Phase transitions for Gödel incompleteness

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Abstract

Gödel's first incompleteness result from 1931 states that there are true assertions about the natural numbers which do not follow from the Peano axioms. Since 1931 many researchers have been looking for natural examples of such assertions and breakthroughs have been obtained in the seventies by Jeff Paris [30] (in part jointly with Leo Harrington [31] and Laurie Kirby [20]) and Harvey Friedman [33,34] who produced first mathematically interesting independence results in Ramsey theory (Paris) and well-order and well-quasi-order theory (Friedman).

In this article we investigate Friedman style principles of combinatorial well-foundedness for the ordinals below ε_0. These principles state that there is a uniform bound on the length of decreasing sequences of ordinals which satisfy an elementary recursive growth rate condition with respect to their Gödel numbers.

For these independence principles we classify (as a part of a general research program) their phase transitions, i.e. we classify exactly the bounding conditions which lead from provability to unprovability in the induced combinatorial well-foundedness principles.

As Gödel numbering for ordinals we choose the one which is induced naturally from Gödel's coding of finite sequences from his classical 1931 paper on his incompleteness results.

This choice makes the investigation highly non trivial but rewarding and we succeed in our objectives by using an intricate and surprising interplay between analytic combinatorics and the theory of descent recursive functions. For obtaining the required bounds on count functions for ordinals we use a classical 1961 Tauberian theorem by Parameswaran which apparently is far remote from Gödel's theorem.

Key words: Analytic combinatorics, proof-theoretic ordinals, phase transitions, Gödel incompleteness, Friedman-style independence results, first order Peano arithmetic, Tauberian theory, additive number theory, multiplicative number theory
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1 Introduction

Phase transition is a type of behaviour wherein small changes of a parameter of a system cause dramatic shifts in some globally observed behaviour of the system, such shifts being usually marked by a sharp 'threshold point'. (An everyday life example of such thresholds are ice melting and water boiling temperatures.) This kind of phenomena nowadays occurs throughout many mathematical and computational disciplines: statistical physics [9], evolutionary graph theory [4], percolation theory [18], computational complexity [10], artificial intelligence [26], etc.

This paper is part of a general research program on phase transition thresholds for Gödel incompleteness results. The underlying idea is roughly speaking as follows. Let us assume that A is a given assertion in the language of first order Peano arithmetic (PA) which is parametrized with a non negative rational number r and that $A(r)$ is true for all values of r. Let us further assume that $A(r)$ is unprovable for large enough values of r and that this property is monotone in the following sense; if $r < s$ and $A(r)$ does not follow from PA then $A(s)$ also does not follow from PA. Moreover assume that for small enough values of r the assertion $A(r)$ does follow from PA. In this situation there will be a phase transition threshold $\rho \in \mathbb{R}$ given by the resulting Dedekind cut. Determining ρ will in general provide valuable information about the general question:

What makes a true assertion A unprovable from PA*?*

For Kruskal's theorem the critical value for ρ is given by $\frac{\ln(2)}{\ln(\alpha)}$ (where α has numerical value $2.95576\ldots$) is Otter's tree constant (which is currently not known to be rational or algebraic) [28,37]).
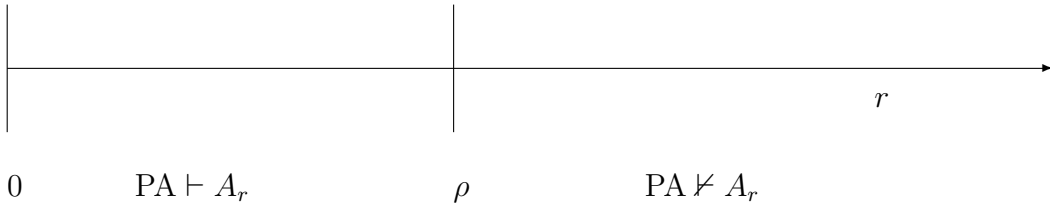
In a more general context we may assume that A depends on a function parameter f for a number-theoretic function f. We may assume that A_f is always true for any f and provable if f is very slow growing. Moreover we may assume similarly as above that if A_f is provable in PA and g is eventually dominated by f that then A_g is provable in PA too. Moreover we assume that A_f becomes unprovable in PA if f grows reasonably fast. Determining the threshold for f


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will in general again provide valuable information about the general question:

What makes a true assertion A unprovable from PA*?*

One a more refined level our results will have various implications in first order proof theory regarding (one) consistency of PA as indicated in example statements after the proofs of Theorems 6 and 7.

We obtained in the meantime a series of results in this respect concerning unprovability results for ordinals [24,37,38,40,41,43,45,46], well quasi orders [12,37,40,45,46] and Ramseyan statements (in the style of Paris Harrington or Kanamori McAloon) [6,8,21,27,39,42,45]. In particular it turned out that e.g. the largeness condition in the Paris Harrington assertion [31] emerges naturally from finite Ramsey theory [39,14]. It is further somewhat surprising that the phase transition threshold related to the Kanamori McAloon theorem for fixed dimension $d \geq 2$ is different from the corresponding threshold related to the Paris Harrington assertion for the same fixed dimension d since these statements in their original form are equivalent over $I\Delta_0 + (\exp)$ (according to an unpublished preprint of Jeff Paris). (For a general and recent survey on unprovability results and a rather comprehensive bibliography on this subject we refer the reader to [5].)
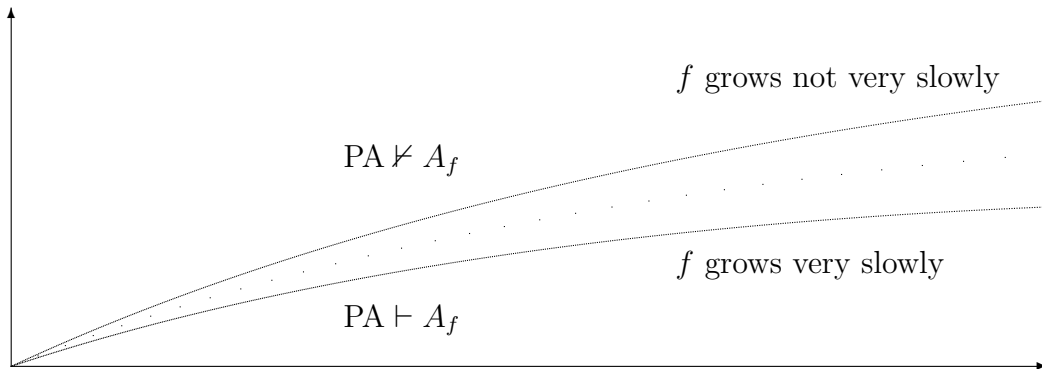
In this paper we determine phase transition thresholds for Friedman style assertions about the combinatorial well-foundedness of ε_0. It is well known by Gentzen [16] that PA does not prove the well-foundedness of ε_0. Even sharper, PA does not prove that there are no primitive recursive descending sequences through ε_0. Friedman refined this by showing that PA does not prove that there is no elementary recursive descending chain of ordinals through ε_0. In [15] it is even sharper shown (by Friedman and Sheard) that there is a uniform bound on the lengths of decreasing sequences of ordinals below ε_0 when their corresponding term-complexities are bounded by an elementary function. (When term-complexity is measured by length the elementary bounding function even can be chosen as a linear function [33,34].)

In this article we consider phase transition thresholds for such bounding functions. If we go for sharp results, the consideration will depend on the choice

of the term-complexity function and so we choose one of the most natural complexity functions for the ordinals below ε_0. We simply take Gödel's coding from his classical paper [17] on "*Über eine formal unentscheidbare Eigenschaft der principia mathematica und verwandter Systeme*" which is also the typical coding used in texts on recursion theory. It turns out that this choice makes the investigation challenging and there is (at least as we can judge) no a priori guess possible regarding the resulting thresholds.

One reason is that the Gödel coding depends on the prime numbers. Another reason is that thresholds are intrinsically related to deep questions on asymptotic enumeration. Luckily all these problems can be overcome by analytic combinatorics and Tauberian theory (in particular Parameswaran's Tauberian theorem) and we are able to obtain a full solution. The resulting approach is rather flexible and versatile and can be used to deal with a large class of natural codings of ordinals.

