Constructions and characterisations of (semi)partial geometries

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Preface

Mathematics is often defined as the study of quantity, magnitude, and relations between numbers or symbols. Mathematics first arose from the practical need to measure time and to count. Thus, the history of mathematics begins with the origins of numbers and recognition of the dimensions and properties of space and time. The earliest evidence of primitive forms of counting occurs in notched bones and scored pieces of wood and stone. Early uses of geometry are revealed in patterns found on ancient cave walls and pottery. As civilizations arose in Asia and the Near East, the field of mathematics evolved. Both sophisticated number systems and basic knowledge of arithmetic, geometry, and algebra began to develop. The Greeks were the first to develop a truly mathematical spirit. They were interested not only in the applications of mathematics but in its philosophical significance, which was especially appreciated by Plato.

Twentieth-century mathematics is highly specialized and abstract. In addition to purely theoretical developments, devices such as high-speed computers influenced both the content and the research of mathematics. Among the areas of mathematical research that were developed in the 20th century are abstract algebra, non-Euclidean geometry (finite and infinite), abstract analysis, mathematical logic, and the foundations of mathematics.

Mathematics is everywhere throughout modern life. Baking a cake or building a house involves the use of numbers, geometry, measures, and space. The design of precision instruments, the development of new technologies, and advanced computers all use technical mathematics. In particular finite mathematics turns out to be very useful for the development of digital information technology, cryptography, mobile phones and the internet.

There are two major divisions of mathematics: pure and applied. Applied mathematics develops tools and techniques for solving specific problems of business and engineering. Pure mathematics investigates the subject solely for its theoretical interest, for its beautiful constructions, as if it is a vast crystal palace that we are building. Very often mathematical theories are used only later in applications.

This thesis is the result of four years of research in the crystal palace. Some of the content is already published or is submitted to international journals with refereeing systems (see [11, 27, 28, 29, 43]). Other work is being prepared for publication (see [44, 45]). Since this is a thesis about geometry, I included a few figures in order to illustrate some geometrical constructions. Some of them can be found in miniature on the cover of this thesis.

In the **first chapter** we give some preliminary results that are important for the following chapters. The story starts with the theory of finite *graphs*, which are studied in many areas of mathematics. This is because graphs are very useful and they are general. They are useful because mathematical concepts can be defined easy when we use a schematic representation of binary relations, that is graphs. And so graphs found their origin in applied mathematics, in particular in the modelling of networks. Graphs are general because there exist many binary relations in a set. Therefore a classification of all finite graphs is quite unlikely. This can quickly change as soon as we suppose some regularity or symmetry. Sometimes general conditions put onto an object can lead to the uniqueness of the object. An example of conditions that we put onto the structure of a graph is the concept of a *strongly regular graph*. Under some conditions a graph can carry the structure of a finite geometry. The fruitful links between graphs and geometries are well studied (see for example [13] and [22]). In 1963, Bose [2] introduced *partial geometries* generalising the generalized quadrangles introduced by Tits [113]. Semipartial geometries were introduced by Debroey and Thas [40] in 1978 and generalise both the partial quadrangles (introduced by Cameron [18]) and the partial geometries. There exist several links between on the one hand these graphs and geometries and on the other hand other parts of mathematics such as coding theory and design theory.

The connection between graph theory and geometry that we study here is the following. When we take the point graph of a (semi)partial geometry, then we obtain a strongly regular graph. Conversely we can obtain a (semi)partial geometry out of a strongly regular graph under some specific conditions. In the **second chapter** we investigate some of these graphs which are candidates to carry the structure of a geometry. We obtain several non-existence results having, for example, consequences to existence of *exterior sets* of a quadric. We also obtain positive results, namely a new class of semipartial geometries.

In the **third chapter** we discuss new construction methods for (semi)partial geometries. First we develop the theory of the so called *point derived* semipartial geometries. Next we introduce the concept of a *perp-system* and we prove that it yields partial geometries, strongly regular graphs, *two-weight codes*, *maximal arcs* and *k-ovoids*. We also give some examples, one of them yielding a new $\mathrm{pg}(8, 20, 2)$.

Up to now there are eight partial geometries $\mathrm{pg}(7, 8, 4)$ known. Their point graphs as well as their block graphs are all related to the triality quadric $\mathrm{Q}^+(7, 2)$. In **chapter four** we prove that some of these graphs are faithfully geometric, that is they are the point graph of (up to isomorphism) exactly one partial geometry. We investigate the relations among some of these partial geometries. Generalizing our results for general dimensions, we construct two new families of partial geometries.

Two-weight codes and projective sets having two intersection sizes with respect to hyperplanes are equivalent objects and they define strongly regular graphs. In **chapter five** we construct projective sets that have the same intersection numbers with respect to hyperplanes as the hyperbolic quadric $Q^+(2m-1,q)$. We investigate these sets; we prove that if $q=2$ the corresponding strongly regular graphs are switching equivalent and that they contain subconstituents that are point graphs of partial geometries. If $m=4$ then some of the corresponding partial geometries $\mathrm{pg}(7,8,4)$ are embeddable in Steiner systems $\mathrm{S}(2,8,120)$.

Finally, for many of the known examples of (semi)partial geometries, the points and lines of the geometry are the points and lines of a projective or affine space. The classification of partial geometries in $\mathrm{PG}(n,q)$ and $\mathrm{AG}(n,q)$ is done. The projective full embedding of generalized quadrangles is done by Buekenhout and Lefèvre [14], while De Clerck and Thas [33] did this for proper partial geometries. The affine embedding of partial geometries is done by Thas [99]. De Clerck, Debroey and Thas determined the full embeddings of proper semipartial geometries in $\mathrm{PG}(n,q)$ [34, 42, 110]. Although in [41] Debroey and Thas classified the proper semipartial geometries with a full embedding in $\mathrm{AG}(n,q)$ for $n=2$ and 3, for $n>3$ there is no such classification. In the **last chapter** we give new characterizations of some $(0,\alpha)$-geometries fully embedded in $\mathrm{AG}(n,q)$. As a consequence we obtain that a $(0,\alpha)$-geometry, with $\alpha \neq 1,2$, which is fully embedded in $\mathrm{AG}(n,q)$ is the linear representation model $T^*_{n-1}(\mathcal{K})$. An other model of a semipartial geometry fully embedded in $\mathrm{AG}(4,q)$, q even, due to Hirschfeld and Thas, is the $\mathrm{spg}(q-1,q^2,2,2q(q-1))$ constructed by projecting the quadric $Q^-(5,q)$ from a point of $\mathrm{PG}(5,q)\setminus Q^-(5,q)$ onto a hyperplane of $\mathrm{PG}(5,q)$. This semipartial geometry is characterised amongst the $\mathrm{spg}(q-1,q^2,2,2q(q-1))$ (of which there is an infinite family of non-classical examples due to Brown) by its full embedding in $\mathrm{AG}(4,q)$.

Chapter 1

Preliminary results

In this chapter we give a few definitions and results that are important for the following chapters. We used mainly "Distance regular graphs" of Brouwer, Cohen and Neumaier [5], "Some classes of Rank 2 Geometries" of De Clerck and Van Maldeghem [38] and "A course in combinatorics" of van Lint and Wilson [77].

1.1 Some general definitions from graph theory

A *graph* $\Gamma = (V, E)$ consists of a finite set V, whose elements we call *vertices*, together with a set E of *edges*, where an edge is a subset of the set of cardinality two. In the language of graph theory, our graphs are *undirected* (we do not allow edges to be ordered pairs), without *loops* (we do not permit the two vertices defining an edge to be equal), and without *multiple edges* (a given pair of vertices can define at most one edge). As we shall be concerned with finite graphs only, we shall use the word *graph* in the sense of finite graphs.

Graph theory began life in the area of applicable mathematics, in the modeling of networks of very general kind, in first instance in 1736 with the problem of the bridges of Köningsgberg (see for example [77]).

A graph is called *complete* if every pair of vertices are *adjacent* (that is are contained in an edge). We call a vertex which is adjacent to the vertex x, a *neighbour* of x. The *complement* of the graph Γ is the graph Γ^c whose vertices are those of Γ and whose edge set is the complement of the edge set of Γ, that is $V^{|2|} \setminus E$. A *path of length i* joining two vertices x and y of a graph Γ is a sequence $x = x_0, x_1, \ldots, x_i = y$ of vertices such that x_{j-1} is adjacent to x_j, $j = 1, \ldots, i$. Being joined by a path is an equivalence relation. The *distance* $d(x, y)$ of two vertices x and y is the length of a shortest path (called *geodesic*) from x to y. The *diameter* of Γ denoted by diam(Γ), is the maximal distance occurring in Γ. The set of vertices at distance i from a vertex x of Γ is denoted by $\Gamma_i(x)$, $i = 1, \ldots,$diam(Γ). Often we denote $\Gamma_1(x)$ simply by $\Gamma(x)$ and we call it the *first subconstituent* of the graph Γ. Similarly we denote $\Gamma_2(x)$ by $\Delta(x)$

and we call it the *second subconstituent* of Γ. If x is a vertex of a graph Γ, we let $\delta(x)$ denote the *valency* of x which is the number of edges containing x, or equivalently, the number $|\Gamma(x)|$ of vertices adjacent to x, where $|X|$ denotes the size of a set X. If all the vertices have the same valency, the graph is called *regular*, and the common valency is the *valency of the graph*. If we denote this number with the letter k, then the graph Γ is called regular of *degree* k. We call a graph *connected* if for every two vertices x and y in a graph Γ, there exists a path of finite length from x to y. A *clique* of a graph Γ is a subset of vertices of Γ of which any two of them are *joined* (that is any two of them are adjacent). A *coclique* of Γ is a subset of vertices of which no two are adjacent. An *isomorphism* of the graph $\Gamma = (V, E)$ onto the graph $\Gamma' = (V', E')$ is a bijection of V onto V', such that adjacent vertices in Γ are mapped on adjacent vertices in Γ'. An isomorphism from Γ onto itself is called an *automorphism*.

1.2 Strongly regular graphs

1.2.1 Definitions

The vast area of graph theory is mainly concerned with questions about general relations on a set. This generality usually means that results obtained are not powerful enough, to have useful consequences in other fields of finite geometry. An interesting class of graphs having a lot of connections with other mathematical theories, is the class of strongly regular graphs.

A *strongly regular graph*, denoted by $\mathrm{srg}(v, k, \lambda, \mu)$, is a graph Γ with v vertices, which is regular of degree k ($k < v-1$) and such that any two adjacent vertices have exactly λ common neighbours, while any two distinct non-adjacent vertices have exactly μ ($0 < \mu < k$) common neighbours.

The conditions on k, λ and μ exclude complete graphs, disconnected graphs and their complements. A strongly regular graph Γ has diameter two. A lot of examples of strongly regular graphs are known, see for instance [9, 66].

The *adjacency matrix* A of a graph Γ with v vertices $1, \ldots, v$, is the $v \times v$ (0,1)–matrix with entries $a_{ij} = a_{ji} = 1$ if and only if the vertices i and j are adjacent. Clearly A is symmetric with zeros on the diagonal. The *Bose-Mesner algebra* $\mathcal{U}$ of a strongly regular graph Γ is the three-dimensional algebra generated by I, J and A (with J the $v \times v$ matrix with all of the entries equal to 1). Bose and Mesner [3] have put the link between strongly regular graphs and linear algebra.

1.2.2 Elementary results

The regularity conditions of strongly regular graphs allow us to tell a lot about its parameters. This yields necessary conditions for the existence of $\mathrm{srg}(v, k, \lambda, \mu)$. We will summarise the most important ones in the next theorem. For their proofs see [5, 22, 77].

Theorem 1.1 *If Γ is an* $\mathrm{srg}(v,k,\lambda,\mu)$ *then the following holds:*

1. $k(k-\lambda-1)=\mu(v-k-1)$.

2. *Its complement is an* $\mathrm{srg}(v,v-k-1,v-2k+\mu-2,v-2k+\lambda)$.

3. *If A is the adjacency matrix of Γ, then*

$$AJ=kJ, \qquad A^2+(\mu-\lambda)A+(\mu-k)I=\mu J,$$

and A has three eigenvalues k,

$$r=\frac{\lambda-\mu+\sqrt{(\lambda-\mu)^2+4(k-\mu)}}{2}, l=\frac{\lambda-\mu-\sqrt{(\lambda-\mu)^2+4(k-\mu)}}{2},$$

$(r>l)$ with multiplicities respectively

$$1, f=\frac{-k(l+1)(k-l)}{(k+rl)(r-l)}, g=\frac{k(r+1)(k-r)}{(k+rl)(r-l)};$$

f and g clearly must be integers.

4. *The eigenvalues $r>0$ and $l<0$ are both integers, except for one family of graphs, the conference graphs, which are* $\mathrm{srg}(2k+1,k,\frac{k}{2}-1,\frac{k}{2})$. *For a conference graph the number of vertices can be written as a sum of squares, and the eigenvalues are $\frac{-1+\sqrt{v}}{2}$ and $\frac{-1-\sqrt{v}}{2}$.*

5. *The two Krein conditions:*
   - $(r+1)(k+r+2rl)\le(k+r)(l+1)^2$,
   - $(l+1)(k+l+2rl)\le(k+l)(r+1)^2$.

6. *The two absolute bounds:*
   - $v\le\frac{1}{2}f(f+3)$, *and if there is no equality in the first Krein condition then* $v\le\frac{1}{2}f(f+1)$;
   - $v\le\frac{1}{2}g(g+3)$ *and if there is no equality in the second Krein condition then* $v\le\frac{1}{2}g(g+1)$.

7. *The claw bound. If $\mu\ne l^2$ and $\mu\ne l(l+1)$ then $2(r+1)\le l(l+1)(\mu+1)$.*

8. *The Hoffman bound.*
   - *If C is a clique of Γ, then $|C|\le 1-\frac{k}{l}$, with equality if and only if every vertex $x\notin C$ has the same number of neighbours (namely $\frac{\mu}{-l}$) in C.*
   - *If C is a coclique of Γ, then $|C|\le v(1-\frac{k}{l})^{-1}$, with equality if and only if every vertex $x\notin C$ has the same number of neighbours (namely $-l$) in C.*

Remark

Unlike the first part of the Hoffman bound, the second part is also valid for regular graphs (see for example [5]).

1.2.3 Seidel switching

Let $\Gamma = (V, E)$ be a graph and let X be a set of vertices of Γ and let X^c be the complementary set of vertices, that is $V \setminus X$. *Seidel switching* Γ with respect to X is the operation of replacing all edges (respectively non-edges) of Γ between X and X^c by non-edges (respectively edges) leaving the edges inside X and X^c. Seidel switching was introduced by Seidel in [76]. Seidel switching will also be called *switching* in the sequel. Graphs which can be mapped to each other by Seidel switching are called *switching equivalent.*

Theorem 1.2 ([**88**]) *Let Γ be an* $\mathrm{srg}(v, k, \lambda, \mu)$*, then the switched graph with respect to a set X is again an* $\mathrm{srg}(v, k, \lambda, \mu)$ *if and only if*

(i) $\lambda + \mu = 2k - \frac{v}{2}$ *and*

(ii) each vertex of X (respectively X^c) is adjacent in Γ with precisely $\frac{|X^c|}{2}$ (respectively $\frac{|X|}{2}$) vertices in X^c (respectively X).

1.2.4 Matrix techniques

The Bose-Mesner algebra of a strongly regular graph allows us to use some valuable and powerful techniques of matrix theory. Hoffman [65] was the first to investigate this link using eigenvalue techniques. For a good survey we refer to [5] and [52]. Here we only state two important results that we need in the following chapters. Define a *principal (square) submatrix* of an $n \times n$ matrix A to be a matrix obtained by crossing out any i rows and the corresponding i columns of A, where $1 \leq i \leq n - 1$.

Theorem 1.3 ([**52**]) *Let A be a real symmetric matrix of order n and let B denote a principal submatrix of order m. Let the eigenvalues of A be $\theta_1 \geq \cdots \geq \theta_n$ and let the eigenvalues of B be $\eta_1 \geq \cdots \geq \eta_m$. Then*

$$\theta_{n-m+j} \leq \eta_j \leq \theta_j, (1 \leq j \leq m).$$

In this case we say that the eigenvalues of B *interlace* the eigenvalues of A. Theorem 1.3 is called the *interlacing theorem.* If for some integer l we have $\eta_j = \theta_j$ for $1 \leq j \leq l$, then we say that the interlacing is *tight*.

Theorem 1.4 ([**52**]) *Let Γ be a graph and let $\Pi = \{X_1, \ldots, X_m\}$ be a partition of its vertex set into non-empty parts. Let B_{ij} denote the average number of neighbours in X_j of a vertex in X_i. Then the eigenvalues of the matrix B interlace those of Γ. If the interlacing is tight, then each vertex in X_i has precisely B_{ij} neighbours in X_j.*

1.3 Strongly regular graphs and two-weight codes

1.3.1 Codes

The problem tackled by the theory of error-correcting codes is to send a message over a noisy channel in which some distortion may occur, so that errors can be corrected but the price paid in loss of speed is not too great. We now give a brief introduction to coding theory. For more information we refer to [22].
A *code* C of *length* n over an alphabet F_q of size q is a subset $C \subseteq F_q^n$ of the set of all n-tuples with components from F_q. If q is a prime power and F_q is the finite field $\mathbb{F}_q$ of order q, then *linear codes* are linear subspaces of the n-dimensional vector space $\mathbb{F}_q^n$. A k-dimensional subspace of $\mathbb{F}_q^n$ is a linear $[n,k]$ code over $\mathbb{F}_q$. The elements of a code are called the *codewords*. A *generator matrix* of a linear $[n,k]$ code C is any matrix of rank k over $\mathbb{F}_q$ with rows from C. The *Hamming distance* between any two words is the number of coordinates in which they differ, that is the number of errors required to change one into the other. A linear code C is *e-error correcting* if the Hamming distance between any two codewords is at least $2e+1$. The *distance* between a vector $x \in \mathbb{F}_q^n$ and a linear code C over $\mathbb{F}_q$ is the minimal number of coordinates in which x differs from a word of C. The *weight* of a word of a linear code is the number of coordinates in which the word differs from the zero vector. A linear code is called a *two-weight code* with weights w_1 and w_2 if the weight of every word equals either the constant value w_1 or w_2 $(w_1 < w_2)$.
To maximise the transmission rate one needs as many codewords as possible. The optimum is obtained when the closed balls of radius e centered at the codewords fill the space of all words without any overlap. A code with this property is called *perfect e-error correcting*.
Since perfect codes are rather rare, we consider the *uniformly packed codes*. An e-error correcting linear code C of length n over $\mathbb{F}_q$ is called uniformly packed with parameters α and β if and only if for every word $x \in \mathbb{F}_q^n$ we have: (i) if x has distance e to C, then x has distance $e+1$ to exactly α codewords, where $\alpha < \frac{(n-e)(q-1)}{e+1}$; (ii) if x has distance more than e to C, then x has distance $e+1$ to exactly β codewords.

Remarks

1. Since $\alpha = \frac{(n-e)(q-1)}{e+1}$ would imply that C is perfect, a uniformly packed code can be considered as a code with a lot of codewords but which is not perfect (see [22]).

2. The existence of a 1-error correcting uniformly packed code is equivalent with the existence of a two-weight code (see [22]).

1.3.2 Linear representation of a strongly regular graph

Let $\mathcal{P}$ be a set of points in $\mathrm{PG}(n,q)$. Now embed this $\mathrm{PG}(n,q)$ as a hyperplane Π in a $\mathrm{PG}(n+1,q)$. Define the *linear representation graph* $\Gamma_n^*(\mathcal{P})$ as the graph with vertices the points in $\mathrm{PG}(n+1,q) \setminus \Pi$; two vertices are adjacent whenever the line joining them intersects $\mathcal{P}$. Then $v = q^{n+1}$ and $k = (q-1)|\mathcal{P}|$. Delsarte [46] proved that this graph is strongly regular if and only if there are two integers w_1 and w_2 such that all hyperplanes of Π miss either w_1 or w_2 points of $\mathcal{P}$, and then $\mathcal{P}$ is called a *two-character set*. If this is the case then $\lambda = k-1+(k-qw_1+1)(k-qw_2+1)$ and $\mu = k+(k-qw_1)(k-qw_2)$. When we view the coordinates of the elements of $\mathcal{P}$ as columns of the generator matrix of a code C then the property that hyperplanes miss either w_1 or w_2 points of $\mathcal{P}$ translates into the property that one has a two-weight code C, with weights w_1 and w_2. And so there are very useful links between two-weight codes, strongly regular graphs and two-character sets. For an extensive discussion see [17].

1.4 Generalities on geometries

1.4.1 Geometries

A *(Buekenhout-Tits) geometry* is a graph Γ together with a fixed partition T of its vertex set into cocliques. The *objects* of the geometry are the vertices of Γ; the *type* $\tau(x)$ of an object x is the element of T containing it. Two objects are *incident* when they coincide or are adjacent. A *flag* is a set of pairwise incident objects, that is a clique in Γ. The *rank* r of the geometry is the number $|T|$ of types. A *(point-line) geometry* or an *incidence structure* is a rank 2 geometry where the two types of objects are called *points* and *lines*, that is a triple $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, with $\mathcal{P}$ the (non-empty and finite) set of points, $\mathcal{L}$ the (non-empty and finite) set of lines, and $\mathrm{I} \subseteq (\mathcal{P}\times\mathcal{L})\cup(\mathcal{L}\times\mathcal{P})$ the (symmetric) incidence relation. Two points (respectively lines) are called *collinear* (respectively *concurrent*) if they are incident with the same line (respectively point). A *subgeometry* of a geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a geometry $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ with $\mathcal{P}' \subset \mathcal{P}$, $\mathcal{L}' \subset \mathcal{L}$ and $\mathrm{I}'=\mathrm{I} \cap((\mathcal{P}' \times \mathcal{L}') \cup (\mathcal{L}' \times \mathcal{P}'))$. The *dual* $\mathcal{S}^D$ of a (point-line) geometry $\mathcal{S}$ is obtained by interchanging points and lines and keeping incidence. An *isomorphism* of the geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ onto the geometry $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ is a bijection of $\mathcal{P}$ onto $\mathcal{P}'$, inducing a bijection of $\mathcal{L}$ onto $\mathcal{L}'$, such that incidence is preserved. An isomorphism from $\mathcal{S}$ onto itself is called an *automorphism*. A *spread* of a (point-line) geometry is a partition of the point set of the geometry in disjoint lines of the geometry. For more information on Buekenhout-Tits geometries and point-line geometries we refer to [13, chapter 3].

1.4.2 Designs

A t-(v,k,λ) *design* is a point-line geometry $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ where we call the lines *blocks*, having the following properties: a block is incident with k points; any t points are incident with precisely λ blocks; k distinct points are incident with at most one block; and $v \geq k \geq t$ (so that $\lambda > 0$). If $\lambda = 1$ then one also calls a t-design a *Steiner t-system* which we denote by $S(t,k,v)$.
Designs have their origin in statistics where they are used in experimental design.

Because of their nice regular properties there are a lot of useful links with the theory of strongly regular graphs, *(semi)partial geometries* , and coding theory. For more information about designs and their links we refer to [22].

1.4.3 Finite polar spaces

Classical polar spaces

When we describe models of graphs, designs or (semi)partial geometries, we use a setting. Usually we consider projective spaces and polar spaces over finite fields. We use their properties in order to construct strongly regular graphs, designs and (semi)partial geometries and to prove theorems about these models. Although we assume some preknowledge on the theory of finite fields, projective and polar spaces, we now give some basic definitions and properties. For more information we refer to [19], [61] and [64].
(Finite) polar spaces describe the geometry of (finite) vector spaces carrying a reflexive sesquilinear form or a quadratic form in the same way as (finite) projective spaces describe the geometry of (finite) vector spaces. There are three types of forms with an associated polar space. The space associated with the alternating bilinear form is called the *symplectic* polar space; the one with the Hermitian form is called the *unitary* or *Hermitian* polar space; finally with the quadratic forms is associated the *orthogonal* polar space. These polar spaces are called the *classical polar spaces*.

We now give some basic definitions and basic properties. Let $V(n+1,q)$ be a vector space carrying a reflexive sesquilinear form σ of one of the three types. Note that σ defines uniquely the polarity in the associated projective space $\mathrm{PG}(n,q)$, unless q is even and σ is orthogonal, in which case one has to use the *quadratic form* which we also denote by σ. A subspace W of V is called *totally isotropic* if σ vanishes identically on W, that is $W \subseteq W^\sigma$. In case of an orthogonal polarity, a subspace on which the quadratic form σ vanishes is called a *totally singular* subspace. We often call the maximal totally isotropic subspaces or maximal totally singular subspaces of a polar space $\mathcal{S}$, the *generators* of $\mathcal{S}$.
The classical polar spaces, regarded as a geometry of rank r on the points of the projective space $\mathrm{PG}(n,q)$, whose flags are the totally isotropic or totally singular subspaces (called subspaces for short) have the following properties (see for example [19] for a proof): [P1] each subspace is isomorphic to a $\mathrm{PG}(d,q)$, $d \leq r-1$; [P2] the intersection of any family of subspaces is a subspace; [P3] if W is a subspace of dimension $r-1$, and p a point not in W, then the set of points p' such that the line pp' is totally isotropic or totally singular, is a hyperplane in W, and the union of those lines pp' is a subspace of dimension $r-1$; [P4] there exist two disjoint subspaces of dimension $r-1$.
A geometry consisting of a set of points with a collection of different subsets called subspaces, satisfying the axioms [P1]-[P2]-[P3]-[P4] is called an *abstract polar space of rank* r. Building on the work of Veldkamp [117], Tits proved that all finite abstract polar spaces of rank at least 3 are classical. The generalized quadrangles are the abstract polar spaces of rank 2. The generalized quadrangles

play much the same role in the theory of polar spaces as projective planes do in the theory of projective spaces.

Buekenhout-Shult geometries

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a point-line geometry satisfying the following: [BS1] if p is a point not on a line L, then p is collinear with one or all points of L; [BS2] any line contains at least three points; [BS3] no point is collinear with all others; [BS4] if we define a *singular subspace* A of $\mathcal{S}$ to be a subset of $\mathcal{P}$ such that any two of its points are on a line L of $\mathcal{S}$ which is completely contained in A, then finally we require that any chain (with respect to inclusion) of singular subspaces is finite.
Such an incidence structure is now commonly known as a *Buekenhout-Shult geometry*. Buekenhout and Shult [15] proved that the singular subspaces of a Buekenhout-Shult geometry constitute an abstract polar space, and so they simplified the axioms [P1]-[P2]-[P3]-[P4].

Notation

We shall use the following notations for the finite classical polar spaces: $W_n(q)$ for the polar space arising from a symplectic polarity of $\mathrm{PG}(n,q)$, n odd and $n \geq 3$ (here $r = \frac{n+1}{2}$); $\mathrm{Q}(2n,q)$ for the polar space arising from a non-singular quadric in $\mathrm{PG}(2n,q)$, $n \geq 2$ (here $r = n$); $\mathrm{Q}^+(2n+1,q)$ for the polar space arising from a non-singular hyperbolic quadric in $\mathrm{PG}(2n+1,q)$, $n \geq 1$ (here $r = n+1$); $\mathrm{Q}^-(2n+1,q)$ for the polar space arising from a non-singular elliptic quadric in $\mathrm{PG}(2n+1,q)$, $n \geq 1$ (here $r = n$); $\mathrm{H}(n,q^2)$ for the polar space arising from a non-singular Hermitian variety H in $\mathrm{PG}(n,q^2)$, $n \geq 3$ (here $r = \frac{n+1}{2}$ if n is odd, and $r = \frac{n}{2}$ if n is even);

We now give two basic properties of these finite classical polar spaces For their proofs we refer to [64].

Theorem 1.5 *For q even, the polar space $\mathrm{Q}(2n,q)$ is isomorphic to the polar space $W_{2n-1}(q)$.*

Theorem 1.6 *The number of points of the finite classical polar spaces are given by the following formulae.*

$$\begin{aligned}
|W_n(q)| &= \tfrac{q^{n+1}-1}{q-1}, \\
|\mathrm{Q}(2n,q)| &= \tfrac{q^{2n}-1}{q-1}, \\
|\mathrm{Q}^+(2n+1,q)| &= \tfrac{(q^n+1)(q^{n+1}-1)}{q-1}, \\
|\mathrm{Q}^-(2n+1,q)| &= \tfrac{(q^n-1)(q^{n+1}+1)}{q-1}, \\
|\mathrm{H}(n,q^2)| &= \tfrac{(q^{n+1}+(-1)^n)(q^n-(-1)^n)}{q^2-1}.
\end{aligned}$$

Remark

In the following chapters, if X is a subspace of a projective space equipped with a polarity, then we denote with X^* the polar space of X with respect to the considered polarity.

Ovoids and spreads of classical polar spaces

Let P be a finite classical polar space of rank $r \geq 2$. An *ovoid* $\mathcal{O}$ of P is a point set of P which has exactly one point in common with every maximal totally isotropic or maximal totally singular subspace of P. A *spread* Σ of P is a set of maximal totally isotropic subspaces or maximal totally singular subspaces of P, that partition the point set of P. A spread of a non-singular quadric is also called an *orthogonal spread*.

Theorem 1.7 **([64])** *Let $\mathcal{O}$ be an ovoid and $\mathcal{S}$ be a spread of the finite classical polar space P. Then*

$$\begin{array}{ll}
\textit{for } P = W_n(q), & |\mathcal{O}| = |\mathcal{S}| = q^{\frac{n+1}{2}} + 1, \\
\textit{for } P = \mathrm{Q}(2n, q), & |\mathcal{O}| = |\mathcal{S}| = q^n + 1, \\
\textit{for } P = \mathrm{Q}^+(2n+1, q), & |\mathcal{O}| = |\mathcal{S}| = q^n + 1, \\
\textit{for } P = \mathrm{Q}^-(2n+1, q), & |\mathcal{O}| = |\mathcal{S}| = q^{n+1} + 1, \\
\textit{for } P = \mathrm{H}(2n, q^2), & |\mathcal{O}| = |\mathcal{S}| = q^{2n+1} + 1, \\
\textit{for } P = \mathrm{H}(2n+1, q^2), & |\mathcal{O}| = |\mathcal{S}| = q^{2n+1} + 1.
\end{array}$$

Ovoids and spreads of classical polar spaces turn out to be very useful for the construction of geometries. Unfortunately they do not always exist. For an overview on their existence and non-existence we refer to table 1 and 2 in [109].

Ovoids of $\mathrm{PG}(3, q)$

An *ovoid* of $\mathrm{PG}(3, q)$ is a set of $q^2 + 1$ points, no three collinear. For an introduction to ovoids of $\mathrm{PG}(3, q)$ and their properties see [60, Chapter 16].

1.4.4 (α, β)-geometries

A *partial linear space of order* (s, t) is a geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, satisfying the following axioms: (i) any two points are incident with at most one line and each point is incident with $t + 1$ $(t \geq 1)$ lines; (ii) each line is incident with $s + 1$ $(s \geq 1)$ points. Note that it follows that two lines are incident with at most one point.

Let (x, L) be an *antiflag* of $\mathcal{S}$, that is a non-incident point-line pair. The *incidence number* $\alpha(x, L)$ of the antiflag (x, L) is the number of incident point-line pairs (y, M) such that $x \mathrel{\mathrm{I}} M \mathrel{\mathrm{I}} y \mathrel{\mathrm{I}} L$.

For integers $\alpha, \beta \geq 0$ and $(\alpha, \beta) \neq (0, 0)$ an *(α, β)-geometry of order* (s, t) is a partial linear space of order (s, t) such that the incidence number of any antiflag (x, L) equals either α or β. Although the concept of an (α, β)-geometry was probably known before, to our knowledge the terminology has been used for the first time in [37].

Remark

From axiom [BS1] it follows that the points and lines of a Buekenhout-Shult geometry form a $(1, q+1)$-geometry.

1.4.5 Embedded geometries

A lot of examples of geometries have points and lines in a projective or affine space. A geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is said to be *embedded* in a projective or an affine space if $\mathcal{L}$ is a subset of the set of lines of the space and if $\mathcal{P}$ is the set of all points of the space on these lines. It is also required that the dimension of the space is the smallest possible dimension for an embedding. Note that some authors call this a *full embedding*.

A special type of affine embedding is the *linear representation* of a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of order (s, t) in $\mathrm{AG}(n+1, s+1)$. It is an embedding of $\mathcal{S}$ in $\mathrm{AG}(n+1, s+1)$ such that the line set $\mathcal{L}$ of $\mathcal{S}$ is a union of parallel classes of lines of $\mathrm{AG}(n+1, s+1)$ and hence the point set $\mathcal{P}$ of $\mathcal{S}$ is the point set of $\mathrm{AG}(n+1, s+1)$. The lines of $\mathcal{S}$ define in the hyperplane at infinity Π_∞ a set $\mathcal{K}$ of points of size $t+1$. A common notation for a linear representation of a partial linear space is $T_n^*(\mathcal{K})$. For an extensive discussion see for example [37].

1.4.6 The point and block graph of a geometry

The *point graph* of a partial linear space is the graph whose vertices are the points of the geometry, two distinct vertices being adjacent whenever they are collinear in the partial linear space. Note that the point graph of a partial linear space of order (s, t) is regular of degree $s(t+1)$.

We call a partial linear space *connected* if its point graph is connected. The lines of a partial linear space of order (s, t) yield cliques of size $s+1$ in its point graph. But there may be other cliques of the same size.

The *block graph* of a partial linear space is the graph whose vertices are the lines, and vertices are adjacent if and only if the corresponding lines are concurrent. The block graph of a partial linear space of order (s, t) is regular of degree $t(s+1)$.

Remark

The point graph of the linear representation $T_n^*(\mathcal{K})$ of a geometry is the linear representation of the graph $\Gamma_n^*(\mathcal{K})$.

1.5 Partial geometries

1.5.1 Definitions and properties

A *partial geometry* with parameters s, t, α, which we denote by $\mathrm{pg}(s,t,\alpha)$ is an (α,β)-geometry of order (s,t) such that $\alpha = \beta$ (> 0). A *generalized quadrangle* is a partial geometry with $\alpha = 1$. We denote a generalized quadrangle of order (s,t) by $\mathrm{GQ}(s,t)$.
Partial geometries were introduced by Bose in [2] as a generalisation of generalized quadrangles introduced by Tits [113]. The theory of generalized quadrangles is a vast theory. We refer to the standard references [13, Chapter 9], [84], [116] for more details. Given a $\mathrm{pg}(s,t,\alpha)$, then its dual geometry is a $\mathrm{pg}(t,s,\alpha)$. This is because its axioms are symmetric.
Partial geometries can be divided into four (non-disjoint) classes:

1. the partial geometries with $\alpha = 1$, the generalized quadrangles;
2. the partial geometries with $\alpha = s+1$ or dually $\alpha = t+1$, that is the 2–$(v, s+1, 1)$ designs and their duals;
3. the partial geometries with $\alpha = s$ or dually $\alpha = t$; the partial geometries with $\alpha = t$ are the *Bruck nets* of order $s+1$ and degree $t+1$;
4. the *proper* partial geometries with $1 < \alpha < \min\{s,t\}$.

For the description of some examples of partial geometries we refer to section 1.5.2 and appendix A, and for further references see [26, 38]. For more information about Bruck nets see [13].

Theorem 1.8 ([2]) *The point graph Γ of a $\mathrm{pg}(s,t,\alpha)$ is an*

$$\mathrm{srg}\left((s+1)\frac{st+\alpha}{\alpha}, s(t+1), s-1+t(\alpha-1), \alpha(t+1)\right).$$

A strongly regular graph Γ with the above parameters such that the positive integers s,t,α satisfy $1 \le \alpha \le \min\{s+1, t+1\}$, is called a *pseudo-geometric (s,t,α)–graph*. If the graph Γ is indeed the point graph of at least one partial geometry then Γ is called *geometric*. A graph can be pseudo-geometric for at most one set of values s,t,α and assuming $\alpha \neq s+1$, the cliques of size $s+1$ corresponding to potential lines must have maximal size. However, there can exist several non-isomorphic partial geometries with the same graph as point or block graph. A pseudo-geometric graph is called *faithfully* geometric if and only if there is up to isomorphism exactly one partial geometry with this graph as point graph.
Bose [2] proved that a pseudo-geometric (s,t,α)-graph Γ is geometric if $2(s+1) > t(t+1) + \alpha(t+2)(t^2+1)$. In general this condition is too strong in order to construct partial geometries from a given strongly regular graph.

Cameron, Goethals an Seidel [20] proved that for a pseudo-geometric (s,t,α)-graph Γ satisfying the above inequality of Bose, $t \leq 2\alpha - 1$ holds.
If we translate the conditions for strongly regular graphs in theorem 1.1 in terms of the parameters of a pseudo-geometric (s,t,α)-graph, then this yields the following theorem.

Theorem 1.9 *If Γ is a pseudo-geometric (s,t,α)–graph whose adjacency matrix has eigenvalues k, r and l with multiplicity respectively 1, f and $g = v-f-1$ then:*

1. $r = s - \alpha$, $l = -t-1$, $f = \dfrac{st(s+1)(t+1)}{\alpha(s+t+1-\alpha)}$.
2. *v is an integer, hence α divides $(s+1)st$.*
3. *f and g are integers hence $\alpha(s+t+1-\alpha)$ divides $st(s+1)(t+1)$.*
4. *The two Krein inequalities for strongly regular graphs yield one equality $(s+1-2\alpha)t \leq (s-1)(s+1-\alpha)^2$.*
5. *If C is a clique in Γ of size $s+1$, then any point outside C has exactly α neighbours in C.*

If $\alpha = 1$ and $s \neq 1$, then the Krein inequality is better known as the Higman inequality $t \leq s^2$. In [20] it is proved that any pseudo-geometric $(s, s^2, 1)$–graph is geometric. Note that the McLaughlin graph is a pseudo-geometric $(4, 27, 2)$-graph satisfying equality in the Krein conditions (see [74] for more information). It is still an open question whether this graph is indeed geometric.
The conditions on the parameters of a partial geometry in theorem 1.9 yield non-existence results. For more properties of pseudo-geometric graphs we refer to [38].

1.5.2 The partial geometry $\mathrm{PQ}^+(4n-1, q), q = 2, 3$

In appendix A we give a brief description of the known models of (semi)partial geometries. A partial geometry which turns out to be crucial for some constructions in this thesis is the partial geometry $\mathrm{PQ}^+(4n-1,q)$, $q = 2$ or 3. In chapters 3 and 4 we explain how to get new (semi)partial geometries out of this class by derivation procedures.
The construction of $\mathrm{PQ}^+(4n-1,q)$, $q = 2$ or 3, uses a spread of the hyperbolic quadric $\mathrm{Q}^+(2m-1,q)$, which can only exist if m is even. Given a quadric $\mathrm{Q}^+(4n-1,q)$ in $\mathrm{PG}(4n-1,q)$, it is well-known that the set of generators (which have dimension m) on it, can be divided into two disjoint families $\mathcal{D}_1$ and $\mathcal{D}_2$, each of the same size. Two generators on $\mathrm{Q}^+(4n-1,q)$ belong to the same family if and only if their intersection has an odd dimension. If $D, D' \in \mathcal{D}_i$ $(i = 1$ or $2)$ such that $D \cap D' = \emptyset$ and if X is a hyperplane of D, then the polar space X^* of X with respect to the polarity defined by the quadric intersects D' in a point. Note that the elements of an orthogonal spread are necessarily of one family, say $\mathcal{D}_1$.

Construction for $q = 2$.

Assume $q = 2$, then the existence of an orthogonal spread has been settled by Dye [49]. De Clerck, Dye and Thas [30] constructed an infinite class of partial geometries as follows. Consider the incidence structure $\mathrm{PQ}^+(4n-1,2) = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with $\mathcal{P}$ the set of points of $\mathrm{PG}(4n-1,2)$ not on the quadric; $\mathcal{L}$ the set of all hyperplanes of the elements of a fixed orthogonal spread Σ; $x \mathrm{~I~} L$, $x \in \mathcal{P}$ and $L \in \mathcal{L}$, if and only if x is contained in the polar space L^* of L with respect to $\mathrm{Q}^+(4n-1,2)$. This is a

$$\mathrm{pg}(2^{2n-1}-1, 2^{2n-1}, 2^{2n-2}),$$

and non-isomorphic orthogonal spreads produce non-isomorphic partial geometries.

Construction for $q = 3$.

For $q = 3$, a similar construction is given by Thas [101]. Let $\mathrm{Q}^+(2m-1,q), q$ odd, be the non-singular hyperquadric of $\mathrm{PG}(2m-1,q)$ $(m \geq 2)$. Let x and y be two points in $\mathrm{PG}(2m-1,q) \setminus \mathrm{Q}^+(2m-1,q)$, then x and y are called *equivalent* if and only if there exists a point $z \in \mathrm{PG}(2m-1,q) \setminus \mathrm{Q}^+(2m-1,q)$ such that the lines xz and yz are tangent lines of $\mathrm{Q}^+(2m-1,q)$. The set $\mathrm{PG}(2m-1,q) \setminus \mathrm{Q}^+(2m-1,q)$ is partitioned in two classes which we denote by $E_1^+(2m-1,q)$ and $E_2^+(2m-1,q)$.
Except for the fact that we only are considering half of the points outside the quadric $\mathrm{Q}^+(4n-1,3)$, that is we take one of the sets $E_i^+(4n-1,3)$, $i = 1,2$, to be the point set $\mathcal{P}$, the construction of the partial geometry $\mathrm{PQ}^+(4n-1,3)$ is similar as for the case $q = 2$ considered above. The line set $\mathcal{L}$ is the set of all hyperplanes of the elements of a fixed orthogonal spread Σ of $\mathrm{Q}^+(4n-1,3)$; $x \mathrm{~I~} L$, $x \in \mathcal{P}$ and $L \in \mathcal{L}$, if and only if x is contained in the polar space L^* of L with respect to $\mathrm{Q}^+(4n-1,3)$. We get a possibly infinite family of partial geometries

$$\mathrm{pg}(3^{2n-1}-1, 3^{2n-1}, 2 \cdot 3^{2n-2}).$$

But up to now it is only known that $\mathrm{Q}^+(7,3)$ has an orthogonal spread, even being unique up to isomorphism [83], which yields a $\mathrm{pg}(26,27,18)$.

Notation

We denote a line of $\mathrm{PQ}^+(4n-1,q), q = 2,3$, considered as point set (and so the point set of an affine space) with a capital letter, for example L. If we consider the same line as a hyperplane of an element of Σ then we denote it by π_L. Conversely, given a hyperplane π of an element of Σ, then we denote with L_π the set of points of the affine space being points of the line π in $\mathrm{PQ}^+(4n-1,q), q = 2,3$, that is the intersection of π^* with the point set of $\mathrm{PQ}^+(4n-1,q), q = 2,3$. And so $\pi_{L_\pi} = \pi$ and $L_{\pi_L} = L$.

1.6 Semipartial geometries

A *semipartial geometry* with parameters s, t, α, μ denoted by $\mathrm{spg}(s, t, \alpha, \mu)$, is a $(0, \alpha)$-geometry of order (s, t) $(\alpha > 0)$ such that for any two non-collinear points there are μ (> 0) points collinear with both points. A *partial quadrangle* is a semipartial geometry with $\alpha = 1$. A *proper* semipartial geometry is a semipartial geometry which is not a 2–design (hence $\alpha \leq \min(t+1, s)$) and which is not a partial geometry.

Semipartial geometries were introduced by Debroey and Thas [40] and generalise both the partial quadrangles (introduced by Cameron [18]) and the partial geometries. A semipartial geometry is a partial geometry if and only if $\mu = (t+1)\alpha$. The dual of a semipartial geometry is again a semipartial geometry if and only if $s = t$ or $\mathcal{S}$ is a partial geometry [40].

In the following we assume that a semipartial geometry (and so also a partial quadrangle) is always proper, and we use the word *semipartial geometry* in the sense of proper semipartial geometry. For the description of some examples of semipartial geometries we refer to appendix A, and for further references see [26, 38].

Debroey and Thas [40] introduced the μ-condition in the definition of a semipartial geometry because they wanted the point graph of a semipartial geometry to be strongly regular, in particular they proved the following

Theorem 1.10 ([40]) *The point graph Γ of an $\mathrm{spg}(s, t, \alpha, \mu)$ is an*

$$\mathrm{srg}\left(1 + \frac{(t+1)s(\mu + t(s - \alpha + 1))}{\mu}, s(t+1), s - 1 + t(\alpha - 1), \mu\right).$$

A *pseudo-semigeometric (s, t, α, μ)–graph*, a *semigeometric graph*, and a *faithfully semigeometric graph* are defined similarly as a pseudo-geometric, respectively a geometric, respectively a faithfully geometric graph, but for a pseudo-semigeometric (s, t, α, μ)-graph we also require that $\mu < \alpha(t+1)$. Unlike a pseudo-geometric graph, a graph can be pseudo-semigeometric for more than one set of values s, t, α, μ, and the cliques of size $s+1$ corresponding to potential lines of the semipartial geometry are not necessarily cliques of maximal size.

The conditions for strongly regular graphs of theorem 1.1 translated in terms of the parameters of a pseudo-semigeometric (s, t, α, μ)-graph yield the following theorem (see [40]).

Theorem 1.11 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a semipartial geometry* $\mathrm{spg}(s, t, \alpha, \mu)$, *then*

1. $t \geq s$ *hence* $|\mathcal{L}| = b \geq v$.
2. $D = (t(\alpha - 1) + s - 1 - \mu)^2 + 4((t+1)s - \mu)$ *is either a square or equals 5 (then $\mathcal{S}$ is isomorphic to the pentagon) and*
$$\frac{2(t+1)s + (v-1)(t(\alpha-1) + s - 1 - \mu + \sqrt{D})}{2\sqrt{D}}$$
is an integer.
3. $\alpha^2 \leq \mu < (t+1)\alpha$ *and* $\alpha | \mu$.
4. $\mu | (t+1)st(s+1-\alpha)$.
5. $\alpha | st(t+1)$ *and* $\alpha | st(s+1)$.
6. $\alpha^2 | \mu st$.
7. $\alpha^2 | t((t+1)\alpha - \mu)$.
8. $2 | v(t+1)s$.
9. $3 | v(t+1)s(s-1)$ *and* $3 | v(t+1)st(\alpha - 1)$.
10. $8 | v(t+1)s(s-1)(s-2)$.
11. $8 | v(t+1)s(t(\alpha-1)((t-1)(\alpha-1) - (\alpha-2)) + t(s+1-\alpha)(\mu - 2\alpha + 1))$.

The conditions on the parameters of a semipartial geometry of theorem 1.11 yield non-existence results. For more properties of pseudo-semigeometric graphs we refer to [38].

Chapter 2

Pseudo-(semi)geometric graphs

2.1 From graphs to geometries

We saw in sections 1.5 and 1.6 that the point graph of a (semi)partial geometry is strongly regular. The other way round, if a strongly regular graph has the parameters of the point graph of a (semi)partial geometry then we can try to check whether it is indeed the point graph of a (semi)partial geometry or not. That is, we want to check whether a pseudo-(semi)geometric graph is (semi)geometric or not. Therefore we look for a collection of maximal cliques in a pseudo-(semi)geometric graph that could yield lines of a putative (semi)partial geometry. So the chosen maximal cliques can intersect in at most one point. In general these questions are quite difficult but for some graphs one can do a very detailed study. Especially the graphs related to classical geometrical objects such as quadrics and other polar spaces are candidates for such a study.
Recall that unlike a pseudo-geometric graph, a graph can be pseudo-semigeometric for more than one set of values s, t, α, μ and the cliques of size $s+1$ corresponding to potential lines of the semipartial geometry are not necessarily cliques of maximal size. And so it is more difficult to calculate if an infinite class of graphs is pseudo-semigeometric or not.
By theorem 1.9, a pseudo-geometric (s, t, α)–graph Γ is geometric if and only if every there is a collection $\mathcal{C}$ of maximal cliques of Γ such that every edge is contained in a unique element of $\mathcal{C}$. We can generalize this theorem for pseudo-semigeometric graphs.

Theorem 2.1 *Let Γ be a pseudo-semigeometric (s, t, α, μ)-graph and let $\mathcal{C}$ be a collection of cliques of Γ of size $s+1$ such that every edge is contained in a unique element $\mathcal{C}$. Then Γ is semigeometric if and only if for every $C \in \mathcal{C}$,*

$$| \cup_{x \in C} \Gamma(x)| = \frac{s(s+1)t}{\alpha}.$$

Proof. Let C be a clique of Γ of size $s+1$, and let $p \in \Omega = \cup_{x\in C}\Gamma(x)$. Suppose that p has α_p neighbours in C, then $\alpha_p \geq 1$. Counting ordered pairs (p,r) and ordered triples (p,r,r') with $p \in \Omega$ and $r, r' \in C$ yields

$$\sum_{p\in\Omega} \alpha_p = (s+1)st, \qquad \sum_{p\in\Omega} \alpha_p(\alpha_p - 1) = (s+1)st(\alpha - 1).$$

And so $\sum(\alpha - \alpha_p) = 0$, hence $\alpha_p = \alpha$ for all $p \in \Omega$. The other axioms of a semipartial geometry are straightforward. And so Γ is semigeometric. $\square$

Remark

There are a lot of strongly regular graphs known (see for example [9] and [66]). For some of them we can calculate for which parameters a graph is pseudo-(semi)geometric, in order to obtain a list of candidates of point graphs of (semi)partial geometries. Then we can check whether the graph is (semi)geometric or not. In this chapter we investigate a few examples out of this list. Most attempts to construct a (semi)partial geometry from a pseudo-(semi)geometric graph were not successful. We refer for example to De Clerck, Gevaert and Thas [31], De Clerck and Tonchev [36], Spence [94], and to sections 2.2.3 and 2.3.4. Exceptions however are the sporadic partial geometry of Haemers [53] and the semipartial geometry $\mathrm{SPQ}(6,q), q$ an odd prime or $q \equiv 0$ or $2 \pmod 3$ (see section 2.3.5).
One class of graphs that is studied a lot is the class of collinearity graphs of a polar space. We give a survey of what is known and we add some results.

2.2 The collinearity graphs

2.2.1 Pseudo-geometric graphs

Consider a polar space P of rank at least two. Define the graph $\Gamma(P)$ with vertex set the set of points of the polar space, two vertices being adjacent whenever they are contained in a line of P. It is well known (see for example [66]) that the graphs $\Gamma(P)$ are strongly regular.
Using the parameters of the collinearity graphs one can check that the graph $\Gamma(\mathrm{Q}^-(2m+1,q))$, $m \geq 2$, is a pseudo-geometric

$$(q\frac{q^{m-1}-1}{q-1}, q^m, \frac{q^{m-1}-1}{q-1})\text{–graph.}$$

The graph $\Gamma(\mathrm{Q}(2m,q))$, $m \geq 2$, and the graph $\Gamma(W_{2m-1}(q))$, $m \geq 2$, are both pseudo-geometric

$$(q\frac{q^{m-1}-1}{q-1}, q^{m-1}, \frac{q^{m-1}-1}{q-1})\text{–graphs.}$$

The graph $\Gamma(\mathrm{Q}^+(2m+1,q))$ is pseudo-geometric if and only if $m = 1, 2$. If $m = 1$, then it is a pseudo-geometric $(q, 1, 1)$-graph, if $m = 2$, then it is a pseudo-geometric $(q(q+1), q, q+1)$-graph.
The graph $\Gamma(\mathrm{H}(2m+1, q^2))$, $m \geq 1$, is a pseudo-geometric

$$(q^2 \frac{q^{2m}-1}{q^2-1}, q^{2m-1}, \frac{q^{2m}-1}{q^2-1})\text{–graph},$$

and finally $\Gamma(\mathrm{H}(2m, q^2))$, $m \geq 2$, is a pseudo-geometric

$$(q^2 \frac{q^{2m-2}-1}{q^2-1}, q^{2m-1}, \frac{q^{2m-2}-1}{q^2-1})\text{–graph}.$$

2.2.2 Positive results

The points and lines of the classical polar spaces $\mathrm{Q}(4,q), \mathrm{Q}^+(3,q), \mathrm{Q}^-(5,q)$, $W_3(q), \mathrm{H}(3,q^2)$ and $\mathrm{H}(4,q^2)$ yield generalized quadrangles (see [84]) and so the corresponding graphs are geometric. The points of $\mathrm{Q}^+(5,q)$, together with the planes of one family of generators form a dual 2-design, and so the corresponding graph is geometric. Hence we have the following theorem.

Theorem 2.2 *The graph* $\Gamma(P)$, *with* $P \in \{\mathrm{Q}(4,q), \mathrm{Q}^+(3,q), \mathrm{Q}^-(5,q), \mathrm{Q}^+(5,q), W_3(q), \mathrm{H}(3,q^2), \mathrm{H}(4,q^2)\}$, *is geometric.*

2.2.3 Negative results

Several authors have investigated the collinearity graphs for larger dimensions and obtained non-existence results: Brouwer [unpublished], De Clerck, Gevaert and Thas [31], Mathon [79], Panigrahi [82], Thas [106] and Thomas [112]. Here we remark that if the graph $\Gamma(P)$, with P a polar space in $\mathrm{PG}(n,q)$, is geometric, then the lines of the geometry containing a given point define a spread of a polar space P' in $\mathrm{PG}(n-2,q)$ with P and P' of the same type.

Theorem 2.3 *The graph* $\Gamma(\mathrm{H}(6,4))$ *is not geometric.*

Theorem 2.4 ([31]) *The following graphs are not geometric:*
$\Gamma(W_5(q))$, *q even;* $\Gamma(W_7(q))$; $\Gamma(\mathrm{Q}(6,q))$, *q even;* $\Gamma(\mathrm{Q}(8,q))$, *q even.*

Theorem 2.5 ([79]) *The graph* $\Gamma(\mathrm{Q}^-(9,2))$ *is not geometric.*

Theorem 2.6 ([82]) *The graph* $\Gamma(\mathrm{Q}^-(7,2))$ *is not geometric.*

Theorem 2.7 ([106]) *The following graphs are not geometric:*
$\Gamma(\mathrm{Q}(4n+2,q))$, $n \geq 1$, *q odd;* $\Gamma(\mathrm{H}(2m+1,q^2)), m \geq 2$.

Theorem 2.8 ([112]) *The graph* $\Gamma(W_5(q))$ *is not geometric.*

The following theorem is due to the indications of J. A. Thas [personal communication].

Theorem 2.9 *The graph $\Gamma(\mathrm{Q}(4n,q))$ is not geometric.*

Proof. Let $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ be a pg $\left(q\frac{q^{2n-1}-1}{q-1},q^{2n-1},\frac{q^{2n-1}-1}{q-1}\right)$ having point graph $\Gamma(\mathrm{Q}(4n,q))$. Let π be a plane intersecting $\mathrm{Q}(4n,q)$ in a non-singular conic $\mathrm{Q}(2,q)$. Let $x_0,x_1\in\mathrm{Q}(2,q)$. Then the lines of $\mathcal{S}$ through x_0 and x_1 define two disjoint spreads Σ_0 and Σ_1 of $\mathrm{Q}(4n-2,q)=\pi^*\cap\mathrm{Q}(4n,q)$, with π^* the intersection of the tangent hyperplanes of $\mathrm{Q}(4n,q)$ at x_0 and x_1.
Let the quadric $\mathrm{Q}(4n-2,q)$ be embedded in the non-singular hyperbolic quadric $\mathrm{Q}^+(4n-1,q)$. Denote the two families of generators of $\mathrm{Q}^+(4n-1,q)$ by $\mathcal{D}_1$ and $\mathcal{D}_2$. Then one can easily see that the elements of $\mathcal{D}_1$ containing the respective elements of Σ_0 constitute a spread Λ_0 of $\mathrm{Q}^+(4n-1,q)$, and similarly the elements of $\mathcal{D}_2$ containing the respective elements of Σ_1 constitute a spread Λ_1 of the quadric $\mathrm{Q}^+(4n-1,q)$. An element of $\mathcal{D}_1$ intersects an element of $\mathcal{D}_2$ in a space of even dimension (see for example [64]). An element σ of Λ_0 cannot intersect every element of Λ_1 in exactly a point, indeed $|\sigma|>|\Lambda_1|$. Hence there exists an element $\sigma'\in\Lambda_1$ intersecting σ in at least a plane. This implies that $(\sigma\cap\mathrm{Q}(4n-2,q))\in\Sigma_0$ intersects $(\sigma'\cap\mathrm{Q}(4n-2,q))\in\Sigma_1$ in at least a line, a contradiction because the elements of Σ_0 and Σ_1 are subsets of distinct lines of the partial geometry $\mathcal{S}$ and so intersect in at most one point. □

Theorem 2.10 *If the graph $\Gamma(\mathrm{Q}(2m,q))$, $m\geq 3$, is geometric, then also the graph $\Gamma(\mathrm{Q}^-(2m-1,q))$ is geometric.*

Proof. Let $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ be a pg $\left(q\frac{q^{m-1}-1}{q-1},q^{m-1},\frac{q^{m-1}-1}{q-1}\right)$ having point graph $\Gamma(\mathrm{Q}(2m,q)),m\geq 3$. Consider a hyperplane $\mathrm{PG}(2m-1,q)$ of $\mathrm{PG}(2m,q)$ intersecting $\mathrm{Q}(2m,q)$ in an elliptic quadric $\mathrm{Q}^-(2m-1,q)$.
Let $\mathcal{L}'$ denote the set $\{L\cap\mathrm{Q}^-(2m-1,q)|L\in\mathcal{L}\}$ and let $\mathcal{P}'$ be the point set of $\mathrm{Q}^-(2m-1,q)$. Then $\mathcal{L}'$ is a set of generators of $\mathrm{Q}^-(2m-1,q)$ and $|\mathcal{L}'|=|\mathcal{L}|$. Consider an element e of the edge set E of the graph $\Gamma(\mathrm{Q}^-(2m-1,q))$ and let n_e denote the number of elements of $\mathcal{L}'$ containing the edge e. If $n_e>1$ then this would yield two elements of $\mathcal{L}$ intersecting in more than one point, a contradiction. Hence $n_e=0$ or 1. Counting in two ways the pairs (e,L) with e an edge of $\Gamma(\mathrm{Q}^-(2m-1,q))$ and L an element of $\mathcal{L}'$ containing e yields

$$\sum_{e\in E}n_e=|\mathcal{L}'|\frac{q(q^{m-1}-1)(q^m-1)}{2(q-1)^2}.$$

If we substitute $|\mathcal{L}'|=|\mathcal{L}|=(q^m+1)(q^{m-1}+1)$, then $\sum n_e$ equals the number of edges of $\Gamma(\mathrm{Q}^-(2m-1,q))$. Therefore $n_e=1$ for every e in E. That is every edge of $\Gamma(\mathrm{Q}^-(2m-1,q))$ is contained in a unique element of $\mathcal{L}'$. By theorem 1.9 this implies that the incidence structure $\mathcal{S}'=(\mathcal{P}',\mathcal{L}',\mathrm{I}')$, where incidence is containment, is indeed a partial geometry with point graph $\Gamma(\mathrm{Q}^-(2m-1,q))$. □

Corollary 2.11 *The following graphs are not geometric:* $\Gamma(\mathrm{Q}(8,2))$, $\Gamma(\mathrm{Q}(10,2))$, $\Gamma(W_7(2))$ *and* $\Gamma(W_9(2))$.

