Pseudo-ovals in even characteristic and ovoidal Laguerre planes

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Abstract

Pseudo-arcs are the higher dimensional analogues of arcs in a projective plane: a *pseudo-arc* is a set $\mathcal{A}$ of $(n-1)$-spaces in $\mathrm{PG}(3n-1,q)$ such that any three span the whole space. Pseudo-arcs of size q^n+1 are called *pseudo-ovals*, while pseudo-arcs of size q^n+2 are called *pseudo-hyperovals*. A pseudo-arc is called *elementary* if it arises from applying field reduction to an arc in $\mathrm{PG}(2,q^n)$.

We explain the connection between dual pseudo-ovals and *elation Laguerre planes* and show that an elation Laguerre plane is *ovoidal* if and only if it arises from an elementary dual pseudo-oval. The main theorem of this paper shows that a pseudo-(hyper)oval in $\mathrm{PG}(3n-1,q)$, where q is even and n is prime, such that every element induces a Desarguesian spread, is elementary. As a corollary, we give a characterisation of certain ovoidal Laguerre planes in terms of the derived affine planes.

Keywords: pseudo-ovals, pseudo-hyperovals, Desarguesian spreads, ovoidal Laguerre planes

1 Introduction

The aim of this paper is to characterise elementary pseudo-(hyper)ovals in $\mathrm{PG}(3n-1,q)$ where q is even. We will impose a condition on the considered pseudo-ovals, namely that every element of the pseudo-oval induces a Desarguesian spread. In Subsection 1.1, we provide the necessary background on

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pseudo-arcs and give some motivation for the study of this problem. In Subsection 1.2, we will introduce Desarguesian spreads and field reduction and prove a theorem on the possible intersection of Desarguesian $(n-1)$-spreads in $\mathrm{PG}(2n-1,q)$. In Section 2, we will explain the connection between dual pseudo-ovals and elation Laguerre planes, meanwhile proving a theorem that characterises ovoidal Laguerre planes as those elation Laguerre planes obtained from an elementary dual pseudo-oval. Finally, in Section 3, we give a proof for our main theorem. We end by stating a corollary of our main theorem in terms of ovoidal Laguerre planes.

1.1 Pseudo-arcs

In this paper, all considered objects will be finite. Denote the n-dimensional projective space over the finite field $\mathbb{F}_q$ with q elements, $q=p^h$, p prime, by $\mathrm{PG}(n,q)$.

Definition. A *pseudo-arc* is a set $\mathcal{A}$ of $(n-1)$-spaces in $\mathrm{PG}(3n-1,q)$ such that $\langle E_i, E_j\rangle \cap E_k = \emptyset$ for distinct E_i, E_j, E_k in $\mathcal{A}$.

We see that a pseudo-arc is a set of $(n-1)$-spaces such that any 3 span $\mathrm{PG}(3n-1,q)$; such a set is also called a set of $(n-1)$-spaces in $\mathrm{PG}(3n-1,q)$ *in general position.*

A *partial spread* in $\mathrm{PG}(2n-1,q)$ is a set of mutually disjoint $(n-1)$-spaces in $\mathrm{PG}(2n-1,q)$. Every element E_i of a pseudo-arc $\mathcal{A}$ defines a partial spread

$$\mathcal{S}_i := \{E_1,\ldots,E_{i-1},E_{i+1},\ldots,E_{|\mathcal{A}|}\}/E_i$$

in $\mathrm{PG}(2n-1,q) \cong \mathrm{PG}(3n-1,q)/E_i$ and we say that the element E_i *induces* the partial spread $\mathcal{S}_i$. Since an element E_i induces a partial spread $\mathcal{S}_i$ in $\mathrm{PG}(2n-1,q)$, which has at most q^n+1 elements, a pseudo-arc in $\mathrm{PG}(3n-1,q)$ can have at most q^n+2 elements. Moreover, we have the following theorem of Thas, where a *pseudo-oval* in $\mathrm{PG}(3n-1,q)$ denotes a pseudo-arc of size q^n+1, and a *pseudo-hyperoval* denotes a pseudo-arc of size q^n+2. Note that for $n=1$, these statements reduce to well-known and easy to prove statements.

Theorem 1.1. [13] *A pseudo-arc in* $\mathrm{PG}(3n-1,q)$*, q odd, has at most* q^n+1 *elements. A pseudo-oval in* $\mathrm{PG}(3n-1,q)$*, q even, is contained in a unique pseudo-hyperoval.*

A pseudo-arc is called *elementary* if it arises by applying field reduction to an arc in $\mathrm{PG}(2,q^n)$. *Field reduction* is the concept where a point in $\mathrm{PG}(2,q^n)$ corresponds in a natural way to an $(n-1)$-space of $\mathrm{PG}(3n-1,q)$.

The set of all points of $\mathrm{PG}(2, q^n)$ then correspond to a set of disjoint $(n-1)$-spaces partitioning $\mathrm{PG}(3n-1, q)$, forming a *Desarguesian spread.* For more information on field reduction and Desarguesian spreads we refer to [8]. A pseudo-oval that is obtained by applying field reduction to a conic in $\mathrm{PG}(2, q^n)$ is called a *pseudo-conic.* A pseudo-hyperoval (necessarily in even characteristic) obtained by applying field reduction to a conic, together with its nucleus, is called a *pseudo-hyperconic.*

All known pseudo-ovals and pseudo-hyperovals are elementary, but it is an open question whether there can exist non-elementary pseudo-ovals and pseudo-hyperovals. A natural question to ask is whether we can characterise a pseudo-oval in terms of the partial spreads induced by its elements.

From [3], we know that a partial spread of $\mathrm{PG}(2n-1, q)$ of size q^n can be extended to a spread in a unique way, i.e. the set of points in $\mathrm{PG}(2n-1, q)$ not contained in an element of such a partial spread of size q^n, form an $(n-1)$-space. So by abuse of notation, we say that an element of a pseudo-oval induces a spread instead of a partial spread. Clearly, for an elementary pseudo-oval every induced spread is Desarguesian. The following theorem shows that for q odd, a strong version of the converse also holds.

Theorem 1.2. [5] *If $\mathcal{O}$ is a pseudo-oval in $\mathrm{PG}(3n-1, q)$, q odd, such that for at least one element the induced spread is Desarguesian, then $\mathcal{O}$ is a pseudo-conic.*

The proof of this theorem relies on the theorem of Chen and Kaerlein [6] for Laguerre planes in odd order, which in its turn relies on the theorem of Segre [11] characterising every oval in $\mathrm{PG}(2, q)$, q odd, as a conic. This clearly rules out a similar approach for even characteristic. The characterisation of pseudo-ovals in terms of the induced spreads for even characteristic was posed as Problem A.3.4 in [14].

In this paper, we will prove that the following holds:

Main Theorem. *If $\mathcal{O}$ is a pseudo-oval in $\mathrm{PG}(3n-1, q)$, $q = 2^h$, $h > 1$, n prime, such that the spread induced by every element of $\mathcal{O}$ is Desarguesian, then $\mathcal{O}$ is elementary.*

As a corollary, we prove a similar statement for pseudo-hyperovals.

Corollary 1.3. *Let $\mathcal{H}$ be a pseudo-hyperoval in $\mathrm{PG}(3n-1, q)$, $q = 2^h$, $h > 1$, n prime, such that the spread induced by at least $q^n + 1$ elements of $\mathcal{H}$ is Desarguesian, then $\mathcal{H}$ is elementary.*

It is worth noting that pseudo-ovals in $\mathrm{PG}(3n-1, q)$ are in one-to-one correspondence with a particular type of generalised quadrangles, namely *translation generalised quadrangles* of order (q^n, q^n). In particular if $\mathcal{O}$ is elementary, we have that the corresponding generalised quadrangle is isomorphic to $T_2(O)$, where $\mathcal{O}$ is obtained from O by field reduction. For more information, we refer to [14].