To attack the problem related to the Gödel coding we take advantage by studying first the corresponding additive situation which is provided by term complexities for ordinals which are induced from generalized Mahler partitions. Dealing with such additive norms is usually much simpler (cf., e.g., Burris [7]) and classifying thresholds for the Mahler norms is a useful preparatory step to deal finally with the intricate Gödel numberings.

2 Some analytic combinatorics for ε_0

In this section we determine the asymptotic for count functions emerging from Gödel's coding for the ordinals below ε_0, which in the sequel are denoted by small Greek letters, and for count functions resembling generalized Mahler partitions.

Let $(p_i)_{i=1}^{\infty}$ denote the enumeration of the primes starting with $p_1 = 2$. We put $\lceil 0 \rceil := 1$ and if $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ with $\alpha_1 \geq \ldots \geq \alpha_n$. then we put $\lceil \alpha \rceil := p_1^{\lceil \alpha_1 \rceil} \cdot \ldots \cdot p_n^{\lceil \alpha_n \rceil}$. This is the classical Gödel numbering going back to Gödel 1931 [17]. It also typically appears in textbooks on recursion theory. We put

$$G_\beta(m) := \#\{\alpha < \beta : \lceil \alpha \rceil \leq m\}. \tag{1}$$

(Note that for $\alpha = \omega^\omega$ we get a multiplicative analogue of the Mahler partition function [11].) Getting non trivial bounds on G_β seems difficult but luckily very powerful machinery from analytic combinatorics has already been developed and a seminal paper by Parameswaran [29] can be used to obtain weak asymptotics for G_β. These results are strong enough for the intended applications to phase transitions for Gödel incompleteness. We believe that

even better bounds on the count functions are available by applying the saddle point method together with Cauchy's integral formula a la Dumas and Flajolet [13] but we leave this for the experts in analytic combinatorics.

There are other complexity measures which can be assigned to members of ε_0. The desired property of such a measure $c : \varepsilon_0 \to \mathbb{N}$ is that for any $k \in \mathbb{N}$ and any $\beta < \varepsilon_0$ the number of elements in $\{\alpha < \beta : c(\alpha) \leq k\}$ is finite.

A canonical choice for c is given by the Mahler norm. We may put $M(0) := 0$ and if $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ with $\alpha_1 \geq \ldots \geq \alpha_n$ we may define by recursion

$$M\alpha := 2^{M\alpha_1} + \ldots + 2^{M\alpha_n}.$$

In this case we put

$$M_\beta(n) := \#\{\alpha < \beta : M\alpha = n\}. \tag{2}$$

Again one may ask for the asymptotic of $M_\beta(n)$ as $n \to \infty$. For a certain choice of β there is much information on this problem available from the literature about Mahler partitions. Indeed, $M_{\omega^\omega}(m)$ is the number of Mahler partitions of m in sums of exponentials with base 2 (See, e.g., [11] for a seminal treatment of the related asymptotic). For example, it is well known that

$$\ln(M_{\omega^\omega}(n)) \sim \frac{1}{2\ln(2)} (\ln(n))^2$$

as $n \to \infty$.

We investigate the Mahler norm since it can be used to approximate the Gödel coding. The Mahler norm behaves additively whereas the Gödel coding behaves approximately multiplicatively. It turns out that the Mahler norm behaves more nicely with respect to the asymptotic of the induced count functions and therefore we study it in detail. Later we shall infer results from Mahler norms to Gödel codings.

Now we come to the key result which we use to study the asymptotic of (the Mahler norm and) the Gödel coding. This result by Parameswaran is as follows.

Theorem 1 (Parameswaran [29]) *Suppose that all the following conditions hold:*

(1) *$L(u)$ and $P(u)$ are functions on the non negative reals such that $\int_0^R L(u)du$ and $\int_0^R P(u)du$ exist in the Lebesgue sense for every positive R.*
(2) *$\exp\left(s \int_0^\infty \frac{e^{-su}}{1-e^{-su}} L(u)du\right) = s \int_0^\infty P(u)e^{-su}du$ for all positive s,*
(3) *$\langle N, N^* \rangle$ form a pair of conjugate (for a definition see the next theorem) slowly varying functions,*

(4) N *is non decreasing,*

(5) $\int_0^u \frac{L(t)}{t} dt \sim N(u)$ *as* $u \to \infty$, *and*

(6) $P(u)$ *is non decreasing.*

Then $\ln P(u) \sim \frac{1}{N^*(u)}$ *as* $u \to \infty$.

The notion of conjugateness for slowly varying functions is due to de Bruijn.

Theorem 2 (de Bruijn [11]) *If N is slowly varying, then there is a (asymptotically uniquely determined) slowly varying function N^* (the so called de Bruijn conjugate of N) such that $N^*(x \cdot N(x)) \cdot N(x) \to 1$ as $x \to \infty$ and $N(x \cdot N^*(x)) \cdot N^*(x) \to 1$ as $x \to \infty$.*

Some elementary facts concerning de Bruijn conjugates are listed in the appendix of [3]. We now state our main results. For a compact presentation we use the following notations. We put

$$\ln_{d+1}(x) := \ln(\max\{1, \ln_d(x)\})$$

where $\ln_1(x) = \ln(\max\{1, x\})$. (By convention we therefore have that $\ln_d$ can be 0 but is modified such that it never goes below [or becomes undefined].) Moreover we put

$$\omega_{d+1}(k) := \omega^{\omega_d(k)}$$

where $\omega_0(k) := k$. Also we put as usual $\omega_d := \omega_d(1)$. (The idea is that d counts the hight of the exponential tower so that, for example, $\omega_1(k) = \omega^k$.)

In addition we put

$$\exp_{d+1}(x) := \exp(\exp_d(x))$$

where $\exp_1(x) = \exp(x) = \sum_{i=0}^{\infty} \frac{x^i}{i!}$.

Theorem 3 *(1) If $\beta = \omega^k$ then there exist explicitly calculable constants C_1, C_2 such that*

$$M_\beta(x) \sim C_1 \cdot x^{k-1}, \tag{3}$$

$$G_\beta(x) \sim C_2 \cdot \left(\frac{\ln(x)}{\ln(\ln(x))}\right)^k. \tag{4}$$

(2) If $\beta = \omega_2(k)$ then there exist explicitly calculable constants C_3, C_4 such that

$$\ln(M_\beta(x)) \sim C_3 \cdot (\ln(x))^{k+1}, \tag{5}$$

$$\ln(G_\beta(x)) \sim C_4 \cdot \ln(\ln(x)) \cdot \left(\frac{\ln\ln(x)}{\ln(\ln(\ln(x)))}\right)^k. \tag{6}$$

(3) If $\beta = \omega_d(k)$ and $d \geq 2$ then with the same constants C_3, C_4 as in assertion (2)

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$$\ln_{d-1}(M_\beta(x)) \sim C_3 \cdot \left(\ln_{d-1}(x)\right)^{k+1}, \tag{7}$$

$$\ln_{d-1}(G_\beta(x)) \sim C_4 \cdot \ln_d(x) \cdot \left(\frac{\ln_d(x)}{\ln(\ln_d(x))}\right)^k. \tag{8}$$

PROOF. Let us start with a proof of assertion (1). The asymptotic (3) is more or less well known. Indeed, we may consider $\{\alpha < \omega^k\}$ as a generalized additive number system generated from the additive primes ω^l for $0 \leq l < k$. By Theorem 2.48 in Burris [7] we therefore obtain

$$M_{\omega^k}(x) \sim \frac{1}{(k-1)!} \frac{1}{\prod_{l<k} 2^l} x^{k-1}$$

and equation (3) follows.

Equation (4) of assertion (1) follows from Karamata's theorem (which can be found, for example, in Korevaar's textbook on Tauberian theory [23]) as shown in [41].

