.2.1 Representations: definitions

Let us start with the definition of a representation of an algebra.

Definition 2.9. *A representation $\rho_{\mathbb{V}}$ for an algebra $\mathcal{A}$ on a vector space $\mathbb{V}$ is a homomorphism*

$$\rho_{\mathbb{V}} : \mathcal{A} \to \mathrm{End}(\mathbb{V}).$$

If it is clear from the context, one often uses the name 'representation' for the support $\mathbb{V}$ of the representation instead of the actual homomorphism. The action of $\alpha \in \mathcal{A}$ on a vector $v \in \mathbb{V}$ is also denoted as

$$\rho_{\mathbb{V}}(\alpha)[v] := \alpha \cdot v.$$

Among representations, we define a special type.

Definition 2.10. *A representation $\mathbb{V}$ is called irreducible if there exists no non-trivial subspace $\mathbb{W} \subset \mathbb{V}$ which is invariant under the action of $\mathcal{A}$.*

These irreducible representations actually are the building blocks of all other representations. Finally, we mention the following important result.

Lemma 2.3 (Schur's Lemma)**.** *Suppose $\mathbb{V}$ is an irreducible finite dimensional representation of $\mathcal{A}$ and there exists an endomorphism $\phi \in End(\mathbb{V})$ for which $[\phi, \rho_{\mathbb{V}}(\alpha)] = 0$, for all $\alpha \in \mathcal{A}$. Then there exists a complex number $\lambda \in \mathbb{C}$ such that $\phi = \lambda \mathbf{1}_{\mathbb{V}}$.*

2.2.2 Representations in Clifford analysis

We will now apply this theory to the case of $\mathcal{A}$ being the Clifford algebra $\mathbb{C}_m$. To this end, we shall introduce a new basis for the complex vector space $\mathbb{C}^m$. Since we will often use the truncated half dimension, we introduce the notation

$$n = \left\lfloor \frac{m}{2} \right\rfloor.$$

Definition 2.11. *In even dimension, i.e. for $m = 2n$, the Witt basis of $\mathbb{C}^{2n}$ is given by*

$$\mathfrak{f}_j = \frac{e_j - ie_{j+n}}{2} \text{ and } \mathfrak{f}_j^\dagger = -\frac{e_j + ie_{j+n}}{2},$$

for all $1 \leq j \leq n$. In odd dimension, i.e. for $m = 2n+1$, the Witt basis is complemented by an additional vector, traditionally chosen to be the vector e_m.

The Witt basis vectors satisfy the Grassmann identities

$$\{\mathfrak{f}_i, \mathfrak{f}_j\} = \{\mathfrak{f}_i^\dagger, \mathfrak{f}_j^\dagger\} = 0 \quad \text{and} \quad \{\mathfrak{f}_i, \mathfrak{f}_j^\dagger\} = \delta_{ij},$$

where $\{\cdot,\cdot\}$ denotes the anticommutator. Due to these relations, we get the two Grassmann algebras

$$\Lambda W_n = \mathrm{Alg}_{\mathbb{C}}\{\mathfrak{f}_j : 1 \leq j \leq n\} \quad \text{and} \quad \Lambda W_n^\dagger = \mathrm{Alg}_{\mathbb{C}}\{\mathfrak{f}_j^\dagger : 1 \leq j \leq n\}.$$

Using the Witt basis, we can introduce the mutually commuting idempotents

$$I_j = \mathfrak{f}_j \mathfrak{f}_j^\dagger, \quad \forall j = 1, \ldots, n,$$

which satisfy the relations

$$I_j^2 = I_j, \quad \forall j = 1, \ldots, n \quad \text{and} \quad I_i I_j = I_j I_i, \quad \forall i, j = 1, \ldots, n$$

for all $1 \leq i, j \leq n$. Then the primitive idempotent I is defined as

$$I = \prod_{j=1}^{n} I_j.$$

With these definitions, we can consider representations of the Clifford algebra $\mathbb{C}_m$. In the case where $m = 2n$, we know that $\dim_{\mathbb{C}} \mathbb{C}_{2n}$ equals 2^{2n}. As a matter of fact, we have the isomorphism

$$\mathbb{C}_{2n} \cong \mathbb{C}^{2^n \times 2^n},$$

where $\mathbb{C}^{2^n \times 2^n}$ is the algebra of $2^n \times 2^n$-dimensional matrices over the complex numbers (see e.g. [30]). In this case there exists a unique *irreducible* complex representation, called the space of *Dirac spinors* (or just spinors), denoted by $\mathbb{S}_{2n}$. Moreover, this space is isomorphic to the 2^n-dimensional complex space:

$$\mathbb{S}_{2n} \cong \mathbb{C}^{2^n}.$$

Explicitly, the space of Dirac spinors can be realised as the minimal left ideal in $\mathbb{C}_{2n}$ given by

$$\mathbb{S}_{2n} = \mathbb{C}_{2n} I = \left(\Lambda W_n^\dagger\right) I.$$

Remark 2.3. This is one way to explicitly realise the spinor space in the Clifford algebra $\mathbb{C}_{2n}$. Alternatively, one can define the commuting idempotents $K_j = \mathfrak{f}_j^\dagger \mathfrak{f}_j$, and construct primitive idempotents of the form $J = J_1 \ldots J_n$, where each J_i is either I_i or K_i. It is easily seen that

$$\mathbb{C}_{2n} = \mathbb{C}_{2n} \prod_{j=1}^{n} (\mathfrak{f}_j \mathfrak{f}_j^\dagger + \mathfrak{f}_j^\dagger \mathfrak{f}_j) = \mathbb{C}_{2n} \sum J \tag{2.6}$$

where the last sum is taken over all possible idempotents. This also means that $\mathbb{C}_{2n}$ may in fact be written as the direct sum of 2^n isomorphic copies of the spinor space.

In odd dimension $m = 2n+1$, the situation is a bit more difficult. In this case, we have that

$$\mathbb{C}_{2n+1} \cong \mathbb{C}^{2^n \times 2^n} \oplus \mathbb{C}^{2^n \times 2^n} \cong \mathbb{C}_{2n+2}^{+},$$

so the Clifford algebra is no longer simple. This means that there now exist two (non-equivalent) representations, which will be denoted by $\mathbb{S}_{2n+2}^{\pm}$. In literature, one often calls them *Weyl spinors, or positive and negative spinors.* As a vector space, $\mathbb{S}_{2n+2}^{\pm} \cong \mathbb{C}^{2^n}$. That is why Weyl spinors $\mathbb{S}_{2n+2}^{\pm}$ are generally realised as 'half-spaces' of the space of Dirac spinors $\mathbb{S}_{2n+2}$, hence the (seemingly awkward) index notation. Explicitly, the spaces of Weyl spinors can be constructed as follows. Let us introduce the so-called *chirality operator* (to be seen as a multiplication operator)

$$\theta = i^{(n-1)(n+1)} e_1 e_2 \ldots e_{2n+2} \in \mathbb{C}_{2n+2}^{+}.$$

This operator is in fact an involution ($\theta^2 = 1$) which satisfies the relation $\theta I = I$ and the eigenspaces for the multiplication operator θ on $\mathbb{S}_{2n+2}$ are given by

$$\mathbb{S}_{2n+2}^{\pm} = \frac{1 \pm \theta}{2} \mathbb{S}_{2n+2}.$$

Note that these spaces are both subspaces of $\mathbb{C}_{2n+2}$. We have the isomorphism

$$\varphi : \mathbb{C}_{2n+1} \to \mathbb{C}_{2n+2}^{+} : e_j \mapsto e_j e_{2n+2}. \tag{2.7}$$

Because $[\varphi(e_j), \theta] = 0$ for all $j = 1, \ldots, 2n+1$, we find for all $\psi \in \mathbb{S}_{2n+2}$ that

$$\varphi(e_j) \frac{1 \pm \theta}{2} \psi = \frac{1 \pm \theta}{2} \varphi(e_j) \psi \in \mathbb{S}_{2n+2}^{\pm}.$$

Throughout this thesis, we will almost always restrict ourselves to the case of an **odd dimension**, in order to avoid the parity signs for the spinor spaces.

2.3 Fundamental operators in Clifford analysis

As mentioned before, higher spin Clifford analysis is a function theoretical framework providing concrete models for half-integer finite-dimensional Spin(m)-representations. The introduction of this framework necessitates some fundamental definitions.

We work with vector variables $x \in \mathbb{R}^m$ of the form $x = (x_1, \ldots, x_m)$. The space of $\mathbb{V}$-valued polynomials depending on a vector variable x is denoted by $\mathcal{P}(\mathbb{R}^m, \mathbb{V})$ and its subspace of h-homogeneous polynomials is denoted by $\mathcal{P}_h(\mathbb{R}^m, \mathbb{V})$. Then we have the decomposition

$$\mathcal{P}(\mathbb{R}^m, \mathbb{V}) = \bigoplus_{h=0}^{+\infty} \mathcal{P}_h(\mathbb{R}^m, \mathbb{V}).$$

Remark 2.4. As a rule, in this thesis, $\mathbb{V}$ will be an irreducible representation of Spin(m), which are discussed in section 2.4.

We now introduce the fundamental operators in the Clifford analysis context.

Definition 2.12. *The Euler operator on* $\mathbb{R}^m$ *is defined as*

$$\mathbb{E}_x = \sum_{j=1}^{m} x_j \partial_{x_j}.$$

This operator has the well-known property that it measures the degree of homogeneity of a homogeneous polynomial:

$$\mathbb{E}_x P(x) = hP(x),$$

for all $P(x) \in \mathcal{P}_h(\mathbb{R}^m, \mathbb{C})$.

Definition 2.13. *The Laplace operator on* $\mathbb{R}^m$ *is defined as*

$$\Delta_x = \sum_{j=1}^{m} \partial_{x_j}^2.$$

This operator is the keystone in harmonic analysis. Functions in the kernel of Δ_x are called *harmonic functions*. At the heart of Clifford analysis however, lies the following first-order differential operator, called the *Dirac operator*.

Definition 2.14. *The Dirac operator on* $\mathbb{R}^m$ *is defined as*

$$\partial_x = \sum_{j=1}^{m} e_j \partial_{x_j}.$$

Definition 2.15. *A function* $f(x) \in \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{V})$ *is called monogenic in* x *if it satisfies*

$$\partial_x f(x) = 0.$$

Remark 2.5. We always can choose $\mathbb{V} = \mathbb{S}_{2n}$, the space of Dirac spinors. The Dirac operator acts between the spaces

$$\partial_x : \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S}_{2n}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S}_{2n}).$$

As we have seen in the previous section, the spinor space is irreducible in odd dimensions. However, in even dimensions, we have that $\mathbb{S}_{2n} = \mathbb{S}_{2n}^+ \oplus \mathbb{S}_{2n}^-$ Hence, one should always keep in mind that, in this case, the above mapping splits up in two parts:

$$\partial_x : \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S}_{2n}^\pm) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S}_{2n}^\mp).$$

This mainly is a notational inconvenience, as we always should consider the parity switch, whence we will mostly restrict ourselves to the case of odd dimension in this thesis.

Remark 2.6. Throughout this thesis, we will use additional vector variables denoted by $u_j \in \mathbb{R}^m, j \in \mathbb{N}$. The corresponding Euler operator will be denoted by $\mathbb{E}_j$ rather then $\mathbb{E}_{u_j}$, the Laplace operator by Δ_j instead of Δ_{u_j} and the Dirac operator by ∂_j instead of ∂_{u_j}.

Often, Clifford analysis is considered as a refinement of harmonic analysis, a statement which is justified by the observation that $\Delta_x = -\partial_x^2$. On the other hand the Dirac operator is a generalisation of the Cauchy-Riemann operator in the complex plane, whence Clifford analysis is also seen as a generalisation to higher dimension of the theory of holomorphic functions of one complex variable.

Let us now introduce a polar decomposition. To this end, we will use the following operators.

Definition 2.16. *The Gamma operator (or spherical Dirac operator) is defined as*

$$\Gamma_x = - \sum_{1 \le i < j \le m} e_{ij} L_{ij}^x.$$

It involves the angular operators

$$L_{ij}^x = x_i \partial_{x_j} - x_j \partial_{x_i},$$

for all $1 \le i \ne j \le m$.

Remark 2.7. If we write $x = r\omega$, where $r := |x|$ is the *radial* part of x and $\omega := \frac{x}{|x|} \in S^{m-1}$ is the *angular* part, then it holds that for any function $f(r)$ only depending on the radial part of x, $L^x_{ij} f(r) = 0$, hence the name *angular* operator.

Observe that the Gamma and Euler operators commute. Like the Cauchy-Riemann operator, the Dirac operator also has a polar decomposition:

$$\partial_x = \omega \left(\partial_r + \frac{1}{r}\Gamma_x \right) = \frac{1}{r}\omega \left(\mathbb{E}_x + \Gamma_x \right).$$

Definition 2.17. *The Laplace-Beltrami operator on the sphere S^{m-1} is defined as*

$$\Delta_x^{LB} = \Gamma_x(m - 2 - \Gamma_x) = \sum_{i<j} (L^x_{ij})^2.$$

Note that, while the Gamma operator is not a scalar operator, the Laplace-Beltrami operator is. Furthermore we have a polar decomposition of the Laplace operator:

$$\Delta_x = \partial_r^2 + \frac{m-1}{r}\partial_r + \frac{1}{r^2}\Delta_x^{LB}.$$

In the context of several vector variables, we will also consider the so-called *mixed Laplace-Beltrami operator* given by

$$\Delta^{LB}_{u_1,u_2} = \sum_{i<j} L^{u_1}_{ij} L^{u_2}_{ij}.$$

Remark 2.8. The Laplace-Beltrami operator is in fact a shifted Casimir operator related to the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. More about the Casimir operator will be explained in the next chapter.

Remark 2.9. In inner products, we treat Dirac operators as 'vectors'. As an example, we can define operators of the type

$$\langle u_i, \partial_j \rangle = \sum_{p=1}^{m} u_{ip} \partial_{u_{jp}},$$

and

$$\langle \partial_i, \partial_j \rangle = \sum_{p=1}^{m} \partial_{u_{ip}} \partial_{u_{jp}}.$$

As mentioned before, we often work with functions of several vector variables. Therefore, to conclude this section, we define some polynomial spaces that are important in this thesis.

Definition 2.18. *A function $f : \mathbb{R}^{km} \to \mathbb{C} : (u_1, \ldots, u_k) \mapsto f(u_1, \ldots, u_k)$ is called simplicial harmonic if it satisfies the system*

$$\begin{aligned} \langle \partial_i, \partial_j \rangle f &= 0, \text{ for all } i, j = 1, \ldots, k \\ \langle u_i, \partial_j \rangle f &= 0, \text{ for all } 1 \leq i < j \leq k. \end{aligned}$$

Moreover, the space of simplicial harmonic polynomials homogeneous of degree l_i in u_i is denoted by $\mathcal{H}_{l_1,\ldots,l_k}$, or $\mathcal{H}_\lambda$ for short, where $\lambda = (l_1, \ldots, l_k)$.

Definition 2.19. *A function $f : \mathbb{R}^{km} \to \mathbb{S} : (u_1, \ldots, u_k) \mapsto f(u_1, \ldots, u_k)$ is called simplicial monogenic if it satisfies the system*

$$\begin{aligned} \partial_i f &= 0, \text{ for all } i = 1, \ldots, k \\ \langle u_i, \partial_j \rangle f &= 0, \text{ for all } 1 \leq i < j \leq k. \end{aligned}$$

The space of simplicial monogenic polynomials homogeneous of degree l_i in u_i is denoted by $\mathcal{S}_{l_1,\ldots,l_k}$, or $\mathcal{S}_\lambda$ for short, where $\lambda = (l_1, \ldots, l_k)$.

2.4 Models for irreducible Spin(m)-representations

In this section we will make a classification of finite dimensional Spin(m)-representations. First of all, we need the definition of a representation of a group.

Definition 2.20. *A representation $\rho_{\mathbb{V}}$ of a group G on a vector space $\mathbb{V}$ is a homomorphism*

$$\rho_{\mathbb{V}} : G \to \mathrm{Aut}(\mathbb{V}).$$

We can use the automorphism group in the case of a group representation, as the group structure allows for invertibility, as opposed to an algebra structure. Again, if it is clear from the context, we use the name 'representation' for the support $\mathbb{V}$ of the representation as well.

In this section, we will follow the procedure given in [86] to construct Spin(m)-representations. The group Spin(m) itself is not commutative. There exists however a maximal commuting subgroup of Spin(m), also called the maximal torus. From [47] and [48], we know that, up to equivalence, the unitary irreducible Spin(m)-modules can be labelled by considering the action of this maximal torus of Spin(m):

$$T = \left\{ \exp\left(\frac{1}{2}e_{12}t_1\right)\exp\left(\frac{1}{2}e_{34}t_2\right)\cdots \right. \\ \left. \times \exp\left(\frac{1}{2}e_{(2n-1)(2n)}t_n\right) : t_j \in \mathbb{R}, \forall j = 1,\ldots,n \right\}.$$

Let $\rho_{\mathbb{V}} : \mathrm{Spin}(m) \to \mathrm{Aut}(\mathbb{V})$ be an irreducible representation of Spin(m). If we restrict this representation to the maximal abelian subgroup T of Spin(m), then the space $\mathbb{V}$ splits into weight (sub)spaces generated by eigenvectors w, satisfying

$$\rho_{\mathbb{V}}\left(\exp\left(\frac{1}{2}e_{12}t_1\right)\exp\left(\frac{1}{2}e_{34}t_2\right)\cdots\exp\left(\frac{1}{2}e_{(2n-1)(2n)}t_n\right)\right)w \\ = \exp\left(i(l_1t_1 + l_2t_2 + \ldots + l_nt_n)\right)w, \quad (2.8)$$

where the eigenvalues are determined by the n-tuples $\lambda = (l_1,\ldots,l_n)$, so called *weights*, consisting entirely of either integer or half-integer numbers. This gives rise to two classes of representations. The weights can be ordered lexicographically: $\lambda = (l_1,\ldots,l_n) > \lambda' = (l'_1,\ldots,l'_n)$ if the first non-zero difference $l_i - l'_i$ is positive. As a matter of fact there is a 1-1 correspondence between each irreducible representation $\mathbb{V}$ and the largest weight λ it contains. These largest weights are called *highest weights* and are of the form

$$\begin{aligned} \lambda = (l_1,\ldots,l_n) &\ : \ l_1 \geq l_2 \geq \cdots \geq l_n \text{ if } m = 2n+1, \\ \lambda = (l_1,\ldots,l_n) &\ : \ l_1 \geq l_2 \geq \cdots \geq |l_n| \text{ if } m = 2n, \end{aligned}$$

where all $l_i \in \mathbb{Z}$, or all $l_i - \frac{1}{2} \in \mathbb{Z}$. The generating eigenvector w_λ of this eigenspace is called the *highest weight vector*.

Of particular interest are the highest weights of the form

$$(1,0,\ldots,0),(1,1,0,\ldots,0),\ldots,(1,\ldots,1) \text{ and } \left(\frac{1}{2},\ldots,\frac{1}{2}\right)$$

in the case of odd dimension, and

$$(1,0,\ldots,0),\ldots,(1,\ldots,1),(1,\ldots,1,-1) \text{ and } \left(\frac{1}{2},\ldots,\frac{1}{2}\right),\left(\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2}\right)$$

in case of even dimension.

Remark 2.10. The sets of weights stated above are larger then the sets of fundamental weights (see e.g. [47]).

Each representation of Spin(m) can be built by taking Cartan products of these representations. Essentially, this is the following. Take two irreducible representations $\mathbb{V}_1$ and $\mathbb{V}_2$ with highest weights $\lambda_1 = (l_1, \ldots, l_n)$ and $\lambda_2 = (l'_1, \ldots, l'_n)$, and highest weight vectors w_{λ_1} and w_{λ_2}. Then the tensor product $\mathbb{V}_1 \otimes \mathbb{V}_2$ is also a representation of Spin(m), be it not necessarily irreducible. The highest weight occurring in $\mathbb{V}_1 \otimes \mathbb{V}_2$ is given by $(l_1 + l'_1, \ldots, l_n + l'_n)$, and the corresponding weight space is generated by $w_{\lambda_1} \otimes w_{\lambda_2}$. As a matter of fact, by a theorem of Cartan, we know that the irreducible representation of Spin(m) with this highest weight occurs exactly once in this tensor product. This representation is called the Cartan product of $\mathbb{V}_1$ and $\mathbb{V}_2$, and is often denoted by $\mathbb{V}_1 \boxtimes \mathbb{V}_2$. We now show how these abstract considerations can be made concrete in the language of Clifford analysis (see e.g. [32] and [76]). First, we consider the case of odd dimension, $m = 2n + 1$, afterwards, the case of even dimension, $m = 2n$, as there are some fundamental differences between both cases.

2.4.1 The fundamental representations with half-integer valued highest weights in odd dimension

In this section, we take a look at the irreducible representation corresponding to the fundamental weight $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$. To this end, consider the following action of Spin(m) on the complex Clifford algebra $\mathbb{C}_m$:

$$l : \mathrm{Spin}(m) \to \mathrm{Aut}(\mathbb{C}_m),$$

explicitly given by

$$l(s)a = sa,$$

where $s \in \mathrm{Spin}(m)$ and $a \in \mathbb{C}_m$. However, this representation is not irreducible. Indeed, it is not hard to see that each subspace $\mathbb{C}_m J$ in (2.6) is $l(s)$-invariant. Moreover they are all isomorphic as Spin(m)-modules. The action of the maximal torus (see (2.8)) on the primitive idempotent I gives us

$$\begin{aligned} l(s)I &= \exp\left(\frac{1}{2}\left(t_1 e_{12} + \cdots + t_n e_{(2n-1)(2n)}\right)\right) I \\ &= \exp\left(\frac{i}{2}(t_1 + t_2 + \ldots + t_n)\right) I, \end{aligned}$$

so the highest weight is given by $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$.

2.4.2 The representations with integer valued highest weights in odd dimension

In order to construct models for irreducible representations of Spin(m) with highest weights of the form $\lambda = (l_1, l_2, \ldots, l_n)$, where each entry is integer valued, we need to introduce a new notation. Let us denote

$$\langle u_1 \wedge u_2 \wedge \cdots \wedge u_k, \mathfrak{f}_1 \wedge \mathfrak{f}_2 \wedge \ldots \wedge \mathfrak{f}_k \rangle = \det \begin{pmatrix} \langle u_1, \mathfrak{f}_1 \rangle & \cdots & \langle u_1, \mathfrak{f}_k \rangle \\ \vdots & \ddots & \vdots \\ \langle u_k, \mathfrak{f}_1 \rangle & \cdots & \langle u_k, \mathfrak{f}_k \rangle \end{pmatrix},$$

for all $1 \leq k \leq n$. Also, let us introduce the following unitary representation of Spin(m) on polynomials in k vector variables:

$$H(s)f(u_1, \ldots, u_k) = sf(\bar{s}u_1 s, \ldots \bar{s}u_k s)\bar{s}.$$

We now prove some auxiliary results.

Lemma 2.4. *For all indices $1 \leq a \leq p$, we have the relation*

$$\partial_a = x\partial_{xu_a x}x. \tag{2.9}$$

Proof. We prove this by direct calculation. Let $(u_a)_j$ be the j-th component of the vector variable u_a. From the chain rule and the fact that

$$xu_a x = -2\langle u_a, x\rangle x + |x|^2 u_a,$$

we get that

$$\begin{aligned} \partial_a &= \sum_{j=1}^m e_j \partial_{(u_a)_j} \\ &= \sum_{j=1}^m e_j \sum_{k=1}^m \partial_{(xu_a x)_k} \cdot \frac{\partial}{\partial (u_a)_j} \left(-2\langle u_a, x\rangle x_k + |x|^2 (u_a)_k\right) \\ &= \sum_{j=1}^m e_j \sum_{k=1}^m \partial_{(xu_a x)_k} \cdot (-2x_j x_k + |x|^2 \delta_{jk}) \\ &= |x|^2 \partial_{xu_a x} - 2\langle x, \partial_{xu_a x}\rangle x \\ &= x\partial_{xu_a x} x \ , \end{aligned}$$

which proves the statement. □

Lemma 2.5. *The operators $\langle \partial_a, \partial_b \rangle, \Delta_a$ and $\langle u_a, \partial_b \rangle$ are $H(s)$-invariant for all $1 \leq a \neq b \leq n$.*

Proof. If x is a unit vector, it follows from the previous lemma that

$$x\partial_a x = \partial_{x u_a x}. \tag{2.10}$$

Take a function $f(u_1,\dots,u_n)$. Then, for any $s \in \mathrm{Spin}(m)$, we get due to (2.10) that

$$\begin{aligned} H(s)\Delta_a f(u_1,\dots,u_n) &= \langle \partial_{\bar{s}u_a s}, \partial_{\bar{s}u_a s}\rangle f(\bar{s}u_1 s,\dots,\bar{s}u_n s)\\ &= \bar{s}\Delta_a s H(s) f(u_1,\dots,u_n)\\ &= \Delta_a H(s) f(u_1,\dots,u_n), \end{aligned}$$

because of (2.10) and the fact that any element of Spin(m) can be written as a product of unit vectors. For the operators $\langle \partial_a, \partial_b\rangle$, an entirely similar reasoning can be done. Finally, we get

$$\begin{aligned} H(s)\langle u_a, \partial_b\rangle f(u_1,\dots,u_n) &= \langle \bar{s}u_a s, \partial_{\bar{s}u_b s}\rangle f(\bar{s}u_1 s,\dots,\bar{s}u_n s)\\ &= \bar{s}\langle u_a, \partial_b\rangle s H(s) f(u_1,\dots,u_n)\\ &= \langle u_a, \partial_b\rangle H(s) f(u_1,\dots,u_n). \end{aligned}$$

□

Then one can calculate that for the action of the maximal torus of Spin(m), one has that

$$H(s)\langle u_1\wedge\cdots\wedge u_k, \mathfrak{f}_1\wedge\ldots\wedge\mathfrak{f}_k\rangle = \exp\left(i(t_1+\ldots+t_k)\right)\langle u_1\wedge\cdots\wedge u_k, \mathfrak{f}_1\wedge\ldots\wedge\mathfrak{f}_k\rangle,$$

meaning that $\langle u_1\wedge\cdots\wedge u_k, \mathfrak{f}_1\wedge\ldots\wedge\mathfrak{f}_k\rangle$ is a highest weight vector for the fundamental representation of Spin(m) with highest weight $(1,\dots,1,0,\dots,0)$, containing k nonzero entries. By taking tensor products of these fundamental representations, we find that

$$\begin{aligned} & w_{l_1,\dots,l_n}(u_1,\dots,u_n) = \langle u_1, \mathfrak{f}_1\rangle^{l_1-l_2}\langle u_1\wedge u_2, \mathfrak{f}_1\wedge\mathfrak{f}_2\rangle^{l_2-l_3}\cdots\\ & \times\langle u_1\wedge\cdots\wedge u_{n-1}, \mathfrak{f}_1\wedge\ldots\wedge\mathfrak{f}_{n-1}\rangle^{l_{n-1}-l_n}\langle u_1\wedge\cdots\wedge u_n, \mathfrak{f}_1\wedge\ldots\wedge\mathfrak{f}_n\rangle^{l_n} \end{aligned}$$

is a highest weight vector for the irreducible representation of Spin(m) with highest weight $(l_1,\dots,l_n)$. It is not hard to calculate that the polynomial $w_{l_1,\dots,l_n}(u_1,\dots,u_n)$ is in fact an element of $\mathcal{H}_{l_1,\dots,l_n}$. Since the operators defining simplicial harmonic functions are Spin(m)-invariant (Lemma 2.5), this means that the entire irreducible representation is contained in $\mathcal{H}_{l_1,\dots,l_n}$. However, this is not sufficient to conclude that the space of simplicial harmonic polynomials itself is irreducible. To this end, we need to go back to more abstract representation theory.[1] It holds that, inside the tensor product of fundamental representations, $\ker(\mathcal{C}(H) - C_{l_1,l_2,\dots,l_n})$ is isomorphic to

[1] One could also calculate the dimension of the space of simplicial harmonic polynomials and the irreducible representation with highest weight $(l_1,\dots,l_n)$, and verify that they are equal.

the irreducible representation with highest weight $(l_1, \ldots, l_n)$, where $\mathcal{C}(H)$ is the so-called *Casimir operator* related to the H-action, and $C_{l_1,l_2,\ldots,l_n}$ the eigenvalue of the Casimir operator acting on the irreducible representation with highest weight $(l_1, \ldots, l_n)$. It has been shown in [75] that

$$\begin{aligned}\frac{1}{4}\mathcal{C}(H) &= \sum_{1\le i<j\le n}(L_{ij}^{u_1}+\cdots+L_{ij}^{u_n})^2\\ &= \sum_{j=1}^{n}\Delta_{u_j}^{LB}+2\sum_{1\le i<j\le n}\Delta_{u_i,u_j}^{LB}\\ &= \sum_{j=1}^{n}\Delta_{u_j}^{LB}+2\sum_{1\le i<j\le n}\left(\langle u_i,u_j\rangle\langle\partial_i,\partial_j\rangle-\langle u_j,\partial_i\rangle\langle u_i,\partial_j\rangle+\mathbb{E}_j\right).\end{aligned}$$

The action of $\mathcal{C}(H)$ on $\mathcal{H}_{l_1,\ldots,l_n}$ produces the eigenvalue

$$C_{l_1,\ldots,l_n} = -4\sum_{j=1}^{n} l_j(l_j+m-2j),$$

which is constant for the entire eigenspace $\mathcal{H}_{l_1,\ldots,l_n}$. This immediately proves that $\mathcal{H}_{l_1,\ldots,l_n}$ is in fact a model for the irreducible representation of $\mathrm{Spin}(m)$ with highest weight $(l_1, \ldots, l_n) \in \mathbb{Z}^n$.

Remark 2.11. For notational convenience, we will from now on denote highest weights of the form

$$(l_1, \ldots, l_k, 0, \ldots, 0)$$

as

$$\lambda = (l_1, \ldots, l_k).$$

In other words, we omit redundant zeros. Also, since there is a 1-1 correspondence between irreducible representations and its highest weights, we denote such representations by its highest weight.

2.4.3 Representations with half-integer valued highest weights in odd dimension

Finally, we take a closer look at irreducible $\mathrm{Spin}(m)$-representations with half-integer valued highest weights.

Remark 2.12. For half-integer valued highest weights, we introduce the notation

$$\lambda' = (l_1, \ldots, l_k)' := (l_1 + \frac{1}{2}, \ldots, l_k + \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}).$$

Here, the prime represents the Cartan product with the spinor space $\mathbb{S}$, after we have omitted redundant zeros.

Let us introduce the representation of Spin(m) on polynomials in k vector variables, given by

$$L(s)f(u_1,\ldots,u_k)=sf(\bar{s}u_1s,\ldots\bar{s}u_ks),$$

i.e. $L=H\otimes l$. For this representation, we have the following properties.

Lemma 2.6. *The operators $\langle\partial_a,\partial_b\rangle,\partial_a$ and $\langle u_a,\partial_b\rangle$ are $L(s)$-invariant for all $1\leq a\neq b\leq n$.*

Proof. Take a function $f(u_1,\ldots,u_n)$. Then for any $s\in\mathrm{Spin}(m)$, we get

$$\begin{aligned}L(s)\partial_a f(u_1,\ldots,u_n) &= s\partial_{\bar{s}u_as}f(\bar{s}u_1s,\ldots,\bar{s}u_ns)\\ &= s\bar{s}\partial_a sf(\bar{s}u_1s,\ldots,\bar{s}u_ns)\\ &= \partial_a L(s)f(u_1,\ldots,u_n),\end{aligned}$$

because of (2.10) and the fact that any element of Spin(m) can be written as a product of unit vectors. For the operators $\langle\partial_a,\partial_b\rangle$, the reasoning is as follows:

$$\begin{aligned}L(s)\langle\partial_a,\partial_b\rangle f(u_1,\ldots,u_n) &= s\langle\partial_{\bar{s}u_as},\partial_{\bar{s}u_bs}\rangle f(\bar{s}u_1s,\ldots,\bar{s}u_ns)\\ &= s\bar{s}\langle\partial_a,\partial_b\rangle sf(\bar{s}u_1s,\ldots,\bar{s}u_ns)\\ &= \langle\partial_a,\partial_b\rangle L(s)f(u_1,\ldots,u_n).\end{aligned}$$

Finally, for the last type of operators, we have

$$\begin{aligned}L(s)\langle u_a,\partial_b\rangle f(u_1,\ldots,u_n) &= s\langle\bar{s}u_as,\partial_{\bar{s}u_bs}\rangle f(\bar{s}u_1s,\ldots,\bar{s}u_ns)\\ &= s\bar{s}\langle u_a,\partial_b\rangle sL(s)f(\bar{s}u_1s,\ldots,\bar{s}u_ns)\\ &= \langle u_a,\partial_b\rangle L(s)f(u_1,\ldots,u_n).\end{aligned}$$

□

In view of Section 2.4.1 and 2.4.2, we suggest considering the highest weight vector

$$\begin{aligned}&w'_{l_1,\ldots,l_n}(u_1,\ldots,u_n)=\langle u_1,\mathfrak{f}_1\rangle^{l_1-l_2}\langle u_1\wedge u_2,\mathfrak{f}_1\wedge\mathfrak{f}_2\rangle^{l_2-l_3}\cdots\\ &\times\langle u_1\wedge\cdots\wedge u_{n-1},\mathfrak{f}_1\wedge\ldots\wedge\mathfrak{f}_{n-1}\rangle^{l_{n-1}-l_n}\langle u_1\wedge\cdots\wedge u_k,\mathfrak{f}_1\wedge\ldots\wedge\mathfrak{f}_k\rangle^{l_n}I\end{aligned}\tag{2.11}$$

for the irreducible representation with highest weight $(l_1,\ldots,l_n)'$. The L-action of the maximal torus on this function is given by

$$\begin{aligned}&L(s)w'_{l_1,\ldots,l_n}(u_1,\ldots,u_n)\\ &\quad=\exp\left(i\left(\left(l_1+\frac{1}{2}\right)t_1+\cdots+\left(l_n+\frac{1}{2}\right)t_n\right)\right)w'_{l_1,\ldots,l_n}(u_1,\ldots,u_n),\end{aligned}$$

whence it indeed is a highest weight vector for the irreducible Spin(m)-representation $(l_1, \ldots, l_n)'$. One can easily check that $w'_{l_1,\ldots,l_n}(u_1, \ldots, u_n)$ in fact is an element of $\mathcal{S}_{l_1,\ldots,l_n}$. Since the operators defining simplicial monogenic functions are Spin(m)-invariant (Lemma 2.6), the entire representation is contained in $\mathcal{S}_{l_1,\ldots,l_n}$. In order to prove that this space is irreducible, we take a look at the Casimir operator related to the L-representation:

$$\begin{aligned}\mathcal{C}(L) &= 4\sum_{i<j}(L^{u_1}_{ij} + \cdots + L^{u_n}_{ij} + \frac{1}{2}e_{ij})^2 \\ &= \mathcal{C}(H) + 4\sum_{j=1}^{n}\Gamma_{u_j} - \frac{m(m-1)}{2}.\end{aligned}$$

The action of $\mathcal{C}(L)$ on $\mathcal{S}_{l_1,\ldots,l_n}$ gives the eigenvalue

$$C'_{l_1,\ldots,l_n} = -4\sum_{j=1}^{n} l_j(l_j + m - 2j + 1) - \frac{m(m-1)}{2},$$

which is constant for the entire eigenspace $\mathcal{S}_{l_1,\ldots,l_n}$. So we can conclude that

$$\mathcal{S}_{l_1,\ldots,l_n} \cong (l_1, \ldots, l_n)'.$$

2.4.4 Representations in even dimension

In the even dimensional case ($m = 2n$), the construction of highest weight vectors is similar, except for the fact that there are now two inequivalent spinor spaces which lead to inequivalent basic representations of Spin(m), namely the spinor spaces

$$\mathbb{S}^+_{2n} = \frac{1+\theta}{2}\mathbb{S}_{2n} = \mathbb{C}^+_m I$$

and

$$\mathbb{S}^-_{2n} = \frac{1-\theta}{2}\mathbb{S}_{2n} = \mathbb{C}^-_m I = \mathbb{C}^+_m \mathfrak{f}^\dagger_n I.$$

The weights again are obtained from the action of the maximal torus, and are respectively given by $\left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, \frac{1}{2}, \ldots, -\frac{1}{2}\right)$. It is not hard to see that

$$\begin{aligned}&\langle u_1, \mathfrak{f}_1\rangle^{l_1-l_2}\langle u_1 \wedge u_2, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle^{l_2-l_3}\cdots \\ &\times \langle u_1 \wedge \cdots \wedge u_{n-1}, \mathfrak{f}_1 \wedge \ldots \wedge \mathfrak{f}_{n-1}\rangle^{l_{n-1}-l_n}\langle u_1 \wedge \cdots \wedge u_n, \mathfrak{f}_1 \wedge \ldots \wedge \mathfrak{f}_n\rangle^{l_n}\end{aligned}$$

again is a highest weight vector for the irreducible representation with highest weight $(l_1, l_2, \ldots, l_n)$,

$$\begin{aligned}&\langle u_1, \mathfrak{f}_1\rangle^{l_1-l_2}\langle u_1 \wedge u_2, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle^{l_2-l_3}\cdots \\ &\times \langle u_1 \wedge \cdots \wedge u_{n-1}, \mathfrak{f}_1 \wedge \ldots \wedge \mathfrak{f}_{n-1}\rangle^{l_{n-1}-l_n}\langle u_1 \wedge \cdots \wedge u_n, \mathfrak{f}_1 \wedge \ldots \wedge \mathfrak{f}^\dagger_n\rangle^{l_n}\end{aligned}$$

for the irreducible representation with highest weight $(l_1, l_2, \ldots, -l_n)$,

$$\langle u_1, \mathfrak{f}_1\rangle^{l_1-l_2}\langle u_1\wedge u_2, \mathfrak{f}_1\wedge\mathfrak{f}_2\rangle^{l_2-l_3}\cdots$$
$$\times\,\langle u_1\wedge\cdots\wedge u_{n-1}, \mathfrak{f}_1\wedge\ldots\wedge\mathfrak{f}_{n-1}\rangle^{l_{n-1}-l_n}\langle u_1\wedge\cdots\wedge u_n, \mathfrak{f}_1\wedge\ldots\wedge\mathfrak{f}_n\rangle^{l_n} I$$

is a highest weight vector for the irreducible representation with highest weight $(l_1, l_2, \ldots, l_n)'$, and

$$\langle u_1, \mathfrak{f}_1\rangle^{l_1-l_2}\langle u_1\wedge u_2, \mathfrak{f}_1\wedge\mathfrak{f}_2\rangle^{l_2-l_3}\cdots$$
$$\times\,\langle u_1\wedge\cdots\wedge u_{n-1}, \mathfrak{f}_1\wedge\ldots\wedge\mathfrak{f}_{n-1}\rangle^{l_{n-1}-l_n}\langle u_1\wedge\cdots\wedge u_n, \mathfrak{f}_1\wedge\ldots\wedge\mathfrak{f}_n^\dagger\rangle^{l_n}\mathfrak{f}_n^\dagger I$$

for the irreducible representation with highest weight $(l_1, l_2, \ldots, -l_n - 1)'$.

Fantasy, energy, self-confidence and self-criticism are the characteristic endowments of the mathematician.

Sophus Lie

3

General notions on Lie algebra representations

In this chapter, we discuss some classical Lie algebras. More specifically, we take a look at their root systems, and in most cases their finite-dimensional irreducible representations. First, we give a general introduction after which we focus on the simplest Lie algebra $\mathfrak{sl}(2,\mathbb{K})$, i.e. the algebra of traceless 2×2-matrices over a field $\mathbb{K}$. This algebra is of fundamental importance in order to describe representations of other Lie algebras. Afterwards, we turn our attention to $\mathfrak{sp}(2k,\mathbb{C})$, a Lie algebra for which there exists an elegant realisation in terms of differential operators playing a crucial role in what follows. Finally, we take a look at $\mathfrak{so}(2n,\mathbb{C})$ and $\mathfrak{so}(2n+1,\mathbb{C})$, the complex versions of the Lie algebras of $\mathrm{Spin}(2n)$ and $\mathrm{Spin}(2n+1)$, respectively.

3.1 General concepts

Before starting with specific Lie algebra representations, we introduce some general definitions. Remember that we already defined Lie algebras over a general field $\mathbb{K}$ in Definition 2.6. In this thesis, we restrict ourselves to the case where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$.

Definition 3.1. *A Lie algebra $\mathfrak{g}$ is called abelian if $[X,Y] = 0$ for all $X, Y \in \mathfrak{g}$.*

The definition of an algebra representation was introduced in Definition 2.9. In the case where the algebra is a Lie algebra, the definition is as follows.

Definition 3.2. *A representation $\rho_{\mathbb{V}}$ for a Lie algebra $\mathfrak{g}$ on a vector space $\mathbb{V}$ is a homomorphism*

$$\rho_{\mathbb{V}} : \mathfrak{g} \to \mathrm{End}(\mathbb{V}),$$

for which we have that for all $v \in \mathbb{V}$, $X, Y \in \mathfrak{g}$:

$$\rho_{\mathbb{V}}([X,Y])(v) = [\rho_{\mathbb{V}}(X), \rho_{\mathbb{V}}(Y)](v).$$

The bracket at the right-hand side is given by the commutator.

Suppose that the dimension of $\mathbb{V}$ is n, then $\mathrm{End}(\mathbb{V}) = \mathfrak{gl}(n,\mathbb{K})$, the Lie algebra of $(n \times n)$-matrices over the field $\mathbb{K}$.

A special representation of Lie algebras is the *adjoint representation*.

Definition 3.3. *The adjoint representation* ad *of a Lie algebra $\mathfrak{g}$ is defined as the homomorphism*

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}) : X \mapsto \mathrm{ad}(X),$$

where the action is given by $\mathrm{ad}(X)(Y) = [X,Y]$*, for all $X, Y \in \mathfrak{g}$.*

Remark 3.1. The adjoint representation indeed is a representation, since

$$\begin{aligned} [\mathrm{ad}(X), \mathrm{ad}(Y)](Z) &= [X,[Y,Z]] - [Y,[X,Z]] \\ &= [X,[Y,Z]] + [Y,[Z,X]] \\ &= \mathrm{ad}([X,Y])(Z), \end{aligned}$$

for all $X, Y, Z \in \mathfrak{g}$, due to the Jacobi-identity.

Definition 3.4. *An ideal $\mathfrak{h} \subset \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ is a subalgebra for which*

$$[X,\mathfrak{h}] \subset \mathfrak{h},$$

for all $X \in \mathfrak{g}$. Moreover, an ideal $\mathfrak{h}$ is called maximal in $\mathfrak{g}$ if there exists no ideal $\mathfrak{h}_1$ of $\mathfrak{g}$ such that

$$\mathfrak{h} \subsetneq \mathfrak{h}_1 \subsetneq \mathfrak{g}.$$

Definition 3.5. *A non-abelian Lie algebra $\mathfrak{g}$ is called simple if $\mathfrak{g}$ has no nontrivial ideal (i.e. not 0 or $\mathfrak{g}$ itself).*

Definition 3.6. *For a Lie algebra $\mathfrak{g}$, the derived series $(\mathfrak{g}^{(n)})$ is inductively defined as*

$$\mathfrak{g}^{(0)} = \mathfrak{g} \quad \textit{and} \quad \mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}].$$

In fact, simple and solvable Lie algebras turn out to be the building blocks of general Lie algebras.

Definition 3.7. *An ideal* $\mathfrak{h} \subset \mathfrak{g}$ *is called solvable if there exists* $n \in \mathbb{N}$ *such that* $\mathfrak{h}^{(n)} = 0$.

Definition 3.8. *A Lie algebra* $\mathfrak{g}$ *is called semisimple if* $\mathfrak{g}$ *has no solvable ideals.*

In particular, all simple Lie algebras are semisimple. For semisimple Lie algebras, we have the following theorems concerning representations.

Theorem 3.1 (Jordan decomposition)**.** *Let* $\mathfrak{g}$ *be a semisimple Lie algebra. Then for any element* $X \in \mathfrak{g}$*, the semisimple part* X_s *and the nilpotent part* X_n *also are in* $\mathfrak{g}$.

Theorem 3.2 (Weyl's theorem)**.** *Suppose* $\mathbb{V}$ *is a finite-dimensional representation of a semisimple Lie algebra* $\mathfrak{g}$*, and suppose* $\mathbb{W} \subset \mathbb{V}$ *is a subspace which is invariant under the action of* $\mathfrak{g}$*, then there also exists an invariant subspace* $\mathbb{W}^\perp$ *such that* $\mathbb{V} = \mathbb{W} \oplus \mathbb{W}^\perp$.

Remark 3.2. This theorem tells us that each representation of a semisimple Lie algebra can be decomposed in irreducible representations. The semisimplicity is crucial. Take for example the Lie algebra $\mathfrak{b}_3$ of upper triangular (3×3)-matrices and let the action of an element $X \in \mathfrak{b}_3$ this Lie algebra on a vector $v \in \mathbb{C}^3$ be given by

$$\rho(X)[v] = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{22}y + a_{23}z \\ a_{33}z \end{pmatrix}.$$

The x-axis in $\mathbb{C}^3$ obviously is an invariant subspace. There is however no invariant complement.

Remark 3.3. Without going into too much detail, let us also give an example which proves that the finite dimension is a necessary condition for Weyl's theorem. One can easily check that, if we set $m = 2$, the operator algebra

$$\mathrm{Alg}_{\mathbb{C}} \left\{ |x|^2 \partial_{x_2} - 2x_2 \mathbb{E}_x, \partial_{x_2}, 2\mathbb{E}_x \right\}$$

is isomorphic to the simple Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ (see next section). Then,

$$\mathfrak{sl}(2, \mathbb{C})[x_2],$$

the set of polynomials obtained by repeatedly applying operators of $\mathfrak{sl}(2, \mathbb{C})$ on x_2, is an infinite-dimensional module for $\mathfrak{sl}(2, \mathbb{C})$. Indeed, put $X = |x|^2 \partial_{x_2} - 2x_2 \mathbb{E}_x$ and $Y = \partial_{x_2}$, then applying X and Y repeatedly on x_2 results in the chain depicted in Figure 3.1. Since $X[1] = Y[1] = 0$, we find that $\mathrm{Span}_{\mathbb{C}}\{1\}$ is a submodule of $\mathfrak{sl}(2, \mathbb{C})[x_2]$. However, this submodule has no invariant complement.

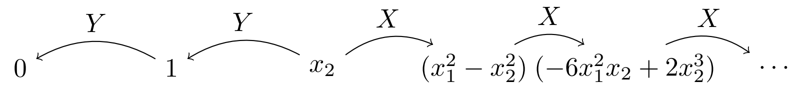


Figure 3.1: Action of X and Y on the eigenspaces of an infinite-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$

3.2 The Lie algebra $\mathfrak{sl}(2,\mathbb{C})$

In this section, we take a look at the special linear algebra $\mathfrak{sl}(2,\mathbb{C})$.

Definition 3.9. *The Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ is the space of all traceless (2×2)-matrices :*

$$\mathfrak{sl}(2,\mathbb{C}) = \{X \in \mathfrak{gl}(2,\mathbb{C}) : \mathrm{Tr}(X) = 0\},$$

where the Lie bracket is given by the commutator.

Define the following (2×2)-matrices:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It is easily seen that we have the following theorem.

Theorem 3.3. *The matrices H, X and Y form a basis for $\mathfrak{sl}(2,\mathbb{C})$.*

As $\mathfrak{sl}(2,\mathbb{C})$ is an algebra, let us take a look at the commutators between these basis elements. We get the relations

$$[X,Y] = H, [H,X] = +2X \text{ and } [H,Y] = -2Y. \tag{3.1}$$

Let $\mathbb{V}$ be a finite-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$. Then we have that the action of H on $\mathbb{V}$ is diagonalisable due to Jordan's theorem, resulting in a decomposition

$$\mathbb{V} = \bigoplus_{\alpha} V_\alpha,$$

where α runs over a *finite* set of complex numbers, such that for all $v \in V_\alpha$, we have that $H(v) := \rho_{\mathbb{V}}(H)(v) = \alpha v$.

Example 3.1. For the adjoint representation of $\mathfrak{sl}(2,\mathbb{C})$, we have the decomposition

$$\begin{aligned} \mathfrak{sl}(2,\mathbb{C}) &= V_{-2} \oplus V_0 \oplus V_2 \\ &= \mathrm{span}_{\mathbb{C}}(Y) \oplus \mathrm{span}_{\mathbb{C}}(H) \oplus \mathrm{span}_{\mathbb{C}}(X), \end{aligned}$$

due to the relations in (3.1). For general finite-dimensional representations of $\mathfrak{sl}(2, \mathbb{C})$, the following lemma holds.

Lemma 3.1. *If v is an eigenvector for the action of H on a finite-dimensional irreducible representation $\mathbb{V}$ of $\mathfrak{sl}(2, \mathbb{C})$ with eigenvalue $\alpha \in \mathbb{C}$, then $X(v)$ is an eigenvector for the action of H with eigenvalue $\alpha + 2$.*

Proof. The action of H on $X(v)$ is given by

$$\begin{aligned} \rho_{\mathbb{V}}(H)(X(v)) &= H(X(v)) \\ &= X(H(v)) + [H, X](v) \\ &= (\alpha + 2)X(v). \end{aligned}$$

This proves the lemma. □

A similar lemma holds for $Y(v)$.

Lemma 3.2. *If v is an eigenvector for the action of H on a finite-dimensional irreducible representation $\mathbb{V}$ of $\mathfrak{sl}(2, \mathbb{C})$ with eigenvalue $\alpha \in \mathbb{C}$, then $Y(v)$ is an eigenvector for the action of H with eigenvalue $\alpha - 2$.*

Proof. The proof is entirely similar to the previous proof. □

From the last two lemmata, we get that the following properties hold for X and Y, and each irreducible finite-dimensional representation $\mathbb{V} = \bigoplus V_\alpha$ of $\mathfrak{sl}(2, \mathbb{C})$:

$$X : V_\alpha \to V_{\alpha+2} \tag{3.2}$$

and

$$Y : V_\alpha \to V_{\alpha-2} \tag{3.3}$$

for each eigenvalue $\alpha \in \mathbb{C}$. This property is of crucial importance for the next theorem.

Theorem 3.4. *For each two eigenvalues α_1 and α_2 for the action of H on a finite-dimensional irreducible representation $\mathbb{V}$ of $\mathfrak{sl}(2, \mathbb{C})$, we have that $\alpha_1 = \alpha_2 \mod 2$.*

Proof. Since $\mathbb{V}$ is irreducible, we have that $\mathbb{V}$ can be constructed by taking an eigenspace V_α, and repeatedly applying X, Y and H on this eigenspace. The theorem then directly follows from (3.2) and (3.3). □

To illustrate this last theorem, we can represent the action of $\mathfrak{sl}(2, \mathbb{C})$ on a finite-dimensional irreducible representation $\mathbb{V}$ as in Figure 3.2, where n is chosen to be the largest number such that $X(v) = 0$ for all $v \in V_n$. In other

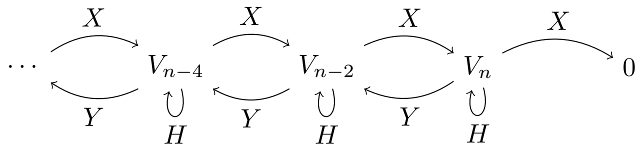


Figure 3.2: Action of X, Y and H on the eigenspaces of a finite-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$

words, n is the largest eigenvalue corresponding to a nontrivial eigenspace. Such n exists, since $\mathbb{V}$ is finite-dimensional. We prove in the next theorem that n must be a positive integer.

Theorem 3.5. *Let $\mathbb{V}$ be a nontrivial finite-dimensional irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$, then it holds that*

$$\mathbb{V} = \text{span}_{\mathbb{C}}\{v, Y(v), Y^2(v), \dots\},$$

with $v \in \mathbb{V}$ such that $X(v) = 0$ and $n \in \mathbb{N}$.

Proof. Since $\mathbb{V}$ is finite-dimensional and irreducible, it follows from Theorem 3.4 that there exists an eigenvector $v \in \mathbb{V}$ for the action of H, with eigenvalue $\alpha \in \mathbb{C}$ such that $\mathbb{R}\text{e}(\alpha)$ is maximal. Then, again since $\mathbb{V}$ is maximal, there exists $p \in \mathbb{N}$ such that $Y^p(v) = 0$. Suppose that p is minimal. Then, from the commutation relations (3.1), we find that

$$0 = X(Y^p(v)) = p(\alpha - p + 1)Y^{p-1}(v).$$

Since we have chosen p to be the smallest integer for which this is true, $Y^{p-1}(v) \neq 0$. $\mathbb{V}$ is nontrivial, so also $p \neq 0$. Hence, we find that $\alpha = p - 1$. ☐

Definition 3.10. *Suppose $\mathbb{V}$ is a finite-dimensional representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, with eigenspace decomposition*

$$\mathbb{V} = \bigoplus_{\alpha} V_\alpha,$$

for the action of H on $\mathbb{V}$. A vector $v \in V_\alpha$ is called a weight vector with weight α. A vector $v \in V_\alpha$, such that $X(v) = 0$ for each $v \in V_\alpha$, is called a highest weight vector of $\mathbb{V}$ with highest weight α, and V_α is the highest weight space.

The theorem tells us that Figure 3.2 translates into Figure 3.3.

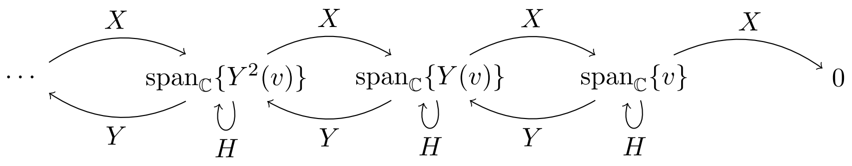


Figure 3.3: Action of X, Y and H on a finite-dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$ with highest weight vector v

Corollary 3.1. *All eigenspaces for the action of H of a finite-dimensional irreducible representation $\mathbb{V}$ of $\mathfrak{sl}(2, \mathbb{C})$, with highest weight $n \in \mathbb{N}$, are 1-dimensional*

Proof. For all $k \in \mathbb{N}$ and $v \in V_n$, $Y^k(v) \in V_{n-2k}$. Invoking Theorem 3.5 then finishes the proof. □

We can combine our findings in another theorem.

Theorem 3.6. *For each $n \in \mathbb{N}$, there exists (up to an isomorphism) a unique irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$. This representation is $(n+1)$-dimensional and the eigenvalues α for the action H on the eigenspaces V_α are given by*

$$\{n, n-2, n-4, \ldots, -(n-4), -(n-2), -n\}.$$

From this theorem, it follows that each irreducible finite-dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$ is uniquely determined by its highest weight n. That is why we will from now on denote such a representation by $\mathbb{V}_n$. The scheme for such a representation is then as in Figure 3.4.

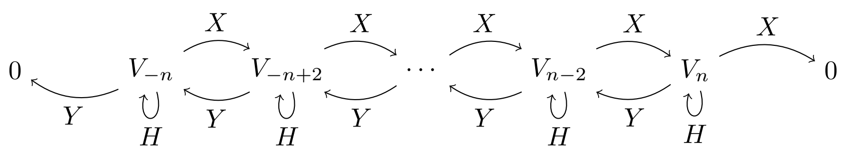


Figure 3.4: Action of X, Y and H on a finite-dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$

Example 3.2. In case of the adjoint representation, where $\mathbb{V} = \mathfrak{sl}(2, \mathbb{C})$, we have the relations $[Y, X] = -H$, and $[Y, [Y, X]] = -2Y$. Then the scheme is given in Figure 3.5.

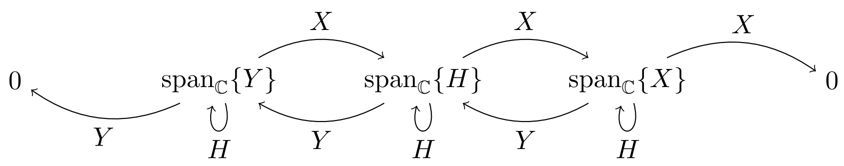


Figure 3.5: Action of X, Y and H on the adjoint representation of $\mathfrak{sl}(2, \mathbb{C})$.

Remark 3.4. In case of the adjoint representation, the weights are also called *roots*, '2' being the *positive root*, and '-2' the *negative root.* X and Y are correspondingly called *positive* and *negative root vectors.* The set of all roots is also called the *root system.*

Note that Figure 3.4 is symmetric. Indeed, suppose that v is a highest weight vector of $\mathbb{V}_n$, then

$$\mathbb{V}_n = \mathrm{span}_{\mathbb{C}}\{v, Y(v), Y^2(v), \ldots, Y^{n-1}(v), Y^n(v)\}.$$

Because of (3.2) and Theorem 3.5, one can also write this as

$$\mathbb{V}_n = \mathrm{span}_{\mathbb{C}}\{X^n(Y^n(v)), X^{n-1}(Y^n(v)), \ldots, X^2(Y^n(v)), X(Y^n(v)), Y^n(v)\},$$

or, by putting $v' = Y^n(v)$,

$$\mathbb{V}_n = \mathrm{span}_{\mathbb{C}}\{X^n(v'), X^{n-1}(v'), \ldots, X^2(v'), X(v'), v'\}.$$

This means that the roles of X and Y as positive and negative root vectors, respectively, can be reversed.

Example 3.3. We have the relations

$$[\langle u_1, \partial_2\rangle, \langle u_2, \partial_1\rangle] = \mathbb{E}_1 - \mathbb{E}_2,$$

$$[\mathbb{E}_1 - \mathbb{E}_2, \langle u_1, \partial_2\rangle] = 2\langle u_1, \partial_2\rangle,$$

and

$$[\mathbb{E}_1 - \mathbb{E}_2, \langle u_2, \partial_1\rangle] = -2\langle u_2, \partial_1\rangle.$$

This means that we have found a model for $\mathfrak{sl}(2, \mathbb{C})$ by putting $X = \langle u_1, \partial_2\rangle$, $Y = \langle u_2, \partial_1\rangle$ and $H = \mathbb{E}_1 - \mathbb{E}_2$. Then $\langle u_1, \mathfrak{f}_1\rangle^{l_1}$, $l_1 \in \mathbb{N}$ is a highest weight vector for the irreducible representation $\mathbb{V}_{l_1}$ of $\mathfrak{sl}(2, \mathbb{C})$ where the weight spaces V_{l_1-2a} are generated by $\langle u_1, \mathfrak{f}_1\rangle^{l_1-a}\langle u_2, \mathfrak{f}_1\rangle^{l_1}$.

3.3 Symplectic Lie algebras

In this section, we take a look at representations of symplectic Lie algebras $\mathfrak{sp}(2k,\mathbb{C})$. Define the $(2k \times 2k)$-matrix M as the block matrix

$$M = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix},$$

where I_k is the $(k \times k)$ unit matrix. Then the symplectic Lie algebra $\mathfrak{sp}(2k,\mathbb{C})$ is defined as the matrix algebra

$$\mathfrak{sp}(2k,\mathbb{C}) = \left\{X \in \mathbb{C}^{2k\times 2k} : X^T M + MX = 0\right\}. \tag{3.4}$$

Suppose X is a block matrix of the form

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A, B, C, D \in \mathbb{C}^{k\times k}$. Then the defining relation in (3.4) tells us that

$$B = B^T, C = C^T \text{ and } A = -D^T. \tag{3.5}$$

In the case of $\mathfrak{sl}(2,\mathbb{C})$, we used H to define the eigenspaces of irreducible representations under the action of H. We determine such a subspace as well in the case of $\mathfrak{sp}(2k,\mathbb{C})$. As elements of this subspace must commute with each other, they all have to be diagonal matrices satisfying the relations (3.5). An obvious basis is the set of matrices

$$H_i := E_{i,i} - E_{k+i,k+i},$$

$E_{i,j}$ being the matrix $(E_{i,j})_{k,l} = \delta_{ik}\delta_{jl}$, for all $1 \le i, j \le k$. This subalgebra is called a Cartan subalgebra and is denoted by

$$\mathfrak{h} = \operatorname{span}_{\mathbb{C}}\{H_i : 1 \le i \le k\}.$$

Correspondingly, take a basis for the dual vector space $\mathfrak{h}^*$, and denote the basis elements by L_i for all $1 \le i \le k$, such that $L_j(H_i) = \delta_{ij}$. Let us now expand the basis for $\mathfrak{h} \subset \mathfrak{sp}(2k,\mathbb{C})$ to a full basis of the symplectic Lie algebra. The first relation in (3.5) gives us the basis elements

$$Y_{i,j} := E_{i,k+j} + E_{j,k+i}$$

and

$$U_i := E_{i,k+i},$$

for all $1 \le i, j \le k$. The second relation in (3.5) gives us the basis elements

$$Z_{i,j} := E_{k+i,j} + E_{k+j,i}$$

and

$$V_i := E_{k+i,i},$$

for all $1 \leq i, j \leq k$. And finally, the third relation in (3.5) determines the basis elements

$$X_{i,j} := E_{i,j} - E_{k+j,k+i},$$

for all $1 \leq i, j \leq k$. Hence we have that

$$\mathfrak{sp}(2k, \mathbb{C}) = \mathrm{span}_{\mathbb{C}}\{H_i, X_{i,j}, Y_{i,j}, Z_{i,j}, U_i, V_i\}.$$

Next, let us determine the roots of $\mathfrak{sp}(2k, \mathbb{C})$. Remember that the roots of $\mathfrak{sl}(2, \mathbb{C})$ are the eigenvalues corresponding to the weight spaces of the adjoint representation. To this end, we have to take a look at the adjoint representation of $\mathfrak{sp}(2k, \mathbb{C})$.

