Automorphisms and Combinatorics of Finite Generalized Quadrangles

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Attribution

Most of the (mathematical) work of this thesis was done before the end of February 2001. However, a large part of the results of Chapter 14 and [45] were (essentially) obtained by Dr. B. De Bruyn and myself during a short visit in Kansas at the end of March 2001, for the occasion of the conference 'Finite Simple Groups, Geometries, Buildings and Related Topics' in honor of E. E. Shult.
Although I introduced the notion of *half pseudo Moufang generalized quadrangle* in 1999 [204], it was only at the very end of 2001 that Prof. Dr. Hendrik Van Maldeghem and I succeeded in completely classifying those objects, a theorem which resulted in Chapter 13 and [220].
Finally, the results of Chapter 11 and [191] were obtained in collaboration with J. A. Thas in the Summer of 2001 during some very pleasant conversations.

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Preface

Buildings and Generalized Polygons

Generalized quadrangles were introduced by J. Tits [221] in his celebrated work on triality as a subclass of a larger class of incidence structures, namely the generalized polygons, in order to understand the Chevalley groups of rank 2. Generalized polygons themselves form a subclass of the class of buildings. We therefore find it convenient to indicate the precise interrelation between buildings and generalized polygons.
The reader does not have to know all the (especially group theoretical) technical details in order to fully understand the purpose of this part of the preface.

Initially, the purpose of the theory of buildings was primarily to understand the exceptional Lie groups from a geometrical point of view. The starting point appeared to be the observation that it is possible to associate with each complex analytic semisimple group a certain well-defined geometry, in such a way that the 'basic' properties of the geometries thus obtained and their mutual relationships can be easily read from the Dynkin diagrams of the corresponding groups. The definition of these geometries was suggested by the following reconstruction method of $\mathbf{PG}(n, \mathbb{C})$ from the projective linear group $\mathbf{PGL}(n+1, \mathbb{C})$:

(i) the linear subspaces of the projective space $\mathbf{PG}(n, \mathbb{C})$ can be represented

by their stabilizers in $\mathbf{PGL}(n+1,\mathbb{C})$ (which, by a theorem of Lie, are the maximal connected non-semisimple subgroups of $\mathbf{PGL}(n+1,\mathbb{C})$);

(ii) the conjugacy classes of these subgroups represent the set of all points, the set of all lines, ..., and the set of all hyperplanes of $\mathbf{PG}(n,\mathbb{C})$[1];

(iii) two linear subspaces are incident if and only if the intersection of the corresponding subgroups contains a maximal connected solvable subgroup of $\mathbf{PGL}(n+1,\mathbb{C})$.

Generalizing the above example, it seemed natural to associate with an arbitrary complex semisimple group G a geometry consisting of a set (the set of maximal parabolic subgroups), partitioned into classes (the conjugacy classes), parametrized by the vertices of the Dynkin diagram M of G, and endowed with an incidence relation as follows: two maximal parabolic subgroups are incident if their intersection contains a maximal connected solvable subgroup (the 'Borel subgroup'). Now we have the following two essential properties:

(D1) let M be the Dynkin diagram, Γ the associated geometry, x an object in Γ, and $v(x)$ the vertex of M corresponding to the class of x; then the *residue* of x in Γ — that is, the geometry consisting of all objects of Γ distinct from x but incident with x, with the partition and incidence relation induced by those by Γ — is the geometry associated with the diagram obtained from M by deleting $v(x)$ and all strokes containing it;

(D2) when one knows the geometries associated with the Dynkin diagrams of rank 2 (that is, those having two vertices), Assertion (D1) 'almost' characterizes that geometry uniquely.

As the theory of algebraic semisimple groups over an arbitrary field emerged, it then became apparent that the same geometrical theory would apply to the group $G(\mathbb{K})$ of rational points of an arbitrary isotropic algebraic semisimple group G defined over any field $\mathbb{K}$, as:

1. such a group $G(\mathbb{K})$ also has 'parabolic subgroups';

2. the conjugacy classes of the maximal parabolic subgroups are parametrized by the vertices of the so-called 'relative Dynkin diagram' of G over $\mathbb{K}$;

[1]For arbitrary complex semisimple groups, the conjugacy classes of the maximal connected non-semisimple subgroups are called *maximal parabolic subgroups*.

3. for the geometry consisting of the set of all maximal parabolic subgroups partitioned into conjugacy classes and endowed with a suitable incidence relation, the relation between residues and subdiagrams which was explained in (D1) holds.

Then J. Tits noticed that all the geometries associated with a diagram consisting of a single stroke with multiplicity $n-2$, $n \geq 3$, satisfied the following important property:

(GP_n) *For any two elements x and y of the geometry, there is a sequence $x = x_0, x_1, \ldots, x_m = y$ of elements so that x_i and x_{i+1} are incident and $x_i \neq x_{i+2}$ for all $i = 0, 1, \ldots, m-2$, with $m \leq n$, and if $m < n$, then the sequence is unique.*

At that point, *geometries of type M* — which are the direct precursors of buildings — were introduced, their definition being directly inspired by the latter observation(s). Property (GP_n) is the main axiom for the *generalized polygons*, see below. Before proceeding with the actual definition of the latter point-line geometries, we first state the definition of building in terms of chamber geometries. It should be noted however that, although (GP_n) will not explicitly turn up in that definition, it will be reflected directly in Property (S), below. A *chamber geometry* is a geometry $\Gamma = (\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_j, I)$ of rank j (so Γ has j different kinds of varieties and I is an incidence relation between the elements such that no two elements belonging to the same $\mathcal{C}_i$, $1 \leq i \leq j$, can be incident) so that the simplicial complex $(\mathcal{C}, X)$, where $\mathcal{C} = \cup_{i=1}^{j} \mathcal{C}_i$ and $S \subseteq \mathcal{C}$ is contained in X if and only if every two distinct elements of S are incident, is a chamber complex (as in, e.g., [229]). A *building* $(\mathcal{C}, X)$ is a thick chamber geometry $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_j, I)$ of rank j, where $\mathcal{C} = \cup_{i=1}^{j} \mathcal{C}_i$, together with a set $\mathcal{A}$ of thin chamber subgeometries (called *apartments*), so that:

(i) every two chambers are contained in some element of $\mathcal{A}$;

(ii) for every two elements Σ and Σ' of $\mathcal{A}$ and every two simplices F and F', respectively contained in Σ and Σ', there exists an isomorphism $\Sigma \mapsto \Sigma'$ which fixes all elements of both F and F'.

If all elements of $\mathcal{A}$ are finite, then the building is called *spherical*. Let us mention for the sake of completeness, that there also is a very subtle graph theoretical definition of the notion of buildings which was observed by E. E. Shult, see [157].

Let $n \geq 1$ be a natural number. A *generalized n-gon* is a point-line geometry $\Gamma = (P, B, I)$ so that the following axioms are satisfied:

(i) Γ contains no ordinary k-gon (as a subgeometry), for $2 \leq k < n$;

(ii) any two elements $x, y \in P \cup B$ are contained in some ordinary n-gon (as a subgeometry) in Γ;

(iii) there exists an ordinary $(n+1)$-gon (as a subgeometry) in Γ.

Note that (i) and (ii) reflect the essence of (GP_n), and that the generalized 3-gons are precisely the projective planes. A geometry Γ which satisfies (i) and (ii) is a *weak generalized n-gon*. If (iii) is not satisfied for Γ, then Γ is called *thin*. Otherwise, it is called *thick*. Sometimes we will speak of 'thick (respectively thin) generalized n-gon' instead of 'thick (respectively thin) weak generalized n-gon'. Each thick generalized n-gon, $n \geq 3$, Γ has an order: there are (not necessarily finite) constants s and t so that each point is incident with $t+1$ lines and each line is incident with $s+1$ points. We then say that Γ 'has order (s,t)'.

The relation between buildings and *generalized polygons* (which are just generalized n-gons for some n), as observed by J. Tits in [225], is now as follows:

(S) *Suppose $(\mathcal{C}, X)$, $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, is a spherical building of rank 2. Then $\Gamma = (\mathcal{C}_1, \mathcal{C}_2, I)$ is a generalized polygon. Conversely, suppose that $\Gamma = (P, B, I)$ is a generalized polygon, where $\mathcal{F}$ is the set of its flags. Then $(P \cup B, \emptyset \cup \{\{v\} \parallel v \in P \cup B\} \cup \mathcal{F})$ is a chamber geometry of rank 2. Declaring the thin chamber geometry corresponding to any ordinary subpolygon an apartment, we obtain a spherical building of rank 2.*

So, in view of (D1), generalized polygons truly are the essential particles of buildings, as was already emphasized by (GP_n) in the precursorial introduction of buildings.
The first time that the notion of generalized polygon appeared in literature, however, was slightly before the introduction of geometries of type M and of buildings. In 1959, in the appendix of his celebrated paper "Sur la trialité et certains groupes qui s' en déduisent" (*Publ. Math. I.H.E.S.* **2**, 13–60) [221], where J. Tits discovered the simple groups of type ${}^3D_4$ by classifying the trialities with at least one absolute point of the D_4-geometry (as in, e.g., [229, 2.4.2]), which arises from the quadric $\mathcal{Q}^+(7, \mathbb{K})$ in $\mathbf{PG}(7, \mathbb{K})$, J. Tits gave a definition of generalized polygons which was suggested by some remarkable properties — notably 4.3.1 and 4.3.2 of [221] — of the geometry $\Gamma^{(i)}$ with point set the absolute i-points and line set the absolute lines of a triality, and with the natural incidence, see [221] or [229, 2.4.3 and 2.4.4]. Examples of generalized 4-gons (or 'generalized quadrangles') were already known for some time as the geometries of points and lines which are contained in a nonsingular quadric of Witt index

2, or in a nonsingular Hermitian variety in dimension 3 or 4, or the geometry which consists of the points and totally isotropic lines of a symplectic polarity in $\mathbf{PG}(3,q)$. These examples are called the 'classical generalized quadrangles', which were first discovered as such by J. Tits in [48]. Their point-line duals are called 'dual classical generalized quadrangles'. (For $n = 6, 8$, there are also 'classical' and 'dual classical' examples defined, see [229].)

'Abstract' generalized quadrangles were already studied under various other names in the specialized literature, however, such as, for example, '4-gonal configurations'.

Some History of Finite Generalized Quadrangles

We now focus mostly on (finite) generalized quadrangles (GQ's), and it is our intention to describe some of the most important developments and subtheories in that subject. We make no claim of being complete, and we alone are responsible for any shortcoming.

Let us first mention that in 1964, the theory of generalized polygons got a first great impulse from the marvelous theorem of W. Feit and G. Higman [58], which asserts that for a finite generalized n-gon Γ, n necessarily must be contained in $\{2, 3, 4, 6, 8, 12\}$ (and in the $(n = 12)$-case, Γ is thin). Also, for finite generalized quadrangles of order (s, t), $s \neq 1 \neq t$, D. G. Higman [74] showed (in 1971) that $t \leq s^2$ and $s \leq t^2$.

Some years later, in 1968, it was Jacques Tits himself who provided the first non-classical examples of (finite) generalized quadrangles (of order q and of order (q, q^2)) in the celebrated book of P. Dembowski [48]. In the meantime, S. E. Payne and J. A. Thas started to systematically investigate especially the combinatorics of finite generalized quadrangles, obtaining many characterizations of the classical examples, but also introducing for instance a theory for ovoids and spreads of generalized quadrangles. The characterizations obtained by various authors of both the classical and the known non-classical examples in the last three decades form one of the highlights in the theory of finite generalized quadrangles. Also, especially by the seminal work of J. A. Thas in that period, a theory became available for subquadrangles (subGQ's) of generalized quadrangles. Spreads, ovoids and subGQ's now play a central role in the combinatorial theory of GQ's. It is also appropriate to mention here that the interrelation between dual nets and generalized quadrangles with a regular

point has appeared to be very fruitful in that context.

Each of the classical or dual classical generalized polygons satisfies the so-called 'Moufang condition'. In their celebrated papers [60, 61], P. Fong and G. M. Seitz classified the finite groups with a split BN-pair of rank 2. Their main theorem infers that a finite generalized polygon is classical (or dual classical) if and only if it satisfies one of (i),(ii) of [229, 5.7.2]. If Γ is a finite Moufang generalized polygon, then it follows easily that Condition (ii) (which is equivalent to (i)), holds, see [229, 5.7.2]. This was first noted by J. Tits in [223]. As a direct corollary of that observation, it follows that, conversely, each finite Moufang generalized polygon is one of the classical or dual classical examples. As the theorem of P. Fong and G. M. Seitz is obtained by applying deep group theoretical results, S. E. Payne and J. A. Thas initiated the search for an 'elementary' proof for the case of the generalized quadrangles, see especially [139, Chapter 9]. They almost completely succeeded in that project, except for one open case. We refer the reader to Chapter 13 and Appendix A for more details. The results obtained by P. Fong and G. M. Seitz inspired a lot of researchers (such as W. M. Kantor, M. Ronan, S. E. Payne, J. A. Thas, J. Tits, H. Van Maldeghem, M. Walker, R. Weiss) to obtain characterizations of the classical and dual classical generalized quadrangles, respectively generalized polygons, in the same spirit, i.e. to characterize those examples by certain transitivity properties. Numerous results were obtained in the last three decades.

J. Tits defined the finite classical generalized quadrangles as being associated to certain classical groups. He did it in such a way though, that they are all (up to duality) fully embedded in finite projective space [48]. By a famous result of F. Buekenhout and C. Lefèvre [29], the converse also holds; each finite generalized quadrangle which is fully embedded in a finite projective space is one of the classical examples. The dual classical Hermitian quadrangle $H(4,q^2)^D$ does not admit such a full embedding, however; this is the reason why we prefer to make a distinction between 'classical' and 'dual classical' generalized quadrangles (respectively generalized polygons). Then J. A. Thas obtained the analogous theorem for full embeddings in finite affine spaces, in [167]. The result of F. Buekenhout and C. Lefèvre seemed to inspire a bunch of researchers (such as P. J. Cameron, F. De Clerck, I. Debroey, P. De Winne, K. J. Dienst, W. M. Kantor, C. Lefèvre, D. Olanda, A. Steinbach, J. A. Thas, H. Van Maldeghem) to develop a deep theory for (various kinds of) embeddings for generalized polygons in general, and also for partial and semipartial geometries, partial quadrangles, polar spaces, etc.. At present, the main contributors are without any doubt J. A. Thas and H. Van Maldeghem, who obtained many

deep results on, e.g., lax embeddings of polar spaces and finite generalized quadrangles, and who are developing a complete theory for embeddings of generalized hexagons, see, e.g., [197, 199, 200].

Then, in 1980, W. M. Kantor [90] introduced the notion of 4-gonal family, which provided a general construction of (elation) generalized quadrangles as group coset geometries. In that same paper, W. M. Kantor introduced the first non-classical family of GQ's of order (s, s^2), denoted $K(s)$, which did not arise as a $T_3(\mathcal{O})$ of Tits for some ovoid $\mathcal{O}$ in $\mathbf{PG}(3, s)$, while studying generalized hexagons and the simple groups $\mathbf{G}_2(s)$.[2] Some time later, S. E. Payne introduced the q-clans, which are certain sets of matrices giving rise to 4-gonal families, and hence also to generalized quadrangles. It were those observations that lead to the first non-classical examples of generalized quadrangles of order (q, q^2), $q > 1$, since the examples of J. Tits and of W. M. Kantor. Also, the results of W. M. Kantor and S. E. Payne suggested the importance of a general theory for elation generalized quadrangles (EGQ's).

Already in 1974, J. A. Thas introduced the notion of 'translation 4-gonal configuration' [165], now better known under the name 'translation generalized quadrangle' (TGQ). Later, S. E. Payne and J. A. Thas developed a complete general theory for these objects, which resulted in Chapter 8 of their monograph [139]. In fact, TGQ's are equivalent to an abelian group which admits a 4-gonal family of W. M. Kantor (and as such, it is a particular kind of EGQ). The GQ's $T_d(\mathcal{O})$ of Tits, $d = 2, 3$, are examples of TGQ's, and S. E. Payne and J. A. Thas generalized the Tits construction (which started from ovals or ovoids in $\mathbf{PG}(2, q)$, respectively $\mathbf{PG}(3, q)$) to a construction of TGQ's from so-called 'generalized ovals' (in $\mathbf{PG}(3n - 1, q)$) and 'generalized ovoids' (in $\mathbf{PG}(2n + m - 1, q)$). That observation lead to the fact that each TGQ has a representation in projective space, and hence Galois theory could be applied to that theory.

Imitating the theory of coordinates for finite projective planes (with planar ternary rings, see M. Hall [69] or Chapter 5 of [81]), S. E. Payne [115] started a general theory of coordinatization for generalized quadrangles in 1974, see also Chapter 11 of [139], but this theory was for a very restricted class of generalized quadrangles. That theory was further developed in S. E. Payne and J. A. Thas [137, 138], in 1975/76, where generalized quadrangles (of order s) 'with symmetry' were coordinatized. In 1988, G. Hanssens and H. Van

[2] At this point, I gratefully acknowledge that J. A. Thas has pointed out to me the importance of Kantor's discovery of the GQ's $K(s)$.

Maldeghem [71] then developed a generalization for arbitrary thick generalized quadrangles (the coordinatizing structure was called a 'quadratic quaternary ring'). Later on, a coordinatization theory was introduced for 'general' generalized polygons, mainly by H. Van Maldeghem and co-authors (see, e.g., H. Van Maldeghem and I. Bloemen [233], and Chapter 3 of [229]). For generalized polygons, coordinates appear to be important tools in proving results for both general and classical or dual classical generalized polygons. However, contrary to the corresponding theory in projective planes, no generalized n-gon with $n > 3$ was first constructed via coordinatization (some have otherwise no elementary description, though).

In 1987, J. A. Thas proved in his celebrated paper on flocks [172], that the algebraic conditions for q planes in $\mathbf{PG}(3,q)$ to define a flock $\mathcal{F}$ of the quadratic cone $\mathcal{K}$ in $\mathbf{PG}(3,q)$, precisely correspond to the Payne-conditions for sets of matrices to be q-clans. So q-clans and flocks of the quadratic cone in $\mathbf{PG}(3,q)$ were proven to be equivalent objects! Translation planes can be constructed from such flocks by constructing an ovoid of the Klein quadric from a flock of the quadratic cone in $\mathbf{PG}(3,q)$; this ovoid corresponds to a line spread of $\mathbf{PG}(3,q)$ via the Klein correspondence, which in turn gives rise to a translation plane (applying the André/Bruck-Bose construction). This was independently observed by both M. Walker [236] and J. A. Thas. Thus, the observation of 1987 of J. A. Thas gave rise to new generalized quadrangles, new translation planes and new spreads (of $\mathbf{PG}(3,q)$). It was also that observation that blew the theory of finite generalized quadrangles up to one of the widest investigated parts of finite geometry, as the (Galois) geometric theory of flocks, the algebraic theory of q-clans and the group theory of 4-gonal families became unified in one and the same framework. Also, in 1990, L. Bader, G. Lunardon and J. A. Thas [5] introduced a procedure (called *derivation*) for constructing new flocks — and hence possibly new generalized quadrangles — from old ones, using BLT-sets. Due to the many connections between flock generalized quadrangles and other geometrical objects, again new flocks, new translation planes, etc., were discovered by that particular process. We should also mention that there is a big theory available on ovals and hyperovals of $\mathbf{PG}(2,q)$ which are associated in some way to flocks of the quadratic cone in $\mathbf{PG}(3,q)$. Amongst others, important contributors to that theory are M. R. Brown, W. Cherowitzo, C. M. O'Keefe, S. E. Payne, T. Penttila and G. F. Royle.
One of the most intensely investigated areas inside the geometrical theory of flock generalized quadrangles is that of those non-classical TGQ's of order (q, q^2), $q > 1$, which arise from flocks (at least one of the derived flocks is then a semifield flock, and q necessarily is odd as the TGQ's are non-classical). Equiv-

alently to that study is the study of the generalized ovoids of $\mathbf{PG}(4n-1,q)$, q odd, which are 'good' at some element. Important results (some of them being very recent) in that theory were obtained by, e.g, L. Bader, S. Ball, A. Blokhuis, D. Ghinelli, N. L. Johnson, M. Lavrauw, G. Lunardon, S. E. Payne, T. Penttila, I. Pinneri, J. A. Thas, H. Van Maldeghem.
In a masterful sequence of papers, notably [177], [181], [182], [183], [186], J. A. Thas then solved several major problems that had remained unsolved for a long time, most notably characterizing the flock generalized quadrangles in terms of Property (G) — which was introduced by S. E. Payne [128] in his well-known work on skew translation generalized quadrangles — characterizing TGQ's, and producing a geometrical construction for all flock GQ's. This latter problem had already been solved for q odd by N. Knarr with a beautiful construction.

Also very recently, M. R. Brown has used the Tits construction of GQ's of order q from ovals in $\mathbf{PG}(2,q)$ to obtain the impressive result that if a plane of $\mathbf{PG}(3,q)$, q even, meets an ovoid of $\mathbf{PG}(3,q)$ in a conic, then the ovoid must be an elliptic quadric [22].

Finally, very recently, R. D. Baker, G. L. Ebert and K. L. Wantz [7] have introduced the notion of *hyperbolic fibration of* $\mathbf{PG}(3,q)$, which is a set of $q-1$ hyperbolic quadrics and two lines which together partition the points of $\mathbf{PG}(3,q)$. These fibrations were introduced in order to study large classes of translation planes. It turns out that a particular type of hyperbolic fibration corresponds to a flock of a quadratic cone. This relationship between hyperbolic fibrations and generalized quadrangles seems not to have exposed new examples of q-clans at the present time, but it emphasizes the importance of flocks and their connection with other geometrical constructs.

The Literature

The main results, up to 1983, on finite generalized quadrangles are contained in the monograph *Finite Generalized Quadrangles* by S. E. Payne and J. A. Thas [139]. Surveys of 'new' developments on this subject in the period 1984-1992, can be found in *Recent developments in the theory of finite generalized quadrangles* [176], 3-*Regularity in generalized quadrangles: a survey, recent results and the solution of a longstanding conjecture* [182], and, finally, *Generalized quadrangles of order* (s,s^2)*: recent results* [184], all by the hand of J. A. Thas. On the more general subject of generalized polygons, we refer to Chapter 5 in

Handbook of Incidence Geometry [28] (Edited by F. Buekenhout) of J. A. Thas, and the monograph *Generalized Polygons* of H. Van Maldeghem [229].
The latter treats generalized hexagons and generalized octagons in more depth than it does generalized quadrangles (especially in view of the fast evolution of the theory of quadrangles); it does not advance the theory of flock quadrangles and related geometrical structures beyond what is surveyed in *N. Johnson and S. E. Payne, Flocks of Laguerre planes and associated geometries* [87]. *The Subiaco Notebook* [130] of S. E. Payne, which is available on the web, gives an almost complete treatment of q-clan geometry for q an even prime power, with special emphasis on the Subiaco generalized quadrangles and their associated ovals.
After the appearance of the aforementioned standard works, it was the purpose of the Academy Contact Forum "Generalized Polygons" of the Royal Flemish Academy of Belgium for Science and the Arts to survey the evolution of the theory of generalized polygons. That conference resulted in the book *Generalized Polygons (Proceedings of the Academy Contact Forum, 20 October 2000)* [47] (Edited by F. De Clerck, L. Storme, J. A. Thas and H. Van Maldeghem), three chapters of which surveyed specifically recent developments in the theory of finite generalized quadrangles, namely *Spreads and Ovoids of Finite Generalized Quadrangles* [23] by M. R. Brown; *Flock Generalized Quadrangles and Related Structures: An Update* [132] by S. E. Payne, and, finally, *Automorphisms and Characterizations of Finite Generalized Quadrangles* [206] by the author of this work.

About this Work

In Chapter 1, we will define all the necessary notions and notation for understanding the sequel; also, many important results will be mentioned which will be frequently used (sometimes without further notice). The aim is to introduce the reader to the theory of (finite) generalized quadrangles.

Chapter 2 contains a detailed synopsis of all known constructions and examples of finite generalized quadrangles, omitting most of the 'small' cases.

In Chapter 3, some (rather) combinatorial results will be obtained on generalized quadrangles with a regular point which will be essential for the sequel. Especially the connection between subquadrangles and subnets, and recognition of symmetries in the latter context, will be exploited many times throughout this work.

We will investigate nonexistence of complete $(st-t/s)$-arcs of generalized quadrangles of order (s,t) in Chapter 4, and will obtain connections with some other purely combinatorial problems, such as the existence problem for complete grids with parameters $s-1, s+1$ in GQ's of order (s,t), $s,t>1$, and the study of (partial) ovoids and (partial) spreads of affine generalized quadrangles[3]. The results of Chapter 4 will be applied in Chapter 12.

The next chapter contains new characterizations of translation generalized quadrangles, and in particular, the strong results of X. Chen and D. Frohardt [35] and of D. Hachenberger [68] will be generalized. Chapter 5 will be essential for this thesis many times, as it provides a fairly general recognition result for translation generalized quadrangles.

Then, in Chapter 6, we present an extensive study of generalized quadrangles which contain concurrent axes of symmetry. The following two problems will be studied: (1) 'What is — in general — the minimal number of distinct axes of symmetry through a point p of a GQ $\mathcal{S}$ forcing $\mathcal{S}^{(p)}$ to be a TGQ?'; (2) 'Given a general TGQ $\mathcal{S}^{(p)}$, what is the minimal number of lines through p such that the translation group is generated by the symmetries about these lines?'.

Motivated then by results of Chapter 6, we introduce the new notion of 'semi quadrangle' in Chapter 7, and we work out a combinatorial study of these geometries. Inequalities, divisibility conditions, standard examples (which in some way all are associated to generalized quadrangles or complete caps in finite projective space), etc. will be obtained. We emphasize that semi quadrangles arise naturally from generalized quadrangles which have some concurrent axes of symmetry, see Chapter 6.

Chapter 8 is the first chapter of this thesis which treats generalized quadrangles with non-concurrent axes of symmetry — the so-called 'span-symmetric generalized quadrangles' — and in that chapter, we solve a longstanding conjecture of 1980 by completely classifying the span-symmetric generalized quadrangles of order s, $s>1$.
That result will also be crucial for the study of this type of generalized quadrangle of general order (s,t), $s,t>1$.

We will proceed with that study in Chapter 9, where it is the main purpose to

[3] In fact, the notions '(partial) ovoids' and '(partial) spreads' of the latter objects will be introduced in that chapter.

classify the generalized quadrangles with distinct translation points, observing that a non-classical GQ of order (s,t), $s \neq 1 \neq t$, with such points necessarily is the translation dual of the point-line dual of a flock generalized quadrangle, with s an odd prime power and $t = s^2$.

Conversely, in Chapter 10, we make the rather peculiar observation that each translation generalized quadrangle of which the translation dual is the point-line dual of a flock generalized quadrangle, always has a line each point of which is a translation point. This crucial (and totally unexpected) observation explains the intrinsic difference between non-classical translation generalized quadrangles of which the point-line dual is a flock generalized quadrangle and their translation duals, if the flock is not a Kantor flock (in which case those quadrangles are isomorphic).
Many well-known open problems concerning automorphisms of generalized quadrangles, derivation of flocks, doubly subtended subquadrangles in generalized quadrangles, etc. will be completely solved.

It is an old and fundamental open problem (from 1974) whether each automorphism of a TGQ $\mathcal{S}^{(\infty)} = T(n,m,q) = T(\mathcal{O})$ of order (q^n, q^m) which fixes the translation point (∞) is induced by an automorphism of the $\mathbf{PG}(2n+m,q)$ in which the TGQ can be represented, which fixes $\mathcal{O} \subset \mathbf{PG}(2n+m-1,q)$. In Chapter 11 (joint work with J. A. Thas), this problem will be completely solved affirmatively. Then, using that observation and results of Chapter 8, Chapter 9 and Chapter 10, we will explain the connection between the automorphism group of a TGQ and the automorphism group of its translation dual (if the latter is defined). To that end, 'generalized oval cones' and 'generalized ovoid cones' will be introduced.

'The Lenz-Barlotti classification for projective planes' (which is an amalgamation of work done by H. Lenz in 1954 and A. Barlotti in 1957) is one of the most important results (and the only known coordinate-free 'general' classification) in the theory of projective planes. In Chapter 12, we present an analogue of this classification for finite generalized quadrangles (based on the possible configurations of axes of symmetry). Essential results of each of the preceding chapters will be utilized for that purpose.
In an appendix to that chapter, the solution of a recent conjecture by W. M. Kantor concerning span-symmetric generalized quadrangles is obtained.

In Chapter 13 (joint work with H. Van Maldeghem), the Moufang condition for (finite) generalized quadrangles will be generalized to the so-called 'pseudo

Moufang condition'. We will then prove that each half pseudo Moufang GQ is classical or dual classical, a theorem which generalizes the celebrated result of P. Fong and G. M. Seitz for (finite) generalized quadrangles.

We will apply our results (and techniques) obtained in, especially, Chapter 9, to the theory of generalized quadrangles with a spread of symmetry and near polygons in Chapter 14 (partially joint work with B. De Bruyn). Many characterizations in that direction will be obtained, the most notable saying that each elation generalized quadrangle of order (q, q^2) which allows a spread of symmetry is isomorphic to the classical GQ $\mathcal{Q}(5, q)$.

In the final chapter, we will obtain a schema ('blueprint') for a (possible) classification program for all translation generalized quadrangles, which is suggested by various results of this thesis obtained before that chapter.

In Appendix A, we will give a proof of the so-called 'Hole in the Moufang Theorem' which only uses the classification of finite groups with a split BN-pair of rank 1 and results obtained in Chapters 9 and 12.

We will obtain an inequality for semi quadrangles in Appendix B, which extends results of Higman/Bose/Shrikhande/Cameron.

In Appendix C some relevant open problems will be discussed.

Appendix D contains a detailed Dutch summary of the most important results obtained in this work.

A final appendix lists the sizes of some groups which are frequently used in this work, and describes the Schur multipliers of some of these groups.

In the beginning of each chapter, the specific paper(s) will be mentioned on which it is based.

* * *

Final Remark

The main contributions of this thesis are

(a) the development of a general theory for span-symmetric generalized quadrangles, and in particular investigating generalized quadrangles with distinct translation points and TGQ's of which the translation dual arises from a flock;

(b) obtaining a blueprint for the classification of all (finite) translation generalized quadrangles, which is suggested by the various results in the context of (a) (and solving some parts of that program completely);

(c) obtaining a Lenz-Barlotti classification for finite generalized quadrangles (extending results of (a), and thus motivating (b));

(d) in view of (c), handling Moufang conditions for finite generalized quadrangles.

We therefore emphasize that the core of this work lies in Chapter 8, Chapter 9, Chapter 10, Chapter 11, Chapter 12, Chapter 13 and Chapter 15. All other chapters will be utilized for these aims (and are therefore necessary).

* * *

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Chapter 1

Introduction

1.1 Finite Generalized Quadrangles

A (finite) *generalized quadrangle (GQ) of order* (s,t), $s \geq 1$ and $t \geq 1$ and $s,t \in \mathbb{N}$, is a point-line incidence structure[1] $\mathcal{S} = (P, B, I)$ in which P and B are disjoint (non-empty) sets of objects called 'points' and 'lines' respectively, and for which I is a symmetric point-line incidence relation satisfying the following axioms.

1. Each point is incident with $t+1$ lines, and two distinct points are incident with at most one line.

2. Each line is incident with $s+1$ points, and two distinct lines are incident with at most one point.

3. If p is a point and L is a line not incident with p, then there is a unique point-line pair (q, M) such that $pIMIqIL$.

[1] See Chapter 7 for a formal definition of 'incidence structure'; this will not be needed here.

If $s = t$, then $\mathcal{S}$ is also said to be 'of order s'.

Recall that 'FGQ' denotes the monograph Finite Generalized Quadrangles *' by S. E. Payne and J. A. Thas [139].*

A *grid*, respectively *dual grid*, is an incidence structure $\Gamma = (P, B, I)$ with $P = \{x_{ij} \parallel i = 0, 1, \ldots, s_1 \text{ and } j = 0, 1, \ldots, s_2\}$, $s_1, s_2 > 0$, respectively $B = \{L_{ij} \parallel i = 0, 1, \ldots, t_1 \text{ and } j = 0, 1, \ldots, t_2\}$, $t_1, t_2 > 0$, with $B = \{L_0, L_1, \ldots, L_{s_1}, M_0, M_1, \ldots, M_{s_2}\}$, respectively $P = \{x_0, x_1, \ldots, x_{t_1}, y_0, y_1, \ldots, y_{t_2}\})$, $x_{ij} I L_k$ if and only if $i = k$, respectively $L_{ij} I x_k$ if and only if $i = k$, and $x_{ij} I M_k$ if and only if $j = k$, respectively $L_{ij} I y_k$ if and only if $j = k$. If Γ is a grid, respectively dual grid, and if s_1 and s_2, respectively t_1 and t_2, are as above, then we say that Γ has *parameters* $s_1 + 1, s_2 + 1$, respectively *parameters* $t_1 + 1, t_2 + 1$.[2] A grid, respectively dual grid, with parameters $s_1 + 1, s_2 + 1$, respectively with parameters $t_1 + 1, t_2 + 1$, is a GQ if and only if $s_1 = s_2$, respectively $t_1 = t_2$. It is clear that the grids, respectively dual grids, with $s_1 = s_2$, respectively $t_1 = t_2$, are the GQ's with $t = 1$, respectively $s = 1$. Sometimes we will speak of an '$(s_1 + 1) \times (s_2 + 1)$-grid', respectively 'dual $(t_1 + 1) \times (t_2 + 1)$-grid', instead of a 'grid with parameters $s_1 + 1, s_2 + 1$', respectively a 'dual grid with parameters $t_1 + 1, t_2 + 1$'.

Let $\mathcal{S} = (P, B, I)$ be a (finite) generalized quadrangle of order (s, t), $s \neq 1 \neq t$. Then $|P| = (s+1)(st+1)$, $|B| = (t+1)(st+1)$ and $s+t$ divides $st(s+1)(t+1)$ (see 1.2.1 and 1.2.2 of FGQ). Also, $s \leq t^2$ [74, 75] and, dually, $t \leq s^2$, and we will refer to both inequalities as being the

"Inequality of Higman",

see Appendix B for a proof of a more general assertion.
There is a point-line duality for GQ's of order (s, t) for which in any definition or theorem the words 'point' and 'line' are interchanged, and also the parameters. Normally, we assume without further notice that the dual of a given theorem or definition has also been given. Often a line will be identified with the set of points incident with it without further notice.
A GQ is called *thick* if every point is incident with more than two lines and if every line is incident with more than two points. Otherwise, a GQ is called *thin*. So a thin GQ of order $(s, 1)$ is just a grid with parameters $s + 1, s + 1$.

[2] In FGQ, a slightly different definition for the parameters of a grid or a dual grid is used. This will not lead to confusion, however.

A *flag* of a GQ is an incident point-line pair.
Let p and q be (not necessarily distinct) points of the GQ $\mathcal{S}$; we write $p \sim q$ and say that p and q are *collinear*, provided that there is some line L so that $pILIq$ (so $p \not\sim q$ means that p and q are not collinear). Dually, for $L, M \in B$, we write $L \sim M$ or $L \not\sim M$ according as L and M are *concurrent* or *non-concurrent*. If $p \neq q \sim p$, the line incident with both is denoted by pq, and if $L \sim M \neq L$, the point which is incident with both is sometimes denoted by $L \cap M$.
For $p \in P$, put

$$p^\perp = \{q \in P \parallel q \sim p\},$$

and note that $p \in p^\perp$. For a pair of distinct points $\{p, q\}$, the *trace* of $\{p, q\}$ is defined as $p^\perp \cap q^\perp$, and we denote this set by $\{p, q\}^\perp$. Then $|\{p, q\}^\perp| = s + 1$ or $t + 1$, according as $p \sim q$ or $p \not\sim q$. More generally, if $A \subseteq P$, $A^\perp$ is defined by

$$A^\perp = \bigcap \{p^\perp \parallel p \in A\}.$$

For $p \neq q$, the *span* of the pair $\{p, q\}$ is $sp(p, q) = \{p, q\}^{\perp\perp} = \{r \in P \parallel r \in s^\perp$ for all $s \in \{p, q\}^\perp\}$. When $p \not\sim q$, then $\{p, q\}^{\perp\perp}$ is also called the *hyperbolic line* defined by p and q, and $|\{p, q\}^{\perp\perp}| = s + 1$ or $|\{p, q\}^{\perp\perp}| \leq t + 1$ according as $p \sim q$ or $p \not\sim q$. If $p \sim q$, $p \neq q$, or if $p \not\sim q$ and $|\{p, q\}^{\perp\perp}| = t + 1$, then we say that the pair $\{p, q\}$ is *regular*. The point p is *regular* provided $\{p, q\}$ is regular for every $q \in P \setminus \{p\}$. Regularity for lines is defined dually. One easily proves that either $s = 1$ or $t \leq s$ if $\mathcal{S}$ has a regular pair of non-collinear points. A point p is *coregular* provided each line incident with p is regular. Dually, one defines *coregular lines*.
If (p, L) is a non-incident point-line pair of a GQ (i.e. an *anti-flag*), then by $[p, L]$ we denote the unique line of the GQ which is incident with p and concurrent with L. Sometimes, we will also use the notation $proj_p L$, and, dually, $proj_L p$.
A *panel* of a generalized quadrangle $\mathcal{S} = (P, B, I)$ is an element (p, L, q) of $P \times B \times P$ for which $pILIq$ and $p \neq q$. Dually, one defines *dual panels*.

Finally, if $\mathcal{S}$ is a GQ, then by $\mathcal{S}^D$ we denote its point-line dual.

1.1.1 Triads and 3-regularity

A *triad* of points, respectively lines, is a triple of pairwise non-collinear points, respectively pairwise non-concurrent lines. Given a triad T, a *center* of T is

just an element of $T^{\perp}$. A triad (of points or lines) is called *centric* if it has at least one center. It is called *unicentric* if there is precisely one center. The following result will often be utilized without further notice.

Theorem 1.1.1 (C. C. Bose and S. S. Shrikhande [17]) *Let $\mathcal{S}$ be a generalized quadrangle with parameters (s,t), $s \neq 1 \neq t$. Then $t = s^2$ if and only if the number of centers of each triad of points is a constant, and if this occurs, the constant is $s+1$. Dually, $s = t^2$ if and only if the number of centers of each triad of lines is a constant, and if this occurs, the constant is $t+1$.*

Suppose $\mathcal{S}$ is a GQ of order (s, s^2), $s \neq 1$. Then for any triad of points $\{p,q,r\}$, $|\{p,q,r\}^{\perp}| = s+1$ by Theorem 1.1.1. Evidently $|\{p,q,r\}^{\perp\perp}| \leq s+1$. We say that $\{p,q,r\}$ is 3-*regular* provided that $|\{p,q,r\}^{\perp\perp}| = s+1$. A point p is 3-*regular* if each triad of points containing p is 3-regular.

The following result will frequently be used, and was first observed in [169].

Theorem 1.1.2 (FGQ, 2.6.1) *Let $\{x,y,z\}$ be a 3-regular triad of the GQ $\mathcal{S} = (P,B,I)$ of order (s,s^2), $s \neq 1$, and let P' be the set of all points incident with lines of the form uv, with $u \in \{x,y,z\}^{\perp} = \mathbf{X}$ and $v \in \{x,y,z\}^{\perp\perp} = \mathbf{Y}$. If L is a line which is incident with no point of $\mathbf{X} \cup \mathbf{Y}$ and if k is the number of points in P' which are incident with L, then $k \in \{0,2\}$ if s is odd and $k \in \{1, s+1\}$ if s is even.*

We also have

Theorem 1.1.3 (FGQ, 1.4.1) *Let $X = \{x_1, x_2, \ldots, x_m\}$, $m \geq 2$, and $Y = \{y_1, y_2, \ldots, y_n\}$, $n \geq 2$, be disjoint sets of pairwise non-collinear points of the GQ $\mathcal{S} = (P,B,I)$ of order (s,t), $s,t \neq 1$, and suppose that $X \subseteq Y^{\perp}$. Then $(m-1)(n-1) \leq s^2$. If equality holds, then one of the following must occur.*

1. *$m = n = s+1$, and each point of $Z = P \setminus (X \cup Y)$ is collinear with precisely two points of $X \cup Y$.*
2. *$m \neq n$. If $m < n$, then s is a divisor of t, $s < t$, $n = t+1$, $m = s^2/t$, and each point of $P \setminus X$ is collinear with either 1 or $t/s+1$ points of Y according as it is or is not collinear with some point of X.*

1.1.2 Property (H) and antiregularity

Suppose p and q are two non-collinear points of the GQ $\mathcal{S} = (P,B,I)$. Then we put

$$cl(p,q) = \{z \in \mathcal{S} \parallel z^\perp \cap \{p,q\}^{\perp\perp} \neq \emptyset\}.$$

A point x is *semiregular* provided that $r \in cl(p,q)$ whenever x is the unique center of $\{p,q,r\}$. A point x has *Property (H)* provided that $r \in cl(p,q)$ if and only if $p \in cl(q,r)$ whenever (p,q,r) is a triad of points in $x^\perp$; we denote the dual notion also by *Property (H)*. Each semiregular point clearly has Property (H).

The pair $\{x,y\}$, $x \not\sim y$, is *antiregular* if

$$|\{x,y\}^\perp \cap z^\perp| \leq 2$$

for all $z \in P \setminus \{x,y\}$. The point x is *antiregular* if $\{x,y\}$ is antiregular for each $y \in P \setminus x^\perp$.

1.1.3 Automorphisms of finite generalized quadrangles

A *collineation* or *automorphism* of a generalized quadrangle $\mathcal{S} = (P,B,I)$ is a permutation of $P \cup B$ which preserves P, B and I. By $Aut(\mathcal{S})$, we denote the full automorphism group of the GQ $\mathcal{S}$.

Two GQ's $\mathcal{S} = (P,B,I)$ and $\mathcal{S}' = (P',B',I')$ are said to be *isomorphic* if there are two bijective maps $\alpha : P \mapsto P'$ and $\beta : B \mapsto B'$ so that if pIL in $\mathcal{S}$, then $p^\alpha I' L^\beta$ in $\mathcal{S}'$; the pair (α,β) is called an *isomorphism* of $\mathcal{S}$ (on)to $\mathcal{S}'$ (or between $\mathcal{S}$ and $\mathcal{S}'$). If $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic, then we write

$$\mathcal{S} \cong \mathcal{S}'.$$

Note. In some standard works, a *correlation* of $\mathcal{S}$ (that is, a permutation of $P \cup B$ which preserves I and interchanges P and B) is also considered as an automorphism of $\mathcal{S}$.

1.2 The Classical and Dual Classical Generalized Quadrangles

1.2.1 The classical generalized quadrangles

Consider a nonsingular quadric $\mathcal{Q}$ of Witt index 2, that is, of projective index 1, in $\mathbf{PG}(3,q)$, $\mathbf{PG}(4,q)$, $\mathbf{PG}(5,q)$, respectively. The points and lines of

the quadric form a generalized quadrangle which is denoted by $\mathcal{Q}(3,q)$, $\mathcal{Q}(4,q)$, $\mathcal{Q}(5,q)$, respectively, and has order $(q,1)$, (q,q), (q,q^2), respectively. As $\mathcal{Q}(3,q)$ is a grid, its structure is trivial.

Recall that $\mathcal{Q}$ has the following canonical form:

1. $X_0X_1 + X_2X_3 = 0$ if $d = 3$;
2. $X_0^2 + X_1X_2 + X_3X_4 = 0$ if $d = 4$;
3. $f(X_0, X_1) + X_2X_3 + X_4X_5 = 0$ if $d = 5$, where f is an irreducible binary quadratic form.

Next, let $\mathcal{H}$ be a nonsingular Hermitian variety in $\mathbf{PG}(3,q^2)$, respectively $\mathbf{PG}(4,q^2)$. The points and lines of $\mathcal{H}$ form a generalized quadrangle $H(3,q^2)$, respectively $H(4,q^2)$, which has order (q^2,q), respectively (q^2,q^3).

The variety $\mathcal{H}$ has the following canonical form:

$$X_0^{q+1} + X_1^{q+1} + \ldots + X_d^{q+1} = 0.$$

The points of $\mathbf{PG}(3,q)$ together with the totally isotropic lines with respect to a symplectic polarity, form a GQ $W(q)$ of order q.

A symplectic polarity of $\mathbf{PG}(3,q)$ has the following canonical form:

$$X_0Y_1 - X_1Y_0 + X_2Y_3 - X_3Y_2.$$

The generalized quadrangles defined in this paragraph are the so-called *classical generalized quadrangles*, see Chapter 3 of FGQ. Their point-line duals are called the *dual classical generalized quadrangles.*

1.2.2 Isomorphisms between the classical and dual classical examples

The following result will be used frequently (sometimes without further notice).

Theorem 1.2.1 (FGQ, 3.2.1, 3.2.2 and 3.2.3) *For the classical generalized quadrangles $\mathcal{Q}(4,q)$, $W(q)$, $\mathcal{Q}(5,q)$ and $H(3,q^2)$, the following isomorphism properties hold:*

(*i*) $\mathcal{Q}(4,q) \cong W(q)^D$;

(*ii*) $\mathcal{Q}(4,q) \cong W(q)$ *if and only if* q *is even;*

(*iii*) $\mathcal{Q}(5,q) \cong H(3,q^2)^D$.

1.2.3 Combinatorial properties of the classical and dual classical generalized quadrangles

PROPERTIES OF $\mathcal{Q}(4,q)$. All lines are regular; all points are regular if and only if q is even; all point are antiregular if and only if q is odd; all points and lines are semiregular and have Property (H).
We also have the following important characterization theorem.

Theorem 1.2.2 (FGQ, 5.2.1) *A GQ of order* s, $s > 1$, *is isomorphic to* $W(s)$ *if and only if each point is regular.*

PROPERTIES OF $\mathcal{Q}(5,q)$. All lines are regular; all points are 3-regular; all points and lines are semiregular and have Property (H).

Theorem 1.2.3 (FGQ, 5.3.3) (*i*) *Let* $\mathcal{S}$ *be a GQ of order* (s,s^2), $s > 1$ *and* s *odd. Then* $\mathcal{S} \cong \mathcal{Q}(5,s)$ *if and only if* $\mathcal{S}$ *has a 3-regular point.*

(*ii*) *Let* $\mathcal{S}$ *be a GQ of order* (s,s^2), s *even. Then* $\mathcal{S} \cong \mathcal{Q}(5,s)$ *if and only if one of the following holds:*

(*a*) *all points of* $\mathcal{S}$ *are 3-regular;*

(*b*) $\mathcal{S}$ *has at least one 3-regular point not incident with some regular line.*

PROPERTIES OF $H(4,q^2)$. For each two distinct non-collinear points x, y we have that $|\{x,y\}^{\perp\perp}| = q+1$; if L and M are non-concurrent lines, then $|\{L,M\}^{\perp\perp}| = 2$, but $\{L,M\}$ is not antiregular; all points and lines have Property (H) and all points but no lines are semiregular.

Theorem 1.2.4 (FGQ, 5.5.1) *A GQ of order* (s^2,s^3), $s > 1$, *is isomorphic to the classical GQ* $H(4,s^2)$ *if and only if every hyperbolic line has at least* $s+1$ *points.*

1.3 Generalized Quadrangles with Small Parameters

Let $\mathcal{S}$ be a finite generalized quadrangle of order (s,t), $1 < s \leq t$. We consider the cases $s = 2, 3, 4$.

THE CASE $s = 2$. If $s = 2$, then $t \in \{2, 4\}$. The GQ of order 2 is unique and is isomorphic to $\mathcal{Q}(4,2)$. The uniqueness of the GQ of order $(2,4)$ was proved independently at least five times, by S. Dixmier and F. Zara [54], J. J. Seidel [153], E. E. Shult [155], J. A. Thas [166] and H. Freudenthal [62].

THE CASE $s = 3$. If $s = 3$, then $t \in \{3, 5, 6, 9\}$. The uniqueness of the GQ of order $(3,5)$ was proved by S. Dixmier and F. Zara in [54]. The uniqueness of the GQ of order $(3,9)$ was proved independently by S. Dixmier and F. Zara [54], and by P. J. Cameron in 1976 (see FGQ). For $s = t = 3$ there are exactly two non-isomorphic GQ's, due independently to S. Dixmier and F. Zara [54] and S. E. Payne [116]. Finally, S. Dixmier and F. Zara proved in [54] that no GQ of order $(3,6)$ exists.

THE CASE $s = 4$. If $s = 4$, then $t \in \{4, 6, 8, 11, 12, 16\}$. Nothing is known about the case $t = 11$ or $t = 12$. In the other cases, unique examples are known, but the uniqueness question is only settled for the case $t = 4$. The proof is due to S. E. Payne [117, 118], with a gap filled by J. Tits in 1983, see also FGQ.

1.4 The Moufang Condition for Generalized Quadrangles

J. Tits [223] defines a generalized quadrangle of *Moufang type*, or a *Moufang generalized quadrangle*, as a generalized quadrangle $\mathcal{S} = (P, B, I)$ in which the following conditions hold:

(M) for any dual panel (L, p, M) of $\mathcal{S}$, the group of all automorphisms of $\mathcal{S}$ fixing L and M pointwise and p linewise is transitive on the lines which are incident with a given point xIL, $x \neq p$, and different from L;

(M′) for any panel (p, L, q) of $\mathcal{S}$, the group of all automorphisms of $\mathcal{S}$ fixing p and q linewise and L pointwise is transitive on the points which are incident with a given line MIp, $M \neq L$, and different from p.

Theorem 1.4.1 (P. Fong and G. M. Seitz [60, 61]) *A finite Moufang GQ is classical or dual classical.*

It is well-known that the converse also holds (see, e.g., Chapter 9 of FGQ):

Each classical or dual classical GQ is Moufang.