1.2 Field reduction, Desarguesian spreads and Segre varieties

We recall the *André/Bruck-Bose representation* of a translation plane of order q^n. Let $\mathcal{S}$ be a $(n-1)$-spread of the projective space $\Sigma_\infty = \mathrm{PG}(2n-1,q)$ and embed Σ_∞ as hyperplane of $\mathrm{PG}(2n,q)$. Consider the following incidence structure $\mathcal{A}(\mathcal{S}) = (\mathcal{P}, \mathcal{L})$, where incidence is natural:

$\mathcal{P}:$ the points of $\mathrm{PG}(2n,q) \setminus \Sigma_\infty$ (the affine points),

$\mathcal{L}:$ the n-spaces of $\mathrm{PG}(2n,q)$ intersecting Σ_∞ exactly in an element of $\mathcal{S}$.

This defines an affine translation plane of order q^n [1, 4]. If the spread $\mathcal{S}$ is Desarguesian, $\mathcal{A}(\mathcal{S})$ is a Desarguesian affine plane $\mathrm{AG}(2,q^n)$. Adding Σ_∞ as the line at infinity, and considering the spread elements as its points, we obtain a projective plane of order q^n.

An $(n-1)$*-regulus* or *regulus* $\mathcal{R}$ in $\mathrm{PG}(2n-1,q)$ is a set of $q+1$ mutually disjoint $(n-1)$-spaces having the property that if a line meets 3 elements of $\mathcal{R}$, then it meets all elements of $\mathcal{R}$. There is a unique regulus through 3 mutually disjoint $(n-1)$-spaces A, B and C in $\mathrm{PG}(2n-1,q)$, let us denote this by $\mathcal{R}(A,B,C)$. Every Desarguesian spread $\mathcal{D}$ has the property that for 3 elements A, B, C in $\mathcal{D}$, the elements of $\mathcal{R}(A,B,C)$ are also contained in $\mathcal{D}$, i.e. $\mathcal{D}$ is *regular* (see also [4]). Moreover, every Desarguesian spread $\mathcal{D}$ clearly has the property that the space spanned by 2 elements of $\mathcal{D}$ is partitioned by elements of $\mathcal{D}$, i.e. $\mathcal{D}$ is *normal*.

We will use the following notation for points of a projective space $\mathrm{PG}(r-1,q^n)$. A point P of $\mathrm{PG}(r-1,q^n)$ defined by a vector $(x_1, x_2, \ldots, x_r) \in (\mathbb{F}_{q^n})^r$ is denoted by $\mathbb{F}_{q^n}(x_1, x_2, \ldots, x_r)$, reflecting the fact that every $\mathbb{F}_{q^n}$-multiple of $(x_1, x_2, \ldots, x_r)$ gives rise to the point P.

An $\mathbb{F}_{q^t}$*-subline* in $\mathrm{PG}(1,q^n)$, where $t|n$, is a set of q^t+1 points in $\mathrm{PG}(1,q^n)$ that is PGL-equivalent to the set $\{\mathbb{F}_{q^n}(1,x) | x \in \mathbb{F}_{q^t}\} \cup \{\mathbb{F}_{q^n}(0,1)\}$. As $\mathrm{PGL}(2,q^n)$ acts sharply 3-transitively on the points of the projective line, we see that any 3 points define a unique $\mathbb{F}_{q^t}$-subline.

We can identify the vector space $(\mathbb{F}_q)^{rn}$ with $(\mathbb{F}_{q^n})^r$, and hence, we can write every point of $\mathrm{PG}(rn-1,q)$ as $\mathbb{F}_q(x_1, x_2, \ldots, x_r)$, where $x_i \in \mathbb{F}_{q^n}$. In this way, by field reduction, a point $\mathbb{F}_{q^n}(x_1, x_2, \ldots, x_r)$ in $\mathrm{PG}(r-1,q^n)$ corresponds to the $(n-1)$-space $\mathbb{F}_{q^n}(x_1, x_2, \ldots, x_r) = \{\mathbb{F}_q(\alpha x_1, \alpha x_2, \ldots, \alpha x_r) | \alpha \in \mathbb{F}_{q^n}\}$ in $\mathrm{PG}(rn-1,q)$.

We will need a lemma on Desarguesian spreads which has a straightforward proof, but we include it for completeness.

Lemma 1.4. *Let $\mathcal{D}_1$ be a Desarguesian $(n-1)$-spread in a $(2n-1)$-dimensional subspace Π of $\mathrm{PG}(3n-1,q)$, let μ be an element of $\mathcal{D}_1$ and let E_1 and E_2 be disjoint $(n-1)$-spaces disjoint from Π such that $\langle E_1, E_2\rangle$ meets Π exactly in the space μ. Then there exists a unique Desarguesian $(n-1)$-spread of $\mathrm{PG}(3n-1,q)$ containing the elements of $\mathcal{D}_1$ and $\mathcal{R}(\mu, E_1, E_2)$.*

Proof. Since $\mathcal{D}_1$ is a Desarguesian spread in Π, we can choose coordinates for Π such that $\mathcal{D}_1 = \{\mathbb{F}_{q^n}(1,x) | x \in \mathbb{F}_{q^n}\} \cup \{\mu = \mathbb{F}_{q^n}(0,1)\}$. We embed Π in $\mathrm{PG}(3n-1,q)$ by mapping a point $\mathbb{F}_q(x_1,x_2)$, $x_1, x_2 \in \mathbb{F}_{q^n}$, of Π to $\mathbb{F}_q(x_1,x_2,0)$. Consider a point P of μ and let ℓ_P denote the unique transversal line through the point P of μ to the regulus $\mathcal{R}(\mu, E_1, E_2)$.

We can still choose coordinates for $n+1$ points in general position in $\mathrm{PG}(3n-1,q) \setminus \Pi$. We will choose these $n+1$ points such that n of them belong to E_1 and one of them belongs to E_2. Consider a set $\{y_i | i = 1,\ldots,n\}$ forming a basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. We may assume that the line ℓ_{P_i} through $P_i = \mathbb{F}_q(0,y_i,0)$ meets E_1 in the point $\mathbb{F}_q(0,0,y_i)$. It follows that $E_1 = \mathbb{F}_{q^n}(0,0,1)$. Moreover, we may assume that ℓ_Q with $Q = \mathbb{F}_q(0, \sum_{i=1}^n y_i, 0)$ meets E_2 in $\mathbb{F}_q(0, \sum_{i=1}^n y_i, \sum_{i=1}^n y_i)$. Since $\mathbb{F}_q(0, \sum_{i=1}^n y_i, \sum_{i=1}^n y_i)$ has to be in the space spanned by the intersection points $R_i = \ell_{P_i} \cap E_2$, it follows that $R_i = \mathbb{F}_q(0,y_i,y_i)$ and consequently, that $E_2 = \mathbb{F}_{q^n}(0,1,1)$.

It is clear that the Desarguesian spread $\mathcal{D} = \{\mathbb{F}_{q^n}(x_1,x_2,x_3) | x_1,x_2,x_3 \in \mathbb{F}_{q^n}\}$ contains the spread $\mathcal{D}_1$ and the regulus $\mathcal{R}(\mu, E_1, E_2)$. Moreover, since a Desarguesian spread is normal, every element of $\mathcal{D}$, not in $\langle E_1, E_2\rangle$ is obtained as the intersection of $\langle E_1, X\rangle \cap \langle E_2, Y\rangle$, where $X, Y \in \mathcal{D}_1$, it is clear that $\mathcal{D}$ is the unique Desarguesian spread satisfying our hypothesis. □

Theorem 1.5. *A set $\mathcal{S}$ of at least 3 points in $\mathrm{PG}(1,q^n)$, $q > 2$, such that any three points of $\mathcal{S}$ determine a subline entirely contained in $\mathcal{S}$, defines an $\mathbb{F}_{q^t}$-subline $\mathrm{PG}(1,q^n)$ for some $t|n$.*

Proof. Without loss of generality, we may choose the points $\mathbb{F}_{q^n}(0,1)$, $\mathbb{F}_{q^n}(1,0)$ and $\mathbb{F}_{q^n}(1,1)$ to be in $\mathcal{S}$. Put $S = \{x \mid \mathbb{F}_{q^n}(1,x) \in \mathcal{S}\}$, clearly $\mathbb{F}_q \subseteq S$.