The basis elements defined above are eigenvectors for the action of $\mathfrak{h}$, in the sense that they are eigenvectors for the action of each H_a, $1 \leq a \leq k$. This means that the weight of the representation is in fact an k-tuple of eigenvalues. Indeed, we have the relations

$$\begin{aligned}
\mathrm{ad}(H_a)(X_{i,j}) &= ((L_i - L_j)(H_a))X_{i,j} = (\delta_{ia} - \delta_{ja})X_{i,j} \\
\mathrm{ad}(H_a)(Y_{i,j}) &= ((L_i + L_j)(H_a))Y_{i,j} = (\delta_{ia} + \delta_{ja})Y_{i,j} \\
\mathrm{ad}(H_a)(Z_{i,j}) &= ((-L_i - L_j)(H_a))Z_{i,j} = (-\delta_{ia} - \delta_{ja})Z_{i,j} \\
\mathrm{ad}(H_a)(U_i) &= ((2L_i)(H_a))U_i = (2\delta_{ia})U_i \\
\mathrm{ad}(H_a)(V_i) &= ((-2L_i)(H_a))V_i = (-2\delta_{ia})V_i.
\end{aligned}$$

For instance, U_i is an element of the weight space for the adjoint representation of $\mathfrak{sp}(2k, \mathbb{C})$ corresponding to the root $(0, \ldots, 0, 2, 0, \ldots, 0)$, with 2 on the i-th position. Note that from these calculations, it follows that we can denote these roots in terms of elements in $\mathfrak{h}^*$.

root vector	root
$X_{i,j}$	$L_i - L_j$
$Y_{i,j}$	$L_i + L_j$
$Z_{i,j}$	$-L_i - L_j$
U_i	$2L_i$
V_i	$-2L_j$

Table 3.1: Root vectors for $\mathfrak{sp}(2k, \mathbb{C})$ and their respective roots

Example 3.4. In the case where $k = 1$, we have the roots $2L_1$ and $-2L_1$. For the adjoint representation, we thus have two eigenspaces, one with eigenvalue '2' and one with eigenvalue '-2'. This is as expected, since $\mathfrak{sp}(2, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C})$. We can graphically represent the roots of $\mathfrak{sp}(2, \mathbb{C})$ by two dots on a single axis (see Figure 3.6).

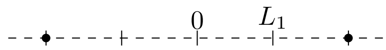


Figure 3.6: The root system of $\mathfrak{sp}(2,\mathbb{C})$

Example 3.5. Consider the case $k = 2$. Then the roots of $\mathfrak{sp}(4,\mathbb{C})$ are given by $\pm L_1 \pm L_2$, $\pm 2L_1$ and $\pm 2L_2$. Also this root system can be graphically represented by means of dots on a 2-dimensional grid (see Figure 3.7).

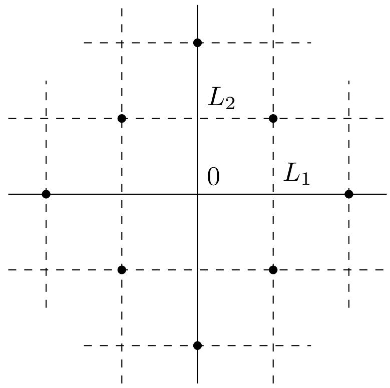


Figure 3.7: The root system of $\mathfrak{sp}(4,\mathbb{C})$

For the adjoint representation, the relations between the roots and the actual weights of the corresponding weight spaces are given in Table 3.2.

root	eigenvalues	root	eigenvalues
$L_1 + L_2$	$(1,1)$	$2L_1$	$(2,0)$
$L_1 - L_2$	$(1,-1)$	$-2L_1$	$(-2,0)$
$-L_1 + L_2$	$(-1,1)$	$2L_2$	$(0,2)$
$-L_1 - L_2$	$(1,1)$	$-2L_1$	$(0,-2)$

Table 3.2: Roots of $\mathfrak{sp}(2k,\mathbb{C})$ and the corresponding eigenvalues for the adjoint representation

We get a decomposition of the form

$$\mathfrak{sp}(2k,\mathbb{C}) = \mathfrak{h} \oplus \bigoplus_{1\le i\le j\le k} \left(\mathfrak{g}_{L_i+L_j} \oplus \mathfrak{g}_{-L_i-L_j}\right) \oplus \bigoplus_{1\le i<j\le k} \left(\mathfrak{g}_{L_i-L_j} \oplus \mathfrak{g}_{-L_i+L_j}\right),$$

where each $\mathfrak{g}_{\pm L_i \pm L_j}$ is the root space corresponding to the root $\pm L_i \pm L_j$. Remember that in the case of $\mathfrak{sl}(2,\mathbb{C})$, we made a distinction between positive and negative roots. In the previous section, we found that there is no unique way to do this. In the case of $\mathfrak{sp}(2k,\mathbb{C})$, this categorisation is done as follows. Consider the real linear functional l defined by k different real numbers c_i such that

$$l(a_1L_1 + \cdots + a_kL_k) = a_1c_1 + \cdots + a_kc_k.$$

Then a root α is considered *positive* if

$$l(\alpha) > 0$$

and *negative* if

$$l(\alpha) < 0.$$

One has to put an ordering on the constants c_i to fix the positive and negative roots. In literature (e.g. [47]), usually the ordering

$$c_1 > c_2 > \cdots > c_k > 0 \tag{3.6}$$

is taken. In this thesis, however, also the ordering

$$c_k < c_{k-1} < \cdots < c_1 < 0 \tag{3.7}$$

is considered (see Chapter 4). This does make a big difference, as for the ordering (3.6), the set of positive roots Δ^+ is given by

$$\Delta^+ = \{L_i + L_j\}_{i\leq j} \cup \{L_i - L_j\}_{i<j}$$

and the set of negative roots Δ^- by

$$\Delta^- = \{-L_i - L_j\}_{i\leq j} \cup \{L_i - L_j\}_{i>j}\,.$$

When assuming ordering (3.7), the set of positive roots is given by

$$\Delta^+ = \{-L_i - L_j\}_{i\leq j} \cup \{L_i - L_j\}_{i<j}$$

and the set of negative roots by

$$\Delta^- = \{L_i + L_j\}_{i\leq j} \cup \{L_i - L_j\}_{i>j}\,.$$

Remark 3.5. The ordering on the c_i can be represented by a hyperplane in the space $\mathfrak{h}^*$, in the sense that all roots on one side of the hyperplane are positive, and the roots on the other side are negative. For instance, in the case of $\mathfrak{sp}(4,\mathbb{C})$ and ordering (3.7), we have that all roots below the thick line are positive, and all roots above it are negative (see Figure 3.8).

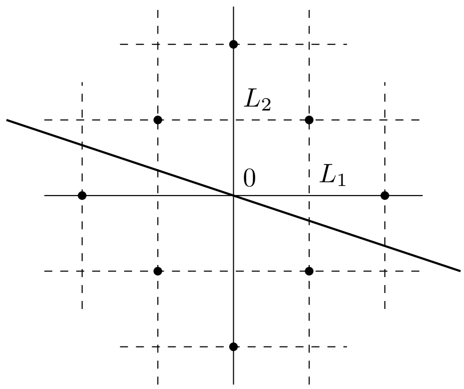


Figure 3.8: The root system of $\mathfrak{sp}(4,\mathbb{C})$

In Clifford analysis, $\mathfrak{sp}(2k,\mathbb{C})$ can be realised as an operator algebra by means of the isomorphism

$$\begin{array}{rcl} X_{i,j} & \mapsto & \langle u_i, \partial_j \rangle \\ Y_{i,j} & \mapsto & \langle u_i, u_j \rangle \\ Z_{i,j} & \mapsto & \langle \partial_i, \partial_j \rangle \\ U_i & \mapsto & |u_i|^2 \\ V_i & \mapsto & \Delta_i, \end{array} \tag{3.8}$$

for all $1 \leq i \neq j \leq k$. Using ordering (3.7), the positive root vectors are given by $\Delta_i, \langle \partial_i, \partial_j \rangle$ and $\langle u_i, \partial_j \rangle$, for all $i < j$. The fact that these are exactly the defining operators of the simplicial harmonic polynomials (Definition 2.18) will be used frequently in this thesis.

3.4 Orthogonal Lie algebras

Finally, we discuss the representations of orthogonal Lie algebras $\mathfrak{so}(m,\mathbb{C})$. In order to define these algebras, we have to make a distinction between even and odd m. In the case where $m = 2n$ is even, define the matrix

$$M = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

Then the special orthogonal Lie algebra $\mathfrak{so}(2n,\mathbb{C})$ is defined as the matrix algebra

$$\mathfrak{so}(2n,\mathbb{C}) = \left\{ X \in \mathbb{C}^{2n\times 2n} : X^T M + MX = 0 \right\}. \tag{3.9}$$

Remark 3.6. Usually, the identity matrix is used to define the special orthogonal algebra instead of M. However, because of the similarities with the symplectic case, we choose this alternative form.

Suppose X is a block matrix of the form

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A, B, C, D \in \mathbb{C}^{n\times n}$. Then the defining relation in (3.9) tells us that

$$B = -B^T, C = -C^T \text{ and } A = -D^T. \tag{3.10}$$

In the case where $m = 2n+1$, we define the matrix M as the $(2n+1) \times (2n+1)$-matrix

$$M = \begin{pmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The special orthogonal Lie algebra is then defined as

$$\mathfrak{so}(2n+1,\mathbb{C}) = \left\{ X \in \mathbb{C}^{(2n+1)\times(2n+1)} : X^T M + M X^T = 0 \right\}. \tag{3.11}$$

If X is a block matrix of the form

$$X = \begin{pmatrix} A & B & E \\ C & D & F \\ G & H & J \end{pmatrix},$$

where $A, B, C, D \in \mathbb{C}^{n\times n}$, $E, F, G^T, H^T \in \mathbb{C}^{n\times 1}$ and $J \in \mathbb{C}^{1\times 1}$. From (3.11), it follows that

$$B = -B^T, C = -C^T, A = -D^T, E = -H^T, F = -G^T \text{ and } J = 0.$$

The Cartan subalgebra of $\mathfrak{so}(m,\mathbb{C})$ can be chosen as

$$\mathfrak{h} = \operatorname{span}_{\mathbb{C}} \left\{ H_i := E_{i,i} - E_{n+i,n+i} : 1 \leq i \leq n \right\}.$$

The dual vector space $\mathfrak{h}^*$ then is spanned by the same basis elements as $\mathfrak{sp}(2k,\mathbb{C})$, namely L_i for all $1 \leq i \leq n$ such that $L_i(H_j) = \delta_{ij}$. This basis of $\mathfrak{h} \subset \mathfrak{so}(m,\mathbb{C})$ can be extended to a full basis of $\mathfrak{so}(m,\mathbb{C})$ as follows. From the relation $A = -D^T$, we get the basis elements

$$X_{i,j} = E_{i,j} - E_{n+j,n+i}, \text{ for all } 1 \leq i,j \leq n.$$

The relations $B = B^T$ and $C = C^T$ yield the basis elements

$$Y_{i,j} = E_{i,n+j} - E_{j,n+i}, \text{ for all } 1 \leq i,j \leq n$$

and

$$Z_{i,j} = E_{n+i,j} - E_{n+j,i}, \text{ for all } 1 \leq i,j \leq n$$

respectively. These matrices $H_i, X_{i,j}, Y_{i,j}$ and $Z_{i,j}$ give us a full basis for the Lie algebra $\mathfrak{so}(2n,\mathbb{C})$. If $m = 2n+1$, we get some extra basis elements from the relations $E = -H^T$ and $F = -G^T$, namely

$$U_i = E_{i,2n+1} - E_{2n+1,n+i}, \text{ for all } 1 \leq i \leq n$$

and

$$V_i = E_{n+i,2n+1} - E_{2n+1,i}, \text{ for all } 1 \leq i \leq n.$$

Hence the orthogonal Lie algebra is spanned by

$$\mathfrak{so}(2n,\mathbb{C}) = \text{span}_{\mathbb{C}} \{H_i, X_{i,j}, Y_{i,j}, Z_{i,j} | 1 \leq i,j \leq n\}$$

and

$$\mathfrak{so}(2n+1,\mathbb{C}) = \text{span}_{\mathbb{C}} \{H_i, X_{i,j}, Y_{i,j}, Z_{i,j}, U_i, V_i | 1 \leq i,j \leq n\}.$$

To determine the roots of $\mathfrak{so}(m,\mathbb{C})$, we take a look at its adjoint representation. Analogously to the case of $\mathfrak{sp}(2k,\mathbb{C})$, we find that

$$\begin{aligned}
\text{ad}(H_a)(X_{i,j}) &= ((L_i - L_j)(H_a))X_{i,j} = (\delta_{ia} - \delta_{ja})X_{i,j} \\
\text{ad}(H_a)(Y_{i,j}) &= ((L_i + L_j)(H_a))Y_{i,j} = (\delta_{ia} + \delta_{ja})Y_{i,j} \\
\text{ad}(H_a)(Z_{i,j}) &= ((-L_i - L_j)(H_a))Z_{i,j} = (-\delta_{ia} - \delta_{ja})Z_{i,j}
\end{aligned}$$

for the even-dimensional case. In the odd-dimensional case, we have the additional relations

$$\begin{aligned}
\text{ad}(H_a)(U_i) &= ((L_i)(H_a))U_i = (\delta_{ia})U_i \\
\text{ad}(H_a)(V_i) &= ((-L_i)(H_a))V_i = (-\delta_{ia})V_i.
\end{aligned}$$

Summarising (see Table 3.3), we find the following root vectors with their corresponding roots. Instead of discussing the algebra $\mathfrak{so}(m,\mathbb{C})$ as a matrix

$m = 2n$

root vector	root
$X_{i,j}$	$L_i - L_j$
$Y_{i,j}$	$L_i + L_j$
$Z_{i,j}$	$-L_i - L_j$

$m = 2n+1$

root vector	root
$X_{i,j}$	$L_i - L_j$
$Y_{i,j}$	$L_i + L_j$
$Z_{i,j}$	$-L_i - L_j$
U_i	L_i
V_i	$-L_j$

Table 3.3: Root vectors for $\mathfrak{so}(m,\mathbb{C})$ and their respective roots

algebra, we can also look at this algebra in a Clifford context. In Theorem

2.1, we already mentioned the Lie algebra $\mathbb{R}_m^{(2)}$ of bivectors in $\mathbb{R}_m$. In fact, the complexified space of bivectors $\mathbb{C}_m^{(2)}$ in $\mathbb{C}_m$ is isomorphic to $\mathfrak{so}(m, \mathbb{C})$. The isomorphism can be given in terms of Witt basis elements:

$$\begin{aligned}\varphi : \mathfrak{so}(m, \mathbb{C}) &\rightarrow \mathbb{C}_m^{(2)} : \\ X_{i,j} &\mapsto \mathfrak{f}_i \mathfrak{f}_j^\dagger \\ Y_{i,j} &\mapsto \mathfrak{f}_i \mathfrak{f}_j \\ Z_{i,j} &\mapsto \mathfrak{f}_i^\dagger \mathfrak{f}_j^\dagger \\ H_i &\mapsto \mathfrak{f}_i \mathfrak{f}_i^\dagger - \frac{1}{2},\end{aligned}$$

in the case where $m = 2n$. If $m = 2n + 1$, this isomorphism is extended by the mappings

$$\begin{aligned}U_i &\mapsto \mathfrak{f}_i e_{2n+1} \\ V_i &\mapsto \mathfrak{f}_i^\dagger e_{2n+1}.\end{aligned}$$

It is easily checked that this indeed is an isomorphism. For instance, we have on the one hand that

$$\begin{aligned}[X_{a,b}, Y_{c,d}] &= \delta_{bc} E_{a,n+d} - \delta_{bd} E_{a,n+c} + \delta_{bd} E_{c,n+a} - \delta_{bc} E_{d,n+a} \\ &= \delta_{bc} Y_{a,d} - \delta_{bd} Y_{a,c},\end{aligned}$$

while on the other hand,

$$\begin{aligned}[\mathfrak{f}_a \mathfrak{f}_b^\dagger, \mathfrak{f}_c \mathfrak{f}_d] &= \mathfrak{f}_a \mathfrak{f}_b^\dagger \mathfrak{f}_c \mathfrak{f}_d - \mathfrak{f}_c \mathfrak{f}_d \mathfrak{f}_a \mathfrak{f}_b^\dagger \\ &= \delta_{bc} \mathfrak{f}_a \mathfrak{f}_d - \delta_{bd} \mathfrak{f}_a \mathfrak{f}_c.\end{aligned}$$

Thus $\varphi([X_{a,b}, Y_{c,d}]) = [\varphi(X_{a,b}), \varphi(Y_{c,d})]$. Similar calculations hold for the other commutation relations. In bivector language, the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{so}(m, \mathbb{C})$ is given by

$$\mathfrak{h} = \text{span} \left\{ H_i = \mathfrak{f}_i \mathfrak{f}_i^\dagger - \frac{1}{2} : 1 \leq i \leq n \right\}.$$

Note that the term $\frac{1}{2}$ that appears might be a bit counter-intuitive, but this is due to the fact that $\mathfrak{f}_j \mathfrak{f}_j^\dagger = \frac{1}{2} - \frac{i}{2} e_j e_{n+j}$ itself is not a bivector.

Remark 3.7. The cardinality of the basis for $\mathfrak{h}$ also is called the **rank** of the Lie algebra. Hence $\mathfrak{so}(2n + 1, \mathbb{C})$ and $\mathfrak{so}(2n, \mathbb{C})$ are both Lie algebras of rank n.

As the Cartan elements H_i mutually commute, their action on any representation $\mathbb{V}$ of $\mathfrak{so}(m, \mathbb{C})$ can be diagonalised simultaneously. This means that finite-dimensional representations of simple Lie algebras can be decomposed as eigenspaces for the subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Each of these subspaces can be identified with a set of n eigenvalues, the so-called weight of the eigenspace. Hence there exists a finite set of weights W such that for each finite-dimensional representation, we have the decomposition

$$\mathbb{V} = \bigoplus_{\lambda \in W} V_\lambda$$

where

$$V_\lambda = \{v \in \mathbb{V} : H_i v = l_i v, 1 \leq i \leq n\},$$

for all $\lambda = (l_1, \ldots, l_n) \in W$.

Let us verify the results above in terms of bivectors and take a look at the adjoint representation of $\mathfrak{so}(m, \mathbb{C})$ first. In the case of $\mathfrak{so}(m, \mathbb{C}) = \mathfrak{so}(2n, \mathbb{C})$, the following relations hold for $p \neq q$:

$$\begin{aligned}
\mathrm{ad}(H_i)(\mathfrak{f}_p\mathfrak{f}_q) &= [H_i, \mathfrak{f}_p\mathfrak{f}_q] = \delta_{ip}\mathfrak{f}_p\mathfrak{f}_q + \delta_{iq}\mathfrak{f}_p\mathfrak{f}_q \\
\mathrm{ad}(H_i)(\mathfrak{f}_p\mathfrak{f}_q^\dagger) &= [H_i, \mathfrak{f}_p\mathfrak{f}_q^\dagger] = \delta_{ip}\mathfrak{f}_p\mathfrak{f}_q^\dagger - \delta_{iq}\mathfrak{f}_p\mathfrak{f}_q^\dagger \\
\mathrm{ad}(H_i)(\mathfrak{f}_p^\dagger\mathfrak{f}_q^\dagger) &= [H_i, \mathfrak{f}_p^\dagger\mathfrak{f}_q^\dagger] = -\delta_{ip}\mathfrak{f}_p^\dagger\mathfrak{f}_q^\dagger - \delta_{iq}\mathfrak{f}_p^\dagger\mathfrak{f}_q^\dagger.
\end{aligned}$$

In other words, $\mathfrak{f}_p\mathfrak{f}_q, \mathfrak{f}_p\mathfrak{f}_q^\dagger$ and $\mathfrak{f}_p^\dagger\mathfrak{f}_q^\dagger$ are eigenvectors for the action of $\mathfrak{h}$. Hence the spaces

$$\mathbb{C}\mathfrak{f}_p\mathfrak{f}_q, \mathbb{C}\mathfrak{f}_p\mathfrak{f}_q^\dagger \quad \text{and} \quad \mathbb{C}\mathfrak{f}_p^\dagger\mathfrak{f}_q^\dagger$$

are *weight spaces*, or since we are discussing the adjoint representation, *root spaces*. Note that these weight spaces can be represented by elements of $\mathfrak{h}^*$ (or roots), similar to the case of $\mathfrak{sl}(2k, \mathbb{C})$, as shown in Table 3.4.

root vector	root
$\mathfrak{f}_p\mathfrak{f}_q$	$L_p + L_q$
$\mathfrak{f}_p\mathfrak{f}_q^\dagger$	$L_p - L_q$
$\mathfrak{f}_p^\dagger\mathfrak{f}_q^\dagger$	$-L_p - L_q$

Table 3.4: Roots of $\mathfrak{so}(2n, \mathbb{C})$

We can combine our findings in the following theorem.

Theorem 3.7. *The orthogonal Lie algebra $\mathfrak{so}(2n, \mathbb{C})$ can, for all $n \geq 2$, be decomposed in 1-dimensional eigenspaces for the action of $\mathfrak{h}$:*

$$\mathfrak{so}(2n, \mathbb{C}) \cong \mathbb{C}_{2n}^{(2)} \cong \mathfrak{h} \oplus \bigoplus_{p<q} \left(\mathfrak{g}_{L_p+L_q} \oplus \mathfrak{g}_{L_p-L_q} \oplus \mathfrak{g}_{-L_p-L_q} \oplus \mathfrak{g}_{-L_p+L_q}\right).$$

with $\mathfrak{g}_{L_p+L_q} = \mathbb{C}\mathfrak{f}_p\mathfrak{f}_q, \mathfrak{g}_{L_p-L_q} = \mathbb{C}\mathfrak{f}_p\mathfrak{f}_q^\dagger$ and $\mathfrak{g}_{-L_p-L_q} = \mathbb{C}\mathfrak{f}_p^\dagger\mathfrak{f}_q^\dagger$.

Example 3.6. The roots of $\mathfrak{so}(4, \mathbb{C})$, given by $L_1 + L_2, L_1 - L_2, -L_1 - L_2$ and $-L_1 + L_2$, can be nicely visualised in the space $\mathfrak{h}^*$ (see Figure 3.9).

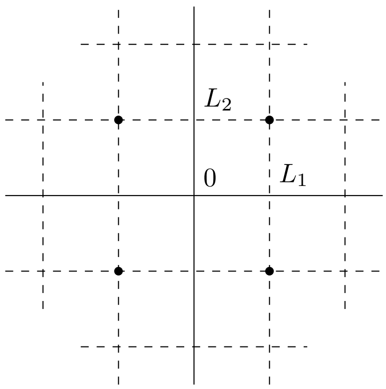


Figure 3.9: The roots and respective root vectors of $\mathfrak{so}(4,\mathbb{C})$

Remark 3.8. One should note that $\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ is not simple but semisimple. This is however not a restriction as the entire theory works for semisimple Lie algebras as well.

In the case where $m = 2n+1$, we have the additional relations

$$\begin{aligned} \mathrm{ad}(H_i)(\mathfrak{f}_p e_{2n+1}) &= \delta_{ip}\mathfrak{f}_p e_{2n+1} \\ \mathrm{ad}(H_i)(\mathfrak{f}_p^\dagger e_{2n+1}) &= -\delta_{ip}\mathfrak{f}_p^\dagger e_{2n+1}. \end{aligned}$$

Hence we get $2n$ additional root spaces

$$\mathfrak{g}_{L_p} := \mathbb{C}\mathfrak{f}_p e_{2n+1} \text{ and } \mathfrak{g}_{-L_p} := \mathbb{C}\mathfrak{f}_p^\dagger e_{2n+1}.$$

Summarising our findings, we get the following theorem.

Theorem 3.8. *The orthogonal Lie algebra $\mathfrak{so}(2n+1,\mathbb{C})$ can for all $n \geq 2$ be decomposed in 1-dimensional eigenspaces for the action of $\mathfrak{h}$:*

$$\begin{aligned} \mathfrak{so}(2n+1,\mathbb{C}) &\cong \mathbb{C}^{(2)}_{2n+1} \\ &\cong \mathfrak{h} \oplus \bigoplus_{p<q} \left(\mathfrak{g}_{L_p+L_q} \oplus \mathfrak{g}_{L_p-L_q} \oplus \mathfrak{g}_{-L_p-L_q} \oplus \mathfrak{g}_{-L_p+L_q}\right) \\ &\quad \oplus \bigoplus_{p=1}^{n} \left(\mathfrak{g}_{L_p} \oplus \mathfrak{g}_{-L_p}\right). \end{aligned}$$

Example 3.7. The roots of $\mathfrak{so}(5,\mathbb{C})$, given by $L_1+L_2, L_1-L_2, -L_1-L_2, -L_1+L_2, L_1, L_2, -L_1$ and $-L_2$, can be nicely visualised in the plane $\mathfrak{h}^*$, see Figure 3.10.

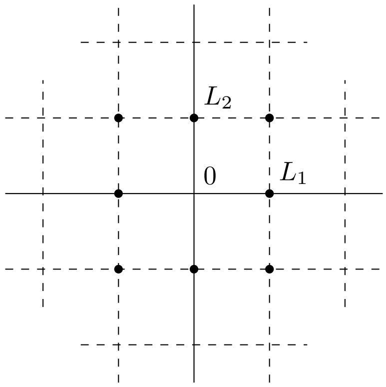


Figure 3.10: The roots of $\mathfrak{so}(5,\mathbb{C}) \cong \mathfrak{sp}(4,\mathbb{C})$

As in the previous section, we want to make a distinction between positive and negative roots. To this end, consider the linear functional $l : \mathfrak{h}^* \to \mathbb{R}$ defined by n different real numbers c_i such that

$$l(a_1L_1 + \cdots + a_nL_n) = a_1c_1 + \cdots + a_nc_n.$$

Then a root α is said to be positive if $l(\alpha) > 0$ and negative if $l(\alpha) < 0$. The ordering on the c_i again fixes the sets of positive and negative roots. In the remainder of this section, we will use the ordering

$$c_1 > c_2 > \cdots > c_n > 0.$$

Then, for the Lie algebra $\mathfrak{so}(2n,\mathbb{C})$, we find that the sets of positive and negative roots are respectively given by

$$\Delta^+ = \{L_i + L_j : 1 \leq i \neq j \leq n\} \cup \{L_i - L_j : 1 \leq i < j \leq n\}$$

and

$$\Delta^- = \{-L_i - L_j : 1 \leq i \neq j \leq n\} \cup \{-L_i + L_j : 1 \leq i < j \leq n\}.$$

In case of the Lie algebra $\mathfrak{so}(2n+1,\mathbb{C})$, they are respectively given by

$$\begin{aligned}\Delta^+ = \{L_i + L_j &: 1 \leq i \neq j \leq n\} \\ &\cup \{L_i - L_j : 1 \leq i < j \leq n\} \cup \{L_i : 1 \leq i \leq n\}\end{aligned}$$

and

$$\begin{aligned}\Delta^- = \{-L_i - L_j &: 1 \leq i \neq j \leq n\} \\ &\cup \{-L_i + L_j : 1 \leq i < j \leq n\} \cup \{-L_i : 1 \leq i \leq n\}.\end{aligned}$$

Root	Eigenvalues of H_1 and H_2
$L_1 + L_2$	$(1, 1)$
$L_1 - L_2$	$(1, -1)$
$-L_1 + L_2$	$(-1, 1)$
$-L_1 - L_2$	$(-1, -1)$

Table 3.5: Roots and corresponding eigenvalues of $\mathfrak{so}(4, \mathbb{C})$

Root	Eigenvalues of H_1 and H_2
$L_1 + L_2$	$(1, 1)$
$L_1 - L_2$	$(1, -1)$
$-L_1 + L_2$	$(-1, 1)$
$-L_1 - L_2$	$(-1, -1)$
L_1	$(1, 0)$
L_2	$(0, 1)$
$-L_1$	$(-1, 0)$
$-L_2$	$(0, -1)$

Table 3.6: Roots and corresponding eigenvalues of $\mathfrak{so}(5, \mathbb{C})$

Example 3.8. As an example, let us summarise the roots of the adjoint representations of $\mathfrak{so}(4, \mathbb{C})$ and $\mathfrak{so}(5, \mathbb{C})$ together with their respective eigenvalues for the action of $\mathfrak{h}$ in Tables 3.5 and 3.6.

One can fix a lexicographical ordering on the 2-tuples of eigenvalues, determining the highest weight $(1, 1)$ of the adjoint representation, with highest weight vector $\mathfrak{f}_1\mathfrak{f}_2$ in both cases $m = 4$ and $m = 5$. This can be generalised to the case of arbitrary m, fixing the highest weight $(1, 1, 0, \dots, 0)$ and highest weight vector $\mathfrak{f}_1\mathfrak{f}_2$.

Example 3.9. As a second example, consider the vector representation

$$\rho_{\mathbb{V}} : \mathfrak{so}(m) \to \mathrm{End}(\mathbb{C}^m),$$

with $\rho_{\mathbb{V}}(X)(v) = [X, v]$ for all $v \in \mathbb{C}^m$ and $X \in \mathfrak{so}(m, \mathbb{C})$. This action is well-defined as a commutator in the Clifford algebra $\mathbb{C}_m$. We have that $\mathbb{C}^{2n} = \mathrm{span}_{\mathbb{C}}\{\mathfrak{f}_1, \dots \mathfrak{f}_n, \mathfrak{f}_1^\dagger, \dots, \mathfrak{f}_n^\dagger\}$ and $\mathbb{C}^{2n+1} = \mathrm{span}_{\mathbb{C}}\{\mathfrak{f}_1, \dots \mathfrak{f}_n, \mathfrak{f}_1^\dagger, \dots, \mathfrak{f}_n^\dagger, e_{2n+1}\}$. The following relations hold:

$$\begin{aligned} \rho_{\mathbb{V}}(H_i)(\mathfrak{f}_p) &= \delta_{ip}\mathfrak{f}_p \\ \rho_{\mathbb{V}}(H_i)(\mathfrak{f}_p^\dagger) &= -\delta_{ip}\mathfrak{f}_p^\dagger, \end{aligned}$$

for $m = 2n$, together with the extra relation

$$\rho_{\mathbb{V}}(H_i)(e_{2n+1}) = 0$$

for $m = 2n+1$. Hence we find the weight space decompositions

$$\mathbb{C}^{2n} = \bigoplus_{j=1}^{n} \left(V_{L_j} \oplus V_{L_j}\right)$$

and

$$\mathbb{C}^{2n} = V_0 \oplus \bigoplus_{j=1}^{n} \left(V_{L_j} \oplus V_{L_j}\right).$$

Ordering the weights lexicographically, we find that this is a representation with highest weight $(1, 0, \ldots, 0)$ and highest weight vector $\mathfrak{f}_1 \in \mathbb{C}^m$ in both cases.

Next, consider the spinor representations

$$\rho_{\mathbb{S}_{2n}} : \mathfrak{so}(2n+1) \to \mathrm{End}(\mathbb{S}_{2n})$$

and

$$\rho_{\mathbb{S}_{2n}^{\pm}} : \mathfrak{so}(2n) \to \mathrm{End}(\mathbb{S}_{2n}^{\pm}),$$

where the explicit realisations were given in Chapter 2:

$$\mathbb{S}_{2n} = \mathbb{C}_m^+ I \cong \mathbb{C}_{2n} I,$$

if $m = 2n+1$, where the isomorphism is given by (2.7) and

$$\mathbb{S}_{2n}^+ = \mathbb{C}_m^+ I, \qquad \mathbb{S}_{2n}^- = \mathbb{C}_m^+ \mathfrak{f}_n^\dagger I \cong \mathbb{C}_m^- I$$

if $m = 2n$. In each of these cases, the action is given by the multiplication. In the odd case $m = 2n+1$, we find that $\mathbb{S}_{2n}$ is spanned by

$$\mathbb{C}_{2n} I = \mathrm{span}_{\mathbb{C}} \left\{ \psi_{i_1,\ldots,i_p} := \mathfrak{f}_{i_1}^\dagger \ldots \mathfrak{f}_{i_p}^\dagger I : 1 \leq i_1 < \cdots < i_p \leq n \right\}.$$

For the action of the Cartan elements, we find that

$$H_k \psi_{i_1,\ldots,i_{2p}} = \frac{1}{2} \psi_{i_1,\ldots,i_{2p}}$$

if $k \notin \{i_1, \ldots, i_{2p}\}$, and

$$H_k \psi_{i_1,\ldots,i_{2p}} = -\frac{1}{2} \psi_{i_1,\ldots,i_{2p}}$$

if $k \in \{i_1, \ldots, i_{2p}\}$. This means that each $\psi_{i_1,\ldots,i_{2p}}$ generates a weight space with weight $\left(\pm\frac{1}{2}, \ldots, \pm\frac{1}{2}\right)$, containing an even number of minus signs. For the other generators, we get that

$$H_k \psi_{i_1,\ldots,i_{2p+1}} = \frac{1}{2} \psi_{i_1,\ldots,i_{2p+1}}$$

if $k \notin \{i_1, \ldots, i_{2p+1}\}$, and

$$H_k \psi_{i_1,\ldots,i_{2p+1}} = -\frac{1}{2}\psi_{i_1,\ldots,i_{2p+1}}$$

if $k \in \{i_1, \ldots, i_{2p+1}\}$. Hence, each $\psi_{i_1,\ldots,i_{2p+1}}$ generates a weight space with highest weight $\left(\pm\frac{1}{2}, \ldots, \pm\frac{1}{2}\right)$, containing an odd number of minus signs. This means that we have found a weight space decomposition

$$\mathbb{S}_{2n} = \bigoplus V_{\left(\pm\frac{1}{2},\ldots,\pm\frac{1}{2}\right)}$$

with highest weight $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ and corresponding highest weight vector I. The representation has dimension 2^n, as was already determined in the previous chapter.

For the case $m = 2n$, we have that

$$\mathbb{S}^+_{2n} = \operatorname{span}_{\mathbb{C}} \left\{ \mathfrak{f}^\dagger_{i_1} \ldots \mathfrak{f}^\dagger_{i_{2p}} I : 1 \leq i_1 < \cdots < i_{2p} \leq n \right\}.$$

Following a similar reasoning as above, the weight space decomposition is given by

$$\mathbb{S}^+_{2n} = \bigoplus V_{\left(\pm\frac{1}{2},\ldots,\pm\frac{1}{2}\right)},$$

where each weight contains an even number of minus signs. The highest weight is given by $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ and the highest weight vector is I. The dimension is 2^{n-1}. For the last spinor representation, we get that

$$\mathbb{S}^-_{2n} = \operatorname{span}_{\mathbb{C}} \left\{ \mathfrak{f}^\dagger_{i_1} \ldots \mathfrak{f}^\dagger_{i_{2p+1}} I : 1 \leq i_1 < \cdots < i_{2p+1} \leq n \right\}.$$

We get the weight space decomposition

$$\mathbb{S}^-_{2n} = \bigoplus V_{\left(\pm\frac{1}{2},\ldots,\pm\frac{1}{2}\right)},$$

where each weight contains an odd number of minus signs. The highest weight is given by $\left(\frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}\right)$, the highest weight vector by $\mathfrak{f}^\dagger_n I$, and the dimension is 2^{n-1}.

Remark 3.9. These results are as expected, since $\mathfrak{so}(m, \mathbb{C})$ is the Lie algebra of $\mathrm{Spin}(m)$, of which the representations were discussed in the previous chapter. However, since representation theory is very important for this thesis, we wanted to do these realisations explicitly for both the Lie groups and the Lie algebra.

With this knowledge of representation theory, we can get to the actual topic of this thesis, higher spin operators. In the next chapter, we use this knowledge to construct these operators.

It seems to be one of the fundamental features of nature that fundamental physical laws are described in terms of a mathematical theory of great beauty and power.

Paul Adrien Maurice Dirac

4 Higher spin operators

In this chapter, we introduce higher spin Dirac and twistor operators, but in order to understand how they are defined, we first take a look at the classical Dirac operator again. Remember that it is was defined as

$$\partial_x = \sum_{j=1}^{m} e_j \partial_{x_j}.$$

This operator plays a fundamental role in Clifford analysis (see e.g. [12, 30]. It is an elliptic conformally invariant first-order differential operator, acting on $\mathbb{S}^{\pm}$-valued functions:

$$\partial_x : \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S}^{\pm}) \rightarrow \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S}^{\mp}).$$

Unless stated otherwise, we will from now on only use **odd dimensions**, in order to avoid the parity signs for the spinor spaces. The entire theory in this thesis can however be generalised to the even-dimensional case as well. Thus, the Dirac operator acts as follows:

$$\partial_x : \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S}) \rightarrow \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S}).$$

The origin of the Dirac operator, studied in Clifford analysis, lies in representation theory and theoretical physics. Actually, the representation theoretical link will be at the basis of the definitions of higher spin operators. It follows from the results in [16, 45] that there exists a set of elliptic conformally invariant first-order differential operators acting on functions with values in more complicated irreducible representations $\mathbb{V}_\lambda$ of the Spin(m)-group. In the second chapter, we have proven that the polynomial spaces $\mathcal{S}_\lambda$ form models for such representations $\mathbb{V}_\lambda$ in the case of a half integer valued highest weight. The *higher spin Dirac operator* (HSD operator) is then defined as the conformally invariant first-order differential operator

$$\mathcal{Q}_\lambda : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda).$$

The representations appearing in the target space and in the space the operator acts on, can however be different from each other. It has been shown in [45, 79] that there only is a limited amount of possibilities for such operators, which we can put into two groups. First, we have the operators

$$\mathcal{T}_\lambda^{(i)} : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda - L_i}),$$

where $\lambda - L_i = (l_1, \ldots, l_{i-1}, l_i - 1, l_{i+1}, \ldots, l_k)$. These are called *higher spin twistor operators* (HST operators). Secondly, we can also have the case

$$\mathcal{T}_\lambda^{*(i)} : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda + L_i}),$$

where $\lambda + L_i = (l_1, \ldots, l_{i-1}, l_i + 1, l_{i+1}, \ldots, l_k)$. We speak of *dual higher spin twistor operators* in this case.

In this section, we will first go through some properties of the classical Dirac operator in Section 1, before moving on to the general higher spin operators, which we will construct in Section 4. In order to construct these operators, we need an object called an extremal projection operator, which is introduced in Sections 2 and 3. Finally, we will prove the conformal invariance of higher spin Dirac and twistor operators in Section 5.

4.1 Dirac operator

In the first section of this chapter, we will mention and prove some of the most important properties of the classical Dirac operator, more specifically its ellipticity, and its conformal invariance.

4.1.1 Ellipticity

Let us introduce the multi-index notation $\alpha = (\alpha_1, \ldots, \alpha_m)$, such that for $\partial = (\partial_{x_1}, \ldots, \partial_{x_m})$, we can use the notation $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_m}^{\alpha_m}$ and for $x = (x_1, \ldots, x_m)$, we write $x^\alpha = x_1^{\alpha_1} \cdots x_m^{\alpha_m}$. The order of such a multi-index is denoted by $|\alpha| = \alpha_1 + \ldots + \alpha_m$.

Definition 4.1. *A linear differential operator L of order k in $\Omega \subset \mathbb{R}^m$ given by*

$$L = \sum_{|\alpha| \leq k} a_\alpha \partial^\alpha,$$

is elliptic if its principal symbol

$$\sum_{|\alpha| \leq k} a_\alpha x^\alpha$$

is invertible for all $x \in \Omega \backslash \{0\}$.

Basically, the principal symbol of a differential operator is obtained by replacing each ∂_{x_j} by x_j. For the classical Dirac operator ∂_x, its principal symbol is thus x, which indeed is invertible, since we know every non-zero vector is. We conclude that the Dirac operator is elliptic.

4.1.2 Conformal invariance

An important property of the Dirac operator is its conformal invariance. We introduce the following definition (see e.g. [64, 34]):

Definition 4.2. *An operator d is a generalised symmetry for a given operator $\mathcal{D}$ if and only if there exists an operator δ such that $\mathcal{D}d = \delta\mathcal{D}$. In case $d = \delta$, we speak of a symmetry, i.e. $[\mathcal{D}, d] = 0$.*

Remark 4.1. Note that, in the definition above, d preserves the kernel of the operator $\mathcal{D}$.

In the setting of Clifford analysis, conformal invariance usually is described in terms of the conformal group $\mathrm{Spin}(1, m+1)$, which can be realised in terms of Vahlen matrices (see e.g. [1, 77]). Indeed, the conformal transformations on $\mathbb{R}^m$ (rotations, translations, dilations and inversions) can be neatly expressed in terms of a (2×2)-matrix containing Clifford numbers satisfying certain conditions. In this section, we will show that the infinitesimal generators of the transformations above generate the conformal algebra $\mathfrak{so}(1, m+1)$, corresponding to the conformal group, when choosing a suitable conformal weight $w \in \mathbb{R}$. Note that this is a 1-graded Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$, where the zero-graded part is given by $\mathfrak{g}_0 = \mathfrak{so}(m, \mathbb{R}) \oplus \mathbb{R}E$ and $\mathfrak{g}_{\pm 1} \cong \mathbb{R}^m$. The element E is the so-called grading element, satisfying $[E, X_a] = aX_a$, with $a \in \{-1, 0, +1\}$, and the grading is encoded in the fact that $[X_a, X_b] \in \mathfrak{g}_{a+b}$ (where it is understood that the commutator is zero when $|a + b| = 2$).

First of all, we have the obvious symmetries:

- *Rotations:* $dL(e_{ij}) = L^x_{ij} - \frac{1}{2}e_{ij}$, where L^x_{ij} are the angular momentum operators defined in Chapter 2. These span the simple part $\mathfrak{so}(m,\mathbb{R}) \subset \mathfrak{g}_0$. We can use the set of bivectors e_{ij} as a basis for $\mathfrak{so}(m,\mathbb{R})$ since $\mathfrak{so}(m,\mathbb{R}) \cong \mathbb{R}^{(2)}_m$.
- *Translations:* $dT(e_j) = \partial_{x_j}$. They can be identified with basis elements of the subspace $\mathfrak{g}_{-1}$.

Remark 4.2. One might not be used to working with infinitesimal operators in the conformal algebra, but more used to the actual conformal transformations defined in the conformal group. To that end, let us explain the connection between both a bit further. The action of an element s of Spin(m) (hence a rotation) is given by

$$L(s)f(x) = sf(\bar{s}xs).$$

Using (2.5), and picking s equal to a generator $e^{e_{ij}}$, one finds that

$$dL(e_{ij})f(x) = \frac{d}{dt}e^{te_{ij}}f(e^{-te_{ij}}xe^{te_{ij}})\bigg|_{t=0} = \left(e_{ij} + 2x_j\partial_{x_i} - 2x_i\partial_{x_j}\right)f(x).$$

A translation over $a \in \mathbb{R}^m$ is given by

$$T(a)(f(x-a)).$$

Applying the same trick gives us

$$dT(e_j) = \frac{d}{dt}f(x+te_j)\bigg|_{t=0} = \partial_{x_j}f(x).$$

The dilations are given by

$$D(\lambda)f(x) = \lambda^{\frac{m-1}{2}}f(\lambda x). \tag{4.1}$$

Hence we get

$$dD(\ln(\lambda))f(x) = \frac{d}{dt}\lambda^{\frac{t(m-1)}{2}}f(\lambda^t x)\bigg|_{t=0} = \ln(\lambda)\left(\mathbb{E}_x + \frac{m-1}{2}\right)f(x). \tag{4.2}$$

It is easily verified that $[\partial_x, dL(e_{ij})] = [\partial_x, dT(e_j)] = 0$. Next, as the operator ∂_x is homogeneous of degree (-1), we also have that

$$\partial_x\left(\mathbb{E}_x + \frac{m-1}{2}\right) = \left(\mathbb{E}_x + \frac{m+1}{2}\right)\partial_x,$$

which means that also the generator of dilatations is a generalised symmetry. Note that the element between brackets at the left-hand side is the grading element $E \in \mathfrak{g}_0$, we found in (4.2) (also note that the shift defines the conformal weight, i.e. $2w = m-1$, which is also clear from (4.1)). Finally, we have the generalised symmetries in the subspace $\mathfrak{g}_{+1}$, which can be defined in terms of the inversion operator.

Definition 4.3. *The inversion operator I_{∂_x} on $\mathbb{S}$-valued functions is defined as*

$$I_{\partial_x} f(x) = \frac{x}{|x|^m} f\left(\frac{x}{|x|^2}\right).$$

It is well-known that I_{∂_x} preserves ∂_x-solutions, see e.g. [77]. The following relation also is important.

Theorem 4.1. *On arbitrary functions $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S})$, one has the operator identity*

$$I_{\partial_x} \partial_x I_{\partial_x} = |x|^2 \partial_x. \tag{4.3}$$

Proof. A straightforward calculation gives us that

$$\begin{aligned}
&I_{\partial_x} \partial_x I_{\partial_x} f(x) \\
=&I_{\partial_x} \partial_x \frac{x}{|x|^m} f\left(\frac{x}{|x|^2}\right) \\
=&I_{\partial_x} \sum_{j=1}^m \left(-\frac{1}{|x|^m} - \frac{m x_j e_j x}{|x|^{m+2}}\right) f\left(\frac{x}{|x|^2}\right) \\
&+ \frac{e_j x}{|x|^m} \left(\partial_{\frac{x_1}{|x|^2}} f\left(\frac{x}{|x|^2}\right) \frac{-2x_1 x_j}{|x|^4} + \cdots + \partial_{\frac{x_j}{|x|^2}} f\left(\frac{x}{|x|^2}\right) \frac{|x|^2 - 2x_j^2}{|x|^4} + \cdots\right) \\
=&\frac{x}{|x|^m} \left(\sum_{j=1}^m \left(-|x|^m - m x_j e_j x |x|^{m-2}\right) + e_j x |x|^{m-2} \left(-2x_j \mathbb{E}_x + |x|^2 \partial_{x_j}\right)\right) f(x) \\
=&|x|^2 \partial_x f(x).
\end{aligned}$$

□

Lemma 4.1. *For all $1 \leq j \leq m$, the operators $I_{\partial_x} \partial_{x_j} I_{\partial_x}$ define generalised symmetries for the Dirac operator ∂_x.*

Proof. Using Theorem 4.1 and the fact that $I_{\partial_x}^2 = -1$, we get that

$$\begin{aligned}
\partial_x \left(I_{\partial_x} \partial_{x_j} I_{\partial_x}\right) &= -I_{\partial_x} |x|^2 \partial_x \partial_{x_j} I_{\partial_x} \\
&= I_{\partial_x} |x|^2 \partial_{x_j} I_{\partial_x}^2 \partial_x I_{\partial_x} = \left(I_{\partial_x} |x|^2 \partial_{x_j} I_{\partial_x} |x|^2\right) \partial_x,
\end{aligned}$$

from which the conclusion follows. The fact that all these operators commute also follows from the fact that $I_{\partial_x}^2 = -1$. □

Remark 4.3. The corresponding action of $I_{\partial_x} \partial_x I_{\partial_x}$ in the conformal group is given by exponentiating this expression. We find that

$$\begin{aligned}
&\exp(I_{\partial_x}\partial_{x_j}I_{\partial_x})\\
&= 1 + I_{\partial_x}\partial_{x_j}I_{\partial_x} - \frac{I_{\partial_x}\partial^2_{x_j}I_{\partial_x}}{2!} + \frac{I_{\partial_x}\partial^3_{x_j}I_{\partial_x}}{3!} - \frac{I_{\partial_x}\partial^4_{x_j}I_{\partial_x}}{4!} + \cdots\\
&= I_{\partial_x}\left(-1 + \partial_{x_j} - \frac{\partial^2_{x_j}}{2!} + \frac{\partial^3_{x_j}}{3!} - \frac{\partial^4_{x_j}}{4!} + \ldots\right)I_{\partial_x}\\
&= -I_{\partial_x}\exp(-\partial_{x_j})I_{\partial_x}.
\end{aligned}$$

This is an inversion, followed by a translation, and subsequently by another inversion.

These generalised symmetries clearly belong to the subspace $\mathfrak{g}_{+1}$, it suffices to note that they are homogeneous of degree $(+1)$. In order to finish the proof of conformal invariance, we have the following result.

Theorem 4.2. *The generalised symmetries constructed above generate a Lie algebra which is isomorphic to* $\mathfrak{so}(1, m+1)$.

Proof. First, we prove that $[\mathfrak{g}_0, \mathfrak{g}_{-1}] \subset \mathfrak{g}_{-1}$. For $\partial_{x_j} \in \mathfrak{g}_{-1}$ and $f(x)$ an arbitrary $\mathbb{S}$-valued function, we get that $2[dL(e_{ab}), \partial_{x_j}]f(x) = (\delta_{bj}\partial_{x_a} - \delta_{aj}\partial_{x_b})f(x)$. In order to prove that $[\mathfrak{g}_{-1}, \mathfrak{g}_{+1}] \subset \mathfrak{g}_0$ and $[\mathfrak{g}_0, \mathfrak{g}_{+1}] \subset \mathfrak{g}_{+1}$, we find that

$$I_{\partial_x}\partial_{x_j}I_{\partial_x} = \frac{1}{2}\left\{I_{\partial_x}e_jI_{\partial_x}, I_{\partial_x}\partial_xI_{\partial_x}\right\} = \frac{1}{2}\left\{\frac{xe_jx}{|x|^2}, |x|^2\partial_x\right\},$$

where we have used the fact that the Euclidean inner product of two vectors $x, y \in \mathbb{R}^m$ can be written as $-2\langle x, y\rangle = \{x, y\}$ in the Clifford algebra $\mathbb{C}_m$. This gives

$$\frac{1}{2}\left\{\frac{xe_jx}{|x|^2}, |x|^2\partial_x\right\} = -|x|^2\partial_{x_j} + x_j(2\mathbb{E}_x + m - 1) - e_j \wedge x,$$

see e.g. [40]. Taking the commutator between generators of $\mathfrak{g}_{-1}$ and $\mathfrak{g}_{+1}$, we get

$$\left[\partial_{x_i}, I_{\partial_x}\partial_{x_j}I_{\partial_x}\right] = \begin{cases} e_ie_j - 2L^x_{ij} & \text{if} \quad i \neq j\\ 2\mathbb{E}_x + m - 1 & \text{if} \quad i = j,\end{cases}$$

which is an element of $\mathfrak{g}_0$ on both occasions. Taking the commutator between generators or $\mathfrak{g}_0$ and $\mathfrak{g}_{+1}$ yields

$$[dL(e_{ij}), I_{\partial_x}\partial_{x_k}I_{\partial_x}] = \delta_{jk}I_{\partial_x}\partial_{x_i}I_{\partial_x} - \delta_{ik}I_{\partial_x}\partial_{x_j}I_{\partial_x},$$

which again shows that $[\mathfrak{g}_0, \mathfrak{g}_{+1}] \subset \mathfrak{g}_{+1}$. This finishes the proof. □

This concludes the proof of the conformal invariance of the Dirac operator. Not that this property also follows from other work, such as [45].Next, we will take a look at higher spin operators in general. We will construct them explicitly within the context of an algebraic concept, called a transvector algebra, which is discussed in the next section.

4.2 Transvector algebras: general notations

The problem of symmetry of an equation is a classical one. For systems of equations of a special form (so-called extremal systems, see further), it is possible in some sense to find the full solution of the problem of describing the symmetry algebras, or transvector algebras as they are called in this thesis. In this section, we introduce the general theory for transvector algebras. Similarly to [90] we do this for both Lie algebras and superalgebras. Further in this chapter, we will then explicitly exploit the general theory in case of a superalgebra. Take a complex Lie (super)algebra $\mathfrak{g}$ and a reductive (super)algebra $\mathfrak{k} \subset \mathfrak{g}$. We then can take a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{k}$ to arrive at the triangular decomposition $\mathfrak{k} = \mathfrak{k}^- \oplus \mathfrak{h} \oplus \mathfrak{k}^+$. The graded subspaces $\mathfrak{k}^+$ and $\mathfrak{k}^-$ then are spanned by the positive and the negative root vectors e_α and $e_{-\alpha}$ respectively, where α runs over the set of positive roots Δ^+ of $\mathfrak{k}$ with respect to $\mathfrak{h}$. If $\mathfrak{k}$ is a Lie superalgebra, we have the decomposition $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$ where the even subalgebra $\mathfrak{k}_0$ defines a classical Lie algebra. If $\mathfrak{k}$ is a classical Lie algebra, $\mathfrak{k}^+ = \mathfrak{k}_0^+$. This thus means that we can also split the positive root system Δ^+ into two subsets: the even subset Δ_0^+, which is the positive root system of $\mathfrak{k}_0$, and the odd root system Δ_1^+. For a classical Lie algebra, $\Delta^+ = \Delta_0^+$. For each positive root $\alpha \in \Delta_0^+$ (resp. $\beta \in \Delta_1^+$), we define the Cartan element $h_\alpha := [e_\alpha, e_{-\alpha}] \in \mathfrak{h}$ (resp. $h_\beta := \{e_\beta, e_{-\beta}\}$). The even positive roots $\alpha \in \Delta_0^+$ have the property that their associated root vectors e_α are suitably normalised (although this is not clear at this moment, this normalisation will not be necessary for the odd roots):

$$\alpha(h_\alpha)e_\alpha := [h_\alpha, e_\alpha] = 2e_\alpha. \tag{4.4}$$

Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping superalgebra of $\mathfrak{g}$ (where we use juxtaposition instead of the tensor product symbol to denote the product), and define $J = \mathcal{U}(\mathfrak{g})\mathfrak{k}^+$ as the left ideal of $\mathcal{U}(\mathfrak{g})$ generated by $\mathfrak{k}^+$. The normaliser of this ideal is then defined as

$$\text{Norm } J = \{u \in \mathcal{U}(\mathfrak{g}) : Ju \subseteq J\}.$$

Obviously, J is an ideal in Norm J so that we can define the quotient algebra

$$S(\mathfrak{g}, \mathfrak{k}) = \text{Norm } J/J,$$

which is known in the literature as the Mickelsson algebra [63]. We then introduce $R(\mathfrak{h})$, the field of fractions of the commutative (enveloping) algebra

$\mathcal{U}(\mathfrak{h})$. We can now extend the universal enveloping superalgebra $\mathcal{U}(\mathfrak{g})$ for $\mathfrak{g}$ to $\mathcal{U}'(\mathfrak{g}) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} R(\mathfrak{h})$, and define the left ideal $J' = \mathcal{U}'(\mathfrak{g})\mathfrak{k}^+$ of $\mathcal{U}'(\mathfrak{g})$, its normaliser Norm $J' = \{u \in \mathcal{U}'(\mathfrak{g}) : J'u \subseteq J'\}$, and the quotient algebra

$$Z(\mathfrak{g}, \mathfrak{k}) = \text{Norm } J'/J'.$$

This algebra is better known as a transvector algebra or Mickelsson-Zhelobenko algebra. The advantage of this algebra over the Mickelsson algebra, lies in the fact that 'division' by elements of the Cartan algebra is well-defined, since it is easy to see that

$$Z(\mathfrak{g}, \mathfrak{k}) = S(\mathfrak{g}, \mathfrak{k}) \otimes_{\mathcal{U}(\mathfrak{h})} R(\mathfrak{h}).$$

In the next chapter, the difference between both algebras will be explained by means of an example. Now that we have formally defined the transvector algebra, we will clarify its algebraic structure. This will be done by means of the so-called *extremal projector* for the Lie superalgebra $\mathfrak{k}$. Roughly speaking, this algebraic object will turn arbitrary root vectors into elements of $\ker(\mathfrak{k}^+)$. In order to define this projector, we need a normal ordering on the set of positive roots Δ^+.

Definition 4.4. *An ordering of a set of positive roots for a Lie (super-) algebra is called normal if any composite root lies between its components.*

For instance, writing the positive root system of $\mathfrak{sl}(3, \mathbb{C})$ as $\Delta^+ = \{\alpha, \beta, \alpha + \beta\}$, the normal orderings are given by $\{\alpha, \alpha+\beta, \beta\}$ and $\{\beta, \alpha+\beta, \alpha\}$, where α and β are the primitive positive roots of $\mathfrak{sl}(3, \mathbb{C})$. Note that the existence of a normal ordering is guaranteed by the following result, a proof of which can be found e.g. in [90].

Lemma 4.2. *For each Lie superalgebra, there exists a (not necessarily unique) normal ordering on the set of positive roots Δ^+.*

Define ρ_0 as half the sum of the even positive roots $\alpha \in \Delta_0^+$, and ρ_1 as half the sum of the odd positive roots $\beta \in \Delta_1^+$. For the even roots $\alpha \in \Delta_0^+$, we define the operators

$$p_\alpha = 1 + \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \frac{e_{-\alpha}^j e_\alpha^j}{(\phi_\alpha + 1)(\phi_\alpha + 2) \cdots (\phi_\alpha + j)},$$

where $\phi_\alpha := h_\alpha + \rho_0(h_\alpha) - \rho_1(h_\alpha) + \delta_{\frac{\alpha}{2}}$, and $\delta_{\frac{\alpha}{2}} = \frac{1}{2}$ if $\frac{\alpha}{2}$ is an element of Δ_1^+, and $\delta_{\frac{\alpha}{2}} = 0$ otherwise ($\rho_1(h_\alpha) = \delta_{\frac{\alpha}{2}} = 0$ for all α in the case of a classical Lie algebra). For the odd roots $\beta \in \Delta_1^+$, we define the operators

$$p_\beta = 1 - \frac{e_{-\beta} e_\beta}{h_\beta + \rho_0(h_\beta) - \rho_1(h_\beta) + \frac{1}{2}\beta(h_\beta)}.$$

Finally, we can define the extremal projector $p_{\mathfrak{k}}$ as

$$p_{\mathfrak{k}} = p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_a}, \tag{4.5}$$

where $\Delta^+ = \{\alpha_1, \alpha_2, \ldots, \alpha_a\}$ and the product is taken in a normal ordering of Δ^+. This is always possible because of Lemma 4.2. The extremal projector satisfies the properties below (see [90]):

Theorem 4.3. *There holds:*

- *The extremal projector $p_{\mathfrak{k}}$ does not depend on the choice for the normal ordering on Δ^+;*
- $e_\alpha p_{\mathfrak{k}} = p_{\mathfrak{k}} e_{-\alpha} = 0$*, for all $\alpha \in \Delta^+$;*
- $p_{\mathfrak{k}}^2 = p_{\mathfrak{k}}$.

This means that the extremal projector projects onto the kernel of all positive root vectors of $\mathfrak{k}$ simultaneously, which is exactly the property we will use further in the chapter, when we construct higher spin analogues of the Dirac operator. In order to explain how the extremal projector can be used to describe the algebraic structure of a transvector algebra $Z(\mathfrak{g}, \mathfrak{k})$, we first note that we have the $\mathfrak{k}$-module decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, since $\mathfrak{k}$ is reductive in $\mathfrak{g}$.

Remark 4.4. The module $\mathfrak{p}$ always is a $\mathfrak{k}$-module. We explicitly check this in the example in the next chapter.

If we denote by $\{e_{\alpha_{a+1}}, \ldots, e_{\alpha_b}\}$ the set of positive root vectors for the complementary module $\mathfrak{p} \subset \mathfrak{g}$, then we have the following theorem (see e.g. [66]).