J. A. Thas, S. E. Payne and H. Van Maldeghem [201] considerably generalized Theorem 1.4.1 by showing that each GQ which satisfies either Property (M) or Property (M′) — such a GQ is generally called *half Moufang* — automatically satisfies both properties:

Theorem 1.4.2 ([201]) *Any finite half Moufang GQ is Moufang.*

1.5 The $T_2(\mathcal{O})$ and $T_3(\mathcal{O})$ of Tits

The first non-classical examples of generalized quadrangles were discovered by J. Tits, and appeared in the monograph of P. Dembowski [48]. Let $\mathcal{O}$ be an oval, respectively ovoid (see Section 1.12 for details on both objects), in $\mathbf{PG}(d,q) = H$, for respectively $d = 2$ and $d = 3$. Embed $\mathbf{PG}(d,q)$ as a hyperplane in $\mathbf{PG}(d+1,q) = H'$, and define a point-line incidence structure $T_d(\mathcal{O})$ as follows:

- The POINTS are of three types.
  - (i) A symbol (∞).
  - (ii) The hyperplanes Π of H' for which $|\Pi \cap \mathcal{O}| = 1$.
  - (iii) The points of $H' \setminus H$.
- The LINES are of two types.
  - (a) The points of $\mathcal{O}$.
  - (b) The lines of $H' \setminus H$ which intersect H in a point of $\mathcal{O}$.
- INCIDENCE is defined as follows:
  - the point (∞) is incident with all the lines of Type (a) and with no other lines;
  - a point of Type (ii) is incident with the unique line of Type (a) contained in it and with all the lines of Type (b) which it contains (as subspaces);

- a point of Type (iii) is incident with the lines of Type (b) that contain it.

Then J. Tits showed $T_d(\mathcal{O})$ to be a GQ of order q, respectively (q, q^2), for $d = 2$, respectively $d = 3$.
We end this section with the following.

Theorem 1.5.1 (FGQ, 3.2.2 and 3.2.4) *(i) A $T_2(\mathcal{O})$ of order q is isomorphic to $\mathcal{Q}(4, q)$ if and only if $\mathcal{O}$ is an irreducible conic of $\mathbf{PG}(2, q)$.*

(ii) A $T_3(\mathcal{O})$ of order (q, q^2) is isomorphic to $\mathcal{Q}(5, q)$ if and only if $\mathcal{O}$ is an elliptic quadric of $\mathbf{PG}(3, q)$.

1.6 Subquadrangles of Generalized Quadrangles

A *subquadrangle*, or also *subGQ*, $\mathcal{S}' = (P', B', I')$ of a GQ $\mathcal{S} = (P, B, I)$ is a GQ for which $P' \subseteq P$, $B' \subseteq B$, and where I' is the restriction of I to $(P' \times B') \cup (B' \times P')$. If $\mathcal{S}' \neq \mathcal{S}$, then $\mathcal{S}'$ is called a *proper subquadrangle* of $\mathcal{S}$.

1.6.1 Parameters and recognition of subquadrangles

Suppose $\mathcal{S}'$ is a proper subquadrangle of order (s', t') of the GQ $\mathcal{S}$ of order (s, t). An *external point* of $\mathcal{S}'$ is a point which is not incident with any line of $\mathcal{S}'$.

Theorem 1.6.1 (FGQ, 2.2.1) *Let $\mathcal{S}'$ be a proper subquadrangle of order (s', t') of the GQ $\mathcal{S}$ of order (s, t). Then either $s = s'$ or $s \geq s't'$. If $s = s'$, then each external point of $\mathcal{S}'$ is collinear with $st' + 1$ mutually non-collinear points of $\mathcal{S}'$;* [3] *if $s = s't'$, then each external point of $\mathcal{S}'$ is collinear with exactly $1 + s'$ points of $\mathcal{S}'$.*

Theorem 1.6.2 (FGQ, 2.2.2) *Let $\mathcal{S}'$ be a proper subquadrangle of the GQ $\mathcal{S}$, where $\mathcal{S}$ has order (s, t) and $\mathcal{S}'$ has order (s, t') (so $t > t'$). Then we have*

1. *$t \geq s$; if $s = t$, then $t' = 1$.*
2. *If $s > 1$, then $t' \leq s$; if $t' = s \geq 2$, then $t = s^2$.*
3. *If $s = 1$, then $1 \leq t' < t$ is the only restriction on t'.*
4. *If $s > 1$ and $t' > 1$, then $\sqrt{s} \leq t' \leq s$ and $s^{3/2} \leq t \leq s^2$.*

[3] This set constitutes an *ovoid* of $\mathcal{S}'$; see Section 1.12.

5. *If $t = s^{3/2} > 1$ and $t' > 1$, then $t' = \sqrt{s}$.*

6. *Let $\mathcal{S}'$ have a proper subquadrangle $\mathcal{S}''$ of order (s, t''), $s > 1$. Then $t'' = 1$, $t' = s$ and $t = s^2$.*

Theorem 1.6.3 (FGQ, 2.3.1) *Let $\mathcal{S}' = (P', B', I')$ be a substructure of the GQ $\mathcal{S}$ of order (s, t) so that the following two conditions are satisfied:*

(*i*) *if $x, y \in P'$ are distinct points of $\mathcal{S}'$ and L is a line of $\mathcal{S}$ such that $xILIy$, then $L \in B'$;*

(*ii*) *each element of B' is incident with $s + 1$ elements of P'.*

Then there are four possibilities:

1. *$\mathcal{S}'$ is a dual grid, so $s = 1$;*

2. *the elements of B' are lines which are incident with a distinguished point of P, and P' consists of those points of P which are incident with these lines;*

3. *$B' = \emptyset$ and P' is a set of pairwise non-collinear points of P;*

4. *$\mathcal{S}'$ is a subquadrangle of order (s, t').*

The following result is now easy to prove.

Theorem 1.6.4 (FGQ, 2.4.1) *Let θ be an automorphism of the GQ $\mathcal{S} = (P, B, I)$ of order (s, t). The substructure $\mathcal{S}_\theta = (P_\theta, B_\theta, I_\theta)$ of $\mathcal{S}$ which consists of the fixed elements of θ must be given by (at least) one of the following:*

(*i*) *$B_\theta = \emptyset$ and P_θ is a set of pairwise non-collinear points;*

(*i*)$'$ *$P_\theta = \emptyset$ and B_θ is a set of pairwise non-concurrent lines;*

(*ii*) *P_θ contains a point x so that $y \sim x$ for each $y \in P_\theta$, and each line of B_θ is incident with x;*

(*ii*)$'$ *B_θ contains a line L so that $M \sim L$ for each $M \in B_\theta$, and each point of P_θ is incident with L;*

(*iii*) *$\mathcal{S}_\theta$ is a grid;*

(*iii*)$'$ *$\mathcal{S}_\theta$ is a dual grid;*

(*iv*) *$\mathcal{S}_\theta$ is a subGQ of $\mathcal{S}$ of order (s', t'), $s', t' \geq 2$.*

1.6.2 Subquadrangles and collineations

Theorem 1.6.5 (FGQ, 8.1.1) *Let θ be a nontrivial collineation of the GQ $\mathcal{S} = (P, B, I)$ of order (s, t), $s \neq 1 \neq t$, fixing the point p linewise (that is, a* whorl about *p, see Section 1.7). Then one of the following must hold for the fixed element structure $\mathcal{S}_\theta = (P_\theta, B_\theta, I_\theta)$.*

1. *$y^\theta \neq y$ for each $y \in P \setminus p^\perp$.*
2. *There is a point y, $y \not\sim p$, for which $y^\theta = y$. Put $V = \{p, y\}^\perp$ and $U = V^\perp$. Then $V \cup \{p, y\} \subseteq P_\theta \subseteq V \cup U$, and $L \in B_\theta$ if and only if L joins a point of V with a point of $U \cap P_\theta$.*
3. *$\mathcal{S}_\theta$ is a subGQ of order (s', t), where $2 \leq s' \leq s/t \leq t$, and hence $t < s$.*

1.6.3 Subquadrangles and 3-regularity

Let $\{x, y, z\}$ be a 3-regular triad of the GQ $\mathcal{S} = (P, B, I)$ of order (s, s^2), $s \neq 1$ and s even. Let P' be the set of all points incident with lines of the form uv, with $u \in \{x, y, z\}^\perp = \mathbf{X}$ and $v \in \{x, y, z\}^{\perp\perp} = \mathbf{Y}$, and let B' be the set of lines L which are incident with at least two points of P'. Then J. A. Thas proves in [169] (see also [139, 2.6.2]) that, with I' the restriction of I to $(P' \times B') \cup (B' \times P')$, the geometry $\mathcal{S}' = (P', B', I')$ is a subGQ of $\mathcal{S}$ of order s. Moreover, $\{x, y\}$ is a regular pair of points of $\mathcal{S}'$, with $\{x, y\}^{\perp'} = \{x, y, z\}^\perp$ and $\{x, y\}^{\perp'\perp'} = \{x, y, z\}^{\perp\perp}$ (with the meaning of " $\perp'$ " being obvious).

1.6.4 A characterization of the GQ $\mathcal{Q}(5, s)$ by subGQ's

Theorem 1.6.6 (FGQ, 5.3.5) *(i) A GQ $\mathcal{S}$ of order (s, t), $s > 1$, is isomorphic to $\mathcal{Q}(5, s)$ if and only if every centric triad of lines is contained in a proper subGQ of order (s, t').*

(ii) A GQ $\mathcal{S}$ of order (s, t), $s > 1$, is isomorphic to $\mathcal{Q}(5, s)$ if and only if for each triad of points $\{x, y, z\}$ with distinct centers u, v, the points x, y, z, u, v are contained in a proper subGQ of order (s, t').

1.7 Elation Generalized Quadrangles and Translation Generalized Quadrangles

A *whorl about a point p* of a GQ $\mathcal{S} = (P, B, I)$ is a collineation of the GQ fixing p linewise. An *elation about the point p* is a whorl about p that fixes no point

of $P \setminus p^\perp$. Dually, one defines *whorls* and *elations about lines*. By definition, the identical permutation is an elation (about every point). If p is a point of the GQ $\mathcal{S}$ for which there exists a group of elations G about p which acts regularly on the points of $P \setminus p^\perp$, then $\mathcal{S}$ is said to be an *elation generalized quadrangle (EGQ)* with *elation* or *base-point* p and *elation* or *base-group* G, and we sometimes write $(\mathcal{S}^{(p)}, G)$ or $\mathcal{S}^{(p)}$ for $\mathcal{S}$.

Note. If $\mathcal{S}$ is an elation generalized quadrangle with elation point (∞) and elation group G, then for the sake of convenience, we will write $(\mathcal{S}^{(\infty)}, G)$ or $\mathcal{S}^{(\infty)}$ instead of, respectively, $(\mathcal{S}^{((\infty))}, G)$ or $\mathcal{S}^{((\infty))}$. The same remark holds for TGQ's, see below, etc..

Let $\mathcal{S}$ be a thick GQ of order (s,t), and suppose L is a line of $\mathcal{S}$. A *symmetry about* L is an automorphism of $\mathcal{S}$ which fixes every line concurrent with L. A line L is an *axis of symmetry* if there is a full group H of size s of symmetries about L. Dually, one defines a *symmetry about a point* and *center of symmetry*. If L is an axis of symmetry and if $U \neq L$ is an arbitrary element of $L^\perp$, then H acts regularly on the points of $U \setminus \{L \cap U\}$ by Theorem 1.6.5.

Remark 1.7.1 Every line of the classical examples $\mathcal{Q}(4,s)$ and $\mathcal{Q}(5,s)$ is an axis of symmetry; this is easily seen since by Section 1.2.3 any line of $\mathcal{Q}(4,s)$, respectively $\mathcal{Q}(5,s)$, is regular, and since every classical GQ is Moufang.

A point of a generalized quadrangle is a *translation point* if every line through it is an axis of symmetry. If a GQ $(\mathcal{S}^{(p)}, G)$ is an EGQ with elation point p, and if each line incident with p is an axis of symmetry, then we say that $\mathcal{S}$ is a *translation generalized quadrangle (TGQ)* with *translation* or *base-point* p and *translation* or *base-group* G (by definition any TGQ is thick). In such a case, G is uniquely defined; G is generated by all symmetries about every line incident with p, and G is the set of all elations about p, see Chapter 8 of FGQ. The elements of G are called *translations*.

TGQ's were introduced by J. A. Thas in [165] for the case $s = t$, and by S. E. Payne and J. A. Thas in FGQ (cf. Chapter 8 of the latter) for the general case.

Theorem 1.7.2 (FGQ, 8.2.4) *Let $\mathcal{S} = (P, B, I)$ be a GQ of order (s,t) with $s \leq t$ and $s > 1$, and let p be a point for which $\{p,x\}^{\perp\perp} = \{p,x\}$ for all $x \in P \setminus p^\perp$. Let G be a group of whorls about p.*

1. *If $y \sim p$, $y \neq p$, and if θ is a nonidentity whorl about p and y, then all points fixed by θ lie on py and all lines fixed by θ meet py.*

2. *If θ is a nonidentity whorl about p, then θ fixes at most one point of $P \setminus p^\perp$.*

3. *If G is generated by elations about p, then G is a group of elations, i.e. the set of elations about p is a group.*

4. *If G acts transitively on $P \setminus p^\perp$ and $|G| > s^2t$, then G is a Frobenius group (cf. Section 1.13.2) on $P \setminus p^\perp$, so that the set of all elations about p is a normal subgroup of G of order s^2t acting regularly on $P \setminus p^\perp$, i.e. $\mathcal{S}^{(p)}$ is an EGQ with some normal subgroup of G as elation group.*

5. *If G is transitive on $P \setminus p^\perp$ and G is generated by elations about p, then $(\mathcal{S}^{(p)}, G)$ is an EGQ.*

Theorem 1.7.3 (FGQ, 8.3.1) *Let $\mathcal{S} = (P, B, I)$ be a GQ of order (s, t), $s, t \neq 1$. Suppose each line through some point p is an axis of symmetry, and let G be the group generated by the symmetries about the lines through p. Then G is elementary abelian and $(\mathcal{S}^{(p)}, G)$ is a TGQ.*

Consequently, we have the following result.

Theorem 1.7.4 (FGQ, 8.3.2) *The translation group of a TGQ is uniquely defined and is abelian.*

For the case $s = t$, the following result of FGQ is available (cf. Chapter 6 of the present work for an alternative short proof).

Theorem 1.7.5 (FGQ, 11.3.5) *Let $\mathcal{S} = (P, B, I)$ be a GQ of order s, with $s \neq 1$. Suppose that there are at least three axes of symmetry through a point p, and let G be the group generated by the symmetries about these lines. Then G is elementary abelian and $(\mathcal{S}^{(p)}, G)$ is a TGQ.*

Theorem 1.7.6 (FGQ, 8.2.3, 8.5.2 and 8.7.2) *Suppose $(\mathcal{S}^{(x)}, G)$ is an EGQ of order (s, t), $s \neq 1 \neq t$. Then $(\mathcal{S}^{(x)}, G)$ is a TGQ if and only if G is an (elementary) abelian group. Also, in such a case there is a prime p and there are natural numbers n and k, where k is odd, such that either $s = t = p^n$ or $s = p^{nk}$ and $t = p^{n(k+1)}$. It follows that G is a p-group. Also, if $p = 2$, then $k = 1$.*

It is strongly believed that if $s \neq t$ and s and t are odd in Theorem 1.7.6, then $k = 1$.

Finally, the following result is a recent result which will appear to be very useful in the sequel.

Theorem 1.7.7 (J. A. Thas [187]) *Suppose $\mathcal{S}^{(p)}$ is a TGQ of order (s, s^2) with s even, which contains at least two classical subGQ's of order s containing the point p, and where these subGQ's have distinct line sets on p. Then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5, s)$.*

Note. If $\mathcal{S}$ is the point-line dual of a TGQ $\mathcal{S}^D$ with base-point p, where p corresponds to L in $\mathcal{S}$, then we also say that $\mathcal{S}$ is a TGQ, and L is called the *base-line* of the TGQ (L is a *translation line*). In the case of EGQ's, we speak of *base-line* or *elation line*. Sometimes we just speak of an EGQ without additional terms when it is clear whether it is w.r.t. a point or a line.

1.8 Skew Translation Generalized Quadrangles

A *skew translation generalized quadrangle (STGQ)* with base-point p and *base-group* G is an EGQ with base-point p and base-group G, such that p is a center of symmetry with the property that G contains the full group (of size t) of symmetries about p.

1.9 4-Gonal Families and Elation Generalized Quadrangles

Suppose $(\mathcal{S}^{(p)}, G)$ is an EGQ of order (s, t), $s \neq 1 \neq t$, with elation point p and elation group G, and let q be a point of $P \setminus p^\perp$. Let $L_0, L_1, \ldots, L_t$ be the lines incident with p, and define r_i and M_i by $L_i I r_i I M_i I q$, $0 \leq i \leq t$. Put $H_i = \{\theta \in G \parallel M_i^\theta = M_i\}$ and $H_i^* = \{\theta \in G \parallel r_i^\theta = r_i\}$, and $\mathcal{J} = \{H_i \parallel 0 \leq i \leq t\}$. Then $|G| = s^2 t$ and $\mathcal{J}$ is a set of $t+1$ subgroups of G, each of order s. Also, for each i, H_i^* is a subgroup of G of order st containing H_i as a subgroup. Moreover, the following two conditions are satisfied:

(K1) $H_i H_j \cap H_k = \mathbf{1}$ for distinct i, j and k;

(K2) $H_i^* \cap H_j = \mathbf{1}$ for distinct i and j.

Conversely, if G is a group of order $s^2 t$ and $\mathcal{J}$ (respectively $\mathcal{J}^*$) is a set of $t+1$ subgroups H_i (respectively H_i^*) of G of order s (respectively of order st), and if the conditions (K1) and (K2) are satisfied, then the H_i^* are uniquely defined by the H_i (H_i^* is sometimes called the *tangent space* at H_i), and $(\mathcal{J}, \mathcal{J}^*)$ is said to be a *4-gonal family of type (s, t) in G*. Sometimes we will also say that $\mathcal{J}$ is a *4-gonal family of type (s, t) in G* if this seems convenient.

Let $(\mathcal{J}, \mathcal{J}^*)$ be a 4-gonal family of type (s,t) in the group G of order s^2t, $s \neq 1 \neq t$. Define an incidence structure $\mathcal{S}(G,\mathcal{J})$ as follows.

- POINTS of $\mathcal{S}(G,\mathcal{J})$ are of three kinds:
  - (i) elements of G;
  - (ii) right cosets $H_i^* g$, $g \in G$, $i \in \{0,1,\ldots,t\}$;
  - (iii) a symbol (∞).
- LINES are of two kinds:
  - (a) right cosets $H_i g$, $g \in G$, $i \in \{0,1,\ldots,t\}$;
  - (b) symbols $[H_i]$, $i \in \{0,1,\ldots,t\}$.
- INCIDENCE. A point g of Type (i) is incident with each line $H_i g$, $0 \leq i \leq t$. A point $H_i^* g$ of Type (ii) is incident with $[H_i]$ and with each line $H_i h$ contained in $H_i^* g$. The point (∞) is incident with each line $[H_i]$ of Type (b). There are no further incidences.

It is straightforward to check that the incidence structure $\mathcal{S}(G,\mathcal{J})$ is a GQ of order (s,t). Moreover, if we start with an EGQ $(\mathcal{S}^{(p)}, G)$ to obtain the family $\mathcal{J}$ as above, then we have that

$$(\mathcal{S}^{(p)}, G) \cong \mathcal{S}(G,\mathcal{J}).$$

For any $h \in G$ let us define θ_h by $g^{\theta_h} = gh$, $(H_i g)^{\theta_h} = H_i gh$, $(H_i^* g)^{\theta_h} = H_i^* gh$, $[H_i]^{\theta_h} = [H_i]$, $(\infty)^{\theta_h} = (\infty)$, with $g \in G$, $H_i \in \mathcal{J}$, $H_i^* \in \mathcal{J}^*$. Then θ_h is an automorphism of $\mathcal{S}(G,\mathcal{J})$ which fixes the point (∞) and all lines of Type (b). If

$$G' = \{\theta_h \parallel h \in G\},$$

then clearly $G' \cong G$ and G' acts regularly on the points of Type (i). Hence, a group of order s^2t admitting a 4-gonal family is an elation group of a suitable elation generalized quadrangle. These results were first noted by W. M. Kantor [90].

We now have the following interesting properties.
Let $(\mathcal{S}^{(p)}, G)$ be an EGQ, and define H_i, with $i = 0,1,\ldots,t$, and $\mathcal{J}$ as above.

Theorem 1.9.1 (FGQ, 8.2.2) *H_i is a group of symmetries about the line L_i if and only if $H_i \trianglelefteq G$ (and hence $\mathcal{S}^{(p)}$ is a TGQ if and only if $H_i \trianglelefteq G$ for each i), only if L_i is a regular line. The line L_i is regular if and only if $H_iH_j = H_jH_i$ for all $H_j \in \mathcal{J}$.*

1.10 q-Clans, Flocks, Flock Generalized Quadrangles, BLT-Sets and Property (G)

1.10.1 q-Clans

Let $\mathbb{F} = \mathbf{GF}(q)$, q any prime power, and put $G = \{(\alpha, c, \beta) \parallel \alpha, \beta \in \mathbb{F}^2, c \in \mathbb{F}\}$. Define a binary operation on G by

$$(\alpha, c, \beta)(\alpha', c', \beta') = (\alpha + \alpha', c + c' + \beta\alpha'^T, \beta + \beta').$$

This makes G into a group whose center is $C = \{(0, c, 0) \in G \parallel c \in \mathbb{F}\}$.

Let $\mathcal{C} = \{A_u \parallel u \in \mathbb{F}\}$ be a set of q distinct upper triangular 2×2-matrices over $\mathbb{F}$. Then $\mathcal{C}$ is called a *q-clan* provided $A_u - A_r$ is *anisotropic* whenever $u \neq r$, i.e. $\alpha(A_u - A_r)\alpha^T = 0$ has only the trivial solution $\alpha = (0, 0)$. For $A_u \in \mathcal{C}$, put $K_u = A_u + A_u^T$. Let

$$A_u = \begin{pmatrix} x_u & y_u \\ 0 & z_u \end{pmatrix}, \qquad x_u, y_u, z_u, u \in \mathbb{F}.$$

For q odd, $\mathcal{C}$ is a q-clan if and only if

$$-\det(K_u - K_r) = (y_u - y_r)^2 - 4(x_u - x_r)(z_u - z_r) \tag{1.1}$$

is a non-square of $\mathbb{F}$ whenever $r, u \in \mathbb{F}$, $r \neq u$. For q even, $\mathcal{C}$ is a q-clan if and only if

$$y_u \neq y_r \text{ and } tr((x_u + x_r)(z_u + z_r)(y_u + y_r)^{-2}) = \mathbf{1} \tag{1.2}$$

whenever $r, u \in \mathbb{F}$, $r \neq u$.

Now we can define a family of subgroups of G by

$$A(u) = \{(\alpha, \alpha A_u \alpha^T, \alpha K_u) \in G \parallel \alpha \in \mathbb{F}^2\},\ u \in \mathbb{F},$$

and

$$A(\infty) = \{(0, 0, \beta) \in G \parallel \beta \in \mathbb{F}^2\}.$$

Then put $\mathcal{J} = \{A(u) \parallel u \in \mathbb{F} \cup \{\infty\}\}$ and $\mathcal{J}^* = \{A^*(u) \parallel u \in \mathbb{F} \cup \{\infty\}\}$, with $A^*(u) = A(u)C$. So

$$A^*(u) = \{(\alpha, c, \alpha K_u) \in G \parallel \alpha \in \mathbb{F}^2\},\ u \in \mathbb{F},$$

and

$$A^*(\infty) = \{(0, c, \beta) \parallel \beta \in \mathbb{F}^2\}.$$

With $G, A(u), A^*(u), \mathcal{J}$ and $\mathcal{J}^*$ as above, the following important theorem is a combination of results of S. E. Payne [120] and W. M. Kantor [90].

Theorem 1.10.1 ([120]; [90]) *The pair $(\mathcal{J}, \mathcal{J}^*)$ is a 4-gonal family for G if and only if $\mathcal{C}$ is a q-clan. Hence if $\mathcal{C}$ is a q-clan, then it defines a GQ of order (q^2, q).*

Now let

$$A = \begin{pmatrix} x & y \\ w & z \end{pmatrix} \quad \text{and} \quad A' = \begin{pmatrix} x' & y' \\ w' & z' \end{pmatrix}$$

be 2×2-matrices over $\mathbb{F}$. Then we say that A and A' are *equivalent*, and we write $A \equiv A'$, provided $x = x'$, $z = z'$, and $y + w = y' + w'$. Then for arbitrary 2×2-matrices B and B' over $\mathbb{F}$, we have that $\alpha B \alpha^T = \alpha B' \alpha^T$ for all $\alpha \in \mathbb{F}^2$, if and only if $B \equiv B'$.

Hence we can also define a q-clan as follows. A *q-clan* is a set

$$\mathcal{C} = \{A_u \equiv \begin{pmatrix} x_u & y_u \\ 0 & z_u \end{pmatrix} \parallel u \in \mathbb{F}\}$$

of q distinct (equivalence classes of) 2×2-matrices over $\mathbb{F}$ for which $A_u - A_r$ is anisotropic whenever $u \neq r$.

1.10.2 Flocks and flock generalized quadrangles

Let $\mathcal{F}$ be a *flock* of the quadratic cone $\mathcal{K}$ with vertex v of $\mathbf{PG}(3,q)$, that is, a partition of $\mathcal{K} \setminus \{v\}$ into q disjoint (irreducible) conics.
In his well-known paper on flock geometry [172], J. A. Thas showed in an algebraic way that (1.1) and (1.2) are exactly the conditions for the planes

$$x_u X_0 + z_u X_1 + y_u X_2 + X_3 = 0$$

of $\mathbf{PG}(3,q)$ to define a flock of the quadratic cone $\mathcal{K}$ with equation $X_0 X_1 = X_2^2$. Hence we have the following theorem.

Theorem 1.10.2 (J. A. Thas [172]) *To any flock $\mathcal{F}$ of the quadratic cone of $\mathbf{PG}(3,q)$ corresponds a GQ $\mathcal{S}$ of order (q^2,q).*

In the rest of this work, we denote by $\mathcal{S}(\mathcal{F})$ the GQ of order (q^2,q) which arises from $\mathcal{F}$ as in Theorem 1.10.2, and such a GQ is called a *flock generalized quadrangle*.

Remark 1.10.3 (i) *Translation planes* (cf. the final section of Chapter 3) can be constructed from flocks by constructing an ovoid of the Klein quadric from a flock of the quadratic cone in $\mathbf{PG}(3,q)$. This ovoid corresponds to a line spread of $\mathbf{PG}(3,q)$ via the Klein correspondence, which in turn gives rise to a translation plane via the André/Bruck-Bose construction. This was independently observed in 1976 by both M. Walker [236] and J. A. Thas. A direct description of the line spread in terms of the associated q-clan was given in [64]. Alternatively, the ovoid was constructed directly from the *BLT-set* (cf. Section 1.10.4) in [101].

(ii) Suppose $\mathcal{F}$ is a flock of the quadratic cone in $\mathbf{PG}(3,q)$, and let $\Pi(\mathcal{F})$ be the translation plane which arises from $\mathcal{F}$ as in (i). If $\Pi(\mathcal{F})$ is a *semifield plane* (see [81]), then $\mathcal{F}$ is called a *semifield flock*. If $\mathcal{F}$ is a semifield flock, then $\mathcal{S}(\mathcal{F})^D$ is a TGQ, see, e.g., [132].

(iii) The point (∞) of a flock GQ $\mathcal{S}(\mathcal{F})$ clearly is a center of symmetry. This fact will be used without further notice throughout this work.

1.10.3 Flock generalized quadrangles and Property (G)

Property (G). Let $\mathcal{S}$ be a generalized quadrangle of order (s, s^2), $s \neq 1$. Let x_1, y_1 be distinct collinear points. We say that the pair $\{x_1, y_1\}$ has *Property (G)*, or that $\mathcal{S}$ has *Property (G) at* $\{x_1, y_1\}$, if every triad $\{x_1, x_2, x_3\}$ of points for which $y_1 \in \{x_1, x_2, x_3\}^\perp$ is 3-regular. The GQ $\mathcal{S}$ has *Property (G) at the line* L, or the line L has *Property (G)*, if each pair of points $\{x, y\}$, $x \neq y$ and $xILy$, has Property (G). If (x, L) is a flag, then we say that $\mathcal{S}$ has *Property (G) at* (x, L), or that (x, L) has *Property (G)*, if every pair $\{x, y\}$, $x \neq y$ and yIL, has Property (G).
Property (G) was introduced by S. E. Payne [128] in connection with generalized quadrangles of order (q^2, q) arising from flocks of quadratic cones in $\mathbf{PG}(3, q)$.

Theorem 1.10.4 (S. E. Payne [128]) *Any flock GQ satisfies Property (G) at its point* (∞).

Now we come to the main theorem of the masterful sequence of papers [177], [181], [183], [182], [186] of J. A. Thas; it is a converse of the previous theorem and the solution of a longstanding conjecture.

Theorem 1.10.5 (J. A. Thas [183]) *Let* $\mathcal{S} = (P, B, I)$ *be a GQ of order* (q^2, q), $q > 1$, *and assume that* $\mathcal{S}$ *satisfies Property (G) at the flag* (x, L). *If* q *is odd, then* $\mathcal{S}$ *is the point-line dual of a flock GQ. If* q *is even and all ovoids* $\mathcal{O}_z$ *(see Section 5 of [183]) are elliptic quadrics, then we have the same conclusion.*

1.10.4 BLT-sets

Let q be an odd prime power, and let $\mathcal{F} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_q\}$ be a flock of the quadratic cone $\mathcal{K}$ in $\mathbf{PG}(3, q)$ with vertex v. Let $\mathcal{K}$ be embedded in a nonsingular quadric $\mathcal{Q}$ of a $\mathbf{PG}(4, q)$ containing $\mathbf{PG}(3, q)$ as a hyperplane, so that $\mathcal{K} = \mathbf{PG}(3, q) \cap \mathcal{Q}$. There are unique points $p_1, p_2, \ldots, p_q$ of $\mathcal{Q}$ for which $\mathcal{C}_i = v^\perp \cap p_i^\perp$, $1 \leq i \leq q$, where '$\perp$' is relative to the GQ defined by $\mathcal{Q}$ [5]. Then the condition that $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_q$ are mutually disjoint is precisely the condition that $V = \{v, p_1, p_2, \ldots, p_q\}$ is a set of $q + 1$ points of $\mathcal{Q}$ such that for $1 \leq i < j \leq q$, (v, p_i, p_j) is a triad of points of the GQ $\mathcal{Q}$ for which $\{v, p_i, p_j\}^\perp = \emptyset$. The main theorem of [5] is that given such a set V, it is also true that for *each* triple (p_i, p_j, p_k), $0 \leq i < j < k \leq q$, no point of $\mathcal{Q}$ is

collinear (in $\mathcal{Q}$) with all three of the points. It follows that each point of $\mathcal{Q} \setminus V$ is collinear with either 0 or 2 points of V [5]. Such a set V of $q+1$ points of $\mathcal{Q}$ is called a *BLT-set of* $\mathcal{Q}$. L. Bader, G. Lunardon and J. A. Thas have showed in [5] that by using the BLT-set V, the flock $\mathcal{F}$ of the quadratic cone $\mathcal{K}$ may be interpreted as one of a set of $q+1$ flocks $\mathcal{F}_0 = \mathcal{F}, \mathcal{F}_1, \ldots, \mathcal{F}_q$ (also called a *BLT-set*) — recall that q is odd [5]. The flock $\mathcal{F}_i$ of the cone $p_i^\perp \cap \mathcal{Q}$ consists of the conic $\mathcal{C}_i$ together with the $q-1$ conics $p_i^\perp \cap p_j^\perp$, $i \neq j$, with $i = 1, 2, \ldots, q$.

1.10.5 Isomorphisms of flocks, BLT-sets and flock generalized quadrangles

We recall some facts, see [132] for a general reference.

- Two flocks $\mathcal{F}_1$ and $\mathcal{F}_2$ of the quadratic cone $\mathcal{C}$ of $\mathbf{PG}(3,q)$ are called *isomorphic* or *equivalent* provided there is an element of $\mathbf{P\Gamma L}(4,q)$ which fixes $\mathcal{C}$ and maps $\mathcal{F}_1$ to $\mathcal{F}_2$.

- In [5] it is proved that if x and y are two points of the BLT-set $\mathcal{P}$, then the flock $\mathcal{F}_x$ arising from x is isomorphic to the flock $\mathcal{F}_y$ arising from y if and only if x and y are in the same orbit of the stabilizer $\mathbf{P\Gamma O}(5,q)_{\mathcal{P}}$ of $\mathcal{P}$ in the appropriate orthogonal group.

- Each of the flocks determined by a given BLT-set (i.e., *derived* from any one of the flocks determined by that BLT-set) gives rise to the same generalized quadrangle [136]; each of these flocks corresponds to a line of the GQ through (∞) (starting from a given flock $\mathcal{F}$, that is, from a given q-clan, each of the q 'new' flocks is obtained by recoordinatizing the GQ $\mathcal{S}(\mathcal{F})$ so as to interchange the line $[A(\infty)]$ and some other line through (∞)).

- Two BLT-sets $\mathcal{P}_1$ and $\mathcal{P}_2$ are *isomorphic*, i.e., there exists an element of $\mathbf{P\Gamma O}(5,q)$ which maps $\mathcal{P}_1$ to $\mathcal{P}_2$, if and only if the generalized quadrangle $\mathcal{S}_1$ arising from $\mathcal{P}_1$ is isomorphic to the generalized quadrangle $\mathcal{S}_2$ arising from $\mathcal{P}_2$. Moreover, if $\mathcal{S}$ is the generalized quadrangle arising from the BLT-set $\mathcal{P}$, then the number of orbits of lines through the base-point (∞) of $\mathcal{S}$ under the action of the subgroup $\mathcal{G}_0$ of collineations of $\mathcal{S}$ that fix the point (∞) is equal to the number of orbits in $\mathcal{P}$ of the stabilizer of $\mathcal{P}$ in $\mathbf{P\Gamma O}(5,q)$.

1.10.6 Geometrical constructions of flock generalized quadrangles

The problem of finding a geometric construction of a flock generalized quadrangle was open for quite some time; N. Knarr [97] found such a construction for q odd, but many researchers came to believe that no such construction could exist for q even. Then J. A. Thas presented a construction for all q in [183], and, furthermore, in [186] it was shown that the construction of N. Knarr can be derived from that of J. A. Thas. As the construction of J. A. Thas implicitly will play a big role in the sequel, we mention it for the sake of completeness.

Let $\mathcal{K}$ be the quadratic cone with vertex v of $\mathbf{PG}(3,q)$. Further, let x be a point of $\mathcal{K} \setminus \{v\}$ and let Π be a plane of $\mathbf{PG}(3,q)$ not containing x. Project $\mathcal{K} \setminus \{x\}$ from x onto Π. Let τ be the tangent plane of $\mathcal{K}$ at the line vx and let $\tau \cap \Pi = L$. Then with the q^2 points of $\mathcal{K} \setminus vx$ there correspond the q^2 points of the affine plane $\Pi \setminus L = \Pi'$. With any point of $vx \setminus \{x\}$ there corresponds the intersection (∞) of vx and Π; with the generators of $\mathcal{K}$ distinct from vx there correspond the lines of Π distinct from L and containing (∞). With the nonsingular conics on $\mathcal{K}$ passing through x there correspond the affine parts of the q^2 lines of Π not passing through (∞), and with the nonsingular conics on $\mathcal{K}$ not passing through x there correspond the $q^2(q-1)$ nonsingular conics of Π which are tangent to L at (∞).

Let $\mathcal{F} = \{\mathcal{C}_1^*, \mathcal{C}_2^*, \ldots, \mathcal{C}_q^*\}$ be a flock of the cone $\mathcal{K}$. Consider the set $\tilde{\mathcal{F}} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{q-1}, M\}$ consisting of the $q-1$ nonsingular conics $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{q-1}$ and the line M of Π, which are obtained by projecting the elements of $\mathcal{F}$ from x onto Π ($\tilde{\mathcal{F}}$ is the 'plane model' of $\mathcal{F}$). So $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{q-1}$ are conics which are mutually tangent at (∞) — with common tangent line L — and M is a line of Π not containing (∞).

Now consider planes $\pi_\infty \neq \Pi$ and $\mu \neq \Pi$ of $\mathbf{PG}(3,q)$, respectively containing L and M. In μ we consider a point r, with $r \notin \Pi \cup \pi_\infty$. Next, let $\mathcal{Q}_i$ be the nonsingular quadric which contains $\mathcal{C}_i$, which is tangent to π_∞ at (∞) and which is tangent to μ at r, with $i = 1, 2, \ldots, q-1$. As $\mathcal{C}_i \cap M = \emptyset$, the quadric $\mathcal{Q}_i$ is elliptic, $1 \leq i \leq q-1$ [183].

Now let $\mathcal{S}$ be the following incidence structure.

- POINTS of $\mathcal{S}$ are of the six following types.

  (a) The $q^3(q-1)$ nonsingular elliptic quadrics $\mathcal{Q}$ containing $\mathcal{Q}_i \cap \pi_\infty =$

$L_\infty^{(i)} \cup M_\infty^{(i)}$ (over $\mathbf{GF}(q^2)$) such that the intersection multiplicity of $\mathcal{Q}_i$ and $\mathcal{Q}$ at (∞) is at least three (these are $\mathcal{Q}_i$, the nonsingular elliptic quadrics $\mathcal{Q} \neq \mathcal{Q}_i$ containing $L_\infty^{(i)} \cup M_\infty^{(i)}$ (over $\mathbf{GF}(q^2)$) and intersecting $\mathcal{Q}_i$ over $\mathbf{GF}(q)$ in a nonsingular conic containing (∞), and the nonsingular elliptic quadrics $\mathcal{Q} \neq \mathcal{Q}_i$ for which $\mathcal{Q} \cap \mathcal{Q}_i$ over $\mathbf{GF}(q^2)$ is $L_\infty^{(i)} \cup M_\infty^{(i)}$ counted twice), with $1 \leq i \leq q-1$.

(b) The q^3 points of $\mathbf{PG}(3,q) \setminus \pi_\infty$.

(c) The q^3 planes of $\mathbf{PG}(3,q)$ not containing (∞).

(d) The $q-1$ sets $\mathbf{Q}_i$, where $\mathbf{Q}_i$ consists of the q^3 quadrics $\mathcal{Q}$ of Type (a) corresponding with $\mathcal{Q}_i$, $1 \leq i \leq q-1$.

(e) The plane π_∞.

(f) The point (∞).

- LINES of $\mathcal{S}$ are of five types.

(i) Let (p, N) be a point-plane flag of $\mathbf{PG}(3,q)$, with $p \notin \pi_\infty$ and $(\infty) \notin N$. Then all quadrics $\mathcal{Q}$ of Type (a) which are tangent to N at p, together with p and N, form a line of Type (i). Any two distinct quadrics of such a line have exactly two points ((∞) and p) in common. The total number of lines of Type (i) is q^5.

(ii) Let $\mathcal{Q}$ be a point of Type (a) which corresponds to the quadric $\mathcal{Q}_i$, $1 \leq i \leq q-1$. If $\mathcal{Q} \cap \pi_\infty = \mathcal{Q}_i \cap \pi_\infty = L_\infty^{(i)} \cup M_\infty^{(i)}$ (over $\mathbf{GF}(q^2)$), then all points $\mathcal{Q}'$ of Type (a) for which $\mathcal{Q}' \cap \mathcal{Q}$ over $\mathbf{GF}(q^2)$ is $L_\infty^{(i)} \cup M_\infty^{(i)}$ counted twice, together with $\mathcal{Q}$ and $\mathcal{Q}_i$, form a line of Type (ii). There are $q^2(q-1)$ lines of Type (ii).

(iii) A line of Type (iii) is a set of q parallel planes of $\mathbf{AG}(3,q) = \mathbf{PG}(3,q) \setminus \pi_\infty$, where the line at infinity does not contain (∞), together with the plane π_∞.

(iv) Lines of Type (iv) are the lines of $\mathbf{PG}(3,q)$ not in π_∞ containing (∞).

(v) $\{(\infty), \pi_\infty, \mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q}_{q-1}\}$ is the unique line of Type (v).

- INCIDENCE of $\mathcal{S}$ is containment.

It is then proved in [183] that $\mathcal{S}$ is a generalized quadrangle which is isomorphic to the point-line dual of the flock GQ $\mathcal{S}(\mathcal{F})$.

1.11 Translation Generalized Quadrangles and $T(n, m, q)$'s

1.11.1 $T(n, m, q)$'s and translation duals of TGQ's

In this paragraph, we introduce the important notion of *translation dual* of a translation generalized quadrangle.

Suppose $H = \mathbf{PG}(2n+m-1, q)$ is the finite projective $(2n+m-1)$-space over $\mathbf{GF}(q)$, and let H be embedded in a $\mathbf{PG}(2n+m, q)$, say H'. Now define a set $\mathcal{O} = \mathcal{O}(n, m, q)$ of subspaces as follows: $\mathcal{O}$ is a set of q^m+1 $(n-1)$-dimensional subspaces of H, denoted $\mathbf{PG}(n-1, q)^{(i)}$, so that

(i) every three generate a $\mathbf{PG}(3n-1, q)$;

(ii) for every $i = 0, 1, \ldots, q^m$, there is a subspace $\mathbf{PG}(n+m-1, q)^{(i)}$ of H of dimension $n+m-1$, which contains $\mathbf{PG}(n-1, q)^{(i)}$ and which is disjoint from any $\mathbf{PG}(n-1, q)^{(j)}$ if $j \neq i$.

If $\mathcal{O}$ satisfies these conditions for $n = m$, then $\mathcal{O}$ is called a *pseudo-oval* or a *generalized oval* or an *[n - 1]-oval* of $\mathbf{PG}(3n-1, q)$. A generalized oval of $\mathbf{PG}(2, q)$ is just an oval (cf. Section 1.12) of $\mathbf{PG}(2, q)$. For $n \neq m$, $\mathcal{O}(n, m, q)$ is called a *pseudo-ovoid* or a *generalized ovoid* or an *[n - 1]-ovoid* or an *egg* of $\mathbf{PG}(2n+m-1, q)$. A [0]-ovoid of $\mathbf{PG}(3, q)$ is just an ovoid (cf. Section 1.12) of $\mathbf{PG}(3, q)$.
The spaces $\mathbf{PG}(n+m-1, q)^{(i)}$ are the *tangent spaces* of $\mathcal{O}(n, m, q)$, or just the *tangents*. Sometimes we will call an $\mathcal{O}(n, n, q)$ also an 'egg' or a 'generalized ovoid' for the sake of convenience.
Generalized ovals were introduced by J. A. Thas in [164], and generalized by S. E. Payne and J. A. Thas in FGQ, cf. Chapter 8.

Then S. E. Payne and J. A. Thas prove in [165, 139] that from any egg $\mathcal{O}(n, m, q)$ there arises a GQ $T(n, m, q) = T(\mathcal{O})$ which is a TGQ of order (q^n, q^m), for some special point (∞). This goes as follows.

- The POINTS are of three types.
  (1) A symbol (∞).
  (2) The subspaces $\mathbf{PG}(n+m, q)$ of H' which intersect H in a $\mathbf{PG}(n+m-1, q)^{(i)}$.
  (3) The points of $H' \setminus H$.

- The LINES are of two types.
  (a) The elements of the egg $\mathcal{O}(n,m,q)$.
  (b) The subspaces $\mathbf{PG}(n,q)$ of $\mathbf{PG}(2n+m,q)$ which intersect H in an element of the egg.
- INCIDENCE is defined as follows: the point (∞) is incident with all the lines of Type (a) and with no other lines; a point of Type (2) is incident with the unique line of Type (a) contained in it and with all the lines of Type (b) which it contains (as subspaces); finally, a point of Type (3) is incident with the lines of Type (b) that contain it.

Conversely, any TGQ can be seen in this way (that is, as a $T(n,m,q)$ associated to an egg $\mathcal{O}(n,m,q)$ in $\mathbf{PG}(2n+m-1,q)$), and whence, the study of translation generalized quadrangles is equivalent to the study of the generalized ovoids.

We already mentioned that for a TGQ of order (s,t), there are natural q, k and n, where k is odd and q is a prime power, so that either $s=t=q^n$ or $s=q^{nk}$ and $t=q^{n(k+1)}$, and if q is even, then $k=1$, see FGQ. Each TGQ $\mathcal{S}$ of order $(s,s^{\frac{k+1}{k}})$ (where $s=q^n$ for some prime power q), with translation point (∞), where k is odd and $s\neq 1$, has a *kernel* $\mathbb{K}$, which is a field with a multiplicative group isomorphic to the group of all collineations of $\mathcal{S}$ fixing the point (∞) and any given point not collinear with (∞) linewise. We have $|\mathbb{K}|\leq s$, see FGQ. The field $\mathbf{GF}(q)$ is a subfield of $\mathbb{K}$ if and only if $\mathcal{S}$ is of type $T(n,m,q)$. Completely similar results are obtained for the case $s=t$.

If $n\neq m$, then by 8.7.2 of [139] the q^m+1 tangent spaces of $\mathcal{O}(n,m,q)$ form an egg $\mathcal{O}^*(n,m,q)$ in the dual space of $\mathbf{PG}(2n+m-1,q)$. So in addition to $T(n,m,q)$ there arises a TGQ $T(\mathcal{O}^*)$, also denoted $T^*(n,m,q)$, or $T^*(\mathcal{O})$. The TGQ $T(\mathcal{O}^*)$ is called the *translation dual* of the TGQ $T(\mathcal{O})$. Now let $n=m$, with q odd. Then similar observations can be made, see e.g. [139, p. 182]. So also in this case there arises a *translation dual* of the TGQ $T(\mathcal{O})$. However, the only known TGQ for $s=t$ odd, is the classical GQ $\mathcal{Q}(4,s)$, which is isomorphic to its translation dual.

Remark 1.11.1 The GQ's $T_3(\mathcal{O})$ and $\mathcal{S}(\mathcal{F})^D$, where $\mathcal{F}$ is a Kantor flock, see Chapter 2, are the only known TGQ's of order (s,t), $s\neq t$, which are isomorphic to their translation dual.

We also have the following theorem.

Theorem 1.11.2 *The TGQ $T(\mathcal{O})$ and its translation dual $T(\mathcal{O}^*)$ have isomorphic kernels.*

Proof. We start from an $\mathcal{O}$ in $\mathbf{PG}(2n+m-1,q)$, where $\mathbf{GF}(q)$ is the kernel of $T(\mathcal{O})$ and either $n \neq m$ or $n = m$ and q odd. Then the egg $\mathcal{O}^*$ is also represented in a $\mathbf{PG}(2n+m-1,q)$, hence we know that $\mathbf{GF}(q)$ is a subfield of the kernel of $T(\mathcal{O}^*)$. Interchanging the role of $\mathcal{O}$ and $\mathcal{O}^*$ then yields the result. ∎

A TGQ $T(\mathcal{O})$ with $t = s^2$, $s = q^n$, is called *good* at an element $\pi \in \mathcal{O}$ (or is *good* at the corresponding line through (∞) of the TGQ) if for every two distinct elements π' and π'' of $\mathcal{O} \setminus \{\pi\}$ the $(3n-1)$-space $\pi\pi'\pi''$ contains exactly q^n+1 elements of $\mathcal{O}$. We also say that π is a *good element* of $T(\mathcal{O})$ or $\mathcal{O}$, and that $\mathcal{O}$ is *good* at its element π. In that case, it is easy to see that $\pi\pi'\pi''$ is skew to the other elements (use for instance [139, 8.7.2(iii)]). If the egg $\mathcal{O}$ contains a good element, then the egg is consequently called *good*.

Note. In the sequel, if $\mathcal{S} = T(\mathcal{O})$ is a TGQ for some translation point x, then by $\mathcal{S}^*$ we will sometimes denote the translation dual $T(\mathcal{O}^*)$ of $T(\mathcal{O})$.

We will often use the notation of this section without further notice.

Theorem 1.11.3 (J. A. Thas [177]) *If the TGQ $\mathcal{S}^{(\infty)}$ contains a good element π, then its translation dual satisfies Property (G) at the corresponding flag $((\infty)', \pi')$.*

Theorem 1.11.4 (FGQ, see the proof of 8.7.4(i)) *A TGQ of order s is isomorphic to a $T_2(\mathcal{O})$ of Tits with $\mathcal{O}$ an oval of $\mathbf{PG}(2,s)$ if and only if $|\mathbb{K}| = s$.*

Theorem 1.11.5 (FGQ, 8.7.4) *Let $(\mathcal{S}^{(p)}, G)$ be the TGQ arising from the generalized ovoid $\mathcal{O} = \mathcal{O}(n, 2n, q)$. Then $\mathcal{S}^{(p)} \cong T_3(\mathcal{O})$ for some ovoid $\mathcal{O}$ of $\mathbf{PG}(3,q^n)$, if and only if one of the following holds:*

- *(i) for a fixed point y, $y \not\sim p$, the group of all whorls about p fixing y has order $q^n - 1$, that is, the kernel has size q^n;*
- *(ii) for each point z not contained in an element of $\mathcal{O}$, the q^n+1 tangent spaces containing z have exactly $(q^n-1)/(q-1)$ points in common;*
- *(iii) each $\mathbf{PG}(3n-1,q)$ containing at least three elements of $\mathcal{O}$ contains exactly q^n+1 elements of $\mathcal{O}$.*

1.12 Some Interesting Point Sets of Generalized Quadrangles and Projective Spaces

1.12.1 Spreads and ovoids in generalized quadrangles

A *spread* of a generalized quadrangle $\mathcal{S}$ is a set $\mathbf{T}$ of mutually non-concurrent lines of $\mathcal{S}$ such that every point of $\mathcal{S}$ is incident with a (necessarily unique) line of $\mathbf{T}$. If the GQ is of order (s,t), then $|\mathbf{T}| = st+1$ [139, 1.8.1], and, conversely, a set of $st+1$ mutually non-concurrent lines of a GQ of order (s,t) is a spread. Dually, one defines *ovoids* of generalized quadrangles.
For a complete survey on spreads (and many new results) of GQ's up to 1993, see [190]. For a survey on results since 1993, see M. R. Brown [23]. We recall the following result, which is due to E. E. Shult [155].