Let us now prove equation (5) from assertion (2). By Remark 2.32 and Theorem 2.48 in Burris [7] we obtain

$$m(x) := \#\{\beta < \omega^k : M(\beta) \leq x\} \sim \frac{1}{k!} \frac{1}{\prod_{l<k} 2^l} x^k.$$

Let

$$n(u) = \sum_{2^l \leq u} M_{\omega^k}(l) = m\left(\frac{\ln(u)}{\ln(2)}\right).$$

Then

$$n(u) \sim \frac{1}{k!(\prod_{l<k} 2^l)(\ln(2))^k}(\ln(u))^k =: L(u).$$

Let

$$C := \frac{1}{k!(\prod_{l<k} 2^l)(\ln(2))^k}.$$

Let

$$N(u) := \int_a^u \frac{L(t)}{t} dt$$

where $a > 0$ is arbitrary but fixed. Then by de l'Hospital's rule

$$N(u) \sim \frac{C}{k+1}(\ln(u))^{k+1}.$$

Let

$$P(u) := \sum_{l \leq u} M_{\omega^{\omega^k}}(l).$$

By Theorem 1 of Parameswaran (or Corollary I* on page 238 of [29]) we obtain

$$\ln(P(u)) \sim \frac{C}{k+1}(\ln(u))^{k+1}.$$

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Moreover this yields $\ln(M_{\omega^{\omega^k}}(u)) \sim \frac{C}{k+1}(\ln(u)^{k+1})$ as indicated on the last page of [29]. So we may put $C_3 := \frac{C}{k+1}$.

Now we prove equation (6) of assertion (2).

We have

$$
\begin{aligned}
&G_{\omega^{\omega^k}}(x) \\
&= \#\{\alpha < \omega^{\omega^k} : \alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n} \geq \alpha_1 \geq \ldots \geq \alpha_n \ \& \ G\alpha \leq x\} \\
&\leq \#\{\langle \alpha_1, \ldots, \alpha_n \rangle : \omega^k > \alpha_1 \geq \ldots \geq \alpha_n \ \& \ 2^{G\alpha_1} \cdot \ldots \cdot 2^{G\alpha_n} \leq x\} \\
&= \#\{\langle \alpha_1, \ldots, \alpha_n \rangle : \omega^k > \alpha_1 \geq \ldots \geq \alpha_n \ \& \ G(\alpha_1) + \ldots + G(\alpha_n) \leq \frac{\ln(x)}{\ln(2)}\}.
\end{aligned}
$$

Now equation (4) of assertion (1) yields

$$
\#\{\langle \alpha_1, \ldots, \alpha_n \rangle : k > \alpha_1 \geq \ldots \geq \alpha_n \ \& \ G(\alpha_1) + \ldots + G(\alpha_n) \leq x\} \sim C\left(\frac{\ln(x)}{\ln(\ln(x))}\right)^k.
$$

Thus Corollary I* of Parameswaran (see [29] page 238) yields

$$
\ln(G_{\omega^{\omega^k}}(x)) \leq \frac{C}{k+1}\ln_2(x) \cdot \left(\frac{\ln_2(x)}{\ln_3(x)}\right)^d.
$$

Let us now prove the reverse inequality. The elementary prime number theorem yields $p_i \leq 6i\ln(i)$ for all i. Hence $G_{\omega^{\omega^k}}(x) \geq \#\{\langle \alpha_1, \ldots, \alpha_n \rangle : \omega^k \geq \alpha_1 \geq \ldots \geq \alpha_n \ \& \ (6n\ln(n)^{G\alpha_1} \cdot \ldots \cdot (6n\ln(n)^{G\alpha_n}) \leq m\}$. Let $Q(x) = \#\{\langle \alpha_1, \ldots, \alpha_n \rangle : \omega^k \geq \alpha_1 \geq \ldots \geq \alpha_n \ \& \ G(\alpha_1) + \ldots + G(\alpha_n) \leq m\}$. We claim that

$$
G_{\omega^{\omega^k}}(x) \geq Q\left(\frac{\ln(x)}{\ln\ln(x)}\right) \tag{9}
$$

for all large x. Indeed, let $g := G(\alpha_1) + \ldots + G(\alpha_n)$. Then $g \leq \frac{\ln(x)}{\ln\ln(x)}$ yields $g \cdot \ln(g \cdot 6 \cdot \ln(g)) \leq \ln(x)$ hence $g \cdot \ln(n \cdot 6 \cdot \ln(n)) \leq \ln(x)$. This proves equation (9).

Therefore

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$$\ln(G_{\omega^{\omega^k}}(x))$$

$$\geq \ln(Q(\frac{\ln(x)}{\ln\ln(x)}))$$

$$\sim C\frac{(\ln(\ln(\frac{\ln(x)}{\ln(\ln(x))})))^{k+1}}{\ln(\ln(\ln(\frac{\ln(x)}{\ln(\ln(x))})))^k}$$

$$\sim C\frac{(\ln(\ln(\ln(x))))^{k+1}}{\ln(\ln(\ln(\ln(x))))^k}.$$

Let us now prove equation (7) of assertion (3) by induction on d. Put

$$m(x) := \#\{\alpha < \omega_d(k) : M\alpha \leq x\}.$$

Let

$$n(u) = \sum_{2^l \leq u} M_{\omega_d(k)}(l) = m(\frac{\ln(u)}{\ln(2)}) =: L(u).$$

The induction hypothesis yields

$$\ln_{d-1}(m(x)) \sim C \cdot (\ln_{d-1}(x))^{k+1} \tag{10}$$

for $C = C_3$. Let

$$N(u) := \int_a^u \frac{L(t)}{t}dt$$

for some arbitrary fixed $a > 0$. Let

$$P(u) := \sum_{l \leq u} M_{\omega_d(k)}(l).$$

We have $N^*(u) \sim \frac{1}{N(u)}$ by Example 2 in Section 3 of Appendix 5 of [3]. By Theorem 1 (of Parameswaran) we therefore obtain

$$\ln(P(u)) \sim N(u).$$

We claim that

$$\ln_d(P(u)) \sim C \cdot (\ln_d(x))^{k+1}. \tag{11}$$

Proof: Pick

$$\epsilon > 0.$$

Then (10) yields

$$m(u) \leq \exp_{d-1}((1 + \frac{\epsilon}{2})C(\ln_{d-1}(u))^k)$$

for large enough u. Hence

$$L(u) \leq \exp_{d-1}((1 + \frac{\epsilon}{2})C(\ln_{d-1}(\frac{\ln(u)}{\ln(2)}))^{k+1}).$$

Thus

$$N(u) \leq \int_a^u \frac{\exp_{d-1}((1 + \frac{\epsilon}{2})C(\ln_{d-1}(\frac{\ln(u)}{\ln(2)}))^{k+1})}{u} du.$$

Put

$$\tilde{N}(u) := \exp_{d-1}((1 + \epsilon)C(\ln_{d-1}(\ln(u)))^{k+1}).$$

Then de l'Hospital's rule yields

$$N(u) = o(\tilde{N}(u))$$

as $u \to \infty$. In particular we obtain that $N(u) \leq \tilde{N}(u)$ for large enough u. Therefore

$$\ln(P(u)) \sim N(u) \leq \tilde{N}(u)$$

for large enough u. Hence

$$P(u) \leq \exp_d((1 + \epsilon)C(\ln_d(u))^{k+1}).$$

By a similar argument we obtain

$$P(u) \geq \exp_d((1 - \epsilon)C(\ln_d(u))^{k+1}).$$

Thus we have shown claim (11). Further

$$\ln_d(M_{\omega_{d+1}(k)}(u)) \sim C(\ln_d(u))^{k+1}(u))$$

follows as indicated on the last page of Parameswaran's paper [29].