Theorem 4.4. *Using the notation above to label weight basis elements, one has that the transvector algebra $Z(\mathfrak{g}, \mathfrak{k})$ is generated by $z_{\pm i} = p_{\mathfrak{k}} e_{\pm\alpha_i}$, for all $i \in \{a+1, \ldots, b\}$.*

Example 4.1. Consider $\mathfrak{g} = \mathfrak{gl}(3, \mathbb{C})$, the general linear algebra of (3×3)-matrices. The standard basis for this algebra then is given by the matrices $E_{ij} = (\delta_{ik}\delta_{jl})_{kl}$, with $1 \leq i, j \leq 3$. For $\mathfrak{k}$, we take the subalgebra $\mathfrak{gl}(2, \mathbb{C})$ generated by the matrices E_{ij}, with $2 \leq i, j \leq 3$. The triangular decomposition of $\mathfrak{gl}(2)$ is then

$$\mathfrak{gl}(2, \mathbb{C}) = \mathfrak{k}^- \oplus \mathfrak{h} \oplus \mathfrak{k}^+ = \mathrm{span}_{\mathbb{C}}\{E_{32}\} \oplus \mathrm{span}_{\mathbb{C}}\{E_{22}, E_{33}\} \oplus \mathrm{span}_{\mathbb{C}}\{E_{23}\}.$$

The positive root vectors of $\mathfrak{gl}(3, \mathbb{C})$ are E_{12}, E_{13} and E_{23}, and the negative root vectors are given by E_{21}, E_{31} and E_{32}. This means that the generators of $Z(\mathfrak{gl}(3, \mathbb{C}), \mathfrak{gl}(2, \mathbb{C}))$ are

$$Z(\mathfrak{gl}(3, \mathbb{C}), \mathfrak{gl}(2, \mathbb{C})) = \mathrm{Alg}_{\mathbb{C}}\left\{p_{\mathfrak{gl}(2,\mathbb{C})} E_{12}, p_{\mathfrak{gl}(2,\mathbb{C})} E_{13}, p_{\mathfrak{gl}(2,\mathbb{C})} E_{21}, p_{\mathfrak{gl}(2,\mathbb{C})} E_{31}\right\},$$

where the elements between brackets are the generators and where the extremal projector for $\mathfrak{gl}(2)$ is given by

$$p_{\mathfrak{gl}(2,\mathbb{C})} = 1 + \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \frac{E_{32}^j E_{23}^j}{(E_{22}-E_{33}+2)(E_{22}-E_{33}+3)\cdots(E_{22}-E_{33}+j+1)}.$$

Note that this particular transvector algebra appeared in [28], in the context of a special type of solutions for the higher spin Dirac operator. This is explained in Chapter 9.

An example for the Lie superalgebra case is given in the next chapter.

4.3 An extremal projector for $\mathfrak{osp}(1,2k)$

In this section, we will apply the techniques from the previous section to the particular case $\mathfrak{g} = \mathfrak{osp}(1,2k+2)$ and $\mathfrak{k} = \mathfrak{osp}(1,2k)$. An elegant model for this Lie superalgebra comes from Clifford analysis in k vector variables $(u_1,\ldots,u_k) \in \mathbb{R}^{k\times m}$ and their corresponding Dirac operators $(\partial_1,\ldots,\partial_k)$, as it is easily verified that

$$\mathfrak{osp}(1,2k) = \mathrm{LS}_{\mathbb{C}}\left\{u_1,\ldots,u_k,\partial_1,\ldots,\partial_k\right\},$$

where the notation should be interpreted as the Lie superalgebra generated by the odd generators given between the braces. This is a subalgebra of the Weyl-Clifford algebra $\mathcal{W}\otimes\mathbb{C}_m$ with

$$\mathcal{W} := \mathrm{Alg}_{\mathbb{C}}\left\{u_{ij},\partial_{u_{ij}} : 1\leq i\leq k, 1\leq j\leq m\right\}.$$

On the other hand, we have the graded decomposition $\mathfrak{osp}(1,2k) = \mathfrak{k}_0 \oplus \mathfrak{k}_1$, where the even part is nothing but the symplectic Lie algebra

$$\mathfrak{k}_0 = \mathfrak{sp}(2k,\mathbb{C}) = \mathfrak{k}_0^+ \oplus \mathfrak{h} \oplus \mathfrak{k}_0^-, \tag{4.6}$$

(see Chapter 3) and where we have introduced the Cartan algebra $\mathfrak{h}\subset\mathfrak{k}$

$$\mathfrak{h} = \mathrm{Alg}_{\mathbb{C}}\left\{H_i := \mathbb{E}_i + \frac{m}{2} : 1\leq i\leq k\right\},$$

and the (suitably normalised, see below) root spaces

$$\mathfrak{k}_0^+ \cup \mathfrak{k}_0^- := \mathrm{span}_{\mathbb{C}}\left\{\langle u_i,\partial_j\rangle, \langle u_j,\partial_i\rangle, -\frac{\langle\partial_a,\partial_b\rangle}{1+\delta_{ab}}, \frac{\langle u_a,u_b\rangle}{1+\delta_{ab}} : 1\leq i<j\leq k, 1\leq a,b\leq k\right\}.$$

Next, we will separate the positive from the negative root vectors in $\mathfrak{k}$ by means of a suitable functional ℓ on $\mathfrak{h}^*$ which then fixes the parity of our roots. To do so, we will *demand* our positive root vectors to be precisely the operators defining the simplicial monogenics (see Definition 2.19). The reason for this is the following: we will use the extremal projector $p_{\mathfrak{k}}$ for $\mathfrak{osp}(1,2k)$, having the property that $e_\alpha p_{\mathfrak{k}} = 0$ for all $\alpha \in \Delta^+$. Choosing the positive root vectors to be the ones defining simplicial monogenics, we will be able to use $p_{\mathfrak{k}}$ in order to define differential operators *preserving* values. Let us therefore choose k real numbers $c_1, c_2, \ldots, c_k$ such that $c_k < \cdots < c_2 < c_1 < 0$, and consider, as in Chapter 3, the linear functional

$$\ell(a_1L_1 + a_2L_2 + \cdots + a_kL_k) := a_1c_1 + a_2c_2 + \cdots + a_kc_k,$$

where we have introduced the standard dual basis $L_i = H_i^*$ for which $L_i(H_j) = \delta_{ij}$ (see e.g. [47]). We then have:

$$\begin{aligned}
\mathfrak{k}_0^+ &= \text{span}_{\mathbb{C}}\left\{-\frac{1}{2}\Delta_a, -\langle\partial_a,\partial_b\rangle, \langle u_i,\partial_j\rangle : 1 \le i < j \le k, 1 \le a \neq b \le k\right\}\\
\mathfrak{k}_0^- &= \text{span}_{\mathbb{C}}\left\{\frac{1}{2}|u_a|^2, \langle u_a,u_b\rangle, \langle u_j,\partial_i\rangle : 1 \le i < j \le k, 1 \le a \neq b \le k\right\}.
\end{aligned}$$

Indeed, we for example have that

$$[H_j, -\langle\partial_a,\partial_b\rangle] = -(\delta_{aj}+\delta_{bj})(-\langle\partial_a,\partial_b\rangle) \Rightarrow \ell(-L_a - L_b) = -(c_a + c_b) > 0,$$

which means that $(-\langle\partial_a,\partial_b\rangle)$ indeed is a positive root vector. Similarly, for the 'mixed' operators one has that

$$[H_j, \langle u_a,\partial_b\rangle] = (\delta_{aj}-\delta_{bj})\langle u_a,\partial_b\rangle \Rightarrow \ell(L_a - L_b) = c_a - c_b.$$

Hence, we get a positive root vector for $a < b$ and a negative root vector for $a > b$. Finally, note that the odd root vectors are given by:

$$\mathfrak{k}_1^+ = \text{span}_{\mathbb{C}}\left\{-\frac{\sqrt{2}}{2}\partial_a\right\} \quad \text{and} \quad \mathfrak{k}_1^- = \text{span}_{\mathbb{C}}\left\{\frac{\sqrt{2}}{2}u_a\right\}.$$

Next, we define for each (even) positive root $\alpha \in \Delta_0^+$ the corresponding Cartan element, through the aforementioned relation $h_\alpha = [e_\alpha, e_{-\alpha}]$:

$$\begin{aligned}
h_{-2L_a} &= \left[-\frac{\Delta_a}{2}, \frac{|u_a|^2}{2}\right] = -\left(\mathbb{E}_a + \frac{m}{2}\right)\\
h_{-L_a-L_b} &= [-\langle\partial_a,\partial_b\rangle, \langle u_a,u_b\rangle] = -\left(m + \mathbb{E}_a + \mathbb{E}_b\right)\\
h_{L_i-L_j} &= [\langle u_i,\partial_j\rangle, \langle u_j,\partial_i\rangle] = \mathbb{E}_i - \mathbb{E}_j.
\end{aligned}$$

Similarly, for the odd positive roots $\beta \in \Delta_1^+$, we get

$$h_{-L_a} = \left\{-\frac{\sqrt{2}}{2}\partial_a, \frac{\sqrt{2}}{2}u_a\right\} = -\left(\mathbb{E}_a + \frac{m}{2}\right).$$

The normalisation requirements for the even positive root vectors given in (4.4) are now satisfied, which, post factum, explains the numerical coefficients and minus signs in our original choices (see above). Indeed, we have:

$$\begin{aligned}\left[-\left(\mathbb{E}_a+\frac{m}{2}\right),-\frac{\Delta_a}{2}\right] &= 2\left(-\frac{\Delta_a}{2}\right)\\ \left[-\left(\mathbb{E}_a+\mathbb{E}_b+m\right),-\langle\partial_a,\partial_b\rangle\right] &= 2\left(-\langle\partial_a,\partial_b\rangle\right)\\ \left[\mathbb{E}_i-\mathbb{E}_j,\langle u_i,\partial_j\rangle\right] &= 2\langle u_i,\partial_j\rangle.\end{aligned}$$

In order to write down an explicit expression for the extremal projector, we first need to calculate the values $\rho_0(h_\alpha)$ and $\rho_1(h_\alpha)$ for each $\alpha\in\Delta^+$:

α	h_α	$\rho_0(h_\alpha)$	$\rho_1(h_\alpha)$
$-L_a$	$-\mathbb{E}_a-\frac{m}{2}$	a	$\frac{1}{2}$
$-2L_a$	$-\mathbb{E}_a-\frac{m}{2}$	a	$\frac{1}{2}$
$-L_a-L_b$	$-\mathbb{E}_a-\mathbb{E}_b-m$	$a+b$	1
L_i-L_j	$\mathbb{E}_i-\mathbb{E}_j$	$j-i$	0

The operators corresponding to the *even* roots then are given by

$$\begin{aligned}p_{-2L_a} &= \sum_{s=0}^{\infty}\frac{1}{4^s s!}\frac{\Gamma(-\mathbb{E}_a-\frac{m}{2}+a+1)}{\Gamma(-\mathbb{E}_a-\frac{m}{2}+a+1+s)}|u_a|^{2s}\Delta_a^s\\ p_{-L_a-L_b} &= \sum_{s=0}^{\infty}\frac{1}{s!}\frac{\Gamma(-\mathbb{E}_a-\mathbb{E}_b-m+a+b)}{\Gamma(-\mathbb{E}_a-\mathbb{E}_b-m+a+b+s)}\langle u_a,u_b\rangle^s\langle\partial_a,\partial_b\rangle^s\\ p_{L_i-L_j} &= \sum_{s=0}^{\infty}\frac{(-1)^s}{s!}\frac{\Gamma(\mathbb{E}_i-\mathbb{E}_j+j-i+1)}{\Gamma(\mathbb{E}_i-\mathbb{E}_j+j-i+1+s)}\langle u_j,\partial_i\rangle^s\langle u_i,\partial_j\rangle^s, \quad (4.7)\end{aligned}$$

whereas the ones corresponding to the *odd* roots are given by

$$p_{-L_a}=1+\frac{u_a\partial_a}{m+2\mathbb{E}_a-2a}.$$

Remark 4.5. Note that p_{-L_a} is independent of the chosen normalisation on the odd root vectors.

In order to construct the extremal projector for $\mathfrak{osp}(1,2k)$, we then need to fix a normal ordering on the set of positive roots. There are two normal orderings we will use in this chapter, given by

$$\begin{aligned}&-L_1,-2L_1,-L_1-L_2,-L_2,-2L_2,-L_1-L_3,\\ &\qquad -L_2-L_3,-L_3,-2L_3,\ldots,-L_k,-2L_k,\\ &L_1-L_2,L_1-L_3,\ldots,L_1-L_k,L_2-L_3,L_2-L_4,\ldots,L_{k-1}-L_k, \quad (4.8)\end{aligned}$$

and

$$\begin{aligned}L_1 - L_2, L_1 - L_3, \ldots, L_1 - L_k, L_2 - L_3, L_2 - L_4, \ldots, L_{k-1} - L_k,\\ - L_k, -2L_k, \ldots, -L_3, -2L_3, -L_2 - L_3,\\ - L_1 - L_3, -L_2, -2L_2, -L_1 - L_2, -L_1, -2L_1. \quad (4.9)\end{aligned}$$

To verify that these are indeed *normal*, it suffices to invoke the definition. For instance, we have that $L_1 - L_3 = 1(L_1 - L_2) + 1(L_2 - L_3)$. Indeed, in both orderings, $L_1 - L_3$ lies between $L_1 - L_2$ and $L_2 - L_3$. In view of our explicit model for $\mathfrak{osp}(1, 2k)$ in terms of Dirac operators and vector variables, taking the product of the operators defined in (4.7) in any of the normal orderings above yields an operator which projects an arbitrary (homogeneous and $\mathbb{S}$-valued) polynomial $P(u_1, \ldots, u_k)$ onto its simplicial monogenic part. This can therefore be used to define the invariant operators we are after.

4.4 Construction of the higher spin operators

We first introduce the *twisted* Dirac operator on $(\mathcal{H}_\lambda \otimes \mathbb{S})$-valued functions, which amounts to letting the Dirac operator act on functions *having the wrong values* (i.e. not $\mathbb{S}$-valued functions in one vector variable x). Twisted operators in general will prove to be important in the determination of the kernel of higher spin Dirac operators later on.

Definition 4.5. *For arbitrary integer-valued highest weights λ for* Spin(m)*, the twisted Dirac operator on $(\mathcal{H}_\lambda \otimes \mathbb{S})$-valued polynomials is defined by*

$$\partial_x^T := \mathbf{1}_\lambda \otimes \partial_x : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_\lambda \otimes \mathbb{S}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_\lambda \otimes \mathbb{S}).$$

For instance, for $\lambda = l_1$, the action of ∂_x^T on any

$$f(x, u) = \sum_{A,B} f_{A,B}(x) H_A(u) \otimes \psi_B,$$

where $H_A(u) \in \mathcal{H}_{l_1}$ and $\psi_B \in \mathbb{S}$ is given by

$$\partial_x^T f(x, u) = \sum_{A,B,j} e_j \partial_{x_j}(f_{A,B}(x)) H_A(u) \otimes \psi_B.$$

This twisted Dirac operator will play an important role in Chapter 8. $\mathcal{H}_\lambda \otimes \mathbb{S}$ is a (highly) *reducible* Spin(m)-representation (see e.g. [39]):

Lemma 4.3. *As a* Spin(m)*-representation, the tensor product $\mathcal{H}_\lambda \otimes \mathbb{S}$ can be decomposed as the direct sum of (at most) 2^k irreducible Spin(m)-modules, each one appearing with multiplicity 1:*

$$\mathcal{H}_\lambda \otimes \mathbb{S} \ \cong \ \bigoplus_{i_1=0}^{1} \cdots \bigoplus_{i_k=0}^{1} (l_1 - i_1, \ldots, l_k - i_k)'. \quad (4.10)$$

Each summand $(l_1 - i_1, \dots l_k - i_k)'$ is contained in the decomposition as long as its highest weight satisfies the dominant weight condition.

This lemma will be proven in Section 8.4.2. From this lemma, it follows that $\mathcal{S}_\lambda$ is a submodule of $\mathcal{H}_\lambda \otimes \mathbb{S}$. All the other modules (seen as function spaces, see Definition 2.19) are isomorphically embedded into the function space $\mathcal{H} \otimes \mathbb{S}$ by means of a non-trivial embedding operator. This essentially is an operator that realises the isomorphism between the space $\mathcal{S}_{\lambda - L_i}$ and its isomorphic copy in $\mathcal{H} \otimes \mathbb{S}$. It can easily be seen that such an embedding operator is needed since the degrees of homogeneity of both spaces don't match. Using the method of constructing conformally invariant operators by means of generalised gradients (see e.g. [45, 79]), one can deduce from the lemma above that the twisted Dirac operator $\mathbf{1}_\lambda \otimes \partial_x$ can be written as the sum of at most $(k+1)$ first-order differential operators: a higher spin Dirac operator $\mathcal{Q}_\lambda$ and (at most) k twistor operators $\mathcal{T}_\lambda^{(j)}$ for $1 \leq j \leq k$, as visualised in Figure 4.1. How this decomposition can be done explicitly, is explained in Chapter 12.

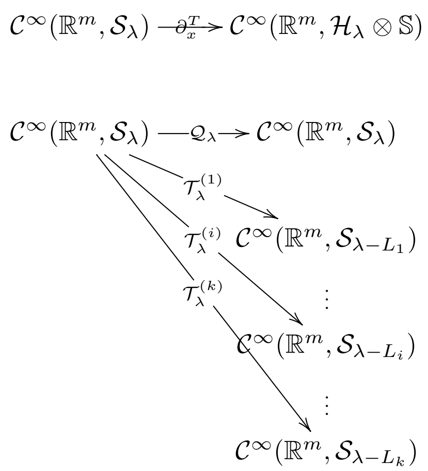


Figure 4.1: Decomposition of the twisted Dirac operator

Note that $\lambda - L_i$ stands for the highest weight $(l_1, \dots, l_i - 1, \dots, l_k)$. Also note that not all summands in (4.10) are reached by an operator, which essentially follows from Fegan's result. From this scheme, it follows that the higher spin Dirac operator must be of the form $\mathcal{Q}_\lambda = \pi[\partial_x^T]$, where π is a projection operator on the space $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda)$. Taking Definition 2.19 into account, it is clear that π is nothing but the extremal projector $p_{\mathfrak{k}}$ for $\mathfrak{k} = \mathfrak{osp}(1, 2k)$. If we choose the normal ordering (4.8), this means that the higher spin Dirac operator can be defined as follows:

$$\mathcal{Q}_\lambda = \big(p_{-L_1}p_{-2L_1}p_{-L_1-L_2}p_{-L_2}p_{-2L_2}\cdots p_{-L_k}p_{-2L_k} \times p_{L_1-L_2}p_{L_1-L_3}\cdots p_{L_{k-1}-L_k}\big)[\partial_x].$$

For each $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda)$, we have that $\langle u_i, \partial_j\rangle \partial_x f = 0$, for all $1 \leq i < j \leq k$, meaning that this expression can be somewhat reduced, in the sense that the operators $p_{L_i-L_j}$ can be omitted:

$$\mathcal{Q}_\lambda = \big(p_{-L_1}p_{-2L_1}p_{-L_1-L_2}p_{-L_2}p_{-2L_2}\cdots p_{-L_k}p_{-2L_k}\big)[\partial_x].$$

Invoking the commutation relations in $\mathfrak{osp}(1, 2k+2)$ and the definition of $\mathcal{S}_\lambda$, it is not hard to see that this expression can be simplified even further. For example, the operator p_{-2L_k} at the end reduces to the identity operator, as Δ_k acts trivially on $\mathcal{S}_\lambda$-valued functions. Similar observations can be made for most of the other projection operators p_{e_α}, which then leads to the following product (ordered, with i increasing from left to right):

$$\mathcal{Q}_\lambda = \big(p_{-L_1}p_{-L_2}\cdots p_{-L_k}\big)[\partial_x] = \prod_{i=1}^{k}\left(1 + \frac{u_i\partial_i}{m + 2\mathbb{E}_i - 2i}\right)[\partial_x], \tag{4.11}$$

which is exactly the result found in [28].

Remark 4.6. Invoking the definition of ellipticity shows us that the higher spin Dirac operator is an *elliptic* operator, since its symbol

$$\prod_{i=1}^{k}\left(1 + \frac{u_i\partial_i}{m + 2\mathbb{E}_i - 2i}\right)[x] \neq 0,$$

for all $x \neq 0$ (see Remark 7.2).

The advantage of the present approach is that we can now also write down explicit expressions for the twistor operators, formally defined for $j \in \{1, \ldots, k\}$ as the unique first-order conformally invariant differential operator

$$\mathcal{T}_\lambda^{(j)} : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda-L_j}).$$

In view of our model for the spaces $\mathcal{S}_\lambda$, it is clear that this operator lowers the degree in u_j by one. There is an obvious rotationally invariant operator which has this effect, given by $\langle \partial_j, \partial_x\rangle$, but since $\langle u_j, \partial_b\rangle\langle \partial_j, \partial_x\rangle f = -\langle \partial_b, \partial_x\rangle f \neq 0$ for functions $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda)$, whenever $b > j$, we again need a suitable projection in order to obtain an operator having the desired mapping properties. Similarly as above, we define

$$\mathcal{T}_\lambda^{(j)} := p_{\mathfrak{k}}[\langle \partial_j, \partial_x\rangle].$$

We will now reduce this expression to a more condensed form (taking into account that we are meant to act on $\mathcal{S}_\lambda$-valued functions), choosing the normal ordering (4.9). As the operator $\langle \partial_j, \partial_x \rangle$ commutes with each of the operators ∂_i and $\langle \partial_a, \partial_b \rangle$, the twistor operator immediately reduces to

$$\mathcal{T}_\lambda^{(j)} = \big(p_{L_1-L_2} p_{L_1-L_3} \cdots p_{L_1-L_k} p_{L_2-L_3} p_{L_2-L_4} \cdots p_{L_{k-1}-L_k}\big)\langle \partial_j, \partial_x \rangle.$$

Next, we show that each of the remaining operators $p_{L_i-L_j}$ reduces to either the identity operator, or the sum of two terms only. First of all, we consider the set of positive roots given by $\{L_j - L_{j+1}, \ldots, L_{k-1} - L_k\}$. Note that this set is empty if $j = k$.

Lemma 4.4. *Given a fixed index j, we have that*

$$\begin{aligned}\big(p_{L_j-L_{j+1}} p_{L_j-L_{j+2}} \cdots p_{L_{k-1}-L_k}\big)&\langle \partial_j, \partial_x \rangle \\ &= \prod_{p=j+1}^{k} \left(1 - \frac{\langle u_p, \partial_j \rangle \langle u_j, \partial_p \rangle}{\mathbb{E}_j - \mathbb{E}_p + p - j + 1}\right) \langle \partial_j, \partial_x \rangle.\end{aligned}$$

Proof. First of all, for $j < a < b$ we have that $\langle u_b, \partial_a \rangle \langle u_a, \partial_b \rangle \langle \partial_j, \partial_x \rangle f = 0$, implying that the operator $p_{L_a-L_b}$ reduces to the identity operator in this case. Next, for all indices $j < b \leq k$, we have that

$$\langle u_b, \partial_a \rangle \langle u_j, \partial_b \rangle \langle \partial_j, \partial_x \rangle f = -\langle u_b, \partial_j \rangle \langle \partial_j, \partial_x \rangle f,$$

which means that

$$\langle u_b, \partial_a \rangle^2 \langle u_a, \partial_b \rangle^2 \langle \partial_j, \partial_x \rangle f = 0.$$

For these indices the operator $p_{L_j-L_b}$ thus reduces to the sum of two operators. □

Lemma 4.5. *Given a fixed index j, we have that*

$$\big(p_{L_1-L_2} \cdots p_{L_{k-1}-L_k}\big)\langle \partial_j, \partial_x \rangle = \prod_{p=j+1}^{k} \left(1 - \frac{\langle u_p, \partial_j \rangle \langle u_j, \partial_p \rangle}{\mathbb{E}_j - \mathbb{E}_p + p - j + 1}\right) \langle \partial_j, \partial_x \rangle,$$

where the product is ordered.

Proof. In view of the previous lemma, it suffices to prove that the product of operators $p_{L_a-L_b}$ with $a < b < j$ reduces to the identity operator. This immediately follows from the observation that the operator $\langle u_a, \partial_b \rangle$ commutes with each of the operators on its right, after which it acts trivially on the $\mathcal{S}_\lambda$-valued function f. □

Finally, this leads to the following:

Definition 4.6. *Given an arbitrary half-integer highest weight λ and an index $1 \leq j \leq k$ such that $\lambda - L_i$ satisfies the dominant weight condition, we have that*

$$\mathcal{T}_\lambda^{(j)} = \prod_{p=j+1}^{k} \left(1 - \frac{\langle u_p, \partial_j \rangle \langle u_j, \partial_p \rangle}{\mathbb{E}_j - \mathbb{E}_p + p - j + 1}\right) \langle \partial_j, \partial_x \rangle. \tag{4.12}$$

Note that the product in this equation again is ordered (p increasing from left to right), as the factors do not commute.

It is crucial to point out that this definition is in a sense still an educated guess: in order to be sure that this is indeed a twistor operator, we need to prove that this operator indeed is conformally invariant. This will be postponed until the next section.

Note that the general theory of generalised gradients (see [45, 79]) also predicts a third type of conformally invariant first-order differential operators, the so-called dual twistor operators, which raise the degree in u_j by one. Using the same reasoning as for the normal twistor operators, one can deduce that they are defined by

$$\mathcal{T}_\lambda^{(j)*} = p_{\mathfrak{k}}[\langle u_j, \partial_x \rangle].$$

For instance, the dual twistor operator

$$\mathcal{T}_{l_1}^{(1)*} : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1}) \rightarrow \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1+1})$$

equals

$$\mathcal{T}_{l_1}^{(1)*} = \left(1 + \frac{u_1 \partial_1}{m + 2\mathbb{E}_1 - 2}\right) \left(1 - \frac{|u_1|^2 \Delta_1}{2(m + 2\mathbb{E}_1 - 4)}\right) \langle u_1, \partial_x \rangle$$

in its reduced form.

4.5 Conformal invariance

In this section, we will explicitly prove that the higher spin Dirac operator and the higher spin twistor operators are conformally invariant. As was pointed out in the previous section, this is in a sense crucial in order to justify the definitions from the previous section: the existence and uniqueness of conformally invariant operators usually comes from geometrical arguments, whereas we merely identified them as generators of a transvector algebra. Also, despite the fact that this symmetry is usually taken for granted, we have not yet encountered an explicit proof in the literature (in the setting of Clifford analysis, where everything is done on flat space $\mathbb{R}^m$). We will prove the conformal invariance in a similar way as we did for the classical Dirac operator. The infinitesimal generators of the conformal transformations will have to generate the algebra $\mathfrak{so}(1, m+1)$, when again choosing a suitable conformal weight $w \in \mathbb{R}$, and using the graded structure of this Lie algebra.

First of all, we have the symmetries:

- *Rotations:* $dL(e_{ij}) = L_{ij}^x + \sum_{p=1}^k L_{ij}^{u_p} - \frac{1}{2}e_{ij}$. These span the simple part $\mathfrak{so}(m) \subset \mathfrak{g}_0$.
- *Translations:* $dT(e_j) = \partial_{x_j}$. They can be identified with the subspace $\mathfrak{g}_{-1}$ (see below for the grading element).

Indeed, it is easily verified that $[\mathcal{Q}_\lambda, dL(e_{ij})] = [\mathcal{Q}_\lambda, dT(e_j)] = 0$. Next, as the operator $\mathcal{Q}_\lambda$ is homogeneous of degree (-1), we also have that

$$\mathcal{Q}_\lambda \left(\mathbb{E}_x + \frac{m-1}{2} \right) = \left(\mathbb{E}_x + \frac{m+1}{2} \right) \mathcal{Q}_\lambda,$$

which means that the generator of dilatations is a generalised symmetry. Note that the element between brackets at the left-hand side is the grading element $E \in \mathfrak{g}_0$ (the shift defines the conformal weight, i.e. $2w = m-1$). The expression of E essentially follows from (4.14) further on. Finally, we have the generalised symmetries in the subspace $\mathfrak{g}_{+1}$, which can be defined in terms of the inversion operator.

Definition 4.7. *The inversion operator $I_{\mathcal{Q}_\lambda}$ on $\mathcal{S}_\lambda$-valued functions is defined as*

$$I_{\mathcal{Q}_\lambda} f(x; u_1, \ldots, u_k) = \frac{x}{|x|^m} f\left(\frac{x}{|x|^2}; \frac{xu_1x}{|x|^2}, \ldots, \frac{xu_kx}{|x|^2} \right).$$

It is well-known that $I_{\mathcal{Q}_\lambda}$ preserves $\mathcal{Q}_\lambda$-solutions, see e.g. [77]. Recently, an alternative proof was given in [37], based on the following results. For a proof, we also refer to that paper.

Lemma 4.6. *On arbitrary functions $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda)$, one has the operator identity*

$$I_{\mathcal{Q}_\lambda} \partial_x I_{\mathcal{Q}_\lambda} = |x|^2 \partial_x - \sum_{i=1}^k [\Gamma_{u_i}, x]. \tag{4.13}$$

Theorem 4.5. *For all $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda)$, we have the equality*

$$I_{\mathcal{Q}_\lambda} \mathcal{Q}_\lambda I_{\mathcal{Q}_\lambda} f = |x|^2 \mathcal{Q}_\lambda f.$$

Remark 4.7. In the case of the Dirac operator, we get that $I_{\partial_x}\partial_x I_{\partial_x} = |x|^2\partial_x$, which is exactly Theorem 4.1.

Lemma 4.7. *For all $1 \leq j \leq m$, the operators $I_{\mathcal{Q}_\lambda}\partial_{x_j}I_{\mathcal{Q}_\lambda}$ define (commuting) generalised symmetries for the higher spin Dirac operator $\mathcal{Q}_\lambda$.*

Proof. Using Theorem 4.5 and the fact that $I_{\mathcal{Q}_\lambda}^2 = -1$, we get that

$$\begin{aligned}\mathcal{Q}_\lambda\big(I_{\mathcal{Q}_\lambda}\partial_{x_j}I_{\mathcal{Q}_\lambda}\big) &= -I_{\mathcal{Q}_\lambda}|x|^2\mathcal{Q}_\lambda\partial_{x_j}I_{\mathcal{Q}_\lambda}\\ &= I_{\mathcal{Q}_\lambda}|x|^2\partial_{x_j}I_{\mathcal{Q}_\lambda}^2\mathcal{Q}_\lambda I_{\mathcal{Q}_\lambda} = \big(I_{\mathcal{Q}_\lambda}|x|^2\partial_{x_j}I_{\mathcal{Q}_\lambda}|x|^2\big)\mathcal{Q}_\lambda,\end{aligned}$$

from which the conclusion follows. The fact that they all commute easily follows from the observation that $I_{\mathcal{Q}_\lambda}^2 = -1$. □

These generalised symmetries clearly belong to the subspace $\mathfrak{g}_{+1}$, it suffices to note that they are homogeneous of degree $(+1)$. In order to finish the proof of conformal invariance, we mention the following:

Theorem 4.6. *The generalised symmetries constructed above generate a Lie algebra which is isomorphic to $\mathfrak{so}(1, m+1)$.*

Proof. In order to prove that $[\mathfrak{g}_0, \mathfrak{g}_{\pm1}] \subset \mathfrak{g}_{\pm1}$, it is easier to switch back to the spin group and to make use of the fact that the derived action is given by

$$dL(e_{ab}) = \frac{d}{dt}L(e^{te_{ab}})\bigg|_{t=0} := \frac{d}{dt}L(s_{ab})\bigg|_{t=0}.$$

For $\partial_{x_j} \in \mathfrak{g}_{-1}$ and $f(x;u_{(k)})$ an arbitrary $\mathcal{S}_\lambda$-valued function (where $u_{(k)}$ was a shorthand notation for the k dummy variables), we then get:

$$\begin{aligned}L(s_{ab})\partial_{x_j}f(x;u_{(k)}) &= s_{ab}\partial_{\langle\bar{s}_{ab}xs_{ab},e_j\rangle}f(\bar{s}_{ab}xs_{ab};\bar{s}_{ab}u_{(k)}s_{ab})\\ &= \partial_{\langle\bar{s}_{ab}xs_{ab},e_j\rangle}L(s_{ab})f(x;u_{(k)}),\end{aligned}$$

from which we immediately see that $2[dL(e_{ab}),\partial_{x_j}] = (\delta_{bj}\partial_{x_a} - \delta_{aj}\partial_{x_b})$. For the generalised symmetries $I_{\mathcal{Q}_\lambda}\partial_{x_j}I_{\mathcal{Q}_\lambda} \in \mathfrak{g}_{+1}$, the reasoning is completely similar (albeit more elaborate to write down). In order to prove that $[\mathfrak{g}_{-1}, \mathfrak{g}_{+1}] \subset \mathfrak{g}_0$ we will make use of the fact that

$$I_{\mathcal{Q}_\lambda}\partial_{x_j}I_{\mathcal{Q}_\lambda} = \frac{1}{2}\left\{I_{\mathcal{Q}_\lambda}e_jI_{\mathcal{Q}_\lambda}, I_{\mathcal{Q}_\lambda}\partial_xI_{\mathcal{Q}_\lambda}\right\} = \frac{1}{2}\left\{\frac{xe_jx}{|x|^2}, |x|^2\partial_x - \sum_{a=1}^{k}[\Gamma_{u_a}, x]\right\},$$

since it holds for the Euclidean inner product of two vectors $x, y \in \mathbb{R}^m$ that $-2\langle x, y\rangle = \{x, y\}$ in the Clifford algebra $\mathbb{C}_m$. The first part, coming from the Dirac operator, gives

$$\frac{1}{2}\left\{\frac{xe_jx}{|x|^2}, |x|^2\partial_x\right\} = -|x|^2\partial_{x_j} + x_j(2\mathbb{E}_x + m - 1) - e_j \wedge x,$$

see e.g. [40]. The other part, coming from the Gamma operators, gives

$$\begin{aligned}\frac{1}{2}\sum_{a=1}^{k}\left\{\frac{xe_jx}{|x|^2},[\Gamma_{u_a},x]\right\} &= \frac{1}{2}\sum_{a=1}^{k}\{x,[e_j,\Gamma_{u_a}]\}\\ &= -2\sum_{a=1}^{k}\big(u_{aj}\langle x,\partial_a\rangle-\langle u_a,x\rangle\partial_{u_{aj}}\end{aligned}$$

Taking the commutator between generators of $\mathfrak{g}_{-1}$ and $\mathfrak{g}_{+1}$, we get

$$\left[\partial_{x_i},I_{\mathcal{Q}_\lambda}\partial_{x_j}I_{\mathcal{Q}_\lambda}\right]=\begin{cases} e_ie_j-2L_{ij}^x-2\sum_{a=1}^k L_{ij}^{u_a} & \text{if} \quad i\neq j\\ 2\mathbb{E}_x+m-1 & \text{if} \quad i=j,\end{cases} \tag{4.14}$$

which is an element of $\mathfrak{g}_0$ in both cases. □

Remark 4.8. The uniqueness of the conformal weight also becomes clear from (4.14) as this shows that only for a particular value of the conformal weight, one indeed gets that the commutation relations are internally defined on the Lie algebra.

This concludes the proof of the conformal invariance of the higher spin Dirac operator $\mathcal{Q}_\lambda$. We conclude this chapter with a similar reasoning for the higher spin twistor operators

$$\mathcal{T}_\lambda^{(p)}:\mathcal{C}^\infty(\mathbb{R}^m,\mathcal{S}_\lambda)\to\mathcal{C}^\infty(\mathbb{R}^m,\mathcal{S}_{\lambda-L_p}).$$

An important difference between $\mathcal{Q}_\lambda$ and $\mathcal{T}_\lambda^{(p)}$ is the conformal weight, which equals

$$w_p=m+l_p-p-\frac{1}{2}$$

for the twistor operator $\mathcal{T}_\lambda^{(p)}$ (see [45]). As a consequence of this difference, the dilatations are generated by multiples of $\mathbb{E}_x+m+l_p-p-\frac{1}{2}$. Since we have the relation

$$\mathcal{T}_\lambda^{(p)}\left(\mathbb{E}_x+m+l_p-p-\frac{1}{2}\right)=\left(\mathbb{E}_x+m+l_p-p+\frac{1}{2}\right)\mathcal{T}_\lambda^{(p)},$$

the dilatations are generalised symmetries for $\mathcal{T}_\lambda^{(p)}$. The rotations $dL(e_{ij})$ and translations $dT(e_i)$ remain unchanged, and it is again easily verified that

$$[\mathcal{T}_\lambda^{(p)},dL(e_{ij})]=[\mathcal{T}_\lambda^{(p)},dT(e_i)]=0,$$

making them actual symmetries. Finally, we have the inversion operator related to $\mathcal{T}_\lambda^{(p)}$, which will then be used to define the missing generalised symmetries:

Definition 4.8. *The inversion operator* $I_{\mathcal{T}_\lambda^{(p)}}$ *on* $\mathcal{S}_\lambda$*-valued functions is defined as*

$$I_{\mathcal{T}_\lambda^{(p)}} f(x;u_1,\ldots,u_k) = \frac{x}{|x|^{2(m+l_p-p)}} f\left(\frac{x}{|x|^2}; \frac{xu_1x}{|x|^2},\ldots,\frac{xu_kx}{|x|^2}\right).$$

Rembember that the inversion operator $I_{\mathcal{Q}_\lambda}$ itself is a generalised symmetry for the HSD operator, as it maps elements of the kernel space of $\mathcal{Q}_\lambda$ to the same space. The same holds for the inversion operators related to the twistor operators. As an example, we first give the proof in case of $k=1$, for the HST operator $\mathcal{T}_{l_1}^{(1)} = \langle \partial_1, \partial_x \rangle$.

Lemma 4.8. *If* $f(x,u_1) \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1}) \cap \ker(\langle \partial_1, \partial_x \rangle)$*, we have that*

$$\langle \partial_1, \partial_x \rangle I_{\mathcal{T}_{l_1}^{(1)}} f = 0.$$

Proof. First of all, note that we have

$$I_{\mathcal{T}_{l_1}^{(1)}} = \frac{1}{|x|^{m+2l_1-2}} I_{\mathcal{Q}_{l_1}}.$$

Also, ∂_1 and $I_{\mathcal{Q}_{l_1}}$ anti-commute. This is easily proven by using the invariance properties of ∂_1 (see e.g. [37]).

$$\begin{aligned}\partial_1 I_{\mathcal{Q}_{l_1}} g(x,u_1) &= \partial_1 \frac{x}{|x|^m} g\left(\frac{x}{|x|^2}, \frac{xu_1x}{|x|^2}\right) \\ &= \frac{x}{|x|} \partial_{\frac{xu_1x}{|x|^2}} \frac{x}{|x|} \frac{x}{|x|^m} g\left(\frac{x}{|x|^2}, \frac{xu_1x}{|x|^2}\right) = -I_{\mathcal{Q}_{l_1}} \partial_1 g(x,u_1).\end{aligned}$$

From this, it follows that

$$\begin{aligned}&-2\langle \partial_1, \partial_x \rangle I_{\mathcal{T}_\lambda^{(1)}} f(x,u_1) = (\partial_1\partial_x + \partial_x\partial_1) \frac{1}{|x|^{m+2l_1-2}} I_{\mathcal{Q}_{l_1}} f(x,u_1) \\ &= \partial_1 \left(\frac{-(m+2l_1-2)x}{|x|^{m+2l_1}} I_{\mathcal{Q}_{l_1}} f(x,u_1) + \frac{1}{|x|^{m+2l_1-2}} \partial_x I_{\mathcal{Q}_{l_1}} f(x,u_1)\right).\end{aligned}$$

Using the fact that $I^2_{\mathcal{Q}_{l_1}} = -1$, (4.13) and the anti-commuting relation above, this reduces to

$$\begin{aligned}\frac{2(m+2l_1-2)}{|x|^{m+2l_1}} \langle x, \partial_1 \rangle I_{\mathcal{Q}_{l_1}} f(x,u_1) &- \frac{2}{|x|^{m+2l_1-2}} I_{\mathcal{Q}_{l_1}} \partial_1 u_1 \langle x, \partial_1 \rangle f(x,u_1) \\ &+ \frac{2}{|x|^{m+2l_1-2}} I_{\mathcal{Q}_{l_1}} |x|^2 \partial_1 \partial_x f(x,u_1).\end{aligned}$$

Since $\langle \partial_1, \partial_x \rangle f = 0$, the last term equals 0. Further using the commutation relation between ∂_1 and u_1, the expression reduces to

$$-2\langle \partial_1, \partial_x \rangle I_{\mathcal{T}^{(1)}_{l_1}} f(x, u_1) = \frac{2(m+2l_1-2)}{|x|^{m+2l_1}} \left(\langle x, \partial_1 \rangle I_{\mathcal{Q}_{l_1}} f + |x|^2 I_{\mathcal{Q}_{l_1}} \langle x, \partial_1 \rangle f \right).$$

Moreover, we have that

$$\langle x, \partial_1 \rangle I_{\mathcal{Q}_{l_1}} f = -\frac{1}{2} \partial_1 x I_{\mathcal{Q}_{l_1}} f = \frac{1}{2} I_{\mathcal{Q}_{l_1}} \partial_u \frac{x}{|x|^2} f = -I_{\mathcal{Q}_{l_1}} \frac{1}{|x|^2} \langle x, \partial_1 \rangle f,$$

thus we get

$$-2\langle \partial_1, \partial_x \rangle I_{\mathcal{T}^{(1)}_{\lambda}} f = \frac{2}{|x|^{m+2l_1}} \left(-I_{\mathcal{Q}_{l_1}} \frac{1}{|x|^2} \langle x, \partial_1 \rangle f + |x|^2 I_{\mathcal{Q}_{l_1}} \langle x, \partial_1 \rangle f \right) = 0.$$

This proves the lemma. □

The general case can be proven similarly.

Lemma 4.9. *If $f(x, u_{(k)}) \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda) \cap \ker \mathcal{T}^{(a)}_\lambda$, then we have that*

$$\mathcal{T}^{(a)}_\lambda) I_{\mathcal{T}^{(a)}_\lambda} f = 0.$$

Proof. First of all, note that we have

$$I_{\mathcal{T}^{(a)}_\lambda} = \frac{1}{|x|^{m+2l_a-2a}} I_{\mathcal{Q}_\lambda}.$$

Also, ∂_a and $I_{\mathcal{Q}_\lambda}$ anticommute. This is easily proved by using the invariance properties of ∂_a (see e.g. [37]).

$$\begin{aligned} \partial_a I_{\mathcal{Q}_\lambda} g(x, u_{(k)}) &= \partial_1 \frac{x}{|x|^m} g\left(\frac{x}{|x|^2}, \frac{x u_{(k)} x}{|x|^2} \right) \\ &= \frac{x}{|x|} \partial_{\frac{x u_a x}{|x|^2}} \frac{x}{|x|} \frac{x}{|x|^m} g\left(\frac{x}{|x|^2}, \frac{x u_{(k)} x}{|x|^2} \right) \\ &= -I_{\mathcal{Q}_\lambda} \partial_a g(x, u_{(k)}). \end{aligned}$$

Using similar invariance properties, one can prove that $p_{\mathfrak{osp}(1,2k)}$ commutes with $I_{\mathcal{Q}_\lambda}$. From this, it follows that

$$\begin{aligned} &-2\langle \partial_a, \partial_x \rangle I_{\mathcal{T}^{(a)}_\lambda} f(x, u_{(k)}) = (\partial_a \partial_x + \partial_x \partial_a) \frac{1}{|x|^{m+2l_a-2a}} I_{\mathcal{Q}_\lambda} f(x, u_{(k)}) \\ &= \partial_a \left(\frac{-(m+2l_a-2a)x}{|x|^{m+2l_a-2a+2}} I_{\mathcal{Q}_\lambda} f(x, u_{(k)}) + \frac{1}{|x|^{m+2l_a-2a}} \partial_x I_{\mathcal{Q}_\lambda} f(x, u_{(k)}) \right). \end{aligned}$$

Applying $p_{\mathfrak{osp}(1,2k)}$, using the fact that $I_{\mathcal{Q}_\lambda}^2 = -1$, (4.13) and the (anti-) commuting relations above, this reduces to

$$\frac{2(m+2l_a-2a)}{|x|^{m+2l_a-2a+2}} p_{\mathfrak{osp}(1,2k)} \langle x, \partial_a \rangle I_{\mathcal{Q}_\lambda} f(x, u_{(k)}) \tag{4.15}$$

$$-\sum_{p=1}^{k} \frac{2}{|x|^{m+2l_a-2a}} I_{\mathcal{Q}_\lambda} p_{\mathfrak{osp}(1,2k)} \partial_a u_p \langle x, \partial_p \rangle f(x, u_{(k)}) \tag{4.16}$$

$$+\frac{1}{|x|^{m+2l_a-2a}} I_{\mathcal{Q}_\lambda} |x|^2 p_{\mathfrak{osp}(1,2k)} \partial_a \partial_x f(x, u_{(k)}). \tag{4.17}$$

Since $f \in \ker \mathcal{T}_\lambda^{(a)}$, (4.17) equals 0. For (4.15), we get that

$$\langle x, \partial_a \rangle I_{\mathcal{Q}_\lambda} f = -\frac{1}{2} \partial_a x I_{\mathcal{Q}_\lambda} f = \frac{1}{2} I_{\mathcal{Q}_\lambda} \partial_a \frac{x}{|x|^2} f = -I_{\mathcal{Q}_\lambda} \frac{1}{|x|^2} \langle x, \partial_a \rangle f.$$

And finally, for (4.16), note that if $p > a$, the properties of the extremal projector guarantee that $p_{\mathfrak{osp}(1,2k)} \langle u_p, \partial_a \rangle f = 0$. Hence the sum can be reduced to

$$\begin{aligned} &\sum_{p=1}^{a} \frac{2}{|x|^{m+2l_a-2a}} I_{\mathcal{Q}_\lambda} p_{\mathfrak{osp}(1,2k)} \partial_a u_p \langle x, \partial_p \rangle f(x, u_{(k)}) \\ &\quad = \sum_{p=1}^{a-1} \frac{4}{|x|^{m+2l_a-2a}} I_{\mathcal{Q}_\lambda} p_{\mathfrak{osp}(1,2k)} \langle x, \partial_a \rangle f(x, u_{(k)}) \\ &\qquad + \frac{2(-m-2a+2)}{|x|^{m+2l_a-2a}} I_{\mathcal{Q}_\lambda} p_{\mathfrak{osp}(1,2k)} \langle x, \partial_a \rangle f(x, u_{(k)}). \end{aligned}$$

Adding the three results gives 0. □

With this definition of inversion operator, we prove the following:

Lemma 4.10. *On arbitrary functions $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda)$, one has the operator identity*

$$\begin{aligned} I_{\mathcal{T}_\lambda^{(p)}} \partial_{x_j} I_{\mathcal{T}_\lambda^{(p)}} &= x \wedge e_j + 2x_j \left(\mathbb{E}_x + m + l_p - p - \frac{1}{2} \right) - |x|^2 \partial_{x_j} \\ &+ 2 \sum_{a=1}^{k} \left(u_{aj} \langle x, \partial_a \rangle - \langle u_a, x \rangle \partial_{u_{aj}} \right). \end{aligned}$$

Proof. Follows from direct calculations on any function $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda)$. □

In the following lemma, the operator $p_{\mathfrak{osp}(1,2k)}$ denotes the extremal projector from (4.12).

Lemma 4.11. *On arbitrary functions $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda)$, we have*

$$\left[\mathcal{T}_\lambda^{(p)}, \sum_{a=1}^{k} \left(u_{aj}\langle x, \partial_a\rangle - \langle u_a, x\rangle \partial_{u_{aj}}\right)\right] = -p_{\mathfrak{osp}(1,2k)} \left(\partial_{x_j}\langle x, \partial_p\rangle - (m + \mathbb{E}_x + \mathbb{E}_p)\partial_{u_{pj}}\right). \quad (4.18)$$

Proof. In view of the fact that $\mathcal{T}_\lambda^{(p)} = p_{\mathfrak{osp}(1,2k)}\langle \partial_x, \partial_p\rangle$, together with

$$p_{\mathfrak{osp}(1,2k)} \sum_{a=1}^{k} \left(u_{aj}\langle x, \partial_a\rangle - \langle u_a, x\rangle \partial_{u_{aj}}\right) = \sum_{a=1}^{k} \left(u_{aj}\langle x, \partial_a\rangle - \langle u_a, x\rangle \partial_{u_{aj}}\right) p_{\mathfrak{osp}(1,2k)},$$

it suffices to calculate the commutator of $\sum_{a=1}^{k} \left(u_{aj}\langle x, \partial_a\rangle - \langle u_a, x\rangle \partial_{u_{aj}}\right)$ and $\langle \partial_x, \partial_p\rangle$. This proves the lemma. □

This brings us to the following result:

Lemma 4.12. *The operators defined in lemma 4.10 are generalised symmetries of the higher spin twistor operator.*

Proof. Joining the above lemmata, we get the relation

$$\mathcal{T}_\lambda^{(p)} I_{\mathcal{T}_\lambda^{(p)}} \partial_{x_j} I_{\mathcal{T}_\lambda^{(p)}} = [x e_j + 2x_j(\mathbb{E}_x + m + l_p - p + 2) - |x|^2 \partial_{x_p} + 2\sum_{a=1}^{k} \left(u_{aj}\langle x, \partial_a\rangle - \langle u_a, x\rangle \partial_{u_{aj}}\right)]\mathcal{T}_\lambda^{(p)},$$

which proves the lemma. □

Finally, we check that we indeed end up with the conformal algebra.

Theorem 4.7. *The generalised symmetries (of first order) for $\mathcal{T}_\lambda^{(p)}$ generate a Lie algebra isomorphic to the conformal Lie algebra $\mathfrak{so}(1, m + 1)$.*

Proof. Recalling the graded structure for $\mathfrak{so}(1, m+1)$, it is obvious to choose

$$\partial_{x_i} \in \mathfrak{g}_{-1} \ , \ I_{\mathcal{T}_\lambda^{(p)}} \partial_{x_j} I_{\mathcal{T}_\lambda^{(p)}} \in \mathfrak{g}_{+1} \ , \ dL(e_{ij}) \in \mathfrak{g}_0.$$

Also, $E = \mathbb{E}_x + m + l_p - p - \frac{1}{2} \in \mathfrak{g}_0$. As is shown in Proposition 4.6, we immediately have the relations $[\mathfrak{g}_0, \mathfrak{g}_{\pm 1}] \subset \mathfrak{g}_{\pm 1}$. Since we already have an expression for $I_{\mathcal{T}_\lambda^{(p)}} \partial_{x_j} I_{\mathcal{T}_\lambda^{(p)}}$ from Lemma 4.10, we get

$$\left[\partial_{x_i}, I_{\mathcal{T}_\lambda^{(p)}} \partial_{x_j} I_{\mathcal{T}_\lambda^{(p)}}\right] = \begin{cases} e_i e_j - 2L_{ij}^x - 2\sum_{a=1}^k L_{ij}^{u_a} & \text{if} \quad i \neq j \\ 2\left(\mathbb{E}_x + m + l_p - p - \frac{1}{2}\right) & \text{if} \quad i = j. \end{cases}$$

This finishes the proof. □

4.6 Conclusion

In this chapter, we have described a way to construct higher spin Dirac and (dual) twistor operators. Also, we have proven the conformal invariance of these operators by means of explicit calculations, verifying that the first-order generalised symmetries generate the conformal Lie algebra $\mathfrak{so}(1, m+1)$. It is important to see that the method used in this chapter does not limit one to the construction of the conformally invariant higher spin *differential* operators, but may also be applied for constructing all other generators of the transvector algebra $Z(\mathfrak{osp}(1, 2k+2), \mathfrak{osp}(1, 2k))$. These are the counterparts of the higher spin differential operators in which ∂_x is replaced by x. Although they were not explicitly used in this chapter, it is clear that they will have to be used when explicitly decomposing the (polynomial) kernel for the operator $\mathcal{Q}_\lambda$ further on.

Mathematics is a game played according to certain simple rules with meaningless marks on paper.

David Hilbert

5

The transvector algebra $Z(\mathfrak{osp}(1,4), \mathfrak{osp}(1,2))$

In the previous chapter, we discovered that general higher spin Dirac and (dual) twistor operators can be constructed using an extremal projector operator for the Lie superalgebra $\mathfrak{osp}(1,2k)$. From this, we noticed that these operators (amongst others) are generators of a so-called transvector algebra. In this section we take a closer look at the transvector algebra $Z(\mathfrak{osp}(1,4), \mathfrak{osp}(1,2))$, which contains the (total) Rarita-Schwinger operator as a generator.

5.1 General setting

Let us recapitulate the construction of the transvector algebra. Since the general construction was rather abstract, we go over it again, this time in a very specific case. We follow the construction method given by Zhelobenko for Lie superalgebras in general in [90], and of which a case study was made by Molev in [66] in the case of classical Lie algebras. Take the Lie superalgebra $\mathfrak{g} = \mathfrak{osp}(1,4)$. We already discussed that this orthosymplectic Lie superalgebra can be realised as an operator algebra generated by

$$\mathfrak{osp}(1,4) = \mathrm{Alg}_{\mathbb{C}} \{x, u, \partial_x, \partial_u\} .$$

This Lie superalgebra can be decomposed into an even and an odd part:

$$\mathfrak{osp}(1,4) = \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

where the even subalgebra is given by

$$\mathfrak{g}_0 = \mathfrak{sp}(4) = \mathrm{Alg}_{\mathbb{C}} \left\{ \Delta_x, \Delta_u, \langle \partial_x, \partial_u \rangle, \langle u, \partial_x \rangle, \langle x, \partial_u \rangle, x^2, u^2, \langle x, u \rangle, \mathbb{E}_x, \mathbb{E}_u \right\},$$

the classical symplectic Lie algebra. On the other hand, we take

$$\mathfrak{k} = \mathfrak{osp}(1,2)$$

as a subalgebra. We have the $\mathfrak{k}$-module decomposition

$$\mathfrak{osp}(1,4) = \mathfrak{osp}(1,2) \oplus \mathfrak{p},$$

where $\mathfrak{osp}(1,2)$ is realised as an operator algebra of the form

$$\mathfrak{osp}(1,2) = \mathrm{LS}_{\mathbb{C}} \left\{ u, \partial_u \right\}.$$

The space $\mathfrak{p}$ indeed is a $\mathfrak{k}$-module, since

$$\begin{aligned} \mathfrak{p} &\cong \mathrm{span}_{\mathbb{C}}\{\langle \partial_u, \partial_x \rangle, \partial_x, \langle u, \partial_x \rangle\} \oplus \mathrm{span}_{\mathbb{C}}\{\langle x, \partial_u \rangle, x, \langle u, x \rangle\} \\ &\quad \oplus \mathrm{span}_{\mathbb{C}}\{\Delta_x\} \oplus \mathrm{span}_{\mathbb{C}}\{|x|^2\} \oplus \mathrm{span}_{\mathbb{C}}\{\mathbb{E}_x + \frac{m}{2}\} \\ &\cong \mathfrak{k}[\langle \partial_u, \partial_x \rangle] \oplus \mathfrak{k}[\langle x, \partial_u \rangle] \oplus \mathfrak{k}[\Delta_x] \oplus \mathfrak{k}[|x|^2] \oplus \mathfrak{k}[\mathbb{E}_x + \frac{m}{2}]. \end{aligned}$$

The action of $\mathfrak{k}$ in the last line is each time given by the Lie superbracket in $\mathfrak{osp}(1,4)$. This means that $\mathfrak{p}$ decomposes as a direct sum of five irreducible $\mathfrak{k}$-modules.

Since $\mathfrak{osp}(1,2)$ is a Lie superalgebra as well, it can be decomposed in an even subalgebra and and odd subspace:

$$\mathfrak{osp}(1,2) = \mathfrak{k}_0 \oplus \mathfrak{k}_1 = \mathfrak{sp}(2) \oplus \mathfrak{k}_1 = \mathfrak{sl}(2) \oplus \mathfrak{k}_1.$$

The even subalgebra is then given by the operator algebra

$$\mathfrak{sp}(2) = \mathfrak{sl}(2) = \mathrm{Alg}_{\mathbb{C}} \left\{ \Delta_u, |u|^2 \right\}.$$

The set of positive roots then considerably reduces, compared to the general case in the previous chapter

$$\mathfrak{k}^+ = \mathfrak{k}_0^+ \cup \mathfrak{k}_1^+ = \mathrm{span}_{\mathbb{C}} \left\{ \Delta_u \right\} \cup \mathrm{span}_{\mathbb{C}} \left\{ \partial_u \right\}.$$

The negative roots are given by

$$\mathfrak{k}^- = \mathfrak{k}_0^- \cup \mathfrak{k}_1^- = \mathrm{span}_{\mathbb{C}} \left\{ |u|^2 \right\} \cup \mathrm{span}_{\mathbb{C}} \left\{ u \right\}.$$

Finally, we choose the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{k}$ as the algebra

$$\mathfrak{h} = \mathrm{Alg}_{\mathbb{C}} \left\{ \mathbb{E}_u + \frac{m}{2} \right\}.$$

Let $\mathcal{U}(\mathfrak{osp}(1,4))$ be the universal enveloping algebra of $\mathfrak{osp}(1,4)$, then we can define

$$J = \mathcal{U}(\mathfrak{osp}(1,4))\mathfrak{k}^+.$$

In this case, this means that J consists of all *words* (or to be more precise, tensor products) in the universal enveloping algebra ending with ∂_u. The normaliser of J may then be defined as

$$\text{Norm } J = \{u \in \mathcal{U}(\mathfrak{osp}(1,4)) : Ju \subseteq J\}.$$

Then J is an ideal of Norm J and the *Mickelsson algebra* $S(\mathfrak{osp}(1,4), \mathfrak{osp}(1,2))$ is defined as the quotient algebra

$$S(\mathfrak{osp}(1,4), \mathfrak{osp}(1,2)) = \text{Norm } J/J.$$

We can also define the following related algebra. Let $R(\mathfrak{h})$ be the field of fractions of the commutative algebra $\mathcal{U}(\mathfrak{h})$ and consider $\mathcal{U}'(\mathfrak{osp}(1,4))$ the extension of $\mathcal{U}(\mathfrak{osp}(1,4))$, defined a follows:

$$\mathcal{U}'(\mathfrak{osp}(1,4)) = \mathcal{U}(\mathfrak{osp}(1,4)) \otimes R(\mathfrak{h}).$$

Elements of $\mathcal{U}(\mathfrak{osp}(1,4))$ will be identified as elements of $\mathcal{U}(\mathfrak{osp}(1,4)) \otimes 1$. Let $J' = \mathcal{U}'(\mathfrak{osp}(1,4))\mathfrak{k}^+$, then we can define the *Mickelsson-Zhelobenko algebra* or *transvector algebra* as the quotient algebra

$$Z(\mathfrak{osp}(1,4), \mathfrak{osp}(1,2)) = \text{Norm } J'/J'.$$

This algebra actually is nothing but the Mickelsson algebra, but now, we are allowed to 'divide' by elements of the field of fractions in the Cartan algebra.

5.2 Extremal projection

The main aim of this chapter is to describe the structure of the transvector algebra $Z(\mathfrak{osp}(1,4), \mathfrak{osp}(1,2))$ in the context of Clifford analysis. We will proceed by constructing all generators of the transvector algebra, and determine the commutation relations between them. In order to do that, we first need an expression for the *extremal projector operator* related to $\mathfrak{osp}(1,2)$. We could, of course, simply take over the expressions from the previous chapter, however we prefer a more intuitive way in this particular case. We will construct the projection operator in two steps, keeping in mind that its purpose is to project onto the kernel space of the set of positive roots, in this case ∂_u. Since $\Delta_u = -\partial_u^2$, we automatically end up in the kernel of the Laplace operator for u as well.