Consider $x, y \in S$, where $x \neq y$ and $xy \neq 0$, then every point of the $\mathbb{F}_q$-subline through the distinct points $\mathbb{F}_{q^n}(0,1)$, $\mathbb{F}_{q^n}(1,x)$ and $\mathbb{F}_{q^n}(1,y)$ has to be contained in $\mathcal{S}$. The points of this subline, different from $\mathbb{F}_{q^n}(0,1)$ are given by $\mathbb{F}_{q^n}(1, x + (y-x)t)$, where $t \in \mathbb{F}_q$. This implies that if x and y are in S, also $(1-t)x + ty$ is in S for all $t \in \mathbb{F}_q$. It easily follows that S is closed under taking linear combinations with elements of $\mathbb{F}_q$, hence, S forms an $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^n}$.

Now consider $x', y' \in S$, $x', y' \neq 0$. We claim that (1) $x'^2/y' \in S$ and (2) $x'^2 \in S$.

If $y'/x' \in \mathbb{F}_q$, our claim (1) immediately follows from the fact that S is an $\mathbb{F}_q$-subspace so we may assume that $y'/x' \in \mathbb{F}_{q^n} \setminus \mathbb{F}_q$. Since $q > 2$, we can consider an element $t \in \mathbb{F}_q$ such that $t(t-1) \neq 0$. Put $z' := y' - (t-1)x'$. Since S is an $\mathbb{F}_q$-subspace, $z' \in S$. It is easy to check that $z' \notin \{0, x'\}$. Every point of the $\mathbb{F}_q$-subline containing distinct points $\mathbb{F}_{q^n}(1,0)$, $\mathbb{F}_{q^n}(1,x')$ and $\mathbb{F}_{q^n}(1,z')$ has to be contained in $\mathcal{S}$, and the points of this subline, different from $\mathbb{F}_{q^n}(1,z')$, are given by $\mathbb{F}_{q^n}(z' - x' + t'x', tx'z')$, where $t' \in \mathbb{F}_q$. This implies that $\frac{t'x'z'}{z'+(t'-1)x'}$ is in S for every $t' \in \mathbb{F}_q$, so also for $t' = t$, which implies that $tx' - \frac{t(t-1)x'^2}{y'} \in S$. Since $tx' \in S$ and $t(t-1) \neq 0$, we conclude that $\frac{x'^2}{y'} \in S$ which proves claim (1). Claim (2) follows immediately from Claim (1) by taking $y = 1 \in \mathbb{F}_q \subseteq S$.

Now let $v, w \in S$ and first suppose that q is odd, then $vw = \frac{1}{2}((v+w)^2 - v^2 - w^2)$, and since S is an $\mathbb{F}_q$-subspace and by claim (2), all terms on the right hand side are in S, so is vw. If q is even, say $q^n = 2^h$, then $v = u^2$ for some $u \in \mathbb{F}_{q^n}$, but since $u = u^{2^h} = v^{2^{h-1}}$, v is contained in S. This implies that $\frac{v}{w} = \frac{u^2}{w} \in S$ by claim (1) and consequently, again by claim (1), $vw = \frac{v^2}{v/w} \in S$. In both cases, we get that S is a subfield of $\mathbb{F}_{q^n}$ and the statement follows. □

Corollary 1.6. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two Desarguesian $(n-1)$-spreads in* $\mathrm{PG}(2n-1, q)$*, $q = p^h$, p prime, $q > 2$, with at least* 3 *elements in common, then $\mathcal{D}_1$ and $\mathcal{D}_2$ share exactly $q^t + 1$ elements for some $t|n$. In particular, if n is prime, then $\mathcal{D}_1$ and $\mathcal{D}_2$ share a regulus or coincide.*

Proof. Let X be the set of common elements of $\mathcal{D}_1$ and $\mathcal{D}_2$. Since a Desarguesian spread $\mathcal{D}$ is regular, it has to contain the regulus defined by any three elements of $\mathcal{D}$, which, since $\mathcal{D}_1$ and $\mathcal{D}_2$ are Desarguesian, implies that the regulus through 3 elements of X is contained in X. Now since X is contained in a Desarguesian spread, X corresponds to a set of points $\mathcal{S}$ in $\mathrm{PG}(1, q^n)$ such that every $\mathbb{F}_q$-subline through 3 points of $\mathcal{S}$ is contained in $\mathcal{S}$. The first part of the statement now follows from Theorem 1.5. The second part follows from the fact that the only divisors of a prime n are 1 and n. □

An $\mathbb{F}_q$-*subplane* of $\mathrm{PG}(2, q^n)$, is a subgeometry $\mathrm{PG}(2, q)$ of $\mathrm{PG}(2, q^n)$, i.e. a set of $q^2 + q + 1$ points and $q^2 + q + 1$ lines in $\mathrm{PG}(2, q^n)$ forming an projective plane, where the point set is PGL-equivalent to the set $\{\mathbb{F}_{q^n}(x_0, x_1, x_2) | (x_0, x_1, x_2) \in (\mathbb{F}_q \times \mathbb{F}_q \times \mathbb{F}_q) \setminus (0,0,0)\}$. If we apply field reduction to the point set of an $\mathbb{F}_q$-subplane, we find a set $\mathcal{S}$ of $q^2 + q + 1$ elements of a Desarguesian spread $\mathcal{D}$. All elements of $\mathcal{S}$ meet a fixed plane of $\mathrm{PG}(3n-1,q)$ and form one system of a Segre variety $\mathbf{S}_{n-1,2}$ (see e.g. [8]). Note that $\mathbf{S}_{n-1,2}$ is contained in $\mathrm{PG}(3n-1,q)$ and consists of two systems of subspaces, one with subspaces of dimension $(n-1)$ and the other consisting of planes. Moreover, every point of $\mathbf{S}_{n-1,2}$ lies on exactly one subspace of each system.

As $\mathrm{PGL}(3,q^n)$ acts sharply transitively on the frames of $\mathrm{PG}(2,q^n)$, we see that 4 points in general position define a unique $\mathbb{F}_q$-subplane of $\mathrm{PG}(2,q^n)$. A similar statement holds for 4 $(n-1)$-spaces in $\mathrm{PG}(3n-1,q)$ in general position. A proof can be found in e.g. [7, Proposition 2.1, Corollary 2.3, Proposition 2.4].

Lemma 1.7. *Four $(n-1)$-spaces in $\mathrm{PG}(3n-1,q)$ in general position are contained in a unique Segre variety $\mathbf{S}_{n-1,2}$.*

2 Laguerre planes

Definition. A *Laguerre plane* is an incidence structure with points $\mathcal{P}$, lines $\mathcal{L}$ and circles $\mathcal{C}$ such that $(\mathcal{P},\mathcal{L},\mathcal{C})$ satisfies the following four axioms:

AX1 Every point lies on a unique line.

AX2 A circle and a line meet in a unique point.

AX3 Through 3 points, no two collinear, there is a unique circle of $\mathcal{C}$.

AX4 If P is a point on a fixed circle C and Q a point, not on the line through P and not on the circle C, then there is a unique circle C' through P and Q, meeting C only in the point P.

In a finite Laguerre plane, every circle contains $s+1$ points for some s; this constant s is called the *order* of the Laguerre plane.

Starting from a point P of a Laguerre plane $\mathbb{L}=(\mathcal{P},\mathcal{L},\mathcal{C})$, we obtain an affine plane $(\mathcal{P}',\mathcal{L}')$, where incidence is inherited from $\mathbb{L}$, as follows.