Equation (8) of assertion (3) is proved by induction on d. Equation (6) of assertion (2) covers the case $d = 1$. Assume $d \geq 2$ and

$$\ln_{d-1}(\#\{\alpha < \omega_d(k) : \lceil \alpha \rceil \leq x\}) \sim C\left(\frac{(\ln_d(x))^{k+1}}{(\ln_{d+1}(x))^k}\right)$$

for $C = C_4$. Then

$$\ln_{d-1}(\#\{\alpha < \omega_d : \ln(2^{\lceil \alpha \rceil}) \leq x\}) \sim C\left(\frac{(\ln_d(x))^{k+1}}{(\ln_{d+1}(x))^k}\right).$$

We may assume (alternatively we may use an ϵ argument as in the proof of 5) that we can find a subset $S \subset \omega_d(k)$ such that

$$\#\{\alpha \in S : \alpha < \omega_d(k) \ \& \ \ln(2^{\lceil \alpha \rceil}) \leq x\} \sim \exp_{d-1}(C \cdot (\frac{(\ln_d(x))^{k+1}}{(\ln_{d+1}(x))^k})).$$

Let

$$L(u) = \exp_{d-1}\left(C \cdot \left(\frac{(\ln_d(u))^{k+1}}{(\ln_{d+1}(u))^k}\right)\right).$$

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and

$$N(u) = \int_a^u \frac{L(u)}{u} du.$$

Then

$$N(u) \sim L(u) \cdot \frac{d}{du}(\exp_{d-1}\left(C \cdot \left(\frac{(\ln_d(u))^{k+1}}{(\ln_{d+1}(u))^k}\right)\right)$$

and $\frac{1}{N^*(u)} \sim N(u)$ (by Example 2 in Section 3 of Appendix 5 of [3]). Thus

$$\begin{aligned}
&\ln(\#\{\langle \alpha_1, \ldots, \alpha_n \rangle : \alpha_1 \geq \ldots \geq \alpha_n \,\&\, \alpha_1, \ldots, \alpha_n \in S \,\& \\
&\quad 2^{\lceil \alpha_1 \rceil} \cdot \ldots \cdot 2^{\lceil \alpha_n \rceil} \leq x\}) \\
&\sim N(\ln(x))
\end{aligned}$$

and thus

$$\begin{aligned}
&\ln(G_{\omega_{d+1}(k)}(x)) \\
&= \ln(\#\{\langle \alpha_1, \ldots, \alpha_n \rangle : \alpha_1 \geq \ldots \geq \alpha_n \,\&\, \alpha_1, \ldots, \alpha_n \in S \,\& \\
&\quad p_1^{\lceil \alpha_1 \rceil} \cdot \ldots \cdot p_n^{\lceil \alpha_n \rceil} \leq \ln(x)\}) \\
&\geq \ln(\#\{\langle \alpha_1, \ldots, \alpha_n \rangle : \alpha_1 \geq \ldots \geq \alpha_n \,\&\, \alpha_1, \ldots, \alpha_n \in S \,\& \\
&\quad 2^{\lceil \alpha_1 \rceil} \cdot \ldots \cdot 2^{\lceil \alpha_n \rceil} \leq x\}) \\
&\sim \exp_{d-1}\left(C\left(\frac{(\ln_{d+1}(x))^{k+1}}{(\ln_{d+2}(x))^k}\right)\right).
\end{aligned}$$

The lower bound is obtained similarly. Indeed, we have

$$\begin{aligned}
&G_{\omega_{d+1}(k)}(x) \\
&\geq \#\{\langle \alpha_1, \ldots, \alpha_n \rangle : \omega_d(k) > \alpha_1 \geq \ldots \geq \alpha_n \,\&\, p_1^{\lceil \alpha_1 \rceil} \cdot \ldots \cdot p_n^{\lceil \alpha_n \rceil} \leq x\} \\
&\geq \#\{\langle \alpha_1, \ldots, \alpha_n \rangle : \omega_d(k) > \alpha_1 \geq \ldots \geq \alpha_n \,\&\, (6n\ln(n))^{\lceil \alpha_1 \rceil + \ldots + \lceil \alpha_n \rceil} \leq x\} \\
&\geq \#\{\langle \alpha_1, \ldots, \alpha_n \rangle : \omega_d(k) > \alpha_1 \geq \ldots \geq \alpha_n \,\&\, \lceil \alpha_1 \rceil + \ldots + \lceil \alpha_n \rceil \leq \frac{\ln(x)}{\ln(\ln(x))}\} \\
&\geq \#\{\langle \alpha_1, \ldots, \alpha_m \rangle : \omega_d(k) > \alpha_1 \geq \ldots \geq \alpha_n \,\&\, \ln(2^{\lceil \alpha_1 \rceil + \ldots + \lceil \alpha_n \rceil}) \leq \frac{\ln(x)}{\ln(\ln(x))}\} \\
&\sim \exp_d\left(C \cdot \left(\frac{(\ln_{d+1}(x))^{k+1}}{(\ln_{d+2}(x))^k}\right)\right)
\end{aligned}$$

since $d \geq 2$.

3   Resulting phase transitions for Gödel incompleteness

As before small Greek letters range over ordinals below ε_0. We assume basic familiarity with these ordinals. (On an intuitive level these ordinals can also be understood as a certain class of unary functions which contains $\lambda x.0$ and which with two functions f, g also contains the function $\lambda x.x^{f(x)} + g(x)$. The ordering of ordinals is then induced on these functions as the ordering provided by eventual domination. More details about this connection can be found, for example, in [40].)

For a limit $\lambda < \varepsilon_0$ let $\lambda[x]$ be the x-th element of the canonical fundamental sequence for λ. This means that if $\lambda = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ where $\alpha_1 \geq \ldots \geq \alpha_n$ and $\alpha_n = \gamma + 1$ then $\lambda[x] := \omega^{\alpha_1} + \cdots + \omega^{\gamma} \cdot x$ and that if $\lambda = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ where $\alpha_1 \geq \ldots \geq \alpha_n$ and α_n is a limit then $\lambda[x] := \omega^{\alpha_1} + \cdots + \omega^{\alpha_n[x]}$. For ε_0 the fundamental sequence is defined via $\varepsilon_0[x] := \omega_x$. We then can define define the Schwichtenberg-Wainer hierarchy of functions F_α for $\alpha \leq \varepsilon_0$ as follows by recursion on ordinals.

$$
\begin{aligned}
F_0(x) &:= x + 1, \\
F_{\alpha+1}(x) &:= F_\alpha^{(x)}(x) \text{ where the upper index denotes number of iterations,} \\
F_\lambda(x) &:= F_{\lambda[x]}(x) \text{ where } \lambda \text{ is a limit.}
\end{aligned}
$$

It is well known that each function F_α is provably recursive in PA if $\alpha < \varepsilon_0$. Moreover it is well known that every PA-provably recursive function is eventually dominated by F_{ε_0} and it is well known that each function F_d is primitive recursive whereas F_ω grows like the Ackermann function.

Let c be a complexity measure for the ordinals below ε_0. Following Harvey Friedman let $\mathrm{CWF}(\beta, f, c)$ be the statement

$$(\forall K)(\exists L)(\forall \alpha_0, \ldots, \alpha_L < \beta) \left((\forall i \leq L)[c(\alpha_i) \leq K + f(i)] \rightarrow (\exists i < L)[\alpha_i \leq \alpha_{i+1}] \right).$$

$\mathrm{CWF}(\varepsilon_0, c, f)$ states the combinatorial well-foundedness of ε_0. If the complexity measure is elementary recursive then (under some mild extra conditions) there will exist an elementary recursive function f such that $\mathrm{PA} \nvdash \mathrm{CWF}(\varepsilon_0, c, f)$. This has been proved by Friedman and Sheard in [15]. (Note that $\mathrm{CWF}(\varepsilon_0, c, f)$ is always true by König's Lemma.) On the other hand it is clear that $\mathrm{PA} \vdash \mathrm{CWF}(\varepsilon_0, c, f)$ for constant functions f and so there will be a phase transition from provability to unprovability for $\mathrm{CWF}(\varepsilon_0, c, f)$. But it is by no means clear where the threshold is located. It is even not at all obvious that an exponential function leads to unprovability in case that c is defined by the Gödel coding.
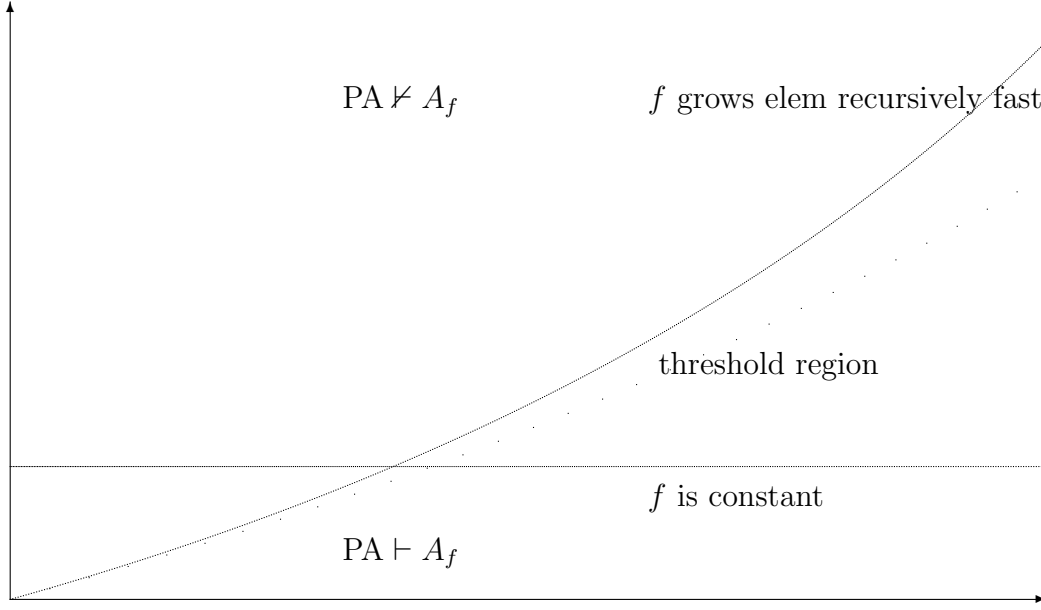
As already mentioned a typical complexity measure is the Mahler norm or the Gödel coding.