5.2.1 Extremal projector of $\mathfrak{sp}(2, \mathbb{C})$

The first step in determining the extremal projector of $\mathfrak{osp}(1, 2)$, is to construct a projection operator that projects any spinor-valued function in a vector variable u onto the kernel of Δ_u, in other words, to determine an extremal projector for $\mathfrak{sp}(2, \mathbb{C})$, the even subalgebra of $\mathfrak{osp}(1, 2)$. Using the isomorphism $\mathfrak{sp}(2, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C})$, we get that

$$\mathfrak{sl}(2, \mathbb{C}) = \mathrm{Alg}_{\mathbb{C}} \left\{ \frac{|u|^2}{2}, -\frac{\Delta_u}{2} \right\}.$$

The extremal projector for this algebra has been determined in [59], in a more intuitive way then the one from the previous section. For completeness, we will give the proof here as well. We know that the space of all polynomials in a vector variable u can be decomposed as

$$\begin{aligned} \mathcal{P}(\mathbb{R}^m, \mathbb{C}) &= \bigoplus_{j=0}^{\infty} |u|^{2j} \mathcal{H}(\mathbb{R}^m, \mathbb{C}) \\ &= \bigoplus_{j=0}^{\infty} \bigoplus_{k=0}^{\infty} |u|^{2j} \mathcal{H}_k(\mathbb{R}^m, \mathbb{C}) \\ &\cong \bigoplus_{k=0}^{\infty} I_k \otimes \mathcal{H}_k(\mathbb{R}^m, \mathbb{C}). \end{aligned}$$

Here, I_k is the irreducible $\mathfrak{sl}(2, \mathbb{C})$-module with weight $-(k + \frac{m}{2})$. So the space $\mathcal{P}(\mathbb{R}^m, \mathbb{C})$ has a multiplicity free decomposition under the action $\mathfrak{sl}(2, \mathbb{C}) \otimes \mathrm{SO}(m)$. Indeed, it is well known that $(\mathfrak{sl}(2, \mathbb{C}), \mathrm{SO}(m))$ is a particular example of a Howe dual pair. Each harmonic polynomial $H(u) \in \mathcal{H}(\mathbb{R}^m, \mathbb{C})$ can be seen as a highest weight vector for a Verma module for $\mathfrak{sl}(2, \mathbb{C})$, which thus is annihilated by the positive root vector $-\frac{1}{2}\Delta_u$. The corresponding $\mathfrak{sl}(2, \mathbb{C})$-module generated by $H_k(u) \in \mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$ is the infinite dimensional module $\bigoplus_{j=0}^{\infty} |u|^{2j} \mathcal{H}_k(\mathbb{R}^m, \mathbb{C})$. Given a polynomial $P(u) \in \mathcal{P}(\mathbb{R}^m, \mathbb{C})$, we then have the unique decomposition

$$P(u) = \sum_{j=0}^{\infty} |u|^{2j} H^{(j)}(u),$$

where each $H^{(j)}(u) \in \mathcal{H}(\mathbb{R}^m, \mathbb{C})$. The harmonic polynomial $H^{(0)}(u)$ is called the *harmonic part* of $P(u)$. The projector we are looking for is the operator that returns $H^{(0)}(u)$ when acting on $P(u)$. It is also known as the *harmonic projection.* One can find the formula in [80], but we will explicitly determine it here for completeness.

Theorem 5.1. *Consider the Fischer orthogonal direct sum of the polynomial space of one vector variable* $\mathcal{P}(\mathbb{R}^m, \mathbb{C}) = \bigoplus_{j=0}^{\infty} |u|^{2j} \mathcal{H}(\mathbb{R}^m, \mathbb{C})$. *The* *harmonic projection* $p_{\mathfrak{sl}(2,\mathbb{C})} : \mathcal{P}(\mathbb{R}^m, \mathbb{C}) \to \mathcal{H}(\mathbb{R}^m, \mathbb{C})$ *can be expressed by*

$$p_{\mathfrak{sl}(2,\mathbb{C})} = \sum_{j=0}^{\infty} \frac{1}{4^j!} \frac{\Gamma(-\mathbb{E}_u - \frac{m}{2} + 2)}{\Gamma(-\mathbb{E}_u - \frac{m}{2} + j + 2)} |u|^{2j} \Delta_u^j.$$

Proof. We start the proof from the assumption that $p_{\mathfrak{sl}(2,\mathbb{C})}$ is of the form

$$p_{\mathfrak{sl}(2,\mathbb{C})} = \sum_{j=0}^{\infty} K_j(\mathbb{E}_u) |u|^{2j} \Delta_u^j,$$

where $K_j(\mathbb{E}_u)$ are unknown functions in $\mathbb{E}_u$. They do not need to be polynomials. We have the commutation relations

$$[\Delta_u, |u|^{2j}] = 4j(\mathbb{E}_u + \frac{m}{2} - j + 1)|u|^{2j-2} \text{ and } \Delta_u \mathbb{E}_u = (\mathbb{E}_u + 2)\Delta_u.$$

Thus, for each $f \in \mathcal{P}(u)$, we find

$$\begin{aligned} & \Delta_u p_{\mathfrak{sl}(2,\mathbb{C})} f \\ = & \sum_{j=0}^{\infty} K_j(\mathbb{E}_u + 2) \Delta_u |u|^{2j} \Delta_u^j f \\ = & \sum_{j=0}^{\infty} \left(4(j+1) K_{j+1}(\mathbb{E}_u + 2) \left(\mathbb{E}_u + \frac{m}{2} - j \right) + K_j(\mathbb{E}_u + 2) \right) \\ & \times |u|^{2j} \Delta_u^{j+1} f. \end{aligned}$$

This equals zero if each coefficient in this sum equals zero, or

$$4(j+1) K_{j+1}(\mathbb{E}_u + 2) \left(\mathbb{E}_u + \frac{m}{2} - j \right) + K_j(\mathbb{E}_u + 2) = 0.$$

This is the case if

$$K_j(\mathbb{E}_u + 2) = \frac{(-1)^j}{4^j j!} \frac{1}{(\mathbb{E}_u + \frac{m}{2} - j + 1) \cdots (\mathbb{E}_u + \frac{m}{2})},$$

or

$$K_j(\mathbb{E}_u) = \frac{1}{4^j j!} \frac{\Gamma(-\mathbb{E}_u - \frac{m}{2} + 2)}{\Gamma(-\mathbb{E}_u - \frac{m}{2} + j + 2)},$$

where $\Gamma(z)$ denotes the Gamma-function. □

This operator has the properties that

$$\Delta_u p_{\mathfrak{sl}(2,\mathbb{C})} = 0, \tag{5.1}$$

$$p_{\mathfrak{sl}(2,\mathbb{C})} |u|^2 = 0. \tag{5.2}$$

Using (5.1), we also find that

$$p_{\mathfrak{sl}(2,\mathbb{C})}^2 = p_{\mathfrak{sl}(2,\mathbb{C})}.$$

5.2.2 Extremal projector of $\mathfrak{osp}(1,2)$

Now that a projection operator has been established which projects each function in u onto the kernel of Δ_u, the next step is to find an operator which projects even further, namely onto the kernel of ∂_u. This will be the extremal projection operator for $\mathfrak{osp}(1,2)$. Inspired by the expression for the Rarita-Schwinger operator, we claim it to take the form

$$p_{\mathfrak{osp}(1,2)} = \left(1 + \frac{u\partial_u}{2\mathbb{E}_u + m - 2}\right) p_{\mathfrak{sl}(2,\mathbb{C})}.$$

Lemma 5.1. *The operator* $p_{\mathfrak{osp}(1,2)} = \left(1 + \frac{u\partial_u}{2\mathbb{E}_u+m-2}\right) p_{\mathfrak{sl}(2,\mathbb{C})}$ *has the property that*

$$\partial_u p_{\mathfrak{osp}(1,2)} f(u) = 0$$

for any function $f(u) \in \mathcal{P}(\mathbb{R}^m, \mathbb{S})$.

Proof. We have that

$$\begin{aligned}
\partial_u \left(1 + \frac{u\partial_u}{2\mathbb{E}_u + m - 2}\right) &= \partial_u + \frac{(-m - 2\mathbb{E}_u - u\partial_u)\partial_u}{m + 2\mathbb{E}_u - 2} \\
&= \partial_u + \frac{-m - 2\mathbb{E}_u}{m + 2\mathbb{E}_u}\partial_u + \frac{u\Delta_u}{m + 2\mathbb{E}_u - 2} \\
&= \frac{u\Delta_u}{m + 2\mathbb{E}_u - 2}.
\end{aligned}$$

Because of (5.1), we then have that

$$\partial_u p_{\mathfrak{osp}(1,2)} = \frac{u\Delta_u}{m + 2\mathbb{E}_u - 2} p_{\mathfrak{sl}(2,\mathbb{C})} = 0,$$

which finishes the proof. □

If $p_{\mathfrak{osp}(1,2)}$ indeed is the extremal projector for $\mathfrak{osp}(1,2)$, it also needs to have the properties that $p_{\mathfrak{osp}(1,2)} u = 0$, and $p^2_{\mathfrak{osp}(1,2)} = p_{\mathfrak{osp}(1,2)}$. For the sake of completeness, we explicitly prove this in the following lemma.

Lemma 5.2. *We have that* $p_{\mathfrak{osp}(1,2)} u = 0$ *and* $p^2_{\mathfrak{osp}(1,2)} = p_{\mathfrak{osp}(1,2)}$.

Proof. Assume that $P(u) \in \mathcal{P}(\mathbb{R}^m, \mathbb{S})$ is a function in one vector variable u. Then we know that there exists a monogenic Fischer decomposition of the form

$$P(u) = \bigoplus_{k=0}^{\infty} u^k M_k(u),$$

where all $M_k(u)$ are monogenic in u. Because of (5.2), we have that

$$p_{\mathfrak{osp}(1,2)} uP = p_{\mathfrak{osp}(1,2)} uM_0.$$

so that we can restrict ourselves to monogenic polynomials in u. Thus, from now on, we assume P monogenic in u. Then

$$\Delta_u uP = 0.$$

Hence,

$$\begin{aligned} p_{\mathfrak{osp}(1,2)} uP &= \left(1 + \frac{u\partial_u}{2\mathbb{E}_u + m - 2}\right) uP \\ &= uP + \frac{u(-m - 2\mathbb{E}_u - u\partial_u)}{m + 2\mathbb{E}_u} P \\ &= uP - uP = 0. \end{aligned}$$

This proves the first part of the lemma. As for the second part of the lemma, due to the fact that $p_{\mathfrak{osp}(1,2)} u = 0$, it directly follows that $p^2_{\mathfrak{osp}(1,2)} = p_{\mathfrak{osp}(1,2)}$ from the expression for the extremal projector. □

Remark 5.1. A straightforward comparison shows that the expression for the extremal projector of $\mathfrak{osp}(1,2)$ we found in this section equals the expression we found in the previous chapter.

5.3 The transvector algebra

Let us recall that

$$\mathfrak{osp}(1,4) = \mathfrak{osp}(1,2) \oplus \mathfrak{p}.$$

Obviously, $\mathfrak{p}$ then is the subspace

$$\mathfrak{p} = \text{span}_{\mathbb{C}} \left\{ (x, \partial_x, |x|^2, \Delta_x, \mathbb{E}_x, \langle \partial_x, \partial_u \rangle, \langle x, u \rangle, \langle x, \partial_u \rangle, \langle u, \partial_x \rangle \right\}. \tag{5.3}$$

In other words, it contains all the root vectors which have x or ∂_x in their expression. The transvector algebra is generated by all $p_{\mathfrak{osp}(1,2)} e_i$, where $\{e_i\}$ is a weight basis of $\mathfrak{p}$ (5.3). Thus, the generators of the Mickelsson-Zhelobenko algebra are given by

$$\begin{aligned} \{&p_{\mathfrak{osp}(1,2)} x, p_{\mathfrak{osp}(1,2)} \partial_x, p_{\mathfrak{osp}(1,2)} |x|^2, p_{\mathfrak{osp}(1,2)} \Delta_x, \\ &p_{\mathfrak{osp}(1,2)} \langle \partial_x, \partial_u \rangle, p_{\mathfrak{osp}(1,2)} \langle x, u \rangle, p_{\mathfrak{osp}(1,2)} \langle x, \partial_u \rangle, p_{\mathfrak{osp}(1,2)} \langle u, \partial_x \rangle\}. \end{aligned}$$

In order to gather more knowledge about the structure of this algebra, we calculate the (anti-)commutation relations between these operators, according to the Lie superbracket. Remember that the transvector algebra is defined as an algebra 'modulo J', so all words ending on ∂_u can be omitted.

5.4 Some properties of the generators

Let us fix the following notations for the generators:

$$\begin{array}{ll} C_x := p_{\mathfrak{osp}(1,2)} x & A_x := p_{\mathfrak{osp}(1,2)} \partial_x \\ C_{x,x} := p_{\mathfrak{osp}(1,2)} |x|^2 & A_{x,x} := p_{\mathfrak{osp}(1,2)} \Delta_x \\ C_{x,u} := p_{\mathfrak{osp}(1,2)} \langle x, u \rangle & A_{x,u} := p_{\mathfrak{osp}(1,2)} \langle \partial_x, \partial_u \rangle \\ S_u := p_{\mathfrak{osp}(1,2)} \langle u, \partial_x \rangle & S_x := p_{\mathfrak{osp}(1,2)} \langle x, \partial_u \rangle . \end{array}$$

As the C-operators increase the degree in a vector variable, they can be seen as *creation operators*. The A-operators decrease the degree of a vector variable, hence they can be seen as *annihilation operators*. The two remaining operators consist of the extremal projector acting on a *skew* Euler operator. In these notations, this means that A_x is the higher spin Dirac operator, and $A_{x,u}$ is a twistor operator (the only one in this case).

Remark 5.2. Note that in general, it is not necessary to look at he full expression of the extremal projector, since the transvector algebra is an algebra 'modulo ∂_u'. This means that, when executing calculations involving the expressions above, all 'words' ending with ∂_u or Δ_u can simply be omitted.

As an illustration of this kind of calculations, we explicitly compute some of the commutation relations between the generating operators:

$$\begin{aligned} & [A_{x,u}, C_{x,x}] \\ &= p_{\mathfrak{osp}(1,2)} \langle \partial_x, \partial_u \rangle |x|^2 - p_{\mathfrak{osp}(1,2)} |x|^2 \langle \partial_x, \partial_u \rangle \\ &= p_{\mathfrak{osp}(1,2)} (\langle \partial_x, \partial_u \rangle |x|^2 - |x|^2 \langle \partial_x, \partial_u \rangle) \\ &= p_{\mathfrak{osp}(1,2)} (2\langle x, \partial_u \rangle + \cancel{|x|^2 \langle \partial_x, \partial_u \rangle} - \cancel{|x|^2 \langle \partial_x, \partial_u \rangle}) \\ &= 2S_x, \end{aligned} \tag{5.4}$$

$$\begin{aligned} & [A_x, C_{x,x}] \\ &= p_{\mathfrak{osp}(1,2)} (-\partial_x x x - |x|^2 \partial_x) \\ &= p_{\mathfrak{osp}(1,2)} (-(-m - 2\mathbb{E}_x - x\partial_x) x - |x|^2 \partial_x) \\ &= p_{\mathfrak{osp}(1,2)} (\cancel{mx} + 2\mathbb{E}_x x + x(-\cancel{m} - 2\mathbb{E}_x - x\partial_x) - |x|^2 \partial_x) \\ &= p_{\mathfrak{osp}(1,2)} (2\mathbb{E}_x x - 2x\mathbb{E}_x + \cancel{|x|^2 \partial_x} - \cancel{|x|^2 \partial_x}) \\ &= p_{\mathfrak{osp}(1,2)} 2x(\cancel{\mathbb{E}_x} + 1 - \cancel{\mathbb{E}_x}) \\ &= 2C_x, \end{aligned} \tag{5.5}$$

$$\begin{aligned} & [A_{x,x}, C_{x,x}] \\ &= p_{\mathfrak{osp}(1,2)} \left(\Delta_x |x|^2 - |x|^2 \Delta_x \right) \\ &= p_{\mathfrak{osp}(1,2)} \left(2(m + 2\mathbb{E}_x) \right) \\ &= 2m + 4\mathbb{E}_x . \end{aligned} \tag{5.6}$$

In general, however, the commutator of two generators will turn out not to be a multiple of one single other generator. The relations are mostly of second order. An example of such a second order relation is given by

$$[A_x, C_{x,u}] = \frac{-2}{m + 2\mathbb{E}_u - 2} S_u C_x.$$

Finally, another interesting relation is the interaction between the HSD operator and the twistor operator:

$$A_{x,u} A_x = \frac{m + 2\mathbb{E}_u + 2}{m + 2\mathbb{E}_u} A_x A_{x,u}.$$

We see that these operators 'nearly' commute, up to a multiplicative factor containing Euler operators and scalars. For the sake of completeness, we list all commutation relations below, in a number of subsequent tables, however without further calculations.

We notice that the annihilation operators commute or 'nearly' commute:

	A_x	$A_{x,x}$	$A_{x,u}$
A_x	0	0	$\frac{2}{m + 2\mathbb{E}_u} A_x A_{x,u}$
$A_{x,x}$	0	0	0
$A_{x,u}$	$\frac{-2}{m + 2\mathbb{E}_u} A_x A_{x,u}$	0	0

Table 5.1: Commutation relations between annihilation operators

Similar to the case of the annihilation operators, also the creation operators commute or 'nearly' commute:

	C_x	$C_{x,x}$	$C_{x,u}$
C_x	0	0	$\frac{-2}{m + 2\mathbb{E}_u - 2} C_{x,u} C_x$
$C_{x,x}$	0	0	0
$C_{x,u}$	$\frac{2}{m + 2\mathbb{E}_u - 2} C_{x,u} C_x$	0	0

Table 5.2: Commutation relations between creation operators

The relations between annihilation and creation operators are mostly of second order:

	C_x	$C_{x,x}$	$C_{x,u}$
A_x	$-m-2\mathbb{E}_x+\frac{4}{m+2\mathbb{E}_u-2}S_xS_u$ $+\frac{4}{m+2\mathbb{E}_u-2}C_{x,u}A_{x,u}$	$2C_x$	$\frac{-2}{m+2\mathbb{E}_u-2}S_uC_x$
$A_{x,x}$	$2A_x$	$2m+4\mathbb{E}_x$	$2S_u$
$A_{x,u}$	$\frac{-2}{m+2\mathbb{E}_u}A_xS_x$	$2S_x$	$m+\mathbb{E}_u+\mathbb{E}_x-\frac{m+2\mathbb{E}_x}{m+2\mathbb{E}_u}$ $-\frac{1}{m+2\mathbb{E}_u}C_xA_x-\frac{2}{m+2\mathbb{E}_u}S_uS_x$

Table 5.3: Commutation relations between annihilation and creation operators

Furthermore, also in the relations with the skew operator, we see second order expressions pop up:

	S_x	S_u
A_x	$\frac{2}{m+2\mathbb{E}_u}C_xA_{x,u}$	$\frac{-2}{m+2\mathbb{E}_u-2}S_uA_x$
$A_{x,x}$	$2A_{x,u}$	0
$A_{x,u}$	0	$\frac{m+2\mathbb{E}_u-1}{m+2\mathbb{E}_u}A_{x,x}-\frac{2}{m+2\mathbb{E}_u}S_xA_{x,u}$
C_x	$\frac{2}{m+2\mathbb{E}_u}C_xS_x$	$\frac{-2}{m+2\mathbb{E}_u-2}C_{x,u}A_x$
$C_{x,x}$	0	$-2C_{x,u}$
$C_{x,u}$	$\frac{m+2\mathbb{E}_u-1}{m+2\mathbb{E}_u}C_{x,x}-\frac{2}{m+2\mathbb{E}_u}C_{x,u}S_x$	0

Table 5.4: Remaining commutation relations

.

Residues arise... naturally in several branches of analysis.... Their consideration provides simple and easy-to-use methods, which are applicable to a large number of diverse questions, and some new results...

Augustin Louis Cauchy

6

Fundamental solution of $\mathcal{Q}_\lambda$

After having constructed, in Chapter 4, the higher spin Dirac operator $\mathcal{Q}_\lambda$, which is unique up to a multiplicative constant and moreover is an elliptic conformally invariant first-order differential operator

$$\mathcal{Q}_\lambda : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda),$$

we are ready to start the search for the fundamental solution of this operator. It is well-known that the construction of its fundamental solution is an important analytical result for any linear differential operator, also in view of establishing integral representation formulae for its null solutions.

In Section 6.1, we will first determine this solution for the low orders (less vector variables) first, before addressing the general fundamental solution at the end of the section. To this end, we will rely on results from distribution theory. In classical Clifford analysis, the Cauchy integral formula has proved to be a corner stone of the function theory: it can be used to decompose arbitrary null solutions for the Dirac operator into homogeneous components and forms the basis to develop boundary value theory. Cauchy integral formulae naturally rely upon the existence of a fundamental solution of the (higher spin) Dirac operator. In Section 6.2 it is explained how a higher spin version of this formula can be obtained. Also, this will lead to a generalised Stokes' and Cauchy-Pompeiu theorem.

6.1 Fundamental solution

Before turning to the fundamental solution of the operator $\mathcal{Q}_\lambda$, in its most general form, we will first consider a few examples to get grip on the general idea behind its construction and properties.

6.1.1 HSD operators of order less than 3

The fundamental solution $N(x)$ for the Laplace operator Δ_x is given by

$$N(x) = \begin{cases} \dfrac{1}{(2-m)A_m|x|^{m-2}} & m > 2 \\ \\ \dfrac{1}{2\pi}\log|x| & m = 2, \end{cases}$$

where A_m is the surface area of the unit sphere S^{m-1} in $\mathbb{R}^m$, i.e.

$$A_m = \frac{2\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})}.$$

In view of the fact that $\Delta_x = -\partial_x^2$, the fundamental solution $E(x)$ for the Dirac operator is easily obtained as

$$\begin{aligned} E(x) = -\partial_x N(x) &= \left\{ \begin{array}{l} -\dfrac{1}{(2-m)A_m}\dfrac{2-m}{2}\displaystyle\sum_{i=1}^m e_i|x|^{-m}2x_i \\ -\dfrac{1}{2\pi}\displaystyle\sum_{i=1}^m e_i\dfrac{1}{|x|^2}x_i \end{array} \right\} \\ &= -\frac{1}{A_m}\frac{x}{|x|^m}, \end{aligned}$$

This function is also called the *Cauchy kernel* and, as a fundamental solution of the Dirac operator, it satisfies the relation

$$\partial_x E(x) = \delta(x),$$

$\delta(x)$ being the delta-distribution. Denoting $\mathbb{R}^m\backslash\{0\}$ by $\mathbb{R}_0^m$, we can say that $E(x)$ is an element of the function space $\mathcal{C}^\infty(\mathbb{R}_0^m, \mathbb{C}_m)$. Because $\mathbb{C}_m$ can be seen as the space of endomorphisms of the (total) spinor space $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ for the multiplicative action ($\mathbb{C}_m\mathbb{S} \subseteq \mathbb{S}$), we have that $E(x) \in \mathcal{C}^\infty(\mathbb{R}_0^m, \mathrm{End}(\mathbb{S}))$.

For the Rarita-Schwinger operator $\mathcal{R}_{l_1}$, the first higher spin generalisation of the Dirac operator, the fundamental solution has been constructed in [20] as

$$E_{l_1}(x; u_1, u_1') = -\frac{1}{A_m}\frac{m+2l_1-2}{m-2}\frac{x}{|x|^{m+2l_1}}K_{l_1}(xu_1x, u_1').$$

Here, $K_{l_1}(u_1, u_1')$ denotes the so-called *reproducing kernel* for l_1-homogeneous monogenic polynomials, which has the property that

$$(K_{l_1}(u_1, u_1'), P_{l_1}(u_1))_{(u_1)} = P_{l_1}(u_1'),$$

where the notation $(.,.)_{(u_1)}$ refers to the *Fischer inner product* on $\mathcal{P}(\mathbb{R}^m, \mathbb{S})$, given by:

$$(f(u_1), g(u_1))_{(u_1)} = \left[f(\partial_1)^\dagger g(u_1)\right]\Big|_{u_1=0}.$$

The fundamental solution then satisfies

$$\mathcal{R}_{l_1} E_{l_1}(x; u_1, u_1') = \delta(x) K_{l_1}(u_1, u_1').$$

This time, the fundamental solution of the operator $\mathcal{R}_{l_1}$ belongs to the function space $\mathcal{C}^\infty(\mathbb{R}^m_0, \mathrm{End}(\mathcal{S}_{l_1}))$.

In full generality, we can therefore expect the fundamental solution of $\mathcal{Q}_\lambda$ to belong to the space $\mathcal{C}^\infty(\mathbb{R}^m_0, \mathrm{End}(\mathcal{S}_\lambda))$.

6.1.2 HSD operators of general order

The main result of this section is the following:

Proposition 6.1. *Let $C_\lambda \in \mathbb{R}$ be a constant. For every $P_\lambda(u_{(p)}) \in \mathcal{S}_\lambda$, the function*

$$E_\lambda(x; u_{(p)}) := C_\lambda |x|^{-m+1} L\left(\frac{x}{|x|}\right) P_\lambda(u_{(p)})$$

belongs to $\mathcal{C}^\infty(\mathbb{R}^m_0, \mathcal{S}_\lambda)$. Furthermore, $E_\lambda(x; u_{(p)})$ belongs to the kernel of the operator $\mathcal{Q}_\lambda$ and has a singularity of degree $(-m+1)$ in $x = 0$.

Remark 6.1. The L-action of a vector is defined as if this vector would be a spin element:

$$L(x)\, P_\lambda(u_{(p)}) = P_\lambda\left(x u_{(p)} x\right),$$

where $u_{(p)} = (u_1, \ldots, u_p)$ and $x u_{(p)} x = (x u_1 x, \ldots, x u_p x)$.

The first step in proving Proposition 6.1 is showing that $E_\lambda(x; u_{(p)})$ belongs to $\mathcal{C}^\infty(\mathbb{R}^m_0, \mathcal{S}_\lambda)$. To do so, we refer to equation (2.9).

Lemma 6.1. *For all $P_\lambda(u_{(p)})$ in $\mathcal{S}_\lambda$, the polynomial $xP_\lambda(xu_{(p)}x)$ belongs to $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda)$. Moreover, we also have that $P_\lambda(xu_{(p)}x) \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_\lambda \otimes \mathbb{S})$.*

Proof. I view of Lemma 2.4, we have that

$$\partial_i x P_\lambda(x u_{(p)} x) = -x|x|^2 \partial_{x u_i x} P_\lambda(x u_{(p)} x) = 0, \qquad (6.1)$$

since $\partial_i P_\lambda(u_{(p)}) = 0$. From (2.9), we have that

$$x\partial_a x = |x|^2 \partial_{xu_a x} |x|^2.$$

Dividing both sides of the equation by $|x|^4$ yields $\partial_{xu_i x} = \frac{1}{|x|^4} x\partial_i x$, leading to

$$\langle xu_i x, \partial_{xu_j x}\rangle \;=\; \frac{1}{|x|^4}\langle xu_i x, x\partial_j x\rangle \;=\; \langle u_i, \partial_j\rangle.$$

Since moreover $\langle u_i, \partial_j\rangle$ and x commute, we can then derive that

$$\langle u_i, \partial_j\rangle x P_\lambda(xu_{(p)}x) = 0, \tag{6.2}$$

for all $1 \le i < j \le m$. The equations (6.1) and (6.2) exactly are the necessary conditions for $xP_\lambda(xu_{(p)}x)$ to be an element of $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda)$. We have that $P_\lambda(xu_{(p)}x) = -\frac{x}{|x|^2} xP_\lambda(xu_{(p)}x)$. Since

$$\langle u_i, \partial_j\rangle P_\lambda(xu_{(p)}x) = -\frac{x}{|x|^2}\langle u_i, \partial_j\rangle x P_\lambda(xu_{(p)}x) = 0$$

and

$$\Delta_i P_\lambda(xu_{(p)}x) = -\frac{x}{|x|^2}\Delta_i x P_\lambda(xu_{(p)}x) = 0,$$

one also has that $P_\lambda(xu_{(p)}x) \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_\lambda \otimes \mathbb{S})$ as a result. □

In [86], it was shown that the irreducible finite-dimensional representation $\mathcal{S}_\lambda$, with highest weight λ, is generated by the highest weight vector

$$\langle u_1, \mathfrak{f}_1\rangle^{l_1-l_2}\langle u_1 \wedge u_2, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle^{l_2-l_3}\cdots\langle u_1 \wedge\cdots\wedge u_p, \mathfrak{f}_1\wedge\cdots\wedge\mathfrak{f}_p\rangle^{l_p} I,$$

where each of these inner products is defined by

$$\begin{aligned}
&\langle u_1 \wedge\cdots\wedge u_k, \mathfrak{f}_1\wedge\cdots\wedge\mathfrak{f}_k\rangle \\
&= \det\begin{pmatrix}\langle u_1, \mathfrak{f}_1\rangle & \cdots & \langle u_1, \mathfrak{f}_k\rangle \\ \vdots & \ddots & \vdots \\ \langle u_k, \mathfrak{f}_1\rangle & \cdots & \langle u_k, \mathfrak{f}_k\rangle\end{pmatrix} \\
&= \sum_{\sigma\in S_k} \operatorname{sgn}(\sigma)\langle u_{\sigma(1)}, \mathfrak{f}_1\rangle\cdots\langle u_{\sigma(k)}, \mathfrak{f}_k\rangle,
\end{aligned}$$

S_k being the symmetric group in k elements. Without loss of generality, we can now choose

$$P_\lambda(u_{(p)}) = \langle u_1, \mathfrak{f}_1\rangle^{l_1-l_2}\langle u_1 \wedge u_2, \mathfrak{f}_1 \wedge \mathfrak{f}_2\rangle^{l_2-l_3}\cdots\langle u_1 \wedge\cdots\wedge u_p, \mathfrak{f}_1\wedge\cdots\wedge\mathfrak{f}_p\rangle^{l_p} I,$$

since all operators in $\mathrm{Alg}_{\mathbb{C}}\{x, \partial_x, u_1, \ldots, u_p, \partial_1, \ldots, \partial_p\}$ are $\mathrm{Spin}(m)$-invariant, and

$$\mathcal{S}_\lambda = \mathrm{Span}_{\mathbb{C}}\{L(s)P_\lambda(u_{(p)}) : s \in \mathrm{Spin}(m)\}.$$

Defining $|\lambda| = l_1 + \cdots + l_p$, this choice for P_λ then leads to

$$\begin{aligned}
&|x|^{-m+1} L\left(\frac{x}{|x|}\right) P_\lambda \\
&= \frac{x\langle xu_1x, \mathfrak{f}_1\rangle^{l_1-l_2} \cdots \langle xu_1x \wedge \cdots \wedge xu_px, \mathfrak{f}_1 \wedge \cdots \wedge \mathfrak{f}_p\rangle^{l_p}}{|x|^{m+2|\lambda|}} I. \qquad (6.3)
\end{aligned}$$

We will now prove the second part of Proposition 6.1, namely that the expression (6.3) indeed belongs the kernel of $\mathcal{Q}_\lambda$ for $|x| \neq 0$. Recalling $p_{\mathfrak{osp}(1,2k)}$ to be the projection operator

$$p_{\mathfrak{osp}(1,2k)} : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_\lambda \otimes \mathbb{S}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda)$$

and the definition of the HSD operator on $\mathcal{S}_\lambda$-valued functions to be $\mathcal{Q}_\lambda = p_{\mathfrak{osp}(1,2k)}\partial_x$, we arrive at

$$\begin{aligned}
&\mathcal{Q}_\lambda(|x|^{-m-2|\lambda|} x P_\lambda(xu_{(p)}x)) \\
&= (m + 2|\lambda|)|x|^{-m-2|\lambda|} p_{\mathfrak{osp}(1,2k)} P_\lambda(xu_{(p)}x) \\
&\quad + |x|^{-m-2|\lambda|} p_{\mathfrak{osp}(1,2k)} \partial_x(xP_\lambda(xu_{(p)}x)) \\
&= |x|^{-m-2|\lambda|} p_{\mathfrak{osp}(1,2k)} \Gamma_x P_\lambda(xu_{(p)}x),
\end{aligned}$$

where we also have invoked the operator identity $\partial_x x = \Gamma_x - \mathbb{E}_x - m$ for the Dirac operator (see Chapter 2). Here, Γ_x denotes the Gamma-operator (the tangential part of the Dirac operator, see e.g. [30]). So we are left with proving the statement

$$p_{\mathfrak{osp}(1,2k)} \Gamma_x P_\lambda(xu_{(p)}x) = 0.$$

Let us recall that, in full generality, we have the following decomposition for spinor-valued polynomials in the variables $(u_{(p)}) \in \mathbb{R}^{pm}$:

$$\mathcal{P}(\mathbb{R}^{pm}, \mathbb{S}) = \mathcal{M}_\lambda \oplus (u_1\mathcal{P}(\mathbb{R}^m, \mathbb{S}) + \cdots + u_p\mathcal{P}(\mathbb{R}^m, \mathbb{S})).$$

The summations between brackets obviously are not direct, but we will only use the fact that the operator $p_{\mathfrak{osp}(1,2k)}$ is the projection operator onto $\mathcal{S}_\lambda \subset \mathcal{M}_\lambda$. So it suffices to work modulo the vector spaces $u_j\mathcal{P}(\mathbb{R}^m, \mathbb{S})$. This means for example that

$$\Gamma_x\langle x, u_j\rangle \bmod u_j\mathcal{P} = u_j \wedge x \bmod u_j\mathcal{P} = \langle x, u_j\rangle \bmod u_j\mathcal{P}.$$

This, and the fact that $\mathfrak{f}_i^2 = 0$, allows us to prove the following:

$$\begin{aligned}
&\left(\Gamma_x\langle x, u_j\rangle\langle x, \mathfrak{f}_i\rangle \mathfrak{f}_i\mathfrak{f}_i^\dagger\right) \bmod u_j\mathcal{P} \\
&= \left((\langle x, u_j\rangle\langle x, \mathfrak{f}_i\rangle + \mathfrak{f}_i \wedge x\langle x, u_j\rangle)\mathfrak{f}_i\mathfrak{f}_i^\dagger\right) \bmod u_j\mathcal{P} \\
&= \langle x, u_j\rangle \left(\langle x, \mathfrak{f}_i\rangle + \frac{1}{2}\mathfrak{f}_i x\right) \mathfrak{f}_i\mathfrak{f}_i^\dagger \bmod u_j\mathcal{P} \\
&= -\frac{1}{2}\langle x, u_j\rangle x\mathfrak{f}_i\mathfrak{f}_i\mathfrak{f}_i^\dagger \bmod u_j\mathcal{P} = 0.
\end{aligned}$$

In view of the fact that $P_\lambda(xu_{(p)}x)$ consists of factors of the form

$$\sum_{\sigma\in S_k}\text{sgn}(\sigma)\langle xu_{\sigma(1)}x,\mathfrak{f}_1\rangle\cdots\langle xu_{\sigma(k)}x,\mathfrak{f}_k\rangle$$
$$=\sum_{\sigma\in S_k}\text{sgn}(\sigma)\prod_{i=1}^{k}\left(|x|^2\langle u_{\sigma(i)},\mathfrak{f}_i\rangle-2\langle x,\mathfrak{f}_i\rangle\langle x,u_{\sigma(i)}\rangle\right).$$

it is clear that the first terms between brackets also will not contribute, since they only depend on the norm of x (on which Γ_x acts trivially). We are now ready to explain why we indeed have that

$$\Gamma_x P_\lambda(xu_{(p)}x) \bmod u_j\mathcal{P}=0.$$

First of all, as Γ_x is a first-order differential operator, it suffices to verify that

$$\Gamma_x\langle xu_1x\wedge\cdots\wedge xu_kx,\mathfrak{f}_1\wedge\cdots\wedge\mathfrak{f}_k\rangle^a I \bmod u_j\mathcal{P}=0,$$

for all $1\leq k\leq p$ and $a\in\mathbb{N}$. In view of the chain rule, it suffices to prove this for $a=1$, which amounts to showing that

$$\Gamma_x\sum_{\sigma\in S_k}\text{sgn}(\sigma)\prod_{i=1}^{k}\left(|x|^2\langle u_{\sigma(i)},\mathfrak{f}_i\rangle-2\langle x,\mathfrak{f}_i\rangle\langle x,u_{\sigma(i)}\rangle\right) \bmod u_j\mathcal{P}=0.$$

But as was explained above, none of these factors will survive, which proves Proposition 6.1. Note that, until now, we have excluded the pointwise singularity of $E_\lambda(x;u_{(p)})$ at $x=0$. In order to investigate this singularity, we use results from distribution theory.

6.1.3 Riesz potentials

Consider the function $x\mapsto|x|^{\alpha-2|\lambda|}xP_\lambda(xu_{(p)}x)$, for a fixed $\alpha\in\mathbb{C}$, where $|\lambda|=l_1+\cdots+l_k$. This obviously is an element of the function space $\mathcal{C}^\infty(\mathbb{R}^m_0,\mathcal{S}_\lambda)$, whence we can consider the action of the HSD operator on it. Using similar calculations as above, we get

$$\mathcal{Q}_\lambda(|x|^{\alpha-2|\lambda|}xP_\lambda(xu_{(p)}x))=-(\alpha+m)|x|^{\alpha-2|\lambda|}xP_\lambda(xu_{(p)}x). \tag{6.4}$$

For $\alpha=-m$, we thus have that $|x|^{\alpha-2|\lambda|}xP_\lambda(xu_{(p)}x)$ belongs to the kernel of the HSD operator $\mathcal{Q}_\lambda$. Furthermore, it clearly has a pointwise singularity of degree $(-m+1)$ in the origin $x=0$. The function defined by

$$x\mapsto|x|^{\alpha-2|\lambda|}xP_\lambda(xu_{(p)}x)$$

is an element of the space of locally integrable functions $L_1^{loc}(\mathbb{R}^m,\mathcal{S}_\lambda)$ if $\Re(\alpha)>-m-1$, so it defines a distribution on the space $\mathcal{D}(\mathbb{R}^m,\mathcal{S}_\lambda)$ of test functions ϕ in $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda)$ with compact support. Consider, for $\Re(\gamma) > -m$, the distribution $|x|^\gamma$, whose action is defined by

$$\langle |x|^\gamma, \phi \rangle = \int_{\mathbb{R}^m} |x|^\gamma \phi(x) dx,$$

for all test functions $\phi(x) \in \mathcal{D}(\mathbb{R}^m)$. We will use the following result, see e.g. [52]:

Lemma 6.2. *The mapping $\gamma \mapsto |x|^\gamma$ can be extended uniquely to a meromorphic mapping from the complex numbers to the space of tempered distributions on $\mathbb{R}^m$ (i.e. holomorphic on $\mathbb{C}$, except for a few isolated points). The poles are the points $\gamma = -m - 2a$ (for all $a \in \mathbb{N}$), and they are all simple.*

Define for $\gamma \in \mathbb{C}\backslash\{m + 2a, -2b : a, b \in \mathbb{N}\}$ the action of the Riesz potential I_x^γ on a test function ϕ as follows:

$$I_x^\gamma \phi := \frac{\Gamma\left(\frac{m-\gamma}{2}\right)}{2^\gamma \pi^{\frac{m}{2}} \Gamma\left(\frac{\gamma}{2}\right)} \phi * |x|^{-m+\gamma},$$

where $*$ is the convolution product on $\mathbb{R}^m$ and $I_x^0 \phi = \lim_{\gamma \to 0} I_x^\gamma \phi = \phi$. Note that he poles of $|x|^{\gamma - m}$ are cancelled by the poles of $\Gamma\left(\frac{\gamma}{2}\right)$. The Riesz potential for $\gamma = 2$ can be seen as an 'inverse' of the Laplace operator Δ_x, because it satisfies the following relation in distributional sense:

$$I_x^\gamma \Delta_x \phi = \Delta_x I_x^\gamma \phi = -I_x^{\gamma-2} \phi.$$

For all $b \in \mathbb{N}$, we have that $I_x^\gamma = (-1)^b \Delta_x^b I_x^{\gamma+2b}$, so if we define

$$I_x^{-2a} := (-1)^a \Delta_x^a \delta(x),$$

where $\delta(x)$ is the Dirac-delta distribution, then this is an analytic continuation of the mapping $\gamma \mapsto I_x^\gamma$ to a holomorphic function with poles in $\{\gamma = m + 2a | a \in \mathbb{N}\}$. These are the poles of $\Gamma\left(\frac{m-\gamma}{2}\right)$. If we reformulate our findings in terms of the distribution $|x|^{-m+\gamma}$, then we can analytically extend the mapping $\gamma \mapsto |x|^{-m+\gamma}$ to $\mathbb{C}\backslash\{-2a : a \in \mathbb{N}\}$, according to Lemma 6.2. Its singularities are simple poles, with residues

$$\begin{aligned}
\text{Res}[|x|^{-m-\gamma}, \gamma = -2a] &= \text{Res}\left[\frac{2^\gamma \pi^{\frac{m}{2}} \Gamma\left(\frac{\gamma}{2}\right)}{\Gamma\left(\frac{m-\gamma}{2}\right)} I_x^\gamma, \gamma = -2a\right] \\
&= \frac{2^{-2a} \pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2} + a\right)} \text{Res}\left[\Gamma\left(\frac{\gamma}{2}\right), \gamma = -2a\right] I_x^{-2a}.
\end{aligned}$$

In view of the fact that

$$\text{Res}\left[\Gamma\left(\frac{\gamma}{2}\right), \gamma = -2a\right] = \lim_{\gamma \to -2a} (\gamma + 2a) \Gamma\left(\frac{\gamma}{2}\right) = 2\frac{(-1)^a}{a!},$$

it then follows that

$$\text{Res}[|x|^{-m-\gamma}, \gamma = -2a] = \frac{2^{-2a+1}\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}+a\right)a!}\Delta_x^a \delta(x).$$

Thus, the mapping $\alpha \mapsto |x|^{\alpha-2|\lambda|} x P_\lambda(xu_{(p)}x)$ is holomorphic in $\mathbb{C}\backslash\{-m + 2|\lambda| - 2a, a \in \mathbb{N}\}$. Moreover, the singularities at $\{-m+2(|\lambda|-1), \ldots, -m+2, -m\}$ are removable. For instance, for the pole $\alpha = -m$, we have that

$$\text{Res}[|x|^{\alpha-2|\lambda|} x P_\lambda(xu_{(p)}x), \alpha = -m] = \lim_{\alpha \to -m} (\alpha + m)|x|^{\alpha-2|\lambda|} x P_\lambda(xu_{(p)}x).$$

Putting $x = r\omega$ with $r = |x|$, this can then be rewritten as

$$\lim_{\alpha \to -m} (\alpha + m) r^{\alpha+1} \omega P_\lambda(\omega u_{(p)}\omega) = 0.$$

Similar calculations can be done for the other singularities. So we have proven the following proposition:

Proposition 6.2. *The mapping $\alpha \mapsto |x|^{\alpha-2|\lambda|} x P_\lambda(xu_{(p)}x)$ can be continued holomorphically in $\mathbb{C}\backslash\{-m - 2a, a \in \mathbb{N}\}$.*

This means that (6.4) holds in distributional sense in $\mathbb{C}$, as long as $\Re(\alpha) > -m - 1$. Hence, with this restriction on α,

$$\begin{aligned}
&\mathcal{Q}_\lambda \left(|x|^{-m-2|\lambda|} x P_\lambda(xu_{(p)}x)\right) \\
&= -\lim_{\alpha \to -m} (\alpha + m)|x|^{\alpha-2|\lambda|} p_{\mathfrak{osp}(1,2k)} P_\lambda(xu_{(p)}x) \\
&= -\text{Res}\left[|x|^{\alpha-2|\lambda|}, \alpha = -m\right] p_{\mathfrak{osp}(1,2k)} P_\lambda(xu_{(p)}x) \\
&= \frac{2^{-2|\lambda|+1}\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}+|\lambda|\right)|\lambda|!} (\Delta_x^{|\lambda|}\delta(x)) p_{\mathfrak{osp}(1,2k)} P_\lambda(xu_{(p)}x). \qquad (6.5)
\end{aligned}$$

Moreover, in view of the fact that $\langle \delta, \varphi \rangle = \varphi(0)$, we get:

$$\begin{aligned}
&\langle (\Delta_x^{|\lambda|}\delta)(p_{\mathfrak{osp}(1,2k)} P_\lambda(xu_{(p)}x)), \phi \rangle \\
&= \langle \Delta_x^{|\lambda|}\delta, (p_{\mathfrak{osp}(1,2k)} P_\lambda(xu_{(p)}x))\phi \rangle \\
&= \langle \delta, \Delta_x^{|\lambda|}((p_{\mathfrak{osp}(1,2k)} P_\lambda(xu_{(p)}x))\phi) \rangle \\
&= \langle \delta, \Delta_x^{|\lambda|}(p_{\mathfrak{osp}(1,2k)} P_\lambda(xu_{(p)}x))\phi + \cdots \rangle,
\end{aligned}$$

where the dots indicate all other terms coming from the action of $\Delta_x^{|\lambda|}$. They can safely be ignored, in view of the fact that we still need to act with the distribution $\delta(x)$, which will make all these terms disappear. We thus get that

$$\begin{aligned}
\langle (\Delta_x^{|\lambda|}\delta)(p_{\mathfrak{osp}(1,2k)} P_\lambda(xu_{(p)}x)), \phi \rangle &= \langle \delta, \Delta_x^{|\lambda|}(p_{\mathfrak{osp}(1,2k)} P_\lambda)\phi \rangle \\
&= \langle \Delta_x^{|\lambda|}(p_{\mathfrak{osp}(1,2k)} P_\lambda)\delta, \phi \rangle.
\end{aligned}$$

This means that formula (6.5) reduces to

$$\begin{aligned}&\mathcal{Q}_\lambda\left(|x|^{-m-2|\lambda|}xP_\lambda(xu_{(p)}x)\right)\\ =\ &\frac{2^{-2|\lambda|+1}\pi^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}+|\lambda|\right)|\lambda|!}\Delta_x^{|\lambda|}(p_{\mathfrak{osp}(1,2k)}P_\lambda(xu_{(p)}x))\delta(x).\end{aligned}$$

In order to calculate the remaining expression $\Delta_x^{|\lambda|}(p_{\mathfrak{osp}(1,2k)}P_\lambda(xu_{(p)}x))$, we first note that Δ_x and $p_{\mathfrak{osp}(1,2k)}$ commute. Next, we introduce the action $H(x)$ by

$$H(x):\mathcal{S}_\lambda\to\mathcal{P}_{2|\lambda|}(x)\otimes\mathcal{H}_\lambda(u_{(p)})\otimes\mathbb{S}:P_\lambda(u_{(p)})\mapsto P_\lambda(xu_{(p)}x).$$

The map $\Delta_x^{|\lambda|}H(x)$ is Spin(m) invariant:

$$\begin{aligned}\Delta_x^{|\lambda|}H(x)L(s)P_\lambda(u_{(p)}) &= \Delta_x^{|\lambda|}sP_\lambda(s^*xss^*u_{(p)}ss^*xs)\\ &= s\Delta_x^{|\lambda|}P_\lambda(s^*xss^*u_{(p)}ss^*xs)\\ &= L(s)\Delta_x^{|\lambda|}H(x)P_\lambda(u_{(p)}).\end{aligned}$$

The image of the Spin(m)-invariant map $\Delta_x^{|\lambda|}H(x)$ equals $\mathcal{S}_\lambda$. According to Schur's lemma, there must therefore exist a constant C_λ such that

$$\Delta_x^{|\lambda|}H(x)P_\lambda(u_{(p)})=C_\lambda P_\lambda(u_{(p)}). \tag{6.6}$$

Let us determine the constant C_λ explicitly. We do this by complexifying the variables u_j and choosing a specific value for them: $u_j:=e_j+ie_{n+j}$, for all $j=1,\ldots,n$. Then our calculations simplify a great deal, since $\langle u_i,\mathfrak{f}_j\rangle=\delta_{ij}$ and thus $P_\lambda(u_{(p)})=I^\pm$. Furthermore,

$$\begin{aligned}&\langle xu_1x\wedge\cdots\wedge xu_kx,\mathfrak{f}_1\wedge\cdots\wedge\mathfrak{f}_k\rangle\\ =\ &|x|^{2k}\langle u_1\wedge\cdots\wedge u_k,\mathfrak{f}_1\wedge\cdots\wedge\mathfrak{f}_k\rangle\\ &-2\sum_{j=1}^k|x|^{2k-2}\langle u_j,x\rangle\langle(u_1\wedge\cdots\wedge u_k)_j,\mathfrak{f}_1\wedge\cdots\wedge\mathfrak{f}_k\rangle\\ =\ &|x|^{2k}-2\sum_{j=1}^k|x|^{2k-2}\langle u_j,x\rangle\langle x,\mathfrak{f}_j\rangle\\ =\ &|x|^{2k-2}(x_{k+1}^2+\cdots+x_n^2+x_{n+k+1}^2+\cdots+x_m^2).\end{aligned}$$

Putting $x^{(j)}=(0,\ldots,0,x_j,\ldots,x_n,0,\ldots,0,x_{n+j},\ldots,x_m)$, we then get that

$$P_\lambda(xu_{(p)}x)=|x|^{2|\lambda|-l_1}|x^{(2)}|^{2(l_1-l_2)}\cdots|x^{(p)}|^{2(l_{p-1}-l_p)}|x^{(p+1)}|^{2l_p}I^\pm.$$

Together with the relation

$$\Delta_x^a|x|^{2b}=\sum_{j=0}^{\min(a,b)}\binom{a}{j}2^{2j}j!\binom{b}{j}\frac{\Gamma\left(\frac{m}{2}+\mathbb{E}_x-b+a+j\right)}{\Gamma\left(\frac{m}{2}+\mathbb{E}_x-b+a\right)}|x|^{2(b-j)}\Delta_x^{a-j},$$

which follows from the fact that $[\Delta_x, |x|^2] = 2m + 4\mathbb{E}_x$, we find that

$$\begin{aligned}&\Delta_x^{|\lambda|} P_\lambda(xu_{(p)}x)\\ = \;& 2^{2|\lambda|}|\lambda|! \frac{\Gamma\left(\frac{m}{2}+|\lambda|\right)}{\Gamma\left(\frac{m}{2}+l_1\right)} \frac{\Gamma\left(\frac{m}{2}+l_1-1\right)}{\Gamma\left(\frac{m}{2}+l_2-1\right)} \cdots\\ &\times \frac{\Gamma\left(\frac{m}{2}+l_{k-1}-k+1\right)}{\Gamma\left(\frac{m}{2}+l_k-k+1\right)} \frac{\Gamma\left(\frac{m}{2}+l_k-k\right)}{\Gamma\left(\frac{m}{2}-k\right)} P_\lambda(xu_{(p)}x).\end{aligned}$$

This then leads to the following conclusion:

$$\begin{aligned}&\mathcal{Q}_\lambda\left(|x|^{-m-2|\lambda|}xP_\lambda(xu_{(p)}x)\right)\\ = \;& \frac{\pi^{\frac{m}{2}}2^{-|\lambda|+1}}{\Gamma\left(\frac{m}{2}+|\lambda|\right)|\lambda|!} \frac{2^{|\lambda|}|\lambda|!\Gamma\left(\frac{m}{2}+|\lambda|\right)}{\left(\frac{m}{2}+l_1-1\right)\cdots\left(\frac{m}{2}+l_k-k\right)\Gamma\left(\frac{m}{2}-k\right)} P_\lambda(u_{(p)})\delta(x)\\ = \;& -A_m \prod_{j=1}^{p} \frac{m-2j}{m+2l_j-2j} P_\lambda(u_{(p)})\delta(x).\end{aligned}$$

To conclude our findings we formulate the following theorem.

Theorem 6.1. *Defining the constant C_λ by*

$$C_\lambda = -\frac{1}{A_m}\prod_{j=1}^{k}\frac{m+2l_j-2j}{m-2j},$$

the distribution

$$e_\lambda(x) := C_\lambda |x|^{-m+1} L\left(\frac{x}{|x|}\right) \in \mathcal{C}^\infty(\mathbb{R}^m_0, \mathrm{End}(\mathcal{S}_\lambda))$$

satisfies, for every $P_\lambda \in \mathcal{S}_\lambda$, in distributional sense

$$\mathcal{Q}_\lambda e_\lambda(x) P_\lambda = \delta(x) P_\lambda.$$

Remark 6.2. This constant also was found in the case of the Rarita-Schwinger operator (see e.g. [83]).

Let us then introduce the notation $(.,.)_{(u_{(p)})}$ for the Fischer inner product on $\mathcal{P}_\lambda$, which is defined as follows:

$$\begin{aligned}&(f(u_1,\ldots,u_p), g(u_1,\ldots,u_p))_{(u_{(p)})}\\ = \;& \left[f(\partial_1,\ldots,\partial_p)^\dagger g(u_1,\ldots,u_p)\right]\Big|_{u_1=\cdots=u_p=0}.\end{aligned}$$

In order to obtain a fundamental solution of $\mathcal{Q}_\lambda$, we then let the distribution $e_\lambda(x)$ act on the reproducing kernel $K_\lambda(u_{(p)}, u'_{(p)})$ for $\mathcal{S}_\lambda$, satisfying the defining relation

$$(K_\lambda(u_{(p)}, u'_{(p)}), P_\lambda(u_{(p)}))_{(u_{(p)})} = P_\lambda(u'_{(p)}),$$

for each $P_\lambda(u_{(p)}) \in \mathcal{S}_\lambda$.

Definition 6.1. *The fundamental solution of the operator $\mathcal{Q}_\lambda$ is defined as*

$$E_\lambda(x; u_{(p)}, u'_{(p)}) := e_\lambda(x) K_\lambda(u_{(p)}, u'_{(p)}).$$

This finishes the proof of the fundamental solution of the higher spin Dirac operators. With this fundamental solution, one can take a look at applications of this property, which will be done in the next section.

6.2 Basic integral formula

Now that we have constructed the fundamental solution, we can prove the main integral formulae in higher spin Clifford analysis. Define the volume element $dx = dx_1 \wedge \cdots \wedge dx_m$ and the surface element $d\sigma_x = \sum_{j=1}^{m} (-1)^{j-1} e_j d\hat{x}_j$, where $d\hat{x}_j = dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge dx_m$.

Theorem 6.2. *Let $\Omega' \subset \mathbb{R}^m$ and $\overline{\Omega} \subset \Omega'$. Then for $f(x)$ and $g(x) \in \mathcal{C}^\infty(\Omega', \mathcal{S}_\lambda)$, where we will not mention the variables $u_{(p)}$ to avoid overloaded notations, we have the following formulae (for arbitrary $y \in \Omega'$)*

(i) (Stokes' theorem)

$$\int_\Omega \left[-(\mathcal{Q}_\lambda g(x), f(x))_{(u_{(p)})} + (g(x), \mathcal{Q}_\lambda f(x))_{(u_{(p)})} \right] dx = \int_{\partial\Omega} (g(x), p_{\mathfrak{osp}(1,2k)}(d\sigma_x) f(x))_{(u_{(p)})}.$$

(ii) (Cauchy-Pompeiu)

$$-\int_{\partial\Omega} (E_\lambda(y-x), p_{\mathfrak{osp}(1,2k)}(d\sigma_x) f(x))_{(u_{(p)})} + \int_\Omega (E_\lambda(y-x), \mathcal{Q}_\lambda f(x))_{(u_{(p)})} dx = \begin{cases} f(y) & y \in \Omega \\ 0 & y \notin \overline{\Omega}. \end{cases}$$

(iii) (Cauchy integral formula) If $\mathcal{Q}_\lambda f = 0$ in Ω', one has

$$-\int_{\partial\Omega} (E_\lambda(y-x), p_{\mathfrak{osp}(1,2k)}(d\sigma_x) f(x))_{(u_{(p)})} = \begin{cases} f(y) & y \in \Omega \\ 0 & y \notin \overline{\Omega}, \end{cases}$$

where $p_{\mathfrak{osp}(1,2k)}(d\sigma_x) f(x)$ is an $\mathcal{S}_\lambda$-valued $(m-1)$-form.

Proof. Let $f(x), g(x) \in \mathcal{C}^\infty(\Omega', \mathcal{S}_\lambda)$. The classical Stokes' formula for the Dirac operator (e.g. [30]) leads to

$$\int_\Omega \left[-(\partial_x g(x))^\dagger f(x) + g(x)^\dagger (\partial_x f(x)) \right] dx = \int_{\partial\Omega} g(x)^\dagger d\sigma_x f(x).$$

This identity still depends on the vector variables $u_{(p)} \in \mathbb{R}^{pm}$. To obtain the generalised Stokes' theorem for the operator $\mathcal{Q}_\lambda$, it is sufficient to take the Fischer inner product with respect to $u_{(p)}$, since we have that

$$\begin{aligned}
(\mathcal{Q}_\lambda g(x), f(x))_{(u_{(p)})} &= (p_{\mathfrak{osp}(1,2k)} \partial_x g(x), f(x))_{(u_{(p)})} \\
&= (\partial_x g(x), f(x))_{(u_{(p)})}, \\
(g(x), \mathcal{Q}_\lambda f(x))_{(u_{(p)})} &= (p_{\mathfrak{osp}(1,2k)} \partial_x f(x), g(x))^\dagger_{(u_{(p)})} \\
&= (\partial_x f(x), g(x))^\dagger_{(u_{(p)})} \\
&= (g(x), \partial_x f(x))_{(u_{(p)})},
\end{aligned}$$

since $\Delta_i g(x) = \langle \partial_i, \partial_x \rangle g(x) = 0$ and

$$(g(x), p_{\mathfrak{osp}(1,2k)}(d\sigma_x) f(x))_{(u_{(p)})} = (g(x), (d\sigma_x) f(x))_{(u_{(p)})}.$$

The Cauchy-Pompeiu formula for the operator $\mathcal{Q}_\lambda$ then is obtained from Stokes' formula, by substituting $g(x; u_{(p)}) = E_\lambda(y - x; u_{(p)}, u'_{(p)})$. We then get

$$\begin{aligned}
&\int_\Omega \Big[-(\delta(y-x) K(u_{(p)}, u'_{(p)}), f(x; u_{(p)})) \\
&\qquad + (E_\lambda(y-x; u_{(p)}, u'_{(p)}), \mathcal{Q}_\lambda f(x; u_{(p)}))_{(u_{(p)})} \Big] dx \\
&\qquad = \int_{\partial\Omega} (E_\lambda(y-x; u_{(p)}, u'_{(p)}), p_{\mathfrak{osp}(1,2k)}(d\sigma_x) f(x; u_{(p)}))_{(u_{(p)})} \\
\Leftrightarrow \quad & \int_\Omega \Big[-\delta(y-x) f(x; u'_{(p)}) + (E_\lambda(y-x; u_{(p)}, u'_{(p)}), \mathcal{Q}_\lambda f(x; u_{(p)}))_{(u_{(p)})} \Big] dx \\
&\qquad = \int_{\partial\Omega} (E_\lambda(y-x; u_{(p)}, u'_{(p)}), p_{\mathfrak{osp}(1,2k)}(d\sigma_x) f(x; u_{(p)}))_{(u_{(p)})}.
\end{aligned}$$

In order to further simplify these integrals, we invoke the definition of the fundamental solution and proceed as follows:

$$\begin{aligned}
&-f(y, u'_{(p)}) + \int_\Omega (e_\lambda(y-x) K_\lambda(u_{(p)}, u'_{(p)}), \mathcal{Q}_\lambda f(x; u_{(p)}))_{(u_{(p)})} dx \\
&= \int_{\partial\Omega} (e_\lambda(y-x) K_\lambda(u_{(p)}, u'_{(p)}), p_{\mathfrak{osp}(1,2k)}(d\sigma_x) f(x; u_{(p)}))_{(u_{(p)})}.
\end{aligned}$$

Using the fact that

$$\left(L\left(\frac{x}{|x|}\right) P(u_{(p)}), R(u_{(p)}) \right)_{(u_{(p)})} = - \left(P(u_{(p)}), L\left(\frac{x}{|x|}\right) R(u_{(p)}) \right)_{(u_{(p)})},$$

for any $P(u_{(p)}), R(u_{(p)}) \in \mathcal{S}_\lambda^\pm$, we can now rewrite these expressions as

$$\begin{aligned}
\Leftrightarrow\quad & -f(y, u'_{(p)}) - \int_\Omega e_\lambda(y-x)(K_\lambda(u_{(p)}, u'_{(p)}), \mathcal{Q}_\lambda f(x; u_{(p)}))_{(u_{(p)})} dx \\
& = -\int_{\partial\Omega} e_\lambda(y-x)(K_\lambda(u_{(p)}, u'_{(p)}), p_{\mathfrak{osp}(1,2k)}(d\sigma_x) f(x))_{(u_{(p)})} \\
\Leftrightarrow\quad & f(y, u'_{(p)}) + \int_\Omega e_\lambda(y-x)\mathcal{Q}_\lambda f(x; u'_{(p)}) dx \\
& = +\int_{\partial\Omega} e_\lambda(y-x) p_{\mathfrak{osp}(1,2k)}(d\sigma_x) f(x; u'_{(p)}).
\end{aligned}$$

The fact that $f \in \ker \mathcal{Q}_\lambda$ immediately gives us the Cauchy integral formula. □

6.3 Conclusion

Using the theory of Riesz distributions and techniques coming from representation theory, the fundamental solution $E_\lambda(x; u_{(p)}, u'_{(p)})$ for the higher spin Dirac operator $\mathcal{Q}_\lambda$ was determined in this chapter. This fundamental solution allowed us to prove a generalised version of the classical integral formulae (Stokes' theorem, the Cauchy-Pompeiu theorem and the Cauchy integral formula), which are the basic steps towards developing a function theory for the operator $\mathcal{Q}_\lambda$.