$\mathcal{P}'$: the points of $\mathcal{P}$, different from P and not collinear with P,

$\mathcal{L}'$: (1) the lines of $\mathcal{L}$ not through P,
(2) the elements of $\mathcal{C}$ through P.

The obtained affine plane $(\mathcal{P}',\mathcal{L}')$ is called the *derived affine plane* at P.

Definition. A finite *ovoidal* Laguerre plane with points $\mathcal{P}$, lines $\mathcal{L}$ and circles $\mathcal{C}$ is a Laguerre plane that can be constructed from a cone $\mathcal{K}$ as follows. Consider a cone $\mathcal{K}$ in $\mathrm{PG}(3,q)$ with vertex the point V and base an oval in a plane H, not containing V. Incidence is natural.

$\mathcal{P}$: the points of $\mathcal{K} \setminus \{V\}$,

$\mathcal{L}$: the *generators* of $\mathcal{K}$, i.e. the lines of $\mathrm{PG}(3, q)$, lying on $\mathcal{K}$,

$\mathcal{C}$: the plane sections of $\mathcal{K}$, not containing V.

For later use, we will consider the dual model in $\mathrm{PG}(3, q)$ of the definition of an ovoidal Laguerre plane obtained from the cone $\mathcal{K}$ with vertex V and base an oval A, embedded in $\mathrm{PG}(3, q)$. Let H denote the plane which is the dual of the point V in $\mathrm{PG}(3, q)$. Let $\overline{A}$ denote the dual (in $\mathrm{PG}(2, q)$) of the oval A contained in H. It is not hard to see that we find the following incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{C})$:

$\mathcal{P}$: planes different from H and meeting H in a line of $\overline{A}$,

$\mathcal{L}$: the lines in H belonging to $\overline{A}$,

$\mathcal{C}$: the points of $\mathrm{PG}(3, q)$ not contained in H (the affine points).

We will denote the ovoidal Laguerre plane that is obtained in this way by $L(\overline{A})$.

Definition. The *classical* Laguerre plane of order q is an ovoidal Laguerre plane, obtained from a quadratic cone $\mathcal{K}$ in $\mathrm{PG}(3, q)$, i.e. a cone whose base is a conic.

Remark. A Laguerre plane is called *Miquelian* if for each eight pairwise different points A, B, C, D, E, F, G, H it follows from $(ABCD)$, $(ABEF)$, $(BCFG)$, $(CDGH)$, $(ADEH)$ that $(EFGH)$, where $(PQRS)$ denotes that P, Q, R, S are on a common circle. By a theorem of van der Waerden and Smid a Laguerre plane is Miquelian if and only if it is classical [15] and we, as well as many others, use the term 'Miquelian Laguerre plane' instead of 'classical Laguerre plane'.

It follows from Segre's theorem that an ovoidal Laguerre plane of odd order is necessarily Miquelian.

For later use, we will also introduce the *plane model* of the Miquelian Laguerre plane of even order q (for more information we refer to [2]). Consider a point N in $\mathrm{PG}(2, q)$, q even. Since three points together with a nucleus determine a unique conic, one can easily count that there are exactly $q^3 - q^2$ conics in $\mathrm{PG}(2, q)$, q even, all having the same point N as their nucleus. The plane model of the Miquelian Laguerre plane is the following incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{C})$ embedded in $\mathrm{PG}(2, q)$, q even, with natural incidence.

$\mathcal{P}$: the points of $\mathrm{PG}(2, q)$ different from N,

$\mathcal{L}$: the lines of $\mathrm{PG}(2, q)$ containing N,

$\mathcal{C}$: the q^2 lines of $\mathrm{PG}(2, q)$ not containing to N and the $q^3 - q^2$ conics in $\mathrm{PG}(2, q)$ having N as their nucleus.

Remark. One can easily deduce this model from the standard cone model obtained from a quadratic cone $\mathcal{K}$ with vertex V and base a conic $\mathcal{C}$ by projecting the cone $\mathcal{K}$ from a point on the line through V and the nucleus of $\mathcal{C}$ on a plane.

The *kernel* K of a Laguerre plane $\mathbb{L}$ is the subgroup of $\mathrm{Aut}(\mathbb{L})$ consisting of all automorphisms which map a point P onto a point collinear with P, for every point P of $\mathbb{L}$. In other words, K is the elementwise stabiliser of lines of $\mathbb{L}$.

Lemma 2.1. (see e.g. [12, Theorem 1]) *The order of the kernel K of a Laguerre plane $\mathbb{L}$ of order s divides $s^3(s-1)$. Moreover, $|K| = s^3(s-1)$ if and only if $\mathbb{L}$ is ovoidal.*

Definition. A Laguerre plane $\mathbb{L}$ is an *elation Laguerre plane* if its kernel K acts transitively on the circles of $\mathbb{L}$.

We denote the dual of a subspace M or a set of subspaces $\mathcal{O}$ of $\mathrm{PG}(3n-1, q)$ by $\overline{M}$ and $\overline{\mathcal{O}}$.

A dual pseudo-oval $\overline{\mathcal{O}}$ in $\mathrm{PG}(3n-1, q)$ gives rise to an elation Laguerre plane $L(\overline{\mathcal{O}})$ in the following way. Embed $H_\infty = \mathrm{PG}(3n-1, q)$ as a hyperplane in $\mathrm{PG}(3n, q)$ and define $L(\overline{\mathcal{O}})$ to be the incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{C})$ with natural incidence and:

$\mathcal{P}$: $2n$-spaces meeting H_∞ in an element of $\overline{\mathcal{O}}$,

$\mathcal{L}$: elements of $\overline{\mathcal{O}}$,

$\mathcal{C}$: points of $\mathrm{PG}(3n, q)$ not in H_∞ (the affine points).

It is not hard to check that this incidence structure defines a Laguerre plane of order q^n and that the group of perspectivities with axis H_∞ in $\mathrm{P\Gamma L}(3n, q)$ induces a subgroup of the kernel of $L(\overline{\mathcal{O}})$ that acts transitively on the circles of $L(\overline{\mathcal{O}})$. So $L(\overline{\mathcal{O}})$ is indeed an elation Laguerre plane.

In [12], Steinke showed the converse: every elation Laguerre plane can be constructed from a dual pseudo-oval.

Theorem 2.2. [12] *A finite Laguerre plane $\mathbb{L}$ is an elation Laguerre plane if and only if $\mathbb{L} \cong L(\overline{\mathcal{O}})$ for some dual pseudo-oval $\overline{\mathcal{O}}$.*

More explicitely, it is shown that a Laguerre plane of order q^n with kernel of order $q^{3n}(q-1)$ can be obtained from a dual pseudo-oval in $\mathrm{PG}(3n-1, q)$.

We show in Theorem 2.4 that every elementary dual pseudo-oval gives rise to an ovoidal Laguerre plane and vice versa. In order to prove this, we need the following lemma.

Lemma 2.3. *Let $\mathbb{L}$ be an ovoidal Laguerre plane of order q^n, then there is a unique subgroup T of order q^{3n} in the kernel K of $\mathbb{L}$.*

Proof. Consider the dual model for an ovoidal Laguerre plane. Every perspectivity in $\mathrm{P\Gamma L}(4, q^n)$ with axis H_∞ induces an element of K. Since the group of perspectivities with axis H_∞ has order $q^{3n}(q^n - 1)$, which equals the order of K by Lemma 2.1, it follows that every element of K corresponds to a perspectivity. The group G_{el} consisting of all elations in $\mathrm{PG}(3, q^n)$ with axis H_∞ is a normal subgroup of the group of all perspectivities with axis H_∞ and has order q^{3n}.

Let S be a subgroup of K of order q^{3n}, $q = p^h$, p prime, then S is a Sylow p-subgroup and since all Sylow p-subgroups are conjugate and G_{el} is normal in K, $S = G_{el}$. □

Theorem 2.4. *A finite elation Laguerre plane $\mathbb{L}$ is ovoidal if and only if $\mathbb{L} \cong L(\overline{\mathcal{O}})$ where $\overline{\mathcal{O}}$ is an elementary dual pseudo-oval in* $\mathrm{PG}(3n - 1, q)$.