Another example for a complexity measure is provided by the *length norm* $|\cdot|$ which is recursively defined by $|0| := 0$ and $|\alpha| = n + |\alpha_1| + \cdots + |\alpha_n|$ if $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ and $\alpha_1 \geq \ldots \geq \alpha_n$. Corresponding phase transitions have been classified in [37]. (The asymptotic of the corresponding count functions for the length norm has been classified rather sharply by Petrogradsky [32] during his investigations on Lie algebras.)

We also use $|i|$ to denote the binary lengths of the natural number i. It will be clear from the context whether we use use $|i|$ for the ordinal norm of i or the length of i. (Typically it will be the latter.)

Theorem 4 *Let*
$$f_\alpha(i) := \exp_{F_\alpha^{-1}(i)}(\sqrt{\ln_{F_\alpha^{-1}(i)}(i)}).$$
(Note that by convention all iterated logarithms are well-defined.) Then the following phase transition dichotomy holds for $\mathrm{CWF}(\varepsilon_0, f, M)$.

(1) *If* $\alpha < \varepsilon_0$ *then*
$$\mathrm{PA} \vdash \mathrm{CWF}(\varepsilon_0, f_\alpha, M).$$

(2) *If* $\alpha = \varepsilon_0$ *then*
$$\mathrm{PA} \nvdash \mathrm{CWF}(\varepsilon_0, f_\alpha, M).$$

PROOF. We first prove assertion (1). Assume that $\varepsilon_0 > \alpha_0 > \ldots > \alpha_n$ is a

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given sequence such that $M(\alpha_i) \leq K + \exp_{F_\alpha^{-1}(i)}(\sqrt{\ln_{F_\alpha^{-1}(i)}(i)})$ for any i with $0 \leq i \leq n$. Put $N := F_\alpha(K) \cdot 2$. We claim that $n \leq N$. If this is proved we are done because $K \mapsto F_\alpha(K)$ is provably recursive in PA.

Assume for a contradiction that $n > N$. The inequality $M\alpha_0 \leq K$ yields $\alpha_0 \leq \omega_{K'}$ for some $K' < K$. In fact K' is $\ln^\star(K)$ (where $\ln^\star$ is the inverse of the superexponential function) but this is not important for the argument. Now choose any i with $N/2 \leq i \leq N$ and consider the condition

$$M(\alpha_i) \leq K + \exp_{F_\alpha^{-1}(i)}(\sqrt{\ln_{F_\alpha^{-1}(i)}(i)}).$$

The function $n \mapsto \exp_n(\sqrt{\ln_n(i)})$ is decreasing in n but $i \mapsto \exp_n(\sqrt{\ln_n(i)})$ is increasing in i. Therefore

$$M(\alpha_i) \leq K + \exp_{F_\alpha^{-1}(N/2)}(\sqrt{\ln_{F_\alpha^{-1}(N/2)}(N)}).$$

hence

$$M(\alpha_i) \leq K + \exp_K(\sqrt{\ln_K(N)}) =: k.$$

Therefore $\alpha_i \in \{\alpha < \omega_{K'} : M(\alpha) \leq k\} =: S$. Since this is true for any i with $N/2 \leq i \leq N$ we obtain that the cardinality of S exceeds $N/2$. By equation (7) of assertion (3) of Theorem 3 (which is provable in RCA_0 in the appropriate way) we know that there is a primitive recursive function p such that the cardinality of $\{\alpha < \omega_K : M(\alpha) \leq l\}$ is bounded from above through $\exp_{K-1}((\ln_{K-1}(l))^2)$ for all $l \geq p(K)$. We may assume that F_α grows faster than p by assuming that α is sufficiently large.

We have

$$\#S \leq \exp_{K'-1}((\ln_{K'-1}(k))^2).$$

A contradiction follows if we can show

$$\exp_{K'-1}((\ln_{K'-1}(k))^2) < \frac{N}{2}.$$

and for this it is sufficient to have

$$(\ln_{K'-1}(k))^2 < \ln_{K'-1}(N/2). \tag{12}$$

By assuming that α is large enough we may assume that

$$F_\alpha(l) \geq \exp_l(2 \cdot l + 4) \tag{13}$$

for all l. Now fix an m such that $\exp_K(2K + m) \leq N/2 \leq \exp_K(2K + m + 1)$.

The assumption (13) yields $m \geq 4$ hence

$$\exp_{K-K'+1}(\sqrt{2K + m + 1} + 1) < \exp_{K-K'+1}(2K + m - 1).$$

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This yields

$$\exp_{K-K'+1}(\sqrt{2K+m+1}+1) < \exp(\frac{1}{2}\exp_{K-K'}(2K+m))$$

and this gives the desired (12).

We now prove assertion (2). Here we will apply a renormalization procedure to approximate (from above) the threshold from known thresholds.

Recall that $|\alpha|$ denotes the length norm of α. Let

$$D(K) := \max\{L : (\exists \alpha_0, \dots, \alpha_L)[\varepsilon_0 > \alpha_0 > \alpha_1 > \dots > \alpha_L \ \& \ (\forall i)[|\alpha_i| \leq K+i]\}.$$

Then $K \mapsto D(K)$ is not provably recursive in PA according to Friedman's theory of descent recursive functions [34]. By Friedman's theory we may therefore pick an elementary function p such that $D(p(l)) > F_{\varepsilon_0}(l)$ for all l.

Given K choose a (unprovable) long sequence β_i such that $\varepsilon_0 > \beta_0 > \dots > \beta_M$ where $M = F_{\varepsilon_0}(K)$ and $|\beta_i| \leq K + i$ for $0 \leq i \leq M$. Then $\omega_K > \beta_0$.

For any $\alpha < \omega_K$ we have that $M\alpha \leq 2_K(|\alpha|)$ and therefore $M\beta_i \leq 2_K(K+i)$ for $0 \leq i \leq M$. By applying renormalization K times we may assume that $\omega_K > \beta_0 > \dots > \beta_M$ and $M\beta_i \leq K + i$ for $0 \leq i \leq M$.

The procedure follows the pattern of the following lines of proof, in which we show how to compress linear bound on growth rates by a sublinear bound. Each compression will allow for one application of a ln function. This technique is exemplified in [46]. Alternatively an inspection of [33,34] also yields that a linear bound leads to an unprovable assertion.

Let us assume without loss of generality that p is now a primitive recursive function such that

$$\#\{\alpha < \omega_K : M\alpha \geq l\} \geq \exp_{K-1}(\frac{1}{4}(\ln_{K-1}(l)^2)$$

for all $l \geq p(K)$. (This can by achieved by equation (7) of assertion (3) of Theorem 3 which is provable in RCA_0 in the appropriate way.) We may further assume that $D(p(l)) > F_{\varepsilon_0}(l)$ for all l by the general theory of descent recursive functions. We are going to define a sequence

$$\omega_{K+4} > \alpha_0 > \dots > \alpha_{F_{\varepsilon_0}(K)}$$

such that

$$M\alpha_i \leq p(K) + \exp_{F_{\varepsilon_0}^{-1}(i)}(\sqrt{\ln_{F_{\varepsilon_0}^{-1}(i)}(i)}).$$

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This is sufficient to prove the assertion. For $i \le p(K)$ we may put

$$\alpha_i = \omega_{K+3} + p(K) - i.$$

Now assume that $i > p(K)$. Put

$$k(i) = \exp_{F_{\varepsilon_0}^{-1}(2^{|i|-1})}\left(\sqrt{\ln_{F_{\varepsilon_0}^{-1}(2^{|i|-1})}(2^{|i|})}\right). \tag{14}$$

Then $|i| = |j|$ implies $k(i) = k(j)$.