Many who have had an opportunity of knowing any more about mathematics confuse it with arithmetic, and consider it an arid science. In reality, however, it is a science which requires a great amount of imagination.

Sofia Kovalevskaya

7

Cauchy-Kovalevskaya extensions

The Cauchy-Kovalevskaya (CK) theorem for linear differential operators has a long history. In the sense of the classical Dirac operator, this theorem provides a way to construct monogenic functions (or functions in the kernel of the Dirac operator), starting from functions in one variable less. In the first section of this chapter, we will repeat the proof given in e.g. [30], a proof which we generalise in the second section. In the third section, we will then use this extension to determine the dimension of the polynomial kernel of $\mathcal{Q}_\lambda$.

7.1 Classical Cauchy-Kovalevskaya extension

Before we prove a general CK-extension theorem in the context of HSD operators, let us recapitulate the classical case. One way of constructing monogenic functions in $\mathbb{R}^m$ is by extending a real-analytic function in some open connected domain Ω^* in $\mathbb{R}^{m-1}$, one dimension lower. The classical formulation of the problem is as follows.

> *Given a real-analytic function f^* in $\Omega^* \subset \mathbb{R}^{m-1}$, does there exist a monogenic function f in an open neighbourhood Ω of Ω^* in $\mathbb{R}^m$ such that $f|_{\Omega^*} = f^*$?*

The answer to this question is positive; we recall the main lines of the constructive proof. Consider $\mathbb{R}^{m-1}$ as the hyperplane with equation $x_m = 0$ in $\mathbb{R}^m$. We introduce the vector variable $x^* = (x_1, \ldots, x_{m-1}) \in \Omega^*$, and the corresponding Dirac operator in $\mathbb{R}^{m-1}$, given by

$$\partial_{x^*} = \sum_{j=1}^{m-1} e_j \partial_{x_j}.$$

This means that we have rewritten $x = x^* + x_m e_m \in \mathbb{R}^{m-1} \oplus \mathbb{R} = \mathbb{R}^m$, and $\partial_x = \partial_{x^*} + e_m \partial_{x_m}$. Let Ω be an open connected and x_m-normal neighbourhood of Ω^* in $\mathbb{R}^m$. The concept of x_m-normality signifies that for each $x \in \Omega$, the line segment $\{x + te_m : t \in \mathbb{R}\} \cap \Omega$ is connected and exactly contains one point in Ω^*. The function f that needs to be found now has to satisfy the conditions

- $\partial_x f = 0$ in Ω;
- $f(x)|_{x_m=0} = f^*(x^*)$.

We have that

$$\partial_x f = 0 \Leftrightarrow \partial_{x_m} f = -e_m \partial_{x^*} f.$$

This partial differential equation has the (formal) solution

$$\begin{aligned} f(x) &= e^{-x_m e_m \partial_{x^*}} f^*(x^*) \\ &= \sum_{j=0}^{\infty} \frac{(-x_m)^j}{j!} (e_m \partial_{x^*})^j f^*(x^*). \end{aligned}$$

Remark 7.1. Take $P_k^*(x^*) \in \mathcal{P}_k(\mathbb{R}^{m-1}, \mathbb{S})$. Then P_k may be expressed as

$$P_k^*(x^*) = \sum_{|\alpha|=k} x^{*\alpha} a_\alpha,$$

where α is the multi index $(\alpha_1, \ldots, \alpha_{m-1})$, $\alpha_a \in \mathbb{C}$, and $x^{*\alpha} = x_1^{\alpha_1} \cdots x_{m-1}^{\alpha_{m-1}}$. The CK-extension of this function is then given by

$$P_k(x) = \sum_{j=0}^{\infty} \frac{(-x_m)^j}{j!} (e_m \partial_{x^*})^j P_k^*(x^*).$$

Since $(e_m \partial_{x^*})^j P_k^*(x^*) = 0$ for all $j > k$, this sum reduces to

$$P_k(x) = \sum_{j=0}^{k} \frac{(-x_m)^j}{j!} (e_m \partial_{x^*})^j P_k^*(x^*),$$

whence $P_k(x)$ is a homogeneous monogenic polynomial of degree k in $\mathbb{R}^m$. Conversely, if $P_k(x)$ is a homogeneous monogenic polynomial of degree k in $\mathbb{R}^m$, its restriction $P(x^*,0)$ to $\mathbb{R}^{m-1}$ is a homogeneous polynomial of degree k. This means that the CK-extension actually gives us the isomorphism

$$\mathcal{S}_k(\mathbb{R}^m,\mathbb{S}) \cong \mathcal{P}_k(\mathbb{R}^{m-1},\mathbb{S}),$$

seen as a vector spaces over $\mathbb{C}$.

7.2 Generalised Cauchy-Kovalevskaya extension

In the previous section, we recalled the CK-extension related to the Dirac operator, which is one of the traditional methods for constructing monogenic functions (null solutions of the Dirac operator) from analytic functions in one variable less. In this section, we investigate if a similar method can be established to construct polynomial solutions of HSD operators. Remember that the HSD operator is defined as follows:

$$\mathcal{Q}_\lambda = p_{\mathfrak{osp}(1,2k)}\partial_x = \prod_{j=1}^{k}\left(1+\frac{u_j\partial_j}{m+2\mathbb{E}_i-2i}\right)\partial_x : \mathcal{C}(\mathbb{R}^m,\mathcal{S}_\lambda) \to \mathcal{C}(\mathbb{R}^m,\mathcal{S}_\lambda).$$

Similar to the classical case, we define

$$x^* = \sum_{j=1}^{m-1} e_j x_j \text{ and } \partial_{x^*} = \sum_{j=1}^{m-1} e_j\partial_{x_j}.$$

Then we can define the following operator on $\mathbb{R}^{m-1}$ as

$$\mathcal{Q}_\lambda^* = p_{\mathfrak{osp}(1,2k)}\partial_{x^*} : \mathcal{C}^\infty(\mathbb{R}^{m-1},\mathcal{S}_\lambda) \to \mathcal{C}^\infty(\mathbb{R}^{m-1},\mathcal{S}_\lambda).$$

Note that, while the operator is defined on functions in $m-1$ variables, $\mathcal{S}_\lambda \subset \mathcal{P}(\mathbb{R}^{km},\mathbb{S})$. For any function f, it holds that

$$f \in \ker_h \mathcal{Q}_\lambda \Leftrightarrow \partial_{x_m} f = -(p_{\mathfrak{osp}(1,2k)}e_m)^{-1}\mathcal{Q}_\lambda^* f.$$

Obviously, in order to be able to write this, the operator $p_{\mathfrak{osp}(1,2k)}e_m$ needs to be invertible, which is the subject of the following lemma. The unique solution to this first-order differential equation is given by

$$\begin{aligned} f(x) &= e^{-x_m(p_{\mathfrak{osp}(1,2k)}e_m)^{-1}\mathcal{Q}_\lambda^*} f^*(x^*) \\ &= \sum_{j=0}^{\infty}\frac{(-x_m)^j}{j!}((p_{\mathfrak{osp}(1,2k)}e_m)^{-1}\mathcal{Q}_\lambda^*)^j f^*(x^*). \end{aligned}$$

Lemma 7.1. *The operator $p_{\mathfrak{osp}(1,2k)}e_m$ is invertible.*

Proof. From [39, Theorem 6], we have that

$$\Delta^n = \mathcal{Q}_\lambda \left[\sum_{\lambda \in B(\lambda)} c(\mu,\lambda) G_{\mu,\lambda}[\partial_x] \Delta^{n-|\mu,\lambda|-1} G_{\lambda,\mu}[\partial_x] \right] \mathcal{Q}_\lambda,$$

where $G_{\mu,\lambda}$ is a product of twistor operators, $c(\mu,\lambda)$ are constants, $G_{\lambda,\mu}[\partial_x]$ is the inverse operator of $G_{\mu,\lambda}[\partial_x]$, $\lambda = (l_1,\ldots,l_k)$, $\mu = (\mu_1,\ldots,\mu_k)$,

$$|\mu,\lambda| = \sum_{i=1}^{k} |\mu_i - l_i|,$$

and where

$$B(\lambda) = [l_2,l_1] \times [l_3,l_2] \times \cdots \times [l_k,l_{k-1}] \times [0,l_k].$$

Replacing ∂_x by e_m, we get

$$(-1)^n = p_{\mathfrak{osp}(1,2k)}[e_m] \left[\sum_{\lambda \in B(\lambda)} c(\mu,\lambda) G_{\mu,\lambda}[e_m] (-1)^{n-|\mu,\lambda|-1} G_{\lambda,\mu}[e_m] \right] \times p_{\mathfrak{osp}(1,2k)}[e_m].$$

One can choose $n = l_1 + 1$, the smallest number for which this equality holds. This proves that $p_{\mathfrak{osp}(1,2k)}[e_m]$ indeed is invertible. □

Remark 7.2. The same reasoning can also be used to prove the ellipticity of the HSD operator. Replacing ∂_x by x instead of e_m shows the invertibility of the symbol of the HSD operator for all $x \neq 0$, since any non-zero vector x itself is invertible. This is exactly the requirement for ellipticity.

The following corollary is crucial.

Corollary 7.1. *As vector spaces over $\mathbb{C}$, the following isomorphism holds:*

$$\ker_h \mathcal{Q}_\lambda \cong \mathcal{P}_h(\mathbb{R}^{m-1},\mathbb{C}) \otimes \mathcal{S}_\lambda,$$

Proof. The generalised CK-extension implies that each

$$f \in \ker_h \mathcal{Q}_\lambda \cap \mathcal{P}_h(\mathbb{R}^m,\mathcal{S}_\lambda)$$

is in $(1-1)$-correspondence with $f^*(x^*) = f(x^*,0) \in \mathcal{P}_h(\mathbb{R}^{m-1},\mathcal{S}_\lambda)$. □

From this corollary, it follows that

$$\dim \ker_h(\mathcal{Q}_\lambda) = \dim(\mathcal{P}_h(\mathbb{R}^{m-1},\mathbb{C}))\dim(\mathcal{S}_\lambda).$$

To find a way to calculate the dimension of the kernel space of $\mathcal{Q}_\lambda$, we must thus find the dimension of $\mathcal{S}_\lambda$.

7.3 Dimension formula for $\ker_h(\mathcal{Q}_\lambda)$

In order to calculate the dimension of the kernel space of $\mathcal{Q}_\lambda$, we have to find the dimension of $\mathcal{S}_\lambda$ as a vector space over $\mathbb{C}$. We make use of the Weyl dimension formula (e.g. [47]), stating that

$$\dim \mathcal{S}_\lambda = \frac{\prod_{\alpha\in\Delta^+}\langle \lambda+\delta, \alpha\rangle}{\prod_{\alpha\in\Delta^+}\langle \delta, \alpha\rangle}, \tag{7.1}$$

where $\delta = \left(n-\frac{1}{2}, n-\frac{3}{2}, \ldots, \frac{1}{2}\right)$ is half the sum of the positive weights, $\langle\cdot,\cdot\rangle$ is the Killing form, and Δ^+ the positive root system of $\mathfrak{so}(m)$

$$\begin{aligned}\Delta^+ = \{&(1,\pm1,0,\ldots,0),(1,0,\pm1,0,\ldots,0),\ldots,(0,\ldots,0,1,\pm1),\\ &(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1)\}.\end{aligned}$$

The denominator of the dimension formula is then given by

$$\begin{aligned}
&\prod_{\alpha\in\Delta^+}\langle\delta,\alpha\rangle\\
&= \left(n-\frac{1}{2}+n-\frac{3}{2}\right)\left(n-\frac{1}{2}+n-\frac{5}{2}\right)\cdots\left(n-\frac{1}{2}+\frac{1}{2}\right)\\
&\quad\times\left(n-\frac{3}{2}+n-\frac{5}{2}\right)\left(n-\frac{3}{2}+n-\frac{7}{2}\right)\cdots\left(n-\frac{3}{2}+\frac{1}{2}\right)\\
&\quad\times\cdots\left(\frac{3}{2}+\frac{1}{2}\right)\\
&\quad\times\left(n-\frac{1}{2}-n+\frac{3}{2}\right)\left(n-\frac{1}{2}-n+\frac{5}{2}\right)\cdots\left(n-\frac{1}{2}-\frac{1}{2}\right)\\
&\quad\times\left(n-\frac{3}{2}-n+\frac{5}{2}\right)\left(n-\frac{3}{2}-n+\frac{7}{2}\right)\cdots\left(n-\frac{3}{2}-\frac{1}{2}\right)\\
&\quad\times\cdots\left(\frac{3}{2}-\frac{1}{2}\right)\\
&\quad\times\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\frac{1}{2},
\end{aligned}$$

which reduces to

$$(2n-2)!(2n-4)!\cdots 2!\frac{(2n-1)(2n-3)\cdots 1}{2^n} = \frac{1}{2^n}\prod_{j=1}^{n}(2j-1)!.$$

The numerator of (7.1) is obtained as follows:

$$\prod_{\alpha\in\Delta^+}\langle\lambda+\delta,\alpha\rangle = \prod_{1\le i<j\le k}(l_i+l_j+2n-i-j+2)(l_i-l_j+j-i)$$
$$\times\prod_{j=1}^{k}\frac{(l_i+2n-k-i+1)!}{(l_i+k-i)!}\prod_{j=1}^{n-k}(2j-1)!\ .$$

This leads to

$$\dim(\mathcal{S}_\lambda) = 2^n \prod_{1\le i<j\le k}(l_i+l_j+2n-i-j+2)(l_i-l_j+j-i)$$
$$\times\prod_{j=1}^{k}\binom{l_j+2n-k-j+1}{2n-2k+1}\frac{(2n-2k+1)!}{(2n-2k+2j-1)!}. \quad (7.2)$$

We can implement this in Maple by means of the code

```
dimensionFormula:= proc(HW,k,n)
local a, i, j, l, m;
for i from 1 to k-1 do
for j from i+1 to k do
a:=a*(HW[i]+HW[j]+2*n-i-j+2)*(HW[i]-HW[j]+j-i):
od;
od;
for l  from 1 to k do
a:=a*(HW[l]+2*n-k-l+1)!/((2*n-2*k+1)!*(HW[l]+k-l)!)
od;
for m from 0 to k-2 do
a:=(a*(2*n-2*k+1)!)/((2*n-1-2*m)!):
od;
a;
end proc:
```

In this code, HW represents λ as a $(1\times k)$-matrix containing the truncated highest weight entries.

Using this formula, we can now calculate the dimension of the h-homogeneous kernel of $\mathcal{Q}_\lambda$:

$$\dim(\ker_h\mathcal{K}_\lambda) = \dim(\mathcal{P}_h(\mathbb{R}^{m-1},\mathbb{C}))\dim(\mathcal{S}_\lambda) = \binom{h+m-2}{h}\times\dim(\mathcal{S}_\lambda). \quad (7.3)$$

This result will be important in the dimension analysis in Chapter 10.

Example 7.1. In the case where $k = 1$, the formula tells us that the dimension of the h-homogeneous kernel of $\mathcal{R}_{l_1}$ equals

$$2^n \binom{h+2n-1}{h} \binom{l_1+2n-1}{l_1}.$$

This is in accordance with the decomposition found in [20].

Example 7.2. In the case where $k = 2$, (7.3) gives us the dimension of the kernel of the HSD operator studied in [82]:

$$\begin{aligned} \dim(\ker_h \mathcal{K}_{l_1,l_2}) = 2^n &\binom{h+2n-1}{h} \binom{l_1+2n-2}{l_1+1} \\ &\times \binom{l_2+2n-3}{l_2} \frac{(l_1+l_2+2n-1)(l_1-l_2+1)}{(2n-1)(2n-2)}. \end{aligned}$$

What we know is not much. What we do not know is immense.

Pierre-Simon Laplace

8 Twisted higher spin Dirac operators

Rotationally invariant differential operators usually are constructed using the Stein-Weiss method [79]: this means that one acts with the gradient operator ∇ on functions $f(x)$ taking values in the desired representation $\mathbb{V}$ for the orthogonal Lie algebra, adding a suitable conformal weight in case the resulting operators are meant to be conformally invariant. In view of the fact that $\nabla f(x)$ transforms as an element of $\mathbb{C}^m \otimes \mathbb{V}$ under $\mathfrak{so}(m)$, it then suffices to make a suitable projection on the irreducible summands appearing in the tensor product (roughly speaking: 'vectors times monogenics' in our case of interest) in order to find invariant operators. In these cases, i.e. for half-integer highest weight representations, there is an alternative: instead of acting with the gradient one can also act with the Dirac operator ∂_x, herewith fully exploiting the Clifford multiplication.

From the point of view of invariant operators, the Dirac operator is, strictly speaking, only defined on spinor-valued functions, i.e. for $\mathbb{V}$ the spinor space. This is why the method we will develop in this chapter, as the title suggests, is referred to as the *twisted Dirac operator method.* The word 'twisted' captures the idea that the Dirac operator acts on functions taking values in the 'wrong' space. One is then again led to a tensor product, which is however different from the one mentioned above (roughly speaking: 'harmonics times spinors'). In this chapter, we will show that HSD operators can also be constructed using twisted operators 'of a lower order' (to be clarified in what follows): instead of working with the gradient or Dirac operator, we thus choose yet another operator acting on functions with values in $\mathbb{V}$, again reducing the construction of invariant operators to a tensor product. However, as we will see, this time the tensor product will only contain two relevant components, which is considerably less than the number of components obtained using e.g. the twisted Dirac operator. In a sense, this approach leads to an inductive pattern, which was already mentioned in [28], although there it was a purely formal observation, which we now elaborate from the representation theoretical point of view. This inductive pattern will allow us to find a full decomposition of the kernel of the HSD operator $\mathcal{Q}_\lambda$, the main target of this thesis.

In the first section of this chapter, we will introduce the twisted Dirac operator and its use in higher spin analysis. In Section 2, the twisted RS operators are defined and used to obtain an alternative realisation for the operator $\mathcal{Q}_{l_1,l_2}$. We have chosen to include this operator as a special case, in order to illustrate the more general procedure of Section 3, in which the operators $\mathcal{Q}_\lambda$ are obtained through an inductive procedure involving twisted HSD operators.

8.1 The twisted Dirac operator

In this section we recall the twisted Dirac operator, which can be defined on $\mathcal{H}_\lambda \otimes \mathbb{S}$-valued functions (for arbitrary λ) and enables us to define the HSD operators, see Definition 4.5. Note that ∂_x^T is well defined, as Δ_i and $\langle u_i, \partial_j \rangle$ commute with ∂_x^T for all $1 \leq i, j \leq k$. In order to explain why this operator is so useful to define HSD operators acting on $\mathcal{S}_\lambda$-valued polynomials, we invoke Lemma 4.3, stating that

$$\mathcal{H}_\lambda \otimes \mathbb{S} \ \cong \ \bigoplus_{i_1=0}^{1} \cdots \bigoplus_{i_k=0}^{1} (l_1 - i_1, \ldots, l_k - i_k)',$$

where each summand $(l_1 - i_1, \ldots l_k - i_k)'$ is contained in the decomposition as long as its highest weight satisfies the dominant weight condition $l_1 - i_1 \geq l_2 - i_2 \geq \cdots \geq l_k - i_k$. We will prove this result in the technical section at the end of this chapter. Note that it follows from this lemma that $\mathcal{S}_\lambda$ is a submodule of $\mathcal{H}_\lambda \otimes \mathbb{S}$ with multiplicity 1, which precisely is at the origin of Definition 4.5. Also, note that all other modules are isomorphically embedded into the tensor product, but we will omit the embedding factor (unless explicitly mentioned). Using the method of constructing conformally invariant operators by means of generalised gradients (see e.g. [45, 79]), one can deduce from the lemma above that the twisted Dirac operator $\mathbf{1}_\lambda \otimes \partial_x$ can be written as the sum of at most $|\lambda|+1$ first-order differential operators: a HSD operator $\mathcal{Q}_\lambda$ and (at most) $|\lambda|$ twistor operators $\mathcal{T}_\lambda^{(i)}$, defined as the unique first-order differential operators mapping $\mathcal{S}_\lambda$-valued functions to $\mathcal{S}_{\lambda-L_i}$-valued functions. Here, $\lambda - L_i$ stands for $(l_1, \ldots, l_i - 1, \ldots, l_k)$. Each of these operators is defined through the scheme in Figure 8.1:

$$\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda) \xrightarrow{\partial_x^T} \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_\lambda \otimes \mathbb{S}) \tag{8.1}$$

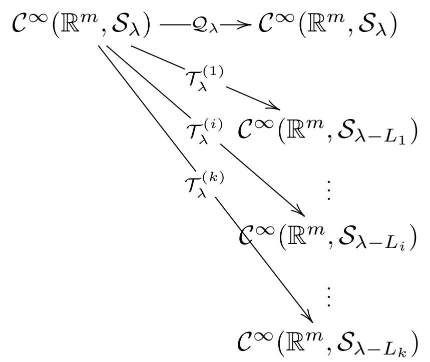


Figure 8.1: Classical decomposition of the twisted Dirac operator

Remark 8.1. Note that there are *less* twistor operators than summands in the decomposition of Lemma 4.3, which follows from Fegan's result on existence. More about this decomposition can be found in Chapter 12.

Before we turn our attention to arbitrary twisted HSD operators, we will illustrate our approach by means of a low order example: for $k = 1$ we encounter the classical Dirac operator, the decomposition of which leads to the definition of the RS operator. Here, Lemma 4.3 reduces to the classical Fischer decomposition (this time including the embedding factor):

$$\mathcal{H}_{l_1} \otimes \mathbb{S} \; = \; \mathcal{S}_{l_1} \oplus u_1 \mathcal{S}_{l_1 - 1}. \tag{8.2}$$

Since

$$f(x;u) \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1}) \xrightarrow{\partial_x^T} \partial_x^T f(x;u) \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{l_1} \otimes \mathbb{S}),$$

we only need to project $\partial_x^T f$ onto the space $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1})$ in order to obtain an expression for the RS operator $\mathcal{Q}_{l_1}$, in view of Lemma 4.3. Schematically, this can be shown as in Figure 8.2 Here, we can see that the twisted Dirac operator decomposes into two operators. These are two natural invariant operators acting on the $\mathcal{S}_{l_1}$-valued functions under consideration: the Rarita-Schwinger operator $\mathcal{Q}_{l_1}$ and a twistor operator $\mathcal{T}_{l_1}^{(1)}$ (see e.g. [20]).

$$\begin{array}{ccc}
\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1}) & \xrightarrow{\partial_x^T} & \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{l_1} \otimes \mathbb{S}) \\
= & & \cong \\
\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1}) & \xrightarrow{\mathcal{Q}_{l_1}} & \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1}) \\
& \searrow^{\mathcal{T}^{l_1}_{l_1-1}} & \oplus \\
& & \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1-1})
\end{array}$$

Figure 8.2: New decomposition of the twisted Dirac operator

Remark 8.2. As k increases, it will be harder to obtain a projection operator on $\mathcal{S}_\lambda$, since the scheme (8.1) will split into more components.

Remark 8.3. When using ∂_x to construct $\mathcal{Q}_{l_1}$, one actually uses the natural operator acting on functions taking values in the space "with one dummy variable less" (here: spinors). This observation has inspired us to follow a similar approach for more general values.

As illustrated above, explicit realisations for the HSD operators can be obtained by decomposing the twisted Dirac operator. However, our main aim is to eventually describe the polynomial null-solutions for general HSD operators. As k increases, the complexity of this kernel space increases as well (see e.g. [15]). This is why we will use a different approach to construct HSD operators in the remainder of this chapter (using recursion), which will then lead to an alternative method to determine null-solutions. Instead of using the twisted Dirac operator, we will construct the HSD operators using twisted HSD operators of *lower order*. This will be illustrated in the next section, where we will use the twisted RS operator, in order to define the HSD operator $\mathcal{Q}_{l_1,l_2}$ $(k = 2)$.

8.2 The twisted Rarita-Schwinger operator

The main aim of this section is to construct the HSD operators $\mathcal{Q}_{l_1,l_2}$ using twisted RS operators. The classical RS operator is the HSD operator of order 1 (where 'order' refers to the number of nontrivial entries in the highest weight of the representations), defined on $\mathcal{S}_{l_1}$-valued functions $f(x; u_1)$. It is however clear that the operator $\mathcal{Q}_{l_1}$, given by

$$\mathbf{1}_\mathbb{V} \otimes \mathcal{Q}_{l_1} = \mathbf{1}_\mathbb{V} \otimes \left(1 + \frac{u_1 \partial_1}{m + 2l_1 - 2}\right) \partial_x,$$

can act on any function space of the form $\mathcal{C}^\infty(\mathbb{R}^m, \mathbb{V} \otimes \mathcal{S}_{l_1})$. Just as for the Dirac operator, this will then lead to the twisted RS operator. In this chapter, we will take $\mathbb{V} = \mathcal{H}_{l_2}$. The reason for this is that we eventually want to determine the expression for $\mathcal{Q}_{l_1,l_2}$ starting from the twisted RS operator. This choice will prove to be very useful.

Definition 8.1. *For any highest weight* $(\mu_1, \ldots, \mu_{k-1})$ *with* $l_1 \geq \mu_1$*, we define the twisted RS operator as*

$$\begin{aligned}\mathcal{Q}_{l_1}^T = \mathbf{1}_{(\mu_1,\ldots,\mu_{k-1})} \otimes \mathcal{Q}_{l_1} :\\ \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{\mu_1,\ldots,\mu_{k-1}} \otimes \mathcal{S}_{l_1}) \rightarrow \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{\mu_1,\ldots,\mu_{k-1}} \otimes \mathcal{S}_{l_1}).\end{aligned} \quad (8.3)$$

Note that we have chosen not to include the highest weight $(\mu_1, \ldots, \mu_{k-1})$ *in the symbol for the twisted RS operator to avoid overloaded notations, although the precise definition obviously depends on the choice of these integers.*

Remember that $\mathcal{Q}_{l_1,l_2}$ acts on functions taking values in $\mathcal{S}_{l_1,l_2}$. Recalling the meaning of this space as a vector space containing polynomial solutions to systems of differential equations, it is easily seen that the spin-module $\mathcal{S}_{l_1,l_2}$ is a subspace of $\mathcal{H}_{l_2} \otimes \mathcal{S}_{l_1}$. So we can determine $\mathcal{Q}_{l_1,l_2}$ by letting the twisted RS operator act on functions in the space $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1,l_2})$ and projecting the result on the very same space afterwards. To prove the uniqueness of this projection, we need to prove that $\mathcal{S}_{l_1,l_2}$ is contained in $\mathcal{H}_{l_2} \otimes \mathcal{S}_{l_1}$ with multiplicity 1. This is the subject of the following theorem, the proof of which is rather technical and because of this reason is postponed till Section 8.4.1.

Theorem 8.1. *For each pair of integers* $l_1 \geq l_2 > 0$*, we have that* $\mathcal{S}_{l_1,l_2}$ *and* $\mathcal{S}_{l_1,l_2-1}$ *are* Spin(m)*-submodules of* $\mathcal{H}_{l_2} \otimes \mathcal{S}_{l_1}$ *with multiplicity* 1.

Remark 8.4. Note here, that by 'submodule', we mean that there is an isomorphic copy embedded in the tensor product as the degrees of homogeneity obviously do not agree. So one needs a nontrivial embedding operator. This will be the case throughout this thesis.

Let us then start from an arbitrary function $f(x; u_1, u_2) \in \mathcal{C}^\infty(\mathbb{R}, \mathcal{S}_{l_1,l_2})$. After applying the twisted RS operator (8.3), we obviously get that

$$\mathcal{Q}_{l_1}^T f = \left(1 + \frac{u_1 \partial_1}{m + 2l_1 - 2}\right) \partial_x f \in \ker(\partial_1).$$

It is no longer true that $\mathcal{Q}_{l_1}^T f \in \ker(\partial_2, \langle u_1, \partial_2 \rangle)$ but we do have that

$$\mathcal{Q}_{l_1}^T f \in \ker(\partial_1, \Delta_2, \langle u_1, \partial_2 \rangle^2),$$

where Δ_2 stands for the Laplace operator in the variable u_2. This can easily be seen: due to the fact that $\mathcal{Q}_{l_1}^T$ projects on the kernel of ∂_1, Δ_2 commutes with $\mathcal{Q}_{l_1}^T$ and

$$\langle u_1, \partial_2 \rangle \mathcal{Q}_{l_1}^T f = -\frac{u_1 \partial_2}{m + 2l_1 - 2} \partial_x f.$$

Since we know that the result is harmonic in the variable u_2, we can use the monogenic decomposition (8.2):

$$\mathcal{Q}_{l_1}^T f = F_{l_2} + u_2 F_{l_2-1},$$

where F_{l_2} and F_{l_2-1} are both monogenic in u_2. Applying ∂_2 to both sides of the equation, calculations involving the explicit expression for $\mathcal{Q}_{l_1}$ lead to

$$\begin{aligned} F_{l_2-1} &= -\frac{\partial_2 \mathcal{Q}_{l_1}^T f}{m+2l_2-2} \\ &= -\frac{(m+2l_1-2)\partial_2 - 2\langle u_1, \partial_2\rangle \partial_1}{(2l_1+m-2)(2l_2+m-2)} \partial_x f \\ &= 2\frac{2l_1+m}{(2l_1+m-2)(2l_2+m-2)} \langle \partial_2, \partial_x\rangle f. \end{aligned} \tag{8.4}$$

Observe that the operator $\langle \partial_2, \partial_x\rangle$ appearing here is, up to a multiplicative constant, the twistor operator $\mathcal{T}_{l_1,l_2}^{(2)}$, which is the unique first-order differential operator acting between the following spaces:

$$\langle \partial_2, \partial_x\rangle : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1,l_2}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1,l_2-1}).$$

Remember that this operator also appeared in the scheme 8.1 (for $k = 2$).

Defining $\pi_{l_1}[u_2]$ as the projection of the multiplication operator u_2 on the kernel of the Dirac operator ∂_1, gives rise to a mapping

$$\pi_{l_1}[u_2] := \left(1 + \frac{u_1\partial_1}{m+2l_1-2}\right) u_2 : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1,l_2-1}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{l_2} \otimes \mathcal{S}_{l_1}).$$

We can prove the following proposition.

Proposition 8.1. *For all integers $l_1 \geq l_2 > 0$ and for all $f(x; u_1, u_2) \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1,l_2})$ with $\langle \partial_2, \partial_x\rangle f \neq 0$, there exists a unique constant $\gamma_{l_2} \in \mathbb{R}$ such that*

$$\mathcal{Q}_{l_1}^T f = \phi_0 + \gamma_{l_2} \pi_{l_1}[u_2] \langle \partial_2, \partial_x\rangle f,$$

with ϕ_0 satisfying

$$\partial_1 \phi_0 = \partial_2 \phi_0 = \langle u_1, \partial_2\rangle \phi_0 = 0,$$

whence $\phi_0 = \mathcal{Q}_{l_1,l_2} f$. This constant is given by

$$\gamma_{l_2} = \frac{2}{2l_2+m-4}.$$

Proof. Let us consider a function $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1,l_2})$ with $\langle \partial_2, \partial_x\rangle f \neq 0$, and define ϕ_0 by

$$\phi_0 := \mathcal{Q}_{l_1}^T f - \gamma_{l_2} \pi_{l_1}[u_2] \langle \partial_2, \partial_x\rangle f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{l_2} \otimes \mathcal{S}_{l_1}),$$

where the constant γ_2 is to be fixed in such a way that ϕ_0 indeed satisfies the requirements mentioned above. In view of the fact that $\mathcal{Q}_{l_1}^T f \in \ker \Delta_2$, we can use (8.2) to arrive at

$$\mathcal{Q}_{l_1}^T f = F_{l_2} + u_2 F_{l_2-1},$$

where both functions F_i are homogeneous of degree i and monogenic in u_2. Applying the Dirac operator ∂_2 on both expressions for $\mathcal{Q}_{l_1}^T f$ yields:

$$-(2l_2+m-2)F_{l_2-1} = \partial_2\phi_0 - \gamma_{l_2}\frac{(2l_1+m)(2l_2+m-2)}{2l_1+m-2}\langle\partial_2, \partial_x\rangle f.$$

Using equation (8.4), we thus get that

$$\partial_2\phi_0 = \left(\gamma_{l_2}\frac{(2l_1+m)(2l_2+m-2)}{2l_1+m-2} - \frac{2(2l_1+m)}{2l_1+m-2}\right)\langle\partial_2, \partial_x\rangle f.$$

If we choose

$$\gamma_{l_2} = \frac{2}{2l_2+m-4},$$

the proposition is seen to hold. □

Remark 8.5. If $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1,l_2}) \cap \ker\langle\partial_2, \partial_x\rangle$, the proposition above reduces to

$$\mathcal{Q}_{l_1}^T f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1,l_2}).$$

Note that the operator $\gamma_{l_2}\pi_{l_1}[u_2]\langle\partial_2, \partial_x\rangle$ is nothing but the operator $\mathcal{T}_{l_1,l_2}^{(2)} = \langle\partial_2, \partial_x\rangle$ and an embedding factor $\gamma_{l_2}\pi_{l_1}[u_2]$, which means that we get the scheme in Figure 8.3 for the action of $\mathcal{Q}_{l_1}^T$ (up to isomorphic embeddings):

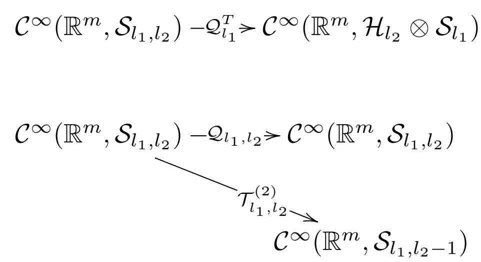


Figure 8.3: Decomposition twisted RS operator

Due to Theorem 8.1, this decomposition is unique.

Remark 8.6. From [82], we know that

$$\mathcal{Q}_{l_1,l_2} = \left(1 + \frac{u_1\partial_1}{m+2l_1-2}\right)\left(1 + \frac{u_2\partial_2}{m+2l_2-4}\right)\partial_x.$$

This indeed is the form we found in Proposition 8.1.

8.3 The twisted higher spin Dirac operator

Similar to the derivation of $\mathcal{Q}_{l_1,l_2}$ in the previous section, we can now obtain an explicit realisation for the most general HSD operators, using a related twisted HSD operator of 'lower order'. Suppose $\lambda = (l_1, \ldots, l_k)$, with $l_1 \geq \cdots \geq l_k > 0$. The standard HSD operator in k vector variables was defined as the first-order differential operator:

$$\mathcal{Q}_\lambda : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda).$$

As for the twisted RS operator, we introduce the following definition.

Definition 8.2. *The twisted HSD operator $\mathcal{Q}_\lambda^T$ is the operator*

$$\mathcal{Q}_\lambda^T = \mathbf{1}_\mathbb{V} \otimes \mathcal{Q}_\lambda : \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{V} \otimes \mathcal{S}_\lambda) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{V} \otimes \mathcal{S}_\lambda), \tag{8.5}$$

acting on $\mathbb{V} \otimes \mathcal{S}_\lambda$-valued functions on $\mathbb{R}^m$. Note that we again prefer not to mention this space $\mathbb{V}$ explicitly in the symbol for the twisted HSD operator, in order to avoid overloaded notations.

Once again, note that the difference between the ordinary HSD operator and its twisted version lies in the values of the functions $f(x)$ these operators are meant to act on. The 'twisted' refers to the fact that this operator acts on a 'bigger' space than the canonical domain of the ordinary HSD operator. Now, let $\lambda^+ = (l_1, \ldots, l_k, l_{k+1})$ be a dominant highest weight. The $+$-sign hereby suggests that we have added an entry to λ. In order to use this twisted HSD operator to construct more complicated HSD operators, one must choose $\mathbb{V}$ in such a way that $\mathcal{S}_{\lambda^+} \subset \mathbb{V} \otimes \mathcal{S}_\lambda$. We will prove that $\mathbb{V} = \mathcal{H}_{l_{k+1}}$, with $l_k \geq l_{k+1} > 0$, fits this purpose.

In view of our polynomial models, it is easily seen that $\mathcal{S}_{\lambda^+}$ indeed is a subspace of the tensor product $\mathcal{H}_{l_{k+1}} \otimes \mathcal{S}_\lambda$. In order to obtain a HSD operator $\mathcal{Q}_{\lambda^+}$ which is well-defined, we need to prove that $\mathcal{S}_{\lambda^+}$ is contained in this tensor product with multiplicity one.

Theorem 8.2. *Defining the (dominant) highest weights*

$$\lambda^- = (l_1, \ldots, l_k, l_{k+1} - 1) \quad \textit{and} \quad \lambda^+ = (l_1, \ldots, l_k, l_{k+1}),$$

both $\mathcal{S}_{\lambda^-}$ and $\mathcal{S}_{\lambda^+}$ are contained in $\mathcal{H}_{l_{k+1}} \otimes \mathcal{S}_{l_1,\ldots,l_k}$ as a submodule with multiplicity 1.

Again, the technical proof of this theorem is postponed till the last section of this chapter. Let us now consider a function $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda^+})$. When applying the twisted HSD operator $\mathcal{Q}_\lambda^T$ on this function, we get that

$$\mathcal{Q}_\lambda^T f = \left(\prod_{i=1}^k \left(1 + \frac{u_i \, \partial_i}{m + 2l_i - 2i} \right) \right) \partial_x f$$

is an element of $\ker(\partial_1, \ldots, \partial_k, \langle u_1, \partial_2\rangle, \ldots, \langle u_{k-1}, \partial_k\rangle)$. However, it does no longer belong to $\ker(\partial_{k+1}, \langle u_k, \partial_{k+1}\rangle)$. Denoting the Laplace operator in u_{k+1} by Δ_{k+1}, one can easily prove that

$$\mathcal{Q}_\lambda^T f \in \ker(\Delta_{k+1}, \langle u_k, \partial_{k+1}\rangle^2),$$

since Δ_{k+1} commutes with $\mathcal{Q}_\lambda^T$, and

$$\langle u_k, \partial_{k+1}\rangle \mathcal{Q}_\lambda^T f = -\prod_{i=1}^{k-1}\left(1 + \frac{u_i\,\partial_i}{m+2l_i-2i}\right)\frac{u_k\partial_{k+1}}{m+2l_k-2k}\partial_x f.$$

This implies that $\mathcal{Q}_\lambda^T f$ takes values in $\mathcal{H}_{l_{k+1}} \otimes \mathcal{S}_\lambda$ and explains our choice for $\mathbb{V}$ in (8.5). In order to derive the explicit expression for the HSD operator $\mathcal{Q}_{\lambda^+}$, we thus need the projection of $\mathcal{Q}_\lambda^T f$ on $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda^+})$. Since we know that $\mathcal{Q}_\lambda^T f$ is harmonic in u_{k+1}, we can use the monogenic Fischer decomposition (8.2) in this variable:

$$\mathcal{Q}_\lambda^T f = F_{l_{k+1}} + u_{k+1}F_{l_{k+1}-1},$$

where both functions F_j are monogenic of degree j in the variable u_{k+1}. Applying ∂_{k+1} to both sides of this equation then yields

$$F_{l_{k+1}-1} = -\frac{\partial_{k+1}\mathcal{Q}_\lambda^T f}{m+2l_{k+1}+2}.$$

Further calculations on the right-hand side of the latter expression lead to

$$\begin{aligned} F_{l_{k+1}-1} &= -\frac{1}{m+2l_{k+1}+2}\prod_{i=1}^{k}\frac{m+2l_i-2(i-1)}{m+2l_i-2i}\partial_{k+1}\partial_x f \\ &= \frac{2}{m+2l_{k+1}+2}\prod_{i=1}^{k}\frac{m+2l_i-2(i-1)}{m+2l_i-2i}\langle\partial_{k+1}, \partial_x\rangle f. \end{aligned} \quad (8.6)$$

Remember that the operator $\langle\partial_{k+1}, \partial_x\rangle$ is, up to a multiplicative constant, equal to the twistor operator defined by

$$\mathcal{T}_{\lambda^+}^{(k)} : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda^+}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda^-}).$$

Next, let us define $\pi_\lambda[u_{k+1}]$ as

$$\pi_\lambda[u_{k+1}] := \prod_{i=1}^{k}\left(1 + \frac{u_i\partial_i}{m+2l_i-2i}\right)u_{k+1}.$$

In other words: this is the simplicial monogenic projection of the multiplication operator u_{k+1}, defined by means of

$$\pi_\lambda[u_{k+1}] : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda^-}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{l_{k+1}} \otimes \mathcal{S}_\lambda).$$

One can then prove the following generalisation of Proposition 8.1.

Proposition 8.2. *For all integers $l_1 \geq l_2 \geq \ldots \geq l_{k+1} > 0$ and for all $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda^+})$ with $\langle \partial_{k+1}, \partial_x \rangle f \neq 0$, there exists a unique constant $\gamma_{l_{k+1}} \in \mathbb{R}$ such that*

$$\mathcal{Q}_\lambda^T f = \phi_0 + \gamma_{l_{k+1}} \pi_\lambda [u_{k+1}] \langle \partial_{k+1}, \partial_x \rangle f,$$

with $\phi_0 \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda^+})$, whence $\phi_0 = \mathcal{Q}_{\lambda^+} f$. The constant is given by

$$\gamma_{l_{k+1}} := \frac{2}{2l_{k+1} + m - 2(k+1)}.$$

Proof. Define a function ϕ_0 by

$$\phi_0 := \mathcal{Q}_\lambda^T f - \gamma_{l_{k+1}} \pi_\lambda [u_{k+1}] \langle \partial_{k+1}, \partial_x \rangle f,$$

where the constant $\gamma_{l_{k+1}}$ is to be fixed in such a way that ϕ_0 meets the requirements of the proposition. The classical Fischer decomposition in the vector variable u_1 yields

$$\mathcal{Q}_\lambda^T f = F_{l_{k+1}} + u_{k+1} F_{l_{k+1}-1},$$

since $\mathcal{Q}_\lambda^T f \in \ker \Delta_{k+1}$. Applying the Dirac operator ∂_{k+1} to both sides of the equality gives us that

$$-(2l_{k+1} + m - 2) F_{l_{k+1}-1} =$$
$$\partial_{k+1}\phi_0 - \gamma_{l_{k+1}} \prod_{i=1}^{k} \left(\frac{2l_i + m - 2(i-1)}{2l_i + m - 2i} \right) (m + 2l_{k+1} - 2k) \langle \partial_{k+1}, \partial_x \rangle f.$$

Using (8.6), we then get that

$$\partial_{k+1}\phi_0 =$$
$$\big(\gamma_{l_{k+1}}(2l_{k+1} + m - 2(k+1)) - 2\big) \prod_{i=1}^{k} \left(\frac{2l_i + m - 2(i-1)}{2l_i + m - 2i} \right) \langle \partial_{k+1}, \partial_x \rangle f.$$

Choosing

$$\gamma_{l_{k+1}} = \frac{2}{2l_{k+1} + m - 2(k+1)},$$

the proposition is directly seen to hold. □

Schematically, we can represent this decomposition as follows:

$$\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda^+}) \xrightarrow{\mathcal{Q}_\lambda^T} \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{l_{k+1}} \otimes \mathcal{S}_\lambda)$$

$$\begin{array}{ccc} \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda^+}) & \xrightarrow{\mathcal{Q}_{\lambda^+}} & \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda^+}) \\ & \searrow^{\langle \partial_{k+1}, \partial_x \rangle} & \\ & & \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda^-}) \end{array}$$

Remark 8.7. In case $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda+}) \cap \ker\langle \partial_{k+1}, \partial_x \rangle$, the proposition above immediately reduces to

$$\mathcal{Q}_\lambda^T f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda+}).$$

8.4 Technical proofs

In this section, we have gathered the proofs of technical results which were omitted in the preceding text.

8.4.1 Representations of order 2

It was proven in [56] that

Lemma 8.1. *For all integers $l_1 \geq l_2 > 0$, we have that $\mathcal{H}_{l_1} \otimes \mathcal{H}_{l_2}$ decomposes as*

$$(l_1, 0, \cdots, 0) \otimes (l_2, 0, \cdots, 0) \cong \bigoplus_{i=0}^{l_2} \bigoplus_{j=0}^{i} (l_1 + l_2 - 2i + j, j, 0, \cdots, 0),$$

where each highest weight refers to an irreducible representation of the spin group.

We can nicely represent the right-hand side of the theorem above in Figure 8.4.

$$\begin{array}{rcl} (l_1, l_2) \oplus (l_1 - 1, l_2 - 1) \oplus & \cdots & \oplus (l_1 - l_2 + 1, 1) \oplus (l_1 - l_2, 0) \\ & \vdots & \\ (l_1 + l_2 - j, j) \oplus & \cdots & \oplus (l_1 + l_2 - 2j, 0) \\ & \vdots & \\ (l_1 + l_2 - 1, 1) & \oplus & (l_1 + l_2 - 2, 0) \\ & & \\ & (l_1 + l_2, 0) & \end{array} \tag{8.7}$$

Figure 8.4: Decomposition of the tensor product $\mathcal{H}_{l_1} \otimes \mathcal{H}_{l_2}$

In this decomposition, each summand appears with multiplicity 1. Using this lemma, Theorem 2 from Section 4 can now be proven.

Theorem 8.3. *For each pair of integers $l_1 \geq l_2 > 0$, we have that $\mathcal{S}_{l_1,l_2}$ and $\mathcal{S}_{l_1,l_2-1}$ are Spin(m)-submodules of $\mathcal{H}_{l_2} \otimes \mathcal{S}_{l_1}$ with multiplicity* 1.

Proof. In view of our polynomial models, it is easily seen that $\mathcal{S}_{l_1,l_2}$ is a subset of $\mathcal{H}_{l_2} \otimes \mathcal{S}_{l_1}$. It is multiplicity-free, since

$$\mathcal{H}_{l_2} \otimes \mathcal{S}_{l_1} \subset \mathcal{H}_{l_1} \otimes \mathcal{H}_{l_2} \otimes \mathbb{S},$$

and $\mathcal{S}_{l_1,l_2}$ only appears as a submodule of $\mathcal{H}_{l_1,l_2} \otimes \mathbb{S}$, using (8.7) and (4.10). The vector space $\mathcal{S}_{l_1,l_2-1}$ however, is a submodule of $\mathcal{H}_{l_1} \otimes \mathcal{H}_{l_2} \otimes \mathbb{S}$ with multiplicity 2, since it is both a submodule of $\mathcal{H}_{l_1,l_2} \otimes \mathbb{S}$ and $\mathcal{H}_{l_1+1,l_2-1} \otimes \mathbb{S}$, again using (8.7) and (4.10). On the other hand, we also have that

$$\mathcal{H}_{l_1} \otimes \mathcal{H}_{l_2} \otimes \mathbb{S} \cong (\mathcal{H}_{l_2} \otimes \mathcal{S}_{l_1}) \oplus (\mathcal{H}_{l_2} \otimes \mathcal{S}_{l_1-1}).$$

If one can prove that $\mathcal{H}_{l_2} \otimes \mathcal{S}_{l_1-1}$ has $\mathcal{S}_{l_1,l_2-1}$ as a submodule with multiplicity 1, then the theorem is proven. This indeed is the case since

$$\mathcal{H}_{l_1-1} \otimes \mathcal{H}_{l_2} \otimes \mathbb{S} \cong (\mathcal{H}_{l_2} \otimes \mathcal{S}_{l_1-1}) \oplus (\mathcal{H}_{l_2} \otimes \mathcal{S}_{l_1-2}),$$

and the module $\mathcal{S}_{l_1,l_2-1}$ is contained in the subrepresentation $\mathcal{H}_{l_1,l_2-1} \otimes \mathbb{S}$ of the tensor product $(\mathcal{H}_{l_1-1} \otimes \mathcal{H}_{l_2}) \otimes \mathbb{S}$ with multiplicity 1. It is however not a submodule of $\mathcal{H}_{l_2} \otimes \mathcal{S}_{l_1-2}$, since

$$\mathcal{H}_{l_2} \otimes \mathcal{S}_{l_1-2} \subset \mathcal{H}_{l_1-2} \otimes \mathcal{H}_{l_2} \otimes \mathbb{S},$$

and $\mathcal{S}_{l_1,l_2-1}$ is not a submodule of $\mathcal{H}_{l_1-2} \otimes \mathcal{H}_{l_2} \otimes \mathbb{S}$, again because of (8.7) and (4.10). □

8.4.2 Representations of order k

Let $\mathbb{V}$ be an arbitrary representation of Spin(m), or its Lie algebra $\mathfrak{so}(m)$. Denote by Γ_λ the finite-dimensional irreducible representation with highest weight λ. The multiplicity of Γ_λ in $\mathbb{V}$ is denoted by $m_\lambda(\mathbb{V})$, and the multiplicity of an arbitrary weight μ in Γ_λ is denoted by $n_\mu(\Gamma_\lambda)$. We will use the following result (see e.g. [52]).

Theorem 8.4. *If ν is a dominant weight such that $m_\nu(\Gamma_\lambda \otimes \Gamma_\mu) > 0$, then there is a weight μ' of Γ_μ such that $\nu = \lambda + \mu'$. Moreover, if this is the case, then we at the same time have that $m_\nu(\Gamma_\lambda \otimes \Gamma_\mu) \leq n_{\lambda-\nu}(\Gamma_\mu)$.*

This theorem is needed to prove Lemma 4.3. Recall that $\lambda = (l_1, \ldots, l_k)$ is the highest weight for $\mathcal{H}_\lambda$ as $\mathfrak{so}(m, \mathbb{C})$-representation, and $(0)'$ for $\mathbb{S}$. Let ν be a dominant integral weight corresponding to one or more vector spaces in the decomposition of $\mathcal{H}_\lambda \otimes \mathbb{S}$, i.e. $n_\nu(\mathcal{H}_\lambda \otimes \mathbb{S}) > 0$. Then by Theorem 8.4, there exists a weight s of $\mathbb{S}$ such that $\nu = \lambda + s$ and

$$n_\nu(\mathcal{H}_\lambda \otimes \mathbb{S}) \leq m_s(\mathbb{S}) = 1.$$

Hence

$$n_\nu(\mathcal{H}_\lambda \otimes \mathbb{S}) = 1.$$

All possible weights ν are given by

$$\left(l_1 \pm \frac{1}{2}, \ldots, l_k \pm \frac{1}{2}, \pm\frac{1}{2}, \ldots, \pm\frac{1}{2}\right).$$

Because ν is dominant integral, we only have to deal with the cases

$$\left(l_1 \pm \frac{1}{2}, \ldots, l_k \pm \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right).$$

The representations corresponding to these highest weights occur exactly once in $\mathcal{H}_\lambda \otimes \mathbb{S}$. This finishes the proof of Lemma 4.3.

Put $\lambda^- = (l_1, l_2, \ldots, l_k, l_{k+1} - 1)$ and $\lambda^+ = (l_1, l_2, \ldots, l_k, l_{k+1})$, as above. In view of definitions 2.18 and 2.19, it is easily seen that $\mathcal{S}_{\lambda^+}$ indeed is contained as a submodule in the tensor product $\mathcal{H}_{l_{k+1}} \otimes \mathcal{S}_\lambda$, since all polynomials in $\mathcal{S}_{\lambda^+}$ indeed are simplicial monogenic in the first k variables and harmonic in u_{k+1}. Recalling the definition of the projection operator $\pi_\lambda[u_{k+1}]$, it is also clear that

$$\prod_{i=1}^{k}\left(1 + \frac{u_i \partial_i}{m + 2l_i - 2i}\right)[u_{k+1}]\mathcal{S}_{\lambda^-} \subset \mathcal{H}_{l_{k+1}} \otimes \mathcal{S}_{l_1,\ldots,l_k},$$

whence $\mathcal{S}_{\lambda^-}$ is, up to an embedding factor, also contained in $\mathcal{H}_{l_{k+1}} \otimes \mathcal{S}_{l_1,\ldots,l_k}$. We then proceed with the proof of Theorem 8.5 to obtain uniqueness (in two parts).

Theorem 8.5. *The vector space $\mathcal{S}_{\lambda^-}$ is contained as a submodule inside the tensor product $\mathcal{H}_{l_{k+1}} \otimes \mathcal{S}_{l_1,\ldots,l_k}$ with multiplicity* 1.

We already have shown that $m_{\lambda^-}(\mathcal{H}_{l_{k+1}} \otimes \mathcal{S}_{l_1,\ldots,l_k}) > 0$. Putting $\lambda = (l_1, \ldots, l_k)'$, $\mu = (l_{k+1})$ and $\nu = (\lambda^-)'$ in Theorem 8.4, we then need to prove that $n_{\lambda-\nu}(\Gamma_\mu) = 1$, which will lead to the first part of Theorem 2. We see that $\lambda - \nu = (0, \ldots, 0, -l_{k+1} + 1, 0, \ldots, 0)$, where the nonzero element is on the $(k+1)$-th position. Due to the action of the Weyl-group, we know that

$$n_{(0,\ldots,0,-l_{k+1}+1,0,\ldots,0)}(\mathcal{H}_{l_{k+1}}) = n_{(l_{k+1}-1)}(\mathcal{H}_{l_{k+1}}).$$

In order to calculate the multiplicity of the weight $(l_{k+1} - 1)$ in the Spin(m)-representation $\mathcal{H}_{l_{k+1}}$, we make use of Freudenthal's formula, which we state in the following theorem (in a form adapted to our needs).

Theorem 8.6. *Let Γ_λ be an irreducible representation with highest weight λ for $\mathfrak{g} = \mathfrak{so}(m)$. The multiplicity $n_\mu(\Gamma_\lambda)$ of the weight μ in Γ_λ is given recursively by*

$$\big(2\langle\lambda-\mu,\mu+\delta\rangle + \|\lambda-\mu\|^2\big)\, n_\mu(\Gamma_\lambda) = 2\sum_{\alpha\in\Delta^+}\sum_{a\geq 1}\langle\mu+\alpha a,\alpha\rangle n_{\mu+\alpha a}(\Gamma_\lambda).$$

Here, δ stands for half the sum of the positive roots, $\langle\cdot,\cdot\rangle$ is the Killing form and Δ^+ is the set of positive roots.

In our case, i.e. for the Lie algebra $\mathfrak{so}(m) = \mathfrak{so}(2n+1)$, we get

$$\begin{aligned}\Delta^+ = \{&(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1),\\ &(1,1,0,\ldots,0),(1,0,1,0,\ldots,0),\ldots,(0,\ldots,0,1,1),\\ &(1,-1,0,\ldots,0),(1,0,-1,0,\ldots,0),\ldots,(0,\ldots,0,1,-1)\}.\end{aligned} \quad (8.8)$$

and the Killing form is the standard inner product. Thus

$$\delta = (n-\frac{1}{2}, n-\frac{3}{2},\ldots,\frac{1}{2}),$$

with $m = 2n+1$. Putting $\lambda = (l_{k+1},0,\ldots,0)$ and $\mu = (l_{k+1}-1,0,\ldots,0)$, the left-hand side of the Freudenthal formula becomes $2(l_{k+1}+n-1)n_\mu(\Gamma_\lambda)$. In order to determine the right-hand side of the equality, it suffices to note that the only non-trivial $\mu+\alpha a$ that will appear in the sum are of the form

$$(l_{k+1},0,\ldots,0),(l_{k+1}-1,1,0,\ldots,0),\ldots,(l_{k+1}-1,0,\ldots,0,1),$$

where α is $(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,0,\ldots,0,1)$ respectively, and $a=1$. There are no other possibilities, in view of the following classical result.

Theorem 8.7. *If $\lambda = (\lambda_1,\ldots,\lambda_k)$ is the highest weight of an irreducible representation Γ_λ, and $\mu = (\mu_1,\ldots,\mu_k)$ is a non-trivial weight in Γ_λ, then*

$$\sum_{i=1}^k |\mu_i| \leq \sum_{i=1}^k |\lambda_i|.$$

We then get that

$$\begin{aligned}&2\sum_{\alpha\in\Delta^+}\sum_{a\geq 1}\langle\mu+\alpha a,\alpha\rangle n_{\mu+\alpha a}(\Gamma_\lambda)\\ &= \quad 2l_{k+1}n_{(l_{k+1})}(\Gamma_{(l_{k+1})}) + 2\sum_{i=2}^n n_{(l_{k+1}-1)+L_i}(\Gamma_{l_{k+1}}),\end{aligned}$$

where $L_i = (0,\ldots,0,1,0,\ldots,0)$, with 1 on the i-th position. It now suffices to show that for each $2\leq i\leq n$ one also has that $n_{(l_{k+1}-1)+L_i}(\Gamma_{l_{k+1}}) = 1$

in order to complete our proof. This indeed is the case, as follows from Freudenthal's formula again. Putting $\lambda = (l_{k+1})$ and $\mu = (l_{k+1} - 1) + L_i$, the left-hand side of the formula becomes $2(l_{k+1}+i-2)n_{(l_{k+1}-1)+L_i}(\Gamma_{(l_{k+1})})$. For the right-hand side, the only possible values for $\mu + \alpha a$ in the sum are $(l_{k+1}), (l_{k+1} - 1) + L_1, \ldots, (l_{k+1} - 1) + L_{i-1}$. Induction on i indeed gives us that

$$n_{(l_{k+1}-1)+L_i}(\Gamma_{(l_{k+1})}) = 1,$$

for each $2 \leq i \leq n$.

Finally, we also prove the remaining part of Theorem 8.5.

Theorem 8.8. *The vector space $\mathcal{S}_{\lambda^+}$ is contained as a submodule inside the tensor product $\mathcal{H}_{l_{k+1}} \otimes \mathcal{S}_{l_1,\ldots,l_k}$ with multiplicity* 1.

Proof. We once again make use of Theorem 8.4. Putting $\lambda = (l_1, \ldots, l_k)'$, $\mu = (l_{k+1})$ and $\nu = (\lambda^+)'$, this theorem states that

$$m_{(\lambda^+)'}(\Gamma_{(l_1,\ldots,l_k)'} \otimes \Gamma_{(l_{k+1})}) \leq n_{(0,\ldots,0,-l_{k+1},0,\ldots,0)}(\Gamma_{(l_{k+1})}).$$

Due to the action of the Weyl group, we have

$$n_{(0,\ldots,0,-l_{k+1},0,\ldots,0)}(\Gamma_{(l_{k+1})}) = n_{(l_{k+1})}(\Gamma_{(l_{k+1})}) = 1,$$

as this is the multiplicity of the highest weight space. □

A mathematician is a device for turning coffee into theorems.

Paul Erdos

9

Type A solutions of $\mathcal{Q}_\lambda$

This chapter is a major step towards the main goal of this thesis, decomposing the polynomial kernel space of the higher spin Dirac operators in irreducible representations for the spin group. More specifically, we introduce a special class of solutions and investigate their connection with transvector algebras of type $Z(\mathfrak{gl}(k+1,\mathbb{C}),\mathfrak{gl}(k,\mathbb{C}))$.

9.1 Type A solutions of higher spin operators

As in any function theory, the study of polynomial solutions of the involved differential operators plays a crucial role, due to the fact that these are often used to decompose arbitrary solutions belonging to appropriate function spaces. We will therefore take a closer look at this in the cases of the higher spin Dirac operators found in (4.11).

Similarly to what has been done for the Rarita-Schwinger operator and its generalisations, we will study so-called *type A solutions.* Indeed, in general two types of homogeneous polynomial solutions of higher spin Dirac operators are to be distinguished: either the polynomial belongs to the kernel of the (twisted) Dirac operator ∂_x, or the operator ∂_x essentially maps its values to one of the other summands inside the tensor product $\mathcal{H}_\lambda \otimes \mathbb{S}$, meaning that the projection operator $p_{\mathfrak{osp}(1,2k)}$ then acts trivially. The latter are the so-called *type B solutions*, which can be characterised in terms of twistor operators (see e.g. [20] for the case of the Rarita-Schwinger operator). Type A solutions are then obviously polynomials in the kernel of ∂_x as well as in the respective kernels of the operators $\partial_1, \ldots, \partial_k$; hence they must belong to the space

$$\mathcal{M}_{l_0,\ldots,l_k} := \{P \in \mathcal{P}_{l_0,\ldots,l_k}(u_0,\ldots,u_k) : \partial_0 P = \cdots = \partial_k P = 0\}\,,$$

where l_0 is an additional degree of homogeneity in $x \equiv u_0$. However, in order to ensure that these solutions have the correct values (i.e., simplicial monogenics), we have to consider the following subspace.

