Ghent University
Faculty of Sciences

Semilinear and semiquadratic conjunctive aggregation functions

Tarad Jwaid

Thesis submitted in fulfillment of the requirements for the degree of Doctor (Ph.D.) of Mathematics

Academic year 2013-2014

Supervisors: Prof. dr. Bernard De Baets
Faculty of Bioscience Engineering
Department of Mathematical Modelling,
Statistics and Bioinformatics
Ghent University, Belgium

Prof. dr. Hans De Meyer
Faculty of Sciences
Department of Applied Mathematics,
Computer Science and Statistics
Ghent University, Belgium

Examination committee: Prof. dr. Andreas Weiermann (chairman)

Dr. Glad Deschrijver
Prof. dr. Anna Kolesárová
Prof. dr. Manuel Úbeda-Flores
Prof. dr. David Vyncke

Dean: Prof. dr. Herwig Dejonghe

Rector: Prof. dr. Anne De Paepe

Tarad Jwaid

Semilinear and semiquadratic conjunctive aggregation functions

Thesis submitted in fulfillment of the requirements for the degree of

Doctor (Ph.D) of Mathematics

Academic year 2013-2014

Please refer to this work as follows:
Tarad Jwaid (**2014**). *Semilinear and semiquadratic conjunctive aggregation functions*, PhD Thesis, Faculty of Sciences, Ghent University, Ghent, Belgium.

ISBN 978-90-5989-722-9

Acknowledgements

I acknowledge the presence of God who created me and gave me this rare privilege to achieve my dream of attaining the highest qualification. This thesis is an output of several years of research that has been done since I came to Ghent. Since that time, I have worked with many people whose support and collaboration in various and diverse ways contributed to this great success of my thesis. It is a pleasure to convey my gratitude to them all in my humble acknowledgment. I am highly indebted to my supervisors Prof. dr. Bernard De Baets and Prof. dr. Hans De Meyer who taught and supervised me during these years of research. Bernard and Hans, it is a great honor to work with you. Without any doubt, your efforts were putting me on the right path. I will never forget your guidance and the help you gave me even during weekends, holidays and all other opportunities "Heel Hartelijk Bedankt".

I sincerely thank the chairman of the jury committee, Prof. dr. Andreas Weiermann and the other jury members, Dr. Glad Deschrijver, Prof. dr. Anna Kolesárová, Prof. dr. Manuel Úbeda-Flores and Prof. dr. David Vyncke.

I wish to express my profound appreciation to my colleagues at KERMIT for the friendly atmosphere and cooperation. I sincerely thank Ali Demir for his kind help of Dutch translation of the thesis summary. I wish also to express my appreciation to my colleagues at BIOMATH for the friendly atmosphere.

I would like to thank the administrative and technical staff in the department, Rose, Annie, Timpe, Ruth, Tinne, Karel and Jan.

I would like to express my gratitude to all my Syrian friends in Belgium and their families who have helped me during my study, especially M. Hamshou, M. Alchihab, Abd Al- Karim, M. Al- Shukor, N. Al- Shukor, Raki, M. Alakch, H. Kadoni, Fateh, Ehab, Abo Jameel, Moursy, M. Khlosy, M. Al- Mohamad and many other friends of the Syrian community in Ghent. I also wish to send my sincere gratitude to the Ministry of Higher Education (especially Mrs. Eyman and Heba) in Syria who supported me to pursue my stay and education in Belgium.

I am very grateful to my mother 'Sara' and my father 'Ibrahim'. Their prayers, passionate encouragements and generosities have followed me everywhere to give me a lot of power. My deepest gratitude goes to my sisters and my brothers. Specific gratitude goes to my sister 'Azeza' and my brother 'Motaz' who passed away during my stay in Belgium, God have mercy on them. I wish to send my best regards to my wife's family especially my mother and father-in-low. I wish all of you a prosperous life full of happiness and health. My lovely wife 'Hana' and my adorable children 'Sara' and 'Ibrahim', you were the main supporters of me along

my entire PhD thesis. I am deeply grateful for your patience and sacrifices. I hope I can compensate you with all my love for all the moments which I spent far away from you.

Gent, September 2014
Tarad Jwaid

Table of contents

List of symbols

$\mathcal{D}$ — The set of diagonal functions (defined on page 21)

$\mathcal{D}_{\mathrm{A}}$ — The set of $[0,1] \to [0,1]$ functions that satisfy the first two conditions of the definition of a diagonal function (defined on page 22)

$\mathcal{D}_{\mathrm{S}}$ — The set of $[0,1] \to [0,1]$ functions that satisfy the first three conditions of the definition of a diagonal function (defined on page 22)

$D_{\mathrm{S}}^{\mathrm{ac}}$ — The set of absolutely continuous $[0,1] \to [0,1]$ functions that satisfy the first three condition of the definition of a diagonal function (defined on page 22)

$\mathcal{O}$ — The set of opposite diagonal functions (defined on page 23)

$\mathcal{O}_{\mathrm{S}}$ — The set of $[0,1] \to [0,1]$ functions that satisfy the first condition of the definition of an opposite diagonal function (defined on page 24)

$O_{\mathrm{S}}^{\mathrm{ac}}$ — The set of absolutely continuous $[0,1] \to [0,1]$ functions that satisfy the first condition of the definition of an opposite diagonal function (defined on page 24)

Z — The zero set of an aggregation function (defined on page 29)

δ — The generic symbol of a diagonal function (defined on page 20)

ω — The generic symbol of an opposite diagonal function (defined on page 22)

ρ — The generic symbol of the population version of Spearman (defined on page 16)

γ — The generic symbol of the population version of Gini (defined on page 16)

τ — The generic symbol of the population version of Kendall (defined on page 16)

λ_{UU} — The generic symbol of the upper-upper tail dependence (defined on page 17)

λ_{UL} — The generic symbol of the upper-lower tail dependence (defined on page 17)

λ_{LU} — The generic symbol of the lower-upper tail dependence (defined on page 17)

λ_{LL} — The generic symbol of the lower-lower tail dependence (defined on page 17)

λ_{δ} — Auxiliary function defined on page 61

μ_δ — Auxiliary function defined on page 63

ψ_δ — Auxiliary function defined on page 120

ξ_δ — Auxiliary function defined on page 120

λ_ω — Auxiliary function defined on page 83

μ_ω — Auxiliary function defined on page 83

ψ_ω — Auxiliary function defined on page 140

ξ_ω — Auxiliary function defined on page 140

$\vartheta_{\delta,\omega}$ — Auxiliary function defined on page 151

$\psi_{\delta,\omega}$ — Auxiliary function defined on page 151

φ — Auxiliary function defined on page 92

$\widehat{\varphi}$ — Auxiliary function defined on page 92

ψ — Auxiliary function defined on page 92

$\widehat{\psi}$ — Auxiliary function defined on page 92

Preface

The study of aggregation functions has become one of the core activities in several areas of research, as can be seen from the vast number of papers, monographs [2, 6, 9, 48] and summer schools on the topic. Their importance can be seen in applied mathematics (e.g., probability theory, statistics, fuzzy set theory), computer science (e.g., artificial intelligence, operations research), as well as in many applied fields (image processing, decision making, control theory, information retrieval, finance, etc.).

The word *aggregation* [6, 48] refers to the process of combining several input values into a single representative output value and the function that performs this process is called an *aggregation function*. The input values depend on the field of application. For instance, in fuzzy set theory, they can be degrees of membership, truth values, intensities of preference, and so on. For this reason, aggregation functions play an important role in many applications of fuzzy set theory, such as fuzzy modelling, fuzzy logic [58], preference modelling [28, 54, 101] and similarity measurement [26]. Their most prominent use is as fuzzy logical connectives [6].

Special classes of aggregation functions are of particular interest, such as semi-copulas [12, 41, 44], triangular norms [2, 75], quasi-copulas [47, 60, 78] and copulas [2, 88]. They are all conjunctors, in the sense that they extend the classical Boolean conjunction. Semi-copulas have recently gained importance in reliability theory, fuzzy set theory and multi-valued logic [3, 34, 45, 59]. Triangular norms are the most popular operations for modelling the intersection in fuzzy set theory [75, 48]. Quasi-copulas and copulas are widely studied. For instance, quasi-copulas appear in fuzzy set theoretical approaches to preference modelling and similarity measurement [25, 26, 28, 52]. Due to Sklar's theorem [99], copulas have received ample attention from researchers in probability theory and statistics [61, 64].

The arithmetic mean is an example of an aggregation function and it has been used over centuries in several areas of research. This provides us an idea how old the existence of aggregation functions is. Although the existence of aggregation functions is rather old, they have been buried until recently. The arrival of computers in the eighties has created the appropriate circumstances where they become present. Hence, since the eighties, aggregation functions have become a genuine research field, rapidly developing, but in a rather scattered way since aggregation functions are rooted in many different fields.

Modern technologies have helped the researchers to produce a massive amount of data based on observations. In order to allow more flexible modelling techniques, new methods to construct aggregation functions are being proposed continuously in the literature. Some methods are based on transformations [1, 17, 29, 76, 77, 86]. In other words, one starts from a given aggregation function and by applying an appropriate transformation, the resulting function is an aggregation function. Some other methods are based on composing aggregation functions [10, 48]. Several construction methods apply linear or quadratic interpolation to various types of partial information, such as given sections (horizontal, vertical, diagonal, etc.) [4, 17, 21, 42, 48, 49, 50, 91].

In this work, we mainly focus on construction methods of aggregation functions of the latter type. This dissertation is organized as follows.

1. In Chapter 1, we provide a general introduction.

2. In Part I, we provide several construction methods based on linear interpolation. In Chapter 2, we consider the linear interpolation on segments connecting the upper boundary curve of the zero-set of an aggregation function to the point $(1, 1)$, while we consider the linear interpolation on segments connecting the diagonal (resp. opposite diagonal) of the unit square to the points $(0, 1)$ and $(1, 0)$ (resp. $(0, 0)$ and $(1, 1)$) in Chapter 3. Rather than using the upper boundary curve of the zero-set, we consider in Chapter 4 any curve from a semi-copula determined by a strict negation operator. Instead of using the linear interpolation on segments connecting a line in the unit square to the corners of the unit square, we consider in Chapter 5 the linear interpolation on segments that are perpendicular to the diagonal or opposite diagonal of the unit square. We involve in Chapter 6 both the diagonal and the opposite diagonal of the unit square in the linear interpolation procedure.

3. In Part II, we provide several construction methods based on quadratic interpolation. In Chapter 7, we generalize lower semilinear copulas by considering the quadratic interpolation on segments connecting the diagonal of the unit square to the sides of the unit square. In Chapter 8, we complete the results of Chapter 7, and generalize the results of Chapter 6 by considering all the possible horizontal and vertical quadratic interpolations on segments connecting the diagonal or/and opposite diagonal section of the unit square to the sides of the unit square.

4. Finally, general conclusions are drawn.

Most of our work presented in this dissertation has already been published or submitted for publication in peer-reviewed international journals. Chapters 2, 3, 4, 5, 6, 7 and 8 have been described in [70], [66], [69], [67], [65], [71] and [68], respectively.

1 General introduction

1.1. Aggregation functions

1.1.1. Basic definitions

Aggregation functions have become very popular over the last years due to their wide range of applications in several areas of research. Their main role appears in applied sciences, such as image processing, decision making, control theory, information retrieval, etc. [6, 30, 48]. In general, aggregation functions are used to convert finitely many input values into a single representative output value. These input values can represent experimental observations, intensities of preferences, statistical data, probabilities, etc. The output value enables us to describe and predict experimental phenomena, to classify objects and species and make appropriate decisions. The aggregation process requires the input values as well as the output value to belong to the same numerical interval. Two properties are fundamental for any aggregation function A. They coincide in the point $\mathbf{0} = (0, 0, \cdots, 0)$ as well as in the point $\mathbf{1} = (1, 1, \cdots, 1)$, and they are increasing. In fuzzy set theory, for instance, such properties can be seen as follows. The input vectors $\mathbf{0}$ and $\mathbf{1}$ represent no membership and full membership. Hence, it is natural to assign $A(\mathbf{0}) = 0$ and $A(\mathbf{1}) = 1$. Consider the two input vectors $(b, a, \cdots, a)$ and $(c, a, \cdots, a)$, with $b \leq c$, representing intensities of preferences. Hence, it is natural to consider $A(b, a, \cdots, a) \leq A(c, a, \cdots, a)$. Due to the increasingness of the aggregation functions involved, it is often possible to rescale the input values as well as the output values to the unit interval.

Definition 1.1. *An n-ary aggregation function A is a $[0,1]^n \to [0,1]$ function satisfying the following minimal conditions:*

(i) boundary conditions*: $A(\mathbf{0}) = 0$ and $A(\mathbf{1}) = 1$;*

(ii) monotonicity*: for any $\mathbf{x}, \mathbf{y} \in [0,1]^n$ such that $\mathbf{x} \leq \mathbf{y}$, it holds that $A(\mathbf{x}) \leq A(\mathbf{y})$.*

Well-known examples of aggregation functions are:

1. the arithmetic mean:

$$\mathrm{AM}(x_1, x_2, \cdots, x_n) = \frac{x_1 + x_2 + \cdots + x_n}{n},$$

2. the geometric mean:

$$\mathrm{GM}(x_1, x_2, \cdots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n}\,,$$

3. harmonic mean:

$$\mathrm{HM}(x_1, x_2, \cdots, x_n) = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}\,,$$

4. minimum:

$$T_{\mathbf{M}}(x_1, x_2, \cdots, x_n) = \min(x_1, x_2, \cdots, x_n)\,,$$

5. maximum:

$$S_{\mathbf{M}}(x_1, x_2, \cdots, x_n) = \max(x_1, x_2, \cdots, x_n)\,,$$

6. product:

$$T_{\mathbf{P}}(x_1, x_2, \cdots, x_n) = x_1 x_2 \cdots x_n\,,$$

7. bounded sum:

$$T_{\mathbf{L}}(x_1, x_2, \cdots, x_n) = \min(1, x_1 + x_2 + \cdots + x_n)\,.$$

The maximum and minimum aggregation functions have been used to classify aggregation functions into four main classes:

1. averaging,
2. conjunctive,
3. disjunctive,
4. mixed.

Definition 1.2. *Let $A : [0,1]^n \to [0,1]$ be an aggregation function. Then*

(i) *A is called* averaging *if*

$$\min(\mathbf{x}) \leq A(\mathbf{x}) \leq \max(\mathbf{x})$$

for any $\mathbf{x} \in [0,1]^n$.

(ii) *A is called* conjunctive *if*

$$A(\mathbf{x}) \leq \min(\mathbf{x})$$

for any $\mathbf{x} \in [0,1]^n$.

(iii) *A is called* disjunctive *if*

$$A(\mathbf{x}) \geq \max(\mathbf{x})$$

for any $\mathbf{x} \in [0,1]^n$.

(iv) *Any aggregation function that does not satisfy one of the above inequalities is called* mixed *aggregation function.*

The aggregation functions A^u and A^l given by

$$A^u(\mathbf{x}) = \begin{cases} 0 & \text{, if } \max(\mathbf{x}) = 0\,, \\ 1 & \text{, otherwise,} \end{cases}$$

$$A^l(\mathbf{x}) = \begin{cases} 1 & \text{, if } \min(\mathbf{x}) = 1\,, \\ 0 & \text{, otherwise,} \end{cases}$$

are respectively the greatest and the smallest aggregation function, i.e. for any aggregation function A, it holds that

$$A^l \leq A \leq A^u\,.$$

Throughout the dissertation, we restrict our attention mostly to binary aggregation functions. A (binary) aggregation function A is an increasing $[0,1]^2 \to [0,1]$ function that preserves the bounds, i.e. $A(0,0) = 0$ and $A(1,1) = 1$. Obviously, this definition has to be complemented by a variety of additional properties depending on the field of application.

1.1.2. Properties and facts

Let $A : [0,1]^2 \to [0,1]$ be an aggregation function.

(i) A has $a \in [0,1]$ as absorbing element if

$$A(x,a) = A(a,x) = a$$

for any $x \in [0,1]$.

(ii) A has $b \in [0,1]$ as neutral element if

$$A(x,b) = A(b,x) = x$$

for any $x \in [0,1]$.

(iii) A is *commutative* (or symmetric) if

$$A(x,y) = A(y,x)$$

for any $x, y \in [0,1]$.

(iv) A is *associative* if

$$A(A(x,y),z) = A(x,A(y,z))$$

for any $x,y,z \in [0,1]$.

(v) A is *continuous in the first variable* if

$$\lim_{x\to x_0} A(x,y) = A(x_0,y)$$

for any $x_0, y \in [0,1]$.

(vi) A is *continuous in the second variable* if

$$\lim_{y\to y_0} A(x,y) = A(x,y_0)$$

for any $x, y_0 \in [0,1]$.

(vii) A is *continuous* if it is continuous in each variable.

(viii) A is 1-Lipschitz continuous if

$$|A(x',y') - A(x,y)| \le |x'-x| + |y'-y|$$

for any $x,x',y,y' \in [0,1]$.

(ix) A is 2-increasing if

$$V_A([x,x']\times[y,y']) = A(x,y) - A(x',y) - A(x,y') + A(x',y') \ge 0 \tag{1.1}$$

for any $x,x',y,y' \in [0,1]$ such that $x \le x'$ and $y \le y'$. V_A is called the A-volume of the rectangle $[x,x']\times[y,y']$.

Note that A-volumes are additive, i.e. when a rectangle is decomposed into a number of rectangles then the A-volume of the original rectangle is equal to the sum of the A-volumes of all the rectangles in its decomposition.

If an aggregation function A has 1 as neutral element, then due to its increasingness, it has 0 as absorbing element as well. The 1-Lipschitz continuity of an aggregation function implies its continuity. The 2-increasingness of an aggregation function A that has 1 as neutral element implies its 1-Lipschitz continuity [48]. Note that a function $G : [0,1]^2 \to [0,1]$ that has 0 as absorbing element and 1 as neutral element, and satisfies the 2-increasingness, is an aggregation function. Moreover, G is 1-Lipschitz continuous.

1.1.3. Subclasses of aggregation functions

Conjunctors

Definition 1.3. *An aggregation function A is called a* conjunctor *if it has* 0 *as absorbing element.*

Conjunctors are used to extend the classical Boolean conjunction. The aggregation functions J^u and A_l with $J^u(x,y) = 0$ whenever $\min(x,y) = 0$, and $J^u(x,y) = 1$ elsewhere, are respectively the greatest and the smallest conjunctor, i.e. for any conjunctor J, it holds that

$$A_l \leq J \leq J^u \,.$$

Semi-copulas

Definition 1.4. *An aggregation function A is called a* semi-copula *if it has* 1 *as neutral element.*

The notion of a semi-copula appeared for the first time in the literature in the field of reliability theory. Semi-copulas turn out to be appropriate tools for capturing the relation between multivariate aging and dependence [3, 34]. The functions $T_{\mathbf{M}}$ and $T_{\mathbf{D}}$, with $T_{\mathbf{D}}(x,y) = \min(x,y)$ whenever $\max(x,y) = 1$, and $T_{\mathbf{D}}(x,y) = 0$ elsewhere, are semi-copulas. Moreover, they are respectively the greatest and the smallest semi-copula, i.e. for any semi-copula S, it holds that

$$T_{\mathbf{D}} \leq S \leq T_{\mathbf{M}} \,.$$

Any semi-copula is a conjunctor, and hence, the class of semi-copulas is a subclass of the class of conjunctors. The conjunctor $J : [0,1]^2 \to [0,1]$ defined by $J(x,y) = xy^2$ is not a semi-copula. Consequently, the class of semi-copulas is a proper subclass of the class of conjuntors.

Triangular norms

Definition 1.5. *An aggregation function A with neutral element* 1 *is called a* triangular norm *if it is commutative and associative.*

Triangular norms (t-norms for short) are the most popular operations for modelling the intersection in fuzzy set theory. The functions $T_{\mathbf{M}}$, $T_{\mathbf{P}}$ and $T_{\mathbf{D}}$ are examples of t-norms. $T_{\mathbf{D}}$ is called the drastic t-norm. Any t-norm is a semi-copula, and hence, the class of t-norms is a subclass of the class of semi-copulas. The semi-copula $S : [0,1]^2 \to [0,1]$ defined by $S(x,y) = xy \max(x,y)$ is not a t-norm. Consequently, the class of t-norms is a proper subclass of the class of semi-copulas.

Continuous Archimedean t-norms

Let T be a t-norm and $x \in]0,1[$. A T-power of x is defined by

$$x^{(1)} = x \quad \text{and} \quad x^{(n+1)} = T(x^{(n)}, x)\,,$$

where $n \in \mathbb{N}_0$.

Definition 1.6. *A t-norm T is called* Archimedean *if for any $(x,y) \in]0,1[^2$ there exists an $n \in \mathbb{N}_0$ such that*

$$x^{(n)} < y\,.$$

In the following proposition we recall an equivalent condition for a t-norm T to be Archimedean when T is continuous.

Proposition 1.1. [48] *A continuous t-norm T is Archimedean if and only if*

$$T(x,x) < x$$

for any $x \in]0,1[$.

The t-norm $T_{\mathbf{P}}$ is Archimedean while $T_{\mathbf{M}}$ is not. The t-norm $T_{\mathbf{D}}$ is an Archimedean t-norm that is not continuous. Note also that the t-norm $T_{n\mathrm{M}}$ given by

$$T_{n\mathrm{M}}(x,y) = \begin{cases} 0 & \text{, if } x+y \leq 1\,, \\ \min(x,y) & \text{, if } x+y > 1\,, \end{cases}$$

is neither continuous nor Archimedean [53].

Let $t : [0,1] \to [0,\infty]$ be a strictly decreasing continuous function satisfying $t(1) = 0$. The function $t^{(-1)} : [0,\infty] \to [0,1]$ defined by

$$t^{(-1)}(x) = \begin{cases} t^{-1}(x) & \text{, if } x \in [0, t(0)]\,, \\ 0 & \text{, otherwise}\,, \end{cases}$$

is called *the pseudo-inverse* of the function t. In fact, any continuous Archimedean t-norm can be represented by means of a strictly decreasing continuous $[0,1] \to [0,\infty]$ function.

Theorem 1.1. [2] *A continuous t-norm T is Archimedean if and only if there exists a strictly decreasing continuous $[0,1] \to [0,\infty]$ function t satisfying $t(1) = 0$ such that*

$$T(x, y) = t^{(-1)}(t(x) + t(y))$$

for any $x, y \in [0,1]$.

The function t is called an *additive generator*. Additive generators of the t-norms $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$ are respectively defined by $t_1(x) = -\log(x)$ and $t_2(x) = 1 - x$.

Quasi-copulas

Definition 1.7. *An aggregation function A with neutral element 1 is called a* quasi-copula *if it is 1-Lipschitz continuous.*

Quasi-copulas appear in fuzzy set theoretical approaches to preference modelling and similarity measurement. The 1-Lipschitz continuity of a quasi-copula implies its continuity. Note that any quasi-copula is a semi-copula, and hence, the class of quasi-copulas is a subclass of the class of semi-copulas. The function $S : [0,1]^2 \to [0,1]$ defined by

$$S(x, y) = \begin{cases} 0 & \text{, if } (x, y) \in [0, 1/2] \times [0, 1[\,, \\ \min(x, y) & \text{, otherwise,} \end{cases}$$

is a semi-copula, but it is not a quasi-copula [6]. Consequently, the class of quasi-copulas is a proper subclass of the class of semi-copulas. The functions $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ are quasi-copulas. Moreover, they are respectively the greatest and the smallest quasi-copula, i.e. for any quasi-copula Q, it holds that

$$T_{\mathbf{L}} \leq Q \leq T_{\mathbf{M}} \,.$$

Copulas

Definition 1.8. *An aggregation function A with neutral element 1 is called a* copula *if it is 2-increasing.*

The notion of a copula appeared for the first time in probability theory and statistics. Copulas turn out to be appropriate tools for linking a joint distribution function with its margins. Due to Sklar's theorem, this fact can been seen as follows. For a joint distribution function H with margins F and G, there exists a copula C such that

$$H(x, y) = C(F(x), G(y)) \,.$$

The copula C is unique if F and G are continuous; otherwise it is unique on $RanF \times RanG$. The functions $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ are also copulas. They are called the

Fréchet-Hoeffding upper and lower bounds: for any copula C it holds that

$$T_{\mathbf{L}} \leq C \leq T_{\mathbf{M}} \,.$$

A third important copula is the product copula $T_{\mathbf{P}}$. The 2-increasingness of a copula implies its 1-Lipschitz continuity. Hence, any copula is continuous. Moreover, any copula is a quasi-copula. Consequently, the class of copulas is a subclass of the class of quasi-copulas. The function $Q : [0,1]^2 \to [0,1]$ defined by

$$Q(x,y) = \begin{cases} \min(x, y, 1/3, x+y-2/3) & \text{, if } 2/3 \leq x+y \leq 4/3\,, \\ \max(x+y-1, 0) & \text{, otherwise,} \end{cases}$$

is quasi-copula, but it is not a copula [88]. Consequently, the class of copulas is a proper subclass of the class of quasi-copulas. Every associative copula is a t-norm (the commutativity can be obtained from the continuity [75]), while every 1-Lipschitz t-norm is a copula.

The relation between the above subclasses of binary aggregation functions is represented in Figure 1.1.

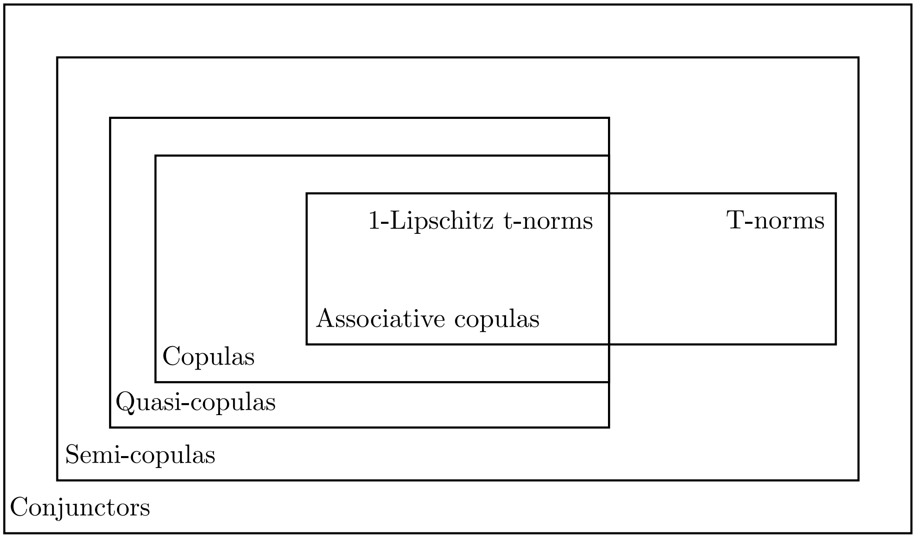


Figure 1.1: An illustration of the inclusions and intersections between the above subclasses of binary aggregation functions.

1.1.4. Copulas

Some families of copulas

Some families of copulas are of our interest in this dissertation. The Yager family [2] of copulas is given by

$$C_\lambda^{\mathbf{Y}}(x,y) = \begin{cases} T_{\mathbf{M}}(x,y) & \text{, if } \lambda = \infty \\ \max(0, 1-((1-x)^\lambda + (1-y)^\lambda)^{\frac{1}{\lambda}}) & \text{, if } \lambda \in [1,\infty[\,. \end{cases} \tag{1.2}$$

Any member of the Yager family has the property of being linear on each segment connecting a point from the upper boundary curve of its zero-set to the point $(1,1)$. The Yager family of copulas is a family of t-norms as well. Moreover, for any $\lambda \in [0,\infty]$, the function $C_\lambda^{\mathbf{Y}}$ defined in (1.2) is a t-norm. Two members of the Yager family with their contour plots are shown in Figure 1.2.

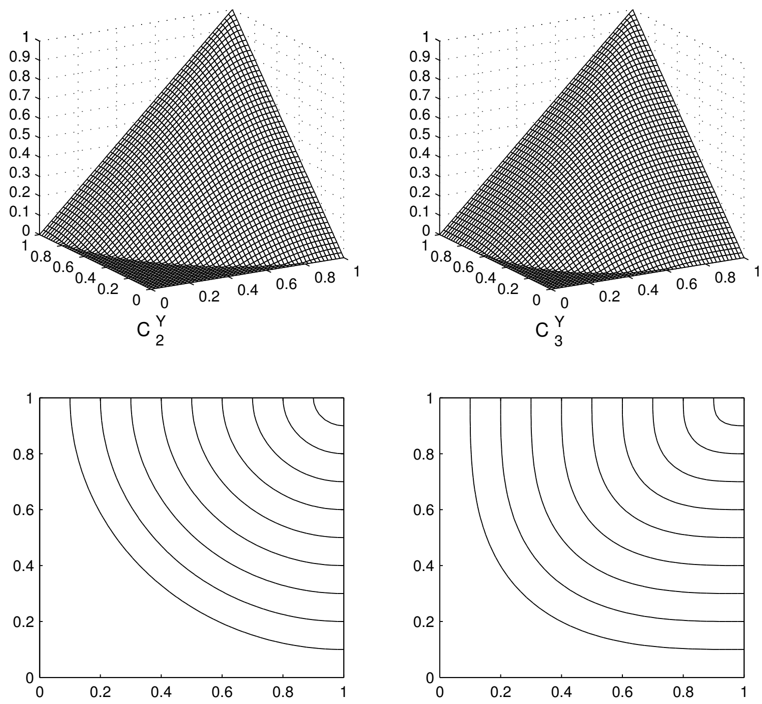


Figure 1.2: The 3D plots of two members of the Yager family of copulas with their contour plots.

Another important family is the Farlie–Gumbel–Morgenstern family [88]. This family is given by

$$C_\lambda^{\mathbf{FGM}}(x,y) = xy + \lambda xy(1-x)(1-y)\,,$$

with $\lambda \in [-1,1]$. The Farlie–Gumbel–Morgenstern family contains all copulas that are quadratic in both variables. The product copula is the only copula that is linear in both variables. The only member of the Farlie–Gumbel–Morgenstern family that is a t-norm is $T_{\mathbf{P}}$. Two members of the Farlie–Gumbel–Morgenstern family with their contour plots are shown in Figure 1.3.

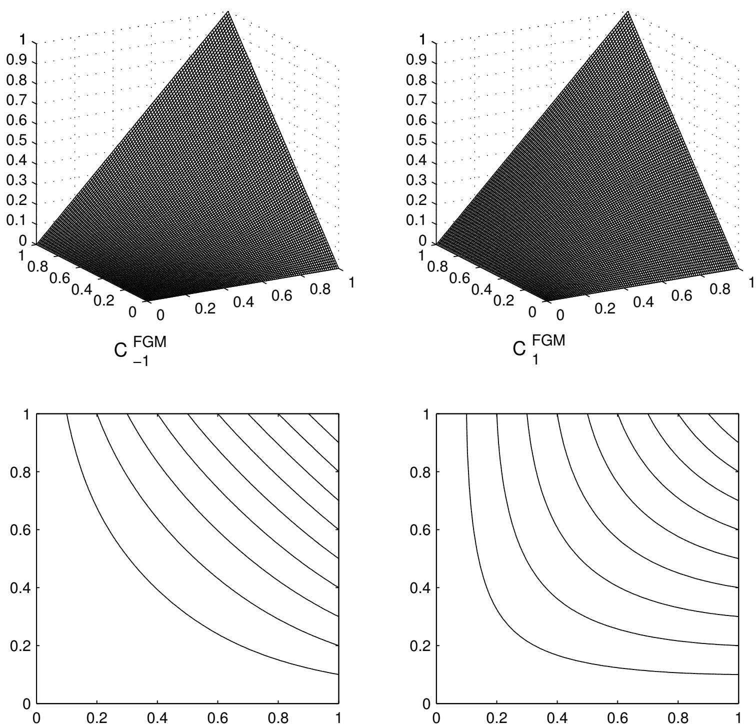


Figure 1.3: The 3D plots of two members of the Farlie–Gumbel–Morgenstern family with their contour plots.

A third important family of copulas is the Ali–Mikhail–Haq family [88]. This family is given by

$$C_\lambda^{\mathbf{AMH}}(x,y) = \frac{xy}{1-\lambda(1-x)(1-y)}\,,$$

with $\lambda \in [-1,1]$.

The Ali–Mikhail–Haq family has been encountered in the literature when constructing copulas based on the algebraic relationship between the joint distribution function and its margins [88]. The Ali–Mikhail–Haq family of copulas is a family of t-norms as well. Two members of the Ali–Mikhail–Haq family with their contour plots are shown in Figure 1.4.

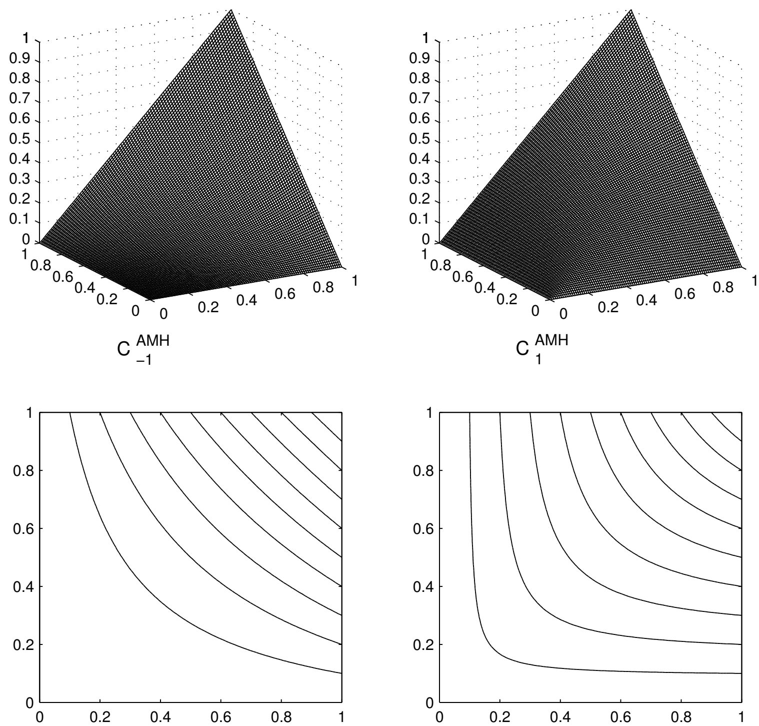


Figure 1.4: The 3D plots of two members of the Ali–Mikhail–Haq family of copulas with their contour plots.

A fourth important family of copulas is the Mayor–Torrens family [88]. This family is given by

$$C_\lambda^{\mathbf{MT}}(x,y) = \begin{cases} \max(x+y-\lambda,0) & \text{, if } \lambda \in]0,1] \text{ and } (x,y) \in [0,\lambda]^2 \,, \\ \min(x,y) & \text{, otherwise.} \end{cases}$$

The Mayor–Torrens family of copulas is a family of t-norms as well. This family has the property of being the only family that satisfies the following equality

$$C(x,y) = \max(C(\max(x,y),\max(x,y)) - |x-y|, 0)\,,$$

for any $x, y \in [0,1]$. Two members of the Mayor–Torrens family with their contour plots are shown in Figure 1.5.

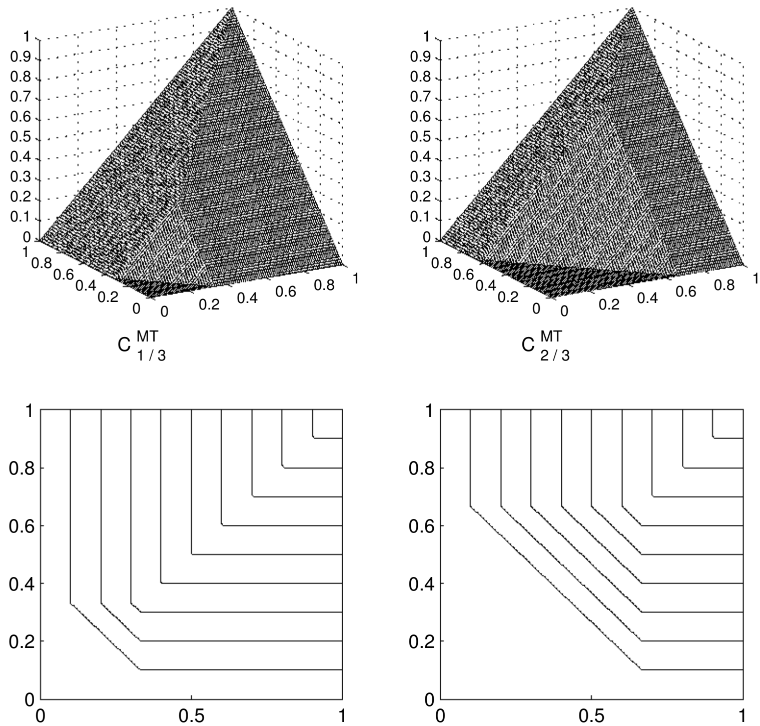


Figure 1.5: The 3D plots of two members of the Mayor–Torrens family of copulas with their contour plots.

Absolutely continuous and singular copulas

Let $\mathcal{B}([0,1]^2)$ be the class of Borel subsets of $[0,1]^2$. Any copula C induces on $\mathcal{B}([0,1]^2)$ a measure μ_C defined by

$$\mu_C([x,x'] \times [y,y']) = V_C([x,x'] \times [y,y'])$$

for any rectangle $[x, x'] \times [y, y'] \in [0,1]^2$. In view of the Lebesgue decompostion theorem [31], it holds that $\mu_C = \mu_C^{\mathrm{ac}} + \mu_C^{\mathrm{s}}$, where μ_C^{ac} is a measure on $\mathcal{B}([0,1]^2)$ that is absolutely continuous w.r.t. the Lebesgue measure and μ_C^{s} is a measure on $\mathcal{B}([0,1]^2)$ that is singular w.r.t. the Lebesgue measure. Therefore, for any copula C, it holds that

$$C = C_{\mathrm{ac}} + C_{\mathrm{s}}\,,$$

where

$$C_{\mathrm{ac}}(x,y) = \mu_C^{\mathrm{ac}}([0,x] \times [0,y]) \quad \text{and} \quad C_{\mathrm{s}}(x,y) = \mu_C^{\mathrm{s}}([0,x] \times [0,y])\,.$$

The function C_{ac} (resp. C_{s}) is called the *absolutely continuous component* (resp. *singular component*) of C.

Definition 1.9. *Let C be a copula.*

(i) *C is called* absolutely continuous *if $C = C_{\mathrm{ac}}$.*

(ii) *C is called* singular *if $C = C_{\mathrm{s}}$.*

If a copula C is absolutely continuous, then it holds that

$$C(x,y) = \int_0^1 \int_0^1 \frac{\partial^2 C(s,t)}{\partial s \partial t} \mathrm{d}s\mathrm{d}t\,,$$

for any $(x,y) \in [0,1]^2$, and C has a density function given by $\frac{\partial^2 C(s,t)}{\partial s \partial t}$. The copulas $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ are singular, while the copula $T_{\mathbf{P}}$ is absolutely continuous. Any member of the Farlie–Gumbel–Morgenstern (resp. Ali–Mikhail–Haq) family of copulas is absolutely continuous. Several methods to construct absolutely continuous copulas have been introduced in the literature [15, 33, 46].

In the next proposition we recall a sufficient condition for the singularity of a copula. To this end we need the definition of the support of a copula. The *support* of a copula C is the complement of the union of all (non-degenerated) open rectangles of the unit square such that the C-volume of the closed rectangle is equal to zero. Hence, a point belongs to the support of C if any rectangle to which the point is internal, has a positive C-volume. Note that if $\frac{\partial^2 C(x,y)}{\partial x \partial y} = 0$ for some point (x,y), then it does not belong to the support of C. The support of the copula $T_{\mathbf{M}}$ (resp. $T_{\mathbf{L}}$) is the diagonal (resp. opposite diagonal) of the unit square, while the support of the copula $T_{\mathbf{P}}$ is the whole unit square.

Proposition 1.2. [31] *If a copula C is supported on a set with Lebesgue measure zero, then C is singular.*

Example 1.1. *Consider convex sums of $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$, i.e. $C_\lambda = \lambda T_{\mathbf{M}} + (1-\lambda)T_{\mathbf{L}}$, with $\lambda \in [0,1]$. Clearly, the support of C_λ consists of the diagonal and opposite diagonal of the unit square for any $\lambda \in]0,1[$. For $\lambda = 1$ (resp. $\lambda = 0$), the support of C_λ is the diagonal (resp. opposite diagonal) of the unit square. Hence, the support of C_λ has Lebesgue measure zero for any $\lambda \in [0,1]$. Due to Proposition 1.2, the copula C_λ is a singular copula for any $\lambda \in [0,1]$.*

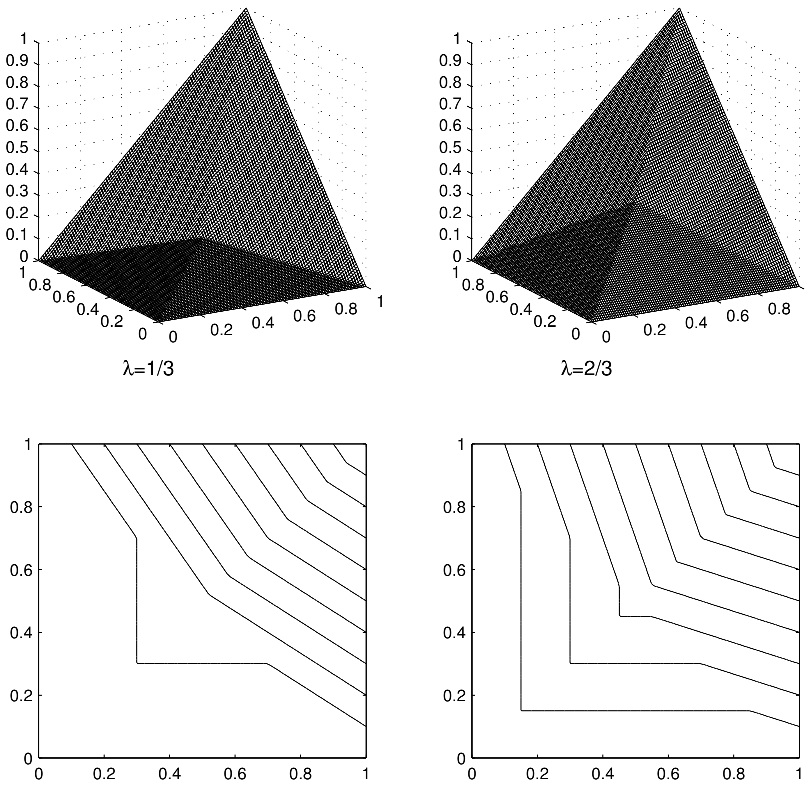


Figure 1.6: The 3D plots of two convex sums of $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ with their contour plots.

The converse of Proposition 1.2 is not necessarily true [31].

Transformations of copulas

For a given function $\kappa : [0,1]^2 \to \mathbb{R}$, the transformations π, φ, φ_1, φ_2, σ, σ_1 and σ_2 [57, 72] produce the following $[0,1]^2 \to \mathbb{R}$ functions defined by

$$\begin{aligned}
\pi(\kappa)(x,y) &= \kappa(y,x)\,,\\
\varphi(\kappa)(x,y) &= x+y-1+\kappa(1-x,1-y)\,,\\
\varphi_1(\kappa)(x,y) &= y-\kappa(1-x,y)\,,\\
\varphi_2(\kappa)(x,y) &= x-\kappa(x,1-y)\,,\\
\sigma\,(\kappa)(x,y) &= x+y-1+\kappa(1-y,1-x)\,,\\
\sigma_1(\kappa)(x,y) &= x-\kappa(1-y,x)\,,\\
\sigma_2(\kappa)(x,y) &= y-\kappa(y,1-x)\,.
\end{aligned} \tag{1.3}$$

The transformations φ, φ_2, σ, σ_1 and σ_2 can be generated by using only the transformations π and φ_1 [57]. If the function κ is a (quasi-)copula, then all of its above transforms are (quasi-)copulas as well [57, 72]. The transform $\varphi(C)$ of a copula C is called *the survival copula*, while the transforms $\varphi_1(C)$ and $\varphi_2(C)$ of a copula C are called the x-flip and y-flip of C [19, 22]. Such transformations have a probabilistic interpretation.

Proposition 1.3. [88] *Let X and Y be two continuous random variables whose dependence is modelled by a copula C_{XY}, and let f (resp. g) be a monotone function on RanX (resp. RanY).*

1. *If f is strictly increasing and g is strictly decreasing, then*
$$C_{f(X)g(Y)} = \varphi_2(C_{XY})\,.$$

2. *If f is strictly decreasing and g is strictly increasing, then*
$$C_{f(X)g(Y)} = \varphi_1(C_{XY})\,.$$

3. *If f and g are strictly decreasing, then*
$$C_{f(X)g(Y)} = \varphi(C_{XY})\,.$$

Definition 1.10. [88] *Let C be a copula. Then*

1. *C is called* symmetric *if*
$$C = \pi\,(C)\,.$$

2. *C is called* opposite symmetric *if*
$$C = \sigma\,(C)\,. \tag{1.4}$$

Symmetric copulas model the dependence between exchangeable random variables. In practice, however, non-exchangeability [5] of random variables is more frequently encountered. Often the degree of non-symmetry of a copula C is expressed by means of the so-called degree of non-exchangeability $\mu_{+\infty}(C)$ with respect to the $L_{+\infty}$ distance [89], defined as

$$\mu_{+\infty}(C) = 3 \sup_{(x,y)\in[0,1]^2} |C(x,y) - C(y,x)| \,. \tag{1.5}$$

The scaling factor 3 ensures that the maximum degree of non-exchangeability is equal to 1. Recently, Durante et al. [35] have made an in-depth study of this and other measures of non-exchangeability.

A symmetric copula is opposite symmetric if and only if it coincides with its survival copula. Any member of the Yager, Farlie–Gumbel–Morgenstern, Ali–Mikhail–Haq or Mayor–Torrens family of copulas is symmetric. Any member of the Farlie–Gumbel–Morgenstern family of copulas is opposite symmetric.

Dependence measures

Another property of a bivariate random vector is the degree of concordance of the two random variables. It is expressed by means of a so-called measure of association. The three most frequently encountered such measures are Spearman's rho, Gini's gamma and Kendall's tau [88].

Let X and Y be two continuous random variables whose dependence is modelled by a copula C.

1. The population version of Spearman's ρ_C for X and Y is given by

$$\rho_C = 12 \int_0^1 \int_0^1 C(x,y)\,\mathrm{d}x\mathrm{d}y - 3\,.$$

2. The population version of Gini's γ_C for X and Y is given by

$$\gamma_C = 4 \int_0^1 C(x, 1-x)\mathrm{d}x - 4 \int_0^1 (x - C(x,x))\mathrm{d}x\,.$$

3. The population version of Kendall's τ_C for X and Y is given by

$$\tau_C = 4 \iint_{[0,1]^2} C(x,y)\mathrm{d}C(x,y) - 1 = 1 - 4 \int_0^1 \int_0^1 \frac{\partial C}{\partial x}(x,y)\frac{\partial C}{\partial y}(x,y)\mathrm{d}x\mathrm{d}y\,.$$

Table 1.1: Spearman's rho, Gini's gamma and Kendall's tau of the copulas $T_{\mathbf{M}}$, $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$.

C	ρ_C	γ_C	τ_C
$T_{\mathbf{M}}$	1	1	1
$T_{\mathbf{P}}$	0	0	0
$T_{\mathbf{L}}$	-1	-1	-1

The relationship between Spearman's rho and Kendall's tau has been studied in detail in [56]. For the copulas $T_{\mathbf{M}}$, $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$, the above measures are listed in Table 1.1. For some members of the Farlie–Gumbel–Morgenstern family of copulas and Ali–Mikhail–Haq family of copulas the above measures are listed in Table 1.2.

Table 1.2: Spearman's rho, Gini's gamma and Kendall's tau of some members of the families $C_\lambda^{\mathbf{FGM}}$ and $C_\lambda^{\mathbf{AMH}}$.

λ	C_λ	ρ_{C_λ}	γ_{C_λ}	τ_{C_λ}
-1	$C_{-1}^{\mathbf{FGM}}$	-0.333333	-0.266667	-0.222222
	$C_{-1}^{\mathbf{AMH}}$	-0.271065	-0.21586	-0.181726
0	$C_0^{\mathbf{FGM}}$	0	0	0
	$C_0^{\mathbf{AMH}}$	0	0	0
1	$C_1^{\mathbf{FGM}}$	0.333333	0.266667	0.222222
	$C_1^{\mathbf{AMH}}$	0.478418	0.381976	0.333333

Some other important measures of association are the upper-upper (λ_{UU}), lower-lower (λ_{LL}), upper-lower (λ_{UL}), and lower-upper (λ_{LU}) tail dependence. Let X and Y be two continuous random variables whose dependence is modelled by a copula C, and let x_t and y_t be the $100t$-th percentiles of X and Y for any $t \in]0,1[$. Then λ_{UU}, λ_{LL}, λ_{UL} and λ_{LU} are defined by

$$\lambda_{UU} = \lim_{t\to 1^-} \mathrm{Prob}\{Y > y_t \mid X > x_t\} = \lim_{t\to 1^-} \frac{1-2t+C(t,t)}{1-t}\,,$$

$$\lambda_{LL} = \lim_{t\to 0^+} \mathrm{Prob}\{Y < y_t \mid X < x_t\} = \lim_{t\to 0^+} \frac{C(t,t)}{t}\,,$$

$$\lambda_{UL} = \lim_{t\to 1^-} \mathrm{Prob}\{Y < y_{1-t} \mid X > x_t\} = \lim_{t\to 1^-} \frac{1-t-C(t,1-t)}{1-t}\,,$$

$$\lambda_{LU} = \lim_{t\to 0^+} \mathrm{Prob}\{Y > y_{1-t} \mid X < x_t\} = \lim_{t\to 0^+} \frac{1-C(t,1-t)}{t}\,,$$

(if the limits exist) [64, 102]. The above tail dependences are used in the literature to model the dependence between extreme events [98].

Probabilistic properties of copulas

Definition 1.11. *Let X and Y be two continuous random variables whose dependence is modelled by a copula C_{XY}. Then*

1. *C_{XY} is* positive quadrant dependent *(PQD) if $C_{XY} \geq T_{\mathbf{P}}$,*
2. *C_{XY} is* negative quadrant dependent *(NQD) if $C_{XY} \leq T_{\mathbf{P}}$.*

A member $C_\lambda^{\mathbf{FGM}}$ of the Farlie–Gumbel–Morgenstern family is PQD (resp. NQD) if and only if $\lambda \geq 0$ (resp. $\lambda \leq 0$). A member $C_\lambda^{\mathbf{AMH}}$ of the Ali–Mikhail–Haq family family is PQD (resp. NQD) if and only if $\lambda \geq 0$ (resp. $\lambda \leq 0$)

Proposition 1.4. [88] *Let X and Y be two continuous random variables whose dependence is modelled by a copula C_{XY}. Then*

1. *X and Y are independent if and only if $C_{XY} = T_{\mathbf{P}}$,*
2. *$Y = f(X)$, where f is strictly increasing, if and only if $C_{XY} = T_{\mathbf{M}}$,*
3. *$Y = f(X)$, where f is strictly decreasing, if and only if $C_{XY} = T_{\mathbf{L}}$.*

Archimedean copulas

Definition 1.12. *A copula C is called* Archimedean *if there exists a convex strictly decreasing continuous $[0,1] \to [0,\infty]$ function t satisfying $t(1) = 0$ such that*

$$C(x,y) = t^{(-1)}(t(x) + t(y))$$

for any $x, y \in [0,1]$.

Any member $C_\lambda^{\mathbf{Y}}$ of the Yager family is an Archimedean copula with additive generator t_λ defined by $t_\lambda(x) = (1-x)^{1/\lambda}$. Any member $C_\lambda^{\mathbf{AMH}}$ of the Ali–Mikhail–Haq family of copulas is an Archimedean copula with additive generator t_λ defined by $t_\lambda(x) = \log\left(\frac{1-\lambda(1-x)}{x}\right)$.

Archimedean copulas are also 1-Lipschitz continuous t-norms. For a copula C, the strict inequality

$$C(x,x) < x \tag{1.6}$$

for any $x \in]0,1[$ is a necessary condition for C to be Archimedean, but it is not sufficient in general. The only Archimedean copula of the Mayor–Torrens family of copulas is $C_1^{\mathbf{MT}} = T_{\mathbf{L}}$ [48].

Some types of convexity and concavity of copulas

Definition 1.13. [88] *A copula C is called* concave *if the inequality*

$$C(\lambda a + (1-\lambda)c, \lambda b + (1-\lambda)d) \geq \lambda C(a,b) + (1-\lambda)C(c,d) \tag{1.7}$$

holds for any $a, b, c, d, \lambda \in [0,1]$.

If the converse inequality holds, then the copula C is called *convex*. The copula $T_{\mathbf{M}}$ (resp. $T_{\mathbf{L}}$) is the only concave (resp. convex) copula [88]. This shows that the above definition is strong. Therefore, new types of concavity (resp. convexity), such as quasi-concavity (resp. quasi-convexity) and Schur-concavity (resp. Schur-convexity), have been proposed in the literature.

Definition 1.14. [88] *Let C be a copula. Then*

1. *C is called* quasi-concave *if the inequality*

$$C(\lambda a + (1-\lambda)c, \lambda b + (1-\lambda)d) \geq \min(C(a,b), C(c,d))$$

   holds for any $a, b, c, d, \lambda \in [0,1]$.

2. *C is called* quasi-convex *if the inequality*

$$C(\lambda a + (1-\lambda)c, \lambda b + (1-\lambda)d) \leq \max(C(a,b), C(c,d))$$

   holds for any $a, b, c, d, \lambda \in [0,1]$.

Note that the only quasi-convex copula is $T_{\mathbf{L}}$ [88], while the class of quasi-concave copulas is a wide class. In the next proposition we recall a necessary and sufficient condition for quasi-concavity of copulas. First we need to introduce the upper boundary curve of a level set of a copula C. Let C be a copula and $t \in [0,1[$. The function whose graph is the upper boundary curve of the t-level set $\{(x,y) \in [0,1]^2 \mid C(x,y) = t\}$ is denoted as $L_{t,C}$, i.e.

$$L_{t,C}(x) = \sup \{y \in [0,1] \mid C(x,y) = t\},$$

for any $x \in [0,1]$.

Proposition 1.5. [2] *A copula C is quasi-concave if and only if $L_{t,C}$ is convex for any $t \in [0,1[$.*

Definition 1.15. *A copula C is called* Schur-concave *[40, 43, 87] if the inequality*

$$C(x,y) \leq C(\lambda x + (1-\lambda)y, (1-\lambda)x + \lambda y) \tag{1.8}$$

holds for any $x, y, \lambda \in [0,1]$.

If the converse inequality holds, then the copula C is called *Schur-convex*. Note that the only Schur-convex copula is again $T_{\mathbf{L}}$, while the class of Schur-concave copulas is a wide class.

Ordinal sums

The notion of an ordinal sum has appeared in the algebraic structure of posets and lattices [14] as well as of semigroups [8]. In the framework of aggregation functions, ordinal sums have been considered mainly with some subclasses of aggregation functions such as t-norms and copulas. Let $\{J_i\}$ denote a partition of $[0,1]$, that is, a (possibly infinite) collection of closed, non-overlapping (except at common endpoints) nondegenerate intervals $J_i = [a_i, b_i]$ whose union is $[0,1]$. Let $\{C_i\}$ be a collection of copulas with the same indexing as $\{J_i\}$. Then the ordinal sum of $\{C_i\}$ with respect to $\{J_i\}$ is the copula given by

$$C(x,y) = \begin{cases} a_i + (b_i - a_i) C_i\left(\dfrac{x-a_i}{b_i-a_i}, \dfrac{y-a_i}{b_i-a_i}\right) & \text{, if } (x,y) \in [a_i,b_i]^2\,, \\ \min(x,y) & \text{, otherwise}\,. \end{cases}$$

Any member of Mayor–Torrens family of copulas is an ordinal sum of $\{T_{\mathbf{L}}, T_{\mathbf{M}}\}$ with respect to $\{[0,\lambda],[\lambda,1]\}$. Any copula is a trivial ordinal sum of itself with respect to $\{[0,1]\}$. A copula that can be represented not only by the trivial ordinal sum is called a *proper ordinal sum*.

Proposition 1.6. [88] *Let C be a copula. Then C is an ordinal sum if and only if there exists a $t \in]0,1[$ such that $C(t,t) = t$.*

For any member $C_\lambda^{\mathbf{Y}}$, with $\lambda < \infty$, it holds that

$$C_\lambda^Y(t,t) < t$$

for any $t \in]0,1[$. Hence, any member $C_\lambda^{\mathbf{Y}}$, with $\lambda < \infty$, of the Yager family is not a proper ordinal sum.

1.2. Diagonal sections and opposite diagonal sections

The diagonal section of a $[0,1]^2 \to [0,1]$ function F is the function $\delta_F : [0,1] \to [0,1]$ defined by $\delta_F(x) = F(x,x)$. In order to characterize the diagonal section of (quasi-)copulas, the following class of functions was considered. A diagonal function [36, 38] is a function $\delta : [0,1] \to [0,1]$ satisfying the following properties:

(D1) $\delta(0) = 0$, $\delta(1) = 1$;

(D2) δ is increasing;

(D3) for any $x \in [0,1]$, it holds that $\delta(x) \leq x$;

(D4) δ is 2-Lipschitz continuous, i.e. for any $x, x' \in [0,1]$, it holds that

$$|\delta(x') - \delta(x)| \leq 2|x' - x|\,.$$

The functions $\delta_{T_{\mathbf{M}}}(x) = x$ and $\delta_{T_{\mathbf{L}}}(x) = \max(2x-1, 0)$ are examples of diagonal functions. Moreover, for any diagonal function δ, it holds that

$$\delta_{T_{\mathbf{L}}} \leq \delta \leq \delta_{T_{\mathbf{M}}}\,.$$

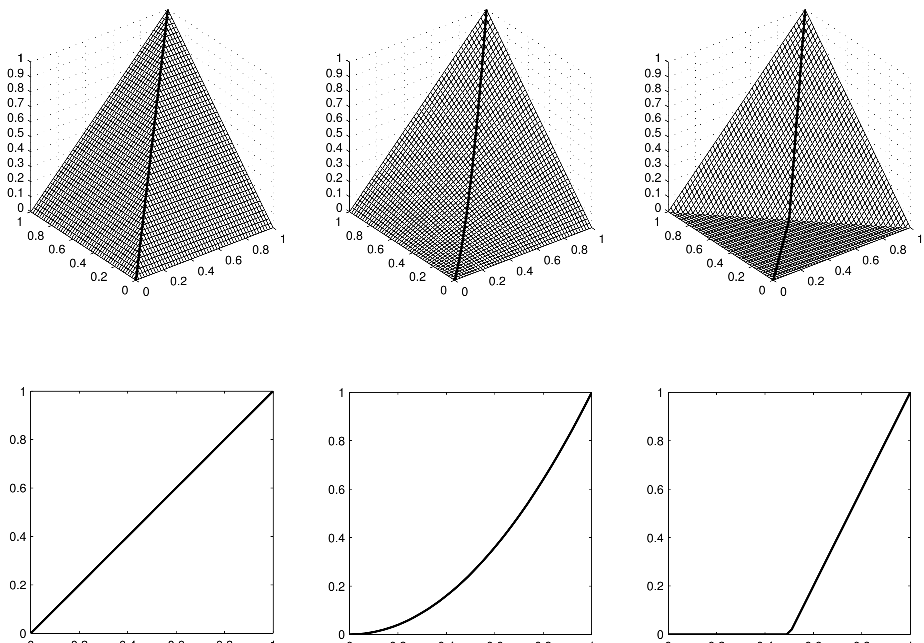


Figure 1.7: The 3D plots of the copulas $T_{\mathbf{M}}$, $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$ with the 2D plots of their diagonal section.

The copula $T_{\mathbf{M}}$ is the only copula with diagonal section $\delta_{T_{\mathbf{M}}}$. The set of all diagonal functions is denoted by $\mathcal{D}$. The diagonal section δ_C of a (quasi-)copula C is a diagonal function. Conversely, for any diagonal function δ there exists at least one copula C with diagonal section $\delta_C = \delta$. For example, the function $K_\delta : [0,1]^2 \to [0,1]$, defined by

$$K_\delta(x,y) = \min(x, y, (\delta(x) + \delta(y))/2)\,, \tag{1.9}$$

is a copula with diagonal section δ. Moreover, K_δ is the greatest symmetric copula with diagonal section δ [36, 39, 90]. The Bertino copula B_δ defined by

$$B_\delta(x,y) = \min(x,y) - \min\{t - \delta(t) \mid t \in [\min(x,y), \max(x,y)]\}\,, \tag{1.10}$$

is the smallest copula with diagonal section δ [7, 55, 73]. Note that B_δ is symmetric. Copulas with a given diagonal section are important tools for modelling upper-upper and lower-lower tail dependence, which can be expressed as

$$\lambda_{UU} = 2 - \delta'_C(1^-) \quad \text{and} \quad \lambda_{LL} = \delta'_C(0^+)\,.$$

The set of all $[0,1] \to [0,1]$ functions that satisfy properties **D1**–**D3** is denoted by $\mathcal{D}_\mathrm{S}$; the subset of *absolutely continuous* functions in $\mathcal{D}_\mathrm{S}$ is denoted by $\mathcal{D}_\mathrm{S}^\mathrm{ac}$.

Note that for a function $\delta \in \mathcal{D}_\mathrm{S}$, the function C_δ defined by (1.9) has neutral element 1 if and only if $\delta(x) \geq 2x - 1$ for any $x \in [1/2, 1]$. In fact, the last inequality holds for the class of diagonal sections of quasi-copulas and copulas. Therefore, for a given $\delta \in \mathcal{D}_\mathrm{S}$, the function C_δ defined by (1.9) need not be a semi-copula in general. In order to characterize the diagonal section of semi-copulas, the class $\mathcal{D}_\mathrm{S}$ was considered. The diagonal section δ_S of a semi-copula S belongs to $\mathcal{D}_\mathrm{S}$. Conversely, for any $\delta \in \mathcal{D}_\mathrm{S}$ there exists at least one semi-copula C with diagonal section $\delta_C = \delta$. For example, the function S_δ, defined by

$$S_\delta(x,y) = \begin{cases} \min(\delta(x), \delta(y)) & \text{, if } x, y \in [0,1[\,, \\ \min(x,y) & \text{, otherwise,} \end{cases} \tag{1.11}$$

is a semi-copula with diagonal section δ.

The set of all $[0,1] \to [0,1]$ functions that satisfy properties **D1** and **D2** is denoted as $\mathcal{D}_\mathrm{A}$. In order to characterize the diagonal section of aggregation functions, the class $\mathcal{D}_\mathrm{A}$ was considered. The diagonal section δ_A of an aggregation function A belongs to $\mathcal{D}_\mathrm{A}$. Conversely, for any $\delta \in \mathcal{D}_\mathrm{A}$ there exists at least one an aggregation function A with diagonal section $\delta_A = \delta$. For example, the function $A_\delta : [0,1]^2 \to [0,1]$, defined by

$$A_\delta(x,y) = \frac{\delta(x) + \delta(y)}{2}\,, \tag{1.12}$$

is an aggregation function with diagonal section δ.

Similarly, the opposite diagonal section of a $[0,1]^2 \to [0,1]$ function F is the function $\omega_F : [0,1] \to [0,1]$ defined by $\omega_F(x) = F(x, 1-x)$. In order to characterize the opposite diagonal section of (quasi-)copulas, the following class of functions was considered. An opposite diagonal function [23, 24] is a function $\omega : [0,1] \to [0,1]$ satisfying the following properties:

(OD1) for any $x \in [0,1]$, it holds that $\omega(x) \leq \min(x, 1-x)$;

(OD2) ω is 1-Lipschitz continuous, i.e. for any $x, x' \in [0,1]$, it holds that

$$|\omega(x') - \omega(x)| \leq |x' - x| \ .$$

The functions $\omega_{T_{\mathbf{M}}}(x) = \min(x, 1-x)$ and $\omega_{T_{\mathbf{L}}}(x) = 0$ are examples of opposite diagonal functions. Moreover, for any opposite diagonal function ω, it holds that

$$\omega_{T_{\mathbf{L}}} \leq \omega \leq \omega_{T_{\mathbf{M}}} \ .$$

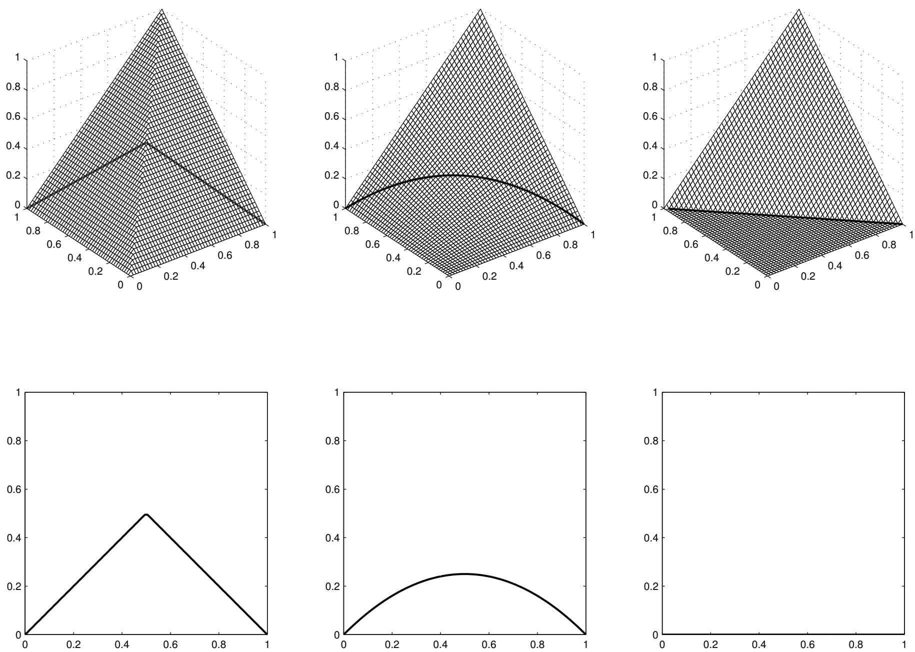


Figure 1.8: The 3D plots of the copulas $T_{\mathbf{M}}$, $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$ with the 2D plots of their opposite diagonal section.

The copula $T_{\mathbf{L}}$ is the only copula with opposite diagonal section $\omega_{T_{\mathbf{L}}}$. The set of all opposite diagonal functions is denoted by $\mathcal{O}$. The opposite diagonal section ω_C of a (quasi-)copula C is an opposite diagonal function. Conversely, for any opposite diagonal function ω there exists at least one copula C with opposite diagonal section $\omega_C = \omega$. For example, the function $F_\omega : [0,1]^2 \to [0,1]$, defined by

$$F_\omega(x,y) = T_{\mathbf{L}}(x,y) + \min\{\omega(t) \mid t \in [\min(x,1-y), \max(x,1-y)]\}\,, \qquad (1.13)$$

is a copula with opposite diagonal section [73]. Moreover, F_ω is the greatest copula with opposite diagonal section. Note that F_ω is opposite symmetric. Copulas with a given opposite diagonal section are important tools for modelling upper-lower and lower-upper tail dependence, which can be expressed as

$$\lambda_{UL} = 1 + \omega_C'(1^-) \quad \text{and} \quad \lambda_{LU} = 1 - \omega_C'(0^+)\,.$$

The set of all $[0,1] \to [0,1]$ functions that satisfy condition (**OD1**) is denoted by $\mathcal{O}_{\mathrm{S}}$; the subset of *absolutely continuous* functions in $\mathcal{O}_{\mathrm{S}}$ is denoted by $\mathcal{O}_{\mathrm{S}}^{\mathrm{ac}}$.

The opposite diagonal section ω_{S} of a semi-copula S belongs to $\mathcal{O}_{\mathrm{S}}$. Conversely, for any function $\omega \in \mathcal{O}_{\mathrm{S}}$ there exists at least one semi-copula S with opposite diagonal section $\omega_S = \omega$. For example, the function $S_\omega : [0,1]^2 \to [0,1]$ defined by

$$S_\omega(x,y) = \begin{cases} 0 & \text{, if } x+y<1\,, \\ \omega(x) & \text{, if } x+y=1\,, \\ \min(x,y) & \text{, if } x+y>1\,, \end{cases} \qquad (1.14)$$

is a semi-copula with opposite diagonal section ω.

In general, any $[0,1] \to [0,1]$ function can be the opposite diagonal section of an aggregation function. For instance, for a function $\omega : [0,1] \to [0,1]$, the function $A_\omega : [0,1]^2 \to [0,1]$ defined by

$$A_\omega(x,y) = \begin{cases} 0 & \text{, if } x+y<1\,, \\ \omega(x) & \text{, if } x+y=1\,, \\ 1 & \text{, if } x+y>1\,, \end{cases} \qquad (1.15)$$

is always an aggregation function with opposite diagonal section ω.

1.3. Semilinear and semiquadratic aggregation functions

Several methods to construct conjunctive aggregation functions have been introduced in the literature. Some of these methods are based on linear or quadratic interpolation on segments connecting lines in the unit square to the sides of the unit square. Such lines can be the diagonal, the opposite diagonal, a horizontal straight line, a vertical straight line or the graph that represents a decreasing function. We introduce the notions of semilinear and semiquadratic aggregation functions that generalize all aggregation functions that are obtained based on such methods. We denote the (linear) segment with endpoints $\mathbf{x}, \mathbf{y} \in [0,1]^n$ as

$$\langle \mathbf{x}, \mathbf{y} \rangle = \{\theta \mathbf{x} + (1-\theta)\mathbf{y} \mid \theta \in [0,1]\} .$$

A continuous function $f : [0,1] \to [0,1]$ is called *piecewise linear* if its graph consists of segments only.

Definition 1.16. *An aggregation function A is called* semilinear *(resp.* semiquadratic*) if for any $\mathbf{x} \in [0,1]^2$, there exists $\mathbf{y} \in [0,1]^2$, $\mathbf{y} \neq \mathbf{x}$ such that A is linear (resp. quadratic) on the segment $\langle \mathbf{x}, \mathbf{y} \rangle$.*

All piecewise linear aggregation functions (in particular, $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$) are semilinear copulas since all their horizontal and vertical sections are piecewise linear. The product copula $T_{\mathbf{P}}$ is semilinear, as all its horizontal and vertical sections are linear [88]. Any member of Yager family of copulas is also semilinear since its radial sections are piecewise linear [2]. Any member of Farlie–Gumbel–Morgenstern family of copulas is also semiquadratic since its horizontal and vertical sections are quadratic [88]. In this dissertation, we introduce several methods to construct semilinear and semiquadratic aggregation functions.

Throughout this dissertation, we use the following conventions.

1. We mean by the statement "a function $G : [0,1]^2 \to [0,1]$ satisfies the boundary conditions of a (semi-, quasi-)copula" or the statement "a function $G : [0,1]^2 \to [0,1]$ satisfies the first condition of the definition of a (semi-, quasi-)copula" that G has 0 as absorbing element and 1 as neutral element.

2. From Chapter 6 on, we restrict our attention to the class of copulas and we respectively use the notations M, W and Π instead of the notations $T_{\mathbf{M}}$, $T_{\mathbf{L}}$ and $T_{\mathbf{P}}$.

PART I

METHODS BASED ON LINEAR INTERPOLATION

2 Conic aggregation functions

2.1. Introduction

The zero-set of a binary aggregation function is of particular interest in this chapter. In the case of t-norms, for instance, the discovery of Fodor's nilpotent minimum t-norm [53] has instigated the study of the zero-set of left-continuous t-norms [62, 63, 79, 80, 81, 83, 84]. The boundary curve of the zero-set is in this case formed by an involutive negator [82]. Characteristic for the aggregation functions $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ is that their graph is constituted from their zero-set and linear segments connecting the upper boundary curve of this zero-set to the point $(1,1,1)$. For this reason, they are called conic, and all conic t-norms have been characterized as belonging to the Yager family of t-norms [2]. The purpose of this chapter is to study conic aggregation functions in general, inspired by the above graphical interpretation of $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$, and lay bare the connection with the corresponding zero-sets. It fits in a broader study of aggregation functions whose surface consists of linear segments [4, 20, 21, 38, 65] or contains such linear segments as the result of a transformation [17].

This chapter is organized as follows. In the next section we give the definition of a conic aggregation function. In Section 2.3 we restrict our attention to the class of binary conic aggregation functions and we recall the characterization of conic t-norms in Section 2.4. In Sections 2.5–2.7, we characterize the classes of conic quasi-copulas, conic copulas and conic copulas supported on a set with Lebesgue measure zero. For conic copulas, we provide simple expressions for Spearman's ρ, Gini's γ and Kendall's τ in Section 2.8. We conclude the chapter with a discussion of some aggregations of conic (quasi-)copulas.

2.2. Conic aggregation functions

The *zero-set* Z_A of an aggregation function A is the inverse image of the value 0, i.e.

$$Z_A := A^{-1}(\{0\}) = \{\mathbf{x} \in [0,1]^n \mid A(\mathbf{x}) = 0\} .$$

Since $A(1,\ldots,1) = 1$, Z_A is a proper subset of $[0,1]^n$. A point $\mathbf{x} = (x_1, ..., x_n) \in Z_A$ is called a *weakly undominated point* if there exists no $\mathbf{y} = (y_1, ..., y_n) \in Z_A$ such that $y_1 > x_1$, $y_2 > x_2$, ..., $y_n > x_n$. In case $n = 2$, we will refer to the set of weakly undominated points of the zero-set of a continuous aggregation function as the *upper boundary curve* of the zero-set.

Let $(X, \leq)$ be a partially ordered set. A subset $Y \subseteq X$ is called a *lower set* (of X) if for all $x, y \in X$ such that $y \leq x$ and $x \in Y$ it holds that $y \in Y$. Due to the increasingness of an aggregation function A it holds for any $\mathbf{x}, \mathbf{y} \in [0,1]^n$ such that $\mathbf{y} \leq \mathbf{x}$ and $A(\mathbf{x}) = 0$ that also $A(\mathbf{y}) = 0$, i.e. Z_A is a lower set of $[0,1]^n$. Moreover, if A is continuous, then Z_A is a closed lower set of $[0,1]^n$.

Suppose that 0 is the absorbing element of A, i.e. $A(x_1, ..., x_n) = 0$ whenever $0 \in \{x_1, ..., x_n\}$. Then A has no zero-divisors, i.e. $A(x_1, ..., x_n) = 0$ implies $0 \in \{x_1, ..., x_n\}$, if and only if $Z_A = Z_*$, with $Z_* = [0,1]^n \setminus]0,1]^n$.

Now we state the general definition of a conic function.