Let $M_i := \{\beta < \omega_{K+2} : M(\beta) \le k(i)\}$ and enum_{M_i} be the enumeration function for M_i. Thus $\mathrm{enum}_{M_i}(l)$ is the l-th member of M_i with respect to $<$.

Now put

$$\alpha_i := \omega_{K+2} \cdot \beta_{|i|} + \mathrm{enum}_{M_i}(2^{|i|} - i)$$

for $p(K) < i < M$.

Assume first that α_i is well defined. Then $\alpha_i > \alpha_{i+1}$ for all $i < M$.

It is easy to verify that $M(\omega_K \cdot \gamma) \le 2_K \cdot M\gamma$ for any γ. We may assume that

$$F_{\varepsilon_0}^{-1}(i) \ge 2,$$

that

$$\exp_2(\sqrt{\ln_2(i)} - 1) \ge |i|^2,$$

and that

$$\exp_{K+1}\left(\frac{1}{16}(\ln_{K+1}(2^{|i|-1}))^2\right) \ge 2^{|i|-1}$$

for $i > p(K)$. Therefore using the inequality $a^2 + b^2 \ge 2ab$ we obtain:

$$\begin{aligned} M\alpha_i &\le 2_{K+2} \cdot (K + |i|) + k(i) \\ &\le 2_{K+2} \cdot K + 2_{K+2}^2 + |i|^2 + k(i) \\ &\le 2_{K+2} \cdot K + 2_{K+2}^2 + \exp_{F_{\varepsilon_0}^{-1}(i)}\left(\sqrt{\ln_{F_{\varepsilon_0}^{-1}(i)}(i)}\right). \end{aligned}$$

We still have to show that $\mathrm{enum}_{M_i}(2^{|i|} - i)$ is well defined. For this it suffices to show that the cardinality of M_i is not smaller than $2^{|i|-1}$.

We know that $D(p(K)) \ge F_{\varepsilon_0}(K)$ and thus we obtain $F_{\varepsilon_0}^{-1}(i) \le K$ for all $i \le F_{\varepsilon_0}(K)$. Hence $k(i) \ge \exp_K(\sqrt{\ln_K(2^{|i|-1})})$. Therefore

$$\#M_i \geq \exp_{K+1}(\frac{1}{4}(\ln_{K+1}(\exp_K(\sqrt{\ln_K}(2^{|i|-1})))^2)$$

$$= \exp_{K+1}(\frac{1}{4}(\ln(\sqrt{\ln_K(2^{|i|-1})}))^2)$$

$$= \exp_{K+1}(\frac{1}{16}(\ln_{K+1}(2^{|i|-1}))^2)$$

$$\geq 2^{|i|-1}$$

since $i \geq p(K)$.

Let $I\Sigma_d$ be the fragment of PA where the induction scheme is restricted to formulas with at most d quantifiers. Then it is well known that F_α is provably recursive in $I\Sigma_d$ iff $\alpha < \omega_d$.

Theorem 5 *Let $d \geq 1$. Let*

$$f_\alpha(i) := \exp_{d-1}(\sqrt[F_\alpha^{-1}(i)]{\ln_{d-1}(i)}).$$

Then the following phase transition dichotomy holds for $\mathrm{CWF}(\omega_{d+1}, f, M)$.

(1) If $\alpha < \omega_d$ then

$$I\Sigma_d \vdash \mathrm{CWF}(\omega_{d+1}, f_\alpha, M).$$

(2) If $\alpha = \omega_d$ then

$$I\Sigma_d \nvdash \mathrm{CWF}(\omega_{d+1}, f_\alpha, M).$$

PROOF. We first prove assertion (1). Assume that $\omega_{d+1} > \alpha_0 > \ldots > \alpha_n$ is a given sequence such that $M(\alpha_i) \leq K + \exp_{d-1}(\sqrt[F_\alpha^{-1}(i)]{\ln_{d-1}(i)})$ for $0 \leq i \leq n$. Put $N := F_\alpha(K) \cdot 2$. We may assume that $K \geq 3$. We claim that $n \leq N$. If this is proved we are done because $K \mapsto F_\alpha(K)$ is provably recursive in $I\Sigma_d$.

Assume for a contradiction that $n > N$. The inequality $M\alpha_0 \leq K$ yields $\alpha_0 \leq \omega_d(K')$ for some $K' < K - 1$.

Now consider any i with $N/2 \leq i \leq N$. Consider

$$M(\alpha_i) \leq K + \exp_{d-1}(\sqrt[F_\alpha^{-1}(i)]{\ln_{d-1}(i)}).$$

The function $n \mapsto \exp_{d-1}(\sqrt[n]{\ln_{d-1}(i)})$ is decreasing in n but the function $i \mapsto \exp_{d-1}(\sqrt[n]{\ln_{d-1}(i)})$ is increasing in i. Therefore

$$M(\alpha_i) \leq K + \exp_{d-1}(\sqrt[K]{\ln_{d-1}(N)}) =: k$$

hence

$$\alpha_i \in \{\alpha < \omega_d(K') : M(\alpha) \leq k\} =: S.$$

Since this is true for any i with $N/2 \leq i \leq N$ we obtain that the cardinality of S exceeds $N/2$. By equation (8) of assertion (3) of Theorem 3 (which is provable in RCA_0 in the appropriate way) we know that there is a primitive recursive function p such that the cardinality of $\{\alpha < \omega_d(K') : M(\alpha) \leq l\}$ is bounded from above through $\exp_{d-1}((\ln_{d-1}(l))^{K'+1})$ for all $l \geq p(k)$. We may assume that F_α grows faster than p by assuming α sufficiently large. We may further assume that $N \geq \exp_{d-1}(K^{K^2})$.

Then we obtain $\exp_{d-1}((\ln_{d-1}(k))^{K'+1})$ as upper bound for the cardinality of S. A contradiction follows if we can show

$$\exp_{d-1}((\ln_{d-1}(k))^{K'+1}) < N/2. \tag{15}$$

Since $k < \exp_{d-1}\left(K + \sqrt[K]{\ln_{d-1}(N)}\right)$ for this it is sufficient to have

$$\exp_{d-1}\left((K + \sqrt[K]{\ln_{d-1}(N)})^{K'+1}\right) < N/2.$$

Now fix an r such that

$$\exp_{d-1}(r) \leq N < \exp_{d-1}(r+1). \tag{16}$$

Then the claim (15) follows from $(\sqrt[K]{(r+1)} + K)^{K'+1} < r$ which is a consequence of $N \geq \exp_{d-1}(K^{K^2})$, $K \geq 3$ and (16).

We now prove assertion (2). Again we will apply a renormalization procedure to approximate (from above) the threshold from known thresholds.

Again recall that $|\alpha|$ denotes the length norm of α. Let

$$D(K) := \max\{L : (\exists \alpha_0, \ldots, \alpha_L)\big[\omega_{d+1} > \alpha_0 > \ldots > \alpha_n \ \& \ (\forall i)[|\alpha_i| \leq K + i]\big]\}.$$

Then $K \mapsto D(K)$ is not provably recursive in $\mathrm{I}\Sigma_d$ according to Friedman's theory of descent recursive functions. We thus assume that $D(p(l)) > F_{\omega_d}(l)$ for some elementary recursive function p.

Given K choose a (unprovable) long sequence β_i' such that $\omega_d(K) > \beta_0' > \ldots > \beta_M'$ where $M = D(K)$ and $|\beta_i'| \leq K + i$ for $0 \leq i \leq M$.

For any $\alpha < \omega_d(K)$ we have that $M\alpha \leq 2_d(N\alpha)$ and therefore $M\beta_i' \leq 2_d(K+i)$ for $0 \leq i \leq M$. By applying renormalization K times we find a sequence β_i with $\omega_{d+1} > \beta_0 > \ldots > \beta_M$ and $M\beta_i \leq K + i$ for $0 \leq i \leq M$.