Proof. Note that the graphs $\Gamma(\mathrm{Q}(2m,q))$ and $\Gamma(W_{2m-1}(q))$ are isomorphic for q even. Theorems 2.5, 2.6 and 2.10 prove the result. □

2.3 The graphs of Wilbrink

2.3.1 Definitions

In [9] Wilbrink generalised a construction of Metz for strongly regular graphs. Let $\mathrm{Q}(2m,q)$ be a non-singular parabolic quadric in $\mathrm{PG}(2m,q)$, $m \geq 2$. Let us define a graph $\mathrm{WIL}^-(2m,q)$ (respectively $\mathrm{WIL}^+(2m,q)$) with vertex set the set of hyperplanes meeting $\mathrm{Q}(2m,q)$ in a non-singular elliptic (respectively hyperbolic) quadric. Two vertices H_1 and H_2 are adjacent if and only if their corresponding quadrics are *tangent*, that is if $H_1 \cap H_2 \cap \mathrm{Q}(2m,q)$ is degenerate. Using the parameters of these graphs (see from [9]) one can check that the graph $\mathrm{WIL}^-(2m,q)$ is a pseudo-semigeometric

$$\left(q^{m-1}-1, q^m, 2q^{m-2}, 2q^{m-1}(q^{m-1}-1)\right)\text{–graph}$$

and it is not pseudo-geometric. Note that the graph of Metz coincides with the graph $\mathrm{WIL}^-(4,q)$.
The complement of the graph $\mathrm{WIL}^-(2m,3)$ is a pseudo-geometric

$$\left(\frac{3^m+1}{2}, 3^{m-1}-1, \frac{3^{m-1}+1}{2}\right)\text{–graph.}$$

The complement of the graph $\mathrm{WIL}^+(2m,3)$ is a pseudo-geometric

$$\left(\frac{3^m-1}{2}, 3^{m-1}-1, \frac{3^{m-1}-1}{2}\right)\text{–graph.}$$

Note that when q is odd, we can obtain an other model of the graphs $\mathrm{WIL}^-(2m,q)$ and $\mathrm{WIL}^+(2m,q)$ (see section 2.3.3).
In the same paper [9] an other construction of Wilbrink is given. Later on we show its connection with the above graphs. Let $\mathrm{Q}(2m,5)$ be a non-singular parabolic quadric in $\mathrm{PG}(2m,5)$, $m \geq 2$. Define the graph $\mathrm{WIL}'^-(2m,5)$ (respectively $\mathrm{WIL}'^+(2m,5)$) with vertex set the set of non-isotropic points x of $\mathrm{PG}(2m,5)$ such that x^* meets $\mathrm{Q}(2m,5)$ in an elliptic (respectively hyperbolic) quadric. Two vertices x and y are adjacent if and only if $x \in y^*$ (and so $y \in x^*$). The graph $\mathrm{WIL}'^-(2m,5)$ is not pseudo-geometric but the graph $\mathrm{WIL}'^+(2m,5)$ is a pseudo-geometric

$$\left(\frac{5^m-1}{2}, 5^{m-1}-1, \frac{5^{m-1}-1}{2}\right)\text{–graph.}$$

2.3.2 Two sets of points off the quadric

In section 1.5.2 we introduced the sets $E_1^+(2m+1,q)$ and $E_2^+(2m+1,q)$ corresponding with the hyperbolic quadric $\mathrm{Q}^+(2m+1,q), q$ odd. Now we will give an alternative definition of them and we generalize this definition for the elliptic quadric $\mathrm{Q}^-(2m+1,q)$ and the parabolic quadric $\mathrm{Q}(2m,q)$.

Let Q be a non-degenerate quadric of $\mathrm{PG}(N,q)$, q odd. Assume N is even. Then we define the *internal* (respectively *external*) points of $\mathrm{Q}(2m,q)$ to be the points of $\mathrm{PG}(2m,q) \setminus \mathrm{Q}(2m,q)$ whose polar space intersects $\mathrm{Q}(2m,q)$ in a non-singular elliptic (respectively hyperbolic) quadric. This corresponds with the points of $\mathrm{PG}(2m,q)$ where the quadratic form takes as value a non-square (respectively a square). Let $E_1(2m,q)$ (respectively $E_2(2m,q)$) denote the set of interior (respectively exterior) points, then

$$|E_1(2m,q)| = \frac{q^m(q^m-1)}{2}, \qquad |E_2(2m,q)| = \frac{q^m(q^m+1)}{2}.$$

Assume N is odd. Consider the quadric $\mathrm{Q}^\square(2m+1,q)$, where the symbol $\square$ denotes either $+$ or $-$. Define the set $E_1^\square(2m+1,q)$ (respectively $E_2^\square(2m+1,q)$) corresponding with $\mathrm{Q}^\square(2m+1,q)$ to be the set of points of $\mathrm{PG}(2m+1,q)$ where the quadratic form takes as value a non-square (respectively a square). Then we obtain that

$$|E_1^\square(2m+1,q)| = |E_2^\square(2m+1,q)| = \frac{q^m(q^{m+1}-(\square 1))}{2}.$$

Remarks

1. Note that the polar space of a point off $\mathrm{Q}^+(2m+1,q)$ always intersects the quadric in a $\mathrm{Q}(2m,q)$, and so the sets $E_1^+(2m+1,q)$ and $E_2^+(2m+1,q)$ cannot be distinguished on a geometrical way. Therefore we do not call them the internal and external points anymore. Similarly for the sets $E_1^-(2m+1,q)$ and $E_2^-(2m+1,q)$.

2. Thas [101] gave an alternative description of the point sets $E_i(2m,q)$, $E_i^+(2m+1,q)$ and $E_i^-(2m+1,q)$, $i=1,2$, using the projection of quadrics.

   The point set $E_i^\square(2m+1,q)$.

   Let $\square$ denote either the symbol $+$ or $-$. Embed $\mathrm{Q}^\square(2m+1,q)$ in a quadric $\mathrm{Q}(2m+2,q)$ of $\mathrm{PG}(2m+2,q)$, q odd. Let H denote the hyperplane of $\mathrm{PG}(2m+2,q)$ containing the quadric $\mathrm{Q}^\square(2m+1,q)$. Let p denote the polar point of H with respect to $\mathrm{Q}(2m+2,q)$. Since q is odd we have $p \notin H$. Then for some $i \in \{1,2\}$, the point set of $E_i^\square(2m+1,q) \cup \mathrm{Q}^\square(2m+1,q)$ is the projection of the points of $\mathrm{Q}(2m+2,q)$, from the point p onto H.

The point set $E_i(2m,q)$.

Embed $\mathrm{Q}(2m,q)$ in an elliptic quadric $\mathrm{Q}^-(2m+1,q)$ of $\mathrm{PG}(2m+1,q)$, q odd. Let H denote the hyperplane of $\mathrm{PG}(2m+1,q)$ containing $\mathrm{Q}(2m,q)$ and let p denote the polar point of H with respect to $\mathrm{Q}^-(2m+1,q)$. Again since q is odd we have that $p \notin H$. Then the point set of $E_1(2m,q) \cup \mathrm{Q}(2m,q)$ is the projection of the points of $\mathrm{Q}^-(2m+1,q)$, from the point p onto H.

Finally embed $\mathrm{Q}(2m,q)$ in an hyperbolic quadric $\mathrm{Q}^+(2m+1,q)$ of $\mathrm{PG}(2m+1,q)$, q odd. Let H denote the hyperplane of $\mathrm{PG}(2m+1,q)$ containing $\mathrm{Q}(2m,q)$ and let p denote the polar point of H with respect to $\mathrm{Q}^+(2m+1,q)$. Since q is odd, $p \notin H$. Then the point set of $E_2(2m,q) \cup \mathrm{Q}(2m,q)$ is the projection of the points of $\mathrm{Q}^+(2m+1,q)$, from the point p onto H.

The following theorem is well known.

Theorem 2.12 *Let E_i denote one of the point sets $E_i(2m,q)$, $E_i^+(2m+1,q)$ or $E_i^-(2m+1,q)$, q odd, $i=1,2$, and let Q denote the corresponding quadric. Then*

(i) each secant line of the quadric Q has $\frac{q-1}{2}$ points in E_1 and the same number of points in E_2;

(ii) a passant or an exterior line of the quadric Q has half of its points in E_1 and the other half in E_2;

(iii) a tangent line of the quadric Q has all of its points but the tangent point in either E_1 or E_2.

Lemma 2.13 *Let the symbol $\square$ denote either $+$ or $-$. Let Ω be a hyperplane of $\mathrm{PG}(2m+1,q), q$ odd, intersecting the quadric $\mathrm{Q}^\square(2m+1,q)$ in a $\mathrm{Q}(2m,q)$. Then $\Omega \cap E_i^\square(2m+1,q)$ equals either $E_i(2m,q)$ (and then we call Ω of* type *i) or $E_j(2m,q)$ (and then we call Ω of* type *j), $i,j \in \{1,2\}, i \neq j$.*

Proof. Recall the alternative description of the point sets $E_i^\square(2m+1,q), i \in \{1,2\}$, of the quadric $\mathrm{Q}^\square(2m+1,q)$, q odd. Let H denote the hyperplane of $\mathrm{PG}(2m+2,q)$ intersecting the quadric $\mathrm{Q}(2m+2,q)$ in $\mathrm{Q}^\square(2m+1,q)$. Then the point set $E_i^\square(2m+1,q)$ is the projection from the point p of the points of $\mathrm{Q}(2m+2,q) \setminus \mathrm{Q}^\square(2m+1,q)$ onto H, where p denotes the polar point of H with respect to $\mathrm{Q}(2m+2,q)$. That is, a point x of $E_i^\square(2m+1,q)$ corresponds with the pair of points (x',x'') in the intersection of the projective line $\langle p,x\rangle$ with $\mathrm{Q}(2m,q) \setminus \mathrm{Q}^\square(2m-1,q)$ (see figure 2.1).
Let Ω be a hyperplane of H, intersecting the quadric $\mathrm{Q}^\square(2m+1,q)$ in a $\mathrm{Q}(2m,q)$. Let Q denote the intersection of $\mathrm{Q}(2m+2,q)$ with $\langle p,\Omega\rangle$. Then the points of the quadric Q will be projected onto a subset of $E_i^\square(2m+1,q)$ (see figure 2.1). Moreover if $Q=\mathrm{Q}^-(2m+1,q)$ then $\Omega \cap E_i^\square(2m+1,q) = E_1(2m,q)$ while if $Q=\mathrm{Q}^+(2m+1,q)$ then we obtain $\Omega \cap E_i^\square(2m+1,q) = E_2(2m,q)$. $\square$

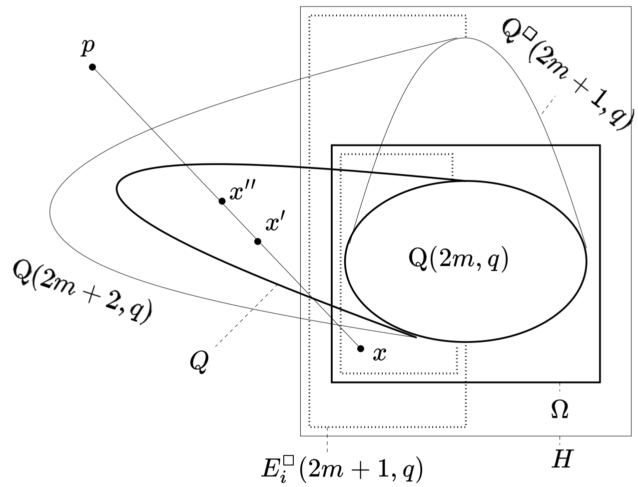

Figure 2.1: The intersection of $E_i^{\square}(2m+1,q)$ with a hyperplane of $\mathrm{PG}(2m+1,q)$

2.3.3 The tangency graphs on the internal or external points of a parabolic quadric.

Let $\mathrm{Q}(2m,q)$ be a non-singular quadric in $\mathrm{PG}(2m,q)$, q odd, $m \geq 2$. Let H be a hyperplane of $\mathrm{PG}(2m,q)$ meeting $\mathrm{Q}(2m,q)$ in a non-singular elliptic (respectively hyperbolic) quadric. Then the polar point H^* of H with respect to $\mathrm{Q}(2m,q)$ is an element of $E_1(2m,q)$ (respectively $E_2(2m,q)$). Conversely with a point p of $E_1(2m,q)$ (respectively $E_2(2m,q)$) there corresponds a hyperplane p^* of $\mathrm{PG}(2m,q)$ meeting $\mathrm{Q}(2m,q)$ in a non-singular elliptic (respectively hyperbolic) quadric. Let H_1 and H_2 be two hyperplanes of $\mathrm{PG}(2m,q)$ both intersecting $\mathrm{Q}(2m,q)$ in a non-singular elliptic (respectively hyperbolic) quadric. Then the intersection of the quadric $H_1 \cap \mathrm{Q}(2m,q)$ with the quadric $H_2 \cap \mathrm{Q}(2m,q)$ is degenerate if and only if the points H_1^* and H_2^* are contained in a tangent line of $\mathrm{Q}(2m,q)$. And so if q is odd, we obtain a second model for the graph $\mathrm{WIL}^-(2m,q)$ (respectively $\mathrm{WIL}^+(2m,q)$). Since we mainly work in the second model, we call this graph, the *tangency graph* on the internal (respectively external) points of a parabolic quadric.

Remark

For $q = 3$, adjacency in the graph $\mathrm{WIL}^-(2m,3)$ (that is being contained in a tangent line of the quadric $\mathrm{Q}(2m,3)$) translates into being non-orthogonal with respect to the quadric $\mathrm{Q}(2m,q)$, and so non-adjacency translates into orthogonality. This explains the link between the graphs $\mathrm{WIL}^-(2m,3)$ and $\mathrm{WIL}^+(2m,3)$ on the one hand and the graphs $\mathrm{WIL}'^-(2m,5)$ and $\mathrm{WIL}'^-(2m,5)$ on the other hand.

2.3.4 Negative results

Definition

An *exterior set* with respect to a quadric Q is a set X of points such that any line joining two elements of X is a passant with respect to the quadric. If $Q = \mathrm{Q}^+(2m-1,q)$ then one easily proves that $|X| \leq \frac{q^m-1}{q-1}$; if equality holds, then the exterior set is called *maximal*.

Overview

De Clerck and Thas [35] proved that maximal exterior sets with respect to $\mathrm{Q}^+(2m-1,q)$, q even, only exist in $\mathrm{PG}(3,q)$ and $\mathrm{PG}(5,q)$. The complete classification of all maximal exterior sets of $\mathrm{Q}^+(2m-1,q), m \geq 2$, is done by Thas (see [104, 105] for more details).

Dye [50] proved that the size of an exterior set X with respect to $\mathrm{Q}(2m,2)$ is smaller than or equal to $2m+1$; the size of an exterior set X with respect to $\mathrm{Q}^+(2m+1,2)$ is smaller than or equal to $2m+1, 2m+1, 2m+3, 2m+2$ according as m is congruent modulo 4 to 0,1,2 or 3 respectively; the size of an exterior set X with respect to $\mathrm{Q}^-(2m+1,2)$ is smaller than or equal to $2m+3, 2m+2, 2m+1, 2m+1$ according as m is congruent modulo 4 to 0,1,2 or 3 respectively. Moreover a classification of sets achieving these bounds is given.

In this section we prove an upper bound for the size of exterior sets with respect to non-singular quadrics in $\mathrm{PG}(N,3)$.

Theorem 2.14 *Let X be an exterior set with respect to a quadric Q. Then $|X| \leq 4m+2$ for the quadric $\mathrm{Q}(2m,3)$ and $|X| \leq 4m$ for the quadrics $\mathrm{Q}^-(2m-1,3)$ and $\mathrm{Q}^+(2m-1,3)$.*

Proof. Let Q denote the quadric $\mathrm{Q}(2m,3)$ and let X be an exterior set with respect to Q. Let $X \cap E_i(2m,3) = X_i = \{x_1^i, \ldots, x_{\eta_i}^i\}, i = 1,2$. As two points in $E_i(2m,3)$ $(i=1,2)$ are on a passant with respect to $\mathrm{Q}(2m,3)$, whenever they are orthogonal, it follows that $x_2^i \in (x_1^i)^* \cap E_i(2m,3)$ where $(x_1^i)^*$ is the polar hyperplane of x_1^i with respect to the quadric $\mathrm{Q}(2m,3)$. Note that $(x_1^i)^* \cap Q$ is a $\mathrm{Q}^-(2m-1,3)$ for $i=1$ and a $\mathrm{Q}^+(2m-1,3)$ for $i=2$. As X_i is a cap in $\mathrm{PG}(2m,3)$, we may conclude that for $j \geq 2$, $\cap_{k=1}^{j-1}(x_k^i)^* = \mathrm{PG}(2m+1-j,3)$ and $x_j^i \in \mathrm{PG}(2m+1-j,3) \cap E_i(2m,3)$. Assume $j = 2m$. When $L = \mathrm{PG}(1,3)$ is an exterior line with respect to the quadric Q, then $|L \cap E_1(2m,3)| = 2$, and when $L = \mathrm{PG}(1,3)$ is a secant line with respect to the quadric Q, then $|L \cap E_2(2m,3)| = 1$. Therefore $\eta_i = |X_i| \leq 2m+1$, $i = 1,2$. And so $|X| = |X_1| + |X_2| \leq 4m+2$.

Similarly we obtain that $|X| \leq 4m$ for the quadrics $\mathrm{Q}^-(2m-1,3)$ and $\mathrm{Q}^+(2m-1,3)$. $\square$

Corollary 2.15 *The complements of the graphs* $\mathrm{WIL}^-(2m,3)$*,* $m \geq 2$*, and* $\mathrm{WIL}^+(2m,3)$*,* $m \geq 3$*, are not geometric.*

Proof. As two vertices in the graph $\mathrm{WIL}^-(2m,3)$ are non-adjacent whenever they are on a passant with respect to the quadric, a line of the putative partial geometry $\mathrm{pg}(\frac{3^m+1}{2}, 3^{m-1}-1, \frac{3^{m-1}+1}{2})$ with point graph the complement of $\mathrm{WIL}^-(2m,3)$, can be viewed as the intersection of an exterior set of the quadric with the set $E_1(2m,3)$. Hence from the proof of theorem 2.14 it follows that $\frac{3^m+1}{2} \leq 2m$, which is a contradiction if $m \geq 2$.
Similarly a line of a putative partial geometry $\mathrm{pg}(\frac{3^m-1}{2}, 3^{m-1}-1, \frac{3^{m-1}-1}{2})$, with point graph the complement of the graph $\mathrm{WIL}^+(2m,3)$ can be viewed as the intersection of an exterior set of the quadric with the set $E_2(2m,3)$ and hence $\frac{3^m-1}{2} \leq 2m$, which is again a contradiction if $m \geq 3$. □

Theorem 2.16 *The graph* $\mathrm{WIL}'^+(2m,5)$*,* $m \geq 2$*, is not geometric.*

Proof. Let $C = \{x_0, \ldots, x_{\frac{5^m-1}{2}}\}$ be a maximal clique of the graph $\mathrm{WIL}'^+(2m,5)$. Since any two points in C are orthogonal, it follows that $x_2 \in (x_1)^* \cap E_2(2m,5)$. Since a line through an exterior point x of $\mathrm{Q}(2m,5)$ intersects x^* in exactly one point we have that C is a cap in $\mathrm{PG}(2m,5)$, hence for $j \geq 2$ we obtain $\cap_{k=1}^{j-1}(x_k)^* = \mathrm{PG}(2m+1-j,5)$ and $x_j \in \mathrm{PG}(2m+1-j,5) \cap E_2(2m,5)$. Assume $j = 2m$, then $L = \mathrm{PG}(1,5)$ is either a secant line or an exterior line of the quadric $\mathrm{Q}(2m,5)$ and so $|L \cap E_2(2m,5)| = 1$ or 2, from which follows that $\frac{5^m-1}{2} \leq 2m$, a contradiction. □

2.3.5 Positive results

In [62] Hirschfeld and Thas constructed a semipartial geometry $\mathrm{TQ}(4,q)$ arising from the projection of an elliptic quadric $\mathrm{Q}^-(5,q)$ from a point of $\mathrm{PG}(5,q) \setminus \mathrm{Q}^-(5,q)$ onto a hyperplane of $\mathrm{PG}(5,q)$ (see also section 6.5.1). The point graph of this semipartial geometry is the graph $\mathrm{WIL}^-(4,q)$. And so we have the following.

Theorem 2.17 ([62]) *The graph* $\mathrm{WIL}^-(4,q)$ *is semigeometric.*

When q is even then $\mathrm{TQ}(4,q)$ is embedded in the affine space $\mathrm{AG}(4,q)$. In section 6.5.3 we characterize the semipartial geometry $\mathrm{TQ}(4,q)$, q even, by its parameters and its embedding in $\mathrm{AG}(4,q)$.
Next we investigate the graph $\mathrm{WIL}^-(2m,q), q$ odd, for general dimensions. Under the condition of the existence of an orthogonal spread of $\mathrm{Q}(2m,q)$ we will obtain a new class of semipartial geometries which we will call $\mathrm{SPQ}(2m,q)$.

Lemma 2.18 *Let σ be a generator of* $\mathrm{Q}(2m,q), m \geq 2$*, q odd, and let X be a hyperplane of σ, then for the polar space X^* of X,*

$$|X^* \cap E_1(2m,q)| = \frac{(q-1)q^m}{2}, \qquad |X^* \cap E_2(2m,q)| = \frac{(q+1)q^m}{2}.$$

Proof. Note that $X^* \cap \mathrm{Q}(2m, q)$ is a cone with vertex X and base a conic. Hence there are $q+1$ generators $\sigma = \pi_X^0, \ldots, \pi_X^q$ through X contained in $\mathrm{Q}(2m, q)$.
If $x \in E_2(2m, q)$ then $x^* \cap \mathrm{Q}(2m, q)$ is a $\mathrm{Q}^+(2m-1, q)$ whose generators are also generators of $\mathrm{Q}(2m, q)$; while for $x \in E_1(2m, q)$, $x^* \cap \mathrm{Q}(2m, q)$ is a $\mathrm{Q}^-(2m-1, q)$ which does not contain a generator of $\mathrm{Q}(2m, q)$.
Recall the properties of projective lines of $\mathrm{PG}(2m, q)$ with respect to the sets $E_1(2m, q)$ and $E_2(2m, q)$ from theorem 2.12. Then each of the q m-dimensional spaces $\Pi_i = \langle \sigma, \pi_X^i \rangle \subset X^*, i = 1, \ldots, q$, intersects $E_j(2m, q)$, $j = 1, 2$ in the points of $\frac{q-1}{2}$ affine $(m-1)$-dimensional space $L_X^{i,j,k}$ having X at infinity, $k = 1, \ldots, \frac{q-1}{2}$. We denote the $q(q-1)$ projective $(m-1)$-dimensional spaces $L_X^{i,j,k} \cup X$ by $P_X^{i,j,k}$, $i = 1, \ldots, q$, $j = 1, 2$, $k = 1, \ldots, \frac{q-1}{2}$.
Since σ^* and $\langle \sigma, \pi_X^i \rangle, i = 1, \ldots, q$, are different m-dimensional spaces intersecting each other in the $(m-1)$-dimensional space σ, we obtain that each of the $q(q-1)$ spaces $P_X^{i,j,k}$, $i = 1, \ldots, q$, $j = 1, 2$, $k = 1, \ldots, \frac{q-1}{2}$, intersects the q $(m-1)$-dimensional spaces Q_X^l, $l = 1, \ldots, q$, through X in σ^* only in X. Note that π_X^i, Q_X^i, and $P_X^{i,j,k}$, $i = 1, \ldots, q$, $j = 1, 2$, $k = 1, \ldots, \frac{q-1}{2}$, are the q^2+q+1 $(m-1)$-dimensional spaces through X that are contained in X^*, and that $\sigma^* \cap E_1(2m, q) = \emptyset$. This implies that $\pi_X^i \subset \mathrm{Q}(2m, q)$ and $P_X^{i,1,k} \setminus X \subset E_1(2m, q)$ and $P_X^{i,2,k} \setminus X \subset E_2(2m, q)$ and $Q_X^i \setminus X \subset E_2(2m, q)$, $i = 1, \ldots, q$, $k = 1, \ldots, \frac{q-1}{2}$. The result now follows. $\square$

Construction

First note that the parabolic quadric $\mathrm{Q}(4n, q)$ in $\mathrm{PG}(4n, q)$ with q odd has no spreads (see [106]). The existence of spreads of $\mathrm{Q}(4n-2, q)$, $n > 2$ and q odd, is still open, whereas $\mathrm{Q}(6, q)$ with q an odd prime or $q \equiv 0$ or $2 \pmod 3$ has spreads (see for example [107]).
Let Σ be an orthogonal spread of $\mathrm{Q}(4n-2, q), n \geq 2$, q odd. Let $\mathrm{Hyp}(\Sigma)$ denote the set of hyperplanes of the elements of Σ. For every element X of $\mathrm{Hyp}(\Sigma)$, let L_X^i, $i = 1, \ldots, \frac{q(q-1)}{2}$, denote the $(2n-2)$-dimensional affine subspaces in $X^* \cap E_1(4n-2, q)$ having X at infinity (see figure 2.2 for the 6-dimensional case). Define the incidence structure $\mathrm{SPQ}(4n-2, q) = (E_1(4n-2, q), \mathcal{L}, \in)$ with

$$\mathcal{L} = \{L_X^i : i = 1, \ldots, \frac{q(q-1)}{2}, X \in \ \mathrm{Hyp}(\Sigma)\}.$$

Theorem 2.19 *The incidence structure* $\mathrm{SPQ}(4n-2, q)$ *is an*

$$\mathrm{spg}(q^{2n-2}-1, q^{2n-1}, 2 \cdot q^{2n-3}, 2 \cdot q^{2n-2}(q^{2n-2}-1)),$$

having point graph $\mathrm{WIL}^-(4n-2, q)$.

Proof. Obviously $s = q^{2n-2} - 1$. When X and Y are $(2n-3)$-dimensional spaces in different elements of Σ, then the corresponding L_X^i and L_Y^j, $i, j = 1, \ldots, \frac{q(q-1)}{2}$, intersect in at most one point because Σ is a spread. Obviously

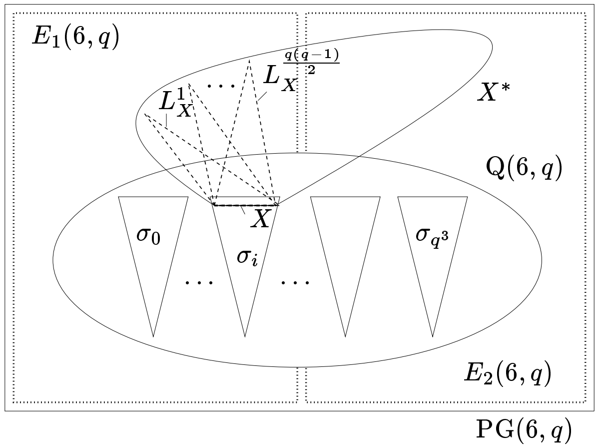

Figure 2.2: The construction of $\mathrm{SPQ}(6,q)$

L_X^i and L_X^j, $i,j = 1,\ldots,\frac{q(q-1)}{2}$, $i \neq j$, are disjoint. Suppose that X and Y are two different hyperplanes of $\sigma \in \Sigma$, and suppose that there exist an L_X^i and L_Y^j intersecting in a point x. Then $X \subset x^*$ and $Y \subset x^*$ hence $\langle X, Y\rangle = \sigma \subset x^*$ which is a contradiction since x is point of $E_1(4n-2,q)$. Therefore L_X^i and L_Y^j are disjoint.
When X is a hyperplane of $\sigma \in \Sigma$, then by lemma 2.18,

$$\cup_{i=1}^{\frac{q(q-1)}{2}} L_X^i = X^* \cap E_1(4n-2,q).$$

Hence for $x \in E_1(4n-2,q)$ and $\sigma \in \Sigma$ we have that $\langle x, x^* \cap \sigma\rangle \setminus (x^* \cap \sigma)$ is a line of $\mathrm{SPQ}(4n-2,q)$. Hence $t = |\Sigma| - 1 = q^{2n-1}$, and $\mathrm{SPQ}(4n-2,3)$ is a partial linear space with point graph the strongly regular graph $\mathrm{WIL}^-(4n-2,q)$.
Let (x,L) be an antiflag of $\mathrm{SPQ}(4n-2,q)$. Consider the $(2n-3)$-dimensional space $X \subset \mathrm{Q}(4n-2,q)$ at infinity of L. Suppose that $x \in X^* \setminus L$. Take $y \in L$, then $\langle x, y\rangle$ must be an exterior line to the quadric. Therefore $\alpha(x,L) = 0$, while for a point $z \in E_1(4n-2,q) \setminus X^*$, z^* intersects X in a $(2n-4)$-dimensional space. Therefore the space $\langle z, L\rangle$ intersects $\mathrm{Q}(4n-2,q)$ in a (singular) quadric Q, and Q is not equal to X. Hence z is collinear in $\mathrm{SPQ}(4n-2,q)$ with at least one element of L. We now have the following. The set $\mathcal{L}$ is a collection of cliques of the pseudo-semigeometric

$$(q^{2n-2}-1, q^{2n-1}, 2\cdot q^{2n-3}, 2\cdot q^{2n-2}(q^{2n-2}-1))\text{–graph}$$

$\Gamma = \mathrm{WIL}^-(4n-2,q)$ such that every edge is contained in a unique clique $L_X^i \in \mathcal{L}$ of size q^{2n-2}. Moreover for a clique $L_X^i \in \mathcal{L}$ we have

$$|\cup_{x\in L_X^i}\Gamma(x)| \quad = \quad |E_1(4n-2,q)\setminus X^*|$$

$$= \frac{q^{2n}(q^{2n-2}-1)}{2}$$
$$= \frac{s(s+1)t}{\alpha}.$$

By theorem 2.1 this implies that $\mathrm{WIL}^-(4n-2,q)$ is semigeometric. □

Remarks

1. Assume $q=3$. Then collinearity in $\mathrm{SPQ}(4n-2,3)$ translates to non-orthogonality, and so it is easy to prove immediately that the incidence number of an antiflag (x, L_X) of $\mathrm{SPQ}(4n-2,3)$ equals 0 or $2\cdot 3^{2n-3}$. When $x \in X^* \setminus L_X$ then $\alpha(x, L_X)=0$ and when $x \in E_1 \setminus X^*$ then $x^* \cap L_X$ is an affine $(2n-3)$-dimensional space $A_{(x,L)}$. Hence
$$\alpha(x, L_X) = |L_X \setminus A_{(x,L)}| = 2\cdot 3^{2n-3}.$$

2. Assume $q=3$. Then we can describe the line set of the semipartial geometry $\mathrm{SPQ}(4n-2,3)$ in the following way. Let Σ be an orthogonal spread of $\mathrm{Q}(4n-2,3), n\geq 2$. For every $(2n-1)$-dimensional subspace X of an element σ of Σ, let $\sigma=\pi_X^0, \pi_X^1, \pi_X^2, \pi_X^3$ denote the four generators of the quadric through X. Then the set $\mathcal{L}$ is the collection of the $(2n-2)$-dimensional affine subspaces $L_X^i = \langle \sigma, \pi_X^i\rangle \cap E_1(4n-2,3)$, $i=1,2,3$. Note that in section 2.4 we give yet another description of $\mathrm{SPQ}(4n-2,3)$.

3. The construction of the semipartial geometry $\mathrm{SPQ}(4n-2,q), q$ odd, depends on the existence of orthogonal spreads. Different spreads yield different geometries. Take for example the quadric $\mathrm{Q}(6,3)$. It has exactly three non-isomorphic spreads [R. Mathon, private communication], yielding at least three non-isomorphic semipartial geometries of type $\mathrm{SPQ}(6,3)$, which are $\mathrm{spg}(8,27,6,144)$.

4. Let σ be an element of the orthogonal spread Σ of $\mathrm{Q}(4n-2,q), n\geq 2$, q odd. Let $\mathrm{Hyp}(\sigma)$ denote the set of hyperplanes σ. Then from the proof of theorem 2.19 we see that the set
$$\{L_X^i : i=1,\ldots,\frac{q(q-1)}{2}, X \in \mathrm{Hyp}(\sigma)\},$$
is a spread of the semipartial geometry $\mathrm{SPQ}(4n-2,q)$.

5. We published lemma 2.18 and theorem 2.19 for $q=3$ in [43], but the changes that have to be made for general q odd are minimal. Note that the parameters of the semipartial geometry $\mathrm{SPQ}(4n-2,q)$, q odd, are new.

SPG-systems

Recently Thas [108] generalized the infinite class semipartial geometries $\mathrm{SPQ}(4n-2,q)$, q odd, to a class of (semi)partial geometries that includes several other known examples. He generalized the concept of an SPG-regulus of a polar space P to an SPG-system of P (see section 3.2.1 for the definition of an SPG-regulus).

Let $\mathrm{Q}(2n+2,q)$, $n \geq 1$, be a non-singular quadric of $\mathrm{PG}(2n+2,q)$. An *SPG-system* of $\mathrm{Q}(2n+2,q)$ is a set $\mathcal{D}$ of $(n-1)$-dimensional totally singular subspaces of $\mathrm{Q}(2n+2,q)$ such that the elements of $\mathcal{D}$ on any non-singular elliptic quadric $\mathrm{Q}^-(2n+1,q) \subset \mathrm{Q}(2n+2,q)$ constitute a spread of the quadric $\mathrm{Q}^-(2n+1,q)$. Let $\mathrm{Q}^+(2n+1,q)$ be a non-singular hyperbolic quadric of $\mathrm{PG}(2n+1,q)$, $n \geq 1$. An *SPG-system* of $\mathrm{Q}^+(2n+1,q)$ is a set $\mathcal{D}$ of $(n-1)$-dimensional totally singular subspaces of $\mathrm{Q}^+(2n+1,q)$ such that the elements of $\mathcal{D}$ on any non-singular quadric $\mathrm{Q}(2n,q) \subset \mathrm{Q}^+(2n+1,q)$ constitute a spread of $\mathrm{Q}(2n,q)$. Let $\mathrm{H}(2n+1,q)$ be a non-singular Hermitian variety of $\mathrm{PG}(2n+1,q)$, $n \geq 1$, q a square. An *SPG-system* of $\mathrm{H}(2n+1,q)$ is a set $\mathcal{D}$ of $(n-1)$-dimensional totally singular subspaces of $\mathrm{H}(2n+1,q)$ such that the elements of $\mathcal{D}$ on any non-singular Hermitian variety $\mathrm{H}(2n,q) \subset \mathrm{H}(2n+1,q)$ constitute a spread of $\mathrm{H}(2n,q)$.

One can prove that in each case the number of elements in $\mathcal{D}$ equals the number of points of the polar space.

The construction by Thas of the semipartial geometry is as follows. Let P be one of the above polar spaces, that is $\mathrm{Q}(2n+2,q)$, $\mathrm{Q}^+(2n+1,q)$, $\mathrm{H}(2n+1,q)$ $(n \geq 1)$. Let $\mathrm{PG}(d,q)$ be the ambient space of P. Hence in the first case $d = 2n+2$, in the other two cases $d = 2n+1$. Let $\mathcal{D}$ be an SPG-system of P and let P be embedded in a non-singular polar space $\bar{P}$ with ambient space $\mathrm{PG}(d+1,q)$ of the same type as P and with projective index n. Hence for $P = \mathrm{Q}(2n+2,q)$, we have $\bar{P} = \mathrm{Q}^-(2n+3,q)$; for $P = \mathrm{Q}^+(2n+1,q)$, we have $\bar{P} = \mathrm{Q}(2n+2,q)$ and for $P = \mathrm{H}(2n+1,q)$, we have $\bar{P} = \mathrm{H}(2n+2,q)$. If $\bar{P}$ is not symplectic and $y \in \bar{P}$, then let τ_y be the tangent hyperplane of $\bar{P}$ at y; if $\bar{P}$ is symplectic and θ is the corresponding symplectic polarity of $\mathrm{PG}(d+1,q)$, then let $\tau_y = y^\theta$ for any $y \in \mathrm{PG}(d+1,q)$.

For $y \in \bar{P} \setminus P$ let $\bar{y}$ be the set of all points z of $\bar{P} \setminus P$ for which $\tau_z \cap P = \tau_y \cap P$. Note that no two distinct points of $\bar{y}$ are collinear in $\bar{P}$. If P is orthogonal then $|\bar{y}| = 2$ except when $P = \mathrm{Q}^+(2n+1,q)$ and q even, in which case $|\bar{y}| = 1$. If P is Hermitian then $|\bar{y}| = \sqrt{q}+1$.

Let ξ be any maximal totally singular subspace of $\bar{P}$, not contained in P, such that $\xi \cap P \in \mathcal{D}$ and let $y \in \xi \setminus P$. Further let $\bar{\xi}$ be the set of all maximal totally singular subspaces η of $\bar{P}$, not contained in P, for which $\xi \cap P = \eta \cap P$ and $\eta \cap \bar{y} \neq \emptyset$.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be the incidence structure with $\mathcal{P} = \{\bar{y} \| y \in \bar{P} \setminus P\}$; $\mathcal{L}$ contains all the sets $\bar{\xi}$ as defined above; if $\bar{y} \in \mathcal{P}$ and $\bar{\xi} \in \mathcal{L}$ then $\bar{y} \mathrel{\mathrm{I}} \bar{\xi}$ if and only if for some $z \in \bar{y}$ and some $\eta \in \bar{\xi}$, one has that $z \in \eta$.

In [108] it is proved that this incidence structure is a $(0,\alpha)$-geometry of order (s,t) with $s+1 = q^n$ and $t+1$ the number of elements in a spread of P. The parameter α equals to q^{n-1} times the number of points of $\bar{P}$ in any set $\bar{y} \in \mathcal{P}$. In particular Thas [108] proved the following.

1. If P is the polar space $\mathrm{Q}(2n+2, q)$ then $\mathcal{S}$ is a semipartial geometry $\mathrm{spg}(q^n - 1, q^{n+1}, 2q^{n-1}, 2q^n(q^n - 1))$.
2. If P is the polar space $\mathrm{Q}^+(2n+1, q)$ then the point graph $\Gamma(\mathcal{S})$ is strongly regular if and only if $q = 2$ or $q = 3$. In these cases $\mathcal{S}$ is a partial geometry.
3. If P is the polar space $\mathrm{H}(2n+1, q^2)$ then $\mathcal{S}$ is a semipartial geometry $\mathrm{spg}(q^{2n} - 1, q^{2n+1}, q^{2n-2}(q+1), q^{2n-1}(q^{2n} - 1)(q+1))$.

Remarks

1. Let P be the polar space $\mathrm{Q}(2n+2, q)$. Then the corresponding geometry will be denoted by $\mathrm{TQ}(2n+2, q)$. If $n = 1$ the SPG-system is the complete set of points of $\mathrm{Q}(4, q)$ and the semipartial geometry was known before, it is the semipartial geometry of Metz, see [38]. Assume $n = 2$. It is proved in [108] that there are exactly two SPG-systems on $\mathrm{Q}(6, q)$. One arises from a spread of $\mathrm{Q}(6, q)$, the other arises from the classical general hexagon of order q, in which case we denote the corresponding semipartial geometry by $\mathrm{SPH}(q)$. For $n \geq 3$, any spread of $\mathrm{Q}(2n+2, q)$ defines an SPG-system. Such a spread is known to exist if q is even, or if $n = 2$ and q is an odd prime or $q \equiv 0$ or $2 \pmod 3$. In the previous section we gave a construction of the semipartial geometry $\mathrm{SPQ}(4n-2, q)$ under the condition of existence of an orthogonal spread. This semipartial geometry is isomorphic to $\mathrm{TQ}(2n+2, q)$.

   Actually, from the proof of theorem 2.19 one can see that in order to construct a semipartial geometry with point graph $\mathrm{WIL}^-(6, q)$ it is sufficient to have a set Ω of projective lines of the quadric $\mathrm{Q}(6, q)$ with the property that the elements of Ω on any $\mathrm{Q}^-(5, q)$ contained in the quadric $\mathrm{Q}(6, q)$, constitute a spread of $\mathrm{Q}^-(5, q)$, that is Ω is an SPG-system of the quadric $\mathrm{Q}(6, q)$.

2. Let P be the polar space $\mathrm{Q}^+(2n+1, q)$; $q = 2$ or 3. If $n = 2m - 1$ is odd and $q = 2$ then $\mathrm{Q}^+(2n+1, 2)$ has a spread and the partial geometry is isomorphic to the partial geometry $\mathrm{PQ}^+(4m-1, 2)$.

   If $n = 2m - 1$ is odd and $q = 3$ then the partial geometry is isomorphic to the partial geometry $\mathrm{PQ}^+(4m-1, 3)$ of Thas, which only exists if $\mathrm{Q}^+(4m-1, 3)$ has a spread. Recall that the existence of such a spread is open for $m \geq 3$.

3. Let P be the polar space $\mathrm{H}(2n+1, q)$. The geometry will be denoted by $\mathrm{TH}(2n+1, q)$. Unfortunately, if $n \geq 2$ then no SPG-system of $\mathrm{H}(2n+1, q)$ is known. If $n = 1$, then $\mathcal{D}$ is the set of points of $\mathrm{H}(3, q)$ and the semipartial geometry is the one of Thas as described in [38].

From the theory of SPG-systems we obtain the following.

Theorem 2.20 (**[108]**) *The graph* $\mathrm{WIL}^-(4n-2,q)$ *is semigeometric for* $n=2$, *or* $q=2^h, h \geq 1$ *and* $n \geq 3$.

2.4 From $\mathrm{PQ}^+(4n-1,3)$ to $\mathbf{SPQ}(4n-2,3)$

In the next theorem we show that there is a link between the partial geometry $\mathrm{PQ}^+(4n-1,3)$ and the semipartial geometry $\mathrm{SPQ}(4n-2,3)$.

Theorem 2.21 *Let* H *be a hyperplane of* $\mathrm{PG}(4n-1,3), n \geq 2$, *intersecting* $\mathrm{Q}^+(4n-1,3)$ *in a* $\mathrm{Q}(4n-2,3)$ *such that* H *is of type 1. Then the set of points of the semipartial geometry* $\mathrm{SPQ}(4n-2,3)$ *is the set of points in* H *of the partial geometry* $\mathrm{PQ}^+(4n-1,3)$*; the lines of* $\mathrm{SPQ}(4n-2,3)$ *are the non-empty intersections of the lines of* $\mathrm{PQ}^+(4n-1,3)$ *with* H.

Proof. First of all, note that H will intersect any spread $\Sigma = \{\sigma_1, \ldots, \sigma_{3^{2n-1}}\}$ of $\mathrm{Q}^+(4n-1,3)$ in a spread

$$\Sigma' = \{\sigma_i' = \sigma_i \cap H | i = 1, \ldots, 3^{2n-1}\}$$

of $\mathrm{Q}(4n-2,3)$. Since H is a type 1 hyperplane, the point set of $\mathrm{SPQ}(4n-2,3)$ is indeed the intersection of H with the point set of $\mathrm{PQ}^+(4n-1,3)$ (see lemma 2.13).
Since in H the set $E_1(4n-2,3)$ is the set of the internal points of the quadric $\mathrm{Q}(4n-2,3)$, the polar space x^* of a point $x \in H \cap E_1(4n-2,3)$ with respect to the quadric $\mathrm{Q}(4n-2,3)$ intersects the quadric in a $\mathrm{Q}^-(4n-3,3)$. Since the maximal subspaces of $\mathrm{Q}^-(4n-3,3)$ are $(2n-3)$-dimensional, the intersection of $E_1(4n-2,3)$ with the polar space of an element σ_i' of Σ' is empty. If X is a $(2n-3)$-dimensional subspace of an element σ_i' of Σ', then the three lines $L_X^j, j = 1,2,3$, of $\mathrm{SPQ}(4n-2,3)$ corresponding with X are the intersections of H with the three lines of $\mathrm{PQ}^+(4n-1,3)$ corresponding to the $(2n-2)$- dimensional spaces π_j through X which are contained in σ_i and $\pi_j \neq \sigma_i', j = 1,2,3$. □

Remark

1. The set $\Delta(x)$ of points which are not collinear with a point x of the partial geometry $\mathrm{PQ}^+(4n-1,3)$, is contained in x^*. Since $|\Delta(x)| = |E_1(2m,3)|$, the polar hyperplane x^* of x is always of type 1, no matter if the point set $\mathcal{P}$ of $\mathrm{PQ}^+(4n-1,3)$ is the set $E_1^+(4n-1,3)$ or $E_2^+(4n-1,3)$.

2. In order to make the difference with an orthogonal spread, we call a spread of a partial geometry (respectively semipartial geometry) also a *pg-spread* (respectively *spg-spread*). If Φ is a any pg-spread of the partial geometry $\mathrm{PQ}^+(4n-1,3)$, then theorem 2.21 implies that the restriction of Φ to the secant hyperplane H is an spg-spread of $\mathrm{SPQ}^+(4n-2,3)$. Note that the partial geometry $\mathrm{PQ}^+(4n-1,3)$ has spreads which have interesting regularity conditions (see section 4.2.1).

3. In section 3.1 we discuss an other connection between the partial geometry $\mathrm{PQ}^+(4n-1,3)$ and the semipartial geometry $\mathrm{SPQ}(4n-2,3)$ which we call *derivation with respect to a point*. Moreover, the links between these geometries allows us to prove results about the point and block graphs of $\mathrm{SPQ}(4n-1,3)$ (see section 4.6).

Chapter 3

New construction methods

3.1 Point derivation

3.1.1 Construction

Definitions

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a partial geometry pg(s, t, α). Let p be any point of $\mathcal{S}$, let $p^\perp$ be the set of points of $\mathcal{S}$ collinear with p (including p) and let $\mathcal{L}(p)$ denote the set of lines of $\mathcal{S}$ that contain p. Then consider the following incidence structure $\mathcal{S}_p = (\mathcal{P}_p, \mathcal{L}_p, \mathrm{I}_p)$ with $\mathcal{P}_p = \mathcal{P} \setminus p^\perp$, with $\mathcal{L}_p = \mathcal{L} \setminus \mathcal{L}(p)$ and with I_p=I $\cap((\mathcal{P}_p \times \mathcal{L}_p) \cup (\mathcal{L}_p \times \mathcal{P}_p))$. We call $\mathcal{S}_p$ the geometry *derived from* $\mathcal{S}$ *with respect to the point* p. In this case we also call $\mathcal{S}_p$ *point derived* from $\mathcal{S}$.

Theorem 3.1 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a* pg(s, t, α) *and let p be a point of $\mathcal{S}$ such that for any triad $\{p, y, z\}$ of non-collinear points, the set $\{p, y, z\}^\perp$ of points in $\mathcal{S}$ collinear with p, y and z has constant cardinality η. Then the derived geometry $\mathcal{S}_p = (\mathcal{P}_p, \mathcal{L}_p, \mathrm{I}_p)$ with respect to p is an*

$$\mathrm{spg}(s - \alpha, t, \beta, \alpha(t + 1) - \eta)$$

if and only if $\forall L \in \mathcal{L}, \forall x \in \mathcal{P}_p : |L \cap p^\perp \cap x^\perp| \in \{\alpha, \alpha - \beta\}$.

Proof. Since for every antiflag (p, L) of $\mathcal{S}$, $\alpha(p, L) = \alpha$, we immediately have that the number of points on a line of $\mathcal{S}_p$ equals $s - \alpha + 1$ while the number of lines through a point of $\mathcal{S}_p$ remains $t + 1$. And so $\mathcal{S}_p$ is a partial linear space of order $(s - \alpha, t)$.
Let (x, L) be an antiflag of $\mathcal{S}_p$, then $\alpha(x, L) = 0$ if and only if in $\mathcal{S}$ we have $|L \cap p^\perp \cap x^\perp| = \alpha$ and $\alpha(x, L) = \beta$ if and only if in $\mathcal{S}$ $|L \cap p^\perp \cap x^\perp| = \alpha - \beta$. A line through p or x always intersects $p^\perp \cap x^\perp$ in α points.
Let y, z be two different non-collinear points of $\mathcal{S}_p$. Then p, y, z are three different non-collinear points of $\mathcal{S}$. Hence there are a constant number η of points that are collinear with all three of p, y, z. Since $\mathcal{S}$ is a partial geometry there

are $\alpha(t+1)$ points of $\mathcal{S}$ collinear with both y and z. Hence there are $\alpha(t+1)-\eta$ points of $\mathcal{S}$ that are collinear with y and z but that are non-collinear with p and so these $\alpha(t+1)-\eta$ points are contained in the point set of $\mathcal{S}_p$. This proves the result. □

Lemma 3.2 *Assume that $\mathcal{S}_p$ is an* $\mathrm{spg}(s-\alpha, t, \beta, \mu)$ *derived from a* $\mathrm{pg}(s,t,\alpha)$ *$\mathcal{S}$ with respect to a point p of $\mathcal{S}$. Let p, y, z be a triad of non-collinear points of $\mathcal{S}$, and assume $\eta = |\{p,y,z\}^\perp|$. Then*

$$\mu = \frac{\alpha t(t+1)(s-\alpha)(s-\alpha-\beta+1)}{st(s-\alpha+1)-\alpha(s-\alpha)(t+1)-\alpha} \tag{3.1}$$

$$\eta = \frac{\alpha(t+1)(\alpha(t-1)+(s-\alpha)(\beta t-\alpha))}{st(s-\alpha+1)-\alpha(s-\alpha)(t+1)-\alpha} \tag{3.2}$$

$$\beta = \frac{(s-\alpha)(\alpha(t+1)(\alpha-\eta)+\eta st)-\alpha^2(t^2-1)+\eta(st-\alpha)}{\alpha t(t+1)(s-\alpha)} \tag{3.3}$$

are positive integers.

Proof. The second subconstituent of the geometric graph $\Gamma(\mathcal{S})$ is a semigeometric graph $\Gamma(\mathcal{S}_p)$. Hence we can calculate the parameters of the graph $\Gamma(\mathcal{S}_p)$ in terms of s, t, α, β and μ. Solving the equality $k(k-\lambda-1)=\mu(v-k-1)$ for the strongly regular graph $\Gamma(\mathcal{S}_p)$ in terms of μ yields (see theorem 1.1)

$$(s-\alpha)(t+1)t(s-\alpha-\beta+1) = \mu\left(\frac{st(s-\alpha+1)}{\alpha} - (s-\alpha)(t+1) - 1\right).$$

and the first result follows.
Since $\mu = \alpha(t+1)-\eta$ we can solve equation (3.1) in terms of η in order to obtain (3.2). Finally solve equation (3.2) in terms of β and equation (3.3) follows. □

3.1.2 Partial quadrangles and generalized quadrangles

Point derivation of partial geometries is a generalisation of the known theory of constructing partial quadrangles from generalized quadrangles (see [38]). Let p be a point of a generalized quadrangle $\mathcal{S}$ and let $\mathcal{S}_p$ denote the derived geometry with respect to p. Since $\alpha = 1$, we immediately obtain that

$$\forall L \in \mathcal{L}, \forall x \in \mathcal{P}_p : |L \cap p^\perp \cap x^\perp| \in \{0,1\}.$$

Hence $\beta = \alpha = 1$. And so we do not need this as a condition anymore in order to obtain that the point derived geometry $\mathcal{S}_p$ is a $(0,1)$-geometry.
The fact that $\mathcal{S}_p$ is also a partial quadrangle is equivalent to requiring that the second subconstituent of the point graph $\Gamma(\mathcal{S})$ of a generalized quadrangle $\mathcal{S}$ is strongly regular (see [38]). This is true if and only if the generalized quadrangle $\mathcal{S}$ has order (s, s^2) and so $\eta = |\{x,y,z\}^\perp| = s+1$, where $\{x,y,z\}$ is a triad of non-collinear points of $\mathcal{S}$ (see [4, 18]). Note that an element of $\{x,y,z\}^\perp$ is called a *center*. And $\eta = s+1$ implies that the μ-value of the partial quadrangle equals $s(s-1)$. We can summarise the above in the following.

Theorem 3.3 (**[38]**) *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a* GQ(s,t) *such that any triad $\{x, y, z\}$ of non-collinear points has a constant number of centers. Then $(s,t) = (s, s^2)$ and the derived geometry $\mathcal{S}_p = (\mathcal{P}_p, \mathcal{L}_p, \mathrm{I}_p)$ with respect to a point p of $\mathcal{S}$ is a partial quadrangle of order $(s-1, s^2)$ with $\mu = s(s-1)$.*

Remark

When a generalized quadrangle $\mathcal{S}$ has order (s, s^2), this implies equality in the Krein condition, better known in this case as the Higman condition. And this is equivalent with the fact that the second subconstituent is strongly regular (see [20]). Note that the first subconstituent of the point graph of a generalized quadrangle always is a union of cliques.
For a proper partial geometry, it is not obvious to know in terms of its parameters when the second subconstituent of a strongly regular graph is again strongly regular. There are for example ten strongly regular graphs srg$(26, 15, 8, 9)$. Sometimes the second subconstituent is an srg$(10, 6, 3, 4)$ (the complement of the unique Petersen graph), but sometimes it is not [Haemers, private communication].
We could require a stronger condition, namely that both subconstituents are strongly regular. Then the parameters of the point graph $\Gamma(\mathcal{S})$ of a pg(s,t,α) $\mathcal{S}$ satisfy equality in the Krein condition (see [20]). This could severely restrict the possible parameter values of a partial geometry, but none of the point graphs of the known proper partial geometries satisfies this equality. Recall that the McLaughlin graph is a pseudo-geometric graph satisfying equality in the Krein conditions (see [74] for more information). It is still an open question whether this graph is indeed geometric.

3.1.3 The semi-classical generalized quadrangle of Tits

First note that many generalized quadrangles of order (s, s^2) are known and in all of them s is a prime power q.
A first example is the semi-classical generalized quadrangle of Tits $T_3(\mathcal{O})$ (see [113] for a description). It is a generalized quadrangle GQ(q, q^2) satisfying the properties of theorem 3.3. If we derive with respect to the special point (∞) then the resulting partial quadrangle has a linear representation in AG$(4,q)$: it is the partial quadrangle $T_3^*(\mathcal{O})$, with $\mathcal{O}$ an ovoid in the hyperplane Π_∞ at infinity of AG$(4,q)$. If we derive with respect to an other point then it yields other partial quadrangles that might be non-isomorphic to $T_3^*(\mathcal{O})$. We refer to [37, 38] for more information about this example and its many interesting properties and characterisations.

Remark

De Clerck and Van Maldeghem [37] proved that a $(0,1)$-geometry $\mathcal{S}_p$ which is point derived from a generalized quadrangle $\mathcal{S}$ of order (s,t) satisfies the following property: if L and M are 2 disjoint lines of $\mathcal{S}_p$ then there are either

$0, s-1$ or s lines of $\mathcal{S}_p$ concurrent to both L and M. Using this property they gave a geometric characterisation of the partial quadrangle $T_3^*(\mathcal{O})$.
If we try to generalize the above property for point derivation of partial geometries, then this property becomes very complicated since we have to consider a lot of different cases. As a consequence, this is the reason why we did not investigate when a partial geometry has a point derived dual semipartial geometry.