Theorem 1.12.1 (FGQ, 1.8.3) *A GQ $\mathcal{S}$ of order (s,t), $s \neq 1 \neq t$ and $t > s^2 - s$, has no ovoid. Dually, a GQ $\mathcal{S}$ of order (s,t), $s \neq 1 \neq t$ and $s > t^2 - t$, has no spread.*

Two spreads $\mathbf{T}$ and $\mathbf{T}'$, respectively ovoids $\mathcal{O}$ and $\mathcal{O}'$, of a GQ $\mathcal{S}$ are said to be *isomorphic* if there is an automorphism of $\mathcal{S}$ which maps $\mathbf{T}$ onto $\mathbf{T}'$, respectively $\mathcal{O}$ onto $\mathcal{O}'$.

Suppose $\mathbf{T}$ is a spread of the GQ $\mathcal{S}$ of order (s,t), $s,t > 1$. Then $\mathbf{T}$ is *Hermitian* or *regular* or *normal* if for every two distinct lines L and M of $\mathbf{T}$, the pair $\{L,M\}$ is regular (so $|\{L,M\}^{\perp\perp}| = s+1$) and $\{L,M\}^{\perp\perp} \subseteq \mathbf{T}$. Let $\mathbf{T}$ be a spread of $\mathcal{S}$. Then $\mathbf{T}$ is *locally Hermitian* or *semiregular* or *seminormal* w.r.t. the line L if for every line $M \neq L$ of $\mathbf{T}$, the pair $\{L,M\}$ is regular and $\{L,M\}^{\perp\perp} \subseteq \mathbf{T}$. Dually, we define *(locally) Hermitian ovoids*, *(semi)regular ovoids* and *(semi)normal ovoids*.

We now present some of the known results on spreads and ovoids of generalized quadrangles.

Theorem 1.12.2 (FGQ, 3.4.1, 3.4.2 and 3.4.3; see also [190] for (v))

(*i*) *The GQ $\mathcal{Q}(4,q)$ always has ovoids. It has spreads if and only if q is even.*

(*ii*) *The GQ $T_2(\mathcal{O})$ of Tits always has ovoids.*

(*iii*) *The GQ $\mathcal{Q}(5,q)$ has spreads but no ovoids.*

(*iv*) *The GQ $T_3(\mathcal{O})$ of Tits has no ovoid but always has spreads.*

(v) *Each TGQ $T(\mathcal{O})$, where $\mathcal{O}$ is good at some element π, has spreads.*

(vi) *The GQ $H(4,q^2)$ has no ovoid. For $q=2$ it has no spread.*

(vii) *The GQ $\mathcal{P}(\mathcal{S},x)$ of S. E. Payne always has spreads. It has an ovoid if and only if $\mathcal{S}$ has an ovoid containing x.*

Remark 1.12.3 It should be noted that:

- (i),(iii) and (vi) dualize, respectively, to $W(q)$, $H(3,q^2)$ and $H(4,q^2)^D$;
- (ii) generalizes the first part of (i), (iv) generalizes (iii), and (v) generalizes the second part of (iv);
- (v) applies to the Roman GQ's, the dual Kantor GQ's and the Penttila-Williams GQ (see Chapter 2);
- we will give a proof of (v) in the 'odd characteristic case' which is independent (and totally different) from that which appeared in [190], see (essentially) the appendix of Chapter 9 and the main result of Chapter 10, or Section 12.6 of Chapter 12.

1.12.2 Subtended and doubly subtended ovoids

Suppose $\mathcal{S}$ is a GQ of order (s,t), $s \neq 1 \neq t$, which contains a subGQ $\mathcal{S}'$ of order (s,t'), $t' > 1$, and let z be a point of $\mathcal{S} \setminus \mathcal{S}'$. Then we know by Theorem 1.6.1 that z is collinear with the points of an ovoid $\mathcal{O}_z$ of $\mathcal{S}'$. We say that $\mathcal{O}_z$ is 'subtended by z', and that $\mathcal{O}_z$ is a *subtended ovoid*. Now suppose that $t = s^2$ and that $t' = s$. By Theorem 1.1.3, it immediately follows that each span of non-collinear points of $\mathcal{S}$ has size 2. Hence a subtended ovoid $\mathcal{O}_z$ of $\mathcal{S}'$, where $z \in \mathcal{S} \setminus \mathcal{S}'$, can be subtended by at most one point of $\mathcal{S} \setminus \mathcal{S}'$ different from z. If this is the case, then we say that $\mathcal{O}_z$ is *doubly subtended*. If each subtended ovoid of $\mathcal{S}'$ is doubly subtended, then $\mathcal{S}'$ is called a *doubly subtended subGQ* of $\mathcal{S}$.

1.12.3 Translation ovals and translation ovoids

Suppose $\mathcal{O}$ is an ovoid of a GQ $\mathcal{S}$ of order (s,t), $s \neq 1 \neq t$. Let x be a point of $\mathcal{O}$, and suppose L is incident with x. Then $|(\mathcal{O} \setminus \{x\}) \cap y^\perp| = t$ for any point $y \neq x$ on L. Denote $(\mathcal{O} \setminus \{x\}) \cap y^\perp$ by $V(\mathcal{O},x,y)$. We call $\mathcal{O}$ a *translation ovoid w.r.t. the flag* (x,L) if there is an automorphism group $G_{x,L}$ of $\mathcal{S}$ which fixes $\mathcal{O}$ (as a set), which fixes x linewise and L pointwise, and which acts transitively on the points of $V(\mathcal{O},x,y)$ for each $y \neq x$ on L. We call $\mathcal{O}$ a *translation ovoid*

w.r.t. the point x if $\mathcal{O}$ is a translation ovoid w.r.t. the flag (x, M) for each line MIx. Translation ovoids (of generalized $2n$-gons, $n = 2, 3$) were introduced by I. Bloemen, J. A. Thas and H. Van Maldeghem in [15].

Suppose $T(\mathcal{O})$ is a TGQ of order (q^n, q^{2n}) with translation point (∞), and assume that $\mathcal{S}'$ is a subGQ of $T(\mathcal{O})$ of order q^n which contains (∞). Suppose z is a point of $T(\mathcal{O}) \setminus \mathcal{S}'$, and consider $\mathcal{O}_z$. Then it is easy to see that $\mathcal{O}_z$ is a translation ovoid of $\mathcal{S}'$ w.r.t. (∞). Suppose that $\mathcal{O}$ is good at some element π, and suppose that $\pi \in \mathcal{S}'$ (as a line), and let q be odd. Then $\mathcal{S}' \cong \mathcal{Q}(4, q^n)$. It was shown in J. A. Thas [180] and G. Lunardon [105] that there is a canonical connection between translation ovoids of $\mathcal{Q}(4, q)$, q odd, and semifield flocks of the quadratic cone of $\mathbf{PG}(3, q)$. Hence the following result.

Translation ovoids of $\mathcal{Q}(4, q)$, q *odd, w.r.t. a point and semifield flocks of the quadratic cone of* $\mathbf{PG}(3, q)$ *are equivalent objects.*

So there appears to be a strong interaction between TGQ's $T(\mathcal{O})$ of order (q^n, q^{2n}), q odd, $\mathcal{O}$ good at some element, translation ovoids of $\mathcal{Q}(4, q^n)$ w.r.t. a point, and semifield flocks.

An *oval* of a (finite) projective plane Π of order n is a set of $n + 1$ points of Π no three of which are collinear.

Theorem 1.12.4 (B. Segre [151]) *Each oval of* $\mathbf{PG}(2, q)$, q *odd, is an irreducible conic.*

Hence by Theorem 1.5.1 (i), each $T_2(\mathcal{O})$ of order q, q odd, is isomorphic to $\mathcal{Q}(4, q)$.

A *hyperoval* of Π is a set of $n + 2$ points of Π no three of which are collinear. If a projective plane of order n admits a hyperoval, then n is even, see [81]. A *regular hyperoval* of $\mathbf{PG}(2, q)$ is just a conic of $\mathbf{PG}(2, q)$ together with its nucleus [81]. A *translation oval* $\mathcal{O}$ *w.r.t. a point* $p \in \mathcal{O}$ of $\mathbf{PG}(2, q)$, is an oval $\mathcal{O}$ such that there is a group H of automorphisms of $\mathbf{PG}(2, q)$ which stabilizes $\mathcal{O}$, which fixes the unique tangent of $\mathcal{O}$ at p pointwise, and which acts regularly on $\mathcal{O} \setminus \{p\}$. Then we have the following result, see Chapter 12 of FGQ.

A $T_2(\mathcal{O})$ *of Tits of order* q *is self-dual if and only if* $\mathcal{O}$ *is a translation oval of* $\mathbf{PG}(2, q)$ *(and then* q *is even).*

An *ovoid* $\mathcal{O}$ of $\mathbf{PG}(3,q)$, $q > 2$, is a set of q^2+1 distinct points no three of which are collinear; an *ovoid* of $\mathbf{PG}(3,2)$ is a set of 5 points no four of which are coplanar. If $q > 2$, then an ovoid is a maximal sized set with that property. The following theorem is (independently) due to A. Barlotti [8] and G. Panella [111].

Theorem 1.12.5 ([8]; [111]) *Each ovoid of* $\mathbf{PG}(3,q)$, q *odd, is an elliptic quadric.*

Hence by Theorem 1.5.1 (ii), each $T_3(\mathcal{O})$ of order (q,q^2), q odd, is isomorphic to $\mathcal{Q}(5,q)$.

1.13 Some More Group Theory

1.13.1 Permutation group theory

Let X be a set and let G be a group acting faithfully on X. Then the pair (X,G) is said to have *permutation rank* n, $n > 1$, if G acts transitively on X and if the stabilizer G_x of some element x of X in G has exactly n orbits in X. A rank 2 group is the same as a *2-transitive group*. The group G acts *n-transitively*, $n \geq 2$, on X if G acts transitively on X and if G_x acts $(n-1)$-transitively on $X \setminus \{x\}$ for some $x \in X$.
The group G acts *semiregularly* on X if no nontrivial element of G fixes a point of X (in the finite case, there follows that $|G|$ divides $|X|$). The group G acts *regularly* on X if it acts semiregularly and transitively on X (in the finite case, this is equivalent to saying that $|X| = |G|$ and that G acts semiregularly on X).

1.13.2 Frobenius groups

Suppose (X,G) is a permutation group which satisfies the following properties:

- G acts transitively but not regularly on X;
- there is no nontrivial element of G with more than one fixed point in X.

Then (X,G) is a *Frobenius group* (or G is a *Frobenius group* in its action on X). Define $N \subseteq G$ by:

$$N = \{g \in G \parallel f(g) = 0\} \cup \{\mathbf{1}\},$$

where $f(g)$ is the number of fixed points of g in X. Then N is called the *Frobenius kernel* of G (or of (X, G)), and we have the following well-known result.

Theorem 1.13.1 (Theorem of Frobenius) *N is a normal regular subgroup of G.*

Final Remark

The only result in Chapter 1 which will not be used explicitly is the geometrical construction of flock generalized quadrangles of order (q^2, q) of J. A. Thas (for all q). Implicitly however, this construction plays an important role in the proof of the result of J. A. Thas which asserts that a thick generalized quadrangle of order (q, q^2) which satisfies Property (G) at a flag is the point-line dual of a flock generalized quadrangle, provided the quadrangle satisfies an extra condition in the even characteristic case. To that end, S. E. Payne remarked in [132] that

"No work on generalized quadrangles which does not contain the construction of J. A. Thas could be complete".

Chapter 2

The Known Generalized Quadrangles

In this chapter, it is our purpose to provide a census of the known finite thick generalized quadrangles. In some cases, the size of the automorphism group is given (mostly based on S. E. Payne [129]). We will treat in much greater detail those TGQ's which arise from flocks, and their translation duals. We omit a discussion on 'sporadic' examples[1]; only the Penttila-Williams TGQ and its translation dual are considered in that context. We also provide an explicit form for the Penttila-Williams GQ, which is taken from K. Thas [218].

2.1 The Classical and Dual Classical Examples

The thick classical and dual classical examples are $H(4,q^2)$, $H(3,q^2)$, $W(q)$, $\mathcal{Q}(4,q)$, $\mathcal{Q}(5,q)$ and $H(4,q^2)^D$, q an arbitrary prime power. For more details, see Section 1.2.[2]

[1] A detailed account is contained in S. E. Payne [132].

[2] A good reference on the size of their respective automorphism groups is [229].

2.2 The $T_2(\mathcal{O})$ and $T_3(\mathcal{O})$ of J. Tits

See Section 1.5. If $\mathcal{S}$ is isomorphic to a $T_d(\mathcal{O})$ with $q+1$ points on a line and if $\mathcal{S}$ is non-classical, $d = 2, 3$, then

$$Aut(\mathcal{S}) \cong \mathbf{P\Gamma L}(d+1, q)_{\mathcal{O}},$$

where $\mathcal{O} \subseteq \mathbf{PG}(2,q) \subseteq \mathbf{PG}(3,q)$ if $d = 2$, and $\mathcal{O} \subseteq \mathbf{PG}(3,q) \subseteq \mathbf{PG}(4,q)$ if $d = 3$ (cf. [108]; see also Chapter 11).

2.3 The Generalized Quadrangles of Order $(s-1, s+1)$ and $(s+1, s-1)$

For each prime power q, R. W. Ahrens and G. Szekeres [1] constructed GQ's of order $(q-1, q+1)$. For q even, these examples were found independently by M. Hall, Jr. [70]. S. E. Payne [112] found a construction method which included all these examples and which produced some additional ones for q even, see [113, 114]. These examples yield the only known cases of GQ's of order (s,t) with $s \neq 1$ and $t \neq 1$, in which s and t are not powers of the same prime.

2.3.1 The GQ's $AS(q)$ of order $(q-1, q+1)$ of R. W. Ahrens and G. Szekeres, q an odd prime power

Define a point-line incidence structure $AS(q) = (P, B, I)$, q an odd prime power, as follows.

- The POINTS of P are the points of the affine 3-space $\mathbf{AG}(3,q)$.
- The LINES of B are the following curves of $\mathbf{AG}(3,q)$:

  (i) $x = \sigma$, $y = a$, $z = b$,

  (ii) $x = a$, $y = \sigma$, $z = b$,

  (iii) $x = c\sigma^2 - b\sigma + a$, $y = -2c\sigma + b$, $z = \sigma$,

  where the parameter σ ranges over $\mathbf{GF}(q)$ and where a, b, c are arbitrary elements of $\mathbf{GF}(q)$.
- INCIDENCE is the natural one.

Then $AS(q)$ is a GQ of order $(q-1, q+1)$ [1].

2.3.2 The GQ's S_{xy}^- of order $(q+1, q-1)$, $q = 2^h$

Let $\mathcal{O}$ be a hyperoval in $\mathbf{PG}(2,q)$, $q = 2^h$, and let $\mathbf{PG}(2,q) = H$ be embedded as a plane in $\mathbf{PG}(3,q) = H'$. Let x and y be distinct points of $\mathcal{O}$. The following GQ S_{xy}^- of order $(q+1, q-1)$ can then be constructed [112, 123].

- The POINTS are of three types:
  - (i) the points of $H' \setminus H$;
  - (ii) the planes through x not containing y;
  - (iii) the planes through y not containing x.
- The LINES are just those lines of H' which are not contained in H and which meet $\mathcal{O} \setminus \{x, y\}$ (necessarily in a unique point).
- INCIDENCE is inherited from H'.

2.3.3 The GQ's $T_2^*(\mathcal{O})$ of order $(q-1, q+1)$, $q = 2^h$

Let $\mathcal{O}$ be a hyperoval in $\mathbf{PG}(2,q)$, $q = 2^h$, and let $\mathbf{PG}(2,q) = H$ be embedded as a plane in $\mathbf{PG}(3,q) = H'$. Define an incidence structure $T_2^*(\mathcal{O})$ by taking

- for POINTS just those points of $H' \setminus H$;
- for LINES just those lines of H' which are not contained in H and which meet $\mathcal{O}$ (necessarily in a unique point);
- for INCIDENCE just that inherited from H'.

Then $T_2^*(\mathcal{O})$ is a GQ of order $(q-1, q+1)$ [1, 70].

2.3.4 The GQ's $\mathcal{P}(\mathcal{S}, x)$ of S. E. Payne

In this section, we recall the beautiful construction of S. E. Payne [112].

Let s be a regular point of the GQ $\mathcal{S} = (P, B, I)$ of order s, $s > 1$. Define an incidence structure $\mathcal{P}(\mathcal{S}, x) = \mathcal{S}' = (P', B', I')$ as follows:

- The POINT SET P' is the set $P \setminus x^\perp$.
- The LINES of B' are of two types:
  - the elements of Type (a) are the lines of B which are not incident with x;

- the elements of Type (b) are the hyperbolic lines $\{x, y\}^{\perp\perp}$ where $y \not\sim x$.

- INCIDENCE I' is containment (regarding a line of $\mathcal{S}$ as a set of points).

Then $\mathcal{S}'$ is a GQ of order $(s-1, s+1)$.

2.4 TGQ's which Arise from Flocks

2.4.1 Kantor semifield generalized quadrangles

Let $\mathcal{K}$ be the quadratic cone with equation $X_0X_1 = X_2^2$ of $\mathbf{PG}(3,q)$, q odd. Then the q planes π_t with equation

$$tX_0 - mt^\sigma X_1 + X_3 = 0,$$

where $t \in \mathbf{GF}(q)$, m is a given non-square in $\mathbf{GF}(q)$ and σ a given automorphism of $\mathbf{GF}(q)$, define a flock $\mathcal{F}$ of $\mathcal{K}$; see [172]. All the planes π_t contain the exterior point $(0,0,1,0)$ of $\mathcal{K}$. The flock is *linear*, that is, all the planes π_t contain a common line, if and only if $\sigma = \mathbf{1}$. Conversely, every nonlinear flock $\mathcal{F}$ of $\mathcal{K}$ for which the planes of the q conics share a common point, is of the type just described, see [172]. The corresponding GQ $\mathcal{S}(\mathcal{F})$ was first discovered by W. M. Kantor, and is called the *Kantor semifield (flock) generalized quadrangle*. The kernel $\mathbb{K}$ is the fixed field of σ, see [125, 148]. This GQ is a TGQ for some base-line. The following was shown by Payne in [128].

Theorem 2.4.1 (S. E. Payne [128]) *Suppose a TGQ $\mathcal{S} = T(\mathcal{O})$ is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$, $\mathcal{F}$ a Kantor flock. Then $T(\mathcal{O})$ is isomorphic to its translation dual $T(\mathcal{O}^*)$.*

We also have the following, which is due to J. A. Thas and H. Van Maldeghem [196].

Theorem 2.4.2 ([196]) *Suppose that the TGQ $T(\mathcal{O})$, with $\mathcal{O} = \mathcal{O}(n, 2n, q)$ and q odd, is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$, where the point (∞) of $\mathcal{S}(\mathcal{F})$ corresponds to the line η of Type (b) of $T(\mathcal{O})$. Then $T(\mathcal{O})$ is good at the element η if and only if $\mathcal{F}$ is a Kantor flock.*

Note. In the rest of this work, we will write 'Kantor (flock) GQ' instead of 'Kantor semifield (flock) GQ'. This cannot lead to confusion as this is the only

class of GQ's discovered by W. M. Kantor which we consider.

One flock arises by derivation (see Chapter 10).

If $\mathcal{S} = \mathcal{S}(\mathcal{F})$ is a Kantor GQ, then $Aut(\mathcal{S})$ acts triply transitively on the lines through (∞) [129]. Suppose that $\mathcal{F}$ is non-linear, and put $q = p^e$ for the prime p. If $\sigma^2 \neq \mathbf{1}$, then

$$|Aut(\mathcal{S})| = q^6(q+1)(q-1)^2 2e,$$

and if $\sigma^2 = \mathbf{1}$, $\sigma \neq \mathbf{1}$, then

$$|Aut(\mathcal{S})| = q^6(q+1)(q-1)^2 4e,$$

see S. E. Payne [129].

2.4.2 Roman generalized quadrangles

Let $\mathcal{K}$ be the quadratic cone with equation $X_0X_1 = X_2^2$ of $\mathbf{PG}(3,q)$, with $q = 3^r$ and $r > 2$. Then the q planes π_t with equation

$$tX_0 - (mt + m^{-1}t^9)X_1 + t^3X_2 + X_3 = 0,$$

$t \in \mathbf{GF}(q)$, m a given non-square in $\mathbf{GF}(q)$, define a flock $\mathcal{F}$ of $\mathcal{K}$ which is called the *Ganley flock*; see, e.g., [128]. The corresponding GQ $\mathcal{S}(\mathcal{F})$ is a TGQ for some base-line, and so the dual $\mathcal{S}(\mathcal{F})^D$ of $\mathcal{S}(\mathcal{F})$ is isomorphic to some $T(\mathcal{O})$. By [125, 148], the kernel $\mathbb{K}$ is isomorphic to $\mathbf{GF}(3)$. S. E. Payne [128] shows the following.

Theorem 2.4.3 *$T(\mathcal{O})$ is not isomorphic to its translation dual $T(\mathcal{O}^*)$.*

Also, he proves that $T(\mathcal{O}^*)$ is a TGQ which does not arise from a flock. In [128], the GQ's $T(\mathcal{O}^*)$ were called the *Roman generalized quadrangles.*

We present the Roman GQ's also under the form of their 4-gonal family; this will be very convenient, as it illustrates a more general 'coordinatization method' which will be introduced in Chapter 10.

Let n be any non-square of $\mathbb{F} = \mathbf{GF}(q)$, $q = 3^r$, $r > 2$. For $t \in \mathbb{F}$, $\gamma \in \mathbb{F}^2 = \mathbb{F} \times \mathbb{F}$, put

$$\hat{g}_t(\gamma) = \gamma \begin{pmatrix} t & 0 \\ 0 & -nt \end{pmatrix} \gamma^T + \left[\gamma \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \gamma^T\right]^{1/3} + \left[\gamma \begin{pmatrix} 0 & 0 \\ 0 & -n^{-1}t \end{pmatrix} \gamma^T\right]^{1/9}.$$

Define $\hat{f} : \mathbb{F}^2 \times \mathbb{F}^2 \longrightarrow \mathbb{F}$ by

$$\hat{f}(\alpha, \gamma) = \alpha \begin{pmatrix} -1 & 0 \\ 0 & n \end{pmatrix} \gamma^T + \left[\alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \gamma^T\right]^{1/3} + \left[\alpha \begin{pmatrix} 0 & 0 \\ 0 & n^{-1} \end{pmatrix} \gamma^T\right]^{1/9}.$$

Then for all $d, t, u \in \mathbb{F}$ and $\alpha, \gamma \in \mathbb{F}^2$:

1. $\hat{f}$ is biadditive and symmetric;
2. $\hat{g}_t(\alpha + \gamma) - \hat{g}_t(\alpha) - \hat{g}(\gamma) = \hat{f}(t\alpha, \gamma) = \hat{f}(t\gamma, \alpha)$;
3. $\hat{g}_{t+u}(\alpha) = \hat{g}_t(\alpha) + \hat{g}_u(\alpha)$;
4. $\hat{g}_t(d\gamma) = \hat{g}_{td^2}(\gamma)$.

Put $G = \mathbb{F}^4 = \{(r, c, b, d) \parallel r, c, b, d \in \mathbb{F}\}$ with coordinatewise addition. Define subgroups in the following way: $B(\infty) = \{(r, 0, 0, 0) \in G \parallel r \in \mathbb{F}\}$; $B^*(\infty) = \{(r, 0, \gamma) \in G \parallel r \in \mathbb{F}, \gamma \in \mathbb{F}^2\}$. For $\gamma \in \mathbb{F}^2$, write $B(\gamma) = \{(-\hat{g}_c(\gamma), c, -c\gamma) \in G \parallel c \in \mathbb{F}\}$. Put $\gamma = (g_1, g_2) \in \mathbb{F}^2$. Then $B^*(\gamma) = \{(r, c, b, d) \in G \parallel c, b, d \in \mathbb{F}$ with $r = (cg_1^2 - cng_2^2 - bg_1 + dng_2) + (cg_1g_2 + bg_2 + dg_1)^{1/3} + (-cn^{-1}g_2^2 + dn^{-1}g_2)^{1/9}\}$.
Then $\mathcal{J} = \{B(\gamma) \parallel \gamma \in \mathbb{F}^2 \cup \{\infty\}\}$ is a 4-gonal family for G with tangent space $B^*(\gamma)$ at $B(\gamma)$. So, in the usual way we have a TGQ $\mathcal{S} = (\mathcal{S}^{(\infty)}, G) = \mathcal{S}(G, \mathcal{J})$ of order (s, s^2), and $\mathcal{S}$ is isomorphic to the Roman GQ of order (q, q^2) (cf. [128]).

Recall that $q \geq 27$. Then $\mathcal{S}$ and $\mathcal{S}^*$ are non-classical, see [128], and by [129], we have that

$$|Aut(\mathcal{S}^*)| = q^6(q-1)2r.$$

We emphasize that $\mathcal{S}^*$ is the point-line dual of the Ganley flock GQ.

Two flocks arise by derivation (see Chapter 10).

The size of the full automorphism group of $\mathcal{S}$ will be obtained in Chapter 10.

2.4.3 The sporadic Penttila-Williams generalized quadrangle

In this paragraph, it is our goal to give an 'explicit' description of the sporadic *Penttila-Williams TGQ* of order $(3^5, 3^{10})$, as for the Roman TGQ's in the previous section. The method is taken from S. E. Payne [128].

Let $q = 3^5$. The q planes π_t with equation

$$tX_0 + 2t^9X_1 + t^{27}X_2 + X_3 = 0,$$

$t \in \mathbf{GF}(q)$, define a semifield flock of the quadratic cone with equation $X_0X_1 = X_2^2$ of $\mathbf{PG}(3,q)$. The flock, which is called the *Penttila-Williams flock*, was constructed by L. Bader, G. Lunardon and I. Pinneri in [4], using the Penttila-Williams ovoid of $\mathcal{Q}(4,3^5)$ defined in [145], and the corresponding GQ, that is, the translation dual of $\mathcal{S}(\mathcal{F})^D$, is therefore referred to as the *(sporadic) Penttila-Williams generalized quadrangle*. The kernel of the Penttila-Williams GQ is isomorphic to $\mathbf{GF}(3)$ [132].

Define a 2×2-matrix A_t, $t \in \mathbf{GF}(q) = \mathbb{F}$, with $q = 3^5$, as

$$A_t = \begin{pmatrix} t & t^{27} \\ 0 & 2t^9 \end{pmatrix}.$$

Then with $A(t) = \{(\alpha, \alpha A_t \alpha^T, \alpha(A_t + A_t^T)) \parallel \alpha \in \mathbb{F}^2\}$ and $A(\infty) = \{(0,0,\beta) \parallel \beta \in \mathbb{F}^2\}$, $\mathcal{J} = \{A(\infty)\} \cup \{A(t) \parallel t \in \mathbb{F}\}$ is a 4-gonal family of type (q^2, q). This is the 4-gonal family which yields the TGQ $\mathcal{S}(\mathcal{F})^D$ with $\mathcal{F}$ the Penttila-Williams flock. We now determine $(\mathcal{S}(\mathcal{F})^D)^*$.

Suppose that $G = \{(a,b,c,d) \parallel a,b,c,d \in \mathbb{F}\}$ is a group provided with coordinatewise addition. Put $tr(r_0 r + c_0 \hat{g}_r(\gamma)) = 0$ for all $r \in \mathbb{F}$, where $\hat{g}_r(\gamma) = \gamma A_r \gamma^T$, and where $c_0 \in \mathbb{F}$ is fixed. Then with $\gamma = (g_0, g_1)$, we have that $tr(r_0 r + c_0(g_o^2 r + g_0 g_1 r^{27} + 2g_1^2 r^9)) = tr(r(r_0 + c_0 g_0^2) + r^9(2g_1^2 c_0) + r^{27}(c_0 g_0 g_1)) = 0$ for all r if and only if $r_0 = -c_0 g_0^2 - (2g_1^2 c_0)^{1/9} - (c_0 g_0 g_1)^{1/27}$. Now put $\hat{A}(\infty) = \{(r,0,0,0) \in G \parallel r \in \mathbb{F}\}$, and $\hat{A}(\gamma) = \{(-\hat{g}_c(\gamma), c, -c\gamma) \in G \parallel c \in \mathbb{F}\}$, where

$$\hat{g}_t(\gamma) = \gamma \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} \gamma^T + \left[\gamma \begin{pmatrix} 0 & 0 \\ 0 & 2t \end{pmatrix} \gamma^T\right]^{1/9} + \left[\gamma \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \gamma^T\right]^{1/27}.$$

Then $\mathcal{J} = \{\hat{A}(\infty)\} \cup \{\hat{A}(\gamma) \parallel \gamma \in \mathbb{F}^2\}$ is a 4-gonal family such that the GQ $\mathcal{S}(G, \mathcal{J})$ is a TGQ of order (q, q^2) for which the translation dual is the point-line dual of the flock GQ $\mathcal{S}(\mathcal{F})$ with $\mathcal{F}$ the Penttila-Williams flock.

Two flocks arise by derivation (see Chapter 10).

2.5 The Known Flock GQ's of Order (q^2, q), q Odd (which Are not TGQ's)

Throughout this section we assume that $\mathbb{K} = \mathbf{GF}(q)$, q an odd prime power. Traditionally, the known examples of flock quadrangles have been given in terms of the associated q-clans. In [144] T. Penttila gave a new construction of some flock for $q \equiv \pm 1 \mod 10$ as a BLT-set (which still lacks a satisfactory description as a q-clan). The newest infinite family for q odd has $q = 3^e$ and was discovered by M. Law and T. Penttila as a generalization of an example with $q = 27$ that was studied via computer (cf. [103]). That infinite family appears in [102]. Below we give a census of all these families along with some information about the associated flocks, and the automorphism group of the corresponding GQ.

As q is odd in this section, we give the q-clans as symmetric matrices of the form

$$A_t = \begin{pmatrix} t & f(t) \\ f(t) & g(t) \end{pmatrix}, \quad t \in \mathbb{K}; \quad f, g : \mathbb{K} \to \mathbb{K}.$$

(For the purpose of belonging to a q-clan, a matrix $A = \begin{pmatrix} x & y \\ z & u \end{pmatrix}$ is completely determined by three quantities: x, u and $z + y$. For q odd, one may require that A be symmetric.)

The corresponding BLT-set $\mathcal{P}$ of the quadric

$$\mathcal{Q} : X_4X_0 + X_1X_3 + X_2^2 = 0$$

is then given by

$$\mathcal{P} = \{(1, t, f(t), -g(t), tg(t) - (f(t))^2)\} \cup \{(0, 0, 0, 0, 1)\}.$$

Recall that $\mathcal{C} = \{A_t \parallel t \in \mathbb{K}\}$ is a q-clan if and only if

$$-\det(A_s - A_t) = (f(s) - f(t))^2 - (s-t)(g(s) - g(t))$$

is a non-square of $\mathbb{K} = \mathbf{GF}(q)$ whenever $s \neq t$. As in S. E. Payne [132], we follow the presentation and naming conventions of the recent survey by N. Johnson and S. E. Payne [87]. In the case where a BLT-set gives rise to more than one flock, the family is named simply by concatenating the names given to the non-isomorphic flocks. For a given q-clan $\mathcal{C}$, respectively BLT-set $\mathcal{P}$, the associated generalized quadrangle is denoted $\mathcal{S}(\mathcal{C})$, respectively $\mathcal{S}(\mathcal{P})$. We also indicate how many non-isomorphic flocks are associated with the generalized quadrangle. No comment indicates that only one flock occurs.

From here on, the present chapter is based on S. E. Payne [132].

2.5.1 Classical: For all q and with $x^2 + bx + c$ irreducible over $\mathbb{K}$

$$\mathcal{C} = \left\{ \begin{pmatrix} t & \frac{1}{2}bt \\ \frac{1}{2}bt & ct \end{pmatrix} \| \in \mathbb{K} \right\}.$$

2.5.2 FTW: For $q \equiv -1 \mod 3$

$$\mathcal{C} = \left\{ \begin{pmatrix} t & \frac{3}{2}t^2 \\ \frac{3}{2}t^2 & 3t^3 \end{pmatrix} \| \, t \in \mathbb{K} \right\}.$$

We have that

$$|Aut(\mathcal{S})| = q^6(q+1)(q-1)^2 e,$$

if $\mathcal{S}$ is non-classical [129] and $q = p^e$, p a prime, and $Aut(\mathcal{S})$ is triply transitive on the lines through (∞).

2.5.3 K_2/JP: For $q \equiv \pm 2 \mod 5$

$$\mathcal{C} = \left\{ \begin{pmatrix} t & \frac{5}{2}t^3 \\ \frac{5}{2}t^3 & 5t^5 \end{pmatrix} \| \, t \in \mathbb{K} \right\}.$$

Two flocks arise.

There holds that

$$|Aut(\mathcal{S})| = q^5(q-1)^2 2e,$$

if $\mathcal{S}$ is non-classical [129], $q = p^e$ for the prime p, and $Aut(\mathcal{S})$ has two orbits on the lines through (∞).

2.5.4 K_3/BLT: For $q = 5^e$, with k a non-square of $\mathbb{K}$

$$\mathcal{C} = \left\{ \begin{pmatrix} t & 3t^2 \\ 3t^2 & k^{-1}t(1+kt^2)^2 \end{pmatrix} \parallel t \in \mathbb{K} \right\}.$$

Two flocks arise.

If $q > 5$, then $\mathcal{S}$ is non-classical, and by [129] it holds that

$$|Aut(\mathcal{S})| = q^6(q-1)2e.$$

2.5.5 Fi: For all odd $q \geq 5$

The Fisher flock has a q-clan representation (first discovered by J. A. Thas [172]; first given in explicit q-clan form in [127]) which is described in detail in [87]. However, it is rather involved and not as elegant as the BLT representation of T. Penttila in [144], which we will give here.

Let $\mathbb{K} = \mathbf{GF}(q) \subseteq \mathbf{GF}(q^2) = \mathbb{K}'$, q odd and $q \geq 5$. Let η be a primitive element of $\mathbb{K}'$, and put $\eta = \zeta^{q-1}$ (so η has multiplicative order $q+1$). Put $T(x) = x + \bar{x}$, where $\bar{x} = x^q$, and consider

$$V = \{(x, y, a) \parallel x, y \in \mathbb{K}', a \in \mathbb{K}\}$$

as a 5-dimensional vector space over $\mathbb{K}$. Define a map $\mathcal{Q} : V \to \mathbb{K}$ by

$$\mathcal{Q}(x, y, a) = x^{q+1} + y^{q+1} - a^2.$$

Then $\mathcal{Q}$ is a quadratic form on V of which the polar form f is given by $f((x, y, a), (z, w, b)) = T(x\bar{z}) + T(y\bar{w}) - 2ab$. The BLT-set $\mathcal{P}$ is now given by

$$\mathcal{P} = \{(\eta^{2j}, 0, 1) \parallel 1 \leq j \leq \frac{q+1}{2}\} \cup \{(0, \eta^{2j}, 1) \parallel 1 \leq j \leq \frac{q+1}{2}\}.$$

For non-classical Thas-Fisher GQ's, we have that

$$|Aut(\mathcal{S})| = q^5(q+1)^2(q-1)2e,$$

where $q = p^e$, p a prime, see [129].

2.5.6 Pe: For all $q \equiv \pm 1 \mod 10$

This example was discovered by T. Penttila [144]. With the same notation as in the preceding example, the BLT-set is given by

$$\mathcal{P} = \{(2\eta^{2j}, \eta^{3j}, \sqrt{5}) \in V \parallel 0 \leq j \leq q\}.$$

The group of the BLT-set acts transitively on its points [132], thus there arises only one flock.

2.5.7 LP: For all $q = 3^e$

For $t \in \mathbb{K}$ with $q = 3^e$ and n a fixed non-square of $\mathbb{K}$, let

$$A_t = \begin{pmatrix} t & t^4 + nt^2 \\ t^4 + nt^2 & -n^{-1}t^9 + t^7 + n^2t^3 - n^3t \end{pmatrix}.$$

Then $\mathcal{C} = \{A_t \parallel t \in \mathbb{K}\}$ is a q-clan. This construction is due to M. Law and T. Penttila, see [102]. The full collineation group of $\mathcal{S}(\mathcal{C})$ is determined in [133], with some very interesting consequences for flocks. Let us recall some facts about those collineation groups (cf. [133]).

- The lines $[A(\infty)]$ and $[A(0)]$ each yield orbits of size one.
- For each $\sigma \in Aut(\mathbb{K})$ and each choice of ± 1 there is a collineation of $\mathcal{S}(\mathcal{C})$ mapping $[A(t)]$ to $[A(\bar{t})]$, where

$$\bar{t} = \pm n^{\frac{1-\sigma}{2}} t^\sigma.$$

- If the orbit of $[A(t)]$ has size h, then the stabilizer of $[A(t)]$ has size $\frac{2e}{h}$. So the flocks corresponding to $[A(\infty)]$ and $[A(0)]$ both have stabilizers of size $2e$.
- In general there are many isomorphism classes of flocks, including some with stabilizers of size $2e$ and others with stabilizers of size one.

2.6 Infinite Families of Flocks with q Even

For q even, the connection between flocks, spreads, q-clans, generalized quadrangles of order (q^2, q), subquadrangles of order q and herds of ovals is given in detail in the survey [87]. As $q = 2^e$, not only is there a generalized quadrangle $\mathcal{S}(\mathcal{C})$ associated with the q-clan $\mathcal{C}$ as well as a spread of $\mathbf{PG}(3, q)$, but there is also a family of $q+1$ subquadrangles of $\mathcal{S}(\mathcal{C})$ each having order q and a so-called *herd* of ovals in $\mathbf{PG}(2, q)$. For a detailed account on both latter objects, see S. E. Payne [132].

We now give a listing of the known q-clans along with an indication of how many inequivalent flocks arise (by recoordinatization, see S. E. Payne [132]).

2.6.1 The classical case

The q-clan is

$$\mathcal{C} = \left\{ \begin{pmatrix} t^{\frac{1}{2}} & t^{\frac{1}{2}} \\ 0 & \kappa t^{\frac{1}{2}} \end{pmatrix} \parallel t \in \mathbb{K} \right\},$$

where $\kappa \in \mathbb{K}$ with $tr(\kappa) = \mathbf{1}$ is fixed.

2.6.2 The FTWKB-examples, $q = 2^e$, e odd

Suppose $q = 2^e$, e odd. The q-clan is given by

$$\mathcal{C} = \left\{ \begin{pmatrix} t^{\frac{1}{4}} & t^{\frac{1}{2}} \\ 0 & \kappa t^{\frac{3}{4}} \end{pmatrix} \parallel t \in \mathbb{K} \right\}.$$

The flocks arise by the geometrical construction of J. C. Fisher and J. A. Thas [59]; the corresponding translation planes were discovered by M. Walker [236] (using flocks) and independently by D. Betten [12]. The associated GQ's were discovered by W. M. Kantor [90] essentially via q-clans. These examples are non-classical if $e \geq 2$, in which case still only one flock arises [134].

2.6.3 The examples of S. E. Payne, $q = 2^e$, e odd

Suppose $q = 2^e$, e odd. The q-clan is

$$\mathcal{C} = \left\{ \begin{pmatrix} t^{\frac{1}{6}} & t^{\frac{1}{2}} \\ 0 & t^{\frac{5}{6}} \end{pmatrix} \| \, t \in \mathbb{K} \right\}.$$

The GQ is classical if and only if $q = 2$ and FTWKB if and only if $q = 8$. For $q \geq 32$ two flocks arise [129].

2.6.4 The Subiaco and Adelaide geometries

The Subiaco examples were first given by W. E. Cherowitzo, T. Penttila, I. Pinneri and G. F. Royle [37] as q-clans. They exist for all $q = 2^e$ and were new for $q \geq 32$, except that certain of the smaller examples had been found by computer and the general construction was obtained in pieces. Since their construction is rather technical, since a rather complete review of them is in [87], and since the new construction of the Adelaide geometries is via a technique that gives both the Subiaco and the Adelaide (as well as the classical) examples, we give only this new version and refer the reader to [37] (also see [142]) for the original constructions. This new construction was discovered during an attempt to generalize the cyclic construction of [143]. However, the problem of giving a direct connection between the Adelaide construction and that in [143] seems to be open.

Let $\mathbb{K}' = \mathbf{GF}(q^2)$ be a quadratic extension of $\mathbb{K} = \mathbf{GF}(q)$, $q = 2^e$. Let $\mathbf{1} \neq \beta \in \mathbb{K}'$ satisfy $\beta^{q+1} = \mathbf{1}$, and put $T(x) = x + x^q$ for all $x \in \mathbb{K}'$. Let $a \in \mathbb{K}$ and $f, g : \mathbb{K} \to \mathbb{K}$ be defined by:

$$a = \frac{T(\beta^m)}{T(\beta)} + \frac{1}{T(\beta^m)} + 1;$$

$$f(t) = f_{m,\beta}(t) = \frac{T(\beta^m)(t+1)}{T(\beta)} + \frac{T((\beta t + \beta^q)^m)}{T(\beta)(t + T(\beta)t^{\frac{1}{2}} + 1)^{m-1}} + t^{\frac{1}{2}},$$

and

$$ag(t) = ag_{m,\beta}(t) = \frac{T(\beta^m)}{T(\beta)}t + \frac{T((\beta^2 t + 1)^m)}{T(\beta)T(\beta^m)(t + T(\beta)t^{\frac{1}{2}} + 1)^{m-1}} + \frac{1}{T(\beta^m)}t^{\frac{1}{2}}.$$

Put

$$\mathcal{C} = \mathcal{C}_{m,\beta} = \left\{ \begin{pmatrix} f(t) & t^{\frac{1}{2}} \\ 0 & ag(t) \end{pmatrix} \parallel t \in \mathbb{K} \right\}.$$

Then W. E. Cherowitzo, C. M. O'Keefe and T. Penttila [36] prove the following.

- If $m \equiv \pm 1 \mod q+1$, then $\mathcal{C}$ is the classical q-clan for all $q = 2^e$ and for all β.
- If $q = 2^e$ with e odd and $m \equiv \pm\frac{q}{2} \mod q+1$, then $\mathcal{C}$ is the example first found as a q-clan by W. M. Kantor and which gives the Fisher-Thas-Walker flock.
- If $q = 2^e$ with $m = \pm 5$, then $\mathcal{C}$ is the *Subiaco q-clan* for all β such that if λ is a primitive element of $\mathbb{K}$ and $\beta = \lambda^{k(q-1)}$, then $q+1$ does not divide km.
- If $q = 4^e > 4$ and $m \equiv \pm\frac{q-1}{3} \mod q+1$, then for all β, $\mathcal{C}$ is a q-clan called the *Adelaide q-clan*.

THE SUBIACO EXAMPLES. Given $q = 2^e$, the q-clan is unique up to equivalence, giving just one flock in all cases.

THE ADELAIDE EXAMPLES. These are constructed for $q = 2^e$ with e even. For the examples with $q = 4^k$, $k \leq 8$, that were studied by computer, the group acts transitively on the lines through (∞). S. E. Payne conjectures that this must be true in general.

Chapter 3

Nets and Generalized Quadrangles with a Regular Point

Suppose $\mathcal{S}$ is a generalized quadrangle of order (s,t), $s,t \neq 1$, with a regular point. Then there is a net which arises from this regular point. We will prove that if such a net $\mathcal{N}$ has a proper subnet with the same degree as the net $\mathcal{N}$, then it must be an affine plane of order t. Also, this affine plane induces a proper subquadrangle of order t containing the regular point, and we necessarily have that $s = t^2$. This result has many applications; we give one example. We will obtain that a GQ $\mathcal{S}$ of order s, $s > 1$, is an STGQ with base-point p if and only if p is an elation point which is regular.

The results of this chapter are taken from *K. Thas, A theorem concerning nets arising from generalized quadrangles with a regular point* [209], which appears in *Designs, Codes and Cryptography.*

MISCELLANEOUS REMARK

The contents of Chapter 3 and Chapter 4 will mainly serve as tools for the rest of this work (although [207], which resulted in Chapter 4, was independently written of that idea).

3.1 Generalities

A (finite) *net of order* k (≥ 2) *and degree* r (≥ 2) is an incidence structure $\mathcal{N} = (P, B, I)$ satisfying the following properties.

1. Each point is incident with r lines and two distinct points are incident with at most one line.
2. Each line is incident with k points and two distinct lines are incident with at most one point.
3. If p is a point and L a line not incident with p, then there is a unique line M incident with p and not concurrent with L.

A net of order k and degree r has k^2 points and kr lines. For any net we have $k \geq r - 1$ [11, 7.1,7.4,7.5], and if $k = r - 1$ then $\mathcal{N}$ is precisely an *affine plane*. Dually, one defines *dual nets* and *dual affine planes*, respectively. Similarly as in the case of affine planes, one defines *parallel classes* of a net. *Subnets* and *automorphisms* of a net are defined in the usual sense.

Theorem 3.1.1 (FGQ, 1.3.1) *Let p be a regular point of a GQ $\mathcal{S} = (P, B, I)$ of order (s, t), $s \neq 1 \neq t$. Then the incidence structure with point set $p^\perp \setminus \{p\}$, with line set the set of spans $\{q, r\}^{\perp\perp}$, where q and r are non-collinear points of $p^\perp \setminus \{p\}$, and with the natural incidence, is the dual of a net of order s and degree $t + 1$. If in particular $s = t$, there arises a dual affine plane of order s. Also, in the case $s = t$, the incidence structure π_p with point set $p^\perp$, with line set the set of spans $\{q, r\}^{\perp\perp}$, where q and r are different points in $p^\perp$, and with the natural incidence, is a projective plane of order s.*

Suppose $\mathcal{S}$ is a GQ of order (s, t), $s > 1, t > 1$, and let p be a regular point of $\mathcal{S}$. Then by $\mathcal{N}_p$ we will denote the net which is the dual of the dual net

corresponding to p, denoted $\mathcal{N}_p^*$, as described in Theorem 3.1.1.[1] A similar notation will be used for nets and dual nets which arise from regular lines in generalized quadrangles.

It seems a very interesting problem to focus on the connection between nets and generalized quadrangles with a regular point. As a first illustration of this idea, we summarize some recent results of J. A. Thas.

First, we introduce the *Axiom of Veblen* for dual nets $\mathcal{N}^* = (P, B, I)$.

Axiom of Veblen. If $L_1 I x I L_2, L_1 \neq L_2, M_1 \backslash\!\!\backslash x \backslash\!\!\backslash M_2$, and if the line L_i is concurrent with the line M_j for all $i, j \in \{1, 2\}$, then M_1 is concurrent with M_2.

An example of a dual net $\mathcal{N}^*$ which is not a dual affine plane and which satisfies the Axiom of Veblen is the dual net H_q^n, $n > 2$, which is constructed as follows:

- the POINTS of H_q^n are the points of $\mathbf{PG}(n, q)$ not in a given subspace $\mathbf{PG}(n-2, q) \subseteq \mathbf{PG}(n, q)$;
- the LINES of H_q^n are the lines of $\mathbf{PG}(n, q)$ which have no point in common with $\mathbf{PG}(n-2, q)$;
- the INCIDENCE in H_q^n is the natural one.

The net which is the dual of H_q^n will be denoted by $(H_q^n)^D$ in the sequel of this piece of work [2].

By the following theorem these dual nets H_q^n are characterized by the Axiom of Veblen.

Theorem 3.1.2 (J. A. Thas and F. De Clerck [189]) *Let $\mathcal{N}^*$ be a dual net with $s+1$ points on any line and $t+1$ lines through any point, where $t+1 > s$. If $\mathcal{N}^*$ satisfies the Axiom of Veblen, then $\mathcal{N}^* \cong H_q^n$ with $n > 2$ (hence $s = q$ and $t+1 = q^{n-1}$).*

Theorem 3.1.3 (J. A. Thas and H. Van Maldeghem [196]) *Let $\mathcal{S}$ be a GQ of order (q^2, q), q even, satisfying Property (G) at the point x. Then x is*

[1] In our notation, it would be more convenient to use the notation $\mathcal{N}_p^D$ instead of $\mathcal{N}_p^*$, but we follow the existing literature at this point to avoid confusion.

[2] We find that this notation is clearer than $(H_q^n)^*$, although we use the latter notation for point-line dual in the case of 'abstract' nets and dual nets arising from regular points or lines in GQ's.

regular for $\mathcal{S}$ and the dual net $\mathcal{N}_x^*$ defined by x satisfies the Axiom of Veblen. Consequently $\mathcal{N}_x^* \cong H_q^3$.

Theorem 3.1.4 (J. A. Thas and H. Van Maldeghem [196]) *Let $\mathcal{S}^{(p)}$ be a TGQ of order (s, s^2), $s \neq 1$, with base-point p. Then the dual net $\mathcal{N}_L^*$ defined by the regular line L, with LIp, satisfies the Axiom of Veblen if and only if the egg $\mathcal{O}(n, 2n, q)$ which corresponds to $\mathcal{S}^{(p)}$ is good at its element π which corresponds to L.*

Theorem 3.1.5 (J. A. Thas and H. Van Maldeghem [196]) *Let $\mathcal{S}(\mathcal{F})$ be a flock GQ of order (s^2, s), $s > 1$ and s odd, with special point (∞). Then the dual net $\mathcal{N}_{(\infty)}^*$ defined by the regular point (∞), satisfies the Axiom of Veblen if and only if $\mathcal{F}$ is a Kantor flock.*

Let $\mathcal{N} = (\overline{P}, \overline{B}, \overline{I})$ be a net of order k and degree r (where the notation is obvious). Further, let R be a line of $\mathcal{N}$ and and let $\mathcal{P}$ be the parallel class of $\overline{B}$ containing R, that is, $\mathcal{P}$ consists of R and the $k-1$ lines not concurrent with R. An automorphism θ of $\mathcal{N}$ is called a *transvection with axis R* if either $\theta = \mathbf{1}$ or if $\mathcal{P}$ is the set of all fixed lines of θ and R is the set of all fixed points of θ. The net $\mathcal{N}$ is a *$\mathcal{P}$-net* if for any two non-parallel lines $M, N \in \overline{B} \setminus \mathcal{P}$ there is some transvection with axis belonging to $\mathcal{P}$ and mapping M onto N. The notion of $\mathcal{P}$-net was introduced by J. A. Thas in [185].
In particular, let $\mathcal{N} = (\overline{P}, \overline{B}, \overline{I})$ be an affine plane of order k and let $\mathcal{D}$ be the corresponding projective plane. Then $\mathcal{N}$ is a $\mathcal{P}$-net if and only if the point z of $\mathcal{D}$ defined by $\mathcal{P}$, is a translation point of $\mathcal{D}$; see D. Hughes and F. Piper [81].
If $\mathcal{N}$ is the dual of H_q^n, then it is easy to check that $\mathcal{N}$ is a $\mathcal{P}$-net for any parallel class $\mathcal{P}$.