The procedure follows the pattern of this proof where we show how to compress linear growth by sub linear growth. Each compression will allow for one application of a ln function.

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By equation (7) of assertion (3) of Theorem 3 we may (similarly as before) assume that there is a primitive recursive function p and a constant C such that

$$\#\{\alpha < \omega_d(K+2) : M\alpha \geq l\} \geq \exp_{d-1}\left(C(\ln_{K-1}(l))^{K+3}\right)$$

for all $l \geq p(K)$.

We are going to define a sequence

$$\omega_d(K+4) > \alpha_0 > \ldots > \alpha_{F_{\omega_d}(K)}$$

such that

$$M\alpha_i \leq p(K) + \exp_{d-1}\left({}^{F_{\omega_d}^{-1}(i)}\!\!\sqrt{\ln_{d-1}(i)}\right)$$

for $i \leq F_{\omega_d}(K)$. This is sufficient to prove the assertion. For $i \leq p(K)$ we may put $\alpha_i = \omega_d(K+3) + p(K) - i$. Now assume that $i > p(K)$. Put

$$k(i) = \exp_{d-1}\left({}^{F_{\omega_d}^{-1}(2^{|i|-1})}\!\!\sqrt{\ln_{d-1}(2^{|i|})}\right). \tag{17}$$

Then $|i| = |j|$ implies $k(i) = k(j)$.

Let $M_i := \{\beta < \omega_d(K+2) : M(\beta) \leq k(i)\}$ and enum_{M_i} be the enumeration function for M_i. Thus $\mathrm{enum}_{M_i}(l)$ is the l-th member of M_i with respect to $<$.

Now put $\alpha_i := \omega_d(K+2) \cdot \beta_{|i|} + \mathrm{enum}_{M_i}(2^{|i|} - i)$ for $p(n) < i < M$.

Assume first that α_i is well defined. Then $\alpha_i > \alpha_{i+1}$ for all $i < M$.

We have that $M(\omega_d(K) \cdot \gamma) \leq 2_d(K) \cdot M\gamma$ for any γ. We may assume that

$$F_{\omega_d}^{-1}(i) \geq 2,$$

that

$$\exp_2(\sqrt{\ln_2(i)} - 1) \geq |i|^2$$

and that

$$\exp_{d-1}(\frac{C}{4}\ln_{d-1}(2^{|i|-1})^{\frac{K+3}{K}}) \geq 2^{|i|-1}$$

for $i > p(K)$. Therefore using the inequality $a^2 + b^2 \geq 2ab$ we obtain:

$$\begin{aligned}
M\alpha_i &\leq 2_d(K+2) \cdot (K + |i|) + k(i) \\
&\leq 2_d(K+2) \cdot K + (2_d(K+2))^2 + |i|^2 + k(i) \\
&\leq 2_d(K+2) \cdot K + (2_d(K+2))^2 + \exp_{d-1}\left({}^{F_{\omega_d}^{-1}(i)}\!\!\sqrt{\ln_{d-1}(i)}\right).
\end{aligned}$$

We still have to show that $\mathrm{enum}_{M_i}(2^{|i|} - i)$ is well defined. For this it suffices to show that the cardinality of M_i is not smaller than $2^{|i|}$.

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We know that $D(p(K)) \geq F_{\omega_d}(K)$ and thus we obtain $F_{\omega_d}^{-1}(i) \leq K$ for all $i \leq F_{\omega_d}(K)$. Hence $k(i) \geq \exp_{d-1}(\sqrt[K]{\ln_{d-1}(2^{|i|-1})})$ for $i \leq F_{\omega_d}(K)$.

Therefore

$$
\begin{aligned}
\#M_i &\geq \exp_{d-1}\left(C\left(\ln_{d-1}(\exp_{d-1}(\sqrt[K]{\ln_{d-1}(2^{|i|-1})}))\right)^{K+3}\right) \\
&= \exp_{d-1}\left(C(\sqrt[K]{\ln_{d-1}(2^{|i|})})^{K+3}\right) \\
&= \exp_{d-1}\left(C(\ln_{d-1}(2^{|i|}))^{\frac{K+3}{K}}\right) \\
&\geq 2^{|i|-1}
\end{aligned}
$$

since $i \geq p(K)$.

Now we turn to the phase transition thresholds for the Gödel coding. The provable versions follow easily from the inequality $\lceil \alpha \rceil \geq 2^{M\alpha}$. As reverse inequality we have for $\alpha < \omega_d$ that $\lceil \alpha \rceil \leq \exp_d((\ln_{d-1}(M\alpha) + \exp_d(d))^{2D+2})$ but we have been unable to deduce the unprovability results from this. Nevertheless the unprovability results can be proved analogously as for the Mahler norms. The proofs will be more intricate but the essential pattern is as before. So we will be more brief in the proofs.

Theorem 6 *Let*

$$
f_\alpha(i) := \exp\left(\exp_{F_\alpha^{-1}(i)}(\sqrt{\ln_{F_\alpha^{-1}(i)}(i)})\right).
$$

Then the following phase transition dichotomy holds for $\mathrm{CWF}(\varepsilon_0, f, \lceil \cdot \rceil)$.

(1) If $\alpha < \varepsilon_0$ then
$$
\mathrm{PA} \vdash \mathrm{CWF}(\varepsilon_0, f_\alpha, \lceil \cdot \rceil).
$$
(2) If $\alpha = \varepsilon_0$ then
$$
\mathrm{PA} \nvdash \mathrm{CWF}(\varepsilon_0, f_\alpha, \lceil \cdot \rceil).
$$

PROOF. We first prove assertion (1). Basically the claim follows from assertion (1) of Theorem 4 and the inequality $2^{M\alpha} \leq \lceil \alpha \rceil$. Indeed, assume that we have a given sequence α_i with $\lceil \alpha \rceil_i \leq K + \exp(\exp_{F_\alpha^{-1}(i)}(\sqrt{\ln_{F_\alpha^{-1}(i)}i}))$. Then $M\alpha_i \leq \frac{1}{\ln(2)}(K + \exp_{F_\alpha^{-1}(i)}(\sqrt{\ln_{F_\alpha^{-1}(i)}i})) \leq K^2 + \exp_{F_\alpha^{-1}(i)+1}(\sqrt{\ln_{F_\alpha^{-1}(i)+1}i})$. The length of such a sequence can be bounded (provably so in PA) according to assertion (1) of Theorem 4.

Now let us proof assertion (2). By a lemma of Friedman we obtain that

$$
D(K) := \max\{L : (\exists \alpha_0, \ldots, \alpha_L)\left[\varepsilon_0 > \alpha_0 > \ldots > \alpha_L \ \& \ (\forall i)[\lceil \alpha_i \rceil \leq K + 2^{2^i}]\right]\}
$$

is not provably recursive in PA. This can also be inferred from the corresponding result using the Mahler norm M and using the estimate $\lceil \alpha \rceil \leq \exp_d((\ln_{d-1}(M\alpha) + \exp_d(d))^{2d+2})$ for $\alpha < \omega_d$. (But as already mentioned we have not been able to use this bound for a direct proof of assertion (2).) By logarithmic compression one shows easily that the following function E is also not provably recursive in PA. (The transition from 2^{2^i} to 2^i follows along the same pattern as the following proof where we obtain the sharp sub exponential threshold from an exponential bound on the growth rate. Further details are very similar to a corresponding procedure used in [46].) Let us define

$$E(K) := \max\{L : (\exists \alpha_0, \ldots, \alpha_L)\big[\varepsilon_0 > \alpha_0 > \ldots > \alpha_L \ \& \ (\forall i \leq L)[\lceil \alpha_i \rceil \leq K + 2^i]\big]\}.$$