3.1.4 The partial geometry $\mathrm{PQ}^+(4n-1,3)$

Lemma 3.4 *For any triad $\{x_1, x_2, x_3\}$ of non-collinear points of the partial geometry $\mathrm{PQ}^+(4n-1,3)$ the set $\{x_1, x_2, x_3\}^\perp$ has constant cardinality*

$$\eta = 4 \cdot 3^{2n-2}(3^{2n-2}+1).$$

Proof. Two points are collinear in $\mathrm{PQ}^+(4n-1,3)$ if and only if they are non-orthogonal with respect to the quadric $\mathrm{Q}^+(4n-2,3)$ [70]. A line L_i^k of $\mathrm{PQ}^+(4n-1,3)$ through x_i ($k \in \{0, \ldots, 3^{2n-1}\}$) can be identified with a $(2n-1)$-dimensional space intersecting the quadric in a $(2n-2)$-dimensional space π_i^k which is a hyperplane of an element σ_k of an orthogonal spread Σ of the quadric $\mathrm{Q}^+(4n-1,3)$. And so the lines L_i^k, $k = 0, \ldots, 3^{2n-1}, i = 1,2,3,$ of $\mathrm{PQ}^+(4n-1,3)$ through x_i define a spread

$$\Sigma_i = \{\pi_i^0, \ldots, \pi_i^{3^{2n-1}}\}$$

of the quadric $\mathrm{Q}_i(4n-2,3) = x_i^* \cap \mathrm{Q}^+(4n-1,3)$.
Since the projective line $\langle x_1, x_2\rangle$, is an exterior line of $\mathrm{Q}^+(4n-1,3)$ we obtain that $x_1^* \cap x_2^* \cap \mathrm{Q}^+(4n-1,3)$ is a non-singular elliptic quadric $\mathrm{Q}_{1,2}^-(4n-3,3)$ and the lines of $\mathrm{PQ}^+(4n-1,3)$ through x_1 and x_2 define a spread

$$\Sigma_{1,2} = \{X_{1,2}^0 = \pi_1^0 \cap \pi_2^0, \ldots, X_{1,2}^{3^{2n-1}} = \pi_1^{3^{2n-1}} \cap \pi_2^{3^{2n-1}}\}$$

of the quadric $\mathrm{Q}_{1,2}^-(4n-3,3)$.
Since an exterior line contains exactly two points of $\mathrm{PQ}^+(4n-1,q)$, the plane $\langle x_1, x_2, x_3\rangle$ is a plane intersecting $\mathrm{Q}^+(4n-1,3)$ in a non-singular conic. Therefore $x_1^* \cap x_2^* \cap x_3^* \cap \mathrm{Q}^+(4n-1,3)$ is a non-singular parabolic quadric $\mathrm{Q}_{1,2,3}(4n-4,3)$ (see figure 3.1 for the seven-dimensional case).
Suppose that $\mathrm{Q}_{1,2,3}(4n-4,3)$ contains a elements of the spread $\Sigma_{1,2}$. Then counting in two different ways the ordered pairs (v, X) with $v \in \mathrm{Q}_{1,2,3}(4n-4,3)$ and $X \in \Sigma_{1,2}$ such that $v \in X$, implies

$$\frac{3^{4n-4}-1}{2} = a\left(\frac{3^{2n-2}-1}{2}\right) + (3^{2n-1}+1-a)\frac{3^{2n-3}-1}{2}.$$

¿From which follows that $a = 4$. Without loss of generality, let $X_{1,2}^0, \ldots, X_{1,2}^3$ denote the 4 elements of $\Sigma_{1,2}$ contained in $\mathrm{Q}_{1,2,3}(4n-4,3)$. Let $P_{1,j}^k$ denote

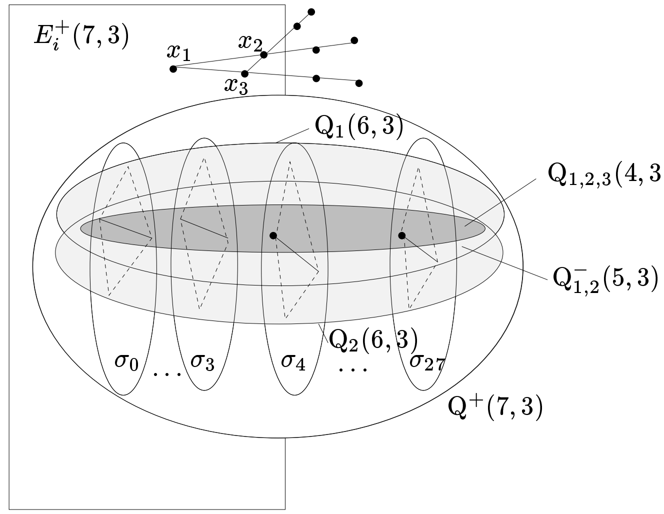

Figure 3.1: A triad $\{x_1, x_2, x_3\}$ of non-collinear points of $\mathrm{PQ}^+(7,3)$

the $(2n-2)$-dimensional affine space $x_j^* \cap L_1^k$, $j = 2,3$. Then the points of L_1^k ($k \in \{0, \dots, 3^{2n-1}\}$) collinear with x_j are the points of $L_1^k \setminus P_{1,j}^k$, $j = 2,3$. Then for $k = 4, \dots, 3^{2n-1}$ we obtain that $L_1^k \cap x_2^* \cap x_3^* = P_{1,2}^k \cap P_{1,3}^k$ is a $(2n-3)$-dimensional affine space and the points of L_1^k that are collinear with both x_2 and x_3 are the $4 \cdot 3^{2n-3}$ points of $L_1^k \setminus (P_{1,2}^k \cup P_{1,3}^k)$. For $k = 0, \dots, 3$ we obtain that $L_1^k \cap x_2^* \cap x_3^* = P_{1,2}^k = P_{1,3}^k$. Therefore the points of L_1^k (k=0,...,3) that are collinear with both x_2 and x_3 are the $2 \cdot 3^{2n-2}$ points of $L_1^k \setminus P_{1,2}^k$. Hence in total there are

$$4 \cdot 3^{2n-3}(3^{2n-1} - 3) + 8 \cdot 3^{2n-2} = 4 \cdot 3^{2n-2}(3^{2n-2} + 1)$$

points of $\mathrm{PQ}^+(4n-1,3)$ collinear with all three of x_1, x_2 and x_3. □

Remarks

1. Lemma 3.4 is not true for the partial geometry $\mathrm{PQ}^+(4n-1,2)$. In this case collinearity translates into orthogonality instead of non-orthogonality but actually this is only a minor problem. The fact that a triad of non-collinear points spans either a projective line (and so an exterior line) or a (secant) plane is crucial.

2. From lemmas 3.2 and 3.4 we obtain that $\mathrm{PQ}^+(4n-1,3)$ is good a candidate to have a point derived semipartial geometry with the following parameters:

$$\mathrm{spg}(3^{2n-2} - 1, 3^{2n-1}, 2 \cdot 3^{2n-3}, 2 \cdot 3^{2n-2}(3^{2n-2} - 1)).$$

Lemma 3.5 *Let x_1 and x_2 be two non-collinear points of* $\mathrm{PQ}^+(4n-1,3)$*, then a line of the geometry missing x_1 and x_2, intersects the set $\{x_1,x_2\}^\perp$ of points of* $\mathrm{PQ}^+(4n-1,3)$ *collinear with both x_1 and x_2, in either $2\cdot 3^{2n-2}$ or $4\cdot 3^{2n-3}$ points.*

Proof. From the proof of lemma 3.4 we see that if $x_3 \notin \{x_1,x_2\}^\perp$, then there are exactly 4 lines through x_3 intersecting the set $\{x_1,x_2\}^\perp$ in $2\cdot 3^{2n-2}$ points, while there are exactly $3^{2n-1}-3$ lines through x_3 intersecting $\{x_1,x_2\}^\perp$ in $4\cdot 3^{2n-3}$ points. □

Corollary 3.6 *The partial geometry* $\mathrm{PQ}^+(4n-1,3)$ *has a point derived semipartial geometry*

$$\mathcal{S}_p = \mathrm{spg}(3^{2n-2}-1, 3^{2n-1}, 2\cdot 3^{2n-3}, 2\cdot 3^{2n-2}(3^{2n-2}-1)).$$

Proof. The theorem is an immediate consequence of theorem 3.1. Note that by lemma 3.4 the point graph of $\mathcal{S}_p$ is strongly regular with

$$\mu = 2\cdot 3^{2n-2}(3^{2n-2}-1).$$

On the other hand we have by lemma 3.5 that for any line L of $\mathcal{S}$ and any point x of $\mathcal{S}_p$, L intersects $p^\perp \cap x^\perp$ in either $2\cdot 3^{2n-2}$ or $4\cdot 3^{2n-3}$ points. Hence in $\mathcal{S}_p$ the incidence number $\alpha(x,L)$ of any antiflag (x,L) is 0 or $2\cdot 3^{2n-3}$, and the result follows. □

Remarks

1. In section 2.3.5 we constructed a semipartial geometry having the same parameters as the point derived semipartial geometry $\mathcal{S}_p$ of corollary 3.6, namely $\mathrm{SPQ}(4n-2,3)$. Consider a point p of $\mathrm{PQ}^+(4n-1,3)$. Recall that collinearity in the partial geometry translates to being non-orthogonal with respect to $\mathrm{Q}^+(4n-1,3)$ and that $p \notin H = p^*$. Let $\mathcal{P}$ denote the point set of $\mathrm{PQ}^+(4n-1,3)$ and let $\mathcal{P}_p$ denote the point set of $\mathcal{S}_p$. Since $|p^\perp| = |\mathcal{P}\setminus H|$, we obtain that $p^\perp = \mathcal{P}\setminus H$. Therefore $\mathcal{P}_p = H\cap\mathcal{P}$. And so the point set of $\mathcal{S}_p$ is the point set of $\mathrm{SPQ}(4n-2,3)$. It is also clear that lines of $\mathcal{S}_p$ are the non-empty intersections of the lines of $\mathrm{PQ}^+(4n-1,3)$ with H. And so the semipartial geometry $\mathrm{SPQ}(4n-2,3)$ is the geometry which is point derived semipartial geometry.

2. Ivanov and Spectorov proved in [67] that every partial quadrangle which is an $\mathrm{spg}(q-1,q^2,1,q(q-1))$, is point derived and is uniquely extendible to a generalized quadrangle $\mathcal{S}$. More generally, given a point derived semipartial geometry $\mathcal{S}_p$, we want to reconstruct the partial geometry $\mathcal{S}$ in order to find new partial geometries. When $\mathcal{S}_p = \mathrm{SPQ}(6,3)$ then we can easily find back $\mathcal{S} = \mathrm{PQ}^+(7,3)$ by embedding the spread Σ of $\mathrm{Q}(6,3)$ into a spread Σ' of $\mathrm{Q}^+(7,3)$ by considering elements σ_i' of one system of

generators of $Q^+(7,3)$ such that every σ'_i contains one element σ_i of the spread Σ of $Q(6,3)$, $i = 0, \ldots, 27$. Note that there could be other partial geometries having the same parameters as $PQ^+(7,3)$, or even having the same point graph, that yield after derivation the semipartial geometry $SPQ(6,3)$.

3. The semipartial geometry SPH(3) (see section 2.3.5 for a description of this geometry) has the same point graph and so the same parameters as $SPQ(6,3)$. If the semipartial geometry SPH(3) would be point derived with respect to a partial geometry $\mathcal{S}$, then it is natural to look for a partial geometry with the same point graph as the partial geometry $PQ^+(7,3)$. In this case the lines of $\mathcal{S}$ cannot be contained in three-dimensional subspaces of $PG(7,3)$ since this would imply $\mathcal{S}$ to be isomorphic with $PQ^+(7,3)$ (see theorem 4.7), which would imply SPH(3) to be isomorphic with $SPQ(6,3)$, a contradiction. It is an open question whether SPH(3) is a point derived semipartial geometry.

4. The partial geometries which are spread derived from $PQ^+(4n-1,3)$ have the same parameters as the partial geometry $PQ^+(4n-1,3)$ (see section 4.2.2 for more information), and so they are candidates for point derivation. In theorem 4.14 we prove that the partial geometries $\mathcal{S}'_1(n)$, $\mathcal{S}'_2$, $\mathcal{S}'_3$ (and so as well $\mathcal{S}''_3$) have no point derived semipartial geometry.

3.2 Perp-systems

3.2.1 Introduction

In [102] a new construction method for semipartial geometries is introduced. An *SPG-regulus* is a set $\mathcal{R}$ of r-dimensional subspaces $\pi_1, \ldots, \pi_k$, $k > 1$, of $PG(N,q)$ satisfying the following conditions.

(SPG-R1) $\pi_i \cap \pi_j = \emptyset$ for all $i \neq j$.

(SPG-R2) If a $PG(r+1,q)$ contains π_i then it has a point in common with either 0 or $\alpha(>0)$ spaces in $\mathcal{R} \setminus \{\pi_i\}$; if this $PG(r+1,q)$ has no point in common with π_j for all $j \neq i$, then it is called a *tangent space* of $\mathcal{R}$ at π_i.

(SPG-R3) If x is a point of $PG(N,q)$ which is not contained in an element of $\mathcal{R}$, then it is contained in a constant number θ (≥ 0) of tangent $(r+1)$-dimensional spaces of $\mathcal{R}$.

Embed $PG(N,q)$ as a hyperplane Π in $PG(N+1,q)$, and define an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of points and lines as follows. Points of $\mathcal{S}$ are the points of $PG(N+1,q) \setminus \Pi$. Lines of $\mathcal{S}$ are the $(r+1)$-dimensional subspaces of $PG(N+1,q)$ which contain an element of $\mathcal{R}$, but are not contained in Π. Incidence is that of $PG(N+1,q)$. Thas [102] proved that $\mathcal{S}$ is a semipartial geometry

$$\mathrm{spg}(q^{r+1}-1, k-1, \alpha, (k-\theta)\alpha).$$

If $\theta = 0$, then $\mu = k\alpha$, and hence $\mathcal{S}$ is a partial geometry

$$\mathrm{pg}(q^{r+1} - 1, k - 1, \alpha).$$

In appendix A we give more information about SPG-reguli.

Remark

Recently, Thas [108] proved that if $N = 2r + 2$ and a set $\mathcal{R} = \{\pi_1, \ldots, \pi_k\}$ of r-dimensional spaces in $\mathrm{PG}(2r + 2, q)$ satisfies (SPG-R1) and (SPG-R2) then

$$\alpha(k(q^{r+2}-1)-(q^{2r+3}-1)) \leq k^2(q^{r+1}-1)-k(q^{2r+2}+q^{r+1}-2)+q^{2r+3}-1. \quad (3.4)$$

If equality holds then $\mathcal{R}$ is an SPG-regulus, and conversely.

Let ρ be a polarity in $\mathrm{PG}(N, q)$ $(N \geq 2)$. Let r $(r \geq 2)$ be the rank of the related polar space P. A *partial m-system* M of P, with $0 \leq m \leq r - 1$, is a set $\{\pi_1, \ldots, \pi_k\}$ $(k > 1)$ of totally singular m-dimensional spaces of P such that no maximal totally singular space containing π_i has a point in common with an element of $M \setminus \{\pi_i\}, i = 1, 2, \ldots, k$. If the set M is maximal then M is called an *m-system*. For m=0, the m-system is an ovoid of P. For $m = r - 1$, with r the rank of P, the m-system is a spread of P. The size of an m-system M for each of the classical finite polar spaces P is the same as the size of a spread or the size of an ovoid of the polar space P (see theorem 1.7 for the size of a spread or an ovoid of P). Actually the fact that $|M|$ is independent of m gives us the explanation why an ovoid and a spread of a finite classical polar space P have the same size. For more information we refer to [91, 93].

Remarks

1. Hamilton and Quinn [58] showed that maximal arcs in symplectic translation planes may be obtained from certain m-systems of finite symplectic polar spaces. Many new examples of maximal arcs are then constructed and new examples of m-systems are also constructed in $\mathrm{Q}^-(2n+1, q)$ and $W_{2n+1}(q)$.

2. Recently Luyckx [78] proved that any m-system of the classical polar spaces $W_{2n+1}(q), \mathrm{Q}^-(2n + 1, q)$ and $\mathrm{H}(2n, q^2)$ is an SPG-regulus, and so it gives rise to a semipartial geometry. For spreads of $\mathrm{Q}^-(2n + 1, q)$ or $\mathrm{H}(2n, q^2)$, this was already observed by Thas in [102]. Unfortunately, most of the known m-systems of these polar spaces do not yield new semipartial geometries. For example the new m-systems of the polar spaces $\mathrm{Q}^-(2n + 1, q)$ and $W_{2n+1}(q)$ of Hamilton and Quinn yield semipartial geometries that are isomorphic to some know ones.

   Hamilton and Mathon [57] found by computer new m-systems of the polar space $W_{2n+1}(q)$, $n \leq 4$. Some of them yield indeed new semipartial geometries, but their parameters are not new. In the same article Hamilton and Mathon classified the m-systems of the finite polar spaces

$W_{2n+1}(2), \mathrm{Q}^-(2n+1,2), \mathrm{Q}^+(2n+1,2)$ and $\mathrm{Q}(2n+2,2)$ for $n = 1,2,3$ and 4. Moreover they obtain improvements of the non-existence results on m-systems of $W_{2n+1}(q), \mathrm{Q}^-(2n+1,q)$ and $\mathrm{H}(2n,q^2)$.

As remarked in [78], Offer discovered a new class of spreads of the generalized hexagon $\mathrm{H}(2^{2h})$, which yields a new class of 1-systems of the parabolic quadric $\mathrm{Q}(6,2^{2h})$. By projection from the nucleus of $\mathrm{Q}(6,2^{2h})$ onto a 5-dimensional subspace not containing the nucleus, a new class of 1-systems of $W_5(2^{2h})$ is obtained. Hence a new class of semipartial geometries is obtained, but again their parameters are not new.

3.2.2 Perp-systems and partial geometries

We introduce an object which has very strong connections with m-systems and SPG-reguli, not only because of the geometrical construction but also because of other similarities of their properties such as bound (3.4) for SPG-reguli.
Again, let ρ be a polarity of $\mathrm{PG}(N,q)$. Define a *partial perp-system* $\mathcal{R}(r)$ to be any set $\{\pi_1,\ldots,\pi_k\}$ of k $(k>1)$ mutual disjoint r-dimensional subspaces of $\mathrm{PG}(N,q)$ such that no π_i^ρ meets an element of $\mathcal{R}(r)$. Hence each π_i is non-singular with respect to ρ. Note that the definition of a partial perp-system implies that $N \geq 2r+1$.

Theorem 3.7 *Let $\mathcal{R}(r)$ be a partial perp-system of $\mathrm{PG}(N,q)$ equipped with a polarity ρ. Then*

$$|\mathcal{R}(r)| \leq \frac{q^{\frac{N-2r-1}{2}}(q^{\frac{N+1}{2}}+1)}{q^{\frac{N-2r-1}{2}}+1}. \tag{3.5}$$

Proof. Consider a partial perp-system $\mathcal{R}(r) = \{\pi_1,\ldots,\pi_k\}$ (with $k>1$) of r-dimensional subspaces π_i of $\mathrm{PG}(N,q)$.
We count in two different ways the number of ordered pairs (p_i,π^ρ), $\pi \in \mathcal{R}(r)$ and p_i a point of π^ρ. If t_i is the number of $(N-r-1)$-dimensional spaces π^ρ $(\pi \in \mathcal{R}(r))$ containing p_i then

$$\sum t_i = |\mathcal{R}(r)| \cdot \frac{q^{N-r}-1}{q-1}.$$

Next we count in two different ways the number of ordered triples (p_i,π^ρ,π'^ρ), with $\pi,\pi' \in \mathcal{R}(r)$ $(\pi \neq \pi')$ and p_i a point of $\pi^\rho \cap \pi'^\rho$. Then we obtain

$$\sum t_i(t_i-1) = |\mathcal{R}(r)|(|\mathcal{R}(r)|-1)\frac{q^{N-2r-1}-1}{q-1}.$$

The number of points p_i equals

$$|I| = \frac{q^{N+1}-1}{q-1} - |\mathcal{R}(r)| \cdot \frac{q^{r+1}-1}{q-1}.$$

Then the inequality $|I|\sum t_i^2 - (\sum t_i)^2 \geq 0$ yields after some calculation the bound in the statement of the theorem. □

Corollaries

1. If equality holds in (3.5) then $\mathcal{R}(r)$ is called a *perp-system* . This is equivalent to the fact that every point p_i of $\mathrm{PG}(N,q)$ not contained in an element of $\mathcal{R}(r)$ is incident with a constant $\bar{t}$ of $(N-r-1)$-dimensional spaces π^ρ with
$$\bar{t} = \frac{\sum t_i}{|I|} = q^{\frac{N-2r-1}{2}}.$$

2. Assume that $N = 2r+1$, then a perp-system contains $\frac{q^{r+1}+1}{2}$ elements. In this case q has to be odd and every point not contained in an element of the perp-system is incident with exactly one space π^ρ $(\pi \in \mathcal{R}(r))$, which is an r-dimensional space.

3. Assume $N > 2r+1$, then the right hand side of (3.5) is an integer if and only if $\frac{N+1}{N-2r-1}$ is an odd integer, say $2l+1$. This is equivalent to
$$N = 2r+1+\frac{r+1}{l}.$$
Hence l divides $r+1$ and
$$2r+1 \leq N \leq 3r+2. \tag{3.6}$$

4. If N is even then equality in (3.5) implies that q is a square.

5. Assume that $N = 3r+2$, then a perp-system contains
$$q^{\frac{r+1}{2}}(q^{r+1} - q^{\frac{r+1}{2}} + 1)$$
elements. Hence if r is even then q has to be a square.

Theorem 3.8 *Let $\mathcal{R}(r)$ be a perp-system of $\mathrm{PG}(N,q)$ equipped with a polarity ρ and let $\overline{\mathcal{R}(r)}$ denote the union of the point sets of the elements of $\mathcal{R}(r)$. Then $\overline{\mathcal{R}(r)}$ has two intersection sizes with respect to hyperplanes.*

Proof. Suppose that p is a point of $\mathrm{PG}(N,q)$ which is not contained in an element of $\mathcal{R}(r)$. Then p is incident with $q^{\frac{N-2r-1}{2}}$ $(N-r-1)$-dimensional spaces π^ρ $(\pi \in \mathcal{R}(r))$. Therefore the hyperplane p^ρ contains
$$\begin{aligned} h_1 &= \frac{q^{r+1}-1}{q-1}q^{\frac{N-2r-1}{2}} + \frac{q^r-1}{q-1}\left(|\mathcal{R}(r)| - q^{\frac{N-2r-1}{2}}\right) \\ &= q^{\frac{N-1}{2}} + \frac{q^r-1}{q-1}\cdot|\mathcal{R}(r)| \end{aligned}$$
points of $\overline{\mathcal{R}(r)}$.

Suppose that p is contained in an element of $\mathcal{R}(r)$. Since all elements of $\mathcal{R}(r)$ are non-singular, we obtain that p^ρ contains

$$\begin{aligned} h_2 &= \frac{q^r-1}{q-1} \cdot |\mathcal{R}(r)| \\ &= h_1 - q^{\frac{N-1}{2}} \end{aligned}$$

points of $\overline{\mathcal{R}(r)}$. □

Remarks

1. Theorem 3.8 implies that $\overline{\mathcal{R}(r)}$ yields a projective two-weight code and a strongly regular graph $\Gamma^*(\overline{\mathcal{R}(r)})$ (see section 1.3). Some calculations yield that this graph is a pseudo-geometric

$$\left(q^{r+1}-1, \frac{q^{\frac{N-2r-1}{2}}(q^{\frac{N+1}{2}}+1)}{q^{\frac{N-2r-1}{2}}+1}-1, \frac{q^{r+1}-1}{q^{\frac{N-2r-1}{2}}+1}\right)\text{-graph.}$$

2. Recall that the existence of the perp-system $\mathcal{R}(r)$ implies that $\frac{N+1}{N-2r-1}$ is odd, say $2l+1$, which implies that $\frac{2(r+1)}{N-2r-1} = 2l$, hence is even; so

$$\frac{q^{r+1}-1}{q^{\frac{N-2r-1}{2}}+1}$$

is a positive integer.

Theorem 3.9 *Let $\mathcal{R}(r)$ be a perp-system of* $\mathrm{PG}(N,q)$ *equipped with a polarity ρ, and let $\overline{\mathcal{R}(r)}$ denote the union of the point sets of the elements of $\mathcal{R}(r)$. Then the graph $\Gamma^*(\overline{\mathcal{R}(r)})$ is geometric.*

Proof. The vertices of $\Gamma^*(\overline{\mathcal{R}(r)})$ are the points of $\mathrm{PG}(N+1,q) \setminus \mathrm{PG}(N,q)$. The incidence structure $\mathcal{S}$ with this set of points as point set and with lines the $(r+1)$-dimensional subspaces of $\mathrm{PG}(N+1,q)$ which contain an element of $\mathcal{R}(r)$ but that are not contained in $\mathrm{PG}(N,q)$, is a partial linear space with point graph $\Gamma^*(\overline{\mathcal{R}(r)})$. Since the point graph $\Gamma^*(\overline{\mathcal{R}(r)})$ of $\mathcal{S}$ is pseudo-geometric the last part of theorem 1.9 implies that $\mathcal{S}$ is a partial geometry. □

Remark

The following theorem proves that $\mathcal{R}(r)$ is an SPG-regulus with $\theta = 0$, and by [102] this implies the result of theorem 3.9 as well.

Theorem 3.10 *Let $\mathcal{R}(r)$ be a perp-system of* $\mathrm{PG}(N,q)$ *equipped with a polarity ρ, then $\mathcal{R}(r)$ is an SPG-regulus with $\theta = 0$.*

Proof. By definition of a perp-system $\mathcal{R}(r)$ of $\mathrm{PG}(N,q)$, the first condition (SPG-R1) is automatically satisfied.
Consider an $(r+1)$-dimensional subspace Ω of $\mathrm{PG}(N,q)$, containing a fixed element π_i of $\mathcal{R}(r)$. Suppose that $|\Omega \cap \overline{\mathcal{R}(r)} \setminus \pi_i| = \alpha_\Omega$. Note that α_Ω can be zero. We now count ordered pairs (x, H) such that x is a point of $\overline{\mathcal{R}(r)} \setminus \Omega$ and H is a hyperplane of $\mathrm{PG}(N,q)$ containing both x and Ω. From the proof of theorem 3.8 we can see that only hyperplanes H with $H^\rho \notin \overline{\mathcal{R}(r)}$, can contain an element of $\mathcal{R}(r)$ and so only such a hyperplane H can contain Ω. Moreover, in this case $|H \cap \overline{\mathcal{R}(r)}| = h_1$ (see theorem 3.8). Hence counting the ordered pairs (x, H) in two ways implies

$$(|\overline{\mathcal{R}(r)}| - |\pi_i| - \alpha_\Omega)\frac{q^{N-r-2}-1}{q-1} = \frac{q^{N-r-1}-1}{q-1}(h_1 - |\pi_i| - \alpha_\Omega).$$

Substituting $h_1 = q^{\frac{N-1}{2}} + \frac{q^r-1}{q-1} \cdot |\mathcal{R}(r)|$ and $|\overline{\mathcal{R}(r)}| = \frac{q^{r+1}-1}{q-1} \cdot |\mathcal{R}(r)|$ yields after some calculations

$$\frac{q^{N-r-2}-q^r}{q-1} \cdot |\mathcal{R}(r)| - q^{\frac{N-1}{2}} \cdot \frac{q^{N-r-1}-1}{q-1} + q^{N-r-2}\left(\frac{q^{r+1}-1}{q-1} + \alpha_\Omega\right) = 0.$$

After substitution of $|\mathcal{R}(r)| = \frac{q^{\frac{N-2r-1}{2}}(q^{\frac{N+1}{2}}+1)}{q^{\frac{N-2r-1}{2}}+1}$, one can check that indeed

$$\alpha_\Omega = \frac{q^{r+1}-1}{q^{\frac{N-2r-1}{2}}+1}.$$

This implies that a $\mathrm{PG}(r+1,q)$ containing an element π_i of $\mathcal{R}(r)$ has a point in common with $\alpha = \alpha_\Omega$ spaces in $\mathcal{R}(r) \setminus \{\pi_i\}$, and so there are no tangent $(r+1)$-dimensional spaces. Therefore $\mathcal{R}(r)$ satisfies condition (SPG-R2) and condition (SPG-R3) with $\theta = 0$. □

3.2.3 Perp-systems and intersections with generators

In section 3.2.5 we will come back to these partial geometries for the extremal cases of N. However we will first discuss a few properties that are similar to those for m-systems.
Assume that the polarity ρ is a non-singular symplectic polarity in $\mathrm{PG}(N,q)$, hence N is odd. Let $|\Sigma(W_N(q))|$ denote the number of generators of the symplectic polar space $W_N(q)$. Then, as for m-systems, we can calculate the intersection of a perp-system with a generator of $W_N(q)$.

Theorem 3.11 *Let $\mathcal{R}(r)$ be a perp-system of the finite classical polar space $W_N(q)$ and let $\overline{\mathcal{R}(r)}$ denote the union of the point sets of the elements of $\mathcal{R}(r)$. Then for any maximal totally isotropic subspace G of $W_N(q)$*

$$|G \cap \overline{\mathcal{R}(r)}| = \frac{q^{\frac{N-2r-1}{2}}(q^{r+1}-1)}{(q^{\frac{N-2r-1}{2}}+1)(q-1)}.$$

Proof. It is well known that $|\Sigma(W_N(q))| = (q^{\frac{N+1}{2}}+1)\cdot|\Sigma(W_{N-2}(q))|$ (see for instance [64]).
We count in two ways the number of ordered pairs (p, G_i) with $p \in \overline{\mathcal{R}(r)}$ and G_i a maximal totally isotropic subspace of the polar space $W_N(q)$ such that $p \in G_i$. If $t_i = |G_i \cap \overline{\mathcal{R}(r)}|$ then

$$\begin{aligned}\sum t_i &= |\mathcal{R}(r)|\cdot\frac{q^{r+1}-1}{q-1}\cdot|\Sigma(W_{N-2}(q))| \\ &= \frac{q^{\frac{N-2r-1}{2}}(q^{\frac{N+1}{2}}+1)(q^{r+1}-1)}{(q^{\frac{N-2r-1}{2}}+1)(q-1)}(q^{\frac{N-1}{2}}+1)\cdot|\Sigma(W_{N-4}(q))|.\end{aligned}$$

Next we count in two ways the number of ordered triples (p, p', G_i), with p and p' different points of $\overline{\mathcal{R}(r)}$ contained in the maximal totally isotropic subspace G_i. Then we obtain

$$\begin{aligned}\sum t_i(t_i-1) &= |\mathcal{R}(r)|\frac{q^{r+1}-1}{q-1}\left(\frac{q^r-q}{q-1}+(|\mathcal{R}(r)|-1)\frac{q^r-1}{q-1}\right)|\Sigma(W_{N-4}(q))| \\ &= |\mathcal{R}(r)|\frac{q^{r+1}-1}{q-1}\left(|\mathcal{R}(r)|\frac{q^r-1}{q-1}-1\right)|\Sigma(W_{N-4}(q))|.\end{aligned}$$

And so

$$\begin{aligned}\sum t_i^2 &= \frac{q^{\frac{N-2r-1}{2}}(q^{\frac{N+1}{2}}+1)(q^{r+1}-1)}{(q^{\frac{N-2r-1}{2}}+1)(q-1)}\cdot \\ &\quad\left(q^{\frac{N-1}{2}}+q\frac{q^{\frac{N-2r-1}{2}}(q^{\frac{N+1}{2}}+1)(q^r-1)}{(q^{\frac{N-2r-1}{2}}+1)(q-1)}\right)\cdot|\Sigma(W_{N-4}(q))|.\end{aligned}$$

Finally we obtain for the cardinality of the index set

$$|I| = |\Sigma(W_N(q))| = (q^{\frac{N+1}{2}}+1)(q^{\frac{N-1}{2}}+1)\cdot|\Sigma(W_{N-4}(q))|.$$

Some calculations now show that $|I|\sum t_i^2 - (\sum t_i)^2 = 0$. Therefore

$$t_i = \bar{t} = \frac{\sum t_i}{|I|} = \frac{q^{\frac{N-2r-1}{2}}(q^{r+1}-1)}{(q^{\frac{N-2r-1}{2}}+1)(q-1)}.$$

□

Remarks

1. Theorem 3.11 is not valid for the other finite classical polar spaces $P = \mathrm{Q}^+(2m+1,q), \mathrm{Q}^-(2m+1,q), \mathrm{Q}(2m,q)$, or $\mathrm{H}(n,q^2)$, since a generator of P is always disjoint from an element of a perp-system of P.

2. Let P be a finite classical polar space of rank $n \geq 2$. In [103] Thas introduced the concept of a *k-ovoid* of P, that is a point set $\mathcal{K}$ of P such

that each generator of P contains exactly k points of $\mathcal{K}$. Note that a k-ovoid with $k = 1$ is an ovoid. By theorem 3.11 a perp-system $\mathcal{R}(r)$ of $W_N(q)$ yields a k-ovoid with

$$k = \frac{q^{\frac{N-2r-1}{2}}(q^{r+1}-1)}{(q^{\frac{N-2r-1}{2}}+1)(q-1)}.$$

In section 3.2.5 we will give an example of a perp-system $\mathcal{R}(1)$ in $W_5(3)$ yielding a new 3-ovoid.

3.2.4 Perp-systems arising from a given one

The next lemma is commonly known but we give a proof for the sake of completeness.

Lemma 3.12 *Let B be a non-degenerate reflexive sesquilinear form on the vector space $V(N+1, q^n)$ of dimension $N+1$ over the field $\mathrm{GF}(q^n)$, and let T be the trace map from $\mathrm{GF}(q^n)$ to $\mathrm{GF}(q)$. Then the map $B' = T \circ B$ is a non-degenerate reflexive sesquilinear form on the vector space $V((N+1)n, q)$.*

Proof. The fact that B' is sesquilinear on $V((N+1)n, q)$ follows immediately from B being sesquilinear and T being additive.
Assume that x is some non-zero element of the vector space $V(N+1, q^n)$. Then the map $y \mapsto B(x, y)$ maps the vector space onto $\mathrm{GF}(q^n)$. Since there exist elements of $\mathrm{GF}(q^n)$ that have non-zero trace, there must be some y such that $T \circ B(x, y) \neq 0$. Hence B' is non-degenerate.
It remains to be shown that B' is reflexive, that is $B'(x, y) = 0$ implies that $B'(y, x) = 0$. By the classification of the non-degenerate reflexive sesquilinear forms, we obtain that B is either symmetric ($B(x, y) = B(y, x)$), anti-symmetric ($B(x, y) = -B(y, x)$) or Hermitian ($B(x, y) = B(y, x)^\sigma$ for some $\sigma \in \mathrm{Aut}(\mathrm{GF}(q))$. In the first and second case it is obvious that $B' = T \circ B$ is reflexive. When B is Hermitean, then

$$B'(x, y) = T \circ B(x, y) = T \circ (B(y, x)^\sigma) = (T \circ B(y, x))^\sigma = B'(y, x)^\sigma,$$

and so is reflexive. □

Remark

It is well known that a non-degenerate reflexive sesquilinear form on a vector space $V(N+1, q^n)$ gives rise to a polarity of the related projective space $\mathrm{PG}(N, q^n)$ and conversely. Note however that a polarity of the projective space $\mathrm{PG}((N+1)n - 1, q)$ obtained from a polarity of $\mathrm{PG}(N, q^n)$ by composition with the trace map is not necessarily of the same type as the original polarity. For instance a Hermitean polarity may under certain conditions give rise to an orthogonal polarity. See [92, Section 9] for examples.

Theorem 3.13 *Let $\mathcal{R}(r)$ be a perp-system with respect to some polarity of* $\mathrm{PG}(N, q^n)$ *then there exists a perp-system $\mathcal{R}'((r+1)n-1)$ with respect to some polarity of* $\mathrm{PG}((N+1)n-1, q)$.

Proof. Let ρ be a polarity of $\mathrm{PG}(N, q^n)$ such that $\mathcal{R}(r)$ is a perp-system with respect to ρ. Let B be the non-degenerate reflexive sesquilinear form on $V(N+1, q^n)$ associated with ρ. Then $B' = T \circ B$ induces as in the previous lemma, a polarity of $\mathrm{PG}((N+1)n-1, q)$. The elements of $\mathcal{R}(r)$ can be considered as $((r+1)n-1)$-dimensional subspaces of $\mathrm{PG}((N+1)n-1, q)$. Denote this set of subspaces by $\mathcal{R}'((r+1)n-1)$. We show that $\mathcal{R}'((r+1)n-1)$ is a perp-system of $\mathrm{PG}((N+1)n-1, q)$ with respect to the polarity ρ' induced by B'.
First note that the size of $\mathcal{R}'((r+1)n-1)$ is the correct size to be a perp-system of $\mathrm{PG}((N+1)n-1, q)$. Then consider an element M of $\mathcal{R}(r)$ and let M' be the corresponding element of $\mathcal{R}'((r+1)n-1)$. The tangent space $M^{\perp_B}$ of M is defined to be

$$M^{\perp_B} = \{x \in \mathrm{PG}(N, q^n) \| B(x, y) = 0 \text{ for all } y \in M\}.$$

It has projective dimension $N - r - 1$ over $\mathrm{GF}(q^n)$, and considered as a subspace of $\mathrm{PG}((N+1)n-1, q)$ it has projective dimension $(N-r)n-1$. Also

$$M'^{\perp_{B'}} = \{x \in \mathrm{PG}((N+1)n-1, q) \| B'(x, y) = 0 \text{ for all } y \in M'\}$$

has projective dimension $(N-r)n-1$ over $\mathrm{GF}(q)$. Now if x is such that $B(x, y) = 0$ then $B'(x, y) = 0$, so it follows that the tangent space of M' with respect to B' is exactly the tangent space of M with respect to B considered as a subspace of $\mathrm{PG}((N+1)n-1, q)$. Hence since $M^{\perp_B}$ is disjoint from the set of points of elements of $\mathcal{R}(r)$, also $M'^{\perp_{B'}}$ is disjoint from the set of points of elements of $\mathcal{R}'((r+1)n-1)$. $\square$

Remark

It is possible to calculate the type of the polar space obtained by taking the trace of a reflexive sesquilinear form (cf. [92]). But in some sense perp-systems do not care about the type of the underlying polar space since the size of a perp-system is only dependent on the dimension of the projective space it is embedded in. Actually the perp-system $\mathcal{R}(1)$ in $\mathrm{PG}(5, 3)$ that we describe in the next session is related to a symplectic polarity as well as to an elliptic one.

3.2.5 Examples

We recall, see inequality (3.6), that if $\mathcal{R}(r)$ is perp-system in $\mathrm{PG}(N, q)$ then $2r + 1 \leq N \leq 3r + 2$. We do not know examples for N not equal to one of the bounds.

Perp-systems in $\mathrm{PG}(2r+1,q)$

Assume that $N = 2r+1$, then a perp-system $\mathcal{R}(r)$ in $\mathrm{PG}(2r+1,q)$ yields a

$$\mathrm{pg}\left(q^{r+1}-1, \frac{q^{r+1}-1}{2}, \frac{q^{r+1}-1}{2}\right),$$

which is a Bruck net of order q^{r+1} and degree $\frac{q^{r+1}+1}{2}$.
Note that q is odd and that a Bruck net of order q^{r+1} and degree $\frac{q^{r+1}+1}{2}$ coming from a perp system $\mathcal{R}(r)$ in $\mathrm{PG}(2r+1,q)$ is in fact a net that is embeddable in an affine plane of order q^{r+1}. Actually, assume that Φ is an r-spread of $\mathrm{PG}(2r+1,q)$, then $|\Phi| = q^{r+1}+1$ and taking half of the elements of Φ yields a net with requested parameters. However this does not immediately imply that there exist a polarity ρ such that these $\frac{q^{r+1}+1}{2}$ elements form a perp system with respect to ρ. However examples do exist. Take in $\mathrm{AG}(2,q^{r+1})$ only those lines with slope a square. Let ν be any non-square in $\mathrm{GF}(q^{r+1})$, then the mapping $x \mapsto \nu x$ $(x \neq 0)$ extended with $0 \mapsto \infty \mapsto 0$ is an involution and hence a polarity on the line at infinity $\mathrm{PG}(1,q^{r+1})$ of $\mathrm{AG}(2,q^{r+1})$. Using theorem 3.13 this yields a perp-system $\mathcal{R}'(r)$ in $\mathrm{PG}(2r+1,q)$.

Perp-systems in $\mathrm{PG}(3r+2,q)$

We will now describe perp-systems $\mathcal{R}(r)$ in $\mathrm{PG}(3r+2,q)$. Note that the partial geometry related to such a perp-system is a

$$\mathrm{pg}(q^{r+1}-1, (q^{r+1}+1)(q^{\frac{r+1}{2}}-1), q^{\frac{r+1}{2}}-1).$$

Such a partial geometry has the parameters of a partial geometry of type $T_2^*(\mathcal{K})$ with $\mathcal{K}$ a maximal arc of degree $q^{\frac{r+1}{2}}$ in a projective plane $\mathrm{PG}(2,q^{r+1})$ (see appendix A for a description of the geometry $T_2^*(\mathcal{K})$). As we will see in the sequel there do exist partial geometries related to perp-systems and isomorphic to a $T_2^*(\mathcal{K})$ while there exist partial geometries coming from perp-systems $\mathcal{R}(r)$ in $\mathrm{PG}(3r+2,q)$ that are not isomorphic to a $T_2^*(\mathcal{K})$.

Example 1. A perp-system $\mathcal{R}(0)$ in $\mathrm{PG}(2,q^2)$ equipped with a polarity ρ is equivalent to a self-polar maximal arc $\mathcal{K}$ of degree q in the projective plane $\mathrm{PG}(2,q^2)$; that is for every point $p \in \mathcal{K}$, the line p^ρ is an exterior line with respect to the set $\mathcal{K}$. The corresponding partial geometry $T_2^*(\mathcal{K})$ is a $\mathrm{pg}(q^2-1,(q^2+1)(q-1),q-1)$. Note that a necessary condition for the existence of a maximal arc of degree d in $\mathrm{PG}(2,q)$ is $d|q$; this condition is sufficient for q even [47, 98], while non-trivial maximal arcs (i.e. $d<q$) do not exist for q odd [1].

Self-polar maximal arcs do exist as is proven in the next lemma.

Lemma 3.14 *In $\mathrm{PG}(2,q^2)$ there exists a self-polar maximal arc of degree q for all even q.*

Proof. We show that certain maximal arcs constructed by Denniston admit a polarity. In the sequel the Desarguesian plane $\mathrm{PG}(2,2^e)$ is represented via homogeneous coordinates over the Galois field $\mathrm{GF}(2^e)$. Let $\xi^2+\alpha\xi+1$ be an irreducible polynomial over $\mathrm{GF}(2^e)$, and let $\mathcal{F}$ be the set of conics given by the pencil

$$F_\lambda : x^2+\alpha xy+y^2+\lambda z^2=0, \quad \lambda \in \mathrm{GF}(2^e)\cup\{\infty\}.$$

Then F_0 is the point $(0,0,1)$, F_∞ is the line $z^2=0$ (which we shall call the *line at infinity*). Every other conic in the pencil is non-degenerate and has nucleus F_0. Further, the pencil is a partition of the points of the plane. For convenience, this pencil of conics will be referred to as the *standard pencil*.

Denniston [47] showed that if A is an additive subgroup of $\mathrm{GF}(2^e)$ of order n, then the set of points of all F_λ for $\lambda\in A$ form a $\{2^e(n-1)+n;n\}$–arc $\mathcal{K}$, that is a maximal arc of degree n in $\mathrm{PG}(2,2^e)$.

In [56, Theorem 2.2.4], Hamilton showed that if $\mathcal{F}$ is the standard pencil of conics, A an additive subgroup of $\mathrm{GF}(2^e)$, and $\mathcal{K}$ the Denniston maximal arc in $\mathrm{PG}(2,2^e)$ determined by A and $\mathcal{F}$, then the dual maximal arc $\mathcal{K}'$ of $\mathcal{K}$ has points determined by the standard pencil and additive subgroup

$$A'=\{\alpha^2 s \| s\in \mathrm{GF}(2^e)^* \text{ and } T(\lambda s)=0 \quad \forall\lambda\in A\}\cup\{0\},$$

where T denotes the trace map from $\mathrm{GF}(2^e)$ to $\mathrm{GF}(2)$.

In the case when e is even and $\mathrm{GF}(q^2)=\mathrm{GF}(2^e)$, it follows that if A is the additive group of $\mathrm{GF}(q)$ then $A'=\alpha^2 A$. Simple calculations then show that the homology matrix

$$H=\begin{pmatrix}\alpha & 0 & 0\\ 0 & \alpha & 0\\ 0 & 0 & 1\end{pmatrix}$$

is a collineation that maps the Denniston maximal arc determined by A to that determined by A'. Furthermore, $HH^{-t}=H^{-t}H=I$, where H^{-t} is the inverse transpose of H. It follows that the function, mapping the point (x,y,z) to the line with coordinate $(x,y,z)H$, is a polarity that maps the Denniston maximal arc of degree q determined by the additive group A of $\mathrm{GF}(q)$ to its set of external lines. □

By expanding over a subfield we can obtain an SPG-regulus (with $\theta=0$) from a maximal arc $\mathcal{K}$, but the corresponding partial geometry is isomorphic to $T_2^*(\mathcal{K})$.

A self-polar maximal arc of degree q^n in $\mathrm{PG}(2,q^{2n})$ is a perp-system $\mathcal{R}(0)$. Applying theorem 3.13 gives a perp-system with $r=n-1$ in $\mathrm{PG}(3n-1,q^2)$ and a perp-system with $r=2n-1$ in $\mathrm{PG}(6n-1,q)$.

Example 2. A perp-system $\mathcal{R}(1)$ in $\mathrm{PG}(5,q)$ equipped with a polarity ρ will yield a

$$\mathrm{pg}(q^2-1,(q^2+1)(q-1),q-1).$$

Mathon found by computer search such a system M in $\mathrm{PG}(5,3)$ yielding a $\mathrm{pg}(8,20,2)$.

Following Mathon we represent the set M as follows. A point of $\mathrm{PG}(5,3)$ is given as a triple abc where a, b and c are in the range 0 to 8. Taking the base 3 representation of each digit then gives a vector of length 6 over $\mathrm{GF}(3)$. Each of the following columns of 4 points corresponds to a line of the set in $\mathrm{PG}(5,3)$.

300	330	630	310	610	440	540	470	570	713	813	343	843
100	103	203	201	101	707	137	134	404	831	531	741	351
700	763	563	821	421	387	827	684	254	157	657	407	717
400	433	833	511	711	247	377	514	674	344	144	184	264

373	773	723	823	353	453	383	583
451	641	381	671	881	761	571	461
177	267	867	187	537	347	227	217
704	424	174	564	214	224	834	654

This set M is the unique perp-system with respect to a symplectic polarity in $\mathrm{PG}(5,3)$ but also with respect to an elliptic orthogonal polarity. The set has many interesting properties.

(*i*) If we consider the set M as the unique perp-system with respect to the symplectic polarity $W_5(3)$ in $\mathrm{PG}(5,3)$, then theorem 3.9 yields that any generator of $W_5(3)$ (being a plane) intersects the point set of M in 3 points.

(*ii*) The stabilizer of M in PG(5,3) has order 120 and has two orbits on M containing 6 and 15 lines, respectively. Hence the group of the $\mathrm{pg}(8,20,2)$ has order 120*729, acts transitively on the points and has two orbits on the lines. Since each line of M generates a spread of lines in $\mathrm{pg}(8,20,2)$ it contains a parallelism. The group is isomorphic to the sharply 3-transitive group on 6 points generated by

$$(12453),(16)(23),(1345).$$

(*iii*) There are 7 solids S_i in PG(5,3) which contain 3 lines of M each. The S_i meet in a common line L (disjoint from the lines of M).

(*iv*) Every point of $\mathrm{PG}(5,3)\setminus M$ is incident with a unique line with 3 points in M. These 280 lines meet each of the 21 lines of M 40 times and each pair of lines 4 times, hence forming a 2-(21,3,4) design.

(v) The set M contains exactly 21 lines of $\mathrm{PG}(5,3)$, these lines form a partial spread. $\mathrm{PG}(5,3) \setminus M$ contains exactly 21 solids of $\mathrm{PG}(5,3)$, these solids intersect mutually in a line, and there are exactly 3 solids through any point of $\mathrm{PG}(5,3) \setminus M$. An exhaustive computer search established that any set of 21 solids in $\mathrm{PG}(5,3)$ satisfying the above properties is isomorphic to the complement of our set M.

Remarks

1. When we look for a perp-system $\mathcal{R}(1)$ in the polar space $W_5(3)$, then with the aid of our geometrical arguments, the computer search only uses a few seconds of computer time. Unfortunately so far we were not able to give a computer free construction of this perp-system M.
2. The related partial geometry $\mathrm{pg}(8,20,2)$ has the same parameters as one of type $T_2^*(\mathcal{K})$, with $\mathcal{K}$ a maximal arc of degree 3 in $\mathrm{PG}(2,9)$ which can not exist by [1], but that was already proved for this small case by Cossu [23].
3. The graph $\Gamma_5^*(\overline{M})$ which is a $\mathrm{srg}(729,168,27,42)$ seems to be new although at least one graph with the same parameters has been known before, namely the strongly regular graph corresponding to the two-character set of Gulliver [51]. The construction of Gulliver is not completely computer free neither. He uses the so called *heuristic optimization with a local search* and the so called *greedy algorithm* in order to speed up his search until it only uses a few seconds of computer time. Note that by computer Mathon [private communication] found in total 10 non-isomorphic $\mathrm{srg}(729,168,27,42)$ using decompositions or fusions of cyclotomic schemes and other group schemes.

Chapter 4

Spread derived partial geometries

4.1 The partial geometry $\mathrm{PQ}^+(4n-1,q), q=2,3$

Recall the definition of the partial geometry $\mathrm{PQ}^+(4n-1,q)$ which is a

$$\mathrm{pg}(q^{2n-1}-1, q^{2n-1}, (q-1)q^{2n-2}), q=2,3.$$

Given an orthogonal spread Σ of the quadric $\mathrm{Q}^+(4n-1,q)$, $q=2,3$. The point set $\mathcal{P}$ of the geometry is the set of all points of $\mathrm{PG}(4n-1,q), q=2,3$, where the quadratic form takes value 1 (or equivalently where the quadratic form takes value -1). This implies for $q=2$ that $\mathcal{P}$ is the set of points off the quadric while for $q=3$ we obtain half of this set, namely $E_1^+(4n-1,3)$ or $E_2^+(4n-1,3)$. The line set $\mathcal{L}$ of $\mathrm{PQ}^+(4n-1,q), q=2,3$, is the set of all hyperplanes of the elements of Σ. Finally $x \mathrm{\ I\ } L$, $x \in \mathcal{P}$ and $L \in \mathcal{L}$, if and only if x is contained in the polar space L^* of L with respect to $\mathrm{Q}^+(4n-1,2)$. For $q=2$, the existence of an orthogonal spread has been settled by Dye [49] and for $q=3$ it is only known that $\mathrm{Q}^+(7,3)$ has an orthogonal spread, even being unique up to isomorphism [83], which yields a $\mathrm{pg}(26,27,18)$.
Also recall that for $q=2$ or 3, being adjacent in the point graph of the partial geometry $\mathrm{PQ}^+(4n-1,q)$ corresponds with being contained in a tangent line. When $q=2$, adjacency translates into orthogonality while for $q=3$, adjacency becomes non-orthogonality.
Finally recall the notation for a line of $\mathrm{PQ}^+(4n-1,q), q=2,3$, considered as the point set of an affine space with a capital letter, for example L, while we denote the same line as a hyperplane of an element of Σ by π_L. And conversely, given a hyperplane π of an element of Σ, then we denote with L_π the set of points of the affine space being points of the line π in $\mathrm{PQ}^+(4n-1,q), q=2,3$, that is the intersection of π^* with the point set of $\mathrm{PQ}^+(4n-1,q), q=2,3$. And so $\pi_{L_\pi}=\pi$ and $L_{\pi_L}=L$.

Remark

Two different lines π_1 and π_2 of $\mathrm{PQ}^+(4n-1,q), q = 2,3$, are concurrent if and only if $\pi_1^* \cap \pi_2 = \emptyset$, if and only if $\pi_1 \cap \pi_2^* = \emptyset$ (see [31] for $q = 2$ and lemma 4.6 for $q = 3$). We often use this property in the constructions of this chapter.

4.2 Spread derivation

4.2.1 Spread derived partial geometries

In section 3.1 we introduced a new method for constructing semipartial geometries derived from partial geometries with respect to a point of the partial geometry. The partial geometry $\mathrm{PQ}^+(7,3)$ turned out to be a good candidate for point derivation. In section 4.6 we study the point and block graph of its point derived semipartial geometry $\mathrm{SPQ}(6,3)$. But first we investigate an other type of derivation, introduced by De Clerck [25] (which is based on [81]) and which we will call *spread derivation*. Again the same family $\mathrm{PQ}^+(4n-1,q), q = 2,3$, of partial geometries turns out to be a good candidate for spread derivation.

Construction

Let Φ be a pg–spread of a pg(s,t,α) $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, that is a (maximal) set of $\frac{st}{\alpha} + 1$ lines partitioning the point set. Assume $t > 1$ and let L be any line of $\mathcal{L} \setminus \Phi$. Let Φ_L be the set of $s+1$ lines of Φ intersecting L. Then L is called *regular with respect to* Φ if and only if there exists a set of $s+1$ lines $\mathcal{L}(L) = \{L_0 = L, L_1, \ldots, L_s\}$ that partitions the point set $\mathcal{P}(\Phi_L)$ of Φ_L, and each element of $\mathcal{L} \setminus (\mathcal{L}(L) \cup \Phi)$ is intersecting Φ_L in at most α points.

Properties

It was proved in [25] that if a pg(s,t,α) $\mathcal{S}$ has a regular line L with respect to a pg–spread Φ, then $t \geq s+1$. If $t = s+1$ then every line M not being an element of the pg–spread Φ neither of $\mathcal{L}(L)$ intersects $\mathcal{P}(\Phi_L)$ in α points.
Now assume that Φ is a pg–spread of a pg$(s, s+1, \alpha)$ such that every line is regular with respect to Φ. Then $\mathcal{L} \setminus \Phi$ is partitioned in $\frac{s(s+1)}{\alpha} + 1$ sets $\mathcal{L}_i$ $(i = 1, \ldots, \frac{s(s+1)}{\alpha} + 1)$ each containing $s+1$ mutually skew lines.
The spread Φ is called a *replaceable spread* and can be used to construct the following incidence structure $\mathcal{S}_\Phi = (\mathcal{P}_\Phi, \mathcal{L}_\Phi, \mathrm{I}_\Phi)$. The elements of $\mathcal{P}_\Phi$ are on the one hand the points of $\mathcal{S}$ and on the other hand the sets $\mathcal{L}_i$ $(i = 1, \ldots, \frac{s(s+1)}{\alpha}+1)$; and $\mathcal{L}_\Phi = \mathcal{L} \setminus \Phi$; finally $p \ \mathrm{I}_\Phi \ L$ is defined by $p \ \mathrm{I} \ L$ if $p \in \mathcal{P}$ and by $L \in p$ if $p \in \{\mathcal{L}_i \mid i = 1, \ldots, \frac{s(s+1)}{\alpha} + 1\}$. Generalizing a construction of Mathon and Street [81], De Clerck [25] proved that $\mathcal{S}_\Phi$ is a pg$(s+1, s, \alpha)$. The partial geometry $\mathcal{S}_\Phi$ (and its dual) is called a *partial geometry derived from* $\mathcal{S}$ *with respect to the spread* Φ; for short we shall call $\mathcal{S}_\Phi$ (and its dual) *spread derived* from $\mathcal{S}$ with respect to Φ.

Theorem 4.1 *The set $\phi = \{\mathcal{L}_i \mid i = 1, \dots, \frac{s(s+1)}{\alpha} + 1\}$ is a replaceable spread of $\mathcal{S}_\Phi^D$. The spread derived partial geometry $(\mathcal{S}_\Phi^D)_\phi$ is isomorphic to the partial geometry $\mathcal{S}^D$.*

Proof. By definition, the set ϕ is an ovoid in the partial geometry $\mathcal{S}_\Phi$, and hence a pg–spread of the dual geometry $\mathcal{S}_\Phi^D$.
Consider $\mathcal{S}_\Phi^D$. In order not to create confusion we call the elements of $\mathcal{L} \setminus \Phi$ the $\mathcal{S}_\Phi^D$–points while the elements of $\mathcal{P} \cup \phi$ are called the $\mathcal{S}_\Phi^D$–lines. Obviously, the line set of $(\mathcal{S}_\Phi^D)_\phi$ equals $\mathcal{P}$. Hence we only need to prove that the new points of $(\mathcal{S}_\Phi^D)_\phi$ correspond to the lines of the pg–spread Φ of $\mathcal{S}$.
A new point of $(\mathcal{S}_\Phi^D)_\phi$ can be obtained by taking an $\mathcal{S}_\Phi^D$–line p_0 $(\in \mathcal{P})$ and by considering $\phi_{p_0} = \{\mathcal{L}_0, \dots, \mathcal{L}_s\} \subset \phi$. In $\mathcal{S}$ this corresponds to a point p_0 and to the set $\{\mathcal{L}_0, \dots, \mathcal{L}_s\}$ with $\mathcal{L}_i = \{L_0^{(i)}, \dots, L_s^{(i)}\}$ $(0 \le i \le s)$. Let $L_0^{(i)}$ $(0 \le i \le s)$ be the $s+1$ lines of $\mathcal{S}$ incident with p_0 and not contained in Φ. Then p_0 is a line of $(\mathcal{S}_\Phi^D)_\phi$ incident with the points $L_0^{(0)}, \dots, L_0^{(s)}$.
Let $M = \{p_0, p_1, \dots, p_s\}$ be the element through p_0 of the spread Φ of $\mathcal{S}$. The incidences can be taken such that $L_j^{(i)} \in \mathcal{L}_i$ $(1 \le j \le s)$ intersects M in the point p_j. In $\mathcal{S}_\Phi^D$, the set $\{p_0, p_1, \dots, p_s\}$ is a set of $s+1$ mutually disjoint $\mathcal{S}_\Phi^D$-lines each of them intersecting $\mathcal{L}_0, \dots, \mathcal{L}_s$, that is the set $M = \{p_0, p_1, \dots, p_s\}$ is a new point of $(\mathcal{S}_\Phi^D)_\phi$. Hence to each element of the spread Φ of $\mathcal{S}$ there corresponds a unique new point of $(\mathcal{S}_\Phi^D)_\phi$. From this follows that ϕ is a replaceable pg–spread of $(\mathcal{S}_\Phi^D)$ and that $(\mathcal{S}_\Phi^D)_\phi \cong \mathcal{S}^D$. □

Remarks

1. It has been checked by computer (see [81]) that $\mathrm{PQ}^+(7,2)$ has exactly 3 types of replaceable spreads yielding (after dualizing) 3 non-isomorphic partial geometries $\mathrm{pg}(7,8,4)$. De Clerck [25] proved this result geometrically. Assume $\Sigma = \{\sigma_0, \dots, \sigma_8\}$, then the geometric construction of the three types of replaceable spreads is as follows.