Theorem 3.1.6 (J. A. Thas [185]) *Let $(\mathcal{S}^{(p)}, G)$ be a TGQ of order (s, t), $s \neq 1 \neq t$. Then for any line L incident with p, the dual net $\mathcal{N}_L^*$ defined by L is the dual of a $\mathcal{P}$-net $\mathcal{N}_L$ with $\mathcal{P}$ the parallel class of $\mathcal{N}_L$ defined by the point p.*

Theorem 3.1.7 (J. A. Thas [185]) *Let $\mathcal{S}$ be a GQ of order (s, t), $s \neq 1 \neq t$, with coregular point p. If for at least one line L incident with p the dual net $\mathcal{N}_L^*$ is the dual of a $\mathcal{P}$-net $\mathcal{N}_L$ with $\mathcal{P}$ the parallel class of $\mathcal{N}_L$ defined by p, then $\mathcal{S}$ is a TGQ with base-point p.*

Corollary 3.1.8 (J. A. Thas [185]) *Let $\mathcal{S}$ be a GQ of order s, $s \neq 1$, with coregular point p. If for at least one line L incident with p the corresponding projective plane π_L is a translation plane (see Section 3.4) with as translation line the set of all lines of $\mathcal{S}$ incident with p, then $\mathcal{S}$ is a TGQ with base-point p.*

Corollary 3.1.9 (J. A. Thas [185]) *Let $\mathcal{S}$ be a GQ of order s, $s \neq 1$, with coregular point p. If for at least one line L incident with p the projective plane π_L is Desarguesian, then $\mathcal{S}$ is a TGQ with base-point p. If in particular s is odd, then $\mathcal{S}$ is isomorphic to the classical GQ $\mathcal{Q}(4,s)$.*

Corollary 3.1.10 (J. A. Thas [185]) *Let $\mathcal{S}$ be a GQ of order (q^2,q), $q \neq 1$, with regular point x for which the dual net $\mathcal{N}_x^*$ defined by x satisfies the Axiom of Veblen. If x is incident with a coregular line L, then $\mathcal{S}$ is a TGQ with base-line L.*

Corollary 3.1.11 (J. A. Thas [185]) *Let $\mathcal{S}$ be a GQ of order (q^2,q), q even, satisfying Property (G) at the point x. If x is incident with a coregular line L, then $\mathcal{S}$ is a TGQ with base-line L.*

Corollary 3.1.12 (J. A. Thas [185]) *Let $\mathcal{S}(\mathcal{F})$ be a GQ of order (q^2,q), q even, arising from a flock $\mathcal{F}$. If the point (∞) of $\mathcal{S}(\mathcal{F})$ is collinear with a regular point x, with $(\infty) \neq x$, then $\mathcal{S}(\mathcal{F})$ is isomorphic to the classical GQ $H(3,q^2)$.*

Corollary 3.1.13 (J. A. Thas [185]) *Let $\mathcal{S} = (P,B,I)$ be a GQ of order (s,t), $s \neq 1 \neq t$, having a coregular point x. If for a fixed line L, with xIL, and any two lines M,N, with $M \not\sim L$ and $L \sim N \not\sim M$, there is a proper subquadrangle $\mathcal{S}'$ of $\mathcal{S}$ of order (s,t'), with $t' \neq 1$, containing L,M,N, then $t' = s = \sqrt{t}$, $\mathcal{S}$ is a TGQ with base-point x, and $\mathcal{S}'$ is a TGQ with base-point x. For s even $\mathcal{S}$ satisfies Property (G) at the flag (x,L) and for s odd the translation dual $\mathcal{S}^*$ of $\mathcal{S}$ satisfies Property (G) at the flag (x',L'), with x' the base-point of $\mathcal{S}^*$ and L' the line of $\mathcal{S}^*$ corresponding to the line L of $\mathcal{S}$. For s odd the subquadrangles $\mathcal{S}'$ are isomorphic to $\mathcal{Q}(4,s)$ and $\mathcal{S}$ is isomorphic to the dual of a GQ $\mathcal{S}(\mathcal{F})$ arising from a flock $\mathcal{F}$. If s is even and if $\mathcal{S}$ is isomorphic to the dual of a GQ $\mathcal{S}(\mathcal{F})$ arising from a flock $\mathcal{F}$, then $\mathcal{S} \cong \mathcal{Q}(5,s)$.*

3.2 Nets and Subquadrangles

Part of the proof of the following theorem is contained in Section 7 of [196]. However, we bring it in a slightly different and more general context in order to characterize subnets of nets which are attached to regular points or regular lines of generalized quadrangles.

Theorem 3.2.1 *Suppose $\mathcal{S} = (P,B,I)$ is a GQ of order (s,t), $s,t \neq 1$, with a regular point p. Let $\mathcal{N}_p$ be the net which arises from p, and suppose $\mathcal{N}_p'$ is a subnet of the same degree as $\mathcal{N}_p$. Then we have the following possibilities:*

1. *$\mathcal{N}'_p$ coincides with $\mathcal{N}_p$;*

2. *$\mathcal{N}'_p$ is an affine plane of order t and $s = t^2$; also, from $\mathcal{N}'_p$ there arises a proper subquadrangle of $\mathcal{S}$ of order t having p as a regular point.*

If, conversely, $\mathcal{S}$ has a proper subquadrangle of order (s', t), $s' \neq 1$, containing the point p, then it is of order t, and hence $s = t^2$. Also, there arises a proper subnet of $\mathcal{N}_p$ which is an affine plane of order t.

Proof. First suppose that $\mathcal{S}$ contains a proper subquadrangle $\mathcal{S}'$ of order (s', t), where $s', t \neq 1$, containing the point p. Then p is also regular in $\mathcal{S}'$ and since $s' \neq 1$, it follows that $s' \geq t$. By Theorem 1.6.2 this implies that $s' = t$ and that $s = t^2$. By Theorem 3.1.1, the net $\mathcal{N}'_p$ arising from the point p in $\mathcal{S}'$ is an affine plane of order t, and this net is clearly a subnet of the net which arises from the point p in $\mathcal{S}$.
Conversely, suppose that $\mathcal{N}_p$ is the net which arises from the regular point p in the GQ $\mathcal{S}$, and assume that it contains a proper subnet $\mathcal{N}'_p$ of the same degree. In the following, we identify points of the net with the corresponding spans of points in the GQ, and we use the same notation.
Suppose $P_1, P_2, \ldots, P_k$ are the points of $\mathcal{N}'_p$, define a point set $P' \subseteq P$ of $\mathcal{S}$ as consisting of the points of $[\bigcup P_i] \cup [\bigcup P_i^{\perp}]$, and define $B' \subseteq B$ as the set of all lines of $\mathcal{S}$ through a point of P'. Then it is not hard to check that the following properties are satisfied for the geometry $\mathcal{S}' = (P', B', I')$, with $I' = I \cap [(P' \times B') \cup (B' \times P')]$:

1. any point of P' is incident with $t + 1$ lines of B';

2. if two lines of B' intersect in $\mathcal{S}$, then they also intersect in $\mathcal{S}'$.

Then by the dual of Theorem 1.6.3, $\mathcal{S}'$ is a proper subquadrangle of $\mathcal{S}$ of order (s', t), $s' \neq 1$, and by analogy to the beginning of this proof, we have that $s' = t$ and $s = t^2$. Also, the affine plane of order t which arises from the regular point p of this subquadrangle is the subnet $\mathcal{N}'_p$. ■

This result has several corollaries.

Corollary 3.2.2 *A net $\mathcal{N}$ which is attached to a regular point of a GQ contains no proper subnet of the same degree as $\mathcal{N}$, other than (possibly) an affine plane.*

■

Corollary 3.2.3 *Suppose p is a regular point of the GQ $\mathcal{S}$ of order (s,t), $s,t \neq 1$, and let $\mathcal{N}_p$ be the corresponding net. If $s \neq t^2$, then $\mathcal{N}_p$ contains no proper subnet of degree $t+1$.*

■

The following corollary tells us that nets which arise from a regular point of a GQ and which do not contain affine planes are very 'irregular'.

Corollary 3.2.4 *Let p be a regular point of a GQ $\mathcal{S}$ of order (s,t), $s,t \neq 1$, and suppose $\mathcal{N}_p$ is the corresponding net. Moreover, suppose $s \neq t^2$. If u,v and w are distinct lines of $\mathcal{N}_p$, for which $w \not\sim u \sim v$, then these lines generate the whole net (under the taking of spans).*

Proof. Consider the points of $p^\perp \setminus \{p\}$ which correspond to the lines u,v,w of $\mathcal{N}_p$, and denote them respectively in the same way. Then by Theorem 1.6.3, u,v and w generate a (not necessarily proper) subGQ $\mathcal{S}'$ of $\mathcal{S}$ of order (s',t), where $s' > 1$. By Theorem 3.2.1 this implies that $\mathcal{S}' = \mathcal{S}$, since $s \neq t^2$. Hence u,v and w, as lines of $\mathcal{N}_p$, generate $\mathcal{N}_p$. ■

3.3 Nets and Symmetries

We now apply the preceding results to the theory of symmetries of generalized quadrangles.
Let us first obtain

Lemma 3.3.1 *Suppose $\mathcal{S}$ is a GQ of order (s,s^2), $s \neq 1$, and suppose that $\mathcal{S}'$ and $\mathcal{S}''$ are two proper subquadrangles of $\mathcal{S}$ of order s.*
Then one of the following possibilities occurs:

1. *$\mathcal{S}' \cap \mathcal{S}''$ is a set of s^2+1 pairwise non-collinear points (i.e. an ovoid) of $\mathcal{S}'$ and $\mathcal{S}''$;*
2. *$\mathcal{S}' \cap \mathcal{S}''$ consists of a point p of $\mathcal{S}'$ (and $\mathcal{S}''$), together with all lines of $\mathcal{S}'$ (and $\mathcal{S}''$) through this point, and all points of $\mathcal{S}'$ (and $\mathcal{S}''$) incident with these lines;*
3. *$\mathcal{S}' \cap \mathcal{S}''$ is a GQ of order $(s,1)$;*
4. *$\mathcal{S}' = \mathcal{S}''$.*

Proof. Every line of a GQ of order (s, s^2), $s \neq 1$, intersects any subGQ of order s. The proof now follows from Theorem 1.6.3 and a simple counting argument. ■

Theorem 3.3.2 *Suppose $\mathcal{S}$ is a generalized quadrangle of order (s,t), $s,t \neq 1$, and suppose ϕ is a nontrivial whorl about a regular point p. Also, suppose ϕ fixes distinct points q, r and u of $p^\perp \setminus \{p\}$ for which $q \sim r$ and $q \not\sim u$. Then we have one of the following possibilities.*

1. *We have that $s = t^2$ and $\mathcal{S}$ contains a proper subquadrangle $\mathcal{S}'$ of order t. Moreover, if ϕ is not an elation, then $\mathcal{S}'$ is fixed pointwise by ϕ.*
2. *ϕ is a nontrivial symmetry about p.*

Proof. It is clear that if v and w are non-collinear points of $p^\perp$ which are fixed by a whorl about p, then every point of the span $\{v, w\}^{\perp\perp}$ is also fixed by the whorl. Now suppose $\mathcal{N}_p$ is the net which arises from p, and suppose that $\mathcal{N}_p'$ is the (not necessarily proper) subnet of $\mathcal{N}_p$ of order $t+1$ which is generated by u, q and r. Then every point of $\mathcal{N}_p'$ is fixed by ϕ by the previous observation. If $\mathcal{N}_p'$ is proper, then by Theorem 3.2.1 it is an affine plane of order t and $s = t^2$. Also, there arises a proper subquadrangle $\mathcal{S}'$ of $\mathcal{S}$ of order t. If ϕ is not an elation, then by Theorem 1.6.5 it follows that there is a proper subquadrangle $\mathcal{S}_\phi$ of order (s', t), $s' \neq 1$, which is fixed pointwise (and then also linewise) by ϕ. Since $\mathcal{S}_\phi$ has a regular point, we have that $s' \geq t$. By Theorem 1.6.2, $\mathcal{S}'$ is necessarily of order t. From Lemma 3.3.1 now follows that $\mathcal{S}_\phi = \mathcal{S}'$.

If $\mathcal{N}_p' = \mathcal{N}_p$, then every point of $p^\perp$ is fixed by ϕ. Since ϕ is not the identity, it follows from Theorem 1.6.5 that ϕ is an elation and hence a symmetry about p. ■

Remark 3.3.3 It may be clear to the reader that if ϕ and $\mathcal{S}' = \mathcal{S}_\phi$ are as in (1) of Theorem 3.3.2, then ϕ is an involution which acts semiregularly on the points and lines of $\mathcal{S} \setminus \mathcal{S}_\phi$ (cf. Section 1.12 of Chapter 1). Hence $\mathcal{S}_\phi$ is a doubly subtended subGQ of $\mathcal{S}$.

Notation. By $gcd(n, m)$, with $m, n \in \mathbb{N}$, we denote the greatest common divisor of m and n.

Theorem 3.3.4 *Let $(\mathcal{S}^{(p)}, G)$ be an EGQ of order (s,t), $s,t \neq 1$, and suppose p is a regular point. Moreover, suppose that $gcd(s-1, t) = 1$. Then we have one of the following:*

1. *there is a full group of order t of symmetries about p which is completely contained in G, and hence $(\mathcal{S}^{(p)}, G)$ is a skew translation generalized quadrangle;*

2. *$\mathcal{S}$ contains a proper subGQ of order t (for which p is a regular point), and consequently $s = t^2$.*

Proof. Suppose $\mathcal{S} = (P, B, I) = (\mathcal{S}^{(p)}, G)$ satisfies the desired properties, and assume that $(\mathcal{S}^{(p)}, G)$ is not a skew translation generalized quadrangle (note that since the GQ contains a regular point, we know that $t \leq s$). Then there is no full group of symmetries of size t about p which is completely contained in the elation group G. Consider an arbitrary point q of $P \setminus p^\perp$; then the subgroup H of G which fixes every element of $\{q, p\}^\perp$ has order t, and acts regularly on $\{q, p\}^{\perp\perp} \setminus \{p\}$. Now consider an arbitrary but nontrivial element θ of H of prime-power order, say h. Then h is a divisor of t. Next, suppose L is an arbitrary line through p, and suppose X is the set of $s - 1$ points on L which are different from p and not contained in $q^\perp$. Then it is clear that θ, and also every element of $\langle\theta\rangle$, has at least one fixed point u in X, as $gcd(s - 1, t) = 1$. By the proof of Theorem 3.2.1, the points of $\{p, q\}^\perp$ together with u generate — as lines of $\mathcal{N}_p$ — a subnet $\mathcal{N}_\theta$ of $\mathcal{N}_p$ of the same degree as $\mathcal{N}_p$, and every point of $p^\perp$ which corresponds to a line of $\mathcal{N}_\theta$ is fixed by θ. Thus, if $\mathcal{N}_\theta = \mathcal{N}_p$, then θ is a nontrivial symmetry about p. Suppose that for *every* such $\theta \in H$ of prime-power order, the net $\mathcal{N}_\theta$ coincides with $\mathcal{N}_p$; then it is clear that H is a full group of symmetries about p of order t, since a finite group is always generated by its elements of prime-power order (the composition of symmetries about the same point still gives a symmetry about this point). Moreover, H is completely contained in G, a contradiction since this would imply that $(\mathcal{S}^{(p)}, G)$ is a skew translation GQ.
Hence $\mathcal{N}_p$ has a proper subnet $\mathcal{N}_{\theta'}$ for some $\theta' \in H$ of prime-power order, and by Theorem 3.2.1, $\mathcal{N}_{\theta'}$ is an affine plane of order t. Also, $s = t^2$ and there arises a proper subquadrangle $\mathcal{S}'$ of $\mathcal{S}$ of order t which contains the point p. It is clear that p is also regular in $\mathcal{S}'$. Thus the proof is complete. ∎

Remark 3.3.5 The collineation θ' induces a nontrivial symmetry about p of $\mathcal{S}'$.

The following result is a consequence of Theorem 3.3.4, and provides a characterization theorem of skew translation generalized quadrangles of order s, $s > 1$.

Corollary 3.3.6 *Suppose $(\mathcal{S}^{(p)}, G)$ is an EGQ of order s, $s > 1$, and suppose that the elation point p is regular. Then there is a full group of symmetries $\mathcal{C}$ about p, and $\mathcal{C}$ is completely contained in G, hence $(\mathcal{S}^{(p)}, G)$ is an STGQ with base-point p and base-group G.*

Proof. Directly from Theorem 3.3.4. ∎

3.4 Generalized Quadrangles with a Regular Point and Translation Nets

The following theorem shows that skew translation generalized quadrangles provide special nets. Recall that a *translation net* $\mathcal{N}$ is a net for which there is a group G of automorphisms of $\mathcal{N}$ each element of which fixes each parallel class of $\mathcal{N}$, and so that G acts regularly on the points of $\mathcal{N}$. If $\mathcal{N}$ has order k and degree r, $k, r \geq 2$, and if $k = r - 1$, then $\mathcal{N}$ is a *translation plane.*

Theorem 3.4.1 *Suppose $(\mathcal{S}^{(p)}, G)$ is a skew translation generalized quadrangle of order (s,t), $s, t \neq 1$, with elation point p. Then the net $\mathcal{N}_p$ is a translation net.*

Proof. Suppose $\mathcal{C}$ is the full group of order t of symmetries about p. If we consider the action of G on the net $\mathcal{N}_p$, then it is clear that $\mathcal{C}$ is precisely the kernel of this action, hence the group $G/\mathcal{C}$ acts as a faithful automorphism group on $\mathcal{N}_p$. Also, it is clear that $G/\mathcal{C}$ acts semiregularly on the points of $\mathcal{N}_p$ ($\mathcal{C}$ is the only subgroup of G which contains nontrivial elements fixing a point of $\mathcal{N}_p$). Moreover, the group $G/\mathcal{C}$ has size s^2, and hence $G/\mathcal{C}$ acts regularly on the points of $\mathcal{N}_p$. Thus, since every parallel class of $\mathcal{N}_p$ is fixed by $G/\mathcal{C}$, it follows that $\mathcal{N}_p$ is a translation net. ∎

Remark 3.4.2 If we put $s = t$ in Theorem 3.4.1, then $\mathcal{N}_p$ is a translation plane, s is the power of a prime and $G/\mathcal{C}$ is elementary abelian; see e.g. [81, p. 100].

Chapter 4

Complete $(st - t/s)$-Arcs in Generalized Quadrangles of Order (s, t)

Let $\mathcal{S}$ be a finite generalized quadrangle of order (s, t), $s \neq 1 \neq t$. A k-arc $\mathcal{K}$ is a set of k mutually non-collinear points. For any k-arc of $\mathcal{S}$ we have $k \leq st+1$; if $k = st+1$, then $\mathcal{K}$ is an ovoid of $\mathcal{S}$. A k-arc is complete if it is not contained in a k'-arc with $k' > k$. In FGQ it is proved that an $(st - m)$-arc, where $-1 \leq m < t/s$, is always contained in a uniquely defined ovoid, hence it is a natural question to ask whether or not complete $(st - t/s)$-arcs exist. In this chapter, we prove that the classical GQ $H(4, q^2)$ has no complete $(q^5 - q)$-arcs. We also obtain that a GQ $\mathcal{S}$ of order s, $s \neq 1$, with a regular point has no complete $(s^2 - 1)$-arcs, except when $s = 2$, i.e. $\mathcal{S} \cong \mathcal{Q}(4, 2)$, and in that case there is a unique example. As a by-product no known GQ of even order s with $s > 2$ can have complete (s^2-1)-arcs. Also, we prove that a GQ of order (s, s^2), $s \neq 1$, cannot have complete $(s^3 - s)$-arcs unless $s = 2$, i.e. $\mathcal{S} \cong \mathcal{Q}(5, 2)$, in which case there is a unique example (up to isomorphism). Next we will realize a connection between complete grids with parameters $s - 1, s + 1$ as substruc-

tures of GQ's of order s, $s \neq 1$, and complete (s^2-1)-arcs, and we will proceed with a classification of the quadrangles of order (s,t), where $s \neq 1 \neq t$, which contain complete grids with parameters $s-1, s+1$. Finally, we will introduce (partial) ovoids and (partial) spreads in affine generalized quadrangles, and we will prove nonexistence results by using some of the aforementioned results.
In an appendix, we will derive some new bounds for partial spreads of the Hermitian quadrangle $H(3,q^2)$ (which are in some cases stronger than the known ones).

The results of this chapter up to Section 4.3 appeared in *Journal of Combinatorial Theory, Series A* in the note *K. Thas, Nonexistence of Complete $(st - t/s)$-Arcs in Generalized Quadrangles of Order (s,t), I* [207]. The appendix is taken from [202].

4.1 k-Arcs in Generalized Quadrangles

A *k-arc* $\mathcal{K}$ of a GQ $\mathcal{S}$ of order (s,t), $s \neq 1 \neq t$, is a set of k mutually non-collinear points. One easily observes that $k \leq st+1$ (see, e.g., FGQ), and if $k = st+1$, then $\mathcal{K}$ is an ovoid of $\mathcal{S}$. A (k-)arc is also called a *partial ovoid* (with k points). Dually one defines *partial spreads*. A k-arc is *complete* if it is not contained in a k'-arc with $k' > k$. Dually, one defines *dual k-arcs* and *complete dual k-arcs*. The following theorem is an important observation.

Theorem 4.1.1 (FGQ, 2.7.1) *An $(st-m)$-arc in a GQ of order (s,t), where $-1 \leq m < t/s$ and $s \neq 1 \neq t$, is always contained in a uniquely defined ovoid.*

Considering Theorem 4.1.1, it is a natural question to ask whether or not complete $(st - t/s)$-arcs exist.
Let us first recall some notions and results concerning complete $(st-t/s)$-arcs. Let $\mathcal{K}$ be a complete $(st - t/s)$-arc in the GQ $\mathcal{S} = (P, B, I)$ of order (s,t), $s \neq 1 \neq t$. If B' is the set of lines incident with no point of $\mathcal{K}$, P' the set of points on the lines of B', and I' the restriction of I to points of P' and lines of B', then $\mathcal{S}' = (P', B', I')$ is a subquadrangle of $\mathcal{S}$ of order $(s, t/s)$ (see [139, 2.7.2]). We denote this subGQ by $\mathcal{S}(\mathcal{K})$.

The following result is taken from J. A. Thas [168], see also 1.4.2 (ii) of FGQ.

Theorem 4.1.2 (J. A. Thas [168]; see also 1.4.2 (ii) of FGQ) *Suppose $\mathcal{S}$ is a generalized quadrangle of order (s,t), $s,t \neq 1$ and $s \neq t$, and let $\{x,y\}^{\perp\perp}$ be a hyperbolic line of size $p+1$, where $pt = s^2$. Then every point of $\mathcal{S}$ not in $cl(x,y)$ is collinear with $t/s + 1 = s/p + 1$ points of $\{x,y\}^{\perp}$.*

Remark 4.1.3 If $s = t$ and the hyperbolic line $\{x, y\}^{\perp\perp}$ has size $s+1$, that is, $\{x, y\}$ is regular, then $cl(x, y)$ coincides with the point set of $\mathcal{S}$. If $|\{x, y\}^{\perp\perp}| = p + 1$, $x \not\sim y$, with $pt = s^2$, then $\mathcal{S} \setminus cl(x, y) \neq \emptyset$ if and only if $s \neq t$.

Lemma 4.1.4 *Let $\mathcal{S}$ be a GQ of order (s, t), where $s \neq 1 \neq t$. Assume that $\mathcal{S}$ contains a complete $(st - t/s)$-arc $\mathcal{K}$ and that $\mathcal{S}(\mathcal{K})$ is the corresponding GQ of order $(s, t/s)$.*

1. *Then every point off $\mathcal{K}$ is collinear with either $t - t/s$ points of $\mathcal{K}$ or $t+1$ points of $\mathcal{K}$, according to whether this point is contained in $\mathcal{S}(\mathcal{K})$ or not.*

2. *If x and y are distinct non-collinear points of $\mathcal{S}$ and $\{x, y\}^{\perp\perp} \cap \mathcal{S}(\mathcal{K}) \neq \emptyset$, then $|\{x, y\}^{\perp} \cap \mathcal{S}(\mathcal{K})| = t/s + 1$.*

Proof. Easy. ∎

4.2 Nonexistence of Complete $(st - t/s)$-Arcs

Theorem 4.2.1 *Let $\mathcal{S}$ be a GQ of order (s, t), $s \neq 1 \neq t$, and suppose $\mathcal{K}$ is a complete $(st - t/s)$-arc of $\mathcal{S}$. If $\{x, y\}^{\perp\perp} = \mathsf{H}$ is a hyperbolic line of $\mathcal{S}$ of size $p + 1$ with $pt = s^2$, then either $|\mathsf{H} \cap \mathcal{K}| \in \{0, 1\}$, or $\mathcal{S}$ is isomorphic to $\mathcal{Q}(4, 2)$ or $\mathcal{Q}(5, 2)$.*

Proof. Suppose that $|\mathcal{K} \cap \mathsf{H}| = r + 1$, and suppose that $r > -1$. Since no two points of $\mathcal{K}$ are collinear, we have that $\mathsf{H}^{\perp} \cap \mathcal{K} = \emptyset$. Since $\mathcal{K}$ is a complete $(st - t/s)$-arc, we know that $\mathcal{S}$ contains a proper subquadrangle $\mathcal{S}(\mathcal{K})$ of order $(s, t/s)$, which has empty intersection with $\mathcal{K}$. Suppose that $|\mathsf{H} \cap \mathcal{S}(\mathcal{K})| = p' + 1$, with $-1 \leq p' < p$ (note that we can assume that $p' \neq p$ because otherwise H has empty intersection with $\mathcal{K}$). We count the number N of pairs (u, q), with $u \in \{x, y\}^{\perp}$, $q \sim u$ and $q \in \mathcal{K} \setminus \{x, y\}^{\perp\perp}$, in two ways.
First of all, suppose that $p' = -1$.
The number of points of $\mathcal{K} \setminus \{x, y\}^{\perp\perp}$ which are collinear with a point of $\{x, y\}^{\perp\perp}$ — that is, the number of elements of $(\mathcal{K} \setminus \{x, y\}^{\perp\perp}) \cap cl(x, y)$ — is $(p - r)(t + 1)$, and every such point is clearly collinear with exactly one point of $\{x, y\}^{\perp}$. Every point of $\mathcal{K} \setminus cl(x, y)$ is collinear with $t/s + 1 = s/p + 1$ points of $\{x, y\}^{\perp}$ by Theorem 4.1.2. Hence, it follows directly that $N = (p - r)(t + 1) + (st - t/s - r - 1 - (p - r)(t + 1))(t/s + 1)$.
Also, if $\mathsf{H}^{\perp} \cap \mathcal{S}(\mathcal{K}) = \emptyset$, then it is clear that $N = (t + 1)(t - r)$, and hence, since $t/s = s/p$, we have that $(t + 1)(t - r) = (p - r)(t + 1) + (st - t/s - r -$

$1 - (p - r)(t + 1))(s/p + 1)$.
Some simplifying yields

$$(r - 1)t + (rs - 1)t/p = 2t/s + s + 1, \tag{4.1}$$

and in this equality, $r = 0$ is not possible. If $s = t$, there would follow that $s = p$ and (4.1) would lead to $2rs = 2s + 4$, or $s = 2$ and $r = 2$. By Section 1.3 there follows that $\mathcal{S} \cong \mathcal{Q}(4, 2)$. If $s \neq t$, then by Theorem 4.1.2, $s|t$ and $p|s$. But then the left-hand side of (4.1) is divisible by s/p while the right-hand side is only divisible by s/p if $s = p = t$, a contradiction.
Now suppose $\mathsf{H}^\perp \cap \mathcal{S}(\mathcal{K}) \neq \emptyset$, and assume that $|\mathsf{H}^\perp \cap \mathcal{S}(\mathcal{K})| = t' + 1$ with $t' > -1$.
Then it is clear that $(t - t')(t - r) + (t' + 1)(t - t/s - r - 1) = (p - r)(t + 1) + (st - t/s - r - 1 - (p - r)(t + 1))(t/s + 1)$, or that

$$t + t/s + s + t/p = rt + rst/p + t' + tt'/s. \tag{4.2}$$

Since $t \geq s$, it is clear that $r \geq 2$ implies that $t' < 0$, a contradiction. Hence $r \leq 1$. Suppose that $r = 1$. Then $s = t' + (s - 1)t/p + (t' - 1)t/s$. If $t' > 0$, then it is clear that the only possibility is that $s = t = p$ and $t' = 1$. This case is excluded since each line of $\mathcal{S}(\mathcal{K})$ containing a point of $\mathcal{S}(\mathcal{K}) \cap \mathsf{H}^\perp$ also contains a point of H, contradicting $p' = -1$. Now suppose that $t' = 0$. Then $p = (s - 1)t/s - 1$. The case $s = t$ is clearly not possible, so we can suppose that $s|t$ with $s \neq t$ (and thus $p < s$). Hence $p \geq 2(s - 1) - 1 = 2s - 3$, and there follows that $s > 2s - 3$. Thus, $s = 2$, and by Section 1.3, there follows that $\mathcal{S} \cong \mathcal{Q}(5, 2)$.
Next suppose that $p' \neq -1$.
Note that we have that $|\mathsf{H}^\perp \cap \mathcal{S}(\mathcal{K})| = t/s + 1$. By Lemma 4.1.4, there follows that $N = (t/s + 1)(t - t/s - r - 1) + (t - t/s)(t - r)$. Also, in the same way as in the case $p' = -1$, we have that $N = (p - p' - r - 1)(t + 1) + (p' + 1)(t - s/p) + (st - s/p - r - 1 - (p - p' - r - 1)(t + 1) - (p' + 1)(t - s/p))(s/p + 1)$. Thus, using the fact that $pt = s^2$, we have that

$$t - (s/p)^2 - t/s = rst/p - s + (s/p)^2 p' + rt + sp'/p. \tag{4.3}$$

The last equality clearly is impossible if $r > 0$ since the left-hand side is smaller than t and the right-hand side is larger than t. Hence $|\mathcal{K} \cap \mathsf{H}| \leq 1$. ∎

The theorem has a lot of interesting corollaries.

Theorem 4.2.2 *The classical generalized quadrangle $\mathcal{S} = H(4, q^2)$ has no complete $(q^5 - q)$-arcs.*

Proof. Every hyperbolic line of $H(4, q^2)$ has size $q+1$, see Section 1.2.3. Now suppose that $\mathcal{K}$ is a complete $(q^5 - q)$-arc of $\mathcal{S}$. Then for every two distinct points x and y on $\mathcal{K}$ there holds that $|\{x, y\}^{\perp\perp}| = q + 1$, in contradiction with Theorem 4.2.1. ∎

As a nice corollary, we have the following upper bound for partial ovoids of the Hermitian quadrangle $H(4, q^2)$.

Theorem 4.2.3 *If $\mathcal{K}$ is a partial ovoid of $H(4, q^2)$, then we have that $|\mathcal{K}| \leq q^5 - q - 1$.*

Proof. By Theorem 1.12.2 $H(4, q^2)$ has no ovoids. The theorem then follows from Theorem 4.2.2 and Theorem 4.1.1. ∎

Remark 4.2.4 In P. Govaerts, L. Storme and H. Van Maldeghem [66], an improvement of Theorem 4.2.3 is obtained; there it is shown that $|\mathcal{K}| \leq q^5 - \frac{4}{3}q + 2$ if $q > 2$.

The following theorem completely solves the problem under consideration for all GQ's of order (s, s^2), $s > 1$.

Theorem 4.2.5 *Let $\mathcal{S}$ be a GQ of order (s, s^2), $s \neq 1$. Then $\mathcal{S}$ has no complete $(s^3 - s)$-arcs unless $s = 2$, i.e. $\mathcal{S} \cong \mathcal{Q}(5, 2)$. In that case there is a unique example up to isomorphism.*

Proof.

1. Every hyperbolic line in a GQ of order (s, s^2) (with $s \neq 1$) has size 2, hence we can apply Theorem 4.2.1 to conclude that $\mathcal{S} \cong \mathcal{Q}(5, 2)$.

2. Fix a subGQ $\mathcal{S}'$ of order 2 of $\mathcal{S} \cong \mathcal{Q}(5, 2)$ (which is isomorphic to the classical GQ $\mathcal{Q}(4, 2)$), and let v be a point of $\mathcal{S}$ outside $\mathcal{S}'$. Then $v^\perp$ intersects $\mathcal{S}'$ in the points of an ovoid $\mathcal{O}$ of $\mathcal{S}'$, see Theorem 1.6.1, and it is well-known that this ovoid is subtended by v and exactly one other point of $\mathcal{S} \setminus \mathcal{S}'$, say v'. It is clear that $\mathcal{K} = \{v\} \cup ((v')^\perp \setminus (\{v'\} \cup \mathcal{O}))$ and $\mathcal{K}' = \{v'\} \cup (v^\perp \setminus (\{v\} \cup \mathcal{O}))$ are two disjoint 6-arcs in $\mathcal{S} \setminus \mathcal{S}'$ (note that $\mathcal{K} \cup \mathcal{K}'$ is the point set of $\mathcal{S} \setminus \mathcal{S}'$). As $\mathcal{S}$ has no ovoid (see, e.g., the appendix of this chapter or Theorem 1.12.2) the 6-arcs $\mathcal{K}$ and $\mathcal{K}'$ are

complete, and moreover $\mathcal{S}(\mathcal{K}) = \mathcal{S}(\mathcal{K}') = \mathcal{S}'$. It is easy to show that for given $\mathcal{S}'$ and v in $\mathcal{S} \setminus \mathcal{S}'$, $\mathcal{K}$ is the unique 6-arc $\overline{\mathcal{K}}$ containing v for which $\mathcal{S}' = \mathcal{S}(\overline{\mathcal{K}})$. It is well-known that the stabilizer of $\mathcal{S}(\mathcal{K}) \cong \mathcal{Q}(4,2)$ in the automorphism group G of $\mathcal{S} \cong \mathcal{Q}(5,2)$ acts transitively on the points of $\mathcal{S} \setminus \mathcal{S}'$ (this follows easily from the fact that $\mathcal{S}$ is Moufang, and hence G acts transitively on the complete 6-arcs of $\mathcal{S}$ since the automorphism group of $\mathcal{S}$ acts transitively on the subGQ's of order 2. ∎

Remark 4.2.6 It is clear that the number of distinct complete 6-arcs in $\mathcal{Q}(5,2)$ equals two times the number of distinct $\mathcal{Q}(4,2)$'s in $\mathcal{Q}(5,2)$, and hence this number equals 72.

Note. The second part of the proof of Theorem 4.2.5 is taken from K. Thas [202]. In [27], it is proved that $H(3,q^2)$ has no spreads, and that every partial spread has at most $q^3 - q$ lines. In [173], J. A. Thas proves that such a partial spread has at most $q^3 - q^2 + q + 1$ lines, see the appendix of this chapter, and the first bound is sharper than the second if and only if $q = 2$. For more on the last bound, see [202] or the appendix of this chapter.

Consider again the case $p' \neq -1$ in the proof of Theorem 4.2.1; then $t - (s/p)^2 - t/s = rst/p - s + (s/p)^2 p' + rt + sp'/p$ and $r = 0$. Then with $t/s = s/p$ and after dividing by t/s, we obtain that

$$s - 1 = \frac{s}{p}(p' + 1) + p' - p.$$

Since $p' < p$, the last equality clearly only is possible when $p' = p - 1$. Hence $|\mathsf{H} \cap \mathcal{S}(\mathcal{K})| = p = s^2/t$, and since $\mathcal{S}(\mathcal{K})$ is of order $(s, t/s)$, there follows that $s^2/t \leq t/s + 1$, so $s^3 \leq t^2 + st$. In particular, the case $s = t$ is excluded when $s > 2$.

There is an immediate corollary of this observation.

Theorem 4.2.7 *Suppose $\mathcal{S}$ is a GQ of order s, $s > 2$, with a regular point p. Then $\mathcal{S}$ contains no complete $(s^2 - 1)$-arcs.*

Proof. Suppose $\mathcal{K}$ is a complete $(s^2 - 1)$-arc of the GQ $\mathcal{S}$ of order s, $s > 2$. Then $\mathcal{S}(\mathcal{K})$ is a grid with parameters $s + 1, s + 1$. If the regular point p is a point of either $\mathcal{S}(\mathcal{K})$ or $\mathcal{K}$, then we can take x on $\mathcal{K}$ and y in $\mathcal{S}(\mathcal{K})$ ($p \in \{x, y\}$), and so the assertion clearly follows from the previous observation. Suppose

$p \in \mathcal{S} \setminus (\mathcal{K} \cup \mathcal{S}(\mathcal{K}))$. Let $x \in p^\perp \cap \mathcal{K}$ and $y \in p^\perp \cap \mathcal{S}(\mathcal{K})$. Then $\{x, y\}$ is a regular pair of points, and $|\{x, y\}^{\perp\perp} \cap \mathcal{K}| = 1$ and $\{x, y\}^{\perp\perp} \cap \mathcal{S}(\mathcal{K}) \neq \emptyset$. The result now follows from the observation. ■

The following corollary, which we want to present as a theorem, is rather general.

Theorem 4.2.8 *The dual of the GQ $T_2(\mathcal{O})$ of order q, $q > 2$, has no complete $(q^2 - 1)$-arcs. In particular, the classical GQ $W(q)$ has no complete $(q^2 - 1)$-arcs if $q \neq 2$. The $T_2(\mathcal{O})$ of Tits of order q has no complete $(q^2 - 1)$-arcs if q is even and $q > 2$.*

Proof. The dual of $T_2(\mathcal{O})$ always has a regular point as $T_2(\mathcal{O})$ always has regular lines; see [139, 3.3.2]. The GQ $T_2(\mathcal{O})$ of Tits of order q, q even, always has a regular point; see [139, 3.3.2]. ■

Remark 4.2.9 Recall that if q is even, there holds that $W(q) \cong \mathcal{Q}(4, q)$.

Note. For $W(q)$, q even, stronger results are known, see e.g. the dual results of G. Tallini [162].

For $q = 2$, we have the following result.

Theorem 4.2.10 *Let $\mathcal{S}$ be a GQ of order 2, i.e. suppose that $\mathcal{S}$ is isomorphic to the classical GQ $\mathcal{Q}(4, 2)$. Then $\mathcal{S}$ has a unique complete 3-arc.*

Proof. The group of automorphisms of $\mathcal{Q}(4, 2)$, $Aut(\mathcal{S})$, acts transitively on the pairs of non-collinear points, hence we only look at the number of non-isomorphic complete 3-arcs through two given non-collinear points x and y. First of all, we note that $|\{x, y\}^\perp| = 3$ and that $|\{x, y\}^{\perp\perp}| = 3$ since every point of $\mathcal{Q}(4, 2)$ is regular. Put $\{x, y\}^{\perp\perp} = \{x, y, z\} = \mathcal{K}$. Then $\mathcal{K}$ is a complete 3-arc. Since every point of $\mathcal{S}$ is incident with a line which contains two points of $\{x, y\}^\perp \cup \{x, y\}^{\perp\perp}$, it is easy to show that $\mathcal{K}$ is the unique complete 3-arc containing $\{x, y\}$. ■

Note. The number of complete 3-arcs of $\mathcal{Q}(4, 2)$ is $\frac{15.8}{6} = 20$.

4.3 Complete $(st-t/s)$-Arcs in the Known GQ's of Order (s,t), $s \neq 1 \neq t$

Suppose $\mathcal{S}$ is a known GQ of order (s,t), $s \neq 1 \neq t$. Then we have that $t \in \{s-2, s^{2/3}, \sqrt{s}, s, s+2, s^{3/2}, s^2\}$. If $\mathcal{S}$ has a complete $(st-t/s)$-arc, then necessarily s is a divisor of t, and hence $t \geq s$, i.e. $t \in \{s, s+2, s^{3/2}, s^2\}$. If $\mathcal{S}$ is of order $(s, s+2)$, only the case $s=2$ is allowed, and $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5,2)$. The only known GQ of order $(s, s^{3/2})$, $s > 1$, is isomorphic to the classical GQ $H(4,s)$ (where s is the square of some prime power). Now suppose $\mathcal{K}$ is a complete (s^2-1)-arc of a GQ $\mathcal{S}$ of order s, $s > 2$. Then $\mathcal{S}(\mathcal{K})$ is a grid with parameters $s+1, s+1$, and hence $\mathcal{S}$ contains a regular pair of (non-concurrent) lines. The only known GQ's of order s which have a regular pair of (non-concurrent) lines are the $T_2(\mathcal{O})$ of Tits and the dual of $T_2(\mathcal{O})$ for s even, and by Theorem 4.2.8 we only have to consider the case where s is odd. In that case, by the theorem of B. Segre [151] (cf. Theorem 1.12.4), the oval $\mathcal{O}$ is a conic, and hence $T_2(\mathcal{O}) \cong \mathcal{Q}(4,q)$, q odd.
The following theorem is a direct corollary of the preceding considerations and provides a first version of a classification result for GQ's of order (s,t) with complete $(st-t/s)$-arcs, $s \neq 1 \neq t$.

Theorem 4.3.1 *Let $\mathcal{S}$ be a known GQ of order (s,t) with $s \neq 1 \neq t$, and suppose that $\mathcal{S}$ has a complete $(st-t/s)$-arc $\mathcal{K}$. Then we necessarily have one of the following possibilities.*

1. $\mathcal{S} \cong \mathcal{Q}(4,2)$ *and up to isomorphism there is a unique example.*
2. $\mathcal{S} \cong \mathcal{Q}(5,2)$ *and up to isomorphism $\mathcal{K}$ is unique.*
3. $\mathcal{S} \cong \mathcal{Q}(4,q)$ *with q odd.*

■

In fact, there is an example of a complete 8-arc in $\mathcal{Q}(4,3)$. This example will be obtained in the next section.

4.4 Complete Dual $(s-1) \times (s+1)$-Grids and Complete (s^2-1)-Arcs in GQ's of Order s, $s > 1$

We start this section with an easy observation. Consider an $s \times (s+1)$-grid Γ in a GQ $\mathcal{S}$ of order (s,t), $s,t > 1$. Suppose $\{L_1, L_2, \ldots, L_s, M_1, M_2, \ldots, M_{s+1}\}$

is the line set of Γ, where $M_i \not\sim M_j$ if $i \neq j$, $1 \leq i,j \leq s+1$ and $L_k \not\sim L_r$ if $1 \leq k \neq r \leq s$. Suppose $M,N,O \in \{M_1, M_2, \ldots, M_{s+1}\}$ are arbitrary but distinct, and suppose mIM so that $m \not\mid L_i$ for $i = 1,2,\ldots,s$. Then one observes that $m, proj_N m, proj_O m$ are on the same line, since otherwise $m, proj_N m, proj_O m$ are the points of a triangle of $\mathcal{S}$ (as clearly $proj_N m = proj_N(proj_O m)$), contradiction. Hence

Theorem 4.4.1 *If Γ is an $s\times(s+1)$-grid in a GQ $\mathcal{S}$ of order (s,t), $s \neq 1 \neq t$, then Γ is contained in an $(s+1)\times(s+1)$-grid. Hence $\mathcal{S}$ contains a regular pair of lines, and $s \leq t$.*

■

We call a $k\times(s+1)$-grid Γ of the GQ $\mathcal{S}$ *complete* if there is no $k' > k$ so that Γ is contained in a $k'\times(s+1)$-grid. Dually, we define *complete dual $k\times(t+1)$-grids*. Also, if Γ is an $(s-1)\times(s+1)$-grid, then by $\mathcal{R}$, respectively $\mathcal{R}'$, we denote the set of $s-1$, respectively $s+1$, mutually non-concurrent lines of Γ. Observe the following.

Theorem 4.4.2 *Let $\mathcal{S} = (P,B,I)$ be a GQ of order (s,t), $s,t>1$, which has a complete $(s-1)\times(s+1)$-grid. Then $s \leq t$.*

Proof. Let X be the set of points which are on the lines of $\mathcal{R}'$, and let Y be the set of points which are incident with a line of $\mathcal{R}'$ but not with a line of $\mathcal{R}$. Let $z \in P \setminus X$. It is clear that z is incident with $s-1$ lines which are incident with a point of $X \setminus Y$, and that at most two lines incident with z intersect Y. Hence we have that $t \geq s$ or $t = s-1$. The latter cannot occur as it contradicts the standard divisibility condition for GQ's. ■

Theorem 4.4.3 *Let $\mathcal{S} = (P,B,I)$ be a GQ of order (s,t), $s,t>1$, and let Γ be a complete $(s-1)\times(s+1)$-grid of $\mathcal{S}$. If $s=t$, then $\mathcal{S}$ contains a complete dual (s^2-1)-arc. Also, in that case $\mathcal{S}$ contains a spread.*

Proof. Suppose that X is the set of points which are on the lines of $\mathcal{R}'$, and let Y be the set of points which are incident with a line of $\mathcal{R}'$ but not with a line of $\mathcal{R}$. Consider $z \in P \setminus X$. By projecting z onto the lines of $\mathcal{R}$, z is incident with $s-1$ lines which are incident with a point of $X \setminus Y$, and that at most two lines incident with z intersect Y. As a direct corollary of this argument, we have that, as Γ is a complete $(s-1)\times(s+1)$-grid, each line of $B \setminus (\mathcal{R} \cup \mathcal{R}')$ contains at most two distinct points of X, respectively Y (hence each such line hits at most two distinct lines of $\mathcal{R}'$). Suppose L is a line of

$B \setminus (\mathcal{R} \cup \mathcal{R}')$ which is incident with $y \in Y$. Put $\mathcal{R}' = \{L_0, L_1, \ldots, L_s\}$, and suppose that yIL_0. By projecting y onto $L_1, L_2, \ldots, L_s$, the fact that $s = t$ implies that $L \in \{proj_y L_1, proj_y L_2, \ldots, proj_y L_s\}$. Hence each such line L hits *precisely* two distinct lines of $\mathcal{R}'$. A direct corollary is that each point of $P \setminus X$ is incident with precisely one line of $\mathcal{S}$ which does not contain a point of X. Hence, the set $\mathbf{T}'$ of all lines of $\mathcal{S}$ which have empty intersection with X is a partial spread of $\mathcal{S}$ of size $s^2 - s$. It is now clear that $\mathbf{T}' \cup \mathcal{R}$ is a complete dual $(s^2 - 1)$-arc, and that $\mathbf{T}' \cup \mathcal{R}'$ is a spread of $\mathcal{S}$. ■

Corollary 4.4.4 *The GQ $\mathcal{Q}(4,3)$ has complete 8-arcs.*

Proof. Suppose x and y are non-collinear points of $\mathcal{Q}(4,3)$. Then $\{x,y\}^{\perp} \cup \{x,y\}^{\perp\perp}$ is a complete dual 2×4-grid. Now apply the dual of Theorem 4.4.3. ■

Of course, the last corollary dualizes to $W(3)$.

Remark 4.4.5 Suppose that $\mathcal{S} = (P, B, I)$ is a GQ of order (s,t), $s,t > 1$, and let Γ be a complete $(s-1) \times (s+1)$-grid of $\mathcal{S}$. Use the notation of the proof of Theorem 4.4.3. By the proof of that theorem, the geometry $\mathcal{S}' = (P', B', I')$, where $P' = Y$, B' is the set of lines of $\mathcal{S}$ which intersect Y in two points (note that $\mathcal{R}' \subseteq B'$), and where $I' = I \cap ((P' \times B') \cup (B' \times P'))$, is a dual grid with parameters $s+1, s+1$ ('non-collinearity' in $P' = Y$ is an equivalence relation).

Assume that $\mathcal{S} = (P, B, I)$ is a GQ of order (s,t), $s,t > 1$, and that Γ is a complete $(s-1) \times (s+1)$-grid of $\mathcal{S}$. If L is a line of $\mathcal{S}$ which is not concurrent with a line of Γ, then we call L an *exterior line of* Γ. Note the following corollary of Theorem 4.4.3.

Corollary 4.4.6 *Let $\mathcal{S} = (P, B, I)$ be a GQ of order (s,t), $s,t > 1$, and let Γ be a complete $(s-1) \times (s+1)$-grid of $\mathcal{S}$. Suppose X is as before. If each point of $P \setminus X$ is incident with a constant number of exterior lines of Γ, then $s = t$ and the conclusion of Theorem 4.4.3 holds.*

Proof. Each point of Y is on $t - s$ lines which intersect X in exactly one point. So if $t > s$ there are points of $P \setminus X$ on exactly $t - s$ exterior lines, and for any $t \geq s$ there are points of $P \setminus X$ on precisely $t - s + 1$ exterior lines. Hence if each point of $P \setminus X$ is incident with a constant number of exterior lines of Γ, then $s = t$. ■

Corollary 4.4.7 *Let $\mathcal{S} = (P, B, I)$ be a GQ of order (s, t), $s, t > 1$, and let Γ be a complete $(s-1) \times (s+1)$-grid of $\mathcal{S}$. Let X be as before. If there is a group of automorphisms of $\mathcal{S}$ which fixes Γ and which acts transitively on $P \setminus X$, then $s = t$, $\mathcal{S}$ contains a complete dual $(s^2 - 1)$-arc and $\mathcal{S}$ contains a spread.*

Proof. Immediate. ∎

Remark 4.4.8 Suppose $\mathcal{S} = (P, B, I)$ is a GQ of order (s, t), $s, t > 1$, which has a complete $(s-1) \times (s+1)$-grid Γ, and suppose that $s \neq t$. Assume that X is as before. In view of the proof of Theorem 4.4.3, it could be interesting to investigate the following point-line geometry Γ':

(a) POINTS are the points of $\mathcal{S} \setminus X$;

(b) LINES are those lines of $\mathcal{S}$ which are not incident with a point of X;

(c) INCIDENCE is inherited from $\mathcal{S}$.