Given K put $M := E(K)$. Find $\varepsilon_0 > \beta_0 > \ldots > \beta_M$ such that $\lceil \beta_i \rceil \leq K + 2^i$. From this we define a long descending sequence of ordinals α_i which satisfy the growth condition. Note that $\beta_0 < \omega_K$. Define $\alpha_i := \omega_{K+3} + p(K) - i$ for small i as in Theorem 4 for some suitable prim. rec. function p. Let

$$h(i) := F_{\varepsilon_0}^{-1}(2^{|i|-1}).$$

Define for large enough i

$$M_i := \{\alpha < \omega_{K+2} : \lceil \alpha \rceil \leq \exp_{h(i)+1}\big(\sqrt{\ln_{h(i)}(2^{|i|})}\big)\}$$

and

$$\alpha_i := \omega_{K+2} \cdot \beta_{|i|} + \text{enum}_{M_i}(2^{|i|} - i).$$

For estimating the Gödel number of α_i we use the following technical facts which are easily proved by induction on ordinals:

$$\lceil \omega_d \cdot \alpha \rceil \leq \exp_d(d) \cdot \lceil \alpha \rceil^2$$

$$\lceil \alpha + \beta \rceil \leq \lceil \alpha \rceil \cdot \exp(\ln(\lceil \beta \rceil) \cdot \max\{2 \cdot \ln\ln(\lceil \beta \rceil + 2), 2 \cdot \ln\ln(\lceil \alpha \rceil + 2)\}).$$

Let

$$e = \exp_{h(i)+1}\big(\sqrt{\ln_{h(i)}(i)}\big)$$

and

$$f = \ln(\max\{\ln\ln(e), \exp_{K+2}(K+2) \cdot (\lceil \beta_{|i|} \rceil)^2\}).$$

Then

$$\lceil \alpha_i \rceil \leq \exp_{K+2}(K+2) \cdot (\lceil \beta_{|i|} \rceil)^2 \cdot \exp(\ln(e) \cdot (2 \cdot f + 2))$$
$$\leq \exp_{K \cdot 2}(K) + \exp(\exp_{f(i)}(\sqrt{\ln_{f(i)}(i)})).$$

The α_i are strictly decreasing. To prove their well-definedness compute with equation (8) of assertion (3) of Theorem 3

$$\#M_i \geq \#\{\alpha < \omega_{K+2} : \lceil \alpha \rceil \leq \exp_{h(i)+1}(\sqrt{\ln_{h(i)}(2^{|i|})})\}$$

$$\geq \exp_{K+1}\Big(\frac{\ln_{K+2}(\exp_{K+1}(\sqrt{\ln_K(2^{|i|})})))^2}{\ln_{K+3}(\exp_{K+1}(\sqrt{\ln_K(2^{|i|})}))}\Big)$$

$$\geq \exp_{K+1}((\frac{1}{4}\ln_{K+1}(2^{|i|}))^2) \geq 2^{|i|-1}.$$

Using classical results from proof theory (see, for example, [15]) one may now conclude that (provably in $I\Sigma_1$) $\mathrm{CWF}(\varepsilon_0, f_{\varepsilon_0}, \lceil \cdot \rceil)$ implies the one consistency of PA. Moreover for $\alpha < \varepsilon_0$ $\mathrm{CWF}(\varepsilon_0, f_\alpha, \lceil \cdot \rceil)$ does not imply the one consistency of PA (over $I\Sigma_1$).

Recall that $I\Sigma_d$ is the fragment of PA where the induction scheme is restricted to formulas with at most d quantifiers.

Theorem 7 *Let $d \geq 1$. Let*

$$f_\alpha(i) := \exp\Big(\exp_{d-1}\big(\sqrt[F_\alpha^{-1}(i)]{\ln_{d-1}(i)}\big)\Big).$$

Then the following phase transition dichotomy holds for $\mathrm{CWF}(\omega_{d+1}, f, \lceil \cdot \rceil)$.

(1) If $\alpha < \omega_d$ then

$$I\Sigma_d \vdash \mathrm{CWF}(\omega_{d+1}, f_\alpha, \lceil \cdot \rceil).$$

(2) If $\alpha = \omega_d$ then

$$I\Sigma_d \nvdash \mathrm{CWF}(\omega_{d+1}, f_\alpha, \lceil \cdot \rceil).$$

PROOF. Assertion (1) follows again easily by $\lceil \alpha \rceil \geq 2^{M\alpha}$ and assertion (1) of Theorem 4.

Now we prove assertion (2). Let

$$D(K) := \max\{L : (\exists \alpha_0, ..., \alpha_L)\big[\omega_{d+1} > \alpha_0 > ... > \alpha_L \ \& \ (\forall i)[M\alpha_i \leq K+\ln(f_{\omega_d}(i))]\big]\}.$$

Then $K \mapsto D(K)$ is not provably recursive in PA.

Given K put $M := D(K)$. Without loss of generality we assume $M \geq F_{\omega_d}(K)$. Find $\omega_{d+1} > \beta_0 > \ldots > \beta_M$ such that $\lceil \beta_i \rceil \leq K + 2^i$. Then $\beta_0 < \omega_d(K)$. Let $h(i) := F_{\omega_d}^{-1}(2^{|i|-1})$. Define for large i

$$M_i := \{\alpha < \omega_d(K \cdot 2) : \lceil \alpha \rceil \leq \exp_d\big(\sqrt[h(i)]{\ln_{d-1}(2^{|i|})}\big)\}$$

and

$$\alpha_i = \omega_d(K \cdot 2) \cdot \beta_{|i|} + \mathrm{enum}_{M_i}(2^{|i|} - i).$$

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Let $e = \exp_d(\sqrt[h(i)]{\ln_{d-1}(i)})$. Then

$$\lceil \alpha_i \rceil \leq \exp_d(d) \cdot (\lceil \beta_{|i|} \rceil)^2 \cdot 2^{2 \cdot \ln(e) \cdot \max\{\ln\ln(e)), \ln\ln(\lceil \beta_{|i|} \rceil + 2)\}}$$
$$\leq \exp_{2d}(d) + \exp_d(\sqrt[F_{\omega_d}^{-1}(i)]{\ln_{d-1}(i)}).$$

Then α_i strictly decreasing. To check that α_i is well-defined compute with equation (8) of assertion (3) of Theorem 3

$$\#M_i \geq \#\{\alpha < \omega_d(K \cdot 2) : \lceil \alpha \rceil \leq \exp_d(\sqrt[h(i)]{\ln_{d-1}(2^{|i|})})\}$$
$$\geq \exp_{d-1}\Big(\frac{(\ln_d(\exp_d(\sqrt[K]{\ln_{d-1}(2^{|i|})})))^{2K+1}}{(\ln_{d+1}(\exp_d(\sqrt[K]{\ln_{d-1}(2^{|i|})})))^{2K}}\Big)$$
$$\geq \exp_{d-1}((\ln_{d-1}(i))^{\frac{2K}{K}}) \geq 2^{|i|}.$$

Using classical results from proof theory one may now conclude that (provably in IΣ_1) CWF($\omega_{d+1}, f_{\omega_d}, \lceil \cdot \rceil$) implies the one consistency of IΣ_d. Moreover for $\alpha < \omega_d$ CWF($\omega_{d+1}, f_\alpha, \lceil \cdot \rceil$) does not imply the one consistency of IΣ_d (over IΣ_1).

In a sequel paper we will exploit our investigations to prove (in a joint project with A.R. Woods) zero one laws for segments of ε_0.

In addition we plan to investigate further the analytic properties of M_α and G_α with J.P. Bell.

Conjecture: Fix $\alpha < \varepsilon_0$ such that $\alpha \geq \omega^\omega$. Choose any sentence Φ in the first order language of linear orders. Then

$$\lim_{n \to \infty} \frac{\#\{\beta < \alpha : \beta \models \Phi \ \& \ \lceil \beta \rceil \leq n\}}{\#\{\beta < \alpha : \lceil \beta \rceil \leq n\}} \in \{0, 1\}$$

and

$$\lim_{n \to \infty} \frac{\#\{\beta < \alpha : \beta \models \Phi \ \& \ M\beta \leq n\}}{\#\{\beta < \alpha : M\beta \leq n\}} \in \{0, 1\}.$$

Intuitively this means that the probability that Φ holds on (the set of predecessors of) a randomly chosen ordinal below α is either zero or one. For $\alpha = \varepsilon_0$ we further expect a limit law, i.e. that the corresponding limits will exist in the interval $[0, 1]$.

Question: What is the connection between ordinal count functions and Beckmann's dynamic ordinals [2] from bounded arithmetic? According to Theorem 3 they seem to be closely related.

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