Definition 2.1. *Let $Z \subset [0,1]^n$ be a closed lower set containing Z_*. We define the function $A_Z : [0,1]^n \to [0,1]$ as follows:*

(i) $A_Z(\mathbf{1}) = 1$*;*

(ii) $A_Z(\mathbf{x}) = 0$ *for any* $\mathbf{x} \in Z$*;*

(iii) *for any weakly undominated point* $\mathbf{x} \in Z$*, the function* A_Z *is linear on the segment* $\langle \mathbf{x}, \mathbf{1} \rangle$*.*

The function A_Z is called a *conic function with zero-set Z.*

Remark 2.1. *Note that conic functions are well defined. Indeed, for any fixed* $\mathbf{x} \in [0,1]^n \setminus (Z \cup \{\mathbf{1}\})$*, let*

$$\lambda = \inf\{\mu \in \mathbb{R} \mid \mu\mathbf{x} + (1-\mu)\mathbf{1} \in Z\} .$$

Then $\mathbf{z_x} = \lambda\mathbf{x} + (1-\lambda)\mathbf{1}$ *is the unique weakly undominated point such that the segment* $\langle \mathbf{z_x}, \mathbf{1} \rangle$ *contains* $\mathbf{x}$*. Hence,* $A_Z(\mathbf{x}) = \frac{\lambda-1}{\lambda} \in]0,1[$ *.*

Theorem 2.1. *Let $Z \subset [0,1]^n$ be a closed lower set containing Z_*. Then the conic function A_Z is continuous.*

Proof. Consider the set $U(Z)$ of the weakly undominated points of Z, i.e.

$$U(Z) = \{\mathbf{u} \mid \mathbf{u} \text{ is the greatest element of } Z \text{ on some segment } \langle \mathbf{x}, \mathbf{1} \rangle\} .$$

It clearly holds that $U(Z)$ is a compact subset of $[0,1]^n$ such that for any $\mathbf{u} \in U(Z)$, the segment $\langle \mathbf{u}, \mathbf{1} \rangle$ does not contain any other point in $U(Z)$, and for any $\mathbf{x} \in [0,1]^n \setminus (Z \cup \{\mathbf{1}\})$, there exists a unique $\mathbf{z_x}$ such that $\mathbf{x} \in \langle \mathbf{z_x}, \mathbf{1} \rangle$. Due to the definition of A_Z, it holds that $A_Z(\mathbf{x}) = 0$ if $\mathbf{x} \in Z$, $A_Z(\mathbf{x}) \in]0,1[$ if $\mathbf{x} \notin Z \cup \{\mathbf{1}\}$ and $A_Z(\mathbf{1}) = 1$. Moreover, as $\mathbf{1}$ is not contained in $U(Z)$ and using any L_p-distance d (e.g. L_1 or the Euclidean distance), the distance from $\mathbf{1}$ to $U(Z)$ is positive, i.e. $a = d(\mathbf{1}, U(Z)) > 0$. Furthermore, for any $\mathbf{x} \in [0,1]^n \setminus (Z \cup \{\mathbf{1}\})$, it holds that

$$A_Z(\mathbf{x}) = \frac{d(\mathbf{x}, \mathbf{z_x})}{d(\mathbf{1}, \mathbf{z_x})} = 1 - \frac{d(\mathbf{1}, \mathbf{x})}{d(\mathbf{1}, \mathbf{z_x})} \geq 1 - \frac{d(\mathbf{1}, \mathbf{x})}{a} .$$

Therefore, for any sequence $(\mathbf{x}_m)$ of points in $[0,1]^n$ such that $\lim \mathbf{x}_m = \mathbf{1}$, it holds that $\lim A_Z(\mathbf{x}_m) = 1$, i.e. the function A_Z is continuous at $\mathbf{1}$. Obviously, A_Z is continuous on $Z \setminus U(Z)$ and for points in $U(Z)$ the lower semicontinuity of A_Z holds.

Now consider a sequence $(\mathbf{x}_m)$ in $([0,1]^n \setminus (Z \cup \{\mathbf{1}\})) \cup U(Z)$ such that $\lim \mathbf{x}_m = \mathbf{x}$. If the sequence $(\mathbf{z}_{\mathbf{x}_m})$ converges to $\mathbf{z}_{\mathbf{x}}$, then

$$\lim A_Z(\mathbf{x}_m) = \lim \frac{d(\mathbf{x}_m, \mathbf{z}_{\mathbf{x}_m})}{d(\mathbf{1}, \mathbf{z}_{\mathbf{x}_m})} = \frac{d(\mathbf{x}, \mathbf{z}_{\mathbf{x}})}{d(\mathbf{1}, \mathbf{z}_{\mathbf{x}})} = A_Z(\mathbf{x})\,.$$

Suppose that $\lim \mathbf{z}_{\mathbf{x}_m} \neq \mathbf{z}_{\mathbf{x}}$ (either it is another point in $U(Z)$ or it does not exist). In both cases, due to the compactness of $U(Z)$, there exists a subsequence $(\mathbf{x}_{m_k})$ such that $\lim \mathbf{z}_{\mathbf{x}_{m_k}} = \mathbf{u} \neq \mathbf{z}_{\mathbf{x}}$ and all the points $\mathbf{x}_{m_k}$ are on the segment $\langle \mathbf{z}_{\mathbf{x}_{m_k}}, \mathbf{1}\rangle$. Now consider a hyperplane τ containing the point $\mathbf{1}$ and separating the remainder of the segment $\langle \mathbf{z}_{\mathbf{x}}, \mathbf{1}\rangle$ from $\langle \mathbf{u}, \mathbf{1}\rangle$. Evidently, there exists a k_0 such that for all $k \geq k_0$, the segment $\langle \mathbf{z}_{\mathbf{x}_{m_k}}, \mathbf{1}\rangle$ is on the same side of τ as the segment $\langle \mathbf{u}, \mathbf{1}\rangle$. Hence, $(\mathbf{x}_{m_k})$ cannot converge to $\mathbf{x}$, which being on the segment $\langle \mathbf{z}_{\mathbf{x}}, \mathbf{1}\rangle$, is just on the opposite side of τ. Therefore, convergence of the sequence $(\mathbf{x}_m)$ from $([0,1]^n \setminus (Z \cup \{\mathbf{1}\})) \cup U(Z)$ to a point $\mathbf{x} \in ([0,1]^n \setminus (Z \cup \{\mathbf{1}\})) \cup U(Z)$ also implies $\lim \mathbf{z}_{\mathbf{x}_m} = \mathbf{z}_{\mathbf{x}}$. Hence,

$$A_Z(\mathbf{x}) = \frac{d(\mathbf{x}, \mathbf{z}_{\mathbf{x}})}{d(\mathbf{1}, \mathbf{z}_{\mathbf{x}})}\,,$$

and A_Z is continuous on $[0,1]^n \setminus (Z \cup \{\mathbf{1}\})$ and upper semicontinuous on $U(Z)$. From the above analysis, the continuity of A_Z is clear. □

Theorem 2.2. *Let $Z \subset [0,1]^n$ be a closed lower set containing Z_*. Then the conic function A_Z is a continuous aggregation function with absorbing element* 0.

Proof. As the boundary conditions are trivially fulfilled, it suffices to prove that A_Z is increasing. Consider $\mathbf{x}, \mathbf{y} \in [0,1]^n$ and suppose w.l.o.g. that $\mathbf{x} = (x_1, a, ..., a)$ and $\mathbf{y} = (y_1, a, ..., a)$, with $x_1 < y_1$. If $\mathbf{x} \in Z$ then $A_Z(\mathbf{x}) = 0 \leq A_Z(\mathbf{y})$. Suppose that $\mathbf{x} \notin Z$ and $\mathbf{x} \neq \mathbf{1}$. Let $\mathbf{z}_{\mathbf{x}} = (u_1, \ldots, u_n)$ and $\mathbf{z}_{\mathbf{y}} = (v_1, \ldots, v_n)$ be the unique weakly undominated points corresponding to $\mathbf{x}$ and $\mathbf{y}$, respectively, i.e. there exist $\alpha, \beta \in [0,1]$ such that

$$\mathbf{x} = \alpha \mathbf{z}_{\mathbf{x}} + (1-\alpha)\mathbf{1} \quad \text{ and } \quad \mathbf{y} = \beta \mathbf{z}_{\mathbf{y}} + (1-\beta)\mathbf{1}\,.$$

Then $A_Z(\mathbf{x}) = 1-\alpha$ and $A_Z(\mathbf{y}) = 1-\beta$. Suppose that $\alpha < \beta$. For any $i \in \{2, ..., n\}$, it holds that

$$\alpha u_i + 1 - \alpha = a = \beta v_i + 1 - \beta\,,$$

which implies that $u_i < v_i$. Since $\mathbf{z}_{\mathbf{x}}$ and $\mathbf{z}_{\mathbf{y}}$ are weakly undominated points, it must hold that $u_1 > v_1$, which contradicts the fact that

$$\alpha u_1 + 1 - \alpha = x_1 < y_1 = \beta v_1 + 1 - \beta .$$

Therefore, it holds that $\alpha \geq \beta$, or equivalently, $A_Z(\mathbf{x}) \leq A_Z(\mathbf{y})$, whence A_Z is an aggregation function.

Finally, we show that 0 is the absorbing element of A_Z. Consider $\mathbf{x} \in [0,1]^n$ such that $x_i = 0$ for some $i \in \{1, \dots, n\}$. If $\mathbf{x} \in Z$, then it holds that $A_Z(\mathbf{x}) = 0$. If $\mathbf{x} \notin Z$, then there exists $\alpha \in [0,1]$ such that $\mathbf{x} = \alpha \mathbf{z}_{\mathbf{x}} + (1-\alpha)\mathbf{1}$. Hence, $x_i = 0 = \alpha u_i + 1 - \alpha$, whence $\alpha = 1$, i.e. $A_Z(\mathbf{x}) = 1 - \alpha = 0$. □

Inspired by the above proposition, the conic function A_Z will be called a *conic aggregation function with zero-set Z*. Evidently, if $Z_1 \subseteq Z_2$, then $A_{Z_1} \geq A_{Z_2}$. Hence, the greatest conic aggregation function A_{Z_*} is the n-ary version of the minimum t-norm $T_{\mathbf{M}}$ given by $T_{\mathbf{M}}(\mathbf{x}) = \min(x_1, \dots, x_n)$, for any $\mathbf{x} \in [0,1]^n$. In contrast, there is no greatest proper closed lower set of $[0,1]^n$, and hence, there is no smallest conic aggregation function.

The following proposition is a straightforward consequence of Theorems 2.1 and 2.2.

Proposition 2.1. *Let Z be a proper subset of $[0,1]^n$. Then Z is the zero-set of a conic aggregation function A_Z with absorbing element 0 if and only if Z is a closed lower set containing Z_*.*

Example 2.1. *Let $Z = \{(x_1, \dots, x_n) \in [0,1]^n \mid x_1 + \cdots + x_n \leq n-1\}$. The set Z is a closed lower set containing Z_*. The corresponding conic aggregation function is the n-ary version of the Łukasiewicz t-norm given by*

$$T_{\mathbf{L}}(\mathbf{x}) = \max(x_1 + \cdots + x_n - n + 1, 0) ,$$

for any $\mathbf{x} \in [0,1]^n$. The weakly undominated point $\mathbf{z}_{\mathbf{x}}$ corresponding to a point $\mathbf{x} \notin Z$ is here given by

$$\mathbf{z}_{\mathbf{x}} = \left(\frac{n - 1 - x_2 - \cdots - x_n}{n - x_1 - \cdots - x_n}, \dots, \frac{n - 1 - x_1 - \cdots - x_{n-1}}{n - x_1 - \cdots - x_n} \right) .$$

We conclude this section by studying some aggregations of conic aggregation functions.

Proposition 2.2. *For any two conic aggregation functions A_{Z_1} and A_{Z_2}, it holds that the aggregation functions $\max(A_{Z_1}, A_{Z_2})$ and $\min(A_{Z_1}, A_{Z_2})$ are also conic aggregation functions, with respective zero-sets $Z_1 \cap Z_2$ and $Z_1 \cup Z_2$. In other words, the class of conic aggregation functions is closed under maximum and minimum.*

Proof. Let $Z = Z_1 \cap Z_2$. Clearly, the aggregation function A_Z is given by

(i) If $\mathbf{x} \in Z$, then $A_Z(\mathbf{x}) = 0$.

(ii) If $\mathbf{x} \notin (Z \cup \{\mathbf{1}\})$ and $\mathbf{z_x} \in Z_1$ (here $\mathbf{z_x}$ is taken w.r.t. Z), then $A_Z(\mathbf{x}) = A_{Z_1}(\mathbf{x}) \geq A_{Z_2}(\mathbf{x})$.

(iii) If $\mathbf{x} \notin (Z \cup \{\mathbf{1}\})$ and $\mathbf{z_x} \in Z_2$ (here $\mathbf{z_x}$ is taken w.r.t. Z), then $A_Z(\mathbf{x}) = A_{Z_2}(\mathbf{x}) \geq A_{Z_1}(\mathbf{x})$.

Since the intersection of two closed lower sets of $[0,1]^n$ containing Z_* is again a closed lower set of $[0,1]^n$ containing Z_*, the function A_Z is a conic aggregation function, and coincides with $\max(A_{Z_1}, A_{Z_2})$.

Similarly, one can prove that $\min(A_{Z_1}, A_{Z_2})$ is a conic aggregation function with zero-set $Z_1 \cup Z_2$. □

Proposition 2.3. *For any two distinct conic aggregation functions A_{Z_1} and A_{Z_2} and $\lambda \in]0,1[$, the aggregation function $\lambda A_{Z_1} + (1-\lambda) A_{Z_2}$ is never a conic aggregation function.*

Proof. Suppose that $\lambda A_{Z_1} + (1-\lambda) A_{Z_2}$ is a conic aggregation function. Obviously, its zero-set is given by $Z_1 \cap Z_2$, which implies, due to Proposition 2.2 and the uniqueness of a conic aggregation function with a given zero-set, that

$$\lambda A_{Z_1} + (1-\lambda) A_{Z_2} = \max(A_{Z_1}, A_{Z_2}),$$

which is impossible since $\lambda \in]0,1[$. □

Example 2.2. *The zero-set of the aggregation function $(T_{\mathbf{M}} + T_{\mathbf{L}})/2$ is Z_*. Since*

$$\frac{T_{\mathbf{M}} + T_{\mathbf{L}}}{2} \neq T_{\mathbf{M}},$$

the former is not a conic aggregation function.

Remark 2.2. *Let A_Z be an n-ary conic aggregation function. Then A_Z is given by*

(i) $A_Z(\mathbf{1}) = 1$*;*

(ii) $A_Z(\mathbf{x}) = 0$ *for any* $\mathbf{x} \in Z$*;*

(iii) *if* $\mathbf{x} = (x_1, ..., x_n) \notin (Z \cup \{\mathbf{1}\})$ *and* $x_i^0 \neq 1$ *for some* $i \in \{1, ..., n\}$*, then it holds that*

$$A_Z(\mathbf{x}) = \frac{x_i - x_i^0}{1 - x_i^0} \tag{2.1}$$

with $\mathbf{x}_0 = (x_1^0, ..., x_n^0)$ *the unique weakly undominated point corresponding to* $\mathbf{x}$.

In case multiple such i exist, Eq. (2.1) always leads to the same value.

2.3. Binary conic aggregation functions

From here on, we will deal with binary aggregation functions only, and omit the adjective 'binary'. Obviously, a conic aggregation function A_Z is commutative if and only if its zero-set Z is symmetric, i.e. $(x, y) \in Z$ if and only if $(y, x) \in Z$. The next proposition expresses that a closed lower set of $[0, 1]^2$ containing Z_* is determined by a decreasing function. Let d be the smallest $x \in [0, 1]$ such that $(x, 0)$ is a weakly undominated point, and d' be the smallest $y \in [0, 1]$ such that $(0, y)$ is a weakly undominated point.

Proposition 2.4. *Let Z be a closed lower set of $[0, 1]^2$ containing Z_*. Then there exists a decreasing function $f : [0, d] \to [0, 1]$, such that*

$$Z = \{(x, y) \in [0, 1]^2 \mid x \in [0, d] \text{ and } y \leq f(x)\} \cup Z_* \,.$$

Note that $d = 0$ if and only if $Z = Z_*$; also, if $d = 0$, then $f(d) = 0$. In order to make it meaningful to talk about a function $f : [0, d] \to [0, 1]$, we will therefore assume that $d > 0$, i.e. $A_Z \neq T_{\mathbf{M}}$; then it also holds that $f(0) > 0$. Obviously, the function f is right-continuous at 0 and $f(x) > 0$ for any $x \in [0, d[$.

Since the zero-set of a conic aggregation function is determined by a function f, when convenient, we will refer to such an aggregation function as A_f. The following result is an immediate observation.

Proposition 2.5. *A conic aggregation function A_f has neutral element 1 if and only if*

(i) $f(x) < 1$ *for any* $x \in]0, d]$*;*

(ii) $d < 1$ *or* $(d = 1$ *and* $f(d) = 0)$*.*

The graph of a conic aggregation function A_Z is constituted from its zero-set and segments connecting the upper boundary curve of its zero-set (containing the graph of f) to the point $(1, 1, 1)$.

Suppose that the upper boundary curve of the zero-set of a conic aggregation function A_Z contains a segment determined by the points (x_1, y_1) and (x_2, y_2), then A_Z is linear on the triangle $T = \Delta_{\{(x_1,y_1),(x_2,y_2),(1,1)\}}$. This situation is depicted in Figure 2.1.

For any $(x, y) \in T$, it holds that

$$A_Z(x, y) = ax + by + c \,. \tag{2.2}$$

Furthermore,

$$\begin{aligned} ax_1 + by_1 + c &= 0 \\ ax_2 + by_2 + c &= 0 \\ a + b + c &= 1\,. \end{aligned}$$

Solving this system of linear equations, we obtain

$$A_Z(x,y) = \frac{(y_1 - y_2)x + (x_2 - x_1)y + x_1y_2 - x_2y_1}{y_1 - y_2 + x_2 - x_1 + x_1y_2 - x_2y_1} \tag{2.3}$$

on the triangle considered.

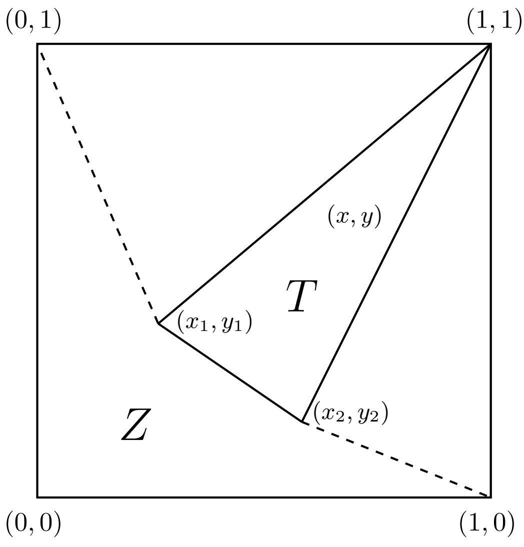


Figure 2.1: Example of a zero-set with a piecewise linear upper boundary curve

Example 2.3. *Let* $Z = \{(x,y) \in [0,1]^2 \mid \min(x,y) \leq \frac{1}{4}\}$. *Here* $d = d' = 1$ *and*

$$f(x) = \begin{cases} 1 & \text{, if } x \leq 1/4\,, \\ 1/4 & \text{, if } x > 1/4\,. \end{cases}$$

The corresponding conic aggregation function A_Z *is given by*

$$A_Z(x,y) = (1/3)\max(\min(4x-1, 4y-1), 0) \tag{2.4}$$

and is depicted in Figure 2.2. Note that A_Z *does not have neutral element 1.*

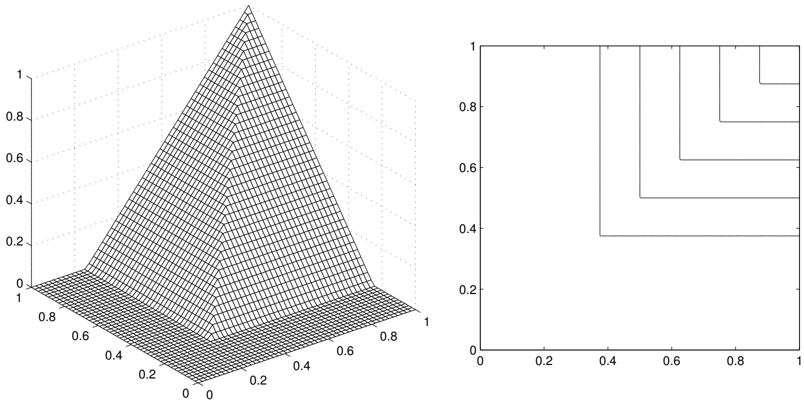


Figure 2.2: Graph and contour plot of a conic aggregation function

Example 2.4. *Let* $Z = \{(x, y) \in [0, 1]^2 \mid \max(x, y) \leq \frac{1}{2}\} \cup Z_*$*. Here* $d = d' = 1/2$ *and* $f(x) = 1/2$ *for any* $x \in [0, 1/2]$*. The corresponding conic aggregation function* A_Z *is given by*

$$A_Z(x, y) = \min(x, y, \max(2x - 1, 2y - 1, 0)) \tag{2.5}$$

and is depicted in Figure 2.3. Note that A_Z *has neutral element 1.*

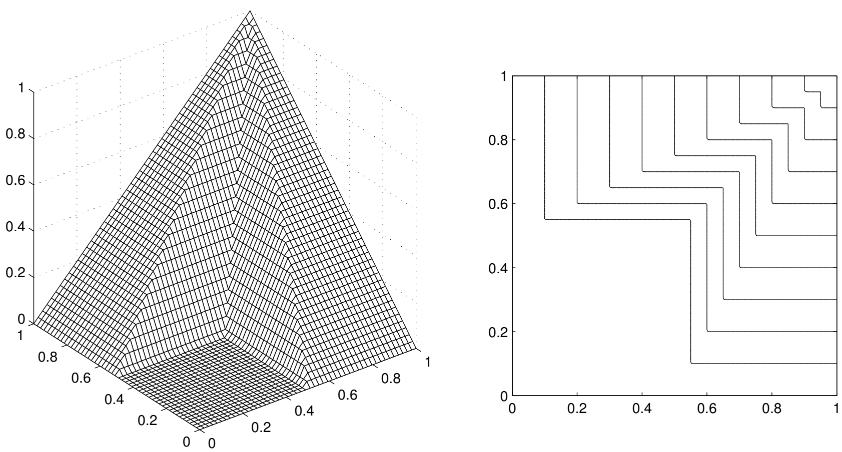


Figure 2.3: Graph and contour plot of a conic aggregation function

2.4. Conic t-norms

Conic t-norms have already been investigated in the literature [2].

Proposition 2.6. *Every associative conic aggregation function has neutral element* 1.

Proof. Consider an associative conic aggregation function A_f. Consider $x \in \,]0,1[$ and let (x_0, y_0) be the unique weakly undominated point corresponding to the point $(x,1)$ (and hence also to the point $(A_f(x,1),1)$). Then from Eq. (2.1) it follows that

$$A_f(x,1) = \frac{x - x_0}{1 - x_0} \qquad \text{and} \qquad A_f(A_f(x,1),1) = \frac{A_f(x,1) - x_0}{1 - x_0}\,.$$

On the other hand, the associativity of A_f leads to $A_f(x,1) = A_f(x, A_f(1,1)) = A_f(A_f(x,1),1)$, and therefore $A_f(x,1) = x$. Similarly, it follows that $A_f(1,x) = x$. □

Corollary 2.1. *A conic aggregation function is a t-norm if and only if it is associative.*

Proof. Follows from the above proposition and the fact that any associative continuous aggregation function with neutral element 1 is also commutative (see Chapter 1). □

It is easy to check that for every conic t-norm T different from $T_{\mathbf{M}}$, the condition $\delta(x) = T(x,x) < x$ is satisfied for any $x \in \,]0,1[\,$. Since a conic t-norm is continuous, it therefore must be Archimedean. The following theorem expresses that the only conic t-norms are the elements of the Yager family of t-norms.

Theorem 2.3. [2] *A t-norm T is conic if and only if either $T = T_{\mathbf{M}}$ or there exists $\lambda \in \,]0,\infty[$ such that*

$$T(x,y) = C_\lambda^{\mathbf{Y}}(x,y)\,, \tag{2.6}$$

for any $(x,y) \in [0,1]^2$.

2.5. Conic quasi-copulas

As mentioned before, both $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ are conic. Note that the zero-set of $T_{\mathbf{L}}$ is given by

$$Z^* = \{(x,y) \in [0,1]^2 \mid x + y \leq 1\}\,.$$

Hence, this is the greatest closed lower set that can be considered for constructing a conic quasi-copula.

In the next proposition, we show that the zero-set of a conic quasi-copula is determined by a strictly decreasing and continuous function.

Proposition 2.7. *Let $Z_* \subset Z \subseteq Z^*$ be the zero-set of a conic quasi-copula Q_Z with corresponding function $f : [0,d] \to [0,1]$. Then $f(d) = 0$ and f is strictly decreasing and continuous.*

Proof. Suppose that the upper boundary curve of Z contains some vertical segment as part of a line $x = x_0$ with $x_0 \in]0,1]$. Consider $x, x', y \in [0,1]$ with $x \leq x'$ such that the points (x_0, y_0) and (x_0, y_1) are the unique weakly undominated points corresponding to the points (x', y) and (x, y), respectively. This situation is depicted in Figure 2.4. The increasingness and 1-Lipschitz continuity of Q_Z imply

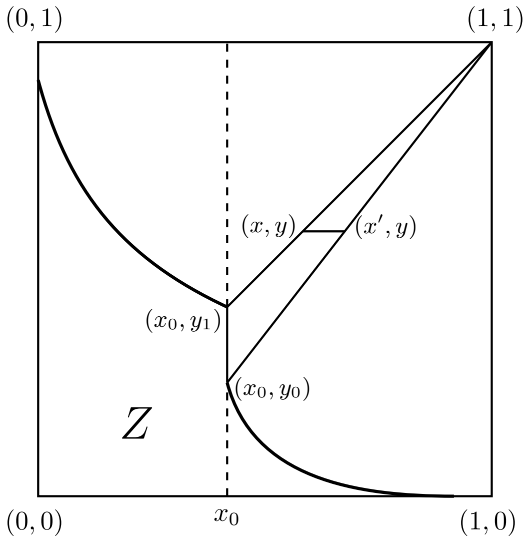


Figure 2.4: Illustration for the proof of Proposition 2.7.

that

$$Q_Z(x', y) - Q_Z(x, y) \leq x' - x\,,$$

or equivalently,

$$\frac{x' - x_0}{1 - x_0} - \frac{x - x_0}{1 - x_0} \leq x' - x\,.$$

The latter implies that $x_0 = 0$. Hence, the function f is continuous.

Similarly one can prove that the upper boundary curve of Z does not contain any horizontal segment as part of a line $y = y_0$ with $y_0 \in]0,1]$. Hence, the function f is strictly decreasing.

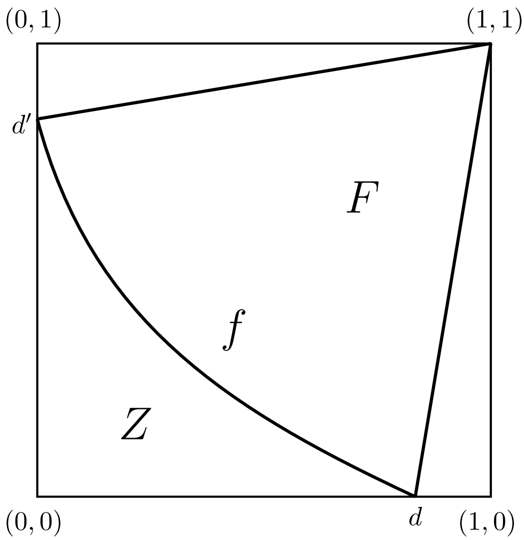


Figure 2.5: Illustration of the set F corresponding to a conic quasi-copula.

As Q_Z has neutral element 1, Proposition 2.5 implies that $d < 1$ or ($d = 1$ and $f(d) = 0$). As $Z_* \neq Z$, it holds that $d > 0$. Consider $d \in]0, 1[$, then $f(d) > 0$ would imply the existence of a vertical segment on the upper boundary curve of Z. The above proof for the continuity implies that this is impossible and thus $f(d) = 0$. □

As a consequence of the previous proposition, the zero-set in Example 2.4 cannot be the zero-set of a conic quasi-copula.

For a function f satisfying the conditions of Proposition 2.7, we introduce the following notations

$$\begin{aligned}
\Delta_{d'} &= \Delta_{\{(0,d'),(0,1),(1,1)\}} \\
\Delta_{d} &= \Delta_{\{(d,0),(1,0),(1,1)\}} \\
F &= [0,1]^2 \setminus (Z \cup \Delta_{d'} \cup \Delta_d)\,.
\end{aligned}$$

The set F is depicted in Figure 2.5. The conic quasi-copula can then be expressed as follows:

$$Q_Z(x,y) = \begin{cases} 0 & \text{, if } (x,y) \in Z\,, \\[2ex] \dfrac{y - f(x_0)}{1 - f(x_0)} & \text{, if } (x,y) \in F\,, \\[2ex] \min(x,y) & \text{, otherwise.} \end{cases} \tag{2.7}$$

Next we characterize all the subsets of the unit square that can be the zero-set of a conic quasi-copula.

Theorem 2.4. *Let Z be a closed lower set of $[0,1]^2$ such that $Z_* \subset Z \subseteq Z^*$ with corresponding function $f : [0,d] \to [0,1]$. The conic aggregation function A_f is a quasi-copula if and only if*

(i) $f(d) = 0$*;*

(ii) f *is strictly decreasing and continuous;*

(iii) *the function* $\varphi_1 :]0,d[\to [0,1]$ *defined by* $\varphi_1(x) = \frac{f(x)}{1-x}$ *is decreasing;*

(iv) *the function* $\varphi_2 :]0,d[\to [0,1]$ *defined by* $\varphi_2(x) = \frac{x}{1-f(x)}$ *is increasing.*

Proof. Suppose that conditions (i)–(iv) are satisfied. According to Proposition 2.5, the strict decreasingness of f and condition (i) imply that A_f has neutral element 1. To prove that A_f is a quasi-copula, we need to show that it is 1-Lipschitz continuous. Recall that the 1-Lipschitz continuity is equivalent to the 1-Lipschitz continuity in each variable. We prove that A_f is 1-Lipchitz continuous in the first variable. For any $x, x', y \in [0,1]$ such that $x \le x'$, we need to show that

$$A_f(x', y) - A_f(x, y) \le x' - x\,. \tag{2.8}$$

Let us denote $\mathbf{b} := (x, y)$ and $\mathbf{b}' := (x', y)$. We distinguish the following cases:

(a) If $\mathbf{b}, \mathbf{b}' \in Z$, then $A_f(x', y) - A_f(x, y) = 0 \le x' - x$;

(b) If $\mathbf{b}, \mathbf{b}' \in \Delta_d \cup \Delta_{d'}$, then

$$A_f(x', y) - A_f(x, y) = \min(x', y) - \min(x, y) \le x' - x\,;$$

(c) If $\mathbf{b}, \mathbf{b}' \in F$, then suppose that $\mathbf{b}_f = (x_0, f(x_0))$ and $\mathbf{b}'_f = (x_1, f(x_1))$ are the unique weakly undominated points such that $\mathbf{b}$, $\mathbf{b}_f$ and $(1,1)$, as well as $\mathbf{b}'$, $\mathbf{b}'_f$ and $(1,1)$, are collinear. Thus, condition (2.8) is equivalent to the inequality

$$\frac{y - f(x_1)}{1 - f(x_1)} - \frac{y - f(x_0)}{1 - f(x_0)} \le x' - x\,. \tag{2.9}$$

Using the collinearity, it follows that

$$x' - x = (1 - y)\left(\frac{1 - x_0}{1 - f(x_0)} - \frac{1 - x_1}{1 - f(x_1)}\right).$$

Therefore, inequality (2.9) is equivalent to

$$\frac{1}{1 - f(x_0)} - \frac{1}{1 - f(x_1)} \le \frac{1 - x_0}{1 - f(x_0)} - \frac{1 - x_1}{1 - f(x_1)}\,,$$

or equivalently,

$$\varphi_2(x_1) - \varphi_2(x_0) \geq 0\,,$$

which is satisfied due to condition (iv).

The proof that A_f is 1-Lipchitz continuous in the second variable is similar and uses condition (iii). Consequently, the aggregation function A_f is a conic quasi-copula.

Now suppose that the function A_f is a quasi-copula. Proposition 2.7 yields (i) and (ii). Consider arbitrary $x_1, x_2 \in]0, d[$ such that $x_1 \leq x_2$, and let (x, y), (x', y) be two points in F such that $(x_1, f(x_1))$ and $(x_2, f(x_2))$ are the corresponding weakly undominated points. The 1-Lipschitz continuity of A_f in the first variable implies that $\varphi_2(x_2) - \varphi_2(x_1) \geq 0$. Hence, condition (iv) follows. The 1-Lipschitz continuity of A_f in the second variable implies condition (iii). □

Example 2.5. *Let $f : [0, \frac{1}{2}] \to [0, 1]$ be the function defined by*

$$f(x) = \min\left(\frac{1-x}{2}, 1 - 2x\right)\,.$$

All the conditions in Theorem 2.4 are satisfied. The corresponding conic quasi-copula is given by

$$Q_f(x, y) = \min\left(x, y, \max\left(0, \frac{x + 2y - 1}{2}, \frac{2x + y - 1}{2}\right)\right)\,.$$

2.6. Conic copulas

In this section, we characterize all the subsets of the unit square that can be the zero-set of a conic copula.

Proposition 2.8. *Consider a conic aggregation function $A_f \neq T_{\mathbf{M}}$ such that f is piecewise linear and $f(d) = 0$. Then A_f is a copula if and only if f is convex.*

Proof. Consider a conic aggregation function A_f such that f is piecewise linear and $f(d) = 0$, and its zero-set Z.

Suppose that f is convex, then the fact that it is decreasing and $f(d) = 0$ implies that $f(x) < 1$ for all $x \in]0, d[$. Hence, A_f has neutral element 1. We only need to show that A_f is 2-increasing. Due to the additivity of volumes, it suffices to consider a number of cases.

Consider a rectangle $[x, x'] \times [y, y'] \subseteq [0, 1]^2$. If this rectangle is included in Z, then its A_f-volume equals 0. Also if the points (x, y') and (x', y) are located on the upper boundary curve of Z, then it holds that

$$V_{A_f}([x,x'] \times [y,y']) = A_f(x',y') \geq 0\,.$$

The study of the 2-increasingness on the remaining part of the unit square is equivalent to the study of this property on each polygon enclosed by two consecutive segments of the upper boundary curve of Z and the point $(1,1)$. Let us consider the polygon G determined by the points $\mathbf{b}_1 := (x_1,y_1)$, $\mathbf{b}_2 := (x_2,y_2)$, $\mathbf{b}_3 := (x_3,y_3)$ and the point $(1,1)$, as illustrated in Figure 2.6. The aggregation function A_f is linear on the triangle $\Delta_1 := \Delta_{\{\mathbf{b}_1,\mathbf{b}_2,(1,1)\}}$ as well as on the triangle $\Delta_2 := \Delta_{\{\mathbf{b}_2,\mathbf{b}_3,(1,1)\}}$. Hence, if the rectangle $[x,x'] \times [y,y']$ is included in Δ_1 or Δ_2, its A_f-volume equals 0.

Finally, suppose that the segment connecting the points (x,y) and (x',y') is a subset of the segment connecting the points $\mathbf{b}_2$ and $(1,1)$ (this situation is also depicted in Figure 2.6). Using Eq (2.3), the nonnegativity of $V_{A_f}([x,x'] \times [y,y'])$ is then equivalent to:

$$\frac{y_1 - y_2}{y_1 - y_2 + x_2 - x_1 + x_1y_2 - x_2y_1} - \frac{y_2 - y_3}{y_2 - y_3 + x_3 - x_2 + x_2y_3 - x_3y_2} \geq 0\,, \quad (2.10)$$

or, equivalently,

$$(y_1 - y_2)(x_3 - x_2 + x_2y_3 - x_3y_2) - (y_2 - y_3)(x_2 - x_1 + x_1y_2 - x_2y_1) \geq 0\,.$$

Some elementary manipulations yield

$$y_1(x_3-x_2)-y_2(x_3-x_2)+y_3(x_2-x_1)-y_1y_2(x_3-x_2)-y_2y_3(x_2-x_1)+y_2^2(x_3-x_1) \geq 0\,,$$

or, equivalently,

$$y_1(x_3 - x_2)(1 - y_2) - y_2(x_3 - x_1)(1 - y_2) + y_3(x_2 - x_1)(1 - y_2) \geq 0\,.$$

Dividing by $(1 - y_2)$, the latter inequality becomes

$$y_1(x_3 - x_2) - y_2(x_3 - x_1) + y_3(x_2 - x_1) \geq 0\,,$$

or, equivalently,

$$y_1(x_3 - x_2) - y_2(x_3 - x_2 + x_2 - x_1) + y_3(x_2 - x_1) \geq 0\,.$$

It easily follows that the above inequality is equivalent to

$$\frac{y_3 - y_2}{x_3 - x_2} \geq \frac{y_2 - y_1}{x_2 - x_1}\,, \quad (2.11)$$

which is satisfied due to the convexity of the function f. The converse part of the

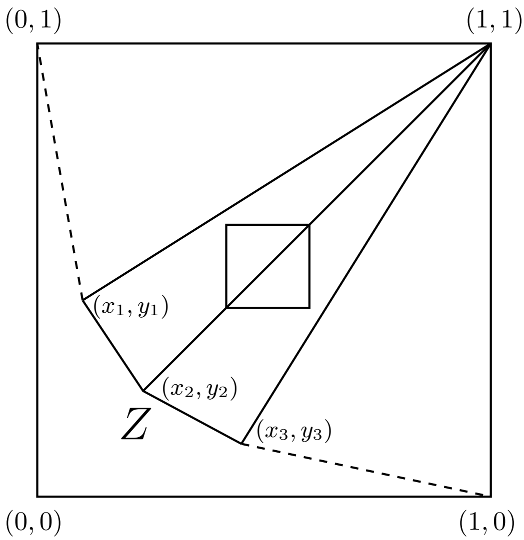


Figure 2.6: Illustration for the proof of Proposition 2.8

proof can be done in a similar way. □

Next we characterize all the subsets of the unit square that can be the zero-set of a conic copula. To this end, we need the following lemma.

Lemma 2.1. *Let C_f be a conic copula and $\alpha, \beta \in]0, \infty[$ such that $\alpha < \beta$. Consider three points $\mathbf{b}_1 := (x_1, f(x_1))$, $\mathbf{b}_2 := (x_2, f(x_2))$ and $\mathbf{b}_3 := (x_3, f(x_3))$ such that $x_1 < x_2 < x_3$ and the segments $\langle \mathbf{b}_1, (1,1) \rangle$, $\langle \mathbf{b}_2, (1,1) \rangle$ and $\langle \mathbf{b}_3, (1,1) \rangle$ have slope α, $\sqrt{\alpha\beta}$ and β, respectively. Then it holds that*

(i) *there exists a rectangle $[x, x'] \times [y, y']$ such that the segment connecting the points (x, y) and (x', y') is a subset of the segment $\langle \mathbf{b}_2, (1,1) \rangle$ and the points (x, y') and (x', y) are located on the segments $\langle \mathbf{b}_1, (1,1) \rangle$ and $\langle \mathbf{b}_3, (1,1) \rangle$ respectively.*

(ii) *the point $\mathbf{b}_2$ is below the segment $\langle \mathbf{b}_1, \mathbf{b}_3 \rangle$.*

Proof. A simple geometric argumentation shows that points $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ with the desired properties always exist. Observation (i) follows from the fact that for such points, we can always find a rectangle $[x, x'] \times [y, y']$ of which the main diagonal is a subset of the segment with slope $\gamma = \sqrt{\alpha\beta}$ and the points (x, y') and (x', y) are located on the segments with slopes α and β, respectively.

To prove assertion (ii), we consider the function $g : [x_1, x_3] \to [0, 1]$ such that g is linear on the interval $[x_1, x_2]$ as well as on the interval $[x_2, x_3]$. Consider the rectangle $[x, x'] \times [y, y']$ from the first part of the proof. Since C_f is a copula, it follows that

$$V_{C_{Z'}}([x,x']\times[y,y']) = V_{C_f}([x,x']\times[y,y']) \geq 0\,,$$

where $C_{Z'}$ is a conic function that has the graph of the function g as part of the upper boundary curve of its zero-set Z'. Due to (2.10), it holds that the function g is convex, or equivalently, the point $\mathbf{b}_2$ lies below the segment $\langle \mathbf{b}_1, \mathbf{b}_3 \rangle$, which completes the proof. □

Theorem 2.5. *Let Z be a closed lower set of $[0,1]^2$ such that $Z_* \subset Z \subseteq Z^*$ with corresponding function $f : [0,d] \to [0,1]$. The conic aggregation function A_f is a copula if and only if*

(i) $f(d) = 0$;

(ii) *f is convex.*

Proof. Suppose that conditions (i) and (ii) are satisfied. To prove that A_f is a copula we need to show its 2-increasingness. Due to the additivity of volumes, it suffices to consider a number of cases. Let $R = [x,x']\times[y,y'] \subseteq [0,1]^2$.

(a) If R is located in Δ_d, $\Delta_{d'}$ or Z, then $V_{A_f}(R) = 0$.

(b) If (x,y') and (x',y) are located on the upper boundary curve of Z, then again

$$V_{A_f}(R) = A_f(x',y') \geq 0\,.$$

(c) If the main diagonal of R is a subset of the segment connecting the points $(0,d')$ and $(1,1)$ (the case when the main diagonal is a subset of the segment connecting the points $(d,0)$ and $(1,1)$ is analogous), then it holds that

$$V_{A_f}([x,x']\times[y,y']) = x' - A_f(x',y) \geq 0\,.$$

(d) If R is located in F, then let $\mathbf{b_1} = (x_1, f(x_1))$, $\mathbf{b_2} = (x_2, f(x_2))$, $\mathbf{b_3} = (x_3, f(x_3))$ and $\mathbf{b_4} = (x_4, f(x_4))$ be the weakly undominated points corresponding to the vertices of this rectangle. The points $\mathbf{b_1}$, $\mathbf{b_2}$, $\mathbf{b_3}$ and $\mathbf{b_4}$ together with $(0,d')$ and $(d,0)$, determine a convex piecewise linear function $h : [0,d] \to [0,1]$ such that $h(x_i) = f(x_i)$ for any $i \in \{1,2,3,4\}$. This situation is illustrated in Figure 2.7 when the main diagonal of this rectangle is a subset of the segment connecting the weakly undominated point corresponding to (x,y) and the point $(1,1)$. Due to Proposition 2.8, the conic aggregation function A_h is a conic copula. Therefore,

$$V_{A_f}([x,x']\times[y,y']) = V_{A_h}([x,x']\times[y,y']) \geq 0\,.$$

Hence, the A_f-volume of any rectangle is nonnegative, which implies that A_f is a copula.

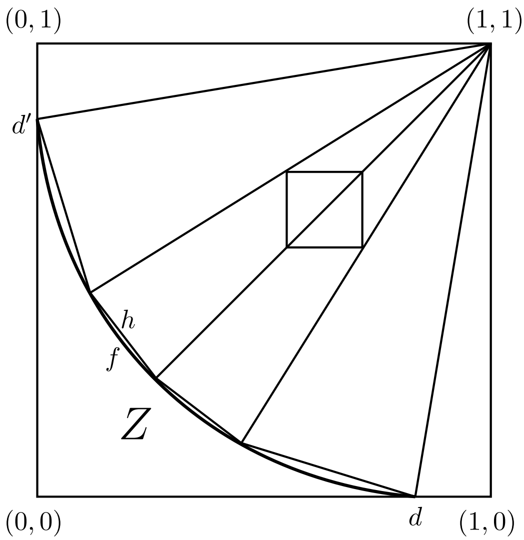


Figure 2.7: Illustration for the proof of Theorem 2.5.

Conversely, suppose that A_f is a copula. In view of Theorem 2.4, it suffices to show that f is convex. Suppose that it is not convex, i.e. there exist $x < y < z$ such that the point $(y, f(y))$ is above the segment connecting the points $(x, f(x))$ and $(z, f(z))$. Since f is continuous there exists $\epsilon > 0$ such that for any $x' \in [y-\epsilon, y+\epsilon]$ the point $(x', f(x'))$ is above the segment connecting the points $(x, f(x))$ and $(z, f(z))$, which contradicts Lemma 2.1. Thus, the function f must be convex. $\square$

Remark 2.3.

(i) *Since f is right-continuous at 0, decreasing and $f(d) = 0$, the convexity of f implies that f is strictly decreasing and continuous.*

(ii) *As any conic copula C_f is a conic quasi-copula, the convexity of f implies conditions (iii)-(iv) of Theorem 2.4.*

(iii) *As associative copulas are (1-Lipschitz) t-norms, the class of associative conic copulas is also characterized by Theorem 2.3.*

As the function f in Example 2.5 is not convex, the corresponding conic quasi-copula Q_f is a proper quasi-copula.

Example 2.6. *For each $\lambda \in]0, \infty[$, let $f_\lambda : [0,1] \to [0,1]$ represent the boundary curve of the zero-set of the Yager t-norm $C^{\mathbf{Y}}_\lambda$, i.e. $f_\lambda(x) = 1 - (1-(1-x)^\lambda)^{\frac{1}{\lambda}}$. It is easily verified that the function f_λ is convex if and only if $\lambda \geq 1$. Hence, $C^{\mathbf{Y}}_\lambda$ is a conic copula for any $\lambda \geq 1$.*

Example 2.7. *Let $f : [0, \frac{1}{2}] \to [0,1]$ be the function defined by $f(x) = (1-2x)^2$. All the conditions in Theorem 2.5 are satisfied. The corresponding conic copula is given by*

$$C_f(x,y) = \begin{cases} 0 & , \text{ if } y \leq (1-2x)^2 \text{ and } x \leq 1/2\,, \\ \dfrac{4x(1-x)-1+y}{4(1-x)-1+y} & , \text{ if } y > (1-2x)^2 \text{ and } y \geq 2x-1\,, \\ \min(x,y) & , \text{ otherwise}\,. \end{cases}$$

Note that, for any conic copula C_f, the convexity of f implies that the upper boundary curve of the t-level set is convex for any $t \in [0,1[$. Hence, the following corollary is clear.

Corollary 2.2. *Any conic copula is quasi-concave.*

2.7. Conic copulas supported on a set with Lebesgue measure zero

We characterize in this section conic copulas that are supported on a set with Lebesgue measure zero on the basis of their zero-set. To this end, we need the following lemma.

Lemma 2.2. *Let $C_f \neq T_{\mathbf{M}}$ be a conic copula. Then*

(i) *the graph of the function f is a subset of the support;*

(ii) *if the upper boundary curve of the zero-set of C_f contains two consecutive segments with common point $\mathbf{b}$, then the segment $\langle \mathbf{b}, (1,1)\rangle$ is a subset of the support.*

Proof. For any rectangle $[x,x'] \times [y,y']$ such that the points (x,y') and (x',y) are located on the graph of the function f, the 2-increasingness implies that

$$V_{C_f}([x,x'] \times [y,y']) = C_f(x',y') > 0\,.$$

Thus (i) follows.

Let $\mathbf{b}$, $\mathbf{b}_1$ and $\mathbf{b}_2$ be three distinct points on the upper boundary curve of the zero-set of C_f such that $\langle \mathbf{b}_1, \mathbf{b}\rangle$ and $\langle \mathbf{b}, \mathbf{b}_2\rangle$ are two segments. Let R be a rectangle such that its main diagonal is a subset of the segment $\langle \mathbf{b}, (1,1)\rangle$. If $V_{C_f}(R) = 0$, then due to (2.10), the points $\mathbf{b}$, $\mathbf{b}_1$ and $\mathbf{b}_2$ are located on the same segment, which is a contradiction. Hence, $V_{C_f}(R) > 0$ and (ii) follows. $\square$

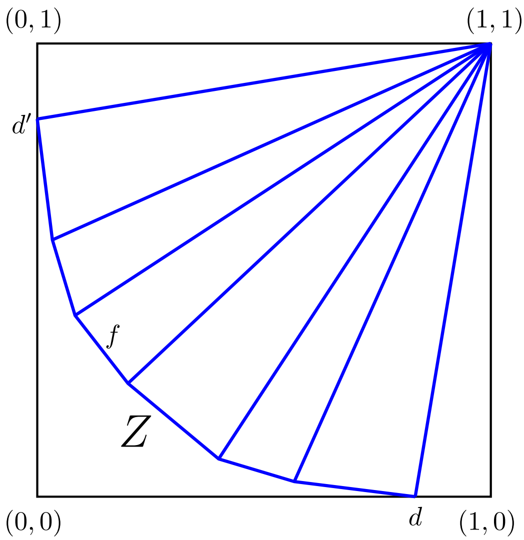


Figure 2.8: The support of a conic copula that is supported on a set with Lebesgue measure zero (case $d, d' < 1$).

In Figure 2.8, the support of a conic copula C_f with a piecewise linear function f is shown.

Proposition 2.9. *A conic copula $C_f \neq T_{\mathbf{M}}$ is supported on a set with Lebesgue measure zero if and only if the function f is piecewise linear.*

Proof. Let C_f be a conic copula with a piecewise linear function f. Due to Lemma 2.2, the support of C_f is constituted from the graph of the function f and all segments connecting the point $(1, 1)$ and a point on the graph of f connecting two consecutive segments of this graph. Since the surface of C_f consists of triangles, it holds that $\frac{\partial^2 C(u,v)}{\partial u \partial v} = 0$ in all other points. Therefore, the conic copula C_f is supported on a set with Lebesgue measure zero.

Conversely, let C_f be supported on a set with Lebesgue measure zero and suppose that the function f is not piecewise linear, i.e. there exists an interval $[m, n] \subseteq [0, d]$ such that the graph of the restriction of f to $[m, n]$ does not contain any segment. Let S be the subset of the unit square enclosed by the graph of the function f between the points $(m, f(m))$ and $(n, f(n))$ and the segments connecting the latter points to $(1, 1)$. Consider a rectangle R located in S such that $V_{C_f}(R) = 0$. It then holds that $V_{C_f}(R_1) = 0$ for any rectangle $R_1 \subseteq R$. Choose a rectangle $R_1 = [x, x'] \times [y, y'] \subseteq R$ such that its diagonal is a subset of the segment $\langle(1, 1), (x_2, f(x_2))\rangle$ with $m < x_2 < n$. Let $(x_1, f(x_1))$ and $(x_3, f(x_3))$ be the two points on the graph of f such that the points (x, y') and (x', y) are respectively located on the segments $\langle(1, 1), (x_1, f(x_1))\rangle$ and $\langle(1, 1), (x_3, f(x_3))\rangle$. Since $V_{C_f}(R_1) = 0$, inequality (2.10) implies that the points $(x_1, f(x_1))$, $(x_2, f(x_2))$ and $(x_3, f(x_3))$ are located on the same segment, which contradicts the fact that f does not contain any segment on the interval $[m, n]$. Hence, for any rectangle R located in S, it holds that $V_{C_f}(R) > 0$, i.e. S is a subset of the support of C_f with non-zero Lebesgue measure. This contradicts the fact that C_f is supported on a set with Lebesgue measure zero. $\square$

As a result of the above proposition, a conic copula that is supported on a set with Lebesgue measure zero is related to a piecewise linear function f and hence, to a (possibly infinite) sequence of points $\mathbf{b}_1 = (x_1, y_1)$, $\mathbf{b}_2 = (x_2, y_2)$, …, $\mathbf{b}_n = (x_n, y_n)$ and $\mathbf{b}_0 = (0, 1)$, $\mathbf{b}_{n+1} = (1, 0)$, so that $0 \leq x_1 < x_2 < \ldots < x_n < 1$ and $1 > y_1 > \ldots > y_n \geq 0$ and

$$\frac{y_i - y_{i-1}}{x_i - x_{i-1}} \leq \frac{y_{i+1} - y_i}{x_{i+1} - x_i} \,.$$

As the function f in Example 2.7 is not piecewise linear, the corresponding conic copula C_f is not supported on a set with Lebesgue measure zero.

In the Yager family, only $T_{\mathbf{L}}$ and $T_{\mathbf{M}}$ are supported on a set with Lebesgue measure zero.

Since any copula that is supported on a set with Lebesgue measure zero is singular, the following corollary is clear.

Corollary 2.3. *Any conic copula C_f with a piecewise linear function f is singular.*

Example 2.8. *For each $\lambda \in \,]0, 1/2]$, let $f_\lambda : [0,1] \to [0,1]$ be the function defined by*

$$f_\lambda(x) = \begin{cases} \dfrac{\lambda - 1}{\lambda} x + 1 & , \text{ if } x \leq \lambda \,, \\ \dfrac{\lambda}{\lambda - 1}(x - 1) & , \text{ if } x > \lambda \,. \end{cases}$$

The function f_λ is convex and piecewise linear for any $\lambda \in \,]0, 1/2]$. The corresponding family of singular conic copulas is given by

$$C_\lambda(x, y) = \begin{cases} \max\left(y - \dfrac{\lambda}{1 - \lambda}(1 - x), 0\right) & , \text{ if } y \leq x \,, \\ \max\left(x - \dfrac{\lambda}{1 - \lambda}(1 - y), 0\right) & , \text{ otherwise} \,. \end{cases}$$

Example 2.9. *For each $c \in [0,1]$, let $f : [0,c] \to [0,1]$ be the function defined by $f(x) = c - x$. The function f is convex and linear for any $c \in [0,1]$. The corresponding family of singular conic copulas is given by*

$$C_c(x,y) = \min\left(x, y, \max\left(0, \frac{x+y-c}{2-c}\right)\right) .$$

This family was introduced in [21].

2.8. Dependence measures

In this section, we derive compact formulae for Spearman's rho, Gini's gamma and Kendall's tau of two continuous random variables whose dependence is modelled by a conic copula C_f. They can be expressed in terms of the function f.

Proposition 2.10. *Let X and Y be two continuous random variables that are coupled by a conic copula $C_f \neq T_{\mathbf{M}}$ and let $a \in]0,d[$ be the unique value such that $f(a) = a$.*

(i) *The population version of Spearman's ρ_{C_f} for X and Y is given by*

$$\rho_{C_f} = 1 - 4 \int_0^d f(x)\,\mathrm{d}x .$$

(ii) *The population version of Gini's γ_{C_δ} for X and Y is given by*

$$\gamma_{C_\delta} = 2\left(\left(\frac{1-d}{2-d}\right)^2 + \left(\frac{1-d'}{2-d'}\right)^2 - a\right)$$
$$+ 4 \int_0^d \left(\frac{(1-x-f(x))(1-f(x)-f'(x)(1-x))}{(2-x-f(x))^3}\right) \mathrm{d}x .$$

where f' is the left (or right) derivative of f.

(iii) *The population version of Kendall's τ_{C_f} for X and Y is given by*

$$\tau_{C_f} = 1 - 2 \int_0^d \frac{f'(x)}{(1-x)f'(x) - 1 + f(x)}\,\mathrm{d}x ,$$

where f' is the left (or right) derivative of f.

Proof. The integral of C_f over the unit square is the volume below its surface. Given the geometrical fact that the volume of a conic body equals one third of the product of the area of its base and its height, (i) follows immediately.

In order to find γ_{C_δ}, we need to compute

$$I_1 = \int_0^1 \omega_{C_f}(x)\,\mathrm{d}x \quad \text{and} \quad I_2 = \int_0^1 (x - \delta_{C_f}(x))\,\mathrm{d}x\,.$$

Using formula (2.7), δ_{C_f} and ω_{C_f} are given by

$$\delta_{C_f}(x) = \begin{cases} 0 & \text{, if } x \le a\,, \\ \dfrac{x-a}{1-a} & \text{, if } x \ge a\,, \end{cases}$$

$$\omega_{C_f}(x) = \begin{cases} \dfrac{1-x-f(x_0)}{1-f(x_0)} & \text{, if } \frac{1-d'}{2-d'} \le x \le \frac{1}{2-d}\,, \\ \min(x, 1-x) & \text{, if } x \le \frac{1-d'}{2-d'} \text{ or } x \ge \frac{1}{2-d}\,, \end{cases}$$

where $(x_0, f(x_0))$ is the weakly undominated point corresponding to $(x, 1-x)$. Simple elementary manipulations show

$$I_2 = \frac{a}{2} \quad \text{and} \quad I_1 = \frac{1}{2}\left(\frac{1-d'}{2-d'}\right)^2 + \frac{1}{2}\left(\frac{1-d}{2-d}\right)^2 + \int_{\frac{1-d'}{2-d'}}^{\frac{1}{2-d}} \omega_{C_f}(x)\,\mathrm{d}x$$

Since $(x_0, f(x_0))$, $(x, 1-x)$ and $(1,1)$ are collinear, it holds that

$$x = \frac{1-f(x_0)}{2-x_0-f(x_0)}\,.$$

Computing $\mathrm{d}x$ and ω_{C_f}, it follows that

$$\int_{\frac{1-d'}{2-d'}}^{\frac{1}{2-d}} \omega_{C_f}(x)\,\mathrm{d}x = \int_0^d \left(\frac{(1-x-f(x))(1-f(x)-f'(x)(1-x))}{(2-x-f(x))^3}\right)\mathrm{d}x\,,$$

where f' is the left (or right) derivative of the function f, which, due to convexity of f, exists everywhere on the interval $]0, d]$ (or $[0, d[$). In 0 (or in d) we can take the limit without influencing the result of the integration over the interval $[0, d]$. Note also that the left and right derivatives coincide, except possibly on a countable subset. Hence, the choice of derivative does not affect the result of the integration.

Substituting I_1 and I_2 in the expression for γ_{C_f}, (ii) follows.

In order to find τ_{C_f}, we need to compute

$$I = \iint\limits_{[0,1]^2} C_f(x,y)\,\mathrm{d}C_f(x,y)\,.$$

Suppose first that C_f is supported on a set with Lebesgue measure zero. Due to Proposition 2.9, the function f is piecewise linear, i.e. there exists an $n \in \mathbb{N}$ such that the graph of the function f is constituted from segments $\langle \mathbf{b}_{i-1}, \mathbf{b}_i \rangle$, $i \in \{1, ..., n\}$, with $\mathbf{b}_0 = (0, y_0)$ and $\mathbf{b}_n = (x_n, 0)$. Due to Lemma 2.2, the mass of C_f is distributed uniformly on the segments $\langle \mathbf{b}_{i-1}, \mathbf{b}_i \rangle$, $i \in \{1, ..., n\}$ and on the segments $\langle \mathbf{b}_j, (1,1) \rangle$, $j \in \{0, ..., n\}$. Let a_i, $i \in \{1, ..., n\}$, and b_j, $j \in \{0, ..., n\}$, be the mass distributed respectively on the segment $\langle \mathbf{b}_{i-1}, \mathbf{b}_i \rangle$ and $\langle \mathbf{b}_j, (1,1) \rangle$. For each segment $\langle \mathbf{b}_{i-1}, \mathbf{b}_i \rangle$, the conic copula C_f attains the value 0, therefore the integral I can be written as

$$I = \sum_{i=1}^{n} \frac{a_i}{x_i - x_{i-1}} \int\limits_{x_{i-1}}^{x_i} 0 \,\mathrm{d}x + \sum_{j=0}^{n} \frac{b_j}{1 - x_j} \int\limits_{x_j}^{1} \frac{x - x_j}{1 - x_j} \,\mathrm{d}x = \frac{1}{2} \sum_{j=0}^{n} b_j\,.$$

Since the total mass is equal to one, it holds that

$$\sum_{i=1}^{n} a_i + \sum_{j=0}^{n} b_j = 1\,,$$

or equivalently,

$$\sum_{j=0}^{n} b_j = 1 - \sum_{i=1}^{n} a_i\,.$$

Hence, the parameter τ is given by

$$\tau_{C_f} = 1 - 2 \sum_{i=1}^{n} a_i\,. \tag{2.12}$$

For each $i \in \{1, ..., n\}$, it holds that

$$a_i = V_{C_f}([x_{i-1}, x_i] \times [y_i, y_{i-1}]) = C_f(x_i, y_{i-1})\,.$$

Using (2.3), we obtain

$$C_f(x_i, y_{i-1}) = \frac{(y_{i-1} - y_i)(x_i - x_{i-1})}{y_{i-1} - y_i + x_i - x_{i-1} + x_{i-1} y_i - x_i y_{i-1}}\,.$$

Hence, (2.11) can be expressed as

$$\tau_{C_f} = 1 - 2\sum_{i=1}^{n} \frac{(y_{i-1} - y_i)(x_i - x_{i-1})}{y_{i-1} - y_i + x_i - x_{i-1} + x_{i-1}y_i - x_i y_{i-1}}\,. \tag{2.13}$$

Let us denote $x_{i-1} = x$, $y_{i-1} = f(x)$, $x_i = x + \mathrm{d}x$ and $y_i = f(x) + f'(x)\mathrm{d}x$, where f' is the left (or right) derivative of the function f.

By letting $\max \mathrm{d}x$ approach 0, it holds that

$$\sum_{i=1}^{n} \frac{(y_{i-1} - y_i)(x_i - x_{i-1})}{y_{i-1} - y_i + x_i - x_{i-1} + x_{i-1}y_i - x_i y_{i-1}}$$

converges to

$$\int\limits_0^d \frac{f'(x)}{(1-x)f'(x) - 1 + f(x)}\,\mathrm{d}x\,.$$

Substituting this result in (2.12), (iii) follows. □

Example 2.10.

(i) *For $\lambda \in]0, 1/2]$, let C_λ be the conic copula given in Example 2.8. Then*

$$\rho_{C_\lambda} = \tau_{C_\lambda} = 1 - 4\lambda \qquad \textit{and} \qquad \gamma_{C_c} = \frac{1 - 4\lambda + 2\lambda^2}{1-\lambda}.$$

(ii) *For $c \in [0, 1]$, let C_c be the conic copula given in Example 2.9. Then*

$$\rho_{C_c} = 1 - 2c^2\,, \qquad \gamma_{C_c} = \frac{4(1-c)}{(2-c)^2} - c \qquad \textit{and} \qquad \tau_{C_c} = \frac{2-3c}{2-c}\,.$$

The results are listed in Table 2.1.

2.9. Aggregation of conic (quasi-)copulas

In this section we study some aggregations of conic quasi-copulas and conic copulas. We formulate a lemma and two immediate propositions.

Let $f_i : [0, d_i] \to [0, 1]$, $i \in \{1, 2\}$, be two strictly decreasing continuous functions such that $f_i(d_i) = 0$. We temporarily extend these functions to $[0, 1]$ by setting $f_i(x) = 0$ for any $x \in]d_i, 1]$. We define the function

$$f_{\max} : [0, \max(d_1, d_2)] \to [0, 1]$$

Table 2.1: Spearman's rho, Gini's gamma and Kendall's tau of the conic copulas C_c.

c	f_c	ρ_{C_c}	γ_{C_c}	τ_{C_c}
0	0	1	1	1
0.2	$0.2 - x$	0.920000	0.787654	0.777778
0.4	$0.4 - x$	0.680000	0.537500	0.500000
0.6	$0.6 - x$	0.280000	0.216327	0.142857
0.8	$0.8 - x$	−0.280000	−0.244444	−0.333333
1	$1 - x$	−1	−1	−1

by $f_{\max}(x) = \max(f_1(x), f_2(x))$. Similarly, we define the function

$$f_{\min} : [0, \min(d_1, d_2)] \to [0, 1]$$

by $f_{\min}(x) = \min(f_1(x), f_2(x))$.

Lemma 2.3. *Using the above notations, it holds that*

(i) *if the functions f_1 and f_2 satisfy condition (iii), resp. (iv), of Theorem 2.4, then also $f_{\max}$ and $f_{\min}$ satisfy condition (iii), resp. (iv);*

(ii) *if the functions f_1 and f_2 are convex, then also $f_{\max}$ is convex.*

Proposition 2.11. *For any two conic quasi-copulas Q_{f_1} and Q_{f_2}, it holds that*

(i) *the functions $\max(Q_{f_1}, Q_{f_2})$ and $\min(Q_{f_1}, Q_{f_2})$ are also conic quasi-copulas, i.e. the class of conic quasi-copulas is closed under maximum and minimum;*

(ii) *the corresponding functions are given by $f_{\max}$ and $f_{\min}$, respectively.*

Proposition 2.12. *For any two conic copulas C_{f_1} and C_{f_2}, it holds that:*

(i) *the function $\min(C_{f_1}, C_{f_2})$ is also a conic copula, i.e. the class of conic copulas is closed under minimum;*

(ii) *the corresponding function is given by $f_{\max}$.*

In general, the maximum of two conic copulas need not be a conic copula. For instance, let C_{f_1} and C_{f_2} be two conic copulas with f_1 and f_2 as depicted in Figure 2.9. Obviously, the function $f_{\min}$ is not convex, and thus $\max(C_{f_1}, C_{f_2})$ is a proper conic quasi-copula.

Since the function f determining a conic quasi-copula Q_f can always be written as the infimum of a family $(f_i)_{i \in I}$ of convex functions, any conic quasi-copula Q_f can

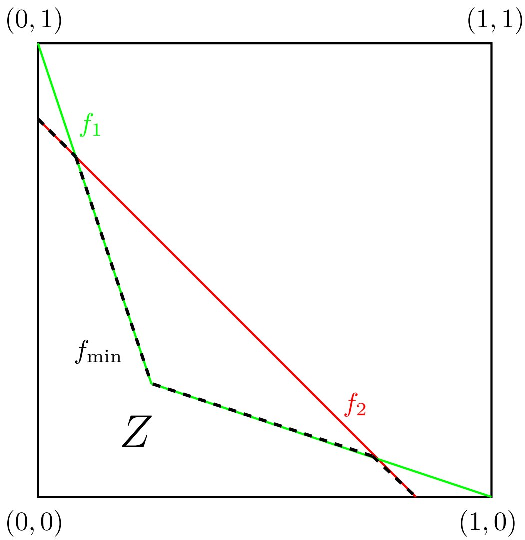


Figure 2.9: An example of the graph of $f_{\min}$

be written as

$$Q_f = \sup_{i \in I} C_{f_i} \,,$$

where C_{f_i} are conic copulas.

3 Biconic aggregation functions

3.1. Introduction

The surface of the aggregation functions $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ is constituted from their zero-set and linear segments connecting the upper boundary curve of their zero-set to the point $(1, 1, 1)$. In the previous chapter, this observation has led to the notion of conic aggregation functions. Characteristic for the aggregation functions $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ is also that their surface is constituted from linear segments connecting their diagonal section to the points $(0, 1, 0)$ and $(1, 0, 0)$. Similarly, their surface is constituted from linear segments connecting their opposite diagonal section to the points $(0, 0, 0)$ and $(1, 1, 1)$. Inspired by these observations, we introduce a new method to construct aggregation functions. These aggregation functions are constructed by linear interpolation on segments connecting the diagonal (resp. opposite diagonal) of the unit square to the points $(0, 1)$ and $(1, 0)$ (resp. $(0, 0)$ and $(1, 1)$).

This chapter is organized as follows. In Section 3.2 we introduce the definition of a biconic function with a given diagonal section and characterize the class of biconic aggregation functions. In Sections 3.3–3.6, we characterize the classes of biconic semi-copulas, biconic quasi-copulas, biconic copulas and biconic copulas supported on a set with Lebesgue measure zero. For biconic copulas, we provide simple expressions for Spearman's rho, Kendall's tau and Gini's gamma in Section 3.7. In Section 3.8, we study the aggregation of biconic (semi-, quasi-) copulas. The class of biconic functions with a given opposite diagonal section is introduced in Section 3.9.

3.2. Biconic functions with a given diagonal section

Biconic functions with a given diagonal section are constructed by linear interpolation on segments connecting the diagonal of the unit square to the points $(0, 1)$ and $(1, 0)$. Let $\delta \in \mathcal{D}_{\mathrm{A}}$ and $\alpha, \beta \in [0, 1]$. The function $A_\delta^{\alpha,\beta} : [0, 1]^2 \to [0, 1]$ defined

by

$$A_\delta^{\alpha,\beta}(x,y) = \begin{cases} \alpha(x-y) + (1+y-x)\,\delta\left(\dfrac{y}{1+y-x}\right) & \text{, if } y \le x\,, \\ \\ \beta(y-x) + (1+x-y)\,\delta\left(\dfrac{x}{1+x-y}\right) & \text{, otherwise,} \end{cases} \tag{3.1}$$

where the convention $\frac{0}{0} := 0$ is adopted, is well defined. This function is called *a biconic function with a given diagonal section* since it satisfies the boundary conditions

$$A_\delta^{\alpha,\beta}(0,1) = \beta \quad \text{and } A_\delta^{\alpha,\beta}(1,0) = \alpha\,,$$

and $A_\delta^{\alpha,\beta}(t,t) = \delta(t)$ for any $t \in [0,1]$, and since it is linear on segments connecting the points (t,t) and $(0,1)$ as well as on segments connecting the points (t,t) and $(1,0)$. In the following proposition, we characterize the elements of $\mathcal{D}_\mathrm{A}$ for which the corresponding biconic function is an aggregation function.

Let us introduce the following notations

$$\begin{aligned} I_1 &= \{(x,y) \in [0,1]^2 \mid y \le x\} \\ I_2 &= \{(x,y) \in [0,1]^2 \mid x \le y\} \\ D &= I_1 \cap I_2\,. \end{aligned}$$

Proposition 3.1. *Let $\delta \in \mathcal{D}_\mathrm{A}$ and $\alpha, \beta \in [0,1]$. The function $A_\delta^{\alpha,\beta}$ defined in (3.1) is an aggregation function if and only if*

(i) *the functions $\lambda_{\delta,\alpha}, \lambda_{\delta,\beta} :]0,1] \to \mathbb{R}$, defined by*

$$\lambda_{\delta,\alpha}(x) = \frac{\delta(x)-\alpha}{x}\,, \quad \lambda_{\delta,\beta}(x) = \frac{\delta(x)-\beta}{x}\,,$$

are increasing;

(ii) *the functions $\mu_{\delta,\alpha}\,, \mu_{\delta,\beta} : [0,1[\to \mathbb{R}$, defined by*

$$\mu_{\delta,\alpha}(x) = \frac{\delta(x)-\alpha}{1-x}\,, \quad \mu_{\delta,\beta}(x) = \frac{\delta(x)-\beta}{1-x}\,,$$

are increasing.

Proof. Suppose conditions (i) and (ii) are satisfied. The function $A_\delta^{\alpha,\beta}$ clearly satisfies $A_\delta^{\alpha,\beta}(0,0) = 0$ and $A_\delta^{\alpha,\beta}(1,1) = 1$. It suffices to prove the increasingness of $A_\delta^{\alpha,\beta}$ in each variable. We prove the increasingness of $A_\delta^{\alpha,\beta}$ in the first variable (the proof of the increasingness in the second variable is similar). Let $(x,y), (x',y) \in [0,1]^2$ such that $x \le x'$.

If $(x,y),(x',y) \in I_1$, the increasingness of $A_\delta^{\alpha,\beta}$ is equivalent to

$$\alpha(x'-y)+(1+y-x')\,\delta\left(\frac{y}{1+y-x'}\right)-\alpha(x-y)-(1+y-x)\,\delta\left(\frac{y}{1+y-x}\right)\geq 0\,,$$

or, equivalently,

$$(1+y-x')\left(\delta\left(\frac{y}{1+y-x'}\right)-\alpha\right)-(1+y-x)\left(\delta\left(\frac{y}{1+y-x}\right)-\alpha\right)\geq 0\,.$$

Denoting $u=\frac{y}{1+y-x}$ and $u'=\frac{y}{1+y-x'}$, the above inequality becomes

$$y(\lambda_{\delta,\alpha}(u')-\lambda_{\delta,\alpha}(u))\geq 0\,.$$

Since $x \leq x'$, it is clear that $u \leq u'$. Hence, the last inequality holds due to the increasingness of the function $\lambda_{\delta,\alpha}$.

If $(x,y),(x',y) \in I_2$, the increasingness of $A_\delta^{\alpha,\beta}$ is equivalent to

$$\beta(y-x')+(1+x'-y)\,\delta\left(\frac{x'}{1+x'-y}\right)-\beta(y-x)-(1+x-y)\,\delta\left(\frac{x}{1+x-y}\right)\geq 0\,,$$

or, equivalently,

$$(1+x'-y)\left(\delta\left(\frac{x'}{1+x'-y}\right)-\beta\right)-(1+x-y)\left(\delta\left(\frac{x}{1+x-y}\right)-\beta\right)\geq 0\,.$$

Denoting $v=\frac{x}{1+x-y}$ and $v'=\frac{x'}{1+x'-y}$, the above inequality becomes

$$(1-y)(\mu_{\delta,\beta}(v')-\mu_{\delta,\beta}(v))\geq 0\,.$$

Since $x \leq x'$, it is clear that $v \leq v'$. Hence, the last inequality holds due to the increasingness of the function $\mu_{\delta,\beta}$.

The remaining case is when $(x,y) \in I_2$ and $(x',y) \in I_1 \setminus D$. The two previous cases then imply that $A_\delta^{\alpha,\beta}(x',y)-A_\delta^{\alpha,\beta}(x,y)=$

$$\left(A_\delta^{\alpha,\beta}(x',y)-A_\delta^{\alpha,\beta}(y,y)\right)+\left(A_\delta^{\alpha,\beta}(y,y)-A_\delta^{\alpha,\beta}(x,y)\right)\geq 0\,.$$

Similarly, one can prove that the increasingness of the functions $\lambda_{\delta,\beta}$ and $\mu_{\delta,\alpha}$ implies that A_δ is increasing in the second variable.

Conversely, suppose that $A_\delta^{\alpha,\beta}$ is an aggregation function. Let $x,x' \in [0,1]$ such that $x \leq x'$, and $y \in [0,1]$ such that $y \leq x$. It then holds that

$$y \leq \frac{x(1+y)-y}{x} \leq \frac{x'(1+y)-y}{x'}\,.$$

The increasingness of A_δ in the first variable implies that

$$A_\delta^{\alpha,\beta}\left(\frac{x'(1+y)-y}{x'},y\right)-A_\delta^{\alpha,\beta}\left(\frac{x(1+y)-y}{x},y\right)\geq 0\,.$$

After some elementary manipulations, the last inequality becomes

$$\alpha\left(\frac{x'-y}{x'}\right)+\frac{y}{x'}\delta(x')-\alpha\left(\frac{x-y}{x}\right)-\frac{y}{x}\delta(x)\geq 0\,,$$

or, equivalently,

$$y(\lambda_{\delta,\alpha}(x')-\lambda_{\delta,\alpha}(x))\geq 0\,.$$

Hence, the function $\lambda_{\delta,\alpha}$ is increasing. In the same way, it follows that the function $\mu_{\delta,\beta}$ is increasing.