Proof. Let $\mathbb{L}$ be an elation Laguerre plane. By Theorem 2.2, $\mathbb{L}$ is isomorphic to $L(\overline{\mathcal{O}})$, where $\overline{\mathcal{O}}$ is a dual pseudo-oval in $\mathrm{PG}(3n - 1, q)$, for some q and n such that the order of $\mathbb{L}$ is q^n. So it remains to show that $L(\overline{\mathcal{O}})$ is ovoidal if and only if $\overline{\mathcal{O}}$ is elementary. In view of the definition of an ovoidal Laguerre plane, using the dual setting, we will show that $L(\overline{\mathcal{O}})$ is isomorphic to $L(\overline{A})$ if and only if the dual pseudo-oval $\overline{\mathcal{O}}$ in $\mathrm{PG}(3n - 1, q)$ is obtained from the dual oval $\overline{A}$ in $\mathrm{PG}(2, q^n)$ by field reduction.

First suppose that the dual pseudo-oval $\overline{\mathcal{O}}$ in $\mathrm{PG}(3n - 1, q)$ is obtained from a dual oval, say $\overline{A}$, in $\mathrm{PG}(2, q^n)$ by field reduction. Apply field reduction to the points, lines and circles of $L(\overline{A})$, then the obtained incidence structure $\mathbb{L}^*$, contained in $\mathrm{PG}(4n - 1, q)$ is isomorphic to $L(\overline{A})$. If we intersect the points, lines and circles of $\mathbb{L}^*$ with a fixed $3n$-dimensional subspace of $\mathrm{PG}(4n - 1, q)$, through the $(3n - 1)$-space containing the field reduced elements of $\overline{A}$, then the obtained structure is clearly isomorphic to the points, lines and circles from $L(\overline{\mathcal{O}})$.

Now, let $\mathbb{L} = (\mathcal{P}, \mathcal{L}, \mathcal{C})$ be a Laguerre plane that on the one hand is isomorphic to $L(\overline{\mathcal{O}})$ (call this *model 1*) and on the other hand isomorphic to $L(\overline{A})$ (call this *model 2*). As before, the elementwise stabiliser of the lines in the automorphism group $\mathrm{Aut}(\mathbb{L})$ of $\mathbb{L}$ (the kernel of $\mathbb{L}$) is denoted by K.

From model 1, we know that the group of elations in $\mathrm{PG}(3n, q)$, with axis the hyperplane H_∞ which contains the elements of $\overline{\mathcal{O}}$, induces a subgroup of K of order q^{3n}, likewise, from model 2, we know that the group of elations in $\mathrm{PG}(3, q^n)$ with axis the hyperplane H which contains the elements of $\overline{A}$ induces a subgroup of K of order q^{3n}. By Lemma 2.3 these induced subgroups are the same, denote this group by T. Consider the stabiliser of a point P

in T. From model 2, we have that T_P has order q^{2n}, the number of elations with axis H fixing a plane of $\mathrm{PG}(3,q^n)$, intersecting H in a line of $\overline{A}$. In model 1, the elements of T_P correspond to elations of $\mathrm{PG}(3n-1,q)$ fixing a $2n$-space intersecting H_∞ in an element of $\overline{\mathcal{O}}$.

The group T corresponds to the elations in $\mathrm{PG}(3,q^n)$ (model 1), hence T forms a 3-dimensional vector space over $\mathbb{F}_{q^n}$. Equivalently, the group T corresponds to the elations in $\mathrm{PG}(3n-1,q)$ (model 2), hence also forms a $3n$-dimensional vector space over $\mathbb{F}_q$. Since T_P in both models is normalised by the perspectivities, we see that T_P forms a 2-dimensional vector subspace $W = V(2,q^n)$ (model 1) and a $2n$-dimensional vector subspace $W' = V(2n,q)$ (model 2) (see also [10]). Clearly, since W and W' correspond to the same vector space, W' is obtained from W by field reduction. Choose for every line ℓ_i of L, one point $P_i \in \ell_i$. Since a point P_i lies on a unique line ℓ_i of L, T_{P_i} can be identified with the line ℓ_i. Considering this projectively, we get that for all $i = 1,\ldots,q^n+1$, the subgroup T_{P_i}, which forms a 2-dimensional vector space over $\mathbb{F}_{q^n}$ and a $2n$-dimensional vector space over $\mathbb{F}_q$, is identified on one hand to an element of $\overline{\mathcal{O}}$ (model 1) and on the other hand to a line of $\overline{A}$ (model 2). This implies that $\overline{\mathcal{O}}$ is obtained from $\overline{A}$ by field reduction. $\square$

From this we can easily deduce the following corollaries.

Corollary 2.5. *A finite elation Laguerre plane $\mathbb{L}$ is Miquelian if and only if $\mathbb{L} \cong L(\overline{\mathcal{O}})$ where $\overline{\mathcal{O}}$ is a dual pseudo-conic in $\mathrm{PG}(3n-1,q)$.*

Corollary 2.6. *Let $\overline{\mathcal{H}}$ be a dual pseudo-hyperoval containing an element $\overline{E}$ such that $L(\overline{\mathcal{O}})$, where $\overline{\mathcal{O}} = \overline{\mathcal{H}} \setminus \overline{E}$, is Miquelian, then $\mathcal{H}$ is a pseudo-hyperconic with E as the field reduced nucleus.*

Proof. By Corollary 2.5, $\overline{\mathcal{O}}$ is obtained by applying field reduction to a dual conic $\overline{\mathcal{C}}$ in $\mathrm{PG}(2,q^n)$. The dual conic $\overline{\mathcal{C}}$ in $\mathrm{PG}(2,q^n)$ uniquely extends to a dual hyperconic by adding its dual nucleus line $\overline{N}$. This shows that $\overline{\mathcal{O}}$ can be extended to a dual pseudo-hyperoval by the $(2n-1)$-space obtained by applying field reduction to the line $\overline{N}$. Since Theorem 1.1 shows that this extension is unique, we see that the element E is the $(n-1)$-space obtained by applying field reduction to the nucleus N of the conic $\mathcal{C}$, and hence, $\mathcal{H}$ is a pseudo-hyperconic. $\square$

3 Towards the proof of the main theorem

Recall that we will prove the following:

Main Theorem. *If $\mathcal{O}$ is a pseudo-oval in* $\mathrm{PG}(3n-1,q)$, $q=2^h$, $h>1$, n *prime, such that the spread induced by every element of $\mathcal{O}$ is Desarguesian, then $\mathcal{O}$ is elementary.*

We know from Theorem 1.1 that a pseudo-oval $\mathcal{O}$ in even characteristic extends in a unique way to a pseudo-hyperoval $\mathcal{H}$ and for the proof of our main theorem, we will work with $\mathcal{H}$, the unique pseudo-hyperoval extending $\mathcal{O}$.

We will split the proof of the Main Theorem in two cases. In Subsection 3.1 we will consider pseudo-hyperovals having a specific property (P1) and we will prove that they are always elementary. In Subsection 3.2 we will consider dual pseudo-hyperovals satisfying a property (P2), and again we show that they are elementary. Finally, in Subsection 3.3 we see that if a pseudo-oval $\mathcal{O}$, such that every element induces a Desarguesian spread, extends to a pseudo-hyperoval $\mathcal{H}$ which does not meet property (P1), then its dual $\overline{\mathcal{H}}$ necessarily meets (P2), which implies that $\mathcal{O}$ is elementary.

3.1 Case 1

In this subsection, we will consider a pseudo-hyperoval $\mathcal{H}$ having the following property:

(P1): there exist four elements E_i, $i=1,\dots,4$ of $\mathcal{H}$, such that

- (i) the induced spreads $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$ are Desarguesian,
- (ii) the unique $\mathbf{S}_{n-1,2}$ through E_1, E_2, E_3 and E_4 does not contain $q+2$ elements of $\mathcal{H}$.