   - A first type of pg-spread, which we denote by Φ_1, consists of all planes of an element σ_i of Σ.
   - Another type of pg-spread, which we denote by Φ_2, equals $V \cup W$, with V the set of planes passing through a point z of the quadric and, assuming $z \in \sigma_0$, with $W = \{M_i = z^* \cap \sigma_i \mid i = 1, \dots, 8\}$.
   - Finally let σ' be an element of $\mathcal{D}_1 \setminus \Sigma$. Then $\sigma' \cap \sigma_i$, $i = 0, \dots, 8$ is either empty or is a projective line. Without loss of generality we may assume that $\sigma' \cap \sigma_i$ is a projective line l_i, $i = 0, \dots, 4$. The set $\Phi_3 = \{\pi_{ij} \mid i = 0, \dots, 4; j = 0, 1, 2\}$, with π_{ij}, $j = 0, 1, 2$, a plane of σ_i through l_i, is a pg-spread of $\mathrm{PQ}^+(7,2)$, which is clearly not isomorphic to the other two.

   These three types of spreads are the replaceable spreads of the partial geometry $\mathrm{PQ}^+(7,2)$ (see [25]), yielding three non-isomorphic partial geometries $\mathrm{pg}(8,7,4)$, which we denote by $\mathrm{PQ}^+_{\Phi_i}(7,2)$ and later on after dualizing by $\mathcal{S}_i$, that is $\mathrm{PQ}^+_{\Phi_i}(7,2) = \mathcal{S}_i^D$, $i = 1,2,3$. The reason why we dualize is because after we have derived the partial geometry, we want to obtain a candidate for a new spread derivation and for this derivation we need again a $\mathrm{pg}(s, s+1, \alpha)$.

Recall that non-isomorphic orthogonal spreads of $\mathrm{Q}^+(4n-1,q)$ yield non-isomorphic partial geometries of type $\mathrm{PQ}^+(4n-1,q), q = 2,3$. Similarly non-isomorphic replaceable spreads of type Φ_i yield non-isomorphic partial geometries of type $\mathcal{S}_i$, for the relevant i. Mathon [private communication] checked by computer that $\mathrm{PQ}^+(7,2)$ has exactly one replaceable spread of each of the three types, yielding in total three non-isomorphic partial geometries $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$;

De Clerck [25] proved that the pg-spreads of type Φ_2 and of type Φ_3 are replaceable only in the 7-dimensional case, while he generalised the construction of the spread of type Φ_1 of $\mathrm{PQ}^+(4n-1,2)$ in order to obtain a new infinite family of

$$\mathrm{pg}(2^{2n-1}-1, 2^{2n-1}, 2^{2n-2}),$$

which we call $\mathcal{S}_1(n)$.

2. De Clerck [25] also generalized the construction of the replaceable spread Φ_1 for the partial geometry $\mathrm{PQ}^+(4n-1,3)$, and he generalized Φ_2 and Φ_3 for $\mathrm{PQ}^+(7,3)$. Mathon [private communication] checked by computer that $\mathrm{PQ}^+(7,3)$ has exactly one replaceable spread of type Φ_1, exactly one of type Φ_2 and two replaceable spreads of type Φ_3, yielding in total four non-isomorphic partial geometries $\mathrm{pg}(26,27,18)$, which we call $\mathcal{S}_1'$, $\mathcal{S}_2'$, $\mathcal{S}_3'$ and $\mathcal{S}_3''$. Similarly as for $q = 2$, the partial geometry $\mathcal{S}_1'$ can be generalized in $\mathrm{PG}(4n-1,3)$ and in that case we call it $\mathcal{S}_1'(n)$.

4.2.2 Overview

Since we use a lot of notations we give a brief overview of the partial geometries considered in this chapter. For general dimensions, that is in $\mathrm{PG}(4n-1,2)$, we have the geometries (of type) $\mathcal{S}_0(n), \mathcal{S}_1(n), \mathcal{S}_4(n)$ and $\mathcal{S}_5(n)$. When $n = 2$ we denote $\mathcal{S}_i = \mathcal{S}_i(2)$, $i = 0,\ldots,7$.
Also in this chapter we will use points for projective points while we will use $\mathcal{S}$-points for the points of the partial geometry $\mathcal{S}$. In the same way we will use lines and $\mathcal{S}$-lines.
In [81] Mathon and Street have constructed by computer seven new partial geometries $\mathrm{pg}(7,8,4)$ by starting from the partial geometry $\mathcal{S}_0 = \mathrm{PQ}^+(7,2)$ and by using spread derivation with respect to a suitable replaceable spread. In the sequel we give a geometric construction of them. The following scheme, taken from [81], shows how the eight partial geometries $\mathrm{pg}(7,8,4)$ are related to each other. The labeled arrow $\overset{\Phi_i}{\longleftrightarrow}$ means that the partial geometries are related under derivation with respect to the replaceable spread (of type) Φ_i (or ϕ_i, see theorem 4.1) and after dualizing.

$$\begin{array}{ccccccccccccc}
\mathcal{S}_2 & \overset{\Phi_2}{\longleftrightarrow} & \mathcal{S}_0 & \overset{\Phi_1}{\longleftrightarrow} & \mathcal{S}_1 & \overset{\Phi_4}{\longleftrightarrow} & \mathcal{S}_4 & \overset{\Phi_6}{\longleftrightarrow} & \mathcal{S}_6 & \overset{\Phi_7}{\longleftrightarrow} & \mathcal{S}_7 \\
 & & \Phi_3 \updownarrow & & \Phi_5 \updownarrow & & & & & & \\
 & & \mathcal{S}_3 & & \mathcal{S}_5 & & & & & &
\end{array}$$

Mathon and Street give in [81] information on the order of the automorphism groups of the geometries as well as information on the point and block graphs of these geometries. They remarked that the point graphs Γ_i of the geometries $\mathcal{S}_i, i = 1, 2, 3, 4$, were isomorphic graphs and their block graphs all are different. Actually the graph $\Gamma = \Gamma_i$, $i = 1, 2, 3, 4$, was not a new graph, it is the complement of the graph constructed in [8]. It is an element of the class of graphs called the *graphs on a quadric with a hole*. Such a graph has vertex set the points of a quadric $\mathrm{Q}^+(2m-1, q) \setminus G$, G a generator of the quadric and vertices x and y are defined to be adjacent whenever $\langle x, y\rangle$ is contained in $\mathrm{Q}^+(2m-1, q) \setminus G$. This graph is strongly regular for general dimensions and general q. Klin and Reichard [72, 86] found, again by computer, but independently from Mathon and Street, that the complement of the graph on the quadric $\mathrm{Q}^+(7, 2)$ with a hole, is indeed the point graph of exactly four partial geometries $\mathrm{pg}(7, 8, 4)$, namely $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$ and $\mathcal{S}_4$. In section 4.5 we prove this result geometrically. Moreover we prove that the graph on the quadric with a hole in $\mathrm{PG}(4n-1, 2)$ is always geometric, namely it is the point graph of the partial geometry $\mathcal{S}_4(n)$.
Recall that the geometrical constructions of $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2$ and $\mathcal{S}_3$ are known (see [25, 30]). In section 4.3 we investigate the geometries $\mathcal{S}_4, \mathcal{S}_5$ and $\mathcal{S}_6$ for which there was no geometrical construction known yet. All these geometries as well as their duals are related to the triality quadric $\mathrm{Q}^+(7, 2)$ and often our proofs rely on the special properties of this quadric.
Generalizing our construction for general dimensions, that is in $\mathrm{PG}(4n-1, 2)$, we construct two new classes of partial geometries $\mathcal{S}_4(n)$ and $\mathcal{S}_5(n)$, which are

$$\mathrm{pg}(2^{2n-1} - 1, 2^{2n-1}, 2^{2n-2}).$$

Hence, from the eight known $\mathrm{pg}(7, 8, 4)$, four of them, namely $\mathcal{S}_i$, $i = 0, 1, 4, 5$, are the smallest member of an infinite class, namely $\mathcal{S}_i(n)$, $i = 0, 1, 4, 5$ (where we define $\mathcal{S}_0(n)$ to be the partial geometry $\mathrm{PQ}^+(4n-1, 2)$). And so it turns out that not $\mathcal{S}_0(n)$, but $\mathcal{S}_1(n)$ can be considered as the "central" partial geometry in the above scheme.

4.3 The partial geometries $\mathcal{S}_4(n), \mathcal{S}_5(n)$ and $\mathcal{S}_6$

4.3.1 The partial geometries $\mathcal{S}_4(n)$ and $\mathcal{S}_5(n)$

Theorem 4.2 *The geometry $\mathcal{S}_1(n) = \mathrm{pg}(2^{2n-1} - 1, 2^{2n-1}, 2^{2n-2})$ has at least three replaceable spreads yielding after dualizing the partial geometries $\mathcal{S}_0(n)$, $\mathcal{S}_4(n)$ and $\mathcal{S}_5(n)$.*

Proof. Let Σ denote the orthogonal spread of the hyperbolic quadric $\mathrm{Q}^+(4n-1, 2)$ used to construct the partial geometry $\mathcal{S}_0(n)$. Let Φ_1 be the first replaceable spread of $\mathcal{S}_0(n)$, that is it consists of all hyperplanes of an element σ_0 of Σ. From the proof of the replaceability of the spread Φ_1 in [25] one can see that for a line π_L of $\mathcal{S}_0(n)$ which is not contained in Φ_1, the new point $\mathcal{L}(\pi_L)$ consists of all hyperplanes of σ_0 that do not contain the point $\pi_L^* \cap \sigma_0$. Therefore we can identify this new point $\mathcal{L}(\pi_L)$ of $(\mathcal{S}_1(n))^D$ with the point $\pi_L^* \cap \sigma_0$. And so we obtain the following easy description of the partial geometry $\mathcal{S}_1(n)$. The line set $\mathcal{L}_1(n)$ of $\mathcal{S}_1(n)$ is the union of the points of $\mathrm{PG}(4n-1,2) \setminus \mathrm{Q}^+(4n-1,2)$ with the points of σ_0. The $\mathcal{S}_1(n)$–points are hyperplanes of elements of $\Sigma \setminus \{\sigma_0\}$. An $\mathcal{S}_1(n)$–point P is incident with the $\mathcal{S}_1(n)$–lines in $P^* \cap \mathcal{L}_1(n)$. This implies that two $\mathcal{S}_1(n)$–lines that are contained in σ_0 are never concurrent; two $\mathcal{S}_1(n)$–lines l and m contained in $\mathrm{PG}(4n-1,2) \setminus \mathrm{Q}^+(4n-1,2)$ are concurrent if and only if $\langle l, m\rangle$ is a tangent line of the quadric $\mathrm{Q}^+(4n-1,2)$ with tangent point not contained in σ_0 (note that $\langle l,m\rangle$ is a tangent line if and only if $l \in m^*$ or equivalently if and only if $m \in l^*$); an $\mathcal{S}_1(n)$–line l contained in σ_0, is concurrent with an $\mathcal{S}_1(n)$–line m contained in $\mathrm{PG}(4n-1,2) \setminus \mathrm{Q}^+(4n-1,2)$, if and only if $\langle l,m\rangle$ is a secant line of $\mathrm{Q}^+(4n-1,2)$.

Part I. By theorem 4.1 the points of σ_0 form a replaceable $\mathcal{S}_1(n)$–spread ϕ_1 and $(\mathcal{S}_1(n))_{\phi_1} = (\mathcal{S}_0(n))^D$.

Part II. A second replaceable spread Φ_4 is obtained as follows. Let π_0 be a hyperplane of an element σ_0 of the orthogonal spread Σ of the hyperbolic quadric $\mathrm{Q}^+(4n-1,2)$. Let $L_0 = L_{\pi_0}$, then

$$\Phi_4 = L_0 \cup \pi_0,$$

that is the union of the 2^{2n-1} points in L_0 and the $2^{2n-1}-1$ points of π_0 which is obviously a pg-spread of $\mathcal{S}_1(n)$.

Let the $\mathcal{S}_1(n)$–line l be a point of $\sigma_0 \setminus \pi_0$. Then $(\Phi_4)_l$ consists of the 2^{2n-1} points in L_0 and hence $\mathcal{L}(l)$ consists of the 2^{2n-1} points of $\sigma_0 \setminus \pi_0$.

Let the $\mathcal{S}_1(n)$–line l' be a point of $\mathrm{PG}(4n-1,2) \setminus (\mathrm{Q}^+(4n-1,2) \cup L_0)$. Let π_1 be the $\mathcal{S}_0(n)$–line, incident (in $\mathcal{S}_0(n)$) with l', where π_1 is a hyperplane of σ_0. Let Y denote the $(2n-3)$-dimensional space $\pi_0 \cap \pi_1$ and let P_1 denote the affine $(2n-2)$–dimensional space $\langle Y, l'\rangle \setminus Y$. Let π_2 denote the $(2n-2)$–dimensional space through Y in σ_0 different from π_0 and π_1, and let $L_i = L_{\pi_i}$, $i = 1,2$.

On the one hand $(\Phi_4)_{l'}$ consists of the 2^{2n-2} points in $\pi_0 \setminus Y$ and on the other hand it consists of the 2^{2n-2} points in $L_0 \cap (l')^*$. Note that we already defined P_1 above. Define P_i, $i = 0$ or 2, to be the affine $(2n-2)$–dimensional space $L_i \setminus (l')^*$. Let P_i' denote the other affine $(2n-2)$–dimensional space in $L_i \setminus P_i$ with Y at infinity, $i = 0,1,2$. And so for $i = 0$ or 2, there follows $P_i' = L_i \cap (l')^*$. Since $P_j' \subset Y^*$, we obtain for $j = 0$ or 2, that

$$\begin{aligned} L_j \cap P_1^* &= L_j \cap \langle l', Y\rangle^* \\ &= (L_j \cap (l')^*) \cap Y^* \end{aligned}$$

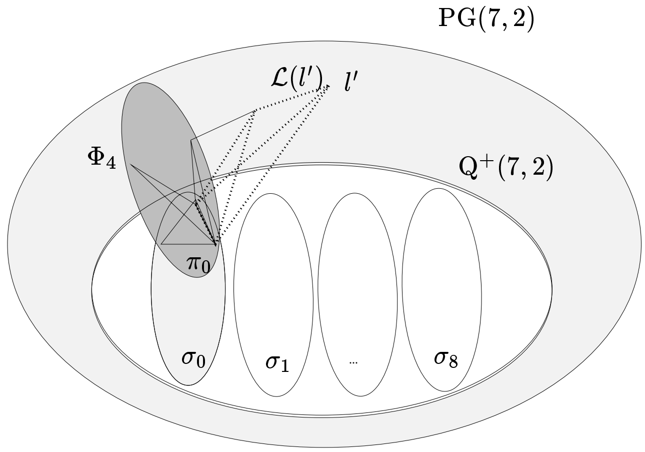

Figure 4.1: Construction of the partial geometry $\mathcal{S}_4$

$$\begin{aligned} &= P'_j \cap Y^* \\ &= P'_j. \end{aligned}$$

In the $(2n+1)$-dimensional space Y^* we have that $\langle P_1, P'_0 \rangle$ intersects π_2^* in a $(2n-2)$-dimensional space of $\mathrm{Q}^+(4n-1,q)$ through Y. Therefore $\langle P_1, P_0 \rangle$ intersects π_2^* in $L_2 \cup Y$, and $\langle P_1, P_0 \rangle \cap L_2$ is an affine $(2n-2)$-dimensional space, in particular $\langle P_1, P_0 \rangle \cap L_2 = P_2$. And so $\langle P_0, P_1, P_2 \rangle = (\cup_{i=0}^2 P_i) \cup Y$ is a $(2n-1)$-dimensional space intersecting the quadric only in Y. A line through a point y of P_i and a point z of P_j cannot intersect the quadric $\mathrm{Q}^+(4n-1,2)$ since it intersects P_k ($i,j,k \in \{0,1,2\}$, i,j,k all different), that is $\langle y, z \rangle$ is an external line of the quadric. Therefore $(\cup_{i=0}^2 P_i) \cup Y$ is a spread of the partial geometry $\mathcal{S}_1(n)$ (which turns out to be Φ_5, see below). Since $L_0 \cap P_i^* = L_0 \cap P_j^* \subset (\Phi_4)_{l'}$, $i,j = 1,2$, we obtain that $\mathcal{L}(l')$ consists of the 2^{2n-1} points in $P_1 \cup P_2$ (see figure 4.1 for the 7-dimensional case; the set $\mathcal{L}(l')$ is drawn by the dotted line). It follows that Φ_4 is indeed replaceable.

Part III. We now claim that the elements of the third replaceable pg-spread Φ_5 of $\mathcal{S}_1(n)$ are the points of the set

$$\Phi_5 = \cup_{i=0}^2 P_i \cup Y.$$

Let the $\mathcal{S}_1(n)$–line l be a point in $\cup_{i=0}^2 (L_i \setminus P_i)$. Without loss of generality we may assume $l \in L_0 \setminus P_0$, then $(\Phi_5)_l = P_1 \cup P_2$ and hence $\mathcal{L}(l)$ consists of the 2^{2n-2} points in $\pi_0 \setminus Y$ and the 2^{2n-2} points in $P'_0 = L_0 \setminus P_0$.

Let the $\mathcal{S}_1(n)$–line l' be a point of $\sigma_0 \setminus Y$; without loss of generality we may assume $l' \in \pi_0$. Then $(\Phi_5)_{l'} = (\Phi_5)_l$ and $\mathcal{L}(l') = \mathcal{L}(l)$.

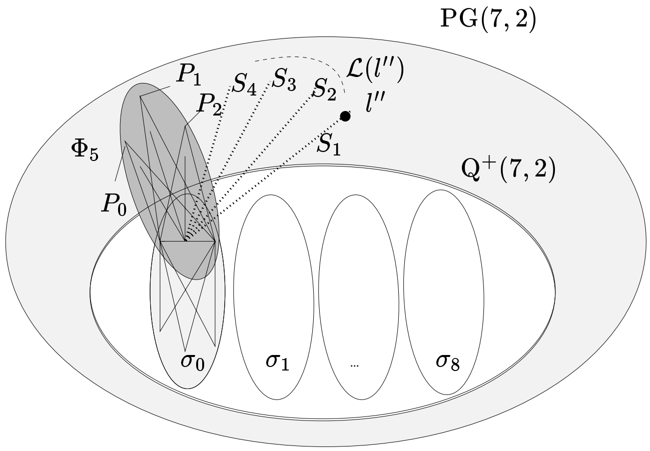

Figure 4.2: Construction of the partial geometry $\mathcal{S}_5$

Note that from the proof of the replaceability of Φ_4 one can see that there are actually 2 spreads of $\mathcal{S}_1(n)$ contained in $\cup_{i=0}^2 \pi_i^*$ intersecting each other in Y. Indeed the choice of P_1 in L_1 in the proof above, defines uniquely the other spaces P_i, $i = 0$ and 2. Let us denote these 2 spreads of $\mathcal{S}_1(n)$ as $\Phi_5^i(Y)$, $i = 1, 2$. We both call them pg-spreads of type five.

Let the $\mathcal{S}_1(n)$–line l'' be a point of $\mathcal{P}_0(n) \setminus (\cup_{i=0}^2 L_i)$. Then l'' is contained in an $\mathcal{S}_0(n)$–line L such that π_L does not contain Y and hence intersects each of the π_i in a $(2n-3)$-dimensional space X_i and $X_i \cap Y$ is a $(2n-4)$-dimensional space Z, $i = 0, 1, 2$. Then $P_i \cap (l'')^*$ is an affine $(2n-3)$-dimensional space Y_i, and $P_i \setminus (l'')^*$ is an affine $(2n-3)$-dimensional space Y_i'. Note that both Y_i and Y_i' have the space Z at infinity, $i = 0, 1, 2$.

Hence $(\Phi_5)_{l''}$ consists of the $4 \times 2^{2n-3}$ points in $\cup_{i=0}^2 Y_i \cup Y \setminus Z$. In order to find $\mathcal{L}(l'')$ we now look for an $\mathcal{S}_1(n)$–spread Ψ containing $\cup_{i=0}^2 Y_i' \cup Z$, and such that by construction Ψ also contains the $(2n-3)$-dimensional space $S_1 = \langle Z, l'' \rangle$. Put $\omega_1 = \pi_L$.

¿From the two type five pg-spreads intersecting in X_i, exactly one, say $\Phi_5^1(X_i)$, intersects $\cup_{j=0}^2 P_j$ in the $(2n-3)$-dimensional spaces Y_j and exactly one, call it $\Phi_5^2(X_i)$, will intersect $\cup_{j=0}^2 P_j$ in the $(2n-3)$-dimensional spaces Y_j'. Then $S_1 = \langle Z, l'' \rangle = \Phi_5^2(X_0) \cap \Phi_5^2(X_1) \cap \Phi_5^2(X_2)$. Let $\omega_2, \omega_3, \omega_4$ denote the $(2n-2)$-dimensional spaces in σ_0 containing Z but different from $\pi_0, \pi_1, \pi_2, \pi_L = \omega_1$, then using a similar argument we obtain a $(2n-3)$-dimensional space S_i in ω_i^* intersecting the quadric in Z, $i = 2, 3, 4$. But ω_i, $i = 2, 3, 4$, intersects ω_1 in a $(2n-3)$-dimensional space which is one of the X_j, $j \in \{0, 1, 2\}$, which already defined a type five pg-spread $\Phi_5^2(X_j)$ above. Therefore $S_i^* \cap S_1 = Z$, $i = 2, 3, 4$. Because of a similar argument we obtain that $S_i^* \cap S_j = Z$, $1 \le i, j \le 4$ and $i \ne j$.

And by construction we have $S_i^* \cap Y_j' = Z$, $1 \leq i \leq 4$ and $0 \leq j \leq 2$. Hence $\mathcal{L}(l'') = \cup_{i=1}^4 S_i$ (see figure 4.2 for the 7-dimensional case; the set $\mathcal{L}(l'')$ is drawn by the dotted line). It follows that Φ_5 is indeed replaceable. □

Remarks

1. Mathon [private communication] checked by a computer search that the three types of replaceable spreads of $\mathcal{S}_1 = \mathcal{S}_1(2)$ from theorem 4.2 each yield exactly one partial geometry while the three types of replaceable spreads of $\mathcal{S}_1(3)$ yield in total 6 partial geometries.

2. We have that the spread Φ_4 of $\mathcal{S}_1(n)$, which we call a type 4 pg-spread, must consist of the points in a $(2n-1)$-dimensional space Π of $\mathrm{PG}(4n-1,q)$ intersecting the quadric $\mathrm{Q}^+(4n-1,q)$ in a $(2n-2)$-dimensional space π contained in an element of the orthogonal spread Σ and such that $\Pi \subset \pi^*$. ¿From the proof of theorem 4.2 we obtain a similar easy description for the type 5 pg-spread. Every type 5 pg-spread of $\mathcal{S}_1(n)$ must consist of the points in a $(2n-1)$-dimensional space Π of $\mathrm{PG}(4n-1,2)$ intersecting the quadric $\mathrm{Q}^+(4n-1,2)$ in a $(2n-3)$-dimensional space Y contained in an element of the orthogonal spread Σ and such that $\Pi \subset Y^*$. Conversely, if Π is a $(2n-1)$-dimensional subspace of $\mathrm{PG}(4n-1,2)$ intersecting the quadric $\mathrm{Q}^+(4n-1,2)$ in a $(2n-3)$-dimensional space Y such that Y is contained in an element σ_0 of the orthogonal spread Σ and such that $\Pi \subset Y^*$. Then Π is a type 5 pg-spread of $\mathcal{S}_1(n)$.

3. Let us consider the partial geometry $\mathcal{S}_1'(n)$, that is we consider the case $q=3$. Then the easy description of $\mathcal{S}_1'(n)$ is similar as for $\mathcal{S}_1(n)$. Let σ_0 be an element of the orthogonal spread Σ of $\mathrm{Q}^+(4n-1,3)$. The line set $\mathcal{L}_1'(n)$ of $\mathcal{S}_1'(n)$ is the union of the points of $E_m^+(4n-1,3)$, $m \in \{1,2\}$, with the points of σ_0. The $\mathcal{S}_1'(n)$–points are hyperplanes of elements of $\Sigma \setminus \{\sigma_0\}$. An $\mathcal{S}_1'(n)$–point P is incident with the $\mathcal{S}_1(n)'$–lines in $P^* \cap \mathcal{L}_1'(n)$. Let Π be a $(2n-1)$-dimensional space of $\mathrm{PG}(4n-1,3)$ intersecting the quadric $\mathrm{Q}^+(4n-1,3)$ in a $(2n-3)$-dimensional space Y contained in an element of the orthogonal spread Σ of $\mathrm{Q}^+(4n-1,3)$ and such that $\Pi \subset Y^*$. Since an external line of $\mathrm{Q}^+(4n-1,3)$ has half of its points in $E_1^+(4n-1,3)$ and half of its points in $E_2^+(4n-1,3)$, only half of the points of Π are $\mathcal{S}_1'(n)$ lines. And so the points of Π even do not form a pg-spread of $\mathcal{S}_1'(n)$ anymore.

4. From [25, Theorem 4] one can observe that the spreads Φ_2 and Φ_3 are related to each other. Indeed, Φ_3 appears in $\mathcal{S}_0$ for the construction of a new point of $\mathcal{S}_2$. In the same way we observe the appearance in $\mathcal{S}_1(n)$ of Φ_5 in the construction of a new line of $\mathcal{S}_4(n)$.

5. The set Ψ from part III of the proof of theorem 4.2 turns out to be an $\mathcal{S}_1(n)$–spread consisting of 7 $(2n-3)$-dimensional spaces intersecting each other and intersecting the quadric $\mathrm{Q}^+(4n-1,2)$ in a $(2n-4)$-dimensional

space $Z \subset \sigma_0$, with $\sigma_0 \in \Sigma$. Namely, put $S_{5+i} = Y_i'$, $i = 0, 1, 2$. Also put $\omega_{5+i} = \pi_i$, $i = 0, 1, 2$, then we obtain the $\mathcal{S}_1(n)$-spread.

$$\Psi = \cup_{i=1}^{7} S_i \cup Z,$$

with $S_i \subset \omega_i^* \cap \mathrm{PG}(4n-1, 2) \setminus \mathrm{Q}^+(4n-1, 2)$.

4.3.2 Easy descriptions of $\mathcal{S}_4(n)$ and $\mathcal{S}_5(n)$

The partial geometry $\mathcal{S}_4(n)$

The partial geometry $\mathcal{S}_4(n)$ has an easy description in the following way. We recall some concepts which appeared in theorem 4.2. Let Σ be an orthogonal spread of the quadric $\mathrm{Q}^+(4n-1, q)$ and let $\sigma_0 \in \Sigma$. Let the $(2n-2)$-dimensional space π_0 be a hyperplane of σ_0. Put $L_0 = L_{\pi_0}$. Then $\mathcal{S}_4(n)$ has as point set

$$\mathcal{P}_4(n) = (\mathrm{PG}(4n-1, 2) \setminus (\mathrm{Q}^+(4n-1, 2) \cup L_0)) \cup (\sigma_0 \setminus \pi_0).$$

And $\mathcal{S}_4(n)$ has three types of lines:

type (i)**:** π_0;

type (ii)**:** the hyperplanes of elements of $\Sigma \setminus \{\sigma_0\}$;

type (iii)**:** the point sets of affine $(2n-1)$-dimensional spaces that we obtain in the following way. Consider $p \in \mathcal{P}_4(n)$. Then the type (iii) line A containing p consists of the points in $\mathcal{P}_4(n) \cap \langle L_0 \setminus p^*, p\rangle$. Note that from theorem 4.2 it follows that the construction of a type (iii) line is independent of the choice of p in A.

Incidence is as follows: π_0 is incident with the points of $\sigma_0 \setminus \pi_0$; a type (ii) line π is incident with the points of $\pi^* \cap \mathcal{P}_4(n)$; a type (iii) line is incident with the points contained in it.

The partial geometry $\mathcal{S}_5(n)$

The partial geometry $\mathcal{S}_5(n)$ has an easy description in the following way. Consider the notations above. Note that in the proof of the replaceability of Φ_5, we can identify $\mathcal{L}(l)$ and $\mathcal{L}(l')$ with the corresponding $(2n-2)$-dimensional space π_i of σ_0 through Y, $i \in \{0, 1, 2\}$. Consider a $(2n-1)$-dimensional space Π of $\mathrm{PG}(4n-1, 2)$ intersecting the quadric $\mathrm{Q}^+(4n-1, 2)$ in a $(2n-3)$-dimensional space Y, such that $Y \subset \sigma_0 \in \Sigma$ and $\Pi \subset Y^*$. Then $\mathcal{S}_5(n)$ has point set

$$\mathcal{P}_5(n) = (\mathrm{PG}(4n-1, 2) \setminus (\mathrm{Q}^+(4n-1, 2) \cup \Pi)) \cup (\sigma_0 \setminus Y)$$

and three types of lines:

type (i)**:** the spaces $\pi_j, j = 0, 1, 2$;

type (ii)**:** the hyperplanes of elements of $\Sigma \setminus \{\sigma_0\}$;

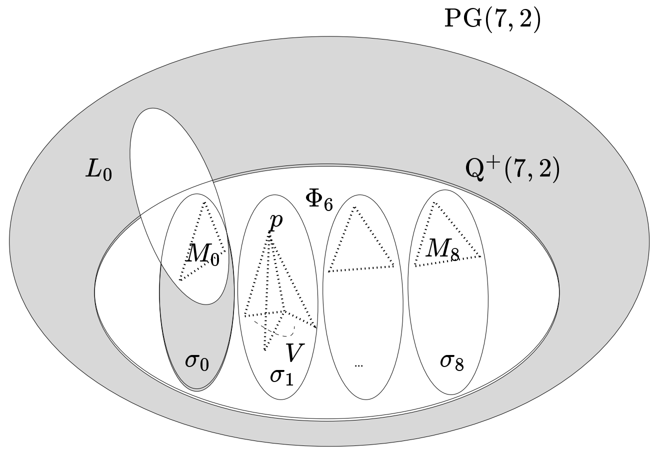

Figure 4.3: The spread Φ_6

type (iii)**:** a collection of 4 affine $(2n-3)$-dimensional spaces S_i having the same $(2n-4)$-dimensional space $Z \subset \sigma_0$ in $Q^+(4n-1, q)$ at infinity. We obtain them in the following way. Consider $p \in \mathcal{P}_5(n) \setminus (\cup_{i=0}^{2} L_{\pi_i})$. Put $Z = p^* \cap Y$ and let ω_i denote the hyperplanes of σ_0 through Z but not through Y, $i = 1, 2, 3, 4$. Let $S_1 = \langle p, Z \rangle$. For $i = 2, 3, 4$, let S_i be the unique affine $(2n-3)$–dimensional space contained in $\omega_i^* \cap (\mathcal{P}_5(n) \setminus \sigma_0)$ determined by S_1 having Z at infinity such that $\cup_{i=1}^{4} S_i$ determines a partial $\mathcal{S}_1(n)$-spread (see the proof of theorem 4.2 for further details). Then the type (iii) line containing p is $\cup_{i=1}^{4} S_i$. Note that from theorem 4.2 it follows that the construction of a type (iii) line is independent of the choice of $p \in \cup_{i=1}^{4} S_i$.

Incidence is as follows: the type (i) line π_i is incident with the points of $(L_{\pi_i} \setminus \Pi) \cup (\pi_i \setminus Y)$, $i = 0, 1, 2$; a type (ii) line π is incident with the points of $\pi^* \cap \mathcal{P}_5(n)$; a type (iii) line is incident with the points contained in it.

4.3.3 The partial geometry $\mathcal{S}_6$

As the partial geometry $\mathcal{S}_4(n)$ has an easy description, it is possible to give a geometric construction of the replaceable spreads of this geometry. We use the notations introduced in the proof of theorem 4.2. Hence π_0 is the hyperplane of σ_0 whose points are elements of Φ_4. Consider a point $p \in \pi_0^* \cap \sigma_i$, $i \in \{1, \ldots, 2^{2n-1}\}$, say $p \in \sigma_1$. Then $\Phi_6 = V \cup W$, with V the set of hyperplanes of σ_1 containing p and with W the set $\{M_i = p^* \cap \sigma_i \mid i = 0, 2, 3, \ldots, 2^{2n-1}\}$ is easily seen to be a spread of $\mathcal{S}_4(n)$. Note that $\pi_0 = M_0$ (see figure 4.3 for the construction of Φ_6).
However, contrary to the spreads Φ_4 and Φ_5 of $\mathcal{S}_1(n)$, the spread Φ_6 is only replaceable if $n = 2$. From the construction of the spread Φ_6 and the proof of

theorem 4.3 it is clear that there is some similarity between the spread Φ_6 of $\mathcal{S}_4(n)$ and the spread Φ_2 of $\mathcal{S}_0(n)$.

Theorem 4.3 *The pg-spread Φ_6 of $\mathcal{S}_4(n)$ is replaceable if and only if $n = 2$.*

Proof. Let π_L be a line of type (ii) of $\mathcal{S}_4(n)$ such that $\pi_L \in \sigma_1 \setminus V$, then π_L^* contains a point of $\sigma_0 \setminus \pi_0$. The 2^{2n-1} elements of $(\Phi_6)_L$ are the $2^{2n-1} - 1$ elements of W not contained in σ_0 together with the plane π_0. The 2^{2n-1} elements of $\mathcal{L}(L)$ are the hyperplanes of σ_1 not contained in V. Indeed they all intersect $\sigma_0 \setminus \pi_0$ and therefore are incident in $\mathcal{S}_4(n)$ with π_0 and obviously the elements of $\mathcal{L}(L)$ intersect the other elements of $(\Phi_6)_L$. No other line is completely contained in the union of the elements of $(\Phi_6)_L$. Hence π_L is regular with respect to Φ_6.

Assume that $\pi_{L'}$ is a line of type (ii) of $\mathcal{S}_4(n)$ such that $\pi_{L'}$ is a hyperplane of σ_i not in W, $i \in \{2, 3, \ldots, 2^{2n-1}\}$, and such that $(\pi_{L'})^* \cap \pi_0 = \emptyset$. Without loss of generality we may assume that $\pi_{L'}$ is a hyperplane of σ_2. Then $\pi_{L'}$ intersects M_2 in a $(2n-3)$-dimensional space that we call H_2. Let $\pi_{L'}^* \cap \mathrm{Q}^+(4n-1, 2) = \langle \sigma_2, \sigma \rangle$, with $\sigma \in \mathcal{D}_2$. Then $\sigma \cap (\sigma_2 \cap \Phi_6) = \pi_{L'} \cap M_2 = H_2$, while $\sigma \cap (\sigma_i \cap \Phi_6)$ for $i \in \{0, 1, 3, 4, \ldots, 2^{2n-1}\}$ is either empty or a point. Since $\pi_{L'} \cap p^* \neq \emptyset$ we have that $(\pi_{L'})^* \cap \sigma_1 = \sigma_1 \cap \sigma$ is a point r_1. And since $\pi_{L'}$ is not contained in p^*, r_1 is different form p. Recall that $(\pi_{L'})^* \cap \pi_0 = \emptyset$ hence $\sigma \cap \pi_0 = \emptyset$.
Note that $\sigma \cap p^* \cap \sigma_0 = \emptyset$. Then without loss of generality we may assume that the 2^{2n-2} points of $(\sigma \cap p^*) \setminus H_2$ are the points $r_i = \sigma \cap \sigma_i$, $i = 1, 3, 4, \ldots 2^{2n-2}+1$. Then $\pi_0 \in (\Phi_6)_{L'}$ and the other elements of $(\Phi_6)_{L'}$ are the $2^{2n-2} - 1$ elements M_i, $2^{2n-2}+2 \leq i \leq 2^{2n-1}$, together with the 2^{2n-2} elements of V not containing r_1.
The lines of $\mathcal{S}_4(n)$ that are not concurrent to π_L' but concurrent to all elements of $(\Phi_6)_{L'}$ should be hyperplanes of $\sigma_2, \ldots, \sigma_{2^{2n-2}+1}$. The hyperplane in σ_2 is the one through H_2 and different from $\pi_{L'}$ and M_2. For $i = 3, \ldots, 2^{2n-2} + 1$ we have that $M_2^* \cap \sigma_i$ is a point p_i and recall that $\pi_{L'}^* \cap \sigma_i$ is a point r_i. Then $\langle p_i, r_i \rangle$ is a projective line of $M_i \in \Phi_6$. For $i = 3, \ldots, 2^{2n-2} + 1$, the $2^{2n-2} - 2$ hyperplanes of σ_i through $\langle p_i, q_i \rangle$ and different from M_i are lines of $\mathcal{S}_4(n)$ each intersecting all the elements of $(\Phi_6)_{L'}$. Hence the total number of lines of $\mathcal{S}_4(n)$ that are not contained in Φ_6 but that are contained in the point set of $(\Phi_6)_{L'}$ equals

$$2 + (2^{2n-2} - 2)(2^{2n-2} - 1).$$

Suppose that out of these lines we can select a set $\mathcal{L}(L')$ of 2^{2n-1} lines that are disjoint in $\mathcal{S}_4(n)$. Then there still remain $2 + (2^{2n-2} - 2)(2^{2n-2} - 1) - 2^{2n-1}$ lines of $\mathcal{S}_4(n)$ that are not contained in $\mathcal{L}(L')$ nor in Φ_6, but that intersect $(\Phi_6)_{L'}$ in more than $\alpha = 2^{2n-2}$ points. By definition of a replaceable spread, this implies that if Φ_6 is replaceable then

$$2 + (2^{2n-2} - 2)(2^{2n-2} - 1) - 2^{2n-1} = 0,$$

that is $n = 2$.

Therefore, from now on we suppose that $n=2$. Then H_2 is a projective line and the 3-dimensional space $\eta=\langle p, r_1, H_2\rangle \in \mathcal{D}_1$. Also $\eta\cap\sigma_i$ is the projective line H_i, $i=1,\ldots,5$, while $\eta\cap\sigma_j=\emptyset$, $j=0,6,7,8$. The set $\mathcal{L}(L')$ consists of the planes of σ_i through H_i different from M_i, $i=2,\ldots,5$ (note that for such a plane N, indeed $N^*\cap\pi_0=\emptyset$). Hence $\pi_{L'}$ is regular with respect to Φ_6 in $\mathcal{S}_4(n)$ if and only if $n=2$.

Assume that $\pi_{L''}$ is a line of type (ii) of $\mathcal{S}_4$ such that $\pi_{L''}$ is a plane of σ_i not in W and such that $(\pi_{L''})^*\cap\pi_0\neq\emptyset$, $2\le i\le 8$. Without loss of generality, we may assume that $\pi_{L''}$ is a plane of σ_2. If $r_1'=(\pi_{L''})^*\cap\sigma_1$, then $\eta'=\langle p, r_1', \pi_{L''}\cap M_2\rangle$ intersects π_0 in a projective line H_0. Without loss of generality we may assume that $\eta'\cap\sigma_i$ is a projective line H_i, $i=0,\ldots,4$. Then the elements of $(\Phi_6)_{L''}$ are the elements M_i, $i=5,\ldots,8$, together with the elements of V not containing r_1'. The set $\mathcal{L}(L'')$ consists of the planes of σ_i through H_i different from M_i, $i=2,3,4$, together with the 2 lines of $\mathcal{S}_4$ of type (iii) through H_0. Note that these 2 lines of type (iii) are disjoint lines and cover the same point set as the 2 $\mathcal{S}_0$-lines L_{π_i} corresponding with the 2 planes $\pi_i\neq\pi_0$ of σ_0 through $H_0, i=1,2$. We conclude that $\pi_{L''}$ is regular with respect to Φ_6 in $\mathcal{S}_4$.

For a line L''' of type *(iii)* we can reverse the arguments for the type *(ii)* line L'' above, in order to obtain that also L''' is regular with respect to Φ_6 in $\mathcal{S}_4$. □

Corollary 4.4 *The partial geometry $\mathcal{S}_4=\mathrm{pg}(7,8,4)$ has at least two replaceable spreads yielding after dualizing the partial geometries $\mathcal{S}_1$ and $\mathcal{S}_6$.*

Proof. By theorem 4.1, one of the replaceable spreads is ϕ_4, which yields $(\mathcal{S}_4)_{\phi_4}=\mathcal{S}_1^D$. Theorem 4.3 now yields the result. □

Remark

The replaceable spread Φ_6 yields the partial geometry $\mathcal{S}_6$. Although it is possible to give a geometric description of this partial geometry in terms of subspaces of $\mathrm{PG}(7,2)$, the description itself is not as easy as the former ones. The computer aided results in [81] tell us that the partial geometry $\mathcal{S}_6$ has a replaceable spread Φ_7 yielding a new partial geometry $\mathcal{S}_7$, with a small automorphism group (of order 21). As $\mathcal{S}_6$ has no nice geometric description we do not give a description of Φ_7.

4.4 The point (block) graph of $\mathrm{PQ}^+(7,q), q=2,3$

4.4.1 Overview

Even when the point graph of a given partial geometry is faithfully geometric, there is no guarantee that the block graph is also faithfully geometric. In [31] it has been proved that the point graph of the partial geometry $\mathrm{PQ}^+(7,2)$ is faithfully geometric. In [82] Panigrahi gives another proof of this result and

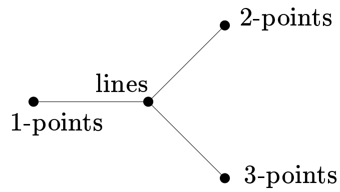

Figure 4.4: Triality

she proves using combinatorial arguments that the block graph of the partial geometry $\mathrm{PQ}^+(7,2)$ is faithfully geometric. In sections 4.4 and 4.5 we extend this result for more graphs related to the quadric $\mathrm{Q}^+(7,2)$, and we give for the result of Panigrahi a shorter proof based on the triality property of the quadric $\mathrm{Q}^+(7,2)$. Moreover we extend some results for the case $q = 3$.

4.4.2 The triality quadric

The quadric $\mathrm{Q}^+(7,q)$ is known as the *triality quadric*, and we use its special properties to prove our results. Consider the two systems $\mathcal{D}_1$ and $\mathcal{D}_2$ of generators. Then

$$|\mathcal{D}_1| = |\mathcal{D}_2| = (q+1)(q^2+1)(q^3+1) = \frac{(q^3+1)(q^4-1)}{q-1},$$

which is equal to the number of points on the quadric $\mathrm{Q}^+(7,q)$. Define the following geometry Ω of rank 4. The 1-points of Ω are the points of $\mathrm{Q}^+(7,q)$; the 2-points are the elements of $\mathcal{D}_1$; the 3-points are the members of $\mathcal{D}_2$; the lines are the lines of $\mathrm{Q}^+(7,q)$; incidence is containment, reverse containment, or meeting in a plane of $\mathrm{Q}^+(7,q)$. Then we have a rank 4 geometry (see figure 4.4 for its diagram), such that interchanging the i-points and j-points, $i,j \in \{1,2,3\}$ does not change the isomorphism type of the geometry. In fact, there always exists an isomorphism of Ω of order 3 mapping the i-points on the $(i+1)$-points, (modulo 3). Such a map is called a *triality* and was introduced by Tits [113]. Let τ be a triality; then $\mathrm{Q}^+(7,q)^\tau \cong \mathrm{Q}^+(7,q)$ and τ transforms an ovoid $\mathcal{O}$ of $\mathrm{Q}^+(7,q)$ into a spread Σ of the quadric $\mathrm{Q}^+(7,q)^\tau$. For further references on triality see [64].

4.4.3 The point graph of $\mathrm{PQ}^+(7,q), q = 2,3$

The following theorem was implicitly proved by De Clerck, Gevaert and Thas.

Theorem 4.5 ([31]) *The point graph of the partial geometry* $\mathrm{PQ}^+(7,2)$ *is faithfully geometric.*

Remark

When $q=2$, then adjacency in the point graph of $\mathrm{PQ}^+(7,2)$ translates into orthogonality with respect to the quadric $\mathrm{Q}^+(7,2)$. This implies that for a maximal clique $C=\{x_0,\ldots,x_7\}$ of the point graph of $\mathrm{PQ}^+(7,2)$, C is contained in $\cap_{i=0}^7 x_i^*$, that is the intersection of eight 6-dimensional subspaces of $\mathrm{PG}(7,2)$. This is only possible when $\cap_{i=0}^7 x_i^*$ is a 3-dimensional space Π intersecting the quadric in a plane π and then $C=\Pi\setminus\pi$ (see [31]). When $q=3$, then adjacency in the point graph of $\mathrm{PQ}^+(7,3)$ translates into non-orthogonality with respect to the quadric $\mathrm{Q}^+(7,3)$ instead of orthogonality. This implies that it is not obvious anymore that maximal cliques in the graph correspond to 3-dimensional affine spaces. And so we cannot generalise theorem 4.5 but we do obtain a characterisation (see theorem 4.7).

Lemma 4.6 *Let $\mathcal{S}$ be a partial geometry with point graph Γ, the point graph of $\mathrm{PQ}^+(7,3)$. Suppose that an $\mathcal{S}$-line L containing two $\mathcal{S}$-points contains all the points of the affine line connecting them. Then two $\mathcal{S}$-lines L_1 and L_2 can be identified with the planes π_{L_1} and π_{L_2} of $\mathrm{Q}^+(7,3)$ such that L_1 and L_2 are intersecting lines in $\mathcal{S}$ if and only if $\pi_{L_1}\cap\pi^*_{L_2}=\emptyset$ (or equivalently $\pi^*_{L_1}\cap\pi_{L_2}=\emptyset$).*

Proof. Suppose that an $\mathcal{S}$-line L containing two $\mathcal{S}$-points x and y contains the points of the affine line $\langle x,y\rangle\setminus\mathrm{Q}^+(7,3)$ connecting them. Obviously L is not equal to $\langle x,y\rangle\setminus\mathrm{Q}^+(7,3)$. Consider a point z of $L\setminus\langle x,y\rangle$. Since an $\mathcal{S}$-line containing two $\mathcal{S}$-points contains the points of the affine line connecting them, the affine plane $\langle x,y,z\rangle\setminus\mathrm{Q}^+(7,3)\subset L$. Since $|L|=27$, we can choose a point u of $L\setminus\langle x,y,z\rangle$. And again we obtain that the 3-dimensional affine space $\langle x,y,z,u\rangle\setminus\mathrm{Q}^+(7,3)\subset L$. Since $|\langle x,y,z,u\rangle\setminus\mathrm{Q}^+(7,3)|=|L|=27$, these sets are equal. Let π_L denote the plane $\langle x,y,z,u\rangle\cap\mathrm{Q}^+(7,3)$. Conversely, if π is a plane of $\mathrm{Q}^+(7,3)$ of type π_L then we denote the corresponding $\mathcal{S}$-line with L_π. Let L_1 and L_2 be two $\mathcal{S}$-lines intersecting in the $\mathcal{S}$-point x. By the above, the $\mathcal{S}$-line L_i is an affine 3- dimensional space having a plane π_{L_i} on $\mathrm{Q}^+(7,3)$ at infinity, $i=1,2$. We have that $\pi_{L_1}\cap\pi_{L_2}=\emptyset$ otherwise L_1 and L_2 intersect in more than one point, a contradiction. Let $(\pi_{L_i})^*\cap\mathrm{Q}^+(7,3)=\sigma_i^1\cup\sigma_i^2$, with $\sigma_i^j\in\mathcal{D}_j$, $i,j=1,2$. Let $k\in\{1,2\}$ and $l\in\{1,2\}\setminus\{k\}$. Suppose that $p\in\sigma_k^1\cap\pi_{L_l}$. Since $p\in x^*$ and $\pi_{L_k}\in x^*$, this implies that the 3-dimensional space $\langle p,\pi_{L_k}\rangle\subset x^*\cap\mathrm{Q}^+(7,3)=\mathrm{Q}(6,3)$, a contradiction. Similarly we prove that $\sigma_k^2\cap\pi_{L_l}=\emptyset$. Hence $\pi^*_{L_k}\cap\pi_{L_l}=\emptyset$.
Conversely, suppose that $\pi^*_{L_k}\cap\pi_{L_l}=\emptyset$. Since the dimension of $\langle\pi_{L_1},\pi^*_{L_2}\rangle$ equals the dimension of $\langle L_1^*,L_2^*\rangle$ which equals 7, we have that $\pi^*_{L_1}\cap\pi^*_{L_2}$ is a projective line intersecting each of the four 3-dimensional spaces through π_{L_i} in $\pi^*_{L_i}, i=1,2$, in a point. Since $\pi^*_{L_i}\cap\mathcal{P}=L_i, i=1,2$, with $\mathcal{P}$ the point set of $\mathcal{S}$, this proves the result. □

Theorem 4.7 *Let $\mathcal{S}$ be a partial geometry with the same point graph Γ as $\mathrm{PQ}^+(7,3)$ and suppose that an $\mathcal{S}$-line L of containing two $\mathcal{S}$-points contains the points of the affine line connecting them. Then $\mathcal{S}$ is isomorphic to $\mathrm{PQ}^+(7,3)$.*

By lemma 4.6, an $\mathcal{S}$-line L can be identified with a 3-dimensional affine space intersecting the quadric in a plane π_L.
Recall that for two different concurrent $\mathcal{S}$-lines M and N there holds $\pi_M \cap \pi_N = \emptyset$. Consider an $\mathcal{S}$-line L and let $\pi_L^* \cap \mathrm{Q}^+(7,3) = \sigma_1 \cup \sigma_2$, $\sigma_i \in \mathcal{D}_i$, $i = 1,2$. Let y be an $\mathcal{S}$-point which is not contained in L. Then $y^* \cap \pi_L$ is a projective line Y and $y^* \cap \sigma_i$ is a plane ω_i through Y, $i = 1,2$. Let $L_0, \ldots, L_{17}$ denote the 18 lines through y intersecting L. Then by lemma 4.6 we have $\pi_{L_i} \cap (\omega_1 \cup \omega_2) = \emptyset, i = 0, \ldots 17$, and for the 10 other lines $L_{18}, \ldots, L_{27}$ through y, $\pi_{L_i} \cap (\omega_1 \cup \omega_2) \neq \emptyset, i = 18, \ldots 27$. Let n_i be the number of planes of $\{\pi_{L_i} | i = 18, \ldots, 27\}$ intersecting $\omega_1 \cup \omega_2$ in exactly i points. No π_{L_i} contains a point of both $\omega_1 \setminus Y$ and $\omega_2 \setminus Y$ since a line connecting two such points must be a secant line of $\mathrm{Q}^+(7,3)$, $i = 18, \ldots, 27$. And so $\pi_{L_i} \cap (\omega_1 \cup \omega_2)$ is a point, a projective line or a plane $i = 18, \ldots, 27$. Therefore only n_1, n_4 and n_{13} are non-zero and $n_1 + n_4 + n_{13} = 10$. By lemma 4.6 the planes π_{L_i}, $i = 18, \ldots, 27$, are disjoint hence $n_4 \leq 2$ and $n_{13} \leq 1$. Since the planes $\pi_{L_i}, i = 0, \ldots, 27$, partition $y^* \cap \mathrm{Q}^+(7,3)$, the planes $\pi_{L_i}, i = 18, \ldots, 27$, partition $\omega_1 \cup \omega_2$. Therefore $n_1 + 4n_4 + 13n_{13} = 22$. If $n_{13} = 0$ then $n_1 = 6$ and $n_4 = 4$, a contradiction. Hence $n_{13} = 1$, $n_1 = 9$ and $n_4 = 0$. This implies that one plane, say $\pi_{L_{18}}$, coincides with, say ω_1, and the remaining 9 intersect ω_2 in a point. Hence $\pi_{L_{18}}$ intersects Y and so π_L in a line while $\pi_{L_0}, \ldots, \pi_{L_{17}}$ and $\pi_{L_{19}}, \ldots, \pi_{L_{27}}$ are disjoint from Y and so from π_L.
Let A denote the set of planes defined by the $\mathcal{S}$-lines. Then the above implies that two different planes of A are either disjoint or they intersect in a line. We will now show that the elements of A can be divided into 28 sets A_i of size 40, such that the A_i, $i = 0, \ldots, 27$, induce an orthogonal spread of $\mathrm{Q}^+(7,3)$.
Consider the following equivalence relation $\sim$ on A. For $\pi_1, \pi_2 \in A$, $\pi_1 \sim \pi_2$ if and only if $\pi_1 = \pi_2$ or $\pi_1 \cap \pi_2$ is a projective line. Consider an element π_{L_i} of A and let A_i denote the equivalence class of $\sim$ defined by π_{L_i}. Let σ_i^j denote the element of $\mathcal{D}_j$ through π_{L_i}, $j = 1,2$. ¿From the above it follows that for an $\mathcal{S}$-point y of $\mathcal{S}$ not in the line L_i of $\mathcal{S}$, there is exactly one $\mathcal{S}$-line L_y such that $\pi_{L_y} \in A_i$ and π_{L_y} is contained in either σ_i^1 or σ_i^2. We count in two ways the ordered pairs (y, π_{L_y}) with y and π_{L_i} as above. Then $1053 = (|A_i| - 1)27$, hence $|A_i| = 40$. Let $\pi_j \in A_i$ and $\pi_j \neq \pi_{L_i}$ such that $\pi_j \in \sigma_i^j$ $j = 1,2$. Then $\pi_1 \cap \pi_{L_i} = \pi_2 \cap \pi_{L_i}$. Since A_i contains 40 elements, there are planes $\pi_3 \in A_i$ such that $\pi_3 \neq \pi_{L_i}$ and $\pi_j \cap \pi_{L_i} \neq \pi_3 \cap \pi_{L_i}$, hence π_3 intersects π_j, $j = 1,2$, in a point, a contradiction. Therefore all elements of A_i are contained in either σ_i^1 or σ_i^2, say σ_i^1. Let $\pi_{L_k} \in A \setminus A_i$. Since each element of A_i is disjoint from each element of A_k, $i, k = 0, \ldots, 27$, $i \neq k$, this set A_k consists of all planes of σ_k^1 with $\sigma_k^1 \cap \sigma_i^1 = \emptyset$. Hence $\Sigma = \{A_0, \ldots, A_{27}\}$ is an orthogonal spread of $\mathrm{Q}^+(7,3)$. Since all spreads of $\mathrm{Q}^+(7,3)$ are isomorphic [83], $\mathcal{S}$ is isomorphic to $\mathrm{PQ}^+(7,3)$. □

4.4.4 The block graph of $\mathrm{PQ}^+(7,q)$, $q = 2,\ 3$

Kantor [70] proved that the block graph Γ' of $\mathrm{PQ}^+(7,q)$, $q = 2$ or 3, is the graph $\Gamma^c(\mathrm{Q}^+(7,q))$ with vertices the points on the hyperbolic quadric $\mathrm{Q}^+(7,q)$, two vertices being adjacent if and only if they are on a secant of the quadric. He also proved that if $n \neq 2$ then the block graph of the partial geometry $\mathrm{PQ}^+(4n-1,q)$, $q=2$ or 3, is not isomorphic to the graph $\Gamma^c(\mathrm{Q}^+(4n-1,q))$. Note that the graph $\Gamma^c(\mathrm{Q}^+(2m-1,q))$ is a pseudo-geometric

$$(q^{m-1}, q^{m-1}-1, (q-1)q^{m-2})\text{–graph},$$

for any q. The graph $\Gamma^c(\mathrm{Q}^+(3,q))$, is the complement of the $(q+1)\times(q+1)$–grid, hence $\Gamma^c(\mathrm{Q}^+(3,q))$ is geometric if and only if there exists a projective plane of order $q+1$. It is not known whether $\Gamma^c(\mathrm{Q}^+(5,q))$, $q \geq 4$, is geometric. The graph $\Gamma^c(\mathrm{Q}^+(5,2))$ is a pseudo-geometric $(4,3,2)$-graph but a pg$(4,3,2)$ does not exist (see for instance [24]). As explained in [82], it can be read off from the computer aided results of M. Hall, Jr. and R. Roth in [55] that $\Gamma^c(\mathrm{Q}^+(5,3))$ is not geometric. As remarked in [82] the graph $\Gamma^c(\mathrm{Q}^+(2m-1,q))$ with $m \geq 5$ is not geometric for $q=2$, but the question is still open for $q>2$. Hence, the fact that the graph $\Gamma^c(\mathrm{Q}^+(7,q))$ is geometric for $q=2,\ 3$, is quite remarkable indeed.