Similarly, one can prove that the increasingness of A_δ in the second variable implies the increasingness of the functions $\lambda_{\delta,\beta}$ and $\mu_{\delta,\alpha}$, which completes the proof. $\square$

Inspired by the above proposition, the biconic function $A_\delta^{\alpha,\beta}$ is called *a biconic aggregation function with a given diagonal section.*

Example 3.1. *Consider the diagonal section of $T_\mathbf{M}$. Obviously, conditions (i) and (ii) of Proposition 3.1 are satisfied. The resulting biconic aggregation function is a Choquet integral* [14, 27], *i.e.*

$$A_{\delta_{T_\mathbf{M}}}^{\alpha,\beta}(x,y)=\begin{cases}\alpha x+(1-\alpha)y & ,\ \textit{if } y\leq x\,,\\[2ex] (1-\beta)x+\beta y & ,\ \textit{otherwise.}\end{cases}$$

Taking $\beta=1-\alpha$, the resulting biconic aggregation function is a weighted arithmetic mean, i.e.

$$A_{\delta_{T_\mathbf{M}}}^{\alpha,1-\alpha}(x,y)=\alpha x+(1-\alpha)y\,.$$

Lemma 3.1. *Let $A_\delta^{\alpha,\beta}$ be a biconic aggregation function. Then the inequality*

$$\max(\alpha x,\beta x)\leq\delta(x)\leq\min(\alpha+(1-\alpha)x,\beta+(1-\beta)x)\,,\tag{3.2}$$

holds for any $x\in[0,1]$.

Proof. The proof is immediate due to the increasingness of the functions $\lambda_{\delta,\alpha}$, $\lambda_{\delta,\beta}$, $\mu_{\delta,\alpha}$ and $\mu_{\delta,\beta}$. $\square$

Now we identify the functions in $\mathcal{D}_\mathrm{A}$ which characterize the extreme biconic aggregation functions with fixed α and β. Let $\alpha,\beta\in[0,1]$ and consider the functions $\underline{\delta}^{\alpha,\beta}, \overline{\delta}^{\alpha,\beta} : [0,1] \to [0,1]$ defined by

$$\underline{\delta}^{\alpha,\beta}(x) = \begin{cases} \max(\alpha x, \beta x) & , \text{ if } x < 1\,, \\ 1 & , \text{ if } x = 1\,, \end{cases}$$

$$\overline{\delta}^{\alpha,\beta}(x) = \begin{cases} \min(\alpha + (1-\alpha)x, \beta + (1-\beta)x) & , \text{ if } x > 0\,, \\ 0 & , \text{ if } x = 0\,. \end{cases}$$

Obviously, $\underline{\delta}^{\alpha,\beta}, \overline{\delta}^{\alpha,\beta} \in \mathcal{D}_{\mathrm{A}}$ and the conditions of Proposition 3.1 are satisfied. Note also that for any two biconic aggregation functions $A^{\alpha,\beta}_{\delta_1}$ and $A^{\alpha,\beta}_{\delta_2}$, it holds that $A^{\alpha,\beta}_{\delta_1} \leq A^{\alpha,\beta}_{\delta_2}$ if and only if $\delta_1 \leq \delta_2$. The following proposition is then obvious.

Proposition 3.2. *Let $A^{\alpha,\beta}_{\delta}$ be a biconic aggregation function. Then it holds that*

$$A^{\alpha,\beta}_{\underline{\delta}^{\alpha,\beta}} \leq A^{\alpha,\beta}_{\delta} \leq A^{\alpha,\beta}_{\overline{\delta}^{\alpha,\beta}}\,.$$

Example 3.2. *The functions $\underline{\delta}^{0,0}$ and $\overline{\delta}^{0,0}$ are given by*

$$\underline{\delta}^{0,0}(x) = \begin{cases} 0 & , \text{ if } x < 1\,, \\ 1 & , \text{ if } x = 1\,, \end{cases}$$

and $\overline{\delta}^{0,0} = \delta_{T_{\mathbf{M}}}$. The corresponding biconic aggregation functions are respectively the smallest t-norm, i.e. $A^{0,0}_{\underline{\delta}^{0,0}} = T_{\mathbf{D}}$, and the greatest t-norm, i.e. $A^{0,0}_{\overline{\delta}^{0,0}} = T_{\mathbf{M}}$.

Example 3.3. *The functions $\underline{\delta}^{1,1}$ and $\overline{\delta}^{1,1}$ are given by $\underline{\delta}^{1,1} = \delta_{T_{\mathbf{M}}}$ and*

$$\overline{\delta}^{1,1}(x) = \begin{cases} 1 & , \text{ if } x > 0\,, \\ 0 & , \text{ if } x = 0\,. \end{cases}$$

The corresponding biconic aggregation functions are respectively the smallest aggregation function with neutral element 0, *i.e.* $A^{1,1}_{\underline{\delta}^{1,1}}(x,y) = \max(x,y)$, *and the greatest aggregation function with neutral element* 0, *i.e.* $A^{1,1}_{\overline{\delta}^{1,1}}(x,y) = \max(x,y)$ *whenever* $\min(x,y) = 0$, *and* $A^{1,1}_{\overline{\delta}^{1,1}}(x,y) = 1$ *elsewhere.*

Example 3.4. *The functions $\underline{\delta}^{1,0}$, $\overline{\delta}^{1,0}$, $\underline{\delta}^{0,1}$ and $\overline{\delta}^{0,1}$ all coincide with $\delta_{T_{\mathbf{M}}}$. The corresponding biconic aggregation functions coincide with the projection to the first and second coordinate* [6, 9], i.e. $A^{1,0}_{\underline{\delta}^{1,0}}(x,y) = A^{1,0}_{\overline{\delta}^{1,0}}(x,y) = x$ and $A^{0,1}_{\underline{\delta}^{0,1}}(x,y) = A^{0,1}_{\overline{\delta}^{0,1}}(x,y) = y$.

Remark 3.1. *Evidently, a biconic aggregation function $A_\delta^{\alpha,\beta}$ is continuous if and only if δ is continuous. The functions $\underline{\delta}^{\alpha,\beta}$ and $\overline{\delta}^{\alpha,\beta}$ need not be continuous in general. In fact, the only case in which they are both continuous is when*

$$\max(\alpha,\beta) = 1 \text{ and } \min(\alpha,\beta) = 0\,.$$

However, as Example 3.4 shows, it then holds that

$$\underline{\delta}^{\alpha,\beta} = \delta = \overline{\delta}^{\alpha,\beta} = \delta_{T_{\mathbf{M}}}\,,$$

and $A_\delta^{\alpha,\beta}$ coincides with one of the projections.

Proposition 3.3. *Let $\delta \in \mathcal{D}_{\mathrm{A}}$. The function $A_\delta^{\alpha,\beta}$ defined in (3.1)*

(i) *is commutative if and only if $\alpha = \beta$;*

(ii) *has 0 as absorbing element if and only if $\alpha = \beta = 0$;*

(iii) *has 1 as neutral element if and only if $\alpha = \beta = 0$.*

Proof. The proof is trivial. □

From here on, we will only consider biconic functions with a given diagonal section that have 1 as neutral element, i.e. $\alpha = \beta = 0$. We then abbreviate $A_\delta^{0,0}$ as A_δ. In this case, A_δ is symmetric and is given by

$$A_\delta(x,y) = \begin{cases} (1+y-x)\,\delta\left(\dfrac{y}{1+y-x}\right) & \text{, if } y \le x\,, \\[2ex] (1+x-y)\,\delta\left(\dfrac{x}{1+x-y}\right) & \text{, otherwise.} \end{cases} \tag{3.3}$$

Suppose that the diagonal section of a biconic aggregation function A_δ is linear on the interval $[x_1, x_2]$. From the definition of A_δ, it follows that A_δ is linear on the triangle $\Delta_1 = \Delta_{\{(x_1,x_1),(x_2,x_2),(1,0)\}}$ as well as on the triangle $\Delta_2 := \Delta_{\{(x_1,x_1),(x_2,x_2),(0,1)\}}$. This situation is depicted in Figure 3.1.

For any $(x,y) \in \Delta_1$, it holds that

$$A_\delta(x,y) = ax + by + c\,. \tag{3.4}$$

Furthermore,

$$\begin{aligned} ax_1 + bx_1 + c &= \delta(x_1) \\ ax_2 + bx_2 + c &= \delta(x_2) \\ a + c &= 0\,. \end{aligned}$$

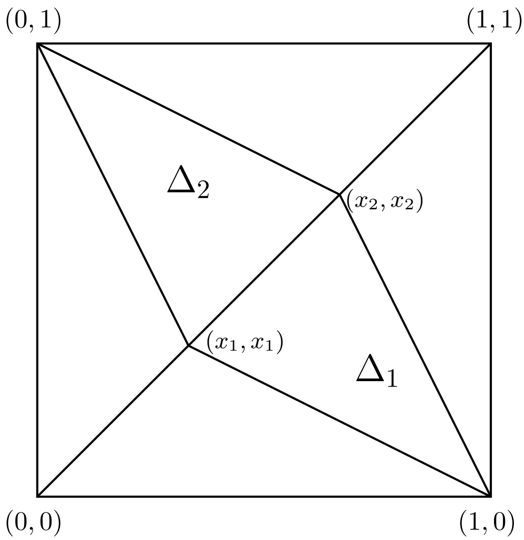


Figure 3.1: An illustration for the triangles Δ_1 and Δ_2.

Solving this system of linear equations and using the symmetry of A_δ, we obtain

$$A_\delta(x,y) = \begin{cases} \dfrac{rx+sy-r}{t} & \text{, if } (x,y) \in \Delta_1\,, \\[2ex] \dfrac{sx+ry-r}{t} & \text{, if } (x,y) \in \Delta_2\,, \end{cases} \tag{3.5}$$

where

$$\begin{aligned} r &= x_1\delta(x_2) - x_2\delta(x_1) \\ s &= (1-x_1)\delta(x_2) - (1-x_2)\delta(x_1) \\ t &= x_2 - x_1\,. \end{aligned}$$

3.3. Biconic semi-copulas with a given diagonal section

Here, we characterize the elements of $\mathcal{D}_{\mathrm{S}}$ for which the corresponding biconic function is a semi-copula.

Proposition 3.4. *Let $\delta \in \mathcal{D}_{\mathrm{S}}$. The function A_δ defined in (3.3) is a semi-copula if and only if the function $\lambda_\delta :]0,1] \to [0,\infty[$, defined by $\lambda_\delta(x) = \frac{\delta(x)}{x}$, is increasing.*

Proof. One easily verifies that for $\delta \in \mathcal{D}_{\mathrm{S}}$, the function $\xi_\delta : [0,1[\to [0,\infty[$ defined by $\xi_\delta(x) = \frac{\delta(x)}{1-x}$ is increasing. Due to Proposition 3.1, the proof is then immediate. $\square$

Example 3.5. *Consider the diagonal functions $\delta_{T_\mathbf{M}}$ and $\delta_{T_\mathbf{L}}$. Clearly, the functions $\lambda_{\delta_{T_\mathbf{M}}}$ and $\lambda_{\delta_{T_\mathbf{L}}}$, defined in Proposition 3.4, are increasing. The corresponding biconic semi-copulas are respectively $T_\mathbf{M}$ and $T_\mathbf{L}$.*

Example 3.6. *Consider the diagonal function $\delta_\theta(x) = x^{1+\theta}$ with $\theta \in [0,1]$. Clearly, the function λ_{δ_θ}, defined in Proposition 3.4, is increasing for any $\theta \in [0,1]$. The corresponding family of biconic semi-copulas is given by*

$$C_\theta(x,y) = \begin{cases} \dfrac{y^{1+\theta}}{(1+y-x)^\theta} & \text{, if } y \le x\,, \\[2ex] \dfrac{x^{1+\theta}}{(1+x-y)^\theta} & \text{, otherwise.} \end{cases}$$

Proposition 3.5. *Let A_δ be a biconic semi-copula and suppose that $\delta(x_0) = x_0$ for some $x_0 \in]0,1[$. Then it holds that $\delta(x) = x$ for any $x \in [x_0,1]$.*

Proof. Suppose that A_δ is a biconic semi-copula and suppose further that $\delta(x_0) = x_0$ for some $x_0 \in]0,1[$. The function λ_δ, defined in Proposition 3.4, is increasing. Therefore, $\lambda_\delta(x) \ge \lambda_\delta(x_0) = 1$ for any $x \in [x_0,1]$. Using the fact that $\delta(x) \le x$ for any $x \in [x_0,1]$, it must hold also that $\lambda_\delta(x) \le 1$. Hence, $\lambda_\delta(x) = 1$ for any $x \in [x_0,1]$. Consequently, $\delta(x) = x$ for any $x \in [x_0,1]$. $\square$

3.4. Biconic quasi-copulas with a given diagonal section

Here, we characterize the diagonal functions for which the corresponding biconic function is a quasi-copula.

Lemma 3.2. *Let $\delta \in \mathcal{D}$. Then it holds that*

(i) *the function $\nu_\delta :]0,1] \to [2,\infty[$, defined by $\nu_\delta(x) = \frac{1+\delta(x)}{x}$, is decreasing;*

(ii) *the function $\phi_\delta : [0,1/2[\cup]1/2,1] \to \mathbb{R}$, defined by $\phi_\delta(x) = \frac{\delta(x)}{1-2x}$, is increasing on the interval $[0,1/2[$ and on the interval $]1/2,1]$.*

Proof. (i) Consider $\delta \in \mathcal{D}$ and consider arbitrary $x, x' \in]0,1]$ such that $x < x'$.

Since $\delta(x') - \delta(x) \le 2(x'-x)$ and $\delta(x) \ge 2x-1$, it holds that

$$\frac{\delta(x') - \delta(x)}{x'-x} \le 2 \le \frac{1+\delta(x)}{x}\,.$$

The latter inequality implies

$$(\delta(x') - \delta(x))x \leq (1 + \delta(x))(x' - x),$$

whence

$$(1 + \delta(x'))x - (1 + \delta(x))x' \leq 0\,,$$

or, equivalently,

$$xx'(\nu_\delta(x') - \nu_\delta(x)) \leq 0\,.$$

Hence, the decreasingness of ν_δ follows.

(ii) Consider now arbitrary $x, x' \in [0, 1/2[$ such that $x < x'$.

Since δ is increasing and $1 - 2x > 0$, the following inequality holds

$$(\delta(x') - \delta(x))(1 - 2x) + 2(x' - x)\delta(x) \geq 0\,.$$

Simple processing yields

$$\delta(x')(1 - 2x) - \delta(x)(1 - 2x') \geq 0\,,$$

or, equivalently,

$$(1 - 2x)(1 - 2x')(\phi_\delta(x') - \phi_\delta(x)) \geq 0\,.$$

Hence, the increasingness of ϕ_δ on the interval $[0, 1/2[$ follows. Similarly, one can prove the increasingness of ϕ_δ on the interval $]1/2, 1]$.

□

Proposition 3.6. *Let $\delta \in \mathcal{D}$. Then the function $A_\delta : [0,1]^2 \to [0,1]$ defined in (3.3) is a quasi-copula if and only if*

(i) *the function λ_δ, defined in Proposition 3.4, is increasing;*

(ii) *the function $\mu_\delta : [0, 1[\to [0, 1]$, defined by $\mu_\delta(x) = \frac{x - \delta(x)}{1 - x}$, is increasing.*

Proof. We use the same notations as in Proposition 3.1. Suppose that conditions (i) and (ii) are satisfied. Due to Proposition 3.4, the function A_δ is increasing. Therefore, to prove that A_δ is a quasi-copula, we need to show that it is 1-Lipschitz continuous. Recall that the 1-Lipschitz continuity is equivalent to the 1-Lipschitz continuity in each variable. Since A_δ is symmetric, it is sufficient to show that A_δ is 1-Lipschitz continuous in the first variable. Let $(x, y), (x', y) \in [0, 1]^2$ such that $x \leq x'$. We need to show that

$$A_\delta(x', y) - A_\delta(x, y) \leq x' - x\,. \tag{3.6}$$

We distinguish several cases. If $(x,y),(x',y) \in I_1$, then inequality (3.6) is equivalent to

$$y\,(\nu_\delta(u') - \nu_\delta(u)) \leq 0\,.$$

Due to Lemma 3.2(i) the last inequality always holds.

If $(x,y),(x',y) \in I_2$, then inequality (3.6) is equivalent to

$$(1-y)(\mu_\delta(v') - \mu_\delta(v)) \geq 0\,,$$

which holds due to condition (ii).

The remaining case is when $(x,y) \in I_2$ and $(x',y) \in I_1 \setminus D$. The two previous cases then imply that

$$A_\delta(x',y) - A_\delta(x,y) = (A_\delta(x',y) - A_\delta(y,y)) + (A_\delta(y,y) - A_\delta(x,y)) \leq x' - x\,.$$

Consequently, A_δ is a biconic quasi-copula.

Conversely, suppose that A_δ is a quasi-copula. Proposition 3.4 implies condition (i). Let $x,x' \in [0,1[$ such that $x \leq x'$, and $y \in [0,1]$ such that $y \geq x'$. Let us consider the following notations

$$b = \frac{(1-y)x}{1-x}\,, \quad b' = \frac{(1-y)x'}{1-x'}\,.$$

Since $x \leq x' \leq y$, it holds that $0 \leq b \leq b' \leq y$. The 1-Lipschitz continuity of A_δ in the first variable implies

$$A_\delta(b',y) - A_\delta(b,y) \leq b' - b\,,$$

or, equivalently,

$$(1-y)(\mu_\delta(x') - \mu_\delta(x)) \geq 0\,.$$

Hence, condition (ii) follows, which completes the proof. □

Proposition 3.7. *Let A_δ be a biconic quasi-copula. Then it holds that*

(i) *if $\delta(x_0) = x_0$ for some $x_0 \in]0,1[$, then $A_\delta = T_{\mathbf{M}}$;*

(ii) *if $\delta(x_0) = 2x_0 - 1$ for some $x_0 \in [1/2,1[$, then $\delta(x) = 2x-1$ for any $x \in [x_0,1]$.*

Proof. Suppose that A_δ be a biconic quasi-copula and suppose further that $\delta(x_0) = x_0$ for some $x_0 \in]0,1[$. Due to Proposition 3.5, it holds that $\delta(x) = x$ for any $x \in [x_0,1]$. Since A_δ is a biconic quasi-copula, it holds that the function μ_δ defined in Proposition 3.6 is increasing. Therefore, $\mu_\delta(x) \leq \mu_\delta(x_0) = 0$ for any $x \in [0,x_0]$. Hence, $\delta(x) \geq x$ for any $x \in [0,x_0]$.

Using the fact that $\delta(x) \leq x$ for any $x \in [0,1]$, it must hold that $\delta(x) = x$ for any $x \in [0, x_0]$. Based on the above discussion, it holds that $\delta(x) = x$ for any $x \in [0,1]$. Since $T_{\mathbf{M}}$ is the only quasi-copula with $\delta_{T_{\mathbf{M}}}$ as diagonal section, it holds that $A_\delta = T_{\mathbf{M}}$.

Assertion (ii) can be proved similarly using the increasingness of the function μ_δ on the interval $[x_0, 1]$. □

Example 3.7. *Consider the diagonal functions in Example 3.6. Clearly, the functions λ_δ and μ_δ, defined in Propositions 3.4 and 3.6, are increasing. The corresponding family of biconic semi-copulas is a family of biconic quasi-copulas.*

Example 3.8. *Consider the diagonal function δ defined by*

$$\delta(x) = \begin{cases} 0 & , \text{ if } x \leq \frac{1}{6}\,, \\ 2x - \dfrac{1}{3} & , \text{ if } \frac{1}{6} \leq x \leq \frac{1}{4}\,, \\ \dfrac{2}{3}x & , \text{ if } \frac{1}{4} \leq x \leq \frac{3}{4}\,, \\ 2x - 1 & , \text{ otherwise.} \end{cases}$$

Clearly, the function λ_δ, defined in Proposition 3.4, is increasing. Note also that the function μ_δ, defined in Proposition 3.6, is not increasing. Hence, the corresponding biconic function A_δ is a proper biconic semi-copula and is given by

$$A_\delta(x,y) = \begin{cases} 0 & , \text{ if } y \leq x \leq 1-5y \text{ or } x \leq y \leq 1-5x\,, \\ \dfrac{1}{3}(x+5y-1) & , \text{ if } \max(y, 1-5y) \leq x \leq 1-3y\,, \\ \dfrac{2}{3}y & , \text{ if } \max(y, 1-3y) \leq x \leq \frac{3-y}{3}\,, \\ \dfrac{1}{3}(y+5x-1) & , \text{ if } \max(x, 1-5x) \leq y \leq 1-3x\,, \\ \dfrac{2}{3}x & , \text{ if } \max(x, 1-3x) \leq y \leq \frac{3-x}{3}\,, \\ x+y-1 & , \text{ otherwise.} \end{cases}$$

The diagonal function and the corresponding biconic semi-copula are depicted in Figure 3.2. Consequently, the class of biconic quasi-copulas with a given diagonal section is a proper subclass of the class of biconic semi-copulas with a given diagonal section.

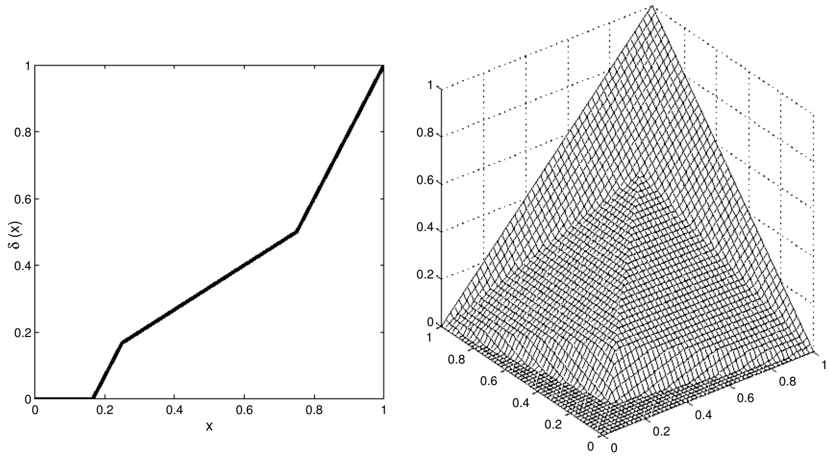


Figure 3.2: The diagonal function and the corresponding biconic semi-copula of Example 3.8.

3.5. Biconic copulas with a given diagonal section

Here, we characterize the diagonal functions for which the corresponding biconic function is a copula. Next, we characterize the piecewise linear diagonal functions for which the corresponding biconic function is a copula. To this end, we need the following lemma.

Lemma 3.3. *Let A_δ be a biconic function such that δ is linear on the interval $[x_1, x_2]$ as well as on the interval $[x_2, x_3]$. Let $R = [x, x'] \times [y, y']$ be a rectangle located in the triangle $\Delta_{\{(x_1,x_1),(x_3,x_3),(1,0)\}}$ such that its opposite diagonal is a subset of the segment $\langle (x_2, x_2), (1, 0)\rangle$. Then it holds that $V_{A_\delta}(R) \geq 0$ if and only if δ is convex on the interval $[x_1, x_3]$.*

Proof. Applying Eq. (3.5) to both triangles $\Delta_1 := \Delta_{\{(x_1,x_1),(x_2,x_2),(1,0)\}}$ and $\Delta_2 := \Delta_{\{(x_2,x_2),(x_3,x_3),(1,0)\}}$ (as depicted in Figure 3.3(a)), it follows that

$$V_{A_\delta}(R) = (x' - x)\left(\frac{r'}{t'} - \frac{r}{t}\right) ,$$

where r and t are as in Eq. (3.5) and

$$\begin{aligned} r' &= x_2\delta(x_3) - x_3\delta(x_2) \\ t' &= x_3 - x_2 \, . \end{aligned}$$

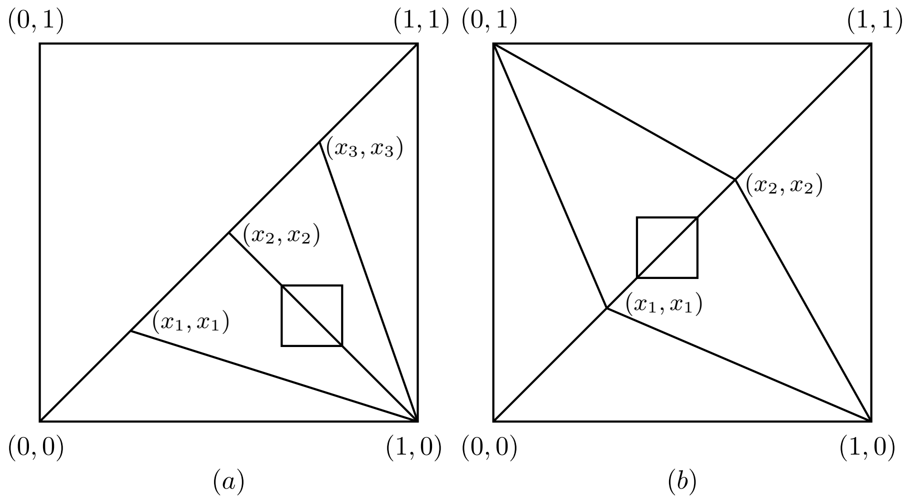


Figure 3.3: Illustration for the proofs of Lemma 3.3 and Proposition 3.8

The nonnegativity of $V_{A_\delta}(R)$ is equivalent to

$$\frac{r'}{t'} - \frac{r}{t} \geq 0 \,.$$

Substituting the expressions for r, r', t and t', the latter inequality can be written as

$$x_2 \left(\frac{\delta(x_3) - \delta(x_2)}{x_3 - x_2} - \frac{\delta(x_2) - \delta(x_1)}{x_2 - x_1} \right) \geq 0 \,,$$

or, equivalently,

$$\frac{\delta(x_3) - \delta(x_2)}{x_3 - x_2} - \frac{\delta(x_2) - \delta(x_1)}{x_2 - x_1} \geq 0 \,, \tag{3.7}$$

i.e. δ is convex on the interval $[x_1, x_3]$. □

Proposition 3.8. *Let δ be a piecewise linear diagonal function. Then the function $A_\delta : [0,1]^2 \to [0,1]$ defined in (3.3) is a copula if and only if δ is convex.*

Proof. First suppose that δ is convex. To prove that A_δ is a copula, we need to show its 2-increasingness. Since δ is piecewise linear, the surface of A_δ consists of triangles of the type $\Delta_{\{(x,x,\delta(x)),(y,y,\delta(y)),(0,1,0)\}}$ and of the type $\Delta_{\{(x,x,\delta(x)),(y,y,\delta(y)),(1,0,0)\}}$. Note that any rectangle in the unit square can obviously be decomposed into a number of rectangles that are either located entirely in one of triangles $\Delta_{\{(x,x),(y,y),(0,1)\}}$ or $\Delta_{\{(x,x),(y,y),(1,0)\}}$, have their diagonal along the diagonal of the unit square or have their opposite diagonal along one of the edges of these triangles. Due to the additivity of volumes, it suffices to consider a restricted number of cases. Consider a rectangle $R := [x, x'] \times [y, y'] \subseteq [0,1]^2$.

(i) If R is located in one of the above triangles, then its A_δ-volume is 0 since A_δ is linear on the considered triangle.

(ii) If the opposite diagonal of R is along one of the edges of the above triangles, then we can consider three points $\mathbf{b}_1 := (x_1, x_1)$, $\mathbf{b}_2 := (x_2, x_2)$ and $\mathbf{b}_3 := (x_3, x_3)$ such that δ is linear on the interval $[x_1, x_2]$ as well as on the interval $[x_2, x_3]$ (see Figure 3.3(a)).

Suppose that the opposite diagonal of R is a subset of the segment $\langle \mathbf{b}_2, (1,0)\rangle$ (the case when the opposite diagonal of R is a subset of the segment $\langle \mathbf{b}_2, (0,1)\rangle$ is identical due to the symmetry of A_δ). Due to Lemma 3.3, it follows that $V_{A_\delta}(R) \geq 0$.

(iii) If the diagonal of R is along the diagonal of the unit square, then we can consider two points $\mathbf{b}_1 := (x_1, x_1)$ and $\mathbf{b}_2 := (x_2, x_2)$ such that δ is linear on the interval $[x_1, x_2]$. Suppose that the diagonal of R is a subset of the segment $\langle \mathbf{b}_1, \mathbf{b}_2\rangle$ (see Figure 3.3(b)). Applying Eq. (3.5), it follows that

$$V_{A_\delta}(R) = (x' - x)\left(\frac{s-r}{t}\right),$$

where s is as in Eq. (3.5). We distinguish two subcases:

(a) If $x_2 \leq 1/2$ or $x_1 \geq 1/2$, then the nonnegativity of $V_{A_\delta}(R)$ is equivalent to $s - r \geq 0$. Substituting the expressions for r and s, the latter inequality becomes

$$(1-2x_1)(1-2x_2)(\phi_\delta(x_2) - \phi_\delta(x_1)) \geq 0\,. \tag{3.8}$$

Due to Lemma 3.2(ii), ϕ_δ is increasing on the interval $[0, 1/2[$ and on the interval $]1/2, 1]$, whence the latter inequality follows.

(b) If $x_1 \leq 1/2 \leq x_2$, then with $\delta(1/2) = \frac{s-r}{2t}$, it follows that

$$V_{A_\delta}(R) = (x' - x)\left(\frac{s-r}{t}\right) = 2(x'-x)\delta(1/2) \geq 0\,. \tag{3.9}$$

Conversely, suppose that A_δ is a copula. Lemma 3.3 implies inequality (3.7) for any two consecutive segments $\langle (x_1, \delta(x_1)), (x_2, \delta(x_2))\rangle$ and $\langle (x_2, \delta(x_2)), (x_3, \delta(x_3))\rangle$ of the graph of δ with $x_1 < x_2 < x_3$. Consequently, δ is convex. $\square$

Lemma 3.4. *Let C_δ be a biconic copula and $m_1, m_2 \in\,]-\infty, 0[$ such that $m_1 > m_2$. Consider three points $\mathbf{b}_1 := (x_1, x_1)$, $\mathbf{b}_2 := (x_2, x_2)$ and $\mathbf{b}_3 := (x_3, x_3)$ such that $0 \leq x_1 < x_2 < x_3 \leq 1$ and the segments $\langle \mathbf{b}_1, (1,0)\rangle$, $\langle \mathbf{b}_2, (1,0)\rangle$ and $\langle \mathbf{b}_3, (1,0)\rangle$ have slope m_1, $-\sqrt{m_1 m_2}$ and m_2, respectively. Then it holds that*

(i) *there exists a rectangle $[x, x'] \times [y, y']$ such that the segment connecting the points (x, y') and (x', y) is a subset of the segment $\langle \mathbf{b}_2, (1,0)\rangle$ and the points*

(x, y) and (x', y') are located on the segments $\langle \mathbf{b}_1, (1,0)\rangle$ and $\langle \mathbf{b}_3, (1,0)\rangle$ respectively.

(ii) *the point $(x_2, \delta(x_2))$ lies below or on the segment $\langle (x_1, \delta(x_1)), (x_3, \delta(x_3))\rangle$.*

Proof. A simple geometric argumentation shows that points $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ with the desired properties always exist. Observation (i) follows from the fact that for such points, we can always find a rectangle $[x, x'] \times [y, y']$ of which the opposite diagonal is a subset of the segment with slope $m = -\sqrt{m_1 m_2}$ and the points (x, y) and (x', y') are located on the segments with slopes m_1 and m_2, respectively.

To prove assertion (ii), we consider the function $h : [x_1, x_3] \to [0, 1]$ that is linear on the interval $[x_1, x_2]$ as well as on the interval $[x_2, x_3]$ and coincides with δ in the points x_1, x_2 and x_3. Consider the rectangle $[x, x'] \times [y, y']$ from the first part of the proof. Since C_δ is a copula, it follows that

$$V_{A_{\delta^*}}([x, x'] \times [y, y']) = V_{C_\delta}([x, x'] \times [y, y']) \geq 0\,,$$

where A_{δ^*} is a biconic function such that δ^* coincides with h on the interval $[x_1, x_3]$. Due to Lemma 3.3, it holds that the function h is convex, or equivalently, the point $(x_2, \delta(x_2))$ lies below or on the segment $\langle (x_1, \delta(x_1)), (x_3, \delta(x_3))\rangle$, which completes the proof. ☐

Theorem 3.1. *Let $\delta \in \mathcal{D}$. Then the function $A_\delta : [0, 1]^2 \to [0, 1]$ defined in (3.3) is a copula if and only if δ is convex.*

Proof. Suppose that δ is convex. To prove that A_δ is a copula, we need to show the 2-increasingness. Due to the additivity of volumes, it suffices to consider a restricted number of cases. Consider a rectangle $R := [x, x'] \times [y, y'] \subseteq [0, 1]^2$.

(i) If $R \subseteq I_1$ (the case when $R \subseteq I_2$ is identical due to the symmetry of A_δ), then let $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ and $\mathbf{b}_4$ be four (possibly coinciding) points on the diagonal of the unit square such that the points (x, y), (x, y'), (x', y) and (x', y') are respectively located on the segments $\langle (1, 0), \mathbf{b}_1\rangle$, $\langle (1, 0), \mathbf{b}_2\rangle$, $\langle (1, 0), \mathbf{b}_3\rangle$ and $\langle (1, 0), \mathbf{b}_4\rangle$ (see Figure 3.4).

The points $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ and $\mathbf{b}_4$, together with $(0, 0)$ and $(1, 1)$, determine a piecewise linear convex diagonal function δ_1 such that $\delta_1(x_i) = \delta(x_i)$ for any $i \in \{1, 2, 3, 4\}$. Due to Proposition 3.8, the biconic function A_{δ_1} is a biconic copula. Therefore,

$$V_{A_\delta}(R) = V_{A_{\delta_1}}(R) \geq 0\,.$$

(ii) If $R = [x, y] \times [x, y]$ with $y \leq 1/2$ (the case when $x \geq 1/2$ can be proved similarly), then it holds that

$$V_{A_\delta}(R) = \delta(x) + \delta(y) - 2A_\delta(x, y)\,.$$

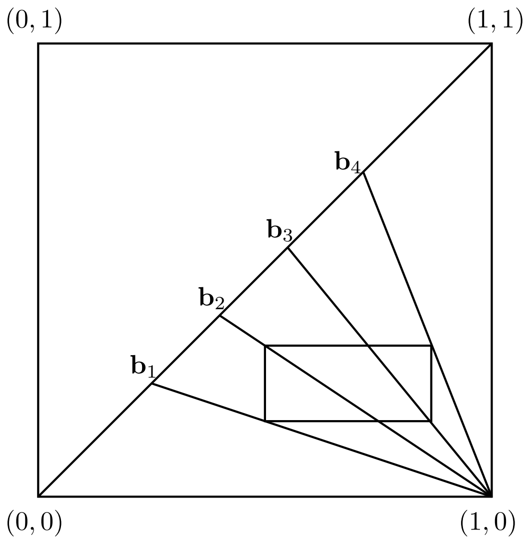


Figure 3.4: Illustration for the proof of Theorem 3.1.

Substituting the expression of $A_\delta(x, y)$ and denoting $v = \frac{x}{1+x-y}$, the latter equation becomes

$$V_{A_\delta}(R) = \delta(x) + \delta(y) - 2x\frac{\delta(v)}{v} = 2\left(\frac{\delta(x)+\delta(y)}{2} - x\frac{\delta(v)}{v}\right) .$$

Since $x \leq y \leq 1/2$, it holds that $x \leq v \leq \frac{x+y}{2}$, whence

$$V_{A_\delta}(R) \geq 2\left(\frac{\delta(x)+\delta(y)}{2} - \delta\left(\frac{x+y}{2}\right)\right) .$$

Due to the convexity of δ, the right-hand side of the latter inequality is nonnegative and therefore, $V_{A_\delta}(R) \geq 0$.

Consequently, the 2-increasingness of A_δ holds, and A_δ is a copula.

Conversely, suppose that A_δ is a copula and suppose further that δ is not convex, i.e. there exist $x < y < z$ such that the point $(y, \delta(y))$ is above the segment connecting the points $(x, \delta(x))$ and $(z, \delta(z))$. Since δ is continuous, there exists $\epsilon > 0$ such that for any $x' \in [y - \epsilon, y + \epsilon]$ the point $(x', \delta(x'))$ is above the segment connecting the points $(x, \delta(x))$ and $(z, \delta(z))$, which contradicts Lemma 3.4. Thus, δ must be convex. □

Since for a biconic copula C it holds that its diagonal section δ_C is convex, it either holds that $\delta_C = \delta_{T_\mathbf{M}}$ or $\delta_C(x) < x$ for any $x \in]0, 1[$. Hence, there do not exist proper ordinal sum biconic copulas with a given diagonal section.

Example 3.9. *Consider the diagonal functions in Example 3.6. Clearly, δ is convex for any $\theta \in [0,1]$. The corresponding family of biconic semi-copulas is a family of biconic copulas.*

Example 3.10. *Consider the diagonal function of a Ali–Mikhail–Haq copula, i.e. $\delta_\theta(x) = \frac{x^2}{1-\theta(1-x)^2}$ for any $x \in [0,1]$, with $\theta \in [-1,1]$. Clearly, δ_θ is convex for any $\theta \in [-1,1]$. The corresponding family of biconic copulas is given by*

$$C_\theta(x,y) = \begin{cases} \dfrac{y^2(1+y-x)}{(1+y-x)^2-\theta(1-x)^2} & , \textit{ if } y \le x\,, \\[2ex] \dfrac{x^2(1+x-y)}{(1+x-y)^2-\theta(1-y)^2} & , \textit{ otherwise.} \end{cases}$$

Example 3.11. *Consider the diagonal function δ given by*

$$\delta(x) = \begin{cases} 0 & , \textit{ if } x \le \frac{1}{4}\,, \\[1ex] \dfrac{1}{3}(4x-1) & , \textit{ if } \frac{1}{4} \le x \le \frac{2}{5}\,, \\[1ex] \dfrac{1}{2}x & , \textit{ if } \frac{2}{5} \le x \le \frac{2}{3}\,, \\[1ex] 2x-1 & , \textit{ otherwise.} \end{cases}$$

Clearly, the functions μ_δ and ξ_δ, defined in Proposition 3.6, are increasing. Note also that δ is not convex. Hence, A_δ is a proper biconic quasi-copula and is given by

$$A_\delta(x,y) = \begin{cases} 0 & , \textit{ if } y \le x \le 1-3y \textit{ or } x \le y \le 1-3x\,, \\[1ex] \dfrac{1}{3}(x+3y-1) & , \textit{ if } \max(y, 1-3y) \le x \le \frac{2-3y}{2}\,, \\[1ex] \dfrac{1}{2}y & , \textit{ if } \max(y, \frac{2-3y}{2}) \le x \le \frac{2-y}{2}\,, \\[1ex] \dfrac{1}{3}(y+3x-1) & , \textit{ if } \max(x, 1-3x) \le y \le \frac{2-3x}{2}\,, \\[1ex] \dfrac{1}{2}x & , \textit{ if } \max(x, \frac{2-3x}{2}) \le y \le \frac{2-x}{2}\,, \\[1ex] x+y-1 & , \textit{ otherwise.} \end{cases}$$

The diagonal function and the corresponding biconic quasi-copula are depicted in Figure 3.5. Consequently, the class of biconic copulas with a given diagonal section is a proper subclass of the class of biconic quasi-copulas with a given diagonal section.

In the following lemma, we study the opposite symmetry of a biconic copula with

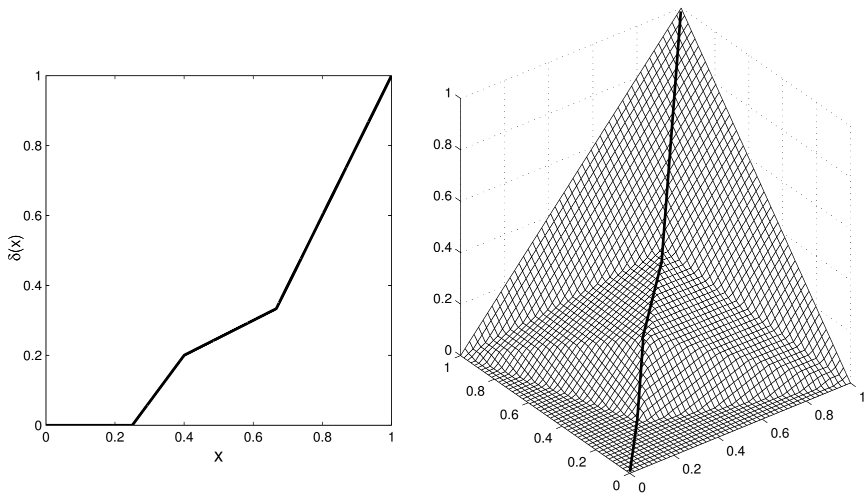


Figure 3.5: The diagonal function and the corresponding biconic quasi-copula of Example 3.11.

a given diagonal section.

Lemma 3.5. *A biconic copula C_δ with a given diagonal section δ is opposite symmetric (also called radially symmetric) if and only if the function $f(x) = x - \delta(x)$ is symmetric with respect to the point $(1/2, 1/2)$, i.e. $\delta(x) - \delta(1-x) = 2x - 1$ for any $x \in [0, 1/2]$.*

Proof. Let C_δ be a biconic copula. Let $(x, y) \in I_1$ (the case $(x, y) \in I_2$ can be proved similarly). Using the notation $z = \frac{y}{1+y-x}$, Eq. (1.4) is equivalent to

$$\delta(z) - \delta(1-z) = 2z - 1\,. \tag{3.10}$$

i.e. the function $f(x) = x - \delta(x)$ is symmetric with respect to the point $(1/2, 1/2)$. □

Now, we lay bare the associativity of biconic copulas.

Proposition 3.9. *$T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ are the only associative biconic copulas (1-Lipschitz t-norms) with a given diagonal section.*

Proof. Let C_δ be a biconic copula. Then its diagonal section δ is a convex diagonal function. Hence, the right (resp. left) derivative of δ exists everywhere on the interval $[0, 1[$ (or $]0, 1]$) [93, 95]. We further assume that C_δ is associative. Then for any $0 \le \epsilon \le \delta(x) \le x \le 1$, it holds that

$$C_\delta(\epsilon, C_\delta(x, x)) = C_\delta(C_\delta(\epsilon, x), x)\,.$$

Let us introduce the notations

$$u = \frac{\epsilon}{1+\epsilon-\delta(x)}, \quad v = \frac{\epsilon}{1+\epsilon-x}.$$

It then holds that

$$C_\delta(\epsilon, C_\delta(x,x)) = (1+\epsilon-\delta(x))\delta(u) \tag{3.11}$$

and

$$C_\delta(C_\delta(\epsilon,x),x) = (1+(1+\epsilon-x)\delta(v)-x)\delta\left(\frac{(1+\epsilon-x)\delta(v)}{1+(1+\epsilon-x)\delta(v)-x}\right). \tag{3.12}$$

Expanding the right-hand side of Eq. (3.11) in powers of ϵ around 0 (by taking the partial derivative with respect to ϵ and $\delta(0)=0$, and by setting $\epsilon = 0$ and $x = 0$), we obtain

$$C_\delta(\epsilon, C_\delta(x,x)) = \delta'(0)\epsilon + O(\epsilon^2),$$

where $\delta'(0)$ is the right derivative at 0. Similarly, expanding the right-hand side of Eq. (3.12) in powers of ϵ around 0 (by taking the partial derivative with respect to ϵ and $\delta(1) = 1$, and by setting $\epsilon = 0$ and $x = 1$), it holds that

$$C_\delta(C_\delta(\epsilon,x),x) = (\delta'(0))^2\epsilon + O(\epsilon^2).$$

It follows that either $\delta'(0) = 1$ or $\delta'(0) = 0$. In the former case, since δ is convex and $\delta(t) \leq t$ for any $t \in [0,1]$, it follows that $\delta(t) = t$ for all $t \in [0,1]$, whence $C_\delta = T_\mathbf{M}$.

Similarly, the associativity of C_δ implies that for $0 < x \leq 1-\epsilon \leq 1$, it holds that

$$C_\delta(C_\delta(x, 1-\epsilon), 1-\epsilon) = C_\delta(x, C_\delta(1-\epsilon, 1-\epsilon)).$$

We denote the left derivative of δ at 1 as $\delta'(1)$. Expanding the left-hand side in powers of ϵ around 0, we obtain

$$C_\delta(C_\delta(x, 1-\epsilon), 1-\epsilon) = x + 2(1-\delta'(1))\epsilon + O(\epsilon^2),$$

whereas the right-hand side is expanded as

$$C_\delta(x, C_\delta(1-\epsilon, 1-\epsilon)) = x + (1-\delta'(1))\delta'(1)\epsilon + O(\epsilon^2).$$

It follows that either $\delta'(1) = 1$ or $\delta'(1) = 2$. Since δ is a convex function, the former case again yields $C_\delta = T_\mathbf{M}$.

Finally, the associativity of C_δ implies that for any $0 \le \epsilon < \frac{1}{4}$, the equality

$$C_\delta(C_\delta(\frac{1}{2}+\epsilon,\frac{1}{2}+\epsilon),1-\epsilon) = C_\delta(C_\delta(\frac{1}{2}+\epsilon,1-\epsilon),\frac{1}{2}+\epsilon)\,,$$

holds. We denote the right derivative of δ at $\frac{1}{2}$ as $\delta'(\frac{1}{2})$. Expanding the left-hand side in powers of ϵ around 0 yields

$$C_\delta(C_\delta(\frac{1}{2}+\epsilon,\frac{1}{2}+\epsilon),1-\epsilon) = \delta(\frac{1}{2}) + (1+\delta'(\frac{1}{2}) - \delta'(1))\epsilon + O(\epsilon^2)\,,$$

whereas expanding the right-hand side yields

$$C_\delta(C_\delta(\frac{1}{2}+\epsilon,1-\epsilon),\frac{1}{2}+\epsilon) = \delta(\frac{1}{2}) + \left(\frac{1}{2}(3-\delta'(1))\delta'(\frac{1}{2}) - (\delta'(1)-1)\delta(\frac{1}{2})\right)\epsilon + O(\epsilon^2)\,.$$

Putting $\delta'(1)=2$, it follows that

$$\delta'(\frac{1}{2}) = 2(1-\delta(\frac{1}{2}))\,.$$

Hence, the (right) tangent of the graph of δ at the point $(\frac{1}{2},\delta(\frac{1}{2}))$, which is determined by the linear function

$$y = \delta(\frac{1}{2}) + \delta'(\frac{1}{2})(x-\frac{1}{2})\,,$$

passes through the point $(1,1)$. Since δ is convex, its graph must lie above this tangent line on the interval $[\frac{1}{2},1]$, which leads to a contradiction unless it coincides entirely with this tangent line on the interval $[1/2,1]$, i.e.

$$\delta(x) = \delta(\frac{1}{2}) + \delta'(\frac{1}{2})(x-\frac{1}{2})\,,$$

for all $x \in [\frac{1}{2},1]$. Since $\delta'(1)=2$, it follows that $\delta(\frac{1}{2})=0$, henceforth also that $C_\delta = T_{\mathbf{L}}$.

Since both $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ are biconic associative copulas with a given diagonal section, we conclude that these two extreme copulas are the only biconic associative copulas with a given diagonal section. $\square$

Corollary 3.1. *$T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ are the only associative biconic quasi-copulas with a given diagonal section.*

We conclude this section by establishing the intersection between the set of biconic copulas and conic copulas with the same diagonal section. Conic copulas were introduced in Chapter 2 and their construction is based on linear interpolation on segments connecting the upper boundary curve of the zero-set to the point $(1,1)$.

Lemma 3.6. *Let δ be a diagonal function and suppose that $\theta_1 = 1 - 2^{-\frac{1}{\theta}}$ with $\theta \in [1, \infty[$ is the maximum value such that $\delta(\theta_1) = 0$. Then the biconic copula C_δ has the zero set Z_{C_δ} given by*

$$Z_{C_\delta} = \{(x, y) \in [0,1]^2 \mid y \leq f_\theta(x)\} ,$$

where the function $f_\theta : [0,1] \to [0,1]$ is given by

$$f_\theta(x) = \begin{cases} (1 - 2^{\frac{1}{\theta}})^{-1}x + 1 & , \text{ if } x \leq 1 - 2^{-\frac{1}{\theta}} , \\ \\ (1 - 2^{\frac{1}{\theta}})(x - 1) & , \text{ if } x \geq 1 - 2^{-\frac{1}{\theta}} . \end{cases} \tag{3.13}$$

Proof. Let C_δ be a biconic copula and suppose that $\theta_1 = 1 - 2^{-\frac{1}{\theta}}$ with $\theta \in [1, \infty[$ is the maximum value such that $\delta(\theta_1) = 0$. Due to the definition of a biconic copula, it holds that C_δ is linear on the segment $\langle(\theta_1, \theta_1), (1, 0)\rangle$ as well as on the segment $\langle(\theta_1, \theta_1), (0, 1)\rangle$. Hence, these two segments form the upper boundary curve of the zero-set of C_δ. A simple computation shows that the function f_θ in (3.13) represents the considered segments. □

Due to the above lemma and the definition of a conic copula, the following proposition is immediate.

Proposition 3.10. *The only copulas that are at the same time conic and biconic with the same given diagonal section, are the members of the following family*

$$C_\theta(x, y) = \begin{cases} \max(y + (1 - x)(1 - 2^{\frac{1}{\theta}}), 0) & , \text{ if } y \leq x , \\ \\ \max(x + (1 - y)(1 - 2^{\frac{1}{\theta}}), 0) & , \text{ otherwise} \end{cases} \tag{3.14}$$

where $\theta \in [1, \infty[$. This family of copulas was introduced in Chapter 2.

3.6. Biconic copulas supported on a set with Lebesgue measure zero

We characterize in this section biconic copulas that are supported on a set with Lebesgue measure zero. To this end, we need the following proposition.

Proposition 3.11. *Let C_δ be a biconic copula with a piecewise linear diagonal section δ. Suppose that $d \in [0, 1/2]$ is the maximum value such that $\delta(d) = 0$, and $d^* \in [1/2, 1]$ is the minimum value such that $\delta(d^*) = 2d^* - 1$. Then the support of C_δ consists of:*

(i) *the segment $\langle(d, d), (d^*, d^*)\rangle$;*

(ii) *the segments* $\langle(x,x),(1,0)\rangle$ *and* $\langle(x,x),(0,1)\rangle$, *for any* x *such that the graph of* δ *contains two consecutive segments with common point* $(x,\delta(x))$.

Proof. From Proposition 3.7 it follows that $\delta(x) = 2x-1$ for any $x \in [d^*,1]$. Note that if $d = d^* = 1/2$, then $C_\delta = T_{\mathbf{L}}$ and the support is given by the segment $\langle(1,0),(0,1)\rangle$. More generally, if δ is piecewise linear, then it suffices to consider a number of cases to prove assertion (i):

(a) Let $\langle(x_1,x_1),(x_2,x_2)\rangle$, with $d \le x_1 < x_2 \le 1/2$, be a segment such that δ is linear on the interval $[x_1,x_2]$. For any rectangle $R = [x,y]\times[x,y]$ such that $x_1 \le x < y \le x_2$, it holds that

$$V_{C_\delta}(R) = (1-2x_1)(1-2x_2)(\phi_\delta(x_2)-\phi_\delta(x_1))\,,$$

where ϕ_δ is the function defined in Lemma 3.2. Since ϕ_δ is increasing on the interval $[0,1/2]$, it holds that $V_{C_\delta}(R) \ge 0$. If $V_{C_\delta}(R) = 0$, then due to the increasingness of ϕ_δ, it holds that ϕ_δ is constant on the interval $[x_1,x_2]$, i.e. there exists $c \ge 0$ such that $\delta(x) = c(1-2x)$ on the interval $[x_1,x_2]$. The increasingness of δ then implies that $c = 0$, which is a contradiction with the fact that d is the maximum value such that $\delta(d) = 0$ and hence, $V_{C_\delta}(R) > 0$.

(b) Similarly, one can prove that for any segment $\langle(x_1,x_1),(x_2,x_2)\rangle$, with $1/2 \le x_1 < x_2 \le d^*$, such that δ is linear on the interval $[x_1,x_2]$, it holds that $V_{C_\delta}(R) > 0$ for any rectangle $R = [x,y]\times[x,y]$ such that $x_1 \le x < y \le x_2$.

Since the support is closed, assertion (i) follows.

Next, we prove assertion (ii). Let $\mathbf{b}_1 := (x_1,x_1)$, $\mathbf{b}_2 := (x_2,x_2)$ and $\mathbf{b}_3 := (x_3,x_3)$, with $d \le x_2 \le d^*$, be three distinct points such that δ is linear on the interval $[x_1,x_2]$ as well as on the interval $[x_2,x_3]$, and δ is not linear on the interval $[x_1,x_3]$. Let $R \subseteq [0,1]^2$ be a rectangle such that its opposite diagonal is a subset of the segment $\langle(x_2,x_2),(1,0)\rangle$. If $V_{C_\delta}(R) = 0$, then due to inequality (3.7), δ is linear on the interval $[x_1,x_3]$, a contradiction. Hence, $V_{C_\delta}(R) > 0$. Consequently, the segment $\langle(x_2,x_2),(1,0)\rangle$ is a subset of the support. Due to the symmetry of C_δ, the segment $\langle(x_2,x_2),(0,1)\rangle$ is a subset of the support as well, hence, (ii) follows.

Since the surface of C_δ consists of triangles (see the proof of Proposition 3.8), it holds that $\frac{\partial^2 C(u,v)}{\partial u \partial v} = 0$ in all other points. $\square$

Example 3.12. *Consider the diagonal function δ given by*

$$\delta(x) = \begin{cases} 0 & , \textit{ if } x \leq \frac{1}{3}\,, \\ x - 1/3 & , \textit{ if } \frac{1}{3} \leq x \leq \frac{2}{3}\,, \\ 2x - 1 & , \textit{ otherwise.} \end{cases}$$

Clearly, δ is a piecewise linear convex function. The support of the corresponding copula is depicted in Figure 3.6(a).

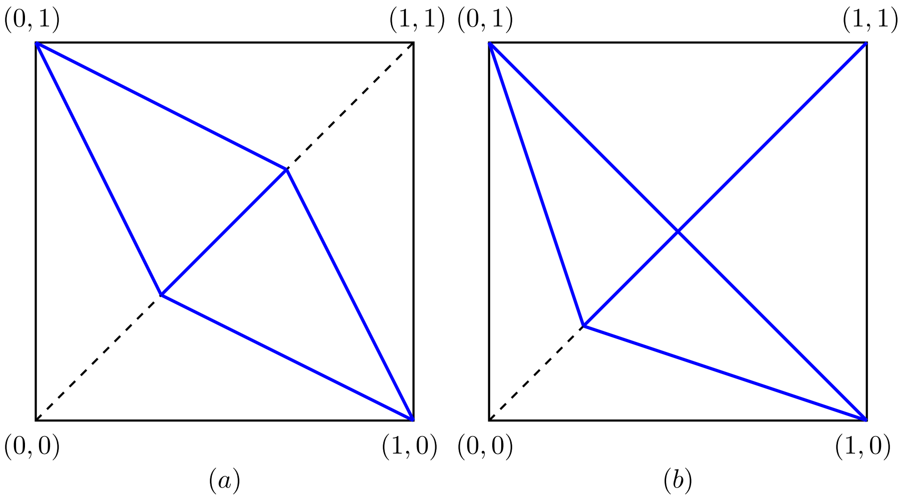


Figure 3.6: The support of the biconic copulas given in Example 3.12 (a) and Example 3.13 (b).

Example 3.13. *Consider the diagonal function δ given by*

$$\delta(x) = \begin{cases} 0 & , \textit{ if } x \leq \frac{1}{4}\,, \\ x - \dfrac{1}{4} & , \textit{ if } \frac{1}{4} \leq x \leq \frac{1}{2}\,, \\ \dfrac{1}{2}(3x - 1) & , \textit{ otherwise.} \end{cases}$$

Clearly, δ is a piecewise linear convex function. The support of the corresponding copula is depicted in Figure 3.6(b).

Theorem 3.2. *Let C_δ be a biconic copula. Then it holds that C_δ is supported on a set with Lebesgue measure zero if and only if δ is piecewise linear.*

Proof. Let C_δ be a biconic copula with a piecewise linear diagonal section δ. From Proposition 3.11, it follows that C_δ is supported on a set with Lebesgue measure zero. Conversely, let C_δ be supported on a set with Lebesgue measure zero and suppose that δ is not piecewise linear, i.e. there exists an interval $[d_1, d_2]$ such that the graph of the restriction of δ to $[d_1, d_2]$ does not contain any segment. Consider the triangle $\Delta_{d_1,d_2} = \Delta_{\{(d_1,d_1),(d_2,d_2),(0,1)\}}$. Consider a rectangle R located in Δ_{d_1,d_2} such that $V_{C_\delta}(R) = 0$. It then holds that $V_{C_\delta}(R_1) = 0$ for any rectangle $R_1 \subseteq R$. Choose a rectangle $R_1 = [x, x'] \times [y, y'] \subseteq R$ such that its opposite diagonal is a subset of the segment $\langle(0,1),(x_2,x_2)\rangle$ with $d_1 < x_2 < d_2$. Let (x_1, x_1) and (x_3, x_3) be the two points on the diagonal of the unit square such that the points (x, y) and (x', y') are respectively located on the segments $\langle(0,1),(x_1,x_1)\rangle$ and $\langle(0,1),(x_3,x_3)\rangle$. Since $V_{C_\delta}(R_1) = 0$, inequality (3.7) implies that the points $(x_1, \delta(x_1))$, $(x_2, \delta(x_2))$ and $(x_3, \delta(x_3))$ are located on the same segment, which contradicts the fact that δ does not contain any segment on the interval $[d_1, d_2]$. Hence, $V_{C_\delta}(R) > 0$ for any rectangle located in $S = \Delta_{d_1,d_2} \setminus \{\langle(d_1,d_1),(d_2,d_2)\rangle \cup \langle(d_1,d_1),(0,1)\rangle \cup \langle(d_2,d_2),(0,1)\rangle\}$, i.e. S is a subset of the support of C_δ with non-zero Lebesgue measure, a contradiction. $\square$

Since any copula that is supported on a set with Lebesgue measure zero is singular, the following corollary is clear.

Corollary 3.2. *Any biconic copula C_δ with a piecewise linear diagonal section δ is singular.*

Example 3.14. *The family of biconic copulas given in (3.14) is a family of singular biconic copulas.*

3.7. Dependence measures

In this section, we derive compact formulae for Spearman's rho, Gini's gamma and Kendall's tau of two continuous random variables whose dependence is modelled by a biconic copula C_δ. These parameters can be expressed in terms of the function δ.

Proposition 3.12. *Let X and Y be two continuous random variables that are coupled by a biconic copula C_δ.*

(i) *The population version of Spearman's ρ_{C_δ} for X and Y is given by*

$$\rho_{C_\delta} = 8 \int_0^1 \delta(x)\, dx - 3\,.$$

(ii) *The population version of Gini's* γ_{C_δ} *for* X *and* Y *is given by*

$$\gamma_{C_\delta} = 4\int_0^1 \delta(x)dx - 2(1-\delta(1/2))\,.$$

(iii) *The population version of Kendall's* τ_{C_δ} *for* X *and* Y *is given by*

$$\tau_{C_\delta} = 1 - 4\int_0^1 (\delta'(x)x - \delta(x))(\delta'(x)(1-x) + \delta(x))\,dx\,,$$

where δ' *is the left (or right) derivative of* δ*.*

Proof. The integral of C_δ over the unit square is the volume below its surface. Since C_δ is symmetric, we consider twice the volume over the region I_1. In fact the volume over I_1 can be seen as a conic body with the area of its base equal to $\int_0^1 \delta(x)\,dx$ and height equal to 1. Recalling the geometrical fact that the volume of a conic body equals one third of the product of the area of its base and its height, (i) follows immediately.

The expression for γ_{C_δ} can be rewritten as

$$\gamma_{C_\delta} = 4\left[\int_0^1 \omega_{C_\delta}(x)dx - \int_0^1 (x-\delta(x))dx\right]\,,$$

where ω_{C_δ} is the opposite diagonal section of C_δ. Since C_δ is biconic, ω_{C_δ} is given by

$$\omega_{C_\delta}(x) = \begin{cases} 2x\delta(1/2) & \text{, if } x \le 1/2\,, \\ 2(1-x)\delta(1/2) & \text{, if } x \ge 1/2\,. \end{cases}$$

Computing $\int_0^1 \omega_{C_\delta}(x)dx$, (ii) follows.

In order to find τ_{C_δ}, we need to compute

$$I = \int_0^1\int_0^1 \frac{\partial C}{\partial x}(x,y)\frac{\partial C}{\partial y}(x,y)dxdy\,.$$

As C_δ is symmetric, it holds that $I = 2\tilde{I}$ with $\tilde{I}$ the integral over the region I_1, i.e.

$$\tilde{I} = \int_0^1 \int_0^x \frac{\partial C}{\partial x}(x,y) \frac{\partial C}{\partial y}(x,y) dy dx \,.$$

Computing the partial derivatives and using the notation $u = \frac{y}{1+y-x}$, it holds that

$$\tilde{I} = \int_0^1 (1-x) \left(\int_0^x u^2 \left(\frac{\delta(u)}{u}\right)' \left(\frac{\delta(u)}{1-u}\right)' du \right) dx \,. \tag{3.15}$$

Since δ is convex, the right and left derivatives exist almost everywhere [93, 95]. Note also that the left and right derivatives coincide, except possibly on a countable subset. Hence, the choice of derivative does not affect the result of the integration [100]. We then use the notation δ' for the right derivative of δ.

Consider the function $\psi :]0,1[\to \mathbb{R}$ given by

$$\psi(x) = \int_0^x u^2 \left(\frac{\delta(u)}{u}\right)' \left(\frac{\delta(u)}{1-u}\right)' du \,.$$

Substituting $\psi(x)$ in Eq. (3.15), it holds that

$$\tilde{I} = \int_0^1 (1-x)\psi(x) dx \,.$$

By integrating by parts, it holds that

$$I = 2\tilde{I} = \int_0^1 (\delta'(x)x - \delta(x))(\delta'(x)(1-x) + \delta(x)) dx \,.$$

Substituting in the expression for τ_{C_δ}, (iii) follows. □

Example 3.15. *Let δ be the diagonal function given in Example 3.6. Then*

$$\rho_{C_\delta} = \frac{2-3\theta}{2+\theta}\,, \quad \gamma_{C_\delta} = \frac{-2\theta + 2^{-\theta}(2+\theta)}{2+\theta} \quad \textit{and} \quad \tau_{C_\delta} = \frac{3-4\theta}{3+2\theta}\,.$$

We computed the values of Spearman's rho, Gini's gamma and Kendall's tau by means of the expressions given in Proposition 3.12. The results are listed in Table 3.1.

Table 3.1: Spearman's rho, Gini's gamma and Kendall's tau of the biconic copulas C_{δ_θ} with diagonal section $\delta_\theta(t) = t^{\theta+1}$.

θ	δ_θ	$\rho_{C_{\delta_\theta}}$	$\gamma_{C_{\delta_\theta}}$	$\tau_{C_{\delta_\theta}}$
0	t	1	1	1
0.2	$t^{1.2}$	0.636364	0.688732	0.647059
0.4	$t^{1.4}$	0.333333	0.424525	0.368421
0.6	$t^{1.6}$	0.076923	0.198215	0.142857
0.8	$t^{1.8}$	-0.142857	0.002920	-0.043478
1	t^2	-0.333333	-0.166667	-0.200000

3.8. Aggregation of biconic (semi-, quasi-)copulas

In this section, we study the aggregation of biconic semi-copulas, quasi-copulas and copulas. We formulate one lemma and two immediate propositions.

Lemma 3.7. *The sets $\mathcal{D}_{\mathrm{S}}$ and $\mathcal{D}$ are closed under minimum, maximum and convex sums.*

Proposition 3.13. *Let $\delta_1, \delta_2 \in \mathcal{D}_{\mathrm{S}}$ (resp. $\mathcal{D}$) and $\theta \in [0,1]$. If C_{δ_1} and C_{δ_2} are biconic semi-copulas (resp. quasi-copulas), then also $\min(C_{\delta_1}, C_{\delta_2})$, $\max(C_{\delta_1}, C_{\delta_2})$ and $\theta C_{\delta_1} + (1-\theta)C_{\delta_2}$ are biconic semi-copulas (resp. quasi-copulas). The corresponding diagonal sections are given by $\delta_{\min} = \min(\delta_1, \delta_2)$, $\delta_{\max} = \max(\delta_1, \delta_2)$ and $\theta\delta_1 + (1-\theta)\delta_2$, respectively.*

Consequently, the class of biconic semi-copulas with a given diagonal section and the class of biconic quasi-copulas with a given diagonal section are closed under minimum, maximum and convex sums.

Proposition 3.14. *Let $\delta_1, \delta_2 \in \mathcal{D}$ and $\theta \in [0,1]$. If C_{δ_1} and C_{δ_2} are biconic copulas, then also $\max(C_{\delta_1}, C_{\delta_2})$ and $\theta C_{\delta_1} + (1-\theta)C_{\delta_2}$ are biconic copulas. The corresponding diagonal sections are given by $\delta_{\max}$ and $\theta\delta_1 + (1-\theta)\delta_2$, respectively.*

Consequently, the class of biconic copulas with a given diagonal section is closed under maximum and convex sums. Hence, the class of biconic copulas with a given diagonal section is not join-dense in the class of biconic quasi-copulas with a given diagonal section in contrast to the general case [92]. In general, the minimum of two biconic copulas with a given diagonal section need not be a biconic copula. For instance, let C_{δ_1} and C_{δ_2} be two biconic copulas with δ_1 and δ_2 as depicted in Figure 3.7. Obviously, the function $\delta_{\min}$ is not convex, and thus $\min(C_{\delta_1}, C_{\delta_2})$ is a proper biconic quasi-copula.

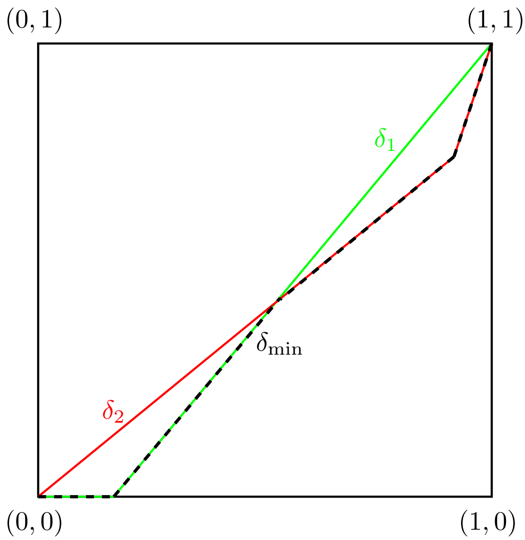


Figure 3.7: An example of the graph of $\delta_{\min}$

Since the diagonal function δ determining a biconic quasi-copula Q_δ can always be written as the infimum of a family $(\delta_i)_{i\in I}$ of convex functions, any biconic quasi-copula Q_δ can be written as

$$Q_\delta = \inf_{i\in I} C_{\delta_i}\,,$$

where C_{δ_i} are biconic copulas. Hence, the class of biconic copulas with a given diagonal section is meet-dense in the class of biconic quasi copulas with a given diagonal section.

3.9. Biconic functions with a given opposite diagonal section

In this section, we introduce biconic functions with a given opposite diagonal section. Their construction is based on linear interpolation on segments connecting the opposite diagonal of the unit square and the points $(0,0)$ and $(1,1)$.

Let $\omega : [0,1] \to [0,1]$ and $\alpha, \beta \in [0,1]$. The function $A_\omega^{\alpha,\beta} : [0,1]^2 \to [0,1]$ defined

by

$$A_\omega(x,y) = \begin{cases} \alpha(1-x-y) + (x+y)\,\omega\left(\dfrac{x}{x+y}\right) & \text{, if } x+y \le 1\,, \\[2ex] \beta(x+y-1) + (2-x-y)\,\omega\left(\dfrac{1-y}{2-x-y}\right) & \text{, otherwise} \end{cases} \tag{3.16}$$

is well defined. This function is called *a biconic function with a given opposite diagonal section* since it satisfies the boundary conditions

$$A_\omega^{\alpha,\beta}(0,0) = \alpha \text{ and } A_\omega^{\alpha,\beta}(1,1) = \beta\,,$$

and $A_\omega^{\alpha,\beta}(t,1-t) = \omega(t)$ for any $t \in [0,1]$, and since it is linear on segments connecting the points $(t,1-t)$ and $(0,0)$ as well as on segments connecting the points $(t,1-t)$ and $(1,1)$. Evidently, for a biconic function $A_\omega^{\alpha,\beta}$, the boundary conditions $A_\omega^{\alpha,\beta}(0,0) = 0$ and $A_\omega^{\alpha,\beta}(1,1) = 1$ imply that $\alpha = 0$ and $\beta = 1$. We then abbreviate $A_\omega^{0,1}$ as A_ω, with

$$A_\omega(x,y) = \begin{cases} (x+y)\,\omega\left(\dfrac{x}{x+y}\right) & \text{, if } x+y \le 1\,, \\[2ex] x+y-1 + (2-x-y)\,\omega\left(\dfrac{1-y}{2-x-y}\right) & \text{, otherwise.} \end{cases} \tag{3.17}$$

Clearly, when $\omega \in \mathcal{O}_{\mathrm{S}}$, the function A_ω defined in (3.17) has 1 as neutral element and therefore, if A_ω is an aggregation function then, it is also a semi-copula.

Let us introduce the following notations

$$\begin{aligned} J_1 &= \{(x,y) \in [0,1]^2 \mid x+y \le 1\} \\ J_2 &= \{(x,y) \in [0,1]^2 \mid x+y \ge 1\} \\ O &= J_1 \cap J_2\,. \end{aligned}$$

In the next proposition, we characterize the functions in $\mathcal{O}_{\mathrm{S}}$ for which the corresponding biconic function is a biconic aggregation function.

Proposition 3.15. *Let $\omega \in \mathcal{O}_{\mathrm{S}}$. Then the function $A_\omega : [0,1]^2 \to [0,1]$ defined in (3.17) is an aggregation function if and only if*

(i) *the functions $\lambda_\omega\,, \rho_\omega :]0,1] \to [0,1]$, defined by $\lambda_\omega(x) = \frac{\omega(x)}{x}$, $\rho_\omega(x) = \frac{1-\omega(x)}{x}$, are decreasing;*

(ii) *the functions $\mu_\omega, \xi_\omega : [0,1[\to [0,1]$, defined by $\mu_\omega(x) = \frac{\omega(x)}{1-x}$, $\xi_\omega = \frac{x-\omega(x)}{1-x}$, are increasing.*

Proof. Suppose conditions (i) and (ii) are satisfied. The function A_ω defined in

(3.17) clearly satisfies the boundary conditions of an aggregation function. We prove the increasingness of A_ω in the first variable (the proof of the increasingness in the second variable is similar). Let $(x,y),(x',y) \in [0,1]^2$ such that $x \leq x'$. If $(x,y),(x',y) \in J_1$, the increasingness of A_ω is equivalent to

$$(x'+y)\,\omega\left(\frac{x'}{x'+y}\right) - (x+y)\,\omega\left(\frac{x}{x+y}\right) \geq 0\,.$$

Using the notations $u = \frac{x}{x+y}$ and $u' = \frac{x'}{x'+y}$, the last inequality is equivalent to

$$y\left(\frac{\omega(u')}{1-u'} - \frac{\omega(u)}{1-u}\right) \geq 0\,,$$

or, equivalently,

$$y(\mu_\omega(u') - \mu_\omega(u)) \geq 0\,. \tag{3.18}$$

Since $x \leq x'$, it holds that $u \leq u'$ and therefore inequality (3.18) holds due to the increasingness of the function μ_ω.

If $(x,y),(x',y) \in J_2$, the increasingness of A_ω is equivalent to

$$x'+y-1+(2-x'-y)\,\omega\left(\frac{1-y}{2-x'-y}\right)-x-y+1-(2-x-y)\,\omega\left(\frac{1-y}{2-x-y}\right) \geq 0\,.$$

Using the notations $v = \frac{1-y}{2-x-y}$ and $v' = \frac{1-y}{2-x'-y}$, the last inequality is equivalent to

$$(2-x'-y)(\omega(v')-1) - (2-x-y)(\omega(v)-1) \geq 0\,.$$

Simple processing yields,

$$(1-y)\left(\frac{1-\omega(v)}{v} - \frac{1-\omega(v')}{v'}\right) \geq 0\,,$$

or, equivalently,

$$(1-y)(\rho_\omega(v) - \rho_\omega(v')) \geq 0\,. \tag{3.19}$$

Since $x \leq x'$, it holds that $v \leq v'$ and therefore inequality (3.19) holds due to the decreasingness of the function ρ_ω.

The remaining case is when $(x,y) \in J_1$ and $(x',y) \in J_2 \setminus O$. The two previous cases then imply that

$$A_\omega(x',y) - A_\omega(x,y) = (A_\omega(x',y) - A_\omega(1-y,y)) + (A_\omega(1-y,y) - A_\omega(x,y)) \geq 0\,.$$

Similarly, one can prove that the increasingness of λ_ω and the decreasingness of ξ_ω imply that A_δ is increasing in the second variable.

Conversely, suppose that A_ω is an aggregation function. Let $x,x' \in [0,1]$ such that

$x \leq x'$ and choose $y \in [0,1]$ such that $x' + y \leq 1$. It then holds that

$$\frac{xy}{1-x} + y \leq \frac{x'y}{1-x'} + y \leq 1\,.$$

The increasingness of A_ω in the first variable implies

$$A_\omega\left(\frac{x'y}{1-x'}, y\right) - A_\omega\left(\frac{xy}{1-x}, y\right) \geq 0\,.$$

After some elementary manipulations, the last inequality becomes

$$y(\mu_\omega(x') - \mu_\omega(x)) \geq 0\,.$$

Since x and x' are arbitrary in $[0,1]$, the increasingness of μ_ω follows. Similarly, the decreasingness of the function ρ_ω can be proved using the increasingness of A_ω in the first variable.

The decreasingness of λ_ω and the increasingness of ξ_ω can be obtained using the increasingness of A_ω in the second variable. Therefore, conditions (i) and (ii) follow, which completes the proof. □

Let A_ω be a biconic function with opposite diagonal section ω. The function A', defined by

$$A' = \varphi_2(A)\,, \tag{3.20}$$

where φ_2 is the transformation defined in (1.3), is again a biconic function whose diagonal section $\delta_{A'}$ is given by $\delta_{A'}(x) = x - \omega(x)$. This transformation permits us to derive in a straightforward manner the conditions that have to be satisfied by an opposite diagonal function to obtain a biconic quasi-copula (resp. copula), which has that opposite diagonal function as opposite diagonal section. Using Proposition 3.6 and Theorem 3.1, the following two propositions are immediate.

Proposition 3.16. *Let $\omega \in \mathcal{O}$. Then the function $A_\omega : [0,1]^2 \to [0,1]$ defined in (3.17) is a quasi-copula if and only if the functions λ_ω and μ_ω, defined in Proposition 3.15, are decreasing and increasing, respectively.*

For the class of biconic semi-copulas with a given opposite diagonal section belonging to $\mathcal{O}$, one can easily see that the functions ρ_ω and ξ_ω are decreasing and increasing, respectively. Hence, any biconic semi-copula with a given opposite diagonal section $\omega \in \mathcal{O}$ is a biconic quasi-copula. Consequently, the class of biconic semi-copulas with a given opposite diagonal section coincides with the class of biconic quasi-copulas with a given opposite diagonal when the opposite diagonal section is an opposite diagonal function.