Theorem 3.1. *Consider a pseudo-hyperoval $\mathcal{H}$ in* $\mathrm{PG}(3n-1,q)$, $q=2^h$, $h>1$, n *prime, satisfying Property (P1), then $\mathcal{H}$ is elementary.*

Proof. Let $E_1,\dots,E_4$ be the four elements obtained from the hypothesis that $\mathcal{H}$ satisfies Property (P1). Denote the $(n-1)$-space $\langle E_1,E_2\rangle\cap\langle E_3,E_4\rangle$ by μ. The spreads $\mathcal{S}_1$ and $\mathcal{S}_2$ can be seen in $\langle E_3,E_4\rangle=\mathrm{PG}(2n-1,q)$. By Property (P1), $\mathcal{S}_1$ and $\mathcal{S}_2$ are Desarguesian. Since by definition E_3, E_4 and μ are contained in $\mathcal{S}_1$ and $\mathcal{S}_2$, and $\mathcal{S}_1$ and $\mathcal{S}_2$ are Desarguesian and hence regular, the $q+1$ elements of the unique regulus $\mathcal{R}(\mu,E_3,E_4)$ through E_3, E_4 and μ are contained in $\mathcal{S}_1$ and $\mathcal{S}_2$. We claim that $\mathcal{S}_1=\mathcal{S}_2$.

We see that μ, E_1, E_2 are elements of the spread $\mathcal{S}_3$ considered in $\langle E_1,E_2\rangle$. By Property (P1), $\mathcal{S}_3$ is Desarguesian, hence, regular, so every element of $\mathcal{R}(\mu,E_1,E_2)$ is contained in $\mathcal{S}_3$. Because $q>2$, we may take an element X of $\mathcal{R}(\mu,E_1,E_2)$, different from E_1, E_2 and μ.

Since $X \in \mathcal{S}_3$, the space $\langle X, E_3\rangle$ contains an element, say E_5, of $\mathcal{H}$. The $(2n-1)$-space $\langle E_1, E_5\rangle$ meets $\langle E_3, E_4\rangle$ in an $(n-1)$-space Y, that is by construction contained in $\mathcal{S}_1$. Let $\mathcal{D}$ be the unique Desarguesian spread obtained from Theorem 1.4, through $\mathcal{S}_1$ and E_1, E_2. Since $E_5 = \langle X, E_3\rangle \cap \langle Y, E_1\rangle$ and a Desarguesian spread is normal, we see that $E_5 \in \mathcal{D}$. This holds for every element $E_i \in \mathcal{H}$ contained in $\langle Z, E_3\rangle$ with $Z \in \mathcal{R}(\mu, E_1, E_2)$; let $E_5, \ldots, E_{q+2}$ be these elements of $\mathcal{H}$.

Now consider the $(n-1)$-spaces $T_i := \langle E_2, E_i\rangle \cap \langle E_3, E_4\rangle$, with $i = 5, \ldots, q+2$. The spaces T_i by definition belong to $\mathcal{S}_2$ (considered in $\langle E_3, E_4\rangle$). But since E_2, E_i, E_3, E_4 are elements of $\mathcal{D}$, T_i is an element of $\mathcal{D}$ and since $\mathcal{D} \cap \langle E_3, E_4\rangle = \mathcal{S}_1$, $T_i \in \mathcal{S}_1$.

So the spreads $\mathcal{S}_1$ and $\mathcal{S}_2$ contain $\mathcal{R}(\mu, E_3, E_4)$ and all elements T_i. Suppose that all elements T_i, $i = 5, \ldots, q+2$ are contained in $\mathcal{R}(\mu, E_3, E_4)$. Let P be a point of μ, let ℓ be the unique transversal line through P to the regulus $\mathcal{R}(\mu, E_1, E_2)$ and let m be the unique transversal line through P to the regulus $\mathcal{R}(\mu, E_3, E_4)$. It is clear that the plane $\langle \ell, m\rangle$ is a plane of the second system of the unique $\mathbf{S}_{n-1,2}$, say $\mathcal{B}$, through E_1, E_2, E_3, E_4. This implies that all elements T_i, as well as the elements of $\mathcal{R}(\mu, E_1, E_2)$ are contained in $\mathcal{B}$.

The element E_i, $i = 5, \ldots, q+2$ is obtained as $\langle T_i, E_2\rangle \cap \langle Z, E_3\rangle$, for some $Z \in \mathcal{R}(\mu, E_1, E_2)$. Now it is clear that $\mathbf{S}_{n-1,2}$ has the property that an $(n-1)$-space that is obtained as the intersection of the span of two elements of $\mathbf{S}_{n-1,2}$ is contained in $\mathbf{S}_{n-1,2}$. Since T_i, E_2, Z, E_3 are $(n-1)$-spaces of $\mathcal{B}$, E_i is in $\mathcal{B}$, for all $i = 1, \ldots, q+2$. This implies that $\mathcal{B}$ contains $q+2$ elements of $\mathcal{B}$, a contradiction since $\mathcal{H}$ satisfies Property (P1).

Since $\mathcal{S}_1$ and $\mathcal{S}_2$ have more elements in common than the elements of the regulus $\mathcal{R}(\mu, E_3, E_4)$, using the fact that n is prime, we see that Corollary 1.6 proves our claim.

Since $\mathcal{S}_1 = \mathcal{S}_2$, every element E of $\mathcal{H}$, different from E_1, E_2, E_3, E_4 can be written as $\langle E_1, U\rangle \cap \langle E_2, V\rangle$, where U, V are elements of $\mathcal{S}_1 = \mathcal{S}_2$. Since the Desarguesian spread $\mathcal{D}$ is normal, it follows that $E \in \mathcal{D}$ for all $E \in \mathcal{H}$. Since $\mathcal{H}$ is contained in a Desarguesian spread, $\mathcal{H}$ is elementary. $\square$

3.2 Case 2

In this subsection, we will use the following theorem on hyperovals.

Theorem 3.2. [9, Theorem 11, Remark 5] *Let $\mathcal{O}$ be an oval of* $\mathrm{PG}(2, q^n)$, *$q > 2$ even. Let N be the unique point extending $\mathcal{O}$ to a hyperoval. Then $\mathcal{O}$ is a conic if and only if every triple of distinct points of $\mathcal{O}$ together with N lie in an $\mathbb{F}_q$-subplane that meets $\mathcal{O}$ in $q+1$ points.*

In the proof of this case we will work in the dual setting, so we need the following lemma on dual pseudo-(hyper)ovals.

Lemma 3.3. *Let $\mathcal{O}$ be a pseudo-oval in* $\mathrm{PG}(3n-1,q)$ *such that every element $E_i \in \mathcal{O}$, $i = 1,\ldots,q^n+1$ induces a Desarguesian spread $\mathcal{S}_i$, then the dual pseudo-oval $\overline{\mathcal{O}}$ has the property that for every element $\overline{E_i}$, the set of intersections $\{\overline{E_j} \cap \overline{E_i} | j \neq i\}$ forms a partial spread in $\overline{E_i}$ uniquely extending to a Desarguesian spread and vice versa. The analoguous statement holds for pseudo-hyperovals.*

Proof. An element of $\mathcal{S}_i$, say E_1/E_i equals $\langle E_1, E_i\rangle/E_i$. This space can be identified with $\langle E_1, E_i\rangle$ and its dual $\overline{\langle E_1, E_i\rangle}$, which equals $\overline{E_1} \cap \overline{E_i}$. This implies that the set $\{E_1,\ldots,E_{i-1},E_{i+1},\ldots,E_{q^n+1}\}/E_i$ extends to a Desarguesian spread of $\mathrm{PG}(2n-1,q)$ if and only if $\{\overline{E_1} \cap \overline{E_i},\ldots,\overline{E_{i-1}} \cap \overline{E_i}, \overline{E_{i+1}} \cap \overline{E_i},\ldots,\overline{E_{q^n+1}} \cap \overline{E_i}\}$ extends to a Desarguesian spread. The same reasoning holds for pseudo-hyperovals. □

By abuse of notation, we say that an element $\overline{E_i}$ of a dual pseudo-hyperoval $\overline{\mathcal{H}} = \{\overline{E_1},\ldots,\overline{E_{q^n+2}}\}$ *induces* the spread $\overline{\mathcal{S}}_i := \{\overline{E_j} \cap \overline{E_i} | j \neq i\}$. Then Lemma 3.3 states that $\mathcal{S}_i$ is Desarguesian if and only if $\overline{\mathcal{S}}_i$ is Desarguesian. Also, we write $\overline{\mathbf{S}_{n-1,2}}$ for the set of $(2n-1)$-spaces in $\mathrm{PG}(3n-1,q)$ that is obtained by dualising the system of $(n-1)$-spaces of $\mathbf{S}_{n-1,2}$. In the case that $n = 3$, both systems have spaces of dimension 2, so we dualise the system of planes that contains the elements E_1, E_2, E_3, E_4 used to define the Segre variety $\mathbf{S}_{2,2}$.