Theorem 4.8 *The block graph of* $\mathrm{PQ}^+(7,q), q=2$ *or* 3, *is faithfully geometric.*

Proof. Unless stated otherwise, let q be 2 or 3. Let $\mathcal{S}$ be a partial geometry pg$(q^3, q^3-1, (q-1)q^2)$ with point graph Γ', the block graph of $\mathrm{PQ}^+(7,q)$. Recall that Kantor [70] proved that Γ' is the graph $\Gamma^c(\mathrm{Q}^+(7,q))$. Hence, maximal cliques of Γ' are certain ovoids of $\mathrm{Q}^+(7,q)$ (that is sets of q^3+1 mutually non-orthogonal singular points).
We will now construct an identification map Υ that maps the points and lines of the partial geometry $\mathcal{S}'$ with point graph Γ' on a partial geometry $(\mathcal{S}')^\Upsilon$ with point graph $(\Gamma')^\Upsilon = \Gamma$, the point graph of $\mathrm{PQ}^+(7,q)$.
Let l_0 be a vertex of the graph, hence a point on the quadric $\mathrm{Q}^+(7,q)$. Let $\{\mathcal{O}^{(0)}, \mathcal{O}^{(1)}, \ldots, \mathcal{O}^{(q^3-1)}\}$ be the set of $\mathcal{S}'$-lines incident with l_0. Hence $\mathcal{O}^{(i)}$, $i=0,\ldots,q^3-1$, is an ovoid of the quadric $\mathrm{Q}^+(7,q)$. Let $\Sigma^{(0)}$ be an orthogonal spread consisting of elements of $\mathcal{D}_1$ and let τ be the triality such that $(\mathcal{O}^{(0)})^\tau = \Sigma^{(0)}$. Note that all ovoids (hence all spreads) of $\mathrm{Q}^+(7,q)$ are isomorphic (see [69, 83]), from which follows that up to isomorphism, τ is uniquely defined. Define the orthogonal spread $\Sigma^{(i)} = (\mathcal{O}^{(i)})^\tau$, $i=1,\ldots,q^3-1$. Let the set of points on $\mathcal{O}^{(0)}$ be $\{l_0^{(0)} = l_0, l_1^{(0)}, \ldots, l_{q^3}^{(0)}\}$. Let $(l_i^{(0)})^\tau = \sigma_i^{(0)}, i=0,\ldots,q^3$, hence

$$\Sigma^{(0)} = \{\sigma_0^{(0)}, \ldots, \sigma_{q^3}^{(0)}\}.$$

The set

$$(\mathcal{O}^{(0)})^{\tau^2} = (\Sigma^{(0)})^\tau = \Sigma'^{(0)} = \{\sigma_0'^{(0)}, \ldots, \sigma_{q^3}'^{(0)}\}$$

is a spread consisting of elements of $\mathcal{D}_2$ and such that $\sigma_i^{(0)} \cap \sigma_i'^{(0)}$ is a plane $\pi_i^{(0)}$, while $\sigma_i^{(0)} \cap \sigma_j'^{(0)}$, $i,j=0,\ldots,q^3$, $i \neq j$, is a point. Let Υ be the mapping (uniquely) defined by $(l_i^{(0)})^\Upsilon = \pi_i^{(0)}, i=0,\ldots,q^3$.

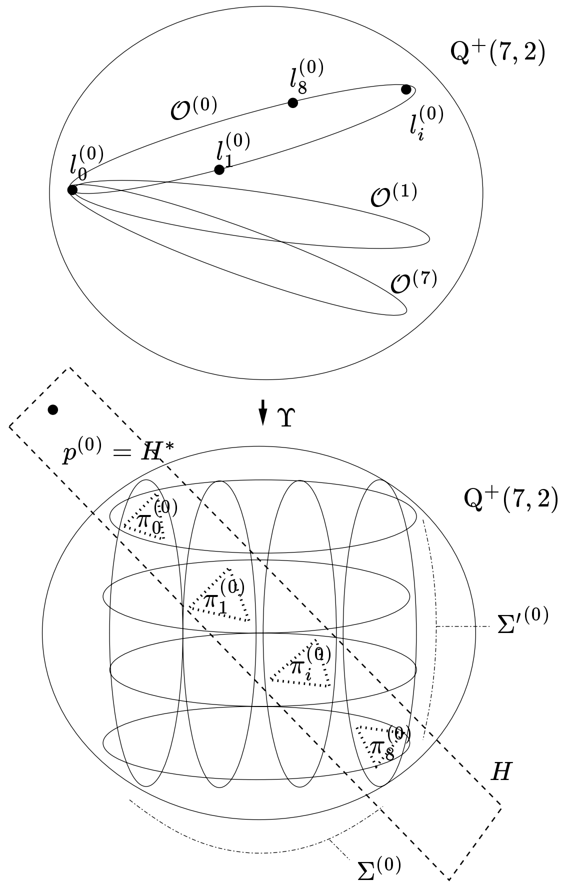

Figure 4.5: The mapping Υ

Consider a hyperplane of $\mathrm{PG}(7,q)$ intersecting $\mathrm{Q}^+(7,q)$ in a quadric $\mathrm{Q}(6,q)$. Let $A = \{\pi_0, \ldots, \pi_{q^3}\}$ denote a spread of planes of $\mathrm{Q}(6,q)$. Consider the polar space π_i^* of π_i $(i = 0, \ldots, q^3)$ in $\mathrm{PG}(7,q)$ with respect to $\mathrm{Q}^+(7,q)$. Then $\pi_i^* \cap \mathrm{Q}^+(7,q) = \sigma_i^1 \cup \sigma_i^2$ where σ_i^j is a generator of $\mathrm{Q}^+(7,q)$ in $\mathcal{D}_j$, $i = 0, \ldots, q^3$, $j = 1, 2$. Moreover $\{\sigma_0^j, \ldots, \sigma_{q^3}^j\}$ is an orthogonal spread of $\mathrm{Q}^+(7,q)$, $j \in \{1,2\}$. Since the triality τ is uniquely defined, we have $(\sigma_i^1)^\tau = \sigma_i^2$, $i = 0, \ldots, q^3$. Since all spreads of $\mathrm{Q}^+(7,q)$ are isomorphic, starting with a spread Σ of $\mathrm{Q}^+(7,q)$, we can obtain a spread of a quadric $\mathrm{Q}(6,q) \subset \mathrm{Q}(7,q)$ by applying triality on the elements of Σ and then considering its intersections. Let $H(\Sigma)$ denote the hyperplane of $\mathrm{PG}(7,q)$ containing the quadric $\mathrm{Q}(6,q)$ corresponding with Σ.

Put $p^{(0)} = H(\Sigma^{(0)})^*$, then $(p^{(0)})^*$ intersects each element $\sigma_i^{(0)}$ (and $\sigma_i'^{(0)}$), $i = 0, \ldots, q^3$, of the spreads $\Sigma^{(0)}$ (and $\Sigma'^{(0)}$) in the plane $\pi_i^{(0)}$. Hence $p^{(0)}$

is uniquely defined by the set

$$\{\pi_i^{(0)} \| i = 0, \ldots, q^3\} = \{(l_i^{(0)})^\Upsilon \| i = 0, \ldots, q^3\}$$

and we may put $p^{(0)} = \left(\mathcal{O}^{(0)}\right)^\Upsilon$ (see figure 4.5 for $q = 2$).
Let $\mathcal{O}^{(j)}$, $j \neq 0$, be another $\mathcal{S}'$-line through l_0. Then the spread $\Sigma^{(j)} = (\mathcal{O}^{(j)})^\tau$ intersects $\Sigma^{(0)}$ in $\sigma_0^{(0)}$, while $\Sigma'^{(j)} = (\mathcal{O}^{(j)})^{\tau^2}$ intersects $\Sigma'^{(0)}$ in $\sigma_0'^{(0)}$. The same construction as above yields a set $\{\pi_i^{(j)} \mid i = 0, \ldots, q^3\}$ with $\pi_0^{(j)} = \pi_0^{(0)}$ and this set uniquely defines a point $p^{(j)}$. And so we can put $p^{(j)} = \left(\mathcal{O}^{(j)}\right)^\Upsilon$.
For $q = 2$ it is clear that this point $p^{(j)}$ is a point of $L_0^{(0)} = L_{\pi_0^{(0)}}$ but different from $p^{(0)}$. When $q = 3$ then $(p^{(0)})^* \cap (p^{(j)})^* \cap \mathrm{Q}^+(7,3)$, $j = 1, \ldots, 26$, contains the plane $\pi^{(0)}$ and so this intersection is degenerate. This implies that $p^{(0)}$ and $p^{(j)}$ are contained in a tangent line of the quadric $\mathrm{Q}^+(7,3)$. Therefore they are contained in the same set $E_m^+(7,3)$, $m \in \{1,2\}$. In both cases it now follows that the q^3 ovoids $\mathcal{O}^{(j)}$, $j = 0, \ldots, q^3 - 1$, through l_0, are mapped by Υ on the q^3 points of the affine space $L_0^{(0)} = L_0$. And for $q = 3$ we have $L_0 \subset E_m^+(7,3)$.
Let $l_i^{(0)}$ be any other point on $\mathcal{O}^{(0)}$; then by the same reasoning, the 27 lines of $\mathcal{S}'$ through $l_i^{(0)}$ are mapped by Υ on the 27 points of $L_{\pi_i^{(0)}} = L_i^{(0)} \subset E_m^+(7,3)$.
Using connectivity of the geometry we have again $L_i^{(0)} \subset E_m^+(7,3)$ for $q = 3$.
The identification map Υ maps the points and lines of the partial geometry $\mathcal{S}'$ with point graph Γ' on a geometry $\left(\mathcal{S}'\right)^\Upsilon$ with point graph $\left(\Gamma'\right)^\Upsilon = \Gamma$, such that the lines of $\left(\mathcal{S}'\right)^\Upsilon$ are 3-dimensional affine spaces with a plane of $\mathrm{Q}^+(7,q)$ at infinity. Since $\left(\mathcal{S}'\right)^\Upsilon$ is a partial linear space with a pseudo-geometric point graph, $\left(\mathcal{S}'\right)^\Upsilon$ is a partial geometry. By theorems 4.5 and 4.7 the geometry $\left(\mathcal{S}'\right)^\Upsilon$ is isomorphic to $\mathrm{PQ}^+(7,q)$. And so $\mathcal{S}'$ is the dual of $\mathrm{PQ}^+(7,q)$. This implies that Γ' is faithfully geometric. □

4.5 The point (block) graph of $\mathcal{S}_i^{(\prime)}, i = 1, \ldots, 4$

Mathon and Street remarked in [81] that the point graphs Γ_i of the geometries $\mathcal{S}_i, i = 1,2,3,4$, were isomorphic graphs and their block graphs all are different. In this section we prove these results geometrically. Moreover we prove that the point graphs of $\mathcal{S}_1'$, $\mathcal{S}_2'$, $\mathcal{S}_3'$ (and so $\mathcal{S}_3''$) are isomorphic as well.

4.5.1 The identification map Υ

Lemma 4.9 *Let q be 2 or 3. There exists a natural bijection Υ between some subspaces of a $\mathrm{Q}_1^+(7,q)$ and some subspaces of a quadric $\mathrm{Q}_2^+(7,q)$ in the following way. Let X_k^l denote a subspace of $\mathrm{Q}_k^+(7,q)$ of dimension l $(l = 0,1,2,3)$, and X_k^4 an orthogonal spread of $\mathrm{Q}_k^+(7,q)$, $k = 1,2$. Then the ordered 5-tuple $(X_i^0, X_i^1, X_i^2, X_i^3, X_i^4)$ with $X_i^0 \in X_i^1 \subset X_i^2 \subset X_i^3 \in X_i^4$, is mapped onto the ordered 5-tuple $(X_j^2, X_j^1, X_j^0, X_j^3, X_j^4)$ with $X_j^0 \in X_j^1 \subset X_j^2 \subset X_j^3 \in X_j^4$, $i, j \in \{1,2\}, i \neq j$.*

Proof. Let q be 2 or 3. In the proof of theorem 4.8, we introduced the mapping Υ which turns out to be an anti-isomorphism between $\mathrm{PQ}^+(7,q)^D$ and $\mathrm{PQ}^+(7,q)$ and hence defines a bijection between the vertices of the block graph Γ' of $\mathrm{PQ}^+(7,q)$ and those maximal cliques of the point graph Γ of $\mathrm{PQ}^+(7,q)$ that define lines of $\mathrm{PQ}^+(7,q)$. As $\Gamma' = \Gamma^c(\mathrm{Q}^+(7,q))$, we can regard Υ as a bijection, which we also denote by Υ, between the points of a hyperbolic quadric $\mathrm{Q}_1^+(7,q)$ and those planes of a hyperbolic quadric $\mathrm{Q}_2^+(7,q)$ contained in the elements σ_i ($i = 0,\ldots,q^3$) of the orthogonal spread Σ of $\mathrm{Q}_2^+(7,q)$.
The inverse mapping Υ^{-1} maps the planes of one element σ_i of the orthogonal spread Σ (defining a pg-spread of $\mathrm{PQ}^+(7,q)$) to a set of points of $\mathrm{PQ}^+(7,q)^D$ defining a pg-ovoid in this geometry and hence, as $\Gamma' = \Gamma^c(\mathrm{Q}_1^+(7,q))$, defining a generator of the quadric, which we call $(\sigma_i)^{\Upsilon^{-1}}$. Moreover disjoint generators are mapped onto disjoint generators. Hence the image of the set of all planes of the elements of Σ under Υ^{-1} is the set of points of an orthogonal spread of the quadric $\mathrm{Q}_1^+(7,q)$, which we call $\Sigma^{\Upsilon^{-1}}$.
We can regard Υ^{-1} as a mapping of the points on $\mathrm{Q}_2^+(7,q)$ onto the planes of elements of the orthogonal spread $\Sigma^{\Upsilon^{-1}}$ by identifying a point $p \in \sigma_i$ on $\mathrm{Q}_2^+(7,q)$ with the planes of σ_i through p, hence p is mapped onto a plane $p^{\Upsilon^{-1}}$ of $(\sigma_i)^{\Upsilon^{-1}}$.
Let τ_1 be an element of an orthogonal spread of $\mathrm{Q}_1^+(7,q)$ and let τ_2 denote its image under Υ (defined by the images of the points of τ_1). Let X_1 be a projective line contained in τ_1. Let $p_0,\ldots,p_q$ denote the points of X_1 and $\pi_0,\ldots,\pi_q$ the planes of τ_1 through X_1. We can regard Υ as a mapping of X_1 onto a projective line X_2 of $\mathrm{Q}_2^+(7,q)$ which is the intersection of the planes $p_0^\Upsilon,\ldots,p_q^\Upsilon$ of $\mathrm{Q}_2^+(7,q)$. Moreover, X_2 is the line of $\mathrm{Q}^+(7,q)$ containing the points $\pi_0^\Upsilon,\ldots,\pi_q^\Upsilon$ of $\mathrm{Q}_2^+(7,q)$. The rest of the lemma now follows immediately. □

Remark

Let q be 2 or 3. Consider $i,j \in \{1,2\}, i \neq j$. When X is a line or a plane of $\mathrm{Q}_i^+(7,q)$ that is not contained in an element of the orthogonal spread Σ_i of $\mathrm{Q}_i^+(7,q)$, $i = 1,2$, then there does not exist a natural bijection anymore between X and an element of a class of subspaces of $\mathrm{Q}_j^+(7,q)$. And then we define the image of such a projective line or plane in $\mathrm{Q}_i^+(7,q)$ under the above bijection using its point set.
Let π_i be a plane of $\mathrm{Q}_i^+(7,q)$ ($i \in \{1,2\}$) intersecting an element σ_i of Σ_i in a projective line X_i of σ_i and q^2 other elements of Σ_i in a point x_i^k ($1 \leq k \leq q^2$). Let the ordered 4-tuple $(x_j^k, X_j, \sigma_j, \Sigma_j)$, $j \in \{1,2\} \setminus \{i\}$, denote the image of $(x_i^k, X_i, \sigma_i, \Sigma_i)$ in $\mathrm{Q}_j^+(7,q)$ ($1 \leq k \leq q^2$). Then the image of π_i under the bijection is a set of planes having $q+1$ elements in σ_j through the projective line X_j, and q^2 elements in the intersection of the 5-dimensional space $\langle \sigma_j, x_j^k \rangle$ with $\mathrm{Q}_j^+(7,q)$, $k \in \{1,\ldots,q^2\}$.
In theorem 4.10 we use this extended definition of the bijection Υ mapping a structure associated with the quadric $\mathrm{Q}_1^+(7,q)$ onto a structure associated with the quadric $\mathrm{Q}_2^+(7,q)$.

4.5.2 The point graph of $\mathcal{S}_i^{(')}, i = 1, 2, 3$

First let us recall that the partial geometry $\mathcal{S}_i$ is the dual of the spread derived partial geometry $\mathrm{PQ}^+_{\Phi_i}(7, 2)$. Similarly $\mathcal{S}'_i$ is the dual of $\mathrm{PQ}^+_{\Phi_i}(7, 3)$, $i = 1, 2, 3$.

Theorem 4.10 *The point graph of the partial geometry $\mathcal{S}_i^{(')}$, $i = 1, 2, 3$, is isomorphic to the complement of the graph on a quadric with a hole.*

Proof. Let $q = 2$ or 3 and consider the orthogonal spread Σ of the quadric $\mathrm{Q}_2^+(7, q)$ used to construct the partial geometry $\mathrm{PQ}^+(7, q)$. Let Γ_i denote the point graph of $\mathcal{S}_i$, $i = 1, 2, 3$, when $q = 2$ and let it denote the point graph of $\mathcal{S}'_i$, $i = 1, 2, 3$, when $q = 3$. Consider the graph $\Gamma' = \Gamma^c(\mathrm{Q}_1^+(7, q))$ being the block graph of $\mathrm{PQ}^+(7, q)$. A pg-spread Φ of the partial geometry $\mathrm{PQ}^+(7, q)$ yields in the graph Γ' a maximal co-clique of $q^3 + q^2 + q + 1$ vertices and so this co-clique defines a generator $\mathcal{A}_i$ of $\mathrm{Q}_1^+(7, q)$. As by construction we delete the elements of a pg-spread Φ from the line set of $\mathrm{PQ}^+(7, q)$, it immediately follows that the vertex set of Γ_i is the set of points on the quadric $\mathrm{Q}_1^+(7, q)$ not contained in the "hole" $\mathcal{A}_i$. Referring to the constructions of the replaceable spreads Φ_i $(i = 1, 2, 3)$ in [25] one easily checks that $\mathcal{A}_1$ is an element $(\sigma)^{\Upsilon^{-1}}$ of the orthogonal spread $(\Sigma)^{\Upsilon^{-1}}$, $\mathcal{A}_2$ is the unique element of $\mathcal{D}_2$ containing the plane $z^{\Upsilon^{-1}}$ and $\mathcal{A}_3 = (\sigma')^{\Upsilon^{-1}}$. Note that contrary to $\mathcal{A}_2$ the generators $\mathcal{A}_1$ and $\mathcal{A}_3$ are elements of $\mathcal{D}_1$.
Let p be an "old point" of $\mathrm{PQ}^+_{\Phi_i}(7, q)$ (that is a point of $\mathrm{PQ}^+(7, q)$), then the lines of $\mathrm{PQ}^+_{\Phi_i}(7, q)$ through p are mapped by Υ^{-1} on the q^3 points of an ovoid of $\mathrm{Q}_1^+(7, q)$ minus its intersection point with $\mathcal{A}_i$. Let p' be a "new point", that is p' is of the form $\mathcal{L}(L)$ where L is a line of $\mathrm{PQ}^+(7, q)$ not contained in the pg-spread Φ_i. Since $\mathcal{L}(L) \cup (\Phi_i \setminus (\Phi_i)_L)$ is a pg-spread of $\mathrm{PQ}^+(7, q)$, its image under Υ^{-1} must be a co-clique of $\Gamma^c(\mathrm{Q}_1^+(7, q))$ and so a generator G of $\mathrm{Q}_1^+(7, q)$ intersecting $\mathcal{A}_i$ in the $q^2 + q + 1$ elements of the image of the elements of $(\Phi_i \setminus (\Phi_i)_L)$ under Υ^{-1}. Hence G intersects $\mathcal{A}_i$ in a plane. This implies that the lines of $\mathrm{PQ}^+_{\Phi_i}(7, q)$ through the "new point" p' are mapped by Υ^{-1} on the q^3 points of an affine space.
If we regard p' as a collection of planes of $\mathrm{Q}_2^+(7, q)$ then we define $(p')^{\Upsilon^{-1}}$ as the collection of images of elements of p' under Υ^{-1}. Note that p' can be identified with a plane $\pi(p')$ of $\mathcal{A}_i$ such that $(p')^{\Upsilon^{-1}} = \mathcal{A}'_i(\pi(p')) \setminus \pi(p')$ where $\mathcal{A}'_i(\pi(p'))$ is the unique element of the opposite class $\mathcal{D}_l$ of generators (hence $l = 2$ if $i = 1$ or 3 and $l = 1$ if $i = 2$) containing $\pi(p')$.
Hence vertices in Γ_i are adjacent whenever they are non-collinear in $\mathrm{Q}_1^+(7, q)$ or they are contained in a line of $\mathrm{Q}_1^+(7, q)$ intersecting $\mathcal{A}_i$ in a point. This shows that the three graphs Γ_i $(i = 1, 2, 3)$ are isomorphic to the complement of the graph on a quadric with a hole $\mathcal{A}_i$ (see [8] for a description of this graph). □

4.5.3 The point graph of $\mathcal{S}_4(n)$

In theorem 4.10 we proved that the graph of the quadric with a hole in $\mathrm{PG}(7, 2)$ and $\mathrm{PG}(7, 3)$ is geometric. We now prove this result in $\mathrm{PG}(4n - 1, 2)$, but note that we consider a different geometry.

Theorem 4.11 *The point graph of the partial geometry $\mathcal{S}_4(n)$ is isomorphic to the complement of the graph on a quadric with a hole.*

Proof. Consider the easy description of $\mathcal{S}_4(n)$ of section 4.3.2 and its notations. Let Σ be the orthogonal spread of the quadric $\mathrm{Q}^+(4n-1,2)$ and let $\sigma_0 \in \Sigma$. Let π_0 be a hyperplane of σ_0. Put $L_0 = L_{\pi_0}$. Then the complement of the point graph Γ_4 of $\mathcal{S}_4(n)$ has the following description. Vertices are points of

$$V = (\mathrm{PG}(4n-1,2) \setminus (\mathrm{Q}^+(4n-1,2) \cup L_0)) \cup (\sigma_0 \setminus \pi_0);$$

two vertices x and y are adjacent if and only if the projective line $\langle x, y\rangle$ is completely contained in the set V (that is if and only if either $\langle x, y\rangle$ is an exterior line of $\mathrm{Q}^+(4n-1,2)$ not intersecting L_0, or $\langle x, y\rangle$ is a tangent line of the quadric with tangent point contained in $\sigma_0 \setminus \pi_0$.
For any hyperplane π of σ_0, let G_π denote the second generator of $\mathrm{Q}^+(4n-1,2)$ through π. Now map the points of $G_\pi \setminus \pi$ onto the points of the line L_π of $\mathcal{S}_0(n)$ by a bijection η_π in such a way that for $x, y \in G_\pi$ we have

$$\langle x^{\eta_\pi}, y^{\eta_\pi}\rangle \cap \sigma_0 = \langle x, y\rangle \cap \sigma_0.$$

Since the set $\mathcal{G} = \{G_\pi \setminus \pi \| \pi \text{ a hyperplane of } \sigma_0\}$ defines a partition of the points of $\mathrm{Q}^+(4n-1,2) \setminus \sigma_0$, and the set $S = \{L_\pi \| \pi \text{ a hyperplane of } \sigma_0\}$ defines a pg-spread of $\mathcal{S}_0(n)$, there exists a bijection η (defined by the η_π) between the points of $\mathrm{Q}^+(4n-1,2) \setminus \sigma_0$ and the points of $\mathrm{PG}(4n-1,2) \setminus \mathrm{Q}^+(4n-1,2)$.
Consider the graph Γ on the points of $\mathrm{Q}^+(4n-1,2) \setminus G_{\pi_0}$; two vertices x_0 and x_1 are adjacent if and only if the line $\langle x_0, x_1\rangle$ is contained in $\mathrm{Q}^+(4n-1,2) \setminus G_{\pi_0}$. And so Γ is the graph on the quadric $\mathrm{Q}^+(4n-1,2)$ with the hole G_{π_0}. Let x_0 and x_1 be two adjacent vertices of Γ such that $\langle x_0, x_1\rangle$ is disjoint from σ_0. Note that since x_0 and x_1 are adjacent in Γ, $\langle x_0, x_1\rangle$ is disjoint from G_{π_0} as well. Let G_i denote the element of $\mathcal{G}$ containing x_i, $i = 0, 1$. Then $X = G_1 \cap G_2 \cap \sigma_0$ is a projective line intersecting π_0 in a point. Let π'_2, denote the hyperplane of σ_0 through X different from $\pi'_j = G_j \cap \sigma_0, j = 0, 1$. Then $\langle x_0^\eta, x_1^\eta\rangle$ intersects $L_{\pi'_2}$ in a point. And so $\langle x_0^\eta, x_1^\eta\rangle$ is an exterior line of $\mathrm{Q}^+(4n-1,q)$ which is disjoint from L_0.
Let x_0 and x_1 be two adjacent vertices of Γ_4 such that $\langle x_0, x_1\rangle$ intersects $\sigma_0 \setminus \pi_0$. Then it is contained in the same element G of $\mathcal{G}$ and so x_0^η and x_1^η are contained in the same element L of S. And so $\langle x_0^\eta, x_1^\eta\rangle$ is a tangent line of $\mathrm{Q}^+(4n-1,q)$ with tangent point $\langle x_0^\eta, x_1^\eta\rangle \cap \sigma_0 = \langle x_0, x_1\rangle \cap \sigma_0 \notin \pi_0$.
Therefore η is a bijection of the vertices of the graph Γ to the vertices of the graph Γ_4 such that adjacency becomes non-adjacency and conversely. This proves the result. $\square$

4.5.4 The block graph of $\mathcal{S}_i, i = 1, 2, 3$

Theorem 4.12 *The block graph Γ'_i of the partial geometry $\mathcal{S}_i, i = 1, 2, 3,$ is faithfully geometric.*

Proof. Take $i \in \{1,2,3\}$ and let $\mathcal{S}_{\Phi_i}$ be any partial geometry pg(8, 7, 4) with point graph Γ'_i, the block graph of the partial geometry $\mathcal{S}_i$, where Φ_i denotes the replaceable pg-spread of $\mathrm{PQ}^+(7,2)$ used to construct $\mathcal{S}_i$. Let L be a line of $\mathcal{S}_{\Phi_i}$ containing the points $x_0, \ldots, x_8$. Two "old" points of L, that is points of $\mathrm{PQ}^+(7,2)$, are contained in a tangent line of the quadric with tangent point not in an element of Φ_i. Without loss of generality let $x_0, \ldots, x_\eta$ denote the "old" points of L. Since "new" points of $\mathcal{S}_{\Phi_i}$ are never collinear and since $\eta \leq 7$ we obtain that $\eta = 7$ and there is exactly one "new" point in L, namely x_8.
Being contained in a tangent line translates into orthogonality with respect to the quadric $\mathrm{Q}^+(7,2)$, therefore we obtain that $\{x_0, \ldots, x_7\}$ is contained in the subspace $\cap_{i=0}^{7} x_i^*$ of PG(7, 2). Since every two elements of $\{x_0, \ldots, x_7\}$ are contained in a tangent line of the quadric we see similarly as in [31] that $\cap_{i=0}^{7} x_i^*$ is a three dimensional space intersecting the quadric in a hyperplane π_L. Moreover $\pi_L \notin \Phi_i$. Put

$$\Omega = \Phi_i \cup \{\pi_L \| L \text{ is a line of } \mathcal{S}_{\Phi_i}\}.$$

Then we obtain a set of 135 planes of $\mathrm{Q}^+(7,2)$ satisfying the conditions of lemma 3 and 4 of [31]. By theorem 8 of [31] we have that the elements of $\Omega \setminus \Phi_i$ are the hyperplanes of the elements of a unique spread Σ, that are not contained in the set Φ_i. And so maximal cliques yielding lines of $\mathcal{S}_{\Phi_i}$ are uniquely defined.
Since the partial geometries $\mathcal{S}_i^D = \mathrm{PQ}^+_{\Phi_i}(7,2)$ $(i = 1,2,3)$ are all different (see [25]) this implies that the graphs Γ_i $(i = 1,2,3)$ are all different and they are faithfully geometric. □

Remark

1. Maximal cliques of $\mathrm{PQ}^+(7,3)$ are not necessarily contained in a 3-dimensional subspace of PG(7, 3). Therefore we cannot generalize theorem 4.12 for the case $q = 3$.

2. Since there is no easy description for the dual of $\mathcal{S}_4$ in terms of subspaces, we do not investigate its block graph.

4.6 The semipartial geometry SPQ(6, 3)

4.6.1 The block graph of SPQ(6, 3)

The point graph of SPQ(6, 3) yields at least two non-isomorphic semipartial geometries, namely SPQ(6, 3) and SPH(3), the example of Thas (see section 2.3.5) and so it is not faithfully semigeometric.
The dual of a semipartial geometry is again a semipartial geometry if and only if either $s = t$ or it is a partial geometry. Hence the block graph of SPQ(6, 3) cannot be faithfully semigeometric since it is not semigeometric. However we can prove the following theorem.

Theorem 4.13 *Let $\mathcal{S}'$ be a partial linear space with point graph the block graph of* SPQ(6, 3). *Then $\mathcal{S}'$ is isomorphic with the dual of* SPQ(6, 3).

Proof. Recall that Kantor [70] proved that the block graph Γ' of $\mathrm{PQ}^+(7,3)$ is the non-collinearity graph of the quadric $\mathrm{Q}^+(7,3)$, and that lines of $\mathrm{PQ}^+(7,3)^D$ are ovoids of $\mathrm{Q}^+(7,3)$. If we consider $\mathrm{SPQ}(6,3)$ as the geometry which is derived from $\mathrm{PQ}^+(7,3)$ with respect to a point p of $\mathrm{PQ}^+(7,3)$ then $\mathrm{SPQ}(6,3)^D$ is the geometry derived from $\mathrm{PQ}^+(7,3)^D$ with respect to the ovoid $\mathcal{O}$ corresponding with p. And so we obtain the following description of $\mathrm{SPQ}(6,3)^D$. The point set of $\mathrm{SPQ}(6,3)^D$ is the point set of $\mathrm{Q}^+(7,3)\backslash\mathcal{O}$; its line set corresponds with certain ovoids of $\mathrm{Q}^+(7,3)$ not intersecting the fixed ovoid $\mathcal{O}$; incidence is containment. Let $\mathcal{S}'$ be a partial linear space with point graph the block graph of $\mathrm{SPQ}(6,3)$. Let $\mathcal{L}$ denote the line set of $\mathrm{PQ}^+(7,3)^D$; $\mathcal{L}'$ the line set of $\mathcal{S}'$ and $\mathcal{L}''$ the set of ovoids of $\mathcal{L}$ intersecting $\mathcal{O}$, including $\mathcal{O}$. Since $\mathrm{PQ}^+(7,3)^D$ is a partial geometry with $\alpha = 18$, every point r of $\mathrm{Q}^+(7,3) \setminus \mathcal{O}$ is contained in exactly 18 elements $\mathcal{O}_1, \ldots, \mathcal{O}_{18}$ of $\mathcal{L}''$. And the 9 ovoids $\mathcal{O}'_1, \ldots, \mathcal{O}'_9$ of $\mathcal{L}'$ though r cover the same points as the 9 ovoids through r being lines of $\mathrm{SPQ}(6,3)^D$. Hence $\mathcal{O}'_i, i = 1 \ldots, 9$, intersects $\mathcal{O}_j, j = 1, \ldots, 18$, only in r. Moreover the union of the point set of $\cup_{i=1}^{18}\mathcal{O}_i$ with the point set of $\cup_{j=1}^{9}\mathcal{O}'_j$ covers all the points adjacent to r in the graph Γ', the block graph of $\mathrm{PQ}^+(7,3)$. And so the geometry with point set $\mathrm{Q}^+(7,3)$ and line set $\mathcal{L}'\cup\mathcal{L}''$ and natural incidence is a partial linear space $\mathcal{S}'''$ of order $(27,26)$ with pseudo-geometric point graph. By theorem 1.11 this implies that $\mathcal{S}'''$ is a partial geometry $\mathrm{pg}(27,26,18)$. Since $\mathcal{S}'''$ has point graph the one of $\mathrm{PQ}^+(7,3)^D$, theorem 4.8 implies that $\mathcal{S}'''$ is isomorphic to $\mathrm{PQ}^+(7,3)^D$, and so $\mathcal{L}' \cup \mathcal{L}'' = \mathcal{L}$.
Given the ovoid $\mathcal{O}$, we uniquely obtain the point set of $\mathcal{S}'$, namely $\mathrm{Q}^+(7,3) \setminus \mathcal{O}$ and then its line set $\mathcal{L}'$ is also uniquely determined namely $\mathcal{L}' = \mathcal{L}\backslash\mathcal{L}''$. Therefore $\mathcal{S}'$ is isomorphic with the dual of $\mathrm{SPQ}(6,3)$. □

Remarks

1. The proof of theorem 4.13 yields a model of a dual semipartial geometry on the quadric $\mathrm{Q}^+(7,3)$ with a "hole", where the hole is now an ovoid $\mathcal{O}$.
2. Note that all spreads and therefore all ovoids of $\mathrm{Q}^+(7,3)$ are isomorphic (see [83]).

4.6.2 Spread derivation and point derivation

By lemma 3.2, the possible parameter values of a partial geometry with a point derived semipartial geometry are quite restricted. In corollary 3.6 we proved that the partial geometry $\mathrm{PQ}^+(4n-1,3)$ has a point derived semipartial geometry. Other candidates of partial geometries that could have a point derived semipartial geometry, are partial geometries which have the same parameters as $\mathrm{PQ}^+(4n-1,3)$, such as the partial geometries that are spread derived from $\mathrm{PQ}^+(4n-1,3)$. In the next theorem we prove that the answer is negative for $\mathcal{S}'_1(n)$, $\mathcal{S}'_2$, $\mathcal{S}'_3$ (and so as well for $\mathcal{S}''_3$).

Theorem 4.14 *The partial geometries $\mathcal{S}'_1(n)$, $\mathcal{S}'_2$ and $\mathcal{S}'_3$ have no point derived semipartial geometry.*

Proof. Let $\mathcal{S}$ denote one of the three spread derived partial geometries $\mathrm{PQ}^+_{\Phi_1}(4n-1,3) = (\mathcal{S}'_1(n))^D$ or $\mathrm{PQ}^+_{\Phi_i}(7,3) = (\mathcal{S}'_i)^D$, $i = 2,3$, from [25]. And let Φ denote the replaceable spread of $\mathrm{PQ}^+(4n-1,3)$ corresponding with $\mathcal{S}$. Consider the "old" points (that is points of $\mathrm{PQ}^+(4n-1,3)$) and the "new points" (that is the sets $\mathcal{L}_j$) of $\mathcal{S}$ (see [25]). Let Σ denote the orthogonal spread used to construct $\mathrm{PQ}^+(4n-1,3)$. Note that the partial geometry $\mathcal{S}^D$ has the same parameters as $\mathrm{PQ}^+(4n-1,3)$. Suppose that $\mathcal{S}^D$ has a point derived semipartial geometry

$$(\mathcal{S}^D)_p = \mathrm{spg}(3^{2n-2}-1, 3^{2n-1}, 2\cdot 3^{2n-3}, 2\cdot 3^{2n-2}(3^{2n-2}-1)).$$

Then by theorem 3.1, for any three disjoint $\mathcal{S}$-lines π_1, π_2, and π_3, there are $\eta = 4\cdot 3^{2n-2}(3^{2n-2}+1)$ lines of $\mathcal{S}$ intersecting all three of π_1, π_2, and π_3. Suppose that π_1, π_2, and π_3 are hyperplanes of the same element $\sigma \in \Sigma$ but are not contained in the pg-spread Φ. Also suppose that $\pi_1 \cap \pi_2 \cap \pi_3$ is a $(2n-3)$-dimensional subspace of σ, such that π_1, π_2, and π_3 are disjoint in the geometry $\mathcal{S}$. From the description of the replaceable spread Φ it follows that such $\pi_i, i = 1,2,3$, exist.
¿From the proof of lemma 4.6 one can see that two $\mathcal{S}$-lines π_1 and π_2 intersect in an "old" point if and only if $\pi_1 \cap \pi_2^* = \emptyset$ (or equivalently $\pi_1^* \cap \pi_2 = \emptyset$). Therefore the number of $\mathcal{S}$-lines that intersect all three of π_1, π_2, and π_3 in an "old" point is smaller than

$$|\sigma \setminus (\cup_{i=1}^3 \pi_i)| \cdot |\Sigma \setminus \{\sigma\}| = 3^{4n-3}.$$

Since every $\mathcal{S}$-line π_i, $i = 1,2,3$, contains exactly one new $\mathcal{S}$-point x_i, and since there are $3^{2n-1}+1$ lines of $\mathcal{S}^D$ through x_i, there follows that

$$\eta \le 3^{4n-3} + 3(3^{2n-1}+1),$$

a contradiction. □

Chapter 5

Two-weight codes and Steiner systems

5.1 Two-character sets and partial geometries

5.1.1 A classical example

We have given in chapter 4, in detail the descriptions of the partial geometries $\mathcal{S}_i$ for $i = 0, \ldots, 5$, and of their duals for $i = 0, 1, 2, 3$. Moreover we have explained their connection with the triality quadric $\mathrm{Q}^+(7, 2)$. In the next sections we will use them regularly, especially their point graphs. In this section we discuss the connection between the point sets of these partial geometries and subsets of $\mathrm{PG}(7, q)$ having two intersection numbers with respect to hyperplanes.

Consider the graph $\Gamma^*_{2m-1}(\mathrm{Q}^+(2m-1, q))$ with a linear representation. As $\mathrm{Q}^+(2m-1, q)$ is a two-character set in $\mathrm{PG}(2m-1, q)$ with respect to hyperplanes, the graph $\Gamma^*_{2m-1}(\mathrm{Q}^+(2m-1, q))$ is strongly regular, more precisely it is an

$$\mathrm{srg}(q^{2m}, (q^{m-1}+1)(q^m-1), q^{2m-2}+q^m-q^{m-1}-2, q^{m-1}(q^{m-1}+1)).$$

Its $(0, 1)$-adjacency matrix has eigenvalues $k = (q^{m-1}+1)(q^m-1)$, $r = q^m - q^{m-1} - 1$ and $l = -q^{m-1} - 1$. Note that the corresponding two-weight code has weights

$$w_1 = q^{2m-2}, \qquad w_2 = q^{m-1}(q^{m-1}+1).$$

See [17] for more information about two-character sets, strongly regular graphs and two-weight codes.

5.1.2 Hyperbolic quasi-quadrics

A *quasi-quadric* [32] in $\mathrm{PG}(N,q)$ is a set of points that has the same intersection numbers with respect to hyperplanes as a non-degenerate quadric in that space. And so a *hyperbolic quasi-quadric* in $\mathrm{PG}(2m-1,q)$ is a set of points that has the same intersection numbers with respect to hyperplanes as a non-degenerate hyperbolic quadric in that space.
Of course, non-degenerate quadrics themselves are examples of quasi-quadrics, but other examples exist. Tonchev [114] has found by computer search all sets of points in $\mathrm{PG}(5,2)$ with the same intersection numbers with respect to hyperplanes as the elliptic and hyperbolic quadric. In [32] a geometrical construction of these sets as well as generalizations are given.

Remark

Hyperbolic quasi-quadrics have two sizes of intersection with hyperplanes and so are two-character sets.
The point set of $\mathrm{Q}^+(2m-1,q)$ is a subset of $\mathrm{PG}(2m-1,q)$ having intersection numbers

$$h_1 = |\mathrm{Q}(2m-2,q)| = \frac{q^{2m-2}-1}{q-1},$$

$$h_2 = |p\mathrm{Q}^+(2m-3,q)| = \frac{q(q^{m-1}-1)(q^{m-2}+1)}{q-1} + 1 = h_1 + q^{m-1},$$

with respect to hyperplanes. Here $p\mathrm{Q}^+(2m-3,q)$ denotes a cone with vertex a point p and basis $\mathrm{Q}^+(2m-3,q)$. In this section we construct three other subsets of $\mathrm{PG}(2m-1,q)$ with the same intersection numbers h_1 and h_2 as above with respect to hyperplanes, that is we construct three new hyperbolic quasi-quadrics.

Theorem 5.1 *Let π be an $(m-2)$-dimensional space of $\mathrm{Q}^+(2m-1,q)$. Suppose that Π is an $(m-1)$-dimensional subspace contained in π^* and intersecting the quadric in π. If $q>2$ then consider an other $(m-1)$-dimensional subspace Π' contained in π^* and intersecting the quadric in π. Let G and G' denote the two generators of the quadric containing π. Then*

(i) the point set of $\mathcal{Q}_4 = (\mathrm{Q}^+(2m-1,q) \setminus G) \cup \Pi$, and

(ii) the point set of $\mathcal{Q}'_4 = (\mathrm{Q}^+(2m-1,q) \setminus (G \cup G')) \cup (\Pi \cup \Pi')$,

have the same two intersection numbers with respect to hyperplanes as the point set of $\mathrm{Q}^+(2m-1,q)$.

Proof. (i) Consider the set $\mathcal{Q}_4$ (see figure 5.1 for $\mathcal{Q}_4$ in $\mathrm{PG}(7,q)$). First note that G and G' are generators of a different class. By construction it is enough to consider the intersection of the hyperplanes of $\mathrm{PG}(2m-1,q)$ with π^*. If a hyperplane H contains π^* then the elements of $G \setminus \pi$ are replaced by the elements of $\Pi \setminus \pi$ and so the intersection size does not change, that is

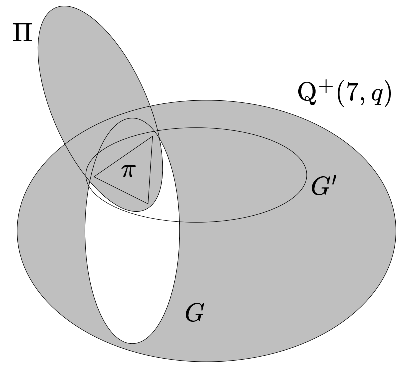

Figure 5.1: The point set $\mathcal{Q}_4$ in PG$(7,q)$

$|H\cap\mathcal{Q}_4| = |H\cap \mathrm{Q}^+(2m-1,q)|$. Hence, we may now assume that H intersects π^* in an $(m-1)$-dimensional subspace. If $H\cap\pi^* = G$, then H is a tangent hyperplane of the quadric and hence will intersect $\mathrm{Q}^+(2m-1,q)$ in a cone $p\mathrm{Q}^+(2m-3,q)$ with vertex a point p and basis $\mathrm{Q}^+(2m-3,q)$ and

$$|H\cap\mathcal{Q}_4| = |pQ^+(2m-3,q)| - |G\setminus\pi| = |\mathrm{Q}(2m-2,q)|.$$

If $H\cap\pi^* = G'$ then $H\cap\mathcal{Q}_4 = H\cap\mathrm{Q}^+(2m-1,q) = pQ^+(2m-3,q)$, and so the intersection size does not change. If $H\cap\pi^* = \Pi$ then H is a secant hyperplane of the quadric and hence

$$|H\cap\mathcal{Q}_4| = |\mathrm{Q}(2m-2,q)| + |\Pi\setminus\pi| = |pQ^+(2m-3,q)|.$$

For $q>2$, consider any $(m-1)$-dimensional subspace Π' in π^* intersecting the quadric in π, $\Pi'\neq\Pi$. If $H\cap\pi^* = \Pi'$ then H is a secant hyperplane of the quadric and hence

$$|H\cap\mathcal{Q}_4| = |\mathrm{Q}(2m-2,q)|.$$

The final possibility is that $H\cap\pi^*$ is an $(m-1)$-dimensional space intersecting each of the $q+1$ $(m-1)$-dimensional spaces through π contained in π^*, in an $(m-2)$-dimensional space P_i with $P_i\cap\pi = H\cap\pi = Y$ $(i=0,\ldots,q)$ an $(m-3)$-dimensional space. Without loss of generality, suppose that $P_0\subset G$ and $P_1\subset\Pi$. Then the elements of $P_0\setminus Y$ are replaced by the elements of $P_1\setminus Y$ and so the intersection size does not change. Hence $|H\cap\mathcal{Q}_4| = |H\cap\mathrm{Q}^+(2m-1,q)|$.

(ii) First note that we assume $q>2$. Consider the set $\mathcal{Q}_4'$ (see figure 5.2 for $\mathcal{Q}_4'$ in PG$(7,q)$). If H contains π^* then the elements of $(G\cup G')\setminus\pi$ are replaced by the elements of $(\Pi\cup\Pi')\setminus\pi$ and so the intersection size does not change. Assume that H intersects π^* in an $(m-1)$-dimensional subspace. If $H\cap\pi^*$ equals G or G', then H is a tangent hyperplane of the quadric and

$$|H\cap\mathcal{Q}_4'| = |pQ^+(2m-3,q)| - |G\setminus\pi| = |pQ^+(2m-3,q)| - |G'\setminus\pi| = |\mathrm{Q}(2m-2,q)|.$$

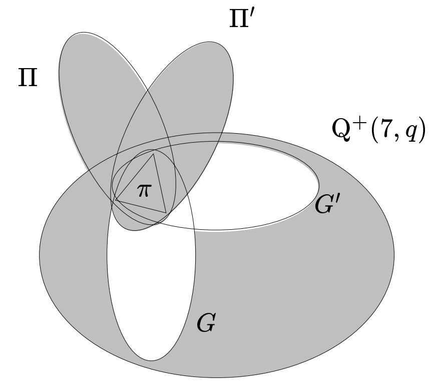

Figure 5.2: The point set $\mathcal{Q}_4'$ in PG$(7, q)$

If $H \cap \pi^* = \Pi$ or Π' then H is a secant hyperplane of the quadric and hence

$$|H \cap \mathcal{Q}_4'| = |\mathrm{Q}(2m-2,q)| + |\Pi \setminus \pi| = |\mathrm{Q}(2m-2,q)| + |\Pi' \setminus \pi| = |pQ^+(2m-3,q)|.$$

For $q > 3$, consider any $(m-1)$-dimensional subspace Π'' in π^* intersecting the quadric in π, $\Pi'' \neq \Pi, \Pi'$. If $H \cap \pi^* = \Pi''$ then H is a secant hyperplane of the quadric and hence

$$|H \cap \mathcal{Q}_4'| = |\mathrm{Q}(2m-2,q)|.$$

Finally suppose that $H \cap \pi^*$ is an $(m-1)$-dimensional space intersecting each of the $q+1$ $(m-1)$-dimensional spaces through π and contained in π^*, in an $(m-2)$-dimensional space P_i with $P_i \cap \pi = H \cap \pi = Y$, $i = 0, \ldots, q$, an $(m-3)$-dimensional space. Without loss of generality, suppose that $P_0 \subset G$, $P_1 \subset G'$, $P_2 \subset \Pi$ and $P_3 \subset \Pi'$. Then the elements of $(P_0 \cup P_1) \setminus Y$ are replaced by the elements of $(P_2 \cup P_3) \setminus Y$ and hence $|H \cap \mathcal{Q}_4'| = |H \cap \mathrm{Q}^+(2m-1,q)|$. □

Remark

If $m = 2n$ and $q = 2$, then the complement in PG$(2m-1, 2)$ of the set $\mathcal{Q}_4$ is the point set of the partial geometry $\mathcal{S}_4(n)$.

Theorem 5.2 *Let Y be an $(m-3)$-dimensional space contained in a generator G of $\mathrm{Q}^+(2m-1,q)$. Suppose that Π is an $(m-1)$-dimensional subspace contained in Y^* intersecting the quadric in Y. Then the point set of*

$$\mathcal{Q}_5 = (\mathrm{Q}^+(2m-1,q) \setminus G) \cup \Pi,$$

has the same two intersection numbers with respect to hyperplanes as the point set of $\mathrm{Q}^+(2m-1,q)$.

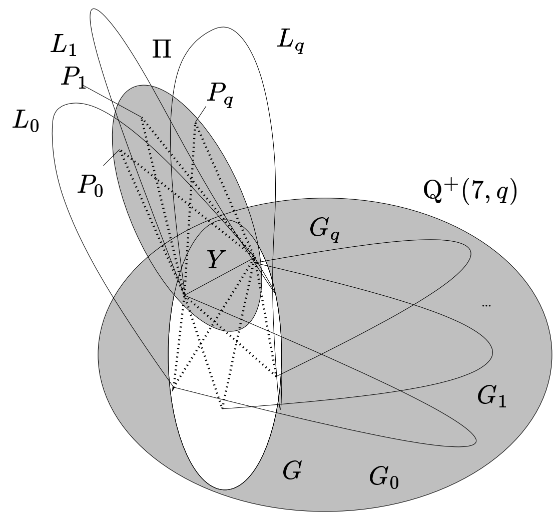

Figure 5.3: The point set $\mathcal{Q}_5$ in $\mathrm{PG}(7,q)$

Proof. Let π_i, $i = 0, \ldots, q$, denote the $q+1$ $(m-2)$-dimensional spaces of G through Y and let G_i denote the second generator of the quadric containing π_i. Note that

$$Y^* \cap \mathrm{Q}^+(2m-1,q) = (\cup_{i=0}^{q} G_i) \cup G = Y\mathrm{Q}^+(3,q),$$

that is a cone with vertex Y and basis a $\mathrm{Q}^+(3,q)$. Since $\Pi \subset Y^*$ we can choose an $(m-1)$-dimensional space L_i contained in π_i^* and intersecting $\mathrm{Q}^+(2m+1,q)$ only in π_i, such that $L_i \cap \Pi$ is an $(m-2)$-dimensional space P_i (see figure 5.3 for the seven-dimensional case).
As in the previous theorem, it is enough to consider the intersection of any hyperplane of $\mathrm{PG}(2m-1,q)$ with Y^*, the polar space of Y with respect to the quadric $\mathrm{Q}^+(2m-1,q)$. If a hyperplane H contains Y^* then by construction the elements of $G \backslash Y$ are replaced by the elements of $\Pi \backslash Y$ and hence the intersection size does not change. Assume that $H \cap Y^* = \pi_i^*$ for some $i \in \{0, \ldots, q\}$, then H is a tangent hyperplane and

$$|H \cap \mathcal{Q}_5| = |pQ^+(2m-3,q)| - |G \setminus Y| + |P_i| = |\mathrm{Q}(2m-2,q)|.$$

Another possibility is that H intersects Y^* in an m-dimensional space M intersecting each of the m-dimensional spaces π_i^* in an $(m-1)$-dimensional space $A_i, i = 0, \ldots, q$. Suppose that $A_i = G$, $i \in \{0, \ldots, q\}$, then since the $A_j = M \cap \pi_j^*$, $j \in \{0, \ldots, q\} \backslash \{i\}$, are also $(m-1)$-dimensional we obtain that $M = G$, a contradiction since M is m-dimensional. Therefore all A_i intersect G in an $(m-2)$-dimensional space ω, $i = 0, \ldots, q$. Suppose that $\omega = \pi_i$ for some i, say π_0. Then M cannot contain Π and so it will intersect Π in an $(m-2)$-dimensional space through Y and so it must be one of the P_j, $j \in \{0, \ldots, q\}$. Hence the

q^{m-2} points of $\omega \setminus Y$ will be replaced by the q^{m-2} points of $(M \cap \Pi) \setminus Y = P_j \setminus Y$. Suppose that $\omega \neq \pi_i$ for all $i = 0, \ldots, q$, then A_i intersects L_i in a affine $(m-2)$-dimensional space that intersects P_i in a affine $(m-3)$-dimensional space Y_i with $Z = \omega \cap Y$ at infinity, $i = 0, \ldots, q$. And so the

$$\frac{q^{m-1}-1}{q-1} - \frac{q^{m-3}-1}{q-1} = (q+1)q^{m-3}$$

points in $\omega \setminus Y$ will be replaced by the $(q+1)q^{m-3}$ points in $(\cup_{i=0}^{q} Y_i) \setminus Z$. Hence the set $\mathcal{Q}_5 = (\mathrm{Q}^+(2m-1,q) \setminus G) \cup \Pi$ has the same two intersection numbers with respect to hyperplanes as the point set of $\mathrm{Q}^+(2m-1,q)$. □

Remarks

1. If $m = 2n$ and $q = 2$, then the complement in $\mathrm{PG}(2m-1,2)$ of the set $\mathcal{Q}_5$ of theorem 5.2 is the point set of the partial geometry $\mathcal{S}_5(n)$.

2. It is known that sets of points in a projective space having two intersection numbers with respect to hyperplanes yield a lot of other geometrical objects: the three sets $\mathcal{Q}_4$, $\mathcal{Q}_4'$ and $\mathcal{Q}_5$ of theorems 5.1 and 5.2 give rise to new strongly regular graphs, two-weight codes, difference sets, ... (see [17, 48, 71] for more information).

3. The strongly regular graphs $\Gamma^*_{2m-1}(\mathcal{P})$, with $\mathcal{P}$ a set in $\mathrm{PG}(2m-1,q)$ having the same intersection numbers with respect to hyperplanes as the set $\mathrm{PG}(2m-1,q) \setminus \mathrm{Q}^+(2m-1,q)$ have $\lambda = \mu$ if and only if $q = 2$. It is commonly known that the incidence structure $\mathcal{D}$ with point set the set of vertices of a strongly regular graph $\mathrm{srg}(v,k,\lambda,\lambda)$, which we denote by Γ, and blocks the first subconstituents of the vertices of Γ is a $2-(v,k,\lambda)$ design. Hence the graphs $\Gamma^*_{2m-1}(\mathcal{P})$ with $\mathcal{P}$ the complement in $\mathrm{PG}(2m-1,2)$ of a hyperbolic quasi-quadric yield

$$2-(2^{2m}, 2^{m-1}(2^m-1), 2^{m-1}(2^{m-1}-1))$$

designs.

4. In [8] it was proved that the point set $\mathrm{Q}^+(2m-1,q) \setminus G$, with G a generator of the quadric is a subset of $\mathrm{PG}(2m-1,q)$ having intersection numbers

$$h_1 = \frac{q^{2m-2}-q^{m-1}}{q-1}, \qquad h_2 = \frac{q^{2m-2}}{q-1},$$

with respect to hyperplanes. And so its complement

$$(\mathrm{PG}(2n-1,q) \setminus \mathrm{Q}^+(2m-1,q)) \cup G$$

has intersection numbers

$$h_1 = \frac{q^{2m-1}-q^{2m-2}+q^{m-1}-1}{q-1}, \qquad h_2 = \frac{q^{2m-1}-q^{2m-2}-1}{q-1}.$$

Therefore $(\mathrm{PG}(2n-1,q) \setminus \mathrm{Q}^+(2m-1,q)) \cup G$ has the same intersection numbers as the quadric $\mathrm{Q}^+(2m-1,q)$ if and only if $q=2$. In other words the point set of $(\mathrm{PG}(2m-1,2) \setminus \mathrm{Q}^+(2m-1,2)) \cup G$ is a hyperbolic quasi-quadric.

5.1.3 More two-character sets

In theorems 5.1 and 5.2 we generalised the point set of $\mathcal{S}_4(n)$ and $\mathcal{S}_5(n)$ for general dimensions and general q. Now we generalize it again for q odd, that is we consider half of the points outside a non-degenerate hyperbolic quadric, namely the sets $E_1^+(2m-1,q)$ and $E_2^+(2m-1,q)$.
The following result is well known, although up to our knowledge it never appeared in literature.