Proposition 3.17. *Let $\omega \in \mathcal{O}$. Then the function $A_\omega : [0,1]^2 \to [0,1]$ defined in (3.17) is a copula if and only if ω is concave.*

Example 3.16. *Consider the opposite diagonal functions $\omega_{T_\mathbf{M}}$ and $\omega_{T_\mathbf{L}}(x)$. Obviously, $\omega_{T_\mathbf{M}}$ and $\omega_{T_\mathbf{L}}$ are concave functions. The corresponding biconic copulas are respectively $T_\mathbf{M}$ and $T_\mathbf{L}$.*

Example 3.17. *Consider the opposite diagonal function $\omega_{T_\mathbf{P}}(x) = x(1-x)$. Obviously, $\omega_{T_\mathbf{P}}$ is concave. The corresponding biconic copula is given by*

$$C_{\omega_{T_\mathbf{P}}}(x,y) = \begin{cases} \dfrac{xy}{x+y} & , \text{ if } x+y \leq 1\,, \\[2ex] \dfrac{x(2-x-y)-(1-y)^2}{2-x-y} & , \text{ otherwise}\,. \end{cases}$$

We now focus on the symmetry and opposite symmetry properties of biconic copulas with a given opposite diagonal section.

Proposition 3.18. *Let C_ω be a biconic copula. Then it holds that*

(i) *C_ω is opposite symmetric;*

(ii) *C_ω is symmetric if and only if ω is symmetric with respect to the point $(1/2, 1/2)$, i.e. $\omega(x) = \omega(1-x)$ for any $x \in [0, 1/2]$.*

Proof. Let C_ω be a biconic copula. Assertion (i) is clear. We discuss the case when $x+y \leq 1$ (the case $x+y \geq 1$ can be proved similarly). Using the notation $z = \frac{x}{x+y}$, the symmetry property of C_ω is equivalent to

$$\omega(z) = \omega(1-z)\,,$$

i.e. ω is symmetric with respect to the point $(1/2, 1/2)$, whence (ii) follows. □

We conclude this section by finding the intersection between the class of biconic copulas with a given opposite diagonal section and the class of biconic copulas with a given diagonal section and the class of conic copulas.

Proposition 3.19. *Let C be a biconic copula with a given opposite diagonal section and suppose further that C is a biconic copula with a given diagonal section. Then it holds that C is a member of convex sums of $T_\mathbf{M}$ and $T_\mathbf{L}$.*

Proof. Suppose that C is a biconic copula with a given opposite diagonal section ω and suppose further that C is a biconic copula with a given diagonal section δ. Due to the construction method of biconic copulas with a given diagonal (resp. opposite diagonal) section, δ and ω must be piecewise linear and are given by

$$\delta(x) = \begin{cases} 2x\omega(1/2) & \text{, if } x \leq 1/2\,, \\ 2x - 1 + 2(1-x)\omega(1/2) & \text{, if } x \geq 1/2\,, \end{cases}$$

$$\omega(x) = \begin{cases} 2x\delta(1/2) & \text{, if } x \leq 1/2\,, \\ 2(1-x)\delta(1/2) & \text{, if } x \geq 1/2\,. \end{cases}$$

Since δ and ω are the diagonal and opposite diagonal sections of C, it holds that $\delta(1/2) = \omega(1/2)$. Using the notation $\theta = 2\delta(1/2) = 2\omega(1/2)$, δ and ω can be rewritten as

$$\delta(x) = \theta\delta_{T_{\mathbf{M}}}(x) + (1-\theta)\delta_{T_{\mathbf{L}}}(x)\,, \quad \omega(x) = \theta\omega_{T_{\mathbf{M}}}(x) + (1-\theta)\omega_{T_{\mathbf{L}}}(x)\,.$$

Recalling that any biconic copula with a given diagonal (resp. opposite diagonal) section is uniquely determined by its diagonal (resp. opposite diagonal) section, our assertion follows. □

Let C_ω be a biconic copula. Due to the definition of C_ω, the only possible zero-sets are

$$Z_{C_\omega} = Z_{T_{\mathbf{M}}} = [0,1]^2 \setminus]0,1]^2$$

and

$$Z_{C_\omega} = Z_{T_{\mathbf{L}}} = \{(x,y) \in [0,1]^2 \mid x + y \leq 1\}\,.$$

Recalling that every conic copula is uniquely determined by its zero-set (see Chapter 2), the following proposition is clear.

Proposition 3.20. *Let C_ω be a biconic copula with a given opposite diagonal section ω and suppose further that C_ω is a conic copula. Then it holds that $C_\omega = T_{\mathbf{M}}$ or $C_\omega = T_{\mathbf{L}}$.*

4 Upper conic, lower conic and biconic semi-copulas

4.1. Introduction

Several methods to construct (semi-, quasi-)copulas have been introduced in the literature. Some of these methods start from given sections. Such sections can be the diagonal section and/or the opposite diagonal section [23] (see also Chapters 2 and 3), or a horizontal section and/or a vertical section [37, 74, 97]. All of the above methods use sections that are determined by straight lines in the unit square, such as the diagonal, the opposite diagonal, a horizontal line or a vertical line. In the present chapter, we consider sections that are determined by a curve in the unit square, which represents a strict negation operator.

For any strict negation operator $N : [0, 1] \to [0, 1]$, the surface of the semi-copula $T_{\mathbf{M}}$ is constituted from (linear) segments connecting the points $(0, 0, 0)$ and $(a, N(a), \min(a, N(a)))$ as well as segments connecting the points $(a, N(a), \min(a, N(a)))$ and $(1, 1, 1)$, with $N(a) \leq a$, and segments connecting the points $(0, 0, 0)$ and $(a, N(a), \min(a, N(a)))$ as well as segments connecting the points $(a, N(a), \min(a, N(a)))$ and $(1, 1, 1)$, with $N(a) \geq a$. This observation has motivated the construction presented in this chapter.

This chapter is organized as follows. In the following section, we recall some definitions and facts concerning convexity and generalized convexity. In Section 4.3, we introduce the class of upper conic functions with a given section. In Sections 4.4 and 4.5, we characterize upper conic semi-copulas, upper conic quasi-copulas and upper conic copulas with a given section. In Sections 4.6 (resp. 4.7), we introduce in a similar way the classes of lower conic (resp. biconic) functions with a given section and characterize lower conic (resp. biconic) semi-copulas, lower conic (resp. biconic) quasi-copulas and lower conic (resp. biconic) copulas with a given section.

4.2. Convexity and generalized convexity

Convexity plays a key role in the characterization of some classes of semilinear copulas, such as conic copulas (see Chapter 2) and biconic copulas (see Chapter 3). A more general type of convexity, called *generalized convexity*, has been introduced in the literature and has been used, for instance, to characterize the comparability of two quasi-arithmetic means [11, 94].

We denote an open, half-open or closed interval in $\mathbb{R}$ with lower endpoint a and upper endpoint b as $I(a,b)$.

A function $h : I(a,b) \to \mathbb{R}$ is called *convex* (on $I(a,b)$) [93] if the inequality

$$h(\lambda x + (1-\lambda)y) \leq \lambda h(x) + (1-\lambda)h(y) \tag{4.1}$$

holds for any $x, y \in I(a,b)$ and any $\lambda \in [0,1]$. If the converse inequality holds, then the function h is called *concave*. In the next proposition, we state an equivalent formulation of convexity.

Proposition 4.1. [93] *A function $h : I(a,b) \to \mathbb{R}$ is convex if and only if*

$$\begin{vmatrix} 1 & x & h(x) \\ 1 & y & h(y) \\ 1 & z & h(z) \end{vmatrix} \geq 0\,, \tag{4.2}$$

for any $x, y, z \in I(a,b)$ such that $x < y < z$.

Proposition 4.2. *A function $h : I(a,b) \to \mathbb{R}$ is concave if and only if*

$$\begin{vmatrix} 1 & 1-x & h(x) \\ 1 & 1-y & h(y) \\ 1 & 1-z & h(z) \end{vmatrix} \geq 0\,, \tag{4.3}$$

for any $x, y, z \in I(a,b)$ such that $x < y < z$.

The notion of convexity can be further generalized as follows.

Definition 4.1. *Let $\nu : I(a,b) \to \mathbb{R}$ be a function and $\xi : I(a,b) \to \mathbb{R}$ be a strictly monotone continuous function. Then ν is called convex (resp. concave) w.r.t. ξ if the function $\nu \circ \xi^{-1}$ is convex (resp. concave) on the interval $\xi(I(a,b))$.*

This definition generalizes the one given in [11, 94], where both functions ν and ξ were considered to be strictly increasing and continuous.

Example 4.1. *Let $\nu, \xi : [0,1] \to [0,1]$ be defined by $\nu(x) = (1-x)^2$ and $\xi(x) = 1 - x^2$. Clearly, ν and ξ are strictly decreasing and continuous. One easily verifies that $\nu \circ \xi^{-1}(x) = (1-\sqrt{1-x})^2$ and $\xi \circ \nu^{-1}(x) = 1 - (1-\sqrt{x})^2$. Hence, ν is convex w.r.t. ξ, while ξ is concave w.r.t. ν.*

In the following propositions, we state equivalent formulations of generalized convexity (resp. concavity).

Proposition 4.3. [11] *Let $\nu : I(a,b) \to \mathbb{R}$ be a function and $\xi : I(a,b) \to \mathbb{R}$ be a strictly increasing continuous function. Then*

(i) ν is convex w.r.t. ξ if and only if

$$\begin{vmatrix} 1 & \xi(x) & \nu(x) \\ 1 & \xi(y) & \nu(y) \\ 1 & \xi(z) & \nu(z) \end{vmatrix} \geq 0\,, \tag{4.4}$$

or, equivalently,

$$\frac{\nu(z)-\nu(y)}{\xi(z)-\xi(y)} \geq \frac{\nu(y)-\nu(x)}{\xi(y)-\xi(x)}\,, \tag{4.5}$$

for any $x, y, z \in I(a,b)$ such that $x < y < z$.

(ii) ν is concave w.r.t. ξ if and only if the converse of inequality (4.4) holds for any $x, y, z \in I(a,b)$ such that $x < y < z$.

Similarly, one can obtain the following proposition.

Proposition 4.4. *Let $\nu : I(a,b) \to \mathbb{R}$ be a function and $\xi : I(a,b) \to \mathbb{R}$ be a strictly decreasing continuous function. Then*

(i) ν is convex w.r.t. ξ if and only if

$$\begin{vmatrix} 1 & \xi(x) & \nu(x) \\ 1 & \xi(y) & \nu(y) \\ 1 & \xi(z) & \nu(z) \end{vmatrix} \leq 0\,, \tag{4.6}$$

or, equivalently,

$$\frac{\nu(z)-\nu(y)}{\xi(z)-\xi(y)} \leq \frac{\nu(y)-\nu(x)}{\xi(y)-\xi(x)}\,, \tag{4.7}$$

for any $x, y, z \in I(a,b)$ such that $x < y < z$.

(ii) ν is concave w.r.t. ξ if and only if the converse of inequality (4.6) holds for any $x, y, z \in I(a,b)$ such that $x < y < z$.

4.3. Upper conic functions with a given section

In this section, we introduce the definition of an upper conic function with a given section. A function $N : [0,1] \to [0,1]$ is called *a negation operator* if it is decreasing, and satisfies $N(0) = 1$ and $N(1) = 0$. A negation operator N is called strict if it is continuous and strictly decreasing; a strict negation operator N is called *strong* if it is involutive (see e.g. [82]). A strict negation operator N has exactly one fixed point $a \in]0,1[$, i.e. $N(a) = a$.

For a strict negation operator N, we introduce the following subsets of $[0,1]^2$ (see Figure 4.1):

$$\begin{aligned} S_N &= \{(x,y) \in [0,1]^2 \mid y < N(x)\} \\ F_N &= [0,1]^2 \setminus S_N \,. \end{aligned}$$

The sets S_N and F_N are depicted in Figure 4.1. Let C be a semi-copula and $g : [0,1] \to [0,1]$ be defined by $g(x) = C(x, N(x))$. Then the function $A^u_{N,C} : [0,1]^2 \to [0,1]$ defined by

$$A^u_{N,C}(x,y) = \begin{cases} C(x,y) & \text{, if } (x,y) \in S_N\,, \\ 1 - \dfrac{1-g(x_1)}{1-N(x_1)}(1-y) & \text{, if } (x,y) \in F_N \text{ and } y \neq 1\,, \\ x & \text{, if } y = 1\,, \end{cases} \tag{4.8}$$

where $(x_1, N(x_1))$ is the unique point such that (x,y) is located on the segment $\langle (x_1, N(x_1)), (1,1) \rangle$ (see Figure 4.1), is well defined. The function $A^u_{N,C}$ is called *an upper conic function with section* (N,g) since $A^u_{N,C}(t, N(t)) = g(t)$ for any $t \in [0,1]$, and it is linear on any segment $\langle (t, N(t)), (1,1) \rangle$ in F_N. Note that the collinearity of the points $(x_1, N(x_1))$, (x,y) and $(1,1)$ implies that

$$1 - \frac{1-g(x_1)}{1-N(x_1)}(1-y) = 1 - \frac{1-g(x_1)}{1-x_1}(1-x)\,.$$

This equality ensures the continuity of $A^u_{N,C}$ on F_N. Note also that $A^u_{N,C}$ is continuous if and only if C is continuous on the closure of S_N.

4.4. Upper conic semi-copulas and quasi-copulas with a given section

In this section, we characterize upper conic semi-copulas and quasi-copulas with a given section. For an upper conic function $A^u_{N,C}$, this characterization involves the use of the functions $\varphi, \widehat{\varphi}, \psi, \widehat{\psi} :]0,1[\to \mathbb{R}$ defined by

$$\varphi(x) = \frac{x}{N(x)}\,, \quad \widehat{\varphi}(x) = \frac{1-x}{1-N(x)}\,,$$

$$\psi(x) = \frac{g(x)}{N(x)}\,, \quad \widehat{\psi}(x) = \frac{1-g(x)}{1-N(x)}\,.$$

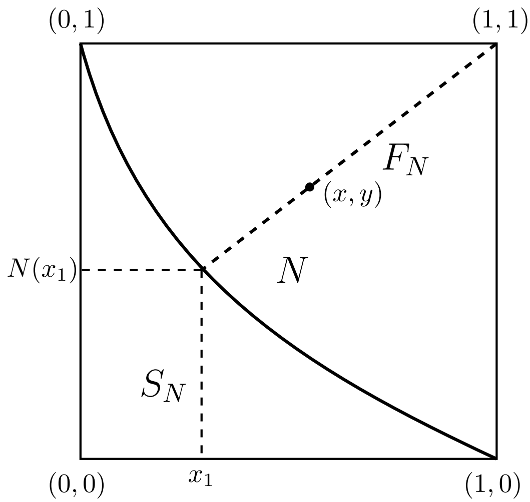


Figure 4.1: Illustration of the subsets F_N and S_N corresponding to a strict negation operator N.

Note that the strict decreasingness of N implies the strict increasingness of φ and the strict decreasingness of $\widehat{\varphi}$.

Proposition 4.5. *Let $N : [0, 1] \to [0, 1]$ be a strict negation operator and C be a semi-copula. Then the function $A^u_{N,C}$ defined by (4.8) is a semi-copula if and only if*

(i) *the function $\widehat{\psi}$ is decreasing;*

(ii) *the function $\dfrac{\widehat{\psi}}{\widehat{\varphi}}$ is increasing.*

Proof. Suppose that conditions (i) and (ii) are satisfied. To prove that $A^u_{N,C}$ is a semi-copula, it suffices to prove its increasingness. Since C is a semi-copula, it suffices to prove the increasingness of $A^u_{N,C}$ in each variable on F_N. We prove the increasingness of $A^u_{N,C}$ in the first variable (the proof of the increasingness in the second variable is similar). Let $(x, y), (x', y) \in F_N$ such that $x \leq x'$ and let $(x_1, N(x_1))$ and $(x_2, N(x_2))$ be the unique points such that (x, y) and (x', y) are located on the segments $\langle (x_1, N(x_1)), (1, 1)\rangle$ and $\langle (x_2, N(x_2)), (1, 1)\rangle$, respectively. The increasingness of $A^u_{N,C}$ in the first variable is then equivalent to

$$(1 - y)\left(\frac{1 - g(x_1)}{1 - N(x_1)} - \frac{1 - g(x_2)}{1 - N(x_2)}\right) = (1 - y)(\widehat{\psi}(x_1) - \widehat{\psi}(x_2)) \geq 0\,. \qquad (4.9)$$

Since $x_1 \leq x_2$ and $\widehat{\psi}$ is decreasing, inequality (4.9) immediately follows.

Conversely, suppose that $A^u_{N,C}$ is a semi-copula. Consider arbitrary $x_1, x_2 \in]0, 1[$

such that $x_1 \leq x_2$, and let (x, y) and (x', y) be two points in F_N that are located on the segments $\langle (x_1, N(x_1)), (1, 1)\rangle$ and $\langle (x_2, N(x_2)), (1, 1)\rangle$, respectively. The increasingness of $A^u_{N,C}$ in the first variable implies that

$$A^u_{N,C}(x', y) - A^u_{N,C}(x, y) \geq 0 \,, \tag{4.10}$$

or, equivalently,

$$\widehat{\psi}(x_1) - \widehat{\psi}(x_2) \geq 0 \,.$$

Hence, the decreasingness of $\widehat{\psi}$ follows. Similarly, the increasingness of $A^u_{N,C}$ in the second variable implies the increasingness of $\dfrac{\widehat{\psi}}{\widehat{\varphi}}$. □

Before characterizing upper conic quasi-copulas, we provide some properties of sections of quasi-copulas. These properties are direct consequences of the increasingness and 1-Lipschitz continuity of quasi-copulas.

Proposition 4.6. *Let $N : [0, 1] \to [0, 1]$ be a strict negation operator. Let C be a quasi-copula and $g : [0, 1] \to [0, 1]$ be defined by $g(x) = C(x, N(x))$. Then it holds that*

(i) $\max(0, x + N(x) - 1) \leq g(x) \leq \min(x, N(x))$, *for any $x \in [0, 1]$;*

(ii) $N(x') - N(x) \leq g(x') - g(x) \leq x' - x$, *for any $x, x' \in [0, 1]$ such that $x \leq x'$.*

Proof. Assertion (i) follows from the bounds on quasi-copulas. Let $x, x' \in [0, 1]$ such that $x \leq x'$. Since C is increasing, it holds that

$$C(x, N(x)) \leq C(x', N(x)) \text{ and } C(x', N(x')) \leq C(x', N(x)) \,.$$

Since C is 1-Lipschitz continuous, it holds that

$$C(x', N(x)) - C(x, N(x)) \leq x' - x \;\text{ and }\; C(x', N(x)) - C(x', N(x')) \leq N(x) - N(x') \,.$$

Using the above inequalities, it follows that

$$\begin{aligned} N(x') - N(x) \leq C(x', N(x')) - C(x', N(x)) &\leq C(x', N(x')) - C(x, N(x)) \\ = g(x') - g(x) \leq C(x', N(x)) - C(x, N(x)) &\leq x' - x \,. \end{aligned}$$

Hence, assertion (ii) follows. □

In fact, assertion (ii) of Proposition 4.6 implies the decreasingness of $\widehat{\psi}$ and the increasingness of $\dfrac{\widehat{\psi}}{\widehat{\varphi}}$.

Proposition 4.7. *Let $N : [0,1] \to [0,1]$ be a strict negation operator with fixed point a. Let C be a quasi-copula and $g : [0,1] \to [0,1]$ be defined by $g(x) = C(x, N(x))$. Then it holds that*

(i) *the functions $\widehat{\psi}$ and $\dfrac{\widehat{\psi}}{\widehat{\varphi}}$ are decreasing and increasing, respectively;*

(ii) *the function $\zeta : [0,a[\,\cup\,]a,1] \to \mathbb{R}$ defined by $\zeta(x) = \dfrac{x-g(x)}{x-N(x)}$ is decreasing on the interval $[0,a[$ as well as on the interval $]a,1]$.*

Proof. Consider arbitrary $x, x' \in]0,1[$ such that $x < x'$. Since $N(x') - N(x) \leq g(x') - g(x) \leq x' - x$ and $1 - g(x) \geq \max(1-x, 1-N(x))$, it holds that

$$\frac{g(x)-g(x')}{N(x)-N(x')} \leq 1 \leq \frac{1-g(x)}{1-N(x)} \quad \text{and} \quad \frac{g(x')-g(x)}{x'-x} \leq 1 \leq \frac{1-g(x)}{1-x}\,.$$

The latter inequalities imply that

$$(g(x)-g(x'))(1-N(x)) \leq (1-g(x))(N(x)-N(x'))$$

and

$$(g(x')-g(x))(1-x) \leq (1-g(x))(x'-x)\,.$$

Some elementary manipulations yield

$$(1-N(x))(1-g(x')) - (1-N(x'))(1-g(x)) \leq 0$$

and

$$(1-x')(1-g(x)) - (1-x)(1-g(x')) \leq 0\,,$$

or, equivalently,

$$(1-N(x))(1-N(x'))(\widehat{\psi}(x') - \widehat{\psi}(x)) \leq 0$$

and

$$(1-x)(1-x')\left(\frac{\widehat{\psi}(x)}{\widehat{\varphi}(x)} - \frac{\widehat{\psi}(x')}{\widehat{\varphi}(x')}\right) \leq 0\,.$$

Hence, the decreasingness of $\widehat{\psi}$ and the increasingness $\dfrac{\widehat{\psi}}{\widehat{\varphi}}$ follow, i.e. assertion (i) follows.

Similarly, one can prove assertion (ii). $\square$

Corollary 4.1. *Let $N : [0,1] \to [0,1]$ be a strict negation operator and C be a quasi-copula. Then the function $A^u_{N,C}$ defined by (4.8) is a semi-copula.*

Proposition 4.8. *Let $N : [0,1] \to [0,1]$ be a strict negation operator and C be a quasi-copula. Then the function $A^u_{N,C}$ defined by (4.8) is a quasi-copula if and only if*

(i) *the function $\widehat{\psi} - \widehat{\varphi}$ is increasing;*

(ii) *the function $\dfrac{\widehat{\psi} - 1}{\widehat{\varphi}}$ is decreasing.*

Proof. Suppose that conditions (i) and (ii) are satisfied. Due to Corollary 4.1, the function $A^u_{N,C}$ is a semi-copula. Therefore, to prove that $A^u_{N,C}$ is a quasi-copula, we need to show that it is 1-Lipschitz continuous. Recall that the 1-Lipschitz continuity is equivalent to the 1-Lipschitz continuity in each variable. Since C is a quasi-copula and $A^u_{N,C}$ is continuous, it is sufficient to show its 1-Lipschitz continuity in each variable on F_N. We prove the 1-Lipschitz continuity of $A^u_{N,C}$ in the first variable (the proof of the 1-Lipschitz continuity in the second variable is similar). Let $(x,y),(x',y) \in F_N$ such that $x \leq x'$ and suppose that $(x_1, N(x_1))$ and $(x_2, N(x_2))$ are the unique points such that (x,y) and (x',y) are located on the segments $\langle (x_1, N(x_1)), (1,1)\rangle$ and $\langle (x_2, N(x_2)), (1,1)\rangle$, respectively. The 1-Lipschitz continuity of $A^u_{N,C}$ in the first variable is equivalent to

$$(1-y)\left(\frac{1-g(x_1)}{1-N(x_1)} - \frac{1-g(x_2)}{1-N(x_2)}\right) \leq x' - x\,. \tag{4.11}$$

Since the points $(x_1, N(x_1))$, (x,y) and $(1,1)$ as well as the points $(x_2, N(x_2))$, (x',y) and $(1,1)$ are collinear, it follows that

$$x' - x = (1-y)\left(\frac{1-x_1}{1-N(x_1)} - \frac{1-x_2}{1-N(x_2)}\right).$$

Therefore, inequality (4.11) is equivalent to

$$\frac{1-g(x_1)}{1-N(x_1)} - \frac{1-g(x_2)}{1-N(x_2)} \leq \frac{1-x_1}{1-N(x_1)} - \frac{1-x_2}{1-N(x_2)}\,,$$

or, equivalently,

$$(\widehat{\psi}(x_2) - \widehat{\varphi}(x_2)) - (\widehat{\psi}(x_1) - \widehat{\varphi}(x_1)) \geq 0\,.$$

Since $x_1 \leq x_2$ and $\widehat{\psi} - \widehat{\varphi}$ is increasing, the above inequality immediately follows.

Conversely, suppose that $A^u_{N,C}$ is a quasi-copula. Consider arbitrary $x_1, x_2 \in \,]0,1[$ such that $x_1 \leq x_2$, and let (x,y) and (x',y) be two points in F_N that are located on the segments $\langle (x_1, N(x_1)), (1,1)\rangle$ and $\langle (x_2, N(x_2)), (1,1)\rangle$, respectively. The

1-Lipschitz continuity of $A^u_{N,C}$ in the first variable implies that

$$A^u_{N,C}(x', y) - A^u_{N,C}(x, y) \leq x' - x\,, \tag{4.12}$$

or, equivalently,

$$(\widehat{\psi}(x_2) - \widehat{\varphi}(x_2)) - (\widehat{\psi}(x_1) - \widehat{\varphi}(x_1)) \geq 0\,.$$

Hence, the increasingness of $\widehat{\psi} - \widehat{\varphi}$ follows. Similarly, the 1-Lipschitz continuity of $A^u_{N,C}$ in the second variable implies the decreasingness of $\dfrac{\widehat{\psi} - 1}{\widehat{\varphi}}$. □

Example 4.2. *Let $N : [0,1] \to [0,1]$ be a strict negation operator and $C = T_{\mathbf{M}}$. One easily verifies that the conditions of Proposition 4.8 are satisfied. Moreover, the corresponding upper conic function is $T_{\mathbf{M}}$ itself.*

Example 4.3. *Let $N : [0,1] \to [0,1]$ be a strict negation operator such that $N(x) \leq 1 - x$ for any $x \in [0,1]$, and $C = T_{\mathbf{L}}$. One easily verifies that the conditions of Proposition 4.5 are satisfied and the corresponding upper conic function $A^u_{N,C}$ is a semi-copula. On the other hand, due to Proposition 4.8, $A^u_{N,C}$ is a quasi-copula if and only if the functions $\dfrac{N(x)}{1-x}$ and $\dfrac{x}{1-N(x)}$ are decreasing and increasing on the interval $]0,1[$, respectively.*

Example 4.4. *Let $N : [0,1] \to [0,1]$ be a strict negation operator such that $N(x) \geq 1 - x$ for any $x \in [0,1]$, and $C = T_{\mathbf{L}}$. One easily verifies that the conditions of Proposition 4.8 are satisfied. Moreover, the corresponding upper conic function is $T_{\mathbf{L}}$ itself.*

Example 4.5. *Let $N : [0,1] \to [0,1]$ be a strict negation operator and $C = T_{\mathbf{P}}$. One easily verifies that the conditions of Proposition 4.8 are satisfied and the corresponding upper conic function $A^u_{N,C}$ is a quasi-copula, and hence, a semi-copula. Consequently, when the considered semi-copula is $T_{\mathbf{P}}$, the class of upper conic semi-copulas and the class of upper conic quasi-copulas coincide.*

Proposition 4.9. *Let $A^u_{N,C}$ be an upper conic quasi-copula. Then it holds that*

(i) *if $g(x_0) = x_0$ for some $x_0 \in]0,1[$, then $g(x) = x$ for any $x \in [0, x_0]$;*

(ii) *if $g(x_0) = N(x_0)$ for some $x_0 \in]0,1[$, then $g(x) = N(x)$ for any $x \in [x_0, 1]$.*

Proof. Suppose that $A^u_{N,C}$ is an upper conic quasi-copula and suppose further that $g(x_0) = x_0$ for some $x_0 \in]0,1[$. Since $A^u_{N,C}$ is an upper conic quasi-copula, it holds that the function $\widehat{\psi} - \widehat{\varphi}$ is increasing. Therefore, $\widehat{\psi}(x) - \widehat{\varphi}(x) \leq \widehat{\psi}(x_0) - \widehat{\varphi}(x_0) = 0$ for any $x \in]0, x_0]$. Hence, $g(x) \geq x$ for any $x \in]0, x_0]$. Since $g(x) \leq x$ for any $x \in [0,1]$ and $g(0) = 0$, it must hold that $g(x) = x$ for any $x \in [0, x_0]$, and hence, assertion (i) follows.

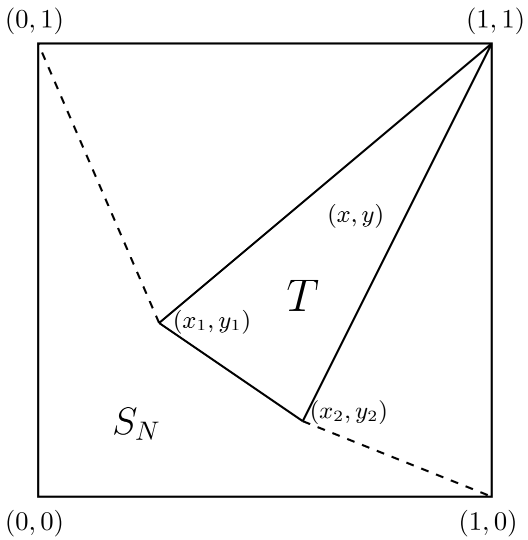


Figure 4.2: An illustration of the triangle T.

Assertion (ii) can be proved similarly using the decreasingness of the function $\dfrac{\widehat{\psi}-1}{\widehat{\varphi}}$ on the interval $[x_0, 1[$. $\square$

4.5. Upper conic copulas with a given section

4.5.1. The case of an arbitrary semi-copula C

Suppose that the graph of the strict negation operator N contains a segment determined by the points $(x_1, N(x_1))$ and $(x_2, N(x_2))$, with $x_1 < x_2$. Suppose further that the function g is linear on the interval $[x_1, x_2]$. Let us introduce the notations $y_i = N(x_i)$ and $z_i = g(x_i)$ for $i \in \{1, 2\}$. From the definition of $A^u_{N,C}$, it follows that $A^u_{N,C}$ is linear on the triangle $T := \Delta_{\{(x_1,y_1),(x_2,y_2),(1,1)\}}$. This configuration is depicted in Figure 4.2.

For any $(x, y) \in T$, it holds that

$$A^u_{N,C}(x,y) = ax + by + c\,. \tag{4.13}$$

Furthermore,

$$\begin{aligned} ax_1 + by_1 + c &= z_1 \\ ax_2 + by_2 + c &= z_2 \\ a + b + c &= 1\,. \end{aligned}$$

Solving this system of linear equations, we obtain

$$A^u_{N,C}(x,y) = \frac{rx + sy + t}{u}\,, \tag{4.14}$$

where

$$\begin{aligned} r &= z_2 - z_1 + y_1 - y_2 + y_2 z_1 - y_1 z_2 \\ s &= z_1 - z_2 + x_2 - x_1 + x_1 z_2 - x_2 z_1 \\ t &= x_1 y_2 - x_2 y_1 - y_2 z_1 + y_1 z_2 - x_1 z_2 + x_2 z_1 \\ u &= x_2 - x_1 + y_1 - y_2 + x_1 y_2 - x_2 y_1\,. \end{aligned}$$

Lemma 4.1. *For any $v_1, v_2 \in [x_1, x_2]$ it holds that*

$$\frac{\widehat{\psi}(v_2) - \widehat{\psi}(v_1)}{\widehat{\varphi}(v_2) - \widehat{\varphi}(v_1)} = \frac{r}{u}\,.$$

Proof. The proof is a matter of elementary manipulations. □

Next, we characterize the upper conic copulas $A^u_{N,C}$ when the functions N and g are piecewise linear. To this end, we need the following proposition.

Proposition 4.10. *Let $A^u_{N,C}$ be an upper conic function such that N and g are linear on the interval $[x_1, x_2]$ as well as on the interval $[x_2, x_3]$. Let R be a rectangle located in the polygon enclosed by the points $(x_1, N(x_1))$, $(x_2, N(x_2))$, $(x_3, N(x_3))$ and $(1,1)$ such that its diagonal is a subset of the segment $\langle (x_2, N(x_2)), (1,1) \rangle$. Then it holds that $V_{A^u_{N,C}}(R) \geq 0$ if and only if $\widehat{\psi}$ is convex w.r.t. $\widehat{\varphi}$ on the interval $[x_1, x_3]$.*

Proof. Consider the rectangle $R = [x, x'] \times [y, y']$ depicted in Figure 4.3. Let us introduce the notations $y_i = N(x_i)$ and $z_i = g(x_i)$ for $i \in \{1, 2, 3\}$. The $A^u_{N,C}$-volume of this rectangle is given by

$$V_{A^u_{N,C}}(R) = (x - x') \left(\frac{r'}{u'} - \frac{r}{u} \right)\,,$$

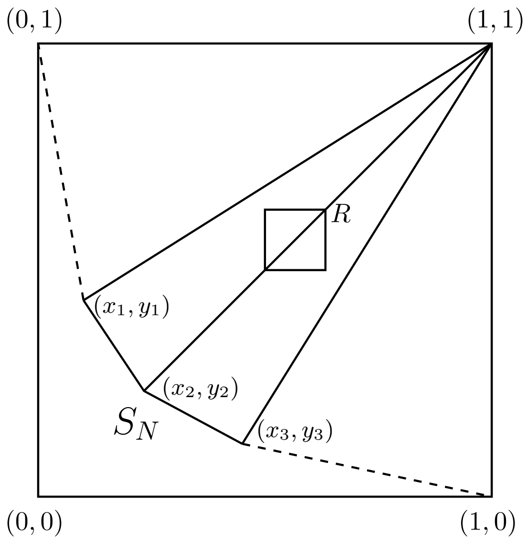


Figure 4.3: An illustration for the proof of Proposition 4.10.

where

$$\begin{aligned} r' &= z_3 - z_2 + y_2 - y_3 + y_3 z_2 - y_2 z_3 \\ u' &= x_3 - x_2 + y_2 - y_3 + x_2 y_3 - x_3 y_2 \,. \end{aligned}$$

The nonnegativity of $V_{A^u_{N,C}}(R)$ is equivalent to

$$\frac{r'}{u'} - \frac{r}{u} \leq 0\,. \tag{4.15}$$

From Lemma 4.1, it follows that

$$\frac{r}{u} = \frac{\widehat{\psi}(x_2) - \widehat{\psi}(x_1)}{\widehat{\varphi}(x_2) - \widehat{\varphi}(x_1)} \quad \text{and} \quad \frac{r'}{u'} = \frac{\widehat{\psi}(x_3) - \widehat{\psi}(x_2)}{\widehat{\varphi}(x_3) - \widehat{\varphi}(x_2)}\,.$$

Therefore, inequality (4.15) is equivalent to

$$\frac{\widehat{\psi}(x_3) - \widehat{\psi}(x_2)}{\widehat{\varphi}(x_3) - \widehat{\varphi}(x_2)} \leq \frac{\widehat{\psi}(x_2) - \widehat{\psi}(x_1)}{\widehat{\varphi}(x_2) - \widehat{\varphi}(x_1)}\,. \tag{4.16}$$

In fact, inequality (4.16) is equivalent to the convexity of $\widehat{\psi}$ w.r.t. $\widehat{\varphi}$ on the interval $[x_1, x_3]$. This can be seen as follows. Since $\widehat{\varphi}$ is strictly decreasing, we use Proposition 4.4(i) to prove that $\widehat{\psi}$ is convex w.r.t. $\widehat{\varphi}$. Consider arbitrary $x, y, z \in [x_1, x_3]$ such that $x < y < z$. Suppose first that inequality (4.16) is satisfied. To prove that $\widehat{\psi}$ is convex w.r.t. $\widehat{\varphi}$, we distinguish the following subcases:

(a) If $x < y < z \leq x_2$ or $x_2 \leq x < y < z$, then it holds that

$$\frac{\widehat{\psi}(z) - \widehat{\psi}(y)}{\widehat{\varphi}(z) - \widehat{\varphi}(y)} = \frac{\widehat{\psi}(y) - \widehat{\psi}(x)}{\widehat{\varphi}(y) - \widehat{\varphi}(x)} \,.$$

(b) If $x \leq x_2 \leq y < z$, then using Lemma 4.1, it follows that the inequality

$$\frac{\widehat{\psi}(z) - \widehat{\psi}(y)}{\widehat{\varphi}(z) - \widehat{\varphi}(y)} \leq \frac{\widehat{\psi}(y) - \widehat{\psi}(x)}{\widehat{\varphi}(y) - \widehat{\varphi}(x)}$$

is equivalent to

$$\frac{\widehat{\psi}(y) - \widehat{\psi}(x_2)}{\widehat{\varphi}(y) - \widehat{\varphi}(x_2)} \leq \frac{\widehat{\psi}(y) - \widehat{\psi}(x_2)}{\widehat{\varphi}(y) - \widehat{\varphi}(x)} + \frac{\widehat{\psi}(x_2) - \widehat{\psi}(x)}{\widehat{\varphi}(y) - \widehat{\varphi}(x)} \,.$$

After some elementary manipulations and using (4.16), the latter inequality becomes

$$\frac{\widehat{\psi}(y) - \widehat{\psi}(x_2)}{\widehat{\varphi}(y) - \widehat{\varphi}(x_2)} = \frac{\widehat{\psi}(x_3) - \widehat{\psi}(x_2)}{\widehat{\varphi}(x_3) - \widehat{\varphi}(x_2)} \leq \frac{\widehat{\psi}(x_2) - \widehat{\psi}(x)}{\widehat{\varphi}(x_2) - \widehat{\varphi}(x)} = \frac{\widehat{\psi}(x_2) - \widehat{\psi}(x_1)}{\widehat{\varphi}(x_2) - \widehat{\varphi}(x_1)} \,,$$

which always holds.

(c) The case $x < y \leq x_2 \leq z$ can be handled similarly.

The converse of the proof is immediate. □

Remark 4.1. *Let $A^u_{N,C}$ be an upper conic function such that $g(x) = 0$ for any $x \in [0,1]$. Then inequality (4.16) is equivalent to the convexity of N on the interval $[x_1, x_3]$.*

Proof. Setting $g(x) = 0$, inequality (4.16) is equivalent to

$$\frac{\frac{1}{1-N(x_3)} - \frac{1}{1-N(x_2)}}{\frac{1-x_3}{1-N(x_3)} - \frac{1-x_2}{1-N(x_2)}} \leq \frac{\frac{1}{1-N(x_2)} - \frac{1}{1-N(x_1)}}{\frac{1-x_2}{1-N(x_2)} - \frac{1-x_1}{1-N(x_1)}} \,,$$

or, equivalently,

$$\frac{N(x_3) - N(x_2)}{x_2 - x_3 + N(x_3) - N(x_2) + x_3 N(x_2) - x_2 N(x_3)}$$
$$\leq \frac{N(x_2) - N(x_1)}{x_1 - x_2 + N(x_2) - N(x_1) + x_2 N(x_1) - x_1 N(x_2)} \,.$$

Setting $N(x_i) = y_i$ for any $i \in \{1, 2, 3\}$, the latter inequality is equivalent to

$$\frac{y_1 - y_2}{y_1 - y_2 + x_2 - x_1 + x_1 y_2 - x_2 y_1} - \frac{y_2 - y_3}{y_2 - y_3 + x_3 - x_2 + x_2 y_3 - x_3 y_2} \geq 0 \,,$$

which is exactly inequality (2.10) in Chapter 2. The latter inequality has led to the convexity of the function $f = N$ on the interval $[x_1, x_3]$. $\square$

Proposition 4.11. *Let $N : [0,1] \to [0,1]$ be a piecewise linear strict negation operator and C be a copula such that g is piecewise linear. Then the function $A^u_{N,C}$ defined by (4.8) is a copula if and only if*

(i) *the function $\widehat{\psi}$ is convex w.r.t. $\widehat{\varphi}$;*

(ii) *for any $x, x' \in [0,1]$ such that $x \leq x'$, it holds that*

$$C(x, N(x')) + A^u_{N,C}(x', N(x)) \geq g(x) + g(x')\,.$$

Proof. Suppose that conditions (i) and (ii) are satisfied. Since $A^u_{N,C}$ satisfies the boundary conditions of a semi-copula, we need to show its 2-increasingness. Since N and g are piecewise linear, the set F_N consists of triangles of the type $\Delta = \Delta_{\{(u,N(u)),(v,N(v)),(1,1)\}}$ such that g is linear on the interval $[u, v]$ (see Figure 4.2). Due to the additivity of volumes, it suffices to consider a restricted number of cases. Consider a rectangle $R := [x, x'] \times [y, y'] \subseteq [0,1]^2$.

(a) Suppose that $R \subseteq F_N$. We distinguish the following subcases:

  (1) Suppose that R is included in a triangle of type Δ. Then $V_{A^u_{N,C}}(R) = 0$.

  (2) Suppose that the diagonal of R is along the edge shared by two triangles of type Δ. Using Proposition 4.10, condition (i) implies the positivity of $V_{A^u_{N,C}}(R)$.

(b) Suppose that R is included in S_N. Then it holds that $V_{A^u_{N,C}}(R) = V_C(R) \geq 0$.

(c) Suppose that the corners (x, y') and (x', y) of R are located on the graph of N, i.e. $R = [x, x'] \times [N(x'), N(x)]$. Using condition (ii), the positivity of $V_{A^u_{N,C}}(R)$ immediately follows.

Conversely, suppose that $A^u_{N,C}$ is a copula. Proposition 4.10 implies that $\widehat{\psi}$ is convex w.r.t. $\widehat{\varphi}$ on $]0,1[$ and hence, condition (i) follows. Let $x, x' \in [0,1]$ such that $x \leq x'$ and consider the rectangle $R = [x, x'] \times [N(x'), N(x)]$. Since $V_{A^u_{N,C}}(R)$ is positive, it then follows that

$$V_{A^u_{N,C}}(R) = C(x, N(x')) - g(x) - g(x') + A^u_{N,C}(x', N(x) \geq 0\,,$$

and hence, condition (ii) follows. $\square$

The above result can be generalized for any strict negation operator N and any copula C. To this end, we first need to construct a class of $[0,1]^2 \to [0,1]$ functions C^* starting from a given copula C. Consider $x_0 = 0 < x_1 < \cdots < x_n = 1$ and $y_0 = 0 < y_1 < \cdots < y_m = 1$. In the points (x_i, y_j) with $i \in \{0, \ldots, n\}$, $j \in \{0, \ldots, m\}$, we set $C^*(x_i, y_j) = C(x_i, y_j)$; in other words, C^* coincides with C

on the given grid. On any rectangle $R_{i,j} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ with $i \in \{0, \ldots, n-1\}$, $j \in \{0, \ldots, m-1\}$, the function C^* is defined to be linear on the triangle $\Delta_{\{(x_i,y_j),(x_i,y_{j+1}),(x_{i+1},y_j)\}}$ as well as on the triangle $\Delta_{\{(x_i,y_{j+1}),(x_{i+1},y_{j+1}),(x_{i+1},y_j)\}}$. We show in the following proposition that such a function C^* is a copula.

Proposition 4.12. *Consider $x_0 = 0 < x_1 < \ldots < x_n = 1$ and $y_0 = 0 < y_1 < \ldots < y_m = 1$. For any copula C, the function $C^* : [0,1]^2 \to [0,1]$ defined by* $C^*(x,y) =$

$$\begin{cases} C(x,y) & , \text{ if } (x,y) = (x_i, y_j)\,, \\ a_{i,j}x + b_{i,j}y + c_{i,j} & , \text{ if } (x,y) \in R_{i,j} \text{ and } y \leq \frac{y_j - y_{j+1}}{x_{i+1}-x_i}(x - x_i) + y_{j+1}\,, \\ a'_{i,j}x + b'_{i,j}y + c'_{i,j} & , \text{ if } (x,y) \in R_{i,j} \text{ and } y > \frac{y_j - y_{j+1}}{x_{i+1}-x_i}(x - x_i) + y_{j+1}\,, \end{cases} \tag{4.17}$$

where $i \in \{0, \ldots, n-1\}$, $j \in \{0, \ldots, m-1\}$, $R_{i,j} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ and

$$\begin{aligned} a_{i,j} &= \frac{C(x_{i+1},y_j) - C(x_i,y_j)}{x_{i+1}-x_i}\,, & a'_{i,j} &= \frac{C(x_{i+1},y_{j+1}) - C(x_i,y_{j+1})}{x_{i+1}-x_i}\,, \\ b_{i,j} &= \frac{C(x_i,y_{j+1}) - C(x_i,y_j)}{y_{j+1}-y_j}\,, & b'_{i,j} &= \frac{C(x_{i+1},y_{j+1}) - C(x_{i+1},y_j)}{y_{j+1}-y_j}\,, \\ c_{i,j} &= C(x_i,y_j) - a_{i,j}x_i - b_{i,j}y_j\,, & c'_{i,j} &= C(x_{i+1},y_{j+1}) - a'_{i,j}x_{i+1} - b'_{i,j}y_{j+1}\,, \end{aligned}$$

is a copula.

Proof. Since C is a copula, it holds that C^* satisfies the boundary conditions of a semi-copula. Therefore, to prove that C^* is a copula, it suffices to show its 2-increasingness. Let $R = [x,x'] \times [y,y'] \subseteq [0,1]^2$. Due to the additivity of volumes, it suffices to consider that R is located in a rectangle $R_{i,j}$, with $i \in \{0, \ldots, n-1\}$ and $j \in \{0, \ldots, m-1\}$.

(a) If R is located in the triangle $\Delta_{(x_i,y_j),(x_i,y_{j+1}),(x_{i+1},y_j)}$ or in the triangle $\Delta_{(x_i,y_{j+1}),(x_{i+1},y_{j+1}),(x_{i+1},y_j)}$, then $V_{C^*}(R) = 0$.

(b) If the opposite diagonal of R is a subset of the segment $\langle (x_i, y_{j+1}), (x_{i+1}, y_j) \rangle$, then it holds that

$$V_{C^*}(R) = (x'-x)(a'_{i,j} - a_{i,j}) = \frac{x'-x}{x_{i+1}-x_i} V_C(R_{i,j}) \geq 0\,.$$

□

Remark 4.2. *The construction of the copula C^* is related to the orthogonal grid construction of copulas [16] and the construction of piecewise linear aggregation functions based on triangulation [21].*

Lemma 4.2. *Let $N : [0,1] \to [0,1]$ be a strict negation operator and C be a copula. If N and g are linear on the interval $[a,b] \subseteq]0,1[$, then the function $\widehat{\psi} \circ \widehat{\varphi}^{-1}$ is linear on the interval $[\widehat{\varphi}(b), \widehat{\varphi}(a)]$ as well.*

Proof. Since N and g are linear on the interval $[a,b]$, there exist $a_1, a_2, b_1, b_2 \in \mathbb{R}$ such that $N(x) = a_1 x + b_1$ and $g(x) = a_2 x + b_2$ for any $x \in [a,b]$. Note that $a_1 + b_1 \neq 1$ due to fact that $b < 1$. Some elementary manipulations show that

$$\widehat{\psi} \circ \widehat{\varphi}^{-1}(x) = \frac{1 - a_2 - b_2}{1 - a_1 - b_2} + \frac{a_2(1 - b_1) - a_1(1 - b_2)}{1 - a_1 - b_1}\, x\,,$$

i.e. $\widehat{\psi} \circ \widehat{\varphi}^{-1}$ is linear on the interval $[\widehat{\varphi}(b), \widehat{\varphi}(a)]$. □

Theorem 4.1. *Let $N : [0,1] \to [0,1]$ be a strict negation operator and C be a copula. Then the function $A^u_{N,C}$ defined by (4.8) is a copula if and only if*

(i) *the function $\widehat{\psi}$ is convex w.r.t. $\widehat{\varphi}$;*

(ii) *for any $x, x' \in [0,1]$ such that $x \leq x'$, it holds that*

$$C(x, N(x')) + A^u_{N,C}(x', N(x)) \geq g(x) + g(x')\,.$$

Proof. Suppose that conditions (i) and (ii) are satisfied. Since $A^u_{N,C}$ satisfies the boundary conditions of a semi-copula, we need to show its 2-increasingness. Due to the additivity of volumes, it suffices to consider a restricted number of cases. Consider a rectangle $R := [x, x'] \times [y, y'] \subseteq [0,1]^2$.

(i) If $R \subseteq F_N$, then let $\mathbf{b_1} = (x_1, N(x_1))$, $\mathbf{b_2} = (x_2, N(x_2))$, $\mathbf{b_3} = (x_3, N(x_3))$ and $\mathbf{b_4} = (x_4, N(x_4))$ be four (possibly coinciding) points on the graph of N such that the points (x, y), (x, y'), (x', y) and (x', y') are located on the segments $\langle \mathbf{b_1}, (1,1)\rangle$, $\langle \mathbf{b_2}, (1,1)\rangle$, $\langle \mathbf{b_3}, (1,1)\rangle$ and $\langle \mathbf{b_4}, (1,1)\rangle$, respectively (see Figure 4.4). Let $\mathbf{c_1} = (x_1, g(x_1))$, $\mathbf{c_2} = (x_2, g(x_2))$, $\mathbf{c_3} = (x_3, g(x_3))$ and $\mathbf{c_4} = (x_4, g(x_4))$ be the points on the graph of g corresponding to $\mathbf{b_1}$, $\mathbf{b_2}$, $\mathbf{b_3}$ and $\mathbf{b_4}$, respectively. The points $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ and $\mathbf{b}_4$, together with $(0,1)$ and $(1,0)$, determine a piecewise linear strict negation operator N_1 such that $N(x_i) = N_1(x_i)$ for any $i \in \{1,2,3,4\}$. The points $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$ and $\mathbf{c}_4$, together with $(0,0)$ and $(1,0)$, also determine a piecewise linear function g_1 such that $g(x_i) = g_1(x_i) = C_1(x_i, N(x_i))$ for any $i \in \{1,2,3,4\}$, for some copula C_1 (due to Proposition 4.12, such a copula C_1 always exists). Let us introduce the functions $\widehat{\varphi}_1$ and $\widehat{\psi}_1$ defined by

$$\widehat{\varphi}_1(x) = \frac{1 - x}{1 - N_1(x)}\,, \quad \widehat{\psi}_1(x) = \frac{1 - g_1(x)}{1 - N_1(x)}\,.$$

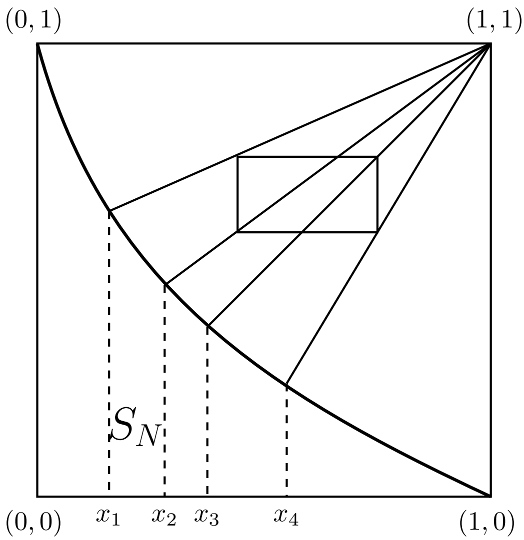


Figure 4.4: Illustration for the proof of Theorem 4.1.

Due to Lemma 4.2, the function $\widehat{\psi_1} \circ \widehat{\varphi_1}^{-1}$ is piecewise linear. Since $\widehat{\psi}$ is convex w.r.t. $\widehat{\varphi}$ it holds that

$$\begin{aligned} \frac{\widehat{\psi}_1(x_{i+1}) - \widehat{\psi}_1(x_i)}{\widehat{\varphi}_1(x_{i+1}) - \widehat{\varphi}_1(x_i)} &= \frac{\widehat{\psi}(x_{i+1}) - \widehat{\psi}(x_i)}{\widehat{\varphi}(x_{i+1}) - \widehat{\varphi}(x_i)} \\ \leq \frac{\widehat{\psi}(x_i) - \widehat{\psi}(x_{i-1})}{\widehat{\varphi}(x_i) - \widehat{\varphi}(x_{i-1})} &= \frac{\widehat{\psi}_1(x_i) - \widehat{\psi}_1(x_{i-1})}{\widehat{\varphi}_1(x_i) - \widehat{\varphi}_1(x_{i-1})} \end{aligned}$$

for any $i \in \{2, 3\}$, i.e. $\widehat{\psi}_1$ is convex w.r.t. $\widehat{\varphi}_1$ on the interval $[x_1, x_4]$. Consider now $x_\beta \in]0, 1[$ such that $x_\beta < x_1$. Let us introduce the notations $x_2' = \widehat{\varphi}(x_2)$, $x_1' = \widehat{\varphi}(x_1)$ and $x_\beta' = \widehat{\varphi}(x_\beta)$. Let us further introduce the functions h and h_1 defined by $h = \widehat{\psi} \circ \widehat{\varphi}^{-1}$ and $h_1 = \widehat{\psi}_1 \circ \widehat{\varphi}_1{}^{-1}$. Since $\widehat{\psi}$ is convex w.r.t. to $\widehat{\varphi}$ it follows that h is convex, and hence,

$$\begin{aligned} \frac{h_1(x_\beta') - h_1(x_1')}{x_\beta' - x_1'} \leq \frac{h(x_\beta') - h_1(x_1')}{x_\beta' - x_1'} &= \frac{h(x_\beta') - h(x_1')}{x_\beta' - x_1'} \\ \leq \frac{h(x_1') - h(x_2')}{x_1' - x_2'} &= \frac{h_1(x_1') - h_1(x_2')}{x_1' - x_2'}, \end{aligned}$$

i.e. $\widehat{\psi}_1$ is convex w.r.t. $\widehat{\varphi}_1$ on $[x_\beta, x_2]$. Similarly, one proves that $\widehat{\psi}_1$ is convex w.r.t. $\widehat{\varphi}_1$ on $[x_3, x_\alpha]$ with $x_4 < x_\alpha < 1$. Therefore, $\widehat{\psi}_1$ is convex w.r.t. $\widehat{\varphi}_1$. Due to Proposition 4.11, the upper conic function $A^u_{N_1,C_1}$ is a copula. Therefore,

$$V_{A^u_{N,C}}(R) = V_{A^u_{N_1,C_1}}(R) \geq 0\,.$$

(ii) The proof of the cases when $R \subseteq S_N$ or when the corners (x, y') and (x', y) of R are located on the graph of N is similar to the corresponding ones in the proof of Proposition 4.11.

Conversely, suppose that $A^u_{N,C}$ is a copula and suppose further that $\widehat{\psi}$ is not convex w.r.t. $\widehat{\varphi}$, i.e. there exist $0 < x < y < z < 1$ such that

$$\frac{\widehat{\psi}(z) - \widehat{\psi}(y)}{\widehat{\varphi}(z) - \widehat{\varphi}(y)} > \frac{\widehat{\psi}(y) - \widehat{\psi}(x)}{\widehat{\varphi}(y) - \widehat{\varphi}(x)}\,.$$

Since $\widehat{\psi}$ and $\widehat{\varphi}$ are continuous, there exists $\epsilon > 0$ such that for any $x' \in [y-\epsilon, y+\epsilon]$, it holds that

$$\frac{\widehat{\psi}(z) - \widehat{\psi}(x')}{\widehat{\varphi}(z) - \widehat{\varphi}(x')} > \frac{\widehat{\psi}(x') - \widehat{\psi}(x)}{\widehat{\varphi}(x') - \widehat{\varphi}(x)}\,.$$

Due to Proposition 4.10, any rectangle located in the polygon enclosed by the points $(x, N(x))$, $(y, N(y)$, $(z, N(z))$ and $(1, 1)$ has a negative $A^u_{N,C}$-volume, a contradiction. The proof of condition (ii) is similar to the previous one. □

To conclude this section, we discuss the lower-lower and upper-upper tail dependences of upper conic copulas. Note that, for an upper conic copula $A^u_{N,C}$, its lower-lower tail dependence obviously coincides with the lower-lower tail dependence of C, while its upper-upper tail dependence is given by

$$\lambda_{UU} = 2 - \frac{1 - g(a)}{1 - a}\,,$$

where a is the fixed point of N.

4.5.2. The case of the product copula

We now focus on upper conic functions $A_{N,C}$ when the considered semi-copula C is the product copula, i.e. $C = T_{\mathbf{P}}$.

Lemma 4.3. *Let $N : [0, 1] \to [0, 1]$ be a strict negation operator. Then the function $A^u_{N,T_{\mathbf{P}}}$ defined by (4.8) satisfies the inequality*

$$A^u_{N,T_{\mathbf{P}}}(x, y) \geq T_{\mathbf{P}}(x, y)\,, \tag{4.18}$$

for any $(x, y) \in [0, 1]^2$.

Proof. Since $C = T_{\mathbf{P}}$, it suffices to prove inequality (4.18) for any $(x,y) \in F_N$, i.e.

$$1 - \frac{1 - x_1 N(x_1)}{1 - x_1}(1-x) \geq xy\,, \tag{4.19}$$

where $(x_1, N(x_1))$ is the unique point such that (x,y) is located on the segment $\langle (x_1, N(x_1)), (1,1) \rangle$. Since the points $(x_1, N(x_1))$, (x,y) and $(1,1)$ are collinear, it follows that

$$y = 1 - \frac{1 - N(x_1)}{1 - x_1}(1-x)\,.$$

Inequality (4.19) is then equivalent to

$$1 - \frac{1 - x_1 N(x_1)}{1 - x_1}(1-x) \geq x\left(1 - \frac{1 - N(x_1)}{1 - x_1}(1-x)\right)\,,$$

or, equivalently, $x \geq x_1$, which always holds. □

Consequently, any upper conic copula $A^u_{N,T_{\mathbf{P}}}$ is positive quadrant dependent (PQD) [88].

Proposition 4.13. *Let $N : [0,1] \to [0,1]$ be a strict negation operator. Then the function $A^u_{N,T_{\mathbf{P}}}$ defined by (4.8) is a copula if and only if the function $\widehat{\varphi}$ is convex.*

Proof. Condition (i) of Theorem 4.1 is equivalent to

$$\frac{\frac{1-zN(z)}{1-N(z)} - \frac{1-yN(y)}{1-N(y)}}{\widehat{\varphi}(z) - \widehat{\varphi}(y)} \leq \frac{\frac{1-yN(y)}{1-N(y)} - \frac{1-xN(x)}{1-N(x)}}{\widehat{\varphi}(y) - \widehat{\varphi}(x)}\,, \tag{4.20}$$

for any $x, y, z \in]0,1[$ such that $x < y < z$. Some elementary manipulations yield

$$1 + \frac{z-y}{\widehat{\varphi}(z) - \widehat{\varphi}(y)} \leq 1 + \frac{y-x}{\widehat{\varphi}(y) - \widehat{\varphi}(x)}\,,$$

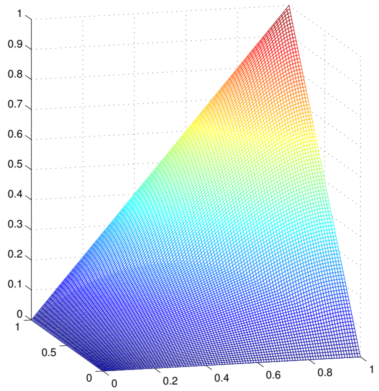
or, equivalently,

$$\frac{\widehat{\varphi}(z) - \widehat{\varphi}(y)}{z-y} \geq \frac{\widehat{\varphi}(y) - \widehat{\varphi}(x)}{y-x}\,,$$

for any $x, y, z \in]0,1[$ such that $x < y < z$, i.e. $\widehat{\varphi}$ is convex. Due to Lemma 4.3, it holds that

$$T_{\mathbf{P}}(x, N(x')) + A^u_{N,T_{\mathbf{P}}}(x', N(x)) \geq T_{\mathbf{P}}(x, N(x')) + T_{\mathbf{P}}(x', N(x)) \geq g_{T_{\mathbf{P}}}(x) + g_{T_{\mathbf{P}}}(x')\,,$$

for any $x, x' \in]0,1[$ such that $x \leq x'$, i.e. condition (ii) of Theorem 4.1 always holds. □



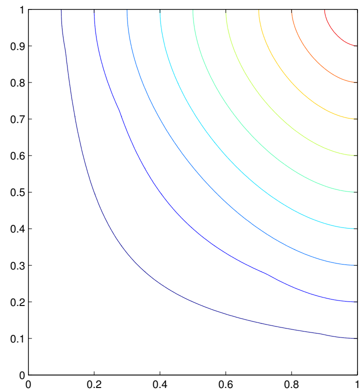


Figure 4.5: The 3D plot and contour plot of the copula of Example 4.6.

Example 4.6. *Let $N : [0,1] \to [0,1]$ be the strict negation operator defined by $N(x) = 1 - x$. The function $\widehat{\varphi}$ is then given by $\widehat{\varphi}(x) = \frac{1-x}{x}$. One easily verifies that $\widehat{\varphi}$ is convex, and hence, the corresponding upper conic function $A^u_{N,T_\mathbf{P}}$ is a copula and is given by*

$$A^u_{N,T_\mathbf{P}}(x,y) = \begin{cases} xy & , \text{ if } y \leq 1-x\,, \\ x+y-1+\dfrac{(1-x)(1-y)}{2-x-y} & , \text{ otherwise.} \end{cases}$$

The 3D plot and contour plot of $A^u_{N,\Pi}$ are depicted in Figure 4.5.

In the next proposition, we show that the convexity of a strict negation operator N is a sufficient condition for the convexity of the function $\widehat{\varphi}$.

Proposition 4.14. *Let $N : [0,1] \to [0,1]$ be a strict negation operator. If N is convex, then the function $\widehat{\varphi}$ is convex.*

Proof. Since N is continuous, it holds that $\widehat{\varphi}$ is continuous on the interval $]0,1[$. Therefore, in order to prove the convexity of $\widehat{\varphi}$, it suffices [93] to show that

$$\widehat{\varphi}\left(\frac{x+y}{2}\right) \leq \frac{\widehat{\varphi}(x)+\widehat{\varphi}(y)}{2}\,, \qquad \text{for any } x,y \in]0,1[\,.$$

Let $x,y \in]0,1[$ and suppose w.l.o.g that $x < y$. Since N is convex, it holds that

$$\widehat{\varphi}\left(\frac{x+y}{2}\right) = \frac{2-x-y}{2\left(1-N\left(\frac{x+y}{2}\right)\right)} \leq \frac{2-x-y}{2-N(x)-N(y)}\,.$$

In order to complete the proof, we need to show that

$$\frac{2-x-y}{2-N(x)-N(y)} \le \frac{2-x-y-N(x)(1-y)-N(y)(1-x)}{2(1-N(x))(1-N(y))} = \frac{\widehat{\varphi}(x)+\widehat{\varphi}(y)}{2}.$$

After some elementary manipulations, the latter inequality is equivalent to

$$x-y-N(x)+N(y)+yN(x)-xN(y) \le 0.$$

After adding and subtracting the term $xN(x)$, the latter inequality becomes

$$(x-y)(1-N(x))+(N(y)-N(x))(1-x) \le 0.$$

Since $x<y$ and taking into account that N is decreasing, the latter inequality clearly holds. □

Proposition 4.15. *Let $N:[0,1]\to[0,1]$ be a strict negation operator. If N is convex, then the function $A^u_{N,T_\mathbf{P}}$ defined by (4.8) is a copula.*

In fact, the convexity of N is not a necessary condition in general. This can be seen in the following example.

Example 4.7. *Let $N:[0,1]\to[0,1]$ be the strict negation operator defined by $N(x)=1-x^2$. The function $\widehat{\varphi}$ is then given by $\widehat{\varphi}(x)=\dfrac{1-x}{x^2}$. One easily verifies that N is concave, while the function $\widehat{\varphi}$ is convex. Hence, the function $A^u_{N,T_\mathbf{P}}$ defined by (4.8) is a copula; it is given by*

$$A^u_{N,T_\mathbf{P}}(x,y)=\begin{cases} xy & \text{, if } y\le 1-x^2\,,\\ x-\dfrac{\left(y-1+\sqrt{(1-y)(5-4x-y)}\right)^3}{4(1-x)(3-2x-y-\sqrt{(1-y)(5-4x-y)})} & \text{, otherwise}\,. \end{cases}$$

Note that, for an upper conic copula $A^u_{N,T_\mathbf{P}}$, the upper-upper tail dependence is given by $\lambda_{UU}=1-a$, where a is the fixed point of N.

4.6. Lower conic functions with a given section

In this section, we introduce the definition of a lower conic function with a given section. Let N be a strict negation operator. Let C be a semi-copula and $g:[0,1]\to[0,1]$ be defined by $g(x)=C(x,N(x))$. Then the function $A^l_{N,C}:[0,1]^2\to[0,1]$

defined by

$$A^l_{N,C}(x,y) = \begin{cases} \dfrac{g(x_0)}{N(x_0)}y & \text{, if } (x,y) \in S_N \text{ and } y \neq 0\,, \\ C(x,y) & \text{, if } (x,y) \in F_N\,, \\ 0 & \text{, if } y = 0\,, \end{cases} \tag{4.21}$$

where $(x_0, N(x_0))$ is the unique point such that (x, y) is located on the segment $\langle(0,0),(x_0,N(x_0))\rangle$, is well defined. The function $A^l_{N,C}$ is called *a lower conic function with section* (N, g) since $A^l_{N,C}(t, N(t)) = g(t)$ for any $t \in [0,1]$, and it is linear on any segment $\langle(0,0),(t,N(t))\rangle$ in S_N. Note that the collinearity of the points $(x_0, N(x_0))$, (x, y) and $(0, 0)$ implies that

$$\frac{g(x_0)}{N(x_0)}y = \frac{g(x_0)}{x_0}x\,.$$

This equality ensures the continuity of $A^l_{N,C}$ on S_N. Note also that $A^l_{N,C}$ is continuous if and only if C is continuous.

Using the same techniques as before the following propositions can be proved.

Proposition 4.16. *Let $N : [0,1] \to [0,1]$ be a strict negation operator and C be a semi-copula. Then the function $A^l_{N,C}$ defined by (4.21) is a semi-copula if and only if the functions ψ and $\dfrac{\psi}{\varphi}$ are increasing and decreasing, respectively.*

Proposition 4.17. *Let $N : [0,1] \to [0,1]$ be a strict negation operator and C be a quasi-copula. Then the function $A^l_{N,C}$ defined by (4.21) is a quasi-copula if and only if*

(i) *the functions ψ and $\dfrac{\psi}{\varphi}$ are increasing and decreasing, respectively;*

(ii) *the functions $\psi - \varphi$ and $\dfrac{\psi - 1}{\varphi}$ are decreasing and increasing, respectively.*

Proposition 4.18. *Let $N : [0,1] \to [0,1]$ be a strict negation operator and C be a copula. Then the function $A^l_{N,C}$ defined by (4.21) is a copula if and only if*

(i) *the function ψ is concave w.r.t. φ;*

(ii) *for any $x, x' \in [0,1]$ such that $x \leq x'$, it holds that*

$$C(x, N(x')) + A^l_{N,C}(x', N(x)) \geq g(x) + g(x')\,.$$

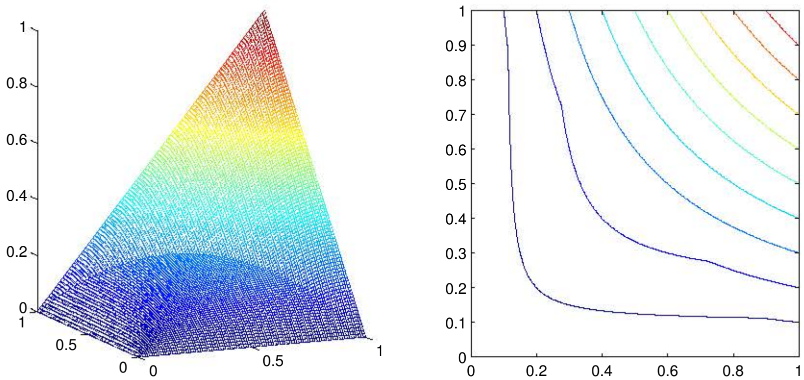


Figure 4.6: The 3D plot and contour plot of the copula of Example 4.8.

Proposition 4.19. *Let $N : [0,1] \to [0,1]$ be a strict negation operator. Then the function $A^l_{N,T_{\mathbf{P}}}$ defined by (4.21) is a copula if and only if the function φ is convex.*

Example 4.8. *Let $N : [0,1] \to [0,1]$ be the strict negation operator defined by $N(x) = 1 - x$. The function φ is then given by $\varphi(x) = \frac{x}{1-x}$. One easily verifies that φ is convex, and hence, the corresponding lower conic function $A^l_{N,T_{\mathbf{P}}}$ is a copula and is given by*

$$A^l_{N,T_{\mathbf{P}}}(x,y) = \begin{cases} \dfrac{xy}{x+y} & , \textit{ if } y \leq 1-x\,, \\[2ex] xy & , \textit{ otherwise.} \end{cases}$$

The 3D plot and contour plot of $A^l_{N,\Pi}$ are depicted in Figure 4.6.

Note that, for a lower conic copula $A^l_{N,C}$, its upper-upper tail dependence coincides with the upper-upper tail dependence of C, while its lower-lower tail dependence is given by

$$\lambda_{LL} = \frac{g(a)}{a}\,,$$

where a is the fixed point of N. Note also that $\lambda_{LL} = a$ when $C = T_{\mathbf{P}}$.

4.7. Biconic functions with a given section

In this section, we introduce the definition of a biconic function with a given section. Let N be a strict negation operator. Let C be a semi-copula and $g : [0,1] \to [0,1]$ be defined by $g(x) = C(x, N(x))$. Then the function $A^b_{N,C} : [0,1]^2 \to [0,1]$ defined

by

$$A^b_{N,C}(x,y) = \begin{cases} \dfrac{g(x_0)}{N(x_0)}y & \text{, if } (x,y) \in S_N \text{ and } y \neq 0\,, \\ 1 - \dfrac{1-g(x_1)}{1-N(x_1)}(1-y) & \text{, if } (x,y) \in F_N \text{ and } y \neq 1\,, \\ \min(x,y) & \text{, otherwise,} \end{cases} \tag{4.22}$$

where $(x_0, N(x_0))$ (resp. $(x_1, N(x_1))$) is the unique point such that (x,y) is located on the segment $\langle(0,0),(x_0,N(x_0))\rangle$ (resp. $\langle(x_1,N(x_1)),(1,1)\rangle$), is well defined. The function $A^b_{N,C}$ is called *a biconic function with section* (N,g) since $A^b_{N,C}(t,N(t)) = g(t)$ for any $t \in [0,1]$, and it is linear on any segment $\langle(0,0),(t,N(t))\rangle$ (resp. $\langle(t,N(t)),(1,1)\rangle$ in S_N (resp. F_N).

Using the same technique as before, the following propositions can be proved.

Proposition 4.20. *Let $N : [0,1] \to [0,1]$ be a strict negation operator and C be a semi-copula. Then the function $A^b_{N,C}$ defined by (4.22) is a semi-copula if and only if the functions ψ and $\dfrac{\widehat{\psi}}{\widehat{\varphi}}$ are increasing, and the functions $\dfrac{\psi}{\varphi}$ and $\widehat{\psi}$ are decreasing.*

Proposition 4.21. *Let $N : [0,1] \to [0,1]$ be a strict negation operator and C be a quasi-copula. Then the function $A^b_{N,C}$ defined by (4.22) is a quasi-copula if and only if the conditions of Propositions 4.6 and 4.17 are satisfied.*

Proposition 4.22. *Let $N : [0,1] \to [0,1]$ be a strict negation operator with fixed point a and C be a copula. Let $a \in]0,1[$. Then the function $A^b_{N,C}$ defined by (4.22) is a copula if and only if*

(i) *the function $\widehat{\psi}$ is convex w.r.t. $\widehat{\varphi}$;*

(ii) *the function ψ is concave w.r.t. φ;*

(iii) *the function $\zeta :\ [0,a[\cup]a,1] \to \mathbb{R}$ defined by $\zeta(x) = \dfrac{x-g(x)}{x-N(x)}$ is decreasing on the interval $[0,a[$ as well as on the interval $]a,1]$.*

Note that condition (iii) is always satisfied when g is a section of a quasi-copula C. Therefore, the following corollary is immediate.

Corollary 4.2. *Let $N : [0,1] \to [0,1]$ be a strict negation operator and C be a copula. Then the function $A^b_{N,C}$ defined by (4.22) is a copula if and only if*

(i) *the function $\widehat{\psi}$ is convex w.r.t. $\widehat{\varphi}$;*

(ii) *the function ψ is concave w.r.t. φ.*

Proposition 4.23. *Let $N : [0,1] \to [0,1]$ be a strict negation operator. Then the function $A^b_{N,T_{\mathbf{P}}}$ defined by (4.22) is a copula if and only if the functions φ and $\widehat{\varphi}$ are convex.*

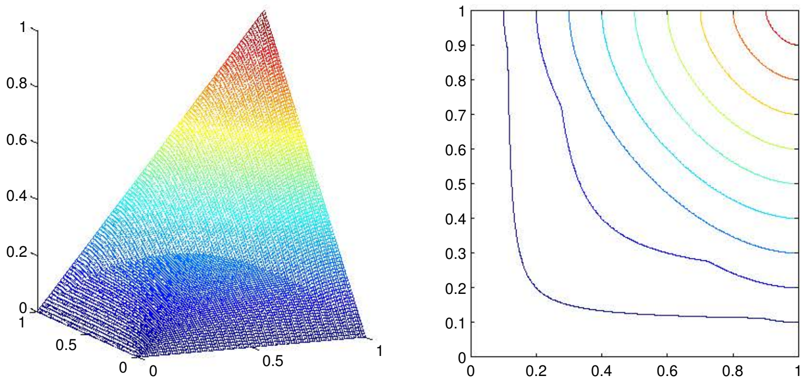


Figure 4.7: The 3D plot and contour plot of the copula of Example 4.9.

Example 4.9. *Let $N : [0,1] \to [0,1]$ be the strict negation operator defined by $N(x) = 1-x$. The functions φ and $\widehat{\varphi}$ are then given by $\varphi(x) = \frac{x}{1-x}$ and $\widehat{\varphi}(x) = \frac{1-x}{x}$. One easily verifies that φ and $\widehat{\varphi}$ are convex, and hence, the corresponding biconic function $A^b_{N,T_\mathbf{P}}$ is a copula and is given by*

$$A^b_{N,T_\mathbf{P}}(x,y) = \begin{cases} \dfrac{xy}{x+y} & , \textit{ if } y \leq 1-x\,, \\[2ex] x+y-1+\dfrac{(1-x)(1-y)}{2-x-y} & , \textit{ otherwise.} \end{cases}$$

The 3D plot and contour plot of $A^b_{N,\Pi}$ are depicted in Figure 4.7.