We know that the $(n-1)$-spaces of $\mathbf{S}_{n-1,2}$ correspond to the points of an $\mathbb{F}_q$-subplane π of $\mathrm{PG}(2,q^n)$, and are exactly the elements of a Desarguesian spread meeting a fixed plane. By considering the field reduction of the lines of the $\mathbb{F}_q$-subplane π we can also see that $\overline{\mathbf{S}_{n-1,2}}$ consists of q^2+q+1 $(2n-1)$-spaces in $\mathrm{PG}(3n-1,q)$ each meeting a fixed plane in a different line of this plane.

Suppose now the dual pseudo-hyperoval $\overline{\mathcal{H}}$ has an element $\overline{E_1}$ such that $\overline{E_1}$ and $\overline{\mathcal{H}}$ satisfy the following properties:

(P2): (i) $\overline{E_1}$ induces a Desarguesian spread,

(ii) for any three elements $\overline{E_2}, \overline{E_3}, \overline{E_4}$ of $\overline{\mathcal{H}} \setminus \{\overline{E_1}\}$, the unique $\overline{\mathbf{S}_{n-1,2}}$ through $\overline{E_1}, \overline{E_2}, \overline{E_3}$ and $\overline{E_4}$ contains $q+2$ elements of $\overline{H}$.

Note that in the following lemma, we do not require n to be prime.

Lemma 3.4. *Let $\mathcal{H}$ be a pseudo-hyperoval in* $\mathrm{PG}(3n-1,q)$*, $q = 2^h$, $h > 1$. Assume that*

- *the spread induced by a subset $\mathcal{T}$ of q^n+1 elements of $\mathcal{H}$ is Desarguesian,*
- *$\overline{\mathcal{H}}$ satisfies Property (P2) for some element $\overline{E_1}$ of $\overline{\mathcal{T}}$,*

then the following statements hold:

(i) the elation Laguerre plane $L(\overline{\mathcal{O}})$ where $\overline{\mathcal{O}} = \overline{\mathcal{H}} \backslash \{\overline{E_1}\}$ is isomorphic to the Laguerre plane $(\mathcal{P}', \mathcal{L}', \mathcal{C}')$ embedded in π, with natural incidence, given by

$\mathcal{P}'$: the lines of π different from ℓ_∞,
$\mathcal{L}'$: the points of ℓ_∞,
$\mathcal{C}'$: the q^{2n} point-pencils of π not containing ℓ_∞ and $q^{3n}-q^{2n}$ dual ovals such that ℓ_∞ extends all of them to a dual hyperoval,

where π is the Desarguesian projective plane $\mathrm{PG}(2, q^n)$ obtained from the André/Bruck-Bose construction obtained from the spread $\overline{\mathcal{S}_1}$ and ℓ_∞ is the line of π corresponding to $\overline{E}_1$.

(ii) a dual oval $\overline{A}$ of the set $\mathcal{C}'$ is a dual conic with ℓ_∞ as its nucleus line.

(iii) $L(\overline{\mathcal{O}})$ is Miquelian.

Proof. (i) Embed the space $\mathrm{PG}(3n-1, q)$, containing $\overline{\mathcal{O}}$, as a hyperplane H_∞ in $\mathrm{PG}(3n, q)$. Recall that $L(\overline{\mathcal{O}})$ is the incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{C})$, with natural incidence, embedded in $\mathrm{PG}(3n, q)$ as follows:

$\mathcal{P}$: the $2n$-spaces meeting H_∞ in an element of $\overline{\mathcal{O}}$,
$\mathcal{L}$: the elements of $\overline{\mathcal{O}}$,
$\mathcal{C}$: the points of $\mathrm{PG}(3n, q)$ not contained in H_∞ (the affine points).

Consider a $2n$-space Π of $\mathrm{PG}(3n, q)$ intersecting H_∞ in $\overline{E_1}$. The elements of $\overline{\mathcal{O}}$ intersect $\overline{E_1}$ in the Desarguesian spread $\overline{\mathcal{S}_1}$. It follows that the (projective) André/Bruck-Bose construction in Π, using $\overline{\mathcal{S}_1}$, defines a Desarguesian projective plane $\pi \cong \mathrm{PG}(2, q^n)$. The elements of $\overline{\mathcal{S}_1}$ correspond to the points of a line ℓ_∞ of π. By intersecting the elements of $L(\overline{\mathcal{O}})$ with Π, we find the representation $(\mathcal{P}', \mathcal{L}', \mathcal{C}')$ of the Laguerre plane $L(\overline{\mathcal{O}})$ in the Desarguesian plane π as given in the statement. For this, we identify every circle of $\mathcal{C}$ with the q^n+1 elements of $\mathcal{P}$ it contains and consider their intersection with Π. Then, an affine point contained in Π corresponds to a point-pencil of π not containing ℓ_∞. An affine point not contained in Π will also correspond to a set of q^n+1 lines of π, different from ℓ_∞. However, since such an affine point does not belong to Π, any three of these lines will have empty intersection, hence they form a dual oval. Moreover, these q^n+1 lines intersect the line ℓ_∞ all in a different point, therefore each dual oval extends uniquely to a dual hyperoval by adding the line ℓ_∞.

(ii) Consider the affine point P of $\mathrm{PG}(3n,q)\backslash\Pi$ corresponding to $\overline{A}$. Consider three lines ℓ_1,ℓ_2,ℓ_3 of $\overline{A}$. These correspond to three elements of $\overline{\mathcal{H}}$, say $\overline{E_2},\overline{E_3}$ and $\overline{E_4}$. Now, since $\overline{\mathcal{H}}$ satisfies Property (P2), we find that the unique $\overline{\mathbf{S}_{n-1,2}}$, say $\mathcal{B}$, through the 4 $(2n-1)$-spaces $\overline{E_1}$, $\overline{E_2},\overline{E_3}$ and $\overline{E_4}$ contains $q+2$ elements of $\overline{\mathcal{H}}$.

The element $\overline{E_1}$ is contained in $\mathcal{B}$, and the projection from P of the q^2+q $(2n-1)$-spaces of $\mathcal{B}$, different from $\overline{E_1}$, onto the space Π (used in the André/Bruck-Bose construction) corresponds to q^2+q lines of the plane π. Every such projected line intersects ℓ_∞ in a point which corresponds to one of the $q+1$ elements of the unique regulus in $\overline{E_1}$ through $\overline{E_1}\cap\overline{E_2}$, $\overline{E_1}\cap\overline{E_3}$ and $\overline{E_1}\cap\overline{E_4}$. This implies that the set of $(2n-1)$-spaces $\mathcal{B}$ corresponds to the set of lines of an $\mathbb{F}_q$-subplane in the Desarguesian plane π, which contains $\ell_\infty,\ell_1,\ell_2,\ell_3$ and $q-2$ other lines of $\overline{A}$. Since this is true for every choice of three distinct lines ℓ_1,ℓ_2,ℓ_3 of $\overline{A}$, by Theorem 3.2, $\overline{A}$ is a dual conic with ℓ_∞ as its nucleus line.