Theorem 5.3 *The point sets $E_i^+(2m-1,q)$ and $E_i^-(2m-1,q)$, q odd, $i \in \{1,2\}$, both have the following two intersection numbers with respect to hyperplanes:*

$$h_1 = \frac{q^{m-1}(q^{m-1}-1)}{2}, \qquad h_2 = \frac{q^{m-1}(q^{m-1}+1)}{2} = h_1 + q^{m-1}.$$

Proof. Let Q denote the hyperquadric $\mathrm{Q}^+(2m-1,q)$ or $\mathrm{Q}^-(2m-1,q)$ in $\mathrm{PG}(2m-1,q)$, q odd. Let H be a hyperplane of $\mathrm{PG}(2m-1,q)$ intersecting the quadric Q in a parabolic quadric $\mathrm{Q}(2m-2,q)$. Then by lemma 2.13, there follows that

$$|H \cap E_i^+(2m-1,q)| = |H \cap E_i^-(2m-1,q)|$$

which equals $|E_i(2m-2,q)|$ or $|E_j(2m-2,q)|$, with $i,j \in \{1,2\}, i \neq j$, that is

$$\frac{q^{m-1}(q^{m-1}-1)}{2} \text{ or } \frac{q^{m-1}(q^{m-1}+1)}{2}.$$

Now suppose that H intersects the quadric Q in a degenerate quadric. Then $H \cap Q = pQ'$, that is a cone with vertex the point p and base a hyperquadric Q' of $\mathrm{PG}(2m-3,q)$, with Q' and Q of the same type. Tangent lines are completely contained in $E_i^+(2m-1,q)$ or $E_i^-(2m-1,q)$, $i \in \{1,2\}$. Counting the number of tangent lines at Q through p in H we obtain that

$$|H \cap E_i^+(2m-1,q)| = q|E_i^+(2m-3,q)| = \frac{q^{m-1}(q^{m-1}-1)}{2} = h_1,$$

$$|H \cap E_i^-(2m-1,q)| = q|E_i^-(2m-3,q)| = \frac{q^{m-1}(q^{m-1}+1)}{2} = h_2.$$

$\square$

Theorem 5.4 *Let π be an $(m-2)$-dimensional space of $\mathrm{Q}^+(2m-1,q)$, q odd. Consider the two generators G and G' of $\mathrm{Q}^+(2m-1,q)$ that are contained in π^* and let Π be an $(m-1)$-dimensional space in π^*, intersecting the quadric in π, such that $\Pi \setminus \pi$ is also contained in $E_i^+(2m-1,q)$. When $q>3$ then let Π' be an other $(m-1)$-dimensional space in π^*, intersecting the quadric in π such that $\Pi' \setminus \pi$ is contained in $E_i^+(2m-1,q)$, $i \in \{1,2\}$. Then*

(i) the point set of $\mathcal{Q}_4'' = (E_i^+(2m-1,q) \setminus \Pi) \cup (G \setminus \pi)$, and

(ii) the point set of $\mathcal{Q}_4''' = (E_i^+(2m-1,q) \setminus (\Pi \cup \Pi')) \cup ((G \cup G') \setminus \pi)$,

have the same two intersection numbers with respect to hyperplanes as the point set of $E_i^+(2m-1,q)$.

Proof. (i) Choose $i \in \{1,2\}$. Again we only consider the intersection of the hyperplanes of $\mathrm{PG}(2m-1,q)$ with π^*. If a hyperplane H contains π^* then the elements of $\Pi \setminus \pi$ are replaced by the elements of $G \setminus \pi$ and so the intersection size does not change.
Assume that H intersects π^* in an $(m-1)$-dimensional subspace. If $H \cap \pi^* = G$, then H is a tangent hyperplane of the quadric and therefore H will intersect $\mathrm{Q}^+(2m-1,q)$ in a cone $p\mathrm{Q}^+(2m-3,q)$ with vertex a point p and with basis a $\mathrm{Q}^+(2m-3,q)$ and

$$|H \cap \mathcal{Q}_4''| = q|E_i^+(2m-3,q)| + |G \setminus \pi| = |E_2(2m-2,q)|.$$

If $H \cap \pi^* = G'$ then

$$|H \cap \mathcal{Q}_4''| = q|E_i^+(2m-3,q)| = |E_1(2m-2,q)|.$$

If $H \cap \pi^* = \Pi$ then H is a secant hyperplane of the quadric. Suppose that $H \cap E_i^+(2m-1,q) = E_1(2m-2,q)$. Let x be a point of $\Pi \setminus \pi$, then the polar space x^* of x in H with respect to the quadric $\mathrm{Q}(2m-2,q) = H \cap \mathrm{Q}^+(2m-1,q)$ is an elliptic quadric $\mathrm{Q}^-(2m-3)$ which contains the $(m-2)$-dimensional space π, a contradiction. Therefore $H \cap E_i^+(2m-1,q) = E_2(2m-2,q)$, and so

$$|H \cap \mathcal{Q}_4''| = |E_2(2m-2,q)| - |\Pi \setminus \pi| = |E_1(2m-2,q)|.$$

Consider any $(m-1)$-dimensional subspace Π' in π^* intersecting the quadric in π, $\Pi' \neq \Pi$. If $H \cap \pi^* = \Pi'$ and $(\Pi' \setminus \pi) \subset E_i^+(2m-1,q)$, then similarly as above $H \cap E_i^+(2m-1,q) = E_2(2m-2,q)$ and so

$$|H \cap \mathcal{Q}_4''| = |E_2(2m-2,q)|.$$

If $(\Pi' \setminus \pi) \not\subset E_i^+(2m-1,q)$, then $H \cap E_i^+(2m-1,q) = E_1(2m-2,q)$ and so

$$|H \cap \mathcal{Q}_4''| = |E_1(2m-2,q)|.$$

The final possibility is that $H \cap \pi^*$ is an $(m-1)$-dimensional space intersecting each of the $q+1$ $(m-1)$-dimensional spaces through π and contained in π^*, in an $(m-2)$-dimensional space P_i with $P_i \cap \pi = H \cap \pi = Y$ an $(m-3)$-dimensional space, $i = 0, \ldots, q$. Without loss of generality, suppose that $P_0 \subset G$ and $P_1 \subset \Pi$. Then the elements of $P_0 \setminus Y$ are replaced by the elements of $P_1 \setminus Y$ and so the intersection size does not change.

The proof of (ii) is a combination of arguments of the second part of the proof of theorem 5.1 and the arguments of (i) above. □

Remarks

1. In the same way as the sets $\mathcal{Q}_4$, $\mathcal{Q}_4'$ and $\mathcal{Q}_5$ of theorems 5.1 and 5.2, the sets $\mathcal{Q}_4''$ and $\mathcal{Q}_4'''$ of theorem 5.4 give rise to new strongly regular graphs, two-weight codes, difference sets,... [17, 48, 71].

2. Let Y be an $(m-3)$-dimensional space contained in a generator G of $\mathrm{Q}^+(2m-1,q)$, q odd. Suppose that Π is an $(m-1)$-dimensional subspace contained in Y^* intersecting the quadric in Y. Then half of the points of Π are contained in $E_1^+(2m-1,q)$ and half of them are contained in $E_2^+(2m-1,q)$. And so we cannot generalise the point set $\mathcal{Q}_5$ of theorem 5.2 using the sets $E_i^+(2m-1,q)$, $i = 1, 2$, in order to obtain a two-character set.

5.2 Subconstituents and partial geometries

In this section we describe strongly regular graphs with a linear representation, that have extreme regularity. One or both of its strongly regular subconstituents carries the structure of a partial geometry.
The Krein bound is reached for the graph $\Gamma^*_{2m-1}(\mathrm{Q}^+(2m-1,q)$ if and only if $q = 2$. This implies that the first and second subconstituents of this graph are strongly regular [20]. And so the same is true for graphs having the same parameters as $\Gamma^*_{2m-1}(\mathrm{Q}^+(2m-1,2)$, namely for

$$\mathrm{srg}(2^{2m}, (2^{m-1}+1)(2^m-1), 2^{2m-2}+2^{m-1}-2, 2^{m-1}(2^{m-1}+1)),$$

that is if $\mathcal{P}$ is a hyperbolic quasiquadric in $\mathrm{PG}(2m-1,2)$, then $\Gamma^*_{2m-1}(\mathcal{P})$ (as well as the complement of this graph) has strongly regular subconstituents.

Lemma 5.5 *Let $\mathcal{P}_0(n) = \mathrm{PG}(4n-1,2) \setminus \mathrm{Q}^+(4n-1,2)$. Then each first subconstituent of the graph $\Gamma^*_{4n-1}(\mathcal{P}_0(n))$ is the complement of the point graph $\Gamma_0(n)$ of $\mathcal{S}_0(n)$; each second subconstituent is the block graph $\Gamma_0'(n)$ of $\mathcal{S}_0(n)$ if and only if $n = 2$.*

Proof. First note that indeed the point set of $\mathcal{P}_0(n)$ is the point set of $\mathcal{S}_0(n)$, and the point set of the complement of $\mathcal{P}_0(2)$ in $\mathrm{PG}(7,2)$ corresponds indeed with the line set of $\mathcal{S}_0$. Since $\mathcal{P}_0(n)$ is the complement of a hyperbolic quadric $\mathrm{Q}^+(4n-1,2)$, equality in the Krein bound is reached. Therefore the first and second subconstituents of the graph $\Gamma^*_{4n-1}(\mathcal{P}_0(n))$ are strongly regular. Also

note that the automorphism group of $\Gamma^*_{4n-1}(\mathcal{P}_0(n))$ acts transitive on its vertex set.
Let x be a vertex of the graph $\Gamma^*_{4n-1}(\mathcal{P}_0(n))$ and let y and z be different vertices adjacent to x. Let X denote the projective line which is the intersection of $\langle x, y, z\rangle$ with the hyperplane Π at infinity, and put $x_1 = X \cap \langle x, y\rangle$, $x_2 = X \cap \langle x, z\rangle$ and $x_3 = X \cap \langle y, z\rangle$. Suppose X is a tangent line of the quadric $\mathrm{Q}^+(4n-1, 2)$, then $\langle y, z\rangle$ intersects Π in the tangent point $x_3 \notin \mathcal{P}_0(n)$ and therefore y and z are non-adjacent in $\Gamma^*_{4n-1}(\mathcal{P}_0(n))$. But x_1 and x_2, which we identify with y and z, are contained in a line of $\mathcal{S}_0(n)$, hence they are indeed adjacent in the point graph $\Gamma_0(n)$ of $\mathcal{S}_0(n)$. Suppose that X is an exterior line of $\mathrm{Q}^+(4n-1, 2)$, then $x_3 \in \mathcal{P}_0(n)$ and therefore y and z are adjacent in $\Gamma^*_{4n-1}(\mathcal{P}_0(n))$. As x_1 and x_2 are never contained in a line of $\mathcal{S}_0(n)$ it follows that y and z are indeed non-adjacent in $\Gamma_0(n)$. This proves the first part of the theorem.
Recall that Kantor has proved that the block graph of $\mathcal{S}_0(n)$ is the non-collinearity graph of the quadric $\mathrm{Q}^+(4n-1, 2)$ if and only if $n = 2$ [70]. Therefore we only need to consider the case $n = 2$. Let y and z be different vertices non-adjacent to x. If $X \subset \mathrm{Q}^+(7, 2)$, then $x_3 \notin \mathcal{P}_0(2)$ hence y and z are non-adjacent in $\Gamma^*_7(\mathcal{P}_0(2))$, and indeed x_1 and x_2 are non-adjacent in the block graph $\Gamma'_0(2)$ of $\mathcal{S}_0$. If X is a secant line of $\mathrm{Q}^+(7, 2)$ then $x_3 \in \mathcal{P}_0(2)$ and hence y and z are adjacent in $\Gamma^*_7(\mathcal{P}_0(2))$, and indeed x_1 and x_2 are adjacent in $\Gamma'_0(2)$. □

Lemma 5.6 *Let $\mathcal{P}_1(n) = \mathrm{Q}^+(4n-1, 2) \setminus G$, with G a generator of the quadric. Then each first subconstituent of the graph $\Gamma^*_{4n-1}(\mathcal{P}_1(n))$ is the complement of the point graph $\Gamma_1(n)$ of $\mathcal{S}_1(n)$ if and only if $n = 2$; each second subconstituent is the block graph $\Gamma'_1(n)$ of $\mathcal{S}_1(n)$.*

Proof. First note that by theorem 4.10, there follows that the point set of $\mathcal{P}_1(2)$ is indeed the point set of $\mathcal{S}_1$. By the easy description of $\mathcal{S}_1(n)$ in the proof of theorem 4.2 there follows that the point set of the complement of $\mathcal{P}_1(n)$ in $\mathrm{PG}(4n-1, 2)$ corresponds indeed with the line set of $\mathcal{S}_1(n)$. Also recall that the point set of $(\mathrm{PG}(2m-1, q) \setminus \mathrm{Q}^+(2m-1, q)) \cup G$, with G a generator of $\mathrm{Q}^+(2m-1, q)$, is a hyperbolic quasi-quadric if and only if $q = 2$. And so the first and second subconstituents of the graph $\Gamma^*_{4n-1}(\mathcal{P}_1(n))$ are strongly regular.
For the second subconstituent, recall that in the block graph of $\mathcal{S}_1(n)$ a vertex $u \in \mathrm{PG}(4n-1, 2) \setminus \mathrm{Q}^+(4n-1, 2)$ is adjacent with a vertex $w \in G$ whenever $w \notin u^*$. Let y and z be non-adjacent to x in $\Gamma^*_{4n-1}(\mathcal{P}_1(n))$. Let X, x_1, x_2, x_3 be as in the proof of lemma 5.5 but considered with respect to $\mathcal{P}_1(n)$. If X is either a tangent line with tangent point on $\mathrm{Q}^+(4n-1, 2) \setminus G$ or a secant line containing a point of G, then $x_3 \in \mathcal{P}_1(n)$ and hence y and z are adjacent in $\Gamma^*_{4n-1}(\mathcal{P}_1(n))$, and indeed x_1 and x_2 are adjacent in $\Gamma'_1(n)$, the block graph of $\mathcal{S}_1(n)$. If X is either a line of G, or an exterior line, or a tangent line intersecting G then $x_3 \notin \mathcal{P}_1(n)$ and hence y and z are non-adjacent in $\Gamma^*_{4n-1}(\mathcal{P}_1(n))$, and indeed this implies that x_1 and x_2 are never adjacent in $\Gamma'_1(n)$. This proves the first part of the theorem.
By Kantor [70] we only need to consider the case $n = 2$ for the first subconstituent of the graph $\Gamma^*_{4n-1}(\mathcal{P}_1(n))$. Let x be a vertex of the graph $\Gamma^*_7(\mathcal{P}_1(2))$

and let y and z be two vertices adjacent to x. Again let X, x_1, x_2, x_3 be as in the proof of lemma 5.5 but considered with respect to $\mathcal{P}_1(2)$. If X is a line of $\mathrm{Q}^+(7,2)$ intersecting the generator G or is a secant line having no points in G, then $x_3 \notin \mathcal{P}_1(2)$ and hence y and z are non-adjacent in $\Gamma_7^*(\mathcal{P}_0(2))$, and indeed x_1 and x_2 are adjacent in the point graph $\Gamma_1(2)$ of $\mathcal{S}_1$, which is the complement of the graph on the quadric with a hole. If X is a line of $\mathrm{Q}^+(7,2)$ not intersecting G, then $x_3 \in \mathcal{P}_1(2)$ and hence y and z are adjacent in $\Gamma_7^*(\mathcal{P}_0(2))$, and indeed x_1 and x_2 are non-adjacent in $\Gamma_1(2)$. □

Remark

Since the graph $\Gamma_1(2)$ of lemma 5.6 is also the point graph of the partial geometries $\mathcal{S}_2$ and $\mathcal{S}_3$ (see theorem 4.10), lemma 5.6 obviously implies that each first subconstituent of the graph $\Gamma_7^*(\mathcal{P}_1(2))$ also is the complement of the point graph of $\mathcal{S}_2$ and $\mathcal{S}_3$. But note that no second subconstituent can be the block graph of $\mathcal{S}_2$ or $\mathcal{S}_3$, since the block graphs of $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ are all different.

Lemma 5.7 *Let $\mathcal{P}_4(n)$ be the point set of $\mathcal{S}_4(n)$. Then each first subconstituent of the graph $\Gamma_{4n-1}^*(\mathcal{P}_4(n))$ is the complement of the point graph $\Gamma_4(n)$ of $\mathcal{S}_4(n)$.*

Proof. Recall the description of $\mathcal{S}_4(n)$ and its notations. Then one can easily see that $\mathcal{P}_4(n)$ is the complement in $\mathrm{PG}(4n-1,2)$ of the set $\mathcal{Q}_4$ of theorem 5.1, which is a quasi-quadric, and so the first and second subconstituents of the graph $\Gamma_{4n-1}^*(\mathcal{P}_4(n))$ are strongly regular.
Let x be a vertex of the graph $\Gamma_{4n-1}^*(\mathcal{P}_4(n))$ and let y and z be two vertices adjacent to x. Let X, x_1, x_2, x_3 be as in the proof of lemma 5.5 but considered with respect to $\mathcal{P}_4(n)$. If X is a line of σ_0 intersecting π_0 in x_3 then y and z are non-adjacent in $\Gamma_{4n-1}^*(\mathcal{P}_4(n))$, and indeed x_1 and x_2 are contained in the type (i) line of $\mathcal{S}_4(n)$. If X is an exterior line intersecting L_0 in the point x_3 or if X is a tangent line intersecting π_0 in x_3, then y and z are non-adjacent in $\Gamma_{4n-1}^*(\mathcal{P}_4(n))$, and indeed x_1 and x_2 are contained in a type (iii) line of $\mathcal{S}_4(n)$. If X is a tangent line intersecting $\mathrm{Q}^+(4n-1,2) \setminus \sigma_0$ in x_3 or if X is a secant line intersecting σ_0, then y and z are non-adjacent in $\Gamma_{4n-1}^*(\mathcal{P}_4(n))$, and indeed x_1 and x_2 are contained in a type (ii) line of $\mathcal{S}_4(n)$. The last possibility is if X is an exterior line not intersecting L_0 or is a tangent line intersecting $\sigma_0 \setminus \pi_0$. Then all its points are contained in $\mathcal{P}_4(n)$ and therefore y and z are adjacent in $\Gamma_{4n-1}^*(\mathcal{P}_4(n))$, while x_1 and x_2 are never contained in a line of $\mathcal{S}_4(n)$. □

Lemma 5.8 *Let $\mathcal{P}_5(n)$ be the point set of $\mathcal{S}_5(n)$. Then each first subconstituent of the graph $\Gamma_{4n-1}^*(\mathcal{P}_5(n))$ is the complement of the point graph of $\mathcal{S}_5(n)$.*

Proof. Recall the description of $\mathcal{S}_5(n)$, and its notations. Then $\mathcal{P}_5(n)$ is the complement in $\mathrm{PG}(4n-1,2)$ of the set $\mathcal{Q}_5$ of theorem 5.2, which is a quasi-quadric, and so the first and second subconstituents of the graph $\Gamma_{4n-1}^*(\mathcal{P}_5(n))$ are strongly regular.

Let $x, y, z, X, x_1, x_2, x_3$ be as in the proof of lemma 5.5 but considered with respect to $\mathcal{P}_5(n)$. If X is a line of σ_0 intersecting Y in x_3 or if X is a line in $(L_i \setminus P_i) \cup Y$ $(i \in \{0,1,2\})$ intersecting Y in x_3, then y and z are non-adjacent in $\Gamma^*_{4n-1}(\mathcal{P}_5(n))$, and indeed x_1 and x_2 are contained in a type (i) line of $\mathcal{S}_5(n)$. If X is line of σ_0 which is disjoint from Y, then y and z are adjacent in $\Gamma^*_{4n-1}(\mathcal{P}_5(n))$ and x_1 and x_2 are indeed never contained in a line of $\mathcal{S}_5(n)$. If X is an exterior line intersecting Π in the point x_3 or if X is a tangent line disjoint from $\cup_{i=0}^2 L_i$ and intersecting Y in x_3, then y and z are non-adjacent in $\Gamma^*_{4n-1}(\mathcal{P}_5(n))$, and indeed x_1 and x_2 are contained in a type (iii) line of $\mathcal{S}_5(n)$. If X is a tangent line intersecting $\mathrm{Q}^+(4n-1,2) \setminus \sigma_0$ in x_3 or if X is a secant line intersecting σ_0, then y and z are non-adjacent in $\Gamma^*_{4n-1}(\mathcal{P}_5(n))$, and indeed x_1 and x_2 are contained in a type (ii) line of $\mathcal{S}_5(n)$. The last possibility is when X is an exterior line not intersecting Π or is a tangent line intersecting $\sigma_0 \setminus Y$ in which case all its points are contained in $\mathcal{P}_5(n)$ and therefore y and z are adjacent in $\Gamma^*_{4n-1}(\mathcal{P}_5(n))$, while x_1 and x_2 are never contained in a line of $\mathcal{S}_5(n)$. $\square$

Remark

There is no nice description known of the block graphs $\Gamma'_i(n)$ of $\mathcal{S}_i(n)$, $i = 4$ or 5, but probably each second subconstituent of $\Gamma^*_{4n-1}(\mathcal{P}_i(n))$ is not the graph $\Gamma'_i(n)$, which implies that the second subconstituents of $\Gamma^*_{4n-1}(\mathcal{P}_i(n))$ will be new strongly regular graphs.

5.3 Switching of graphs and spread deriving of partial geometries

In the previous section we described a strongly regular graph $\Gamma^*_i = \Gamma^*_7(\mathcal{P}_i(2))$ with a linear representation, and we proved that its first strongly regular subconstituent carries the structure of a partial geometry $\mathcal{S}_i$, $i = 0, 1, 4, 5$. In this section we prove that these graphs Γ^*_i and the corresponding partial geometries $\mathcal{S}_i$ are linked to each other such that the following scheme is commutative for $i = 0, 4$ or 5.

$$\begin{array}{ccc} \Gamma^*_1 & \overset{sw-co}{\longleftrightarrow} & \Gamma^*_i \\ \updownarrow & & \updownarrow \\ \mathcal{S}_1 & \overset{de-du}{\longleftrightarrow} & \mathcal{S}_i \end{array}$$

The labeled arrow $\overset{sw-co}{\longleftrightarrow}$ means that the graphs are related under switching and after taking the complementary graph. The labeled arrow $\overset{de-du}{\longleftrightarrow}$ means that the partial geometries are related under spread derivation with respect to a suitable derivable spread and after dualising. The arrow $\updownarrow$ means that the graph Γ^*_j has a linear representation with the point set of $\mathcal{S}_j$ in the hyperplane at infinity, $j = 0, 1, 4, 5$.

The partial geometries $\mathcal{S}_0(n), \mathcal{S}_4(n)$ and $\mathcal{S}_5(n)$ are all three derived with respect to a suitable replaceable pg-spread from the partial geometry $\mathcal{S}_1(n)$ whose line set corresponds to the set $\mathcal{Q}_1$ of points of $\mathrm{PG}(4n-1,2)$ that are not the points of a quadric with a hole, that is they are the points of $(\mathrm{PG}(4n-1,2) \setminus \mathrm{Q}^+(4n-1,2)) \cup G$, with G a generator of the quadric. Let $\mathcal{Q}_i$ be the set of points in $\mathrm{PG}(4n-1,2)$ that are not points of the partial geometry $\mathcal{S}_i(n)$ $(i = 0,4,5)$. We will prove that the graphs $\Gamma^*_{4n-1}(\mathcal{Q}_i)$ $(i = 0,4,5)$ are switching equivalent to the graph $\Gamma^*_{4n-1}(\mathcal{Q}_1)$. Actually we will prove that this is not only true for the dimensions $4n-1$ but for any odd dimension $2m-1$.

Theorem 5.9 *Let $\mathrm{Q}^+(2m-1,2)$ be a hyperbolic quadric in a hyperplane $H = \mathrm{PG}(2m-1,2)$ of $\mathrm{PG}(2m,2)$. Let $\mathcal{Q}_1$ be the set of points in*

$$(H \setminus \mathrm{Q}^+(2m-1,2)) \cup G,$$

with G a generator of the quadric. Then the graph $\Gamma^*_{2m-1}(\mathcal{Q}_1)$ which is an

$$\mathrm{srg}(2^{2m}, (2^{m-1}+1)(2^m-1), 2^{2m-2}+2^{m-1}-2, 2^{m-1}(2^{m-1}+1)),$$

is switching equivalent to the following strongly regular graphs, which all have the same parameters as the graph $\Gamma^*_{2m-1}(\mathcal{Q}_1)$:

1. *$\Gamma^*_{2m-1}(\mathcal{Q}_0)$ with $\mathcal{Q}_0 = \mathrm{Q}^+(2m-1,2)$;*
2. *$\Gamma^*_{2m-1}(\mathcal{Q}_4)$ with $\mathcal{Q}_4 = (\mathrm{Q}^+(2m-1,2) \setminus G) \cup \Pi$, with Π an $(m-1)$-dimensional subspace of H intersecting the quadric in an $(m-2)$-dimensional subspace π of G and with $\Pi \subset \pi^*$;*
3. *$\Gamma^*_{2m-1}(\mathcal{Q}_5)$ with $\mathcal{Q}_5 = (\mathrm{Q}^+(2m-1,2) \setminus G) \cup \Pi$, with Π an $(m-1)$-dimensional subspace of H intersecting the quadric in an $(m-3)$-dimensional subspace Y of G and with $\Pi \subset Y^*$.*

Proof. First of all we remark that the graphs satisfy indeed condition (i) of theorem 1.2, namely $\lambda + \mu = 2k - \frac{v}{2}$. For each of the graphs $\Gamma^*_{2m-1}(\mathcal{Q}_i)$, $i = 0,4,5$, we have to find a switching set X_i in the vertex set V of $\Gamma^*_{2m-1}(\mathcal{Q}_1)$ which satisfies condition (ii) of theorem 1.2, namely each vertex of X_i (resp. X_i^c) is adjacent to precisely $\frac{|X_i^c|}{2}$ (resp. $\frac{|X_i|}{2}$) vertices in X_i^c (resp. X_i).
Note that as the automorphism group of $\Gamma^*_{2m-1}(\mathcal{Q}_1)$ acts transitive on the set V of vertices, we can take any vertex x and we will denote by Γ_1 and Γ_2 the first and second subconstituent with respect to x of the graph $\Gamma^*_{2m-1}(\mathcal{Q}_1)$. In the rest of the proof we will denote for any vertex $y \in V \setminus \{x\}$ the point $\langle x, y\rangle \cap H$ by y_∞.

Part I. Let X_0 be the set $\{x\} \cup \{y \| y_\infty \in G\}$. Since G is an $(m-1)$-dimensional subspace of H and so $\langle x, G\rangle \setminus H$ is a m-dimensional affine space, we obtain that X_0 is a clique in the graph $\Gamma^*_{2m-1}(\mathcal{Q}_1)$ of cardinality 2^m and so $|V \setminus X_0| = 2^{2m} - 2^m$. As $|\Gamma_1 \setminus X_0| = |\Gamma_2| = 2^{2m-1} - 2^{m-1}$, the vertex x is adjacent to half of the vertices in $V \setminus X_0$.

Let $y \in X_0 \setminus \{x\}$, then since X_0 is a clique we obtain that y is adjacent to $\lambda - |X_0 \setminus \{x, y\}| = 2^{2m-2} - 2^{m-1}$ vertices of $\Gamma_1 \setminus X_0$. Also y is adjacent to $k - 1 - \lambda = 2^{2m-2}$ vertices of Γ_2, hence y is adjacent to $2^{2m-1} - 2^{m-1} = \frac{|V \setminus X_0|}{2}$ vertices in $V \setminus X_0$.

Let z be a vertex of $\Gamma_1 \setminus X_0$. Then $y \in X_0 \setminus \{x\}$ will be adjacent to z if and only if the line $\langle y_\infty, z_\infty \rangle$ is a tangent to the quadric $\mathrm{Q}^+(2m-1, 2)$ hence if and only if $y_\infty \in z_\infty^* \cap G$. As $z_\infty^* \cap G$ is a hyperplane of G and hence contains $2^{m-1} - 1$ points, and as z is adjacent to x, in total z is adjacent to 2^{m-1} vertices of X_0 which is indeed half of its vertices.

Finally assume that u is a vertex of Γ_2, then u is adjacent to $y \in X_0 \setminus \{x\}$ if and only if $\langle y_\infty, u_\infty \rangle$ $(y_\infty \in G)$ is a secant to the quadric. Hence y is adjacent to u if and only if y_∞ is a point of G which is not in the polar hyperplane of u_∞, hence y_∞ is a point of the affine $(m-1)$-dimensional subspace $G \setminus u_\infty^*$ which implies that u is adjacent to 2^{m-1} vertices of X_0 which is indeed half of its vertices. Hence X_0 is a switching set and the switched graph is indeed $\Gamma_{2m-1}^*(\mathcal{Q}_0)$ with $\mathcal{Q}_0 = \mathrm{Q}^+(2m-1, 2)$.

Part II. Let Π be an $(m-1)$-dimensional subspace of $\mathrm{PG}(2m-1, 2)$ intersecting the quadric in an $(m-2)$-dimensional subspace π of G with $\Pi \subset \pi^*$, and let X_4 be the set $\{x\} \cup \{y \| y_\infty \in \Pi\}$. Then X_4 is again a clique in the graph $\Gamma_{2m-1}^*(\mathcal{Q}_1)$ of cardinality 2^m and similarly as above there follows that x is adjacent to half of the vertices of $V \setminus X_4$.

Let y be vertex of $X_4 \setminus \{x\}$ then similarly as above we obtain that y is adjacent to half of the vertices in $V \setminus X_4$.

Let u be vertex such that $u_\infty \in G \setminus \pi$, then $y \in X_4 \setminus \{x\}$ is adjacent to u if and only if $y_\infty \in \pi$, hence as u is adjacent to the vertex x this implies that u is adjacent to 2^{m-1} vertices of X_4 which is half of its cardinality.

Let u be a vertex such that $u_\infty \in \mathcal{Q}_1 \setminus (G \cup \Pi)$, then $y \in X_4 \setminus \{x\}$ is adjacent to u if and only if either $y_\infty \in \pi \cap u_\infty^*$ or $y_\infty \in (\Pi \setminus \pi) \setminus u_\infty^*$ from which follows again that u is adjacent to $|\pi \cap u_\infty^*| + |(\Pi \setminus \pi) \setminus u_\infty^*| + |\{x\}|$ vertices of X_4 that is half of the vertices of X_4.

Let G' denote the second generator of $\mathrm{Q}^+(2m-1, 2)$ through π. Let $u \in \Gamma_2$ such that u_∞ is a point of $\mathrm{Q}^+(2m-1, 2) \setminus (G \cup G')$, and assume u is adjacent to $y \in X_4 \setminus \{x\}$. If $y_\infty \in \pi$ then $\langle y_\infty, u_\infty \rangle$ is a secant to the quadric, that is $y_\infty \in \pi \setminus u_\infty^*$ which is an $(m-2)$-dimensional affine space. If $y_\infty \in \Pi \setminus \mathrm{Q}^+(2m-1, 2)$ then $\langle y_\infty, u_\infty \rangle$ is a tangent to the quadric, that is $y_\infty \in (u_\infty^* \cap \Pi) \setminus \mathrm{Q}^+(2m-1, 2)$ which is an $(m-2)$-dimensional affine subspace of $\Pi \setminus \pi$. From this follows that in total u is adjacent indeed to 2^{m-1} vertices of X_4.

Let $u \in \Gamma_2$ such that u_∞ is a point of $G' \setminus \pi$, and assume u is adjacent to $y \in X_4 \setminus \{x\}$. Then y_∞ cannot be an element of π since $\langle u_\infty, y_\infty \rangle$ must be a secant line of the quadric. In π^* every line through u_∞ and a point of $\Pi \setminus \pi$ intersects $G \setminus \pi$ in a point. This implies that $y_\infty \in G \setminus \Pi$ which is an $(m-1)$-dimensional affine space. From this follows that u is adjacent indeed to 2^{m-1} vertices of X_4. Hence we have proved that X_4 is a switching set.

Part III. Let Π be an $(m-1)$-dimensional subspace of $\mathrm{PG}(2m-1,2)$ intersecting the quadric in an $(m-3)$-dimensional subspace Y of G with $\Pi \subset Y^*$, and let X_5 be the set $\{x\} \cup \{y \| y_\infty \in \Pi\}$. Then X_5 is again a clique in the graph $\Gamma^*_{2m-1}(\mathcal{Q}_1)$ of cardinality 2^m and similarly as above there follows that x is adjacent to half of the vertices of $V \setminus X_5$.

Let y be vertex of $X_5 \setminus \{x\}$ then similarly as above we obtain that y is adjacent to half of the vertices in $V \setminus X_5$.

Let u be vertex such that $u_\infty \in G \setminus Y$, then $y \in X_5 \setminus \{x\}$ is adjacent to u if and only if either $y_\infty \in Y$ or $y_\infty \in u^*_\infty \cap \Pi$. And so Y yields $2^{m-2}-1$ vertices in X_5 adjacent to u, while $u^*_\infty \cap (\Pi \setminus \pi)$ yields 2^{m-2} vertices in X_5 adjacent to u. As u is adjacent to x, this gives in total 2^{m-1} vertices of X_5 which is half of its cardinality.

Let $\pi_i, i=0,1,2$, denote the hyperplanes of G through Y. Let L_i denote the point set of $\pi^*_i \setminus \mathrm{Q}^+(2m-1,2)$ and G_i the second generator of $\mathrm{Q}^+(2m-1,2)$ through π_i, $i=0,1,2$. Then every $L_i, i=0,1,2$, intersects Π in a different $(m-2)$-dimensional affine space P_i having Y at infinity.

Let u be a vertex such that $u_\infty \in \mathcal{Q}_1 \setminus (G \cup \Pi \cup (\cup_{i=0}^2 L_i))$, then $y \in X_5 \setminus \{x\}$ is adjacent to u if and only if either $y_\infty \in u^*_\infty \cap Y$ or $y_\infty \in (\Pi \setminus Y) \setminus u^*_\infty$ from which follows again (as u is adjacent to x) that u is adjacent to half of the vertices of X_5. Let u be a vertex such that $u_\infty \in (\cup_{i=0}^2 L_i)) \setminus \Pi$, say $u_\infty \in L_0$. Then $y \in X_5 \setminus \{x\}$ is adjacent to u if and only if either $y_\infty \in Y$ or $y_\infty \in P_0$ from which follows again (as u is adjacent to x), u is adjacent to half of the vertices of X_5.

Let $u \in \Gamma_2$ such that u_∞ is a point of $\mathrm{Q}^+(2m-1,2) \setminus (\cup_{i=0}^2 G_i)$, and assume that u is adjacent to $y \in X_5 \setminus \{x\}$. If $y_\infty \in Y$ then $\langle y_\infty, u_\infty \rangle$ is a secant to the quadric, that is $y_\infty \in Y \setminus u^*_\infty$ which is an $(m-3)$-dimensional affine subspace of Y. If $y_\infty \in \Pi \setminus \mathrm{Q}^+(2m-1,2)$ then $\langle y_\infty, u_\infty \rangle$ is a tangent line to the quadric, that is $y_\infty \in (u^*_\infty \cap \Pi) \setminus \mathrm{Q}^+(2m-1,2)$ and so this yields $2^{m-2}+2^{m-3}$ vertices of X_5. From this follows that u is adjacent indeed to 2^{m-1} vertices of X_5.

Let $u \in \Gamma_2$ such that u_∞ is a point of $(\cup_{i=0}^2 G_i) \setminus G)$. Without loss of generality assume $u_\infty \in G_0$. And assume that u is adjacent to $y \in X_5 \setminus \{x\}$. Then y_∞ cannot be an element of Y since $\langle u_\infty, y_\infty \rangle$ must be a secant line of the quadric. A line through u_∞ and a point of $\Pi \setminus Y$ intersects $G \setminus Y$ if and only if this point is contained in $L_0 \cap \Pi$. And so y_∞ can be contained in the set $L_0 \cap \Pi$. This yields 2^{m-2} vertices in X_5 adjacent to u. Finally suppose that $y_\infty \in \Pi \setminus (G \cup L_0)$, then $\langle y_\infty, u_\infty \rangle$ is a tangent line of the quadric or $y_\infty \in u^*_\infty \cap \Pi$. This yields 2^{m-2} other vertices in X_5 adjacent to u. In total this gives us 2^{m-1} vertices in X_5 adjacent to u. Hence we have proved that X_5 is a switching set. $\square$

Corollary

The set $\mathcal{Q}_1 = (\mathrm{PG}(4n-1,2) \setminus \mathrm{Q}^+(4n-1,2)) \cup G$, with G a generator of the quadric is the line set of the partial geometry $\mathcal{S}_1(n)$. From the above theorem it follows that the replaceable pg-spreads Φ_j $(j = 0,4,5)$, that yield the partial geometries $\mathcal{S}_j(n)$ are subsets Φ_j of $\mathcal{Q}_1$ such that for any vertex x of $\Gamma^*_{4n-1}(\mathcal{Q}_1)$ the set

$$X_j = \{x\} \cup \{y \| \langle x,y\rangle \cap \mathrm{PG}(4n-1,2) \in \Phi_j\},$$

is a switching set of $\Gamma^*_{4n-1}(\mathcal{Q}_1)$, moreover the second subconstituent $\Gamma_2(x)$ of the switched graph is the point graph of the partial geometry $\mathcal{S}_j(n)$ $(j = 0,4,5)$.

5.4 Steiner systems and partial geometries

One of the links between designs and partial geometries that we investigate is the following. Let $\mathcal{S}$ denote a partial geometry having parameters s, t, α and having point set $\mathcal{P}$ and line set $\mathcal{L}$. We want to know if there exists a Steiner 2-system $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ such that $\mathcal{L} \subset \mathcal{B}$. Remark that the existence of such an embedding does not depend on the structure of $\mathcal{P}$, but only on the collinearity graph of $\mathcal{S}$ and its line size $s+1$. We actually look for a collection $\mathcal{L}'$ of $(s+1)$-cocliques in the point graph Γ of $\mathcal{S}$ that cover all non-collinear pairs of points exactly once. Then the partial geometry $\mathcal{S}$ is embedded in the Steiner 2-system

$$\mathrm{S}\left(2, s+1, \frac{(s+1)(st+\alpha)}{\alpha}\right).$$

Brouwer, Haemers and Tonchev proved some divisibility conditions on the parameters of a partial geometry which is embeddable in a Steiner 2-system [7]. Since the partial geometry $\mathrm{PQ}^+(4n-1,3)$, and its spread derivations, and their duals cannot have an embedding by these divisibility conditions, we will only consider the case $q = 2$ in this section. The partial geometry $\mathrm{pg}(7,8,4)$ $\mathcal{S}_0 = \mathrm{PQ}^+(7,2)$ is embeddable into a $\mathrm{S}(2,8,120)$ [7]. We give a short proof of this result and will show that $\mathcal{S}_0$ is embeddable into at least four $\mathrm{S}(2,8,120)$. We will also prove that the spread derived partial geometries $\mathcal{S}_i$ $(i = 1,2,3,4)$ are also embeddable into an $\mathrm{S}(2,8,120)$.

Theorem 5.10 *The partial geometry $\mathcal{S}_0$ is embeddable in at least four non-isomorphic Steiner 2-system $\mathrm{S}(2,8,120)$.*

Proof. Consider the set $\mathcal{O}_i$ of 120 ovoids of $\mathrm{Q}^+(7,2)$ being the lines of the partial geometry $\mathcal{S}_i$ $(i = 1,2,3,4)$ (see theorems 4.10 and 4.11). Recall that $\mathcal{S}_i$ has point set $\mathrm{Q}^+(7,2) \setminus \mathcal{A}$, where $\mathcal{A}$ is a certain generator of $\mathrm{Q}^+(7,2)$. Each ovoid $O \in \mathcal{O}_i$ intersects $\mathcal{A}$ in a point p. Consider the third point $q \in \mathrm{PG}(7,2) \setminus Q^+(7,2)$ of the line $\langle p, p'\rangle$, where $p' \in O \setminus \{p\}$. Let us call the set of eight points we obtain in this way O'. Then every pair of points (p'_1, p'_2) of $\mathrm{Q}^+(7,2) \setminus \mathcal{A}$ which are on a secant line is mapped onto a pair of points (q_1, q_2) of $\mathrm{PG}(7,2) \setminus \mathrm{Q}^+(7,2)$ which

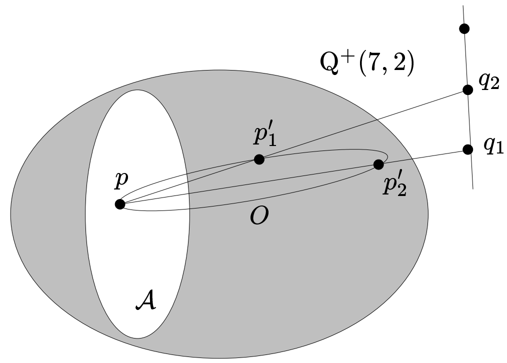

Figure 5.4: The mapping of the pair (p_1', p_2') onto the pair (q_1, q_2)

are on an exterior line (see figure 5.4). Note that

$$|\mathrm{Q}^+(7,2) \setminus \mathcal{A}| = |\mathrm{PG}(7,2) \setminus \mathrm{Q}^+(7,2)|,$$

and so this mapping defines a bijection between these sets. Since every two non-collinear points of $\mathrm{Q}^+(7,2) \setminus \mathcal{A}$ are contained in exactly one element O of $\mathcal{O}_i$, every two non-adjacent vertices in the collinearity graph of $\mathcal{S}_0$ are contained in exactly one block O'. Hence the 120 sets O' are the 120 extra blocks we are looking for in order to obtain the Steiner system $\mathrm{S}(2,8,120)$.
Since the partial geometries $\mathcal{S}_i, i = 1,2,3,4$, are non-isomorphic (see [81]) the sets $\mathcal{O}_i$ of the $\mathcal{S}_i$ are not the same ($i = 1,2,3,4$). In the Steiner system $\mathrm{S}(2,8,120)$ constructed above, we can distinguish the old blocks from the new blocks. Since the old blocks are different for different i, the three corresponding Steiner systems are not isomorphic. □

Theorem 5.11 *The partial geometry $\mathcal{S}_i$ ($i = 1,2,3,4$) is embeddable in a Steiner 2-system* $\mathrm{S}(2,8,120)$.

Proof. Recall the descriptions of the $\mathcal{S}_i$ ($i = 1,2,3,4$) using the quadric with a hole from theorems 4.10 and 4.11. The extra blocks that we are looking for are cocliques of size 8 in its point graph Γ such that every two points of the quadric on a line in $\mathrm{Q}^+(7,2) \setminus \mathcal{A}$ are in a unique block. Hence we need to find subsets U of generators $\mathcal{B}$ of $\mathrm{Q}^+(7,2)$, such that $\mathcal{A} \cap \mathcal{B} = \emptyset$, and such that these subsets are mutually intersecting in at most one point. Indeed when we find those sets U, then the 120 ovoids obviously will intersect U in at most one point and each affine 3-dimensional space on a plane of $\mathcal{A}$ will also intersect U in at most one point.
We will define a mapping from pairs of points inside $\mathrm{Q}^+(7,2) \setminus \mathcal{A}$ which are on a line of $\mathrm{Q}^+(7,2) \setminus \mathcal{A}$ onto pairs of points outside the quadric which are on a tangent line.

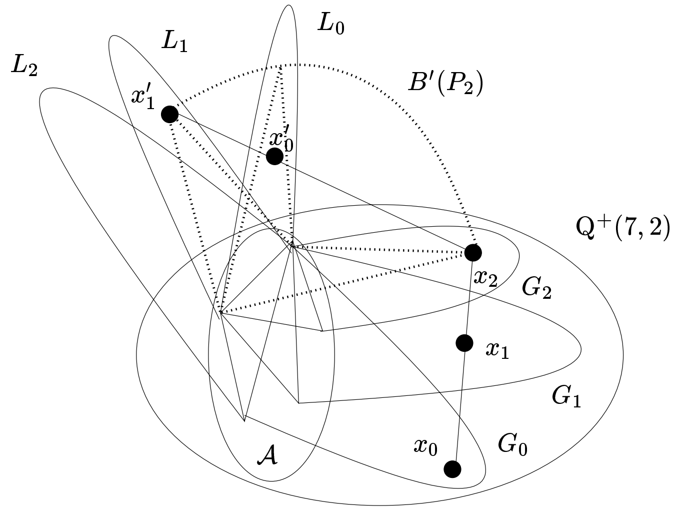

Figure 5.5: The mapping of the pair (x_0, x_1) onto the pair (x_0', x_1')

Let x_0, x_1 be two points on a line of $\mathrm{Q}^+(7,2) \setminus \mathcal{A}$. Let x_2 denote the third point of this line. Let $\pi_i = x_i^* \cap \mathcal{A}$ $(i = 0,1,2)$ then π_0, π_1, π_2 must intersect in a line X of $\mathcal{A}$. Define the affine 3-dimensional spaces $G_i = (\pi_i^* \cap \mathrm{Q}^+(7,2)) \setminus \mathcal{A}$ and $L_i = \pi_i^* \setminus \mathrm{Q}^+(7,2)$ $(i = 0,1,2)$. Let P_i denote the plane $\langle x_i, X\rangle \subset \mathrm{Q}^+(7,2)$ $(i = 0,1,2)$. Now consider the generator $B(P_2) \in \mathcal{D}_1$ through P_2 and define $B'(P_2) = P_2{}^* \setminus \mathrm{Q}^+(7,2)$. Then $B(P_2)$ intersects G_i in the plane P_i $(i = 0,1,2)$. Since $\langle P_2, \pi_i\rangle$ is a three-dimensional space, $P_2^* \cap \pi_i^*$ is also a three-dimensional space, $i = 0,1,2$. This implies that $B'(P_2)$ intersects L_i in an affine plane P_i' $(i = 0,1)$ and $B'(P_2)$ is disjoint from L_2. Choose $x_0' \in P_0'$ to be the image of x_0. Then the image of x_1 is defined to be $\langle x_0', x_2\rangle \cap P_1'$ (see figure 5.5). Next map the three other intersection points $\langle x_0, p\rangle \cap (P_1 \setminus X)$, with $p \in P_2 \setminus X$, onto the corresponding points $\langle x_0', p\rangle \cap P_1'$. Interchanging the role of x_0 and x_1 (where the image of x_1 is known by the above) we can also map the points of $P_0 \setminus X$ onto the points of the affine plane P_0'. Similarly we can interchange the role of x_0 and x_1 with any other point of $P_0 \setminus X$ and $P_1 \setminus X$. This is well defined since no points are mapped twice onto different images because $G_i \cap (p_j)^* = G_i \cap X^* \cap (p_j)^* = G_i \cap (P_j)^*$ (where $p_j \in P_j \setminus X$ and $i, j = 0, 1$, $i \neq j$), and similarly $L_i \cap (p_j')^* = L_i \cap X^* \cap (p_j')^* = L_i \cap (P_j')^*$ (where $p_j' \in P_j'$ and $i, j = 0, 1$, $i \neq j$).

Define $P_2' = L_2 \setminus \langle P_0', L_1 \setminus P_1'\rangle$. Then $(P_i')^* \cap L_j = P_j'$ (where $i, j \in \{0,1,2\}$ and $i \neq j$). Hence interchanging the indices $i \in \{0,1,2\}$ in the above construction, and applying the same construction again will not map points onto different images twice, hence the mapping is still well defined.

The only thing we still need to prove is that this construction extends all over $Q^+(7,2)$. Assume $x \in \cup_{i=0}^2 P_i$, say $x \in P_0$. For $y \in (\cup_{i=0}^2 G_i) \setminus \mathcal{A}$ the construction obviously extends. And so take $y \in Q^+(7,2) \setminus (\cup_{i=0}^2 G_i \cup \mathcal{A})$ such that $\langle x, y\rangle$ is a line of $Q^+(7,2) \setminus \mathcal{A}$. Then by the above we know that x is mapped onto a fixed point x' of P_0'. Let $\pi = y^* \cap \mathcal{A}$ and $G = (\pi^* \cap Q^+(7,2)) \setminus \mathcal{A}$ and $L = \pi^* \setminus Q^+(7,2)$. Using the construction above we know that the points of the affine plane $x^* \cap G$ are mapped onto the points of the affine plane $x'^* \cap L$ and the points of $P = y^* \cap G_0$ are mapped onto the points of $P' = y'^* \cap L_0$. Since $y \notin \cup_{i=0}^2 G_i$, it follows that $\pi \cap X$ is a point m and the point of the affine line $P_0 \cap P = \langle x, m\rangle \setminus \{x, m\}$ is mapped twice (earlier in the above and now again) onto the only and therefore same point of the affine line $P_0' \cap P' = \langle x', m\rangle \setminus \{x', m\}$. Using connectivity of the graph Γ we obtain that the construction of the mapping extends on a well defined way all over $Q^+(7,2)$.
Let Σ denote an orthogonal spread of $Q^+(7,2)$ such that $\mathcal{A} \in \Sigma$ and consider the corresponding partial geometry $PQ^+(7,2)$. Every non-edge of Γ is mapped onto an edge of the point graph of $PQ^+(7,2)$ which is contained in a unique line of $PQ^+(7,2)$ being an affine 3-dimensional space having its plane at infinity in an element of $\Sigma \setminus \{\mathcal{A}\}$. Reversing the mapping gives us the 120 extra blocks we were looking for. □

Remarks

For the Steiner systems $S(2,8,120)$ containing the partial geometry $\mathcal{S}_i, i = 1,2,3,4$, we can distinguish the old blocks from the new blocks. Since the old blocks are different for different i, the four corresponding Steiner systems $S(2,8,120)$ are not isomorphic.

Chapter 6

Embeddings of $(0, \alpha)$-geometries in affine spaces

6.1 Overview

For many of the known examples of $(0, \alpha)$-geometries, the points and lines of the geometry are the points and lines of a projective or affine space. And so it is an interesting question to try determine all of them, or to obtain characterizations.
There exists a complete classification of partial geometries (fully) embedded in a projective space (see [14] for the generalized quadrangles and [33] for $\alpha > 1$). The classification of partial geometries embeddable in an affine space is also known [100]. In the case of generalized quadrangles some sporadic embeddings even occur. The complete classification of semipartial geometries embeddable in a projective space is known for $\alpha > 1$ and for $s > 2$ [34, 42, 110]. If $\mathcal{S}$ is a semipartial geometry with $s = \alpha = 2$, then $\mathcal{S}$ is a *cotriangle space*, and those are classified [88, 90]. However, as explained in [54], it is impossible to classify the projective embeddings of all cotriangle spaces, it is only known for dimensions 3 and 4 [42, 110]. The embedding of a semipartial geometry in an affine space is unsolved. The classification is only known for dimensions 2 and 3 [41].
(Proper) semipartial geometries $\mathrm{spg}(s, t, \alpha, \mu)$ cannot be embedded in the affine plane $\mathrm{AG}(2, q)$ [41]. If a semipartial geometry is embedded in $\mathrm{AG}(3, q)$ then it is the pentagon (trivial case), $T_2^*(\mathcal{B})$ with $\mathcal{B}$ a Baer subplane of the plane π_∞ at infinity of $\mathrm{AG}(3, q)$, or $T_2^*(\mathcal{U})$ with $\mathcal{U}$ a unital of π_∞. Note that in the last two cases q must be a square. The known embeddings of semipartial geometries in $\mathrm{AG}(4, q)$ are $T_3^*(\mathcal{O})$ with $\mathcal{O}$ an ovoid of the hyperplane Π_∞ at infinity of $\mathrm{AG}(4, q)$, $T_3^*(\mathcal{B})$ with $\mathcal{B}$ a Baer subspace of Π_∞ and the Hirschfeld-Thas model of $\mathrm{TQ}(4, q)$ when q is even. When $n > 4$ the only known affine embedding is the linear representation model $T_{n-1}^*(\mathcal{B})$ with $\mathcal{B}$ a Baer subspace of the hyperplane

at infinity. For more information see [37, 38].
Among the affine embeddings of (semi)partial geometries, especially the linear representations are well studied (see [37] for an overview). There exist several characterizations of the known examples. For example in [37] De Clerck and Van Maldeghem give a geometric characterization of the partial quadrangle $T_3^*(\mathcal{O})$. In [16] Calderbank proves that existence of a linear representation $T_n^*(\mathcal{K})$ of a proper partial quadrangle implies the existence of an integer solution of the diophantine equation

$$y^2 = 4q^{\frac{a}{2}} + 4q + 1$$

where the parameter a is a function of the dimension n. And so the following cases occur.

1. The partial quadrangle $T_3^*(\mathcal{O})$ with $\mathcal{O}$ an ovoid of the hyperplane Π_∞ at infinity of $\mathrm{AG}(4, q)$. Note that $T_3^*(\mathcal{O})$ is an $\mathrm{spg}(q-1, q^2, 1, q(q-1))$ (see [18]).

2. Suppose $q = 3$ and assume that $\mathcal{K}$ is not an ovoid. Then $\mathcal{K}$ is either an 11-cap in PG(4,3) (see for instance [85] for a description) the partial quadrangle $T_4^*(\mathcal{K})$ has parameters $s = 2, t = 10, \mu = 2$, or $\mathcal{K}$ is the unique 56-cap in PG(5,3) in which case the partial quadrangle has parameters $s = 2, t = 55, \mu = 20$. This 56-cap was first constructed by Segre [87].

3. Suppose $q = 4$. Then either $\mathcal{K}$ is an ovoid in PG(3,4) or it is a 78-cap in PG(5,4) such that each external point is on 7 secants, or a 430-cap in PG(6,4) such that each external point is on 55 secants. If $\mathcal{K}$ is a 78-cap, the partial quadrangle $T_5^*(\mathcal{K})$ has parameters $s = 3, t = 77, \mu = 14$. At least one example exists and was discovered by Hill [59]. If $\mathcal{K}$ is a 430-cap then the partial quadrangle has parameters $s = 3, t = 429, \mu = 110$. Up to now however, the existence of such a cap is not known.

4. Suppose $q \geq 5$. Then it was proved by Tzanakis and Wolfskill [115] that the partial quadrangle has to be $T_3^*(\mathcal{O})$ with $\mathcal{O}$ an ovoid.

Remarks

1. The Hirschfeld-Thas model of $\mathrm{TQ}(4, q)$ is not a linear representation and in [62] it is shown to arise from the projection of a $\mathrm{Q}^-(5, q)$, q even, from a point of $\mathrm{PG}(5, q) \setminus \mathrm{Q}^-(5, q)$ (see section 6.5.1 for more information) onto a hyperplane of $\mathrm{PG}(5, q)$. For q odd this construction still yields a semipartial geometry with the same parameters but the model of the construction is not embedded in $\mathrm{AG}(4, q)$. Other semipartial geometries with the same parameters are due to Brown [10], and they are constructed from the generalized quadrangles of Kantor. In section 6.5.3 we characterize the semipartial geometry $\mathrm{TQ}(4, q)$, q even, by its parameters and its embedding in $\mathrm{AG}(4, q)$.

2. We obtain characterizations of the linear representation $T^*_{n-1}(\mathcal{K})$ of a $(0,\alpha)$-geometry, $\alpha > 1$ (see section 6.3). This has consequences for the dual semipartial geometries embedded in an affine space (see section 6.4). A complete classification of all linear representations $T^*_{n-1}(\mathcal{K})$ of $(0,\alpha)$-geometries in $\mathrm{AG}(n,q)$ is unlikely since $\mathcal{K}$ is not necessarily a set of points of $\mathrm{PG}(n,q)$ having two intersection sizes with hyperplanes, but a set of points such that a projective line intersects it in 0, 1 or $\alpha+1$ points.

6.2 Behavior at infinity

Let $\mathcal{S} = (\mathcal{P},\mathcal{L},\mathrm{I})$ be a proper $(0,\alpha)$-geometry embedded in $\mathrm{AG}(n,q)$, with $n \geq 3$. Let Π_∞ denote the hyperplane at infinity of $\mathrm{AG}(n,q)$. The line set of $\mathcal{S}$ is a subset of the line set of $\mathrm{AG}(n,q)$, which in turn is a subset of the line set of $\mathrm{PG}(n,q)$, the projective completion of $\mathrm{AG}(n,q)$. Thus a line of $\mathcal{S}$ will be said to intersect Π_∞ in the point of Π_∞ incident with the corresponding line in $\mathrm{PG}(n,q)$.
Subspaces of dimension d of $\mathrm{AG}(n,q)$, $d \in \{1,\ldots,n-1\}$, are called *parallel* if they determine the same $(d-1)$-dimensional space in Π_∞.
For a point x of $\mathcal{S}$, let θ_x denote the set of $t+1$ points in Π_∞ determined by the intersection of Π_∞ with the lines of $\mathcal{S}$ through x.
Let (x,L) be an antiflag of $\mathcal{S}$. Define $M = \langle x,L\rangle \cap \Pi_\infty$ and $p = L \cap \Pi_\infty$. If $\alpha(x,L) = 0$, then M is either a tangent of θ_x at p or an external line of θ_x, while for $\alpha(x,L) = \alpha$, we obtain that either $p \notin \theta_x$ and M intersects θ_x in α points, or $p \in \theta_x$ and M intersects θ_x in $\alpha+1$ points. Hence any line M of Π_∞ intersects θ_x in $0, 1, \alpha$ or $\alpha+1$ points. A line of Π_∞ intersecting θ_x in $0, 1, \alpha$ or $\alpha+1$ points will be referred to as an external line, tangent, α-secant or $(\alpha+1)$-secant, respectively.

6.3 The linear representation of a $(0,\alpha)$-geometry

6.3.1 Connected components of the subspace geometry

Definitions

For a subspace $\Pi = \mathrm{AG}(d,q)$ of $\mathrm{AG}(n,q)$ $(2 \leq d \leq n-1)$, let $\mathcal{P}_\Pi$ be the set $\Pi \cap \mathcal{P}$ and let $\mathcal{L}_\Pi$ be the (non-empty) set of lines of $\mathcal{S}$ completely contained in Π. Define the *subspace geometry* $\mathcal{S}_\Pi$ to be the incidence structure $(\mathcal{P}_\Pi, \mathcal{L}_\Pi, \mathrm{I}_\Pi)$, where I_Π is the natural incidence. A *connected component* $\mathcal{C}_\Pi$ of $\mathcal{S}_\Pi$ is a subgeometry of $\mathcal{S}_\Pi$ that is connected. An *isolated point* p of $\mathcal{S}_\Pi$ is a point of $\mathcal{S}_\Pi$ such that no line of $\mathcal{S}_\Pi$ is incident with p.