In fact, similar results concerning the above classes can be obtained when we consider sections of semi-copulas that are determined by a strictly increasing $[0,1] \to [0,1]$ function $\tilde{N}$ such that $\tilde{N}(0) = 0$ and $\tilde{N}(1) = 1$. Consider for instance a biconic function $A^b_{N,C}$ and let $\tilde{N}, \tilde{g}_C : [0,1] \to [0,1]$ be defined by $\tilde{N}(x) = 1 - N(x)$ and $\tilde{g}_C(x) = x - g(x)$. Note that $\tilde{N}$ is strictly increasing and satisfies $\tilde{N}(0) = 0$ and $\tilde{N}(1) = 1$. The function $A^b_{\tilde{N},\tilde{g}_C} : [0,1]^2 \to [0,1]$ defined by

$$A^b_{\tilde{N},\tilde{g}_C} = \varphi_2(A^b_{N,g})\,, \tag{4.23}$$

where φ_2 is the transformation defined in (1.3), is a biconic function with section $(\tilde{N}, \tilde{g})$, and it is linear on any segment connecting a point from the graph of $\tilde{N}$ to the point $(0,1)$ as well as on any segment connecting a point from the graph of $\tilde{N}$ to the point $(1,0)$.

Proposition 4.24. *Let $N : [0,1] \to [0,1]$ be a strict negation operator and C be a copula. Then the function $A^{\tilde{b}}_{\tilde{N},\tilde{g}}$ defined by (4.23) is a copula if and only if*

(i) *the function $\dfrac{\tilde{g}_C}{\tilde{N}}$ is convex w.r.t. $\dfrac{x-1}{\tilde{N}}$;*

(ii) *the function $\dfrac{\tilde{g}_C}{1-\tilde{N}}$ is convex w.r.t. $\dfrac{x}{1-\tilde{N}}$.*

5 Ortholinear and paralinear semi-copulas

5.1. Introduction

In the previous chapters, we have considered the linear interpolation on segments connecting points from a line in the unit square to the corners of the unit square. We introduce in this chapter semi-copulas that are constructed by linear interpolation on segments that are perpendicular (resp. parallel) to the diagonal of the unit square.

The surface of the semi-copula $T_{\mathbf{M}}$ is constituted from (linear) segments connecting the points $(2a, 0, 0)$ and $(a, a, \delta_{T_{\mathbf{M}}}(a))$ as well as segments connecting the points $(0, 2a, 0)$ and $(a, a, \delta_{T_{\mathbf{M}}}(a))$, with $0 \leq a \leq 1/2$, and segments connecting the points $(2a-1, 1, 2a-1)$ and $(a, a, \delta_{T_{\mathbf{M}}}(a))$ as well as segments connecting the points $(1, 2a-1, 2a-1)$ and $(a, a, \delta_{T_{\mathbf{M}}}(a))$, with $1/2 \leq a \leq 1$. Note that the surface of the semi-copula $T_{\mathbf{M}}$ is also constituted from segments connecting the points $(0, 1-2a, 0)$ and $(a, 1-a, \omega_{T_{\mathbf{M}}}(a))$ as well as segments connecting the points $(2a, 1, 2a)$ and $(a, 1-a, \omega_{T_{\mathbf{M}}}(a))$, with $0 \leq a \leq 1/2$, and segments connecting the points $(2a-1, 0, 0)$ and $(a, 1-a, \omega_{T_{\mathbf{M}}}(a))$ as well as segments connecting the points $(1, 2(1-a), 2(1-a))$ and $(a, 1-a, \omega_{T_{\mathbf{M}}}(a))$, with $1/2 \leq a \leq 1$. The above observation has motivated the present construction.

This chapter is organized as follows. In the following section, we introduce ortholinear functions. In Sections 5.3–5.6, we characterize the classes of ortholinear semi-copulas, ortholinear quasi-copulas, ortholinear copulas and ortholinear copulas supported on a set with Lebesgue measure zero. For ortholinear copulas, we provide simple expressions for Spearman's rho, Gini's gamma and Kendall's tau in Section 5.7. In Section 5.8, we study the aggregation of ortholinear (semi-, quasi-)copulas. The class of paralinear functions is introduced in Section 5.9.

5.2. Ortholinear functions

Ortholinear functions are constructed by linear interpolation on segments that are perpendicular to the diagonal of the unit square. The linear interpolation scheme of this type of function on some segments is depicted in Figure 5.1.

Let us introduce the notation $z = \frac{x+y}{2}$. Let us further consider the subtriangles T_1,

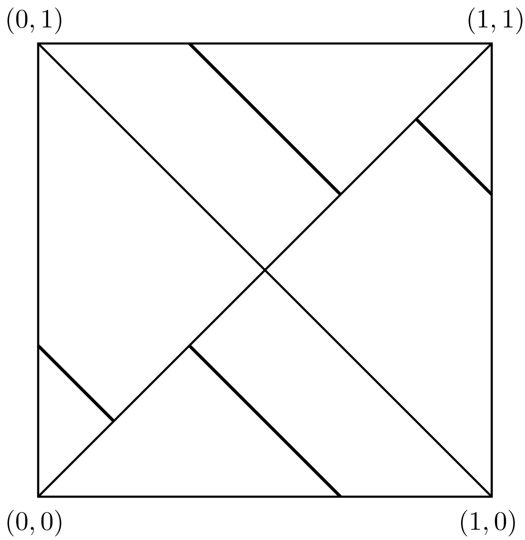


Figure 5.1: Some segments on which an ortholinear function is linear.

T_2, T_3 and T_4 of the unit square (see Figure 5.2) given by

$$\begin{aligned}
T_1 &:= \{(x, y) \in [0, 1]^2 \mid 0 \le y \le 1/2 \text{ and } y \le x \le 1 - y\}\,,\\
T_2 &:= \{(x, y) \in [0, 1]^2 \mid 0 \le x \le 1/2 \text{ and } x \le y \le 1 - x\}\,,\\
T_3 &:= \{(x, y) \in [0, 1]^2 \mid 1/2 \le y \le 1 \text{ and } 1 - y \le x \le y\}\,,\\
T_4 &:= \{(x, y) \in [0, 1]^2 \mid 1/2 \le x \le 1 \text{ and } 1 - x \le y \le x\}\,.
\end{aligned}$$

Let $\delta \in \mathcal{D}_{\mathrm{S}}$. The function $A_\delta : [0, 1]^2 \to [0, 1]$ given by

$$A_\delta(x, y) = \begin{cases} y\dfrac{\delta(z)}{z} & \text{, if } (x, y) \in T_1\,,\\[2ex] x\dfrac{\delta(z)}{z} & \text{, if } (x, y) \in T_2\,,\\[2ex] x - (1 - y)\dfrac{z - \delta(z)}{1 - z} & \text{, if } (x, y) \in T_3\,,\\[2ex] y - (1 - x)\dfrac{z - \delta(z)}{1 - z} & \text{, if } (x, y) \in T_4\,, \end{cases} \tag{5.1}$$

is well defined. This function is called *the ortholinear function with diagonal section* δ, since it is linear on segments connecting the points (x, x), $(2x, 0)$ and $(0, 2x)$, with $x \le 1/2$, as well as on segments connecting the points (x, x), $(2x - 1, 1)$ and $(1, 2x - 1)$, with $x \ge 1/2$.

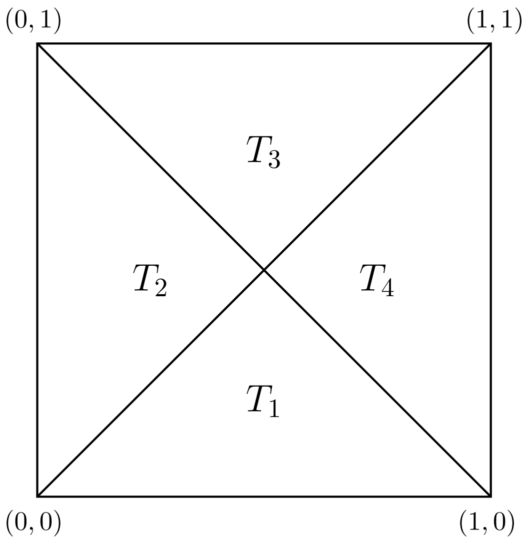


Figure 5.2: Illustration for the triangles T_1, T_2, T_3 and T_4.

Equation (5.1) can be rewritten in a more compact form as

$$A_\delta(x,y) = \begin{cases} \min(x,y)\dfrac{\delta(z)}{z} & \text{, if } (x,y) \in T_1 \cup T_2\,, \\ \min(x,y) - (1-\max(x,y))\dfrac{z-\delta(z)}{1-z} & \text{, if } (x,y) \in T_3 \cup T_4\,. \end{cases}$$

For any ortholinear function, the boundary conditions of a semi-copula always hold. Note that an ortholinear function A_δ is uniquely determined by its diagonal section. Note also that an ortholinear function A_δ is continuous if and only if δ is continuous.

5.3. Ortholinear semi-copulas

In this section, we characterize the elements of $\mathcal{D}_{\mathrm{S}}^{\mathrm{ac}}$ for which the corresponding ortholinear function is a semi-copula. Let us consider the function λ_δ defined as in Chapter 3.

Proposition 5.1. *Let $\delta \in \mathcal{D}_{\mathrm{S}}^{\mathrm{ac}}$. Then the ortholinear function A_δ is a semi-copula if and only if*

(i) *the function λ_δ is increasing on the interval* $]0, 1/2]$*;*

(ii) *the function $\xi_\delta : [0,1] \to [0,1]$, defined by $\xi_\delta(x) = (1-x)(x-\delta(x))$, is decreasing on the interval* $[1/2, 1[$ *.*

Proof. Suppose conditions (i) and (ii) are satisfied. To prove that A_δ is a semi-copula it suffices to prove its increasingness in each variable. Since A_δ is symmetric, it suffices to prove its increasingness in each variable on $T_2 \cup T_3$. We prove that A_δ is increasing in the second variable (the proof of the increasingness in the first variable is similar).

Let $(x, y), (x, y') \in T_2 \cup T_3$ such that $y \leq y'$. Let us introduce the notation $z' = \frac{x+y'}{2}$.

If $(x, y), (x, y') \in T_2$, then the increasingness of A_δ is equivalent to

$$x\frac{\delta(z')}{z'} - x\frac{\delta(z)}{z} = x(\lambda_\delta(z') - \lambda_\delta(z)) \geq 0\,. \tag{5.2}$$

Since $z \leq z'$ and λ_δ is increasing on the interval $]0, 1/2]$, inequality (5.2) immediately follows.

If $(x, y), (x, y') \in T_3$, then the increasingness of A_δ is equivalent to

$$-(1-y')\frac{z' - \delta(z')}{1-z'} + (1-y)\frac{z - \delta(z)}{1-z} \geq 0\,,$$

or, equivalently,

$$\frac{(1-y)}{(1-z)^2}\xi_\delta(z) - \frac{(1-y')}{(1-z')^2}\xi_\delta(z') \geq 0\,.$$

Since $y \leq y'$, it holds that $z \leq z'$. Using the fact that ξ_δ is decreasing on the interval $[1/2, 1[$, it then follows that

$$\frac{(1-y)}{(1-z)^2}\xi_\delta(z) - \frac{(1-y')}{(1-z')^2}\xi_\delta(z') \geq \frac{(1-y')}{(1-z')^2}(\xi_\delta(z) - \xi_\delta(z')) \geq 0\,.$$

If $(x, y) \in T_2$ and $(x, y') \in T_3$, then the preceding cases imply that $A_\delta(x, y') - A_\delta(x, y) =$

$$(A_\delta(x, y') - A_\delta(x, 1-x)) + (A_\delta(x, 1-x) - A_\delta(x, y)) \geq 0\,.$$

Conversely, suppose that A_δ is a semi-copula. Let $y, y' \in]0, 1/2]$ such that $y \leq y'$ and $x \in [0, 1]$ such that $x \leq y$ and $x + y' \leq 1$. Clearly, the points $(x, 2y - x)$ and $(x, 2y' - x)$ are located in T_2. The increasingness of A_δ in the second variable implies

$$A_\delta(x, 2y' - x) - A_\delta(x, 2y - x) \geq 0\,, \tag{5.3}$$

or, equivalently,

$$x(\lambda_\delta(y') - \lambda_\delta(y)) \geq 0\,.$$

Hence, the increasingness of λ_δ on the interval $]0, 1/2]$ follows.

Let $y, y' \in [1/2, 1[$ such that $y < y'$ and $x \in [0, 1]$ such that $x \leq y$ and $x + y \geq 1$.

The increasingness of A_δ in the second variable implies

$$(1-y)\frac{z-\delta(z)}{1-z} - (1-y')\frac{z'-\delta(z')}{1-z'} \geq 0\,. \tag{5.4}$$

Dividing by $y'-y$ and taking the limit $y' \to y$, inequality (5.4) becomes

$$\frac{z-\delta(z)}{1-z} - \frac{1}{2}(1-y)\left(\frac{z-\delta(z)}{1-z}\right)' \geq 0\,,$$

where the derivative exists. Setting $x = y$, the last inequality is equivalent to

$$2y - 1 - \delta(y) + (1-y)\delta'(y) \geq 0\,,$$

or, equivalently, $\xi'_\delta(y) \leq 0$, where the derivative exists. Since δ is absolutely continuous, it holds that ξ_δ is absolutely continuous. The fact that $\xi'_\delta(y) \leq 0$, where the derivative exists, on the interval $[1/2, 1[$, then implies that ξ_δ is decreasing on the interval $[1/2, 1[$. □

Example 5.1. *Consider the diagonal functions $\delta_{T_\mathbf{M}}$ and $\delta_{T_\mathbf{L}}$. Clearly, $\delta_{T_\mathbf{M}}$ and $\delta_{T_\mathbf{L}}$ belong to $\mathcal{D}_\mathrm{S}^\mathrm{ac}$. One easily verifies that the functions $\lambda_{\delta_{T_\mathbf{M}}}$ and $\lambda_{\delta_{T_\mathbf{L}}}$ are increasing on the interval $]0, 1/2]$, and that the functions $\xi_{\delta_{T_\mathbf{M}}}$ and $\xi_{\delta_{T_\mathbf{L}}}$ are decreasing on the interval $[1/2, 1[$. The corresponding ortholinear semi-copulas are $T_\mathbf{M}$ and $T_\mathbf{L}$, respectively.*

Example 5.2. *Consider the diagonal function $\delta_\theta(x) = x^{1+\theta}$ with $\theta \in [0, 1]$. Clearly, $\delta_\theta \in \mathcal{D}_\mathrm{S}^\mathrm{ac}$. One easily verifies that the function λ_{δ_θ} is increasing on the interval $]0, 1/2]$ for any $\theta \in [0, 1]$, and that the function ξ_{δ_θ} is decreasing on the interval $[1/2, 1[$ for any $\theta \in [0, 1]$. The corresponding family of ortholinear semi-copulas is given by*

$$A_\theta(x, y) = \begin{cases} \min(x, y) z^\theta & , \text{ if } (x, y) \in T_1 \cup T_2\,, \\ \min(x, y) - (1 - \max(x, y))\dfrac{z(1-z^\theta)}{1-z} & , \text{ if } (x, y) \in T_3 \cup T_4\,. \end{cases}$$

Proposition 5.2. *Let A_δ be an ortholinear semi-copula such that $\delta \in \mathcal{D}_\mathrm{S}^\mathrm{ac}$ and suppose that $\delta(x_0) = x_0$ for some $x_0 \in]0, 1[$. Then it holds that $\delta(x) = x$ for any $x \in [x_0, 1]$.*

Proof. Suppose that A_δ is an ortholinear semi-copula and suppose further that $\delta(x_0) = x_0$ for some $x_0 \in]0, 1/2]$. The function λ_δ, defined in Proposition 5.1, is increasing on the interval $]0, 1/2]$. Therefore, $\lambda_\delta(x) \geq \lambda_\delta(x_0) = 1$ for any $x \in [x_0, 1/2]$. Hence, $\lambda_\delta(x) = 1$ for any $x \in [x_0, 1/2]$. Similarly, the decreasingness of the function ξ_δ, defined in Proposition 5.1, implies that $\delta(x) = x$ for any $x \in [1/2, 1]$. In case $x_0 \in]1/2, 1[$ the decreasingness of ξ_δ is sufficient to prove the required result. Consequently, $\delta(x) = x$ for any $x \in [x_0, 1]$. □

5.4. Ortholinear quasi-copulas

In this section, we characterize the elements of $\mathcal{D}$ for which the corresponding ortholinear function is a quasi-copula. Let us consider the function μ_δ defined as in Chapter 3.

Proposition 5.3. *Let $\delta \in \mathcal{D}$. Then the ortholinear function A_δ is a quasi-copula if and only if*

(i) *the function λ_δ, and the function $\psi_\delta :\ [0,1] \to [0,1]$, defined by*

$$\psi_\delta(x) = x(x - \delta(x))\,,$$

are increasing on the interval $]0, 1/2]$*;*

(ii) *the functions μ_δ, and the function $\xi_\delta :\ [0,1] \to [0,1]$, defined by*

$$\xi_\delta(x) = (1 - x)(x - \delta(x))\,,$$

are respectively increasing and decreasing on the interval $[1/2, 1[$.

Proof. Suppose conditions (i) and (ii) are satisfied. Due to Proposition 5.1, the function A_δ is increasing. Therefore, to prove that A_δ is a quasi-copula, we need to show that it is 1-Lipschitz continuous. Recall that the 1-Lipschitz continuity is equivalent to the 1-Lipschitz continuity in each variable. Since A_δ is symmetric, it is sufficient to show that A_δ is 1-Lipschitz continuous in each variable on $T_2 \cup T_3$. We prove that A_δ is 1-Lipschitz continuous in the first variable on $T_2 \cup T_3$ (the proof of the 1-Lipschitz continuity in the second variable is similar).

Let $(x, y), (x', y) \in T_2 \cup T_3$ such that $x \le x'$. Let us introduce the notation $u = \frac{x'+y}{2}$.

If $(x, y), (x', y) \in T_2$, then the 1-Lipschitz continuity of A_δ is equivalent to

$$x' \frac{\delta(u)}{u} - x \frac{\delta(z)}{z} \le x' - x\,,$$

or, equivalently,

$$\frac{x'}{u^2}\psi_\delta(u) - \frac{x}{z^2}\psi_\delta(z) \ge 0\,.$$

Since $x \le x'$, it holds that $z \le u$. Using the fact that ψ_δ is increasing on the interval $]0, 1/2]$, it then follows that

$$\frac{x'}{u^2}\psi_\delta(u) - \frac{x}{z^2}\psi_\delta(z) \ge \frac{x}{z^2}(\psi_\delta(u) - \psi_\delta(z)) \ge 0\,.$$

If $(x,y),(x',y)\in T_3$, then the 1-Lipschitz continuity of A_δ is equivalent to

$$(1-y)(\mu_\delta(u)-\mu_\delta(z))\geq 0\,. \tag{5.5}$$

Since $z\leq u$ and μ_δ is increasing on the interval $[1/2,1[$, inequality (5.5) immediately follows.

If $(x,y)\in T_2$ and $(x',y)\in T_3$, then the preceding cases imply that $A_\delta(x',y)-A_\delta(x,y)=$

$$(A_\delta(x',y)-A_\delta(1-y,y))+(A_\delta(1-y,y)-A_\delta(x,y))\leq x'-x\,.$$

Conversely, suppose that A_δ is a quasi-copula. Proposition 5.1 implies the increasingness of λ_δ on the interval $]0,1/2]$ and the decreasingness of ξ_δ on the interval $[1/2,1[$. Let $x,x'\in[1/2,1[$ such that $x\leq x'$ and $y\in[0,1]$ such that $x'\leq y$ and $x+y\geq 1$. Clearly, the points $(2x-y,y)$ and $(2x'-y,y)$ are located in T_3. The 1-Lipschitz continuity of A_δ in the first variable implies that

$$A_\delta(2x'-y,y)-A_\delta(2x-y,y)\leq 2(x'-x)\,, \tag{5.6}$$

or, equivalently,

$$(1-y)(\mu_\delta(x')-\mu_\delta(x))\geq 0\,.$$

Hence, the increasingness of μ_δ on the interval $[1/2,1[$ follows.

Let $x,x'\in]0,1/2[$ such that $x<x'$ and $y\in[0,1]$ such that $x'\leq y$ and $x'+y\leq 1$. The 1-Lipschitz continuity of A_δ in the first variable implies that

$$x'\frac{\delta(z')}{z'}-x\frac{\delta(z)}{z}\leq x'-x\,. \tag{5.7}$$

Dividing by $x'-x$ and taking the limit $x'\to x$, inequality (5.7) becomes

$$\frac{\delta(z)}{z}+\frac{1}{2}x\left(\frac{\delta(z)}{z}\right)'\leq 1\,,$$

where the derivative exists. Setting $x=y$, the last inequality is equivalent to

$$2x-\delta(x)-x\delta'(x)\geq 0\,,$$

or, equivalently, $\psi_\delta'(x)\geq 0$, where the derivative exists. Since δ is absolutely continuous, it holds that ψ_δ is absolutely continuous. The fact that $\psi_\delta'(y)\geq 0$, where the derivative exists, on the interval $]0,1/2]$, then implies that ψ_δ is increasing on the interval $]0,1/2]$. $\square$

Example 5.3. *Consider the diagonal functions in Example 5.2. Clearly, conditions (i) and (ii) of Proposition 5.3 are fulfilled. The corresponding family of ortholinear semi-copulas is a family of ortholinear quasi-copulas.*

Proposition 5.4. *Let A_δ be an ortholinear quasi-copula. Then it holds that*

(i) *if $\delta(x_0) = x_0$ for some $x_0 \in]0,1[$, then $A_\delta = T_{\mathbf{M}}$;*

(ii) *if $\delta(x_0) = 2x_0 - 1$ for some $x_0 \in [1/2, 1[$, then $\delta(x) = 2x - 1$ for any $x \in [x_0, 1]$.*

Proof. Suppose that A_δ is an ortholinear quasi-copula and suppose further that $\delta(x_0) = x_0$ for some $x_0 \in]0,1[$. Due to Proposition 5.2, it holds that $\delta(x) = x$ for any $x \in [x_0, 1]$. Since A_δ is an ortholinear quasi-copula, it holds that the function ψ_δ, defined in Proposition 5.3, is increasing on the interval $]0, 1/2]$. Therefore, $\psi_\delta(x) \le \psi_\delta(x_0) = 0$ for any $x \in [0, x_0]$ when $x_0 \le 1/2$. Hence, $\delta(x) \ge x$ for any $x \in [0, x_0]$. In case $x_0 \ge 1/2$, the increasingness of μ_δ, defined in Proposition 5.3, implies that $\delta(x) \ge x$ for any $x \in [1/2, x_0]$. Using the fact that $\delta(x) \le x$ for any $x \in [0, 1]$, it must hold that $\delta(x) = x$ for any $x \in [0, x_0]$. Based on the above discussion, it follows that $\delta(x) = x$ for any $x \in [0, 1]$. Since $T_{\mathbf{M}}$ is the only quasi-copula with $\delta_{T_{\mathbf{M}}}$ as diagonal section, it holds that $A_\delta = T_{\mathbf{M}}$.

Assertion (ii) can be proved similarly using the increasingness of the functions ψ_δ and μ_δ on the intervals $]0, 1/2]$ and $[1/2, 1[$, respectively. □

5.5. Ortholinear copulas

In this section, we characterize the elements of $\mathcal{D}$ for which the corresponding ortholinear function is a copula. Next, we characterize the piecewise linear diagonal functions for which the corresponding ortholinear function is a copula. To this end, we need the following lemmas.

Lemma 5.1. *Let $\delta \in \mathcal{D}_{\mathrm{S}}^{\mathrm{ac}}$ be piecewise linear and consider the corresponding ortholinear function A_δ. Let $x_1 < x_2 < x_3 \le 1/2$ be such that δ is linear on the interval $[x_1, x_2]$ as well as on the interval $[x_2, x_3]$. If $V_{A_\delta}(S) \ge 0$ for any square S that is included in the trapezoid $\Theta_{\{(x_1,x_1),(0,2x_1),(0,2x_3),(x_3,x_3)\}}$ and of which the opposite diagonal is a subset of the segment $\langle(x_2, x_2), (0, 2x_2)\rangle$, then it holds that δ is convex on the interval $[x_1, x_3]$.*

Proof. Let a and b be the slopes of the segments $\langle(x_1, \delta(x_1)), (x_2, \delta(x_2))\rangle$ and $\langle(x_2, \delta(x_2)), (x_3, \delta(x_3))\rangle$, respectively. Let $S = [x, x'] \times [y, y'] \subset [0, 1]^2$ be an arbitrary square that is included in the trapezoid $\Theta_{\{(x_1,x_1),(0,2x_1),(0,2x_3),(x_3,x_3)\}}$ and of which the opposite diagonal is a subset of the segment $\langle(x_2, x_2), (0, 2x_2)\rangle$. This situation is depicted in Figure 5.3(a).

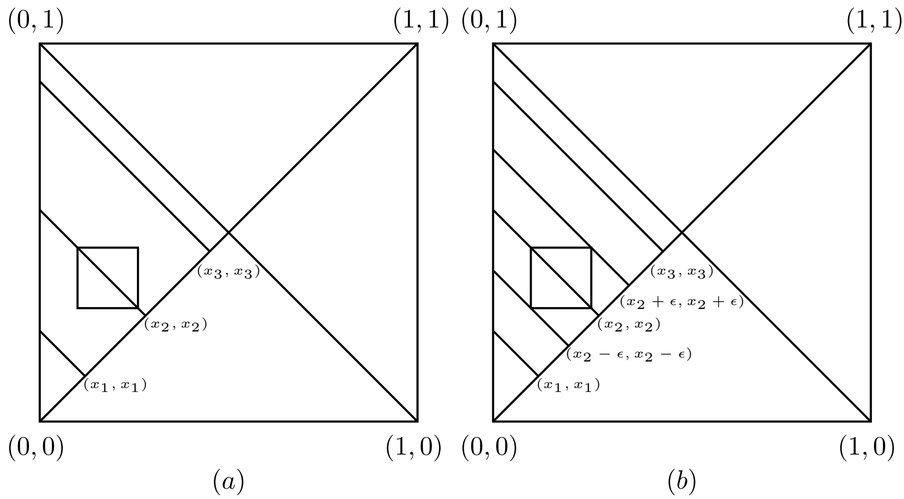


Figure 5.3: Illustration for the proofs of Lemma 5.1 and Proposition 5.5

Since S is a square, it holds that the points (x, y) and (x', y') are located on segments $\langle(x_2 - \epsilon, x_2 - \epsilon), (0, 2(x_2 - \epsilon))\rangle$ and $\langle(x_2 + \epsilon, x_2 + \epsilon), (0, 2(x_2 + \epsilon))\rangle$, for some $\epsilon > 0$, respectively (see Figure 5.3(b)). A simple computation shows that

$$S = [x' - 2\epsilon, x'] \times [2x_2 - x', 2(x_2 + \epsilon) - x'] .$$

The positivity of $V_{A_\delta}(S)$ is equivalent to

$$x' \frac{\delta(x_2 + \epsilon)}{x_2 + \epsilon} + (x' - 2\epsilon) \frac{\delta(x_2 - \epsilon)}{x_2 - \epsilon} - 2(x' - \epsilon) \frac{\delta(x_2)}{x_2} \geq 0 . \tag{5.8}$$

Since δ is linear on the interval $[x_2 - \epsilon, x_2]$ as well as on the interval $[x_2, x_2 + \epsilon]$, it holds that

$$\delta(x_2 - \epsilon) = \delta(x_2) - a\epsilon \text{ and } \delta(x_2 + \epsilon) = \delta(x_2) + b\epsilon .$$

Substituting the above, (5.8) is equivalent to

$$\frac{x' x_2^2 \epsilon(b - a) + \epsilon^2(2ax_2^2 + 2ax_2\epsilon - ax_2x' - bx_2x' + 2\delta(x_2)(x' - x_2 - \epsilon))}{x_2(x_2^2 - \epsilon^2)} \geq 0 ,$$

or, equivalently,

$$x' x_2^2(b - a) + \epsilon(2ax_2^2 + 2ax_2\epsilon - ax_2x' - bx_2x' + 2\delta(x_2)(x' - x_2 - \epsilon)) \geq 0 .$$

Choosing ϵ sufficiently small implies that the sign of the above expression is determined by the first term, i.e. it should hold that

$$x' x_2^2(b - a) \geq 0 , \tag{5.9}$$

or, equivalently, $b \geq a$, i.e. δ is convex on the interval $[x_1, x_3]$. □

Lemma 5.2. *Let $\delta \in \mathcal{D}_{\mathrm{S}}^{\mathrm{ac}}$ be piecewise linear and consider the corresponding ortholinear function A_δ. Let $1/2 \leq x_1 < x_2 < x_3$ be such that δ is linear on the interval $[x_1, x_2]$ as well as on the interval $[x_2, x_3]$. If $V_{A_\delta}(S) \geq 0$ for any square S that is included in the trapezoid $\Theta_{\{(x_1,x_1),(2x_1-1,1),(2x_3-1,1),(x_3,x_3)\}}$ and of which the opposite diagonal is a subset of the segment $\langle(x_2, x_2), (2x_2 - 1, 1)\rangle$, then it holds that δ is convex on the interval $[x_1, x_3]$.*

Proof. The proof is similar to the proof of the previous lemma. □

Lemma 5.3. *Let $\delta \in \mathcal{D}$ be convex. Then it holds that*

(i) *the function λ_δ is increasing;*

(ii) *the function μ_δ is increasing.*

Proof. Assertion (i) follows using the fact that convex functions are star-shaped [85]. Since δ is convex, for any $x, y, z \in [0, 1]$ such that $x < y < z$, it holds that

$$\frac{\delta(y) - \delta(x)}{y - x} \leq \frac{\delta(z) - \delta(x)}{z - x} \,.$$

Setting $z = 1$, assertion (ii) easily follows. □

Proposition 5.5. *Let $\delta \in \mathcal{D}$ be piecewise linear. Then the ortholinear function A_δ is a copula if and only if δ is convex.*

Proof. We first give the proof from right to left. Since A_δ satisfies the boundary conditions of a semi-copula, we need to show its 2-increasingness. Since δ is piecewise linear, the unit square consists of trapezoids of the type $\Theta_1 = \Theta_{\{(u,u),(v,v),(2v,0),(2u,0)\}}$, $\Theta_2 = \Theta_{\{(u,u),(0,2u,),(0,2v),(v,v)\}}$, $\Theta_3 = \Theta_{\{(u,u),(2u-1,1),(2v-1,1),(v,v)\}}$ or $\Theta_4 = \Theta_{\{(u,u),(v,v),(1,2v-1),(1,2u-1)\}}$, such that δ is linear on the interval $[u, v]$ with $v \leq 1/2$ or $1/2 \leq u$ (see Figure 5.4).

Note that any rectangle in the unit square can obviously be decomposed into a number of rectangles that are either located in one of these trapezoids, or are spanning two such trapezoids while having their diagonal along the diagonal of the unit square or having their opposite diagonal along the edge shared by the two trapezoids. Due to the additivity of volumes, it suffices to consider the above cases. Consider a rectangle $R := [x, x'] \times [y, y'] \subseteq [0, 1]^2$.

(i) Suppose that R is included in a trapezoid of type Θ_2 (the case of a trapezoid of type Θ_1 is identical due to the symmetry of A_δ). The positivity of $V_{A_\delta}(R)$ is equivalent to

$$(\lambda_\delta(v) - \lambda_\delta(u))(x' - x)(yy' - xx') \geq 0 \,. \tag{5.10}$$

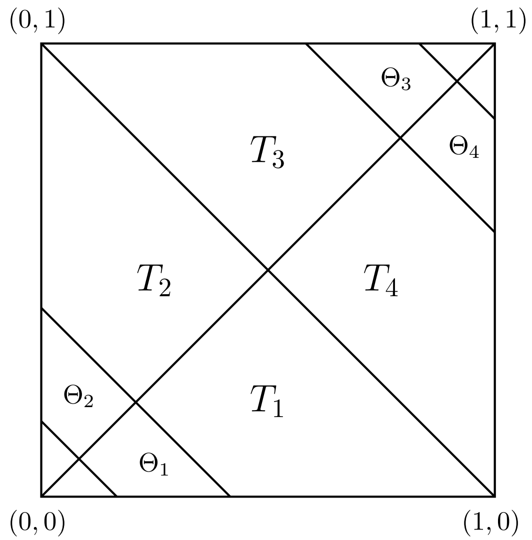


Figure 5.4: Illustration for the proof Proposition 5.5.

Since $R \subseteq T_2$, it holds that $yy' - xx' \geq 0$. Due to Lemma 5.3(i), inequality (5.10) then follows immediately.

(ii) Suppose that R is included in a trapezoid of type Θ_3 (the case of a trapezoid of type Θ_4 is identical due to the symmetry of A_δ). The positivity of $V_{A_\delta}(R)$ is equivalent to

$$(\mu_\delta(v) - \mu_\delta(u))(y' - y)((1-x)(1-x') - (1-y)(1-y')) \geq 0\,. \qquad (5.11)$$

Since $R \subseteq T_3$, it holds that $(1-x)(1-x') - (1-y)(1-y') \geq 0$. Due to Lemma 5.3(ii), inequality (5.11) then follows immediately.

(iii) Suppose that the diagonal of R is along the diagonal of the unit square, i.e. $R = [x, x'] \times [x, x']$. Suppose that $x' \leq 1/2$ (the case when $1/2 \leq x$ is similar). Then it holds that

$$V_{A_\delta}(R) = \delta(x) + \delta(x') - 2A_\delta(x, x') = \delta(x) + \delta(x') - 2\frac{2x}{x+x'}\delta\left(\frac{x+x'}{2}\right)\,.$$

Since $\frac{2x}{x+x'} \leq 1$, it holds that

$$V_{A_\delta}(R) \geq \delta(x) + \delta(x') - 2\delta\left(\frac{x+x'}{2}\right) = 2\left(\frac{\delta(x)+\delta(x')}{2} - \delta\left(\frac{x+x'}{2}\right)\right)\,,$$

which is positive due to the convexity of δ on the interval $[0, 1/2]$.

(iv) Suppose that the opposite diagonal of R is along the edge shared by two trapezoids (either two trapezoids of the same type, or two trapezoids having their edge along the opposite diagonal of the unit square).

(a) Suppose that these two trapezoids are of type Θ_2 (the case of type Θ_1 is similar). Consider $\Theta_2 = \Theta_{\{(u,u),(0,2u),(0,2v),(v,v)\}}$ and $\Theta_2' = \Theta_{\{(v,v),(0,2v),(0,2w),(w,w)\}}$. The rectangle R is then given by $R = [x, x'] \times [2v - x', 2v - x]$. Using the notation $r = \frac{2v-x'+x}{2}$, the positivity of $V_{A_\delta}(R)$ is equivalent to

$$x\frac{\delta(r)}{r} + x'\frac{\delta(2v-r)}{2v-r} - (x+x')\frac{\delta(v)}{v} \geq 0. \tag{5.12}$$

Using the convexity of δ, it follows that

$$\delta(v) \leq \frac{\delta(r) + \delta(2v-r)}{2}.$$

Denoting the left-hand side of inequality (5.12) as c, it then holds that

$$c \geq \frac{(x'-x)(2v-x-x')}{4v}\left(\frac{\delta(2v-r)}{2v-r} - \frac{\delta(r)}{r}\right). \tag{5.13}$$

Since δ is convex, the function λ_δ, defined in Lemma 5.3, is increasing. Using also the facts $v \geq r$ and $2v - x - x' \geq 0$, the right-hand side of inequality (5.13) is positive.

(b) Suppose that these two trapezoids are of type Θ_3 (the case of type Θ_4 is similar). The proof is similar to the previous case.

(c) Suppose that these two trapezoids are of types Θ_2 and Θ_3 (the case of types Θ_1 and Θ_4 is similar).

Consider $\Theta_2 = \Theta_{\{(u,u),(0,2u,),(0,1),(1/2,1/2)\}}$ and $\Theta_3 = \Theta_{\{(1/2,1/2),(0,1),(2w-1,1),(w,w)\}}$. The rectangle R is then given by $R = [x, x'] \times [1-x', 1-x]$ such that $x' \leq 1/2$. Using the notation $r = \frac{1-x'+x}{2}$, it holds that

$$V_{A_\delta}(R) = x\frac{\delta(r)}{r} - 2\delta(1/2)x - 2\delta(1/2)x' + x' - x\frac{1-r-\delta(1-r)}{r}. \tag{5.14}$$

Consider the function $\nu : [0,1] \to \mathbb{R}$ given by

$$\nu(t) = \delta(t) + \delta(1-t) - 2\delta(1/2).$$

Using this function, Eq. (5.14) can be written as

$$V_{A_\delta}(R) = \frac{1}{r}\left(x\nu(r) + (1/2 - \delta(1/2))(x'-x)(1-x-x')\right). \tag{5.15}$$

Since δ is convex, it holds that $\nu(t) \geq 0$ for any $t \in [0, 1/2]$, and therefore the right-hand side of Eq. (5.15) is positive.

Conversely, suppose that A_δ is a copula. Lemma 5.1 implies that δ is convex on the interval $[0, 1/2]$, while Lemma 5.2 implies that δ is convex on the interval $[1/2, 1]$. Let $0 \leq x_1 < 1/2 < x_2 \leq 1$ be such that δ is linear on the interval $[x_1, 1/2]$ as well as on the interval $[1/2, x_2]$, and let a and b be the slopes of the segments $\langle(x_1, \delta(x_1)), (\frac{1}{2}, \delta(\frac{1}{2}))\rangle$ and $\langle(\frac{1}{2}, \delta(\frac{1}{2})), (x_2, \delta(x_2))\rangle$, respectively. Let $S = [x, x'] \times [y, y'] \subseteq [0, 1]^2$ be an arbitrary square such that its opposite diagonal is along the opposite diagonal of the unit square. This situation is depicted in Figure 5.5.

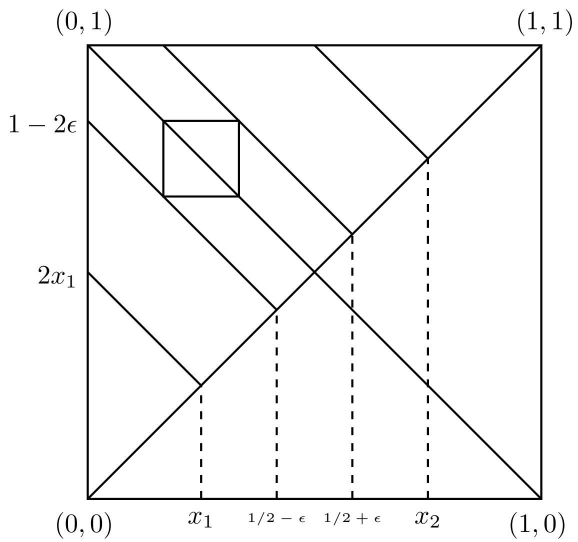


Figure 5.5: Illustration for the proof Proposition 5.5.

Since S is a square, it holds that the points (x, y) and (x', y') are located on segments $\langle(1/2 - \epsilon, 1/2 - \epsilon), (0, 1 - 2\epsilon)\rangle$ and $\langle(1/2 + \epsilon, 1/2 + \epsilon), (2\epsilon, 1)\rangle$, for some $\epsilon > 0$, respectively. A simple computation shows that

$$S = [x, x + 2\epsilon] \times [1 - x - 2\epsilon, 1 - x] .$$

The positivity of $V_{A_\delta}(S)$ is equivalent to

$$x \frac{\delta(1/2 - \epsilon)}{1/2 - \epsilon} + x + 2\epsilon - x \frac{1/2 + \epsilon - \delta(1/2 + \epsilon)}{1/2 - \epsilon} - 4(x + \epsilon)\delta(1/2) \geq 0 . \qquad (5.16)$$

Since δ is linear on the interval $[1/2 - \epsilon, 1/2]$ as well as on the interval $[1/2, 1/2 + \epsilon]$, it holds that

$$\delta(1/2 - \epsilon) = \delta(1/2) - a\epsilon \text{ and } \delta(1/2 + \epsilon) = \delta(1/2) + b\epsilon .$$

Substituting the above, (5.16) is equivalent to

$$x(b-a) + 2(1-2\epsilon-2x)(1/2-\delta(1/2)) \geq 0\,.$$

Taking the limit $x \to 1/2$, and thus $\epsilon \to 0$, the above inequality is equivalent to $b \geq a$. Hence, the convexity of δ at 1/2 follows, which completes the proof. □

Example 5.4. *Consider the diagonal function $\delta_\theta = \theta\delta_{T_{\mathbf{M}}} + (1-\theta)\delta_{T_{\mathbf{L}}}$ with $\theta \in [0,1]$. Clearly, δ_θ is convex and piecewise linear for any $\theta \in [0,1]$. The corresponding family of ortholinear copulas is the family of convex sums of $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$.*

Theorem 5.1. *Let $\delta \in \mathcal{D}$. Then the ortholinear function A_δ is a copula if and only if δ is convex.*

Proof. Suppose that δ is convex. To prove that A_δ is a copula, we need to show its 2-increasingness. Due to the additivity of volumes, it suffices to consider a restricted number of cases. Consider a rectangle $R := [x,x'] \times [y,y'] \subseteq [0,1]^2$.

(i) If $R \subseteq T_2$ (the case when $R \subseteq T_1$ is identical due to the symmetry of A_δ), then let $\mathbf{b_1} = (x_1,x_1)$, $\mathbf{b_2} = (x_2,x_2)$, $\mathbf{b_3} = (x_3,x_3)$ and $\mathbf{b_4} = (x_4,x_4)$ be four (possibly coinciding) points on the diagonal of the unit square such that the points (x,y), (x,y'), (x',y) and (x',y') are located on the segments $\langle(0,2x_1),\mathbf{b_1}\rangle$, $\langle(0,2x_2),\mathbf{b_2}\rangle$, $\langle(0,2x_3),\mathbf{b_3}\rangle$ and $\langle(0,2x_4),\mathbf{b_4}\rangle$, respectively (see Figure 5.6).

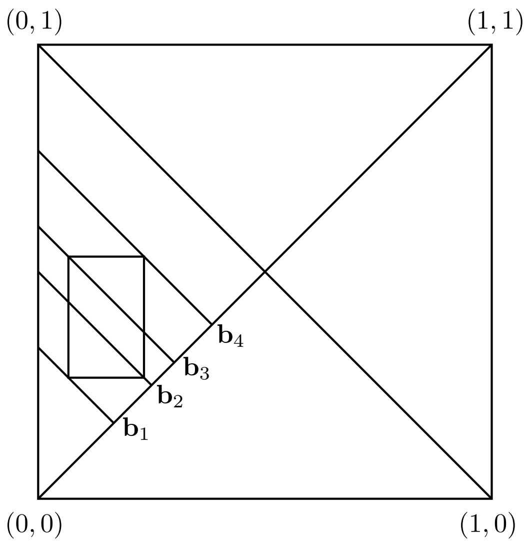


Figure 5.6: An illustration for the proof of Theorem 5.1.

The points $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ and $\mathbf{b}_4$, together with $(0,0)$ and $(1,1)$, determine a piecewise linear convex diagonal function δ_1 such that $\delta_1(x_i) = \delta(x_i)$ for any

$i \in \{1,2,3,4\}$. Due to Proposition 5.5, the ortholinear function A_{δ_1} is an ortholinear copula. Therefore,

$$V_{A_\delta}(R) = V_{A_{\delta_1}}(R) \geq 0 \,.$$

(ii) The proof of the cases when $R \subseteq T_3$ or $R \subseteq T_4$ is similar to the previous one.

The remaining case is when the diagonal (resp. opposite diagonal) of R is along the diagonal (opposite diagonal) of the unit square. The proof of the positivity of $V_{A_\delta}(R)$ in this case is similar as in the proof of Proposition 5.5. Consequently, the 2-increasingness of A_δ holds and, hence A_δ is a copula.

Conversely, suppose that A_δ is a copula and suppose further that δ is not convex on the interval $[0,1/2[$, i.e. there exist $x<y<z$ such that the point $(y,\delta(y))$ is above the segment connecting the points $(x,\delta(x))$ and $(z,\delta(z))$. Since δ is continuous, there exists $\epsilon > 0$ such that for any $x' \in [y-\epsilon, y+\epsilon]$ the point $(x',\delta(x'))$ is above the segment connecting the points $(x,\delta(x))$ and $(z,\delta(z))$, which contradicts Lemma 5.1. Similarly, Lemma 5.2 implies that δ is convex on the interval $]1/2,1]$. Thus, δ is convex on the interval $[0,1/2[$ as well as on the interval $]1/2,1]$. The proof of the convexity of δ at $1/2$ can be done in similar manner. Thus, δ is convex. □

Since for an ortholinear copula C, it holds that its diagonal section δ_C is convex, it either holds that $\delta_C = \delta_{T_{\mathbf{M}}}$ or $\delta_C(x) < x$ for any $x \in]0,1[$. Hence, there do not exist ortholinear copulas that are proper ordinal sums.

Example 5.5. *Consider the diagonal functions $\delta_{T_{\mathbf{M}}}$ and $\delta_{T_{\mathbf{L}}}$. Clearly, the functions $\delta_{T_{\mathbf{M}}}$ and $\delta_{T_{\mathbf{L}}}$ are convex. The corresponding ortholinear copulas are $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$, respectively.*

Example 5.6. *Consider the diagonal function $\delta_\theta(x) = x^{1+\theta}$ with $\theta \in [0,1]$. Clearly, δ_θ is convex for any $\theta \in [0,1]$. The corresponding family of ortholinear functions is a family of ortholinear copulas.*

Example 5.7. *Consider the diagonal function δ defined by*

$$\delta(x) = \begin{cases} 0 & , \text{ if } x \leq \frac{1}{6}\,, \\ \dfrac{1}{5}(6x-1) & , \text{ if } \frac{1}{6} \leq x \leq \frac{1}{3}\,, \\ \dfrac{3}{5}x & , \text{ if } \frac{1}{3} \leq x \leq \frac{2}{3}\,, \\ \dfrac{1}{5}(9x-4) & , \text{ otherwise.} \end{cases}$$

Clearly, the conditions of Proposition 5.3 are satisfied. Consider the rectangle $R = [\frac{1}{4},\frac{1}{2}] \times [\frac{1}{6},\frac{1}{4}]$. One easily verifies that $V_{A_\delta}(R) = -\frac{1}{100}$, and hence, the corresponding ortholinear function is a proper quasi-copula.

Consequently, the class of ortholinear copulas with a given diagonal section is a proper subclass of the class of ortholinear quasi-copulas with a given diagonal section.

Now we lay bare the Schur-concavity [40, 43, 87] of ortholinear copulas.

Proposition 5.6. *Any ortholinear copula is Schur-concave.*

Proof. Suppose that C_δ is an ortholinear copula. Let $\lambda \in [0,1]$ and $(x,y) \in [0,1]^2$. If $(x,y) \in T_1$, then inequality (1.8) is equivalent to

$$y\frac{\delta(z)}{z} \leq (\lambda y + (1-\lambda)x)\frac{\delta(z)}{z}\,,$$

or, equivalently, $(1-\lambda)(y-x) \leq 0$. Since $(x,y) \in T_1$, the latter inequality immediately follows. Similarly, one can prove inequality (1.8) when (x,y) is located in T_2, T_3 or T_4. ☐

5.6. Ortholinear copulas supported on a set with Lebesgue measure zero

We characterize in this section ortholinear copulas that are supported on a set with Lebesgue measure zero. To this end, we need the following proposition.

Proposition 5.7. *Let C_δ be an ortholinear copula with a piecewise linear diagonal section δ. Suppose that $d \in]0,1/2]$ is the maximum value such that $\delta(d)=0$, and $d^* \in [1/2,1[$ is the minimum value such that $\delta(d^*) = 2d^*-1$. Then the support of C_δ consists of:*

(i) *the segment $\langle (d,d),(d^*,d^*)\rangle$;*

(ii) *the trapezoids $\Theta_{\{(2d,0),(0,2d),(0,1),(1,0)\}}$ and $\Theta_{\{(0,1),(1,0),(2d^*-1,1),(1,2d^*-1)\}}$.*

Proof. From Proposition 5.4, it follows that $\delta(x) = 2x-1$ for any $x \in [d^*,1]$. Note that if $d = d^* = 1/2$, then $C_\delta = T_{\mathbf{L}}$ and the support is given by the segment $\langle (1,0),(0,1)\rangle$. More generally, if δ is piecewise linear, then it suffices to consider a number of cases to prove assertion (i):

(a) Let $\langle (x_1,x_1),(x_2,x_2)\rangle$, with $d \leq x_1 < x_2 \leq 1/2$, be a segment such that δ is linear on the interval $[x_1,x_2]$. For any rectangle $R = [x,x'] \times [x,x']$ such that $x_1 \leq x < x' \leq x_2$, it holds that

$$V_{C_\delta}(R) = \frac{(x'-x)}{z}\delta(z)\,.$$

If $V_{C_\delta}(R) = 0$, then it holds that $\delta(z) = 0$, which contradicts the fact that d is the maximum value such that $\delta(d) = 0$, and hence, $V_{C_\delta}(R) > 0$.

(b) Let $\langle(x_1, x_1), (x_2, x_2)\rangle$, with $1/2 \leq x_1 < x_2 \leq d^*$, be a segment such that δ is linear on the interval $[x_1, x_2]$. For any rectangle $R = [x, x'] \times [x, x']$ such that $x_1 \leq x < x' \leq x_2$, it holds that

$$V_{C_\delta}(R) = \frac{(x' - x)}{1 - z}(\delta(z) - (2z - 1)) .$$

If $V_{C_\delta}(R) = 0$, then it holds that $\delta(z) = 2z - 1$, which contradicts the fact that d^* is the minimum value such that $\delta(d^*) = 2d^* - 1$ and hence, $V_{C_\delta}(R) > 0$.

Since the support is closed, assertion (i) follows.

Next, we prove assertion (ii). Let $\mathbf{b}_1 := (x_1, x_1)$ and $\mathbf{b}_2 := (x_2, x_2)$, with $d \leq x_2 \leq d^*$, be two distinct points such that δ is linear on the interval $[x_1, x_2]$, i.e. $\delta(t) = at + b$ for any $t \in [x_1, x_2]$. Let $R \subseteq [0, 1]^2$ be a rectangle. We distinguish two cases:

(a) Suppose that $R \subseteq \Theta_2 = \Theta_{\{(x_1,x_1),(0,2x_1),(0,2x_2),(x_2,x_2)\}}$ and suppose further that $V_{C_\delta}(R) = 0$. Due to inequality (5.10), it holds that $(\lambda_\delta(x_2) - \lambda_\delta(x_1))(yy' - xx') = 0$. Since $d > 0$ and δ is convex it must hold that $\lambda_\delta(x_2) - \lambda_\delta(x_1) > 0$ and it then follows that $x = x' = y = y'$, a contradiction. Hence, $V_{C_\delta}(R) > 0$. Consequently, the trapezoid Θ_2 is a subset of the support. Due to the symmetry of C_δ, it holds that $\Theta_1 = \Theta_{\{(x_1,x_1),(x_2,x_2),(2x_2,0),(2x_1,0)\}}$ is a subset of the support as well.

(b) Suppose that $R \subseteq \Theta_3 = \Theta_{\{(x_1,x_1),(2x_1-1,1),(2x_2-1,1),(x_2,x_2)\}}$ and suppose further that $V_{C_\delta}(R) = 0$. Due to inequality (5.11), it holds that $(\mu_\delta(x_2) - \mu_\delta(x_1))((1 - y)(1 - y') - (1 - x)(1 - x')) = 0$. Since $d^* < 1$ and δ is convex, it must hold that $\mu_\delta(x_2) - \mu_\delta(x_1) > 0$ and it then follows that $x = x' = y = y'$, a contradiction. Hence, $V_{C_\delta}(R) > 0$. Consequently, the trapezoid Θ_3 is a subset of the support. Due to the symmetry of C_δ, it holds that $\Theta_4 = \Theta_{\{(x_1,x_1),(x_2,x_2),(1,2x_2-1),(1,2x_1-1)\}}$ is a subset of the support as well.

Since the support is closed, assertion (ii) follows. $\square$

Corollary 5.1. *Let C_δ be an ortholinear copula with a piecewise linear diagonal section δ. Suppose that $d = 0$ is the maximum value such that $\delta(d) = 0$, and $d^* \in [1/2, 1[$ is the minimum value such that $\delta(d^*) = 2d^* - 1$. If δ is linear on the interval $[0, 1/2]$, then the support of C_δ consists of:*

(i) *the segment $\langle(0, 0), (d^*, d^*)\rangle$;*

(ii) *the trapezoid $\Theta_{\{(0,1),(1,0),(2d^*-1,1),(1,2d^*-1)\}}$.*

Corollary 5.2. *Let C_δ be an ortholinear copula with a piecewise linear diagonal section δ. Suppose that $d \in]0,1/2]$ is the maximum value such that $\delta(d)=0$, and $d^*=1$ is the minimum value such that $\delta(d^*)=2d^*-1$. If δ is linear on the interval $[1/2,1]$, then the support of C_δ consists of:*

(i) *the segment $\langle (d,d),(1,1)\rangle$;*

(ii) *the trapezoid $\Theta_{\{(2d,0),(0,2d),(0,1),(1,0)\}}$.*

Example 5.8. *Consider the diagonal function δ given by*

$$\delta(x)=\begin{cases}0 & \text{, if } x\leq \frac{1}{3}\,,\\ x-1/3 & \text{, if } \frac{1}{3}\leq x\leq\frac{2}{3}\,,\\ 2x-1 & \text{, otherwise.}\end{cases}$$

Clearly, δ is a piecewise linear convex function. The support of the corresponding copula is depicted in Figure 5.7(a).

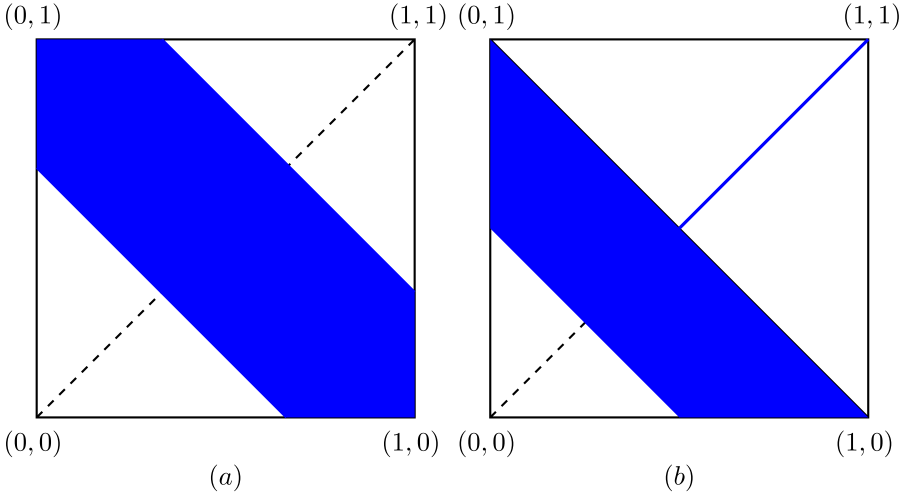


Figure 5.7: The support of the ortholinear copulas given in Example 5.8 (a) and Example 5.9 (b).

Example 5.9. *Consider the diagonal function δ given by*

$$\delta(x)=\begin{cases}0 & \text{, if } x\leq \frac{1}{4}\,,\\ x-\dfrac{1}{4} & \text{, if } \frac{1}{4}\leq x\leq\frac{1}{2}\,,\\ \dfrac{1}{2}(3x-1) & \text{, otherwise.}\end{cases}$$

Clearly, δ is a piecewise linear convex function. The support of the corresponding copula is depicted in Figure 5.7(b).

Proposition 5.8. *Let C_δ be an ortholinear copula. Then it holds that C_δ is supported on a set with Lebesgue measure zero if and only if C_δ is a member of the family of convex sums of $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$.*

Proof. The family of convex sums of $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ is a family of copulas supported on a set with Lebesgue measure zero (see Chapter 1). In Example 5.4, it was shown that the latter family is a family of ortholinear copulas. Therefore, to complete the proof, it suffices to prove the necessity. Let C_δ be an ortholinear copula and suppose further that C_δ is supported on a set with Lebesgue measure zero. Suppose that $d \in [0,1/2]$ is the maximum value such that $\delta(d) = 0$, and $d^* \in [1/2,1]$ is the minimum value such that $\delta(d^*) = 2d^* - 1$. If δ is piecewise linear, then due to Proposition 5.7 it must hold that $d = 0$ and $d^* = 1$ or $d = d^* = 1/2$. Suppose that δ is not piecewise linear, i.e. there exists an interval $[d_1, d_2]$ such that the graph of the restriction of δ to $[d_1, d_2]$ does not contain any segment. Assume w.l.o.g. that $d_2 \leq 1/2$. Let $R = [x, x'] \times [y, y']$ be a rectangle located in the trapezoid $\Theta_2 = \Theta_{\{(d_1,d_1),(0,2d_1),(0,2d_2),(d_2,d_2)\}}$ and let $\mathbf{b_1} = (x_1, x_1)$, $\mathbf{b_2} = (x_2, x_2)$, $\mathbf{b_3} = (x_3, x_3)$ and $\mathbf{b_4} = (x_4, x_4)$ be four (possibly coinciding) points on the diagonal of the unit square such that the points (x, y), (x, y'), (x', y) and (x', y') are located on the segments $\langle (0, 2x_1), \mathbf{b_1} \rangle$, $\langle (0, 2x_2), \mathbf{b_2} \rangle$, $\langle (0, 2x_3), \mathbf{b_3} \rangle$ and $\langle (0, 2x_4), \mathbf{b_4} \rangle$, respectively (see Figure 5.6). The points $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ and $\mathbf{b}_4$, together with $(0,0)$, (d,d), (d^*, d^*) and $(1,1)$, determine a piecewise linear convex diagonal function δ_1 such that $\delta_1(x_i) = \delta(x_i)$ for any $i \in \{1,2,3,4\}$. Due to Proposition 5.5, the ortholinear function A_{δ_1} is an ortholinear copula. Therefore,

$$V_{A_\delta}(R) = V_{A_{\delta_1}}(R) \geq 0 .$$

Furthermore, as in the proof of Proposition 5.7(ii), it holds that $V_{A_{\delta_1}}(R) > 0$. Consequently, the trapezoid Θ_2 is a subset of the support, a contradiction. Hence, δ is piecewise linear. Similarly to the proof of Proposition 5.7(ii) and using the fact that C_δ is supported on a set with Lebesgue measure zero, it follows that δ is linear on the interval $[0,1/2]$ as well as on the interval $[1/2,1]$, i.e.

$$\delta(x) = \theta \delta_{T_{\mathbf{M}}}(x) + (1-\theta)\delta_{T_{\mathbf{L}}}(x) ,$$

with $\theta \in [0,1]$. Recalling that any ortholinear copula is uniquely determined by its diagonal section, our assertion follows. ∎

5.7. Dependence measures

In this section, we derive compact formulae for Spearman's rho, Gini's gamma and Kendall's tau of two continuous random variables whose dependence is modelled by an ortholinear copula C_δ. These parameters can be expressed in terms of the function δ.

Proposition 5.9. *Let X and Y be two continuous random variables whose copula is an ortholinear copula C_δ.*

(i) *The population version of Spearman's ρ_{C_δ} for X and Y is given by*

$$\rho_{C_\delta} = 24 \int_0^{1/2} x(\delta(x) + \delta(1-x))\,\mathrm{d}x - 2\,.$$

(ii) *The population version of Gini's γ_{C_δ} for X and Y is given by*

$$\gamma_{C_\delta} = 4 \int_0^{1} \delta(x)\mathrm{d}x - 2(1 - \delta(1/2))\,.$$

(iii) *The population version of Kendall's τ_{C_δ} for X and Y is given by*

$$\begin{aligned}
\tau_{C_\delta} &= 1 - (4/3)(\delta^2(1/2) - \delta(1/2) + 5) \\
&\quad -(4/3) \int_0^{1/2} x \left(\left(\frac{\mathrm{d}}{\mathrm{d}x}(\delta(x)) \right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}x}\delta(1-x) \right)^2 \right) \mathrm{d}x \\
&\quad +16 \int_0^{1/2} \delta(1-x)\,\mathrm{d}x + (8/3) \int_0^{1/2} \frac{\delta^2(x) + (1-\delta(x))^2}{x}\,\mathrm{d}x\,.
\end{aligned}$$

Proof. In order to find ρ_{C_δ}, we need to compute

$$I = \int_0^1 \int_0^1 C_\delta(x,y)\,\mathrm{d}x\mathrm{d}y\,.$$

As C_δ is symmetric, it holds that $I = 2\tilde{I}$ with $\tilde{I}$ the integral over the region $T_2 \cup T_3$, i.e.

$$\tilde{I} = \int_0^{1/2} \int_x^{1-x} C_\delta(x,y)\mathrm{d}y\mathrm{d}x + \int_{1/2}^{1} \int_{1-y}^{y} C_\delta(x,y)\mathrm{d}x\mathrm{d}y\,.$$

Substituting the expression for $C_\delta(x,y)$, it holds that

$$\int_0^{1/2}\int_x^{1-x} C_\delta(x,y)\mathrm{d}y\mathrm{d}x = 2\int_0^{1/2} x\,\mathrm{d}x \int_x^{1/2} \frac{\delta(z)}{z}\,\mathrm{d}z\,. \tag{5.17}$$

Consider the function $\zeta :]0,1[\to \mathbb{R}$ given by

$$\zeta(x) = \int_0^x \frac{\delta(u)}{u}\,\mathrm{d}u\,.$$

Substituting $\zeta(x)$ in Eq. (5.17) and integrating by parts, it follows that

$$\int_0^{1/2}\int_x^{1-x} C_\delta(x,y)\mathrm{d}y\mathrm{d}x = \int_0^{1/2} x\delta(x)\,\mathrm{d}x\,.$$

Similarly, one can find that

$$\int_{1/2}^{1}\int_{1-y}^{y} C_\delta(x,y)\mathrm{d}x\mathrm{d}y = \int_0^{1/2} x\delta(1-x)\,\mathrm{d}x + 1/24\,.$$

Hence,

$$I = 2\tilde{I} = 2\int_0^{1/2} x(\delta(x)+\delta(1-x))\,\mathrm{d}x + 1/12\,.$$

Substituting in the expression for ρ_{C_δ}, (i) follows.

The expression for γ_{C_δ} can be rewritten as

$$\gamma_{C_\delta} = 4\left[\int_0^1 \omega_{C_\delta}(x)\mathrm{d}x - \int_0^1 (x-\delta(x))\mathrm{d}x\right]\,,$$

where ω_{C_δ} is the opposite diagonal section of C_δ. Since C_δ is ortholinear, ω_{C_δ} is given by

$$\omega_{C_\delta}(x) = \begin{cases} 2x\delta(1/2) & \text{, if } x \le 1/2\,, \\ 2(1-x)\delta(1/2) & \text{, if } x \ge 1/2\,. \end{cases}$$

Computing $\int_0^1 \omega_{C_\delta}(x)\mathrm{d}x$, (ii) follows.

In order to find τ_{C_δ}, we need to compute

$$I = \int_0^1 \int_0^1 \frac{\partial C_\delta}{\partial x}(x,y)\frac{\partial C_\delta}{\partial y}(x,y)\mathrm{d}x\mathrm{d}y\,.$$

Let us introduce the notations

$$I_1 = \int_0^{1/2} \int_x^{1-x} \frac{\partial C_\delta}{\partial x}(x,y)\frac{\partial C_\delta}{\partial y}(x,y)\mathrm{d}x\mathrm{d}y\,,$$

and

$$I_2 = \int_{1/2}^{1} \int_{1-y}^{y} \frac{\partial C_\delta}{\partial x}(x,y)\frac{\partial C_\delta}{\partial y}(x,y)\mathrm{d}x\mathrm{d}y\,.$$

As C_δ is symmetric, it holds that $I = 2\tilde{I}$ with $\tilde{I}$ the integral over the region $T_2 \cup T_3$, i.e.

$$\tilde{I} = I_1 + I_2\,.$$

Computing the partial derivatives, it holds that

$$\begin{aligned}
I_1 &= \int_0^{1/2} \int_x^{1-x} \left(\lambda_\delta(z) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\left(\lambda_\delta(z)\right)\right)\left(\frac{1}{2}x\frac{\mathrm{d}}{\mathrm{d}z}\left(\lambda_\delta(z)\right)\right)\mathrm{d}x\mathrm{d}y \\
&= (1/2)\int_0^{1/2} x\,\mathrm{d}x \int_x^{1/2} \lambda_\delta(z)\frac{\mathrm{d}}{\mathrm{d}z}(\lambda_\delta(z))\,\mathrm{d}z + (1/2)\int_0^{1/2} x\,\mathrm{d}x\int_x^{1/2}\frac{\mathrm{d}}{\mathrm{d}z}(\lambda_\delta(z))^2\,\mathrm{d}z \\
&= (1/12)\delta^2(1/2) + (1/6)\int_0^{1/2} x\left(\frac{\mathrm{d}}{\mathrm{d}x}(\delta(x))\right)^2\mathrm{d}x - (1/3)\int_0^{1/2}\frac{\delta^2(x)}{x}\,\mathrm{d}x\,.
\end{aligned}$$

Similarly, one can find that

$$\begin{aligned}
I_2 = {} & (1/6)\int_0^{1/2} x\left(\frac{\mathrm{d}}{\mathrm{d}x}(\delta(1-x))\right)^2 - (1/3)\int_0^{1/2}\frac{1-\delta(1-x))^2}{x}\,\mathrm{d}x \\
& -2\int_0^{1/2}\delta(1-x)\,\mathrm{d}x + (1/12)\delta^2(1/2) - (1/6)\delta(1/2) + 5/6\,.
\end{aligned}$$

Substituting in the expression for τ_{C_δ}, (iii) follows. $\square$

Table 5.1: Spearman's rho, Gini's gamma and Kendall's tau of the ortholinear copulas C_{δ_θ} with diagonal section $\delta_\theta(t) = t^{\theta+1}$.

θ	δ_θ	$\rho_{C_{\delta_\theta}}$	$\gamma_{C_{\delta_\theta}}$	$\tau_{C_{\delta_\theta}}$
0	t	1	1	1
0.2	$t^{1.2}$	0.667144	0.688732	0.651416
0.4	$t^{1.4}$	0.383928	0.424525	0.380099
0.6	$t^{1.6}$	0.141183	0.198215	0.161747
0.8	$t^{1.8}$	-0.068242	0.002921	-0.017583
1	t^2	-0.250000	-0.166667	-0.166667

Example 5.10. *We reconsider the ortholinear copulas associated with the diagonal functions introduced in Example 5.6. For these copulas we computed the values of Spearman's rho, Gini's gamma and Kendall's tau by means of the expressions given in Proposition 5.9. The results are listed in Table 5.1.*

5.8. Aggregations of ortholinear copulas

In this section we study the aggregation of ortholinear copulas. We formulate two lemmas and two immediate propositions.

Lemma 5.4. *The sets $\mathcal{D}_{\mathrm{S}}^{\mathrm{ac}}$ and $\mathcal{D}$ are closed under minimum, maximum and convex sums.*

Lemma 5.5. *Let C_{δ_1} and C_{δ_2} be two ortholinear functions. Then it holds that*

$$C_{\delta_1} \leq C_{\delta_2} \text{ if and only if } \delta_1 \leq \delta_2 \,.$$

Proposition 5.10. *Let $\delta_1, \delta_2 \in \mathcal{D}_{\mathrm{S}}^{\mathrm{ac}}$ (resp. $\mathcal{D}$) and $\theta \in [0,1]$. If C_{δ_1} and C_{δ_2} are ortholinear semi-copulas (resp. quasi-copulas), then also $\min(C_{\delta_1}, C_{\delta_2})$, $\max(C_{\delta_1}, C_{\delta_2})$ and $\theta C_{\delta_1} + (1-\theta) C_{\delta_2}$ are ortholinear semi-copulas (resp. quasi-copulas). The corresponding diagonal sections are given by $\delta_{\min} = \min(\delta_1, \delta_2)$, $\delta_{\max} = \max(\delta_1, \delta_2)$ and $\theta\delta_1 + (1-\theta)\delta_2$, respectively.*

Consequently, the class of ortholinear semi-copulas and the class of orthogonal quasi-copulas are closed under minimum, maximum and convex sums.

Proposition 5.11. *Let $\delta_1, \delta_2 \in \mathcal{D}$ and $\theta \in [0,1]$. If C_{δ_1} and C_{δ_2} are ortholinear copulas, then also $\max(C_{\delta_1}, C_{\delta_2})$ and $\theta C_{\delta_1} + (1-\theta)C_{\delta_2}$ are ortholinear copulas. The corresponding diagonal sections are given by $\delta_{\max}$ and $\theta\delta_1 + (1-\theta)\delta_2$, respectively.*

Consequently, the class of ortholinear copulas is closed under maximum and convex sums. Hence, the class of orthogonal copulas is not join-dense in the class of ortholinear quasi-copulas in contrast to the general case [92]. In general, the minimum of two ortholinear copulas need not be an ortholinear copula. For instance, let C_{δ_1} and C_{δ_2} be two ortholinear copulas with δ_1 and δ_2 as depicted in Figure 5.8. Obviously, the function $\delta_{\min}$ is not convex, and thus $\min(C_{\delta_1}, C_{\delta_2})$ is a proper ortholinear quasi-copula.

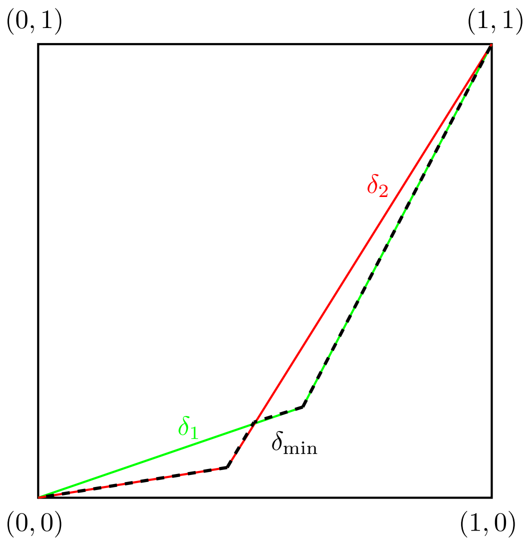


Figure 5.8: An example of the graph of $\delta_{\min}$

Since the diagonal function δ determining an orthogonal quasi-copula Q_δ can always be written as the infimum of a family $(\delta_i)_{i\in I}$ of convex functions, any orthogonal quasi-copula Q_δ can be written as

$$Q_\delta = \inf_{i\in I} C_{\delta_i} \, ,$$

where C_{δ_i} are orthogonal copulas. Hence, the class of orthogonal copulas is meet-dense in the class of orthogonal quasi copulas.

5.9. Paralinear functions

Paralinear functions are constructed by linear interpolation on segments that are parallel to the diagonal of the unit square. The linear interpolation scheme of this type of functions on some segments is depicted in Figure 5.9.

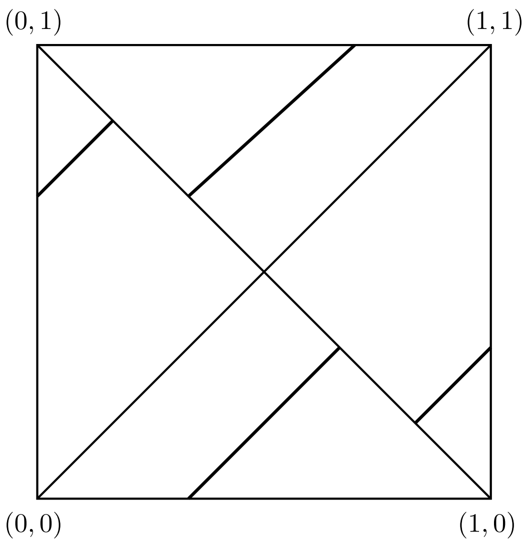


Figure 5.9: Some segments on which a paralinear function is linear.

Let us introduce the notation $v = \frac{1+x-y}{2}$.

Let $\omega \in \mathcal{O}_{\mathrm{S}}$. The function $A_\omega : [0,1]^2 \to [0,1]$ given by

$$A_\omega(x,y) = \begin{cases} y\dfrac{\omega(v)}{1-v} & \text{, if } (x,y) \in T_1\,, \\ x\dfrac{\omega(v)}{v} & \text{, if } (x,y) \in T_2\,, \\ x+y-1+(1-y)\dfrac{\omega(v)}{v} & \text{, if } (x,y) \in T_3\,, \\ x+y-1+(1-x)\dfrac{\omega(v)}{1-v} & \text{, if } (x,y) \in T_4\,, \end{cases} \tag{5.18}$$

is well defined. This function is called *the paralinear function with opposite diagonal section* ω, since it is linear on segments connecting the points $(x, 1-x)$, $(0, 1-2x)$ and $(2x, 1)$, with $x \leq 1/2$, as well as on segments connecting the points $(x, 1-x)$, $(2x-1, 0)$ and $(1, 2(1-x))$, with $x \geq 1/2$.

For any paralinear function, the boundary conditions of a semi-copula always hold. Note that a paralinear function A_ω is uniquely determined by its opposite diagonal section. Note also that a paralinear function A_ω is continuous if and only if ω is continuous. Let us consider the functions λ_ω and μ_ω defined as in Chapter 3.

Proposition 5.12. *Let $\omega \in \mathcal{O}_{\mathrm{S}}^{\mathrm{ac}}$. The paralinear function A_ω is a semi-copula if and only if*

(i) *the function λ_ω, and the function $\psi_\omega : [0,1] \to [0,1]$ defined by*

$$\psi_\omega(x) = x\omega(x)\,,$$

are respectively decreasing and increasing on the interval $]0,1/2]$;

(ii) *the function μ_ω, and the function $\xi_\omega : [0,1] \to [0,1]$ defined by*

$$\xi_\omega(x) = (1-x)\omega(x)\,,$$

are respectively increasing and decreasing on the interval $[1/2,1[$.

Proof. Suppose conditions (i) and (ii) are satisfied. To prove that A_ω is a semi-copula we need to show its increasingness in each variable. We prove the increasingness of A_ω in the second variable (the proof of the increasingness in the first variable is similar). Let $(x,y),(x,y') \in [0,1]^2$ be such that $y \leq y'$. Let us introduce the notation $v' = \frac{1+x-y'}{2}$. If $(x,y),(x,y') \in T_1$, the increasingness of A_ω is equivalent to

$$y'\frac{\omega(v')}{1-v'} - y\frac{\omega(v)}{1-v} \geq 0\,,$$

or, equivalently,

$$\frac{y'}{(1-v')^2}\xi_\omega(v') - \frac{y}{(1-v)^2}\xi_\omega(v) \geq 0\,.$$

Since $x+y \leq x+y' \leq 1$ and the decreasingness of ξ_ω on the interval $[0,1/2]$, it holds that

$$\frac{y'}{(1-v')^2}\xi_\omega(v') - \frac{y}{(1-v)^2}\xi_\omega(v) \geq \frac{y}{(1-v)^2}(\xi_\omega(v') - \xi_\omega(v)) \geq 0\,.$$

If $(x,y),(x,y') \in T_2$, the increasingness of A_ω is equivalent to

$$x\left(\frac{\omega(v')}{v'} - \frac{\omega(v)}{v}\right) \geq 0\,,$$

or, equivalently,

$$x(\lambda_\omega(v') - \lambda_\omega(v)) \geq 0\,. \tag{5.19}$$

Since $y \leq y'$, it holds that $v' \leq v$ and therefore inequality (5.19) holds due to the decreasingness of the function λ_ω on the interval $]0,1/2]$.