For $s = t$, Γ' is just a partial spread of $\mathcal{S}$ with $s^2 - s$ elements.

4.5 Some Remarks about the General Case

For GQ's of order (s, s^2), $s > 1$ and s odd, there is a complete solution to the problem:

Theorem 4.5.1 *A GQ $\mathcal{S} = (P, B, I)$ of order (s, s^2), $s > 1$ and s odd, with a complete $(s-1) \times (s+1)$-grid does not exist.*

Proof. Adopt the notation $P' = Y, B', I', \mathcal{R}, \mathcal{R}'$ from Remark 4.4.5. Then $\mathcal{S}' = (P', B', I')$ is a dual $(s+1) \times (s+1)$-grid. By Theorem 1.1.2, each line of $\mathcal{S}$ which is not incident with a point of Y must hit 0 or 2 lines of B' as s is odd, a contradiction since each line of $\mathcal{R}'$ is concurrent with each line of $\mathcal{R}$, and $\mathcal{R} \subseteq B'$. ∎

For s even, the situation is slightly different. Let $\mathcal{S} = (P, B, I)$ be a GQ of order (s, s^2), $s > 1$ and s even, and assume that Γ is a complete $(s-1) \times (s+1)$-grid of $\mathcal{S}$. Suppose $\mathcal{R}, \mathcal{R}', X, Y, \mathcal{S}' = (P', B', I')$ are as before. As $\mathcal{S}'$ is a dual grid with parameters $s+1, s+1$ and s is even, the geometry $\mathcal{S}''$ which consists of the lines of $\mathcal{S}$ which hit at least two distinct lines of B', and the points of $\mathcal{S}$ on these lines (and the natural incidence), is a subGQ of $\mathcal{S}$ of order s, see Section 1.6.3. As $\mathcal{S}''$ clearly contains $\mathcal{R}$ and $\mathcal{R}'$, there follows that $\mathcal{S}''$ also contains the complete $(s-1) \times (s+1)$-grid Γ. Whence by Theorem 4.4.3, $\mathcal{S}''$ contains a

complete dual (s^2-1)-arc.
Now suppose that $\mathcal{S}''$ is a known GQ. Then by Theorem 4.4.3, $s = 2$, $\mathcal{S}'' \cong W(2)$, and $\mathcal{S}'' \cong \mathcal{Q}(5,2)$ by Section 1.3.
Note that $\mathcal{R}'$ is just a unicentric triad of lines of $\mathcal{S}'$, and if $\mathcal{R} = \{L\}$, then L is the unique center of $\mathcal{R}'$.

4.6 Affine Generalized Quadrangles

Motivated by the results on complete $(st-t/s)$-arcs in generalized quadrangles of order (s,t), $s \neq 1 \neq t$, we now will introduce the notion of *(partial) spreads* and *(partial) ovoids* in *affine generalized quadrangles.*

A *geometrical hyperplane* $\mathcal{H}$ of a generalized quadrangle $\mathcal{S}$ of order (s,t), $s \neq 1 \neq t$, is a subgeometry of $\mathcal{S}$ in the usual sense (this means that $P' \subseteq P$, $B' \subseteq B$ and that I' is the induced incidence), so that each line of $\mathcal{S}$ intersects $\mathcal{H}$ in 1 point, or is completely contained in $\mathcal{H}$. Geometrical hyperplanes of generalized quadrangles are characterized by the following theorem, which is merely a direct corollary of Theorem 1.6.3.

Theorem 4.6.1 *Suppose $\mathcal{S}$ is a generalized quadrangle of order (s,t), $s \neq 1 \neq t$, and suppose $\mathcal{H}$ is a geometrical hyperplane of $\mathcal{S}$. Then $\mathcal{H}$ is given by one of the following.*

(1) *$\mathcal{H}$ is an ovoid of $\mathcal{S}$.*

(2) *$\mathcal{H}$ is a point, together with all the lines incident with that point and all the points on those lines.*

(3) *$\mathcal{H}$ is a (not necessarily thick) subGQ of $\mathcal{S}$ of order $(s,t/s)$ (and hence s divides t).*

Proof. Use Theorem 1.6.3 and easy counting. ∎

An *affine generalized quadrangle (AGQ)* is the complement of a geometrical hyperplane $\mathcal{H}$ of a thick generalized quadrangle $\mathcal{S}$, that is, the natural geometry formed by the points and lines of $\mathcal{S}$ which are not in $\mathcal{H}$. In the following, we will speak of *AGQ's of Type (1),(2),(3)*, according to whether the corresponding geometrical hyperplane is given by respectively (1), (2) or (3) of Theorem 4.6.1. H. Pralle [147] has derived a set of axioms (AGQ1)–(AGQ7) so that each point-line geometry satisfying these axioms arises as the complement of a geometrical hyperplane in some thick generalized quadrangle. Such a point-line geometry

will be called an *abstract affine generalized quadrangle*. From the work of H. Pralle follows that an abstract affine generalized quadrangle Γ corresponds — up to isomorphism — to a unique generalized quadrangle from which it arises. We call this generalized quadrangle the *generalized quadrangle spanned by* Γ, denoted $\mathcal{S} = \mathcal{S}(\Gamma)$. If Γ is an AGQ, then we say that Γ is of *order* (s,t) if the corresponding generalized quadrangle is of order $(s+1,t)$. As in the case of GQ's, we will sometimes speak of an 'affine quadrangle' instead of 'affine generalized quadrangle'.

4.7 Partial Ovoids and Partial Spreads in Affine Generalized Quadrangles

A *partial ovoid* of an AGQ is a set of two by two non-collinear points. An *ovoid* of an AGQ is a partial ovoid such that any line has non-empty intersection with it. Dually, one defines *partial spreads* and *spreads* of AGQ's.

Theorem 4.7.1 *Suppose Γ is an affine generalized quadrangle of order $(s-1,t)$, $s,t>1$. An ovoid of Γ has $st+1$, st or $st-t/s$ points according as Γ is respectively of Type (1), (2) or (3). A spread also contains $st+1$, st or $st-t/s$ lines according as Γ is respectively of Type (1), (2) or (3).*

Proof. In each of the cases, we count the number of flags $(\mathcal{X},\mathcal{Y})$, where $\mathcal{X}$ is an element of the ovoid or spread, and $\mathcal{Y}$ is an element of Γ for which $\mathcal{X}I\mathcal{Y}$ (where 'I' denotes incidence in Γ), in two ways. By $\mathbf{S}$, we denote the ovoid or spread of Γ; by k, we denote $|\mathbf{S}|$. We will look at the three distinct cases as defined by Theorem 4.6.1.

1. Case 1: AGQ's of Type (1). If $\mathbf{S}$ is an ovoid of Γ, then $k(t+1) = (t+1)(st+1)$, and if $\mathbf{S}$ is a spread, we have that $ks = s(st+1)$.

2. Case 2: AGQ's of Type (2). If $\mathbf{S}$ is an ovoid of Γ, then $k(t+1) = (t+1)st$, and if $\mathbf{S}$ is a spread, then $ks = (s+1)(st+1) - (st+s+1)$.

3. Case 3: AGQ's of Type (3). If $\mathbf{S}$ is an ovoid of Γ, then $k(t+1) = (t+1)(st+1) - (t/s+1)(t+1)$, and if $\mathbf{S}$ is a spread, then $ks = (s+1)(st+1) - (s+1)(t+1)$.

■

4.8 Existence of Spreads and Ovoids of AGQ's

Theorem 4.8.1 *An AGQ of Type (2) or (3) always has spreads.*

Proof. Suppose Γ is an affine generalized quadrangle of Type (2) or (3) of order $(s-1,t)$, $s,t>1$, and suppose $\mathcal{H}$ is the corresponding geometrical hyperplane of $\mathcal{S}(\Gamma)$. Consider a line L of $\mathcal{H}$ (note that as Γ is not of Type (1), $\mathcal{H}$ has lines), and suppose $\mathbf{T}$ is the set of all the lines of Γ which intersect L in $\mathcal{S}(\Gamma)$. Then clearly, if Γ is of Type (2), we have that $\mathbf{T}$ contains st elements, and if Γ is of Type (3), then $\mathbf{T}$ has $st - t/s$ lines. Hence the result. ∎

Remark 4.8.2 One could make the spread problem for AGQ's of Type (2) and (3) (of order $(s-1,t)$, $s,t>1$) more interesting by demanding special properties for such a spread. For instance, one could ask the spread to induce a partial spread in $\mathcal{S}(\Gamma)$. In that case, if the AGQ is of Type (2), this would imply that $\mathcal{S}(\Gamma)$ also has spreads (cf. the dual of Theorem 4.1.1), and if the AGQ is of Type (3), $\mathcal{S}(\Gamma)$ would have a partial spread of large size, namely of size $st - t/s$.

Theorem 4.8.3 *Suppose Γ is an AGQ of Type (1), with spanned GQ $\mathcal{S}(\Gamma)$, and where Γ arises from the ovoid $\mathbf{O}$ of $\mathcal{S}(\Gamma)$. Then Γ has a spread $\mathbf{T}$ if and only if it induces a spread of $\mathcal{S}(\Gamma)$.*
So, the study of spreads of AGQ's of Type (1) is equivalent to the study of GQ's which have both spreads and ovoids.

Proof. Suppose Γ is of order $(s-1,t)$, $s,t>1$. We identify lines of Γ with the corresponding lines of $\mathcal{S}(\Gamma)$. Suppose p is a point of $\mathbf{O}$, and suppose p is incident with r lines of $\mathbf{T}$ in $\mathcal{S}(\Gamma)$. Note that in $\mathcal{S}(\Gamma)$, every line of $\mathbf{T}$ intersects $\mathbf{O}$ in precisely one point, and distinct lines of $\mathbf{T}$ can only intersect on $\mathbf{O}$. If $L_0, L_1, \ldots, L_t$ are the lines through p in $\mathcal{S}(\Gamma)$ and if $L_0, L_1, \ldots, L_{r-1}$ are the lines of $\mathbf{T}$ through p in $\mathcal{S}(\Gamma)$, then every line of $\mathbf{T}$ which is not incident with p in $\mathcal{S}(\Gamma)$ intersects a line of $L_r, L_{r+1}, \ldots, L_t$. We obtain that:

$$st + 1 - r \leq (t+1-r)s,$$

hence $r = 1$. The result follows. ∎

Remark 4.8.4 (i) Not a lot is known about GQ's having both spreads and ovoids. If such a GQ $\mathcal{S}$ has order (s,t), $s,t>1$, then by E. E. Shult

[155] and J. A. Thas [168], see Theorem 1.12.1, we have that $t \leq s^2 - s$ and $s \leq t^2 - t$, and hence the GQ's of order (k^i, k^j), with $k \neq 1$ and $(i,j) \in \{(1,2),(2,1)\}$, are trivially excluded. Hence, of the known orders of GQ's, we have that, up to duality, $(s,t) \in \{(s,s),(s,s+2),(s,\sqrt{s^3})\}$. If $\mathcal{S}$ is of order $(s,\sqrt{s^3})$, $s > 1$, then the only known GQ is the classical GQ $H(4,s)$ and this generalized quadrangle does not allow ovoids, see Theorem 1.12.2. Of the other classical GQ's, the only one with the desired property is $\mathcal{Q}(4,s)$ with s even, see Theorem 1.12.2. The $T_2(\mathcal{O})$ of Tits always has ovoids, see Theorem 1.12.2, and several of them also have spreads [26]. The GQ $\mathcal{P}(\mathcal{S},x)$ has spreads and ovoids if and only if $\mathcal{S}$ has an ovoid containing x, see Theorem 1.12.2.

(ii) A geometrical hyperplane $\mathcal{H}$ of a generalized quadrangle has the property that if any two points are collinear in the GQ, they are also collinear in $\mathcal{H}$, and hence the corresponding affine quadrangle will also have this property (actually, that is exactly why the existence problem of ovoids in AGQ's is much harder to investigate than the existence problem of spreads: the line set of an AGQ does not have this property). Hence, if $\mathbf{O}$ is an ovoid of an AGQ Γ, then it will be a (large) partial ovoid of $\mathcal{S}(\Gamma)$.

Suppose that Γ is an AGQ, that $\mathcal{S}(\Gamma)$ is the GQ spanned by Γ and that Γ is of order $(s-1,t)$, $s,t > 1$. Also, $\mathcal{H}$ denotes the geometrical hyperplane of $\mathcal{S}(\Gamma)$ corresponding to Γ. Suppose that Γ has an ovoid $\mathbf{O}$. If Γ is of Type (1), then as $\mathcal{H}$ is an ovoid of $\mathcal{S}(\Gamma)$, it follows that $t \leq s^2 - s$. It is clear that $\mathcal{H}$ and $\mathbf{O}$ are disjoint ovoids of $\mathcal{S}(\Gamma)$, and $O = \mathcal{H} \cup \mathbf{O}$ has the property that any line of $\mathcal{S}(\Gamma)$ meets O in exactly two points. In the sense of J. A. Thas [174], this means that O is a 2-*ovoid*.
Suppose that Γ is of Type (2). Then the point set of $\mathcal{H}$ is of the form $p^\perp$ for some point p of $\mathcal{S}(\Gamma)$. Clearly, there follows that $\mathbf{O} \cup \{p\}$ is an ovoid of $\mathcal{S}(\Gamma)$.

Now suppose that Γ is of Type (3). Then $\mathbf{O}$ is an $(st-t/s)$-arc of $\mathcal{S}(\Gamma)$ of which no point is contained in the subGQ $\mathcal{H}$ of $\mathcal{S}(\Gamma)$ of order $(s,t/s)$. Each point of $\mathcal{S}(\Gamma)$ not on $\mathbf{O}$ is on a line L not in $\mathcal{H}$ but intersecting $\mathcal{H}$, and hence L intersects $\mathbf{O}$, from which follows that $\mathbf{O}$ is a complete $(st-t/s)$-arc of $\mathcal{S} = \mathcal{S}(\Gamma)$, and that $\mathcal{H} = \mathcal{S}(\mathbf{O})$. By Theorem 4.3.1, we then obtain the following.

Theorem 4.8.5 *Suppose that Γ is an AGQ of order $(s-1,t)$ of Type (3), $s,t > 1$, and suppose $\mathcal{S}(\Gamma)$ is a known GQ. Suppose $\mathbf{O}$ is an ovoid of Γ. Then we have one of the following possibilities.*

1. $\mathcal{S}(\Gamma) \cong \mathcal{Q}(4,2)$ *and up to isomorphism*[1], **O** *is the only ovoid of* Γ.

2. $\mathcal{S}(\Gamma) \cong \mathcal{Q}(5,2)$ *and up to isomorphism,* **O** *is the only ovoid of* Γ.

3. $\mathcal{S}(\Gamma) \cong \mathcal{Q}(4,q)$ *with* q *odd.*

Proof. It is clear that, if $Aut(\Gamma)$ is the automorphism group of Γ, and if $\mathcal{H}$ is the geometrical hyperplane of $\mathcal{S}(\Gamma)$ which corresponds to Γ, then $Aut(\mathcal{S})_{\mathcal{H}} \leq Aut(\Gamma)$. The proof now easily follows. ■

Remark 4.8.6 (i) It is not difficult to show that, conversely, if θ is an automorphism of the AGQ Γ, then θ induces an automorphism of $\mathcal{S}(\Gamma)$ which fixes $\mathcal{H}$. Hence $Aut(\Gamma) \cong Aut(\mathcal{S})_{\mathcal{H}}$.

(ii) Note that, if Γ is the AGQ of Type (3) which arises from $\mathcal{Q}(4,3)$, then Γ also contains ovoids, see Corollary 4.4.4.

---

[1] *Automorphisms* of an AGQ, as well as *isomorphic ovoids* of an AGQ, are defined in the usual way.

Appendix: New Bounds for Partial Spreads of Generalized Quadrangles of Order (q^2, q)

We start the appendix with some results which are taken from J. A. Thas [173]. Recall that for any triad $\{L, M, N\}$ of lines of $H(3, q^2)$, we have that $|\{L, M, N\}^{\perp}| = |\{L, M, N\}^{\perp\perp}| = q + 1$.

Theorem 4.8.7 (J. A. Thas [173]) *Let $\mathcal{S}$ be isomorphic to the classical GQ $H(3, q^2)$. If $\mathcal{S}$ has a partial spread $\mathbf{T}$, then*

$$|\mathbf{T}| \leq q^3 - q^2 + q + 1.$$

Corollary 4.8.8 (J. A. Thas [173]) *The classical GQ $H(3, q^2)$ of order (q^2, q) has no spreads.*

Theorem 4.8.9 (J. A. Thas [173]) *Let $\mathcal{S}$ be isomorphic to the classical GQ $H(3, q^2)$. Suppose $\mathbf{T}$ is a partial spread of $\mathcal{S}$ which contains three distinct lines L, M, N for which $\{L, M, N\}^{\perp\perp} \subseteq \mathbf{T}$. Then*

$$|\mathbf{T}| \leq \frac{q^3}{2} + \frac{q}{2} + 1.$$

Note that both Theorem 4.8.7 and Theorem 4.8.9 dualize to partial ovoids of $\mathcal{Q}(5, q)$.

Let us recall that if $\{U, V, W\}$ is a 3-regular triad of lines of a GQ $\mathcal{S}$ of order (s^2, s), $s > 1$, where $\{U, V, W\}^{\perp} = \{L_0, L_1, \ldots, L_s\}$ and $\{U, V, W\}^{\perp\perp} = \{M_0, M_1, \ldots, M_s\}$, then we have that each line of $\mathcal{S}$ which is not in $\{U, V, W\}^{\perp} \cup \{U, V, W\}^{\perp\perp}$, either is incident with a point $L_i \cap M_j$ for some i and j in $\{0, 1, \ldots, s\}$, or intersects exactly two lines of $\{U, V, W\}^{\perp}$ and no lines of $\{U, V, W\}^{\perp\perp}$, or intersects exactly two lines of $\{U, V, W\}^{\perp\perp}$ and no lines of $\{U, V, W\}^{\perp}$.

We now generalize Theorem 4.8.7 to the following result.

Theorem 4.8.10 *Let $\mathcal{S}$ be isomorphic to the classical GQ $H(3,q^2)$. Suppose that $\mathbf{T}$ is a partial spread of $\mathcal{S}$ which contains three distinct lines L, M, N so that $|\{L,M,N\}^{\perp\perp} \cap \mathbf{T}| = r+3$, $r \in \mathbb{N}$, $r \leq q-2$. Then*

$$|\mathbf{T}| \leq q^3 - \frac{q}{2}(q+1)(r+1) + 1.$$

Proof. Let $\{L,M,N\}^{\perp\perp} = L_0, L_1, \ldots, L_q$, set $\{L,M,N\}^{\perp} = M_0, M_1, \ldots, M_q$ and put $\{L,M,N\}^{\perp\perp} \cap \mathbf{T} = X$. Then $|X| = r+3$. For each line U of the $q-r-2$ lines of $\{L,M,N\}^{\perp\perp}$ which are not contained in $\mathbf{T}$, there are at most $q+1$ lines of $\mathbf{T}$ which intersect U (in one point) and no other line of $\{L,M,N\}^{\perp\perp}$. The lines of $\mathbf{T}$ which do not intersect exactly one of these lines U and which do not belong to $\{L,M,N\}^{\perp\perp}$, either intersect $\{L_0, L_1, \ldots, L_q\}$ in 2 lines and no line of $\{M_0, M_1, \ldots, M_q\}$, or they intersect $\{M_0, M_1, \ldots, M_q\}$ in 2 lines and no line of $\{L_0, L_1, \ldots, L_q\}$. We obtain that

$$|\mathbf{T}| \leq (r+3) + (q-r-2)(q+1) + \frac{(q-2-r)(q^2-q)}{2} + \frac{(q+1)(q^2-q)}{2}.$$

Hence the result. ∎

Note that, if $r < r' \leq q-2$, we have that

$$q^3 - \frac{q}{2}(q+1)(r'+1) + 1 \leq q^3 - \frac{q}{2}(q+1)(r+1) + 1.$$

Remark 4.8.11 For $r = 0$, we obtain a weaker result than Theorem 4.8.7. For $r = 1$, however, the bound is already (slightly) stronger. For $r = q-2$, we obtain Theorem 4.8.9.

One also notes that Theorem 4.8.10 dualizes to a result on partial ovoids of $\mathcal{Q}(5,q)$.
As a direct corollary of the proof of Theorem 4.8.10, we have the next result.

Theorem 4.8.12 *Let $\mathcal{S}$ be a GQ of order (s^2,s), $s > 1$, which has a 3-regular triad of lines $\{L,M,N\}$. Suppose that $\mathbf{T}$ is a partial spread of $\mathcal{S}$ so that $|\{L,M,N\}^{\perp\perp} \cap \mathbf{T}| = r+3$, $r \in \mathbb{N}$, $r \leq s-2$. Then*

$$|\mathbf{T}| \leq s^3 - \frac{s}{2}(s+1)(r+1) + 1.$$

Proof.
Immediate. ∎

Corollary 4.8.13 *Theorem 4.8.12 can be stated for each flock GQ $\mathcal{S}(\mathcal{F})$ of order (q^2, q), $q > 1$, and for the point-line dual $T_3(\mathcal{O})^D$ of $T_3(\mathcal{O})$, $\mathcal{O}$ any ovoid $\mathcal{O}$ of $\mathbf{PG}(3, q)$.*

Proof. If $\mathcal{F}$ is a flock of the quadratic cone of $\mathbf{PG}(3, q)$, then $\mathcal{S}(\mathcal{F})^D$ satisfies Property (G) at its line $[\infty]$ (which corresponds to the point (∞) of the flock GQ), and hence $\mathcal{S}(\mathcal{F})$ has 3-regular triads of lines. The point (∞) of $T_3(\mathcal{O})$ is 3-regular. ∎

Chapter 5

Symmetries and Translation Generalized Quadrangles

Suppose $\mathcal{S}$ is a generalized quadrangle of order (s,t), with $s,t \neq 1$, and suppose that L is a line of $\mathcal{S}$. In this chapter we will prove that L is an axis of symmetry if and only if L is regular and if there is a point p on L, a group H of whorls about p, and a line M with $M \sim LIp \not\setminus M$, such that H acts transitively on the points of $M \setminus \{M \cap L\}$. As a corollary, we prove that a line of a generalized quadrangle $\mathcal{S}$ through an elation point p is an axis of symmetry if and only if it is a regular line, and that the group of symmetries of such a line is always completely contained in any elation group corresponding to an elation point on this line. This proves the converse of Theorem 1.9.1. As a nice corollary, we have that an EGQ with elation point p is a TGQ with translation point p if and only if p is coregular. Using the aforementioned results, we prove new characterizations of translation generalized quadrangles, and we improve some characterizations of translation generalized quadrangles obtained by X. Chen and D. Frohardt [35], and D. Hachenberger [68]; we will prove that an

elation generalized quadrangle $\mathcal{S}$ with elation point p and elation group G is a translation generalized quadrangle with translation point p and translation group G if and only if p is incident with at least two regular lines, and if there is an odd number of points on a line, one regular line is sufficient. Also, as an application, we will prove a (new) characterization theorem of the (thick) classical generalized quadrangles arising from a quadric.

The results of this chapter were published in *K. Thas, On symmetries and translation generalized quadrangles, Developments in Mathematics* **3**, *Proceedings of the Fourth Isle of Thorns Conference 'Finite Geometries', 16–21 July 2000, Kluwer Academic Publishers (2001), 333–345* [205].

5.1 A Theorem of X. Chen and D. Frohardt, and a Theorem of D. Hachenberger

Recall the following results:

Theorem 5.1.1 (X. Chen and D. Frohardt [35]) *Let G be a group of order s^2t admitting a 4-gonal family $(\mathcal{J}, \mathcal{J}^*)$ of type (s,t). If there exist two distinct members in $\mathcal{J}$ which are normal subgroups of G, then s and t are powers of the same prime number p and G is an elementary abelian p-group.*

Theorem 5.1.2 (D. Hachenberger [68]) *Let G be a group of order s^2t admitting a 4-gonal family $(\mathcal{J}, \mathcal{J}^*)$ of type (s,t). If G is a group of even order, and if there exists a member of $\mathcal{J}$ which is a normal subgroup of G, then s and t are powers of 2 and G is an elementary abelian 2-group.*

In geometrical terms, Theorem 5.1.1 reads as follows: "Let $(\mathcal{S}^{(x)}, G)$ be an EGQ of order (s,t), $s,t \neq 1$, and suppose that there are at least two axes of symmetry L and M through the elation point x, such that *the full groups of symmetries about L and M are completely contained in G*. Then s and t are powers of the same prime number p and G is an elementary abelian p-group." Hence, by Theorem 1.7.6, $\mathcal{S}^{(x)}$ is a translation generalized quadrangle with translation group G. In geometrical terms, we have the following for Theorem 5.1.2: "Let $(\mathcal{S}^{(x)}, G)$ be an EGQ of order (s,t), with $s,t \neq 1$ and s or t even, and suppose that there is at least one axis of symmetry L through the elation point x, such that *the full group of symmetries about L is completely contained in G*. Then s and t are powers of 2 and G is an elementary abelian 2-group." Thus, $\mathcal{S}^{(x)}$ is a translation generalized quadrangle with translation group G.

5.2 Some Lemmas

Let (p, L, p') be a panel of the GQ $\mathcal{S}$ of order (s, t), $s \neq 1 \neq t$. A (p, L, p')-*collineation* is a whorl about p, L and p'. We present the following result as a lemma:

Lemma 5.2.1 (FGQ, 9.2.1) *Suppose (p, L, p') is a panel of the GQ $\mathcal{S}$ of order (s, t), $s \neq 1 \neq t$, and let θ be a (p, L, p')-collineation. Then θ is a symmetry about L if L is regular.*

■

Corollary 5.2.2 *Suppose $\mathcal{S}$ is a GQ of order (s, t), $s, t \neq 1$, and suppose that L is a line which contains a panel (p, L, p') for which there is a full group of (p, L, p')-collineations of size s. Then L is an axis of symmetry if and only if L is regular.*

■

Lemma 5.2.3 *Suppose $\mathcal{S}$ is a GQ of order (s, t), $s, t \neq 1$, with a whorl θ about a point p, and that L and M are lines such that $M \sim LIp \backslash M$, and for which $M^\theta = M$. Moreover, suppose that L is a regular line, and let q be a point on L, different from p and not on M. Then we have one of the following possibilities:*

1. *q is not fixed by θ;*
2. *q is fixed by θ, and then also every line through q.*

Proof. Suppose that q is fixed by θ, and consider a line QIq, $Q \neq L$. Also, consider an arbitrary point rIM and not on L, and the line $[r, Q] = proj_r Q$. Let NIp be such that $N \sim [r, Q]$, and let the intersection point be r'. Now consider the point r^θ. Since L is a regular line, there holds that the line $[r^\theta, Q]$ also intersects N, say in r''. It is clear that $(r')^\theta = r''$, and that $[r, Q]^\theta = [r^\theta, Q]$. Since the point q is fixed, it follows easily that the line Q also is fixed by θ, and hence the lemma follows. ■

Corollary 5.2.4 *Suppose $\mathcal{S}$ is a GQ of order (s, t), $s, t \neq 1$, and let p be a point, and L and M lines, such that $M \sim LIp \backslash M$. Also, suppose that L is a regular line. Assume that θ is a whorl about p fixing M, and fixing a point q on L, different from p and not on M. Then every line through $r = L \cap M$ is fixed by θ.*

Proof. By Lemma 5.2.3, every line through q is fixed by θ. Since M is fixed, also r is fixed, and by the same lemma the proof follows. ■

Lemma 5.2.5 *Let $\mathcal{S}$ be a GQ of order (s,t), $s,t \neq 1$, and suppose $\theta \neq \mathbf{1}$ is a whorl about distinct collinear points p and q. If moreover, the line pq is regular, then θ is a symmetry about pq.*

Proof. Suppose M is an arbitrary line of $(pq)^\perp$ not through p or q. Suppose U and U' are distinct lines through p, both different from pq, and define the line V, respectively V', by being the unique line of $\{U,M\}^{\perp\perp}$, respectively $\{U',M\}^{\perp\perp}$, which is incident with q. Then $\{U,V\}^{\perp\perp} \cap \{U',V'\}^{\perp\perp} = \{M\}$, and $[\{U,V\}^{\perp\perp}]^\theta = \{U,V\}^{\perp\perp}$, respectively $[\{U',V'\}^{\perp\perp}]^\theta = \{U',V'\}^{\perp\perp}$. Hence $M^\theta = M$, and thus every line of $(pq)^\perp$ is fixed by θ. Hence θ is a symmetry about pq. ∎

Lemma 5.2.6 *Suppose $\mathcal{S}$ is a GQ of order (s,t), with $s,t \neq 1$. Let p be a point of the GQ, and suppose that L and M are lines such that $M \sim LIp \not\in M$, with L a regular line. Suppose that θ is a whorl about p fixing M, and suppose q is a point on L, different from p and not on M, which is also fixed by θ. Then θ is a symmetry about L.*

Proof. By Lemma 5.2.3, all the lines through q (and also $L \cap M$) are fixed by θ. From Lemma 5.2.5, the result follows. ∎

Now suppose that $\mathcal{S}$ is a GQ with parameters (s,t), $s,t \neq 1$. Also, assume that p is a point and LIp a regular line, and again that $M \sim L$ is a line not through p. Suppose θ is a whorl about p which fixes M, and such that $\langle\theta\rangle$ acts semiregularly on the points of M not on L. So $|\langle\theta\rangle|$ is a divisor of s. Consider an arbitrary nontrivial element ϕ of $\langle\theta\rangle$ of prime-power order, and consider the action of $\langle\phi\rangle$ on the points of $X = L \setminus [\{p\} \cup \{L \cap M\}]$. Since $|X| = s-1$ and because of the fact that $s-1$ and s are coprime, there follows immediately that there is a point $x \in X$ for which $x^{\langle\phi\rangle} = \{x\}$ (that is, every element of $\langle\phi\rangle$ fixes x). By Lemma 5.2.6, this implies that $\langle\phi\rangle$ is a group of symmetries about L. Since a finite group is generated by its elements of prime-power order and since ϕ was arbitrary, there follows that also $\langle\theta\rangle$ is a group of symmetries about L (the product of two symmetries about the same line is clearly again a symmetry about this line).
We now obtain

Theorem 5.2.7 *Suppose $\mathcal{S}$ is a GQ of order (s,t), $s,t \neq 1$, and suppose that L is a regular line through the point p. Let M be a line for which $L \sim M \not\in p$. If H is a group of whorls about p which fixes M and which acts transitively on the points of M which are not incident with L, then H contains a full group of symmetries about L of order s (i.e. L is an axis of symmetry).*

Proof. If $|H| = s$, then the statement follows directly from the preceding observation, hence suppose that $|H| > s$. Put $X = M \setminus \{L \cap M\}$. If θ is a nontrivial element of H which fixes at least two points of X, then by Theorem 1.6.5 there follows that $t < s$, a contradiction since L is a regular line. Hence the permutation group (X, H) satisfies the following conditions:

(i) H acts transitively on X;

(ii) for every $x \in X$, we have that $|H_x| \neq 1$, and

(iii) the only element of H which fixes at least two elements of X is the trivial element.

Thus (X, H) is a Frobenius group and hence by Theorem 1.13.1, H contains a normal subgroup N which acts regularly on X, and by the observation preceding Theorem 5.2.7, this group is a full group of symmetries about L of size s.■

There is a nice corollary of Theorem 5.2.7.

Theorem 5.2.8 *1. Suppose $\mathcal{S} = (P, B, I)$ is a GQ of order (s, t), $s, t \neq 1$, and suppose that L is a regular line through the point p. If G is a group of whorls about p which acts transitively on the points of $P \setminus p^\perp$, then L is an axis of symmetry. In particular, suppose that $(\mathcal{S}^{(p)}, G)$ is an EGQ of order (s, t), $s, t \neq 1$, with elation point p and elation group G, and suppose that LIp is a regular line. Then L is an axis of symmetry.*

2. Suppose $\mathcal{S}$ is an EGQ of order (s, t), $s, t \neq 1$, with elation point p. Then LIp is a regular line if and only if L is an axis of symmetry.

■

In Section 8.1 of FGQ, the following property is noted:

The set of all symmetries about some line L of a GQ $\mathcal{S}$ of order (s, t), $s, t \neq 1$, is always a group, and every symmetry about L is an elation about L and about every point on L, hence the group has at most size s; thus, if the group of symmetries about L has size s, then this group is unique.

We hence have the following theorem.

Theorem 5.2.9 *Suppose $\mathcal{S} = (P, B, I)$ is a GQ of order (s, t), $s, t \neq 1$, and suppose G is a group of whorls about the point p which acts transitively on $P \setminus p^\perp$.*

1. *If L is a regular line through p, then L is an axis of symmetry, and the full group of symmetries of size s about L is completely contained in G.*

2. *If LIp is an axis of symmetry, with G_L the full group of symmetries about L, then G_L is completely contained in G.*

In particular, the same statement holds for elation generalized quadrangles.

Proof. From the proof of Theorem 5.2.8, there follows that G contains a group G_L of size s of symmetries about L, and because of the previous remark, this group is unique; Part (1) of the statement follows.
Part (2) follows directly from Part (1) and the fact that an axis of symmetry is a regular line. ∎

We are ready to state the converse of Theorem 1.9.1:

Theorem 5.2.10 *Let $\mathcal{S} = (P, B, I)$ be an EGQ of order (s, t) with elation point p and where $s, t \neq 1$, and suppose q is a point of $P \setminus p^\perp$. Suppose $L_0, L_1, \ldots, L_t$ are the lines through p, and suppose M_i are lines such that $L_i \sim M_i I q$. Let H_i be the subgroup of the elation group G which fixes M_i for all i, and put $\mathcal{J} = \{H_0, H_1, \ldots, H_t\}$. Then H_i is a group of symmetries about the line L_i if and only if $H_i \trianglelefteq G$ (and hence $\mathcal{S}^{(p)}$ is a TGQ if and only if $H_i \trianglelefteq G$ for each i), if and only if L_i is a regular line. The line L_i is regular if and only if $H_i H_j = H_j H_i$ for all $H_j \in \mathcal{J}$.*

∎

We can now give an alternative, more geometrical, definition of translation generalized quadrangles without the use of symmetries, abelian groups or Galois geometries, as follows:

"Suppose $\mathcal{S} = (P, B, I)$ is a generalized quadrangle of order (s, t), $s \neq 1 \neq t$, with a point p so that there is a group of whorls about p which acts transitively on $P \setminus p^\perp$. Then $\mathcal{S}$ is a TGQ with translation point p if and only if every line through p is a regular line."

Remark 5.2.11 This result was proved by J. A. Thas for $s = t$ in [165], see also [139, 8.3.3].

5.3 A Characterization of the (Thick) Classical Generalized Quadrangles Arising from a Quadric

If p is a point of a GQ $\mathcal{S} = (P, B, I)$ of order (s, t), $s, t \neq 1$, such that there is a group of whorls about p which acts transitively on $P \setminus p^\perp$, then we call p a *center of transitivity*. If p is a point of $\mathcal{S}$ for which there is a group of automorphisms of $\mathcal{S}$ which fixes p and which acts transitively on the points of $P \setminus p^\perp$, then p is a *point of transitivity*. Dually, one defines *axis of transitivity* and *line of transitivity*, respectively.

Theorem 5.3.1 *Suppose $\mathcal{S}$ is a GQ of order (s, t), $s, t \neq 1$. Then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(4, s)$ or $\mathcal{Q}(5, s)$ if and only if $\mathcal{S}$ contains a center of transitivity p, a collineation θ of $\mathcal{S}$ for which $p^\theta \not\sim p$, and a regular line.*

Proof. If $\mathcal{S}$ is isomorphic to one of $\mathcal{Q}(4, s)$, $\mathcal{Q}(5, s)$, then any point of $\mathcal{S}$ is a translation point, and hence every point is a center of transitivity. Also, every line is regular.
Conversely, suppose that $\mathcal{S}$ contains a center of transitivity p with a group G_p of whorls about p acting transitively on $P \setminus p^\perp$, a collineation θ of $\mathcal{S}$ for which $p^\theta \not\sim p$, and a regular line. Then it is very easy to see that every point of $\mathcal{S}$ is a center of transitivity, and that $Aut(\mathcal{S})$ acts transitively on the lines of $\mathcal{S}$. Whence every line of $\mathcal{S}$ is a regular line, and by Theorems 1.2.2 and 1.2.1, we have one of the following possibilities:

1. $\mathcal{S}$ is isomorphic to $\mathcal{Q}(4, s)$;
2. $t > s$.

Now consider the last case. Since every line is regular and every point is a center of transitivity, we have by Theorem 5.2.8 that every line of $\mathcal{S}$ is an axis of symmetry. An axis of symmetry L is Moufang for any panel on L, and hence $\mathcal{S}$ is half Moufang. Now, by Theorem 1.4.2 and by Theorem 1.4.1, $\mathcal{S}$ must be classical or dual classical. Since the only two classical or dual classical quadrangles with regular lines are $\mathcal{Q}(4, s)$ and $\mathcal{Q}(5, s)$, the proof is complete.■

Remark 5.3.2 In Chapter 9, We will classify the GQ's satisfying the first and the last 'condition' of Theorem 5.3.1, and for which there is a collineation θ such that $p \neq p^\theta \sim p$, and where the regular line must be an element of $(pp^\theta)^\perp \setminus \{pp^\theta\}$.

5.4 Strong Elation Generalized Quadrangles

We now introduce the notion of 'strong elation generalized quadrangle'. A *strong elation generalized quadrangle (SEGQ)* is a generalized quadrangle $\mathcal{S}$ for which each point is an elation point.

Theorem 5.4.1 *If an SEGQ $\mathcal{S}$ of order (s,t), $s,t \neq 1$, has a regular line, then we have one of the following cases.*

1. $\mathcal{S} \cong \mathcal{Q}(4,s)$.
2. $\mathcal{S} \cong \mathcal{Q}(5,s)$.

Proof. Immediately from Theorem 5.3.1. ∎

It is easy to prove that the automorphism group of a strong elation generalized quadrangle acts transitively on the pairs of non-concurrent lines, see also Chapter 13. Therefore it is by Theorem 5.4.1 sufficient to demand that an SEGQ has a regular pair of lines to conclude that it arises from a quadric. A similar remark could be made about Theorem 5.3.1, and hence we can restate Theorem 5.3.1 as follows.

Theorem 5.4.2 *Suppose $\mathcal{S}$ is a GQ of order (s,t), $s,t \neq 1$. Then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(4,s)$ or $\mathcal{Q}(5,s)$ if and only if $\mathcal{S}$ contains a center of transitivity p, a collineation θ of $\mathcal{S}$ for which $p^\theta \not\sim p$, and a regular pair of lines.*

∎

It seems natural to conjecture that every strong elation generalized quadrangle is classical or dual classical. Using the classification of finite simple groups, F. Buekenhout and H. Van Maldeghem [31] have showed that any finite *distance-transitive generalized polygon* is classical (or dual classical), see also Appendix C. As a corollary for finite generalized quadrangles, there follows that every SEGQ indeed is classical or dual classical. We are however interested in a proof which does not use the aforementioned classification.

Note. The problem of classifying the SEGQ's without the classification of finite simple groups was already suggested by H. Van Maldeghem in [230].

5.5 Improvements of Results of X. Chen and D. Frohardt, and of D. Hachenberger

In Theorem 5.1.1 and Theorem 5.1.2, the group theoretical descriptions have one major restriction; they demand that the groups of symmetries about the

lines must be *completely contained* in the elation group. By our preceding theorems and observations, we can now omit these "contained in"-conditions completely; moreover, we do not even ask that the lines are axes of symmetry, but only that they are regular!

Theorem 5.5.1 *Let $(\mathcal{S}^{(x)}, G)$ be an EGQ of order (s,t), $s,t \neq 1$. If there are two distinct regular lines through the point x, then s and t are powers of the same prime number p, G is an elementary abelian p-group, and hence $\mathcal{S}^{(x)}$ is a TGQ with translation group G.*

Proof. Suppose L and M are regular lines through the point x. Then by Theorem 5.2.9, we know that L and M are axes of symmetry of which the full groups of symmetries of size s are completely contained in G. Hence, by Theorem 5.1.1, the proof follows immediately. ■

Theorem 5.5.2 *Let $(\mathcal{S}^{(x)}, G)$ be an EGQ of order (s,t), $s,t \neq 1$. If there is a regular line through the point x, and G is a group of even order, then s and t are powers of 2, G is an elementary abelian 2-group, and hence $\mathcal{S}^{(x)}$ is a TGQ with translation group G.*

Proof. Suppose L is a regular line through the point x. Then by Theorem 5.2.9, L is an axis of symmetry of which the full group of symmetries of size s is contained in G. Hence, by Theorem 5.1.2, the proof is complete. ■

Remark 5.5.3 By Section 1.7 there follows that if a GQ $(\mathcal{S}^{(x)}, G)$ satisfies the conditions of Theorem 5.5.1 or Theorem 5.5.2, that G is the complete set of elations about x.

Chapter 6

Generalized Quadrangles with Some Concurrent Axes of Symmetry

In FGQ it was proved that a (thick) GQ $\mathcal{S}$ is a TGQ $(\mathcal{S}^{(p)}, G)$ with translation point p if and only if every line through p is an axis of symmetry, that is, if p is a translation point, and then G is precisely the group generated by all symmetries about the lines incident with p, see Theorem 1.7.3. In our Master Thesis [203] we noted that that proof was also valid for all lines through p minus one. This observation is one of the main motivations of the present chapter:

What is — in general — the minimal number of distinct axes of symmetry through a point p of a GQ $\mathcal{S}$ forcing $\mathcal{S}^{(p)}$ to be a TGQ?

There are only such results known for (thick) GQ's of order s, and in that case, three axes of symmetry appears to be sufficient. In an appendix, we will give a short new geometrical proof of this theorem without using the coordinatization method for GQ's — the only known proof of this theorem is contained in

Chapter 11 of FGQ, is rather long and technical and uses this method. For thick GQ's of order (s,t) with $s \neq t$, the problem is a lot harder; we will show that $t-s+3$ axes of symmetries are sufficient in the general case.
In order to study the generalized quadrangles which have some distinct axes of symmetry through some point p, we will introduce Property (T) as follows. An ordered flag (L,p) satisfies Property (T) with respect to L_1, L_2, L_3, where L_1, L_2, L_3 are three distinct lines incident with p and distinct from L, if the following condition is satisfied: if (i,j,k) is a permutation of $(1,2,3)$, if $M \sim L$ and $M \not\mid p$, and if $N \sim L_i$ and $N \not\mid p$ with $M \not\sim N$, then the triads $\{M, N, L_j\}$ and $\{M, N, L_k\}$ are not both centric. TGQ's which satisfy Property (T) for some ordered flag always have order (s, s^2) for some s, and every TGQ of order (s,t) which has a subGQ of order s through the translation point satisfies Property (T) for some ordered flag(s) containing the translation point. Property (T) is closely related to Property (G), and seems to be more general in the case of the translation generalized quadrangles. Moreover, every known translation generalized quadrangle or its translation dual has Property (T). Suppose that the GQ $\mathcal{S}$ satisfies Property (T) for the ordered flag (L,p) w.r.t. the distinct lines L_1, L_2, L_3, all incident with p. Moreover, suppose that L, L_1, L_2, L_3 are axes of symmetry. Then we will show that $\mathcal{S}^{(p)}$ is a TGQ and that the translation group G is generated by the symmetries about L, L_1, L_2, L_3.
We will also study the following related problem:

Given a general TGQ $\mathcal{S}^{(p)}$, what is the minimal number of lines through p such that the translation group is generated by the symmetries about these lines?

We will show that there is a connection between the minimal number of lines through a translation point of a TGQ such that the translation group is generated by the symmetries about these lines, and the kernel of the TGQ; in particular, if $(\mathcal{S}^{(p)}, G)$ is a TGQ of order (s,t), $1 \neq s \neq t \neq 1$, with $(s,t) = (q^{na}, q^{n(a+1)})$, where a is odd and where $\mathbf{GF}(q)$ is the kernel of the TGQ, and if $k+3$ is the minimum number of distinct lines through p such that G is generated by the symmetries about these lines, then we will show that $k \leq n$.

We will also introduce Property (T$'$) as follows. An ordered flag (L,p) satisfies Property (T$'$) with respect to L_1, L_2, L_3, where L_1, L_2, L_3 are distinct lines incident with p and distinct from L, if the following condition holds: if $M \sim L$ and $M \not\mid p$, and if q and q' are distinct arbitrary points on M which are not incident with L, then there is a permutation (i,j,k) of $(1,2,3)$ such that there are lines M_i, M_j, M_k, with $M_r \sim L_r$ and $r \in \{i,j,k\}$, for which $M_j \in \{M_i, M_k, L_j\}^{\perp}$, and such that qIM_i and $q'IM_k$. Another goal of this

chapter is to state elementary combinatorial and group theoretical conditions for a GQ $\mathcal{S}$ such that $\mathcal{S}$ arises from a flock. We will show that a combination of Property (T′) and Property (T) leads to Property (G) for TGQ's $\mathcal{S}$, and hence that it is possible to prove that such a TGQ $\mathcal{S}$ is related to a flock by Theorem 1.10.5. A classification result is eventually obtained.
Many other results will be proved, including a new divisibility condition for GQ's which have a point incident with at least three axes of symmetry, see Section 6.3.

The results up to Section 6.7 are taken from *K. Thas, On generalized quadrangles with some axes of symmetry*[1] [211], which will appear in *Bulletin of the Belgian Mathematical Society — Simon Stevin.*

REMARK

The results of the chapter are also motivated by the following (difficult) problem:

Suppose G is a group which is generated by elations about the same point x of a thick GQ $\mathcal{S}$. When is G a group of elations?

Clearly, this is a very important problem in, e.g., the theory of EGQ's and TGQ's (think of, for instance, Theorem 1.7.3). We will provide several 'structure theorems' where sufficient conditions for such groups will be stated in order to be elation groups.

6.1 Property (T)

We start with defining *Property (T)*.
Suppose $\mathcal{S}$ is a GQ of order (s,t), $s,t \neq 1$, and let p be a point of the GQ.

[1] Which is partially based on [203].

Property (T). An ordered flag (L,p) satisfies *Property (T) with respect to* L_1,L_2,L_3, where L_1,L_2,L_3 are three distinct lines incident with p and distinct from L, if the following condition holds:
if (i,j,k) *is a permutation of* $(1,2,3)$, *if* $M \sim L$ *and* $M \not\mid p$, *and if* $N \sim L_i$ *and* $N \not\mid p$ *with* $M \not\sim N$, *then the triads* $\{M,N,L_j\}$ *and* $\{M,N,L_k\}$ *are not both centric.*
If the ordered flag (L,p) satisfies Property (T) with respect to L_1,L_2,L_3, then we also say that $\mathcal{S}$ *satisfies Property (T) for the ordered flag* (L,p) *w.r.t.* L_1,L_2,L_3.

Recall

Theorem 6.1.1 (FGQ, 8.1.3) *Let* $\mathcal{S}$ *be a GQ of order* (s,t), $s \neq 1 \neq t$, *and suppose that* L *and* M *are distinct concurrent lines. If* α *is a symmetry about* L *and if* β *is a symmetry about* M, *then* $\alpha\beta = \beta\alpha$.

The following theorem will appear to be very useful for the sequel.

Theorem 6.1.2 *Suppose* $\mathcal{S}$ *is a thick GQ of order* (s,t), *and let* p *be a point of* $\mathcal{S}$ *incident with three distinct axes of symmetry. If* G *is the group generated by the symmetries about these lines, then* G *is a group of elations with center* p *of order* s^3.

Proof. Suppose that L_1,L_2,L_3 are three distinct axes of symmetry incident with p, and let G_i be the full group of symmetries about L_i, $i=1,2,3$. With α_1,α_2 and β contained in respectively G_1, G_2 and G_3 (and none of these collineations trivial), suppose that the following holds:

$$\alpha_1\alpha_2 = \beta.$$

If q is not collinear with p, then (q,q^β,q^{α_1}) are the points of a triangle, contradiction. This observation shows us that $|G| = s^3$ (cf. Theorem 6.1.1), and that each element of G is an elation (because no two elements of G have the same action on a point of $P\setminus p^\perp$), and so G is a group of elations about p. ■

In the following, Theorem 6.1.1 will often be used without further notice.
We now obtain

Theorem 6.1.3 *Suppose* $\mathcal{S} = (P,B,I)$ *is a thick GQ of order* (s,t), *and let* $p \in P$ *be a point incident with four distinct axes of symmetry* L_1,L_2,L_3 *and* L_4. *Moreover, suppose that Property (T) holds for the ordered flag* (L_4,p) *w.r.t.*

the lines L_1, L_2, L_3. Then $t = s^2$, every line through p is an axis of symmetry, and $(\mathcal{S}^{(p)}, G)$ is a TGQ, where G is the group generated by all symmetries about L_1, L_2, L_3, L_4.