(iii) We consider the dual $(\mathcal{P}'',\mathcal{L}'',\mathcal{C}'')$ of the incidence structure $(\mathcal{P}',\mathcal{L}',\mathcal{C}')$ and use part (ii) which states that the dual ovals in $\mathcal{C}$ are dual conics. Also note that the dual of the Desarguesian plane π is also Desarguesian. Let the point N be the dual of the line ℓ_∞, then $(\mathcal{P}'',\mathcal{L}'',\mathcal{C}'')$ is given by

- $\mathcal{P}''$: the points of $\mathrm{PG}(2,q^n)$ different from N,
- $\mathcal{L}''$: the lines of $\mathrm{PG}(2,q^n)$ containing N,
- $\mathcal{C}''$: the q^{2n} lines of $\mathrm{PG}(2,q^n)$ not containing N and the $q^{3n}-q^{2n}$ conics in $\mathrm{PG}(2,q^n)$ having N as their nucleus.

This is just the standard plane model for a Miquelian Laguerre plane of even order q^n. □

3.3 The proof of the main theorem

We will first prove a lemma which gives a connection between Properties (P1) and (P2).

Lemma 3.5. *Let $\mathcal{H}$ be a pseudo-hyperoval in $\mathrm{PG}(3n-1,q)$, $q=2^h$, $h>1$, such that there is a subset $\mathcal{O}$ of q^n+1 elements of $\mathcal{H}$ inducing a Desarguesian spread. If $\mathcal{H}$ does not satisfy Property (P1), then $\overline{\mathcal{H}}$ satisfies (P2) for every element of $\overline{\mathcal{O}}$.*

Proof. If the hyperoval $\mathcal{H}$ does not satisfy Property (P1), then clearly, it does not satisfy Property (P1)(ii). So for every 4 elements E_i, $i = 1, \ldots, 4$ of $\mathcal{H}$, the unique $\mathcal{S}_{n-1,2}$ through E_i, $i = 1, \ldots, 4$ contains $q+2$ elements of $\mathcal{H}$. This implies that the unique $\overline{\mathbf{S}_{n-1,2}}$ through $\overline{E_i}$, $i = 1, \ldots, 4$ contains $q+2$ elements of $\overline{\mathcal{H}}$, so $\overline{\mathcal{H}}$ satisfies Property (P2) for all elements of $\overline{\mathcal{O}}$.

□

Theorem 3.6. *If $\mathcal{O}$ is a pseudo-oval in* $\mathrm{PG}(3n-1, q)$*, $q = 2^h$, $h > 1$, n prime, such that the spread induced by every element of $\mathcal{O}$ is Desarguesian, then $\mathcal{O}$ is elementary.*

Proof. By Theorem 1.1, we may consider the unique pseudo-hyperoval $\mathcal{H}$ extending $\mathcal{O}$. Clearly, $\mathcal{H}$ satisfies the conditions of Lemma 3.5. This implies that either $\mathcal{H}$ satisfies Property (P1), and then the statement follows from Theorem 3.1 (and the fact that a subset of an elementary set is elementary), or $\overline{\mathcal{H}}$ satisfies Property (P2) for every element of $\overline{\mathcal{O}}$.

By Lemma 3.4, $L(\overline{\mathcal{O}})$ is Miquelian, and by Lemma 2.6, $\mathcal{H}$ is a pseudo-hyperconic with E corresponding to the nucleus N of a conic $\mathcal{C}$ (hence $\mathcal{O}$ is elementary). Note that only for $q = 4$ this possibility can occur, since it is impossible that the set $\mathcal{C} \cup \{N\} \setminus \{P\}$, where P is a point of $\mathcal{C}$ is again a conic, if $q > 4$. □

As a corollary, we state a similar statement for pseudo-hyperovals.

Corollary 3.7. *Let $\mathcal{H}$ be a pseudo-hyperoval in* $\mathrm{PG}(3n-1, q)$*, $q = 2^h$, $h > 1$, n prime, such that the spread induced by q^n+1 elements of $\mathcal{H}$ is Desarguesian, then $\mathcal{H}$ is elementary.*

Proof. The subset $\mathcal{O}$ of elements inducing a Desarguesian spread is an elementary pseudo-oval by Theorem 3.6, suppose $\mathcal{O}$ is the field reduced oval A. There is a unique element extending $\mathcal{O}$ to a pseudo-hyperoval, so $\mathcal{H} \setminus \mathcal{O}$ must be the element corresponding the unique point of $\mathrm{PG}(2, q^n)$ extending A to a hyperoval. □

Remark. Using a substantial amount of effort, the proof of Theorem 3.1 can be extended to hold for all n, and not only for n prime. However, the conditions (P1) and (P2) become slightly different and hence a modified version of Lemma 3.4 is necessary. For the proof of this modified lemma, we require a more general version of Theorem 3.2 which is unfortunately out of our reach.

3.4 The consequence of the main theorem for Laguerre planes

Lemma 3.8. *A point P of an elation Laguerre plane $\mathbb{L} = L(\overline{\mathcal{O}})$, where $\overline{\mathcal{O}}$ is a dual pseudo-oval in* $\mathrm{PG}(3n-1, q)$*, admits a Desarguesian derivation if and only if the spread $\mathcal{S}$, induced by the line of $L(\overline{\mathcal{O}})$ through P is Desarguesian.*

Proof. Let P be a point of $\mathbb{L}$, then P is a $2n$-space through an element E of $\overline{\mathcal{O}}$. The derived affine plane of order q^n at the point P of $\mathbb{L}$ consists of points $\mathcal{P}'$ and lines $\mathcal{L}'$ obtained as follows:

$\mathcal{P}'$: $2n$-spaces in $\mathrm{PG}(3n, q)$, not in H_∞, through an element of $\overline{\mathcal{O}} \setminus \{E\}$,

$\mathcal{L}'$: points in P not in H_∞, together with the elements of $\overline{\mathcal{O}} \setminus E$.

Now this affine plane clearly extends to a projective plane of order q^n by adding the $q^n + 1$ elements of $\mathcal{S}$ as points and the space E as line at infinity. This projective plane is the dual of the plane obtained from the (projective) André/Bruck-Bose construction starting from $\mathcal{S}$ and hence, is Desarguesian if and only if $\mathcal{S}$ is Desarguesian. □

If $\mathbb{L}$ is a Laguerre plane of odd order, then the main theorem of Chen and Kaerlein [6] states that the existence of one point admitting a Desarguesian derivation forces $\mathbb{L}$ to be Miquelian. The following theorem which is a consequence of our main theorem gives a (much) weaker result in the case of even order Laguerre planes.

Theorem 3.9. *Let $\mathbb{L}$ be a Laguerre plane of order q^n with kernel K, $|K| \geq q^{3n}(q-1)$, n prime, $q > 2$ even. Suppose that for every line of $\mathbb{L}$, there exists a point on that line that admits a Desarguesian derivation, then $\mathbb{L}$ is ovoidal and $|K| = q^{3n}(q^n - 1)$.*

Proof. From the hypothesis on the size of K and Lemma 2.1, we find that q^{3n} divides the order of T, hence, by [12, Theorem 2] $\mathbb{L}$ is an elation Laguerre plane. By Theorem 2.2 $\mathbb{L}$ can be constructed from a dual pseudo-oval $\overline{\mathcal{O}}$ in $\mathrm{PG}(3n-1, q)$, n prime. From Lemma 3.8, we obtain that for every element of $\overline{\mathcal{O}}$ the induced spread is Desarguesian. By Theorem 3.6, $\overline{\mathcal{O}}$ is elementary. By Theorem 2.4 this implies that $\mathbb{L}$ is ovoidal. Finally, this implies by Lemma 2.1 that $|K| = q^{3n}(q^n - 1)$. □

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