If $(x,y),(x,y') \in T_3$, the increasingness of A_ω is equivalent to

$$y' - y + (1-y')\frac{\omega(v')}{v'} - (1-y)\frac{\omega(v)}{v} \geq 0\,,$$

or, equivalently,

$$(1-y)(1-\lambda_\omega(v)) - (1-y')(1-\lambda_\omega(v')) \geq 0\,.$$

Since $y \leq y'$ and the decreasingness of λ_ω on the interval $]0,1/2]$, it holds that

$$(1-y)(1-\lambda_\omega(v)) - (1-y')(1-\lambda_\omega(v')) \geq (1-y')(\lambda_\omega(v') - \lambda_\omega(v)) \geq 0\,.$$

Similarly, one can prove the increasingness of A_ω in the second variable on T_4.

Conversely, suppose that A_ω is a semi-copula. Let $y, y' \in]0,1/2]$ be such that $y \leq y'$, and $x \in [0,1]$ be such that $x + y' \leq 1$ and $y \geq x'$. Clearly, the points $(x, 1+x-2y)$ and $(x, 1+x-2y')$ are located in T_2. The increasingness of A_ω in the second variable implies

$$A_\omega(x, 1+x-2y) - A_\omega(x, 1+x-2y') \geq 0\,, \tag{5.20}$$

or, equivalently,

$$x(\lambda_\omega(y) - \lambda_\omega(y')) \geq 0\,.$$

Hence, the decreasingness of λ_ω on the interval $]0,1/2]$ follows.

Let $x, x' \in]0,1/2]$ such that $x < x'$ and $y \in [0,1]$ such that $y \geq x'$ and $x' + y \leq 1$. Clearly, the points $(2x+y-1, y)$ and $(2x'+y-1, y)$ are located in T_2. The increasingness of A_ω in the first variable implies

$$(2x'+y-1)\frac{\omega(x')}{x'} - (2x+y-1)\frac{\omega(x)}{x} \geq 0\,, \tag{5.21}$$

or, equivalently,

$$2(\omega(x') - \omega(x)) - (1-y)\left(\frac{\omega(x')}{x'} - \frac{\omega(x)}{x}\right) \geq 0\,, \tag{5.22}$$

Dividing by $x' - x$ and taking the limit $x' \to x$, inequality (5.22) becomes

$$2\omega'(x) - (1-y)\left(\frac{\omega(x)}{x}\right)' \geq 0\,,$$

where the derivative exists. Setting $y = 1-x$, the last inequality is equivalent to

$$x\omega'(x) + \omega(x) \geq 0\,,$$

or, equivalently, $\psi'_\omega(x) \geq 0$, where the derivative exists. Since ω is absolutely continuous, it holds that ψ_ω is absolutely continuous. The fact that $\psi'_\omega(x) \geq 0$, where the derivative exists, on the interval $]0,1/2]$, then implies that ψ_ω is increasing on the interval $]0,1/2]$.

Let $x, x' \in [1/2, 1[$ such that $x \leq x'$ and $y \in [0,1]$ such that $x + y' \leq 1$ and $y \leq x$. Clearly, the points $(2x + y - 1, y)$ and $(2x' + y - 1, y)$ are located in T_1. The increasingness of A_ω in the first variable implies

$$A_\omega(2x' + y - 1, y) - A_\omega(2x + y - 1, y) \geq 0\,, \tag{5.23}$$

or, equivalently,

$$y(\mu_\omega(x') - \mu_\omega(x)) \geq 0\,.$$

Hence, the increasingness of μ_ω on the interval $[1/2, 1[$ follows. Similarly, one can prove the increasingness of ψ_ω on the interval $]0, 1/2]$ and the decreasingness of ξ_ω on the interval $[1/2, 1[$, which completes the proof. □

Let A_ω be an ortholinear function with opposite diagonal section ω. The function A', defined by

$$A' = \varphi_2(A)\,, \tag{5.24}$$

where φ_2 is the transformation defined in (1.3), is again an ortholinear function whose diagonal section $\delta_{A'}$ is given by $\delta_{A'}(x) = x - \omega(x)$. This transformation permits to derive in a straightforward manner the conditions that guarantee the existence of a paralinear function. Based on the above discussion the proofs of the following propositions are obvious due to Proposition 5.3 and Theorem 5.1.

Proposition 5.13. *Let $\omega \in \mathcal{O}$. The paralinear function A_ω is a quasi-copula if and only if*

(i) *the function λ_ω, and the function ψ_ω defined in Proposition 5.12, are respectively decreasing and increasing on the interval* $]0, 1/2]$

(ii) *the functions μ_ω, and the function ξ_ω defined in Proposition 5.12, are respectively increasing and decreasing on the interval* $[1/2, 1[$.

Proposition 5.14. *Let $\omega \in \mathcal{O}$. The paralinear function A_ω is a copula if and only if the function ω is concave.*

Example 5.11. *Consider the opposite diagonal functions $\omega_{T_\mathbf{M}}$ and $\omega_{T_\mathbf{L}}$. Clearly, $\omega_{T_\mathbf{M}}$ and $\omega_{T_\mathbf{L}}$ are concave functions. The corresponding ortholinear copulas are $T_\mathbf{M}$ and $T_\mathbf{L}$, respectively.*

Example 5.12. *Consider the opposite diagonal function* $\omega_{T_\mathbf{P}}(x) = x(1-x)$*. Clearly,* $\omega_{T_\mathbf{P}}$ *is concave. The corresponding ortholinear copula is given by*

$$A_{\omega_{T_\mathbf{P}}}(x,y) = \begin{cases} yv & , \text{ if } (x,y) \in T_1 \,, \\ x(1-v) & , \text{ if } (x,y) \in T_2 \,, \\ x-(1-y)v & , \text{ if } (x,y) \in T_3 \,, \\ y-(1-x)(1-v) & , \text{ if } (x,y) \in T_4 \,. \end{cases}$$

We conclude this section by finding the intersection between the class of ortholinear copulas and the class of paralinear copulas.

Proposition 5.15. *Let* C *be a copula. Then it holds that* C *is an ortholinear copula as well as a paralinear copula if and only if* C *is a member of the family of convex sums of* $T_\mathbf{M}$ *and* $T_\mathbf{L}$*.*

Proof. Suppose that C is an ortholinear copula with opposite diagonal section ω and suppose further that C is a paralinear copula with diagonal section δ. Due to the construction method of ortholinear copulas and paralinear copulas, δ and ω must be piecewise linear and are given by

$$\delta(x) = \begin{cases} 2x\omega(1/2) & , \text{ if } x \leq 1/2 \,, \\ 2x-1+2(1-x)\omega(1/2) & , \text{ if } x \geq 1/2 \,, \end{cases}$$

$$\omega(x) = \begin{cases} 2x\delta(1/2) & , \text{ if } x \leq 1/2 \,, \\ 2(1-x)\delta(1/2) & , \text{ if } x \geq 1/2 \,. \end{cases}$$

Since δ and ω are the diagonal and opposite diagonal sections of C, it holds that $\delta(1/2) = \omega(1/2)$. Using the notation $\theta = 2\delta(1/2) = 2\omega(1/2)$, δ and ω can be rewritten as

$$\delta(x) = \theta\delta_{T_\mathbf{M}}(x) + (1-\theta)\delta_{T_\mathbf{L}}(x) \,, \quad \omega(x) = \theta\omega_{T_\mathbf{M}}(x) + (1-\theta)\omega_{T_\mathbf{L}}(x) \,.$$

Recalling that any ortholinear (resp. paralinear) copula is uniquely determined by its diagonal (resp. opposite diagonal) section, our assertion follows. □

6 Semilinear copulas based on horizontal and vertical interpolation

6.1. Introduction

Rather than including one line in the unit square in the linear interpolation procedure as in the previous chapters, we include in this chapter two lines. More specifically, these two lines are the diagonal and the opposite diagonal of the unit square. We restrict our attention in this chapter to the class of copulas.

We introduce in this chapter new families of semilinear copulas. Recently, Durante et al. [38] introduced two families of semilinear copulas with a given diagonal section, which they called lower and upper semilinear copulas. These copulas are obtained by linear interpolation on segments connecting the diagonal and one of the sides of the unit square. Lower and upper semilinear copulas are symmetric. In order to allow for non-symmetric semilinear copulas as well, De Baets et al. [20] have introduced two related families of semilinear copulas with a given diagonal section, called horizontal and vertical semilinear copulas. In the present chapter, we first introduce four families of semilinear copulas with a given opposite diagonal section, called lower-upper, upper-lower, horizontal and vertical semilinear copulas. There is a great similarity between the case of a given opposite diagonal section and that of a given diagonal section (see also [23]), which can be explained by the existence of a transformation that maps copulas onto copulas in such a way that the diagonal is mapped onto the opposite diagonal and vice versa. In the second part of this chapter, we consider the construction of semilinear copulas with given diagonal and opposite diagonal sections. Also here, four new families of semilinear copulas are introduced, called orbital, vertical, horizontal and radial semilinear copulas.

This chapter is organized as follows. In Section 6.2, we recall some essential facts on semilinear copulas with a given diagonal section, while in Section 6.3, we introduce semilinear copulas with a given opposite diagonal section. In Section 6.4, we introduce the four families of semilinear copulas with given diagonal and opposite diagonal sections and provide for each family the conditions to be satisfied by a diagonal and opposite diagonal function such that they can be the diagonal and opposite diagonal sections of a semilinear copula belonging to that family. Finally, in Section 6.5, we derive some interesting properties of the family of orbital semilinear copulas.

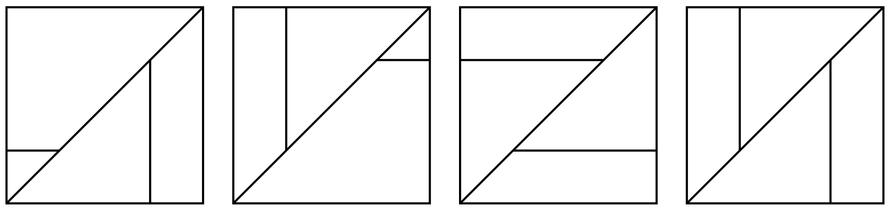


Figure 6.1: Semilinear copulas with a given diagonal section.

6.2. Semilinear copulas with a given diagonal section

Two different methods for constructing semilinear copulas with a given diagonal section have been presented recently. The first method is based on linear interpolation on segments connecting the diagonal with the left and lower side (resp. right and upper side) of the unit square; these symmetric copulas are called lower (resp. upper) semilinear copulas [38]. The second method is based on linear interpolation on segments connecting the diagonal with the lower and upper side (resp. left and right side) of the unit square; these in general non-symmetric copulas are called vertical (resp. horizontal) semilinear copulas [20]. The different interpolation schemes are depicted in Figure 6.1.

We briefly recall the conditions on a diagonal function δ that guarantee the existence of a lower or vertical semilinear copula with δ as diagonal section.

Let us consider the subtriangles I_1 and I_2 of the unit square as in Chapter 3. Let us further consider the functions λ_δ and μ_δ defined as in Chapter 3.

Proposition 6.1. [38] *Let δ be a diagonal function. The function $C^l_\delta : [0,1]^2 \to [0,1]$ defined by*

$$C^l_\delta(x,y) = \begin{cases} y\dfrac{\delta(x)}{x} & , \text{ if } (x,y) \in I_1 \,, \\ x\dfrac{\delta(y)}{y} & , \text{ if } (x,y) \in I_2 \,, \end{cases} \tag{6.1}$$

where the convention $\frac{0}{0} := 0$ is adopted, is a copula with diagonal section δ, called lower semilinear copula with diagonal section δ, *if and only if*

(i) *the function λ_δ is increasing;*

(ii) *the function $\rho_\delta :]0,1] \to [1,\infty[$, defined by $\rho_\delta(x) = \frac{\delta(x)}{x^2}$, is decreasing.*

Proposition 6.2. [20] *Let δ be a diagonal function. The function $C^v_\delta : [0,1]^2 \to [0,1]$ defined by*

$$C^v_\delta(x,y) = \begin{cases} y\dfrac{\delta(x)}{x} & , \text{ if } (x,y) \in I_1 \,, \\ \dfrac{x(y-x)}{1-x} + \dfrac{1-y}{1-x}\,\delta(x) & , \text{ if } (x,y) \in I_2 \,, \end{cases} \tag{6.2}$$

where the convention $\frac{0}{0} := 1$ is adopted, is a copula with diagonal section δ, called vertical semilinear copula with diagonal section δ, *if and only if*

(i) *the function λ_δ is increasing;*

(ii) *the function μ_δ is increasing;*

(iii) *$\delta \geq \delta_\Pi$, i.e. for any $x \in [0,1]$, it holds that $\delta(x) \geq x^2$.*

It can be easily proven that the upper semilinear copula with a given diagonal section can be regarded as a transform of a lower semilinear copula.

Proposition 6.3. *Let δ be a diagonal function and $\hat{\delta}$ be the diagonal function defined by $\hat{\delta}(x) = 2x - 1 + \delta(1-x)$. The function $C^u_\delta : [0,1]^2 \to [0,1]$, defined by*

$$C^u_\delta = \sigma(C^l_{\hat{\delta}}) \,, \tag{6.3}$$

where σ is the transformation defined in (1.3), is a copula with diagonal section δ, called upper semilinear copula with diagonal section δ, *if and only if*

(i) *the function μ_δ is increasing;*

(ii) *the function $\sigma_\delta : \ [0,1[\to [1,\infty[$, defined by $\sigma_\delta(x) = \frac{\delta(x)-x^2}{(1-x)^2}$, is increasing.*

Similarly, the horizontal semilinear copula with a given diagonal section is a transform of a vertical semilinear copula.

Proposition 6.4. *Let δ be a diagonal function. The function $C^v_\delta : [0,1]^2 \to [0,1]$, defined by*

$$C^h_\delta = \pi(C^v_\delta) \,, \tag{6.4}$$

where π is the transformation defined in (1.3), is a copula with diagonal section δ, called horizontal semilinear copula with diagonal section δ, *if and only if C^v_δ is a copula, i.e. under the conditions of Proposition 6.2.*

Note that for any two lower (resp. upper, vertical, horizontal) semilinear copulas C_1 and C_2 it holds that $C_1 \leq C_2$ if and only if $\delta_{C_1} \leq \delta_{C_2}$. Since the function ρ_δ is decreasing, $\rho_\delta(x) \geq \rho_\delta(1) = 1$ for any $x \in]0,1]$. Therefore, $\delta(x) \geq x^2$, for any lower semilinear copula C. Similarly, since the function σ_δ is increasing, $\delta(x) \geq x^2$, for any upper semilinear copula C. Note also that M and Π are examples of copulas that are at the same time lower, upper, vertical and horizontal semilinear copulas. Hence, Π is the smallest semilinear copula (of one of the above four types), i.e.

every semilinear copula with a given diagonal section (of one of the above four types) is positive quadrant dependent.

6.3. Semilinear copulas with a given opposite diagonal section

In analogy with the lower (resp. upper) and vertical (resp. horizontal) semilinear copulas with a given diagonal section δ, we introduce lower-upper (resp. upper-lower) and vertical (resp. horizontal) semilinear copulas with a given opposite diagonal section ω. For instance, the lower-upper semilinear copula is constructed based on linear interpolation on segments connecting the opposite diagonal with the left and upper side of the unit square. See also Figure 6.2 where the four different interpolation schemes are depicted.

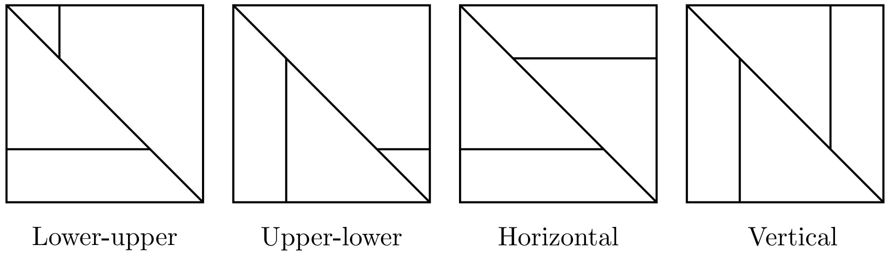


Figure 6.2: Semilinear copulas with a given opposite diagonal section.

Let C_ω be a copula with opposite diagonal section ω. The function C', defined by $C' = \varphi_2(C)$, where φ_2 is the transformation defined in (1.3)), is again a copula whose diagonal section $\delta_{C'}$ is given by $\delta_{C'}(x) = x - \omega(x)$. This transformation permits to derive in a straightforward manner the conditions that guarantee the existence of a semilinear copula (of any of the above types) with a given opposite diagonal section. Let us consider the subtriangles J_1 and J_2 of the unit square as in Chapter 3. Let us further consider the functions λ_ω and defined μ_ω as in Chapter 3.

Proposition 6.5. *Let ω be an opposite diagonal function. The function $C_\omega^{lu} : [0,1]^2 \to [0,1]$ defined by*

$$C_\omega^{lu}(x,y) = \begin{cases} \dfrac{x}{1-y}\,\omega(1-y) & , \text{ if } (x,y) \in J_1\,, \\ x+y-1+\dfrac{1-y}{x}\,\omega(x) & , \text{ if } (x,y) \in J_2\,, \end{cases} \tag{6.5}$$

where the convention $\frac{0}{0} := 0$ is adopted, is a copula with opposite diagonal section ω, called lower-upper semilinear copula with opposite diagonal section ω, *if and* *only if*

(i) *the function* λ_ω *is decreasing;*

(ii) *the function* $\eta_\omega :]0,1] \to [1,\infty[$ *, defined by* $\eta_\omega(x) = \frac{x-\omega(x)}{x^2}$*, is decreasing.*

Proposition 6.6. *Let* ω *be an opposite diagonal function. The function* $C_\omega^v : [0,1]^2 \to [0,1]$*, defined by*

$$C_\omega^v(x,y) = \begin{cases} \dfrac{y}{1-x}\,\omega(x) & , \text{ if } (x,y) \in J_1\,, \\ x+y-1+\dfrac{1-y}{x}\,\omega(x) & , \text{ if } (x,y) \in J_2\,, \end{cases} \tag{6.6}$$

where the convention $\frac{0}{0} := 0$ *is adopted, is a copula with opposite diagonal section* ω*, called* vertical semilinear copula with opposite diagonal section ω*, if and only if*

(i) *the function* λ_ω *is decreasing;*

(ii) *the function* μ_ω *is increasing;*

(iii) $\omega \le \omega_\Pi$*, i.e. for any* $x \in [0,1]$*, it holds that* $\omega(x) \le x(1-x)$*.*

Proposition 6.7. *Let* ω *be an opposite diagonal function and* $\hat{\omega}$ *be the opposite diagonal function defined by* $\hat{\omega}(x) = \omega(1-x)$*. The function* $C_\delta^{ul} : [0,1]^2 \to [0,1]$*, defined by*

$$C_\omega^{ul} = \pi(C_{\hat{\omega}}^{lu})\,, \tag{6.7}$$

is a copula with opposite diagonal section ω*, called* upper-lower semilinear copula with opposite diagonal section ω*, if and only if*

(i) *the function* μ_ω *is increasing;*

(ii) *the function* $\zeta_\omega : [0,1[\to [1,\infty[$ *, defined by* $\zeta_\omega(x) = \frac{1-x-\omega(x)}{(1-x)^2}$*, is increasing.*

Similarly, the horizontal semilinear copula with a given opposite diagonal section is a linear transform of a vertical semilinear copula.

Proposition 6.8. *Let* ω *be an opposite diagonal function and* $\hat{\omega}$ *be the opposite diagonal function defined by* $\hat{\omega}(x) = \omega(1-x)$*. The function* $C_\omega^h : [0,1]^2 \to [0,1]$*, defined by*

$$C_\omega^h = \pi(C_{\hat{\omega}}^v)\,, \tag{6.8}$$

is a copula with opposite diagonal section ω*, called* horizontal semilinear copula with opposite diagonal section ω*, if and only if* $C_{\hat{\omega}}^v$ *is a copula, i.e. under the conditions of Proposition 6.6.*

Note that for any two lower-upper (resp. vertical, upper-lower, horizontal) semilinear copulas C_1 and C_2 it holds that $C_1 \le C_2$ if and only if $\omega_{C_1} \le \omega_{C_2}$. From Propositions 6.5 and 6.6, it follows that $\omega(x) \le x(1-x)$ for any lower-upper, upper-lower, horizontal or vertical semilinear copula. Note also that W and Π are examples of copulas that are at the same time lower-upper, vertical, upper-lower

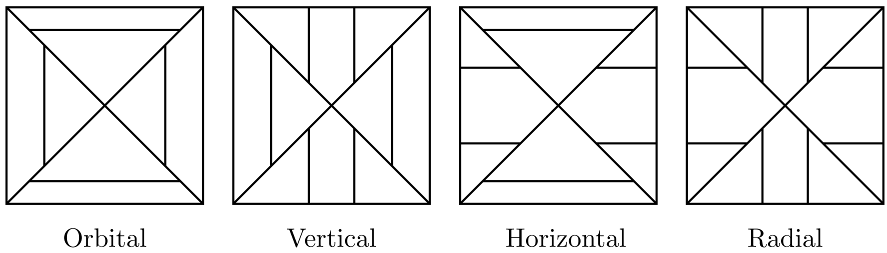


Figure 6.3: Semilinear copulas with given diagonal and opposite diagonal sections.

and horizontal semilinear copulas. Hence, Π is the greatest semilinear copula (of one of the above four types), i.e. every semilinear copula with a given opposite diagonal section (of one of the above four types) is negative quadrant dependent.

6.4. Semilinear copulas with given diagonal and opposite diagonal sections

In this section we introduce four new families of semilinear copulas. Their construction is based on linear interpolation on segments connecting the diagonal and opposite diagonal or connecting the diagonal or opposite diagonal and one of the sides of the unit square. Since in any of the four triangular parts of the unit square delimited by the diagonal and opposite diagonal, we can either interpolate between a point on the diagonal and a point on the opposite diagonal, or between a point on the sides of the unit square and a point on the diagonal or opposite diagonal, there are sixteen possible interpolation schemes. Based on symmetry considerations, we will consider only the four interpolation schemes depicted in Figure 6.3.

Clearly, in general, given a diagonal function δ and an opposite diagonal function ω, there need not exist a copula that has δ as diagonal section and ω as opposite diagonal section. For instance, the diagonal function $\delta(x) = x^2$ and the opposite diagonal function $\omega(x) = \min(x, 1-x)$ cannot be the diagonal and opposite diagonal sections of a copula since $\delta(1/2) = 1/4 \neq 1/2 = \omega(1/2)$.

Let us consider the subtriangles T_1, T_2, T_3 and T_4 of the unit square as in Chapter 5.

Proposition 6.9. *Let δ and ω be diagonal and opposite diagonal functions such that $\delta(1/2) = \omega(1/2)$. The function $C^o_{\delta,\omega} : [0,1]^2 \to [0,1]$, defined by*

$$C^o_{\delta,\omega}(x,y) = \begin{cases} \dfrac{x+y-1}{2y-1}\,\delta(y) + \dfrac{y-x}{2y-1}\,\omega(1-y) & , \text{ if } (x,y) \in T_1 \cup T_3\,, \\ \dfrac{x+y-1}{2x-1}\,\delta(x) + \dfrac{x-y}{2x-1}\,\omega(x) & , \text{ otherwise}\,, \end{cases} \tag{6.9}$$

where the convention $\frac{0}{0} := 1/2$ is adopted, is a copula with diagonal section δ and opposite diagonal section ω, called orbital semilinear copula with diagonal section δ and opposite diagonal section ω, *if and only if:*

(i) *the functions $\vartheta_{\delta,\omega}, \psi_{\delta,\omega} : [0,1/2[\cup]1/2,1] \to [0,1]$, defined by*

$$\vartheta_{\delta,\omega}(x) = \frac{\omega(x) - \delta(x)}{1-2x}\,, \quad \psi_{\delta,\omega}(x) = \frac{\omega(1-x) - \delta(x)}{1-2x}\,,$$

are increasing on the interval $[0,1/2[$ and on the interval $]1/2,1]$;

(ii) *for any $x, x' \in [0,1/2[$, such that $x < x'$, it holds that*

$$\omega(x) + \omega(1-x) \le \frac{\delta(x')(1-2x) - \delta(x)(1-2x')}{x'-x}\,,$$

$$\delta(x) + \delta(1-x) \ge \frac{\omega(x')(1-2x) - \omega(x)(1-2x')}{x'-x}\,;$$

(iii) *for any $x, x' \in]1/2,1]$, such that $x < x'$, it holds that*

$$\omega(x') + \omega(1-x') \le \frac{\delta(x')(1-2x) - \delta(x)(1-2x')}{x'-x}\,,$$

$$\delta(x') + \delta(1-x') \ge \frac{\omega(x')(1-2x) - \omega(x)(1-2x')}{x'-x}\,.$$

Proof. The function $C^o_{\delta,\omega}$ defined in (6.9) clearly satisfies the boundary conditions of a copula. Therefore, it suffices to prove that conditions (i)-(iii) are equivalent to the property of 2-increasingness. Due to the additivity of volumes, we distinguish the following cases. Consider a rectangle $R = [x,x'] \times [y,y'] \subseteq [0,1]^2$.

(a) If $R \subseteq T_2$ such that $x' < 1/2$, then it holds that

$$V_{C^o_{\delta,\omega}}(R) = (\vartheta_{\delta,\omega}(x') - \vartheta_{\delta,\omega}(x))(y'-y)\,.$$

The nonnegativity of $V_{C^o_{\delta,\omega}}(R)$ is clearly equivalent to the increasingness of the function $\vartheta_{\delta,\omega}$ on the intervals $[0,1/2[$. Similarly, the nonnegativity of

$V_{C^{o}_{\delta,\omega}}(R)$ for any rectangle $R \subseteq T_4$ such that $x > 1/2$ is equivalent to the increasingness of the function $\vartheta_{\delta,\omega}$ on the interval $]1/2, 1]$.

(b) If $R \subseteq T_1$ such that $y' < 1/2$, then it holds that

$$V_{C^{o}_{\delta,\omega}}(R) = (\psi_{\delta,\omega}(y') - \psi_{\delta,\omega}(y))(x' - x)\,.$$

The nonnegativity of $V_{C^{o}_{\delta,\omega}}(R)$ is clearly equivalent to the increasingness of the function $\psi_{\delta,\omega}$ on the intervals $[0, 1/2[$. Similarly, the nonnegativity of $V_{C^{o}_{\delta,\omega}}(R)$ for any rectangle $R \subseteq T_3$ such that $y > 1/2$ is equivalent to the increasingness of the function $\psi_{\delta,\omega}$ on the intervals $]1/2, 1]$.

(c) If R is of the type $[x, x'] \times [x, x']$ or of the type $[x, x'] \times [1-x', 1-x]$ (otherwise stated, rectangles whose diagonal or opposite diagonal is situated on the diagonal or opposite diagonal of the unit square), then the nonnegativity of $V_{C^{o}_{\delta,\omega}}(R)$ is equivalent to conditions (ii) and (iii). □

Note that conditions (ii) and (iii) can be reformulated by means of the functions $\vartheta_{\delta,\omega}$ and $\psi_{\delta,\omega}$ in the following way:

(ii') for any $x, x' \in [0, 1/2[$ such that $x < x'$, it holds that

$$\vartheta_{\delta,\omega}(x) + \psi_{\delta,\omega}(x) \leq \frac{\delta(x') - \delta(x)}{x' - x}\,, \quad \psi_{\delta,\omega}(1-x) - \vartheta_{\delta,\omega}(x) \geq \frac{\omega(x') - \omega(x)}{x' - x}\,;$$

(iii') for any $x, x' \in]1/2, 1]$ such that $x < x'$, it holds that

$$\vartheta_{\delta,\omega}(x') + \psi_{\delta,\omega}(x') \geq \frac{\delta(x') - \delta(x)}{x' - x}\,, \quad \psi_{\delta,\omega}(1-x') - \vartheta_{\delta,\omega}(x') \leq \frac{\omega(x') - \omega(x)}{x' - x}\,.$$

In case δ and ω are differentiable functions, the latter two conditions are equivalent with

(ii") for any $x < 1/2$, it holds that

$$\delta'(x) \geq \vartheta_{\delta,\omega}(x) + \psi_{\delta,\omega}(x)\,, \quad \omega'(x) \leq \psi_{\delta,\omega}(1-x) - \vartheta_{\delta,\omega}(x)\,;$$

(iii") for any $x > 1/2$, it holds that

$$\delta'(x) \leq \vartheta_{\delta,\omega}(x) + \psi_{\delta,\omega}(x)\,, \quad \omega'(x) \geq \psi_{\delta,\omega}(1-x) - \vartheta_{\delta,\omega}(x)\,.$$

Example 6.1. *Consider the diagonal function $\delta(x) = x^2$ and the opposite diagonal function $\omega(x) = (1/2)\min(x, 1-x)$. Note that $\delta(1/2) = \omega(1/2) = 1/4$. Clearly, the functions $\vartheta_{\delta,\omega}$ and $\psi_{\delta,\omega}$ are increasing on the intervals $[0, 1/2[$ and $]1/2, 1]$. For* *any* $0 \leq x < x' \leq 1/2$*, it holds that*

$$\vartheta_{\delta,\omega}(x) + \psi_{\delta,\omega}(x) = x < x + x' = \frac{\delta(x') - \delta(x)}{x' - x}$$

and

$$\psi_{\delta,\omega}(1-x) - \vartheta_{\delta,\omega}(x) = 1 - x > \frac{1}{2} = \frac{\omega(x') - \omega(x)}{x' - x},$$

whence condition (ii') is satisfied. Similarly, condition (iii') is satisfied, and therefore the function $C^o_{\delta,\omega}$ *defined in (6.9) is the orbital semilinear copula with diagonal section* δ *and opposite diagonal section* ω*.*

Next, we consider the construction of the horizontal (resp. vertical) semilinear copula with given diagonal and opposite diagonal sections. It is constructed by interpolating in the x-direction (resp. y-direction). As it is again possible to connect the two types by means of a transformation, we will make explicit the conditions to be fulfilled by δ and ω for just one type.

Proposition 6.10. *Let* δ *and* ω *be diagonal and opposite diagonal functions such that* $\delta(1/2) = \omega(1/2)$*. The function* $C^h_{\delta,\omega} : [0,1]^2 \to [0,1]$*, defined by* $C^h_{\delta,\omega}(x,y) =$

$$\begin{cases} \dfrac{x+y-1}{2y-1}\,\delta(y) + \dfrac{y-x}{2y-1}\,\omega(1-y) & \text{, if } (x,y) \in T_1 \cup T_3 \text{ and } (x,y) \neq (1/2,1/2)\,, \\ \dfrac{x}{y}\,\delta(y) & \text{, if } (x,y) \in T_2 \text{ and } y \leq 1/2\,, \\ \dfrac{x}{1-y}\,\omega(1-y) & \text{, if } (x,y) \in T_2 \text{ and } y \geq 1/2\,, \\ x+y-1+\dfrac{1-x}{y}\,\omega(1-y) & \text{, if } (x,y) \in T_4 \text{ and } y \leq 1/2\,, \\ y - \dfrac{1-x}{1-y}\,(y-\delta(y)) & \text{, if } (x,y) \in T_4 \text{ and } y \geq 1/2\,, \end{cases} \tag{6.10}$$

where the convention $\frac{0}{0} := 0$ *is adopted, is a copula with diagonal section* δ *and opposite diagonal section* ω*, called* horizontal semilinear copula with diagonal section δ and opposite diagonal section ω*, if and only if:*

(i) *the function* $\psi_{\delta,\omega}$ *is increasing on the interval* $[0,1/2[$ *and on the interval* $]1/2,1]$*;*

(ii) *the function* λ_δ *is increasing on the interval* $]0,1/2]$ *and the function* λ_ω*is decreasing on the interval* $[1/2,1]$*;*

(iii) *the function* μ_δ *is increasing on the interval* $[1/2,1[$ *and the function* μ_ω *is increasing on the interval* $[0,1/2]$*;*

(iv) *for any* $x \in [0,1/2]$*, it holds that*

$$\min[(1-x)\delta(x) - x\omega(1-x)\,, x\delta(1-x) - (1-x)\omega(x)] \geq 0\,;$$

(v) *for any* $x \in [1/2, 1]$*, it holds that*

$$\min[x(1-2x)-(1-x)\omega(1-x)+x\delta(x)\,,(\delta(1-x)+2x-1)(1-x)-x\omega(x)] \geq 0\,.$$

Proof. The proof is similar to that of Proposition 6.9. □

Example 6.2. *Consider the diagonal function* $\delta(x) = \frac{x}{2-x}$ *and the opposite diagonal function* $\omega(x) = \frac{2}{3}\min(x, 1-x)$*. Note that* $\delta(1/2) = \omega(1/2) = 1/3$*. The first three conditions of Proposition 6.10 are trivially fulfilled. Moreover, for any* $x \in [0, 1/2]$*, it holds that*

$$(1-x)\delta(x) - x\omega(1-x) = \frac{x(3-x)(1-2x)}{3(2-x)} \geq 0$$

and

$$x\delta(1-x) - (1-x)\omega(x) = \frac{x(1-x)(1-2x)}{3(1+x)} \geq 0\,,$$

which implies condition (iv). Condition (v) holds similarly. Therefore the function $C^h_{\delta,\omega}$ *is the horizontal semilinear copula with diagonal section* δ *and opposite diagonal section* ω*.*

The vertical semilinear copula $C^v_{\delta,\omega}$ with diagonal section δ and opposite diagonal section ω is defined by

$$C^v_{\delta,\omega} = \pi(C^h_{\delta,\hat{\omega}})\,, \tag{6.11}$$

with $\hat{\omega}$ the opposite diagonal function defined by $\hat{\omega}(x) = \omega(1-x)$ and $C^h_{\delta,\hat{\omega}}$ the horizontal semilinear copula with diagonal section δ and opposite diagonal section $\hat{\omega}$, provided the latter is properly defined. In fact, the conditions on δ and ω are exactly conditions (i)–(v) of Proposition 6.10.

Finally, we consider the case where the interpolation is done on segments connecting points on the diagonal or opposite diagonal and points on the sides of the unit square. We call a semilinear copula that results from this interpolation scheme a radial semilinear copula.

Proposition 6.11. *Let δ and ω be diagonal and opposite diagonal functions such that $\delta(1/2) = \omega(1/2)$. The function $C^r_{\delta,\omega} : [0,1]^2 \to [0,1]$, defined by*

$$C^r_{\delta,\omega}(x,y) = \begin{cases} \dfrac{y}{x}\,\delta(x) & , \text{ if } (x,y) \in T_1 \text{ and } x \le 1/2\,, \\ \dfrac{y}{1-x}\,\omega(x) & , \text{ if } (x,y) \in T_1 \text{ and } x \ge 1/2\,, \\ \dfrac{x}{y}\,\delta(y) & , \text{ if } (x,y) \in T_2 \text{ and } y \le 1/2\,, \\ \dfrac{x}{1-y}\,\omega(1-y) & , \text{ if } (x,y) \in T_2 \text{ and } y \ge 1/2\,, \\ x+y-1+\dfrac{1-y}{x}\,\omega(x) & , \text{ if } (x,y) \in T_3 \text{ and } x \le 1/2\,, \\ x-\dfrac{1-y}{1-x}\,(x-\delta(x)) & , \text{ if } (x,y) \in T_3 \text{ and } x \ge 1/2\,, \\ y-\dfrac{1-x}{1-y}\,(y-\delta(y)) & , \text{ if } (x,y) \in T_4 \text{ and } y \ge 1/2\,, \\ x+y-1+\dfrac{1-x}{y}\,\omega(1-y) & , \text{ if } (x,y) \in T_4 \text{ and } y \le 1/2\,, \end{cases} \tag{6.12}$$

where the convention $\frac{0}{0} := 0$ is adopted, is a copula with diagonal section δ and opposite diagonal section ω, called radial semilinear copula with diagonal section δ and opposite diagonal section ω, *if and only if*

(i) *the function λ_δ is increasing on $]0,1/2]$, the function ρ_δ is decreasing on $]0,1/2]$, the function λ_ω is decreasing on $]0,1/2]$ and the function η_ω is decreasing on $]0,1/2]$;*

(ii) *the function μ_δ is increasing on $[1/2,1[$, the function σ_δ is increasing on $[1/2,1[$, the function μ_ω is increasing on $[1/2,1[$ and the function ζ_ω is increasing on $[1/2,1[$.*

Note that M, Π and W are examples of copulas that are at the same time orbital, vertical, horizontal and radial semilinear copulas with given diagonal and opposite diagonal sections.

6.5. Properties of orbital semilinear copulas with a given diagonal or opposite diagonal section

In this section we will further study the family of orbital semilinear copulas. It is the only family for which the interpolation in all the triangular parts of the unit square occurs between points on the diagonal and points on the opposite diagonal, and therefore it has no counterpart at all in the families of semilinear copulas which were constructed before by giving either a diagonal or an opposite diagonal section.

It is well known that M (resp. W) is the only copula with diagonal section δ_M (resp. opposite diagonal section ω_W). As these copulas are orbital semilinear copulas, they are obviously the only such copulas with that given behaviour. In general, however, M (resp. W) is not the only copula with opposite diagonal section ω_M (resp. diagonal section δ_W). For instance, the copula F_{δ_W} defined in (1.13) differs from W.

In the context of orbital semilinear copulas, however, M and W take up a unique role again.

Proposition 6.12.

(i) M *is the only orbital semilinear copula which has* ω_M *as opposite diagonal section;*

(ii) W *is the only orbital semilinear copula which has* δ_W *as diagonal section.*

Proof. We will prove (ii), the proof of (i) being similar. Suppose that $C^o_{\delta_W,\omega}$ is the orbital semilinear copula with diagonal section δ_W and a not yet specified opposite diagonal section ω. Note that $\omega(1/2) = \delta_W(1/2) = 0$. From condition (ii) of Proposition 6.9, it follows that for any $x \in [0,1/2[$, it must hold that $\omega(x)+\omega(1-x) = 0$, which implies that $\omega = \omega_W$. Hence, $C^o_{\delta_W,\omega} = C^o_{\delta_W,\omega_W} = W$. $\square$

We now investigate the situation where either the given diagonal function is the diagonal section δ_Π of the product copula Π, or the given opposite diagonal function ω is the opposite diagonal section ω_Π of Π. In fact, we will prove a more general statement by introducing parametrized families of diagonal (resp. opposite diagonal) functions that contain δ_Π (resp. ω_Π).

Proposition 6.13. *Let* δ *and* ω *be differentiable diagonal and opposite diagonal functions such that* $\delta(1/2) = \omega(1/2)$.

(i) *If* $\delta(x) = \alpha x^2 + (1-\alpha)x$ *for arbitrary* $\alpha \in]0,1]$, *then the function* $C^o_{\delta,\omega}$ *given by (6.9) is an orbital semilinear copula if and only if the function* $\phi_\omega : [0,1/2[\cup]1/2,1] \to [-1,1]$, *defined by*

$$\phi_\omega(x) = \frac{2\omega(x) + \alpha(2x^2 - x + 1/2) - 1}{2(1-2x)},$$

is decreasing and α*-Lipschitz continuous on both intervals* $[0,1/2[$ *and* $]1/2,1]$.

(ii) *If* $\omega(x) = \alpha x(1-x)$ *for arbitrary* $\alpha \in]0,1]$, *then the function* $C^o_{\delta,\omega}$ *given by (6.9) is an orbital semilinear copula if and only if the function* $\phi_\delta : [0,1/2[\cup]1/2,1] \to [-1,1]$, *defined by*

$$\phi_\delta(x) = \frac{\delta(x) - \alpha x^2}{1-2x},$$

is increasing and α*-Lipschitz continuous on both intervals* $[0, 1/2[$ *and* $]1/2, 1]$.

Proof. We will prove (ii). The proof of (i) is analogous. Let $C^o_{\delta,\omega}$ be the function defined in (6.9) with $\omega(x) = \alpha x(1-x)$. Note that the functions $\vartheta_{\delta,\omega}$ and $\psi_{\delta,\omega}$ coincide. This function is an orbital semilinear copula if and only if conditions (i)–(iii) of Proposition 6.9 are satisfied. Since δ is differentiable, we use the equivalent conditions (ii') and (iii'). In the case $x \in [0, 1/2[$, it should therefore hold that $\delta'(x) \geq \vartheta_{\delta,\omega}(x) + \psi_{\delta,\omega}(x)$, or

$$\delta'(x) \geq \frac{\omega(x) + \omega(1-x) - 2\delta(x)}{1-2x},$$

which is equivalent to the condition that the function ϕ_δ is increasing on $[0, 1/2[$, as can be readily verified by computing the derivative of ϕ_δ. Furthermore, it is required that $\vartheta_{\delta,\omega}$ is increasing on $[0, 1/2[$, or, equivalently, $\alpha - \phi'_\delta(x) \geq 0$. It follows that ϕ_δ must be α-Lipschitz continuous on $[0, 1/2[$. Since $\phi_\delta(0) = 0$, the increasingness of ϕ_δ implies that $\delta(x) \geq \alpha x^2$ for any $x \in [0, 1/2[$, whence

$$\delta(x) + \delta(1-x) \geq \alpha(x^2 + (1-x)^2) = \omega'(x)(1-2x) + 2\omega(x),$$

which is equivalent to $\omega'(x) \leq \psi_{\delta,\omega}(1-x) - \vartheta_{\delta,\omega}(x)$ for any $x \in [0, 1/2[$. Hence, on the subinterval $[0, 1/2[$, all conditions of Proposition 6.9 are satisfied. In case $x \in]1/2, 1]$, the proof is similar. □

Example 6.3.

(i) *Let* $\delta(x) = \alpha x^2 + (1-\alpha)x$, *with* $\alpha \in]0, 1]$ *and consider the opposite diagonal function*

$$\omega(x) = \left(1 - \frac{\alpha}{2}\right) \min(x, 1-x).$$

The function ϕ_ω *is decreasing and* α*-Lipschitz continuous on* $[0, 1/2[$ *and* $]1/2, 1]$, *whence for any* $\alpha \in]0, 1]$ *the functions* δ *and* ω *are the diagonal and opposite diagonal sections of an orbital semilinear copula.*

(ii) *Let* $\omega(x) = \alpha x(1-x)$, *with* $\alpha \in]0, 1]$, *and consider the diagonal function*

$$\delta(x) = \left(1 - \frac{\alpha}{2}\right) \max(2x - 1, 0) + \frac{\alpha}{2} x.$$

The function ϕ_δ *is increasing and* α*-Lipschitz continuous on* $[0, 1/2[$ *and* $]1/2, 1]$, *whence for any* $\alpha \in]0, 1]$ *the functions* δ *and* ω *are the diagonal and opposite diagonal sections of an orbital semilinear copula.*

To conclude this section, we lay bare the necessary and sufficient conditions on a diagonal function δ guaranteeing that the corresponding Bertino copula is an orbital semilinear copula and we investigate also the symmetry and opposite symmetry properties of orbital semilinear copulas.

Proposition 6.14. *Let δ be a diagonal function, then the corresponding Bertino copula B_δ defined in (1.10) is an orbital semilinear copula if and only if for any $x \in [0, 1/2]$ and any $t \in [x, 1-x]$, it holds that*

$$t - \delta(t) \geq \min(x - \delta(x), 1 - x - \delta(1-x))\,. \tag{6.13}$$

Proof. Let B_δ be the Bertino copula defined (1.10) and suppose that B_δ is an orbital semilinear copula. The opposite diagonal section ω_B of this Bertino copula is given by

$$\omega_B(x) = \min(x, 1-x) - \min\{t - \delta(t) \mid t \in [\min(x, 1-x), \max(x, 1-x)]\}\,.$$

Let C_{δ,ω_B} be the orbital semilinear copula with diagonal section δ and opposite diagonal section ω_B, then it obviously coincides with B_δ. In the triangular sector described by $x \leq y \leq 1-x$, it holds that

$$\frac{x+y-1}{2x-1}\,\delta(x) + \frac{x-y}{2x-1}\,\omega_B(x) = x - \min\{t - \delta(t) \mid t \in [x, y]\}\,.$$

Now, consider the function f defined by $f(t) = t - \delta(t)$, then

$$(x+y-1)\delta(x) + (x-y)(x - \min_{t\in[x,1-x]} f(t)) = (2x-1)(x - \min_{t\in[x,y]} f(t))\,,$$

or equivalently,

$$(1-2x)\min_{t\in[x,y]} f(t) = (1-x-y)\min_{t\in[x,x]} f(t) + (y-x)\min_{t\in[x,1-x]} f(t)\,.$$

Let $t^* \in [x, 1-x]$ be such that $\min_{t\in[x,1-x]} f(t) = f(t^*)$ (note that t^* is not necessarily unique). The above equality, with $y = t^*$, reduces to

$$(1-2x)f(t^*) = (1-x-t^*)f(x) + (t^*-x)f(t^*)\,,$$

which implies that either $t^* = 1-x$ or $f(t^*) = f(x)$. This means that on the interval $[x, 1-x]$ the minimal value of f is attained in at least one of the points $t = x$ or $t = 1-x$, whence condition (6.13) follows. The three other triangular sectors lead to the same condition. The above reasoning can obviously be traversed in the converse direction. □

Corollary 6.1. *Let δ be a diagonal function. If*

(i) *δ is 1-Lipschitz continuous on $[0, 1/2]$,*

(ii) *for every $x \in [0, 1/2]$, it holds that $\delta(1-x) = \delta(x) + 1 - 2x$,*

then the corresponding Bertino copula B_δ is an orbital semilinear copula.

Proof. Condition (ii) expresses that the function $f(t) = t - \delta(t)$ is symmetric w.r.t. $1/2$ and it therefore suffices to show that

$$t - \delta(t) \geq \min(x - \delta(x), 1 - x - \delta(1 - x)) = x - \delta(x)\,,$$

for any $x \in [0, 1/2]$ and any $t \in [x, 1/2]$. As condition (i) simply states that the function $f(t) = t - \delta(t)$ is increasing on $[0, 1/2]$, the latter is trivially fulfilled. □

Proposition 6.15. *Let ω be an opposite diagonal function, then the corresponding copula F_ω defined in (1.13) is an orbital semilinear copula if and only if for any $x \in [0, 1/2]$ and any $t \in [x, 1 - x]$, it holds that*

$$\omega(t) \geq \min(\omega(x), \omega(1 - x))\,. \tag{6.14}$$

Corollary 6.2. *Let ω be an opposite diagonal function. If*

(i) *ω is increasing on $[0, 1/2]$,*

(ii) *for every $x \in [0, 1/2]$ it holds that $\omega(1 - x) = \omega(x)$,*

then the corresponding copula F_ω is an orbital semilinear copula.

Conditions (6.13) and (6.14) respectively express that the function $f(t) = t - \delta(t)$ and the function ω satisfy a restricted form of convexity by considering only intervals symmetric w.r.t. the point $1/2$. Note that δ_W satisfies the conditions of Corollary 6.1, while ω_M satisfies the conditions of Corollary 6.2. This confirms that $B_{\delta_W} = W$ and $F_{\omega_M} = M$ are orbital semilinear copulas.

Proposition 6.16. *Let δ and ω be diagonal and opposite diagonal functions such that $\delta(1/2) = \omega(1/2)$.*

(i) *Let $\delta(x) = \alpha x^2 + (1 - \alpha)x$ for arbitrary $\alpha \in\,]0, 1]$ (see Proposition 6.13), then the smallest orbital semilinear copula with diagonal section δ is the Bertino copula B_δ defined in (1.10).*

(ii) *Let $\omega(x) = \alpha x(1 - x)$ for arbitrary $\alpha \in\,]0, 1]$ (see Proposition 6.13), then the greatest orbital semilinear copula with opposite diagonal section ω is the copula F_ω defined in (3).*

Proof. One easily verifies that the given diagonal function satisfies the sufficient conditions of Corollary 6.1. Hence, the Bertino copula is an orbital semilinear copula. As it is the smallest copula with diagonal section δ, it is obviously also the smallest orbital semilinear copula with this diagonal section. The same reasoning applies to F_ω. □

The following proposition is a matter of direct verification.

Proposition 6.17. *An orbital (resp. radial) semilinear copula $C^o_{\delta,\omega}$ (resp. $C^r_{\delta,\omega}$) is*

(i) *opposite symmetric if and only if the function $f(x) = x - \delta(x)$ is symmetric w.r.t. $x = 1/2$, i.e. $\delta(1-x) = \delta(x) + 1 - 2x$ for any $x \in [0, 1/2]$;*

(ii) *symmetric if and only if ω is symmetric w.r.t. $x = 1/2$, i.e. $\omega(x) = \omega(1-x)$ for any $x \in [0, 1/2]$.*

Under the conditions of Corollary 6.1, the Bertino copula B_δ is an orbital semilinear copula that is both symmetric and opposite symmetric. Similarly, under the conditions of Corollary 6.2, the copula F_ω defined in (1.13) is also an orbital semilinear copula that is both symmetric and opposite symmetric.

PART II

METHODS BASED ON QUADRATIC INTERPOLATION

7 Lower semiquadratic copulas with a given diagonal section

7.1. Introduction

The aim of the present chapter is to propose a method to construct semiquadratic copulas. In Chapter 6, we have studied families of semilinear copulas with a given diagonal section and/or opposite diagonal section. A copula constructed by linear interpolation on segments connecting the diagonal of the unit square to the left and lower side of the unit square is called a lower semilinear copula [38].

In the present chapter, we construct semiquadratic copulas by *quadratic interpolation* on segments connecting the diagonal of the unit square to the left and lower side of the unit square, and call them lower semiquadratic copulas. We unveil the conditions on a diagonal function δ that guarantee the existence of a lower semiquadratic copula with diagonal section δ. Unlike lower semilinear copulas, lower semiquadratic copulas can be not symmetric. Next, we characterize the smallest and the greatest symmetric lower semiquadratic copulas with a given diagonal section. We also characterize the class of continuous differentiable (resp. absolutely continuous) lower semiquadratic copulas. Finally, we provide expressions for the degree of non-exchangeability and the measures of association for various families of lower semiquadratic copulas.

7.2. Lower semiquadratic copulas

For any two $]0,1] \to \mathbb{R}$ functions u and v that are absolutely continuous and satisfy

$$\lim_{\substack{x\to 0\\ 0\leq y\leq x}} y(x-y)u(x) = 0 \quad \text{and} \quad \lim_{\substack{y\to 0\\ 0\leq x\leq y}} x(y-x)v(y) = 0\,, \tag{7.1}$$

and any diagonal function δ, the function $C_\delta^{u,v} : [0,1]^2 \to \mathbb{R}$ defined by:

$$C_\delta^{u,v}(x,y) = \begin{cases} \dfrac{x}{y}\delta(y) - x(y-x)v(y) & \text{, if } 0 < x \leq y\,, \\[2ex] \dfrac{y}{x}\delta(x) - y(x-y)u(x) & \text{, if } 0 < y \leq x\,, \end{cases} \tag{7.2}$$

with $C_\delta^{u,v}(t,0) = C_\delta^{u,v}(0,t) = 0$ for any $t \in [0,1]$, is well defined. Note that the limit conditions on u and v ensure that $C_\delta^{u,v}$ is continuous. The function $C_\delta^{u,v}$ will be called *a lower semiquadratic function* since it satisfies $C_\delta^{u,v}(t,t) = \delta(t)$ for any $t \in [0,1]$, and since it is quadratic in x on $0 \le x \le y \le 1$ and quadratic in y on $0 \le y \le x \le 1$. Obviously, symmetric functions are obtained when $u = v$. Note that for $u = v = 0$, the definition of a lower semilinear function is retrieved.

In the following lemma, we provide a sufficient condition for limit conditions (7.1).

Lemma 7.1. *Let f be a $]0,1] \to \mathbb{R}$ function. If $\lim_{t\to 0} t^2|f(t)| = 0$, then*

$$\lim_{\substack{x\to 0\\ 0\le y\le x}} y(x-y)f(x) = 0 \quad \text{and} \quad \lim_{\substack{y\to 0\\ 0\le x\le y}} x(y-x)f(y) = 0\,.$$

We now investigate the conditions to be fulfilled by the functions u, v and δ such that the lower semiquadratic function $C_\delta^{u,v}$ is a copula. Note that u and v, being absolutely continuous, are differentiable almost everywhere.

Let us consider the subtriangles I_1 and I_2 of the unit square as in Chapter 3.

Proposition 7.1. *Let δ be a diagonal function and let u and v be two absolutely continuous functions that satisfy conditions (7.1). Then the lower semiquadratic function $C_\delta^{u,v}$ defined in (7.2) is a copula with diagonal section δ if and only if*

$$\text{(i)}\ \ u(1) = v(1) = 0\,, \tag{7.3}$$

$$\text{(ii)}\ \ \max\left(u(t) + t\,|u'(t)|\,, v(t) + t\,|v'(t)|\right) \le \left(\frac{\delta(t)}{t}\right)'\,, \tag{7.4}$$

$$\text{(iii)}\ \ u(t) + v(t) \ge t\left(\frac{\delta(t)}{t^2}\right)'\,, \tag{7.5}$$

for any $t \in]0,1]$ where the derivatives exist.

Proof. The boundary conditions $C_\delta^{u,v}(t,1) = t$ and $C_\delta^{u,v}(1,t) = t$ for any $t \in [0,1]$ immediately lead to the conditions $u(1) = v(1) = 0$. Therefore, it suffices to prove that the 2-increasingness of $C_\delta^{u,v}$ is equivalent to conditions (ii) and (iii).

Suppose that $C_\delta^{u,v}$ is 2-increasing. For any rectangle $R = [x,x'] \times [y,y'] \subseteq I_2$, it then holds that $V_{C_\delta^{u,v}}(R) \ge 0$, i.e.

$$(x'-x)\left(\frac{\delta(y')}{y'} - \frac{\delta(y)}{y} - v(y')y' + v(y)y + (x+x')(v(y')-v(y))\right) \ge 0\,,$$

or, equivalently,

$$\frac{\delta(y')}{y'} - \frac{\delta(y)}{y} - v(y')y' + v(y)y + (x+x')(v(y')-v(y)) \ge 0\,. \tag{7.6}$$

Dividing by $y'-y$ and taking the limits $x' \to x$ and $y' \to y$, inequality (7.6)

becomes
$$\left(\frac{\delta(y)}{y}\right)' - v(y) + (2x-y)v'(y) \geq 0\,. \tag{7.7}$$

Since the left-hand side of inequality (7.7) is linear in x, this condition is equivalent to requiring that it holds for $x = 0$ and $x = y$, i.e.

$$\left(\frac{\delta(y)}{y}\right)' - v(y) + yv'(y) \geq 0 \quad \text{and} \quad \left(\frac{\delta(y)}{y}\right)' - v(y) - yv'(y) \geq 0\,,$$

or, equivalently, to
$$v(y) + y\,|v'(y)| \leq \left(\frac{\delta(y)}{y}\right)'\,. \tag{7.8}$$

Similarly, the fact that $V_{C_\delta^{u,v}}(R) \geq 0$ for any rectangle located in I_1 implies that inequality (7.8) also holds for the function u. Hence, condition (ii) follows.

Finally, the fact that $V_{C_\delta^{u,v}}(R) \geq 0$ for any square $R = [x,x'] \times [x,x']$ centered around the main diagonal is equivalent to

$$\begin{aligned} V_{C_\delta^{u,v}}([x,x'] \times [x,x']) &= C_\delta^{u,v}(x,x) + C_\delta^{u,v}(x',x') - C_\delta^{u,v}(x,x') - C_\delta^{u,v}(x',x) \\ &= \delta(x) + \delta(x') - 2\frac{x}{x'}\delta(x') + x(x'-x)(u(x') + v(x')) \geq 0\,. \end{aligned}$$

Dividing by $x(x'-x)$ and taking the limit $x' \to x$, condition (iii) immediately follows.

Now suppose that conditions (ii) and (iii) are satisfied. Due to the additivity of volumes, it suffices to consider a restricted number of cases to prove the 2-increasingness of $C_\delta^{u,v}$. Let $R = [a,b] \times [a',b']$ be a rectangle located in I_2. Since condition (ii) is satisfied, inequality (7.7) follows and it holds that

$$\int_{a'}^{b'} \mathrm{d}y \int_a^b \left(\left(\frac{\delta(y)}{y}\right)' - v(y) + (2x-y)v'(y)\right) \mathrm{d}x \geq 0\,.$$

Computing the above integral, the latter inequality becomes

$$(b-a)\left(\frac{\delta(b')}{b'} - \frac{\delta(a')}{a'} - b'v(b') + a'v(a') + (a+b)v(b') - (a+b)v(a')\right) \geq 0\,,$$

or, equivalently, $V_{C_\delta^{u,v}}(R) \geq 0$. Similarly, one can verify that $V_{C_\delta^{u,v}}(R) \geq 0$ for any rectangle $R = [a,b] \times [a',b']$ located in I_1.

Finally, let $S = [a,b] \times [a,b]$ be a square centered around the main diagonal. Due to condition (iii), it holds that

$$x\left(u(x)+v(x)-x\left(\frac{\delta(x)}{x^2}\right)'\right)\geq 0\,,$$

for any $x\in\,]0,1]$, which implies that

$$\tilde{I}_1=\int_a^b x\left(u(x)+v(x)-x\left(\frac{\delta(x)}{x^2}\right)'\right)\,\mathrm{d}x\geq 0\,.$$

Again using inequality (7.7), it follows that

$$\tilde{I}_2=\int_a^b\int_x^b\left(\left(\frac{\delta(y)}{y}\right)'-v(y)+(2x-y)v'(y)\right)\,\mathrm{d}y\mathrm{d}x\geq 0\,.$$

As inequality (7.7) also holds for the function u, it follows after exchanging the variables x and y that

$$\tilde{I}_3=\int_a^b\int_y^b\left(\left(\frac{\delta(x)}{x}\right)'-u(x)+(2y-x)u'(x)\right)\,\mathrm{d}x\mathrm{d}y\geq 0\,.$$

Computing the above integrals and setting $I=\tilde{I}_1+\tilde{I}_2+\tilde{I}_3$, it follows that

$$I=\delta(a)+\delta(b)-2\frac{a}{b}\delta(b)+a(b-a)(u(b)+v(b))\geq 0\,,$$

or, equivalently,

$$I=V_{C_\delta^{u,v}}(S)\geq 0\,.$$

□

Note that if $u=v=0$, then we retrieve the necessary and sufficient conditions from Proposition 6.1 on δ which guarantee that the copula is a lower semilinear copula (see Chapter 6).

In the next proposition we show an interesting property of lower semiquadratic copulas.

Proposition 7.2. *Let $C_\delta^{u,v}$ be a lower semiquadratic copula. Then it holds that*

$$t^2\max(|u(t)|,|v(t)|)\leq\min(\delta(t),t-\delta(t))\,,\tag{7.9}$$

for any $t\in\,]0,1]$.

Proof. Since $C_\delta^{u,v}$ is a copula, it holds that $C_\delta^{u,v}$ is increasing and 1-Lipschitz continuous in each variable. Let $(x,y),(x',y)\in I_2$ such that $x<x'$. Expressing

that $C_\delta^{u,v}$ is a 1-Lipschitz continuous increasing function leads to the condition

$$0 \leq (x'-x)\left(\frac{\delta(y)}{y} + (x'+x-y)v(y)\right) \leq x'-x\,,$$

or, equivalently,

$$0 \leq \delta(y) + y(x+x'-y)v(y) \leq y\,.$$

Taking the limit $x' \to x$, the latter double inequality becomes

$$0 \leq \delta(y) + y(2x-y)v(y) \leq y\,.$$

For fixed $y \in]0,1]$, this double inequality should hold for any $x \in [0,y]$ and since the expression is linear in x one can equivalently state that the double inequality should hold for $x=0$ and $x=y$, leading to

$$0 \leq \delta(y) - y^2v(y) \leq y \quad \text{and} \quad 0 \leq \delta(y) + y^2v(y) \leq y\,,$$

for any $y \in]0,1]$. Since $0 \leq \delta(y) \leq y$, it must hold for any $y \in]0,1]$ that

$$y^2|v(y)| \leq \min(\delta(y), y-\delta(y))\,.$$

Similarly, expressing that $C_\delta^{u,v}$ is a 1-Lipschitz continuous increasing function in the first variable on I_1 implies that

$$y^2|u(y)| \leq \min(\delta(y), y-\delta(y))\,.$$

Combining the above, condition (7.9) follows. □

Example 7.1. *Let δ_M be the diagonal section of the greatest copula* M, *i.e.* $\delta_\mathrm{M}(t) = t$ *for any* $t \in [0,1]$. *Condition (7.9) on u can be written as*

$$t^2|u(t)| \leq 0 \qquad \text{for any } t \in]0,1]\,,$$

from which it follows that $u = 0$ on $]0,1]$. Similarly, $v = 0$ on $]0,1]$. Since $\lim_{t\to 0} t^2\,|u(t)| = \lim_{t\to 0} t^2\,|v(t)| = 0$, *Lemma 7.1 implies that conditions (7.1) hold. Therefore, the only lower semiquadratic copula with diagonal section δ_M is the (lower semilinear) copula* M *itself. A proper lower semiquadratic copula with diagonal section δ_M does not exist.*

Example 7.2. *Let δ_Π be the diagonal section of the product copula* Π, *i.e.* $\delta_\Pi(t) = t^2$ *for any* $t \in [0,1]$. *Condition (7.9) on u can be written as*

$$|u(t)| \leq 1 \qquad \text{for any } t \in]0,1/2] \qquad \text{and} \qquad |u(t)| \leq \frac{1-t}{t} \qquad \text{for any } t \in [1/2,1]\,.$$

Consider the greatest function u that satisfies these conditions, i.e. $u_g(t) = 1$ if $t \in]0,1/2]$ and $u_g(t) = (1-t)/t$ if $t \in [1/2,1]$. One can verify that u_g does not *satisfy differential condition (7.4) on* $[1/2,1]$, *since* $(1-t)/t+t|(-1/t^2)| = 2/t-1 > 1$ *on* $]1/2,1[$. *The greatest function* u *satisfying conditions (7.3) and (7.4) that is decreasing on* $]0,1]$ *is the solution of the differential equation*

$$u(t) - tu'(t) = 1\,,$$

with boundary condition $u(1) = 0$, *and is given by* $u(t) = 1-t$. *The same observation holds for* v. *Since* $\lim_{t\to 0} t^2(1-t) = 0$, *Lemma 7.1 implies that conditions (7.1) hold. Hence, for* $C^{u,v}_{\delta_\Pi}$ *to be a lower semiquadratic copula the following conditions on* u *and* v *must be satisfied for any* $t \in]0,1]$:

$$\max\left(-1, 1-\frac{1}{t}\right) \leq u(t) \leq 1-t\,, \quad \max\left(-1, 1-\frac{1}{t}\right) \leq v(t) \leq 1-t\,,$$

and

$$u(t) + v(t) \geq 0\,.$$

We conclude that the functions $u(t) = v(t) = 1-t$ *satisfy conditions (7.3)–(7.5) for any* $t \in]0,1]$. *Note that for any two lower semiquadratic functions* C^{u_1,v_1}_δ *and* C^{u_2,v_2}_δ, *it holds that* $C^{u_1,v_1}_\delta \leq C^{u_2,v_2}_\delta$ *if and only if* $u_1 \geq u_2$ *and* $v_1 \geq v_2$. *Thus,* $C^{u,v}_{\delta_\Pi}$, *with* $u(t) = v(t) = 1-t$ *for any* $t \in]0,1]$, *is the smallest lower semiquadratic copula with diagonal section* δ_Π. *Moreover, it is a symmetric copula. Finally, we distinguish two special non-symmetric lower semiquadratic copulas with diagonal section* δ_Π. *They are obtained with* $u(t) = -v(t) = 1-t$ *and* $-u(t) = v(t) = 1-t$, *respectively. One easily verifies that the degree of non-exchangeability* $\mu_{+\infty}$ *(see Chapter 1) for both copulas equals* $2/9$.

For the class of lower semilinear copulas with a given diagonal section, it was shown that the diagonal section of the product copula is the smallest diagonal function that can be considered [37]. In the next two examples, we give examples of lower semiquadratic copulas whose diagonal section is smaller than the diagonal section of the product copula.

Example 7.3. *Let* δ_λ *be a convex sum of the diagonal section of the product copula* Π *and the diagonal section of the smallest copula* W, *i.e.* $\delta_\lambda(t) = \lambda t^2 + (1-\lambda)\max(2t-1,0)$ *for any* $t \in [0,1]$, *with* $\lambda \in [0,1]$. *Let* u_λ *and* v_λ *be defined by*

$$u_\lambda(t) = v_\lambda(t) = \begin{cases} \lambda + \dfrac{1}{3}(7-10\lambda)t & , \textit{if } 0 < t \leq \frac{1}{2}\,, \\[2ex] -(1+2\lambda)\dfrac{t}{3} + \lambda + \dfrac{1-\lambda}{3}\dfrac{1}{t^2} & , \textit{if } \frac{1}{2} \leq t \leq 1\,. \end{cases}$$

Since $\lim_{t\to 0} t^2\,|u_\lambda(t)| = \lim_{t\to 0} t^2\,|v_\lambda(t)| = 0$, *Lemma 7.1 implies that conditions (7.1) hold. One can verify that condition (7.9) holds if and only if* $\lambda \in [0.7,1]$. *The conditions of Proposition 7.1 are satisfied for any* $\lambda \in [0.7,1]$ *and hence,* $C^{u_\lambda v_\lambda}_{\delta_\lambda}$ *is a lower semiquadratic copula for any* $\lambda \in [0.7,1]$.

Example 7.4. *Let δ_λ be the diagonal section of a Farlie–Gumbel–Morgenstern copula, i.e. $\delta_\lambda(t) = t^2(1+\lambda(1-t)^2)$ for any $t \in [0,1]$, with $\lambda \in [-1,1]$. The Farlie–Gumbel–Morgenstern family of copulas contains all copulas that are quadratic in both variables* [88, 96]. *Observe that $\delta_\lambda \leq \delta_\Pi$ if and only if $\lambda \in [-1,0]$. Let u_λ and v_λ be defined by*

$$u_\lambda(t) = v_\lambda(t) = 2|\lambda| t(1-t) \quad \text{for any } t \in]0,1] .$$

Since $\lim_{t\to 0} t^2 |u_\lambda(t)| = \lim_{t\to 0} t^2 |v_\lambda(t)| = 0$, Lemma 7.1 implies that conditions (7.1) hold. One can verify that condition (7.9) holds if and only if $\lambda \in [-1/2, 1/2]$. The conditions of Proposition 7.1 are satisfied for any $\lambda \in [-1/2,1/2]$ and hence, $C_{\delta_\lambda}^{u_\lambda v_\lambda}$ is a lower semiquadratic copula for any $\lambda \in [-1/2,1/2]$. The corresponding family of lower semiquadratic copulas is given by

$$C_{\delta_\lambda}^{u_\lambda, v_\lambda}(x,y) = xy(1 + (1-\max(x,y))(\lambda(1-\max(x,y)) - |2\lambda(y-x)|)) ,$$

with $\lambda \in [-1/2, 1/2]$.

7.3. Extreme lower semiquadratic copulas

We now turn to the problem of identifying for a given diagonal function δ, if possible, the smallest and the greatest functions u and v such that $C_\delta^{u,v}$ defined in (7.2) is a lower semiquadratic copula.

Proposition 7.3. *Let δ be a differentiable diagonal function and let ϕ_δ and ψ_δ be the $]0,1] \to \mathbb{R}$ functions defined by*

$$\phi_\delta(t) = t \int_t^1 \frac{1}{z^2} \left(\frac{\delta(z)}{z}\right)' \mathrm{d}z \quad \text{and} \quad \psi_\delta(t) = \frac{\delta(t)-t}{t^2} . \tag{7.10}$$

Then

(i) *ϕ_δ is an upper bound for the functions u and v (i.e. $C_\delta^{\phi_\delta,\phi_\delta}$ is a lower bound for the lower semiquadratic copulas with diagonal section δ). Moreover, $C_\delta^{\phi_\delta,\phi_\delta}$ is a copula (i.e. it is the smallest lower semiquadratic copula with diagonal section δ) if and only if*

$$\frac{1}{2} t \left(\frac{\delta(t)}{t^2}\right)' \leq \phi_\delta(t) \leq \left(\frac{\delta(t)}{t}\right)' \quad \text{for any } t \in]0,1] . \tag{7.11}$$

(ii) *ψ_δ is a lower bound for the functions u and v (i.e. $C_\delta^{\psi_\delta,\psi_\delta}$ is an upper bound for the lower semiquadratic copulas with diagonal section δ). Moreover, $C_\delta^{\psi_\delta,\psi_\delta}$ is a copula (i.e. it is the greatest lower semiquadratic copula with* *diagonal section δ) if and only if*

$$2\delta(t) - t \leq t\delta'(t) \leq 4\delta(t) - 2t \quad \textit{for any } t \in]0,1]\,. \tag{7.12}$$

Proof. (i): Let f be an absolutely continuous function such that $f(1) = 0$ and $f(t) + t|f'(t)| \leq \left(\frac{\delta(t)}{t}\right)'$ for any $t \in]0,1]$. Since $f(t) - tf'(t) \leq f(t) + t|f'(t)|$ for any $t \in]0,1]$ where the derivative exists, we have

$$f(t) - tf'(t) \leq \left(\frac{\delta(t)}{t}\right)', \tag{7.13}$$

for any $t \in]0,1]$ where the derivatives exist. Since the solution of the initial value problem over $]0,1]$

$$y(t) - ty'(t) = \left(\frac{\delta(t)}{t}\right)', \quad y(1) = 0 \tag{7.14}$$

is given by $y = \phi_\delta$, condition (7.13) yields

$$f(t) - tf'(t) \leq \phi_\delta(t) - t\phi_\delta'(t)\,,$$

for any $t \in]0,1]$ where the derivatives exist. Let ν be the $]0,1] \to \mathbb{R}$ function defined by $\nu(t) = \frac{f(t)-\phi_\delta(t)}{t}$. The function ν is increasing since

$$\nu'(t) = \frac{\phi_\delta(t) - f(t) - t(\phi_\delta'(t) - f'(t))}{t^2} \geq 0\,,$$

for any $t \in]0,1]$ where the derivatives exist. Thus, $\nu(t) \leq \nu(1) = 0$ for any $t \in]0,1]$, whence $f(t) \leq \phi_\delta(t)$ for any $t \in]0,1]$. In particular, we have that ϕ_δ is an upper bound for u and v. Hence, $C_\delta^{u,v} \geq C_\delta^{\phi_\delta,\phi_\delta}$.

Next, we characterize the diagonal functions δ for which the function $C_\delta^{\phi_\delta,\phi_\delta}$ is a copula. Note that $\lim\limits_{t\to 0} t^2(\delta(t)/t)' = 0$, and hence $\lim\limits_{t\to 0} t^2|\phi_\delta(t)| = 0$. Therefore, due to Lemma 7.1, condition (7.1) holds for $u = v = \phi_\delta$. Thus, $C_\delta^{\phi_\delta,\phi_\delta}$ is a copula if and only if the conditions of Proposition 7.1 hold for the functions $u = v = \phi_\delta$. We know that condition (7.3) is satisfied. With regard to condition (7.4), observe that

$$\phi_\delta(t) + t\,|\phi_\delta'(t)| \leq \left(\frac{\delta(t)}{t}\right)' (= \phi_\delta(t) - t\phi_\delta'(t)) \quad \text{for any } t \in]0,1]$$

if and only if $\phi_\delta'(t) \leq 0$ for any $t \in]0,1]$. Since $t\phi_\delta'(t) = \phi_\delta(t) - \left(\frac{\delta(t)}{t}\right)'$, we have that $\phi_\delta'(t) \leq 0$ if and only if $\phi_\delta(t) \leq \left(\frac{\delta(t)}{t}\right)'$. Thus, we conclude that $u = v = \phi$ satisfy condition (7.4) if and only if the second inequality in (7.11) holds.

Moreover, it is immediate that $u = v = \phi_\delta$ satisfy condition (7.5) if and only if $2\phi_\delta(t) \geq t\left(\frac{\delta(t)}{t^2}\right)'$ for any $t \in]0,1]$, which is equivalent to the first inequality in (7.11).

(ii): Let f be an absolutely continuous function such that $f(1) = 0$ and $f(t) + t|f'(t)| \leq \left(\frac{\delta(t)}{t}\right)'$ for any $t \in]0,1]$. Since $f(t) + tf'(t) \leq f(t) + t|f'(t)|$ for any $t \in]0,1]$ where the derivative exists, we have

$$f(t) + tf'(t) \leq \left(\frac{\delta(t)}{t}\right)', \tag{7.15}$$

for any $t \in]0,1]$ where the derivatives exist. Since the solution of the initial value problem over $]0,1]$

$$y(t) + ty'(t) = \left(\frac{\delta(t)}{t}\right)', \quad y(1) = 0\,,$$

is given by $y = \psi_\delta$, condition (7.15) yields

$$f(t) + tf'(t) \leq \psi_\delta(t) + t\psi_\delta'(t)\,,$$

for any $t \in]0,1]$ where the derivatives exist. Let μ be the $]0,1] \to \mathbb{R}$ function defined by $\mu(t) = t(f(t) - \psi_\delta(t))$. The function μ is decreasing since

$$\mu'(t) = f(t) - \psi_\delta(t) + t(f'(t) - \psi_\delta'(t)) \leq 0\,,$$

for any $t \in]0,1]$ where the derivatives exist. Thus, $\mu(t) \geq \mu(1) = 0$ for any $t \in]0,1]$, whence $f(t) \geq \psi_\delta(t)$ for any $t \in]0,1]$. In particular, we have that ψ_δ is a lower bound for u and v. Hence, $C_\delta^{u,v} \leq C_\delta^{\psi_\delta,\psi_\delta}$.

Next, we characterize the diagonal functions δ for which the function $C_\delta^{\psi_\delta,\psi_\delta}$ is a copula. Since $\lim\limits_{t\to 0} t^2|\psi_\delta(t)| = \lim\limits_{t\to 0} |\delta(t) - t| = 0$, Lemma 7.1 implies that conditions (7.1) hold for $u = v = \psi_\delta$. Thus, $C_\delta^{\psi_\delta,\psi_\delta}$ is a copula if and if the conditions of Proposition 7.1 hold for the functions $u = v = \psi_\delta$. We know that condition (7.3) is satisfied. With regard to condition (7.4), observe that

$$\psi_\delta(t) + t|\psi_\delta'(t)| \leq \left(\frac{\delta(t)}{t}\right)' (= \psi_\delta(t) + t\psi_\delta'(t)) \quad \text{for any } t \in]0,1]$$

if and only if $\psi_\delta'(t) \geq 0$ for any $t \in]0,1]$. Since $t\psi_\delta'(t) = \left(\frac{\delta(t)}{t}\right)' - \psi_\delta(t)$, we have that $\psi_\delta'(t) \geq 0$ if and only if $\psi_\delta(t) \leq \left(\frac{\delta(t)}{t}\right)'$. Thus, we conclude that $u = v = \psi$ satisfy condition (7.4) if and only if

$$\frac{\delta(t) - t}{t^2} \leq \frac{t\delta'(t) - \delta(t)}{t^2} \quad \text{for any } t \in]0,1]\,,$$

which is equivalent to the first inequality in (7.12).