Lemma 6.1 *Let $\mathcal{S}$ be a $(0,\alpha)$-geometry of order (s,t), $\alpha \neq 1$, embedded in $\mathrm{AG}(n,q)$, and consider a subspace $\Pi = \mathrm{AG}(d,q)$ of $\mathrm{AG}(n,q)$ $(2 \leq d \leq n-1)$. Then every connected component $\mathcal{C}_\Pi$ of the subspace geometry $\mathcal{S}_\Pi$ is a $(0,\alpha)$-geometry.*

Proof. Consider a connected component $\mathcal{C}_\Pi$ of $\mathcal{S}_\Pi$. Obviously every line of $\mathcal{C}_\Pi$ is incident with $s+1$ points. For each antiflag (x,L) of $\mathcal{C}_\Pi$ we have $\alpha(x,L)=0$ or α. Let x and y be two collinear points of $\mathcal{C}_\Pi$. Let u_x+1, respectively u_y+1 denote the number of lines of $\mathcal{C}_\Pi$ through x, respectively y. Then counting the number of points z of $\mathcal{C}_\Pi$, $z \neq x, z \neq y$, such that z is collinear with both x and y yields $u_x(\alpha-1)+s-1=u_y(\alpha-1)+s-1$. Hence $u_x=u_y$. Since the component $\mathcal{C}_\Pi$ is connected this implies that every point of $\mathcal{C}_\Pi$ is incident with a constant number of lines of $\mathcal{C}_\Pi$. □

Remark

Let $\mathcal{S}$ be a $(0,\alpha)$-geometry, $\alpha \neq 1$, embedded in $\mathrm{AG}(n,q)$, and let π be a plane of $\mathrm{AG}(n,q)$. If the subplane geometry $\mathcal{S}_\pi$ has at least one connected component then either there is exactly one connected component $\mathcal{C}_\pi$ in the subplane geometry $\mathcal{S}_\pi$, or there are e_π ($e_\pi > 0$) connected components each consisting of one line.

Lemma 6.2 *Let $\mathcal{S}$ be a $(0,\alpha)$-geometry, $\alpha \neq 1$, embedded in $\mathrm{AG}(n,q)$, $n>2$, and let π be a plane of $\mathrm{AG}(n,q)$. Then π is one of the following four types:*

type 1: *π only contains a number of isolated points of $\mathcal{S}_\pi$;*

type 2: *π only contains e_π parallel lines of $\mathcal{S}$ (and some isolated points);*

type 3: *the connected component of π is a $\mathrm{pg}(q-1,\alpha,\alpha)$, that is a net;*

type 4: *the connected component of π is a $\mathrm{pg}(2^h-1,1,2), h>0$.*

Proof. Suppose that π contains two intersecting lines of $\mathcal{S}$, then the subplane geometry $\mathcal{S}_\pi$ contains exactly one connected component $\mathcal{C}_\pi$, which is a $(0,\alpha)$-geometry by lemma 6.1. In this case $\mathcal{C}_\pi$ contains an antiflag (x,L) such that $\alpha(x,L)\neq 0$. And so there are exactly α lines through x intersecting L. In the affine plane π there is one affine line M through x parallel with L. If M is a line of $\mathcal{C}_\pi$ then there are $\alpha+1$ lines of $\mathcal{C}_\pi$ incident with x. If not then there are α lines of $\mathcal{C}_\pi$ through x. By lemma 6.1 the number of lines of $\mathcal{C}_\pi$ through a point of $\mathcal{C}_\pi$ is a constant and so it equals either α or $\alpha+1$. Therefore $\mathcal{C}_\pi$ is a $\mathrm{pg}(q-1,\alpha,\alpha)$ or a $\mathrm{pg}(q-1,\alpha-1,\alpha)$. In the last case this yields a dual oval in an affine plane, and therefore we have $\alpha=2$ and $q=2^h, h>0$ (see [100]). Therefore a plane π of $\mathrm{AG}(n,q)$ containing two intersecting lines of $\mathcal{S}$ is of type 3 or 4.
If a plane π of $\mathrm{AG}(n,q)$ does not contain two intersecting lines of $\mathcal{S}$, but it contains at least one line of $\mathcal{S}$, then π is of type 2.
Finally if π does not contain any lines of $\mathcal{S}$ then it is of type 1. □

Lemma 6.3 *Let C be a coclique of the point graph of the semipartial geometry $T_n^*(\mathcal{K})$, $n \geq 2$, that is the linear representation model. Then $|C| < q^n$.*

Proof. Let C be a coclique of the point graph Γ of the semipartial geometry $T_n^*(\mathcal{K})$ and let C' be a clique of Γ corresponding with a line of $T_n^*(\mathcal{K})$, hence $|C'| = q$. Suppose that $|C| \geq q^n$. Then from theorem 1.1 (the Hoffman bound) we know that

$$q^n \leq |C| \leq \frac{v}{1 - \frac{k}{l}}.$$

Therefore $1 - \frac{k}{l} \leq q$. The Hoffman bound for strongly regular graphs also yields

$$q = |C'| \leq 1 - \frac{k}{l}.$$

And so $1 - \frac{k}{l} = q$. By theorem 1.1 equality in the Hoffman bound implies that $\mathcal{S}$ is a partial geometry, a contradiction. Hence $|C| < q^n$. □

Remark

In their proof of the classification of the (proper) semipartial geometries embedded in $\mathrm{AG}(3, q)$, Debroey and Thas [39] classified the linear representation models $T_2^*(\mathcal{K})$ of semipartial geometries. Corollary 6.3 proves their result in a shorter way.

Corollary 6.4 *Consider the linear representation $T_2^*(\mathcal{K})$ of a semipartial geometry. Then $\mathcal{K}$ is either a Baer subplane or a unital of the plane at infinity. In particular q is a square.*

Proof. Let $T_2^*(\mathcal{K})$ be the linear representation of a semipartial geometry. Suppose that the plane at infinity π_∞ contains a line L that does not intersect $\mathcal{K}$. Let x be a point of the semipartial geometry $T_2^*(\mathcal{K})$, then the q^2 affine points of the plane $\langle x, L\rangle$ yield a coclique of size q^2 in the point graph of $T_2^*(\mathcal{K})$, a contradiction by lemma 6.3. And so $\mathcal{K}$ has no exterior lines in π_∞, but only tangents and $(\alpha+1)$-secants. By [95] we obtain that $\mathcal{K}$ is either a Baer subspace, or a unital, or $\mathcal{K}$ consists of all points of π_∞ in which case $T_2^*(\mathcal{K})$ is the design of points and lines of the affine space, a contradiction. Hence q is a square and $\mathcal{K}$ is either a Baer subplane or a unital. □

6.3.2 Semipartial geometries embedded in $\mathrm{AG}(n, q)$ having no planes of type 4

Lemma 6.5 *Assume that x and y are two different points of a $(0, \alpha)$-geometry $\mathcal{S}$, $\alpha \neq 1$, embedded in $\mathrm{AG}(n, q), n > 2$, and suppose that $\mathcal{S}$ has no planes of type 4, then $\theta_x = \theta_y$.*

Proof. Let x and y be two collinear points of $\mathcal{S}$. Suppose that $r \in \theta_x \setminus \theta_y$. Then $\langle x, y\rangle$ and $\langle x, r\rangle$ are intersecting lines of $\mathcal{S}$. Since there are no planes of type 4, these lines are contained in a plane of type 3, that is a net. Therefore there exists a line of $\mathcal{S}$ through y parallel with $\langle x, r\rangle$. And so $r \in \theta_y$, a contradiction. Since the geometry $\mathcal{S}$ is connected the result follows. □

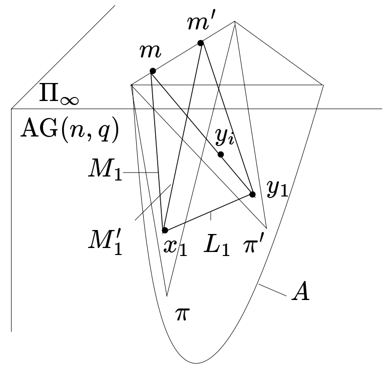

Figure 6.1: The connected component $\mathcal{C}_A$ of $A = \mathrm{AG}(3,q)$ in $\mathcal{S}$

Theorem 6.6 *Let $\mathcal{S}$ be a $(0,\alpha)$-geometry, $\alpha \neq 1$, embedded in $\mathrm{AG}(n,q), n > 2$. Then $\mathcal{S}$ is a linear representation $T^*_{n-1}(\mathcal{K})$ if and only if there are no planes of type* 4.

Proof. Suppose that $\mathcal{S}$ is a linear representation $T^*_{n-1}(\mathcal{K})$ of a $(0,\alpha)$-geometry, $\alpha \neq 1$, $n > 2$. By lemma 6.2 two intersecting lines L and M of $T^*_{n-1}(\mathcal{K})$ are contained in a plane of type 3 or 4. Since $\langle L, M\rangle$ intersects $\mathcal{K}$ in $\alpha+1$ points, the affine plane corresponding with $\langle L, M\rangle$ cannot be of type 4.
Conversely, let $\mathcal{S}$ be a $(0,\alpha)$-geometry embedded in $\mathrm{AG}(n,q)$ with no planes of type 4. Since $\mathcal{S}$ is connected there exist intersecting lines. By lemma 6.2 this implies that they are contained in a plane π of type 3. Suppose that $t = \alpha$, then $\mathcal{S}$ is embedded in π, a contradiction.
Since $t > \alpha$, there must be a subspace $A = \mathrm{AG}(3,q)$ of $\mathrm{AG}(n,q)$ such that the corresponding subspace geometry has a connected component $\mathcal{C}_A$ containing π, but it is not equal to π. By lemma 6.1, the connected component $\mathcal{C}_A$ is a $(0,\alpha)$-geometry of order $(q-1,t')$ with $\alpha < t' \leq t$. In A we consider a plane π' which is parallel to π. Then there must be a line L_1 of $\mathcal{C}_A$ intersecting π in a point x_1 of $\mathcal{C}_A$ and π' in a point y_1 of $\mathcal{C}_A$ (see figure 6.1).
Consider a line M_1 of $\mathcal{C}_A$ through x_1 in π and define $m = M_1 \cap \Pi_\infty$. Then the plane $\langle L_1, M_1\rangle$ contains intersecting lines and so it must be of type 3. Therefore it intersects π' in the q affine points $y_1, \ldots, y_q$ of the line $\langle m, y_1\rangle$. Consider an other line M_1' of $\mathcal{C}_A$ through x_1 in π and define $m' = M_1' \cap \Pi_\infty$. Then $\langle L_1, M_1'\rangle$ is again of type 3 and it intersects π' in the q affine points of the line $\langle m', y_1\rangle$ (see figure 6.1).
Since $\mathcal{C}_A$ is a $(0,\alpha)$-geometry of order $(q-1,t')$, we know that for $i \in \{1,\ldots,q\}$ there is a line L_i of $\mathcal{C}_A$ through y_i and intersecting π in a point x_i of $\mathcal{C}_A$. (Note that it could be that $x_i = x_1$, but this does not matter for the proof.) Since π is a net there exists a line M_i' of $\mathcal{C}_A$ through x_i in π having m' at infinity. The plane $\langle L_i, M_i'\rangle$ contains intersecting lines and so it must be of type 3. Therefore it intersects π' in the q affine points of the line $\langle m', y_i\rangle$. And so the q points of

the affine line $\langle m', y_i\rangle$ are points of the geometry $\mathcal{C}_A$, $i = 1, \ldots, q$. Therefore all of the points of π' are points of $\mathcal{S}$. Considering all the planes parallel with π in A we obtain that all the points of A are points of $\mathcal{S}$, moreover they are points of the connected component $\mathcal{C}_A$. By lemma 6.5, for any two points x and y of $\mathcal{C}_A$ we have that $\theta_x = \theta_y = \mathcal{K}$. And so the restriction of the $(0,\alpha)$-geometry $\mathcal{S}$ to A is a linear representation $T_2^*(\mathcal{K}')$.
If $\mathcal{S}$ is embedded in A, then we are done. If not, then through every point u of A, there exist a line of $\mathcal{S}$ which intersects A only in u. Now consider the three-dimensional affine space A' spanned by a plane π of A and a line L of $\mathcal{S}$ intersecting A in a point u of π. Since the connected component $\mathcal{C}_A$ is a linear representation $T_2^*(\mathcal{K}')$, the plane π is of type 3 and similarly as above we obtain that the restriction of $\mathcal{S}$ to the connected component $\mathcal{C}_{A'}$ is a linear representation. This implies that every point of $\mathrm{AG}(n,q)$ is contained in such a three-dimensional affine space A' such that the connected component $\mathcal{C}_{A'}$ is a linear representation, and so all points of $\mathrm{AG}(n,q)$ are points of $\mathcal{S}$.
By lemma 6.5, for any two points x and y of $\mathcal{S}$ we have that $\theta_x = \theta_y = \mathcal{K}$. And so the semipartial geometry $\mathcal{S}$ is a linear representation $T_{n-1}^*(\mathcal{K})$. □

Corollary 6.7 *A $(0,\alpha)$-geometry embedded in $\mathrm{AG}(n,q), n > 2$, $\alpha \neq 1,2$, is a linear representation $T_{n-1}^*(\mathcal{K})$.*

Proof. Since $\alpha \neq 2$, there are no planes of type 4. Theorem 6.6 yields the result. □

6.4 Dual semipartial geometries in $\mathrm{AG}(n,q)$

In [34], De Clerck and Thas determined all dual semipartial geometries, $\alpha \neq 1$, embeddable in $\mathrm{PG}(n,q)$. In this section we investigate the dual semipartial geometries embeddable in $\mathrm{AG}(n,q)$.

Theorem 6.8 ([34]) *If $\mathcal{S}$ is the dual of a semipartial geometry with $\alpha > 1$, and if $\mathcal{S}$ is embedded in a projective space $\mathrm{PG}(n,q)$, $n \geq 3$, then $n = 3$ and $\mathcal{S}$ is the design of points and lines in $\mathrm{PG}(3,q)$, $\mathcal{S} = H_q^3$, or $\mathcal{S} = NQ^-(3,2)$.*

Theorem 6.9 *If $\mathcal{S}$ is a dual semipartial geometry embedded in $\mathrm{AG}(n,q)$, then $\alpha = 1$ and $\mathcal{S}$ cannot be a linear representation, that is $\mathcal{S}$ is not of type $T_{n-1}^*(\mathcal{K})$.*

Proof. Consider a dual semipartial geometry $\mathcal{S}$ of order $(q-1,t)$ embedded in $\mathrm{AG}(n,q)$, hence the dual geometry $\mathcal{S}^D$ of $\mathcal{S}$ is an $\mathrm{spg}(t, q-1, \alpha, \mu)$. Again let Π_∞ denote the hyperplane at infinity of $\mathrm{AG}(n,q)$.
Suppose that $\alpha \neq 1$. Let L and M be two disjoint lines of $\mathcal{S}$ such that $\langle L, M\rangle$ is a projective plane, that is L and M are parallel lines of $\mathrm{AG}(n,q)$. Then since $\mu \neq 0$ and by lemma 6.2 we have that $\langle L, M\rangle$ is of type 3. And so $\mu = \alpha q$. Therefore $\mathcal{S}^D$ must be a partial geometry, a contradiction.
And so there do not exist lines of $\mathcal{S}$ that are parallel in $\mathrm{AG}(n,q)$, hence the line set of $\mathcal{S}$ can be identified with a subset of the point set of Π_∞. Also there are

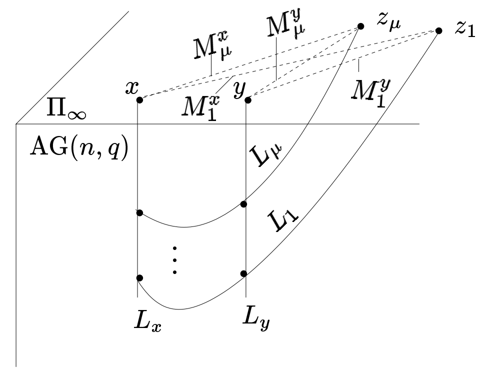

Figure 6.2: The geometry G induced at infinity

no planes of type 3. By lemma 6.2 this implies $\alpha = 2$. Consider the incidence structure G, with point set the set of points of Π_∞ corresponding with a line of $\mathcal{S}$; and with line set the projective lines $\Pi_\infty \cap \pi$ of Π_∞, where π is a type 4 plane of $\mathcal{S}$. Incidence is the natural one. The lines of $\mathcal{S}$ in a type 4 plane π form a dual oval with nucleus the line $\pi \cap \Pi_\infty$. Therefore there are $q+1$ points on a line of G. For a point x of G, let L_x denote the corresponding line of $\mathcal{S}$. Take a point v of $\mathcal{S}$ on L_x, then there are t lines L^i_x of $\mathcal{S}$ incident with v and different from L_x $(i = 1,\ldots,t)$. Then $\langle L_x, L^i_x\rangle$ is a type 4 plane of $\mathcal{S}$ and so it determines a line of G through x $(i = 1,\ldots,t)$. Therefore G is a partial linear space of order $(s', t') = (q, t-1)$.

Let x and y be two non-collinear points of G, hence $\langle L_x, L_y\rangle$ is three dimensional. Then there are μ lines L_i of $\mathcal{S}$ concurrent with both L_x and L_y $(i = 1,\ldots,\mu)$. Since L_x and L_i are intersecting lines of $\mathcal{S}$, they determine a plane of type 4 in $\mathcal{S}$ and so a line M^x_i of G through x $(i = 1,\ldots,\mu)$. Similarly L_y and L_i determine a line M^y_i of G and $M^x_i \cap M^y_i = L_i \cap \Pi_\infty$ $(i = 1,\ldots,\mu)$. There do not exist parallel lines and there do not exist planes of type 3, therefore $M^x_i \neq M^x_j$ and $M^y_i \neq M^y_j$ for $i, j \in \{1,\ldots,\mu\}$ and $i \neq j$. And so in G there are μ points collinear with both x and y, namely $z_i = L_i \cap \Pi_\infty$, $i = 1,\ldots,\mu$ (see figure 6.2). And so the point graph of G is strongly regular.

Let (x, M) be an antiflag of G. Since $\mathcal{S}^D$ is a semipartial geometry we have $t \le q-1$, hence $t' = t-1 \le q-2$, and so there is at least one point y of M which is not collinear with x. Suppose that M is one of the lines M^y_i constructed above $(i \in \{1,\ldots,\mu\})$. Since $M^x_i \neq M^x_j$ and $M^y_i \neq M^y_j$ for $i, j \in \{1,\ldots,\mu\}$ and $i \neq j$, this implies that $\alpha(x, M) = 1$. If M is not one of these lines then $\alpha(x, M) = 0$. If G is a semipartial geometry then $s' \le t'$ or $q \le t'$ a contradiction since $t' \le q-2$. Hence G must be a generalized quadrangle. And so $s' = t' = q$, again a contradiction.

Let $\alpha = 1$ and suppose that $\mathcal{S}$ is a linear representation. Let L and M be two parallel lines of AG(n,q). Then the μ lines of $\mathcal{S}$ collinear with both L and M

are μ parallel lines of the affine plane $\langle L, M\rangle$ intersecting Π_∞ in a point p. But since we have a linear representation, there are q lines of the affine plane $\langle L, M\rangle$ having p at infinity that are points of $\mathcal{S}$. This implies that $\mu = q$, and so $\mathcal{S}^D$ is a partial geometry, a contradiction. $\square$

Remark

By theorem 6.9, if a dual semipartial geometry $\mathcal{S}$ embedded in $\mathrm{AG}(n, q)$, $n \geq 3$, would exist, then $\alpha = 1$ and it cannot be of type $T^*_{n-1}(\mathcal{K})$. Moreover since the lines of $\mathcal{S}$ contained in π cannot form a triangle, a plane π of $\mathrm{AG}(n, q)$ is one of the following two types: *type 1*: by the μ condition of the dual semipartial geometry, π contains μ lines of $\mathcal{S}$ of a first parallel class, and μ lines of $\mathcal{S}$ of a second parallel class, and some isolated points; *type 2*: π contains f_π lines of $\mathcal{S}$ through a point π, with $0 \leq f_\pi < q$, and some isolated points.

6.5 Affine semipartial geometries and projections of quadrics

In this section the embedding in $\mathrm{AG}(4, q)$ of an $\mathrm{spg}(q-1, q^2, 2, 2q(q-1))$ is investigated. The geometry of the semipartial geometry in $\mathrm{AG}(4, q)$ and the geometry in the hyperplane at infinity is determined. By a result of Hirschfeld and Thas [62] this geometry is shown to arise from the projection of an elliptic quadric $\mathrm{Q}^-(5, q)$ from a point of $\mathrm{PG}(5, q) \setminus \mathrm{Q}^-(5, q)$ onto a hyperplane of $\mathrm{PG}(5, q)$.

6.5.1 Semipartial geometries and generalized quadrangles

The following construction can be found in [38]. Let $\mathcal{S}$ be a generalized quadrangle embedded in a projective space $\mathrm{PG}(n, q)$, hence $\mathcal{S}$ is classical and $n = 3, 4$ or 5 ([14]). Let p be a point of $\mathrm{PG}(n, q)$ and let Π be a hyperplane of $\mathrm{PG}(n, q)$ not containing p. Let $\mathcal{P}_1$ be the projection of the point set of $\mathcal{S}$ from p onto Π and let $\mathcal{P}_2$ be the set of points of Π on a tangent through p at $\mathcal{S}$. Consider the incidence structure $\mathcal{S}_p = (\mathcal{P}_p, \mathcal{L}_p, \mathrm{I}_p)$ with $\mathcal{P}_p = \mathcal{P}_1 \setminus \mathcal{P}_2$, $\mathcal{L}_p$ the set of lines of Π with q points in $\mathcal{P}_p$ and incidence I_p inherited from the projective space. For the generalized quadrangles $\mathcal{S} = \mathrm{Q}^-(5, q)$, embedded in $\mathrm{PG}(5, q)$ and $\mathcal{S} = \mathrm{H}(4, q^2)$, embedded in $\mathrm{PG}(4, q^2)$, the incidence structure $\mathcal{S}_p$ is a semipartial geometry.
If $\mathcal{S} = \mathrm{H}(4, q^2)$ and p is a point on $\mathrm{H}(4, q^2)$, then the semipartial geometry $\mathcal{S}_p$ is an $\mathrm{spg}(q^2-1, q^3, q, q^2(q^2-1))$; it is $T^*_2(\mathcal{U})$ where $\mathcal{U}$ is the Hermitian unital in $\mathrm{PG}(2, q^2)$. On the other hand if p is not on $\mathrm{H}(4, q^2)$, then $\mathcal{S}_p$ is an $\mathrm{spg}(q^2-1, q^3, q+1, q(q+1)(q^2-1))$; this example is due to Thas [38].
If $\mathcal{S} = \mathrm{Q}^-(5, q)$ and p is a point on the quadric, then $\mathcal{S}_p$ is the partial quadrangle $T^*_3(\mathcal{O})$, with $\mathcal{O}$ an elliptic quadric in $\mathrm{PG}(3, q)$. However if p is not on the quadric $\mathrm{Q}^-(5, q)$, then $\mathcal{S}_p$ is an $\mathrm{spg}(q-1, q^2, 2, 2q(q-1))$; this construction is due to Hirschfeld and Thas [63]. Following De Clerck [26] this semipartial geometry is denoted $\mathrm{TQ}(4, q)$.

Remarks

1. Another construction of $\mathrm{TQ}(4,q)$ was given by R. Metz [unpublished]. The Metz model has point set the set of non-singular elliptic quadrics $\mathrm{Q}^-(3,q)$ on a non-singular quadric $\mathrm{Q}(4,q)$ of $\mathrm{PG}(4,q)$, and line set the sets of elliptic quadrics which are pairwise tangent at a common point, incidence is the natural one (see [38]). The two models are referred to as the Hirschfeld-Thas model and the Metz model, respectively. Consider the Hirschfeld-Thas model of $\mathrm{TQ}(4,q)$. For q even, the set $\mathcal{P}_p$ is a subset of an $\mathrm{AG}(4,q)$, while for q odd, the set $\mathcal{P}_p$ is a subset of $\mathrm{PG}(4,q)\setminus\mathrm{Q}(4,q)$. Also note that in the case where $q=2$ the geometry $\mathrm{TQ}(4,2)$ is a complete graph.

2. Recently Thas [108] devised a general method for constructing semipartial geometries from the quadric $\mathrm{Q}(2n+2,q)$, $n\geq 1$, which includes the semipartial geometry $\mathrm{TQ}(4,q)$ in the case where $n=1$, as the model of Metz. The new construction method uses the so called SPG systems (see also section 2.3.5).

6.5.2 The Brown construction for semipartial geometries

Construction

In [10] Brown gives the following general construction method for

$$\mathrm{spg}(q-1,q^2,2,2q(q-1)).$$

Let $\mathcal{S}$ be a generalized quadrangle of order (q,q^2) containing a subquadrangle $\mathcal{S}'$ of order q. If x is a point of $\mathcal{S}\setminus\mathcal{S}'$, then each line of $\mathcal{S}$ incident with x is incident with a unique point of $\mathcal{S}'$ and the set $\mathcal{O}_x$ of such points is an ovoid of $\mathcal{S}'$. (An *ovoid* of a generalized quadrangle is a set of points such that each line of the generalized quadrangle is incident with a unique point of the set.) The ovoid $\mathcal{O}_x$ is said to be *subtended* by x. A *rosette* of ovoids of $\mathcal{S}'$ is a set of q ovoids meeting pairwise in a exactly one fixed point of $\mathcal{S}'$. If L is a line of $\mathcal{S}\setminus\mathcal{S}'$, then the ovoids of $\mathcal{S}'$ subtended by the points of $\mathcal{S}\setminus\mathcal{S}'$ incident with L form a rosette of $\mathcal{S}'$.
If for a subtended ovoid $\mathcal{O}_x$ there is a point y of $\mathcal{S}\setminus\mathcal{S}'$, $y\neq x$, such that $\mathcal{O}_y=\mathcal{O}_x$, then $\mathcal{O}_x$ is said to be *doubly subtended*. If each ovoid of $\mathcal{S}'$ subtended by a point of $\mathcal{S}\setminus\mathcal{S}'$ is doubly subtended, then $\mathcal{S}'$ is said to be *doubly subtended* in $\mathcal{S}$. If $\mathcal{S}'$ is doubly subtended in $\mathcal{S}$, then the incidence structure with point set the subtended ovoids of $\mathcal{S}'$; line set the rosettes of subtended ovoids of $\mathcal{S}'$; and incidence containment is a semipartial geometry $\mathrm{spg}(q-1,q^2,2,2q(q-1))$.

Examples

1. The generalized quadrangle $\mathrm{Q}(4,q)$ is doubly subtended in $\mathrm{Q}^-(5,q)$ and the above construction yields the Metz model of $\mathrm{TQ}(4,q)$.

2. For q odd and $\sigma \in \mathrm{Aut}(\mathrm{GF}(q))$ the generalized quadrangle $\mathrm{Q}(4,q)$ is also doubly subtended in a generalized quadrangle of Kantor associated with σ (see [68] for the construction of the generalized quadrangle). Two such generalized quadrangles associated with field automorphisms σ_1 and σ_2, respectively, are isomorphic if and only if $\sigma_1 = \sigma_2$ or $\sigma_1 = \sigma_2^{-1}$, and similarly for the $\mathrm{spg}(q-1, q^2, 2, 2q(q-1))$. In the case where σ is the identity the Kantor construction yields $\mathrm{Q}^-(5,q)$ and the associated semipartial geometry $\mathrm{spg}(q-1, q^2, 2, 2q(q-1))$ is the Metz model of $\mathrm{TQ}(4,q)$. When σ is not the identity the Kantor construction yields a non-classical generalized quadrangle and the $\mathrm{spg}(q-1, q^2, 2, 2q(q-1))$ is not isomorphic to $\mathrm{TQ}(4,q)$.

6.5.3 Characterization of TQ$(4,q)$

In this section, let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be an $\mathrm{spg}(q-1, q^2, 2, 2q(q-1))$ embedded in $\mathrm{AG}(4,q)$. The point graph $\Gamma(\mathcal{S})$ of $\mathcal{S}$ is an

$$\mathrm{srg}(\frac{q^4-q^2}{2}, (q-1)(q^2+1), q^2+q-2, 2q(q-1)).$$

We use the notations of section 6.2 for the geometry in the hyperplane Π_∞ at infinity of $\mathrm{AG}(4,q)$.
Note that for $q=2$, the line set of $\mathcal{S}$ corresponds with the edge set of its point graph, which is a trivial graph. And so in this section we suppose that $q \neq 2$.

Lemma 6.10 *The semipartial geometry $\mathcal{S}$ has a plane of type 4.*

Proof. Suppose that $\mathcal{S}$ has no planes of type 4. Then by theorem 6.6 we have that $\mathcal{S}$ is the linear representation model. Therefore $|\mathcal{P}| = q^4$, which is a contradiction. □

Corollary 6.11 *If $\mathcal{S}$ is an* $\mathrm{spg}(q-1, q^2, 2, 2q(q-1))$ *embedded in* $\mathrm{AG}(4,q)$, *$q \neq 2$, then $q = 2^h, h \geq 2$.*

Proof. Lemmas 6.2 and 6.10 yield the result. □

Lemma 6.12 *Let Γ be a regular subgraph of $\Gamma(\mathcal{S})$ with $\frac{q^3-q^2}{2}$ vertices and with valency $q-1$. Then a vertex of $\Gamma(\mathcal{S}) \setminus \Gamma$ is adjacent to q^2-q vertices of Γ.*

Proof. Let A be the adjacency matrix of the graph $\Gamma(\mathcal{S})$. By theorem 1.1 the matrix A has eigenvalues $(q-1)(q^2+1)$, $q-1$ and $2q-1-q^2$. Consider the matrix B of average row sums of A, with rows and columns partitioned according to the vertex sets of Γ and $\Gamma(\mathcal{S}) \setminus \Gamma$. Then we obtain

$$B = \begin{pmatrix} q-1 & q^2(q-1) \\ q^2-q & (q-1)(q^2-q+1) \end{pmatrix}.$$

Note that the row sums of B must be equal to the row sums of A, that is $(q-1)(q^2+1)$. Then B has eigenvalues $(q-1)(q^2+1)$ and $2q-1-q^2$, and so the interlacing with the eigenvalues of A is tight. By theorem 1.4 each vertex of $\Gamma(\mathcal{S}) \setminus \Gamma$ is adjacent to $q^2 - q$ vertices of Γ. $\square$

Lemma 6.13 *If the semipartial geometry $\mathcal{S}$ has a plane π of type 3, then a line of π determines a unique partition in $q+1$ regular subgraphs of $\Gamma(\mathcal{S})$, each of valency $q-1$ and each with $\frac{q^3-q^2}{2}$ vertices.*

Proof. Let L be a line of $\mathcal{S}$. Let Ω_L denote the set of vertices of $\Gamma(\mathcal{S})$ which are at distance two of any point of L. Then

$$|\Omega_L| = v - (s+1) - \frac{s(s+1)t}{\alpha} = \frac{q^3 - q^2}{2} - q.$$

A vertex z of Ω_L is adjacent to $\frac{(s+1)\mu}{\alpha} = q^2(q-1)$ vertices of $\Gamma(\mathcal{S})$ which have a neighbour in L. Since z is adjacent in total $(q-1)(q^2+1)$ vertices of $\Gamma(\mathcal{S})$, this implies that z is adjacent to exactly $q-1$ vertices of Ω_L. Let Γ_L denote the subgraph of $\Gamma(\mathcal{S})$ induced by the vertices of $\Omega_L \cup L$, then the above implies that Γ_L is a regular subgraph of $\Gamma(\mathcal{S})$, which has valency $q-1$ and which has $\frac{q^3-q^2}{2}$ vertices. Therefore every line L of $\mathcal{S}$ is contained in a unique subgraph of type Γ_L of $\Gamma(\mathcal{S})$.
Let M be a line of $\mathcal{S}$ which is disjoint from the vertex set of Γ_L. By lemma 6.12 every point u of M is adjacent to $q^2 - q$ vertices of Γ_L. Let n_i be the number of vertices of Γ_L adjacent to i points in M. Since $\mathcal{S}$ is a semipartial geometry with $\alpha = 2$ we have $i = 0$ or 2. Counting ordered pairs (u, z) with u a point of M and z a vertex of Γ_L such that u and z are adjacent in $\Gamma(\mathcal{S})$ yields the following two equations

$$n_0 + n_2 = \frac{q^3 - q^2}{2}, \qquad 2n_2 = q(q^2 - q).$$

Therefore $n_0 = 0$ and so every point of Γ_L is adjacent to two points of M. Hence the vertex sets of Γ_L and Γ_M are disjoint.
Suppose that the semipartial geometry $\mathcal{S}$ has a plane π of type 3. Let $L_1, \ldots, L_q$, denote q parallel lines of π. Since the lines of $\mathcal{S}$ contained in π form a net, every point of L_i is adjacent to at least one point of L_j (actually with exactly two points of L_j), $i, j = 1, \ldots, q$, $i \neq j$. And so L_i is disjoint from Γ_{L_j}, $i, j = 1, \ldots, q$, $i \neq j$. From the above this implies that the vertex sets of Γ_{L_i} and Γ_{L_j} are disjoint, $i, j = 1, \ldots, q$, $i \neq j$. Let Ω denote the vertices of $\Gamma(\mathcal{S})$ which are not vertices of Γ_{L_i} $i = 1, \ldots, q$. Then $|\Omega| = \frac{q^3-q^2}{2}$. Let x be a vertex in Ω, then by lemma 6.12, x is adjacent to $q^2 - q$ vertices of Γ_{L_i} in $\Gamma(\mathcal{S})$, $i = 1, \ldots, q$. Since the vertex sets of the Γ_{L_i} are disjoint, this covers already $q(q^2 - q)$ vertices of $\Gamma(\mathcal{S})$ that are adjacent to x. Since the valency of $\Gamma(\mathcal{S})$ equals $(q-1)(q^2+1)$, x is adjacent to $q-1$ vertices in Ω. Let Γ_0 denote the subgraph of $\Gamma(\mathcal{S})$ with vertex set Ω. Then a line L of the type 3 plane π uniquely determines the parallel class $L = L_1, \ldots, L_q$, of lines of $\mathcal{S}$ in π, and so it uniquely determines the graphs Γ_0 and Γ_{L_i}, $i = 1, \ldots, q$, which define a partition of $\mathcal{P}$ in $q+1$ regular subgraphs of $\Gamma(\mathcal{S})$, each of valency $q-1$ and each with $\frac{q^3-q^2}{2}$ vertices. $\square$

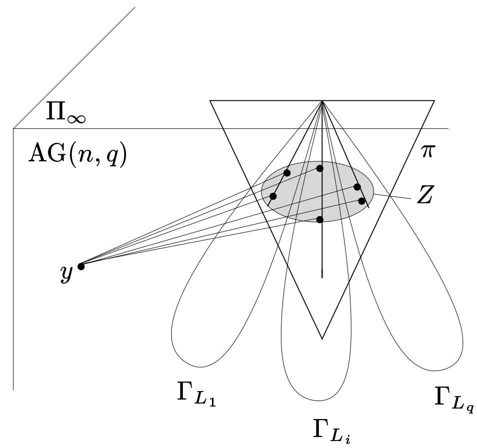

Figure 6.3: The structure of the set Z

Remark

The proof of lemma 6.13 is based on some methods used in [6].

Lemma 6.14 *Let x be a point of the semipartial geometry $\mathcal{S}$. Then θ_x is an ovoid of* Π_∞.

Proof. Let x be a point of the semipartial geometry $\mathcal{S}$ and suppose that M is a three-secant of θ_x. Then $\langle x, M\rangle$ is a plane π of type 3 of $\mathcal{S}$. Consider a parallel class $L_1, \ldots, L_q$, of lines of $\mathcal{S}$ in π, and consider the corresponding graphs Γ_0 and Γ_{L_i}, $i = 1, \ldots, q$, from lemma 6.13. Let y be a point of Γ_0, then by construction of the graphs Γ_{L_i}, y has at least one (and therefore exactly two) neighbours in each L_i, $i = 1, \ldots, q$ (see figure 6.3).
Let Z denote the set of $2q$ neighbours of y in π. Consider an element z of Z. Let K_1, K_2, K_3, denote the three lines of $\mathcal{S}$ in the type 3 plane π that contain z, then (y, K_i) is an antiflag with incidence number two. And so K_i intersects Z in the points z and z_i, $i = 1, 2, 3$.
If a projective line N of π through z which is not a line of $\mathcal{S}$, intersects Z in more than three points, then $\langle y, N\rangle \cap \theta_y > 3$, a contradiction since this implies an antiflag with incidence number greater than 2. Since the $q-2$ lines of π through z which are no lines of $\mathcal{S}$ partition the $2q-4$ points of $Z \setminus \{z, z_1, z_2, z_3\}$, we obtain that such a line must intersect Z in exactly two other points.
Let $N_1, \ldots, N_q$, be q parallel lines of π that are no lines of $\mathcal{S}$. Then the above implies that N_i intersects Z in 0 or 3 points, $i = 1, \ldots, q$. Therefore 3 divides $|Z|$ and so 3 divides q, a contradiction since $q = 2^h$ by corollary 6.11. Therefore θ_x has no three-secants. Since $|\theta_x| = q^2 + 1$, this implies that θ_x is an ovoid of Π_∞. $\square$

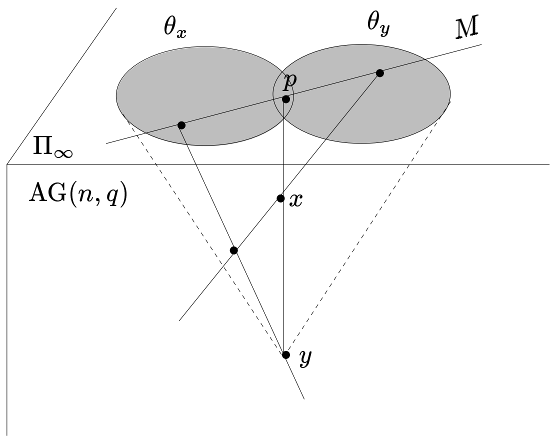

Figure 6.4: Two collinear points

Lemma 6.15 *Let x and y be two collinear points of $\mathcal{S}$, then a line M of Π_∞ incident with $p = \langle x, y\rangle \cap \Pi_\infty$ is either a tangent of both θ_x and θ_y, or it is a secant of both θ_x and θ_y with $M \cap \theta_x \cap \theta_y = \{p\}$.*

Proof. Let M be a line of Π_∞ incident with $p = \langle x, y\rangle \cap \Pi_\infty$. By lemma 6.14 we know that θ_x and θ_y are ovoids of Π_∞ and so $|M \cap \theta_x| \leq 2$ and $|M \cap \theta_y| \leq 2$. If $M \cap \theta_y = \{p\}$ and $|M \cap \theta_x| = 2$, then this contradicts $\alpha = 2$. Hence if $M \cap \theta_y = \{p\}$, then it is also the case that $M \cap \theta_x = \{p\}$, that is, M is a tangent of both θ_x and θ_y. Conversely if M is a secant line of θ_x then it is also a secant line of θ_y.
Now suppose that $|M \cap \theta_x| = |M \cap \theta_y| = 2$, and so $|M \cap \theta_x \cap \theta_y| = 1$ of 2. We now show that the case $|M \cap \theta_x \cap \theta_y| = 2$ does not occur by considering the number of points of $\Gamma(x) \cap \Gamma(y) \setminus \langle x, y\rangle$ in $\langle M, x\rangle$ where M is a line of Π_∞ on p, and Γ denotes the point graph of $\mathcal{S}$. If M is tangent to both θ_x and θ_y, then M is incident with no point of $\theta_x \setminus \{p\}$ and $\langle M, x\rangle$ contains no point of $\Gamma(x) \cap \Gamma(y) \setminus \langle x, y\rangle$. If $|M \cap \theta_x| = |M \cap \theta_y| = 2$ and $|M \cap \theta_x \cap \theta_y| = 1$, then $\langle M, x\rangle$ contains one point of $\Gamma(x) \cap \Gamma(y) \setminus \langle x, y\rangle$ (see figure 6.4). If $|M \cap \theta_x \cap \theta_y| = 2$, then $\langle M, x\rangle$ contains no points of $\Gamma(x) \cap \Gamma(y) \setminus \langle x, y\rangle$. Since $|\theta_x \setminus \{p\}| = q^2$ and

$$|\Gamma(x) \cap \Gamma(y) \setminus \langle x, y\rangle| = \lambda - |\langle x, y\rangle \setminus \{x, y\}| = q^2,$$

the case $|M \cap \theta_x \cap \theta_y| = 2$ does not occur. This proves the result. $\square$

Corollary 6.16 *Let x and y be two collinear points of $\mathcal{S}$, then $|\theta_x \cap \theta_y| = 1$.*

Proof. By lemma 6.15 a projective line of Π_∞ through $p = \langle x, y\rangle \cap \Pi_\infty$, intersects $\theta_x \cap \theta_y$ only in p, and so the result follows. $\square$

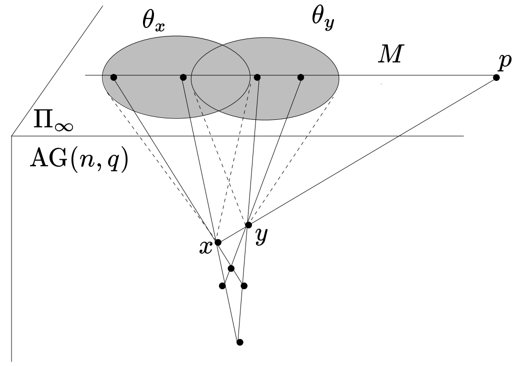

Figure 6.5: Two non-collinear points

Lemma 6.17 *Let x and y be two non-collinear points of $\mathcal{S}$ and define $p = \langle x,y\rangle \cap \Pi_\infty$. Let M be any line of Π_∞ incident with p. Then one of the following is the case:*

(i) *M is secant to both θ_x and θ_y and $M \cap \theta_x \cap \theta_y = \emptyset$;*

(ii) *M is tangent to both θ_x and θ_y at a point of $\theta_x \cap \theta_y$; or*

(iii) *M is external to both θ_x and θ_y.*

Furthermore $\theta_x \cap \theta_y$ is an oval with nucleus p.

Proof. First of all note that $\theta_x \cap \theta_y \neq \emptyset$ since θ_x and θ_y are ovoids by lemma 6.14. Let $r \in \theta_x \cap \theta_y$. Suppose that $\langle r,p\rangle$ is a secant line of at least one of the ovoids θ_x and θ_y. Say, without loss of generality, that $\langle r,p\rangle$ is a secant line of θ_x. Hence $(\theta_x \cap \langle r,p\rangle)\setminus\{r\} = \{u\}$ for some point u. Let $z = \langle u,x\rangle \cap \langle r,y\rangle$. Then x and z are collinear in $\mathcal{S}$ while $|\theta_x \cap \theta_z| \geq 2$, contradicting corollary 6.16. Hence $\langle p,r\rangle$ is a tangent line of both ovoids. Suppose that M is a line of Π_∞ incident with p and intersecting θ_x in the point v and θ_y in the point w, with $v,w \notin \theta_x \cap \theta_y$. Then $\langle v,x\rangle$ intersects $\langle w,y\rangle$, and so $\alpha(y,\langle x,v\rangle) = 2$. This implies that M intersects θ_y in the distinct points w and w', and moreover $w,w' \notin \theta_x \cap \theta_y$. Similarly, since $\alpha(x,\langle y,w\rangle) = 2$, it follows that M intersects θ_x in the distinct points v and v', with $v,v' \notin \theta_x \cap \theta_y$. In other words, M intersects both ovoids in two points outside their intersection. Since $|\Gamma(x) \cap \Gamma(y) \cap \langle x,y,M\rangle| = 4$ (see figure 6.5) and $|\Gamma(x) \cap \Gamma(y)| = \mu = 2q(q-1)$ it follows that there are exactly $q(q-1)/2$ lines incident with p that are secant to both θ_x and θ_y. Since this is the number of secants of an ovoid incident with a point not on the ovoid this means that the set of lines of Π_∞ incident with p and secant to θ_x is also the set of lines incident with p and secant to θ_y.
By corollary 6.11, q is even, and consequently the $q+1$ tangents of θ_x incident with p are contained in a plane π_x on p and similarly the $q+1$ tangents of θ_y

incident with p are contained in a plane π_y. There are two cases to consider: $\pi_x = \pi_y$ and $\pi_x \cap \pi_y$ is a line incident with p. First suppose that $\pi_x = \pi_y$. It follows that the tangents of θ_x incident with p are precisely the tangents of θ_y incident with p with a common point of tangency. Consequently $\theta_x \cap \theta_y$ is an oval of π_x with nucleus p. So in this case $|\theta_x \cap \theta_y| = q+1$.

Now suppose that $\pi_x \cap \pi_y$ is a line L incident with p. The line L is a tangent of both θ_x and θ_y at a point $o \in \theta_x \cap \theta_y$. If M is any other line of π_x incident with p, then by arguments above M must be external to θ_y. From this it follows that π_x is the tangent plane of θ_y at o and similarly π_y is the tangent plane of θ_x at o. Since $\langle p, o\rangle$ is the only line of Π_∞ incident with p that is tangent to both θ_x and θ_y it follows that $\theta_x \cap \theta_y = \{o\}$, and so $|\theta_x \cap \theta_y| = 1$.

It is now shown that the case $|\theta_x \cap \theta_y| = 1$ cannot occur. Suppose that $|\theta_x \cap \theta_y| = 1$. Let M be a line of Π_∞ incident with p and a secant of both θ_x and θ_y. It follows by arguments above that if $\theta_x \cap M = \{v, v'\}$ and $\theta_y \cap M = \{w, w'\}$, then $\{v, v', w, w'\}$ are four distinct points. Let $\{x = x_1, x_2, \ldots, x_q\}$ be the set of q points of $\mathcal{S}$ incident with the line $L = \langle x, v\rangle$. By corollary 6.16, $\theta_{x_i} \cap \theta_{x_j} = \{v\}$ for $i, j \in \{1, \ldots, q\}$, $i \neq j$, and by a consequence of lemma 6.15, the ovoids $\theta_{x_1}, \ldots, \theta_{x_q}$ have a common tangent plane at v, say π_v. It follows that the ovoids $\theta_{x_1}, \ldots, \theta_{x_q}$ partition the points of $\Pi_\infty \setminus \pi_v$ into q sets of size q^2. Without loss of generality assume that y is collinear with the points x_2 and x_3 of L, so by corollary 6.16 we obtain $|\theta_y \cap \theta_{x_2}| = |\theta_y \cap \theta_{x_3}| = 1$. By above arguments it follows that for $i = 4, \ldots, q$, $|\theta_y \cap \theta_{x_i}| = 1$ or $q+1$.

Suppose that $|\pi_v \cap \theta_y| = 1$, then since $v \notin \theta_y$ the ovoids $\theta_{x_1}, \ldots, \theta_{x_q}$ partition the q^2 points of $\theta_y \setminus (\pi_v \cap \theta_y)$ into q sets with size either 1 or $q+1$. This requires $q-1$ sets of size $q+1$ and 1 set of size 1. However $|\theta_{x_i} \cap \theta_y| = 1$ for $i = 1, 2$ and 3, a contradiction. Now suppose that $|\pi_v \cap \theta_y| = q+1$, then since $v \notin \theta_y$ the ovoids $\theta_{x_1}, \ldots, \theta_{x_q}$ partition the $q^2 - q$ points of $\theta_y \setminus (\pi_v \cap \theta_y)$ into q sets with size either 1 or $q+1$. This requires $q-2$ sets of size $q+1$ and 2 sets of size 1, again a contradiction.

It follows that $|\theta_x \cap \theta_y|$ cannot be 1 and so $\pi_x = \pi_y$ and $\theta_x \cap \theta_y$ is an oval of π_x with nucleus p. $\square$

Theorem 6.18 *Let $\mathcal{S}$ be a semipartial geometry* $\mathrm{spg}(q-1, q^2, 2, 2q(q-1))$ *embedded in* $\mathrm{AG}(4,q)$. *Then $q = 2^h$, $\mathcal{S}$ is isomorphic to* $\mathrm{TQ}(4,q)$ *and is embedded as the Hirschfeld-Thas model.*

Proof. Let $\mathcal{S}$ be a semipartial geometry $\mathrm{spg}(q-1, q^2, 2, 2q(q-1))$ embedded in $\mathrm{AG}(4,q)$. If $q = 2$, then $\mathcal{S}$ coincides with its point graph which is the unique complete graph on six vertices and the result follows. Hence we may assume that $q > 2$.

Let $\mathcal{K} = \Pi_\infty \cup \mathcal{P}$, where $\mathcal{P}$ is the point set of $\mathcal{S}$. The intersections of $\mathcal{K}$ with a plane of $\mathrm{PG}(4,q)$ are now considered which will allow the use of a result of Hirschfeld and Thas in [62] in order to prove the theorem. So let π be a plane of $\mathrm{PG}(4,q)$. If $\pi \subset \Pi_\infty$, then $\pi \subset \mathcal{K}$; so suppose that $\pi \not\subset \Pi_\infty$ and that $\pi \cap \Pi_\infty$ is the line M.

Suppose that π contains a point x of $\mathcal{S}$. Then M may either be a secant, tangent or external line of θ_x.

Suppose that M is a secant line of θ_x. This is the case if and only if there exists an antiflag (x, L) of $\mathcal{S}$ contained in π such that $\alpha(x, L) = 2$. By lemma 6.2 the lines of $\mathcal{S}$ in π form a dual oval $\mathcal{D}$ with nucleus M and these are all the lines of $\mathcal{S}$ in π. Let z be any point of $\mathcal{S} \cap \pi$ and not collinear in $\mathcal{S}$ with x. Then by lemma 6.17, M is a secant of θ_z; hence z is incident with exactly two lines of the dual oval $\mathcal{D}$. It follows that $\pi \cap \mathcal{K}$ is a dual hyperoval; or equivalently the complement of a maximal arc of type $(0, \frac{q}{2})$.

Next suppose that $M \cap \theta_x = \{p\}$. Hence M is a tangent of θ_x at p and all points of $\pi \cap \mathcal{S}$ are not collinear in $\mathcal{S}$ with x . If y is such a point of $\mathcal{S}$ on π, then by lemma 6.17 M is a tangent of θ_y at p and so $\langle p, y\rangle$ is a line of $\mathcal{S}$. It follows that lines of $\mathcal{S}$ in π are incident with p and that all points of $\mathcal{S}$ on π are incident with such a line. Let z be a point of $M \setminus \{p\}$, and let N be a secant of θ_x incident with z. By the above the plane $\langle N, x\rangle$ meets $\mathcal{K}$ in a dual hyperoval and since $z \notin \theta_x$ it follows that the line $\langle z, x\rangle$ is not a line of $\mathcal{S}$. Hence $\langle z, x\rangle$ is incident with exactly $\frac{q}{2}$ points of $\mathcal{S}$ and so π meets the line set of $\mathcal{S}$ in exactly $\frac{q}{2}$ lines each intersecting M in p. So M is a tangent of θ_x if and only if π meets $\mathcal{K}$ in the point set of $\frac{q}{2} + 1$ concurrent lines.

Finally suppose that M is an external line of θ_x. Let y be any point of M and let L be a secant of θ_x incident with y. Again by the above the plane $\langle L, x\rangle$ meets $\mathcal{K}$ in a dual hyperoval and since $y \notin \theta_x$ it follows that the line $\langle y, x\rangle$ is not a line of $\mathcal{S}$. Hence the line $\langle x, y\rangle$ is incident with $\frac{q}{2}$ points of $\mathcal{S}$. Hence each line of π incident with x is incident with $\frac{q}{2}$ points of $\mathcal{S}$. If z is any other point of $\mathcal{S}$ in π, then since x and z are not collinear and M is an external line of θ_x it follows by lemma 6.17 that M is also an external line of θ_z. Hence π meets $\mathcal{K}$ in a maximal arc of type $(0, \frac{q}{2})$ which has M as an external line.

By the above discussion a plane section of $\mathcal{K}$ is one of the following sets: (i) a single line; (ii) the entire plane; (iii) a maximal arc of type $(0, \frac{q}{2})$, plus an external line; (iv) a dual hyperoval, or equivalently, the complement of a maximal arc of type $(0, \frac{q}{2})$; or (v) $\frac{q}{2} + 1$ concurrent lines.

¿From this list it follows that with respect to the intersection with lines $\mathcal{K}$ is a set of points of type $(1, \frac{q}{2} + 1, q + 1)$.

Actually, it is possible to show that no planes of type (i) occur, but we do not need this. The set $\mathcal{K}$ does contain plane sections of type (iv), and for $q = 4$, $\mathcal{K}$ has no plane section that is either a unital or a subplane. Hence by [62, Theorem 6] the set $\mathcal{K}$ is the projection of a non-singular hyperbolic quadric of $\mathrm{PG}(5, q)$ onto $\mathrm{PG}(4, q)$, or the projection of a non-singular elliptic quadric of $\mathrm{PG}(5, q)$ onto $\mathrm{PG}(4, q)$. Any plane contained in $\mathcal{K}$ is also contained in Π_∞ which can only be the case if $\mathcal{K}$ is the projection of $\mathrm{Q}^-(5, q)$ onto $\mathrm{PG}(4, q)$. □

Corollary 6.19 *Let $\mathcal{S}$ be an $\mathrm{spg}(q-1, q^2, 2, 2q(q-1))$ embedded in $\mathrm{AG}(4, q)$. Then $q = 2^h$, and for any point x of $\mathcal{S}$, θ_x is an elliptic quadric $\mathrm{Q}^-(3, q)$.*

Remark

Theorem 6.18 characterizes $\mathrm{TQ}(4,q)$, q even, amongst the semipartial geometries $\mathrm{spg}(q-1,q^2,2,2q(q-1))$ by its embedding in $\mathrm{AG}(4,q)$. Hence none of the examples of $\mathrm{spg}(q-1,q^2,2,2q(q-1))$ of Brown constructed using Kantor flocks, nor the $\mathrm{TQ}(4,q)$, q odd, may be embedded in $\mathrm{AG}(4,q)$.
We can rephrase as follows our result for an $\mathrm{spg}(q-1,q^2,2,2q(q-1))$ constructed from a doubly subtended subquadrangle of order q of a generalized quadrangle of order (q,q^2).

Corollary 6.20 *Let $\mathcal{G}$ be a generalized quadrangle of order (q,q^2), $\mathcal{G}'$ a doubly subtended subquadrangle of $\mathcal{G}$ of order q, and $\mathcal{S}$ the $\mathrm{spg}(q-1,q^2,2,2q(q-1))$ constructed from $\mathcal{G}$ and $\mathcal{G}'$. If $\mathcal{S}$ may be embedded in $\mathrm{AG}(4,q)$, then $\mathcal{S}=\mathrm{TQ}(4,q)$, $\mathcal{G}=\mathrm{Q}^-(5,q)$, $\mathcal{G}'=\mathrm{Q}(4,q)$ and $q=2^h$.*

Proof. By theorem 6.18 $\mathcal{S}\cong\mathrm{TQ}(4,q)$ and $q=2^h$. Since $\mathcal{S}$ (in the model of Metz) may be constructed from the doubly subtended subquadrangle $\mathrm{Q}(4,q)$ of $\mathrm{Q}^-(5,q)$, it follows from [10, Theorem 3.3] that $\mathcal{G}=\mathrm{Q}(4,q)$ and $\mathcal{S}$ is the model of Metz in $\mathrm{Q}(4,q)$. By [111, Theorem 7.1] since $\mathrm{Q}(4,q)$ is doubly subtended in $\mathcal{G}$ with all subtended ovoids being elliptic quadrics on $\mathrm{Q}(4,q)$, it follows that $\mathcal{G}=\mathrm{Q}^-(5,q)$. $\square$

Appendix A

The known proper (semi)partial geometries

In this appendix we give a description of the known models of proper (semi)partial geometries, taken from [26] and [38].

A.1 The known proper partial geometries

The partial geometry $\mathcal{S}(\mathcal{K})$

This infinite family was constructed by Thas [97, 98] and independently by Wallis [118]. Define a *maximal arc* $\mathcal{K}$ of a projective plane π to be a non-empty set of k points in the plane such that any line intersects $\mathcal{K}$ in 0 or d points. Then $k = (q+1)(d-1)+1$ (we refer to [13, Chapter 7] for more information on maximal arcs). Let $\mathcal{K}$ be a maximal arc of degree d in a projective plane π of order q, that is a $\{qd-q+d;d\}$–arc. We define the incidence structure $\mathcal{S}(\mathcal{K}) = (\mathcal{P}, \mathcal{L}, \mathrm{I})$. The points of $\mathcal{S}(\mathcal{K})$ are the points of π that are not contained in $\mathcal{K}$. The lines of $\mathcal{S}(\mathcal{K})$ are the lines of π that are incident with d points of $\mathcal{K}$. The incidence is the one of π. Then $\mathcal{S}(\mathcal{K})$ is a partial geometry with parameters $t = q - q/d, s = q - d, \alpha = q - q/d - d + 1$.
As there exist $\{2^{h+m} - 2^h + 2^m; 2^m\}$–arcs, whenever $0 < m < h$, in PG(2,2^h), there exists a class of partial geometries $\mathcal{S}(\mathcal{K})$ with parameters $s = 2^h - 2^m$, $t = 2^h - 2^{h-m}, \alpha = (2^m - 1)(2^{h-m} - 1)$.

The partial geometry $T_2^*(\mathcal{K})$

Let $\mathcal{K}$ be a maximal arc of degree d in $\mathrm{PG}(2,q)$ ($q = p^h, p$ prime). As $\mathcal{K}$ has only passants and d-secants, it will yield a linear representation of a partial geometry in $\mathrm{AG}(3,q)$. This partial geometry $T_2^*(\mathcal{K})$ has parameters $t = (q+1)(d-1)$, $s = q-1, \alpha = d-1$. This infinite family was constructed for the first time by Thas [97, 98].