Proof. Suppose that G_i is the full group of symmetries about the line L_i, $i \in \{1,2,3,4\}$, and consider the group G generated by all symmetries about the lines L_1, L_2, L_3 and L_4. We define G' as the group generated by the symmetries about the lines L_j with $j = 1,2,3$. A general element of G' always can be written in the form $\phi_i\phi_j\phi_k$, with $(i,j,k) = (1,2,3)$ and ϕ_i a symmetry about the line L_i, see Theorem 6.1.1. Also, because of Theorem 6.1.2, G' is an elation group of elations with center p and of size s^3.
Suppose, for a nontrivial symmetry $g \in G_4$ and an elation $\alpha = \alpha_1\alpha_2\alpha_3$ of G', with $\alpha_i \in G_i$, that g and α have the same action on a point q of $P \setminus p^\perp$. Then $q^g = q^\alpha =: q'$, and thus we have that $q' \in [q, L_4]$ (note that $q' \neq q$). It is clear that none of the symmetries α_i are trivial ($i = 1,2,3$), otherwise we would have a composition of at most three symmetries about distinct axes of symmetry with a common intersection point which acts trivially on a point of $P \setminus p^\perp$, a contradiction by Theorem 6.1.2. If we now consider the triads of lines $\{[q, L_4], [q^{\alpha_1}, L_2], L_3\}$ and $\{[q, L_4], [q^{\alpha_1}, L_2], L_1\}$, then the assumption we just made implies that they are both centric, in contradiction with Property (T). Thus we have proved that an element of G' and an element of G_4 can never have the same action on a point of $P \setminus p^\perp$. Therefore, we have that $|G| = s^4$ and that every element of G is an elation with center p. Since $|P \setminus p^\perp| = s^2t$ and since $t \leq s^2$, it follows that $t = s^2$. Also, G acts regularly on the points of $P \setminus p^\perp$, and hence that $(\mathcal{S}^{(p)}, G)$ is an EGQ with elation point p. Since the lines L_i are regular, the proof is complete by Theorem 5.5.1. ∎

Recall, for the following theorem, that if $\mathcal{S}$ is a GQ of order (s, s^2), $s > 1$, and if $\mathcal{S}'$ is a subquadrangle of $\mathcal{S}$ of order s, then every line of $\mathcal{S}$ is either contained in $\mathcal{S}'$, or intersects $\mathcal{S}'$ in exactly one point.

Theorem 6.1.4 *Suppose $\mathcal{S}$ is a thick GQ of order (s,t), and let p be a point incident with four distinct axes of symmetry L_1, L_2, L_3, L three of which are contained in a proper subquadrangle $\mathcal{S}'$ of $\mathcal{S}$ of order (s,t'), but not the fourth, say L. Then (L,p) satisfies Property (T) w.r.t. L_1, L_2, L_3. Thus $t = s^2$ and $(\mathcal{S}^{(p)}, G)$ is a TGQ, with G the group generated by all symmetries about these four lines. Moreover, $\mathcal{S}'$ also is a TGQ (with translation point p).*

Proof. It is clear that an axis of symmetry of a GQ of order (s,t) also is an axis of symmetry of a proper subGQ of order (s,t') which contains this line, $1 \leq t' < t$ and $s \neq 1$. An axis of symmetry of a GQ is a regular line, and since

$t' \neq 1$, Theorem 1.6.2 implies that $s \leq t'$ and hence $t' = s$. Hence $t = s^2$. By Theorem 1.7.5, $\mathcal{S}'$ is a TGQ.
Let L_1, L_2, L_3 be three distinct axes of symmetry of $\mathcal{S}$ through the point p which are contained in $\mathcal{S}'$, and let L_4 be an axis of symmetry of $\mathcal{S}$ through p which is not contained in $\mathcal{S}'$. Suppose $L \sim L_4 \neq L$ is arbitrary, $L \not I p$, and let L' be an arbitrary line of $L_i^\perp \setminus \{L_i\}$ not through p for a fixed i in $\{1,2,3\}$. Suppose that the triads of lines $\{L, L', L_j\}$ and $\{L, L', L_k\}$ are both centric, with $\{j,k\} = \{1,2,3\} \setminus \{i\}$. First suppose that $L' \in \mathcal{S}'$. If M is a center of $\{L, L', L_j\}$, then $|M \cap \mathcal{S}'| \geq 2$, and hence M is a line of $\mathcal{S}'$. It follows that $|L \cap \mathcal{S}'| \geq 2$, and hence L also is a line of $\mathcal{S}'$. This immediately leads to the fact that L_4 is a line of $\mathcal{S}'$, contradiction. The same holds for $\{L, L', L_k\}$.
Next, suppose that $L' \notin \mathcal{S}'$, that $M \in \{L, L', L_j\}^\perp$ and $N \in \{L, L', L_k\}^\perp$. The line L intersects $\mathcal{S}'$ in one point q. Consider a symmetry θ about L_4 which maps the point $q' = L \cap M$ onto q. Then the line M^θ is contained in $\mathcal{S}'$, and hence also $(L')^\theta$. By the first part of this proof, we now obtain a contradiction. Hence Property (T) is satisfied for the ordered flag (L_4, p) w.r.t. the lines L_1, L_2, L_3, and the theorem follows from Theorem 6.1.3. ∎

6.2 Property (T′)

Suppose $\mathcal{S}$ is a GQ of order (s,t), $s, t \neq 1$, and let p be a point of the GQ.

Property (T′). An ordered flag (L,p) satisfies *Property (T′) with respect to the lines* L_1, L_2, L_3, where L_1, L_2, L_3 are different lines incident with p and distinct from L, with the following condition:
if $M \sim L$ and $M \not I p$, and if q and q' are distinct arbitrary points on M which are not incident with L, then there is a permutation (i,j,k) of $(1,2,3)$, such that there are lines M_i, M_j, M_k, with $M_r \sim L_r$ and $r \in \{i,j,k\}$, for which $M_j \in \{M_i, M_k, L_j\}^\perp$, and such that qIM_i and $q'IM_k$.
If the ordered flag (L,p) satisfies Property (T′) w.r.t. the lines L_1, L_2, L_3, then we also say that $\mathcal{S}$ satisfies *Property (T′) for the ordered (L,p) w.r.t. L_1, L_2, L_3.*

We now have

Theorem 6.2.1 *Suppose that $\mathcal{S} = (P, B, I)$ is a thick GQ of order (s,t), and let p be a point of P which is incident with four distinct axes of symmetry L, L_1, L_2, L_3 such that Property (T′) is satisfied for (L,p) w.r.t. the lines L_1, L_2, L_3. If G_L, G_1, G_2, G_3 are the full groups of symmetries about L, L_1, L_2, L_3, respectively, then $G_L \subseteq \langle G_1, G_2, G_3 \rangle$.*

Proof. Put $G = \langle G_1, G_2, G_3 \rangle$. Suppose $M \sim L$ is arbitrary with MIp, and suppose q and q' are distinct arbitrary points on M which are not incident with L. Since Property (T′) is satisfied for the ordered flag (L, p) w.r.t. the lines L_1, L_2, L_3, there is a permutation (i, j, k) of $(1, 2, 3)$, such that there are lines M_i, M_j, M_k for which $M \in \{M_i, M_k, L\}^\perp$, with qIM_i and $q'IM_k$, and $M_j \in \{M_i, M_k, L_j\}^\perp$, and such that $M_r \sim L_r$ with $r \in \{i, j, k\}$. For convenience, put $(i, j, k) = (1, 2, 3)$.
By the considerations above, the following collineations exist:

1. θ_1 is the symmetry about L_1 which sends $q = M \cap M_1$ to $M_1 \cap M_2$;
2. θ_2 is the symmetry about L_2 which sends $M_1 \cap M_2$ to $M_2 \cap M_3$;
3. θ_3 is the symmetry about L_3 which maps $M_2 \cap M_3$ to $q' = M \cap M_3$.

Define the following collineation of $\mathcal{S}$:

$$\theta := \theta_1\theta_2\theta_3.$$

Then θ is an automorphism of $\mathcal{S}$ which is contained in G, and hence θ is an elation about p by Theorem 6.1.2. Also, θ fixes M and maps q onto q'. Now by Theorem 5.2.8, θ is a symmetry about L. It follows now easily that $G_L \subseteq G$ since q and q' were arbitrary. ∎

Remark 6.2.2 By Chapter 5 (see, e.g., Theorem 5.2.8) it is sufficient to ask that L be a regular line to conclude it is an axis of symmetry.

Note. We emphasize at this point that Property (T) and Property (T′) are purely combinatorial properties which are defined without the use of collineations.

6.3 Divisibility Conditions for Generalized Quadrangles with Symmetry

Recall

Theorem 6.3.1 (FGQ, 8.1.2) *If a thick GQ $\mathcal{S}$ has a non-identical symmetry θ about some line, then $st(s+1) \equiv 0 \mod s+t$.*

Then observe

Theorem 6.3.2 *Suppose $\mathcal{S}$ is a GQ of order (s,t), $s \neq 1 \neq t$, and let L, M and N be three different axes of symmetry incident with the same point p. Then $s|t$ and $\frac{t}{s}+1|(s+1)t$.*

Proof. Define G' as the group generated by the symmetries about the lines L, M and N. By Theorem 6.1.2 we have that G' is a group of elations with center p, and the size of G' is s^3. If we consider the permutation group $(P \setminus p^\perp, G')$, then we see that $|G'|$ divides $|P \setminus p^\perp|$, or that $s^3|s^2t$. So t is divisible by s. By Theorem 6.3.1, the theorem now follows. ∎

There is a nice corollary:

Corollary 6.3.3 *Suppose $\mathcal{S} = (P, B, I)$ is a GQ of order (s,t), $s,t \neq 1$, and suppose that $L_0, L_1, \ldots, L_l$ are $l+1$ axes of symmetry incident with the point p. Suppose that the group G which is generated by the symmetries about $L_0, L_1, \ldots, L_l$ acts transitively on $P \setminus p^\perp$. Then s and t have the same parity.*

Proof. If s^2t is odd, then s and t are both odd, so suppose that s^2t is even. Since G acts transitively on $P \setminus p^\perp$, we have that s^2t is a divisor of $|G|$. This means that $|G|$ is even. Now suppose that G is generated by the symmetries about $L_0, L_1, \ldots, L_l$, and denote by $G_0, G_1, \ldots, G_l$ the full groups of symmetries about respectively $L_0, L_1, \ldots, L_l$. (so $G = \langle G_0, G_1, \ldots, G_l \rangle$). By Theorem 1.9.1, $G_iG_j = G_jG_i$ for $i \neq j$, $0 \leq i,j \leq l$. If H and H' are groups, then HH' is a group if and only if $HH' = H'H$. Inductively, if $\{i_0, i_1, \ldots, i_k\} \subseteq \{0, 1, \ldots, l\}$ for a certain k, then $\langle G_{i_0}, G_{i_1}, \ldots, G_{i_k} \rangle = G_{i_0}G_{i_1}\ldots G_{i_k}$. If H and H' are finite groups, then $|HH'| = \dfrac{|H| \times |H'|}{|H \cap H'|}$ and so by the previous property we obtain that

$$|G| = \frac{s^{(l+1)}}{\prod_{i=0}^{l-1} |[G_0G_1\ldots G_i] \cap G_{(i+1)}|}. \tag{6.1}$$

Since $|G|$ is even, we have that s necessarily is even (using Equality (6.1)). As we assumed that G acts transitively on $P \setminus p^\perp$, and as $t \geq s$ because $\mathcal{S}$ has regular lines, $|G| \geq |P \setminus p^\perp| = s^2t \geq s^3$, so $l \geq 2$. By Theorem 6.3.2 we conclude that $s|t$. Hence t is even, and the theorem follows. ∎

6.4 Property (Sub) and Property (T)

We start this section by introducing *Property (Sub)*.

Property (Sub). Suppose $\mathcal{S}$ is a thick GQ of order (s,t), and suppose M_1, M_2, M_3, M_4 are four distinct lines of $\mathcal{S}$, incident with the same point. Then these four lines satisfy *Property (Sub)* if $\mathcal{S}$ does not have a subGQ of order s containing these four lines.

Observe

Theorem 6.4.1 *Suppose $\mathcal{S}$ is a GQ of order (s,t), $s \neq 1 \neq t$, and let M_1, M_2, M_3 and M_4 be four distinct axes of symmetry through the point p of $\mathcal{S}$ such that there is a line $M \in \{M_1, M_2, \ldots, M_4\} = \mathcal{M}$ for which Property (T) is satisfied for the ordered flag (M,p) w.r.t. the lines of $\mathcal{M} \setminus M$. Then these four lines also satisfy Property (Sub).*

Proof. Suppose that the lines M_1, M_2, M_3 and M_4 are contained in a proper subGQ $\mathcal{S}'$ of order s and suppose that $M = M_4$ is such that (M,p) satisfies Property (T) w.r.t. the lines M_1, M_2, M_3. Consider lines L with LIM_4 and L' with $L'IM_1$, both contained in $\mathcal{S}'$ and both not incident with p. The lines M_i, $i = 1,2,3,4$, are axes of symmetry in the quadrangle $\mathcal{S}'$, so they are regular. Now, each triad of lines in $\mathcal{S}'$ which contains one of those lines, is centric (it is easily seen that a pair of lines $\{U,V\}$ in a GQ of order s, $s > 1$, is regular if and only if each triad containing U and V is centric, see [139, 1.3.6 (ii)]). This leads to a contradiction. ∎

Of course, Theorem 6.4.1 could be 'restated' more generally as follows: *Let $\mathcal{S}$ be a GQ of order (s,t), $s \neq 1 \neq t$, and let M_1, M_2, M_3 and M_4 be four distinct lines through the point p of $\mathcal{S}$ so that M_1, M_2 and M_3 are axes of symmetry, and so that that Property (T) is satisfied for the ordered flag (L_4,p) w.r.t. M_1, M_2, M_3. Then these four lines satisfy Property (Sub).*

Theorem 6.4.2 *Suppose that $(\mathcal{S}^{(p)}, G) = (P, B, I)$ is a TGQ of order (s, s^2), $s > 1$, and suppose that $\mathcal{S}$ has a subquadrangle $\mathcal{S}'$ of order s which contains the point p. Then there exist four lines incident with p, such that G is the group generated by the symmetries about these lines.*

Proof. The quadrangle $\mathcal{S}'$ is a TGQ of order s with translation point p, and so there exist three lines in $\mathcal{S}'$ incident with p, such that the translation group G' of $\mathcal{S}'$ is generated by all symmetries about these three lines. Consider a line

L of $\mathcal{S}$ incident with p and not contained in $\mathcal{S}'$. This line intersects $\mathcal{S}'$ only in p. Take an arbitrary line M of $\mathcal{S}$ intersecting L and not incident with p. This line is not contained in $\mathcal{S}'$ as L is not a line of $\mathcal{S}$, and thus it intersects $\mathcal{S}'$ in just one point. Since L is an axis of symmetry, the group G_L of symmetries about L acts transitively on the points of $M \setminus \{L \cap M\}$; it follows now that every G'-orbit in $P \setminus p^\perp$ intersects M in exactly one point. Also, there follows that $G'' = \langle G', G_L \rangle$ has size s^4. Since G'' clearly is a group of elations with center p, the theorem is proved. ∎

Note. It is clear that four is the minimum number of lines such that the translation group of a TGQ of order (s,t), $1 > s \neq t$, is generated by the symmetries about these lines, see e.g. the proof of Theorem 6.1.3.

Suppose that $\mathcal{S}$ is a thick TGQ of order (s,t) which satisfies property (T) for some ordered flag (L,p) w.r.t. the lines L_1, L_2, L_3. Then by Theorem 6.1.3 we have that $t = s^2$. Now suppose that $\mathfrak{C}$ is the class of all thick TGQ's $\mathcal{S}^{(p)}$, where p denotes the translation point, for which the following condition hold:

(C) *If L_1, L_2, L_3, L_4 are different lines through p for which Property (Sub) is satisfied, then there is a line $L \in \{L_1, L_2, L_3, L_4\} = \mathcal{L}$ such that $\mathcal{S}^{(p)}$ satisfies property (T) for the ordered flag (L,p) w.r.t. the lines of $\mathcal{L} \setminus \{L\}$.*

Note that by Theorem 6.4.1, $\mathfrak{C}$ is exactly the class of TGQ's for which Property (T) and Property (Sub) are 'equivalent' (with respect to certain lines incident with p). Of course, each TGQ of order s is an element of $\mathfrak{C}$. Suppose that $\mathcal{S}^{(p)} \in \mathfrak{C}$ is of order (s,t), $s \neq t$, and suppose that $\mathcal{S}^{(p)}$ does *not* satisfy Property (T) for some ordered flag flag (p,L) w.r.t. some three distinct lines through p and different from L. It is easy to observe that such a 4-tuple of lines always exists for $\mathcal{S}^{(p)}$. Then Property (Sub) does not hold, and so there is a subGQ of order s containing those four lines. It follows that $t = s^2$ by Theorem 1.6.2, but moreover, we also have the following essential observation for $\mathcal{S}^{(p)}$: for each four distinct lines U, V, W, Z through p, we have that

(i) either they are contained in a subGQ of $\mathcal{S}^{(p)}$ of order s, or

(ii) for each $O \in \{U, V, W, Z\}$, (O,p) satisfies Property (T) w.r.t. $\{U, V, W, Z\} \setminus \{O\}$.

By Theorem 1.11.5, there now readily follows that $\mathcal{S}^{(p)} \cong T_3(\mathcal{O})$, where $\mathcal{O}$ is an ovoid of $\mathbf{PG}(3,s)$, see the beginning of Section 6.5 for more details.
We hence have the following result.

Theorem 6.4.3 *Let $\mathcal{S}^{(p)}$ be a (thick) element of order (s,t) of $\mathfrak{C}$. Then we have one of the following possibilities:*

(*i*) *$\mathcal{S}^{(p)}$ is a TGQ of order s without any further restrictions;*

(*ii*) *$\mathcal{S}^{(p)} \cong T_3(\mathcal{O})$, where $\mathcal{O}$ is an ovoid in $\mathbf{PG}(3,s)$ (so if s is odd, $\mathcal{S}^{(p)} \cong \mathcal{Q}(5,s)$).*

In particular, there always exist four lines through p which 'determine' the translation group, in the sense that the translation group is generated by the symmetries about those lines.

■

In view of the fact that we want to study the TGQ's which satisfy Property (T) for a certain ordered flag (through the translation point) w.r.t. to certain lines, the latter result seems not be agreeable in that sense, that it does not even 'include' general TGQ's $\mathcal{S} = T(\mathcal{O})$ with $\mathcal{O}$ good at some element. To that end, we will obtain a more general classification result in the next section. A combination of Property (T) and Property (T′) appears to be the right choice.

6.5 Property (T), Property (T′) and Property (G)

We now situate Property (T) in the theory of TGQ's, as follows.

Theorem 6.5.1 *Let $\mathcal{S} = T(n,m,q) = T(\mathcal{O})$ be a TGQ of order (q^n, q^m) for which there exist four distinct elements π_i, $i = 1,2,3,4$, of $\mathcal{O} = \mathcal{O}(n,m,q)$ which generate a $(4n-1)$-space. Then $\mathcal{S}$ is a GQ of order (q^n, q^{2n}) and $\mathcal{S}$ satisfies Property (T) for every ordered flag $(\pi_r, (\infty))$, $r \in \{1,2,3,4\}$, w.r.t. π_i, π_j, π_k, where $\{i,j,k\} = \{1,2,3,4\} \setminus \{r\}$.*

Proof. As usual, we represent $\mathcal{S} = T(\mathcal{O})$ in $\mathbf{PG}(2n+m,q)$, where $\mathcal{O}$ generates a $\mathbf{PG}(2n+m-1,q) \subseteq \mathbf{PG}(2n+m,q)$. Since $4n-1 \leq 2n+m-1$, $\mathcal{S}$ is a GQ of order (q^n, q^{2n}). Now, let $L_k \sim \pi_k$, $k = 1,2,3,4$, be lines of $\mathcal{S}$ such that $\mathcal{V}_3 = \{L_1, L_2, \pi_3\}$ and $\mathcal{V}_4 = \{L_1, L_2, \pi_4\}$ are centric triads with $L_3 \in \mathcal{V}_3^\perp$ and $L_4 \in \mathcal{V}_4^\perp$. Note that we assume that $L_1 \not\sim L_2$. It follows that the n-spaces L_3 and L_4 intersect the $(2n+1)$-dimensional space generated by the n-spaces L_1 and L_2 in a space of minimum dimension 1, thus, if π is the space generated by the n-spaces L_k, $1 \leq k \leq 4$, then π has dimension at most $4n-1$. If we intersect π with $\mathbf{PG}(4n-1,q)$, then we obtain a space with dimension at

most $4n-2$ which contains the π_k's, a contradiction. Hence, for every ordered flag $(\pi_r,(\infty))$, $r=1,2,3,4$, Property (T) is satisfied w.r.t. π_i,π_j,π_k, where $\{i,j,k\}=\{1,2,3,4\}\setminus\{r\}$. ■

The following theorem is a converse of Theorem 6.5.1.

Theorem 6.5.2 *Suppose $\mathcal{S}^{(\infty)}$ is the thick TGQ of order (q^n,q^{2n}) which corresponds to the generalized ovoid $\mathcal{O}$ of $\mathbf{PG}(4n-1,q)$. If $LI(\infty)$ is a line of $\mathcal{S}^{(\infty)}$ such that Property (T) is satisfied for the ordered flag $(L,(\infty))$ with respect to the lines L_1,L_2,L_3 through (∞) (where $|\{L,L_1,L_2,L_3\}|=4$), and if π, respectively π_i, is the element of $\mathcal{O}$ which corresponds to L, respectively L_i, $i=1,2,3$, then $\langle\pi,\pi_1,\pi_2,\pi_3\rangle=\mathbf{PG}(4n-1,q)$.*

Proof. Immediate from the proof of Theorem 6.1.3 and the interpretation in the projective model $T(n,2n,q)$. ■

Theorem 6.5.3 *A TGQ $\mathcal{S}^{(\infty)}=T(\mathcal{O})$ which is good at an element $\pi\in\mathcal{O}$ satisfies Property (T) for the ordered flag $(\pi,(\infty))$ w.r.t. any three distinct lines through (∞) which are different from π, and which generate a $(3n-1)$-space not containing π.*

Proof. If $\mathcal{O}$ is good in some element π, and if the three lines L_1,L_2,L_3 are as above, then π,L_1,L_2,L_3 generate a $\mathbf{PG}(4n-1,q)$. ■

Thus, if $\mathcal{S}=T(\mathcal{O})$ is a TGQ for which $\mathcal{O}$ has a good element, then there are always four lines such that the translation group is generated by the symmetries about these lines.

Corollary 6.5.4 *The $T_3(\mathcal{O})$ of Tits satisfies Property (T) for each ordered flag $(L,(\infty))$ w.r.t. any three distinct lines, all different from L, through (∞).*

Proof. $\mathcal{O}$ is good at every point by Theorem 1.11.5. ■

Lemma 6.5.5 *Suppose that $(\mathcal{S}^{(p)},G)$ is a thick TGQ of order (s,t), $t\geq 3$. Then $\mathcal{S}$ is of order s if and only if there are distinct lines L_1Ip,L_2Ip,L_3Ip such that for every other line LIp, with G_i the full group of symmetries about L_i and G_L the full group of symmetries about L, $i=1,2,3$, the group $\langle G_1,G_2,G_3,G_L\rangle$ has size s^3.*

Proof. It is clear that a TGQ $(\mathcal{S}^{(p)}, G)$ of order s, $s > 1$, has the desired property, since $|G| = s^3$ and since by Theorem 1.7.5, G is generated by the symmetries about three arbitrary distinct lines through p.
Let $(\mathcal{S}^{(p)}, G)$ be a TGQ, and suppose that the required conditions are satisfied. Suppose that G_i is the full group of symmetries about the line L_i, with $i \in \{1, 2, \ldots, t+1\}$ and $L_i Ip$. If $t = 3$, then for any thick TGQ of order (s, t) we have $s = 3$ and so $|G| = s^3$. Hence, let $t \geq 4$. Define the group H_j as $H_j = \langle G_1, G_2, G_3, \ldots, G_j \rangle$ with $j \in \{4, 5, \ldots, t+1\}$. Then $|H_4| = s^3$. Considering that a group generated by the symmetries about three concurrent axes of symmetry is a group of elations of order s^3 about their intersection point (by Theorem 6.1.2), we have that $H_4 = \langle G_1, G_2, G_3 \rangle$. But $H_5 = \langle H_4, G_5 \rangle$, so $H_5 = H_4$, and thus also $H_4 = H_5 = \ldots = H_{t+1}$. Since $H_{t+1} = G$, we have $|G| = s^3$, and hence $\mathcal{S}$ is of order s. ∎

The following theorem implies that Property (T′) is a characteristic property for TGQ's of order s.

Theorem 6.5.6 *Suppose $(\mathcal{S}^{(p)}, G)$ is a thick TGQ of order (s, t), $t \geq 3$. Then $\mathcal{S}$ is of order s if and only if there is a line LIp such that $\mathcal{S}$ satisfies Property (T′) for the ordered flag (L, p) w.r.t. every three distinct lines L_1, L_2, L_3 through p which are different from L.*

Proof. Immediately by Lemma 6.5.5 and Theorem 6.2.1. ∎

Now observe

Theorem 6.5.7 *Suppose that $\mathcal{S}^{(p)} = T(\mathcal{O})$ is a thick TGQ of order (s, t) with $s \neq 1 \neq t$, such that there is a line LIp so that for every three distinct lines L_1, L_2, L_3 through p and different from L, either Property (T) or Property (T′) is satisfied for the ordered flag (L, p) w.r.t. L_1, L_2, L_3, and suppose that there is at least one 3-tuple (M, N, U) of distinct lines incident with p, such that (L, p) satisfies Property (T) w.r.t. (M, N, U). Then $T(\mathcal{O})$ is good at its element L.*

Proof. Since there is at least one 3-tuple (M, N, U) as above such that (L, p) satisfies Property (T) w.r.t. (M, N, U), we have by Theorem 6.1.3 that $t = s^2$. By Theorem 6.5.2, the definitions of Property (T) and Property (T′), Lemma 6.5.5 and Theorem 6.5.6, there follows that the generalized ovoid $\mathcal{O}$ satisfies the condition that for every three distinct lines L_1, L_2, L_3 through p and different from L, the projective space generated by the four corresponding elements of $\mathcal{O}$ either is a $\mathbf{PG}(4n-1, q)$ or a $\mathbf{PG}(3n-1, q)$. Thus every $(3n-1)$-space which is generated by the element π of $\mathcal{O}$ which corresponds to L and two

other elements of $\mathcal{O}$, has the property that it either is disjoint with any other element of $\mathcal{O}$ or completely contains it. Suppose that we denote the elements of $\mathcal{O}$ by $\pi, \pi^1, \ldots, \pi^{q^{2n}}$, where π corresponds to L, and fix for instance π^i, i arbitrary. Then all the $(3n-1)$-spaces of $\mathbf{PG}(4n-1,q)$ which contain π, π^i and an element of $\mathcal{O} \setminus \{\pi, \pi^i\}$ intersect two by two in $\pi\pi^i$ and cover $\mathbf{PG}(4n-1,q)$. By the preceding remarks, every element of $\mathcal{O} \setminus \{\pi, \pi^i\}$ is completely contained in one of these $(3n-1)$-spaces, and is disjoint with any other of these $(3n-1)$-spaces. Since the number of these spaces is q^n+1, every of these $(3n-1)$-spaces contains exactly q^n+1 elements of $\mathcal{O}$. The theorem now follows since i was arbitrary. ∎

Note. It is also possible to prove Theorem 6.5.7 with the use of 8.7.2 of FGQ.

Theorem 6.5.8 *Suppose that $\mathcal{S}^{(p)} = T(\mathcal{O})$ is a thick TGQ of order (s,t) with $s \neq t$, such that there is a line LIp so that for every three distinct lines L_1, L_2, L_3 through p and different from L, either Property (T) or Property (T') is satisfied for the ordered flag (L,p) w.r.t. L_1, L_2, L_3. Then $t = s^2$ and the translation dual $\mathcal{S}^* = T(\mathcal{O}^*)$ of $\mathcal{S}^{(p)}$ satisfies Property (G) for the flag (p', L'), where (p', L') corresponds to (p, L).*

Proof. Immediate by Theorem 6.5.7 and Theorem 1.11.3. ∎

There is a very nice corollary of Theorem 1.10.5 and Theorem 6.5.7.

Corollary 6.5.9 *Suppose that $\mathcal{S}^{(p)} = T(\mathcal{O})$ is a thick TGQ of order (s,t) with $s \neq t$ and s odd, such that there is a line LIp so that for every three distinct lines L_1, L_2, L_3 through p and different from L, either Property (T) or Property (T') is satisfied for the ordered flag (L,p) w.r.t. L_1, L_2, L_3. Then $t = s^2$ and the translation dual $\mathcal{S}^* = T(\mathcal{O}^*)$ of $\mathcal{S}$ is the point-line dual of a flock GQ of order (s^2, s).*

Proof. By Theorem 6.5.8 we have that $t = s^2$, and the translation dual $\mathcal{S}^*$ of $\mathcal{S}$ satisfies Property (G) at its flag (p', L') which corresponds with (p, L). By Theorem 1.10.5, the proof is complete. ∎

6.5.1 Flocks, Property (T) and Property (T')

We recall the following.

Theorem 6.5.10 (J. A. Thas [177]) *Suppose $\mathcal{S}^{(p)} = T(\mathcal{O})$ is a TGQ of order (s, s^2), $s > 1$, for which $\mathcal{O}$ is good at its element π. Then $\mathcal{S}$ contains $s^3 + s^2$ subGQ's of order s which contain the line π. For s odd these subGQ's are isomorphic to the classical GQ $\mathcal{Q}(4, s)$.*

Theorem 6.5.11 (J. A. Thas [187]) *Let $\mathcal{S}^{(p)} = T(\mathcal{O})$ be a TGQ of order (s, s^2), s even, such that $\mathcal{O}$ is good in its element πIp. If $\mathcal{S}$ contains at least one subGQ of order s which is isomorphic to the GQ $\mathcal{Q}(4, s)$ and which contains the line π, then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5, s)$.*

The following recent and very interesting theorem classifies TGQ's arising from flocks in the odd case.

Theorem 6.5.12 (A. Blokhuis, M. Lavrauw and S. Ball [16]) *Let $T(\mathcal{O})$ be a TGQ of order (q^n, q^{2n}), q odd, where $\mathbf{GF}(q)$ is the kernel of the TGQ, and suppose $T(\mathcal{O})$ is the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$, with the additional condition that*

$$q \geq 4n^2 - 8n + 2.$$

Then $T(\mathcal{O})$ is isomorphic to the point-line dual of a Kantor flock GQ.

We are now able to classify the thick TGQ's for which there is a line LIp so that for every three distinct lines L_1, L_2, L_3 through p and different from L, either Property (T) or Property (T′) is satisfied for the ordered flag (L, p) w.r.t. L_1, L_2, L_3, as follows.

Theorem 6.5.13 *Suppose that $\mathcal{S}^{(p)}$ is a thick TGQ of order (s, t), $s \neq 1 \neq t$, such that there is a line LIp so that for every three distinct lines L_1, L_2, L_3 through p and different from L, either Property (T) or Property (T′) is satisfied for the ordered flag (L, p) w.r.t. L_1, L_2, L_3. Then we have the following classification.*

(*i*) *$s = t$ and $\mathcal{S}$ is a TGQ with no further restrictions.*

(*ii*) *$t = s^2$, s is an even prime power, and $\mathcal{O}$ is good at its element π which corresponds to L, where $\mathcal{S} = T(\mathcal{O})$. Also, $\mathcal{S}$ has precisely $s^3 + s^2$ subGQ's of order s which contain the line L, and if one of these subquadrangles is classical, i.e. isomorphic to the GQ $\mathcal{Q}(4, s)$, then $\mathcal{S}$ is classical, that is, isomorphic to the GQ $\mathcal{Q}(5, s)$.*

(*iii*) *$t = s^2$ and $s = q^n$, q odd, where* $\mathbf{GF}(q)$ *is the kernel of the TGQ $\mathcal{S}^{(p)}$, with $q \geq 4n^2 - 8n + 2$, and $\mathcal{S}$ is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ where $\mathcal{F}$ is a Kantor flock.*

(*iv*) *$t = s^2$ and $s = q^n$, q odd, where* $\mathbf{GF}(q)$ *is the kernel of the TGQ $\mathcal{S}^{(p)}$, with $q < 4n^2 - 8n + 2$, and $\mathcal{S}$ is the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ for some flock $\mathcal{F}$.*

Proof. If for every three distinct lines L_1, L_2, L_3 through p and different from L, Property (T′) is always satisfied for the ordered flag (L, p) w.r.t. L_1, L_2, L_3, then by Lemma 6.5.6, $\mathcal{S}$ is a TGQ of order s with no further restrictions. Suppose this is not the case. Then there is a 3-tuple (M, N, U) of distinct lines through p such that Property (T) is satisfied for the ordered flag (L, p) w.r.t. M, N, U ($L \notin \{M, N, U\}$). Hence by Theorem 6.1.3 there follows that $t = s^2$, and if $\mathcal{S} = T(\mathcal{O})$, then by Theorem 6.5.7, $\mathcal{O}$ is good in its element π which corresponds to L. Suppose that s is even. Then (ii) follows from Theorem 6.5.10 and Theorem 6.5.11. Next suppose that s is odd. By Theorem 1.10.5, $T^*(\mathcal{O})$ is the point-line dual of flock GQ $\mathcal{S}(\mathcal{F})$. Then (iii) and (iv) follow from Theorem 6.5.12. ∎

6.6 The General Problem

We start this section with the following result.

Theorem 6.6.1 (FGQ, 9.4.2) *Suppose that $L_0, L_1, \ldots, L_r$, $r \geq 1$, are $r + 1$ lines incident with a certain point p in the GQ $\mathcal{S}$ of order (s, t), $s \neq 1 \neq t$. Suppose that O is the set of points different from p, which are on the lines of $L_0, L_1, \ldots, L_r$, and denote $P \setminus p^\perp$ by Ω. Suppose that G is a group of elations with center p, and suppose G has the property that, if M is an arbitrary line which intersects O in one point m, then G acts transitively on the points of Ω lying on M. If $r > t/s$, then G acts transitively — and so also regularly — on the points of Ω.*

Corollary 6.6.2 (1) *Suppose that $(\mathcal{S}^{(p)}, G)$ is a TGQ of order (s, t), $s, t \neq 1$, and suppose $k > t/s$, $k \in \mathbb{N}$. Then the translation group G is generated by the symmetries about $k+1$ arbitrary lines through the translation point p.*

(2) *Let $\mathcal{S}$ be a thick GQ of order (s,t) with the property that there is a point p incident with at least $s+2$ distinct axes of symmetry, and suppose that the group G generated by all symmetries about $s+2$ axes of symmetry through p is a group of elations. Then $(\mathcal{S}^{(p)}, G)$ is a TGQ.*

(3) *Let $\mathcal{S} = (P, B, I)$ be a GQ of order (s,t), $s \neq 1 \neq t$, with the property that there is a point p incident with at least $s+2$ distinct axes of symmetry, and define G to be the group generated by all symmetries about those lines. Suppose that, for each point $q \in P \setminus \{p\}^{\perp}$, $|\{p,q\}^{\perp\perp}| = 2$. Then $(\mathcal{S}^{(p)}, G)$ is a TGQ.*

(4) *Suppose $\mathcal{S}$ is a GQ of order (s,t), $t > s^2/2$ and $s \neq 1$, with x a point incident with $r+1$ axes of symmetry, $r \geq s+1$. If G is the group generated by all symmetries about these $r+1$ lines, then $(\mathcal{S}^{(x)}, G)$ is a TGQ.*

Proof. The first assertion is immediate. Consider Case (2). From the inequality of Higman follows that $t/s \leq s$, hence $s+1 > t/s$. So, the conditions of Theorem 6.6.1 for the group G and the $s+2$ lines $L_i Ip$ are satisfied. Hence, the group G acts regularly on the points of $P \setminus p^{\perp}$, and $(\mathcal{S}^{(p)}, G)$ is an EGQ. There are at least two regular lines through the elation point p, and by Theorem 5.5.1 (or Theorem 5.1.1) the proof is complete. Suppose we are in Case (3). By Theorem 1.7.2 (v) the conditions of Theorem 6.6.1 are satisfied, which implies that $(\mathcal{S}^{(x)}, G)$ is an EGQ. Theorem 5.5.1 (or Theorem 5.1.1) finishes the proof. It remains to prove (4). If $t > s^2/2$, then each span of non-collinear points containing p has size 2 by Theorem 1.1.3. Whence the result (apply Part (3)). ∎

Note. The assumption '$t > s^2/2$' in Corollary 6.6.2 (4) seems a bit artificial; one would like to demand, for instance, that $t \geq s^2/2$ — probably the only thick GQ which satisfies $t = s^2/2$ is $\mathcal{Q}(4,2)$. However, the standard divisibility condition ($s+t$ divides $st(st+1)$) for GQ's does not exclude all GQ's of order $(s, s^2/2)$ with $s > 2$ (for instance, put $s = 10$).

We are ready to obtain

Theorem 6.6.3 *Suppose $\mathcal{S} = (\mathcal{S}^{(p)}, G)$ is a thick TGQ of order (s,t), and let $\mathbf{GF}(q)$ be the kernel of the TGQ. Next, suppose that $L_1, L_2, \ldots, L_{t+1}$ are the lines incident with p, and let G_i be the group of all symmetries about the line L_i, $i \in \{1, 2, \ldots, t+1\}$. Define k as the minimum number such that $G = \langle G_{i_1}, G_{i_2}, \ldots, G_{i_{(3+k)}} \rangle$, with $\{i_1, i_2, \ldots, i_{(3+k)}\} \subseteq \{1, 2, \ldots, t+1\}$. Then we have the following inequality:*

$$k \leq \log_q \frac{t}{s}.$$

Proof. Denote the groups $\langle G_{i_1}, G_{i_2}, G_{i_3} \rangle$ and $\langle G_{i_1}, G_{i_2}, G_{i_3}, \ldots, G_{i_{(j+3)}} \rangle$ respectively by G'_0 and G'_j, with $j \in \{1, 2, \ldots, k\}$. By Theorem 6.1.2 we have that $|G'_0| = s^3$. Since k is defined as a minimum, we have the following strict chain of groups:

$$G'_0 \leq G'_1 \leq \cdots \leq G'_k = G.$$

Now fix a point $y \in P \setminus p^\perp$, where $\mathcal{S} = (P, B, I)$. The groups G'_i are all groups of elations about p, and hence for the G'_i-orbits $(G')^*_i$ which contain y, we have that $|(G')^*_i| = |G'_i|$, $i = 0, 1, \ldots, k$, and that

$$(G')^*_0 \subset (G')^*_1 \subset \cdots \subset (G')^*_k = P \setminus p^\perp.$$

The TGQ $\mathcal{S}$ is a $T(\mathcal{O})$ for some egg $\mathcal{O}$ in $\mathbf{PG}(2n+m-1, q) \subseteq \mathbf{PG}(2n+m, q)$, where $\mathbf{GF}(q)$ is the kernel of the TGQ. If we interpret the aforementioned strict chain of orbits in $\mathbf{PG}(2n+m-1, q)$, then we obtain a strict chain of affine spaces over $\mathbf{GF}(q)$:

$$\mathbf{AG}'_0 = \mathbf{AG}(3, q) \subset \mathbf{AG}'_1 \subset \cdots \subset \mathbf{AG}'_k = \mathbf{AG}(2n+m, q),$$

and we have that $|\mathbf{AG}'_j| \geq q|\mathbf{AG}'_{j-1}|$ for every $j \in \{1, 2, \ldots, k\}$, which implies that $|G| \geq q^k s^3$. Since $|G| = s^2 t$, the theorem follows. ∎

This leads to one of the main theorems of this chapter, which is a considerable improvement of the best known (general) result.

Theorem 6.6.4 *Let $(\mathcal{S}^{(p)}, G)$ be a TGQ of order (s, t), $1 \neq s \neq t \neq 1$, with $(s, t) = (q^{na}, q^{n(a+1)})$, where $\mathbf{GF}(q)$ is the kernel of the TGQ and where a is odd. If $k+3$ is the minimum number of distinct lines through p such that G is generated by the symmetries about these lines, then*

$$k \leq n.$$

∎

Remark 6.6.5 (An alternative approach for $T_3(\mathcal{O})$) Suppose $(\mathcal{S}^{(p)}, G)$ is a TGQ of order (s,t), $s \neq 1 \neq t$. Then the kernel $\mathbb{K}$ of the TGQ is isomorphic to $\mathbf{GF}(s)$ if and only if $\mathcal{S}$ is a $T_3(\mathcal{O})$ with $\mathcal{O}$ some ovoid of $\mathbf{PG}(3,s)$, see Theorem 1.11.5. From Theorem 6.6.4 immediately follows that there are four distinct lines incident with p such that G is generated by all symmetries about these four lines. Hence, the knowledge of the size of the kernel is already sufficient to completely solve Problem (2) from the introduction for the $T_3(\mathcal{O})$ of Tits.

Part (1) of the following is an analogue of Theorem 6.6.4 in a more general context. Part (2) generalizes the fact that if $\mathcal{S}$ is a (thick) TGQ of order (s,t), then s and t are the powers of the same prime.

Theorem 6.6.6 ('Structure theorem') (1) *Suppose $\mathcal{S}$ is a thick GQ of order (s,t), and let x be a point of $\mathcal{S}$ incident with $r+1$ axes of symmetry $L_0, L_1, \ldots, L_r$. Suppose G is the group generated by all symmetries about the lines L_i, $0 \leq i \leq r$. We denote the full group of symmetries about L_i by G_i, $0 \leq i \leq r$. Define k as the smallest natural number such that $|G| \leq s^3 k$, and suppose $r \geq 2$. Then there are at least $m = r - 2 - \log_p k$ groups of $\{G_0, G_1, \ldots, G_r\}$ which are abelian, and G is generated by the symmetries about at most $3 + \log_p k$ elements of $\{L_0, L_1, \ldots, L_r\}$. Here p is the smallest prime number dividing s.*

(2) *Suppose $\mathcal{S}$ is a GQ of order (s,t), $t > s^2/2$ and $s \neq 1$, and let x be a point which is incident with $r+1$ axes of symmetry $L_0, L_1, \ldots, L_r$, $r \geq 2$, such that the following condition is satisfied.*

- *If G_i is the full group of symmetries about L_i, $i = 0, 1, \ldots, r$, and $H_i = \langle G_j \parallel j \neq i, 0 \leq j \leq r \rangle$, then $G_i \not\subseteq H_i$.*

If $r+1 \geq 3 + \log_p \frac{t}{s}$, then $r+1 = 3 + \log_p \frac{t}{s}$ and $(\mathcal{S}^{(x)}, G)$ is a TGQ. Here p is the smallest prime number dividing s, and $G = \langle G_0, G_1, \ldots, G_r \rangle$. Also, $\frac{t}{s}$ is a power of p, and s and t are powers of p if $t = s^2$.

(3) *Suppose that $\mathcal{S} = (P, B, I)$ is a thick GQ, with x a point which is incident with at least $r+1$ axes of symmetry, $L_0, L_1, \ldots, L_r$, $r \geq 2$. Let G_i be the full group of symmetries about L_i, $i = 0, 1, \ldots, r$. Define $G = \langle G_i \parallel i = 0, 1, \ldots, r \rangle$, and put $H_i = \langle G_j \parallel j \neq i, i, j \in \{0, 1, \ldots, r\} \rangle$. Suppose that G_* is an arbitrary G-orbit in $P \setminus x^\perp$. If for all $i = 0, 1, \ldots, r$, H_i acts transitively on the points of G_*, then G acts regularly on the points of G_*, G is abelian and G is a group of elations about x. The same properties hold for every G-orbit in $P \setminus x^\perp$.*

Proof. (1) By Theorem 6.1.2 we have that s^3 divides $|G|$, and hence $|G| = s^3k$ (where k is as above). Suppose $r+1-m$ is the minimal number of groups of $\{G_0, G_1, \ldots, G_r\}$ which generate G. Then we have by the proof of Theorem 6.6.3 that

$$s^3 p^{r-2-m} \leq |G| = s^3 k,$$

and hence $p^{r-2-m} \leq k$, thus $m \geq r - 2 - \log_p k$.
Next, suppose that $L_0, L_1, \ldots, L_r$ are axes of symmetry through p, indexed in such a way that $G = \langle G_j \parallel m \leq j \leq r \rangle$ (recall that G is generated by all symmetries about $r+1-m$ axes of symmetry incident with x), and define H_i as $H_i = \langle G_j \parallel j \neq i,\ 0 \leq j \leq r \rangle$, $i \in \{0, 1, \ldots, r\}$. Then we have that $G = H_i G_i = G_i H_i$ for every i, $i \in \{0, 1, \ldots, r\}$, since symmetries about different concurrent lines commute, and if $j \in \{0, 1, \ldots, m-1\}$, it follows that $G = H_j G_j = H_j$ ($G_j \leq H_j = G$). Also, as symmetries about distinct concurrent lines commute, every group G_j, with $j \in \{0, 1, \ldots, m-1\}$, is abelian, see the proof of [139, 8.3.1] (this is Theorem 1.7.3). This proves the first part of the result.

(2) From Part (1) and Theorem 6.3.2 follows that $|G| \geq p^{r-2}s^3$. Since $r + 1 \geq 3 + \log_p \frac{t}{s}$, we have that $|G| \geq s^2 t$. Since $t > s^2/2$, it follows by Theorem 1.7.2 that G is a group of elations with center x, hence $|G| = s^2 t$ and $r = 2 + \log_p \frac{t}{s}$. The equality implies that $\frac{t}{s}$ is a power of p, and in particular, if $t = s^2$, then s is a power of p, and then also t. Since $|G| = s^2 t$, $(\mathcal{S}^{(x)}, G)$ is an EGQ, and by, e.g., Theorem 5.5.1 and the fact that $r \geq 1$, we also know that $\mathcal{S}^{(x)}$ is a TGQ.

(3) Suppose $|G_*| = m$; then $m \geq s^3$ by Theorem 6.1.2, and so $|G| = mk$, with $k = |G_y|$ for an arbitrary point $y \in G_*$. In the following y will be fixed, as well as $i \in \{0, 1, \ldots, r\}$. We have $|H_i| = mk'$ with $k' = |(H_i)_y|$, and clearly that $k' | k$, say $k = k'n$. So we have that:

$$|G| = mk = |G_i H_i| = \frac{|G_i| \times |H_i|}{|G_i \cap H_i|} = \frac{smk'}{|G_i \cap H_i|}. \tag{6.2}$$

Hence $s = n|G_i \cap H_i|$, and so $n | s$. Suppose now that p is a prime which divides n; then there exists a $\theta \in G_y$ of order p. Suppose M is a line through y meeting L_i in x_i. The orbits of $\langle \theta \rangle$ on M (seen as a point set) are cycles of length p or length 1, and since p is a divisor of n and of s, we have that there are at

least $p+1$ points incident with M which are fixed by θ. By Theorem 1.6.5, θ has to be the identitity, since $s \leq t$ (as an axis of symmetry is a regular line). It follows that $n = 1$, and that $H_i = G$. Every two symmetries about distinct concurrent lines commute, and since we proved that $G_j H_j = H_j = G$, with $i = 0, 1, \ldots, r$, G_j is commutative for every j. Hence G is abelian, and since G acts transitively on G_*, G also acts regularly on G_*. Finally, suppose that H_* is an arbitrary G-orbit in $P \setminus x^\perp$. Since G is abelian and since G acts transitively on this orbit, we can again conclude that G acts regularly on H_*. This completely proves the assertion. ■

Theorem 6.6.7 *Suppose $\mathcal{S} = (P, B, I)$ is a generalized quadrangle with parameters (s,t), $s \neq 1 \neq t$, and assume that p is a point which is incident with at least three axes of symmetry. Also, suppose that G is the group generated by the symmetries about* every *axis of symmetry through p. Suppose G_* is an arbitrary G-orbit of the permutation group $(P \setminus p^\perp, G)$. Now define the incidence structure $\mathcal{S}'(G_*) = \mathcal{S}' = (P', B', I')$ as follows.*

- *The* POINTS *of P' are of three types:*
  - (1) *the point p;*
  - (2) *the points of G_*;*
  - (3) *any point which is incident with an axis of symmetry through p.*
- *We have two types of* LINES*:*
  - (a) *the axes of symmetry through p;*
  - (b) *the lines of $\mathcal{S}$ which intersect a line of Type (a) and contain at least one point of G_*.*
- *The* INCIDENCE RELATION *$I' \subseteq I$ is the restriction of I to $(P' \times B') \cup (B' \times P')$.*

Then we have the following properties.

1. *There are constants l and k such that any point of the first two types is incident with $l+1$ lines of $\mathcal{S}'$, and every point of the last type is incident with $k+1$ lines.*
2. *A line of $\mathcal{S}'$ contains $s+1$ points of $\mathcal{S}'$.*
3. *$|G_*| = s^2 k$.*

4. *k is divisible by s, and in particular we have that $s \leq k$. Also, $l \leq k$.*

5. *The number of points of $\mathcal{S}'$ is $ks^2 + (l+1)s + 1$, and the number of lines of $\mathcal{S}'$ is $(l+1)(sk+1)$.*

Proof. (1) Let L be an arbitrary line of $\mathcal{S}'$ through p, and consider an arbitrary point qIL, $q \neq p$. Suppose that q is incident with $k+1$ lines of $\mathcal{S}'$. Since G acts transitively on the points of $L \setminus \{p\}$ (p is incident with at least one axis of symmetry), and since G_* is fixed by G, we can conclude that every point of $L \setminus \{p\}$ is incident with $k+1$ lines of $\mathcal{S}'$. Next consider an arbitrary line L' of $\mathcal{S}'$, $L'Ip$, such that $L' \neq L$, and an arbitrary point $q'IL'$, $q' \neq p$ (so q' is a point of $\mathcal{S}'$). If $k'+1$ is the number of lines of $\mathcal{S}'$ incident with q', then we can easily see that $k' \geq k$ (the $k+1$ lines through q' of $\mathcal{S}'$, meeting the $k+1$ lines through q of $\mathcal{S}'$ are also lines of $\mathcal{S}'$), and conversely we have that $k \geq k'$. It follows that there exists a $k \in \mathbb{N}$ such that each point of $\mathcal{S}'$ of Type (3) is incident with $k+1$ lines of $\mathcal{S}'$. Suppose that p is incident with $l+1$ lines of $\mathcal{S}'$, and consider a point p' of G_*. From the definition of $\mathcal{S}'$, we immediatily see that p' is also incident with $l+1$ lines of $\mathcal{S}'$. This proves Part (1) of the theorem.