Definition 9.1. *For all $(k+1)$-tuples of integers $(l_0,\ldots,l_k) \in \mathbb{N}^{k+1}$ satisfying the dominant weight condition $l_0 \geq \cdots \geq l_k$, we define the vector space*

$$\mathcal{M}^s_{l_0,\ldots,l_k} := \{M \in \mathcal{M}_{l_0,\ldots,l_k} : \langle u_1, \partial_2\rangle M = \cdots = \langle u_{k-1}, \partial_k\rangle M = 0\}\,.$$

The space $\mathcal{M}^s_{l_0,\ldots,l_k}$ exactly corresponds to the type A solutions of the higher spin Dirac operator $\mathcal{Q}_\lambda$. We will study the algebraic structure of this vector space by investigating how it decomposes into irreducible modules for the spin group, and which invariant operators can be used to move between different summands inside this Spin(m)-decomposition. As we will explain in the last section, this question again is related to the topic of transvector algebras introduced in Chapter 4.

First of all, recall that the standard general Lie algebra $\mathfrak{gl}(k,\mathbb{C})$ (with $k \geq 2$), spanned by the standard basis elements E_{ij}, $1 \leq i,j \leq k$, i.e. the matrices in $\mathbb{C}^{n\times n}$ for which $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. As was explained in Chapter 2, finite-dimensional irreducible representations for $\mathfrak{gl}(k,\mathbb{C})$ are in one-to-one correspondance with k-tuples $(\lambda_1,\ldots,\lambda_k) \in \mathbb{C}^k$ such that $\lambda_i - \lambda_{i+1} \in \mathbb{Z}^+$, called the highest weight (HW) of the corresponding representation $\mathbb{V}(\lambda)$. By definition, the corresponding highest weight vector (HWV) v_λ satisfies the relations $E_{ii}v_\lambda = \lambda_i v_\lambda$ and $E_{ij}v_\lambda = 0$ for $i < j$. There exists a nice isomorphism between the matrices E_{ij} and the (skew) Euler operators from Clifford analysis, namely $E_{ii} \mapsto \mathbb{E}_i + \frac{m}{2}$ and $E_{ij} \mapsto \langle u_i, \partial_j\rangle$, for all $i,j = 1,\ldots,k$ and $i \neq j$. For $k = 3$, this explicitly yields

$$\begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix} \rightarrow \begin{pmatrix} \mathbb{E}_1 + \frac{m}{2} & \langle u_1, \partial_2\rangle & \langle u_1, \partial_3\rangle \\ \langle u_3, \partial_1\rangle & \mathbb{E}_2 + \frac{m}{2} & \langle u_2, \partial_3\rangle \\ \langle u_3, \partial_1\rangle & \langle u_3, \partial_2\rangle & \mathbb{E}_3 + \frac{m}{2} \end{pmatrix}.$$

Now, observe that the vector space $\mathcal{S}_\lambda$ satisfies, as a whole, the conditions for a HWV; by this we mean that, technically speaking, we have as many copies as the dimension of this vector space. In other words, $\mathcal{S}_\lambda$ generates a $\mathfrak{gl}(k,\mathbb{C})$-module under the action of the negative root vectors, given by

$$\mathbb{V}(l_1,\ldots,l_k)^* \cong \left(l_1 + \frac{m}{2}, \cdots, l_k + \frac{m}{2}\right).$$

The upper index * is a shorthand notation for the shift of the HW over half the dimension, and will frequently be used in what follows. The following can then easily be proven, using the fact that $[\partial_i, E_{pq}] = \delta_{ip}\partial_q$ (for $i \neq j$).

Lemma 9.1. *Each element E_{ij} of the algebra $\mathfrak{gl}(k,\mathbb{C})$ acts as an endomorphism on the (total) space of monogenic polynomials in several variables.*

In other words, if $S(u_1,\ldots,u_k)$ is simplicial monogenic, then each spinor-valued polynomial of the form

$$\left[\sum_{(p_{ij})}\left(\prod_{i,j} E_{ij}^{p_{ij}}\right)\right] S(u_1,\cdots,u_k), \qquad p_{ij}\in\mathbb{N} \tag{9.1}$$

still is monogenic in several variables. The factor between brackets denotes an arbitrary word in the (skew) Euler operators generating $\mathfrak{gl}(k,\mathbb{C})$. This can also be formulated in the following way.

Lemma 9.2. *The elements of the universal enveloping algebra $\mathcal{U}\big(\mathfrak{gl}(k,\mathbb{C})\big)$ preserve the (total) space of monogenic polynomials in k vector variables.*

Moreover, *no other words* in $\mathcal{U}(\mathfrak{osp}(1,2k))$ have this property, which is a crucial observation. To explain what this means, remember that the vector variables $\{u_i\}_{i=1}^k$ and their corresponding Dirac operators $\{\partial_i\}_{i=1}^k$ generate a model for the Lie superalgebra $\mathfrak{osp}(1,2k)$. When decomposing polynomial vector spaces in k vector variables in terms of irreducible modules for the spin group, one needs two pieces of information: highest weights, referring to *which* summands to include, and the so-called embedding factors, referring to *how* to include these summands. Since these factors have to be polynomial invariants, we can easily list all those possibilities: they precisely correspond to products of elements in the algebra $\mathfrak{osp}(1,2k)$, i.e. elements in the algebra $\mathcal{U}\big(\mathfrak{osp}(1,2k)\big)$. For the general invariance theory, we refer to [89]. Next, the well-known PBW-theorem tells us that we can always rearrange these products according to a chosen ordering. Choosing the ordering on the generators of $\mathfrak{osp}(1,2k)$ such that

(i) first all combinations involving the vector variables only are listed

(ii) then all elements in $\mathfrak{gl}(k,\mathbb{C})$ are listed

(iii) finally all combinations involving Dirac operators only are listed,

it follows that the only elements in $\mathcal{U}(\mathfrak{osp}(1,2k))$, which can be used as embedding factors, are elements in $\mathcal{U}(\mathfrak{gl}(k,\mathbb{C}))$. Indeed: combinations involving type (iii) will always act trivially on the space of simplicial monogenics, whereas combinations involving type (i) will always belong to the Fischer complement of the space of monogenic polynomials. The latter statement is based on the fact that

$$\mathcal{P}(\mathbb{R}^{km},\mathbb{S}) = \mathcal{M}(\mathbb{R}^{km},\mathbb{S}) \oplus \Big(u_1\mathcal{P}(\mathbb{R}^{km},\mathbb{S}) + \cdots + u_k\mathcal{P}(\mathbb{R}^{km},\mathbb{S})\Big)$$

the sum between brackets obviously not being direct. We then are lead to the following important conclusion.

Proposition 9.1. *In order to decompose the polynomial vector space $\mathcal{M}_\lambda$ into irreducible modules for the spin group, it suffices to select all weight spaces having the correct degree of homogeneity inside each of the $\mathfrak{gl}(k,\mathbb{C})$-modules $\mathbb{V}(\lambda_1,\cdots,\lambda_k)^*$ generated by the spaces of simplicial monogenics $\mathcal{S}_{\lambda_1,\cdots,\lambda_k}$.*

Example 9.1. Despite the fact that the case $k=2$ is rather trivial, it still is useful to illustrate the procedure described above. Suppose that we want to decompose the vector space $\mathcal{M}_{l_1,l_2}$, $l_1 \geq l_2$. We then need to consider the $\mathfrak{gl}(2,\mathbb{C})$-modules generated by the spaces $\mathcal{S}_{p,q}$, $p \geq q$. The definition of $\mathcal{S}_{p,q}$ yields

$$\mathbb{V}(p,q)^* = \mathcal{S}_{p,q} \oplus \langle u_2,\partial_1\rangle\mathcal{S}_{p,q} \oplus \cdots \oplus \langle u_2,\partial_1\rangle^{p-q}\mathcal{S}_{p,q}$$

where it is easily verified that only a limited number of these modules will contribute to the space $\mathcal{M}_{l_1,l_2}$. Selecting the ones showing the correct degree of homogeneity, we thus indeed have that

$$\mathcal{M}_{l_1,l_2} = \mathcal{S}_{l_1,l_2} \oplus \langle u_2,\partial_1\rangle\mathcal{S}_{l_1+1,l_2-1} \oplus \cdots \oplus \langle u_2,\partial_1\rangle^{l_2}\mathcal{S}_{l_1+l_2,0}.$$

This result was already used in e.g. [20].

In the general case, the procedure becomes more complicated since the weight spaces in arbitrary $\mathfrak{gl}(k,\mathbb{C})$-modules (with $k>2$) occur with higher multiplicity, meaning that also the decomposition for $\mathcal{M}_\lambda$ will no longer be multiplicity-free.

As a direct consequence of Proposition 9.1, techniques from representation theory can be used for $\mathfrak{gl}(k+1,\mathbb{C})$ in order to obtain results on the space $\mathcal{M}^s_{l_0,\ldots,l_k}$.However we should take into account that *not all* $\mathbb{S}$-valued polynomials within the module $\mathbb{V}(l_0,\ldots,l_k)^*$ can be seen as type A solutions of $\mathcal{Q}_\lambda$, since only a specific subspace of it will show the right values. Hence, we still have to intersect the space of monogenics in several variables with the respective kernels of the operators E_{ij}, where $1 < i < j \leq k+1$. Here we need to add a remark on the notations: as we have included the additional vector variable x, formally denoted as u_0 (and ∂_x as ∂_0), the isomorphism between the matrices E_{ij} and the (skew) Euler operators has shifted to

$$E_{ii} \mapsto \mathbb{E}_{i-1} + \frac{m}{2}, \qquad E_{ij} \mapsto \langle u_{i-1},\partial_{j-1}\rangle, \qquad i,j = 1,\ldots,k+1,\ i \neq j.$$

Note that we thus needed to exclude E_{12} (corresponding to $\langle x, \partial_1\rangle$) from the intersection mentioned above, since this operator is not used to define the values. So, not all polynomials of the form (9.1) will contribute to the space $\mathcal{M}^s_{l_0,\ldots,l_k}$.It suffices to realise that the desired polynomials should satisfy the conditions to be a HWV for the algebra $\mathfrak{gl}(k,\mathbb{C})$, whence the language of branching may be used. To this end, we define the subspace $\mathbb{V}(\lambda)^+$ of $\mathbb{V}(\lambda) \equiv \mathbb{V}(\lambda_0,\ldots,\lambda_k)$, containing all HWV of the subalgebra $\mathfrak{gl}(k,\mathbb{C}) \subset \mathfrak{gl}(k+1,\mathbb{C})$:

$$\mathbb{V}(\lambda)^+ = \{\eta \in \mathbb{V}(\lambda) : E_{ij}\eta = 0,\ 1 < i < j \leq k+1\}.$$

Moreover, we introduce a notation for the set of weight spaces in $\mathbb{V}(\lambda)$ realising a copy of the $\mathfrak{gl}(k,\mathbb{C})$-module with highest weight $\mu = (\mu_1,\ldots,\mu_k)$. This means that for each of the elements in the previous set, a subscript μ is added referring to the $\mathfrak{gl}(k,\mathbb{C})$-module for which it actually defines a HWV, viz

$$\mathbb{V}(\lambda)^+_\mu = \{\eta \in \mathbb{V}(\lambda)^+ : E_{ii}\eta = \mu_{i-1}\eta,\ 1 < i \leq k+1\}.$$

As $\mathbb{V}(\lambda)$ is generated by the operators E_{ij} acting on the space $\mathcal{S}_\lambda$, each element $\eta \in \mathbb{V}(\lambda)^+_\mu$ is to be seen as a particular element of the form (9.1), with $S(x,u_1,\ldots,u_k) \in \mathcal{S}_\lambda$. Recall that the dimension of the spaces $\mathbb{V}(\lambda)^+_\mu$ either is 0 or 1, with

$$\dim\left(\mathbb{V}(\lambda)^+_\mu\right) = 1 \Leftrightarrow \lambda_{i-1} - \mu_i \in \mathbb{Z}^+ \text{ and } \mu_i - \lambda_i \in \mathbb{Z}^+, \text{ for all } i = 1,\ldots,k$$

which is called the *betweenness condition*, as it can be represented graphically –at least for integer values of λ_i or integer values shifted over half the dimension– by

$$\lambda_0 \geq \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \geq \lambda_{k-1} \geq \mu_k \geq \lambda_k.$$

In order to explain how this restricts the number of summands which can contribute to the space of type A solutions of a higher spin Dirac operator, let us consider an illustrative example with $k = 2$ and $(\lambda_0,\lambda_1,\lambda_2) = (4,3,1)^*$. According to the branching rules, when considering $\mathbb{V}(\lambda)$ as a $\mathfrak{gl}(2,\mathbb{C})$-module, only the following summands survive:

$$\mathbb{V}(4,3,1)^*\Big|^{\mathfrak{gl}(3,\mathbb{C})}_{\mathfrak{gl}(2,\mathbb{C})} \cong \Big((4,3)^* \oplus (4,2)^* \oplus (4,1)^*\Big) \oplus \Big((3,3)^* \oplus (3,2)^* \oplus (3,1)^*\Big). \quad (9.2)$$

In the above expression each of the terms between brackets stands for a combination of the following form, written in terms of the negative root vectors for $\mathfrak{gl}(3,\mathbb{C})$:

$$\Big(\sum_{a,b,c} E_{21}^a E_{31}^b E_{32}^c\Big)\mathcal{S}_{4,3,1}.$$

Moreover, the result should still belong to $\ker(E_{23})$, with E_{23} the unique positive root vector characterising the algebra $\mathfrak{gl}(2,\mathbb{C}) \subset \mathfrak{gl}(3,\mathbb{C})$. The algebra $\mathfrak{gl}(2,\mathbb{C})$ has Cartan elements E_{22} and E_{33}, meaning that the six couples of integers above are in fact the degrees of homogeneity in (u_1, u_2). In this way limitations on the degree of homogeneity of the embedding factors are obtained. Moreover, looking at the summands above (or at the betweenness condition for the most general case), it is clear that none of the embedding factors will have an effect of the form $(\pm 1, \mp 1)$ on the degree in (u_1, u_2). So there is no need to include the factor E_{32}, which corresponds to the final result having to be in $\ker(E_{23})$. As we will see in the next section, this statement is not yet precise: we will prove that E_{32} *can* occur, but taking into account homogeneities, the embedding factor *as a whole* will always behave as the term $E_{21}^a E_{31}^b$, which, in some sense, is the leading term. For example, in order to have that $(E_{21}^a E_{31}^b)\mathcal{S}_{4,3,1}$ corresponds to $(4, 2)$ we must have that $(3 + a, 1 + b) = (4, 2)$, or $(a, b) = (1, 1)$. In other words: the branching rules tell us which degrees of homogeneity to expect for the (leading term in the) embedding factors.

We may now formulate the following general result.

Proposition 9.2. *For each vector space $\mathcal{S}_{\lambda_0,\cdots,\lambda_k}$, the only summands inside the $\mathfrak{gl}(k+1,\mathbb{C})$-module $\mathbb{V}(\lambda_0,\cdots,\lambda_k)^*$ contributing to the space of type A solutions of the higher spin Dirac operator in k dummy vector variables are of the form*

$$\rho_{d_1,\cdots,d_k}\mathcal{S}_{\lambda_0,\cdots,\lambda_k}$$

where $\rho_{d_1,\cdots,d_k} \in \mathcal{U}\big(\mathfrak{gl}(k+1,\mathbb{C})\big)$ is an embedding factor which is homogeneous of degree $(d_1,\cdots,d_k)$ in $(u_1,\cdots,u_k)$. Moreover, the integers d_j satisfy the following conditions:

$$\lambda_0 \geq \lambda_1 + d_1 \geq \lambda_1 \geq \lambda_2 + d_2 \geq \cdots \geq \lambda_{k-1} \geq \lambda_k + d_k \geq \lambda_k$$

or $0 \leq d_p \leq \lambda_{p-1} - \lambda_p$ (with $1 \leq p \leq k$). These conditions follow from the branching rules.

In the next section, an explicit form for these embedding factors $\rho_{d_1,\cdots,d_k}$ is obtained, using results on raising and lowering operators in transvector algebras. Note that these factors will be unique up to a constant, which follows from the fact that the branching from $\mathfrak{gl}(k+1,\mathbb{C})$ to $\mathfrak{gl}(k,\mathbb{C})$ is multiplicity-free.

Example 9.2. Suppose we want to describe the space $\mathcal{M}^s_{3,1,1}$, i.e. the space of 3-homogeneous type A solutions of the operator $\mathcal{Q}_{1,1}$, studied in [18]. This is the invariant operator acting on spinor-valued functions, see also [72]. Hence we are looking for 3-tuples of integers $(\lambda_0, \lambda_1, \lambda_2)^*$ such that $\rho_{a,b}\,\mathcal{S}_{\lambda_0,\lambda_1,\lambda_2} \subset \mathcal{M}^s_{3,1,1}$, which means that the following conditions have to be satisfied:

$$(\lambda_0 - a - b, \lambda_1 + a, \lambda_2 + b) = (3,1,1) \quad \text{and} \quad \begin{cases} \lambda_0 - \lambda_1 & \geq & a & \geq & 0 \\ \lambda_1 - \lambda_2 & \geq & b & \geq & 0. \end{cases}$$

Now, obviously $(a, b) = (0, 0)$ leads to the summand $\mathcal{S}_{3,1,1} \subset \mathcal{M}^s_{3,1,1}$, as was to be expected, since, in general, the solution $d_1 = \cdots = d_k = 0$ will always be there. Any other solution is non-trivial, which means that $\lambda_0 > 3$. As $\lambda_1 \geq \lambda_2$, the only other possibility is $(a, b) = (0, 1)$. Note that $(\lambda_0, \lambda_1, \lambda_2) = (5, 0, 0)$ is not allowed, as follows from the condition on b. This means that $\mathcal{M}^s_{3,1,1} \cong \mathcal{S}_{3,1,1} \oplus \mathcal{S}_{4,1}$, which corresponds to the results of [18].

Let us now formulate the main conclusion of this section.

Theorem 9.1. *As a module for the spin group, the space $\mathcal{M}^s_{l_0,\cdots,l_k}$, with $l_0 \geq l_1$ decomposes into the following irreducible summands:*

$$\mathcal{M}^s_{l_0,\cdots,l_k} = \bigoplus_{(d_1,\cdots,d_k)} \rho_{d_1,\cdots,d_k} \mathcal{S}_{\lambda_0,\cdots,\lambda_k}$$

where $(\lambda_0, \cdots, \lambda_k)'$ is a dominant weight satisfying

$$(\lambda_0, \lambda_1, \cdots, \lambda_k) = (l_0 + \sum_{i=1}^{k} d_i, l_1 - d_1, \cdots, l_k - d_k)$$

with $l_i - l_{i+1} \geq d_i \geq 0$ for $1 \leq i \leq k-1$ and $0 \leq d_k \leq l_k$. At the same time, this is the decomposition of the space of l_0-homogeneous type A solutions of the operator $\mathcal{Q}_\lambda$.

Proof. First, it follows from the branching rules for $\mathfrak{gl}(k+1, \mathbb{C})$ to $\mathfrak{gl}(k, \mathbb{C})$ that no embedding factor $\rho_{d_1,\cdots,d_k}$ can have a net effect of the form $(\pm p, \mp p)$ on the homogeneity degree in two variables (u_i, u_j), with $i, j \geq 1$ and $p \in \mathbb{N}$. Indeed:

$$(\lambda_1, \cdots, \lambda_k)^* \subset (\lambda_0, \cdots, \lambda_k)^* \Big|^{\mathfrak{gl}(k+1,\mathbb{C})}_{\mathfrak{gl}(k,\mathbb{C})}$$

and any other summand which comes from the branching is obtained by adding positive integers $d_1, \ldots, d_k$ to resp. $\lambda_1, \ldots, \lambda_k$. This implies that the net effect of the factor $\rho_{d_1,\cdots,d_k}$ can always be represented as a leading term of the form $\rho_{d_1,\cdots,d_k} = E_{21}^{d_1} \cdots E_{(k+1)1}^{d_k} + \cdots$, where the numbers $(d_1, \cdots, d_k)$ satisfy the betweenness conditions coming from the branching. If we then fix the numbers $(l_0, \cdots, l_k)$, it suffices to find all the $(k+1)$-tuples $(\lambda_0, \cdots, \lambda_k)$ for which there exist positive integers d_j such that we have an inclusion

$$\rho_{d_1,\cdots,d_k} \mathcal{S}_{\lambda_0,\cdots,\lambda_k} \subset \mathcal{M}_{l_0,\cdots,l_k}.$$

This is only possible if the conditions

$$\Big(\lambda_0 - \sum_{i=1}^{k} d_i, \lambda_1 + d_1, \cdots, \lambda_k + d_k\Big) = (l_0, \cdots, l_k)$$

on the degrees of homogeneity are satisfied, and if moreover

$$\begin{cases} \lambda_0 - \lambda_1 & \geq & d_1 & \geq & 0 \\ \lambda_1 - \lambda_2 & \geq & d_2 & \geq & 0 \\ & \vdots & & & \\ \lambda_{k-1} - \lambda_k & \geq & d_k & \geq & 0. \end{cases}$$

These are the conditions coming from the branching rules. Using the restrictions on the homogeneity, this can also be rewritten as $l_i - l_{i+1} \geq d_i \geq 0$, for all $1 \leq i < k$, and $\lambda_k = l_k - d_k$. This last expression tells us that $0 \leq d_k \leq l_k$. □

9.2 Relation with transvector algebras

The aim of this section is to obtain explicit expressions for the embedding factors $\rho_{d_1,\cdots,d_k}$, i.e. the elements in $\mathcal{U}\big(\mathfrak{gl}(k+1,\mathbb{C})\big)$ realising the decomposition of the space $\mathcal{M}^s_{l_0,\cdots,l_k}$ into irreducible summands under the spin group.

Let us therefore take a look at the Mickelsson algebra $S(\mathfrak{gl}(k+1,\mathbb{C}), \mathfrak{gl}(k,\mathbb{C}))$, constructed as explained in Chapter 4. The generators of this algebra are given by the elements z_{i1} and z_{1i}, $i = 2, \ldots, k+1$:

$$\begin{aligned} z_{i1} &= p_{\mathfrak{gl}(k,\mathbb{C})} E_{i1} \\ &= \sum_{i > i_1 > \cdots > i_s > 1} E_{ii_1} E_{i_1 i_2} \cdots E_{i_{s-1} i_s} E_{i_s 1} (h_i - h_{j_1}) \cdots (h_i - h_{j_r}) \\ z_{1i} &= p_{\mathfrak{gl}(k,\mathbb{C})} E_{1i} \\ &= \sum_{i < i_1 < \cdots < i_s \leq k+1} E_{i_1 i} E_{i_2 i_1} \cdots E_{i_s i_{s-1}} E_{1 i_s} (h_i - h_{j_1}) \cdots (h_i - h_{j_r}). \end{aligned}$$

In these definitions, s runs over nonnegative integers, $h_i = E_{ii} - i + 1$ and $\{j_1, \ldots, j_r\}$ is the complementary subset to $\{i_1, \ldots, i_s\}$ in the set $\{1, \ldots i-1\}$ or $\{i+1, \ldots, k+1\}$. For example, when $k = 3$ we have that

$$z_{41} = E_{41}(h_4 - h_2)(h_4 - h_3) + E_{43}E_{31}(h_4 - h_2) + E_{42}E_{21}(h_4 - h_3) + E_{43}E_{32}E_{21}.$$

The properties of the extremal projector then lead to the following lemma.

Lemma 9.3. *Let $\eta \in \mathbb{V}(\lambda)^+_\mu$, $\mu = (\mu_1, \ldots, \mu_k)$. Then, for any $i = 2, \ldots, k+1$, we have*

$$z_{i1}\eta \in \mathbb{V}(\lambda)^+_{\mu+\delta_{i-1}}, \qquad z_{1i}\eta \in \mathbb{V}(\lambda)^+_{\mu-\delta_{i-1}}$$

where the weight $\mu \pm \delta_{i-1}$ is obtained from μ by replacing μ_{i-1} by $\mu_{i-1} \pm 1$.

This was proven in [66] and explained in Chapter 4. In the present setting of solutions of higher spin operators, the lemma can be reformulated as follows: the operators z_{i1} and z_{1i}, $i = 2, \ldots, k+1$ will map a type A solution of a higher spin Dirac operator to another type A solution (be it for another operator, since the degree of homogeneity will change). More explicitly, the following results hold.

Corollary 9.1. *For every polynomial $P(x; u_{(k)}) \in \mathcal{M}^s_{l_0,\cdots,l_k}$, we have*

$$\begin{aligned}
z_{i1}P(x; u_1, \cdots, u_k) &\in \mathcal{M}^s_{l_0-1,l_1,\cdots,l_{i-2},l_{i-1}+1,l_i,\cdots,l_k}\\
z_{1i}P(x; u_1, \cdots, u_k) &\in \mathcal{M}^s_{l_0+1,l_1,\cdots,l_{i-2},l_{i-1}-1,l_i,\cdots,l_k}.
\end{aligned}$$

Example 9.3. When $k = 2$, we have that

$$\begin{aligned}
z_{21} &= E_{21} = \langle u_1, \partial_x\rangle\\
z_{31} &= E_{32}E_{21} + E_{31}(h_3 - h_2) = \langle u_2, \partial_1\rangle\langle u_1, \partial_x\rangle + \langle u_2, \partial_x\rangle(\mathbb{E}_2 - \mathbb{E}_1 - 1)\\
z_{12} &= E_{32}E_{13} + E_{12}(h_2 - h_3) = \langle u_2, \partial_1\rangle\langle x, \partial_2\rangle + \langle x, \partial_1\rangle(\mathbb{E}_1 - \mathbb{E}_2 + 1)\\
z_{13} &= E_{13} = \langle x, \partial_2\rangle.
\end{aligned} \tag{9.3}$$

In view of Lemma 9.3, z_{31} raises the degree in u_2 by one. Reconsidering the space $\mathcal{M}^s_{3,1,1}$, we can now write its direct sum decomposition in terms of the explicit embedding factors:

$$\mathcal{M}^s_{3,1,1} = \mathcal{S}_{3,1,1} \oplus (\langle u_2, \partial_1\rangle\langle u_1, \partial_x\rangle + \langle u_2, \partial_x\rangle(\mathbb{E}_2 - \mathbb{E}_1 - 1))\mathcal{S}_{4,1}.$$

The Euler operators will only produce multiplicative constants, since they act on homogeneous polynomials. In this way, we also see the aforementioned leading terms in the example, up to a multiplicative constant.

Lemma 9.4. *Let μ satisfy the betweenness condition stated above, and let v_λ be the highest weight vector of the module $\mathbb{V}(\lambda)$. Then the elements*

$$v_\lambda(\mu) := z_{21}^{d_1} \cdots z_{(k+1)1}^{d_k} v_\lambda$$

are nonzero, provided that $(d_1, \cdots, d_k)$ satisfies all conditions of Theorem 9.1. Moreover, the space $\mathbb{V}(\lambda)^+$ is spanned by these elements $v_\lambda(\mu)$.

Example 9.4. As before, take $k = 2$ and $\lambda = (4, 3, 1)^*$, and consider the module $\mathbb{V}(4, 3, 1)^*$ generated by the space $\mathcal{S}_{4,3,1}$. Lemma 9.4 then states that consecutive actions of the operators z_{21} and z_{31} will produce a basis of the space $\mathbb{V}(4, 3, 1)^* \cap \ker\langle u_1, \partial_2\rangle$. More precisely, we obtain the following spaces, corresponding to the 6 possible choices for μ, and the respective spaces of higher spin solutions to which they contribute, see (9.2):

$$\begin{array}{l|l} \mathcal{S}_{4,3,1} & \mathcal{M}^s(4,3,1) \\ z_{21}\mathcal{S}_{4,3,1} & \mathcal{M}^s(3,4,1) \\ z_{31}\mathcal{S}_{4,3,1} & \mathcal{M}^s(3,3,2) \\ z_{21}z_{31}\mathcal{S}_{4,3,1} & \mathcal{M}^s(2,4,2) \\ z_{31}^2\mathcal{S}_{4,3,1} & \mathcal{M}^s(2,3,3) \\ z_{21}z_{31}^2\mathcal{S}_{4,3,1} & \mathcal{M}^s(1,4,3). \end{array}$$

Note however that this is *not* the decomposition of $\mathcal{M}^s_{4,3,1}$. Indeed, using the correct embedding factors, we get that the latter is equal to

$$\mathcal{M}^s_{4,3,1} = \mathcal{S}_{4,3,1} \oplus z_{21}\mathcal{S}_{5,2,1} \oplus z_{21}^2\mathcal{S}_{6,1,1} \oplus z_{31}\mathcal{S}_{5,3,0} \oplus z_{21}z_{31}\mathcal{S}_{6,2,0} \oplus z_{21}^2 z_{31}\mathcal{S}_{7,1,0}.$$

Recall that the embedding factor, as a whole, should behave as $E_{21}^a E_{31}^b$, with this term itself as a leading term. This might not be so obvious from the definitions and lemmas stated above. Note though that the operators z_{i1} actually are defined up to a constant factor. We can also use the corresponding generators of the transvector algebra $Z(\mathfrak{gl}(k+1,\mathbb{C}),\mathfrak{gl}(k,\mathbb{C}))$ by using the field of fractions $R(\mathfrak{h})$. Hence, it is possible to divide z_{i1} by $(h_i - h_{i-1})\ldots(h_i - h_2)$, whence the resulting operators s_{i1} (and likewise s_{1i}) take the form

$$\begin{aligned} s_{i1} &= \sum_{i>i_1>\cdots>i_s>1} E_{ii_1}E_{i_1i_2}\cdots E_{i_{s-1}i_s}E_{i_s1}\frac{1}{(h_i-h_{i_1})\cdots(h_i-h_{i_s})} \\ s_{1i} &= \sum_{i<i_1<\cdots<i_s\leq k+1} E_{ii_1}E_{i_1i_2}\cdots E_{i_{s-1}i_s}E_{i_s1}\frac{1}{(h_i-h_{i_1})\cdots(h_i-h_{i_s})} \end{aligned}$$

or still $s_{i1} = E_{i1} +$ other operators, which proves the statement: it is now easily seen that powers of the operators s_{i1} or s_{1i} indeed behave as the leading terms predicted earlier. For instance, after rescaling, the four operators in (9.3) become

$$\begin{cases} s_{21} &= \langle u_1, \partial_x\rangle \\ s_{31} &= \langle u_2, \partial_x\rangle + \langle u_2, \partial_1\rangle\langle u_1, \partial_x\rangle \dfrac{1}{\mathbb{E}_2 - \mathbb{E}_1 - 1} \\ s_{12} &= \langle x, \partial_1\rangle + \langle u_2, \partial_1\rangle\langle x, \partial_2\rangle \dfrac{1}{\mathbb{E}_1 - \mathbb{E}_2 + 1} \\ s_{13} &= \langle x, \partial_2\rangle. \end{cases}$$

So, the embedding factors defined in Proposition 9.2, are given by

$$\rho_{d_1,d_2,\cdots,d_k} = s_{21}^{d_1}\cdots s_{(k+1)1}^{d_k},$$

in accordance with Lemma 9.4.

Mathematicians may flatter themselves that they possess new ideas which mere human language is as yet unable to express.

James Clerk Maxwell

10 The kernel of $\mathcal{Q}_\lambda$

The aim of the present chapter is to study the vector space of polynomial solutions for arbitrary HSD operators as a spin group module. This is a non-trivial problem, which was already treated for the Rarita-Schwinger operators $\mathcal{R}_{l_1}$ and $\mathcal{Q}_{l_1,l_2}$ in respectively [20] and [15], but still remains an open problem for the most general operator $\mathcal{Q}_{l_1,\ldots,l_k}$. The main problem lies in the fact that the space of polynomial solutions of a general HSD operator shows a completely different structure than the one of the classical Dirac operator. Whereas the space of polynomial solutions for the latter defines a model for an *irreducible* spin representation, solution spaces for the former are highly reducible and need to be decomposed into several summands.

In order to tackle this problem, we repeat the definitions of higher spin Dirac and higher spin twistor operators in Section 1 in order to prove some commutation relations between such operators. Next, we recapitulate the known results for the RS operator and the HSD operator of order two in Section 2 so that we can postulate a conjecture for the general case. In a following step, we treat the third order case, and explain the general approach by means of this example in Section 3, in order to be able to discuss the most general case in Section 4. In the fifth section, an inductive argument will be used to formulate a conjecture describing arbitrary solutions (reducing it to a combinatorial problem), and in Section 6 we will illustrate how this can be verified up to order 3 with the help of computer algebra.

10.1 Higher spin operators

Remember from Chapter 4 that the HSD operators are defined as the operators $p_{\mathfrak{osp}(1,2k)}\partial_x$, and the twistor operators as the operators $p_{\mathfrak{osp}(1,2k)}\langle \partial_x, \partial_a\rangle$, with $1 \leq a \leq k$:

Definition 10.1. *For an arbitrary highest weight $\lambda' = (l_1, \ldots, l_k)'$ with $l_k > 0$, one can define the HSD operators*

$$\mathcal{Q}_\lambda : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda).$$

The (reduced) explicit form for these operators is given by

$$\mathcal{Q}_\lambda = p_{\mathfrak{osp}(1,2k)}\partial_x = \prod_{i=1}^{k}\left(1 + \frac{u_i\partial_i}{m + 2\mathbb{E}_i - 2i}\right)\partial_x.$$

This product is understood to be ordered, with increasing indices from the left to the right.

Definition 10.2. *For an arbitrary half-integer highest weight $\lambda = (l_1, \ldots, l_k)'$ with $l_k > 0$, one can define the HST operators*

$$\mathcal{T}_\lambda^{(j)} : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1,\ldots,l_{j-1},l_j-1,l_{j+1},\ldots,l_k}),$$

provided $l_j > l_{j+1}$. The upper index (j) hereby refers to the variable in which the degree of homogeneity will decrease. Their explicit form is, up to a multiplicative constant, given by

$$\begin{aligned}\mathcal{T}_\lambda^{(j)} &:= p_{\mathfrak{osp}(1,2k)}\langle \partial_x, \partial_j\rangle \\ &= \prod_{p=j+1}^{k}\left(1 + \frac{1}{\mathbb{E}_p - \mathbb{E}_j + j - (p+1)}\langle u_p, \partial_j\rangle\langle u_j, \partial_p\rangle\right)\langle \partial_j, \partial_x\rangle,\end{aligned}$$

for $j < k$ and $\mathcal{T}_\lambda^{(k)} = \langle \partial_k, \partial_x\rangle$. Note that this product is also understood to be ordered, with increasing indices from left to right. The Euler operators in the denominator again automatically introduce constants l_j in case a fixed λ is chosen.

Remark 10.1. Because of the Euler operators appearing in the explicit formulae, the operators $\mathcal{T}_\lambda^{(j)}$ essentially are independent of λ. For this reason, we will from now on also use the notations $\mathcal{T}^{(j)} := \mathcal{T}_\lambda^{(j)}$ for the HST operators, and $\mathcal{Q} := \mathcal{Q}_\lambda$ for HSD operators, unless it is essential to know which space the operators are acting on. This will considerably reduce the notational load. Essentially this means that $\mathcal{Q}$ is in fact the direct sum of all $\mathcal{Q}_\lambda$ and similar for the twistor operators.

In general, these operators have nice commutation relations, stemming from the fact that they are generators of the transvector algebra mentioned earlier. We will explicitly prove those relations which will play a crucial role in what follows.

Lemma 10.1. *For all $a < b$, we have the relation*

$$\mathcal{T}^{(b)}\mathcal{T}^{(a)} = \mathcal{T}^{(a)}\mathcal{T}^{(b)}\frac{\mathbb{E}_a - \mathbb{E}_b + b - a + 1}{\mathbb{E}_a - \mathbb{E}_b + b - a}, \tag{10.1}$$

which means that HST operators commute up to a coefficient in $R(\mathfrak{h})$.

Proof. When expanding the product in the expression of $\mathcal{T}^{(a)}$, we get that

$$\mathcal{T}^{(a)} = \langle \partial_a, \partial_x \rangle + \sum_{a<i_1<\cdots<i_s\leq k} \frac{\langle u_{i_1}, \partial_a\rangle\langle u_{i_2}, \partial_{i_1}\rangle \cdots \langle u_{i_s}, \partial_{i_{s-1}}\rangle\langle \partial_{i_s}, \partial_x\rangle}{(\mathbb{E}_a - \mathbb{E}_{i_1} + i_1 - a)\cdots(\mathbb{E}_a - \mathbb{E}_{i_s} + i_s - a)}. \tag{10.2}$$

Assuming that $a < b$, we notice that because of the properties of the extremal projector and the fact that $\langle \partial_a, \partial_x \rangle$ commutes with each factor of the projector in $\mathcal{T}^{(b)}$ in its simplest form (see Definition 10.2), we get

$$p_{\mathfrak{osp}(1,2k)}\langle \partial_a, \partial_x \rangle p_{\mathfrak{osp}(1,2k)}\langle \partial_b, \partial_x \rangle = p_{\mathfrak{osp}(1,2k)}\langle \partial_a, \partial_x \rangle\langle \partial_b, \partial_x \rangle. \tag{10.3}$$

Still keeping in mind that the extremal projector $p_{\mathfrak{osp}(1,2k)}$ has the property

$$p_{\mathfrak{osp}(1,2k)}\langle u_j, \partial_i \rangle = 0,$$

and using (10.2) for all $i < j$, a straightforward calculation yields

$$p_{\mathfrak{osp}(1,2k)}\langle \partial_b, \partial_x \rangle p_{\mathfrak{osp}(1,2k)}\langle \partial_a, \partial_x \rangle = p_{\mathfrak{osp}(1,2k)}\langle \partial_a, \partial_x \rangle\langle \partial_b, \partial_x \rangle\frac{\mathbb{E}_a - \mathbb{E}_b + b - a + 1}{\mathbb{E}_a - \mathbb{E}_b + b - a}. \tag{10.4}$$

Thus, if $a \leq b$, by combining (10.3) and (10.4), we arrive at

$$\mathcal{T}^{(b)}\mathcal{T}^{(a)} = \mathcal{T}^{(a)}\mathcal{T}^{(b)}\frac{\mathbb{E}_a - \mathbb{E}_b + b - a + 1}{\mathbb{E}_a - \mathbb{E}_b + b - a},$$

as was to be proven. □

Remark 10.2. In Lemma 10.1, when saying that HST operators commute, we merely mean that the expressions of the twistor operators commute up to a coefficient in $R(\mathfrak{h})$, as elements of the transvector algebra. When acting on functions, the operators on the LHS and RHS of (10.1) are in fact different operators, as they act on polynomials of a different degree of homogeneity. The same remark also holds for the following lemma.

Lemma 10.2. *We have the relation*

$$\mathcal{T}^{(a)}\mathcal{Q} = \frac{m+\mathbb{E}_a - 2a}{m+\mathbb{E}_a-2a+2}\mathcal{Q}\mathcal{T}^{(a)},$$

meaning that HST operators and HSD operators commute up to a coefficient in $R(\mathfrak{h})$.

Proof. On the one hand we have that

$$\mathcal{Q}\mathcal{T}^{(a)} = p_{\mathfrak{osp}(1,2k)}\partial_x p_{\mathfrak{osp}(1,2k)}\langle \partial_a, \partial_x\rangle = p_{\mathfrak{osp}(1,2k)}\partial_x\langle \partial_a, \partial_x\rangle,$$

since $\langle u_j, \partial_i\rangle$ and ∂_x commute, and $p_{\mathfrak{osp}(1,2k)}\langle u_j, \partial_i\rangle = 0$ for all $i<j$. On the other hand, a straightforward calculation shows that

$$\begin{aligned} & p_{\mathfrak{osp}(1,2k)}\langle \partial_a, \partial_x\rangle p_{\mathfrak{osp}(1,2k)}\partial_x \\ = \quad & p_{\mathfrak{osp}(1,2k)}\langle \partial_a, \partial_x\rangle \prod_{i=a}^{k}\left(1+\frac{u_i\partial_i}{m+2\mathbb{E}_i-2i}\right)\partial_x \\ = \quad & \frac{m+\mathbb{E}_a-2a}{m+\mathbb{E}_a-2a+2} p_{\mathfrak{osp}(1,2k)}\langle \partial_a, \partial_x\rangle\partial_x, \end{aligned}$$

again using the properties of the extremal projector. □

10.2 Known kernel decompositions

From now on, we will fix an arbitrary highest weight $\lambda' = (l_1, \ldots, l_k)'$ with $k > 2$ and focus on the following problem:

> *How can one decompose the space*
>
> $$\mathcal{K}_{h;\lambda} := \mathcal{P}_h(\mathbb{R}^m, \mathcal{S}_\lambda) \cap \ker \mathcal{Q}_\lambda$$
>
> *as a (highly reducible) module under the regular action of the algebra* $\mathfrak{so}(m)$, *with* $h \in \mathbb{N}$ *the degree of homogeneity of the polynomial solutions (i.e. in* $x \in \mathbb{R}^m$*)?*

Sometimes, it will be necessary to explicitly attach the highest weight as an index (i.e. in vector notation), but in that case we will omit the prime in order not to overload the notations:

$$\mathcal{K}_{h;\lambda} = \mathcal{K}_{h;(l_1,\ldots,l_k)}.$$

Note that we exclude $k \in \{1, 2\}$ as these polynomial kernel spaces have been described in respectively [20] and [15]. We briefly recall these results to illustrate the type of result we are after.

Theorem 10.1. *For all integers $h \geq l_1 > 0$, the kernel of the Rarita-Schwinger operator $\mathcal{R}_{l_1}$ decomposes as follows:*

$$\begin{aligned}
\mathcal{K}_{h;(l_1)} &:= \mathcal{P}_h(\mathbb{R}^m, \mathcal{M}_{l_1}) \cap \ker \mathcal{R}_{l_1} \\
&\cong \left(\mathcal{K}_{h;(l_1)} \cap \ker \mathcal{T}^{(1)}\right) \\
&\quad \oplus \bigoplus_{j_1=1}^{l_1} \big(\mathcal{K}_{h;(l_1)} \cap \ker \left(\mathcal{T}^{(1)}\right)^{j_1+1}\big) / \big(\mathcal{K}_{h;(l_1)} \cap \ker \left(\mathcal{T}^{(1)}\right)^{j_1}\big) \\
&\cong \bigoplus_{j_1=0}^{l_1} \mathcal{M}^s_{h-j_1;l_1-j_1} \\
&= \bigoplus_{i_1=0}^{l_1} \mathcal{M}^s_{h-l_1+i_1;i_1}.
\end{aligned}$$

We can visualise the latter sum as a line of dots on an axis, where each dot represents the space $\mathcal{M}^s_{h-l_1+i_1;i_1}$ (see Figure 10.1).

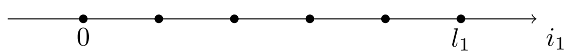


Figure 10.1: The kernel space of $\mathcal{R}_{l_1}$, each dot representing a space $\mathcal{M}^s_{h-l_1+i_1;i_1}$

Remark 10.3. From Theorem 10.1 it follows that this approach only applies to the case where $h \geq l_1$. For the case where $h < l_1$, we refer to Chapter 11.

Theorem 10.2. *For all highest weights $\lambda' = (l_1, l_2)'$ and integers $h \geq l_1+l_2$, the kernel of the HSD operator $\mathcal{Q}_\lambda$ decomposes as follows:*

$$\begin{aligned}
\mathcal{K}_{h;(l_1,l_2)} &:= \mathcal{P}_h(\mathbb{R}^m, \mathcal{S}_\lambda) \cap \ker \mathcal{Q}_\lambda \\
&\cong \left(\mathcal{K}_{h;(l_1,l_2)} \cap \ker \mathcal{T}^{(2)}\right) \\
&\quad \oplus \bigoplus_{j_2=1}^{l_2} \big(\mathcal{K}_{h;(l_1,l_2)} \cap \ker \left(\mathcal{T}^{(2)}\right)^{j_2+1}\big) / \big(\mathcal{K}_{h;(l_1,l_2)} \cap \ker \left(\mathcal{T}^{(2)}\right)^{j_2}\big) \\
&\cong \bigoplus_{j_1=0}^{l_1-l_2} \bigoplus_{j_2=0}^{l_2} \mathcal{M}^s_{h-j_1-j_2;l_1-j_1,l_2-j_2} \\
&= \bigoplus_{i_1=l_2}^{l_1} \bigoplus_{i_2=0}^{l_2} \mathcal{M}^s_{h-l_1-l_2+i_1+i_2;i_1,i_2}.
\end{aligned}$$

The second sum is easier to interpret, as the summation indices i_p represent the degree of homogeneity in u_p of the polynomials in the spaces contained in the direct sum. We can nicely visualise this sum in the rectangular grid shown in Figure 10.2, each dot representing one of the summands:

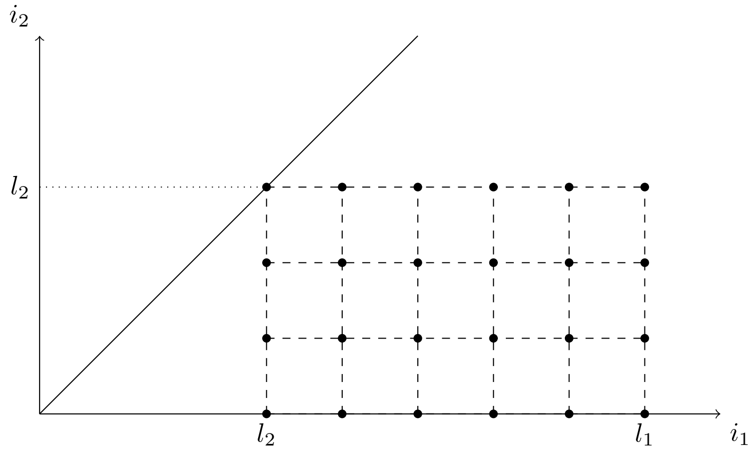


Figure 10.2: The kernel space of $\mathcal{Q}_{l_1,l_2}$, each dot representing a space of type A solutions

By comparing the Figures 10.1 and 10.2, we see a cuboid grid structure appearing in a 1-dimensional and a 2-dimensional space, respectively. This emerging pattern raises the question whether this would be true in general, allowing us to formulate a proposition for general highest weights λ'.

Proposition 10.1. *For all highest weights $\lambda' = (l_1, \dots, l_k)'$ and integers $h \geq l_1 + l_2$, the kernel of the HSD operator $\mathcal{Q}_\lambda$ decomposes as follows:*

$$\begin{aligned} \mathcal{K}_{h;\lambda} &:= \mathcal{P}_h(\mathbb{R}^m, \mathcal{S}_\lambda) \cap \ker \mathcal{Q}_\lambda \\ &\cong \bigoplus_{j_1=0}^{l_1-l_2} \cdots \bigoplus_{j_{k-1}=0}^{l_{k-1}-l_k} \bigoplus_{j_k=0}^{l_k} \mathcal{M}^s_{h-\sum_{p=1}^k j_p; l_1-j_1,\dots,l_k-j_k} \\ &\cong \bigoplus_{i_1=l_2}^{l_1} \cdots \bigoplus_{i_{k-1}=l_k}^{l_{k-1}} \bigoplus_{i_k=0}^{l_k} \mathcal{M}^s_{h-\sum_{p=1}^k (l_p-i_p); i_1,\dots,i_k}. \end{aligned} \tag{10.5}$$

Remark 10.4. The condition $h \geq l_1 + l_2$ is slightly surprising, as one might have expected the condition $h \geq l_1 + \cdots + l_k$ in the most general case. In order to explain why it indeed is the former, it suffices to note that for each term in the direct sum to exists, the condition of Theorem 9.1 should hold, which here translates into $h - \sum_{p=1}^k j_p \geq l_1 - j_1$ or $h \geq l_1 + \sum_{p=2}^k j_p$. Due to the boundaries in the direct sum for k_p ($p \geq 2$), the latter sum cannot exceed l_2.

In the remainder of this chapter, this is what we will investigate. Since the general case might be hard to grasp right away, we will illustrate our approach for the case $k = 3$ first, and discuss $k > 3$ afterwards.

10.3 The case $k = 3$

We will prove that $\ker_h \mathcal{Q}_{l_1,l_2,l_3}$ has a cuboid structure predicted in (10.5), which is visualised in Figure 10.3. Throughout this subsection, we will assume $\lambda = (l_1, l_2, l_3)$, with $l_1 \geq l_2 \geq l_3 \geq 0$, and $h \geq l_1 + l_2$. We will define a grading onto this kernel space, by exploiting the different twistor operators. A first grading will be given by the twistor operator $\mathcal{T}^{(3)}$. To this end, let us introduce the following spaces.

Definition 10.3. *For arbitrary highest weights $\lambda' = (l_1, l_2, l_3)'$ with $l_3 > 0$, we put:*

$$\begin{aligned} \mathcal{K}^{(0)}_{h;\lambda} &:= \mathcal{K}_{h;\lambda} \cap \ker \mathcal{T}^{(3)} \\ \mathcal{K}^{(j_3)}_{h;\lambda} &:= \big(\mathcal{K}_{h;\lambda} \cap \ker (\mathcal{T}^{(3)})^{j_3+1}\big) / \big(\mathcal{K}_{h;\lambda} \cap \ker (\mathcal{T}^{(3)})^{j_3}\big). \end{aligned}$$

We then have the following lemma.

Lemma 10.3. *For all $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda)$, we have that*

$$\left(\mathcal{T}^{(3)}\right)^{l_3+1} f = 0.$$

Proof. Since $\deg_{u_3}(f) = l_3$, and $\mathcal{T}^{(3)}$ lowers the degree in u_3 by 1, this is obviously true. □

Remark 10.5. Keep in mind that a 'power' of a twistor operator is just a notation, as each consecutive twistor operator acts on a different space (since the degree in u_3 is lowered by one each time).

This lemma allows for a decomposition of the form

$$\mathcal{K}_{h;\lambda} \cong \bigoplus_{j_3=0}^{l_3} \mathcal{K}^{(j_3)}_{h;\lambda}. \tag{10.6}$$

Assuming for now that we indeed will get a box structure, this decomposition can be visualised as in Figure 10.4, where the cuboid structure is split

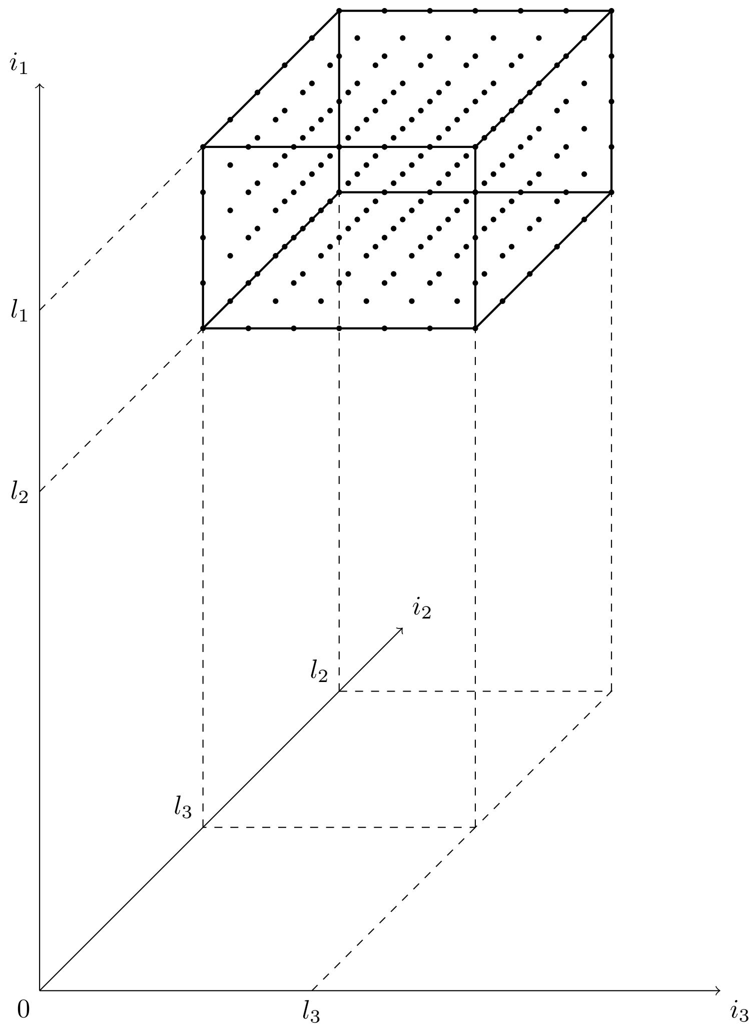


Figure 10.3: The kernel of $\mathcal{Q}_{l_1,l_2,l_3}$, with $(l_1-l_2+1)(l_2-l_3+1)(l_3+1)$ dots, each dot representing a space of type A solutions

into l_3+1 rectangular slices, numbered from 0 to l_3, each slice representing a space $\mathcal{K}_{h;\lambda}^{(j_3)}$ (with $0 \leq j_3 \leq l_3$), counting from the right to the left.

We now have defined a grading on the kernel space of $\mathcal{Q}_{l_1,l_2,l_3}$ using the twistor operator $\mathcal{T}^{(3)}$. From Lemma 10.1, we know that twistor operators commute up to a Cartan factor. This means that we can define a second grading on the slices $\mathcal{K}_{h;\lambda}^{(j_3)}$, this time using the twistor operator $\mathcal{T}^{(2)}$, which

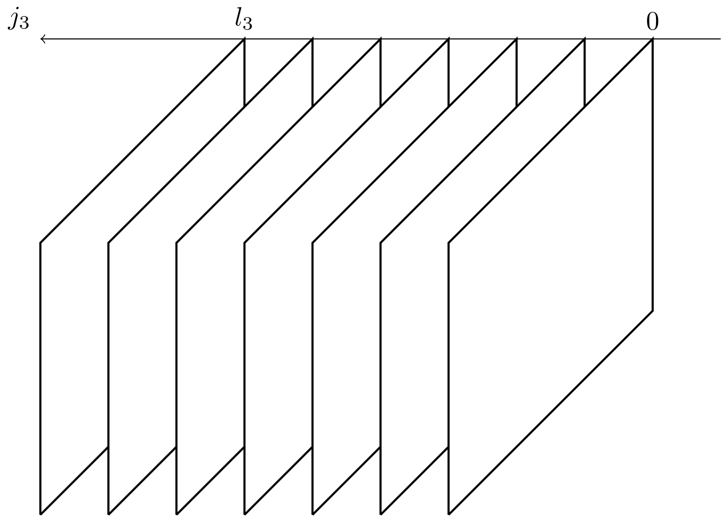


Figure 10.4: A first grading on $\mathcal{K}_{h;l_1,l_2,l_3}$

is independent of the first grading. Let us therefore introduce the following notations:

Definition 10.4. *For arbitrary highest weights $\lambda' = (l_1, l_2, l_3)'$, we put:*

$$\begin{aligned} \mathcal{K}_{h;\lambda}^{(0,j_3)} &:= \mathcal{K}_{h;\lambda}^{(j_3)} \cap \ker \mathcal{T}^{(2)} \\ \mathcal{K}_{h;\lambda}^{(j_2,j_3)} &:= \left(\mathcal{K}_{h;\lambda}^{(j_3)} \cap \ker \left(\mathcal{T}^{(2)}\right)^{j_2+1}\right) / \left(\mathcal{K}_{h;\lambda}^{(j_3)} \cap \ker \left(\mathcal{T}^{(2)}\right)^{j_2}\right). \end{aligned}$$

Remark 10.6. Note that we put the index j_2 before j_3, this is done in accordance with the order of the twistor operators we used to define the gradings, since the action is to be read from the right to the left as well.

Similarly as above, we have the following lemma, the proof of which will be given in the next section, in full generality:

Lemma 10.4. *For all $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda)$, we have that*

$$\left(\mathcal{T}^{(2)}\right)^{l_2-l_3+1} f = 0.$$

We thus have a decomposition of the form

$$\mathcal{K}_{h;\lambda}^{(j_3)} \cong \bigoplus_{j_2=0}^{l_2-l_3} \mathcal{K}_{h;\lambda}^{(j_2,j_3)} \implies \mathcal{K}_{h;\lambda} \cong \bigoplus_{j_3=0}^{l_3} \bigoplus_{j_2=0}^{l_2-l_3} \mathcal{K}_{h;\lambda}^{(j_2,j_3)}.$$

Graphically, the grading on the planes is represented by the dashed line segments in Figure 10.5. Each line segment stands for a space $\mathcal{K}_{h;\lambda}^{(j_2,j_3)}$, where j_3 labels the rectangular slice, and j_2 labels the dashed line segments in the directions of the arrows. Until now, we thus have defined a grading

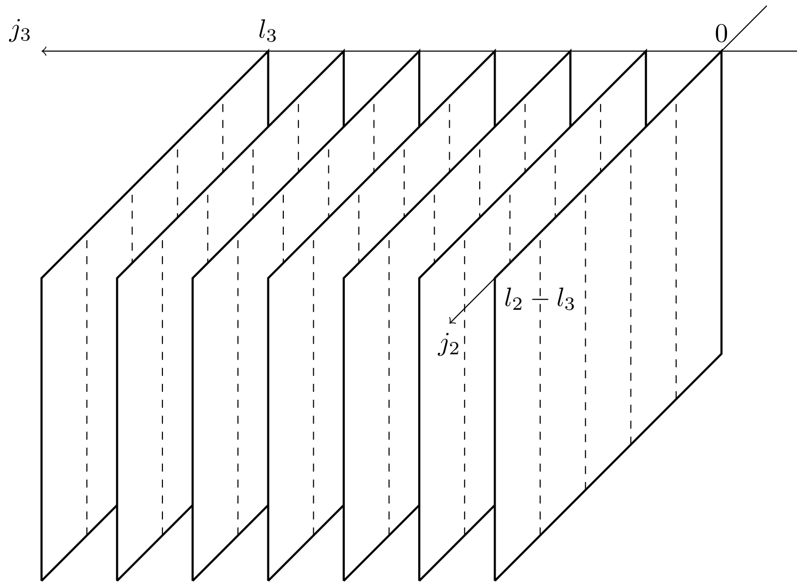


Figure 10.5: A second grading on $\mathcal{K}_{h;l_1,l_2,l_3}$

using $\mathcal{T}^{(3)}$ and $\mathcal{T}^{(2)}$. We can define a third and final grading using the last twistor operator, $\mathcal{T}^{(1)}$, on the 'dashed line segments' $\mathcal{K}_{h;\lambda}^{(j_2,j_3)}$. This gives rise to the following definition:

Definition 10.5. *For arbitrary highest weights $\lambda' = (l_1, l_2, l_3)'$, we put:*

$$\begin{aligned}\mathcal{K}_{h;\lambda}^{(0,j_2,j_3)} &:= \mathcal{K}_{h;\lambda}^{(j_2,j_3)} \cap \ker \mathcal{T}^{(1)}\\ \mathcal{K}_{h;\lambda}^{(j_1,j_2,j_3)} &:= \left(\mathcal{K}_{h;\lambda}^{(j_2,j_3)} \cap \ker \left(\mathcal{T}^{(1)}\right)^{j_1+1}\right) / \left(\mathcal{K}_{h;\lambda}^{(j_2,j_3)} \cap \ker \left(\mathcal{T}^{(1)}\right)^{j_1}\right).\end{aligned}$$

Then, again, the following lemma holds.

Lemma 10.5. *For all $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda)$, we have that*

$$\left(\mathcal{T}^{(1)}\right)^{l_1-l_2+1} f = 0.$$

This thus leads to

$$\mathcal{K}_{h;\lambda}^{(j_2,j_3)} \cong \bigoplus_{j_1=0}^{l_1-l_2} \mathcal{K}_{h;\lambda}^{(j_1,j_2,j_3)} \implies \mathcal{K}_{h;\lambda} \cong \bigoplus_{j_3=0}^{l_3} \bigoplus_{j_2=0}^{l_2-l_3} \bigoplus_{j_1=0}^{l_1-l_2} \mathcal{K}_{h;\lambda}^{(j_1,j_2,j_3)}.$$

Graphically, the final grading on the dashed line segments is depicted by means of the dots in Figure 10.6, each dot representing a space $\mathcal{K}_{h;\lambda}^{(j_1,j_2,j_3)}$.