The partial geometry $T_2^*(\mathcal{K})$ using a maximal arc of degree 2^m, $0 < m < h$, in PG(2,2^h) has parameters $s = 2^h - 1, t = (2^h + 1)(2^m - 1), \alpha = 2^m - 1$.

The partial geometries $\mathbf{PQ}^+(4n-1,q)$, $q = 2$ or 3

For the description of this class of partial geometries we refer to section 1.5.2. Note that there exist several partial geometries with the same parameters as $\mathrm{PQ}^+(4n-1,q)$ which are spread derived from $\mathrm{PQ}^+(4n-1,q)$, $q = 2$ or 3 (see chapter 4 for more information).

The partial geometry from the Hermitian two-graph

A *two-graph* [89] (Ω, Δ) is a pair of a vertex set Ω and a triple set $\Delta \subset \Omega^{(3)}$, such that each 4-subset of Ω contains an even number of triples of Δ. A two-graph is called *regular* whenever each pair of elements of Ω is contained in the same number a of triples of Δ.

Given any graph $\Gamma = (X, \sim)$, one can construct a new graph by using Seidel-switching. It is known [89] that, given v there is a one-to-one correspondence between the two-graphs and the switching classes of graphs on the set of v elements. If the two-graph (Ω, Δ) is regular and if $(\Omega, \sim)$ is any graph in its switching class which has an isolated vertex $\omega \in \Omega$, then $(\Omega \setminus \{\omega\}, \sim)$ is a strongly regular graph.

Let $\mathcal{H}$ be the Hermitian curve in $\mathrm{PG}(2,q)$, q odd, defined by the Hermitian bilinear form $H(x,y)$. The Hermitian two-graph (Ω, Δ) is defined by taking as a vertex set Ω the set of $q^3 + 1$ points of $\mathcal{H}$ and a triple $\{x, y, z\} \in \Omega^{(3)}$ is an element of Δ if and only if $H(x,y)H(y,z)H(z,x)$ is a square (if $q \equiv -1 \pmod 4$) or a non-square (if $q \equiv 1 \pmod 4$) [96]. This two-graph appears to be regular with $a = \frac{(q^2+1)(q-1)}{2}$ and in its switching class there is indeed a graph which has an isolated vertex. This yields a strongly regular graph $\mathcal{H}(q)$ which is an $\mathrm{srg}(q^3, \frac{(q^2+1)(q-1)}{2}, \frac{(q-1)^3}{4} - 1, \frac{(q^2+1)(q-1)}{4})$ and is pseudo-geometric with parameters $s = q - 1, t = \frac{q^2-1}{2}, \alpha = \frac{q-1}{2}$. If $q = 3$ this graph is the point graph of the unique generalized quadrangle of order $(2,4)$. Although it has been proved (computer search) by Spence [94] that $\mathcal{H}(q)$ is not geometric for $q = 5$ and $q = 7$ it is remarkable that the graph is indeed geometric if $q = 3^{2m}$ which has been proved by Mathon; we refer to [80] for more details. A geometric construction of this partial geometry has been given by Kuijken [73].

The sporadic partial geometry of van Lint–Schrijver

Van Lint and Schrijver [75] constructed the following sporadic proper partial geometry. Let β be a primitive element of $\mathbb{F}_{3^4}$. Then $\gamma = \beta^{16}$ is a primitive 5-th root of the unity. Let $\mathcal{P} = \mathbb{F}_{81}$, let $\mathcal{L}$ be the set

$$\{(b, 1 + b, \gamma + b, \gamma^2 + b, \gamma^3 + b, \gamma^4 + b) \| b \in \mathbb{F}_{81}\},$$

I is the natural incidence, namely inclusion. Then $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a partial geometry with $s = t = 5$ and $\alpha = 2$.
Another construction of this geometry is given in [21]. Let C be the *ternary repetition code* of length 6, i.e.

$$C = \{(0,0,0,0,0,0),(1,1,1,1,1,1),(2,2,2,2,2,2)\}.$$

Any coset of C in $\mathbb{F}_3^6$ has a well-defined type i in $\mathbb{F}_3$, that is the sum i of the coordinates of any vector in the coset. Let $\mathcal{A}_i$ be the set of cosets of type i. Define a tripartite graph Γ by joining the coset $C+v$ to the coset $C+v+w$ for each vector w of weight 1. Any element in $\mathcal{A}_i$ has 6 neighbours in $\mathcal{A}_{i+1}$ and 6 in $\mathcal{A}_{i+2}$ (indices taken mod 3). Consider the incidence structure with point set $\mathcal{A}_i$, and line set $\mathcal{A}_{i+1}$, in which incidence is defined by adjacency in Γ. Then this incidence structure is the partial geometry of van Lint–Schrijver.

The sporadic partial geometry of W. Haemers

Haemers [53] constructed another sporadic proper partial geometry. It has parameters $s = 4, t = 17, \alpha = 2$. The point graph Γ however was known before (see for instance [66]). This graph Γ is constructed as follows. The vertices of Γ are the 175 edges of the Hoffman–Singleton graph $HoS(50)$. Two vertices of Γ are adjacent whenever the corresponding edges of $HoS(50)$ have distance two (that is the two edges are disjoint and there exists an edge connecting both). One can prove that this graph is a srg(175,72,20,36), moreover Γ is a pseudo–geometric $(17,4,2)$–graph. Haemers proved that Γ is indeed geometric. First of all we remark that a line of the partial geometry will be a set of 5 disjoint edges pairwise at distance two in the Hoffman–Singleton graph $HoS(50)$. It is easy to see that in a Petersen graph there are 6 such sets. If we can find 105 Petersen graphs in the Hoffman–Singleton graph, then we have the right number of lines. However there are more than 105 Petersen graphs in $HoS(50)$. Haemers was able to find a good subset of 105 special Petersen graphs in the Hoffman–Singleton graph, such that every pentagon of $HoS(50)$ is contained in exactly one such special Petersen graph. Note that any two edges at distance two in $HoS(50)$ are in a unique pentagon, so in a unique special Petersen graph, hence they define a unique set of 5 disjoint edges pairwise at distance two. In other words, the incidence structure of the 175 vertices of Γ and the 630 so-called 1-factors of the special Petersen graphs of $HoS(50)$ has the property that any two adjacent vertices define a unique line. This is enough to conclude that the pseudo–geometric graph Γ indeed is geometric.

Perp-systems

We refer to section 3.2 for the theory about perp-systems and partial geometries. Recall that Mathon [29] found by computer a partial geometry $\mathrm{pg}(8,20,2)$ coming from a perp-system.

A.2 The known semipartial geometries

The partial quadrangles with two points on a line

Let Γ be a strongly regular graph with $\lambda = 0$. Then this graph is a partial quadrangle with $s = 1$ and $t = k - 1$. Up to now the only known examples of such graphs are the pentagon $Pn(5)$, the Petersen graph $Pe(10)$, the Clebsch graph $Cl(16)$, the Hoffman–Singleton graph $HoS(50)$, and the graphs from the Higman–Sims family (i.e. $Gew(56)$, $HS(77)$ and $HS(100)$). The parameter sets (v, k, μ) for these graphs are resp. equal to (5,2,1), (10,3,1), (16,5,2), (50,7,1), (56,10,2), (77,16,4), (100,22,6). All these graphs are uniquely defined by their parameters.

The semipartial geometries $M(r)$, $r \in \{2, 3, 7, 57\}$

The three partial quadrangles with two points on a line such that $\mu = 1$ are better known as *Moore graphs.* These graphs are the graphs with valency $r > 1$, girth 5 (that is they have no 3-cycles nor 4-cycles but they do have 5-cycles) and with the minimum number of vertices, which is $r^2 + 1$. It is known that necessarily $r \in \{2, 3, 7, 57\}$. However a Moore graph with $r = 57$ is not known to exist.

The semipartial geometries $\overline{M(r)}$, $r \in \{2, 3, 7, 57\}$

With each Moore graph Γ there is associated another semipartial geometry, which we will denote by $\overline{M(r)}$. The point set $\mathcal{P}$ is the set of vertices Γ, the line set $\mathcal{L}$ is the set $\{\Gamma(x) \| x \in \mathcal{P}\}$, with $\Gamma(x)$ the set of vertices adjacent to x, I is the natural incidence relation. Then $\overline{M(r)} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a semipartial geometry with parameters $s = t = \alpha = r - 1$, $\mu = (r-1)^2$ ([40]).

The semipartial geometries $U_{2,3}(n)$

Let U be a set of cardinality n. Let $\mathcal{P}$ be the set of pairs, let $\mathcal{L}$ be the set of unordered triples of U, and let I be the inclusion relation. Then $U_{2,3}(n) = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a semipartial geometry with parameters $s = \alpha = 2$, $t = n - 3$, $\mu = 4$ ([40]). The point graph of this geometry is the triangular graph $T(n)$.

The semipartial geometries $LP(n, q)$

Define $\mathcal{P}$ as the set of lines of $\mathrm{PG}(n, q)$ $(n \geq 4)$, $\mathcal{L}$ as the set of planes of $\mathrm{PG}(n, q)$, and I as the inclusion relation. Then $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a semipartial geometry with parameters $s = q(q + 1)$, $t = \frac{q^{n-1}-1}{q-1} - 1$, $\alpha = q + 1$, $\mu = (q + 1)^2$ ([40]).

The semipartial geometries $\overline{W(2n + 1, q)}$

Let σ be a symplectic polarity of $\mathrm{PG}(2n + 1, q)$, $n \geq 1$. Let $\mathcal{P}$ be the point set of $\mathrm{PG}(2n + 1, q)$, $\mathcal{L}$ the set of lines which are not totally isotropic (that is *hyperbolic*) with respect to σ, and I the incidence relation of $\mathrm{PG}(2n+1,q)$. Then $W(2n+1,q) = (\mathcal{P},\mathcal{L},\mathrm{I})$ is a semipartial geometry with parameters $s=q,\ t=q^{2n}-1,\ \alpha=q,\ \mu=q^{2n}(q-1)$ ([40]).

The semipartial geometries $\mathrm{NQ}^+(2n-1,2)$ and $\mathrm{NQ}^-(2n-1,2)$

Let Q be a (non-singular) hyperquadric in $\mathrm{PG}(2n-1,2)$. Let $\mathcal{P}$ be the set of points off the quadric, let $\mathcal{L}$ be the set of non-intersecting lines of Q, and let I be the incidence of $\mathrm{PG}(2n-1,2)$. Then $(\mathcal{P},\mathcal{L},\mathrm{I})$ is a semipartial geometry with parameters $s=\alpha=2,\ t=2^{2n-3}-\varepsilon 2^{n-2}-1,\ \mu=2^{2n-3}-\varepsilon 2^{n-1}$, where $\varepsilon=+1$ for the hyperbolic quadric and $\varepsilon=-1$ for the elliptic quadric (we will denote this geometry by $\mathrm{NQ}^+(2n-1,2)$ and $\mathrm{NQ}^-(2n-1,2)$ respectively). This was first remarked by H. Wilbrink [private communication].

The semipartial geometries $H_q^{(n+1)*}$

This semipartial geometry is defined by taking as point set $\mathcal{P}$ the set of lines of a projective space $\Sigma \cong \mathrm{PG}(n+1,q)$ skew to a fixed projective space $H \cong \mathrm{PG}(n-1,q)$ and as line set $\mathcal{L}$ the set of the planes of Σ which intersect H in exactly one point. This semipartial geometry has parameters $s=q^2-1$, $t=\frac{q^n-1}{q-1}-1,\ \alpha=q,\ \mu=q(q+1)$.

The linear representations of semipartial geometries

Calderbank [16] has given an almost a complete classification of partial quadrangles with a linear representation. His proof is a number theoretic proof. He lists the possible parameter values of the associated strongly regular graph. We refer to section 6.1 for more information.
When $\alpha>1$, then the following models are known. The set $\mathcal{K}$ is a unital $\mathcal{U}$ in the projective plane Π_∞=PG(2,q^2) at infinity, and $T_2^*(\mathcal{U})$ has parameters $s=q^2-1, t=q^3, \alpha=q, \mu=q^2(q^2-1)$.
If $\mathcal{K}$ is a Baer subspace $\mathcal{B}$ of the projective space Π_∞ =PG(n,q^2) at infinity, then $T_n^*(\mathcal{B})$ has parameters $s=q^2-1,\ t=\frac{q^{n+1}-1}{q-1}-1,\ \alpha=q,\ \mu=q(q+1)$. Note that this geometry is isomorphic to $H_q^{(n+2)*}$.

Semipartial geometries and SPG-reguli

In [102] a new construction method for semipartial geometries is introduced using the so called SPG-reguli. We refer to section 3.2.1 for the definition of an SPG-regulus R in $\mathrm{PG}(n,q)$ consisting of r-dimensional spaces, the construction of the corresponding $\mathrm{spg}(q^{r+1}-1,|R|-1,\alpha,\ (|R|-\theta)\alpha)$, and some recent results. A spread R of the non-singular elliptic quadric $\mathrm{Q}^-(2m+3,q)$ $(m\geq 0)$ contains $q^{m+2}+1$ elements (of dimension m) and is always an SPG-regulus. The parameters of the corresponding semipartial geometry which we denote by $\mathrm{RQ}^-(2m+3,q)$, are $s=q^{m+1}-1,\ t=q^{m+2},\ \alpha=q^m,\ \mu=q^{m+1}(q^{m+1}-1)$.

For $m = 0$, this is the partial quadrangle $T_3^*(\mathcal{O})$. For $m = 1$, the semipartial geometry has parameters $s = q^2-1$, $t = q^3$, $\alpha = q, \mu = q^2(q^2-1)$ which also are the parameters of the semipartial geometry $T_2^*(\mathcal{U})$. Indeed $T_2^*(\mathcal{U})$ is isomorphic to the semipartial geometry arising from a *regular* spread R of $\mathrm{Q}^-(5,q)$. However if the spread is *non-regular*, then the associated semipartial geometry is not isomorphic to $T_2^*(\mathcal{U})$. If $m > 1$, and q is even, then the quadric $\mathrm{Q}^-(2m+3,q)$ has spreads, hence this yields new semipartial geometries. If q is odd, no spread of the quadric $\mathrm{Q}^-(2m+3,q)$ $(m > 1)$ is known.
If the non-singular quadric $\mathrm{Q}(2m+2,q)$ (of $\mathrm{PG}(2m+2,q)$), $m \geq 0$, has a spread R, then it is not an SPG-regulus.
If R is a spread of the quadric $\mathrm{Q}^+(2m+1,q)$, $m \geq 1$, then necessarily m is odd, moreover this spread is an SPG-regulus, but the associated semipartial geometry is a net.
Let $\mathrm{H}(n,q^2)$ be a non-singular Hermitian variety of $\mathrm{PG}(n,q^2)$, $n \geq 2$. If n is odd, the Hermitian variety has no spread (see [12] for the case $n = 3$ and [106] for $n \geq 5$). Assume that n is even. Then R is always an SPG-regulus with $m = \frac{1}{2}n - 1$ and $|R| = q^{n+1} + 1$. Hence there corresponds a semipartial geometry $\mathcal{S}$ with parameters $s = q^n - 1$, $t = q^{n+1}$, $\alpha = q^{n-1}$, $\mu = q^n(q^n - 1)$. However if $n = 2$ then this semipartial geometry is $T_2^*(\mathcal{U})$. Unfortunately for $n > 2$ no spread of $\mathrm{H}(n,q^2)$, n even is known. Brouwer [unpublished] proved that $\mathrm{H}(4,4)$ has no spread.

SPG-systems and semipartial geometries

Recently Thas [108] has generalized the concept of SPG-regulus of a polar space P to SPG-systems of P. We refer to section 2.3.5 for more information. His construction includes several known examples, but also a new classes of semipartial geometries, namely the classes $\mathrm{TQ}(2n+2,q)$ and $\mathrm{TH}(3,q^2)$, corresponding to an SPG-system of the quadric $\mathrm{Q}(2n+2,q)$ and an SPG-system of the hermitian variety $\mathrm{H}(3,q^2)$ respectively.

The Brown construction for semipartial geometries

Using the theory of doubly subtended ovoids, Brown [10] constructed a new semipartial geometry with the same parameters as $\mathrm{TQ}(4,q)$, q odd (see section 6.5.2 for more information)

Point derived semipartial geometries

In section 3.1 we introduced the concept of a point derived semipartial geometry. The point derived partial quadrangles are $\mathrm{spg}(s-1,s^2,1,s(s-1))$ and are derived from generalized quadrangles of order (s,s^2). There are a lot of generalized quadrangles of order (s,s^2) known. In all of them s is a prime power q and we will therefore now use q instead of s.
First of all there is the semi-classical example $T_3(\mathcal{O})$, constructed by Tits [113]. If p is the special point ∞ in $T_3(\mathcal{O})$ then the resulting partial quadrangle has

a linear representation in $\mathrm{AG}(4,q)$; it is the partial quadrangle $T_3^*(\mathcal{O})$ with $\mathcal{O}$ an ovoid in the hyperplane Π_∞ at infinity of $\mathrm{AG}(4,q)$. If p is any other point of $T_3(\mathcal{O})$ then the resulting partial quadrangle might be non-isomorphic to $T_3^*(\mathcal{O})$. On the other hand any *flock* of a cone in PG(3,q) implies the existence of a generalized quadrangle of order (q,q^2), and these generalized quadrangles give rise to a lot of non-isomorphic partial quadrangles with parameters $(q-1,q^2,q(q-1))$ (see for example [13, Chapter 9]).
Note that there is another way to construct semipartial geometries from generalized quadrangles by projection of a suitable generalized quadrangle (see section 6.5.1). This construction yields the semipartial geometries $\mathrm{TQ}(4,q)$ and $\mathrm{TH}(3,q^2)$ coming from an SPG-system.

A.3 Parameter list of the known (semi)partial geometries

In tables A.1 and A.2 we give a complete parameter list of all examples of partial and semipartial geometries known so far.

Notation	s	t	α	Remarks and references
$\mathcal{S}(\mathcal{K})$	$2^h - 2^m$	$2^h - 2^{h-m}$	$(2^m - 1)(2^{h-m} - 1)$	$0 < m < h$ and $h \neq 2$, [97, 98]
$T_2^*(\mathcal{K})$	$2^h - 1$	$(2^h + 1)(2^m - 1)$	$2^m - 1$	$0 < m < h$, [97, 98]
$\mathcal{M}_3(n)$	$3^{2n} - 1$	$\frac{1}{2}(3^{4n} - 1)$	$\frac{1}{2}(3^{2n} - 1)$	[80]
$\mathrm{PQ}^+(4n-1,2)$ (and derivations)	$2^{2n-1} - 1$	2^{2n-1}	2^{2n-2}	$1 < n$, [25, 27, 30]
$\mathrm{PQ}^+(7,3)$ (and derivations)	26	27	18	[25, 101]
vL-S	5	5	2	[75]
Haemers	4	17	2	[53]
Mathon	8	20	2	[29]

Table A.1: The known partial geometries (up to duality)

Notation	s	t	α	μ	Remarks and references
$M(r)$	1	$r-1$	1	1	Moore graph $r = 2, 3, 7$
$\overline{M(r)}$	$r-1$	$r-1$	$r-1$	$(r-1)^2$	$r = 2, 3, 7$, [40]
$U_{2,3}(n)$	2	$n-3$	2	4	$n \geq 4$ [40]
$LP(n,q)$	$q(q+1)$	$\frac{q^{n-1}-1}{q-1} - 1$	$q+1$	$(q+1)^2$	$n \geq 4$, [40]
$\overline{W(2n+1,q)}$	q	$q^{2n}-1$	q	$q^{2n}(q-1)$	$n \geq 1$,[40]
$\mathrm{NQ}^+(2n-1,2)$	2	$2^{2n-3} - 2^{n-2} - 1$	2	$2^{2n-3} - 2^{n-1}$	$n \geq 3$, [38]
$\mathrm{NQ}^-(2n-1,2)$	2	$2^{2n-3} + 2^{n-2} - 1$	2	$2^{2n-3} + 2^{n-1}$	$n \geq 3$, [38]
$H_q^{(n+1)*} \cong T_{n-1}^*(\mathcal{B})$	q^2-1	$\frac{q^n-1}{q-1} - 1$	q	$q(q+1)$	$n \geq 3$, [40]
$T_3^*(\mathcal{O})$	$q-1$	q^2	1	$q(q-1)$	[18]
$T_2^*(\mathcal{U})$	q^2-1	q^3	q	$q^2(q^2-1)$	[40]
$\mathrm{TQ}(2n+2,q)$	q^n-1	q^{n+1}	$2q^{n-1}$	$2q^n(q^n-1)$	[43, 108], if $n \geq 3$ then $q = 2^h$ [108] for $n = 1$ see also [10]
$\mathrm{TH}(3,q^2)$	q^2-1	q^3	$q+1$	$q(q+1)(q^2-1)$	[38]
$\mathrm{RQ}^-(2n+3,q)$	$q^{n+1}-1$	q^{n+2}	q^n	$q^{n+1}(q^{n+1}+1)$	q prime power for $n = 1$, $q = 2^h$ for $n \geq 2$ [102]
Gew(56)	1	9	1	2	$v = 56$; Gewirtz graph
HS(77)	1	15	1	4	$v = 77$; Higman-Sims family
HS(100)	1	21	1	6	$v = 100$ Higman-Sims family
$T_4^*(\mathcal{K}(11))$	2	10	1	2	$v = 243$; $\mathcal{K}(11)$ the 11-cap in $\mathrm{PG}(4,3)$
$T_5^*(\mathcal{K}(56))$	2	55	1	20	$v = 729$; $\mathcal{K}(56)$ the 56-cap in $\mathrm{PG}(5,3)$
$T_5^*(\mathcal{K}(78))$	3	77	1	14	$v = 1024$; $\mathcal{K}(78)$ the 78-cap of Hill in $\mathrm{PG}(5,4)$

Table A.2: The known semipartial geometries

Bijlage B

Nederlandstalige samenvatting

B.1 Definities en elementaire resultaten

In vele vakgebieden worden grafen bestudeerd. Dit komt omdat grafen handig zijn en ook algemeen. Ze zijn handig omdat wiskundige begrippen soms gemakkelijker kunnen geformuleerd worden wanneer we gebruik maken van een schematische voorstelling van binaire relaties, van grafen dus. Grafen ontstonden dan ook vanuit de toegepaste wiskunde, vanuit de modellering van netwerken, in eerste instantie met het probleem van de bruggen van Köningsberg (zie [77]). Ze zijn algemeen omdat er zo veel binaire relaties bestaan in een verzameling, maar daarom lijkt een *classificatie* van alle eindige grafen nogal onbegonnen werk. Dit kan veranderen zodra één of andere regelmaat of symmetrie verondersteld wordt. Soms kunnen algemene voorwaarden op een object zelfs leiden tot uniciteit van het object.
Een voorbeeld van voorwaarden op de structuur van een graaf is het begrip *sterk reguliere graaf*. Onder bepaalde voorwaarden kan een sterk reguliere graaf de structuur dragen van een meetkunde. Dit levert een vruchtbare interactie op tussen de grafentheorie en de incidentiemeetkunde. We verwijzen hiervoor bijvoorbeeld naar "*Designs, graphs, codes and their links*" [22], en naar *"Handbook of incidence geometry"* [13], standaardwerken in incidentiemeetkunde.

Definities (p. 1-15)

Na enkele algemene definities uit de grafentheorie, starten we met de definitie van een *sterk reguliere graaf* die we noteren als $\mathrm{srg}(v, k, \lambda, \mu)$. Dit is een reguliere graaf Γ van graad k en met v toppen, zodanig dat (i) voor elke twee adjacente toppen x en y er precies λ toppen bestaan die adjacent zijn met x en y; (ii) voor elke twee niet adjacente toppen x en y er precies μ toppen bestaan die adjacent zijn met x en y.

Een *partiële lineaire ruimte van de orde* (s,t) is een incidentiestructuur $\mathcal{S} = (\mathcal{P},\mathcal{L},\mathrm{I})$ waarbij $\mathcal{P}$ een (eindige) verzameling punten is, $\mathcal{L}$ een (eindige) verzameling rechten, en $\mathrm{I}\subseteq (\mathcal{P}\times\mathcal{L})\cup(\mathcal{L}\times\mathcal{P})$ is de (symmetrische) incidentierelatie, zodanig dat (i) twee punten incident zijn met ten hoogste één rechte; (ii) elk punt incident is met $t+1$ rechten; en zodat (iii) elke rechte incident is met $s+1$ punten. Zij (x,L) een *antivlag* van $\mathcal{S}$, dit wil zeggen een niet incident punt-rechte paar. Dan is het *incidentiegetal* $\alpha(x,L)$ van de antivlag (x,L) het aantal incidente punt-rechte paren (y,M) zodanig dat $x \mathrm{~I~} M \mathrm{~I~} y \mathrm{~I~} L$. Voor gehele getallen $\alpha,\beta\geq 0$ en $(\alpha,\beta)\neq(0,0)$ definiëren we een (α,β)-*meetkunde van orde* (s,t) als een partiële lineaire ruimte van orde (s,t) zodanig dat het incidentiegetal van elke antivlag (x,L) gelijk is aan α of β. Een *partiële meetkunde* met parameters s,t,α, die we noteren als $\mathrm{pg}(s,t,\alpha)$ is een (α,β)-meetkunde van orde (s,t) zodanig dat $\alpha=\beta$ (>0). Een semipartiële meetkunde met parameters s,t,α,μ, die we noteren met $\mathrm{spg}(s,t,\alpha,\mu)$, is een $(0,\alpha)$-meetkunde van orde (s,t) zodanig dat voor elke twee niet collineaire punten er μ (>0) punten bestaan die collineair zijn met beide gegeven punten. In de thesis gaan we er steeds van uit dat de semipartiële meetkunde geen partiële meetkunde is, dit wil zeggen $\mu<\alpha(t+1)$. De *puntgraaf* van een partiële lineaire ruimte $\mathcal{S}$ is de graaf met als toppen de punten van $\mathcal{S}$, twee verschillende toppen heten adjacent wanneer ze collineair zijn.

Voor veel van de gekende voorbeelden van (semi)partiële meetkunden zijn de punten en de rechten van de meetkunde de punten en de rechten van een projectieve of een affiene ruimte. Een incidentiestructuur $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ heet *ingebed* in een projectieve of affiene ruimte als $\mathcal{L}$ een deelverzameling is van de verzameling van rechten van de betreffende ruimte en $\mathcal{P}$ is de verzameling van alle punten van die ruimte op deze rechten. Een bijzondere klasse van $(0,\alpha)$-meetkunden ingebed in $\mathrm{AG}(n,q)$ zijn de zogenaamde *lineaire representaties* $T^*_{n-1}(\mathcal{K})$. Een lineaire representatie van een meetkunde $\mathcal{S}$, is een inbedding van $\mathcal{S}$ in $\mathrm{AG}(n,q)$ zodanig dat de rechtenverzameling $\mathcal{L}$ van $\mathcal{S}$ de unie is van parallelklassen van rechten van $\mathrm{AG}(n,q)$. Een *lineaire representatie* graaf $\Gamma^*_{n-1}(\mathcal{K})$ is de graaf met als toppen de punten van $\mathrm{AG}(n,q)$; twee toppen x en y heten adjacent als de rechte $\langle x,y\rangle$ het hypervlak Π_∞ op oneindig "snijdt" in een element van de puntenverzameling $\mathcal{K}$ van Π_∞. Merk op dat de puntgraaf van een lineaire representatie van een meetkunde $\mathcal{S}$ een lineaire representatie graaf is.

Opmerking

Sterk reguliere grafen werden geïntroduceerd door Bose [2]. De link naar de algebra werd gelegd door Bose en Mesner [3], en de link naar eigenwaardentechnieken door Hoffman [65]. In het eerste hoofdstuk verzamelen we enkele gekende resultaten in verband met sterk reguliere grafen. En met behulp van de lineaire representatie van een sterk reguliere graaf leggen we verbanden naar de codeertheorie.

In 1963 introduceerde Bose [2] het begrip *partiële meetkunde*, als veralgemening van de *veralgemeende vierhoeken* geïntroduceerd door Tits [113] (hier is $\alpha=1$). Semipartiële meetkunden werden geïntroduceerd door Debroey en Thas [40] in

1978, als veralgemening van zowel de partiële vierhoeken (geïntroduceerd door Cameron [18], ook hier is $\alpha = 1$) als de partiële meetkunden.

Definities (paragrafen 1.5 en 1.6 p. 11-15)

Bose [2] en Debroey en Thas [40] bewezen dat de puntgrafen van een partiële en semipartiële meetkunde sterk regulier zijn. Wanneer een gegeven sterk reguliere graaf Γ dezelfde parameters heeft als de puntgraaf van een partiële meetkunde $\mathrm{pg}(s,t,\alpha)$ (dit wil zeggen wanneer we de parameters van een gegeven sterk reguliere graaf Γ op een bepaalde manier kunnen schrijven in functie van positieve gehele getallen s, t, α met $1 \leq \alpha \leq \min\{s+1, t+1\}$) dan heten we deze graaf een *pseudo-meetkundige* (s,t,α)*–graaf*. Wanneer de graaf Γ inderdaad de puntgraaf is van minstens één partiële meetkunde, dan wordt de graaf Γ *meetkundig* genoemd. Nu kunnen er verscheidene niet isomorfe partiële meetkunden bestaan met dezelfde puntgraaf. Een pseudo-meetkundige graaf heten we *uniek meetkundig* als en slechts als er op isomorfisme na precies één partiële meetkunde bestaat met deze gegeven graaf als puntgraaf. Op een gelijkaardige manier definiëren we een *pseudo-semimeetkundige* (s,t,α,μ)*–graaf*, een *semimeetkundige graaf* en een *uniek semimeetkundige graaf*, alleen eisen we nu ook nog dat $\mu < \alpha(t+1)$.

Opmerking (paragrafen 1.5 en 1.6 p. 11-15)

Wanneer we voorwaarden op de parameters van sterk reguliere grafen vertalen naar voorwaarden op de parameters van pseudo-(semi)meetkundige grafen dan bekomen we opnieuw enkele bestaansvoorwaarden.
We geven tenslotte een voorbeeld van een partiële meetkunde, namelijk de oneindige klasse $\mathrm{PQ}^+(4n-1,q)$, $q = 2$ or 3. Deze meetkunde blijkt van groot belang te zijn voor de volgende hoofdstukken.

B.2 Pseudo-(semi)meetkundige grafen

Als we de puntgraaf nemen van een (semi)partiële meetkunde dan krijgen we een sterk reguliere graaf. Omgekeerd willen we, gegeven een pseudo-(semi)meetkundige graaf, nagaan of deze graaf inderdaad de structuur draagt van een (semi-) partiële meetkunde. In het tweede hoofdstuk worden enkele van die kandidaat grafen nader bestudeerd, wat uitmondt in zowel positieve als negatieve resultaten.

Collineariteitsgrafen (paragraaf 2.2.1 p. 18-19)

Een eerste klasse van grafen die we onderzoeken zijn de zogenaamde *collineariteitsgrafen* $\Gamma(P)$ van de klassieke poolruimten $P = W_n(q)$, $\mathrm{Q}(2n,q)$, $\mathrm{Q}^+(2n+1,q)$, $\mathrm{Q}^-(2n+1,q)$ en $\mathrm{H}(n,q^2)$. Deze graaf heeft als toppenverzameling de absolute punten van de poolruimte P; twee toppen heten adjacent als en slechts als ze gelegen zijn op een absolute rechte van P.

Gekende resultaten (paragraaf 2.2.2 p. 19 en stellingen 2.3-2.8 p. 19)

Verscheidene auteurs hebben reeds onderzoek gewijd aan het al dan niet meetkundig zijn van deze grafen: Brouwer [niet gepubliceerd], De Clerck, Gevaert en Thas [31], Mathon [79], Panigrahi [82], Payne en Thas [84], Thas [106] en Thomas [112].
De theorie van de klassieke veralgemeende vierhoeken leert ons dat de graaf $\Gamma(P)$, met $P \in \{\mathrm{Q}(4,q), \mathrm{Q}^+(3,q), \mathrm{Q}^-(5,q), W_3(q), \mathrm{H}(3,q^2), \mathrm{H}(4,q^2)\}$, meetkundig is. Ten slotte is ook de graaf $\Gamma(\mathrm{Q}^+(5,q))$ meetkundig. De volgende grafen zijn niet meetkundig: $\Gamma(\mathrm{Q}(8,q))$, q even; $\Gamma(\mathrm{Q}(6,q))$, q even; $\Gamma(\mathrm{Q}^-(7,2))$; $\Gamma(\mathrm{Q}^-(9,2))$; $\Gamma(\mathrm{Q}(4n+2,q))$, $n \geq 1$, q oneven; $\Gamma(\mathrm{H}(6,4))$; $\Gamma(\mathrm{H}(2m+1,q^2))$, $m \geq 2$; $\Gamma(W_5(q))$; $\Gamma(W_7(q))$.

Stelling 1 (stellingen 2.9 en 2.10 p. 20 en gevolg 2.11 p. 21)

1. *Als de graaf* $\Gamma(\mathrm{Q}(2m,q))$ *meetkundig is, dan is ook* $\Gamma(\mathrm{Q}^-(2m-1,q))$ *meetkundig. In het bijzonder zijn de grafen* $\Gamma(\mathrm{Q}(10,2))$ *en* $\Gamma(W_9(2))$ *niet meetkundig.*

2. *De graaf* $\Gamma(\mathrm{Q}(4n,q))$ *is niet meetkundig.*

De grafen van Wilbrink (paragraaf 2.3 p. 21-26)

Een tweede klasse van grafen die we beschouwen zijn de *grafen van Wilbrink*, die veralgemeningen zijn van de *grafen van Metz* (zie [9]). De belangrijkste graaf hiervan, die we noteren met $\mathrm{WIL}^-(2m,q)$ is een pseudo-semimeetkundige

$$\left(q^{m-1}-1, q^m, 2\cdot q^{m-2}, 2q^{m-1}(q^{m-1}-1)\right)\text{–graaf.}$$

Na een aantal niet bestaansresultaten voor andere grafen van Wilbrink die hun gevolgen hebben voor de zogenaamde *externe verzamelingen van kwadrieken*, komen we tot een positief resultaat: de constructie van een nieuwe klasse van semipartiële meetkunden.

Stelling 2 (stelling 2.19 p. 27) *Zij* Σ *een orthogonale spread van de kwadriek* $\mathrm{Q}(4n-2,q)$, $n \geq 2$, q *oneven. Definieer* $Hyp(\Sigma)$ *als de verzameling van alle hypervlakken van de elementen van* Σ*. Voor elk element* X *van* $Hyp(\Sigma)$*, definieer* L_X^i, $i = 1, \ldots, \frac{q(q-1)}{2}$, *als de* $(2n-2)$*-dimensionale affiene deelruimten bevat in* $X^* \cap E_1(4n-2,q)$ *die* X *op oneindig hebben. Beschouw de incidentiestructuur* $\mathrm{SPQ}(4n-2,q) = (E_1(4n-2,q), \mathcal{L}, \in)$ *waarbij*

$$\mathcal{L} = \{L_X^i : i = 1, \ldots, \frac{q(q-1)}{2}, X \in Hyp(\Sigma)\}.$$

Dan is $\mathrm{SPQ}(4n-2,q)$ *een semipartiële meetkunde*

$$\mathrm{spg}(q^{2n-2}-1, q^{2n-1}, 2\cdot q^{2n-3}, 2\cdot q^{2n-2}(q^{2n-2}-1)),$$

met puntgraaf $\mathrm{WIL}^-(4n-2,q)$*.*

Opmerking

De constructie van deze semipartiële meetkunde $\mathrm{SPQ}(4n-2,q)$, q oneven, hangt af van het bestaan van orthogonale spreads. Verschillende spreads leveren ook verschillende meetkunden. De kwadriek $\mathrm{Q}(4n,q)$ in $\mathrm{PG}(4n,q)$ met q oneven heeft geen spreads (zie [106]). Het bestaan van spreads van $\mathrm{Q}(4n-2,q)$, $n>2$ en q oneven, is een open vraag, terwijl $\mathrm{Q}(6,q)$ met q een oneven priemgetal of $q \equiv 0$ of $2 \pmod 3$ wel spreads heeft (zie [107]).
Wanneer $q=3$, dan blijkt er een verband te bestaan tussen de semipartiële meetkunde $\mathrm{SPQ}(4n-2,3)$ en de partiële meetkunde $\mathrm{PQ}^+(4n-1,3)$. Merk op dat die observatie aanleiding gaf tot de theorie van de *punt-afgeleide* semipartiële meetkunden van het volgende hoofdstuk.

Stelling 3 (stelling 2.21 p. 32) *Zij H een hypervlak van $\mathrm{PG}(4n-1,3)$, $n \geq 2$, dat de kwadriek $\mathrm{Q}^+(4n-1,3)$ snijdt in een $\mathrm{Q}(4n-2,3)$, en zodanig dat H^* een punt is van de partiële meetkunde $\mathrm{PQ}^+(4n-1,3)$. Dan is de puntenverzameling van $\mathrm{SPQ}(4n-2,3)$ de verzameling punten in H van $\mathrm{PQ}^+(4n-1,3)$; de rechten van $\mathrm{SPQ}(4n-2,3)$ zijn de niet ledige doorsneden van de rechten van $\mathrm{PQ}^+(4n-1,3)$ met H.*

B.3 Nieuwe constructiemethodes

We veralgemenen een gekende constructiemethode die semipartiële meetkunden haalt uit bepaalde veralgemeende vierhoeken (zie [37, 38]). We geven ook voorbeelden.

Definitie (paragraaf 3.1.1 p. 35)

Zij $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ een partiële meetkunde $\mathrm{pg}(s,t,\alpha)$. Zij p een punt van $\mathcal{S}$, definieer $p^\perp$ als de verzameling punten van $\mathcal{S}$ die collineair zijn met p (p inbegrepen) en zij $\mathcal{L}(p)$ de verzameling rechten van $\mathcal{S}$ door p. Beschouw dan de volgende incidentie structuur $\mathcal{S}_p=(\mathcal{P}_p,\mathcal{L}_p,\mathrm{I}_p)$ met $\mathcal{P}_p=\mathcal{P}\setminus p^\perp$, $\mathcal{L}_p=\mathcal{L}\setminus\mathcal{L}(p)$ en met $\mathrm{I}_p=\mathrm{I}\cap((\mathcal{P}_p\times\mathcal{L}_p)\cup(\mathcal{P}_p\times\mathcal{L}_p))$. We heten $\mathcal{S}_p$ de meetkunde *afgeleid* van $\mathcal{S}$ *met betrekking tot het punt* p. En dan wordt $\mathcal{S}_p$ *punt-afgeleid* genoemd.

Stelling 4 (stelling 3.1 p. 35) *Zij $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ een $\mathrm{pg}(s,t,\alpha)$ en zij p een punt van $\mathcal{S}$ zodanig dat voor elke drie niet collineaire punten p,y,z van $\mathcal{S}$, de verzameling $\{p,y,z\}^\perp$ van punten in $\mathcal{S}$ die collineair zijn met p,y en z, een constant aantal η elementen bevat. Dan is de afgeleide meetkunde $\mathcal{S}_p=(\mathcal{P}_p,\mathcal{L}_p,\mathrm{I}_p)$ met betrekking tot p een*

$$\mathrm{spg}(s-\alpha,t,\beta,\alpha(t+1)-\eta)$$

als en slechts als $\forall L\in\mathcal{L},\forall x\in\mathcal{P}_p: |L\cap p^\perp\cap x^\perp|\in\{\alpha,\alpha-\beta\}$.

Wanneer we de puntgrafen nemen van deze meetkunden dan bekomen we ook deelbaarheidsvoorwaarden op de parameters s,t,α,β,η.

Voorbeelden (paragraaf 3.1.2 p. 36)

Aangezien de theorie van de punt-afgeleide semipartiële meetkunden een uitbreiding is van een gekende theorie, bekomen we als eerste voorbeeld de puntafleidingen van de semi-klassieke veralgemeende vierhoek van Tits. Een tweede voorbeeld maakt gebruik van het verband tussen de partiële meetkunde $\mathrm{PQ}^+(4n-1,3)$ en de semipartiële meetkunde $\mathrm{SPQ}(4n-2,3)$.

Stelling 5 (gevolg 3.6 p. 40) *De partiële meetkunde* $\mathrm{PQ}^+(4n-1,3)$ *heeft een punt-afgeleide semipartiële meetkunde*

$$\mathcal{S}_p = \mathrm{spg}(3^{2n-2}-1, 3^{2n-1}, 2\cdot 3^{2n-3}, 2\cdot 3^{2n-2}(3^{2n-2}-1)).$$

Definities (paragraaf 3.2 p. 41-53)

In [102] werd er een nieuwe constructiemethode voor semipartiële meetkunden geïntroduceerd. Een *SPG-regulus* is een verzameling $\mathcal{R}$ van r-dimensionale deelruimten $\pi_1,\ldots,\pi_k$, $k>1$, van $\mathrm{PG}(N,q)$ die aan de volgende voorwaarden voldoen: (SPG-R1) $\pi_i \cap \pi_j = \emptyset$ voor alle $i \neq j$; (SPG-R2) Als een $\mathrm{PG}(r+1,q)$ een π_i bevat dan heeft deze $\mathrm{PG}(r+1,q)$ een punt gemeen met ofwel 0 of $\alpha(>0)$ elementen van $\mathcal{R}\setminus\{\pi_i\}$; als deze $\mathrm{PG}(r+1,q)$ disjunct is van alle π_j, $j\neq i$, dan wordt deze ruimte een *raakruimte* genoemd van de SPG-regulus $\mathcal{R}$ in π_i; (SPG-R3) als x een punt is van $\mathrm{PG}(N,q)$ dat niet bevat is in een element van $\mathcal{R}$, dan is het bevat in een constant aantal $(r+1)$-dimensionale raakruimten van $\mathcal{R}$.

Zij ρ een polariteit in $\mathrm{PG}(N,q)$ $(N\geq 2)$ en zij r $(r\geq 2)$ de rank van de corresponderende poolruimte P. Een *partieel m-systeem* M van P [91, 93], met $0\leq m\leq r-1$, is een verzameling $\{\pi_1,\ldots,\pi_k\}$ $(k>1)$ van totaal singuliere m-dimensionale deelruimten van P, zodanig dat geen enkele maximale totaal singuliere deelruimte die π_i bevat, een punt gemeen heeft met een element van $M\setminus\{\pi_i\}$, $i=1,2,\ldots,k$. Wanneer de kardinaliteit van M maximaal is dan wordt M een *m-systeem* genoemd.

We introduceren nu een object dat sterke verbanden heeft met m-systemen en SPG-reguli. Zij ρ een polariteit van $\mathrm{PG}(N,q)$. Definieer een *partieel perp-systeem* $\mathcal{R}(r)$ als een verzameling $\{\pi_1,\ldots,\pi_k\}$ van k $(k>1)$ onderling disjuncte r-dimensionale deelruimten van $\mathrm{PG}(N,q)$ zodat geen enkele π_i^ρ een punt gemeen heeft met een element van $\mathcal{R}(r)$. Wanneer de kardinaliteit van $\mathcal{R}(r)$ maximaal is dan heten we dit een *perp-systeem*, en dit kan aanleiding kan geven tot sterk reguliere grafen, twee-gewichtscodes, k-ovoïden en maximale bogen. Maar het belangrijkste meetkundig object dat afgeleid kan worden uit een perp-systeem is het volgende.

Stelling 6 (stelling 3.9 p. 45) *Zij $\mathcal{R}(r)$ een perp-systeem van $\mathrm{PG}(N,q)$ voorzien van een polariteit ρ, en zij $\overline{\mathcal{R}(r)}$ de verzameling punten van de elementen van $\mathcal{R}(r)$. Dan is de graaf $\Gamma^*(\overline{\mathcal{R}(r)})$ de puntgraaf van een partiële meetkunde*

$$\mathrm{pg}\left(q^{r+1}-1,\ \frac{q^{\frac{N-2r-1}{2}}(q^{\frac{N+1}{2}}+1)}{q^{\frac{N-2r-1}{2}}+1}-1,\ \frac{q^{r+1}-1}{q^{\frac{N-2r-1}{2}}+1}\right).$$

We geven ook enkele voorbeelden, waaronder een perp-systeem dat een nieuwe $\mathrm{pg}(8,20,2)$ oplevert.

B.4 Spread-afgeleide partiële meetkunden

Met behulp van de computer construeerden Mathon en Street [81] zeven nieuwe partiële meetkunden $\mathrm{pg}(7,8,4)$. Ze startten met de partiële meetkunde $\mathcal{S}_0 = \mathrm{PQ}^+(7,2)$ en dan gaan ze telkens de meetkunde afleiden met betrekking tot een geschikte *verwisselbare spread*. Deze constructie heten we dan ook *spread-afleiding*. Zo heeft $\mathcal{S}_0$ precies 3 verwisselbare spreads die na dualisering drie niet isomorfe partiële meetkunden $\mathrm{pg}(7,8,4)$ opleveren. De Clerck [25] bewees dit resultaat meetkundig. In dit hoofdstuk geven we een meetkundige constructie van de andere afgeleide meetkunden. Het volgende overzicht toont aan hoe de acht partiële meetkunden $\mathrm{pg}(7,8,4)$ met elkaar in verband staan. De pijl $\overset{\Phi_i}{\longleftrightarrow}$ betekent dat de partiële meetkunden met elkaar in verband staan door middel van afleiding met betrekking tot de verwisselbare spread Φ_i en na dualisering.

$$\begin{tikzcd}
\mathcal{S}_2 \arrow[r, leftrightarrow, "\Phi_2"] & \mathcal{S}_0 \arrow[r, leftrightarrow, "\Phi_1"] \arrow[d, leftrightarrow, "\Phi_3"'] & \mathcal{S}_1 \arrow[r, leftrightarrow, "\Phi_4"] \arrow[d, leftrightarrow, "\Phi_5"'] & \mathcal{S}_4 \arrow[r, leftrightarrow, "\Phi_6"] & \mathcal{S}_6 \arrow[r, leftrightarrow, "\Phi_7"] & \mathcal{S}_7 \\
& \mathcal{S}_3 & \mathcal{S}_5 & & &
\end{tikzcd}$$

Mathon and Street geven in [81] zowel informatie over de orde van de automorfismegroepen van de meetkunden als informatie over hun punt- en blokgrafen. Ze merken op dat de puntgrafen Γ_i van de meetkunden $\mathcal{S}_i, i = 1,2,3,4$, allen isomorf zijn, terwijl hun blokgrafen allen verschillend zijn. We bewijzen deze resultaten meetkundig.
De meetkundige constructies voor $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2$ en $\mathcal{S}_3$ zijn gekend [25, 30]. In het vierde hoofdstuk onderzoeken we de meetkunden $\mathcal{S}_4, \mathcal{S}_5$ en $\mathcal{S}_6$ waarvoor er nog geen meetkundige constructie gekend was. Merk op dat deze meetkunden en hun dualen in verband staan met de trialiteitskwadriek $\mathrm{Q}^+(7,2)$. De meeste bewijzen in dit hoofdstuk steunen dan ook op de speciale eigenschappen van deze kwadriek. We veralgemenen de constructies van $\mathcal{S}_4$ en $\mathcal{S}_5$ voor algemene dimensies $4n-1$, en zo construeren we twee nieuwe klassen van partiële meetkunden

$$\mathrm{pg}(2^{2n-1}-1, 2^{2n-1}, 2^{2n-2}).$$

Dus, vier van de acht gekende $\mathrm{pg}(7,8,4)$, namelijk $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_4, \mathcal{S}_5$, zijn het kleinste geval van een oneindige klasse.

Daarna beschouwen we de punt- en blokgrafen van deze meetkunden. Zelfs wanneer de puntgraaf van een gegeven partiële meetkunde uniek meetkundig is, dan nog is er geen garantie dat ook de blokgraaf uniek meetkundig is. In [31] werd er bewezen dat de puntgraaf van de partiële meetkunde $\mathrm{PQ}^+(7,2)$ uniek meetkundig is. In [82] geeft Panigrahi een ander bewijs van dit resultaat en met behulp van combinatoriek bewijst ze dat de blokgraaf van de partiële meetkunde $\mathrm{PQ}^+(7,2)$ uniek meetkundig is. We breiden dit resultaat uit voor andere grafen die in verband staan met de trialiteitskwadriek $\mathrm{Q}^+(7,2)$, en we geven een korter bewijs voor het resultaat van Panigrahi. We bewijzen bepaalde resultaten niet alleen voor $q=2$ maar ook voor $q=3$. Ten slotte bestuderen we ook de punt- en blokgraaf van de punt-afgeleide semipartiële meetkunde $\mathrm{SPQ}(6,3)$.

B.5 Twee-gewichtscodes en Steiner systemen

Puntenverzamelingen in een projectieve ruimte die twee intersectiegetallen hebben met betrekking tot hypervlakken (zoals niet ontaarde hyperbolische hyperkwadrieken) induceren veel andere meetkundige objecten: sterk reguliere grafen, twee-gewichtscodes, differentie verzamelingen, ... (zie [17, 48, 71]).

Definitie (paragraaf 5.1.2 p. 82)

Een *quasi-kwadriek* in $\mathrm{PG}(n,q)$ is een verzameling punten met dezelfde intersectiegetallen met betrekking tot hypervlakken als een niet ontaarde kwadriek in die ruimte. Uiteraard zijn kwadrieken zelf voorbeelden, maar er bestaan ook andere (zie bijvoorbeeld [32, 114]).
In het vijfde hoofdstuk construeren we drie nieuwe hyperbolische quasi-kwadrieken.

Stelling 7 (stelling 5.1 p. 82 en stelling 5.2 p. 84)

1. *Zij π een $(m-2)$-dimensionale deelruimte bevat in een generator G van de kwadriek $\mathrm{Q}^+(2m-1,q)$. Stel dat Π een $(m-1)$-dimensionale deelruimte is van π^* die de kwadriek snijdt in π. Dan is de puntenverzameling van $(\mathrm{Q}^+(2m-1,q)\setminus G)\cup\Pi$, een hyperbolische quasi-kwadriek. Wanneer $q>2$, beschouw dan een tweede $(m-1)$-dimensionale deelruimte Π' van π^* die de kwadriek snijdt in π, en zij G' de tweede generator van de kwadriek door π. Dan is puntenverzameling van $(\mathrm{Q}^+(2m-1,q)\setminus(G\cup G'))\cup\Pi\cup\Pi'$ een hyperbolische quasi-kwadriek.*

2. *Zij Y een $(m-3)$-dimensionale deelruimte bevat in een generator G van de kwadriek $\mathrm{Q}^+(2m-1,q)$. Stel dat Π een $(m-1)$-dimensionale deelruimte is van Y^* die de kwadriek snijdt in Y. Dan is de puntenverzameling van $(\mathrm{Q}^+(2m-1,q)\setminus G)\cup\Pi$, een hyperbolische quasi-kwadriek.*

Opmerking (paragrafen 5.2 en 5.3 p. 89-96)

Wanneer $q = 2$, dan bewijzen we dat de lineaire representatie graaf Γ_1^* die correspondeert met de quasi-kwadriek $(\mathrm{PG}(2m-1,2) \setminus \mathrm{Q}^+(2m-1,2)) \cup G$ switching equivalent is met de lineaire representatie graaf Γ_0^* die correspondeert met de kwadriek $\mathrm{Q}^+(2m-1,2)$. De graaf Γ_1^* is ook switching equivalent met de lineaire representatie grafen Γ_4^* en Γ_5^* die corresponderen met de twee quasi-kwadrieken in bovenstaande stelling. Meer nog, als m even is dan geldt voor deze grafen Γ_i^* dat elke tweede sterk reguliere deelgraaf de puntgraaf is van de partiële meetkunde $\mathcal{S}_i$, $i = 0,1,4,5$. De grafen Γ_i^* en de corresponderende partiële meetkunden $\mathcal{S}_i$ staan in verband met elkaar zodanig dat het volgende schema commutatief is voor $i = 0,4,5$.

$$\begin{array}{ccc} \Gamma_1^* & \stackrel{sw-co}{\longleftrightarrow} & \Gamma_i^* \\ \updownarrow & & \updownarrow \\ \mathcal{S}_1 & \stackrel{de-du}{\longleftrightarrow} & \mathcal{S}_i \end{array}$$

De pijl $\stackrel{sw-co}{\longleftrightarrow}$ betekent dat de grafen in verband staan met elkaar onder switching en complementeren, en $\stackrel{de-du}{\longleftrightarrow}$ betekent dat de partiële meetkunden in verband staan met elkaar onder spread-afleiding en dualiseren.

Definitie (paragraaf 1.4.2 p. 6 en paragraaf 5.4 p. 96-99)

Een t-(v,k,λ) *design* bestaat uit een tripel $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ met $\mathcal{P}$ een puntenverzameling, met $\mathcal{B}$ een blokkenverzameling, en I een incidentierelatie met de eigenschap dat elke t punten incident zijn met precies λ blokken, dat elk blok incident is met k punten, en dat k punten incident zijn met ten hoogste één blok. We veronderstellen dat $\mathcal{P}$ en $\mathcal{B}$ niet ledig zijn en dat $v \geq k \geq t$ (zodat $\lambda > 0$). Als $\lambda = 1$ wordt een t-design ook een *Steiner t-systeem* genoemd dat we noteren met $\mathrm{S}(t,k,v)$.

Designs vinden hun oorsprong in de statistiek. Omwille van hun mooie regelmaat bestaan er heel wat verbanden met de theorie van sterk reguliere grafen, (semi)partiële meetkunden, en codeertheorie (zie [22]). Een van de verbanden met partiële meetkunden is de volgende. Zij $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ een partiële meetkunde met parameters s, t, α. Dan willen we onderzoeken of er een Steiner 2-systeem $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ bestaat zodanig dat $\mathcal{L} \subset \mathcal{B}$. De partiële meetkunde $\mathcal{S}$ is dan ingebed in een Steiner 2-systeem

$$\mathrm{S}\left(2, s+1, \frac{(s+1)(st+\alpha)}{\alpha}\right).$$

De partiële meetkunde $\mathrm{pg}(7,8,4)$ $\mathcal{S}_0 = \mathrm{PQ}^+(7,2)$ is inbedbaar in een $\mathrm{S}(2,8,120)$ [7]. We geven een kort bewijs van dit resultaat en we tonen aan dat $\mathcal{S}_0$ inbedbaar is in minstens vier $\mathrm{S}(2,8,120)$. We bewijzen ook dat de afgeleide meetkunden $\mathcal{S}_i$ $(i = 1,2,3)$ inbedbaar zijn in een $\mathrm{S}(2,8,120)$.

B.6 Inbeddingen van $(0, \alpha)$-meetkunden in affiene ruimten

Er bestaat een volledige klassificatie van partiële meetkunden ingebed in $\mathrm{PG}(n, q)$ (zie [14] voor de veralgemeende vierhoeken en zie [33] voor $\alpha > 1$). De klassificatie van partiële meetkunden ingebed in $\mathrm{AG}(n, q)$ is ook gekend (zie [100]). De Clerck, Debroey en Thas bepaalden alle inbeddingen van semipartiële meetkunden in $\mathrm{PG}(n, q)$ [34, 42, 110]. In [41] klassificeerden Debroey and Thas de semipartiële meetkunden ingebed in $\mathrm{AG}(n, q)$ voor $n = 2$ en 3. Maar voor $n > 3$ bestaat geen klassificatie. In het zesde en laatste hoofdstuk van de thesis geven we nieuwe karakterisaties van $(0, \alpha)$-meetkunden die ingebed zijn in $\mathrm{AG}(n, q)$.

Stelling 8 (gevolg 6.7 p. 107) *Een $(0, \alpha)$-meetkunde, $\alpha \neq 1, 2$, ingebed in $\mathrm{AG}(n, q), n > 2$, is een lineaire representatie $T^*_{n-1}(\mathcal{K})$.*

Als gevolg bewijzen we dat voor een duale semipartiële meetkunde $\mathcal{S}$ ingebed in $\mathrm{AG}(n, q)$, geldt dat $\alpha = 1$. Bovendien kan $\mathcal{S}$ geen lineaire representatie zijn. Een ander model van een semipartiële meetkunde ingebed in $\mathrm{AG}(4, q)$, q even, is de $\mathrm{spg}(q-1, q^2, 2, 2q(q-1))$ $\mathrm{TQ}(4, q)$ van Hirschfeld and Thas [62]. Deze meetkunde werd geconstrueerd door projectie van de kwadriek $\mathrm{Q}^-(5, q)$ vanuit een punt van $\mathrm{PG}(5, q) \setminus \mathrm{Q}^-(5, q)$ op een hypervlak. Merk op dat dit geen lineaire representatie is. De semipartiële meetkunde $\mathrm{TQ}(4, q)$, q even, wordt gekarakteriseerd onder de $\mathrm{spg}(q-1, q^2, 2, 2q(q-1))$ (waarvan er een oneindige klasse bestaat van niet klassieke voorbeelden van Brown [10]) door zijn inbedding in $\mathrm{AG}(4, q)$.

Stelling 9 (stelling 6.18 p. 116) *Zij $\mathcal{S}$ een $\mathrm{spg}(q-1, q^2, 2, 2q(q-1))$ ingebed in $\mathrm{AG}(4, q)$. Dan is $q = 2^h$, en is $\mathcal{S}$ isomorf met $\mathrm{TQ}(4, q)$.*

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