(2) Immediate by the definition of $\mathcal{S}'$.

(3) Consider an arbitrary line $L \in B'$ of Type (a); then each point of G_* is incident with a unique line of $\mathcal{S}'$ (of Type (b)) which is concurrent with L. The statement follows easily by Part (1).

(4) If G' is a group generated by all symmetries about three distinct axes of symmetry through a point p in a GQ $\mathcal{S}$, then $|G'| = s^3$ by Theorem 6.1.2, and G' is a group of elations with center p. Thus s^3 is a divisor of $|G_*| = s^2k$. Now suppose that M is a fixed line of $\mathcal{S}'$ of Type (b). Counting the number of pairs (q, M), with qIM a point of G_* lying on a line of $\mathcal{S}'$ not through p, we obtain that

$$s + sl(s-1) + (k-1)s \leq |G_*| = s^2k,$$

or that

$$sl - l \leq sk - k.$$

Hence Part (4) of the theorem follows.

(5) The number of points and lines of $\mathcal{S}'$ follow immediately by (1) and (2).■

Remark 6.6.8 (i) The geometries $\mathcal{S}'$ as above are not always subGQ's. If $\mathcal{S}$ is, for example, a TGQ with $s \neq t$, then this is only always the case if and only if $\mathcal{S}$ is isomorphic to a $T_3(\mathcal{O})$ of Tits.

(ii) In our Master Thesis [203], we axiomatized the incidence structures of Theorem 6.6.7 to what we then called '(l, k)-partial quadrangles'. In some cases they were shown to be generalized quadrangles. In Section 6.8 (see also Section 6.7.1), we will search for a slightly different definition for the natural geometries which arise from generalized quadrangles which have some concurrent axes of symmetry. These geometries will then be investigated in Chapter 7.

We are now ready to state the following answer to Problem (2) of the introduction.

Theorem 6.6.9 *Suppose $\mathcal{S}$ is a GQ of order (s,t), $t \geq s \neq 1$, and let p be a point of $\mathcal{S}$ which is incident with more than $t-s+2$ axes of symmetry. Then $S^{(p)}$ is a translation quadrangle. If $s \neq t$, and G is the group generated by the symmetries about $t-s+2$ arbitrary axes of symmetry through p, then G is the translation group.*

Proof. If $s = t$, then the theorem follows from Theorem 1.7.5, so suppose that $s \neq t$. Consider $r+1$ axes of symmetry through p, with $r = t-s+1$, and suppose that G_* is an arbitrary G-orbit in $P \setminus p^\perp$, with G as above. Since p is incident with at least three axes of symmetry, we have by Theorem 6.3.2 that t is divisible by s, and since $r \geq 2$, we can use Theorem 6.6.7 (in the following we use the same notations as in the proof of that theorem). If $\mathcal{S}'$ is the incidence structure associated to G_* as defined in Theorem 6.6.7, then it follows that $k \geq t-s+1$ since $k \geq l \geq r$. However, by Theorem 6.6.7, $s|k$, and we already remarked that $s|t$. Since $k \leq t$, necessarily $k = t$, thus $|G_*| = s^2k = s^2t$. So G acts transitively on $P \setminus p^\perp$. Since the $t-s+2$ axes were chosen arbitrary, the conditions of Theorem 6.6.6 (3) hold. Hence we can conclude that the abelian group G acts regularly on $P \setminus p^\perp$, and the theorem follows. ■

Using Theorem 5.2.7, Theorem 1.7.5 and Theorem 6.6.9, we immediately have

Theorem 6.6.10 *Let $\mathcal{S}$ be a GQ of order (s,t) with $s \neq 1 \neq t$, and suppose p is a point of $\mathcal{S}$.*

1. *If $s = t$, then $\mathcal{S}$ is a TGQ with translation point p if and only if p is incident with three regular lines L_1, L_2, L_3 for which there are lines M_1, M_2, M_3 such that $L_i \sim M_i \not\sim p$ and such that there are groups G_i of whorls about p which act transitively on the points of $M_i \setminus \{M_i \cap L_i\}$.*

2. *If $s \neq t$, then $\mathcal{S}$ is a TGQ with translation point p if and only if p is incident with $t - s + 3$ regular lines $L_0, L_1, \ldots, L_{t-s+2}$ for which there are lines $M_0, M_1, \ldots, M_{t-s+2}$ such that $L_i \sim M_i \not\sim p$ and such that there are groups G_i of whorls about p which act transitively on the points of $M_i \setminus \{M_i \cap L_i\}$.*

In both cases, the translation group G is the group generated by all the G_i's for all feasible i, and if $s \neq t$, then G is generated by $t - s + 2$ arbitrary G_j's.

■

6.7 Remark on the Structure of Certain Groups which Act on Generalized Quadrangles

In this section, we will generalize a useful theorem of S. E. Payne and J. A. Thas concerning the structure of certain groups which act on generalized quadrangles, namely Theorem 1.7.2. For the sake of convenience, we will recall that theorem here:

Theorem 6.7.1 *Let $\mathcal{S} = (P, B, I)$ be a GQ of order (s, t) with $s \leq t$ and $s > 1$, and let p be a point for which $\{p, x\}^{\perp\perp} = \{p, x\}$ for all $x \in P \setminus p^{\perp}$. Let G be a group of whorls about p.*

1. *If $y \sim p$, $y \neq p$, and if θ is a nonidentity whorl about p and y, then all points fixed by θ lie on py and all lines fixed by θ meet py.*

2. *If θ is a nonidentity whorl about p, then θ fixes at most one point of $P \setminus p^{\perp}$.*

3. *If G is generated by elations about p, then G is a group of elations, i.e. the set of elations about p is a group.*

4. *If G acts transitively on $P \setminus p^{\perp}$ and $|G| > s^2 t$, then G is a Frobenius group on $P \setminus p^{\perp}$, so that the set of all elations about p is a normal subgroup of G of order $s^2 t$ acting regularly on $P \setminus p^{\perp}$, i.e. $\mathcal{S}^{(p)}$ is an EGQ with some normal subgroup of G as elation group.*

5. *If G is transitive on $P \setminus p^\perp$ and G is generated by elations about p, then $(\mathcal{S}^{(p)}, G)$ is an EGQ.*

These observations appear to be crucial tools in many basic theorems for EGQ's and TGQ's, especially in recognition theorems for these quadrangles, see Chapter 8 of FGQ. The result of this section contributes (modestly) to that idea.

6.7.1 Definition of the closure

In view of Theorem 6.6.7, we have found the following notion to be useful in various situations, see, e.g., [203].

Let $\mathcal{S} = (P, B, I)$ be a generalized quadrangle of order (s, t), $s \neq 1 \neq t$, and let $L_0, L_1, \ldots, L_l$ be distinct axes of symmetry of $\mathcal{S}$ which are incident with the point p, $l \geq 2$. Suppose G is the group which is generated by the symmetries about $L_0, L_1, \ldots, L_l$. Let G_* be an arbitrary G-orbit in $P \setminus p^\perp$, and define the point-line geometry $\mathcal{S}(G_*) = (P', B', I')$ as follows:

- POINTS are of three types:
  1. the point p;
  2. the points of G_*;
  3. all points different from p which are incident with lines of Type (a).
- LINES are of two types:
  (a) the lines L' incident with p which have the property that if $L'' \not\parallel p$ is a line of $\mathcal{S}$ which is not skew to G_* and if $L'' \sim L'$, then L'' is incident with s points of G_*;
  (b) the lines of $\mathcal{S}$ which contain at least one point of G_* and which are concurrent with a line of Type (a).
- INCIDENCE is the natural one.

Then following [203], we call $\mathcal{S}(G_*) = (P', B', I')$ the *closure* of G_*, see also Section 6.8.

6.7.2 The structure of certain groups which act on generalized quadrangles

We are ready to prove

Theorem 6.7.2 *Suppose $\mathcal{S} = (P, B, I)$ is a GQ of order (s, t), $s, t \neq 1$ and $s \neq t$, and let p be a point which is incident with $l + 1$ axes of symmetry $L_0, L_1, \ldots, L_l$, $l \geq t/s$. Suppose that G is the group generated by the symmetries about $L_0, L_1, \ldots, L_l$, and suppose that $|G| \geq s^2t$. Assume that each nontrivial element of G has at most two fixed points in $P \setminus p^\perp$. Then the following properties hold.*

1. *The group G acts transitively on $P \setminus p^\perp$ and hence s and t have the same parity.*
2. *If no nontrivial element of G has fixed points in $P \setminus p^\perp$, then $(\mathcal{S}^{(p)}, G)$ is a TGQ with base-group G and base-point p.*
3. *If every non-identical element of G has at most one fixed point in $P \setminus p^\perp$, then $(\mathcal{S}^{(p)}, G)$ is a TGQ.*
4. *The group G contains at least s^2t elations with center p, and when there are exactly s^2t such elations, then $\mathcal{S}^{(p)}$ is a TGQ for some base-group $H \leq G$.*

Proof. (1) First, we recall Burnside's classical theorem. Let (X, G) be a finite permutation group, and suppose k is the number of G-orbits in X. Then we have the following:

$$k|G| = \sum_{g \in G} f(g),$$

where $f(g)$ is the number of fixed points of g in X. Applying this theorem to our situation, we obtain

$$k|G| = s^2t + \sum_{g \neq 1} f(g),$$

and hence that $k|G| < s^2t + |G|2$. Since $|G| \geq s^2t$, we then have that $(k-3)|G| < 0$. The last inequality implies that $k \in \{1, 2\}$, so there are at most two G-orbits in $P \setminus p^\perp$.

Suppose now that G does not act transitively on $P \setminus p^\perp$; this means that there are exactly two orbits in that permutation group, say G_1 and G_2. By Theorem 6.6.7 there exists an $m \in \mathcal{N}$, $s \leq m \leq t$, so that $|G_1| = s^2 m$. Since there are only two G-orbits, it follows that $|G_2| = s^2(t-m)$. Assume that L is a line of $\mathcal{S}$ intersecting G_1, in n points, $0 < n < s$. Let MIp be so that $M \sim L$. Then M is not an axis of symmetry since $n < s$. It is also clear that every line $M' \sim M$ of $\mathcal{S}$, which intersects G_1 in at least one point, does intersect G_1 in exactly n points. Counting the lines of L^G, we then obtain that

$$\frac{s^2 m}{n} \leq st \quad \Rightarrow \quad sm \leq nt. \tag{6.3}$$

Every line of L^G (and in particular the line L) intersects G_2 in precisely $(s-n)$ points, hence

$$s(t-m) \leq (s-n)t \quad \Rightarrow \quad nt \leq sm, \tag{6.4}$$

and thus $sm = nt$. So, every line of $\mathcal{S}$ intersects the G-orbit G_1, respectively G_2, in 0, s or n, respectively s, 0 or $s-n$, points.
Now suppose $L' \not\sim p$ is a line of $\mathcal{S}$ which hits G_1 in s points, and suppose that $\mathcal{S}(G_1)$, respectively $\mathcal{S}(G_2)$, is the closure of G_1, respectively G_2. Suppose k_1+1, respectively k_2+1, is the number of lines in $\mathcal{S}(G_1)$, respectively $\mathcal{S}(G_2)$, through p. Then $k_i \geq t/s$, $i = 1, 2$. Now count the points of G_1, respectively G_2, to obtain

$$|G_1| = s + (m-1)s + sk_1(s-1) + (t-k_1)s(n-1), \tag{6.5}$$

and

$$|G_2| = s + (t-m-1)s + sk_2(s-1) + (t-k_2)s(s-n-1). \tag{6.6}$$

If we then add Equalities (6.5) and (6.6), side by side, we obtain

$$2k_2 s + k_1 ns = s^2 k_1 + snk_2 + st,$$

a contradiction using the facts that $t > s$ (as $\mathcal{S}$ has regular lines and $s \neq t$) and $t \leq s^2$. So there is no line of $\mathcal{S}$ intersecting G_1, in n points with $0 < n < s$. Hence $\mathcal{S}(G_1)$ is a substructure of $\mathcal{S}$ satisfying

(i) each line has $s+1$ points;

(ii) if two points of G_1 are collinear in $\mathcal{S}$, then they are also collinear in $\mathcal{S}(G_1)$.

By Theorem 1.6.2, this means that $\mathcal{S}(G_1)$ is a subGQ of $\mathcal{S}$ of order (s,t'), $t' \geq t/s$. But then it is clear that there must be some line intersecting G_1 in 1 point, contradiction. So, there is only one G-orbit, and G acts transitively on $P \setminus p^\perp$.

(2) If no nontrivial element of G has fixed points in $P \setminus p^\perp$, then G acts regularly on $P \setminus p^\perp$, and $(\mathcal{S}^{(p)}, G)$ is an EGQ. Since G has at least two full groups of symmetries of order s about lines incident with p, this part of the theorem follows by Theorem 1.9.1 and Theorem 5.1.1.

(3) If every non-identical element of G has at most one fixed point in $P \setminus p^\perp$, and if there is at least one nontrivial element which has a fixed point in $P \setminus p^\perp$, then the transitive permutation group $(P \setminus p^\perp, G)$ has the following properties:

1. G acts transitively, but not regularly on $P \setminus p^\perp$;
2. there is no nontrivial element with more than one fixed point in $P \setminus p^\perp$;

Hence $(P \setminus p^\perp, G)$ is a Frobenius group, and the Frobenius kernel N acts regularly on $P \setminus p^\perp$. However, the definition of the Frobenius kernel implies that N contains the symmetries about $L_0, L_1, \ldots, L_l$, and since G is generated by these symmetries, we have that $G = N$, contradiction. Hence G acts regularly on $P \setminus p^\perp$, and Part (2) yields the result.

(4) Let x_i be the number of elements of G which have exactly i fixed points in $P \setminus p^\perp$, $i \in \{0,1,2\}$. Then $|G| = 1 + x_0 + x_1 + x_2$. Applying the identity of Burnside and using the transitivity of the permutation group $(P \setminus p^\perp, G)$, we have that $|G| = s^2t + x_1 + x_2 2$, and so

$$s^2t = 1 + x_0 - x_2. \tag{6.7}$$

Since the identity $\mathbf{1}$ is an elation (by definition), G contains at least s^2t elations with center p. If G would contain exactly s^2t elations with center p, then $x_2 = 0$ because of Equality (6.7). Part (3) now yields the result. ∎

Remark 6.7.3 For the case $s = t$, the theorem is trivially true; recall Theorem 6.1.1 and Theorem 1.7.5.

6.8 Semi Quadrangles

It is not hard to verify that if a TGQ $(\mathcal{S}^{(x)}, G) = (P, B, I)$ of order (s, t), $s \neq 1 \neq t$, has a subGQ $\mathcal{S}'$ of order s which contains the translation point x (note that $\mathcal{S}'$ is then also a TGQ with translation group some subgroup of G), then we have the following fundamental property (which will, in fact, be used at several other stages of this thesis, see, e.g., Chapter 12):

(Clo) *G has a subgroup of order s^3, which is generated by the symmetries about some three distinct lines through x, so that, for some G-orbit G_* in $P\backslash x^\perp$, $\mathcal{S}'$ precisely is the closure of G_*, that is, $\mathcal{S}' = \mathcal{S}(G_*)$.*

Thus, the closure of a general G_*-orbit (in the context of Section 6.7.1) seems the right geometry to study if no subGQ's are available, as was already illustrated a number of times. For an arbitrary subgroup G' of order s^3 of the translation group G, which is generated by the symmetries about three distinct lines through x, one notes the following essential properties for the closure $\mathcal{S}(G_*)$, some of them which were already noted, for instance, in Theorem 6.6.7 (G_* being an arbitrary G-orbit in $P \setminus x^\perp$):

(i) each line of $\mathcal{S}(G_*)$ is incident with $s+1$ lines of $\mathcal{S}(G_*)$;

(ii) each point of $\mathcal{S}(G_*)$ is incident with at least three lines of $\mathcal{S}(G_*)$;

(iii) there are no triangles as subgeometry;

(iv) $\mathcal{S}(G_*)$ contains an ordinary quadrangle and pentagon as subgeometry.

Of course, if one takes the Properties (i)-(ii)-(iii)-(iv) to be the axioms of a point-line geometry Γ, then, as now Γ is not necessarily a subgeometry of a GQ (with $s+1$ points on a line), too many examples arise in order to have defined an 'interesting' geometry. Note, however, that if the following axiom also is satisfied:

(SQ3) *For any two non-collinear points there is at least one point which is collinear with both*;

then the fact that Γ would be a subgeometry (in the usual sense) of a thick GQ $\mathcal{S}$ with $s+1$ points on a line, would imply that Γ is a thick subGQ of $\mathcal{S}$ of order (s, t') (see Theorem 7.2.1 of Chapter 7 for a formal proof of that assertion). Moreover, the point-line geometries satisfying (i)-(ii)-(iii)-(iv)-(SQ3) which have a linear representation in $\mathbf{PG}(n, q)$ (for some q and $n \geq 2$), see Chapter 7, will be equivalent objects as *complete* $(k+1)$*-caps* of $\mathbf{PG}(n, q)$ if

$q > 2$, see Theorem 7.6.3.

We define a *semi quadrangle* as an incidence structure satisfying the Axioms (i)-(ii)-(iii)-(iv)-(SQ3); this definition will be formalized in Chapter 7, and the above observations in this section, but also of Theorem 6.6.7 and of Section 6.7.1, motivate us to systematically study in detail in that chapter.

Remark 6.8.1 Regarding GQ's which have non-concurrent axes of symmetry (the so-called *span-symmetric generalized quadrangles*), we refer to Chapter 8, where a study of those quadrangles will be initiated.

Appendix: a New Short Proof of a Theorem of S. E. Payne and J. A. Thas

We start the appendix with recalling

Theorem 6.8.2 *Suppose $\mathcal{S} = (P, B, I)$ is a thick GQ of order s, and let p be a point of P incident with three distinct axes of symmetry L_1, L_2, L_3. Then every line through p is an axis of symmetry, and so $\mathcal{S}$ is a TGQ.*

The proof of this theorem, mentioned in Chapter 11 of FGQ is not easy, and uses a coordinatization method for GQ's of order s and the theory of planar ternary rings (see Chapter 11 of FGQ). We will give a new geometrical proof without the use of coordinatization (but which uses the result of X. Chen and D. Frohardt, though).

In the following we define G as the group generated by all symmetries about L_1, L_2 and L_3. Furthermore, G_i will be the full group of symmetries about the axis L_i.

Proof of the theorem. By Theorem 6.1.2, G is a group of elations with center p of order s^3, thus, $(\mathcal{S}^{(p)}, G)$ is an EGQ. The theorem now follows from Theorem 5.1.1. ∎

Chapter 7

Semi Quadrangles

As already described in Chapter 6, some particular geometries arise in a natural way in the theory of symmetries of generalized quadrangles and in the theory of translation generalized quadrangles, as certain subgeometries of generalized quadrangles with concurrent axes of symmetry; these subgeometries have interesting automorphism groups. Semi quadrangles axiomatize these geometries.
In the present chapter, we will introduce 'semi quadrangles', which are finite partial linear spaces with a constant number of points on each line, having no ordinary triangles and containing, as minimal circuits, ordinary quadrangles and pentagons, with the additional property that every two non-collinear points are collinear with at least one other point of the geometry. Thick semi quadrangles generalize (thick) partial quadrangles (see [33]). We will emphasize the special situation of the semi quadrangles which are subgeometries of finite generalized quadrangles. We will present several examples of semi quadrangles, most of them arising from generalized quadrangles or partial quadrangles. We will state an inequality for semi quadrangles (the proof of which is written down in Appendix B) which generalizes the inequality of P. J. Cameron [33] for partial quadrangles, and the inequality of D. G. Higman [74, 75] for generalized quadrangles (the proof also gives information about the equality, generalizing a result of C. C. Bose and S. S. Shrikhande [17], see Appendix B). Some other

inequalities and divisibility conditions are computed. Also, we will characterize the linear representations of the semi quadrangles, and we will have a look at the point graphs of semi quadrangles.

The results of this chapter stem from *K. Thas, On Semi Quadrangles* [212], which is to appear in *Ars Combinatoria*.

7.1 Semi Quadrangles

We start with some formal definitions.
An *incidence structure of rank 2* consists of a set of 'points' and a set of 'lines', disjoint, and with a relation — called 'incidence' — between the two sets. As in the case of GQ's, sometimes a line is identified with the set of points incident with it, and we will do this without further notice. The *dual* of an incidence structure is obtained by interchanging the labels 'point' and 'line' (and by interchanging the 'corresponding' parameters). Let $\mathcal{S}$ be a point-line incidence structure. A *path of length d* is a $(d+1)$-tuple of points in which consecutive elements are distinct and collinear. Distances in an incidence structure are measured in the corresponding *incidence graph* (where adjacency is incidence). The *diameter of an incidence structure* is the diameter of its incidence graph, and a finite incidence structure is *connected* if the diameter is finite. An incidence structure is called a *partial linear space* if each point is on at least two (distinct) lines, if all lines are incident with at least two (distinct) points, and if any two distinct points are incident with at most one line (or, equivalently, if any two distinct lines are incident with at most one point). If each two distinct points of a partial linear space are collinear, then it is called a *linear space*. A *semi quadrangle (SQ)* is a partial linear space in which any line is incident with a constant number of points, which contains no ordinary triangles, but contains an ordinary quadrangle and pentagon, and every two non-collinear points are always collinear with at least one common point. It is clear that from this definition it does not necessarily follow that every point is incident with the same number of lines (such as in the case of thick generalized polygons, see [229]). In order to have some more information about these structures, we introduce the *μ-parameters* and the *order* of a semi quadrangle.
Suppose that s, t_i, μ_j, where $1 \leq i \leq n$ and $1 \leq j \leq m$ for nonzero natural numbers n and m, are natural numbers satisfying $s \geq 1$ and $t_i \geq 1$. Then a semi quadrangle of *order* $(s; t_1, t_2, \ldots, t_n)$ and with *μ-parameters* $(\mu_1, \mu_2, \ldots, \mu_m)$ is an incidence structure with the following properties, where we note that (SQ1)

and (SQ2) essentially define the parameters of the SQ, and that (SQ3) and (SQ4) are the main axioms.

(SQ1) The geometry is a partial linear space. Any point is incident with $t_1 + 1$,$t_2 + 1, \ldots$, or $t_n + 1$ lines, and every line is incident with $s + 1$ points. Also, for any $i \in \{1, 2, \ldots, n\}$ there is a point incident with $t_i + 1$ lines.

(SQ2) If two points are not collinear, then there are exactly $\mu_1, \mu_2, \ldots$, or μ_m points collinear with both, and each of these cases occurs.

(SQ3) For any two non-collinear points there is at least one point which is collinear with both (i.e. for each $i = 1, 2, \ldots, m$ there holds that $\mu_i \geq 1$).

(SQ4) The geometry contains an ordinary pentagon and an ordinary quadrangle but no ordinary triangle as subgeometry (hence there is a j for which $\mu_j \geq 2$).

Remark 7.1.1 We emphasize that (SQ2) and (SQ3) should be regarded as different axioms (instead of integrating (SQ3) in (SQ2) by demanding that for every $i = 1, 2, \ldots, m$, $\mu_i \geq 1$). For instance, suppose that $\mathcal{S}$ is a GQ of order (s, t) with $s, t > 2$, and suppose $\mathcal{L}$ is an arbitrary set of k lines with $0 < k < t$. Define a geometry by taking away the lines of $\mathcal{L}$ in the GQ, with the same points as $\mathcal{S}$ and with the natural incidence. Then this geometry satisfies (SQ1), (SQ2) and (SQ4), but not (SQ3).
Other motivatons for this distinction will be clear from the following sections, see e.g. Section 7.3. Also, the reason for demanding that every line has to be incident with a constant number of points is motivated by Theorem 6.6.7, Section 6.7.1, Section 6.8 and the Examples (a) through (f), below.

As for GQ's, if a point p and a line L are incident, we simply write pIL, and if they are not incident, we write $p \not I L$. In the following we agree that $t_1 \leq t_2 \leq \ldots \leq t_n$ and $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_m$. If there are only two possible values for the number of lines through a point, then the SQ is called *near minimal*. Since the parameters t_1, t_n, μ_1, μ_m will play an important role in the following, we call $(s; t_1, t_n)$ the *extremal order* and (μ_1, μ_m) the *extremal μ-parameters*. For a near minimal semi quadrangle, the order and the extremal order coincide. A semi quadrangle is called *thick* if every point is incident with at least three lines and if every line is incident with at least three points. A thick semi quadrangle with $t_1 = t_2 = \ldots = t_n = t$ and $\mu_1 = \mu_2 = \ldots = \mu_m = \mu$ is a thick *partial quadrangle (PQ)* — as defined by P. J. Cameron in [33] — with parameters (s, t, μ) with

$\mu \neq 1$ (the latter notation differs somewhat from that of P. J. Cameron, but in this context it is more convenient), and a thick partial quadrangle with $\mu = t + 1$ is precisely a thick generalized quadrangle with parameters (s, t). Thick generalized quadrangles always contain quadrangles and pentagons. In the case of generalized quadrangles, the condition that the GQ must contain a pentagon is equivalent with the thickness of the GQ, see [229]. This is not the case for semi quadrangles; there are geometries with only two points per line which satisfy all the SQ-conditions. For example, define the geometry $\Gamma = (P, B, I)$ as follows. The point set P consists of six distinct 'letters' a_i, $i \in \{1, 2, \ldots, 6\}$, lines are the sets $\{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\}, \{a_5, a_6\}, \{a_6, a_2\}, \{a_1, a_5\}, \{a_5, a_4\}$, and incidence is the natural one. Then $\mathcal{S}$ is a semi quadrangle. Also, every *strongly regular graph with parameters* (v, k, λ, μ) (see e.g. Chapter 22 of [28]) and with $\mu \geq 2$ and $\lambda = 0$ is a semi quadrangle of order $(1; k - 1)$ and with μ-parameters (μ) (and hence also a partial quadrangle). An example is the unique strongly regular graph with parameters $(16, 5, 0, 2)$, namely the *Clebsch graph*, see [46, p. 440].

Remark 7.1.2 A thick semi quadrangle $\mathcal{S}$ is a thick generalized quadrangle if and only if the following property is satisfied:

(GQ3) *Consider a point p and a line L, p not incident with L. Then there is exactly one line which intersects L and which is incident with p.*

Proof. Immediate. ∎

Notation. Suppose that A is a set of points, respectively lines, of an SQ $\mathcal{S}$. Then $A^{\perp}$ is the set of points, respectively lines, of $\mathcal{S}$ which are collinear, respectively concurrent, with every point, respectively line, of A.

The following theorem shows that a semi quadrangle is loaded with pentagons.

Theorem 7.1.3 *Any anti-flag (p, L) (a non-incident point-line pair) of a semi quadrangle $\mathcal{S}$ which does not satisfy Property (GQ3) is contained in a pentagon.*

Proof. Suppose (p, L) is an anti-flag which does not satisfy Property (GQ3). Suppose u is a point of L and that $x \in \{u, p\}^{\perp}$. Since p is not collinear with any point on L, there is a point v on L that is not collinear with any point on px. Let $y \in \{v, p\}^{\perp}$. Then $y \notin xp, xu$ and hence $uvypx$ is a pentagon which contains L and p. ∎

Theorem 7.1.4 *A geometry $\mathcal{G}$ which satisfies all the SQ-conditions except that there must be a pentagon, automatically contains pentagons if and only if it is not a grid or a dual grid.*

Proof. It is clear that grids and dual grids satisfy all the SQ-conditions except the conditions that there must be a pentagon. If the geometry $\mathcal{G}$ is not a grid or a dual grid and if it satisfies (GQ3), then by Remark 7.1.2, the geometry is a thick generalized quadrangle and hence there are always pentagons. If the geometry $\mathcal{G}$ does not satisfy (GQ3), then it cannot be a grid or a dual grid and applying Theorem 7.1.3, the proof is complete. ∎

7.2 A Motivation for Introducing Semi Quadrangles

In view of Section 6.8 of Chapter 6, we now obtain

Theorem 7.2.1 *Suppose that $\mathcal{S}' = (P', B', I')$ is a subgeometry of a GQ $\mathcal{S} = (P, B, I)$ of order (s, t), with the properties that there are $s' + 1$ points on each line for some s', that there is an ordinary subpentagon in $\mathcal{S}'$ and that (SQ3) is satisfied. Then $s \neq 1 \neq t$, and two points of $\mathcal{S}'$ are collinear if and only if they are collinear in $\mathcal{S}$. If $s' = s$, then $\mathcal{S}'$ is a subGQ of $\mathcal{S}$ of order (s, t') with $t' \neq 1$.*

Proof. That $s \neq 1 \neq t$ follows from the fact that $\mathcal{S}'$ has a pentagon. Suppose p and q are collinear points of $\mathcal{S}'$. Then p and q are also collinear points in $\mathcal{S}$ trivially. Next, suppose p and q are points of $\mathcal{S}'$ which are collinear in $\mathcal{S}$ but not in $\mathcal{S}'$. Then by (SQ3) there is a point x in $\mathcal{S}'$ which is collinear with both p and q. This implies that pxq is a triangle in $\mathcal{S}$ if $x \not I pq$ in $\mathcal{S}$, a contradiction; hence $xIpq$ in $\mathcal{S}'$. It follows that pq is a line of $\mathcal{S}'$, contradiction. Whence two points of $\mathcal{S}'$ are collinear if and only if they are collinear in $\mathcal{S}$. If $s = s'$, then by Theorem 1.6.3 it follows that $\mathcal{S}'$ is a subGQ of $\mathcal{S}$ of order (s, t') with $t' \neq 1$. ∎

7.3 Examples and Constructions of Semi Quadrangles

We only give examples of thick semi quadrangles which are not (always) partial quadrangles, and which are near minimal. All of them are in some way related to generalized quadrangles or partial quadrangles.

We first of all emphasize again that it should be noted that (SQ3) is a very important condition. This will be clearly reflected in the following examples.

(a) Suppose that $\mathcal{S} = (P, B, I)$ is a generalized quadrangle of order (s, t) with $s, t \geq 3$, and suppose p is a point of $\mathcal{S}$ with the property that for every two non-collinear points q, q' of $P \setminus p^\perp$ there holds that

(M) $|\{p, q, q'\}^\perp| < t + 1$.

By 1.7.1 of FGQ there is pair of non-collinear points (x, y) in $P \setminus p^\perp$ with

(Q) $|\{p, x, y\}^\perp| < t$.

Now define the following incidence structure $\mathcal{S}_p = (P_p, B_p, I_p)$: (a) P_p is the set of points of $P \setminus p^\perp$, (b) B_p is the set of all lines of $\mathcal{S}$ not incident with p, and (c) I_p is the restriction of I to $(P_p \times B_p) \cup (B_p \times P_p)$. Then $\mathcal{S}_p$ is a thick semi quadrangle with s points on every line and $t + 1$ lines through every point; Condition (M) implies that (SQ3) is satisfied, (Q) implies the existence of quadrangles and Theorem 7.1.4 yields the existence of a pentagon.

If $\mathcal{S} = (P, B, I)$ is a GQ of order (s, t) with $s, t > 2$ and p an antiregular point, then the geometry $\mathcal{S}_p$ always satisfies Condition (M) and hence Condition (Q), thus $\mathcal{S}_p$ is a semi quadrangle, of which the μ-parameters are contained in $\{t - 1, t, t + 1\}$.

Now specialize, and suppose that $\mathcal{S}^{(x)}$ is a translation generalized quadrangle of order (s, t) with $s, t > 2$ and with translation point x. If $s = t$, we furthermore suppose that s is odd. Then Conditions (M) and (Q) are satisfied, see Chapter 8 of FGQ, and hence $\mathcal{S}^{(x)}$ yields a thick semi quadrangle with a constant number of lines through a point. The semi quadrangles which arise from translation generalized quadrangles in the way described above all have the property that there is an elementary abelian automorphism group (defined in the usual way) which acts regularly on the points of the semi quadrangle. Also, s and t are powers of the same prime p, and there is an odd natural number a and an integer n for which $t = p^{n(a+1)}$ and $s = p^{na}$ if $s \neq t$. If $s = t$ with s odd, then by 1.5.2(v) of FGQ the μ-parameters are given by $(s-1, s+1)$; if $s \neq t$ and if p and a are as above, then the (possible) μ-parameters are $(p^{n(a+1)} - p^n, p^{n(a+1)})$, and $\mathcal{S}_x$ is a partial quadrangle if and only if $a = 1$, and then $\mu = p^{2n} - p^n$.

Let $\mathcal{S}$ be a GQ of order (s, s^2) with $s > 2$. Then by Theorem 1.1.1 every triad of points has $s + 1$ centers (see Section 7.4 for a similar result on SQ's). Now

take an arbitrary point p of $\mathcal{S}$, and consider the geometry $\mathcal{S}_p$. Then $\mathcal{S}_p$ is a partial quadrangle with parameters $(s-1, s^2, s^2-s)$.

(b) Let $\mathcal{S}$ be a GQ of order (s,t) with $s,t>2$, and suppose that $\mathcal{S}'$ is a subGQ of order $(s,t/s)$, with the property that for every two non-collinear points x and y of $\mathcal{S}\setminus\mathcal{S}'$, $|\{x,y\}^\perp \cap \mathcal{S}'| < t+1$. Then every line of $\mathcal{S}$ intersects $\mathcal{S}'$ in 1 or $s+1$ points ($\mathcal{S}'$ is a geometrical hyperplane of $\mathcal{S}$). Next, define a geometry $\mathcal{S}_{\mathcal{S}'} = (P_{\mathcal{S}'}, B_{\mathcal{S}'}, I_{\mathcal{S}'})$ where $B_{\mathcal{S}'}$ is the set of lines of $\mathcal{S}$ which are not contained in $\mathcal{S}'$, $P_{\mathcal{S}'}$ is the set of points of $\mathcal{S}\setminus\mathcal{S}'$, and where $I_{\mathcal{S}'}$ is the natural incidence. Then $\mathcal{S}_{\mathcal{S}'}$ is a thick semi quadrangle of order $(s-1;t)$.
If we put $\mathcal{S}=W(s)$ with $s>2$ even, and $\mathcal{S}'$ is an $(s+1)\times(s+1)$-grid in $\mathcal{S}$, then $\mathcal{S}_{\mathcal{S}'}$ is an example of this construction.

(c) Suppose $\mathcal{S}$ is a GQ of order (s,t) with $s,t>2$, and suppose that $\mathcal{O}$ is an ovoid with the property that for every two non-collinear points x and y of $\mathcal{S}\setminus\mathcal{O}$ there holds that $|\{x,y\}^\perp \cap \mathcal{O}| < t+1$. Define a geometry $\mathcal{S}_{\mathcal{O}} = (P_{\mathcal{O}}, B_{\mathcal{O}}, I_{\mathcal{O}})$ where $B_{\mathcal{O}}$ is the line set of $\mathcal{S}$, $P_{\mathcal{O}} = \mathcal{S}\setminus\mathcal{O}$, and where $I_{\mathcal{O}}$ is the natural incidence. Then $\mathcal{S}_{\mathcal{O}}$ is a thick semi quadrangle of order $(s-1;t)$.

Suppose $\mathcal{O}$ is an ovoid of the classical GQ $W(q)$ of order q, $q>2$. Then every point of $\mathcal{S}$ is regular. By Theorem 1.8.4 of FGQ, $\mathcal{S}_{\mathcal{O}}$ is a semi quadrangle of order $(q-1;q)$ and with μ-parameters $(q-1, q+1)$.

(d) Suppose $\Gamma=(P,B,I)$ is a partial quadrangle with parameters (s,t,μ), where $s,t\geq 3$, and let $\Gamma'=(P',B',I')$ be a *subPQ* (in the obvious sense) of Γ with parameters (s,t',μ'). Then a simple counting argument shows that every line of Γ intersects Γ' if and only if $|P'|\times(t-t') = |B|-|B'|$, that is, if and only if

$$(t-t')(s+1)(1+(t'+1)s(1+\frac{st'}{\mu'})) = (1+(t+1)s(1+\frac{st}{\mu}))(t+1)-$$

$$(1+(t'+1)s(1+\frac{st'}{\mu'}))(t'+1). \qquad (7.1)$$

Note that if we interchange the words 'PQ' and 'GQ', that this condition can be simplified to $t'=t/s$, see Example (b).
Assume Condition (7.1) is satisfied. Furthermore, we suppose that $\mathcal{S}$ has the property that (1) for every two non-collinear points q,q' of $P\setminus p^\perp$ there holds

that $|\{p,q,q'\}^\perp| < \mu$, and that (2) there is a pair of non-collinear points (x,y) in $P\setminus p^\perp$ for which $|\{p,x,y\}^\perp| < \mu-1$. Define a geometry $\Gamma_{\Gamma'} = (P_{\Gamma'}, B_{\Gamma'}, I_{\Gamma'})$ where $B_{\Gamma'}$ is the line set of $\Gamma\setminus\Gamma'$ and $P_{\Gamma'}$ is the set of points of $\Gamma\setminus\Gamma'$, and where $I_{\Gamma'}$ is the natural incidence. Then $\Gamma_{\Gamma'}$ is a semi quadrangle of order $(s-1;t)$. We do not know of any examples.

(e) A *partial ovoid* of a partial quadrangle is a set of mutually non-collinear points. An *ovoid* $\mathcal{O}$ of a partial quadrangle Γ with parameters (s,t,μ) is a set of non-collinear points such that every line is incident with exactly one point of the set. Dually, one defines *partial spreads* and *spreads*[1].
By counting the point-line pairs (p,L) of Γ for which $p\in\mathcal{O}$, pIL with L a line of Γ, in two ways, we obtain $|\mathcal{O}| = \frac{s^2t(t+1)/\mu+(t+1)s+1}{s+1}$. Suppose Γ is a PQ with parameters (s,t,μ) with $s,t>2$, and suppose that $\mathcal{O}$ is an ovoid with the property that for every two non-collinear points x and y of $\Gamma\setminus\mathcal{O}$ there holds that $|\{x,y\}^\perp\cap\mathcal{O}| < \mu$. Also, we demand that there is a pair of non-collinear points (x,y) in $P\setminus p^\perp$ for which $|\{p,x,y\}^\perp| < \mu-1$. Define a geometry $\Gamma_{\mathcal{O}} = (P_{\mathcal{O}}, B_{\mathcal{O}}, I_{\mathcal{O}})$ where $B_{\mathcal{O}}$ is the line set of Γ, $P_{\mathcal{O}} = \Gamma\setminus\mathcal{O}$, and where $I_{\mathcal{O}}$ is the natural incidence. Then $\Gamma_{\mathcal{O}}$ is a thick semi quadrangle of order $(s-1;t)$. We do not know of any concrete examples of such semi quadrangles; we only prove the following nonexistence theorem for ovoids of partial quadrangles.

Theorem 7.3.1 *Consider a partial quadrangle Γ with parameters $(s-1,s^2,s^2-s)$, $s>2$. Then Γ cannot have ovoids.*

Proof. Suppose that $\mathcal{O}$ is an ovoid of Γ. Then

$$|\mathcal{O}| = \frac{(s-1)^2s^2(s^2+1)/(s^2-s)+(s^2+1)(s-1)+1}{s} = s^3.$$

By A. A. Ivanov and S. V. Shpectorov [83], Γ is of the form $\mathcal{S}_p$ (see Example (a)), where $\mathcal{S}$ is a GQ of order (s,s^2), and p is a point of $\mathcal{S}$. Hence $\mathcal{O}\cup\{p\}$ is an ovoid of $\mathcal{S}$, a contradiction by Theorem 1.12.1. ■

Note. By the appendix of Chapter 4 it easily follows that if $\mathcal{O}$ is a partial ovoid of the PQ $\mathcal{S}_p$, then $|\mathcal{O}| \le s^3-s^2+s$.

(f) Suppose $\mathcal{K}$ is a *complete $(t+1)$-cap* of $\mathbf{PG}(n-1,q)$, $n\ge 3$, (see Section 7.6), and embed $\mathbf{PG}(n-1,q)$ in $\mathbf{PG}(n,q)$. Suppose P is the set of points of $\mathbf{PG}(n,q)$ which are not contained in $\mathbf{PG}(n-1,q)$, that B is the set of

[1] In the same way, one could define *(partial) ovoids* for semi quadrangles, and, dually, *(partial) spreads*.

lines L of $\mathbf{PG}(n,q)$ which are not contained in $\mathbf{PG}(n-1,q)$ and for which $|\mathcal{K} \cap L| = 1$. Then the geometry $\mathcal{S} = (P, B, I)$, with I the natural incidence, is a semi quadrangle of order $(q-1;t)$. If $n = 4$ and $\mathcal{K}$ is an ovoid of $\mathbf{PG}(3,q)$, then $\mathcal{S}$ is a partial quadrangle, see [33]. For details and proofs, see Section 7.6.

Remark 7.3.2 (i) The first three constructions given above all arise by taking away a geometrical hyperplane of a GQ. Thus these geometries are all AGQ's. Also, any of the Examples (a),(b),(c),(d),(e) can clearly be generalized in a natural way by considering *geometrical hyperplanes of semi quadrangles* (instead of *geometrical hyperplanes of partial quadrangles*).

(ii) The Examples (a), (b) and (c) are the only thick semi quadrangles which are subgeometries of a GQ with the same number of points on a line as the GQ, minus one.

7.4 Computation of Some Divisibility Conditions, Constants and Inequalities

If a point p of a semi quadrangle is incident with $t_i + 1$ lines, then we denote this by $p \in P_i$, and p is said to have *degree* t_i.
If we write $\lfloor x \rfloor$, with $x \in \mathbb{R}$, then we mean the greatest natural number which is at most x, and with $\lceil x \rceil$ we mean the smallest natural number which is at least x.

7.4.1 Some generalities

Suppose $\mathcal{S}$ is a thick semi quadrangle of order $(s; t_1, t_2, \ldots, t_n)$ and with μ-parameters $(\mu_1, \mu_2, \ldots, \mu_m)$ which is not a generalized quadrangle. Then by the observation we made earlier, $\mathcal{S}$ does not satisfy (GQ3), and hence there is a point-line pair (p, L) so that $p \not I L$ and for which there is no line M satisfying $pIM \sim L$. Suppose $p \in P_j$. By counting the points which are collinear with p and with a point of L in two ways, we obtain $(s+1)\mu_1 \leq (t_j+1)s$, from which follows that $\mu_1 + \frac{\mu_1}{s} \leq t_j + 1$.
Now suppose that $t_i = t$ for all i, and fix a point q. By $N_k = N_k(q)$, we denote the number of points x of $P \setminus q^\perp$ for which there are μ_k points collinear with both x and q. Now we count the number of points in $q^\perp \setminus \{q\}$ in two ways, and we get that

$$\frac{N_1\mu_1 + N_2\mu_2 + \ldots + N_m\mu_m}{st} = (t+1)s, \tag{7.2}$$

and hence we have the following theorem.

Theorem 7.4.1 *Suppose that $\mathcal{S}$ is a semi quadrangle with μ-parameters $(\mu_1, \mu_2, \ldots, \mu_m)$ and with a constant number, $t+1$, of lines through a point. Suppose the N_i are as above. Then*

$$\frac{N_1\mu_1 + N_2\mu_2 + \ldots + N_m\mu_m}{st} = (t+1)s. \tag{7.3}$$

■

7.4.2 An inequality for semi quadrangles

Definition. Suppose $\mathcal{S}$ is a semi quadrangle. A *triad* is a set of three points, respectively lines, two by two non-collinear, respectively non-concurrent. A *center* of a triad $\{\mathcal{U}, \mathcal{V}, \mathcal{W}\}$, where $\mathcal{U}, \mathcal{V}$ and $\mathcal{W}$ are all points or all lines, is an element of $\{\mathcal{U}, \mathcal{V}, \mathcal{W}\}^{\perp}$.

Theorem 7.4.2 *Suppose $\mathcal{S}$ is a semi quadrangle with extremal order $(s; t_1, t_n)$ and with extremal μ-parameters (μ_1, μ_m). Then we have the following inequality.*

$$[(t_1-1)s\mu_1]^2 \leq \mu_m[(\mu_m-1)(\mu_m-2) + (t_n-1)s](\lfloor\frac{(t_n+1)t_n s^2}{\mu_1}\rfloor$$

$$-s(t_1+1) + \mu_m - 1). \tag{7.4}$$

If equality holds, then there is a constant $x_0 = \frac{(t_1-1)s\mu_1}{\lfloor\frac{(t_n+1)t_n s^2}{\mu_1}\rfloor - s(t_1+1)+\mu_m-1}$ such that each triad of points has exactly x_0 centers.
Also, if each triad of points has a constant number of centers, then

$$[(t_n-1)s\mu_m]^2 \geq \mu_1[(t_1-1)s + (\mu_1-1)(\mu_1-2)](\lceil\frac{(t_1+1)t_1 s^2}{\mu_m}\rceil$$

$$-s(t_n+1) + \mu_1 - 1). \tag{7.5}$$

Proof. See Appendix B. ■

Remark 7.4.3 (i) If the dual of the SQ $\mathcal{S}$ is also a semi quadrangle, then the dual statement of Theorem 7.4.2 also holds. In that case $t_1 = t_2 = \ldots = t_n = t$, and for every two non-concurrent lines there is at least one line concurrent with both

(ii) In P. J. Cameron [33] it is proved that a partial quadrangle of order (s, t, μ) has the property that the dual is also a partial quadrangle if and only if $t = s$ or $t + 1 = \mu$.

(iii) Theorem 7.4.2 also holds for several other incidence geometries, see Appendix B.

Corollary 7.4.4 (P. J. Cameron [33]) *Suppose $\mathcal{S}$ is a partial quadrangle with parameters (s, t, μ). Then*

$$\mu(t-1)^2 s^2 \leq [s(t-1) + (\mu-1)(\mu-2)][\frac{(t+1)ts^2}{\mu} - (t+1)s + \mu - 1]. \quad (7.6)$$

Equality holds if and only if the number of points collinear with each of any three pairwise non-collinear points is a constant; if this occurs, the constant is $1 + \frac{(\mu-1)(\mu-2)}{s(t-1)}$.

■

Corollary 7.4.5 (D. G. Higman [74, 75]) *Suppose $\mathcal{S}$ is a generalized quadrangle with parameters (s, t), $s \neq 1 \neq t$. Then $t \leq s^2$ and, dually, $s \leq t^2$.*

■

Corollary 7.4.6 (C. C. Bose and S. S. Shrikhande [17]) *Let $\mathcal{S}$ be a generalized quadrangle with parameters (s, t), $s \neq 1 \neq t$. Then $t = s^2$ if and only if the number of points collinear with every three pairwise non-collinear points is a constant, and if this occurs, the constant is $s + 1$. Dually, $s = t^2$ if and only if the number of lines concurrent with every three pairwise non-concurrent lines is a constant, and if this occurs, the constant is $t + 1$.*

■

7.4.3 Some inequalities

Suppose $\mathcal{S}$ is a semi quadrangle with extremal order $(s; t_1, t_n)$ and with extremal μ-parameters (μ_1, μ_m), and assume that $s \leq t_1$. Suppose b is the number of lines and v is the number of points. Counting the number θ of *flags* of $\mathcal{S}$ (a *flag* is an incident point-line pair), we get that

$$v(t_n + 1) \geq \theta = b(s + 1) \geq v(t_1 + 1) \tag{7.7}$$

Note. If $t_1 = t_n = t$ in (7.7), then $v(t + 1) = b(s + 1)$. If $v = b$, then $t_n \geq s \geq t_1$.

In the case $v = b$, some refinement is possible.

Theorem 7.4.7 *Suppose $\mathcal{S} = (P, B, I)$ is a semi quadrangle of order $(s; t_1, t_2, \ldots, t_n)$ and with μ-parameters $(\mu_1, \mu_m, \ldots, \mu_m)$. If $\mathcal{S}$ has the property that $v := |P| = |B| =: b$, then we have that either $t_1 = t_n = s$ or $t_1 < s < t_n$.*

Proof. Suppose $\mathcal{S}$ is an SQ with equally many points as lines, and suppose $t_1 \neq t_n$ (so $t_1 < t_n$). If we suppose that M_i is the number of points with degree t_i, and if we count the number θ of flags of $\mathcal{S}$, we obtain the following.

$$\theta = b(s + 1) = v(s + 1) = \sum_i M_i(t_i + 1).$$

Since $\sum_i M_i = v$, the theorem easily follows. ∎

Notation. If two (not necessarily distinct) points p and q, respectively lines L and M, of an SQ are collinear, respectively concurrent, then we denote this by $p \sim q$, respectively $L \sim M$.

Theorem 7.4.8 *Suppose $\mathcal{S}$ is a semi quadrangle with extremal order $(s; t_1, t_n)$ and with extremal μ-parameters (μ_1, μ_m). Then we have the following inequalities*

$$s^2 t_1(t_1 + 1) \leq (v - (t_1 + 1)s - 1)\mu_m, \tag{7.8}$$

and

$$s^2 t_n(t_n + 1) \geq (v - (t_n + 1)s - 1)\mu_1, \tag{7.9}$$

and $\mathcal{S}$ is a PQ if and only if equality holds in both (7.8) and (7.9).