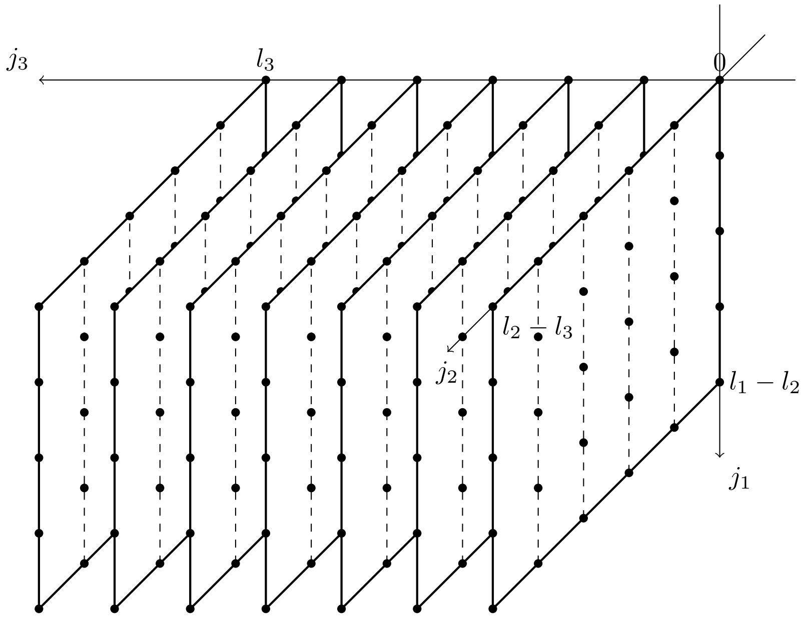


Figure 10.6: A third and final grading on $\mathcal{K}_{h;l_1,l_2,l_3}$

The problem at hand, i.e. describing the kernel space of $\mathcal{Q}_{l_1,l_2,l_3}$, now is reduced to describing the spaces $\mathcal{K}_{h;\lambda}^{(j_1,j_2,j_3)}$.

10.4 The general case

Let us now take a look at the general case. From now on, we will assume $\lambda = (l_1, \ldots, l_k)$ with $l_1 \geq \cdots \geq l_k > 0$. In order to decompose $\mathcal{K}_{h;\lambda}$ for arbitrary half-integer highest weights, we first of all define a grading on the kernel space using the twistor operator $\mathcal{T}^{(k)}$, inspired by Definition 10.3:

Definition 10.6. *For arbitrary highest weights $\lambda' = (l_1, \ldots, l_k)'$ with $l_k >$*

0, *we put:*

$$\begin{aligned}
\mathcal{K}_{h;\lambda}^{(0)} &:= \mathcal{K}_{h;\lambda} \cap \ker \mathcal{T}^{(k)} \\
\mathcal{K}_{h;\lambda}^{(j_k)} &:= \big(\mathcal{K}_{h;\lambda} \cap \ker \left(\mathcal{T}^{(k)}\right)^{j_k+1}\big) / \big(\mathcal{K}_{h;\lambda} \cap \ker \left(\mathcal{T}^{(k)}\right)^{j_k}\big) .
\end{aligned}$$

The following lemma will then again lead to a direct sum decomposition.

Lemma 10.6. *For all $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda)$, we have that*

$$\left(\mathcal{T}^{(k)}\right)^{l_k+1} f = 0.$$

Proof. Since $\deg_{u_k}(f) = l_k$, and $\mathcal{T}^{(k)}$ lowers the degree in u_k by 1, the results obviously follows. □

From Lemma 10.6, and Definition 10.6, we thus have:

$$\mathcal{K}_{h;\lambda} \cong \bigoplus_{j_k=0}^{l_k} \mathcal{K}_{h;\lambda}^{(j_k)}.$$

Remark 10.7. Comparing this to the case $k = 3$, which translated into a decomposition into rectangular slices, this amounts to a decomposition in rectangular hypercuboid slices of codimension 1 (or dimension $k-1$).

Since $\langle \partial_x, \partial_k \rangle = \mathcal{T}_\lambda^{(k)}$ when acting on $\mathcal{S}_\lambda$-valued polynomials, one immediately sees that

$$\mathcal{K}_{h;\lambda}^{(0)} = \mathcal{P}_h(\mathbb{R}^m, \mathcal{S}_\lambda) \cap \ker \mathcal{Q}_{l_1,\ldots,l_{k-1}}^T.$$

In view of the fact that the twisted operator essentially is the HSD operator $\mathcal{Q}_{l_1,\ldots,l_{k-1}}$ acting on $\mathcal{S}_\lambda$-valued functions (see Chapter 8), this clearly suggests using induction on the length of the highest weight of the underlying representation characterising the values.

Lemma 10.7. *Given an arbitrary highest weight $\lambda' = (l_1, \ldots, l_k)'$ with $l_k > 0$, one has:*

$$\langle \partial_x, \partial_k \rangle \mathcal{Q}_\lambda = \frac{m + 2\mathbb{E}_k - 2k}{m + 2\mathbb{E}_k - 2k + 2} \mathcal{Q}_{\lambda - L_k} \langle \partial_x, \partial_k \rangle.$$

Proof. This directly follows from the fact that $\langle \partial_x, \partial_k \rangle = \mathcal{T}_\lambda^{(k)}$, combined with Lemma 10.1. □

This lemma actually tells us that for each $f \in \ker \mathcal{Q}_\lambda$, we have that either $\langle \partial_x, \partial_k \rangle f = 0$, or $\langle \partial_x, \partial_k \rangle f \in \ker \mathcal{Q}_{\lambda - L_k}$. Or more generally, we have the following theorem.

Theorem 10.3. *Given a fixed highest weight $\lambda' = (l_1, \ldots, l_k)'$ and an integer $h \in \mathbb{N}$, the following property holds for all $1 \leq j_k \leq l_k$:*

$$\varphi_{k;j_k} := \langle \partial_x, \partial_k \rangle^{j_k} : \mathcal{K}^{(j_k)}_{h;\lambda} \to \mathcal{K}^{(0)}_{h-j_k;\lambda-j_kL_k}. \tag{10.7}$$

Proof. From Definition 10.6, it follows that the operator $\langle \partial_x, \partial_k \rangle^{j_k}$ maps polynomials in $\mathcal{K}^{(j_k)}_{h;\lambda}$ to elements of the vector space $\ker\langle \partial_x, \partial_k \rangle$. Together with Lemma 10.7, this then proves the assertion. □

Remark 10.8. Note that the target space at the right-hand side of (10.7) contains solutions for the twisted version of the operator $\mathcal{Q}_{l_1,\ldots,l_{k-1}}$, acting on $\mathcal{S}_{\lambda-j_kL_k}$-valued functions.

Remark 10.9. Note that $\langle \partial_x, \partial_k \rangle$ is the twistor operator $\mathcal{T}^{(k)}$, which results in the fact that $\varphi_{k;j_k} = \left(\mathcal{T}^{(k)}\right)^{j_k}$; here we can formally write a power of a twistor operator due to the fact that we use the Euler notations. However, as before, one should bear in mind that each consecutive twistor operator actually acts on a different polynomial space, as the degree of u_k lowers with each consecutive action of $\mathcal{T}^{(k)}$.

Remark 10.10. Here it is crucial to point out that the mappings $\varphi_{k;j_k}$ appearing in the previous proposition are not necessarily surjective, which means that not all irreducible summands in the 0-graded image space at the right-hand side (containing null solutions for a twisted HSD operator) will be present in the j_k-graded subspace at the left-hand side of the arrow.

The question now arises whether an analogue of Theorem 10.3 holds for the other twistor operators as well. This indeed is the case, as we will show in the following theorem, but first we need a lemma.

Lemma 10.8. *For all $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda)$, we have that*

$$\left(\mathcal{T}^{(a)}\right)^{l_a-l_{a+1}+1} f = 0.$$

Proof. Denoting $g = \left(\mathcal{T}^{(a)}\right)^{l_a-l_{a+1}} f$, we have that

$$g \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1,\ldots,l_{a-1},l_{a+1},l_{a+1},\ldots,l_k})$$

where $\deg_{u_a}(g) = \deg_{u_{a+1}}(g) = l_{a+1}$. Remember that $\mathcal{S}_\lambda$ is generated by (2.11). Moreover, the highest weight vector of $\mathcal{S}_{l_1,\ldots,l_{a-1},l_{a+1},l_{a+1},\ldots,l_k}$ is symmetric in u_a and u_{a+1}. Since $\langle u_{a+1}, \partial_a \rangle$ is $\mathrm{Spin}(m)$-invariant, this means that

$$\langle u_{a+1}, \partial_a \rangle g = 0, \tag{10.8}$$

or g is an element in the kernel of an extra operator which is not in the definition of simplicial monogenicity. We find that

$$\mathcal{T}^{(a)}g = \prod_{j=a+1}^{k}\left(1 - \frac{\langle u_j, \partial_a\rangle\langle u_a, \partial_j\rangle}{\mathbb{E}_a - \mathbb{E}_j + j - a + 1}\right)\langle \partial_a, \partial_x\rangle g.$$

Using the relation $\langle u_{a+1}, \partial_a\rangle\langle u_a, \partial_{a+1}\rangle = \langle u_a, \partial_{a+1}\rangle\langle u_{a+1}, \partial_a\rangle + \mathbb{E}_{a+1} - \mathbb{E}_a$, this equals

$$\langle u_a, \partial_{a+1}\rangle\langle u_{a+1}, \partial_a\rangle \prod_{j=a+2}^{k}\left(1 - \frac{\langle u_j, \partial_a\rangle\langle u_a, \partial_j\rangle}{\mathbb{E}_a - \mathbb{E}_j + j - a + 1}\right)\langle \partial_a, \partial_x\rangle g.$$

Inductively running $\langle u_{a+1}, \partial_a\rangle$ through each factor, and using both the fact that $\langle u_{a+1}, \partial_j\rangle g = 0$ for all $j > a+1$ and (10.8), we find that

$$\mathcal{T}^{(a)}g = 0.$$

This proves the lemma. □

This lemma is crucial for what follows, and basically gives us an upper boundary on the number of times a twistor operator can act on $\mathcal{K}_{h,\lambda}$ before the result becomes trivial. Also, let us recall a result that was proven in [39]. It essentially tells us that certain twistor operator compositions are trivial. For the sake of completeness we briefly recall the proof here.

Proposition 10.2. *For a highest weight $\lambda = (l_1, ..., l_{j-2}, l_j, l_j, l_{j+1}, ..., l_k)'$ (note that $l_j = l_{j-1}$) that is fixed, one has that*

$$\mathcal{T}^{(j-1)}_{l_1,...,l_{j-2},l_j,l_j-1,l_{j+1},...,l_k} \circ \mathcal{T}^{(j)}_{l_1,...,l_{j-2},l_j,l_j,l_{j+1},...,l_k} \equiv 0.$$

Proof. First of all, we note that this twistor operator composition connects the following three dominant highest weights:

$$\begin{aligned}(l_1, ..., l_{j-2}, l_j, l_j, l_{j+1}, ..., l_k)' \\ &= (l_1 - 1, ..., l_{j-2} - 1, l_j - 1, l_j - 1, l_{j+1}, ..., l_k)' \\ &\qquad + (1, ..., 1, 1, 1, 0, ..., 0)\end{aligned}$$

$$\begin{aligned}(l_1, ..., l_{j-2}, l_j, l_j - 1, l_{j+1}, ..., l_k)' \\ &= (l_1 - 1, ..., l_{j-2} - 1, l_j - 1, l_j - 1, l_{j+1}, ..., l_k)' \\ &\qquad + (1, ..., 1, 1, 0, 0, ..., 0)\end{aligned}$$

$$\begin{aligned}(l_1, ..., l_{j-2}, l_j - 1, l_j - 1, l_{j+1}, ..., l_k)' \\ &= (l_1 - 1, ..., l_{j-2} - 1, l_j - 1, l_j - 1, l_{j+1}, ..., l_k)' \\ &\qquad + (1, ..., 1, 0, 0, 0, ..., 0).\end{aligned}$$

In other words, the highest weights in the composition all sit inside the tensor product of one fixed highest weight (call it μ) and three fundamental highest weights, which we will denote by ω_+, ω_0 and ω_- respectively (in that order). As a matter of fact, the spaces $\mathcal{S}_\lambda$ associated to the dominant weights at the left-hand side precisely are the Cartan products of the fixed space $\mathcal{S}_\mu$ and the representations $\mathbb{V}_\omega^\pm$ and $\mathbb{V}_\omega^0$. The latter correspond to exterior powers of the fundamental representation, and this means that the sequence of differential operators $\mathbb{V}_\omega^+ \to \mathbb{V}_\omega^0 \to \mathbb{V}_\omega^-$ is nothing but a part of the de Rham sequence for the codifferential operator. If we twist this particular sequence with the representation $\mathcal{S}_\mu$ (i.e. we consider $\mathcal{S}_\mu$-valued forms), we still get a composition which is trivial and so is the restriction and projection of this composition to the Cartan products. Note that the projection on the Cartan products of the tensor product of the de Rham sequence with an irreducible representation does not necessarily lead to the exactness of the operators acting between them. It suffices to consider the tensor product of the de Rham sequence with the standard vector representation to understand why: there may be more then one path connecting the source and target bundles. In this particular case however, the twisting does lead to an exact sequence: this is due to the fact that λ contains at least two weight entries which are equal. Now, in view of the fact that the only first-order operators acting between the Cartan products are the aforementioned twistor operators, we have that the composition

$$\mathcal{T}^{(j-1)}_{l_1,\ldots,l_{j-2},l_j,l_j-1,l_{j+1},\ldots,l_k} \circ \mathcal{T}^{(j)}_{(l_1,\ldots,l_{j-2},l_j,l_j,l_{j+1},\ldots,l_k)}$$

indeed is zero. □

Let us first focus on the space $\mathcal{K}^{(0)}_{h;\lambda}$, which would be the first *hyperslice* (compare to Figure 10.4). Inspired by Definition 10.6, we define the following spaces:

Definition 10.7. *For arbitrary highest weights $\lambda' = (l_1, \ldots, l_k)'$, we put:*

$$\begin{aligned}\mathcal{K}^{(0,j_{i+1},\ldots,j_k)}_{h;\lambda} &:= \mathcal{K}^{(j_{i+1},\ldots,j_k)}_{h;\lambda} \cap \ker \mathcal{T}^{(i)} \\ \mathcal{K}^{(j_i,j_{i+1},\ldots,j_k)}_{h;\lambda} &:= \left(\mathcal{K}^{(j_{i+1},\ldots,j_k)}_{h;\lambda} \cap \ker\left(\mathcal{T}^{(i)}\right)^{j_i+1}\right) \\ &\quad\ / \left(\mathcal{K}^{(j_{i+1},\ldots,j_k)}_{h;\lambda} \cap \ker\left(\mathcal{T}^{(i)}\right)^{j_i}\right).\end{aligned}$$

Remark 10.11. Observe that the number of under indices in the above notations corresponds to the graphical interpretation of the space. For the case $k = 3$, we have for instance that $\mathcal{K}^{(j_3)}_{h;\lambda}$ is a rectangular slice of codimension 1, $\mathcal{K}^{(j_2,j_3)}_{h;\lambda}$ is a line segment of codimension 2, and $\mathcal{K}^{(j_1,j_2,j_3)}_{h;\lambda}$ is a dot of codimension 3. So we might say that it is the codimension of the space, after graphical interpretation.

With this definition, we can state the following proposition.

Proposition 10.3. *The kernel space of $\mathcal{Q}_\lambda$ has the following decomposition:*

$$\mathcal{K}_{h;\lambda} \cong \bigoplus_{j_1=0}^{l_1-l_2} \cdots \bigoplus_{j_{k-1}=0}^{l_{k-1}-l_k} \bigoplus_{j_k=0}^{l_k} \mathcal{K}_{h;\lambda}^{(j_1,j_2,\ldots,j_k)}. \tag{10.9}$$

Proof. Using Lemma 10.8 and Proposition 10.2 we have for all $0 \leq j_k \leq l_k$ that

$$\mathcal{K}_{h;\lambda}^{(j_k)} \cong \bigoplus_{j_{k-1}=0}^{l_{k-1}-l_k} \mathcal{K}_{h;\lambda}^{(j_{k-1},j_k)}$$

and in general, that

$$\mathcal{K}_{h;\lambda}^{(j_{i+1},\ldots,j_k)} \cong \bigoplus_{j_i=0}^{l_i-l_{i+1}} \mathcal{K}_{h;\lambda}^{(j_i,j_{i+1},\ldots,j_k)}.$$

Using this argument inductively on $\mathcal{K}_{h;\lambda}$ we get a full decomposition of the 'hyperrectangle' in 'dots'. □

10.5 Interpreting the spaces $\mathcal{K}_{h;\lambda}^{(j_1,j_2,\ldots,j_k)}$

First, we take a closer look at the component $\mathcal{K}_{h;\lambda}^{(0,\ldots,0)}$ of this direct sum.

Lemma 10.9. *We have the following equality:*

$$\mathcal{K}_{h;\lambda}^{(0,\ldots,0)} = \mathcal{K}_{h;\lambda} \cap \ker \{\langle \partial_x, \partial_i \rangle : i \in \{1,\ldots,k\}\}.$$

Proof. From the definition of $\mathcal{K}_{h;\lambda}^{(0,\ldots,0)}$, we know that any $f \in \mathcal{K}_{h;\lambda}^{(0,\ldots,0)}$ satisfies the relation $\mathcal{T}^{(i)} f = 0$, for all $1 \leq i \leq k$. The k-th twistor operator is defined as

$$\mathcal{T}^{(k)} = \langle \partial_k, \partial_x \rangle,$$

whence $\langle \partial_k, \partial_x \rangle f = 0$. It then follows that

$$\begin{aligned} 0 &= \mathcal{T}^{(k-1)} f \\ &= \left(1 + \frac{1}{\mathbb{E}_k - \mathbb{E}_{k-1} + k - 1 - (k+1)} \langle u_k, \partial_{k-1} \rangle \langle u_{k-1}, \partial_k \rangle \right) \langle \partial_{k-1}, \partial_x \rangle f \\ &= \langle \partial_{k-1}, \partial_x \rangle f, \end{aligned}$$

where we have used that $[\langle u_{k-1}, \partial_k\rangle, \langle \partial_{k-1}, \partial_x\rangle] = -\langle \partial_k, \partial_x\rangle$. Continuing this argument inductively on the twistor operators, we find that

$$\begin{aligned} 0 = \mathcal{T}^{(j)} f &= \prod_{p=j+1}^{k} \left(1 + \frac{1}{\mathbb{E}_p - \mathbb{E}_j + j - (p+1)} \langle u_p, \partial_j\rangle \langle u_j, \partial_p\rangle\right) \langle \partial_j, \partial_x\rangle f \\ &= \langle \partial_j, \partial_x\rangle f. \end{aligned}$$

This proves that

$$\mathcal{K}_{h;\lambda}^{(0,\dots,0)} \subseteq \mathcal{K}_{h;\lambda} \cap \ker\left\{\langle \partial_x, \partial_i\rangle : i \in \{1,\dots,k\}\right\}.$$

From the fact that each twistor operator can be written as (10.2), where each term ends with an operator of the form $\langle \partial_i, \partial_x\rangle$, the inverse inclusion follows, finishing the proof. □

We can then link this space to the type A solutions defined in the previous chapter.

Theorem 10.4. *One has that* $\mathcal{K}_{h;\lambda}^{(0,\dots,0)} = \mathcal{M}_{h;\lambda}^{s}$.

Proof. We have that

$$\mathcal{K}_{h;\lambda}^{(0,\dots,0)} = \mathcal{K}_{h;\lambda} \cap \ker\left\{\langle \partial_x, \partial_i\rangle : i \in \{1,\dots,k\}\right\},$$

so for all $f \in \mathcal{K}_{h;\lambda}^{(0,\dots,0)} \subset \mathcal{P}_h(\mathbb{R}^m, \mathcal{S}_\lambda)$, it holds

$$0 = \mathcal{Q}_\lambda f = \prod_{j=1}^{k} \left(1 + \frac{u_j \partial_j}{2\mathbb{E}_j + m - 2}\right) \partial_x = \partial_x f,$$

since $\partial_i \partial_x = -\partial_x \partial_i - 2\langle \partial_i, \partial_x\rangle$. We thus have proven that

$$\mathcal{K}_{h;\lambda}^{(0,\dots,0)} \subseteq \mathcal{M}_{h;\lambda}^{s}.$$

On the other hand, for all $g \in \mathcal{M}_{h;\lambda}^{s}$,

$$\langle \partial_i, \partial_x\rangle g = \frac{1}{2}(\partial_i \partial_x + \partial_x \partial_i) g = 0,$$

yielding the inverse inclusion

$$\mathcal{M}_{h;\lambda}^{s} \subseteq \mathcal{K}_{h;\lambda} \cap \ker\left\{\langle \partial_x, \partial_i\rangle : i \in \{1,\dots,k\}\right\} = \mathcal{K}_{h;\lambda}^{(0,\dots,0)},$$

which finishes the proof. □

Thus far, we have been able to describe the space $\mathcal{K}_{h;\lambda}^{(0,\dots,0)}$. Let us take a look at the other components of the decomposition (10.9). Let us introduce some operators through the following theorem.

Theorem 10.5. *Given a fixed highest weight $\lambda' = (l_1, \ldots, l_k)'$ and an integer $h \in \mathbb{N}$, one has, for all $1 \leq i \leq k-1$ and all $1 \leq j_i \leq l_i - l_{i+1}$ that the opertor $\varphi_{i;j_i} := \left(\mathcal{T}^{(i)}\right)^{j_i}$ maps between the following two spaces:*

$$\left(\mathcal{K}_{h;\lambda} \cap \ker \left(\mathcal{T}^{(i)}\right)^{j_i+1}\right) / \left(\mathcal{K}_{h;\lambda} \cap \ker \left(\mathcal{T}^{(i)}\right)^{j_i}\right) \to \mathcal{K}_{h-j_i;\lambda-j_iL_i} \cap \ker \mathcal{T}^{(i)}.$$

Proof. This directly follows from Lemma 10.2. Lemma 10.8 provides us with a lower boundary for j for the space $\left(\mathcal{K}_{h;\lambda} \cap \ker \left(\mathcal{T}^{(i)}\right)^{j+1}\right)$ to become trivial. □

Remark 10.12. The operators $\varphi_{i;j_i}$ provide a way to relate functions in $\ker \mathcal{Q}_\lambda \cap \ker \left(\mathcal{T}^{(i)}\right)^{j_i}$ to functions in the kernel of another HSD operator and the kernel of $\mathcal{T}^{(i)}$ itself. In a sense, this operator enables to 'lower' the upper index j_i in Definition 10.7.

From the definition of $\mathcal{K}_{h;\lambda}^{(j_1,j_2,\ldots,j_k)}$, and the properties of the operators $\varphi_{i;j_i}$, we find that

$$\begin{aligned}
\mathcal{K}_{h;\lambda}^{(j_1,j_2,\ldots,j_k)} &\cong \left(\varphi_{1,j_1} \cdots \varphi_{k-1,j_{k-1}} \varphi_{k,j_k} \mathcal{K}_{h,\lambda}\right) \cap \ker \left(\mathcal{T}^{(1)}, \mathcal{T}^{(2)}, \ldots, \mathcal{T}^{(k)}\right) \\
&= \left(\left(\mathcal{T}^{(k)}\right)^{j_k} \left(\mathcal{T}^{(k-1)}\right)^{j_{k-1}} \cdots \left(\mathcal{T}^{(1)}\right)^{j_1} \mathcal{K}_{h,\lambda}\right) \\
&\quad \cap \ker \left(\mathcal{T}^{(1)}, \mathcal{T}^{(2)}, \ldots, \mathcal{T}^{(k)}\right).
\end{aligned}$$

Using the same argument from Lemma 10.9, we get

$$\mathcal{K}_{h;\lambda}^{(j_1,j_2,\ldots,j_k)} \cong \left(\langle \partial_1, \partial_x \rangle^{j_1} \ldots \langle \partial_k, \partial_x \rangle^{j_k} \mathcal{K}_{h,\lambda}\right) \cap \ker \left(\langle \partial_1, \partial_x \rangle, \ldots, \langle \partial_k, \partial_x \rangle\right).$$

Then the properties of the operators $\varphi_{i;j_i}$ tell us that

$$\mathcal{K}_{h;\lambda}^{(j_1,j_2,\ldots,j_k)} \subseteq \mathcal{K}^{(0,\ldots,0)}_{h-\sum_{i=1}^k j_i, l_1-j_1,\ldots,l_k-j_k}.$$

Using Theorem 10.4, we find that

$$\mathcal{K}_{h;\lambda}^{(j_1,j_2,\ldots,j_k)} \subseteq \mathcal{M}^s_{h-\sum_{i=1}^k j_i, l_1-j_1,\ldots,l_k-j_k}. \tag{10.10}$$

We know how a space $\mathcal{M}^s_{h,\lambda}$ decomposes (see Theorem 9.2). This means that if we can prove the inverse inclusion of (10.10), we have found a full decomposition of $\mathcal{K}_{h;\lambda}$. In order to try to prove this, we will count the dimensions of both spaces. To this end, we need the generalised CK-extension found in Chapter 7.

10.6 Open problem

We can use formula (7.2) from the previous subsection to finish the proof of Proposition 10.1. Indeed, we have that

$$\begin{aligned}\mathcal{K}_{h;\lambda} &\cong \bigoplus_{j_1=0}^{l_1-l_2} \cdots \bigoplus_{j_{k-1}=0}^{l_{k-1}-l_k} \bigoplus_{j_k=0}^{l_k} \mathcal{K}_{h;\lambda}^{(j_1,j_2,\ldots,j_k)} \\ &\subseteq \bigoplus_{j_1=0}^{l_1-l_2} \cdots \bigoplus_{j_{k-1}=0}^{l_{k-1}-l_k} \bigoplus_{j_k=0}^{l_k} \mathcal{M}^s_{h-\sum_{p=1}^k j_p;l_1-j_1,\ldots,l_k-j_k}\end{aligned}$$

due to (10.10). If the dimensions of both spaces appearing in this inclusion turn out to be equal, the inclusion turns to an equality, in this way finishing the proof. Corollary 7.1 yields:

$$\dim(\mathcal{K}_{h;\lambda}) = \dim(\mathcal{P}_h(\mathbb{R}^{m-1}))\dim(\mathcal{S}_\lambda).$$

On the other hand, on account of Theorem 9.2, the dimension of the space on the right hand side equals

$$\begin{aligned}&\dim\left(\bigoplus_{j_1=0}^{l_1-l_2} \cdots \bigoplus_{j_{k-1}=0}^{l_{k-1}-l_k} \bigoplus_{j_k=0}^{l_k} \mathcal{M}^s_{h-\sum_{p=1}^k j_p;l_1-j_1,\ldots,l_k-j_k}\right) \\ &= \sum_{j_1=0}^{l_1-l_2} \cdots \sum_{j_{k-1}=0}^{l_{k-1}-l_k} \sum_{j_k=0}^{l_k} \sum_{i_1=0}^{l_1-j_1-l_2+j_2} \\ &\quad \cdots \sum_{i_{k-1}=0}^{l_{k-1}-j_{k-1}-l_k+j_k} \sum_{i_k=0}^{l_k-j_k} \dim\left(\mathcal{S}_{h+\sum_{p=1}^k i_p-j_p,l_1-i_1-j_1,\ldots,l_k-i_p-j_p}\right).\end{aligned}$$

In the case of $k = 1, 2$ and 3 this can be symbolically computed with Maple [62] using the code in Chapter 7, and the respective dimensions are indeed found to be equal:

k	$\dim \mathcal{K}_{h;\lambda}$
1	$2^n \binom{h+2n-1}{h}\binom{l_1+2n-1}{l_1}$
2	$2^n \binom{h+2n-1}{h}\binom{l_1+2n-2}{l_1+1}\binom{l_2+2n-3}{l_2}\frac{(l_1+l_2+2n-1)(l_1-l_2+1)}{(2n-1)(2n-2)}$
3	$2^n \binom{h+2n-1}{h}\binom{l_1+2n-3}{l_1+2}\binom{l_2+2n-4}{l_2+1}\binom{l_3+2n-5}{l_3}$
	$\times\frac{(l_1+l_2+2n-1)(l_1+l_3+2n-2)(l_2+l_3+2n-3)(l_1-l_2+1)(l_1-l_3+2)(l_2-l_3+1)}{(2n-1)(2n-2)(2n-3)^2(2n-4)^2}$

However, in full generality this still remains an open problem.

10.7 Conclusion

In this chapter, we developed an approach to decompose the space $\ker_h \mathcal{Q}_\lambda$ of null solutions for an arbitrary HSD operator using an inductive procedure, exploiting the power of the twistor operators and the twisted version of the HSD operators. Invoking the CK-extension, this reduced the problem to a combinatorial counting argument. For the cases $k \in \{1, 2, 3\}$, the number of dummy variables describing the values of our higher spin fields, this was verified explicitly, whereas the general case seems to be out of grasp at this point.

I do not know what I may appear to the world, but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.

Isaac Newton

11

The kernel of $\mathcal{R}_{l_1}$ and $\mathcal{Q}_{l_1,l_2}$ revisited

In this chapter, we discuss another approach to decompose the kernel of the general HSD operator $\mathcal{Q}_\lambda$ in irreducible Spin(m)-modules. Note that this approach is mainly based on an intuitive reasoning, so there are some conjectures in this chapter which we do not prove. However, we give some examples to strengthen our arguments. In the first section, we explain the reasoning behind our approach, which is based on the generalized CK-extension, discussed in Chapter 7. The results will be proven in the case of the Rarita-Schwinger operator. For the HSD operator of order two, we will test our hypothesis by comparing it to the results given in [82], in order to generalise them for HSD operators of general order. Important is that in contrast to the reasoning in the previous chapter, which only worked when $h \geq l_1 + l_2$, this reasoning works for all degrees of homogeneity h. Note that the h-homogeneous polynomial kernel of the classical Dirac operator is the space of spinor-valued monogenic functions. This space can be written as the Cartan product $\mathcal{H}_h \boxtimes \mathcal{S}$. This is in fact equal to $[\mathcal{H}_h \otimes \mathcal{H}_\lambda] \boxtimes \mathcal{S}$, where $\lambda = (0)$. In this chapter, we make the conjecture that this latter expression is true in general.

11.1 The decomposition of the higher spin kernel revisited

The aim of this section is to arrive at an algorithm which tells us how to decompose the kernel of the higher spin Dirac operators into irreducible summands under the (regular) action of the spin group in an alternative way. We will make use of a higher spin version of the CK-extension which was proven in Chapter 7. Remember that it states that

$$\ker_h \mathcal{Q}_\lambda := \mathcal{P}_h(\mathbb{R}^m, \mathcal{S}_\lambda) \cap \ker \mathcal{Q}_\lambda \cong \mathcal{P}_k(\mathbb{R}^{m-1}, \mathcal{S}_\lambda),$$

for any higher spin Dirac operator $\mathcal{Q}_\lambda$ corresponding to an arbitrary highest weight λ of a half-integer irreducible representation of finite dimension. We can then make use of the classical Fischer decomposition for harmonic polynomials to obtain the direct sum formula

$$\mathcal{P}_h(\mathbb{R}^{m-1}, \mathcal{S}_\lambda) \cong \mathcal{P}_h(\mathbb{R}^{m-1}, \mathbb{C}) \otimes \mathcal{S}_\lambda \cong \left(\bigoplus_{j=0}^{\lfloor \frac{h}{2} \rfloor} \mathcal{H}_{h-2j}(\mathbb{R}^{m-1}, \mathbb{C}) \right) \otimes \mathcal{S}_\lambda.$$

On the other hand, we also have the classical branching rules for harmonic polynomials:

$$\mathcal{H}_h(\mathbb{R}^m, \mathcal{S}_\lambda) \cong \left(\bigoplus_{j=0}^{h} \mathcal{H}_j(\mathbb{R}^{m-1}, \mathbb{C}) \right) \otimes \mathcal{S}_\lambda.$$

Combining both formulae, we arrive at the following result, which is a formal identity, to be understood on the level of isomorphisms. Note that we have omitted the space of values $\mathcal{S}_\lambda$, to shorten the notations:

$$\begin{cases} h = 2\kappa & \mathcal{P}_h(\mathbb{R}^{m-1}) \cong \bigoplus_{j=0}^{\kappa} \mathcal{H}_{h-2j}(\mathbb{R}^m) \backslash \bigoplus_{j=0}^{\kappa-1} \mathcal{H}_{h-(2j+1)}(\mathbb{R}^m) \\ h = 2\kappa+1 & \mathcal{P}_h(\mathbb{R}^{m-1}) \cong \bigoplus_{j=0}^{\kappa} \mathcal{H}_{h-2j}(\mathbb{R}^m) \backslash \bigoplus_{j=0}^{\kappa} \mathcal{H}_{h-(2j+1)}(\mathbb{R}^m). \end{cases}$$

Denoting irreducible Spin(m)-representations by their highest weight, it follows that the decomposition of a (homogeneous) HSD kernel space can be computed as follows:

$$\begin{cases} h = 2\kappa & \mathcal{K}_{h,\lambda} \cong \bigoplus_{j=0}^{\kappa} (h-2j) \otimes \lambda' - \bigoplus_{j=0}^{\kappa-1} (h-2j-1) \otimes \lambda' \\ h = 2\kappa+1 & \mathcal{K}_{h,\lambda} \cong \bigoplus_{j=0}^{\kappa} (h-2j) \otimes \lambda' - \bigoplus_{j=0}^{\kappa} (h-2j-1) \otimes \lambda'. \end{cases} \tag{11.1}$$

Here, the minus sign has to be understood as follows: each tensor product of the form $(a) \otimes \lambda'$ with $a \in \mathbb{N}$, decomposes into a direct sum of highest weights characterising irreducible representations. The minus sign indicates that the summands coming from the second summation have to be omitted from the list of summands generated by the first summation.

Example 11.1. Let us give an example to make this reasoning more clear. Take $h = 4$. The Fischer decomposition yield

$$\mathcal{P}_4(\mathbb{R}^{m-1}, \mathcal{S}_\lambda) \cong \mathcal{H}_4(\mathbb{R}^{m-1}, \mathcal{S}_\lambda) \oplus \mathcal{H}_2(\mathbb{R}^{m-1}, \mathcal{S}_\lambda) \oplus \mathcal{H}_0(\mathbb{R}^{m-1}, \mathcal{S}_\lambda).$$

On the other hand, the branching rules tell us that

$$\begin{aligned}
\mathcal{H}_4(\mathbb{R}^m, \mathcal{S}_\lambda) \cong & \mathcal{H}_0(\mathbb{R}^{m-1}, \mathcal{S}_\lambda) \oplus \mathcal{H}_1(\mathbb{R}^{m-1}, \mathcal{S}_\lambda) \oplus \mathcal{H}_2(\mathbb{R}^{m-1}, \mathcal{S}_\lambda) \\
& \oplus \mathcal{H}_3(\mathbb{R}^{m-1}, \mathcal{S}_\lambda) \oplus \mathcal{H}_4(\mathbb{R}^{m-1}, \mathcal{S}_\lambda) \\
\mathcal{H}_3(\mathbb{R}^m, \mathcal{S}_\lambda) \cong & \mathcal{H}_0(\mathbb{R}^{m-1}, \mathcal{S}_\lambda) \oplus \mathcal{H}_1(\mathbb{R}^{m-1}, \mathcal{S}_\lambda) \oplus \mathcal{H}_2(\mathbb{R}^{m-1}, \mathcal{S}_\lambda) \\
& \oplus \mathcal{H}_3(\mathbb{R}^{m-1}, \mathcal{S}_\lambda) \\
\mathcal{H}_2(\mathbb{R}^m, \mathcal{S}_\lambda) \cong & \mathcal{H}_0(\mathbb{R}^{m-1}, \mathcal{S}_\lambda) \oplus \mathcal{H}_1(\mathbb{R}^{m-1}, \mathcal{S}_\lambda) \oplus \mathcal{H}_2(\mathbb{R}^{m-1}, \mathcal{S}_\lambda) \\
\mathcal{H}_1(\mathbb{R}^m, \mathcal{S}_\lambda) \cong & \mathcal{H}_0(\mathbb{R}^{m-1}, \mathcal{S}_\lambda) \oplus \mathcal{H}_1(\mathbb{R}^{m-1}, \mathcal{S}_\lambda) \\
\mathcal{H}_0(\mathbb{R}^m, \mathcal{S}_\lambda) \cong & \mathcal{H}_0(\mathbb{R}^{m-1}, \mathcal{S}_\lambda).
\end{aligned}$$

Hence, we indeed find that

$$\mathcal{K}_{4,\lambda} \cong \bigoplus_{j=0}^{2} \mathcal{H}_{2j}(\mathbb{R}^m, \mathcal{S}_\lambda) - \bigoplus_{j=0}^{1} \mathcal{H}_{2j+1}(\mathbb{R}^m, \mathcal{S}_\lambda).$$

This means that we are essentially looking for the following abstract result.

> *Given an arbitrary half-integer highest weight λ and an integer $k \in \mathbb{N}$, can we decompose*
>
> $$(k) \otimes \lambda'?$$

Let us prove the decomposition of this tensor product in the case where $k = 1$ in the following section.

11.2 A decomposition of $\mathcal{H}_h \otimes \mathcal{S}_{l_1}$

In this section, we will postulate a theorem which states the irreducible Spin(m) modules in the tensor product $\mathcal{H}_h \otimes \mathcal{S}_{l_1}$. Note that in the main case of this thesis, we only have considered the case where $h \geq l_1$. In this section, however, we intend to give a proof for all cases, which necessitates a case study: $h > l_1$, $h = l_1$, $h < l_1$. In order postulate a conjecture, we will first look at a few examples. We use LiE [87] to determine the decompositions.

Example 11.2. First, consider the case where $h = 5$ and $l_1 = 3$. The irreducible $\mathrm{Spin}(m)$-representations will, as usual be denoted by their highest weight.

$$
(5) \otimes (3)' \cong
\begin{array}{cccccccc}
(8)' & (7)' & (6)' & (5)' & (4)' & (3)' & (2)' & (1)' \\
 & (7,1)' & (6,1)' & (5,1)' & (4,1)' & (3,1)' & (2,1)' & \\
 & & (6,2)' & (5,2)' & (4,2)' & (3,2)' & & \\
 & & & (5,3)' & (4,3)' & & &
\end{array}
$$

Notice that there is a triangular structure, so the question rises whether this is a repeating pattern.

Example 11.3. For the next example, consider $h = 3$ and $l_1 = 5$. Then we have that

$$
(3) \otimes (5)' \cong
\begin{array}{ccccccc}
(8)' & (7)' & (6)' & (5)' & (4)' & (3)' & (2)' \\
 & (7,1)' & (6,1)' & (5,1)' & (4,1)' & (3,1)' & \\
 & & (6,2)' & (5,2)' & (4,2)' & & \\
 & & & (5,3)' & & &
\end{array}
$$

Again, there is a triangular structure, be it a different one than in the previous example.

Example 11.4. For the third example, we take $h = l_1 = 3$. Then the decomposition becomes

$$
(3) \otimes (3)' \cong
\begin{array}{ccccccc}
(6)' & (5)' & (4)' & (3)' & (2)' & (1)' & (0)' \\
 & (5,1)' & (4,1)' & (3,1)' & (2,1)' & (1,1)' & \\
 & & (4,2)' & (3,2)' & (2,2)' & & \\
 & & & (3,3)' & & &
\end{array}
$$

We then postulate the following theorem.

Theorem 11.1. *The decomposition of the* $\mathrm{Spin}(m)$*-representation* $(l_1) \otimes (l_2)'$ *is given by the following direct sums.*

(i) In the case $l_1 > l_2 = 0$*:*

$$(l_1) \otimes (l_2)' \cong (l_1)' \oplus (l_1 - 1)'$$

(ii) In the case $l_1 > l_2 > 0$:

$$(l_1) \otimes (l_2)' \cong \bigoplus_{i=0}^{l_2} \bigoplus_{j=0}^{i} ((l_1 + l_2 - 2i + j, j)' \oplus (l_1 + l_2 - 2i + j - 1, j)')$$

(iii) In the case $l_1 = l_2 = 0$:

$$(l_1) \otimes (l_2)' \cong (0)'$$

(iv) In the case $l_1 = l_2 > 0$:

$$(l_1) \otimes (l_2)' \\ \cong \bigoplus_{i=0}^{l_2-1} \bigoplus_{j=0}^{i} ((l_1 + l_2 - 2i + j, j)' \oplus (l_1 + l_2 - 2i + j - 1, j)') \oplus \bigoplus_{j=0}^{l_2} (j, j)'$$

(v) In the case $l_2 > l_1 = 0$:

$$(l_1) \otimes (l_2)' \cong (l_2)'$$

(vi) In the case $l_2 > l_1 > 0$:

$$(l_1) \otimes (l_2)' \cong \bigoplus_{i=0}^{l_1} \bigoplus_{j=0}^{i} (l_1 + l_2 - 2i + j, j)' \oplus \bigoplus_{i=0}^{l_1} \bigoplus_{j=1}^{i} (l_1 + l_2 - 2i + j, j - 1)'$$

Proof. (i), (iii) and (v) are trivial cases, which obviously are true. For the other cases, let us repeat a general result from (8.4):

$$(l_1) \otimes (l_2) \cong \bigoplus_{i=0}^{\min(l_1, l_2)} \bigoplus_{j=0}^{i} (l_1 + l_2 - 2i + j, j),$$

We will use this result, and combine it with the associativity of the tensor product. We start by proving the case (ii). This will be done by induction on l_2. For the induction basis, set $l_2 = 1$. In this case we have on the one hand that

$$(l_1) \otimes (1) \otimes (0)' = (l_1) \otimes ((1) \otimes (0)') = ((l_1) \otimes (1)') \oplus ((l_1) \otimes (0)') . \quad (11.2)$$

On the other hand, we have that

$$\begin{aligned}
&(l_1)\otimes(1)\otimes(0)' \qquad (11.3)\\
&= ((l_1)\otimes(1))\otimes(0)'\\
&= \left(\bigoplus_{i=0}^{l_2}\bigoplus_{j=0}^{i}(l_1+l_2-2i+j,j)\right)\otimes(0)'\\
&= ((l_1+1,0)\otimes(0)')\oplus((l_1-1,0)\otimes(0)')\oplus((l_1,1)\otimes(0)')\\
&= (l_1+1)'\oplus(l_1)'\oplus(l_1-1)'\oplus(l_1-2)'\\
&\quad \oplus(l_1,1)'\oplus(l_1)'\oplus(l_1-1,1)'\oplus(l_1-1)'
\end{aligned}$$

Since $(l_1)\otimes(0)' = (l_1)'\oplus(l_1-1)'$, we find from (11.2) and (11.4) that

$$(l_1)\otimes(1)' = (l_1+1)'\oplus(l_1)'\oplus(l_1-1)'\oplus(l_1-2)'\oplus(l_1,1)'\oplus(l_1-1,1)'.$$

Since the terms in this direct exactly are the ones mentioned in the statement of the theorem, this proves the induction basis. Now, suppose for the induction hypothesis that (ii) is correct for l_2-1, in other words,

$$\begin{aligned}
&(l_1)\otimes(l_2-1)'\\
&\cong \bigoplus_{i=0}^{l_2-1}\bigoplus_{j=0}^{i}((l_1+l_2-2i+j-1,j)'\oplus(l_1+l_2-2i+j-2,j)')\,. \qquad (11.4)
\end{aligned}$$

On the one side we have the double tensor product

$$(l_1)\otimes(l_2)\otimes(0)' = ((l_1)\otimes(l_2)')\oplus((l_1)\otimes(l_2-1)')\,, \qquad (11.5)$$

while on the other hand,

$$\begin{aligned}
&(l_1)\otimes(l_2)\otimes(0)'\\
&= ((l_1)\otimes(l_2))\otimes(0)'\\
&= \bigoplus_{i=0}^{l_2}\bigoplus_{j=0}^{i}(l_1+l_2-2i+j,j)\otimes(0)'\\
&= \bigoplus_{i=0}^{l_2}\bigoplus_{j=0}^{i}((l_1+l_2-2i+j,j)'\oplus(l_1+l_2-2i+j-1,j)')\\
&\quad \oplus\bigoplus_{i=1}^{l_2}\bigoplus_{j=1}^{i}((l_1+l_2-2i+j,j-1)'\oplus(l_1+l_2-2i+j-1,j-1)')\,.
\end{aligned}$$

Invoking (11.4), and altering the summation indices, we find that

$$\begin{aligned}
&\bigoplus_{i=1}^{l_2}\bigoplus_{j=1}^{i}((l_1+l_2-2i+j,j-1)' \oplus (l_1+l_2-2i+j-1,j-1)') \\
&= \bigoplus_{i=0}^{l_2-1}\bigoplus_{j=0}^{i}((l_1+l_2-1-2i+j,j)' \oplus (l_1+l_2-1-2i+j-1,j)') \\
&= (l_1)\otimes(l_2-1)'
\end{aligned}$$

This finishes the proof of (ii). Next, we will tackle (iv). First, consider the case where $l_1 = l_2 = 1$. On the one hand, we get

$$(1)\otimes(1)\otimes(0)' = ((1)\otimes(1)') \oplus ((1)\otimes(0)')\,.$$

On the other hand,

$$\begin{aligned}
&(1)\otimes(1)\otimes(0)' \\
&= \bigoplus_{i=0}^{1}\bigoplus_{j=0}^{i}(2-2i+j,j)\otimes(0)' \\
&= (2)'\oplus(1)'\oplus(0)'\oplus(1,1)'\oplus(1)'\oplus(0)'.
\end{aligned}$$

Since $(1)\otimes(0)' = (1)'\oplus(0)'$, we find that

$$(1)\otimes(1)' = (2)'\oplus(1)'\oplus(0)'\oplus(1,1)',$$

exactly the components appearing in (iv). In general, we have on the one hand that

$$(l_1)\otimes(l_1)\otimes(0)' = ((l_1)\otimes(l_1)')\oplus((l_1)\otimes(l_1-1)')\,,$$

while this also equals

$$\begin{aligned}
&(l_1)\otimes(l_1)\otimes(0)' \\
&= \bigoplus_{i=0}^{l_2}\bigoplus_{j=0}^{i}(2l_1-2i+j,j)\otimes(0)' \\
&= \bigoplus_{i=0}^{l_2-1}\bigoplus_{j=0}^{i}(2l_1-2i+j,j)'\oplus(2l_1-2i+j-1,j)' \\
&\quad \oplus\bigoplus_{j=0}^{l_2}(j,j)' \\
&\quad \oplus\bigoplus_{i=1}^{l_2}\bigoplus_{j=1}^{i}((2l_1-2i+j,j-1)'\oplus(2l_1-2i+j-1,j-1)')\,.
\end{aligned}$$

Using (ii), and rearranging the summation indices, we find that

$$\bigoplus_{i=1}^{l_2}\bigoplus_{j=1}^{i}((2l_1-2i+j,j-1)'\oplus(2l_1-2i+j-1,j-1)')=(l_1)\otimes(l_1-1)'$$

This finishes the proof of (iv). Finally, we prove (vi) by induction on l_2. As induction basis, we take the case where $l_2=l_1+1$. We get on the one hand that

$$(l_1)\otimes(l_1+1)\otimes(0)'=((l_1)\otimes(l_1+1)')\oplus((l_1)\otimes(l_1)')\,,$$

while also

$$\begin{aligned}&(l_1)\otimes(l_1+1)\otimes(0)' \qquad (11.6)\\ &=\bigoplus_{i=0}^{l_1}\bigoplus_{j=0}^{i}(l_1+1+l_1-2i+j,j)'\oplus(l_1+1+l_1-2i+j-1,j)'\\ &\quad\oplus\bigoplus_{i=1}^{l_1}\bigoplus_{j=1}^{i}(l_1+1+l_1-2i+j,j-1)'\oplus(l_1+1+l_1-2i+j-1,j-1)'.\end{aligned}$$

It holds that

$$\begin{aligned}&\bigoplus_{i=0}^{l_1}\bigoplus_{j=0}^{i}(l_1+1+l_1-2i+j-1,j)'\oplus\bigoplus_{i=1}^{l_1}\bigoplus_{j=1}^{i}(l_1+1+l_1-2i+j-1,j-1)'\\ &=\bigoplus_{i=0}^{l_1-1}\bigoplus_{j=0}^{i}((l_1+l_1-2i+j,j)'\oplus(l_1+l_1-2i+j-1,j)')\oplus\bigoplus_{j=0}^{l_1}(j,j)'\\ &=(l_1)\otimes(l_1)'. \qquad (11.7)\end{aligned}$$

Subtracting (11.7) from (11.6) exactly gives us the terms in (vi). This proves the induction basis. Take as induction hypothesis the case l_2-1, or

$$(l_1)\otimes(l_2-1)'=\bigoplus_{i=0}^{l_1}\bigoplus_{j=0}^{i}(l_1+l_2-1-2i+j,j)'\oplus\bigoplus_{i=1}^{l_1}\bigoplus_{j=1}^{i}(l_1+l_2-2i+j,j-1)'.$$

Using the same argument again, we have that on the one side

$$(l_1)\otimes(l_2)\otimes(0)'=((l_1)\otimes(l_2)')\oplus((l_1)\otimes(l_2-1)')\,,$$

and on the other side

$$\begin{aligned}&(l_2)\otimes(l_1)\otimes(0)'\\ &=\bigoplus_{i=0}^{l_1}\bigoplus_{j=0}^{i}((l_1+l_2-2i+j,j)'\oplus(l_1+l_2-2i+j-1,j)')\\ &\quad\oplus\bigoplus_{i=1}^{l_1}\bigoplus_{j=1}^{i}((l_1+l_2-2i+j,j-1)'\oplus(l_1+l_2-2i+j-1,j-1)')\,.\end{aligned}$$

Using the induction hypothesis, also part (vi) is proven. □

Remark 11.1. The decomposition of $\mathcal{H}_h \otimes \mathcal{S}_{l_1}$ is multiplicity-free in each case.

These direct sums might seem a bit difficult to grasp, but from the examples, we find a triangular structure emerging. This is indeed the case in general, but we need to reorder the summands to see this. A nice visualisation is given on the next page.

In the case where $l_1 > l_2$, we find the triangular structure

$$(l_1) \otimes (l_2)' \cong$$

$$\begin{array}{cccccccc}
(l_1+l_2)' & (l_1+l_2-1)' & (l_1+l_2-2)' & \cdots & \cdots & (l_1-l_2+1)' & (l_1-l_2)' & (l_1-l_2-1)' \\
 & (l_1+l_2-1,1)' & (l_1+l_2-2,1)' & \cdots & \cdots & (l_1-l_2+1,1)' & (l_1-l_2,1)' & \\
 & & (l_1+l_2-2,2)' & \cdots & \cdots & (l_1-l_2+1,2)' & & \\
 & & \ddots & \vdots & \vdots & ⋰ & & \\
 & & & (l_1,l_2)' & (l_1-1,l_2)' & & &
\end{array} \tag{11.8}$$

In the case where $l_1 \leq l_2$, we find the structure

$$(l_1) \otimes (l_2)' \cong$$

$$\begin{array}{ccccccc}
(l_2+l_1)' & (l_2+l_1-1)' & (l_2+l_1-2)' & \cdots & (l_2-l_1+2)' & (l_2-l_1+1)' & (l_2-l_1)' \\
 & (l_2+l_1-1,1)' & (l_2+l_1-2,1)' & \cdots & (l_2-l_1+2,1)' & (l_2-l_1+1,1)' & \\
 & & (l_2+l_1-2,2)' & \cdots & (l_2-l_1+2,2)' & & \\
 & & \ddots & \vdots & ⋰ & & \\
 & & & (l_2,l_1)' & & &
\end{array} \tag{11.9}$$

11.3 Rarita-Schwinger case

In this section, we check whether formula (11.1) in Section 11.1 matches with the known results for the Rarita-Schwinger operator (e.g. [20]). In this case, $\lambda = (l_1)$, so

$$\begin{cases} h = 2\kappa & \mathcal{K}_{h,(l_1)} \cong \bigoplus_{j=0}^{\kappa}(h-2j)\otimes(l_1)' - \bigoplus_{j=0}^{\kappa-1}(h-2j-1)\otimes(l_1)' \\ h = 2\kappa+1 & \mathcal{K}_{h,(l_1)} \cong \bigoplus_{j=0}^{\kappa}(h-2j)\otimes(l_1)' - \bigoplus_{j=0}^{\kappa}(h-2j-1)\otimes(l_1)'. \end{cases}$$

To get a feeling of how the representations in the right sum get cancelled out in the left sum, we give an example.

Example 11.5. Take $h = 5$ and $\lambda = (3)$. Then

$$\begin{aligned} \mathcal{K}_{h,(l_1)} \cong & \bigoplus_{j=0}^{2}(h-2j)\otimes(3)' - \bigoplus_{j=0}^{2}(h-2j-1)\otimes(3)' \\ \cong & (5)\otimes(3)' - (4)\otimes(3)' \oplus (3)\otimes(3)' \\ & - (2)\otimes(3)' \oplus (1)\otimes(3)' - (0)\otimes(3)'. \end{aligned}$$

Using the results from the previous section, we find that the tensor products appearing here decompose as follows:

$$(5)\otimes(3)' \cong$$

$$\begin{array}{cccccccc} (8)' & \underline{(7)'} & (6)' & \underline{(5)'} & (4)' & \underline{(3)'} & (2)' & \underline{(1)'} \\ & (7,1)' & \underline{(6,1)'} & (5,1)' & \underline{(4,1)'} & (3,1)' & \underline{(2,1)'} & \\ & & (6,2)' & \underline{(5,2)'} & (4,2)' & \underline{(3,2)'} & & \\ & & & (5,3)' & \underline{(4,3)'} & & & \end{array}$$

$$(4)\otimes(3)' \cong$$

$$\begin{array}{cccccccc} \underline{(7)'} & \underline{\underline{(6)'}} & \underline{(5)'} & \underline{\underline{(4)'}} & \underline{(3)'} & \underline{\underline{(2)'}} & \underline{(1)'} & \underline{\underline{(0)'}} \\ & \underline{(6,1)'} & \underline{\underline{(5,1)'}} & \underline{(4,1)'} & \underline{\underline{(3,1)'}} & \underline{(2,1)'} & \underline{\underline{(1,1)'}} & \\ & & \underline{(5,2)'} & \underline{\underline{(4,2)'}} & \underline{(3,2)'} & \underline{\underline{(2,2)'}} & & \\ & & & \underline{(4,3)'} & \underline{\underline{(3,3)'}} & & & \end{array}$$

$$(3)\otimes(3)' \cong$$

$$\begin{array}{ccccccc} \underline{\underline{(6)'}} & \utilde{(5)'} & \underline{\underline{(4)'}} & \utilde{(3)'} & \underline{\underline{(2)'}} & \utilde{(1)'} & \underline{\underline{(0)'}} \\ & \underline{\underline{(5,1)'}} & \utilde{(4,1)'} & \underline{\underline{(3,1)'}} & \utilde{(2,1)'} & \underline{\underline{(1,1)'}} & \\ & & \underline{\underline{(4,2)'}} & \utilde{(3,2)'} & \underline{\underline{(2,2)'}} & & \\ & & & \underline{\underline{(3,3)'}} & & & \end{array}$$

$$
\begin{array}{l}
(2)\otimes(3)' \cong \\
\qquad \begin{array}{ccccccccc}
\underset{\sim}{(5)'} & & \underset{-}{(4)'} & & \underset{\sim}{(3)'} & & \underset{-}{(2)'} & & \underset{\sim}{(1)'} \\
& \underset{\sim}{(4,1)'} & & \underset{-}{(3,1)'} & & \underset{\sim}{(2,1)'} & & & \\
& & & & \underset{\sim}{(3,2)'} & & & &
\end{array}
\end{array}
$$

$$
\begin{array}{l}
(1)\otimes(3)' \cong \\
\qquad \begin{array}{ccccc}
\underset{-}{(4)'} & & \underset{\cdots}{(3)'} & & \underset{-}{(2)'} \\
& & \underset{-}{(3,1)'} & &
\end{array}
\end{array}
$$

$$
(0)\otimes(3)' \cong \underset{\cdots}{(3)'}
$$

We observe that it is not possible to cancel all components of a tensor product with a minus sign in one tensor product with a plus sign. However, it seems that all remaining components can be found in the 'largest' tensor product. In the case of the example, that is $(5)\otimes(3)'$. If we take a closer look at the remaining irreducible components in this 'largest' tensor product, we find something that is very similar to (8.4). More specifically, we find that

$$
\mathcal{K}_{5,(3)} \cong [(5)\otimes(3)] \boxtimes (0)',
$$

where the Cartan product should be understood as taking the Cartan product of each irreducible component in $(5)\otimes(3)$ with $(0)'$. This makes us wonder if this is true in general as well.

Remark 11.2. Note that the brackets in $[(h)\otimes(k)]\boxtimes(0)'$ can not be replaced, as there is *no associativity*. Indeed, using LiE, we find for an easy example that

$$
[(1)\otimes(1)]\boxtimes(0)' \cong (2)'\oplus(1,1)'\oplus(0)',
$$

while

$$
(1)\otimes[(1)\boxtimes(0)'] \cong (2)'\oplus(1,1)'\oplus(1)'\oplus(0)'.
$$

Proposition 11.1. *If $h \geq k$, then*

$$
\mathcal{K}_{h,(k)} \cong [(h)\otimes(k)]\boxtimes(0)'
$$

Proof. From [20], we find that

$$
\mathcal{K}_{h,(k)} \cong \bigoplus_{j=0}^{k}\bigoplus_{i=0}^{k-j}\mathcal{S}_{l+k-j-2i,j}.
$$

On the other hand,

$$\mathcal{H}_h \otimes \mathcal{H}_k \cong \bigoplus_{i=0}^{k} \bigoplus_{j=0}^{i} \mathcal{H}_{l+k+j-2i,j}.$$

Rearranging the summation indices shows that the components in the sum are exactly the same up to a Cartan product with $\mathbb{S}$. □

11.4 Higher spin Dirac operators in general

The observations and proof above give rise to the following conjecture for a general HSD operator.

Conjecture 11.1. *For all positive integers h and dominant weights λ*

$$\mathcal{K}_{h,\lambda} \cong [(h) \otimes \lambda] \boxtimes (0)'.$$

We failed to find a proof for this conjecture. However, trying numerous examples suggests that it indeed is true. Let us give one example in the case where $h = 6$ and $\lambda = (4, 2)$. Using LiE again, we find that

$$\begin{array}{l}\bigoplus_{j=0}^{3}(2j) \otimes (4,2)' - \bigoplus_{j=0}^{2}(2j+1) \otimes (4,2)' \cong \\
\qquad \begin{array}{ccccc} (8,0)' & (9,1)' & (10,2)' & & \\ (6,0)' & 2(7,1)' & 2(8,2)' & (9,3)' & \\ (4,0)' & 2(5,1)' & 3(6,2)' & 2(7,3)' & (8,4)' \\ & (3,1)' & 2(4,2)' & 2(5,3)' & (6,4)' \\ & & 2(2,2)' & (3,3)' & (4,4)' \end{array} \\
\oplus \\
\qquad \begin{array}{cccc} (8,1,1)' & (9,2,1)' & & \\ (6,1,1)' & 2(7,2,1)' & 2(8,3,1)' & \\ (4,1,1)' & 2(5,2,1)' & 2(6,3,1)' & (7,4,1)' \\ & 2(3,2,1)' & (4,3,1)' & (5,4,1)' \end{array} \\
\oplus \\
\qquad \begin{array}{ccc} (8,2,2)' & (7,3,2)' & (6,4,2)' \\ (6,2,2)' & (5,3,2)' & \\ (4,2,2)' & & \end{array}\end{array}$$

This indeed equals $[(6) \otimes (4, 2)] \boxtimes (0)'$. Moreover, we can check this result with [82], where this decomposition was obtained as well.

No simplicity of mind, no obscurity of station, can escape the universal duty of questioning all that we believe.

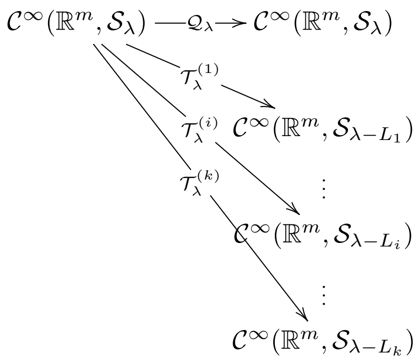
William Kingdon Clifford

12

Decomposition of ∂_x^T

Throughout this thesis, we have regularly mentioned the following scheme.

$$\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda) \xrightarrow{\partial_x^T} \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_\lambda \otimes \mathbb{S})$$



It tells us that the twisted Dirac operator ∂_x^T restricted to the function space $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda)$, can be written as the sum of the HSD operator $\mathcal{Q}_\lambda$ and at most k HST operators. However, this result is not as straightforward as might be expected. As the image space of the twisted Dirac operator is a subspace of $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_\lambda \otimes \mathbb{S})$, and the image spaces of the HST operators are of the form $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda-L_i})$, we see that the degrees of homogeneity are different. This means that the spaces $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda-L_i})$ must be embedded in $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_\lambda \otimes \mathbb{S})$ by means of a non-trivial embedding factor. The first idea would be to multiply with u_i, since that would fix the degrees of homogeneity. However, it is not guaranteed that $u_i\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda-L_i}) \subset \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_\lambda \otimes \mathbb{S})$. Hence we need a projection operator, which projects on the latter space. This will be the extremal projector related to the classical Lie algebra $\mathfrak{sp}(2k, \mathbb{C})$.