Moreover, it is immediate that $u = v = \psi_\delta$ satisfy condition (7.5) if and only if $2\psi_\delta(t) \geq t\left(\frac{\delta(t)}{t^2}\right)'$ for any $t \in]0,1]$, which is equivalent to the second inequality in (7.12). $\square$

Remark 7.1.

(i) *Condition (7.12) can also be written as*

$$\frac{1}{2}t\left(\frac{\delta(t)}{t^2}\right)' \leq \psi_\delta(t) \leq \left(\frac{\delta(t)}{t}\right)' \qquad \textit{for any } t \in]0,1]\,, \tag{7.16}$$

which is the same as condition (7.11), now expressed for the function ψ_δ.

(ii) *Note that $\psi_\delta(t) \leq 0$ for any $t \in]0,1]$.*

(iii) *Moreover, if ϕ_δ satisfies the right inequality in (7.11), i.e. $\phi'_\delta(t) \leq 0$ for any $t \in]0,1]$, it follows from $\phi_\delta(1) = 0$ that $\phi_\delta(t) \geq 0$ for any $t \in]0,1]$.*

Corollary 7.1. *Let δ be a differentiable diagonal function. Then the functions $C_\delta^{\phi_\delta,\phi_\delta}$ and $C_\delta^{\psi_\delta,\psi_\delta}$ are both copulas if and only if*

$$\frac{1}{2}t\left(\frac{\delta(t)}{t^2}\right)' \leq \psi_\delta(t) \leq \phi_\delta(t) \leq \left(\frac{\delta(t)}{t}\right)' \qquad \textit{for any } t \in]0,1]\,. \tag{7.17}$$

Moreover, if $C_\delta^{\phi_\delta,\phi_\delta}$ and $C_\delta^{\psi_\delta,\psi_\delta}$ are copulas, then the function λ_δ defined as in Chapter 3 is increasing, and the function ρ_δ defined as in Chapter 6 is decreasing.

Corollary 7.1 implies that if $C_\delta^{\phi_\delta,\phi_\delta}$ and $C_\delta^{\psi_\delta,\psi_\delta}$ are copulas, then the same diagonal function can be used to construct a lower semilinear copula.

Example 7.5. *Let δ_λ be a convex sum of the diagonal section of the product copula Π and the diagonal section of the greatest copula M, i.e. $\delta_\lambda(t) = \lambda t^2 + (1-\lambda)t$ for any $t \in [0,1]$, with $\lambda \in [0,1]$. Observe that*

$$\phi_{\delta_\lambda}(t) = \lambda(1-t) \qquad \textit{and} \qquad \psi_{\delta_\lambda}(t) = \frac{\lambda(t-1)}{t} \qquad \textit{for any } t \in]0,1]\,.$$

It is immediate that ϕ_{δ_λ} satisfies condition (7.11). Thus, as Proposition 7.3 establishes, $C_{\delta_\lambda}^{\phi_{\delta_\lambda},\phi_{\delta_\lambda}}$ is a lower bound for the lower semiquadratic copulas with diagonal section δ_λ. Moreover, $C_{\delta_\lambda}^{\phi_{\delta_\lambda},\phi_{\delta_\lambda}}$ is a copula for any $\lambda \in [0,1]$. The function ψ_{δ_λ} satisfies the second inequality in (7.12) for any $\lambda \in [0,1]$. However, ψ_{δ_λ} satisfies the first inequality in (7.12) if and only if

$$\frac{\lambda-1}{2} \leq \lambda(t-1) \qquad \textit{for any } t \in]0,1]\,,$$

i.e. if $\lambda \leq 1/3$. Thus, as Proposition 7.3 establishes, $C_{\delta_\lambda}^{\psi_{\delta_\lambda},\psi_{\delta_\lambda}}$ is an upper bound *for the lower semiquadratic copulas with diagonal section* δ_λ *for any* $\lambda \in [0,1]$, *but* $C_{\delta_\lambda}^{\psi_{\delta_\lambda},\psi_{\delta_\lambda}}$ *is a copula only when* $\lambda \in [0,1/3]$.

The following proposition shows that the function ϕ_δ can be used to construct two non-symmetric lower semiquadratic copulas with diagonal section δ.

Proposition 7.4. *Let* δ *be a differentiable diagonal function. Then* $C_\delta^{u,v}$, *with* $u = -v = \phi_\delta$ *or with* $-u = v = \phi_\delta$, *is a lower semiquadratic copula with diagonal section* δ *if and only if*

$$\left(\frac{\delta(t)}{t^2}\right)' \leq 0 \leq \phi_\delta(t) \leq \left(\frac{\delta(t)}{t}\right)' \quad \textit{for any } t \in]0,1]\,. \tag{7.18}$$

Proof. Let $u = \phi_\delta$ and $v = -\phi_\delta$. Since $\lim\limits_{t\to 0} t^2\phi_\delta(t) = \lim\limits_{t\to 0} -t^2\phi_\delta(t) = 0$, Lemma 7.1 implies that conditions (7.1) hold. Clearly, u and v satisfy condition (7.3) in Proposition 7.1. Condition (7.4) in Proposition 7.1 now reads

$$\max\left(\phi_\delta(t) + t\,|\phi_\delta'(t)|\,, -\phi_\delta(t) + t\,|\phi_\delta'(t)|\right) \leq \left(\frac{\delta(t)}{t}\right)' \quad \text{for any } t \in]0,1]\,,$$

which is equivalent to

$$0 \leq \phi_\delta(t) \leq \left(\frac{\delta(t)}{t}\right)' \quad \text{for any } t \in]0,1]\,.$$

This follows from the fact that, as in the proof of Proposition 7.3, the condition

$$\phi_\delta(t) + t\,|\phi_\delta'(t)| \leq \left(\frac{\delta(t)}{t}\right)' \quad \text{for any } t \in]0,1]$$

is equivalent to the second inequality in condition (7.11), and that the latter implies that $\phi_\delta(t) \geq 0$ for any $t \in]0,1]$.

On the other hand, $u = \phi_\delta$ and $v = -\phi_\delta$ satisfy condition (7.5) in Proposition 7.1 if and only if

$$\phi_\delta(t) - \phi_\delta(t) \geq t\left(\frac{\delta(t)}{t^2}\right)' \quad \text{for any } t \in]0,1]\,.$$

i.e.

$$\left(\frac{\delta(t)}{t^2}\right)' \leq 0 \quad \text{for any } t \in]0,1]\,.$$

□

Example 7.6. *Let δ_λ be the diagonal section from Example 7.5. Obviously, δ_λ satisfies the first inequality in (7.18). Since ϕ_{δ_λ} satisfies the third inequality in (7.18) (see Example 7.5) and the second inequality in (7.18) (see Remark 7.1(iii)), it follows that $C^{u,v}_{\delta_\lambda}$, with $u = -v = \phi_{\delta_\lambda}$, is a lower semiquadratic copula with diagonal section δ_λ.*

Example 7.7. *Let $\delta_\alpha(t) = t^{1+\alpha}$ be the diagonal function with parameter $\alpha \in [0,1]$. For values of α outside the unit interval, δ_α is not a diagonal function. Observe that*

$$\left(\frac{\delta_\alpha(t)}{t}\right)' = \alpha t^{\alpha-1} \geq 0\,, \qquad \text{for any } t \in\,]0,1]\,,$$

$$\left(\frac{\delta_\alpha(t)}{t^2}\right)' = (\alpha-1)t^{\alpha-2} \leq 0\,, \qquad \text{for any } t \in\,]0,1]$$

and

$$\phi_{\delta_\alpha}(t) = \alpha t \int_t^1 \frac{z^{\alpha-1}}{z^2}\,\mathrm{d}z = \frac{\alpha}{2-\alpha}[t^{\alpha-1} - t] \quad \text{for any } t \in\,]0,1]\,. \tag{7.19}$$

Since $(\delta_\alpha(t)/t^2)' \leq 0$ and $\phi_{\delta_\alpha}(t) \geq 0$ for any $t \in\,]0,1]$, ϕ_{δ_α} satisfies condition (7.18) of Proposition 7.4 if and only if the third inequality of condition (7.18) holds, i.e. $\alpha - 1 \leq t^{2-\alpha}$ for any $t \in\,]0,1]$ and the latter is trivially satisfied when $\alpha \leq 1$. It follows that $C^{u,v}_{\delta_\alpha}$, with $u = -v = \phi_{\delta_\alpha}$, is a lower semiquadratic copula with diagonal section δ_α.

Example 7.8. *Let $\delta_{\alpha,\beta}$ be the $[0,1] \to [0,1]$ function defined by $\delta_{\alpha,\beta}(t) = t^2 + t^\alpha(1-t)^\beta$ with real parameters $\alpha \geq 1$ and $\beta \geq 1$. It can be verified that in general $\delta_{\alpha,\beta}$ is not a diagonal function. For instance, $\delta_{24,1}$ does not satisfy $\delta'_{24,1}(t) \leq 2$ for any $t \in\,]0,1]$. Note that the first inequality in (7.18) holds if and only if $\alpha \leq 2$. We will only consider the case $\alpha = 2$ from here on. We compute*

$$\phi_{\delta_{2,\beta}}(t) = t \int_t^1 \left(\frac{1}{z^2} + \frac{(1-z)^\beta}{z^2} - \beta\frac{(1-z)^{\beta-1}}{z}\right)\mathrm{d}z\,.$$

Integrating by parts, we obtain

$$\begin{aligned}
\phi_{\delta_{2,\beta}}(t) &= t\left(\frac{(1-t)^\beta}{t} + \int_t^1 \left(\frac{1}{z^2} - 2\beta\frac{(1-z)^{\beta-1}}{z}\right)\mathrm{d}z\right) \\
&= 1 - t + (1-t)^\beta - 2\beta t \int_t^1 \frac{(1-z)^{\beta-1}}{z}\,\mathrm{d}z \\
&= 1 - t + (1-t)^\beta - 2\beta t \int_0^{1-t} \frac{u^{\beta-1}}{1-u}\,\mathrm{d}u\,,
\end{aligned}$$

which, in terms of the incomplete beta function B*, can be written as*

$$\phi_{\delta_{2,\beta}}(t) = 1 - t + (1-t)^\beta - 2\beta t \mathrm{B}[1-t;\beta,0]\,.$$

For $\beta = 1$*, we find*

$$\phi_{\delta_{2,1}}(t) = 2 - 2t + 2t\ln t\,;$$

for $\beta = 2$*, we find*

$$\phi_{\delta_{2,2}} = 2 + t - 3t^2 + 4t\ln t\,;$$

and, for $\beta = 3$*, we find*

$$\phi_{\delta_{2,3}}(t) = 2 + 5t - 9t^2 + 2t^3 + 6t\ln t\,.$$

Now, in each case we should investigate whether the second and third inequalities in condition (7.18) are fulfilled for the functions $\phi_{\delta_{2,\beta}}$*, with* $\beta \in \{1,2,3\}$*. For* $\phi_{\delta_{2,1}}$*, for instance, this amounts to verifying whether for any* $t \in]0,1]$ *it holds that* $0 \leq 2 - 2t + 2t\ln(t) \leq 2 - 2t$*. These inequalities are trivially satisfied. For all cases under consideration, we find that condition (7.18) is satisfied, and Proposition 7.4 can be applied, i.e.* $C^{u,v}_{\delta_{2,\beta}}$*, with* $u = -v = \phi_{\delta_{2,\beta}}$*, is a lower semiquadratic copula with diagonal section* $\delta_{2,\beta}$ *for* $\beta \in \{1,2,3\}$*.*

In contrast to the fact that under condition (7.18) the lower semiquadratic function $C^{u,v}_\delta$, with $u = -v = \phi_\delta$ or with $-u = v = \phi_\delta$, is a lower semiquadratic copula with diagonal section δ, the lower semiquadratic function $C^{u,v}_\delta$, with $u = -v = \psi_\delta$ or with $-u = v = \psi_\delta$, is a lower semiquadratic copula with diagonal section δ if and only if $C^{u,v}_\delta = \mathrm{M}$.

Proposition 7.5. *Let* δ *be a differentiable diagonal function. Then* $C^{u,v}_\delta$*, with* $u = -v = \psi_\delta$ *or with* $-u = v = \psi_\delta$*, is a lower semiquadratic copula with diagonal section* δ *if and only if* $C^{u,v}_\delta = \mathrm{M}$*.*

Proof. As in the proof of Proposition 7.3, ψ_δ satisfies condition (7.4) if and only if the first inequality in condition (7.12) holds. Thus, ψ_δ must be increasing. The function $-\psi_\delta$ satisfies condition (7.4) if and only if

$$-\psi_\delta(t) + t\psi_\delta'(t) \leq \left(\frac{\delta(t)}{t}\right)' \qquad \text{for any } t \in]0,1]\,,$$

which is equivalent to $t - \delta(t) \leq 0$ for any $t \in]0,1]$. Since $\delta(t) \leq t$ for any $t \in [0,1]$, it must hold that $\delta(t) = t$ for any $t \in [0,1]$. Since M is the only copula with diagonal section δ_M, it must hold that $C^{u,v}_\delta = \mathrm{M}$. ☐

Similarly to Proposition 7.4, the functions ϕ_δ and ψ_δ can be used to construct two non-symmetric lower semiquadratic copulas with diagonal section δ.

Proposition 7.6. *Let δ be a differentiable diagonal function. Then $C_\delta^{u,v}$, with $u = \phi_\delta$ and $v = \psi_\delta$ or with $u = \psi_\delta$ and $v = \phi_\delta$, is a lower semiquadratic copula with diagonal section δ if and only if*

$$\frac{t + t\delta'(t) - 3\delta(t)}{t^2} \leq \phi_\delta(t) \leq \left(\frac{\delta(t)}{t}\right)' \qquad \textit{for any } t \in]0,1] \,. \tag{7.20}$$

Proof. Let $u = \phi_\delta$ and $v = \psi_\delta$. In Proposition 7.3, it is shown that u and v satisfy conditions (7.1). Clearly, u and v satisfy condition (7.3) in Proposition 7.1. Condition (7.4) in Proposition 7.1 is equivalent to the second inequality in condition (7.11) and the second inequality in condition (7.16) (or the first inequality in condition (7.12)), i.e.

$$\psi_\delta(t) \leq \phi_\delta(t) \leq \left(\frac{\delta(t)}{t}\right)' \qquad \text{for any } t \in]0,1] \,.$$

On the other hand, $u = \phi_\delta$ and $v = \psi_\delta$ satisfy condition (7.5) in Proposition 7.1 if and only if

$$\phi_\delta(t) + \psi_\delta(t) \geq t\left(\frac{\delta(t)}{t^2}\right)' \qquad \text{for any } t \in]0,1] \,,$$

i.e.

$$\phi_\delta(t) \geq t\left(\frac{\delta(t)}{t^2}\right)' - \psi_\delta(t) = \frac{t + t\delta'(t) - 3\delta(t)}{t^2} \qquad \text{for any } t \in]0,1] \,.$$

□

Example 7.9. *Let δ_λ be the diagonal section from Example 7.5. Clearly, the function ϕ_δ satisfies the second inequality in (7.20) for any $\lambda \in [0,1]$. Moreover, it satisfies the first inequality in (7.20) if and only if*

$$\frac{-1 + 2\lambda - \lambda t}{t} \leq \lambda(1-t) \qquad \textit{for any } t \in]0,1] \,,$$

i.e. if $\lambda \leq 1/2$. Thus, as Proposition 7.6 establishes, $C_{\delta_\lambda}^{\phi_\delta,\psi_\delta}$ is a copula for any $\lambda \in [0,1/2]$.

7.4. Continuous differentiable lower semiquadratic copulas

Condition (iii) in Proposition 7.1 expresses that the $C_\delta^{u,v}$-volume of squares centered around the main diagonal is positive. We now give an alternative interpretation of that condition.

Proposition 7.7. *Let δ be a differentiable diagonal function and $C_\delta^{u,v}$ be a lower semiquadratic copula. Then it holds that*

$$\lim_{x\to y-} \frac{\partial C_\delta^{u,v}(x,y)}{\partial x} \geq \lim_{x\to y+} \frac{\partial C_\delta^{u,v}(x,y)}{\partial x}$$

for any $y \in]0,1[$, and

$$\lim_{y\to x-} \frac{\partial C_\delta^{u,v}(x,y)}{\partial y} \geq \lim_{y\to x+} \frac{\partial C_\delta^{u,v}(x,y)}{\partial y}$$

for any $x \in]0,1[$.

Proof. From (7.2) we compute for $x < y$ that

$$\frac{\partial C_\delta^{u,v}(x,y)}{\partial x} = \frac{\delta(y)}{y} + (2x-y)v(y)\,,$$

and for $x > y$ that

$$\frac{\partial C_\delta^{u,v}(x,y)}{\partial x} = y\left(\frac{\delta(x)}{x}\right)' - yu(x) - y(x-y)u'(x)\,.$$

It follows that

$$\lim_{x\to y-} \frac{\partial C_\delta^{u,v}(x,y)}{\partial x} = \frac{\delta(y)}{y} + yv(y)\,,$$

and

$$\lim_{x\to y+} \frac{\partial C_\delta^{u,v}(x,y)}{\partial x} = y\left(\frac{\delta(y)}{y}\right)' - yu(y)\,,$$

from which it follows that

$$\begin{aligned}
&\lim_{x\to y-} \frac{\partial C_\delta^{u,v}(x,y)}{\partial x} - \lim_{x\to y+} \frac{\partial C_\delta^{u,v}(x,y)}{\partial x} \\
&= y(u(y)+v(y)) + \frac{\delta(y)}{y} - y\left(\frac{\delta(y)}{y}\right)' \\
&= y(u(y)+v(y)) - y^2\left(\frac{\delta(y)}{y^2}\right)' \geq 0\,,
\end{aligned} \tag{7.21}$$

where the last inequality follows from condition (7.5), satisfied by any lower semiquadratic copula. Interchanging the roles of x and y leads to the analogous result for the partial derivatives w.r.t. y. □

Proposition 7.8. *Let δ, u and v be continuous differentiable functions such that $C_\delta^{u,v}$ is a lower semiquadratic copula. Then $C_\delta^{u,v}$ is continuous differentiable in both arguments if and only if*

$$u(t) + v(t) = t \left(\frac{\delta(t)}{t^2} \right)'$$

holds for any $t \in]0, 1]$.

Proof. From (7.2) it is clear that $C_\delta^{u,v}$ is continuous differentiable in both arguments on $[0, 1]^2$ except possibly on the diagonal of the unit square. Due to inequality (7.21), it holds that $C_\delta^{u,v}$ is in both arguments continuous differentiable on the diagonal if and only if (7.5) is realized as an equality for any $t \in]0, 1]$. □

Example 7.10. *Let* δ_Π *be the diagonal section of the product copula* Π. *Since* $(\delta_\Pi(t)/t^2)' = 0$, *all lower semiquadratic copulas in*

$$\{C_{\delta_\Pi}^{u,v} \mid u = -v,\ u(t) = \lambda(1 - t),\ \lambda \in [-1, 1]\},$$

are continuous differentiable in both arguments.

The continuous differentiability does not imply that the horizontal or/and vertical sections of a lower semiquadratic copula are necessarily quadratic functions. The only lower semiquadratic copulas that have quadratic horizontal sections are the symmetric copulas with quadratic sections which were already introduced and characterized in [96].

Proposition 7.9. *Let* $C_\delta^{u,v}$ *be a lower semiquadratic copula with quadratic horizontal (resp. vertical) sections. Then* $C_\delta^{u,v}$ *is symmetric and belongs to the Farlie–Gumbel–Morgenstern family of copulas.*

Proof. A lower semiquadratic copula $C_\delta^{u,v}$ can only have quadratic horizontal sections if u is a quadratic function, i.e. if u is of the form

$$u(t) = u_0 + u_1 t + u_2 t^2,$$

and δ is a quartic function without constant term, i.e.

$$\delta(t) = t(\delta_1 + \delta_2 t + \delta_3 t^2 + \delta_4 t^3),$$

whence also v must be a quadratic function, i.e.

$$v(t) = v_0 + v_1 t + v_2 t^2.$$

Expressing that for any fixed $y \in [0, 1]$, $C_\delta^{u,v}(x, y)$ should be the same quadratic function on the intervals $[0, y]$ and $[y, 1]$, immediately leads to the following equalities:

$$\delta_1 = u_0 = v_0 = 0, \quad u_1 + v_1 = \delta_3, \quad u_2 = v_2 = \delta_4.$$

Expressing that $u(1) = v(1) = 0$, it follows that $u_1 = v_1 = -\delta_4 = \delta_3/2$, whence

$u = v$ and $C_\delta^{u,v}$ is symmetric. Expressing further that $\delta(1) = 1$, leads to $\delta_2 = 1 + \delta_4$. Renaming δ_4 as θ, it follows that

$$\delta(t) = t^2[1 + \theta(1-t)^2] \quad \text{and} \quad u(t) = v(t) = -\theta t(1-t)\,,$$

with θ a real constant. Finally, to be a copula, conditions (7.4) and (7.5) need to be satisfied. The reader can easily verify that such is the case if and only if $|\theta| \le 1$. □

7.5. Absolutely continuous lower semiquadratic copulas

A copula C is absolutely continuous if its density function is given by $\frac{\partial^2 C(x,y)}{\partial x \partial y}$ almost everywhere (see Chapter 1). In the next proposition we characterize the class of absolutely continuous lower semiquadratic copulas.

Proposition 7.10. *Let $C_\delta^{u,v}$ be a lower semiquadratic copula. Then it holds that $C_\delta^{u,v}$ is absolutely continuous if and only if*

$$u(t) + v(t) - t\left(\frac{\delta(t)}{t^2}\right)' = 0\,, \tag{7.22}$$

for any $t \in]0,1]$.

Proof. Let $C = C_\delta^{u,v}$ be a lower semiquadratic copula. Consider a rectangle $R = [x_1, x_2] \times [y_1, y_2] \subseteq I_2$. Let us introduce the following notation

$$I = \int_{x_1}^{x_2} \mathrm{d}x \int_{y_1}^{y_2} \frac{\partial^2 C(x,y)}{\partial x \partial y}\,\mathrm{d}y\,.$$

It then holds that

$$\begin{aligned} I &= \int_{x_1}^{x_2} \mathrm{d}x \int_{y_1}^{y_2} \left(\left(\frac{\delta(y)}{y}\right)' - v(y) - (y - 2x)v'(y)\right)\mathrm{d}y \\ &= (x_2 - x_1)\left(\frac{\delta(y_2)}{y_2} - y_2 v(y_2) - \frac{\delta(y_1)}{y_1} + y_1 v(y_1)\right) \\ &\quad + (x_2 - x_1)(x_2 + x_1)(v(y_2) - v(y_1)) \\ &= C(x_2, y_2) + C(x_1, y_1) - C(x_1, y_2) - C(x_2, y_1) = V_C(R)\,. \end{aligned}$$

Similarly, one can verify that

$$\int_{x_1}^{x_2} \mathrm{d}x \int_{y_1}^{y_2} \frac{\partial^2 C(x,y)}{\partial x \partial y} \, \mathrm{d}y = V_C(R)\,,$$

for any rectangle $R = [x_1, x_2] \times [y_1, y_2]$ located in I_1. Therefore, C is absolutely continuous on I_1 as well as on I_2. Hence, if there exists a singular component, its support must be spread on the main diagonal. Due to the above, it follows that C is absolutely continuous if and only if

$$\int_0^a \mathrm{d}x \int_0^a \frac{\partial^2 C(x,y)}{\partial x \partial y} \, \mathrm{d}y = \delta(a)\,,$$

for any $a \in [0,1]$. Computing the above integral, the latter equality becomes

$$\delta(a) = \int_0^a \left(2\frac{\delta(t)}{t} + t(u(t) + v(t)) \right) \mathrm{d}t\,, \tag{7.23}$$

for any $a \in [0,1]$. Integrating by parts, it holds that

$$\int_0^a 2\frac{\delta(t)}{t} \, \mathrm{d}t = \delta(a) - \int_0^a t^2 \left(\frac{\delta(t)}{t^2} \right)' \mathrm{d}t\,.$$

Substituting in Eq. (7.23), it follows that

$$\int_0^a \left(t(u(t) + v(t)) - t^2 \left(\frac{\delta(t)}{t^2} \right)' \right) \mathrm{d}t = 0\,,$$

for any $a \in [0,1]$. Since condition (7.5) is satisfied, the last equality is equivalent to the condition

$$u(t) + v(t) - t \left(\frac{\delta(t)}{t^2} \right)' = 0\,,$$

for any $t \in]0,1]$.

□

Example 7.11. *The family of copulas given in Proposition 7.9 is a family of absolutely continuous lower semiquadratic copulas. One can easily verify condition (7.22).*

Example 7.12. *Let δ_Π be the diagonal section of the product copula Π. In Example 2, it was shown that $C_{\delta_\Pi}^{u,v}$, with $u = v = 1 - t$ for any $t \in]0,1]$, is the smallest lower semiquadratic copula with diagonal section δ_Π. One can easily verify* *that*

$$u(t) + v(t) = 2(1-t) \neq 0 = t\left(\frac{\delta(t)}{t^2}\right)'$$

for any $t \in\,]0,1[\,$, *and therefore,* $C^{u,v}_{\delta_\Pi}$ *is not absolutely continuous.*

Due to Propositions 7.8 and 7.10, the class of continuous differentiable lower semiquadratic copulas as well as the class of absolutely continuous lower semiquadratic copulas are characterized by realizing condition (7.5) as an equality. Consequently, if u and v are continuous differentiable functions, and $C^{u,v}_\delta$ is an absolutely continuous lower semiquadratic copula, then $C^{u,v}_\delta$ is continuous differentiable.

7.6. Degree of non-exchangeability of lower semiquadratic copulas

In general, the degree of non-exchangeability of a lower semiquadratic copula does not lend itself to a simple expression. However, for the non-symmetric copulas obtained in Propositions 7.4 and 7.6 this is possible.

Proposition 7.11. *Let* δ *be a differentiable diagonal function that satisfies condition (7.18). Then the degree of non-exchangeability of* $C = C^{u,v}_\delta$*, with* $u = -v = \phi_\delta$ *or* $-u = v = \phi_\delta$*, is given by*

$$\mu_{+\infty}(C) = \frac{(t^*)^2}{2}\left(\frac{\delta(t^*)}{t^*}\right)', \tag{7.24}$$

where t^* *is a solution in* $]0,1]$ *of the equation*

$$3\phi_\delta(t) = \left(\frac{\delta(t)}{t}\right)'. \tag{7.25}$$

Proof. We consider the case $u = -v = \phi_\delta$, the case $-u = v = \phi_\delta$ being similar. From the definition of $\mu_{+\infty}(C)$ in (1.5) and the general expression (7.2), it immediately follows that

$$\mu_{+\infty}(C^{u,v}_\delta) = 6 \sup_{0\le x\le y\le 1} x(y-x)\phi_\delta(y)\,.$$

It follows that for fixed $y \in\,]0,1]$ the maximum is attained at the point $(y/2, y)$. Letting y vary, we need to find the value $y^* \in\,]0,1]$ for which the function $y^2\phi_\delta(y)$ attains its maximum on $]0,1]$. Obviously, ϕ_δ being a solution of (7.14), it is differentiable. Hence, y^* is a solution of the equation

$$2y\phi_\delta(y) + y^2\phi'_\delta(y) = 0\,,$$

or, equivalently, of the equation

$$3\phi_\delta(y) - \left(\frac{\delta(y)}{y}\right)' = 0\,.$$

The stated result immediately follows. □

Example 7.13. *Let δ_λ be the diagonal function from Example 7.5 and $C_\lambda = C^{u,v}_{\delta_\lambda}$ with $u = -v = \phi_{\delta_\lambda}$. Equation (7.25) reads*

$$3\lambda(1-t) = \lambda$$

and has $t^* = 2/3$ as solution. It follows that

$$\mu_{+\infty}(C_\lambda) = \frac{2\lambda}{9}.$$

Example 7.14. *Let δ_α be the diagonal function from Example 7.7 and $C_\alpha = C^{u,v}_{\delta_\alpha}$ with $u = -v = \phi_{\delta_\alpha}$. Equation (7.25) reads*

$$\frac{\alpha}{2-\alpha}[t^{\alpha-1} - t] = \frac{\alpha}{3}t^{\alpha-1}$$

and has $t^* = ((1+\alpha)/3)^{1/(2-\alpha)}$ as solution. It follows that

$$\mu_{+\infty}(C_\alpha) = \frac{\alpha}{2}\left(\frac{1+\alpha}{3}\right)^{\frac{1+\alpha}{2-\alpha}}\,.$$

In particular, $\mu_{+\infty}(C_0) = 0$, $\mu_{+\infty}(C_1) = 2/9$ and $\mu_{+\infty}(C_{1/2}) = 1/8$.

Example 7.15. *Let $\delta_{2,1}$, $\delta_{2,2}$ and $\delta_{2,3}$ be the diagonal functions from Example 7.8. The results obtained from (7.24) and (7.25) are summarized in Table 1.*

Table 7.1: Degree of non-exchangeability of the lower semiquadratic copulas $C^{u,v}_{\delta_{2,\beta}}$ with diagonal section $\delta_{2,\beta} = t^2 + t^2(1-t)^\beta$, where $u = -v = \phi_{\delta_{2,\beta}}$.

β	$\delta_{2,\beta}$	$\phi_{\delta_{2,\beta}}$	t^*	$\mu_{+\infty}$
1	$t^2 + t^2(1-t)$	$2 - 2t + 2t\ln t$	0.466411	0.116076
2	$t^2 + t^2(1-t)^2$	$2 + t - 3t^2 + 4t\ln t$	0.711292	0.170157
3	$t^2 + t^2(1-t)^3$	$2 + 5t - 9t^2 + 2t^3 + 6t\ln t$	0.698683	0.204307

As Proposition 7.3 establishes, the functions ϕ_δ and ψ_δ characterize the smallest and the greatest lower semiquadratic copulas with diagonal section δ, respectively. Clearly, under condition (7.20) a lower semiquadratic copula $C^{u,v}_\delta$ with maximal degree of non-exchangeability is obtained when $u = \phi_\delta$ and $v = \psi_\delta$ or when $u = \psi_\delta$ and $v = \phi_\delta$.

Proposition 7.12. *Let δ be a differentiable diagonal function that satisfies condition (7.20). Then the degree of non-exchangeability of $C = C_\delta^{u,v}$, with $u = \phi_\delta$ and $v = \psi_\delta$ or with $u = \psi_\delta$ and $v = \phi_\delta$, is given by*

$$\mu_{+\infty}(C) = \frac{(t^*)^3}{2}\psi'_\delta(t^*)\,, \tag{7.26}$$

where t^* is a solution in $]0,1]$ of the equation

$$3\phi_\delta(t) = \psi_\delta(t) + 2\left(\frac{\delta(t)}{t}\right)'\,. \tag{7.27}$$

Proof. We consider the case $u = \phi_\delta$ and $v = \psi_\delta$, the case $u = \psi_\delta$ and $v = \phi_\delta$ being similar. From the definition of $\mu_{+\infty}(C)$ in (1.5) and the general expression (7.2), it immediately follows that

$$\mu_{+\infty}(C_\delta^{u,v}) = 3 \sup_{0\le x\le y\le 1} x(y-x)(\phi_\delta(y) - \psi_\delta(y))\,.$$

It follows that for fixed $y \in]0,1]$ the maximum is attained at the point $(y/2, y)$. Letting y vary, we need to find the value $y^* \in]0,1]$ for which the function $y^2(\phi_\delta(y) - \psi_\delta(y))$ attains its maximum on $]0,1]$. Since ϕ_δ is a solution of (7.14) and δ is differentiable, the function $\phi_\delta - \psi_\delta$ is differentiable. Hence, y^* is a solution of the equation

$$2y(\phi_\delta(y) - \psi_\delta(y)) + y^2(\phi'_\delta(y) - \psi'_\delta(y)) = 0\,,$$

or, equivalently, of the equation

$$3\phi_\delta(y) - \psi_\delta(y) - 2\left(\frac{\delta(y)}{y}\right)' = 0\,.$$

The stated result immediately follows. □

Example 7.16. *Let δ_λ be the diagonal function from Example 7.5 and $C_\lambda = C_{\delta_\lambda}^{u,v}$ with $u = \phi_{\delta_\lambda}$ and $v = \psi_{\delta_\lambda}$. It was shown in Example 7.9 that $C_{\delta_\lambda}^{u,v}$ is a copula for any $\lambda \in [0,1/2]$. Equation (7.27) reads*

$$3\lambda(1-t) = \frac{\lambda(t-1)}{t} + 2\lambda$$

and has $t^* = 1/\sqrt{3}$ as solution. It follows that

$$\mu_{+\infty}(C_\lambda) = \frac{\lambda}{2\sqrt{3}}\,.$$

Note that for any $\lambda \in [0,1/2]$, it holds that

$$\mu_{+\infty}(C_\lambda) = \frac{\lambda}{2\sqrt{3}} \geq \frac{2\lambda}{9} = \mu_{+\infty}(C_\lambda^{\phi_{\delta_\lambda}, -\phi_{\delta_\lambda}}) .$$

7.7. Measuring the dependence of random variables coupled by a lower semiquadratic copula

We finally want to compute Spearman's rho and Kendall's tau for two continuous random variables whose dependence is modelled by a lower semiquadratic copula.

Proposition 7.13. *Let $C = C_\delta^{u,v}$ be a lower semiquadratic copula. Then the measures of association ρ_C and τ_C are given by*

$$\rho_C = 2\int_0^1 (6t\delta(t) - t^3(u(t)+v(t)))\,\mathrm{d}t - 3$$

and

$$\begin{aligned}\tau_C &= 1 - 2\int_0^1 \left(2t\delta(t)\left(\frac{\delta(t)}{t}\right)' - t\delta(t)(u(t)+v(t))\right)\mathrm{d}t \\ &\quad + \frac{2}{3}\int_0^1 \left(t^2\delta(t)((u'(t)) + v'(t)) - t^3\left(\frac{\delta(t)}{t}\right)'(u(t)+v(t))\right)\mathrm{d}t \\ &\quad + \frac{2}{3}\int_0^1 t^3(u^2(t)+v^2(t))\,\mathrm{d}t\,.\end{aligned}$$

Proof. In order to find ρ_C and τ_C, we need to compute

$$\tilde{I}_\rho = \int_0^1\int_0^1 C(x,y)\,\mathrm{d}x\mathrm{d}y \text{ and } \tilde{I}_\tau = \int_0^1\int_0^1 \frac{\partial C(x,y)}{\partial x}\frac{\partial C(x,y)}{\partial y}\,\mathrm{d}x\mathrm{d}y\,.$$

Decomposing the integrals $\tilde{I}_\rho$ and $\tilde{I}_\tau$, it holds that

$$\tilde{I}_\rho = \int_0^1\int_0^x C(x,y)\,\mathrm{d}y\mathrm{d}x + \int_0^1\int_x^1 C(x,y)\,\mathrm{d}y\mathrm{d}x$$

and

$$\tilde{I}_\tau = \int_0^1 \int_0^x \frac{\partial C(x,y)}{\partial x} \frac{\partial C(x,y)}{\partial y} \, \mathrm{d}y\mathrm{d}x + \int_0^1 \int_x^1 \frac{\partial C(x,y)}{\partial x} \frac{\partial C(x,y)}{\partial y} \, \mathrm{d}y\mathrm{d}x\,.$$

Computing the above integrals and substituting in the expressions for ρ_C and τ_C, the desired result follows. □

Corollary 7.2. *Let δ be a differentiable diagonal function. If δ satisfies condition (7.11), then the measures of association ρ_C and τ_C of $C = C_\delta^{u,v}$, with $u = v = \phi_\delta$, are given by*

$$\rho_C = \frac{72}{5} \int_0^1 t\delta(t)\mathrm{d}t - \frac{19}{5} \tag{7.28}$$

and

$$\tau_C = 1 + \frac{80}{9} \int_0^1 t\delta(t)\phi_\delta(t)\mathrm{d}t - \frac{56}{9} \int_0^1 t\delta(t) \left(\frac{\delta(t)}{t}\right)' \mathrm{d}t\,. \tag{7.29}$$

If δ satisfies condition (7.12), then the measures of association ρ_C and τ_C of $C = C_\delta^{u,v}$, with $u = v = \psi_\delta$, are given by

$$\rho_C = 8 \int_0^1 t\delta(t)\,\mathrm{d}t - \frac{5}{3} \tag{7.30}$$

and

$$\tau_C = 1 + 8 \int_0^1 t\delta(t)\psi_\delta(t)\,\mathrm{d}t\,. \tag{7.31}$$

If δ satisfies condition (7.18), then the measures of association ρ_C and τ_C of $C = C_\delta^{u,v}$, with $u = -v = \phi_\delta$ or $-u = v = \phi_\delta$, are given by

$$\rho_C = 12 \int_0^1 t\delta(t)\,\mathrm{d}t - 3 \tag{7.32}$$

and

$$\tau_C = 1 - \frac{16}{9} \int_0^1 t\delta(t)\phi_\delta(t)\,\mathrm{d}t - \frac{32}{9} \int_0^1 t\delta(t) \left(\frac{\delta(t)}{t}\right)' \mathrm{d}t\,. \tag{7.33}$$

If δ satisfies condition (7.20), then the measures of association ρ_C and τ_C of $C = C_\delta^{u,v}$, with $u = \phi_\delta$ and $v = \psi_\delta$ or with $u = \psi_\delta$ and $v = \phi_\delta$, are given by

$$\rho_C = \frac{56}{5} \int_0^1 t\delta(t)\,\mathrm{d}t - \frac{41}{15} \tag{7.34}$$

and

$$\tau_C = 1 + \frac{40}{9} \int_0^1 t\delta(t)\phi_\delta(t)\,\mathrm{d}t + 4 \int_0^1 t\delta(t)\psi_\delta(t)\,\mathrm{d}t - \frac{28}{9} \int_0^1 t\delta(t) \left(\frac{\delta(t)}{t}\right)' \mathrm{d}t\,. \tag{7.35}$$

Proof. Expressions (7.28), (7.30), (7.32) and (7.34) are obtained by substituting expression (7.2) in the expression for ρ_δ in Proposition 7.13 and using expression (7.10) of ϕ_δ and ψ_δ. The computation is straightforward. To obtain (7.29), (7.31), (7.33) and (7.35) we have used the expression for τ_δ in Proposition 7.13, and also for (7.33) and (7.35) the following equalities between integrals containing ϕ_δ and δ, which can be proven by means of partial integration:

$$\int_0^1 t^3(\phi_\delta(t))^2\,\mathrm{d}t = \frac{1}{3}\int_0^1 t^3\phi_\delta(t)\left(\frac{\delta(t)}{t}\right)'\mathrm{d}t\,,$$

$$\int_0^1 t^3\phi_\delta(t)\left(\frac{\delta(t)}{t}\right)'\mathrm{d}t = \int_0^1 t\delta(t)\left(\frac{\delta(t)}{t}\right)'\mathrm{d}t - 4\int_0^1 t\delta(t)\phi_\delta(t)\,\mathrm{d}t\,.$$

□

Example 7.17. *Let δ_λ be the diagonal function from Example 7.5. Using (7.28)–(7.35), we obtain for $C = C^{u,v}_{\delta_\lambda}$, with $u = v = \phi_{\delta_\lambda}$ and $\lambda \in [0,1]$, that the measures of association are given by*

$$\rho_C = \frac{5-6\lambda}{5}\,, \qquad \tau_C = \frac{9+2\lambda(-6+\lambda)}{9}\,;$$

for $C = C^{u,v}_{\delta_\lambda}$, with $u = v = \psi_{\delta_\lambda}$, and $\lambda \in [0,1/3]$, they are given by

$$\rho_C = \frac{3-2\lambda}{3}\,, \qquad \tau_C = \frac{3+2\lambda(-2+\lambda)}{3}\,;$$

for $C = C^{u,v}_{\delta_\lambda}$, with $u = -v = \phi_{\delta_\lambda}$ or $-u = v = \phi_{\delta_\lambda}$, and $\lambda \in [0,1]$, they are given by

$$\rho_C = 1-\lambda\,, \qquad \tau_C = \frac{45+4\lambda(-15+4\lambda)}{45}\,;$$

and for $C = C^{u,v}_{\delta_\lambda}$, with $u = \phi_{\delta_\lambda}$ and $v = \psi_{\delta_\lambda}$ or with $u = \psi_{\delta_\lambda}$ and $v = \phi_{\delta_\lambda}$, and $\lambda \in [0,1/2]$, they are given by

$$\rho_C = \frac{15-14\lambda}{15}\,, \qquad \tau_C = \frac{(3-2\lambda)^2}{9}\,.$$

8 Semiquadratic copulas based on horizontal and vertical interpolation

8.1. Introduction

The aim of this chapter is to complete the results of the previous chapter and generalize the results of Chapter 6. We first recall lower semiquadratic copulas (see Chapter 7) and introduce in a similar manner three families of semiquadratic copulas with a given diagonal section. Analogously, we introduce four families of semiquadratic copulas with a given opposite diagonal section. There is a great similarity between the case of a given opposite diagonal section and that of a given diagonal section (see also [23]), which can be explained by the existence of a transformation that maps copulas onto copulas in such a way that the diagonal section is mapped onto the opposite diagonal section and vice versa. In the second part of this chapter, we consider the construction of semiquadratic copulas with given diagonal and opposite diagonal sections. Also here, we introduce sixteen families of semiquadratic copulas and, based on a set of transformations given in (1.3), we classify them into six classes.

This chapter is organized as follows. In the next section we introduce lower, upper, horizontal and vertical semiquadratic functions with a given diagonal section and characterize the corresponding families of copulas. In Section 8.2, we introduce in a similar way lower-upper, upper-lower, horizontal and vertical semiquadratic functions with a given opposite diagonal section and characterize the corresponding families of copulas. In Section 8.3, we introduce six classes of semiquadratic functions with given diagonal and opposite diagonal sections and characterize the corresponding classes of copulas.

8.2. Semiquadratic copulas with a given diagonal section (resp. opposite diagonal section)

8.2.1. Classification procedure

In this section we recall a class of semiquadratic copulas and introduce a new class as well. The construction of these copulas is based on quadratic interpolation on segments connecting the diagonal (resp. opposite diagonal) and one of the sides of the unit square. Since in both of the two triangular parts of the unit square

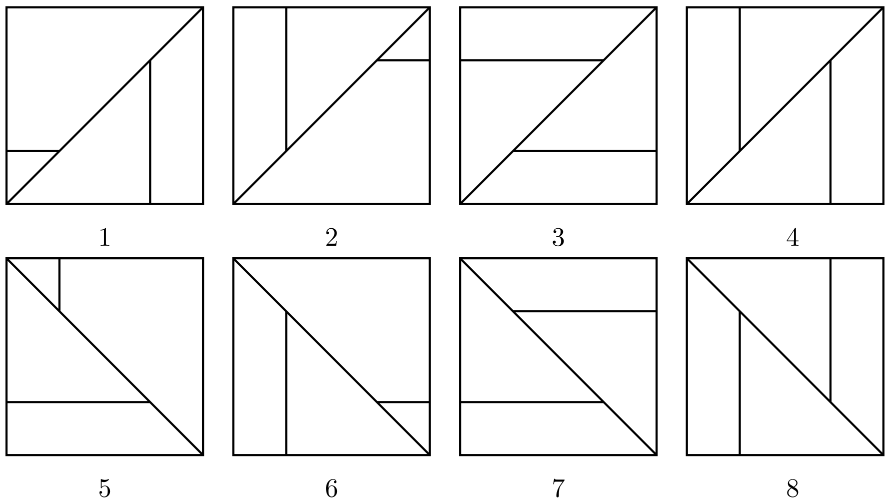


Figure 8.1: Eight possible schemes for horizontal and vertical interpolation when the diagonal (resp. opposite diagonal) section is given.

delimited by the diagonal (resp. opposite diagonal), we can either interpolate horizontally or vertically between a point on the diagonal (resp. opposite diagonal) and a point on the sides of the unit square, there are eight possible interpolation schemes (see Figure 8.1). This quadratic interpolation requires one or two auxiliary functions: a function $f(y)$ (resp. $g(x)$) providing the coefficient of x^2 (resp. y^2) in case of horizontal (resp. vertical) interpolation; the coefficients of the linear terms and the constants are determined by the boundary conditions. We will restrict our attention when characterizing a class of semiquadratic functions to such functions f and g that are absolutely continuous. Note that f and g, being absolutely continuous, are differentiable almost everywhere.

Based on symmetry considerations, we classify the families represented in Figure 8.1 into two classes (see Table 8.1). Using the transformations defined in (1.3), we

Class I	Class II
1	3
2	4
5	7
6	8

Table 8.1: Classification of the families in Figure 8.1 into two classes.

only need to consider one family from each class for characterization. This can be seen as follows. Let us consider the subtriangles I_1 and I_2 of the unit square as in Chapter 3. For a diagonal function δ and two functions $f, g : [0,1] \to \mathbb{R}$, generic members of the first and second families in class I are denoted as ${}^{\mathrm{I}}_{1}C^{f,g}_{\delta}$ and ${}^{\mathrm{I}}_{2}C^{f,g}_{\delta}$, and are given by

$$
{}^{\mathrm{I}}_{1}C^{f,g}_{\delta}(x,y) = \begin{cases} \dfrac{y}{x}\delta(x) + y(y-x)g(x) & , \text{ if } (x,y) \in I_1 \,, \\[2ex] \dfrac{x}{y}\delta(y) + x(x-y)\, f(y) & , \text{ if } (x,y) \in I_2 \,, \end{cases} \tag{8.1}
$$

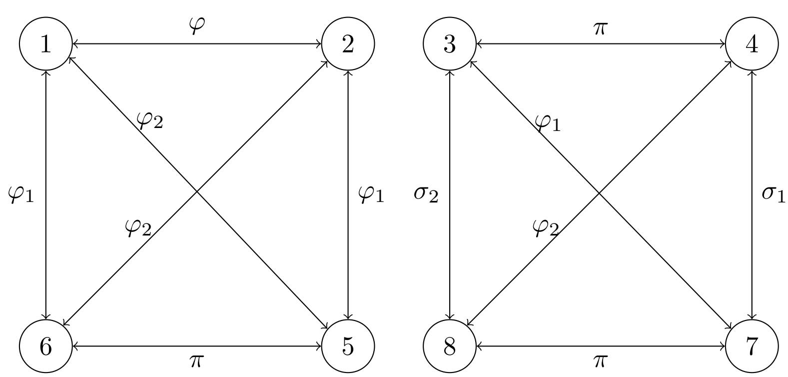
and

$$
{}^{\mathrm{I}}_{2}C^{f,g}_{\delta}(x,y) = \begin{cases} y - \dfrac{y-\delta(y)}{1-y}(1-x) + (1-x)(y-x)f(x) & , \text{ if } (x,y) \in I_1 \,, \\[2ex] x - \dfrac{x-\delta(x)}{1-x}(1-y) + (1-y)(x-y)g(x) & , \text{ if } (x,y) \in I_2 \,. \end{cases}
$$

Consider a diagonal function δ and let δ_1 be the diagonal function defined by $\delta_1(x) = 2x - 1 + \delta(1-x)$. Consider two functions $f, g : [0,1] \to \mathbb{R}$ and let $\hat{f}$ and $\hat{g}$ be the functions defined by $\hat{f}(x) = f(1-x)$ and $\hat{g}(x) = g(1-x)$. One easily verifies that

$$
{}^{\mathrm{I}}_{2}C^{f,g}_{\delta} = \varphi({}^{\mathrm{I}}_{1}C^{\hat{f},\hat{g}}_{\delta_1}) \,.
$$

In Figure 8.2, we illustrate the transformations between the families in the same class.



(a) Class I (b) Class II

Figure 8.2: Transformations between the families from the same class.

8.2.2. Characterization

Class I

For any diagonal function δ and any two functions $f, g : [0,1] \to \mathbb{R}$, the function ${}^{\mathrm{I}}_{1}C^{f,g}_{\delta} : [0,1]^2 \to \mathbb{R}$ defined in (8.1) where the convention $\frac{0}{0} := 0$ is adopted, is a semiquadratic function with diagonal section δ since it satisfies ${}^{\mathrm{I}}_{1}C^{f,g}_{\delta}(t,t) = \delta(t)$ for all $t \in [0,1]$, and it is quadratic in x on I_1 and quadratic in y on I_2. Obviously, symmetric functions are obtained when $f = g$. Note that if the functions f and g are continuous, then ${}^{\mathrm{I}}_{1}C^{f,g}_{\delta}$ is continuous. Note also that for $f = g = 0$, the definition of a lower semilinear function [36] is retrieved. In Chapter 7, we have identified the necessary and sufficient conditions to be fulfilled by the functions δ, f and g (in a slightly more general setting, i.e. for functions f and g not necessarily defined in 0, but satisfying some appropriate limit conditions). Let us consider the function λ_δ defined as in Chapter 3.

Proposition 8.1. [71] *Let δ be a diagonal function and let $f, g : [0,1] \to \mathbb{R}$ be two absolutely continuous functions. Then the semiquadratic function ${}^{\mathrm{I}}_{1}C^{f,g}_{\delta}$ defined in (8.1) is a copula with diagonal section δ if and only if*

(i) $f(1) = g(1) = 0\,,$

(ii) $\max(f(t) + t\,|f'(t)|\,, g(t) + t\,|g'(t)|) \leq \lambda'_\delta(t)\,,$

(iii) $f(t) + g(t) \geq t\left(\frac{\delta(t)}{t^2}\right)'\,,$

for all $t \in \,]0,1]$ where the derivatives exist.

Example 8.1. *Let δ_Π be the diagonal section of the product copula Π, i.e. $\delta_\Pi(t) = t^2$ for all $t \in [0,1]$. Let f and g be defined by $f(t) = g(t) = 1 - t$ for all $t \in \,]0,1]$. One easily verifies that the conditions of Proposition 8.1 are satisfied and hence, ${}^{\mathrm{I}}_{1}C^{f,g}_{\delta_\Pi}$ is a semiquadratic copula with diagonal section δ_Π.*

Class II

For any diagonal function δ and any function $f : [0,1] \to \mathbb{R}$, the function ${}^{\mathrm{II}}_{3}C^{f}_{\delta} : [0,1]^2 \to \mathbb{R}$ defined by ${}^{\mathrm{II}}_{3}C^{f}_{\delta}(x,y) =$

$$\begin{cases} y - \dfrac{1-x}{1-y}\,(y - \delta(y)) + (1-x)(y-x)f(y) & \text{, if } (x,y) \in I_1\,, \\[2ex] \dfrac{x}{y}\,\delta(y) + x(x-y)\,f(y) & \text{, if } (x,y) \in I_2\,, \end{cases} \tag{8.2}$$

where the convention $\frac{0}{0} := 0$ is adopted, is a semiquadratic function with diagonal section δ since it satisfies ${}^{\mathrm{II}}_{3}C^{f}_{\delta}(t,t) = \delta(t)$ for all $t \in [0,1]$, and it is quadratic in x

on I_1 as well as on I_2. Note that for $f = 0$, the definition of a horizontal semilinear function [20] is retrieved.

We now state the conditions to be fulfilled by the functions δ and f such that ${}_3^{\mathrm{II}}C_\delta^f$ is a copula. Let us consider the function μ_δ defined as in Chapter 3.

Proposition 8.2. *Let δ be a diagonal function and let $f : [0,1] \to \mathbb{R}$ be an absolutely continuous function. Then the semiquadratic function ${}_3^{\mathrm{II}}C_\delta^f$ defined in (8.2) is a copula with diagonal section δ if and only if*

(i) $f(0) = f(1) = 0\,,$

(ii) $f(t) + t\,|f'(t)| \leq \lambda'_\delta(t)\,,$

(iii) $f(t) + (1-t)\,|f'(t)| \leq \mu'_\delta(t)\,,$

(iv) $f(t) \geq \frac{t^2-\delta(t)}{t(1-t)}\,,$

for all $t \in]0,1[$ where the derivatives exist.

Proof. Let $C = {}_3^{\mathrm{II}}C_\delta^f$. The boundary conditions $C(t,0) = 0$ and $C(t,1) = t$ for all $t \in [0,1]$ immediately lead to the conditions $f(0) = f(1) = 0$. Therefore, it suffices to prove that the 2-increasingness of C is equivalent to conditions (ii)–(iv).

Suppose that C is 2-increasing. For any rectangle $R = [x_1,x_2] \times [y_1,y_2] \subseteq I_2$, it then holds that $V_C(R) \geq 0$, i.e.

$$(x_2-x_1)\,(\lambda_\delta(y_2) - \lambda_\delta(y_1) - f(y_2)y_2 + f(y_1)y_1 + (x_1+x_2)(f(y_2)-f(y_1))) \geq 0\,,$$

or, equivalently,

$$\lambda_\delta(y_2) - \lambda_\delta(y_1) - f(y_2)y_2 + f(y_1)y_1 + (x_1+x_2)(f(y_2)-f(y_1)) \geq 0\,. \tag{8.3}$$

Dividing by $y_2 - y_1$ and taking the limits $x_2 \to x_1$ and $y_2 \to y_1$, inequality (8.3) becomes

$$\lambda'_\delta(y_1) - f(y_1) + (2x_1 - y_1)f'(y_1) \geq 0\,. \tag{8.4}$$

Since the left-hand side of inequality (8.4) is linear in x_1, this condition is equivalent to requiring that it holds for $x_1 = 0$ and $x_1 = y_1$, i.e.

$$\lambda'_\delta(y_1) - f(y_1) + y_1 f'(y_1) \geq 0 \quad \text{and} \quad \lambda'_\delta(y_1) - f(y_1) - y_1 f'(y_1) \geq 0\,,$$

or, equivalently, to

$$f(y_1) + y_1\,|f'(y_1)| \leq \lambda'_\delta(y_1)\,. \tag{8.5}$$

Hence, condition (ii) follows. Similarly, for any rectangle $R = [x_1,x_2] \times [y_1,y_2] \subseteq I_1$, it holds that $V_C(R) \geq 0$, i.e.

$$(x_2-x_1)\,(\mu_\delta(y_2) - \mu_\delta(y_1) - f(y_2)y_2 + f(y_1)y_1 + (x_1+x_2-1)(f(y_2)-f(y_1))) \geq 0\,,$$

or, equivalently,

$$\mu_\delta(y_2) - \mu_\delta(y_1) - f(y_2)y_2 + f(y_1)y_1 + (x_1 + x_2 - 1)(f(y_2) - f(y_1)) \geq 0\,. \quad (8.6)$$

Dividing by $y_2 - y_1$ and taking the limits $x_2 \to x_1$ and $y_2 \to y_1$, inequality (8.6) becomes

$$\mu'_\delta(y_1) - f(y_1) + (2x_1 - y_1 - 1)f'(y_1) \geq 0\,. \quad (8.7)$$

Since the left-hand side of inequality (8.7) is linear in x_1, this condition is equivalent to requiring that it holds for $x_1 = y_1$ and $x_1 = 1$, i.e.

$$\mu'_\delta(y_1) - f(y_1) - (1 - y_1)f'(y_1) \geq 0 \quad \text{and} \quad \mu'_\delta(y_1) - f(y_1) + (1 - y_1)f'(y_1) \geq 0\,,$$

or, equivalently, to

$$f(y_1) + (1 - y_1)\,|f'(y_1)| \leq \mu'_\delta(y_1)\,. \quad (8.8)$$

Hence, condition (iii) follows.

Finally, the fact that $V_C(R) \geq 0$ for any square $R = [x_1, x_2] \times [x_1, x_2]$ centered around the main diagonal is equivalent to

$$\begin{aligned} V_C(R) &= C(x_1, x_1) + C(x_2, x_2) - C(x_1, x_2) - C(x_2, x_1) \\ &= (x_2 - x_1)\left(\frac{\delta(x_1)}{1 - x_1} + \frac{\delta(x_2)}{x_2} + x_1 f(x_2) + (1 - x_2)f(x_1) - \frac{x_1}{1 - x_1}\right) \geq 0\,, \end{aligned}$$

or, equivalently,

$$\lambda_\delta(x_2) - \mu_\delta(x_1) + x_1 f(x_2) + (1 - x_2)f(x_1) \geq 0\,.$$

Taking the limit $x_2 \to x_1$, condition (iv) immediately follows.

Now suppose that conditions (ii)–(iv) are satisfied. Due to the additivity of volumes, it suffices to consider a restricted number of cases to prove the 2-increasingness of C. Let $R = [a_1, b_1] \times [a_2, b_2]$ be a rectangle located in I_1. Since condition (iii) is satisfied, inequality (8.7) follows and it holds that

$$\int_{a_2}^{b_2} \mathrm{d}y_1 \int_{a_1}^{b_1} \left(\mu'_\delta(y_1) - f(y_1) + (2x_1 - y_1 - 1)f'(y_1)\right) \mathrm{d}x_1 \geq 0\,.$$

Computing the above integral, the latter inequality becomes

$$(b_1 - a_1)\left(\mu_\delta(b_2) - \mu_\delta(a_2) - b_2 f(b_2) + a_2 f(a_2) + (a_1 + b_1 - 1)(f(b_2) - f(a_2))\right) \geq 0\,,$$

or, equivalently, $V_C(R) \geq 0$.

Let $R = [a_1, b_1] \times [a_2, b_2]$ be a rectangle located in I_2. Since condition (ii) is satisfied, inequality (8.4) follows and it holds that

$$\int_{a_2}^{b_2} \mathrm{d}y_1 \int_{a_1}^{b_1} (\lambda'_\delta(y_1) - f(y_1) + (2x_1 - y_1)f'(y_1))\,\mathrm{d}x_1 \geq 0\,.$$

Computing the above integral, the latter inequality becomes

$$(b_1 - a_1)\,(\lambda_\delta(b_2) - \lambda_\delta(a_2) - b_2 f(b_2) + a_2 f(a_2) + (a_1 + b_1)(f(b_2) - f(a_2))) \geq 0\,,$$

or, equivalently, $V_C(R) \geq 0$.

Finally, let $S = [a, b] \times [a, b]$ be a square centered around the main diagonal. Due to condition (iv), it holds that

$$f(x_1) - \frac{x_1^2 - \delta(x_1)}{x_1(1 - x_1)} \geq 0\,,$$

for all $x_1 \in]0, 1]$, which implies that

$$\tilde{I}_1 = \int_a^b \left(f(x_1) - \frac{x_1^2 - \delta(x_1)}{x_1(1 - x_1)} \right) \mathrm{d}x_1 \geq 0\,.$$

Using inequality (8.4), it follows that

$$\tilde{I}_2 = \int_a^b \int_{x_1}^b (\lambda'_\delta(y_1) - f(y_1) + (2x_1 - y_1)f'(y_1))\,\mathrm{d}y_1\mathrm{d}x_1 \geq 0\,.$$

Using inequality (8.7), it follows that

$$\tilde{I}_3 = \int_a^b \int_{y_1}^b (\mu'_\delta(y_1) - f(y_1) + (2x_1 - y_1 - 1)f'(y_1))\,\mathrm{d}x_1\mathrm{d}y_1 \geq 0\,.$$

Computing the above integrals and setting $I = \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3$, it follows that

$$I = (b - a)\,(\lambda_\delta(b) - \mu_\delta(a) + a f(b) + (1 - b) f(a)) \geq 0\,,$$

or, equivalently,

$$I = V_C(S) \geq 0\,.$$

□

Example 8.2. *Let δ_Π be the diagonal section of the product copula. Let f be defined by $f(t) = t(1 - t)$ for all $t \in [0, 1]$. One easily verifies that the conditions of Proposition 8.2 are satisfied and hence, ${}^{\mathrm{II}}_{3}C^f_{\delta_\Pi}$ is a semiquadratic copula with diagonal section δ_Π.*

8.3. Semiquadratic functions with given diagonal and opposite diagonal sections

8.3.1. Classification procedure

In this section we introduce six new classes of semiquadratic copulas. The construction of these classes is based on quadratic interpolation on segments connecting the diagonal and opposite diagonal, or connecting the diagonal or opposite diagonal and one of the sides of the unit square. Since in any of the four triangular parts of the unit square delimited by the diagonal and opposite diagonal, we can either interpolate between a point on the diagonal and a point on the opposite diagonal, or between a point on the sides of the unit square and a point on the diagonal or opposite diagonal, there are sixteen possible interpolation schemes (see Figure 8.3). Based on symmetry considerations, we classify the families represented in Figure 8.3 into six different classes (see Table 8.2).

Class III	Class IV	Class V	Class VI	Class VII	Class VIII
9	10	14	16	20	24
	11	15	17	21	
	12		18	22	
	13		19	23	

Table 8.2: Classification of the families in Figure 8.3 into six classes.

Using the transformations defined in (1.3), we only need to consider one family from each class for characterization. This can be seen as follows. Let us consider the subtriangles T_1, T_2, T_3 and T_4 of the unit square as in Chapter 5.

For a diagonal function δ, opposite diagonal function ω and two functions $f, g : [0,1] \to \mathbb{R}$, generic members of the first and third families in class IV are denoted

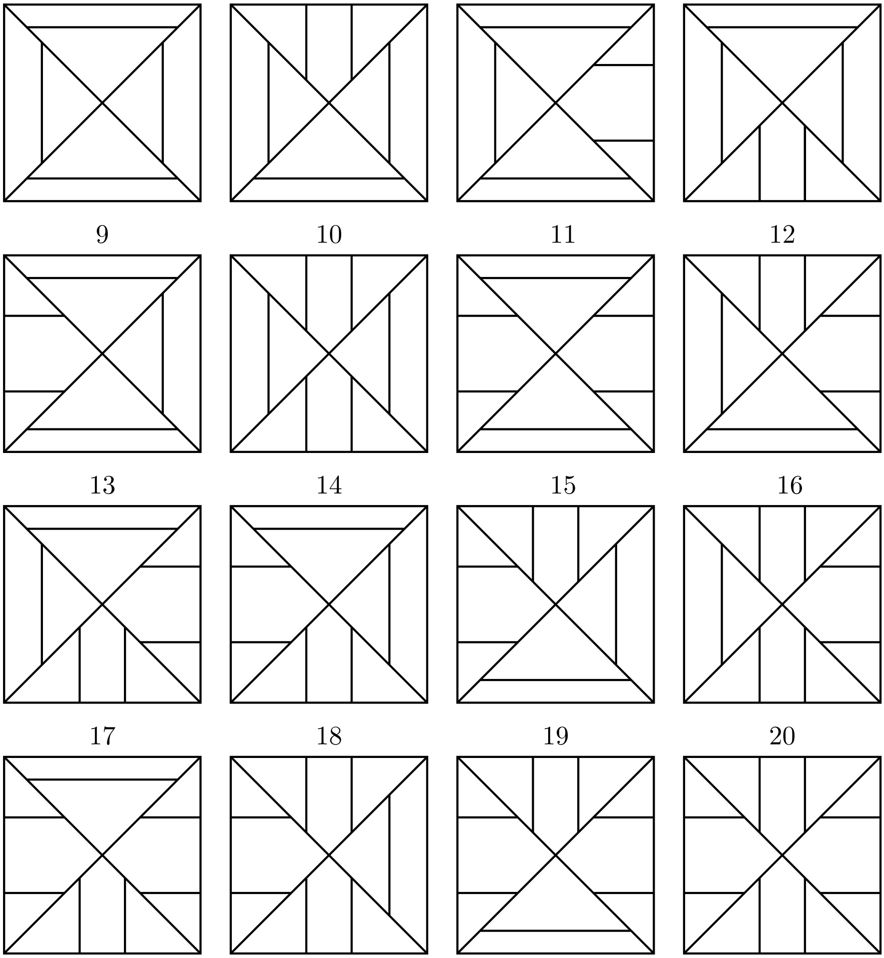


Figure 8.3: Sixteen possible schemes for horizontal and vertical interpolation when the diagonal and opposite diagonal sections are given.

as ${}^{\mathrm{IV}}_{10}C^{f,g}_{\delta,\omega}$ and ${}^{\mathrm{IV}}_{12}C^{f,g}_{\delta,\omega}$, and are given by ${}^{\mathrm{IV}}_{10}C^{f,g}_{\delta,\omega}(x,y) =$

$$\begin{cases} \dfrac{x+y-1}{2y-1}\,\delta(y) + \dfrac{y-x}{2y-1}\,\omega(1-y) - (y-x)(x+y-1)f(y) & \\ \qquad\qquad , \text{ if } (x,y) \in T_1\,, \\ \dfrac{x+y-1}{2x-1}\,\delta(x) - \dfrac{y-x}{2x-1}\,\omega(x) + (y-x)(x+y-1)\,g(x) & \\ \qquad\qquad , \text{ if } (x,y) \in T_2 \cup T_4\,, \\ x+y-1+\dfrac{1-y}{x}\,\omega(x) - (1-y)(x+y-1)g(x) & \\ \qquad\qquad , \text{ if } (x,y) \in T_3 \text{ and } x \leq 1/2\,, \\ x - \dfrac{1-y}{1-x}\,(x-\delta(x)) - (1-y)(y-x)g(x) & , \text{ if } (x,y) \in T_3 \text{ and } x \geq 1/2\,, \end{cases} \tag{8.9}$$

and ${}^{\mathrm{IV}}_{12}C^{f,g}_{\delta,\omega}(x,y) =$

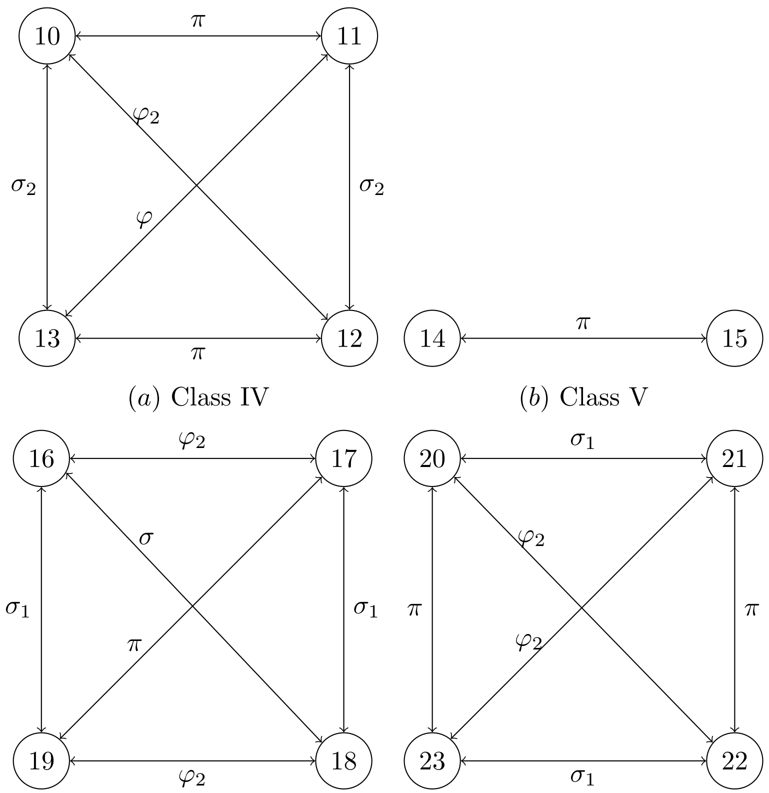
$$\begin{cases} \dfrac{y}{x}\,\delta(x) + y(y-x)g(x) & , \text{ if } (x,y) \in T_1 \text{ and } x \leq 1/2\,, \\ \dfrac{y}{1-x}\,\omega(x) + y\,(x+y-1)g(x) & , \text{ if } (x,y) \in T_1 \text{ and } x \geq 1/2\,, \\ \dfrac{x+y-1}{2x-1}\,\delta(x) - \dfrac{y-x}{2x-1}\,\omega(x) + (y-x)(x+y-1)g(x) & \\ \qquad\qquad , \text{ if } (x,y) \in T_2 \cup T_4\,, \\ \dfrac{x+y-1}{2y-1}\,\delta(y) + \dfrac{y-x}{2y-1}\,\omega(1-y) - (y-x)(x+y-1)f(y) & \\ \qquad\qquad , \text{ if } (x,y) \in T_3\,. \end{cases}$$

Consider a diagonal function δ and an opposite diagonal function ω and let δ_2 be the diagonal function defined $\delta_2(x) = x-\omega(x)$ and $\hat{\omega}$ the opposite diagonal function defined by $\hat{\omega}(x) = x - \delta(x)$. Consider two functions $f,g : [0,1] \to \mathbb{R}$ and let $\hat{f}$ and $\hat{g}$ be the functions defined by $\hat{f}(x) = -f(1-x)$ and $\hat{g}(x) = -g(x)$. One easily verifies that ${}^{\mathrm{IV}}_{12}C^{f,g}_{\delta,\omega} = \varphi_2({}^{\mathrm{IV}}_{10}C^{\hat{f},\hat{g}}_{\delta_2,\hat{\omega}})$. In Figure 8.4, we illustrate the transformations between the families in the same class.

8.3.2. Characterization

Class III

For any diagonal function δ and opposite diagonal function ω such that $\delta(1/2) = \omega(1/2)$, and any two functions $f,g : [0,1] \to \mathbb{R}$, the function ${}^{\mathrm{III}}_{9}C^{f,g}_{\delta,\omega}$ defined by



(a) Class IV (b) Class V

(c) Class VI (d) Class VII

Figure 8.4: Transformations between the families from the same class.

$$
{}_{9}^{\mathrm{III}}C_{\delta,\omega}^{f,g}(x,y) =
$$

$$
\begin{cases} \dfrac{x+y-1}{2y-1}\,\delta(y) + \dfrac{y-x}{2y-1}\,\omega(1-y) - (y-x)(x+y-1)f(y) & \text{, if } (x,y)\in T_1\cup T_3\,, \\ \dfrac{x+y-1}{2x-1}\,\delta(x) - \dfrac{y-x}{2x-1}\,\omega(x) + (y-x)(x+y-1)g(x) & \text{, if } (x,y)\in T_2\cup T_4\,, \end{cases} \tag{8.10}
$$

where the convention $\frac{0}{0} := \delta(1/2)$ is adopted, is a semiquadratic function with diagonal section δ and opposite diagonal section ω. Note that if the functions f and g are continuous, then ${}_{9}^{\mathrm{III}}C_{\delta,\omega}^{f,g}(x,y)$ is continuous. Note also that for $f = g = 0$, the definition of an orbital semilinear function (see Chapter 6).

We now state the conditions to be fulfilled by the functions δ, ω, f and g such that ${}_{9}^{\mathrm{III}}C_{f,g}^{\delta,\omega}$ is a copula.

Let us consider the functions $\varphi_{\delta,\omega}$ and $\psi_{\delta,\omega}$ defined as in Chapter 6.

Proposition 8.3. *Let δ and ω be diagonal and opposite diagonal functions such that $\delta(1/2) = \omega(1/2)$. Let $f, g : [0,1] \to \mathbb{R}$ be two absolutely continuous functions. The function ${}_9^{\mathrm{III}}C_{f,g}^{\delta,\omega}$ defined in (8.10) is a copula with diagonal section δ and opposite diagonal section ω if and only if*

(i) $f(0) = g(0) = f(1) = g(1) = 0\,,$

(ii) $\psi'_{\delta,\omega}(t) - |(1-2t)f'(t)| \geq 0\,,$

(iii) $\varphi'_{\delta,\omega}(t) - |(1-2t)g'(t)| \geq 0\,,$ *for all $t \in\,]0,1]$ where the derivatives exist,*

(iv) *for all $t \in\,]0,1/2[\,$, it holds that*

$$\delta'(t) \geq \varphi_{\delta,\omega}(t) + \psi_{\delta,\omega}(t) + (2t-1)(f(t)+g(t))\,, \text{ and}$$

$$\omega'(t) \leq \psi_{\delta,\omega}(1-t) - \varphi_{\delta,\omega}(t) + (2t-1)(f(1-t)+g(t))\,,$$

where the derivatives exist,

(v) *for all $t \in\,]1/2,1]$, it holds that*

$$\delta'(t) \leq \varphi_{\delta,\omega}(t) + \psi_{\delta,\omega}(t) + (2t-1)(f(t)+g(t))\,, \text{ and}$$

$$\omega'(t) \geq \psi_{\delta,\omega}(1-t) - \varphi_{\delta,\omega}(t) + (2t-1)(f(1-t)+g(t))\,,$$

where the derivatives exist.

Proof. Let $C = {}_9^{\mathrm{III}}C_{f,g}^{\delta,\omega}$. The boundary conditions $C(t,0) = 0$, $C(0,t) = 0$, $C(t,1) = t$ and $C(1,t) = t$ for all $t \in [0,1]$ immediately lead to the conditions $f(0) = g(0) = f(1) = g(1) = 0$. Therefore, it suffices to prove that the 2-increasingness of C is equivalent to conditions (ii)–(v).

Suppose that C is 2-increasing. For any rectangle $R = [x_1,x_2] \times [y_1,y_2] \subseteq T_1 \cup T_3$, it then holds that $V_C(R) \geq 0$, i.e.

$$\begin{aligned} &(x_2-x_1)\left(\frac{\delta(y_1)}{2y_1-1} - \frac{\omega(1-y_1)}{2y_1-1} + f(y_1)(1-x_1-x_2)\right) \\ -\ &(x_2-x_1)\left(\frac{\delta(y_2)}{2y_2-1} - \frac{\omega(1-y_2)}{2y_2-1} + f(y_2)(1-x_1-x_2)\right) \geq 0\,, \end{aligned}$$

or, equivalently,

$$\psi_{\delta,\omega}(y_2) - \psi_{\delta,\omega}(y_1) + (f(y_1) - f(y_2))(1-x_1-x_2) \geq 0\,. \tag{8.11}$$

Dividing by $y_2 - y_1$ and taking the limits $x_2 \to x_1$ and $y_2 \to y_1$, inequality (8.11) becomes

$$\psi'_{\delta,\omega}(y_1) - (1-2x_1)f'(y_1) \geq 0\,. \tag{8.12}$$

Since the left-hand side of inequality (8.12) is linear in x_1, this condition is equivalent to requiring that it holds for $x_1 = y_1$ and $x_1 = 1 - y_1$, i.e.

$$\psi'_{\delta,\omega}(y_1) - (1-2y_1)f'(y_1) \geq 0 \quad \text{and} \quad \psi'_{\delta,\omega}(y_1) + (1-2y_1)f'(y_1) \geq 0\,,$$

or, equivalently,

$$\psi'_{\delta,\omega}(y_1) - |(1-2y_1)f'(y_1)| \geq 0\,. \tag{8.13}$$

Similarly, the fact that $V_C(R) \geq 0$ for any rectangle located in $T_2 \cup T_4$ implies that inequality (8.13) also holds for the functions $\varphi_{\delta,\omega}$ and g. Hence, condition (iii) follows.