12.1 An extremal projector for $\mathfrak{sp}(2k)$

In previous chapters, we already discussed the general approach for constructing an extremal projector for a Lie (super)algebra, and we even constructed it explicitly for $\mathfrak{osp}(1, 2k)$. Here, we need an similar projector, which does not project on simplicial monogenic valued functions, but on simplicial harmonic functions. As already discussed, the ideal candidate to consider in that case is the classical Lie algebra $\mathfrak{sp}(2k, \mathbb{C})$. We will apply the techniques from Section 4.2 to the particular case $\mathfrak{g} = \mathfrak{sp}(2k+2, \mathbb{C})$ and $\mathfrak{k} = \mathfrak{sp}(2k, \mathbb{C})$. An elegant model for this Lie superalgebra comes from Clifford analysis in k vector variables $(u_1, \ldots, u_k) \in \mathbb{R}^{k\times m}$ and their corresponding Dirac operators $(\partial_1, \ldots, \partial_k)$:

$$\mathfrak{sp}(2k) = \mathrm{Alg}_{\mathbb{C}}\left\{-\frac{1}{2}\Delta_a, -\langle\partial_a, \partial_b\rangle, \langle u_i, \partial_j\rangle,\right.$$
$$\left.\frac{1}{2}|u_a|^2, \langle u_a, u_b\rangle, \langle u_j, \partial_i\rangle : 1 \le i < j \le k, 1 \le a \ne b \le k\right\},$$

seen as a subalgebra of the Weyl algebra $\mathcal{W}$. On the other hand, we have a decomposition of the symplectic Lie algebra of the form

$$\mathfrak{k} = \mathfrak{sp}(2k) = \mathfrak{k}^+ \oplus \mathfrak{h} \oplus \mathfrak{k}^-, \tag{12.1}$$

where the Cartan algebra $\mathfrak{h} \subset \mathfrak{k}$ is given by

$$\mathfrak{h} = \mathrm{Alg}_{\mathbb{C}}\left\{H_i := \mathbb{E}_i + \frac{m}{2} : 1 \le i \le k\right\},$$

and the suitably normalised root spaces

$$\mathfrak{k}^+ \cup \mathfrak{k}^- :=$$
$$\mathrm{span}\left(\langle u_i, \partial_j\rangle, \langle u_j, \partial_i\rangle, \langle\partial_a, \partial_b\rangle, \langle u_a, u_b\rangle : 1 \le i < j \le k, 1 \le a, b \le k\right).$$

Next, we will separate the positive from the negative root vectors by means of a suitable functional ℓ on $\mathfrak{h}^*$ which then fixes the parity of our roots. To do so, we will again *demand* our positive root vectors to be precisely the operators defining the simplicial harmonics (see definition 2.18). The reason for this is that the extremal projector $p_{\mathfrak{k}}$ for $\mathfrak{sp}(2k)$ has the property that $e_\alpha p_{\mathfrak{k}} = 0$ for all $\alpha \in \Delta^+$. Let us choose k real numbers $c_1, c_2, \ldots, c_k$ such that $c_k < \cdots < c_2 < c_1 < 0$, and consider the linear functional

$$\ell(a_1L_1 + a_2L_2 + \cdots + a_kL_k) := a_1c_1 + a_2c_2 + \cdots + a_kc_k,$$

where we again have the standard dual basis $L_i = H_i^*$ for which $L_i(H_j) = \delta_{ij}$ (see e.g. [47]). Hence

$$\begin{aligned}
\mathfrak{k}^+ &= \operatorname{span}_{\mathbb{C}}\left\{-\frac{1}{2}\Delta_a, -\langle\partial_a,\partial_b\rangle, \langle u_i,\partial_j\rangle : 1 \le i < j \le k, 1 \le a \ne b \le k\right\}\\
\mathfrak{k}^- &= \operatorname{span}_{\mathbb{C}}\left\{\frac{1}{2}|u_a|^2, \langle u_a,u_b\rangle, \langle u_j,\partial_i\rangle : 1 \le i < j \le k, 1 \le a \ne b \le k\right\}.
\end{aligned}$$

Next, we define for each positive root $\alpha \in \Delta^+$ the corresponding Cartan element, as in the case $\mathfrak{osp}(1,2k)$:

$$\begin{aligned}
h_{-2L_a} &= \left[-\frac{\Delta_a}{2}, \frac{|u_a|^2}{2}\right] = -\left(\mathbb{E}_a + \frac{m}{2}\right)\\
h_{-L_a-L_b} &= [-\langle\partial_a,\partial_b\rangle, \langle u_a,u_b\rangle] = -\left(m + \mathbb{E}_a + \mathbb{E}_b\right)\\
h_{L_i-L_j} &= [\langle u_i,\partial_j\rangle, \langle u_j,\partial_i\rangle] = \mathbb{E}_i - \mathbb{E}_j.
\end{aligned}$$

The normalisation requirements for the even positive root vectors given in (4.4) are now satisfied, which explains the numerical coefficients and minus signs in our original choices (see above). In order to write down an explicit expression for the extremal projector, we first need to calculate the values $\rho_0(h_\alpha)$ for each $\alpha \in \Delta^+$:

α	h_α	$\rho_0(h_\alpha)$
$-2L_a$	$-\mathbb{E}_a - \frac{m}{2}$	a
$-L_a - L_b$	$-\mathbb{E}_a - \mathbb{E}_b - m$	$a+b$
$L_i - L_j$	$\mathbb{E}_i - \mathbb{E}_j$	$j-i.$

The operators corresponding to the roots then are given by

$$\begin{aligned}
p_{-2L_a} &= \sum_{s=0}^{\infty} \frac{1}{4^s s!}\frac{\Gamma(-\mathbb{E}_a - \frac{m}{2} + a + 1)}{\Gamma(-\mathbb{E}_a - \frac{m}{2} + a + 1 + s)}|u_a|^{2s}\Delta_a^s\\
p_{-L_a-L_b} &= \sum_{s=0}^{\infty} \frac{1}{s!}\frac{\Gamma(-\mathbb{E}_a - \mathbb{E}_b - m + a + b + 1)}{\Gamma(-\mathbb{E}_a - \mathbb{E}_b - m + a + b + s + 1)}\langle u_a,u_b\rangle^s\langle\partial_a,\partial_b\rangle^s\\
p_{L_i-L_j} &= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!}\frac{\Gamma(\mathbb{E}_i - \mathbb{E}_j + j - i + 1)}{\Gamma(\mathbb{E}_i - \mathbb{E}_j + j - i + 1 + s)}\langle u_j,\partial_i\rangle^s\langle u_i,\partial_j\rangle^s. \quad (12.2)
\end{aligned}$$

Remark 12.1. Note that $p_{-L_a-L_b}$ is different from the corresponding operator in the case of $\mathfrak{osp}(1,2k)$.

In order to construct the extremal projector for $\mathfrak{sp}(2k)$, we then need to fix a normal ordering on the set of positive roots. For instance, we have the normal orderings

$$\begin{aligned}&-2L_1, -L_1-L_2, -2L_2, -L_1-L_3, -L_2-L_3, -2L_3, \ldots, -2L_k,\\ &L_1-L_2, L_1-L_3, \ldots, L_1-L_k, L_2-L_3, L_2-L_4, \ldots, L_{k-1}-L_k,\end{aligned} \tag{12.3}$$

and

$$\begin{aligned}&L_1-L_2, L_1-L_3, \ldots, L_1-L_k, L_2-L_3, L_2-L_4, \ldots, L_{k-1}-L_k,\\ &-2L_k, \ldots, -2L_3, -L_2-L_3, -L_1-L_3, -2L_2, -L_1-L_2, -2L_1.\end{aligned} \tag{12.4}$$

These are the normal orderings we also found for $\mathfrak{osp}(1,2k)$, where the odd roots are left out. In view of our explicit model for $\mathfrak{sp}(2k,\mathbb{C})$ in terms of Dirac operators and vector variables, taking the product of the operators defined in (12.2) in any of the normal orderings above gives an operator which projects an arbitrary (homogeneous and $\mathbb{C}$-valued) polynomial $P(u_1,\ldots,u_k)$ onto its simplicial harmonic part.

12.2 Embedding factors

Let us elaborate upon the idea of higher spin twistor operators a bit more. Apart from their existence and uniqueness due to the argument of Stein and Weiss, we can look at them in another way. To that end, we have to return to the basics and start again with the twisted Dirac operator:

$$\partial_x^T : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_\lambda \otimes \mathbb{S}).$$

As mentioned earlier, the full tensor product decomposition of $\mathcal{H}_\lambda \otimes \mathbb{S}$ is given by

$$\mathcal{H}_\lambda \otimes \mathbb{S} \cong \bigoplus_{i_1=0}^{1} \cdots \bigoplus_{i_k=0}^{1} (l_1-i_1, \ldots, l_k-i_k)',$$

where each of the highest weights in the sum is kept if it satisfies the dominant weight condition. This means that we can decompose the twisted Dirac operator into at most 2^k first-order differential suboperators, by projecting onto each of the subspaces. However, due to [79], only $k+1$ of them are non-trivial: the higher spin Dirac operator, and k higher spin twistor operators. However, one should be very careful with this *decomposition of the Dirac operator*, since the image spaces of the twistor operators obtained are $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda-L_i})$. Since the image space of the twisted Dirac operator is $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_\lambda \otimes \mathbb{S})$, we will need non-trivial embedding factors $\mathcal{E}_i$:

$$\mathcal{E}_i : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda-L_i}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_\lambda \otimes \mathbb{S}).$$

Essentially, these embedding factors have to raise the degree of homogeneity in the vector variable u_i by one. The logical embedding factor would then be $\mathcal{E}_i = u_i$. However, for each $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda-L_i})$, we have that $\langle u_j, \partial_i \rangle u_i f = u_j f \neq 0$, for $j < i$, so we need to project on the space $\mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_\lambda \otimes \mathbb{S})$. This exactly is what the extremal projector of $\mathfrak{sp}(2k)$, constructed in the previous section, does. If we choose the normal ordering (12.4), the embedding operators are of the form

$$\begin{aligned}\mathcal{E}_i = p_{L_1-L_2} p_{L_1-L_3} \cdots p_{L_1-L_k} p_{L_2-L_3} p_{L_2-L_4} \cdots p_{L_{k-1}-L_k} \\ p_{-2L_k} \cdots p_{-2L_3} p_{-L_2-L_3} p_{-L_1-L_3} p_{-2L_2} - p_{L_1-L_2} p_{-2L_1}.\end{aligned}$$

Take $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{\lambda-\delta_i})$. We then have the relations

- $\Delta_i u_i f = -\partial_i \partial_i u_i f = -\partial_i(-m - 2\mathbb{E}_i - u_i \partial_i) f = 0$,
- $\Delta_j u_i f = u_i \Delta_j f = 0$ for all $i \neq j$,
- $\langle \partial_i, \partial_j \rangle u_i f = \partial_j f + u_i \langle \partial_i, \partial_j \rangle f = 0$ for all $i \neq j$,
- $\langle \partial_j, \partial_l \rangle u_i f = u_i \langle \partial_j, \partial_l \rangle f = 0$, for all $i \neq j$ and $i \neq l$,
- $\langle u_a, \partial_b \rangle u_i f = 0$ for all $b \neq i$ and $a < b$.

Thus, the expressions of the embedding factors reduce to

$$\begin{aligned}\mathcal{E}_i f &= p_{L_{k-1}-L_k} \cdots p_{L_2-L_4} p_{L_2-L_3} p_{L_1-L_k} \cdots p_{L_1-L_3} p_{L_1-L_2} u_i f \\ &= p_{L_{k-1}-L_k} \cdots p_{L_2-L_4} p_{L_2-L_3} p_{L_1-L_k} \cdots p_{L_1-L_i} u_i f.\end{aligned}$$

For all $i > 1$, we have that

$$\langle u_1, \partial_i \rangle u_i f = u_1 f \neq 0 \qquad \text{and} \qquad \langle u_1, \partial_i \rangle^2 u_i f = \langle u_1, \partial_i \rangle u_1 f = 0,$$

whence only the first two terms in the expression of $p_{L_1-L_i}$ act non-trivially. So we get that

$$\mathcal{E}_i f = p_{L_{k-1}-L_k} \cdots p_{L_1-L_{i+1}} \left(1 - \frac{\langle u_i, \partial_1 \rangle \langle u_1, \partial_i \rangle}{\mathbb{E}_1 - \mathbb{E}_i + i - 1 + 1}\right) u_i f.$$

For all $j > i$,

$$\begin{aligned}
&\langle u_1, \partial_j \rangle \left(1 - \frac{\langle u_i, \partial_1 \rangle \langle u_1, \partial_i \rangle}{\mathbb{E}_1 - \mathbb{E}_i + i} \right) u_i f \\
= \; & \frac{1}{\mathbb{E}_1 - \mathbb{E}_i + i} \langle u_1, \partial_j \rangle \langle u_i, \partial_1 \rangle \langle u_1, \partial_i \rangle u_i f \\
= \; & \frac{1}{\mathbb{E}_1 - \mathbb{E}_i + i} \langle u_1, \partial_j \rangle \langle u_i, \partial_1 \rangle u_1 f \\
= \; & \frac{1}{\mathbb{E}_1 - \mathbb{E}_i + i} (-\langle u_i, \partial_j \rangle + \langle u_i, \partial_1 \rangle \langle u_1, \partial_j \rangle) u_1 f \\
= \; & 0.
\end{aligned}$$

We can continue the same reasoning, resulting in the embedding factor

$$\mathcal{E}_i = \prod_{j=i-1}^{1} \left(1 - \frac{\langle u_i, \partial_j \rangle \langle u_j, \partial_i \rangle}{\mathbb{E}_j - \mathbb{E}_i + i - j + 1} \right)$$

in its simplest form. Note that the product is ordered from the highest to the lowest index.

12.3 An explicit decomposition of the twisted Dirac operator

Now that we know the form of the twistor operators and the embedding factors, we can determine the explicit decomposition of the twisted dirac operator

$$\partial_x^T : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_\lambda \otimes \mathbb{S}).$$

This decomposition must be of the form

$$\partial_x^T = \mathcal{Q}_\lambda + \sum_{l=1}^{k} c_l \mathcal{E}_l \mathcal{T}^{(l)}.$$

For shortness of notations, we omit the subindex λ from the twistor operators, as we did in Chapter 10. Indeed, using the Euler operators, the higher spin twistor operators do no longer depend on the space they are acting on. One should keep in mind though that they still depend on the length of the highest weights, which will however always be clear from the context. As for the embedding factors, their expressions do not depend on the representation they are acting on, nor on the length of the respective highest weight. In the expression above, it only remains to determine the constants c_l (in terms of Euler operators). In order to gain insight, we first consider the easy cases. First of all, we of course have the classical Dirac operator

$$\partial_x : \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{S}) \to \mathcal{C}^\infty(\mathbb{R}^m, (0) \otimes \mathbb{S}),$$

which is a trivial case, since there are no twistor operators appearing.

12.3.1 The case $k = 1$

We have that

$$\partial_x^T : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{l_1} \otimes \mathbb{S}).$$

One can easily see that the twisted Dirac operator in this case decomposes as follows:

$$\partial_x^T = \left(1 + \frac{u_1 \partial_1}{m + 2\mathbb{E}_1 - 2}\right) \partial_x + \underbrace{\frac{2}{m + 2\mathbb{E}_1 - 2}}_{(a)} \underbrace{u_1}_{(b)} \underbrace{\langle \partial_1, \partial_x \rangle}_{(c)},$$

where we have

- (a) the constant c_1
- (b) the embedding factor $\mathcal{E}_1$
- (c) the twistor operator $\mathcal{T}^{(1)}$

12.3.2 The case $k = 2$

In this case, the twisted Dirac operator acts as follows:

$$\partial_x^T : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1,l_2}) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{l_1,l_2} \otimes \mathbb{S}).$$

In order to find the explicit decomposition of this twisted Dirac operator, we need to calculate the constants c_1 and c_2 for which

$$\partial_x^T = \mathcal{Q}_{l_1,l_2} + c_1 \mathcal{E}_1 \mathcal{T}^{(1)} + c_2 \mathcal{E}_2 \mathcal{T}^{(2)}.$$

When acting on a function $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{l_1,l_2})$, the right-hand side of this equality is given by

$$\begin{aligned} &\left(1 + \frac{u_1 \partial_1}{m + 2\mathbb{E}_1 - 2}\right) \left(1 + \frac{u_2 \partial_2}{m + 2\mathbb{E}_2 - 4}\right) \partial_x f \\ &+ c_1 u_1 \left(1 - \frac{\langle u_2, \partial_1 \rangle \langle u_1, \partial_2 \rangle}{\mathbb{E}_1 - \mathbb{E}_2 + 2}\right) \langle \partial_1, \partial_x \rangle f \\ &+ c_2 \left(1 - \frac{\langle u_2, \partial_1 \rangle \langle u_1, \partial_2 \rangle}{\mathbb{E}_1 - \mathbb{E}_2 + 2}\right) u_2 \langle \partial_2, \partial_x \rangle f. \end{aligned}$$

This expression equals

$$\begin{aligned}
&\partial_x f - \frac{2}{m+2\mathbb{E}_1-2} u_1\langle \partial_1, \partial_x\rangle f - \frac{2}{m+2\mathbb{E}_2-4} u_2\langle \partial_2, \partial_x\rangle f \\
&\quad + \frac{4}{(m+2\mathbb{E}_1-2)(m+2\mathbb{E}_2-4)} u_1\langle u_2, \partial_1\rangle\langle \partial_2, \partial_x\rangle f \\
&\quad + c_1 u_1\langle \partial_1, \partial_x\rangle f + \frac{c_1}{\mathbb{E}_1-\mathbb{E}_2+1} u_1\langle u_2, \partial_1\rangle\langle \partial_2, \partial_x\rangle f \\
&\quad + c_2 u_2\langle \partial_2, \partial_x\rangle f - \frac{c_2}{\mathbb{E}_1-\mathbb{E}_2+2} u_2\langle \partial_2, \partial_x\rangle f \\
&\quad - \frac{c_2}{\mathbb{E}_1-\mathbb{E}_2+2} u_1\langle u_2, \partial_1\rangle\langle \partial_2, \partial_x\rangle f.
\end{aligned}$$

If we combine terms containing the same 'words' (elements of the universal enveloping algebra), we get

$$\begin{aligned}
&\partial_x f + \left(c_1 - \frac{2}{m+2\mathbb{E}_1-2}\right) u_1\langle \partial_1, \partial_x\rangle f \\
&\quad + \left(c_2 - \frac{2}{m+2\mathbb{E}_2-4} - \frac{c_2}{\mathbb{E}_1-\mathbb{E}_2+2}\right) u_2\langle \partial_2, \partial_x\rangle f \\
&\quad + \left(\frac{4}{(m+2\mathbb{E}_1-2)(m+2\mathbb{E}_2-4)} + \frac{c_1}{\mathbb{E}_1-\mathbb{E}_2+1} - \frac{c_2}{\mathbb{E}_1-\mathbb{E}_2+2}\right) \\
&\quad \times u_1\langle u_2, \partial_1\rangle\langle \partial_2, \partial_x\rangle f
\end{aligned}$$

Here, all coefficients should be zero, except the one of ∂_x. From the second and third coefficient, we get that

$$c_1 = \frac{2}{m+2\mathbb{E}_1-2}$$

and

$$c_2 = \frac{2}{m+2\mathbb{E}_2-4}\,\frac{\mathbb{E}_1-\mathbb{E}_2+2}{\mathbb{E}_1-\mathbb{E}_2+1}.$$

Substituting these in the final coefficient, it is easily checked that it indeed becomes zero. We thus have found:

$$\partial_x^T = \mathcal{Q}_{l_1,l_2} + \frac{2}{m+2\mathbb{E}_1-2}\mathcal{E}_1\mathcal{T}^{(1)} + \frac{2}{m+2\mathbb{E}_2-4}\,\frac{\mathbb{E}_1-\mathbb{E}_2+2}{\mathbb{E}_1-\mathbb{E}_2+1}\mathcal{E}_2\mathcal{T}^{(2)}.$$

12.3.3 General case

We now prove the most general case. Consider the twisted Dirac operator

$$\partial_x^T : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_\lambda \otimes \mathbb{S}),$$

then we want to determine the constants c_l for which

$$\partial_x^T = \mathcal{Q}_\lambda + \sum_{l=1}^{k} c_l \mathcal{E}_l \mathcal{T}^{(l)}.$$

To this end, let us first rewrite some operators. For any $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_\lambda)$, it holds that

$$\begin{aligned}\mathcal{Q}_\lambda f &= \partial_x f \\ &+ \sum_{1\leq a_1<\cdots<a_j\leq k} \left(\prod_{i=1}^{j} \frac{-2}{m+2\mathbb{E}_{a_i}-2a_i}\right) u_{a_1} \left(\prod_{i=1}^{j-1} \langle u_{a_{i+1}}, \partial_{a_i}\rangle\right) \langle \partial_{a_j}, \partial_x\rangle f.\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}\mathcal{T}^{(l)} f &= \langle \partial_l, \partial_x\rangle f \\ &+ \sum_{l=a_1<\cdots<a_i\leq k} \left(\prod_{j=1}^{i} \frac{1}{\mathbb{E}_l - \mathbb{E}_{a_j} + a_j - l + 1}\right) \left(\prod_{j=1}^{i-1} \langle u_{a_{j+1}}, \partial_{a_j}\rangle\right) \langle \partial_{a_i}, \partial_x\rangle f.\end{aligned}$$

So we get that

$$\begin{aligned}\mathcal{E}_l \mathcal{T}^{(l)} = &\prod_{b=l-1}^{1} \left(1 - \frac{\langle u_l, \partial_b\rangle \langle u_b, \partial_l\rangle}{\mathbb{E}_b - \mathbb{E}_l + l - b + 1}\right) u_l \\ &\times \left(\langle \partial_l, \partial_x\rangle + \sum_{l=a_1<\cdots<a_i\leq k} \left(\prod_{j=1}^{i} \frac{1}{\mathbb{E}_l - \mathbb{E}_{a_j} + a_j - l + 1}\right)\right. \\ &\times \left.\left(\prod_{j=1}^{i-1} \langle u_{a_{j+1}}, \partial_{a_j}\rangle\right) \langle \partial_{a_i}, \partial_x\rangle\right).\end{aligned}$$

With these expressions, we can simplify $\mathcal{Q}_\lambda + c_1\mathcal{E}_1\mathcal{T}^{(1)} + \cdots + c_k\mathcal{E}_k\mathcal{T}^{(k)}$. When we use the convention of rewriting each term in this sum in the following form:

$$\text{(coefficient with Euler operators)} \cdot u_{b_1}\langle u_{b_2}, \partial_{b_1}\rangle \cdots \langle u_{b_i}, \partial_{b_{i-1}}\rangle \langle \partial_{b_i}, \partial_x\rangle,$$

with $b_1 < b_2 < \cdots < b_i$, then we get the following coefficients:

- the coefficient of ∂_x equals 1, as we expected;
- the coefficient of $u_1\langle \partial_1, \partial_x\rangle$ is

  $$\frac{-2}{m+2\mathbb{E}_1-2} + c_1,$$

  which has to be zero, whence we find $c_1 = \frac{2}{m+2\mathbb{E}_1-2}$;

- similarly, we find that the coefficient of $u_2\langle\partial_2,\partial_x\rangle$ equals

$$\frac{-2}{m+2\mathbb{E}_2-4}+c_2-\frac{c_2}{\mathbb{E}_1-\mathbb{E}_2+2-1+1}.$$

  This coefficient also has to equal to zero, whence

$$c_2=\frac{2}{m+2\mathbb{E}_2-4}\frac{\mathbb{E}_1-\mathbb{E}_2+2}{\mathbb{E}_1-\mathbb{E}_2+1};$$

- for the coefficient of $u_3\langle\partial_3,\partial_x\rangle$, we find

$$\begin{aligned}&\frac{-2}{m+2\mathbb{E}_3-6}+c_3-\frac{c_3}{\mathbb{E}_1-\mathbb{E}_3+3}-\frac{c_3}{\mathbb{E}_2-\mathbb{E}_3+2}\\&\qquad+\frac{c_3}{(\mathbb{E}_1-\mathbb{E}_3+3)(\mathbb{E}_2-\mathbb{E}_3+2)}\\&=\frac{-2}{m+2\mathbb{E}_3-6}+c_3\left(1-\frac{1}{\mathbb{E}_1-\mathbb{E}_3+3}\right)\left(1-\frac{1}{\mathbb{E}_2-\mathbb{E}_3+2}\right),\end{aligned}$$

  which must equal zero as well, yielding

$$c_3=\frac{2}{m+2\mathbb{E}_3-6}\frac{\mathbb{E}_1-\mathbb{E}_3+3}{\mathbb{E}_1-\mathbb{E}_3+2}\frac{\mathbb{E}_2-\mathbb{E}_3+2}{\mathbb{E}_2-\mathbb{E}_3+1};$$

- last of all, the general coefficient of $u_l\langle\partial_l,\partial_x\rangle$ equals

$$\begin{aligned}&\frac{-2}{m+2\mathbb{E}_l-2l}+c_l\left(1-\frac{1}{\mathbb{E}_1-\mathbb{E}_l+l-1+1}\right)\cdots\\&\qquad\times\left(1-\frac{1}{\mathbb{E}_{l-1}-\mathbb{E}_l+l-(l-1)+1}\right)\end{aligned}$$

  or

$$c_l=\frac{2}{m+2\mathbb{E}_l-2l}\prod_{j=1}^{l-1}\left(\frac{\mathbb{E}_j-\mathbb{E}_l+l-j+1}{\mathbb{E}_j-\mathbb{E}_l+l-j}\right).$$

This completes the decomposition of the twisted Dirac operator.

12.4 Conclusion

In this chapter, we found the explicit decomposition of the twisted Dirac operator in terms of a higher spin Dirac operator and at most k higher spin twistor operators. To achieve this goal, we needed to calculate the embedding operators $\mathcal{E}_i:\mathcal{S}_{\lambda-L_i}\to\mathcal{H}_\lambda\otimes\mathbb{S}$, as well as a simplicial harmonic projection operator $p_{\mathfrak{sp}(2k)}$, which are interesting results in itself.

Nederlandse samenvatting

Binnen de theorie van Riemann-variëteiten bestaat een heel systeem van conforme invariante eerste orde afleidingsoperatoren (zie bvb. [16, 45, 73, 79]). Het doel van deze thesis is de studie van deze operatoren. De bekendste van deze verzameling operatoren is de zogenaamde Dirac-operator. Deze operator beeldt spinorwaardige functies af op functies in dezelfde ruimte. Bij het onderzoeken van deze operatoren is het een belangrijk gegeven dat ze rotatie-invariant zijn. Met andere woorden, de operatoren zijn invariant onder de actie van de spingroep of zijn orthogonale Lie-algebra $\mathfrak{so}(m, \mathbb{C})$. Doorheen de laatste decennia werd de Dirac-operator ook onderzocht vanuit een functietheoretisch perspectief (bvb. de studie van polynomiale oplossingen, integraalrepresentaties, speciale functies etc.). Standaardreferenties voor dit onderzoek zijn o.a. [12, 30, 48]

De laatste jaren is gebleken dat cliffordanalyse een heel elegant kader vormt voor de studie van de bovenstaande functietheoretische problemen, niet alleen voor de Dirac-operator, maar ook voor veralgemeende versies van deze operator. Deze veralgemeende operatoren zijn operatoren die werken op functies die waarden aannemen in algemene irreducibele representaties $\mathbb{V}_\lambda^\pm$ van de spingroep, met *highest weight* $\lambda = \left(l_1 + \frac{1}{2}, \ldots, l_{n-1} + \frac{1}{2}, \pm\frac{1}{2}\right)$, waarbij $n = \left\lfloor \frac{m}{2} \right\rfloor$, met m de dimensie van de onderliggende vectorruimte. Om onze notaties niet nodeloos ingewikkeld te maken beperken we onszelf tot een oneven dimensie m. Dit heeft tot gevolg dat de laatste component van λ enkel $+\frac{1}{2}$ kan zijn en we dus geen $\pm$ bij de representatie hoeven te schrijven. Merk wel op dat, ondanks deze conventie, alle resultaten in deze thesis zonder meer kunnen veralgemeend worden tot het geval van een even dimensie.

Voor elke geschikte keuze van $l_1, \ldots, l_{n-1}$ bestaat er een hogere-spin Dirac-operator $\mathcal{Q}_\lambda$ die gedefinieerd is als

$$\mathcal{Q}_\lambda : \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{V}_\lambda) \to \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{V}_\lambda).$$

De klassieke Dirac-operator is het speciale geval waarbij $l_1 = \cdots = l_{n-1} = 0$ gekozen wordt. De irreducibele representatie $\mathbb{V}_\lambda$ kan dan namelijk gemodelleerd worden door de ruimte van Dirac-spinoren $\mathbb{S}$. Vanuit dit standpunt kan hogere-spin cliffordanalyse beschouwd worden als een veralgemening van de klassieke cliffordanalyse. De oorsprong van de Dirac-operator zelf bevindt zich trouwens in de deeltjesfysica, waar hij door P.A.M. Dirac gebruikt werd om het gedrag van elektronen te beschrijven, fundamentele deeltjes met spingetal $\frac{1}{2}$.

Een belangrijke doorbraak werd gemaakt in [48, 86], waar aangetoond werd dat alle eindigdimensionale irreduciebele representaties $\mathbb{V}_\lambda$ van de spingroep gemodelleerd kunnen worden door bepaalde polynomiale ruimten. Dat betekent dat hogere-spin cliffordanalyse kan steunen op resultaten uit de functietheorie in combinatie met resultaten uit representatietheorie.

Een eerste veralgemening van de Dirac-operator is het geval waarbij $l_1 = 1$ en $l_2 = \cdots = l_{n-1} = 0$ gekozen worden. De resulterende operator is opnieuw bekend in de theoretische fysica. Hij werd voor de eerste maal gebruikt door Rarita en Schwinger, deze keer voor het beschrijven van elementaire fermionische deeltjes met spingetal $\frac{3}{2}$. Binnen de cliffordanalyse zijn verdere veralgemeningen gemaakt, waarbij $\lambda = \left(l_1 + \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$, $l_1 \in \mathbb{N}$ gekozen werd. Deze operatoren kregen de naam Rarita-Schwinger operatoren, naar de auteurs van het oorspronkelijke artikel. De eerste resultaten omtrent deze operatoren komen van de handen van J. Bureš, F. Sommen, V. Souček en P. Van Lancker (zie [20, 21]). In deze artikels worden tal van eigenschappen in verband met deze operatoren bestudeerd, beginnend met hun expliciete gedaante binnen de cliffordanalytische functietheorie als afleidingsoperatoren. Ook werd een fundamentele oplossing gevonden, evenals een volledige decompositie van de polynomiale kern in irreduciebele representaties van de spingroep. Bovendien werd ook bewezen dat de Rarita-Schwinger operatoren conform invariant zijn. De laatste jaren werden zelfs veralgemeningen van de Rarita-Schwinger operatoren op de sfeer bestudeerd in bvb. [83].

Het doel van deze thesis is om een aantal van deze resultaten te veralgemenen voor algemene hogere-spin Dirac-operatoren, met andere woorden voor algemene keuzes van $l_1, \ldots, l_{n-1}$. We gaan nu per hoofdstuk een overzicht geven van de inhoud van deze thesis.

We starten met de basisbegrippen in hoofdstuk 2. Hier zullen we cliffordalgebra's of geometrische algebra's introduceren, samen met definities, eigenschappen en andere belangrijke resultaten die horen bij deze algebra's. Standaardreferenties zijn bijvoorbeeld [12, 30, 48]. Twee groepen worden bediscussieerd, namelijk de spingroep en de pingroep, die dubbele bedekkingen zijn van respectievelijk de speciale orthogonale groep $\mathrm{SO}(m)$ en de orthogonale groep $\mathrm{O}(m)$. Verder zullen we aantonen dat beide groepen gerealiseerd kunnen worden binnen een cliffordalgebra. Aangezien veel resultaten gebruik zullen maken van representatietheoretische argumenten, zullen we ook groeprepresentaties introduceren, samen met een classificatie van de eindigdimensionale irreduciebele representaties van de spingroep.

In hoofdstuk 3 graven we dieper in de representatietheorie. Meer bepaald zullen we kijken naar eindigdimensionale representaties van enkele klassieke Lie-algebra's, aangezien deze enorm belangrijk blijken voor deze thesis. Standaardreferenties zijn hiervoor [47, 52]. In het eerste deel van dit hoofdstuk worden Lie-algebras en hun algemene eigenschappen ingevoerd. Eén specifiek type van Lie-algebra's wordt nader bestudeerd, namelijk de simpele Lie-algebra's. Binnen dit type Lie-algebra's is de eenvoudigste de zogenaamde speciale lineare Lie-algebra $\mathfrak{sl}(2, \mathbb{C})$. Deze algebra is dan ook de perfecte kandidaat om te starten bij het bestuderen van Lie-algebra-representaties. Verder worden ook representaties van symplectische en orthogonale Lie-algebra's geclassificeerd.

In hoofdstuk 4 zetten we een eerste stap in de richting van ons doel om hogere-spin Dirac-operatoren te bestuderen. In dit hoofdstuk wordt een methode ontwikkeld die ons in staat stelt om zulke operatoren te construeren. Hierbij wordt gebruik gemaakt van een algebraïsch concept dat een transvectoralgebra wordt genoemd. Eerst wordt de algemene theorie verduidelijkt, die gebaseerd is op het werk van Zhelobenko, Mickelsson en Molev [90, 63, 66]. Om een verzameling van generatoren te bekomen voor een dergelijke transvectoralgebra wordt een zogenaamde extremale projectieoperator geconstrueerd. Nadien wordt dit hele verhaal vertaald naar cliffordanalyse. De eigenschappen voor extremale projectieoperatoren zullen ons in staat stellen om hogere-spin operatoren te construeren. We zullen twee types hogere-spin operatoren van naderbij bestuderen, namelijk de hogere-spin Dirac-operatoren die al eerder vermeld werden, en de hogere-spin twistor-operatoren. Het is algemeen bekend dat de klassieke Dirac-operator conform invariant is. Daarom wordt in dit hoofdstuk ook de conforme invariantie van deze twee types hogere-spin operatoren bewezen.

Een transvectoralgebra is een een moeilijk en abstract algebraïsch concept. Daarom wordt in hoofdstuk 5 een expliciet voorbeeld uitgewerkt, die de algemene constructie van hoofdstuk 4 verder verklaart. Het doel van dit hoofdstuk is om in dit specifieke voorbeeld de expliciete gedaantes te construeren van de generatoren van de transvectoralgebra.

In de studie van afleidingsoperatoren is het bestaan van een fundamentele oplossing van de operator een belangrijke eigenschap. In hoofdstuk 6 wordt een fundamentele oplossing gezocht voor de hogere-spin Dirac-operator, hierbij gebruik makend van Riesz-potentialen en distributietheorie. Deze fundamentele oplossing wordt dan gebruikt om drie integraalformules te bewijzen voor de hogere-spin Dirac-operator, namelijk de stelling van Stokes, de Cauchy-Pompeiu-formule en de Cauchy-integraalformule.

De klassieke Cauchy-Kovalavskaya-uitbreidingsstelling (bvb. [30]) vertelt ons dat er een isomorfisme bestaat tussen de ruimte van spinorwaardige polynomen in de kern van de Dirac-operator die homogeen zijn van graad

k en de totale ruimte van spinorwaardige polynomen van dezelfde graad in één veranderlijke minder. In hoofdstuk 7 wordt een analogon van deze stelling bewezen voor hogere-spin Dirac-operatoren. Dit zal ons in staat stellen om de dimensie van de h-homogene polynomiale kern te bepalen voor de hogere-spin Dirac-operator.

Het uiteindelijk doel van deze thesis is het vinden van een decompositie van de h-homogene polynomiale kern van de hogere-spin Dirac-operator in irreduciebele spinmodules. Hoofdstuk 8, 9 en 10 zijn hieraan gewijd.

In hoofdstuk 8 introduceren we getwiste operatoren. Dit zijn klassieke operatoren die werken op functies die waarden aannemen in een 'verkeerde' ruimte. Hogere-spin Dirac-operatoren hebben een ingebouwde inductieve structuur, die onthuld wordt in dit hoofdstuk. Eerst en vooral wordt de getwiste Dirac-operator geïntroduceerd, die de eigenschap heeft dat hij geschreven kan worden als de som van een hogere-spin Dirac-operator en hoogstens k twistor-operatoren. Hierbij is k de orde van de hogere-spin Dirac-operator (of nog, het aantal niet triviale componenten in λ). Deze relatie tussen een getwiste operator en een gewone hogere-spin Dirac-operator zal gebruikt worden. We zullen een soorgelijk verband uitwerken tussen een hogere-spin Dirac-operator van orde k en een getwiste hogere-spin Dirac-operator van orde $k-1$. De bewijzen in dit hoofdstuk steunen op argumenten uit de representatietheorie, die verzameld zijn in de laatste sectie van dit hoofdstuk. Belangrijk is dat het verband tussen hogere-spin Dirac-operatoren van verschillende orde suggereert dat er eveneens een relatie is tussen de polynomiale kernen van deze operatoren.

In hoofdstuk 9 tonen we aan dat de ruimte van oplossingen van de hogere-spin Dirac-operator kan geclassificeerd worden in twee deelverzamelingen, zogenaamde type A en type B oplossingen. Er zal worden aangetoond dat de type A oplossingen kunnen worden voorgesteld door een speciale ruimte van veeltermen, die we scheve simpliciaalmonogenen zullen noemen. Deze ruimte is echter geen irreduciebele representatie voor de spingroep, maar we kunnen opnieuw gebruik maken van transvectoralgebra's om tot een decompositie in irreduciebele modulen te komen. Meer zelfs, we zullen aantonen hoe deze modulen kunnen ingebed worden in de ruimten van scheve simpliciaalmonogenen.

In hoofdstuk 10 worden de resultaten van beide voorgaande hoofdstukken samengevoegd. Het belangrijkste resultaat in dit hoofdstuk is dat de verzameling van h-homogene veeltermen in de kern van een hogere-spin Dirac-operator bevat is in de directe som van type A oplossingen (als representaties) van verschillende hogere-spin Dirac-operatoren. Wanneer we voorbeelden uitrekenen merken we dat deze inclusie in werkelijkheid steeds een isomorfisme blijkt te zijn. We zijn er evenwel niet in geslaagd deze observatie in zijn algemeenheid sluitend aan te tonen. Wel hebben we het probleem kunnen herleiden tot een combinatorisch vraagstuk, gebruik makend van dimensie-analyse. Jammer genoeg worden de (symbolische) formules zo groot dat we het combinatorisch probleem niet hebben kunnen oplossen voor algemene orde van de hogere-spin Dirac-operator.

De inductieve aanpak is niet de enige aanpak die men kan gebruiken om de polynomiale kern te beschrijven van hogere-spin Dirac-operatoren. Gebruik makend van de veralgemeende CK-uitbreiding en *branching* regels in hoofdstuk 11, kunnen we de polynomiale kern van de hogere-spin Dirac-operator herschrijven als een directe som van tensorproducten. Dit geeft nog geen volledige decompositie van de kern, aangezien er in de ontbinding dus nog steeds tensorproducten voorkomen, maar toch kan deze aanpak nuttig zijn in verder onderzoek.

In het twaalfde en laatste hoofdstuk van deze thesis bouwen we verder op het feit dat de getwiste Dirac-operator kan geschreven worden als een som van hogere-spin Dirac-operatoren en hoogstens k twistor-operatoren, op een niet-triviale inbeddingsoperator na. In dit hoofdstuk wordt deze ontbinding expliciet bepaald. Met andere woorden, de inbeddingsoperatoren worden worden vastgelegd, hiertoe opnieuw gebruik makend van een geschikte extremale projectieoperator.

List of Figures

List of Tables

Index

List of Symbols

$(.,.)_{(u_1)}$	Fischer inner product
$(\mathfrak{g}^{(n)})$	derived series of a Lie algebra
$[\cdot,\cdot]$	Lie bracket
$\boxtimes$	Cartan product
$\cdot \wedge \cdot$	wedge product or exterior product
$.^*$	reversion or main anti-involution
$.^\dagger$	Hermitean conjugation
$\delta(x)$	delta distribution
Δ^+	set of positive roots
Δ^-	set of negative roots
Δ_j	Laplace operator in u_j
Δ_x	Laplace operator in x
Δ_x^{LB}	Laplace-Beltrami operator
Δ_{u_1,u_2}^{LB}	mixed Laplace-Beltrami operator
$\mathfrak{f}_j, \mathfrak{f}_j^\dagger$	Witt basis elements
$\Gamma(m)$	Clifford group
Γ_x	Gamma operator
$\mathfrak{g}$	Lie algebra
$\mathfrak{h}$	Cartan algebra
$\mathfrak{so}(m,\mathbb{K})$	special orthogonal algebra over the field $\mathbb{K}$
$\hat{.}$	inversion or main involution

$\Lambda\mathbb{R}^m$	Grassmann algebra
λ	weight
$\langle\cdot,\cdot\rangle$	Euclidean inner product
$\mathbb{C}$	complex numbers
$\mathbb{C}_m$	complex Clifford algebra
$\mathcal{C}(H)$	Casimir operator related to the H-representation
$\mathcal{D}^T$	twisted operator
$\mathcal{H}_{l_1,\ldots,l_k}$	simplicial harmonic homogeneous polynomials
$\mathcal{K}_{h;\lambda}$	h-homogeneous kernel of $\mathcal{Q}_\lambda$
$\mathcal{M}^s_\lambda$	skew monogenic polynomials
$\mathcal{P}_h$	h-homogeneous polynomials
$\mathcal{Q}$	higher spin Dirac operator
$\mathcal{Q}_\lambda$	higher spin Dirac operator
$\mathcal{S}_{l_1,\ldots,l_k}$	simplicial monogenic homogeneous polynomials
$\mathcal{T}^{(j)}$	higher spin twistor operator
$\mathcal{T}^{(i)}_\lambda$	higher spin twistor operator
$\mathcal{T}^{*(i)}_\lambda$	dual higher spin twistor operator
$\mathcal{W}$	Weyl algebra
$\mathbb{E}_j$	Euler operator in u_j
$\mathbb{E}_x$	Euler operator in x
$\mathbb{H}$	quaternions
$\mathbb{K}$	general field
$\mathbb{N}$	natural numbers
$\mathbb{R}$	real numbers
$\mathbb{R}^m$	real space of dimension m
$\mathbb{R}_m$	real Clifford algebra
$\mathbb{R}^+_m$	even subalgebra of $\mathbb{R}_m$

$\mathbb{R}_m^{(k)}$	real k-vectors
$\mathbb{S}_{2n}$	space of Dirac spinors
$\mathbb{S}_{2n}^+, \mathbb{S}_{2n}^-$	spaces of Weyl spinors
$\mathbb{V}$	vector space
$\oplus$	direct sum
$\otimes$	tensor product
$\bar{\cdot}$	conjugation
∂_j	Dirac operator in u_j
∂_x	Dirac operator in x
$\rho_{\mathbb{V}}$	representation
$\rho_{\mathbb{V}}$	Lie algebra representation
$\mathfrak{sl}(m, \mathbb{K})$	special linear algebra over the field $\mathbb{K}$
$\mathfrak{sp}(m, \mathbb{K})$	symplectic algebra over the field $\mathbb{K}$
ad	adjoint representation
End($\mathbb{V}$)	Endomorphisms of $\mathbb{V}$
θ	chirality operator
A_m	surface area of the unit sphere S^{m-1} in $\mathbb{R}^m$
$D(1)$	dilation
$d\sigma_x$	surface element (integration)
dD	infinitesimal dilation
dL	infinitesimal rotation
dT	infinitesimal translation
dx	volume element (integration)
$E(x)$	fundamental solution classical Dirac operator
e_i	basis element of m-dimensional real space
I	primitive idempotent
I_n	identity matrix

$I_{\mathcal{D}}$	inversion operator w.r.t. the operator $\mathcal{D}$
$K_{l_1}(u_1, u_1')$	reproducing kernel for l_1-homogeneous monogenic polynomials
$L(e_{ij})$	rotation
L_{ij}^x	angular operator
m	dimension
m_μ	representation multiplicity
n	truncated half dimension
$N(x)$	fundamental solution Laplace operator
n_μ	weight multiplicity
$p_{\mathfrak{g}}$	extremal projector w.r.t the Lie (super) algebra $\mathfrak{g}$
$R(\mathfrak{h})$	field of fractions
$T(e_i)$	translation
V_α	weight space
$Z(\mathfrak{g}, \mathfrak{k})$	transvector algebra or Mickelsson-Zhelobenko algebra
$\mathrm{Aut}(\mathbb{V})$	automorphism group
$\mathrm{O}(m)$	orthogonal group
$\mathrm{Pin}(m)$	pin group
$\mathrm{SO}(m)$	special orthogonal group
$\mathrm{Spin}(m)$	spin group

Bibliography

[1] L. Ahlfors. Clifford numbers and möbius transformations in rn. In J.S.R. Chisholm and A.K. Common, editors, *Clifford Algebras and Their Applications in Mathematical Physics*, volume 183 of *NATO ASI Series*, pages 167–175. Springer Netherlands, 1986.

[2] D. Arnaudon, M. Bauer, and L. Frappat. On Casimir's ghost. *Comm. Math. Phys.*, 187(2):429–439, 1997.

[3] R. M. Ašerova, J. F. Smirnov, and V. N. Tolstoĭ. Projection operators for simple Lie groups. II. General scheme for the construction of lowering operators. The case of the groups SU(n). *Teoret. Mat. Fiz.*, 15(1):107–119, 1973.

[4] R. M. Ašerova, Ju. F. Smirnov, and V. N. Tolstoĭ. Projection operators for simple Lie groups. *Teoret. Mat. Fiz.*, 8(2):255–271, 1971.

[5] R. M. Asherova, Yu. F. Smirnov, and V. N. Tolstoy. On a general analytic formula for U_q(su(3))Clebsch-Gordan coefficients. *Yadernaya Fiz.*, 64(12):2170–2175, 2001.

[6] F. A. Berezin and V. N. Tolstoy. The group with Grassmann structure UOSP(1.2). *Comm. Math. Phys.*, 78(3):409–428, 1980/81.

[7] F. Brackx, J. Bureš, H. De Schepper, D. Eelbode, F. Sommen, and V. Souček. Fundaments of Hermitean Clifford analysis. I. Complex structure. *Complex Anal. Oper. Theory*, 1(3):341–365, 2007.

[8] F. Brackx, J. Bureš, H. De Schepper, D. Eelbode, F. Sommen, and V. Souček. Fundaments of Hermitean Clifford analysis. II. Splitting of h-monogenic equations. *Complex Var. Elliptic Equ.*, 52(10-11):1063–1079, 2007.

[9] F. Brackx, D. Constales, A. Ronveaux, and H. Serras. On the harmonic and monogenic decomposition of polynomials. *J. Symbolic Comput.*, 8(3):297–304, 1989.

[10] F. Brackx, B. De Knock, H. De Schepper, and D. Eelbode. A calculus scheme for Clifford distributions. *Tokyo J. Math.*, 29(2):495–513, 2006.

[11] F. Brackx, H. De Schepper, D. Eelbode, and V. Souček. The Howe dual pair in Hermitean Clifford analysis. *Rev. Mat. Iberoam.*, 26(2):449–479, 2010.

[12] F. Brackx, R. Delanghe, and F. Sommen. *Clifford analysis*, volume 76 of *Research Notes in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1982.

[13] F. Brackx, D. Eelbode, T. Raeymaekers, and L. Van De Voorde. Triple monogenic functions and higher spin Dirac operators. *Internat. J. Math.*, 22(6):759–774, 2011.

[14] F. Brackx, D. Eelbode, and L. Van de Voorde. Higher spin Dirac operators between spaces of simplicial monogenics in two vector variables. *Math. Phys. Anal. Geom.*, 14(1):1–20, 2011.

[15] F. Brackx, D. Eelbode, and L. Van de Voorde. The polynomial null solutions of a higher spin Dirac operator in two vector variables. *Adv. Appl. Clifford Algebr.*, 21(3):455–476, 2011.

[16] T. Branson. Stein-Weiss operators and ellipticity. *J. Funct. Anal.*, 151(2):334–383, 1997.

[17] T Branson and O. Hijazi. Bochner-weitzenböck formulas associated with the rarita-schwinger operators. *International Journal of Mathematics*, 13(02):137–182, 2002.

[18] J. Bureš. The higher spin Dirac operators. In *Differential geometry and applications (Brno, 1998)*, pages 319–334. Masaryk Univ., Brno, 1999.

[19] J. Bureš. Monogenic forms of polynomial type. In *Clifford analysis and its applications (Prague, 2000)*, volume 25 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 39–48. Kluwer Acad. Publ., Dordrecht, 2001.

[20] J. Bureš, F. Sommen, V. Souček, and P. Van Lancker. Rarita-Schwinger type operators in Clifford analysis. *J. Funct. Anal.*, 185(2):425–455, 2001.

[21] J. Bureš, F. Sommen, V. Souček, and P. Van Lancker. Symmetric analogues of Rarita-Schwinger equations. *Ann. Global Anal. Geom.*, 21(3):215–240, 2002.

[22] J. Bureš and V. Souček. Eigenvalues of conformally invariant operators on spheres. *Rend. Circ. Mat. Palermo (2) Suppl.*, (59):109–122, 1999. The 18th Winter School "Geometry and Physics" (Srní, 1998).

[23] A. Čap, J. Slovák, and V. Souček. Bernstein-Gelfand-Gelfand sequences. *Ann. of Math. (2)*, 154(1):97–113, 2001.

[24] W.K. Clifford. Applications of grassmann's extensive algebra. *American Journal of Mathematics*, 1(4):pp. 350–358, 1878.

[25] J. Cnops. Reproducing kernels and conformal mappings in $\mathbf{R}^n$. *J. Math. Anal. Appl.*, 220(2):571–584, 1998.

[26] J. F. Cornwell. Lie algebra bases for the pseudo-orthogonal groups. *Internat. J. Theoret. Phys.*, 12:333–343, 1975.

[27] A. Damiano and D. Eelbode. Invariant operators between spaces of h-monogenic polynomials. *Adv. Appl. Clifford Algebr.*, 19(2):237–251, 2009.

[28] H. De Schepper, D. Eelbode, and T. Raeymaekers. On a special type of solutions of arbitrary higher spin Dirac operators. *J. Phys. A*, 43(32):325208, 13, 2010.

[29] R. Delanghe. Clifford analysis: history and perspective. *Comput. Methods Funct. Theory*, 1(1):107–153, 2001.

[30] R. Delanghe, F. Sommen, and V. Souček. *Clifford algebra and spinor-valued functions*, volume 53 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1992. A function theory for the Dirac operator, Related REDUCE software by F. Brackx and D. Constales, With 1 IBM-PC floppy disk (3.5 inch).

[31] R. Delanghe, F. Sommen, and V. Souček. An explicit realization of spinor spaces and its application to Clifford analysis. *Appl. Anal.*, 45(1-4):95–115, 1992.

[32] R. Delanghe and V. Souček. On the structure of spinor-valued differential forms. *Complex Variables, Theory and Application: An International Journal*, 18(3-4):223–236, 1992.

[33] P. A. M. Dirac. The quantum theory of the electron. I. *Proc. R. Soc. Lond. A*, 117(778):610–624, February 1928.

[34] M. Eastwood. Higher symmetries of the laplacian. *Annals of Mathematics*, 161(3):1645–1665, 2005.

[35] D. Eelbode. Zonal decompositions for spherical monogenics. *Math. Methods Appl. Sci.*, 30(9):1093–1103, 2007.

[36] D. Eelbode and F. L. He. Taylor series in Hermitean Clifford analysis. *Complex Anal. Oper. Theory*, 5(1):97–111, 2011.

[37] D. Eelbode, T. Raeymaekers, and P. Van Lancker. On the fundamental solution for higher spin Dirac operators. *J. Math. Anal. Appl.*, 405(2):555–564, 2013.

[38] D. Eelbode and D. Šmíd. Polynomial invariants for the Rarita-Schwinger operator. In *Hypercomplex analysis*, Trends Math., pages 125–135. Birkhäuser Verlag, Basel, 2009.

[39] D. Eelbode and D. Šmíd. Factorization of Laplace operators on higher spin representations. *Complex Anal. Oper. Theory*, 6(5):1011–1023, 2012.

[40] D. Eelbode, V. Souček, and P. Van Lancker. Gegenbauer polynomials and the fueter theorem. *Complex Variables and Elliptic Equations*, 0(0):1–15, 0.

[41] D. Eelbode and L. Van de Voorde. Higher spin operators and polyharmonic functions. In Junesang Choi, editor, *Finite or Infinite Dimensional Complex Analysis and Applications, 16th International conference, Proceedings*, pages 137–142. Daeyang Printing, 2009.

[42] D. Eelbode and L. Van de Voorde. A toy model for higher spin dirac operators. *Physics of Atomic Nuclei*, 73(2):282–287, 2010.

[43] D. Eelbode and P. Van Lancker. Total Rarita-Schwinger operators in Clifford analysis. *Ann. Global Anal. Geom.*, 42(4):473–493, 2012.

[44] R. J. Farmer and P. D. Jarvis. Representations of orthosymplectic superalgebras. II. Young diagrams and weight space techniques. *J. Phys. A*, 17(12):2365–2387, 1984.

[45] H. D. Fegan. Conformally invariant first order differential operators. *Quart. J. Math. Oxford (2)*, 27(107):371–378, 1976.

[46] P. Franek. Generalized Dolbeault sequences in parabolic geometry. *J. Lie Theory*, 18(4):757–774, 2008.

[47] W. Fulton and J. Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[48] J. E. Gilbert and M. A. M. Murray. *Clifford algebras and Dirac operators in harmonic analysis*, volume 26 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1991.

[49] R. Goodman. Multiplicity-free spaces and Schur-Weyl-Howe duality. In *Representations of real and p-adic groups*, volume 2 of *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, pages 305–415. Singapore Univ. Press, Singapore, 2004.

[50] B. C. Hall. *Lie groups, Lie algebras, and representations*, volume 222 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2003. An elementary introduction.

[51] S. Helgason. *Groups and geometric analysis*, volume 113 of *Pure and Applied Mathematics*. Academic Press Inc., Orlando, FL, 1984. Integral geometry, invariant differential operators, and spherical functions.

[52] D. Hong. Eigenvalues of Dirac and RaritaSchwinger operators. In Rafa Abamowicz, editor, *Clifford Algebras*, volume 34 of *Progress in Mathematical Physics*, pages 201–210. Birkhuser Boston, 2004.

[53] R. Howe. Remarks on classical invariant theory. *Trans. Amer. Math. Soc.*, 313(2):539–570, 1989.

[54] R. Howe, E. Tan, and J. F. Willenbring. Stable branching rules for classical symmetric pairs. *Trans. Amer. Math. Soc.*, 357(4):1601–1626, 2005.

[55] S. Khoroshkin and O. Ogievetsky. Mickelsson algebras and Zhelobenko operators. *J. Algebra*, 319(5):2113–2165, 2008.

[56] A. U. Klimyk. Infinitesimal operators for representations of complex Lie groups and Clebsch-Gordan coefficients for compact groups. *J. Phys. A*, 15(10):3009–3023, 1982.

[57] B. Kostant. A formula for the multiplicity of a weight. *Trans. Amer. Math. Soc.*, 93:53–73, 1959.

[58] S. Kumar. Tensor product decomposition. In *Proceedings of the International Congress of Mathematicians. Volume III*, pages 1226–1261, New Delhi, 2010. Hindustan Book Agency.

[59] R. Lávička, V. Souček, and P. Van Lancker. Orthogonal basis for spherical monogenics by step two branching. *Ann. Global Anal. Geom.*, 41(2):161–186, 2012.

[60] J. Li. *Some Rarita-Schwinger operators*. ProQuest LLC, Ann Arbor, MI, 2011. Thesis (Ph.D.)–University of Arkansas.

[61] P. Lounesto. *Clifford Algebras and Spinors*. Cambridge University Press, second edition, 2001. Cambridge Books Online.

[62] Maplesoft. division of waterloo maple.

[63] J. Mickelsson. Step algebras of semi-simple subalgebras of lie algebras. *Reports on Mathematical Physics*, 4(4):307 – 318, 1973.

[64] W. Miller. *Symmetry and separation of variables*. Encyclopedia of mathematics and its applications. Addison-Wesley Pub. Co., Advanced Book Program, 1977.

[65] A. Molev. Weight bases of Gelfand-Tsetlin type for representations of classical Lie algebras. *J. Phys. A*, 33(22):4143–4158, 2000.

[66] A. I. Molev. *Yangians and classical Lie algebras*, volume 143 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2007.

[67] M. Pilca. A new proof of Branson's classification of elliptic generalized gradients. *Manuscripta Math.*, 136(1-2):65–81, 2011.

[68] M. Plechšmíd. Structure of the kernel of higher spin Dirac operators. *Comment. Math. Univ. Carolin.*, 42(4):665–680, 2001.

[69] W. Rarita and J. Schwinger. On a theory of particles with half-integral spin. *Phys. Rev.*, 60:61–61, Jul 1941.

[70] L. Ross. Representations of graded Lie algebras. *Trans. Amer. Math. Soc.*, 120:17–23, 1965.

[71] I. Sabadini, F. Sommen, and D. Struppa. The Dirac complex on abstract vector variables: megaforms. *Experiment. Math.*, 12(3):351–364, 2003.

[72] V. Severa. Invariant differential operators between spinor-valued forms. *PhD. Thesis, Charles University, Prague*, 1998.

[73] J. Slovák. Natural operators on conformal manifolds. In *Differential geometry and its applications (Opava, 1992)*, volume 1 of *Math. Publ.*, pages 335–349. Silesian Univ. Opava, Opava, 1993.

[74] P. Somberg. Twistor operators on conformally flat spaces. In *Proceedings of the 20th Winter School "Geometry and Physics" (Srní, 2000)*, number 66, pages 179–197, 2001.

[75] F. Sommen. Functions on the spin group. *Advances in Applied Clifford Algebras*, 6(1):37–48, 1996.

[76] F. Sommen. Clifford analysis in two and several vector variables. *Applicable Analysis*, 73(1-2):225–253, 1999.

[77] V. Souček. Clifford analysis for higher spins. In F. Brackx, R. Delanghe, and H. Serras, editors, *Clifford Algebras and their Applications in Mathematical Physics*, volume 55 of *Fundamental Theories of Physics*, pages 223–232. Springer Netherlands, 1993.

[78] V. Souček. Higher spins and conformal invariants in clifford analysis. *Proc. Conf. Seiffen*, 1996.

[79] E. M. Stein and G. Weiss. Generalization of the Cauchy-Riemann equations and representations of the rotation group. *Amer. J. Math.*, 90:163–196, 1968.

[80] V. N. Tolstoy. Projection operator method for quantum groups. In *Special functions 2000: current perspective and future directions (Tempe, AZ)*, volume 30 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 457–488. Kluwer Acad. Publ., Dordrecht, 2001.

[81] V. N. Tolstoy. Fortieth anniversary of extremal projector method for Lie symmetries. In *Noncommutative geometry and representation theory in mathematical physics*, volume 391 of *Contemp. Math.*, pages 371–384. Amer. Math. Soc., Providence, RI, 2005.

[82] Liesbet Van de Voorde. *Higher spin dirac operators in two vector variables in Clifford analysis.* PhD thesis, Ghent University, 2011.

[83] P. Van Lancker. Higher spin fields on smooth domains. In *Clifford analysis and its applications (Prague, 2000)*, volume 25 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 389–398. Kluwer Acad. Publ., Dordrecht, 2001.

[84] P. Van Lancker. Rarita-Schwinger fields in the half space. *Complex Var. Elliptic Equ.*, 51(5-6):563–579, 2006.

[85] P. Van Lancker. The monogenic Fischer decomposition: two vector variables. *Complex Anal. Oper. Theory*, 6(2):425–446, 2012.

[86] P. Van Lancker, F. Sommen, and D. Constales. Models for irreducible representations of Spin(m). *Adv. Appl. Clifford Algebras*, 11(S1):271–289, 2001.

[87] M. A. A. van Leeuwen, A. M. Cohen, and B. Lisser. Lie, a package for lie group computations, 1992.

[88] D. Šmid. A fifth order conformally invariant higher spin operator on the sphere. *AIP Conference Proceedings*, 1479(1):332–335, 2012.

[89] H. Weyl. *The Classical Groups: Their Invariants and Representations.* Princeton University Press, 1939.

[90] D.P. Zhelobenko. Transvector algebras in representation theory and dynamic symmetry. Group theoretical methods in physics, Proc. 3rd Semin., Yurmala/USSR 1985, Vol. 2, 71-93 (1986)., 1986.