The fact that $V_C(R) \geq 0$ for any square $R = [x_1, x_2] \times [x_1, x_2]$ such that $x_2 \leq 1/2$ is equivalent to

$$\begin{aligned} V_C(R) &= C(x_1,x_1) + C(x_2,x_2) - C(x_1,x_2) - C(x_2,x_1) \\ &= \delta(x_1) + \delta(x_2) - 2\frac{x_1+x_2-1}{2x_1-1}\delta(x_1) + \frac{x_2-x_1}{2x_1-1}(\omega(x_1)+\omega(1-x_1)) \\ &\quad +(x_2-x_1)(1-x_1-x_2)(f(x_1)+g(x_1)) \geq 0\,, \end{aligned}$$

or, equivalently,

$$\delta(x_2)-\delta(x_1)+(x_2-x_1)\left(-\varphi_{\delta,\omega}(x_1) - \psi_{\delta,\omega}(x_1) + (1-x_1-x_2)(f(x_1)+g(x_1))\right) \geq 0\,.$$

Dividing by $x_2 - x_1$ and taking the limit $x_2 \to x_1$, it follows that

$$\delta'(x_1) - \varphi_{\delta,\omega}(x_1) - \psi_{\delta,\omega}(x_1) + (1-2x_1)(f(x_1)+g(x_1)) \geq 0\,.$$

or, equivalently,

$$\delta'(x_1) \geq \varphi_{\delta,\omega}(x_1) + \psi_{\delta,\omega}(x_1) + (2x_1-1)(f(x_1)+g(x_1))\,.$$

The fact that $V_C(R) \geq 0$ for any square $R = [x_1, x_2] \times [1-x_2, 1-x_1]$ such that $x_2 \leq 1/2$ is equivalent to

$$\begin{aligned} V_C(R) &= C(x_1,1-x_2) + C(x_2,1-x_1) - C(x_1,1-x_1) - C(x_2,1-x_2) \\ &= \frac{x_2-x_1}{1-2x_1}(\delta(x_1)+\delta(1-x_1)) - 2\frac{x_1+x_2-1}{1-2x_1}\omega(x) \\ &\quad +(x_2-x_1)(x_1+x_2-1)(f(x_1)+g(x_1)) - \omega(x_1) - \omega(x_2) \geq 0\,, \end{aligned}$$

or, equivalently,

$$\begin{aligned} &\omega(x_1) - \omega(x_2) - (x_2-x_1)\left(-\psi_{\delta,\omega}(1-x_1) + \varphi_{\delta,\omega}(x_1)\right) \\ &+\ (x_2-x_1)(x_1+x_2-1)(f(1-x_1)+g(x_1)) \geq 0\,. \end{aligned}$$

Dividing by $x_2 - x_1$ and taking the limit $x_2 \to x_1$, it follows that

$$-\omega'(x_1) + \psi_{\delta,\omega}(1-x_1) - \varphi_{\delta,\omega}(x_1) - (1-2x_1)(f(1-x_1) + g(x_1)) \geq 0\,,$$

or, equivalently,

$$\omega'(x_1) \leq \psi_{\delta,\omega}(1-x_1) - \varphi_{\delta,\omega}(x_1) + (2x_1-1)(f(1-x_1) + g(x_1))\,.$$

Thus, condition (iv) follows. Similarly, the fact that $V_C(R) \geq 0$ for any square $R = [x_1, x_2] \times [x_1, x_2]$ (resp. $R = [x_1, x_2] \times [1-x_2, 1-x_1]$) such that $x_1 \geq 1/2$ implies condition (v).

Now suppose that conditions (ii)–(v) are satisfied. Due to the additivity of volumes, it suffices to consider a restricted number of cases to prove the 2-increasingness of C. Let $R = [a_1, b_1] \times [a_2, b_2]$ be a rectangle located in $T_1 \cup T_2$. Since condition (ii) is satisfied, inequality (8.12) follows and it holds that

$$\int_{a_2}^{b_2} \mathrm{d}y_1 \int_{a_1}^{b_1} \left(\psi'_{\delta,\omega}(y_1) - (1-2x_1)f'(y_1)\right) \mathrm{d}x_1 \geq 0\,.$$

Computing the above integral, the latter inequality becomes

$$(b_1 - a_1)\left(\psi_{\delta,\omega}(b_2) - \psi_{\delta,\omega}(a_2) - (1-a_1-b_1)(f(b_2) - f(a_2))\right) \geq 0\,,$$

or, equivalently, $V_C(R) \geq 0$. Similarly, one can verify that $V_C(R) \geq 0$ for any rectangle $R = [a_1, b_1] \times [a_2, b_2]$ located in $T_2 \cup T_4$. Let $S = [a, b] \times [a, b]$ be a square such that $b \leq 1/2$. Due to condition (iv), it holds that

$$\delta'(x_1) - \varphi_{\delta,\omega}(x_1) - \psi_{\delta,\omega}(x_1) - (2x_1 - 1)(f(x_1) + g(x_1)) \geq 0\,.$$

for all $x_1 \in]0, 1/2]$, which implies that

$$\tilde{I}_1 = \int_a^b (\delta'(x_1) - \varphi_{\delta,\omega}(x_1) - \psi_{\delta,\omega}(x_1) - (2x_1 - 1)(f(x_1) + g(x_1))\,\mathrm{d}x_1 \geq 0\,.$$

Using inequality (8.12), it follows that

$$\tilde{I}_2 = \int_a^b \int_a^{x_1} \left(\psi'_{\delta,\omega}(y_1) - (1-2x_1)f'(y_1)\right) \mathrm{d}y_1 \mathrm{d}x_1 \geq 0\,.$$

As inequality (8.12) also holds for the functions $\varphi_{\delta,\omega}$ and g, it follows after exchanging the variables x_1 and y_1 that

$$\tilde{I}_3 = \int_a^b \int_a^{x_1} \left(\varphi'_{\delta,\omega}(x_1) - (1-2y_1)g'(x_1)\right) \mathrm{d}x_1 \mathrm{d}y_1 \geq 0\,.$$

Computing the above integrals and setting $I = \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3$, it follows that

$$I = \delta(b) - \delta(a) - (a-b)(\varphi_{\delta,\omega}(a) + \psi_{\delta,\omega}(a) - (1-a-b)(f(a)+g(a))) \geq 0\,,$$

or, equivalently,

$$I = V_C(S) \geq 0\,.$$

Similarly, one can verify that $V_C(R) \geq 0$ for any rectangle $R = [a,b] \times [a,b]$ when $a \geq 1/2$. □

Example 8.3. *Suppose that the functions f and g are linear. Condition (i) of Proposition 8.3 implies that $f(x) = g(x) = 0$ for all $x \in [0,1]$. The corresponding family of semiquadratic copulas coincides with the family of orbital semilinear copulas [65].*

Example 8.4. *Consider the diagonal section and the opposite diagonal section of the product copula. Let $f_\lambda, g_\lambda : [0,1] \to \mathbb{R}$ be the functions defined by $f_\lambda(x) = -g_\lambda(x) = \lambda \min(x, 1-x)$ for all $x \in [0,1]$, with $\lambda \in [-1,1]$. One easily verifies that the conditions of Propositions 8.3 are satisfied and the corresponding family of semiquadratic functions ${}_{9}^{\mathrm{III}}C_{\delta_\Pi,\omega_\Pi}^{f_\lambda,g_\lambda}$ is a family of semiquadratic copulas, and is given by*

$${}_{9}^{\mathrm{III}}C_{\delta_\Pi,\omega_\Pi}^{f_\lambda,g_\lambda}(x,y) = \begin{cases} xy - \lambda(y-x)(x+y-1)\min(y,1-y) & , \textit{if } (x,y) \in T_1 \cup T_3\,, \\[2ex] xy - \lambda(y-x)(x+y-1)\min(x,1-x) & , \textit{otherwise}\,, \end{cases}$$

with $\lambda \in [-1,1]$.

Class IV

For any diagonal function δ and opposite diagonal function ω such that $\delta(1/2) = \omega(1/2)$, and any two functions $f, g : [0,1] \to \mathbb{R}$, the function ${}_{10}^{\mathrm{IV}}C_{\delta,\omega}^{f,g} : [0,1]^2 \to [0,1]$ defined in (8.9) where the convention $\frac{0}{0} := 0$ is adopted, is a semiquadratic function with diagonal section δ and opposite diagonal section ω. Note that if the functions f and g are continuous, then ${}_{10}^{\mathrm{IV}}C_{\delta,\omega}^{f,g}$ is continuous.

We now state the conditions to be fulfilled by the functions δ, ω, f and g such that ${}_{10}^{\mathrm{IV}}C_{\delta,\omega}^{f,g}$ is a copula. Let us consider the functions λ_ω and μ_ω defined as in Chapter 3.

Proposition 8.4. *Let δ and ω be diagonal and opposite diagonal functions such that $\delta(1/2) = \omega(1/2)$. Let $f, g : [0,1] \to \mathbb{R}$ be two absolutely continuous functions. The function ${}^{\mathrm{IV}}_{10}C^{f,g}_{\delta,\omega}$ defined in (8.9) is a copula with diagonal section δ and opposite diagonal section ω if and only if*

(i) $f(0) = g(0) = f(1) = g(1) = 0\,,$

(ii) *for all $t \in]0,1[\,$, it holds that*

$$\varphi'_{\delta,\omega}(t) - |(1-2t)f'(t)| \geq 0\,, \quad \psi'_{\delta,\omega}(t) - |(1-2t)g'(t)| \geq 0\,,$$

where the derivatives exist,

(iii) *for all $t \in]0,1/2]$, it holds that*

$$\begin{aligned} -\mu'_{\omega}(t) &\leq g(t) - t\,|g'(t)|\,, \\ \mu'_{\omega}(1-t) &\leq f(t) - t\,|f'(t)|\,, \\ \delta'(t) &\geq \varphi_{\delta,\omega}(t) + \psi_{\delta,\omega}(t) + (2t-1)(f(t)+g(t))\,, \\ \lambda_{\omega}(t) &\leq 1 - \varphi_{\delta,\omega}(t) - (1-t)g(t)\,, \end{aligned}$$

where the derivatives exist,

(iv) *for all $t \in [1/2, 1[\,$, it holds that*

$$\begin{aligned} \mu'_{\delta}(t) &\geq \max(f(t) + (1-t)\,|f'(t)|\,, g(t) + (1-t)\,|g'(t)|)\,, \\ \delta'(t) &\leq \varphi_{\delta,\omega}(t) + \psi_{\delta,\omega}(t) + (2t-1)(f(t)+g(t))\,, \\ \mu_{\delta}(t) &\leq \varphi_{\delta,\omega}(t) + tg(t)\,, \end{aligned}$$

where the derivatives exist.

Class V

For any diagonal function δ and opposite diagonal function ω such that $\delta(1/2) = \omega(1/2)$, and any function $f : [0,1] \to \mathbb{R}$, the function ${}^{\mathrm{V}}_{15}C^{f}_{\delta,\omega} : [0,1]^2 \to [0,1]$ defined by ${}^{\mathrm{V}}_{15}C^{f}_{\delta,\omega}(x,y) =$

$$\begin{cases} \dfrac{x+y-1}{2y-1}\,\delta(y) + \dfrac{y-x}{2y-1}\,\omega(1-y) - (y-x)(x+y-1)f(y) & \\ & , \text{ if } (x,y) \in T_1 \cup T_3\,, \\ \dfrac{x}{y}\,\delta(y) - x(y-x)f(y) & , \text{ if } (x,y) \in T_2 \text{ and } y \leq 1/2\,, \\ \dfrac{x}{1-y}\,\omega(1-y) + x(x+y-1)f(y) & , \text{ if } (x,y) \in T_2 \text{ and } y \geq 1/2\,, \\ x+y-1+\dfrac{1-x}{y}\,\omega(1-y) - (1-x)(x+y-1)f(y) & \\ & , \text{ if } (x,y) \in T_4 \text{ and } y \leq 1/2\,, \\ y - \dfrac{1-x}{1-y}\,(y-\delta(y)) + (1-x)(y-x)f(y) & , \text{ if } (x,y) \in T_4 \text{ and } y \geq 1/2\,, \end{cases} \tag{8.14}$$

where the convention $\frac{0}{0} := 0$ is adopted, is a semiquadratic function with diagonal section δ and opposite diagonal section ω. Note that if the function f is continuous, then ${}^{\mathrm{V}}_{15}C^{f}_{\delta,\omega}$ is continuous. Note also that for $f = 0$, the definition of a horizontal semilinear function (see Chapter 6) is retrieved.

We now state the conditions to be fulfilled by the functions δ, ω and f such that ${}^{\mathrm{V}}_{15}C^{f}_{\delta,\omega}$ is a copula.

Proposition 8.5. *Let δ and ω be diagonal and opposite diagonal functions such that $\delta(1/2) = \omega(1/2)$. Let $f : [0,1] \to \mathbb{R}$ be an absolutely continuous function. The function ${}^{\mathrm{V}}_{15}C^{f}_{\delta,\omega}$ defined in (8.14) is a copula with diagonal section δ and opposite diagonal section ω if and only if*

(i) $f(0) = f(1) = 0\,,$

(ii) *for all $t \in]0,1[$, it holds that*

$$\varphi'_{\delta,\omega}(t) \geq |(1-2t)f'(t)|\,,$$

where the derivatives exist,

(ii) *for all $t \in]0,1/2]$, it holds that*

$$\begin{aligned} \lambda'_\delta(t) &\geq f(t) + t\,|f'(t)|\,, \\ \mu'_\omega(1-t) &\leq f(t) - t\,|f'(t)|\,, \\ \lambda_\delta(t) &\geq \psi_{\delta,\omega}(t) - (1-t)f(t)\,, \\ \lambda_\omega(t) &\leq \psi_{\delta,\omega}(1-t) - (1-t)f(1-t)\,, \end{aligned}$$

where the derivatives exist,

(iii) *for all* $t \in [1/2, 1[$, *it holds that*

$$\begin{aligned} \mu_\delta'(t) &\geq f(t) + t\,|f'(t)|\,, \\ -\lambda_\omega'(1-t) &\leq f(t) - (1-t)\,|f'(t)|\,, \\ \mu_\delta(t) &\leq \psi_{\delta,\omega}(t) + tf(t)\,, \\ \mu_\omega(t) &\leq 1 - \psi_{\delta,\omega}(1-t) - tf(1-t)\,, \end{aligned}$$

where the derivatives exist.

Example 8.5. *Suppose that the function f is linear. Condition (i) of Proposition 8.5 implies that $f(x) = 0$ for all $x \in [0, 1]$. The corresponding family of semiquadratic copulas coincides with the family of horizontal semilinear copulas (see Chapter 6).*

Class VI

For any diagonal function δ and opposite diagonal function ω such that $\delta(1/2) = \omega(1/2)$, and any two functions $f, g : [0,1] \to \mathbb{R}$, the function ${}^{\mathrm{VI}}_{16}C^{f,g}_{\delta,\omega} : [0,1]^2 \to [0,1]$ defined by ${}^{\mathrm{VI}}_{16}C^{f,g}_{\delta,\omega}(x,y) =$

$$\begin{cases} \dfrac{x+y-1}{2y-1}\delta(y) + \dfrac{y-x}{2y-1}\omega(1-y) - (y-x)(x+y-1)f(y) & \\ \qquad\qquad\qquad\qquad\qquad\qquad , \text{ if } (x,y) \in T_1\,, \\ \dfrac{x+y-1}{2x-1}\,\delta(x) - \dfrac{y-x}{2x-1}\,\omega(x) + (y-x)(x+y-1)g(x) & \\ \qquad\qquad\qquad\qquad\qquad\qquad , \text{ if } (x,y) \in T_2\,, \\ x+y-1+\dfrac{1-y}{x}\,\omega(x) - (1-y)(x+y-1)g(x) & \\ \qquad\qquad\qquad\qquad\qquad\qquad , \text{ if } (x,y) \in T_3 \text{ and } x \leq 1/2\,, \\ x - \dfrac{1-y}{1-x}\,(x-\delta(x)) - (1-y)(y-x)g(x) & , \text{ if } (x,y) \in T_3 \text{ and } x \geq 1/2\,, \\ y - \dfrac{1-x}{1-y}\,(y-\delta(y)) + (1-x)(y-x)f(y) & , \text{ if } (x,y) \in T_4 \text{ and } y \geq 1/2\,, \\ x+y-1+\dfrac{1-x}{y}\,\omega(1-y) - (1-x)(x+y-1)f(y) & \\ \qquad\qquad\qquad\qquad\qquad\qquad , \text{ if } (x,y) \in T_4 \text{ and } y \leq 1/2\,, \end{cases} \tag{8.15}$$

where the convention $\frac{0}{0} := 0$ is adopted, is a semiquadratic function with diagonal section δ and opposite diagonal section ω. Note that if the functions f and g are continuous, then ${}^{\mathrm{VI}}_{16}C^{f,g}_{\delta,\omega}$ is continuous.

We now state the conditions to be fulfilled by the functions δ, ω, f and g such that ${}^{\mathrm{VI}}_{16}C^{f,g}_{\delta,\omega}$ is a copula.

Proposition 8.6. *Let δ and ω be diagonal and opposite diagonal functions such that $\delta(1/2) = \omega(1/2)$. Let $f, g : [0,1] \to \mathbb{R}$ be two absolutely continuous functions. The function ${}^{\mathrm{VI}}_{16}C^{f,g}_{\delta,\omega}$ defined in (8.15) is a copula with diagonal section δ and opposite diagonal section ω if and only if*

(i) $f(0) = g(0) = f(1) = g(1) = 0\,,$

(ii) *for all $t \in]0, 1/2]$, it holds that*

$$\begin{aligned}
\varphi'_{\delta,\omega}(t) &\geq |(1-2t)f'(t)|\,,\\
\psi'_{\delta,\omega}(x) &\geq |(1-2x)g'(x)|\,,\\
\lambda'_{\omega}(t) &\leq g(t) - t\,|g'(t)|\,,\\
\delta'(t) &\geq \varphi_{\delta,\omega}(t) + \psi_{\delta,\omega}(t) + (2t-1)(f(t)+g(t))\,,\\
\lambda_{\omega}(t) &\leq 1 - \varphi_{\delta,\omega}(t) - (1-t)g(t)\,,
\end{aligned}$$

where the derivatives exist,

(iii) *for all $t \in [1/2, 1[\,$, it holds that*

$$\begin{aligned}
\mu'_{\delta}(t) &\geq g(x) + (1-x)\,|g'(x)|\,,\\
\delta'(t) &\leq \varphi_{\delta,\omega}(t) + \psi_{\delta,\omega}(t) + (2t-1)(f(t)+g(t))\,,\\
(1-t)^2\left(\frac{2t-1-\delta(t)}{(1-t)^2}\right)' &\leq f(t) + g(t)\,,\\
\mu_{\omega}(t) &\leq 1 - \psi_{\delta,\omega}(1-t) - tf(1-t)\,,
\end{aligned}$$

where the derivatives exist.

Class VII

For any diagonal function δ and opposite diagonal function ω such that $\delta(1/2) = \omega(1/2)$, and any two functions $f, g : [0,1] \to \mathbb{R}$, the function ${}^{\mathrm{VII}}_{20}C^{f,g}_{\delta,\omega} : [0,1]^2 \to [0,1]$

defined by ${}^{\mathrm{VII}}_{20}C^{f,g}_{\delta,\omega}(x,y) =$

$$
\begin{cases}
\dfrac{y}{x}\,\delta(x) + y(y-x)g(x) & \text{, if } (x,y)\in T_1 \text{ and } x\le 1/2\,,\\[2mm]
\dfrac{y}{1-x}\,\omega(x) + y(x+y-1)g(x) & \text{, if } (x,y)\in T_1 \text{ and } x\ge 1/2\,,\\[2mm]
\dfrac{x+y-1}{2x-1}\,\delta(x) - \dfrac{y-x}{2x-1}\,\omega(x) + (y-x)(x+y-1)g(x) & \\
 & \text{, if } (x,y)\in T_2\,,\\[2mm]
x+y-1+\dfrac{1-y}{x}\,\omega(x) - (1-y)(x+y-1)g(x) & \\
 & \text{, if } (x,y)\in T_3 \text{ and } x\le 1/2\,,\\[2mm]
x - \dfrac{1-y}{1-x}\,(x-\delta(x)) - (1-y)(y-x)g(x) & \text{, if } (x,y)\in T_3 \text{ and } x\ge 1/2\,,\\[2mm]
x+y-1+\dfrac{1-x}{y}\,\omega(1-y) - (1-x)(x+y-1)f(y) & \\
 & \text{, if } (x,y)\in T_4 \text{ and } y\le 1/2\,,\\[2mm]
y - \dfrac{1-x}{1-y}\,(y-\delta(y)) + (1-x)(y-x)f(y) & \text{, if } (x,y)\in T_4 \text{ and } y\ge 1/2\,,
\end{cases}
\tag{8.16}
$$

where the convention $\frac{0}{0} := 0$ is adopted, is a semiquadratic function with diagonal section δ and opposite diagonal section ω. Note that if the functions f and g are continuous, then ${}^{\mathrm{VII}}_{20}C^{f,g}_{\delta,\omega}(x,y)$ is continuous.

We now state the conditions to be fulfilled by the functions δ, ω, f and g such that ${}^{\mathrm{VII}}_{20}C^{f,g}_{\delta,\omega}$ is a copula.

Proposition 8.7. *Let δ and ω be diagonal and opposite diagonal functions such that $\delta(1/2) = \omega(1/2)$. Let $f, g : [0,1] \to \mathbb{R}$ be two absolutely continuous functions. The function ${}^{\mathrm{VII}}_{20}C^{f,g}_{\delta,\omega}$ defined in (8.16) is a copula with diagonal section δ and opposite diagonal section ω if and only if*

(i) $f(0) = g(0) = f(1) = g(1) = 0\,,$

(ii) *for all $t \in]0, 1/2]$, it holds that*

$$
\begin{aligned}
\varphi'_{\delta,\omega}(t) &\ge |(1-2t)f'(t)|\,,\\
\lambda'_{\delta}(t) &\ge g(t) + t\,|g'(t)|\,,\\
\lambda'_{\omega}(t) &\le g(t) - t\,|g'(t)|\,,\\
\lambda_{\delta}(t) &\ge \varphi_{\delta,\omega}(t) - (1-t)g(t)\,,\\
\lambda_{\omega}(t) &\le 1 - \varphi_{\delta,\omega}(t) - (1-t)g(t)\,,
\end{aligned}
$$

where the derivatives exist,

(iii) *for all* $t \in [1/2, 1[$, *it holds that*

$$\begin{aligned}
\mu'_\delta(t) &\geq \max(f(t) + (1-t)\,|f'(t)|\,, g(t) + (1-t)\,|g'(t)|)\,,\\
-\mu'_\omega(t) &\leq g(t) - (1-t)\,|g'(t)|\ ,\\
-\lambda'_\omega(1-t) &\leq f(t) - (1-t)\,|f'(t)|\ ,\\
(1-t)^2\left(\frac{2t-1-\delta(t)}{(1-t)^2}\right)' &\leq f(t) + g(t)\,,\\
(1-t)^2\left(\frac{\omega(t)-t}{t^2}\right)' &\geq f(1-t) + g(t)\,,
\end{aligned}$$

where the derivatives exist.

Class VIII

For any diagonal function δ and opposite diagonal function ω such that $\delta(1/2) = \omega(1/2)$, and any two functions $f, g : [0,1] \to \mathbb{R}$, the function ${}^{\mathrm{VIII}}_{24}C^{f,g}_{\delta,\omega} : [0,1]^2 \to [0,1]$ defined by ${}^{\mathrm{VIII}}_{24}C^{f,g}_{\delta,\omega}(x,y) =$

$$\begin{cases}
\dfrac{y}{x}\,\delta(x) + y(y-x)g(x) & \text{, if } (x,y) \in T_1 \text{ and } x \leq 1/2\,,\\
\dfrac{y}{1-x}\,\omega(x) + y(x+y-1)g(x) & \text{, if } (x,y) \in T_1 \text{ and } x \geq 1/2\,,\\
\dfrac{x}{y}\,\delta(y) - x(y-x)f(y) & \text{, if } (x,y) \in T_2 \text{ and } y \leq 1/2\,,\\
\dfrac{x}{1-y}\,\omega(1-y) + x(x+y-1)f(y) & \text{, if } (x,y) \in T_2 \text{ and } y \geq 1/2\,,\\
x+y-1+\dfrac{1-y}{x}\,\omega(x) - (1-y)(x+y-1)g(x) & \\
 & \text{, if } (x,y) \in T_3 \text{ and } x \leq 1/2\,,\\
x - \dfrac{1-y}{1-x}\,(x-\delta(x)) - (1-y)(y-x)g(x) & \text{, if } (x,y) \in T_3 \text{ and } x \geq 1/2\,,\\
y - \dfrac{1-x}{1-y}\,(y-\delta(y)) + (1-x)(y-x)f(y) & \text{, if } (x,y) \in T_4 \text{ and } y \geq 1/2\,,\\
x+y-1+\dfrac{1-x}{y}\,\omega(1-y) - (1-x)(x+y-1)f(y) & \\
 & \text{, if } (x,y) \in T_4 \text{ and } y \leq 1/2\,,
\end{cases} \tag{8.17}$$

where the convention $\frac{0}{0} := 0$ is adopted, is a semiquadratic function with diagonal section δ and opposite diagonal section ω. Note that if the functions f and g are continuous, then ${}^{\mathrm{VIII}}_{24}C^{f,g}_{\delta,\omega}$ is continuous. Note also that for $f = g = 0$, the definition of a radial semilinear function (see Chapter 6) is retrieved.

We now state the conditions to be fulfilled by the functions δ, ω, f and g such that ${}^{\mathrm{VIII}}_{24}C^{f,g}_{\delta,\omega}$ is a copula.

Proposition 8.8. *Let δ and ω be diagonal and opposite diagonal functions such that $\delta(1/2) = \omega(1/2)$. Let $f, g : [0,1] \to \mathbb{R}$ be two absolutely continuous functions. The function ${}^{\mathrm{VIII}}_{24}C^{f,g}_{\delta,\omega}$ defined in (8.17) is a copula with diagonal section δ and opposite diagonal section ω if and only if*

(i) $f(0) = g(0) = f(1) = g(1) = 0\,,$

(ii) *for all $t \in]0, 1/2]$, it holds that*

$$\begin{aligned}
\lambda'_\delta(t) &\geq \max(f(t) + t\,|f'(t)|\,, g(t) + t\,|g'(t)|)\,,\\
\lambda'_\omega(t) &\leq g(t) - t\,|g'(t)|\;,\\
\mu'_\omega(1-t) &\leq f(t) - t\,|f'(t)|\;,\\
t^2\left(\frac{\delta(t)}{t^2}\right)' &\leq f(t) + g(t)\,,\\
(1-t)^2\left(\frac{1-t-\omega(t)}{(1-t)^2}\right)' &\geq f(t) + g(t)\,,
\end{aligned}$$

where the derivatives exist,

(iii) *for all $t \in [1/2, 1[$, it holds that*

$$\begin{aligned}
\mu'_\delta(t) &\geq \max(f(t) + (1-t)\,|f'(t)|\,, g(t) + (1-t)\,|g'(t)|)\,,\\
-\mu'_\omega(t) &\leq g(t) - (1-t)\,|g'(t)|\;,\\
-\lambda'_\omega(1-t) &\leq f(t) - (1-t)\,|f'(t)|\;,\\
(1-t)^2\left(\frac{2t-1-\delta(t)}{(1-t)^2}\right)' &\leq f(t) + g(t)\,,\\
(1-t)^2\left(\frac{\omega(t)-t}{t^2}\right)' &\geq f(1-t) + g(t)\,,
\end{aligned}$$

where the derivatives exist.

Example 8.6. *Suppose that the functions f and g are linear. Condition (i) of Proposition 8.8 implies that $f(x) = g(x) = 0$ for all $x \in [0,1]$. The corresponding family of semiquadratic copulas coincides with the family of radial semilinear copulas (see Chapter 6).*

Example 8.7. *Consider the diagonal section and the opposite diagonal section of the product copula. Let $f_\lambda, g_\lambda : [0,1] \to \mathbb{R}$ be the functions defined by $f_\lambda(x) = -g_\lambda(x) = \lambda x(1-x)$ for all $x \in [0,1]$, with $\lambda \in [-1,1]$. One easily verifies that the conditions of Propositions 8.8 are satisfied and the corresponding family of semiquadratic functions ${}_{24}^{\mathrm{VIII}}C_{\delta_\Pi,\omega_\Pi}^{f_\lambda,g_\lambda}$ is a family of semiquadratic copulas.*

As the product copula is a typical example of all types of semilinear copulas based on horizontal and vertical interpolation, we show in the following example that the Farlie–Gumbel–Morgenstern family of copulas is a typical example of all types of semiquadratic copulas based on horizontal and vertical interpolation. Just as the product copula is the only copula that is linear in both variables, the Farlie–Gumbel–Morgenstern family contains all copulas that are quadratic in both variables [88, 96] (see also Chapter 1).

Example 8.8. *Let δ_λ and ω_λ be the diagonal and opposite sections of a Farlie–Gumbel–Morgenstern copula, i.e. $\delta_\lambda(t) = x^2(1+\lambda(1-x)^2)$ and $\omega_\lambda(t) = x(1-x)(1+\lambda x(1-x))$ for all $t \in [0,1]$, with $\lambda \in [-1,1]$. Let $f_\lambda, g_\lambda : [0,1] \to [0,1]$ be defined by $f_\lambda(x) = g_\lambda(x) = -\lambda x(1-x)$ for all $x \in [0,1]$. One easily verifies that the conditions of Propositions 8.1–8.8 are satisfied and all corresponding semiquadratic functions ${}_{1}^{\mathrm{I}}C_{\delta_\lambda}^{f_\lambda,g_\lambda}$–${}_{24}^{\mathrm{VIII}}C_{\delta_\lambda,\omega_\lambda}^{f_\lambda,g_\lambda}$ coincide with the given Farlie–Gumbel–Morgenstern copula.*

PART III

APPENDICES

General conclusions

In this chapter, the main conclusions that can be drawn from the work in this dissertation are summarized.

We have introduced the class of conic aggregation functions and have characterized the subsets of $[0,1]^n$ that can be the zero-set of a conic aggregation function. We have focused our attention on the binary case, and have identified the necessary and sufficient conditions on the upper boundary curve of the zero-set of a conic aggregation function in order to have a conic quasi-copula or a (singular) conic copula. Moreover, we have investigated basic aggregations, such as minimum, maximum and convex sums, of conic (quasi-)copulas.

We have introduced biconic aggregation functions with a given diagonal (resp. opposite diagonal) section. We have also characterized the classes of biconic semi-copulas, quasi-copulas and copulas with a given diagonal (resp. opposite diagonal) section. The t-norms (resp. copulas) $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ turn out to be the only 1-Lipschitz (resp. associative) biconic t-norms (resp. copulas) with a given diagonal section. Moreover, a copula that is a biconic copula with a given diagonal section as well as with a given opposite diagonal section turns out to be a convex sum of $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$.

We have introduced upper conic, lower conic and biconic functions with a given section. We have also characterized the classes of upper conic, lower conic and biconic (semi-, quasi-)copulas with a given section. Generalized convexity has played an important role when characterizing upper conic, lower conic and biconic copulas with a given section.

We have introduced ortholinear (resp. paralinear) functions. We have also characterized the classes of ortholinear (resp. paralinear) (semi- and quasi-)copulas. Ortholinear copulas supported on a set with Lebesgue measure zero and copulas that are ortholinear as well as paralinear turn out to be a convex sum of $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$.

We have introduced four new types of semilinear copulas and have derived necessary and sufficient conditions on given diagonal and opposite diagonal functions such that a copula of one of the considered types exists that has these functions as diagonal and opposite diagonal sections. The most interesting new copulas are the so-called orbital semilinear copulas which are obtained based on linear interpolation on segments connecting points on the diagonal and opposite diagonal of the unit square solely. The extreme copulas M and W are both orbital semilinear copulas, as well as the product copula Π. Moreover, the smallest copula whose diagonal section coincides with the diagonal section of the product copula and also the greatest copula whose opposite diagonal section coincides with the opposite diagonal section of the product copula turn out to be orbital semilinear copulas different from Π, as follows from Proposition 6.16.

We have introduced the class of lower semiquadratic functions. Moreover, we have identified the necessary and sufficient conditions on a diagonal function and two auxiliary real functions u and v to obtain a copula that has this diagonal function as diagonal section. The class of lower semilinear copulas turns out to be a subclass of lower semiquadratic copulas. Also, we have characterized the extreme lower semiquadratic copulas with a given diagonal section.

We have introduced eight classes of semiquadratic functions with given diagonal and/or opposite diagonal sections. Moreover, we have identified for each class the necessary and sufficient conditions on the given diagonal and/or opposite diagonal functions and two auxiliary real functions f and g to obtain a copula that has these diagonal and/or opposite diagonal functions as diagonal and/or opposite diagonal sections. The use of the transformations π, φ, φ_1, φ_2, σ, σ_1 and σ_2 has considerably eased the effort compared to the semilinear case.

Summary

Summary for Dutch translation

Conjunctive aggregation functions have been extensively used in fuzzy logic and fuzzy set theory. They turn out to be the appropriate operations for modelling the fuzzy logical connective "and". Particular subclasses of conjunctive aggregation functions such as triangular norms (t-norms), semi-copulas, quasi-copulas and copulas have received ample attention from researchers in reliability theory, fuzzy set theory, probability theory and statistics.

Several methods to construct conjunctive aggregation functions have been introduced in the literature. Some of these methods are based on linear or quadratic interpolation on segments connecting lines in the unit square to the sides of the unit square. Such lines can be the diagonal, the opposite diagonal, a horizontal straight line, a vertical straight line or the graph that represents a decreasing function. We introduce the notions of semilinear and semiquadratic aggregation functions that generalize all aggregation functions that are obtained based on such methods. More specifically, an aggregation function A is called *semilinear* (resp. *semi-quadratic*) if for any $(x, y) \in [0, 1]^2$, A is linear (resp. quadratic) in at least one direction.

In this dissertation, we introduce several methods to construct semilinear and semiquadratic aggregation functions.

Conic aggregation functions

Inspired by the notion of conic t-norms, we introduce in this chapter conic aggregation functions. Their construction is based on linear interpolation on segments connecting the upper boundary curve of the zero-set to the point $(1, 1)$. Such aggregation functions are completely characterized by their zero-set, in particular by the upper boundary curve of this zero-set. Special classes of binary conic aggregation functions such as conic quasi-copulas and conic copulas are considered. We provide the necessary and sufficient conditions on the function f that represents the upper boundary curve of the zero-set of a conic aggregation function to obtain a conic (quasi-)copula and conclude that the class of conic copulas is a proper subclass of the class of conic quasi-copulas. Moreover, we characterize the class of conic copulas that are supported on a set with Lebesgue measure zero. The convexity of f plays a key role in characterizing the class of conic copulas. We derive compact formulae for Spearman's rho, Gini's gamma and Kendall's tau of two continuous random variables whose dependence is modelled by a conic copula.

Biconic aggregation functions

Inspired by the previous chapter, we introduce a new method to construct aggregation functions. These aggregation functions are called biconic aggregation functions with a given diagonal (resp. opposite diagonal) section and their construction is based on linear interpolation on segments connecting the diagonal (resp. opposite diagonal) of the unit square to the points $(0, 1)$ and $(1, 0)$ (resp. $(0, 0)$ and $(1, 1)$). Subclasses of biconic aggregation functions such as biconic semi-copulas, biconic quasi-copulas and biconic copulas are studied in detail. We provide the necessary and sufficient conditions on a given diagonal (resp. opposite diagonal) function δ (resp. ω) to obtain a biconic (semi-, quasi-)copula that has δ (resp. ω) as diagonal (resp. opposite diagonal) section. We conclude that the class of biconic copulas is a proper subclass of the class of biconic quasi-copulas. Moreover, the class of biconic quasi-copulas turns out to be a proper subclass of the class of biconic semi-copulas. The convexity (resp. concavity) of the diagonal (resp. opposite diagonal) section plays a key role in characterizing the class of biconic copulas with a given diagonal (resp. opposite diagonal) section. The piecewise linearity of the diagonal section of a biconic copula turns out to be the necessary and sufficient condition to be supported on a set with Lebesgue measure zero. We derive compact formulae for Spearman's rho, Gini's gamma and Kendall's tau of two continuous random variables whose dependence is modelled by a biconic copula with a given diagonal section.

Upper conic, lower conic and biconic semi-copulas with a given section

Inspired by the previous two chapters, we introduce upper conic, lower conic and biconic semi-copulas with a given section. Such semi-copulas are constructed by linear interpolation on segments connecting the graph of a strict negation operator to the points $(0, 0)$ and/or $(1, 1)$. Special classes of upper conic, lower conic and biconic semi-copulas with a given section such as upper conic, lower conic and biconic (quasi-)copulas with a given section are considered. We recall in this chapter the notion of generalized convexity (resp. concavity). This notion plays a key role in characterizing upper conic, lower conic and biconic copulas with a given section. When the given section is taken from the product copula, the convexity of the strict negation operator turns out be a sufficient condition to obtain an upper conic, lower conic or biconic copula with this given section and to conclude that the resulting upper conic, lower conic and biconic copulas are positive quadrant dependent.

Ortholinear and paralinear semi-copulas

Rather than using linear interpolation on segments connecting a line in the unit square to one point or two points in the unit square as in the above chapters, we introduce in this chapter a new method to construct semi-copulas based on linear interpolation on segments that are perpendicular (resp. parallel) to the diagonal of the unit square. These semi-copulas are called ortholinear (resp. paralinear) semi-copulas. We provide the necessary and sufficient conditions on a given diagonal (resp. opposite diagonal) function to obtain an ortholinear (resp. paralinear) (quasi-)copula. We conclude that the class of ortholinear copulas is a proper subclass of the class of biconic quasi-copulas. The convexity (resp. concavity) of the diagonal (resp. opposite diagonal) section plays again a key role in characterizing the class of ortholinear (resp. paralinear) copulas. Ortholinear copulas have the property of being Schur-concave. Convex sums of $T_{\mathbf{M}}$ and $T_{\mathbf{L}}$ are the only ortholinear copulas that are supported on a set with Lebesgue measure zero. We derive compact formulae for Spearman's rho, Gini's gamma and Kendall's tau of two continuous random variables whose dependence is modelled by an ortholinear copula.

Some types of semilinear copulas based on horizontal and vertical interpolation

We first introduce four families of semilinear copulas with a given opposite diagonal section, called lower-upper, upper-lower, horizontal and vertical semilinear copulas. There is a great similarity between the case of a given opposite diagonal section and that of a given diagonal section, which can be explained by the existence of a transformation that maps copulas onto copulas in such a way that the diagonal is mapped onto the opposite diagonal and vice versa. In the second part of this chapter, we consider the construction of semilinear copulas with given diagonal and opposite diagonal sections. Also here, four new families of semilinear copulas are introduced, called orbital, vertical, horizontal and radial semilinear copulas. For each of these families, we provide necessary and sufficient conditions under which given diagonal and opposite diagonal functions can be the diagonal and opposite diagonal sections of a semilinear copula belonging to that family. We focus particular attention on the family of orbital semilinear copulas, which are obtained by linear interpolation on segments connecting the diagonal and opposite diagonal of the unit square.

Lower semiquadratic copulas with a given diagonal section

Inspired by the notion of lower semilinear copulas, introduced by Durante et al. we introduce a new class of copulas. These copulas, called lower semiquadratic copulas, are constructed by quadratic interpolation on segments connecting the diagonal of the unit square to the lower and left boundary of the unit square. Moreover, we unveil the necessary and sufficient conditions on a diagonal function and two auxiliary real functions to obtain a copula that has this diagonal function as diagonal section. Under some mild assumptions, we characterize the smallest and the greatest lower semiquadratic copulas with a given diagonal section. Unlike lower semilinear copulas, lower semiquadratic copulas can be not symmetric. We also characterize the class of continuous differentiable (resp. absolutely continuous) lower semiquadratic copulas. Finally, we provide expressions for the degree of non-exchangeability and the measures of association for various families of lower semiquadratic copulas.

Semiquadratic copulas based on horizontal and vertical interpolation

Generalizing the results in the previous two chapters, we introduce several families of semiquadratic copulas of which the diagonal and/or opposite diagonal sections are given functions. These copulas are constructed by quadratic interpolation on segments connecting the diagonal, opposite diagonal and sides of the unit square; all interpolations are therefore performed horizontally or vertically. For each family we provide the necessary and sufficient conditions on the given diagonal and/or opposite diagonal functions and two auxiliary real functions to obtain a copula that has these diagonal and/or opposite diagonal functions as diagonal and/or opposite diagonal sections. Just as the product copula is a central member of all families of semilinear copulas based on horizontal and vertical interpolation, it turns out that the Farlie-Gumbel-Morgenstern family of copulas is included in all families of semiquadratic copulas introduced and characterized here.

Samenvatting

Samenvatting

Conjunctieve aggregatiefuncties worden uitvoerig gebruikt in de vaaglogica (*fuzzy logic*) en de vaagverzamelingenleer (*fuzzy set theory*). Ze blijken geschikte operatoren te zijn voor het modelleren van de boolese "en". Bijzondere subklassen van conjunctieve aggregatiefuncties zoals t-normen (*triangular norms*), copulas, semi-copulas, en quasi-copulas, werden uitgebreid onderzocht door onderzoekers in de betrouwbaarheidstheorie, de vaagverzamelingenleer, de waarschijnlijkheidstheorie en de statistiek.

In de literatuur werden verschillende methodes voor de constructie van conjunctieve aggregatiefuncties geïntroduceerd. Sommige van deze methodes zijn gebaseerd op lineaire of kwadratische interpolatie op segmenten die lijnen binnen het eenheids vierkant met de zijden ervan verbinden. Dergelijke lijnen kunnen de diagonaal, de nevendiagonaal, een horizontale lijn, een verticale lijn zijn, of een grafiek die een dalende functie voorstelt. We introduceren de begrippen semi-lineaire en semi-kwadratische aggregatiefuncties die alle aggregatiefuncties veralgemenen die bekomen werden gebruikmakend van dergelijke methodes. Meer in het bijzonder wordt een functie $A : [0,1]^2 \to [0,1]$ *semi-lineair* (resp. *semi-kwadratisch*) genoemd als, voor elke $(x,y) \in [0,1]^2$, A lineair (resp. kwadratisch) is in tenminste één richting.

In dit proefschrift introduceren we verschillende methodes voor de constructie van semi-lineaire en semi-kwadratische aggregatiefuncties.

Conische aggegratiefuncties

In dit hoofdstuk introduceren we, geïnspireerd door de notie van t-normen, conische aggregatiefuncties. De constructie van dergelijke functies steunt op lineaire interpolatie op segmenten die de bovenste grenscurve van de nul-set verbinden met het punt $(1,1)$. Dergelijke aggregatiefuncties kunnen volledig gekarakteriseerd worden door hun nul-set, in het bijzonder de bovenste grenscurve van die nul-set. Speciale klassen van binaire aggregatiefuncties, zoals conische copulas en conische quasi-copulas, worden behandeld. We voorzien in de nodige en voldoende voorwaarden waaraan de functie f, die de bovenste grenscurve van de nul-set voorstelt, moet voldoen om een conische (quasi)-copula te bekomen. De conclusie is dat de klasse van conische copulas een echte subklasse van de klasse van conische quasi-copulas vormt. We karakteriseren bovendien de klasse van conische copulas die gedragen worden door een set met Lebesguemaat nul. In het karakteriseringsproces van de klasse van conische copulas speelt de convexiteit van f een sleutelrol. We leiden compacte formules af voor Spearman's rangcorrelatiecoëfficiënt ρ, de Gini-coëfficiënt γ en Kendall's rangcorrelatiecoëfficiënt τ voor twee continue toevalsveranderlijken waarvan de onderlinge afhankelijkheid gemodelleerd wordt door een conische copula.

Biconische aggegratiefuncties

Geïnspireerd door het vorige hoofdstuk, introduceren we in dit hoofdstuk een nieuwe methode voor de constructie van aggregatiefuncties. Deze functies worden biconische aggregatiefuncties met een gegeven (neven)diagonaal genoemd. Hun constructie is gebaseerd op lineaire interpolatie op segmenten die, in het eenheidsvierkant, de diagonaal (resp. nevendiagonaal) met de punten $(0,1)$ en $(1,0)$ (resp. $(0,0)$ en $(1,1)$) verbinden. Subklassen van biconische aggregatiefuncties zoals biconische copulas, biconische semi-copulas en biconische quasi-copulas worden in detail bestudeerd. We voorzien de nodige en voldoende voorwaarden waaraan een gegeven diagonaal- (resp. nevendiagonaal-)functie δ (resp. ω) moet voldoen om een biconische (semi-, quasi-)copula te bekomen die δ (resp. ω) als diagonale (resp. nevendiagonale) sectie heeft. De conclusie is dat de klasse van biconische copulas een echte subklasse vormt van de klasse van biconische quasi-copulas. De convexiteit (resp. concaviteit) van de diagonale (resp. nevendiagonale) sectie speelt een sleutelrol bij de karakterisering van de klasse van biconische copulas met een gegeven (neven-)diagonale sectie. Het blijkt dat de stuksgewijze lineariteit van de diagonale sectie van een biconische copula een nodige en voldoende voorwaarde is om gedragen te kunnen worden door een set met Lebesguemaat nul. We leiden compacte formules af voor Spearman's rangcorrelatiecoëfficiënt ρ, de Gini-coëfficiënt γ en Kendall's rangcorrelatiecoëfficiënt τ voor twee continue toevalsveranderlijken waarvan de onderlinge afhankelijkheid gemodelleerd wordt door een biconische copula met een gegeven diagonale sectie.

Bovenconische, onderconische en biconische semi-copulas met een gegeven sectie

In dit hoofdstuk introduceren we, geïnspireerd door de vorige twee hoofdstukken, bovenconische, onderconische en biconische semi-copulas met een gegeven sectie. Dergelijke semi-copulas worden geconstrueerd door lineaire interpolatie op segmenten die de grafiek van een strikte negatie-operator verbinden met de punten $(0,0)$ en/of $(1,1)$. We behandelen speciale klassen van onderconische en biconische semi-copulas met een gegeven sectie, zoals bovenconische, onderconische en biconische (quasi-)copulas met een gegeven sectie. We herhalen in dit hoofdstuk het begrip veralgemeende convexiteit (resp. concaviteit). Dit speelt een belangrijke rol bij de karakterisering van bovenconische, onderconische en biconische copulas met een gegeven sectie. Wanneer de gegeven sectie genomen wordt van de productcopula, blijkt de convexiteit van de strikte negatie-operator een voldoende voorwaarde te zijn voor het bekomen van een bovenconische, onderconische of biconische copula met deze opgegeven sectie. De resulterende bovenconische, onderconische en biconische copulas blijken positief-kwadrant-afhankelijk te zijn.

Ortholineaire en paralineaire semi-copulas

In plaats van gebruik te maken van lineaire interpolatie op segmenten die, zoals in de vorige hoofdstukken, een lijn in het eenheidsvierkant verbinden met één of twee punten in het eenheidsvierkant, introduceren we hier een nieuwe methode voor de constructie van semi-copulas gebaseerd op lineaire interpolatie op segmenten die loodrecht (resp. parallel) staan t.o.v. het eenheidsvierkant. Deze semi-copulas worden ortholineaire (resp. paralineaire) semi-copulas genoemd. We voorzien de nodige en voldoende voorwaarden waaraan een gegeven diagonaal- (resp. nevendiagonaal-)functie moet voldoen om een ortholineaire (resp. paralineaire) (quasi-)copula te bekomen. We vinden dat de klasse van ortholineaire copulas een echte subklasse is van de klasse van biconische quasi-copulas. The convexiteit (resp. concaviteit) van de diagonale (resp. nevendiagonale) sectie speelt ook nu een sleutelrol bij de karakterisering van de klasse van ortholineaire (resp. paralineaire) copulas. Ortholineaire copulas hebben de eigenschap Schur-concaaf te zijn. De convexe sommen van $T_{\mathbf{M}}$ en $T_{\mathbf{L}}$ zijn de enige ortholineaire copulas die gedragen worden door een set met Lebesgue-maat nul. We leiden compacte formules af voor Spearman's rangcorrelatiecoëfficiënt ρ, de Gini-coëfficiënt γ en Kendall's rangcorrelatiecoëfficiënt τ voor twee continue toevalsveranderlijken waarvan de onderlinge afhankelijkheid gemodelleerd wordt door een ortholineaire copula.

Enkele types van semi-lineaire copulas gebaseerd op horizontale en verticale interpolatie

In het eerste deel van dit hoofdstuk introduceren we vier families van semi-lineaire copulas met een gegeven nevendiagonale sectie, namelijk boven-onder, onder-boven, horizontale en verticale semi-lineaire copulas. Er is een grote overeenkomst tussen de gevallen met een gegeven diagonale of nevendiagonale sectie, hetgeen verklaard wordt door het bestaan van een transformatie die copulas op copulas afbeeldt zodanig dat de diagonaal op de nevendiagonaal wordt afgebeeld en omgekeerd.

In het tweede deel van dit hoofdstuk beschouwen we de constructie van semi-lineaire copulas met gegeven diagonale en nevendiagonale secties. We introduceren hier ook vier nieuwe families van semi-lineaire copulas, namelijk orbitale, radiale, verticale en horizontale semi-lineaire copulas. Voor elk van deze families voorzien we de nodige en voldoende voorwaarden waaronder gegeven diagonaal en nevendiagonaalfuncties de diagonale en nevendiagonale secties kunnen zijn van semi-lineaire copula die tot die families behoren. We schenken bijzondere aandacht aan de familie van orbitale semi-lineaire copulas die bekomen werden door lineaire interpolatie op segmenten die de diagonaal met de nevendiagonaal van het eenheidsvierkant verbinden.

Onder semi-kwadratische copulas met een gegeven diagonale sectie

Geïnspireerd door het begrip semi-lineaire copulas, geïntroduceerd door Durante et al. introduceren we in dit hoofdstuk een nieuwe klasse van copulas. Deze laatste, ondere semi-kwadratische copulas genoemd, worden geconstrueerd door kwadratische interpolatie op segmenten die, in het eenheidsvierkant, de diagonaal met de onder- en linkerzijde verbinden. We voorzien bovendien de nodige en voldoende voorwaarden waaraan de diagonaalfunctie en twee reële hulpfuncties moet voldoen om een copula te bekomen die de diagonaalfunctie als diagonale sectie heeft. Onder een aantal milde aannames karakteriseren we de kleinste en grootste ondere semi-kwadratische copulas met een gegeven diagonale sectie. In tegenstelling tot semi-lineaire copulas, kunnen ondere semi-kwadratische copulas asymmetrisch zijn. Verder karakteriseren we ook de klasse van continu-afleidbare (resp. absoluut continue) ondere semi-kwadratische copulas. We geven tenslotte uitdrukkingen voor de graad van niet-uitwisselbaarheid en voor de associatiematen voor verschillende families van semi-kwadratische copulas.

Semi-kwadratische copulas gebaseerd op horizontale en verticale interpolatie

In dit hoofdstuk veralgemenen we de resultaten uit de vorige twee hoofdstukken, en introduceren we verschillende families van semi-kwadratische copulas waarvan de diagonale en/of nevendiagonale secties gegeven functies zijn. Deze copulas worden geconstrueerd door kwadratische interpolatie op segmenten die, in het eenheidsvierkant, de diagonaal, de nevendiagonaal en de zijden met elkaar verbinden. Alle interpolaties worden derhalve horizontaal of verticaal uitgevoerd.
Voor elke familie voorzien we de nodige en voldoende voorwaarden waaraan de gegeven diagonaal- en/of nevendiagonaalfunctie en twee reële hulpfuncties moeten

voldoen om een copula te bekomen die deze diagonaal- en/of nevendiagonaalfunctie als diagonale en/of nevendiagonale sectie heeft.
Zoals de productcopula een centraal lid is van de familie van semi-lineaire copulas gebaseerd op horizontale en verticale interpolatie, blijkt dat de Farli–Gumbel–Morgenstern familie van copulas vervat is in alle families van semi-kwadratische copulas die hier geïntroduceerd en gekarakteriseerd werden.

Bibliography

[1] I. Aguiló, J. Suñer and J. Torrens, *A construction method of semicopulas from fuzzy negations*, Fuzzy Sets and Systems **226** (2013), 99–114.

[2] C. Alsina, M.J. Frank and B. Schweizer, *Associative Functions: Triangular Norms and Copulas*, World Scientific, Singapore, 2006.

[3] B. Bassan and F. Spizzichino, *Relations among univariate aging, bivariate aging and dependence for exchangeable lifetimes*, Journal of Multivariate Analysis **93** (2005), 313–339.

[4] G. Beliakov, T. Calvo and J. Lazaro, *Pointwise construction of Lipschitz aggregation operators with specific properties*, Internat. J. of Uncertainty, Fuzziness and Knowledge-Based Systems **15** (2007), 193–223.

[5] G. Beliakov, B. De Baets, H. De Meyer, R. Nelsen and M. Úbeda-Flores, *Best-possible bounds on the set of copulas with given degree of non-exchangeability*, Journal of Mathematical Analysis and Applications **417** (2014), 451–468.

[6] G. Beliakov, A. Pradera and T. Calvo, *Aggregation Functions: a Guide for Practitioners*, Studies in Fuzziness and Soft Computing, Vol. 221, Springer, Berlin, 2007.

[7] S. Bertino, *On dissimilarity between cyclic permutations*, Metron **35** (1977), 53–88.

[8] G. Birkhoff, *Lattice Theory*, Third edition, American Mathematical Society Colloquium Publications, Vol. XXV. American Mathematical Society, Providence, R.I. 1967.

[9] T. Calvo, A. Kolesárová, M. Komorníková and R. Mesiar, *Aggregation Operators: Properties, Classes and Construction Methods*. In: *Aggregation Operators*, New trends and Applications, Physica-Verlag, New York, 2002, pp. 3–106.

[10] T. Calvo and A. Pradera, *Double aggregation operators*, Fuzzy Sets and Systems **142** (2004), 15–33.

[11] G.T. Cargo, *Comparable means and generalized convexity*, Journal of Mathematical Analysis and Applications **12** (1965), 387–392.

[12] R. Cerqueti and F. Spizzichino, *Extension of dependence properties to semi-copulas and applications to the mean-variance model*, Fuzzy Sets and Systems **220** (2013), 99–108.

[13] G. Choquet, Theory of capacities, Annales Institute Fourier **5** (1953), 131–295.

[14] A.H. Clifford, *Naturally totally ordered commutative semigroups*, American Journal of Mathematics **76** (1954), 631–646.

[15] E. de Amo, M. Díaz-Carrillo and J. Fernández-Sánchez, *Absolutely continuous copulas with given sub-diagonal section* , Fuzzy Sets and Systems **228** (2013), 105–113.

[16] B. De Baets and H. De Meyer, *Orthogonal grid constructions of copulas*, IEEE Trans. Fuzzy Systems **15** (2007), 1053–1062.

[17] B. De Baets, H. De Meyer and S. Díaz, *On an idempotent transformation of aggregation functions and its application on absolutely continuous Archimedean copulas*, Fuzzy Sets and Systems **160** (2009), 733–751.

[18] B. De Baets, H. De Meyer and J. Fernández-Sánchez, *On the existence of a trivariate copula with given values of a trivariate quasi-copula at several points*, Fuzzy Sets and Systems **228** (2013), 3–14.

[19] B. De Baets, H. De Meyer, J. Kalická and R. Mesiar, *Flipping and cyclic shifting of binary aggregation functions*, Fuzzy Sets and Systems **160** (2009), 752–765.

[20] B. De Baets, H. De Meyer and R. Mesiar, *Asymmetric semilinear copulas*, Kybernetika **43** (2007), 221–233.

[21] B. De Baets, H. De Meyer and R. Mesiar, *Piecewise linear aggregation functions based on triangulation*, Information Sciences **181** (2011), 466–478.

[22] B. De Baets, H. De Meyer and R. Mesiar, *Binary survival aggregation functions*, Fuzzy Sets and Systems **191** (2012), 83–102.

[23] B. De Baets, H. De Meyer and M. Úbeda-Flores, *Opposite diagonal sections of quasi-copulas and copulas*, Internat. J. Uncertainty, Fuzziness and Knowledge-Based Systems **17** (2009), 481–490.

[24] B. De Baets, H. De Meyer and M. Úbeda-Flores, *Constructing copulas with given diagonal and opposite diagonal sections*, Communications in Statistics: Theory and Methods **40** (2011), 828–843.

[25] B. De Baets and J. Fodor, *Additive fuzzy preference structures: the next generation*, in: Principles of Fuzzy Preference Modelling and Decision Making (B. De Baets and J. Fodor, eds.), Academic Press, 2003, pp. 15–25.

[26] B. De Baets, S. Janssens and H. De Meyer, *On the transitivity of a parametric family of cardinality-based similarity measures*, Internat. J. Approximate Reasoning **50** (2009), 104–116.

[27] D. Denneberg, *Non-additive Measure and Integral*, Kluwer, 1994.

[28] S. Díaz, S. Montes and B. De Baets *Transitivity bounds in additive fuzzy preference structures*, IEEE Trans. Fuzzy Systems **15** (2007), 275–286.

[29] A. Dolati, S. Mohseni and M. Úbeda-Flores, *Some results on a transformation of copulas and quasi-copulas*, Information Sciences **257** (2014), 176–182.

[30] D. Dubois and H. Prade, *On the use of aggregation operations in information fusion processes*, Fuzzy Sets and Systems **142** (2004), 143–161.

[31] F. Durante, J. Fernández-Sánchez and C. Sempi, *A note on the notion of singular copula*, Fuzzy Sets and Systems **211** (2013), 120–122.

[32] F. Durante, J. Fernández-Sánchez and M. Úbeda-Flores, *Bivariate copulas generated by perturbations*, Fuzzy Sets and Systems **228** (2013), 137–144.

[33] F. Durante and P. Jaworski, *Absolutely continuous copulas with given diagonal sections*, Communications in Statistics: Theory and Methods **37** (2008), 2924–2942.

[34] F. Durante, E.P. Klement, R. Mesiar and C. Sempi, *Conjunctors and their residual implicators: characterizations and construction methods*, Mediterranean Journal of Mathematics **4** (2007), 343–356.

[35] F. Durante, E. Klement, C. Sempi, M. Úbeda-Flores, *Measures of non-exchangeability for bivariate random vectors*, Statistical Papers **151** (2010), 687–699.

[36] F. Durante, A. Kolesárová, R. Mesiar and C. Sempi, *Copulas with given diagonal sections, novel constructions and applications*, Internat. J. Uncertainty, Fuzziness and Knowledge-Based Systems **15** (2007), 397–410.

[37] F. Durante, A. Kolesárová, R. Mesiar and C. Sempi, *Copulas with given values on a horizontal and a vertical section*, Kybernetika **43** (2008), 209–220.

[38] F. Durante, A. Kolesárová, R. Mesiar and C. Sempi, *Semilinear copulas*, Fuzzy Sets and Systems **159** (2008), 63–76.

[39] F. Durante, R. Mesiar and C. Sempi, *On a family of copulas constructed from the diagonal section*, Soft Computing **10** (2006), 490–494.

[40] F. Durante and P.-L. Papini, *A weakening of Schur-concavity for copulas*, Fuzzy Sets and Systems **158** (2007), 1378–1383.

[41] F. Durante, J.J. Quesada Molina and C. Sempi, *Semicopulas: characterizations and applicability*, Kybernetika **42** (2006), 287–302.

[42] F. Durante, S. Saminger-Platz, P. Sarkoci, *On representations of* 2*-increasing binary aggregation functions*, Information Sciences **178** (2008), 4534–4541.

[43] F. Durante and C. Sempi, *Copulae and Schur-concavity*, Int. Math. J. **3** (2003), 893–905.

[44] F. Durante and C. Sempi, *Semicopulae*, Kybernetika **41** (2005), 315–328.

[45] F. Durante and F. Spizzichino, *Semi-copulas, capacities and families of level sets*, Fuzzy Sets and Systems **161** (2010), 269–276.

[46] A. Erdely and J.M. González-Barrios, *On the construction of families of absolutely continuous copulas with given restrictions*, Communications in Statistics: Theory and Methods **35** (2006), 649–659.

[47] C. Genest, J.J. Quesada Molina, J.A. Rodríguez Lallena and C. Sempi, *A characterization of quasi-copulas*, Journal of Multivariate Analysis **69** (1999), 193–205.

[48] M. Grabisch, J.-L. Marichal, R. Mesiar and E. Pap, *Aggregation Functions*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2009.

[49] M. Grabisch, J.-L. Marichal, R. Mesiar and E. Pap, *Aggregation functions: Construction methods, conjunctive, disjunctive and mixed classes*, Information Sciences **181** (2011), 23–43.

[50] M. Grabisch, J.-L. Marichal, R. Mesiar and E. Pap, *Aggregation functions: Means*, Information Sciences **181** (2011), 1–22.

[51] J. Fernández-Sánchez and M. Úbeda-Flores, *Semi-polynomial copulas*, Journal of Nonparametric Statistics **26** (2014), 129–140.

[52] J. Fernández-Sánchez and M. Úbeda-Flores, *A note on quasi-copulas and signed measures*, Fuzzy Sets and Systems **234** (2014), 109–112.

[53] J. Fodor, *Contrapositive symmetry of fuzzy implications*, Fuzzy Sets and Systems **69** (1995), 141–156.

[54] J. Fodor and M. Roubens, *Fuzzy Preference Modelling and Multicriteria Decision Support*, Kluwer Academic Publishers, Dordrecht, 1994.

[55] G. Fredricks and R. Nelsen, *The Bertino family of copulas.* In: C.M. Cuadras, J. Fortiana and J. Rodríguez Lallena, Eds., Distributions with given Marginals and Statistical Modelling, Kluwer Academic Publishers, Dordrecht, 2002, pp. 81–91.

[56] G. Fredricks and R. Nelsen, *On relationship between Spearman's rho and Kendall's tau for pairs of continuous random variables*, Journal of Statistical Planning and Inference **137** (2007), 2143–2150.

[57] S. Fuchs and K. D. Schmidt, *Asymmetric bivariate copulas, transformations and measure of concordance*, Kybernetika, in press.

[58] P. Hájek, *Metamathematics of Fuzzy Logic*, Kluwer, Dordrecht, 1998.

[59] P. Hájek and R. Mesiar *On copulas, quasicopulas and fuzzy logic*, Soft Computing **12** (2008), 1239–1243.

[60] S. Janssens, B. De Baets and H. De Meyer, *Bell-type inequalities for quasi-copulas*, Fuzzy Sets and Systems **148** (2004), 263–278.

[61] P. Jaworski and T. Rychlik, *On distributions of order statistics for absolutely continuous copulas with applications to reliability*, Kybernetika **44** (2008), 757–776.

[62] S. Jenei, *Structure of left-continuous triangular norms with strong induced negations. (III) Construction and decomposition*, Fuzzy Sets and Systems **128** (2002), 197–208.

[63] S. Jenei, *How to construct left-continuous triangular norms - state of the art*, Fuzzy Sets and Systems **143** (2004), 27–45.

[64] H. Joe, *Multivariate Models and Dependence Concepts*, Chapman & Hall, London, 1997.

[65] T. Jwaid, B. De Baets and H. De Meyer, *Orbital semilinear copulas*, Kybernetika **45** (2009), 1012–1029.

[66] T. Jwaid, B. De Baets and H. De Meyer, *Biconic aggregation functions*, Information Sciences **187** (2012), 129–150.

[67] T. Jwaid, B. De Baets and H. De Meyer, *Ortholinear and paralinear semi-copulas*, Fuzzy Sets and Systems **252** (2014), 76–98.

[68] T. Jwaid, B. De Baets and H. De Meyer, *Semiquadratic copulas based on horizontal and vertical interpolation*, Fuzzy Sets and Systems, doi: 10.1016/j.fss.2014.04.023.

[69] T. Jwaid, B. De Baets, H. De Meyer and R. Mesiar, *The role of generalized convexity in conic copula constructions*, Journal of Mathematical Analysis and Applications, submitted.

[70] T. Jwaid, B. De Baets, J. Kalická and R. Mesiar, *Conic aggregation functions*, Fuzzy Sets and Systems **167** (2011), 3–20.

[71] T. Jwaid, H. De Meyer and B. De Baets, *Lower semiquadratic copulas with a given diagonal section*, Journal of Statistical Planning and Inference **143** (2013), 1355–1370.

[72] J. Kalická, *On some construction methods for 1-Lipschitz aggregation functions*, Fuzzy Sets and Systems **160** (2009), 726–732.

[73] E.P. Klement and A. Kolesárová, *Extension to copulas and quasi-copulas as special 1-Lipschitz aggregation operators*, Kybernetika **41** (2005), 329–348.

[74] E. Klement, A. Kolesárová and R. Mesiar. *Copulas constructed from horizontal sections.* Communications in Statistics: Theory and Methods **36** (2007), 2901–2911.

[75] E.P. Klement, R. Mesiar and E. Pap, *Triangular Norms*, Trends in Logic, Studia Logica Library, Vol. 8, Kluwer Academic Publishers, Dordrecht, 2000.

[76] E.P. Klement, R. Mesiar and E. Pap, *Archimax copulas and invariance under transformations*, C. R. Acad. Sci. Ser. I **340** (2005), 755–758.

[77] E.P. Klement, R. Mesiar and E. Pap, *Transformation of copulas*, Kybernetika **41** (2005), 425–434.

[78] A. Kolesárová, *1-Lipschitz aggregation operators and quasi-copulas*, Kybernetika **39** (2003), 615–629.

[79] K.C. Maes and B. De Baets, *A contour view on uninorm properties*, Kybernetika **42** (2006), 303–318.

[80] K.C. Maes and B. De Baets, *On the structure of left-continuous t-norms that have a continuous contour line*, Fuzzy Sets and Systems **158** (2007), 843–860.

[81] K.C. Maes and B. De Baets, *The triple rotation method for constructing t-norms*, Fuzzy Sets and Systems **158** (2007), 1652–1674.

[82] K.C. Maes and B. De Baets, *Negation and affirmation: the role of involutive negators*, Soft Computing **11** (2007), 647–654.

[83] K.C. Maes and B. De Baets, *Rotation-invariant t-norms: the rotation invariance property revisited*, Fuzzy Sets and Systems **160** (2009), 44–51.

[84] K.C. Maes and B. De Baets, *Rotation-invariant t-norms: where triple rotation and rotation-annihilation meet*, Fuzzy Sets and Systems **160** (2009), 1998–2016.

[85] A.W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and its Applications*, Academic Press Inc. New York, 1979.

[86] R. Mesiar and C. Sempi, *Ordinal sums and idempontents of copulas*, Aequationes Math. **79** (2010), 39–52.

[87] J. Navarro and F. Spizzichino, *Comparisons of series and parallel systems with components sharing the same copula*, Appl. Stoch. Models Bus. Ind. **26** (2010), 775–791.

[88] R. Nelsen, *An Introduction to Copulas*, Springer, New York, 2006.

[89] R. Nelsen, *Extremes of nonexchangeability*, Stat. Papers **48** (2007), 329–336.

[90] R. Nelsen and G. Fredricks, *Diagonal copulas.* In: V. Beneš and J. Štěpán, Eds., Distributions with given Marginals and Moment Problems, Kluwer Academic Publishers, Dordrecht, 1977, pp. 121–127.

[91] R. Nelsen, J. Quesada-Molina, J. Rodríguez-Lallena and M. Úbeda-Flores, *On the construction of copulas and quasi-copula with given diagonal sections*, Insurance: Math. Econ. **42** (2008), 473–483.

[92] R. Nelsen and M. Úbeda-Flores, *The lattice-theoretic structure of sets of bivariate copulas and quasi-copulas*, Comptes Rendus Mathematique **341** (2005), 583–586.

[93] C.P. Niculescu and L.-E. Persson, *Convex Functions and their Applications, A Contemporary Approach*, Springer, Berlin, 2000.

[94] J.A. Palmer, *Relative convexity*, Technical Report, ECE Dept, UCSD, 2003.

[95] J.E. Pecarić, F. Proschan and J. L. Tong, *Convex Functions, Partial Ordering, and Statistical Applications*, Academic Press, Inc. New York, 1992.

[96] J.J. Quesada-Molina and J. Rodríguez-Lallena, *Bivariate copulas with quadratic sections*, Journal of Nonparametric Statistics **5** (1995), 323–337.

[97] J. Rodríguez Lallena, *A class of copulas with piecewise linear horizontal sections*, Journal of Statistical Planning and Inference **139** (2009), 3908–3920.

[98] R. Schmidt and U. Stadtmüller, *Non-parametric estimation of tail dependence*, Scand. J. Statist. **33**(2) (2006), 307–335.

[99] A. Sklar, *Fonctions de répartition à n dimensions et leurs marges*, Publ. Inst. Statist. Univ. Paris **8** (1959), 229–231.

[100] A.E. Taylor, *General Theory of Functions and Integration*, Courier Dover Publications, New York, 1965.

[101] B. Van de Walle, B. De Baets and E. Kerre, *Characterizable preference structures*, Ann. Oper. Res. **80** (1998), 105–136.

[102] M.-H. Zhang, *Modelling total tail dependence along diagonals*, Insurance: Math. Econ. **42** (2008), 73–80.

Curriculum Vitae

Personalia

Name	Tarad Jwaid
Gender	Male
Date of birth	06/12/1980
Place of birth	Aleppo, Syria
Marital situation	Married, 2 childeren
E-mail	Tarad.Jwaid@UGent.be
	Jwaid.Tarad@Gmail.com

Education

University

- 2003 - 2004: Fourth year of Mathematics (Algebra), Aleppo University.
- 2002 - 2003: Third year of Mathematics, Aleppo University.
- 2001 - 2002: Second year of Mathematics, Aleppo University.
- 2000 - 2001: First year of Mathematics, Aleppo University.

Secondary school

- 1997 - 2000: Qensreen School, Aleppo, Syria.

Elementary school

- 1994 - 1997: Alzerba School, Aleppo, Syria.

Employment

- 2004 - 2008: Teacher of mathematics in Alzerba School.

Higher studies

- 2008 - present: PhD student at research unit KERMIT, Department of Mathematical Modelling, Statistics and Bioinformatics, Ghent University.

Scientific output

Publications in international journals (ISI-papers)

- **T. Jwaid**, B. De Baets, H. De Meyer and R. Mesiar, *The role of generalized convexity in conic copula constructions*, Journal of Mathematical Analysis and Applications, submitted.
- **T. Jwaid**, B. De Baets and H. De Meyer, *Semiquadratic copulas based on horizontal and vertical interpolation*, Fuzzy Sets and Systems, doi: 10.1016/j.fss.2014.04.023.
- **T. Jwaid**, B. De Baets and H. De Meyer, *Ortholinear and paralinear semi-copulas*, Fuzzy Sets and Systems **252** (2014), 76–98.
- **T. Jwaid**, H. De Meyer and B. De Baets, *Lower semiquadratic copulas with a given diagonal section*, Journal of Statistical Planning and Inference **143** (2013), 1355–1370.
- **T. Jwaid**, B. De Baets and H. De Meyer, *Biconic aggregation functions*, Information Sciences **187** (2012), 129–150.
- **T. Jwaid**, B. De Baets, J. Kalická and R. Mesiar, *Conic aggregation functions*, Fuzzy Sets and Systems **167** (2011), 3–20.
- **T. Jwaid**, B. De Baets and H. De Meyer, *Orbital semilinear copulas*, Kybernetika **45** (2009), 1012–1029.

Conference proceedings

- **T. Jwaid**, B. De Baets, H. De Meyer and R. Mesiar, Biconic semi-copulas with a given section. Eighth Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT-2013, pp. 565–568), Milano, Italy.
- **T. Jwaid**, B. De Baets and H. De Meyer, On the construction of semi-quadratic copulas. Seventh International Summer School on Aggregation Operators (AGOP-2013, pp. 47–57), Pamplona, Spain.

- H. De Meyer, B. De Baets and **T. Jwaid**, On a class of variolinear copulas. Fourteenth International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU-2012, pp. 171–180), Catania, Italy.
- **T. Jwaid**, B. De Baets, H. De Meyer, Biconic aggregation functions with a given diagonal or opposite diagonal section. Seventh conference of the European Society for Fuzzy Logic and Technology (EUSFLAT-2011, pp. 67–74), Aix-Les-Bains, France.
- **T. Jwaid**, B. De Baets, H. De Meyer, Orthogonal copulas with a given diagonal or opposite diagonal section. Sixth International Summer School on Aggregation Operators (AGOP-2011, pp. 29–34), Benevento, Italy.
- J. Kalicka, V. Jagr, M. Juranova, **T. Jwaid**, and B. De Baets, Conic copulas and quasi-copulas. Fifth International Summer School on Aggregation Operators (AGOP-2009, pp. 151–154), Palma de Mallorca, Spain.
- **T. Jwaid**, B. De Baets and H. De Meyer, Semilinear copulas with given diagonal sections. Fifth International Summer School on Aggregation Operators (AGOP-2009, pp. 89–94), Palma de Mallorca, Spain.

Conference abstracts

- **T. Jwaid**, B. De Baets, H. De Meyer, Variolinear copulas with a given diagonal section. International Student Conference on Applied Mathematics and Informatics (ISCAMI-2012), Malenovice, Czech Republic.
- **T. Jwaid**, B. De Baets, H. De Meyer, Double-conic copulas. Tenth International Conference on Fuzzy Set Theory and Applications (FSTA-2010), Liptovsky Jan, Slovak Republic.
- **T. Jwaid**, B. De Baets and H. De Meyer, Semi-quadratic copulas, Tenth International Conference on Fuzzy Set Theory and Applications (FSTA-2010), Liptovsky Jan, Slovak Republic.

Lectures in conferences

- **T. Jwaid**, B. De Baets, H. De Meyer, Variolinear copulas with a given diagonal section. International Student Conference on Applied Mathematics and Informatics (ISCAMI-2012), Malenovice, Czech Republic.
- **T. Jwaid**, B. De Baets, H. De Meyer, Biconic aggregation functions with a given diagonal or opposite diagonal section. Seventh conference of the European Society for Fuzzy Logic and Technology (EUSFLAT-2011), Aix-Les-Bains, France.

- **T. Jwaid**, B. De Baets, H. De Meyer, Orthogonal copulas with a given diagonal or opposite diagonal section. Sixth International Summer School on Aggregation Operators (AGOP-2011), Benevento, Italy.
- **T. Jwaid**, B. De Baets, H. De Meyer, Double-conic copulas. Tenth International Conference on Fuzzy Set Theory and Applications (FSTA-2010), Liptovsky Jan, Slovak Republic.
- **T. Jwaid**, B. De Baets and H. De Meyer, Semi-quadratic copulas, Tenth International Conference on Fuzzy Set Theory and Applications (FSTA-2010), Liptovsky Jan, Slovak Republic.
- **T. Jwaid**, B. De Baets and H. De Meyer, Semilinear copulas with given diagonal sections. Fifth International Summer School on Aggregation Operators (AGOP-2009), Palma de Mallorca, Spain.

Awards and nominations

- The paper '**On the construction of semiquadratic copulas**' won the best paper award in the Seventh International Summer School on Aggregation Operators (AGOP-2013), Pamplona, Spain.
- The paper '**Biconic semi-copulas with a given section**' was nominated for the best paper award in the Eighth Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT-2013), Milano, Italy.