Proof. The inequalities are immediate by counting in two ways the ordered triples of points (p, q, r) of $\mathcal{S}$, with the property that p, q and r are not on the same line, and that $p \sim q$ and $p \sim r$. If $\mathcal{S}$ is a PQ, then $t_1 = t_n = t$, $\mu_1 = \mu_m = \mu$ and $s^2t(t+1) = (v-(t+1)s-1)\mu$. If equality holds in (7.8) and (7.9), then from $s^2t_1(t_1+1) \leq s^2t_n(t_n+1)$ and $(v-(t_1+1)s-1)\mu_m \geq (v-(t_n+1)s-1)\mu_1$ follows that $s^2t_1(t_1+1) = s^2t_n(t_n+1) = (v-(t_n+1)s-1)\mu_1 = (v-(t_1+1)s-1)\mu_m$, and so $t_1 = t_n$ and $\mu_1 = \mu_m$, that is, $\mathcal{S}$ is a partial quadrangle. ■

7.5 Semi Quadrangles and their Point Graphs

A (simple) *graph* is an incidence structure in which lines are called *edges* and points are called *vertices*, and in which any edge is incident with two points and any two distinct points are incident with at most one edge. Two distinct points incident with the same edge are called *adjacent*, and a graph is *complete* if any two distinct vertices are adjacent. If a vertex v is incident with t edges, then t is called the *valency* of v. The *μ-values* of a graph $\mathcal{G}$ are numbers $\mu_1, \mu_2, \ldots, \mu_m$ such that any two non-adjacent vertices are both adjacent with μ_i vertices for some $1 \leq i \leq m$. The *λ-values* of a graph are numbers $\lambda_1, \lambda_2, \ldots, \lambda_{m'}$ such that any two distinct adjacent points are both adjacent with λ_j points for some $1 \leq j \leq m'$. An *induced subgraph* consists of a subset of points of the point set, together with all the edges joining two points in the subset, and a *(maximal) clique* is a (maximal) complete induced subgraph of a graph. The *point graph* of an incidence geometry is the graph in which two distinct points are adjacent if and only if they are collinear (where the vertices are the points).

Theorem 7.5.1 (P. J. Cameron [33]) *The point graph of a partial quadrangle is strongly regular and has no induced subgraph isomorphic to a complete graph on four points with one edge removed. Conversely, a strongly regular graph with this property is the point graph of a partial quadrangle.*

There is a similar theorem for semi quadrangles.

Theorem 7.5.2 *A graph is the point graph of a semi quadrangle if and only if (a) every μ-value is strictly positive (i.e. the diameter of the graph is at most 2), (b) there is only one λ-value $s-1$, and (c) the graph contains an induced quadrangle, respectively pentagon, and it has no induced subgraph isomorphic to a complete graph on four points with one edge removed. Moreover, if the SQ has order $(s; t_1, t_2, \ldots, t_n)$ and μ-parameters $(\mu_1, \mu_2, \ldots, \mu_m)$, then the λ-value of the graph is $s-1$, the possible μ-values are $\mu_1, \mu_2, \ldots, \mu_m$, and $\{(t_1 +$*

$1)s, (t_2+1)s, \ldots, (t_n+1)s\}$ *is the set of valencies, and, conversely, a graph which satisfies Properties (a), (b) and (c), and which has these parameters, is the point graph of a semi quadrangle of order* $(s; t_1, t_2, \ldots, t_n)$ *and with* μ*-parameters* $(\mu_1, \mu_2, \ldots, \mu_m)$.

Proof. Suppose $\mathcal{S}$ is a semi quadrangle of order $(s; t_1, t_2, \ldots, t_n)$ and with μ-parameters $(\mu_1, \mu_2, \ldots, \mu_m)$, where $n, m \geq 1$. It immediately follows that its point graph has valencies $(t_1+1)s, (t_2+1)s, \ldots, (t_n+1)s$, that the λ-value is $s-1$, and that $\{\mu_1, \mu_2, \ldots, \mu_m\}$ is the set of μ-values. If there would be an induced subgraph isomorphic to a complete graph $pqp'q'$ on four points with one edge $p'q'$ removed, then p' and q' both lie on the line pq, and so they are collinear, a contradiction. The other conditions of (SQ4) are reflected in (c). Now suppose a graph $\mathcal{G}$ has one λ-value $s-1$ and μ-values $\mu_1, \mu_2, \ldots, \mu_m$, and suppose that it satisfies Properties (a) and (c). Any edge $\{p, q\}$ is contained in a unique maximal clique $\mathcal{G}_{pq}$ whose vertex set consists of p, q and all vertices joined to both. Hence $\mathcal{G}_{pq}$ has $s+1$ vertices. If any vertex is contained in t_i+1 maximal cliques, with $i \in \{1, 2, \ldots, m\}$, then it is clear that the vertices and maximal cliques of $\mathcal{G}$ form a semi quadrangle of order $(s; t_1, t_2, \ldots, t_n)$ and with μ-parameters $(\mu_1, \mu_2, \ldots, \mu_m)$. ∎

7.6 Linear Representations

A *linear representation* of a semi quadrangle $\mathcal{S} = (P, B, I)$ is a monomorphism θ of $\mathcal{S}$ into the geometry of points and lines of the affine space $\mathbf{AG}(n, q)$, in such a way that P^θ is the set of all points of $\mathbf{AG}(n, q)$, that B^θ is a union of parallel classes of lines of $\mathbf{AG}(n, q)$, and that each point of L^θ is the image of some point of L for any line L in B. Usually we identify $\mathcal{S}$ with its image $\mathcal{S}^\theta$. Note that any parallel class of lines partitions the point set of $\mathbf{AG}(n, q)$. Since parallel classes of lines in an $\mathbf{AG}(n, q)$ correspond to points of $\mathbf{PG}(n-1, q)$ in a natural way, such a representation $\mathcal{S}^\theta$ defines a set of points $\mathcal{K}$ in $\mathbf{PG}(n-1, q)$. An *r-cap* in $\mathbf{PG}(n-1, q)$ (usually called *r-arc* if $n = 3$) is a set of r points, no three of which are collinear. A line is *secant*, respectively *tangent*, to an r-cap according as it meets the cap in two points, respectively one point.

Theorem 7.6.1 (P. J. Cameron [33]) *1. A subset* $\mathcal{K}$ *of the point set of* $\mathbf{PG}(n-1, q)$ *provides a linear representation of a partial quadrangle with parameters* $(q-1, t, \mu)$ *if and only if it is a* $(t+1)$*-cap with the property that any point not in* $\mathcal{K}$ *lies on* $t - \mu + 1$ *tangents to* $\mathcal{K}$.

2. A subset $\mathcal{K}$ *of the point set of* $\mathbf{PG}(n-1, q)$ *provides a linear representation of a generalized quadrangle* $\mathcal{S}$ *if and only if one of the following occurs:*

(a) $n = 2$ *and* $|\mathcal{K}| = 2$;

(b) $n = 3$, q *is even and* $\mathcal{K}$ *is a hyperoval (a* $(q+2)$*-arc);*

(c) $q = 2$ *and* $\mathcal{K}$ *is the complement of a hyperplane.*

Remark 7.6.2 If $n = 2$ and $|\mathcal{K}| = 2$, then $\mathcal{S}$ is a grid. If $q = 2$ and $\mathcal{K}$ is the complement of a hyperplane, then $\mathcal{S}$ is a dual grid. If $n = 3$, $q > 2$, and $\mathcal{K}$ is a hyperoval, then $\mathcal{S}$ is neither a grid nor a dual grid.

Now suppose $\mathcal{S} = (P, B, I)$ is a semi quadrangle with μ-parameters $(\mu_1, \mu_2, \ldots, \mu_k)$, and suppose $\mathcal{S}$ has a linear representation in an $\mathbf{AG}(n, q)$. If $t + 1$ is the number of parallel classes defined by this representation, then it is first of all clear that $\mathcal{S}$ is of order $(q - 1, t)$ (hence every point of $\mathcal{S}$ is incident with a constant number of lines).
Suppose $\mathcal{V}$ is the set of $t + 1$ points of $\mathbf{PG}(n - 1, q)$ which corresponds to the semi quadrangle, and suppose that three points p, o and r of $\mathcal{V}$ are collinear. Consider an arbitrary affine point x of $\mathbf{AG}(n, q)$, and suppose $y \neq x$ is a point of $\mathbf{AG}(n, q) \cap xr$. Then the lines yp, xo and xr define a triangle which is contained in the semi quadrangle, a contradiction. Hence $\mathcal{V}$ is a $(t + 1)$-cap.

Let $\mathcal{K}$ be the $(t+1)$-cap in $\mathbf{PG}(n-1, q)$ which corresponds to a semi quadrangle $\mathcal{S}$, and suppose p and o are arbitrary points of $\mathbf{AG}(n, q)$ which are non-collinear in $\mathcal{S}$. Then the line po of $\mathbf{PG}(n, q)$ intersects $\mathbf{PG}(n - 1, q)$ in a point r off $\mathcal{K}$. Suppose there are μ_j points of $\mathcal{S}$ collinear (in $\mathcal{S}$) with p and o. Then this means that there are exactly $\mu_j/2$ planes through pq in the projective completion $\mathbf{PG}(n, q)$ of $\mathbf{AG}(n, q)$ which intersect $\mathcal{K}$ in exactly two points, and hence there are precisely $t - \mu_j + 1$ tangents to $\mathcal{K}$ through r.
Since $|\mathbf{AG}(n, q)| = |P|$, *every* point of $\mathbf{PG}(n - 1, q)$ off $\mathcal{K}$ is incident with μ_h tangent lines to $\mathcal{K}$ for a certain $h \in \{1, 2, \ldots, k\}$. It follows that $\mu_h \equiv 0 \mod 2$ for all feasible h. Now suppose o' is such a point which is incident with $t - \mu + 1$ tangents to $\mathcal{K}$ ($\mu \in \{\mu_1, \mu_2, \ldots, \mu_k\}$), and let L be an arbitrary line through o' and not in $\mathbf{PG}(n - 1, q)$. Then every two distinct points x, y on L, $x \neq o' \neq y$, are non-collinear in $\mathcal{S}$, and $|\{x, y\}^{\perp}| = \mu$. Also, there are exactly $\frac{q(q-1)}{2}$ such (non-ordered) pairs on L.

CONDITION (SQ3). The fact that $\mathcal{S}$ satisfies (SQ3) is clearly equivalent with the fact that for every two points p and o of $\mathbf{AG}(n, q)$ for which po does not intersect $\mathcal{K}$, there must be at least one secant to the cap. Hence the cap must be *complete* (by definition).

We now investigate how the existence of a quadrangle, respectively pentagon, is reflected on the linear respresentation.

THE EXISTENCE OF A QUADRANGLE. Let L be a secant to $\mathcal{K}$ and let π be a plane of $\mathbf{PG}(n,q)$ containing L, but not contained in $\mathbf{PG}(n-1,q)$. Then π contains quadrangles of $\mathcal{S}$.

THE EXISTENCE OF A PENTAGON. By Theorem 7.1.3 and the preceding results, the existence of a pentagon in $\mathcal{S}$ is equivalent to the condition that $\mathcal{S}$ is not a grid or a dual grid. By Remark 7.6.2, this is always the case except if one of the following occurs:

1. $n = 2$ and $|\mathcal{K}| = 2$;
2. $q = 2$ and $\mathcal{K}$ is the complement of a hyperplane.

We have proved the following theorem.

Theorem 7.6.3 *1. A subset $\mathcal{K}$ of the point set of $\mathbf{PG}(n-1,q)$, $n \geq 3$, provides a linear representation of a semi quadrangle with μ-parameters $(\mu_1, \mu_2, \ldots, \mu_k)$ if and only if the following conditions are satisfied:*

   (a) it is a complete $(t+1)$-cap for a certain t with the property that any point off $\mathcal{K}$ in $\mathbf{PG}(n-1,q)$ lies on $t - \mu_j + 1$ tangents to $\mathcal{K}$ for some $\mu_j \in \{\mu_1, \mu_2, \ldots, \mu_k\}$, and each possibility occurs;

   (b) If $q = 2$, then $\mathcal{K}$ is not the complement of a hyperplane.

2. If a $(t+1)$-cap $\mathcal{K}$ of $\mathbf{PG}(n-1,q)$ provides a linear representation of the semi quadrangle $\mathcal{S}$, then every point of $\mathcal{S}$ is incident with $t+1$ lines.

3. Suppose $\mathcal{S} = (P, B, I)$ is an SQ with μ-parameters $(\mu_1, \mu_2, \ldots, \mu_k)$ which has a linear representation in $\mathbf{AG}(n,q)$, and define P_j by $P_j = \{\{x,y\} \in P \times P, x \not\sim y \parallel |\{x,y\}^{\perp}| = \mu_j\}$. Then for all j, $|P_j| \equiv 0 \mod (q(q-1)/2)$. Also, each u_h is even $\forall h$.

■

If $n = 3$ and q is even, then the $(t+1)$-cap $\mathcal{K}$ is a hyperoval if r equals $q+2$, and a hyperoval is always complete. Suppose $q \geq 4$. If we consider the semi quadrangle $\mathcal{S}$ which corresponds with $\mathcal{K}$, then $\mathcal{S}$ is the $T^*(\mathcal{K})$ due to R. W. Ahrens and G. Szekeres (see also Theorem 7.6.1). If $n = 4$, $q > 2$, and $\mathcal{K}$ is an ovoid (which is a complete (q^2+1)-cap, see Section 1.12) of $\mathbf{PG}(3,q)$, then the associated semi quadrangle is a partial quadrangle with parameters $(q-1, q^2, q^2-q)$.

7.7 Semi Quadrangles and Complete Caps of Projective Spaces

The following two theorems are direct consequences of Theorem 7.6.3 and the results of Section 7.4.

Theorem 7.7.1 *Suppose $\mathcal{K}$ is a complete $(k+1)$-cap in $\mathbf{PG}(n,q)$, where $n \geq 2$ and where $\mathcal{K}$ is not the complement of a hyperplane if $q = 2$, and let μ, respectively μ', be the integers such that every point of $\mathbf{PG}(n,q) \setminus \mathcal{K}$ is incident with at least $k+1-\mu$, respectively at most $k+1-\mu'$, tangents to $\mathcal{K}$. Then we have the following inequality.*

$$[(k-1)(q-1)\mu']^2 \leq \mu[(\mu-1)(\mu-2)+(k-1)(q-1)](\lfloor\frac{(k+1)k(q-1)^2}{\mu'}\rfloor$$

$$-(q-1)(k+1)+\mu-1).$$

If equality holds, then there is a constant $x_0 = \frac{(k-1)(q-1)\mu'}{\lfloor\frac{(k+1)k(q-1)^2}{\mu'}\rfloor-(q-1)(k+1)+\mu-1}$ such that, if we embed $\mathbf{PG}(n,q)$ as a hyperplane in $\mathbf{PG}(n+1,q)$, for every set $\{p_1,p_2,p_3\}$ of three distinct points in $\mathbf{PG}(n+1,q) \setminus \mathbf{PG}(n,q)$ with the property that $p_ip_j \cap \mathcal{K} = \emptyset$ for every $i \neq j$ in $\{1,2,3\}$, there holds that there are precisely x_0 points r in $\mathbf{PG}(n+1,q) \setminus \mathbf{PG}(n,q)$ for which $|rp_i \cap \mathcal{K}| = 1$ $\forall i = 1,2,3$. Also, if for every set $\{p_1,p_2,p_3\}$ of three distinct points in $\mathbf{PG}(n+1,q) \setminus \mathbf{PG}(n,q)$ with the property that $p_ip_j \cap \mathcal{K} = \emptyset$ for every $i \neq j$ in $\{1,2,3\}$, it is true that there is a constant number of points r in $\mathbf{PG}(n+1,q) \setminus \mathbf{PG}(n,q)$ for which $|rp_i \cap \mathcal{K}| = 1$ $\forall i = 1,2,3$, then we have the following.

$$[(k-1)(q-1)\mu]^2 \geq \mu'[(k-1)(q-1)+(\mu'-1)(\mu'-2)](\lceil\frac{(k+1)k(q-1)^2}{\mu}\rceil$$

$$-(q-1)(k+1)+\mu'-1).$$

■

We do not know whether Theorem 7.7.1 yields new information about complete caps in projective spaces.

Theorem 7.7.2 *Suppose that $\mathcal{K}$ is a complete $(k+1)$-cap in $\mathbf{PG}(n,q)$, , where $n \geq 2$ and where $\mathcal{K}$ is not the complement of a hyperplane if $q = 2$, and let μ, respectively μ', be the integers such that every point of $\mathbf{PG}(n,q) \setminus \mathcal{K}$ is incident with at least $k+1-\mu$, respectively at most $k+1-\mu'$, tangents to $\mathcal{K}$. Then we have the following inequalities*

$$(q-1)^2k(k+1) \leq (q^{n+1} - (k+1)(q-1) - 1)\mu, \tag{7.10}$$

and

$$(q-1)^2k(k+1) \geq (q^{n+1} - (k+1)(q-1) - 1)\mu', \tag{7.11}$$

and equality holds in both cases if and only if $\mu = \mu'$.

■

It follows that

$$k+1 - \frac{(q-1)^2k(k+1)}{(q^{n+1} - (k+1)(q-1) - 1)} \leq k+1-\mu'. \tag{7.12}$$

From [80], we know that since $\mathcal{K}$ is complete, there holds that $k+1-\mu' < \delta(q^{n-1} + q^{n-2} + \ldots + 1 - k)$, where $\delta = 1$ if q is even and $\delta = 2$ otherwise. We conclude that, with $f(k,q,n) := \frac{(q-1)^2k(k+1)}{(q^{n+1}-(k+1)(q-1)-1)} - 1$, that

$$(1+\delta)k < f(k,n,q) + \delta(q^{n-1} + q^{n-2} + \ldots + 1). \tag{7.13}$$

We will come back to the connection between complete caps in $\mathbf{PG}(n,q)$ and semi quadrangles in the future. For an excellent (updated) survey on bounds of complete caps in projective spaces and several related problems, see J. W. P. Hirschfeld and L. Storme [79].

Appendix: Semi $2N$-Gons

A semi quadrangle contains no substructure isomorphic to an ordinary 2-gon or 3-gon. With this property in mind, we could define a *semi $2N$-gon* of *order* $(s; t_1, t_2, \ldots, t_n)$ and with *μ-parameters* $(\mu_1, \mu_2, \ldots, \mu_m)$ to be an incidence structure satisfying (SQ1), and also the following conditions.

1. There is no substructure isomorphic to an ordinary M-gon, for $2 \leq M \leq 2N - 1$.
2. If two distinct points are not contained in a path of length $N - 1$ or less, then they are contained in exactly $\mu_1, \mu_2, \ldots,$ or μ_m paths of length N, where $\mu_j \geq 1$ for every j. Also, each of the cases occurs.
3. There are substructures isomorphic to an ordinary $2N$-gon and an ordinary $(2N + 1)$-gon.

With this definition, a *thick partial $2N$-gon* [33] is just a semi $2N$-gon with $s > 1$, $t_1 = t_2 = \cdots = t_n > 1$ and $\mu_1 = \mu_2 = \cdots = \mu_m$. Also, a generalized $2N$-gon is precisely a semi $2N$-gon with $t_i = \mu_j$ for arbitrary i and j.

Chapter 8

Span-Symmetric Generalized Quadrangles and the Solution of a Longstanding Open Problem

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If a (thick) generalized quadrangle $\mathcal{S}$ has two non-concurrent axes of symmetry, then $\mathcal{S}$ is called a 'span-symmetric generalized quadrangle'. In this chapter, we will prove the twenty-year-old conjecture that every span-symmetric generalized quadrangle of order s, $s \neq 1$, is classical, i.e. isomorphic to the generalized quadrangle $\mathcal{Q}(4, s)$ which arises from a nonsingular parabolic quadric in $\mathbf{PG}(4, s)$.

The results described in this chapter are taken from *K. Thas, Classification of span-symmetric generalized quadrangles of order s* [208], which appeared in

Advances in Geometry.

HISTORICAL REMARK

It was conjectured in 1980 by S. E. Payne that a span-symmetric generalized quadrangle of order $s > 1$ is isomorphic to $\mathcal{Q}(4, s)$. There was a "proof" of this theorem as early as in 1981 by S. E. Payne which appeared in [121], but later on, it was noticed by the author himself that there was a mistake in that proof. The paper was very valuable however, since the author introduced there the '4-gonal bases', see Section 8.2, and proved for instance Theorem 8.2.4 and Theorem 8.2.5, see below.
There was also an unpublished proof by W. M. Kantor. Recently, he has written down that proof, see [94] (which appeared in the same volume of *Advances in Geometry* as [208]).

8.1 The Main Results

In this chapter, we will prove the following main result.

Theorem 8.1.1 *Let $\mathcal{S}$ be a span-symmetric generalized quadrangle of order s, where $s \neq 1$. Then $\mathcal{S}$ is classical, i.e. isomorphic to $\mathcal{Q}(4, s)$.*

This leads to the complete classification of groups which have a *4-gonal basis* (as defined in Section 8.2).

Theorem 8.1.2 *A finite group is isomorphic to $\mathbf{SL}(2, s)$ for some s if and only if it has a 4-gonal basis.*

Both results will be essential for the rest of this thesis, especially for the development of a general theory for span-symmetric generalized quadrangles.

8.2 Span-Symmetric Generalized Quadrangles

Suppose $\mathcal{S}$ is a GQ of order (s, t), $s, t \neq 1$, and suppose L and M are distinct non-concurrent axes of symmetry; then it is easy to see by transitivity

that every line of $\{L, M\}^{\perp\perp}$ is an axis of symmetry, and $\mathcal{S}$ is called a *span-symmetric generalized quadrangle (SPGQ)* with *base-span* $\{L, M\}^{\perp\perp}$. Note that $|\{L, M\}^{\perp\perp}| = s + 1$, as L and M are regular lines.

Let $\mathcal{S}$ be a span-symmetric generalized quadrangle of order (s, t), $s, t \neq 1$, with base-span $\{L, M\}^{\perp\perp}$. Throughout this chapter, we will continuously use the following notation.
First of all, the base-span will always be denoted by $\mathcal{L}$. The group which is generated by all the symmetries about the lines of $\mathcal{L}$ is G, and sometimes we will call this group the *base-group* . This group clearly acts 2-transitively on the lines of $\mathcal{L}$, and fixes every line of $\mathcal{L}^{\perp}$. The set of all the points which are on lines of $\{L, M\}^{\perp\perp}$ is denoted by Ω (of course, Ω is also the set of points on the lines of $\{L, M\}^{\perp}$; we have that $|\{L, M\}^{\perp}| = |\{L, M\}^{\perp\perp}| = s + 1$). We will refer to $\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^{\perp}, I')$, with I' being the restriction of I to $(\Omega \times (\mathcal{L} \cup \mathcal{L}^{\perp})) \cup ((\mathcal{L} \cup \mathcal{L}^{\perp}) \times \Omega)$, as being *the base-grid* .
The following remarkable theorem states that the order of an SPGQ is essentially known.

Theorem 8.2.1 (W. M. Kantor [90]; see 10.7.4 of FGQ) *Suppose $\mathcal{S}$ is a span-symmetric generalized quadrangle of order (s, t), $s, t \neq 1$. Then $t \in \{s, s^2\}$.*

Remark 8.2.2 A proof of Theorem 8.2.1 is contained in Chapter 9.

Theorem 8.2.3 (S. E. Payne [121]; see also 10.7.2 of FGQ) *If $\mathcal{S}$ is an SPGQ of order s, $s \neq 1$, with base-group G, then G acts regularly on the set of $(s + 1)s(s - 1)$ points of $\mathcal{S}$ which are not on any line of $\mathcal{L}$.*

In the sequel, we will come to an analogue of this result for the general case, see Chapter 9.

Let $\mathcal{S}$ be an SPGQ of order $s \neq 1$ with base-span $\mathcal{L}$, and put $\mathcal{L} = \{U_0, U_1, \ldots, U_s\}$. The group of symmetries about U_i is denoted by G_i, $i = 0, 1, \ldots, s$, throughout this chapter. Then one notes the following properties (see [121] and 10.7.3 of FGQ):

1. the groups $G_0, G_1, \ldots, G_s$ form a complete conjugacy class in G, and are all of order s, $s \geq 2$;
2. $G_i \cap N_G(G_j) = \{1\}$ for $i \neq j$;
3. $G_i G_j \cap G_k = \{1\}$ for i, j, k distinct, and

4. $|G| = s^3 - s$.

We say that G is a group with a 4-*gonal basis* $\mathcal{T} = \{G_0, G_1, \ldots, G_s\}$ if these four conditions are satisfied.

It is possible to recover the SPGQ $\mathcal{S}$ of order s, $s > 1$, from the base-group G starting from 4-gonal bases in the following way, see [121, 139].
Suppose G is a group of order $s^3 - s$ with a 4-gonal basis $\mathcal{T} = \{G_0, G_1, \ldots, G_s\}$, and let $G_i^* = N_G(G_i)$ for $i = 0, 1, \ldots, s$. Define a point-line incidence structure $\mathcal{S}_\mathcal{T} = (P_\mathcal{T}, B_\mathcal{T}, I_\mathcal{T})$ as follows.

- $P_\mathcal{T}$ consists of two kinds of POINTS.
  (a) Elements of G.
  (b) Right cosets of the G_i^*'s.
- $B_\mathcal{T}$ consists of three kinds of LINES.
  (i) Right cosets of G_i, $0 \le i \le s$.
  (ii) Sets $M_i = \{G_i^* g \parallel g \in G\}$, $0 \le i \le s$.
  (iii) Sets $L_i = \{G_j^* g \parallel G_i^* \cap G_j^* g = \emptyset, 0 \le j \le s, j \ne i\} \cup \{G_i^*\}$, $0 \le i \le s$.
- INCIDENCE. $I_\mathcal{T}$ is the natural incidence: a line $G_i g$ of Type (i) is incident with the s points of Type (a) contained in it, together with that point $G_i^* g$ of Type (b) containing it. The lines of Types (ii) and (iii) are already described as sets of those points with which they are to be incident.

Then $\mathcal{S}_\mathcal{T} = (P_\mathcal{T}, B_\mathcal{T}, I_\mathcal{T})$ is a GQ of order s which is span-symmetric for the base-span $\{L_0, L_1\}^{\perp\perp}$ [121, 139]. Also, if $\mathcal{S}$ is an SPGQ of order s, $s \ne 1$, with base-span $\mathcal{L}$ and base-group G, and where $\mathcal{T}$ is the corresponding 4-gonal basis, then $\mathcal{S} \cong \mathcal{S}_\mathcal{T}$ [121, 139]. We thus have the following interesting theorem.

Theorem 8.2.4 (S. E. Payne [121]; see also 10.7.8 of FGQ) *An SPGQ of order $s \ne 1$ with given base-span $\mathcal{L}$ is canonically equivalent to a group G of order $s^3 - s$ with a 4-gonal basis $\mathcal{T}$.*

It is also important to recall the following.

Theorem 8.2.5 (S. E. Payne [121]; see also 10.7.9 of FGQ) *Let $\mathcal{S}$ be an SPGQ of order $s \neq 1$, with base-span $\mathcal{L}$. Then every line of $\mathcal{L}^\perp$ is an axis of symmetry.*

This theorem thus yields the fact that for any two distinct lines U and V of $\mathcal{L}^\perp$, the GQ is also an SPGQ with base-span $\{U,V\}^{\perp\perp}$. The corresponding base-group will be denoted by $G^\perp$. It should be emphasized that this property only holds for SPGQ's of order s, $s > 1$.

8.3 Split BN-Pairs of Rank 1

Recent developments in the theory of SPGQ's have shown that the notion of *(finite) split BN-pair of rank* 1 is particularly useful for various aims, as well as the theory of *perfect central extensions of (perfect) groups* [3, 96]. Let us recall that a group with a *split BN-pair of rank* 1 , or sometimes just a *split BN-pair of rank* 1, see e.g. [149, 222], is a (not necessarily faithful) permutation group (X, H) which satisfies the following properties:

(BN1) H acts 2-transitively on X, $|X| > 2$;

(BN2) for every $x \in X$ there holds that the stabilizer of x in H has a normal subgroup H^x which acts regularly on $X \setminus \{x\}$.

Note. In [225], J. Tits has introduced the notion of *Moufang set*, which is essentially the same object as a split BN-pair of rank 1.

The elements of X are the *points* of the split BN-pair of rank 1, and for any x, the group H^x will be called a *root group*. An element of the group which is generated by all the root groups is a *transvection*, and the group H is the *transvection group*. If X is a finite set, then the split BN-pair of rank 1 also is called *finite* (note that, if the permutation group (X, H) is faithful, then H is also finite if X is finite). The following theorem by E. E. Shult [156] and C. Hering, W. M. Kantor and G. M. Seitz [72] classifies all finite groups with a split BN-pair of rank 1 without invoking the classification of the finite simple groups.

Theorem 8.3.1 ([156]; [72]) *Suppose (X, H) is a finite group with a split BN-pair of rank* 1*, and suppose $|X| = s + 1$, with $s \in \mathbb{N}_0$. If H is generated by the root groups of the split BN-pair*[1]*, then H is always one of the following list (up to isomorphism):*

[1] In some papers, see e.g. [206], we did not mention the latter condition explicitly.

(*i*) *a sharply 2-transitive group on* X*;*

(*ii*) $\mathbf{PSL}(2,s)$*;*

(*iii*) *the Ree group* $\mathbf{R}(\sqrt[3]{s})$ *(also sometimes denoted by* ${}^2\mathbf{G_2}(\sqrt[3]{s})$*), with* $\sqrt[3]{s}$ *an odd power of* 3*;*

(*iv*) *the Suzuki group* $\mathbf{Sz}(\sqrt{s})$ *(sometimes denoted by* ${}^2\mathbf{B_2}(\sqrt{s})$*), with* $\sqrt{s}$ *an odd power of* 2*;*

(*v*) *the unitary group* $\mathbf{PSU}(3,\sqrt[3]{s^2})$*,*

each with its natural action of degree $s+1$ *(see [156, 72]).*

Every root group has order s. In the first case, (X,H) is a 2-transitive Frobenius group, and then $s+1$ is the power of a prime (see, e.g., [55]); in all of the other cases, s is the power of a prime. Further, we have that $|\mathbf{PSL}(2,s)| = (s+1)s(s-1)$ or $(s+1)s(s-1)/2$, according as s is even or odd, and the group acts (sharply) 3-transitively on X if and only if s is even; in the other cases, we have that $|\mathbf{R}(\sqrt[3]{s})| = (s+1)s(\sqrt[3]{s}-1)$, $|\mathbf{Sz}(\sqrt{s})| = (s+1)s(\sqrt{s}-1)$, and $|\mathbf{PSU}(3,\sqrt[3]{s^2})| = \frac{(s+1)s(\sqrt[3]{s^2}-1)}{gcd(3,\sqrt[3]{s}+1)}$ (For references on the orders of these groups, see [55], [78, p. 420-421] or Appendix E.)

Remark 8.3.2 (i) The root groups of $\mathbf{PSL}(2,s)$ and of the sharply 2-transitive groups are the only ones to be abelian (note that $\mathbf{Sz}(2)$ also has this property; this group acts sharply 2-transitively, however).

(ii) In the literature, sometimes also the (group theoretical) notation $\mathbf{PSL}_2(s)$, $\mathbf{SL}_2(s)$, etc., is used instead of, respectively, the (projective) notation $\mathbf{PSL}(2,s)$, $\mathbf{SL}(2,s)$, etc. (in various papers, e.g. [206], [208], [210], we used the first notation).

8.4 Remarks about 4-Gonal Bases

Now suppose G is a group of order s^3-s, $s>1$, where s is a power of the prime p, and suppose that G has a 4-gonal basis $\mathcal{T} = \{G_0, G_1, \ldots, G_s\}$. Since the groups G_i all have order s, all these groups are Sylow p-subgroups in G. Since $\mathcal{T}$ is a complete conjugacy class, this means that every Sylow p-subgroup of G is contained in $\mathcal{T}$, and hence G has *exactly* $s+1$ Sylow p-subgroups. Hence we have proved the following easy but important theorem.

Theorem 8.4.1 *Suppose G is a group of order $s^3 - s$, $s > 1$, with s a prime power. Then G can have at most one 4-gonal basis. In particular, if G has a 4-gonal basis, then it is unique.*

■

As a corollary we obtain

Theorem 8.4.2 *Suppose $\mathcal{S}$ is a span-symmetric generalized quadrangle of order s, $s \neq 1$. Then $\mathcal{S}$ is isomorphic to the classical GQ $\mathcal{Q}(4, s)$ if and only if the base-group is isomorphic to $\mathbf{SL}(2, s)$.*

Proof. Suppose that the base-group G is isomorphic to $\mathbf{SL}(2, s)$; then s is a power of a prime and hence by Theorem 8.4.1, $\mathbf{SL}(2, s)$ has at most one 4-gonal basis. Now consider $\mathcal{Q}(4, s)$ and suppose L and M are non-concurrent lines of $\mathcal{Q}(4, s)$. Then L and M are axes of symmetry, and hence $\mathcal{Q}(4, s)$ is span-symmetric for the base-span $\{L, M\}^{\perp\perp}$. In this case, the base-group is well-known to be isomorphic to $\mathbf{SL}(2, s)$, see e.g. [121] and Remark 8.4.3, which proves that $\mathbf{SL}(2, s)$ indeed *has* a 4-gonal basis, which is necessarily unique by Theorem 8.4.1. Hence, by Theorem 8.2.4, there is only one GQ (up to isomorphism) which can arise from $\mathbf{SL}(2, s)$ using 4-gonal bases in the usual sense, and this is $\mathcal{Q}(4, s)$, whence $\mathcal{S} \cong \mathcal{Q}(4, s)$. ■

Remark 8.4.3 The fact that the base-group corresponding to an arbitrary base-span $\mathcal{L}$ of $\mathcal{Q}(4, s)$ (in the usual sense), is isomorphic to $\mathbf{SL}(2, s)$, will also be shown implicitly in the sequel of this chapter; this will in fact be a direct corollary of the main theorem of this chapter.

8.5 Perfect and Universal Central Extensions of Perfect Groups

The following notions and results are taken from [3].
First recall that a *perfect group* is a group G which equals its derived group, denoted G'. Suppose G and H are groups. Then H is called a *central extension* of G if there is a surjective homomorphism

$$\phi : H \longrightarrow G$$

for which $ker(\phi) \leq Z(H)$ ($ker(\phi)$ is the kernel of the homomorphism ϕ, $Z(H)$ is the center of H). Sometimes the pair (H, ϕ) is also called a *central extension* of G. A central extension $(\overline{G}, \xi)$ of a group G is called *universal*, if for any other central extension (H, ξ') of G there exists a unique homomorphism

$$\psi : \overline{G} \longrightarrow H$$

such that the diagram defined by

$$\overline{G} \stackrel{\psi}{\longrightarrow} H \stackrel{\xi'}{\longrightarrow} G$$

and

$$\overline{G} \stackrel{\xi}{\longrightarrow} G$$

commutes. If a group G has a universal central extension $\overline{G}$, then $\overline{G}$ is known to be unique, up to isomorphism.

Theorem 8.5.1 ([3]) *A group G has a universal central extension if and only if it is perfect. The universal central extension of a group is always perfect if it exists.*

Using the preceding remarks and Theorem 8.5.1, it is possible to prove the following well-known result.

Theorem 8.5.2 ([3]) *Suppose G is a perfect group, and suppose $\overline{G}$ is its universal central extension. Furthermore, let H be a perfect group which is a central extension of G. Then there exists a subgroup N of the center $Z(\overline{G})$ of $\overline{G}$, such that*

$$\overline{G}/N \cong H.$$

8.6 The Base-Group G

From now on, we denote by N the kernel of the action of G on the lines of $\mathcal{L}$. The notation of Section 8.2 will be freely used. The following result is crucial:

Theorem 8.6.1 *Suppose $\mathcal{S}$ is a span-symmetric generalized quadrangle of order (s,t), $s,t \neq 1$, with base-span $\mathcal{L}$ and base-group G. Then G/N acts as a sharply 2-transitive group on $\mathcal{L}$, or is isomorphic, as a permutation group, to one of the following:*

(*i*) $\mathbf{PSL}(2,s)$;

(*ii*) *the Ree group* $\mathbf{R}(\sqrt[3]{s})$;

(*iii*) *the Suzuki group* $\mathbf{Sz}(\sqrt{s})$;

(*iv*) *the unitary group* $\mathbf{PSU}(3,\sqrt[3]{s^2})$,

each with its natural action of degree $s+1$.

Proof. The group G (and hence also G/N) is doubly transitive on $\mathcal{L}$, and for every $L \in \mathcal{L}$ the full group of symmetries about L, which acts regularly on $\mathcal{L} \setminus \{L\}$, is a normal subgroup of the stabilizer of L in G. This means that $(\mathcal{L}, G/N)$ is a split BN-pair of rank 1. Theorem 8.3.1 provides the above list of possibilities for G/N, noting that G/N is generated by the normal subgroups mentioned above. ∎

Lemma 8.6.2 *G is a perfect group if G/N does not act sharply 2-transitively on $\mathcal{L}$.*

Proof. Suppose G/N does not act sharply 2-transitively on $\mathcal{L}$. By Theorem 8.6.1, G/N is isomorphic to one of the following: (a) $\mathbf{PSL}(2,s)$; (b) $\mathbf{R}(\sqrt[3]{s})$; (c) $\mathbf{Sz}(\sqrt{s})$; (d) $\mathbf{PSU}(3,\sqrt[3]{s^2})$. All these groups are perfect groups, except when $G/N \cong \mathbf{R}(3)$ [65]. The latter case is not possible by [94], hence if $G/N \cong \mathbf{R}(\sqrt[3]{s})$, then we assume that $s \neq 27$. Assume that G is distinct from its derived group G'. Then since G/N is a perfect group, we have that

$$(G/N)' = G'N/N = G/N,$$

and hence $G'N = G$. First suppose we are in Case (a). If s is even, then $|G| = |\mathbf{PSL}(2,s)|$, and thus $|N| = \{\mathbf{1}\}$. So in that case $G = G'$, a contradiction. If s is odd, then G' is a subgroup of G of index 2. It follows that G and G' have exactly the same Sylow p-subgroups, with s a power of the odd prime p, as s is the largest power of p which divides $|G|$ and $|G'|$. Since here G is generated by its Sylow p-subgroups (by the definition of the base-group G), we infer that $G = G'$, a contradiction. Hence G is perfect.

Now suppose we are in Case (b) or (c). Then $|N| = \frac{s-1}{s^n-1}$ with $n \in \{1/2, 1/3\}$, and hence $|N|$ and s are mutually coprime since $s-1$ and s are mutually coprime. Hence s is a divisor of $|G'|$, since $|G| = \frac{|G'|\times|N|}{|G'\cap N|}$. Thus G and G' have precisely the same Sylow p-subgroups, with s a power of the prime p. Since here G is generated by its Sylow p-subgroups, we conclude that $G = G'$, a contradiction. Finally, assume that we are in the last case. Then $|N| = \frac{gcd(3, \sqrt[3]{s}+1)(s-1)}{\sqrt[3]{s^2}-1}$, and thus it is clear that $|N|$ and s are mutually coprime. The same argument as before yields that $|G'| \equiv 0 \mod s$, and hence that $G = G'$, a contradiction. Consequently G is perfect. ∎

Remark 8.6.3 For $s = 2$, the GQ is isomorphic to $\mathcal{Q}(4,2)$. In this case $G = G/N \cong S_3$ acts sharply 2-transitively on $\mathcal{L}$.

Lemma 8.6.4 *N is contained in the center of G.*

Proof. Clearly N is a normal subgroup of G. Let H be the full group of symmetries about an arbitrary line of $\mathcal{L}$. Then $N \cap H = \{\mathbf{1}\}$ and N and H normalize each other, thus they commute. As G is generated by the symmetries about the lines of $\mathcal{L}$, the lemma follows. ∎

Lemma 8.6.5 *If $\mathcal{S}$ is an SPGQ of order $s \neq 1$ with base-group G and base-span $\mathcal{L}$, then G/N acts either as $\mathbf{PSL}(2,s)$ or as a sharply 2-transitive group on the lines of $\mathcal{L}$.*

Proof. Assume by way of contradiction that G/N does not act as $\mathbf{PSL}(2,s)$ or a sharply 2-transitive group on the lines of $\mathcal{L}$. If $s = 27$, then we assume that $G/N \ncong \mathbf{R}(3)$ as before. First of all, G is a perfect group, and since N is in the center of G, the group G is a perfect central extension of the group G/N which acts on $\mathcal{L}$. The perfect group G/N has a universal central extension $\overline{G/N}$, and $\overline{G/N}$ contains a central subgroup F such that $\overline{G/N}/F \cong G$. We now look at the possible cases.

If $G/N \cong \mathbf{Sz}(\sqrt{s})$, and if $s > 8^2$, then N must be trivial since in that case the Suzuki group has a trivial universal central extension (i.e. $\overline{G/N} \cong G/N$) by [65] p. 302, see also Appendix E, an impossibility since the orders of G and $\mathbf{Sz}(\sqrt{s})$ are not the same if $s > 8^2$. Suppose that $s = 8^2$. Then by [65] p. 302 any perfect central extension H of $\mathbf{Sz}(8)$ satisfies $|H| = 2^k|\mathbf{Sz}(8)|$ for some $k \in \{0,1,2\}$. None of these cases occurs since $|G| = (64)^3 - 64 = 262080$ and since $|\mathbf{Sz}(8)| = 29120$.

If $G/N \cong \mathbf{R}(\sqrt[3]{s})$, then we have exactly the same situation as in the preceding (general) case (i.e. the universal central extension of $\mathbf{R}(\sqrt[3]{s})$ is trivial), compare [65] p. 302 or Appendix E, hence this case is excluded as well.

Finally, assume that $G/N \cong \mathbf{PSU}(3, \sqrt[3]{s^2})$. The universal central extension of $\mathbf{PSU}(3, \sqrt[3]{s^2})$ is known to be $\mathbf{SU}(3, \sqrt[3]{s^2})$, see [65] p. 302, and also, we know that $|\mathbf{SU}(3, \sqrt[3]{s^2})| = gcd(3, \sqrt[3]{s}+1)|\mathbf{PSU}(3, \sqrt[3]{s^2})| = (s+1)s(\sqrt[3]{s^2}-1)$. This provides us with a contradiction since $s > 1$, hence $s - 1 > \sqrt[3]{s^2} - 1$.
This proves the assertion. ∎

Lemma 8.6.6 *If G/N acts as $\mathbf{PSL}(2,s)$, then $G \cong \mathbf{SL}(2,s)$ and $\mathcal{S}$ is classical.*

Proof. The universal central extension of $\mathbf{PSL}(2,s)$ is $\mathbf{SL}(2,s)$, except in the cases $s = 4$ and $s = 9$, compare [65] p. 302, and in general $|\mathbf{SL}(2,s)| = gcd(2, s-1)|\mathbf{PSL}(2,s)| = |G|$. Hence if $s \neq 4, 9$, then G is isomorphic to $\mathbf{SL}(2,s)$, and by Theorem 8.4.2, $\mathcal{S}$ is classical.
There is a unique GQ of order 4, namely $\mathcal{Q}(4,4)$, see Chapter 1, so $s = 4$ gives no problem; in this case, G is isomorphic to $\mathbf{SL}(2,4)$. Finally, suppose that $s = 9$. Then there is only one possible perfect central extension of $G/N \cong \mathbf{PSL}(2,9)$ with size $9^3 - 9 = 234$, namely $\mathbf{SL}(2,9)$, see [65] p. 302. Hence $G \cong \mathbf{SL}(2,9)$, and by Theorem 8.4.2, $\mathcal{S}$ is classical and isomorphic to $\mathcal{Q}(4,9)$. ∎

8.7 The Sharply 2-Transitive Case

Recall that if $\mathcal{S}$ is an SPGQ of order $s \neq 1$ with base-span $\mathcal{L}$, then every line of $\mathcal{L}^\perp$ is an axis of symmetry.

Suppose that G/N acts as a sharply 2-transitive group on the lines of $\mathcal{L}$ in the SPGQ $\mathcal{S}$ of order $s > 1$. Since the lines of $\mathcal{L}^\perp$ are also axes of symmetry, we can assume that the base-group $G^\perp$ corresponding to these lines also acts as a sharply 2-transitive group on $\mathcal{L}^\perp$, because otherwise $G^\perp$ is isomorphic to $\mathbf{SL}(2,s)$, and then $\mathcal{S}$ is classical by Theorem 8.4.2. Hence G and $G^\perp$ contain normal central subgroups N and $N^\perp$ which act trivially on the points of Ω, both of order $s-1$ (where Ω is the set of points on the lines of the base-span). Note that G and $G^\perp$ act regularly on the points of $\mathcal{S}$ not in Ω by Theorem 8.2.3.

Let p be a point and L a line of a projective plane Π. Then Π is said to be (p, L)-*transitive* if the group of all collineations of Π with center p and axis L acts transitively on the points, distinct from p and not on L, of any line through

p. The following theorem is a step in the Lenz-Barlotti classification of finite projective planes, see e.g. [48, 161]; it states that the Lenz-Barlotti class III.2 is empty.

Theorem 8.7.1 (J. C. D. S. Yaqub [238]) *Let Π be a finite projective plane, containing a non-incident point-line pair (x, L) for which Π is (x, L)-transitive, and assume that Π is (y, xy)-transitive for every point y on L. Then Π is Desarguesian.*

As every axis of symmetry L is a regular line, there is a projective plane Π_L canonically associated with L as in Theorem 3.1.1. Hence

Theorem 8.7.2 *Suppose that $\mathcal{S}$ is an SPGQ of order s, where $s \neq 1$, with base-group G and base-span $\mathcal{L}$. Also, let N be the kernel of the action of G on the lines of $\mathcal{L}$, and suppose that G/N acts as a sharply 2-transitive group on the lines of $\mathcal{L}$. Then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(4,2)$ or $\mathcal{Q}(4,3)$.*

Proof. Fix a line L of $\mathcal{L}$, and consider the projective plane Π_L^* of order s, which is the dual of Π_L. Then $\mathcal{L}^\perp$ is a point of Π_L^* which is not incident with L as a line of the plane. For convenience, denote this point by p. Now consider the action of N as a collineation group on Π_L^*. Clearly, this action is faithful (recall that N fixes Ω pointwise). Then, as $|N| = s - 1$ and as N fixes L pointwise and p linewise, the plane Π_L^* is (p, L)-transitive. Now fix an arbitrary line U through p in Π_L^*; then U is a line of $\mathcal{L}^\perp$. If we interpret the group $G_U^\perp$ of all symmetries about U as a collineation group of Π_L^* (this is possible since $G_U^\perp$ fixes L), then $G_U^\perp$ fixes every line through the point $L \cap U$ of Π_L^*. Suppose r is an arbitrary point of Π_L^* on U and different from $L \cap U$. Then, in the GQ, r is of the form $\{U, U'\}^{\perp\perp}$, with U' some line of $L^\perp$ which does not meet U. It is clear that for any symmetry θ about U we have $(\{U, U'\}^{\perp\perp})^\theta = \{U, U'\}^{\perp\perp}$, and thus any element of $G_U^\perp$ as a collineation of Π_L^* fixes every point on the line U. From the fact that $|G_U^\perp| = s$, and that distinct elements of $G_U^\perp$ induce distinct collineations of Π_L^*, it follows that Π_L^* is $(U \cap L, U)$-transitive. Hence by Theorem 8.7.1 the plane Π_L^* is Desarguesian.
Now consider the action of the groups $G_V^\perp$ on Π_L^*, with $V \in \mathcal{L}^\perp$ (and where the notation is obvious). Then $G_V^\perp$ fixes the line L and the point $V \cap L$ and acts regularly on the other points of L. The group $G^\perp = \langle G_V^\perp \parallel V \in \mathcal{L}^\perp \rangle$, as a collineation group of the plane, induces a sharply 2-transitive permutation group on the points of L by our hypothesis. But since the plane Π_L^* is Desarguesian, we also know that the groups $G_V^\perp$, as collineation groups of the plane, generate a $\mathbf{PSL}(2, s)$ on L. For $s > 3$, the fact that $\mathbf{PSL}(2, s)$ acts sharply 2-transitively on L, implies that $|\mathbf{PSL}(2, s)| = (s + 1)s$, a contradiction.

Finally, suppose that $s = 2$, respectively $s = 3$. Then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(4,2)$, respectively to $\mathcal{Q}(4,3)$ (cf. Section 1.3 and the fact that $\mathcal{S}$ has regular lines). ■

8.8 Proof of the Main Theorem

Theorem 8.8.1 *Let $\mathcal{S}$ be an SPGQ of order s, where $s \neq 1$. Then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(4,s)$.*

Proof. Adopt the notation $G, N, G/N, \mathcal{L}$, etc. from above. By Lemma 8.6.5, G/N either acts as a sharply 2-transitive group on $\mathcal{L}$, or as $\mathbf{PSL}(2,s)$. If G/N acts as $\mathbf{PSL}(2,s)$, then $G \cong \mathbf{SL}(2,s)$ and $\mathcal{S}$ is classical by Lemma 8.6.6. If G/N acts sharply 2-transitively on $\mathcal{L}$, then by Theorem 8.7.2, $\mathcal{S} \cong \mathcal{Q}(4,2)$. Whence the result. ■

This leads to the complete classification of groups having a 4-gonal basis:

Theorem 8.8.2 *A finite group is isomorphic to $\mathbf{SL}(2,s)$ for some s if and only if it has a 4-gonal basis.*

■

Chapter 9

Generalized Quadrangles with Two Translation Points

In the present chapter, we will classify the generalized quadrangles with at least two distinct translation points. In order to obtain that main result, we prove many more general theorems which are useful for the theory of span-symmetric generalized quadrangles, and some more general versions of that theorem will be obtained. As a by-product of the proof of our main result, we will show that any span-symmetric generalized quadrangle $\mathcal{S}$ of order (s,t), $s \neq 1 \neq t \neq s$ and s odd, always contains at least $s+1$ classical subquadrangles of order s. We will obtain the analogous result for s even in Chapter 12 (the latter result being inessential here). In an appendix, we obtain an explicit construction of some classes of spreads for the point-line duals of the Kantor flock generalized quadrangles as a second by-product of the proof of our main result.

In Chapter 5, Chapter 6 and Chapter 7, we have studied the generalized quadrangles with some concurrent axes of symmetry in detail. In Chapter 8 we

then completely classified those generalized quadrangles of order $s > 1$ having non-concurrent axes of symmetry. As two of the main goals in this thesis are:

(i) to obtain a classification of generalized quadrangles with axes of symmetry;

(ii) to develop a theory for those generalized quadrangles with non-concurrent axes of symmetry, that is, the span-symmetric generalized quadrangles, ((ii) being a step towards (i)),

we proceed with the study of span-symmetric generalized quadrangles in the 'general' case. Recall at this point that W. M. Kantor (see Theorem 8.2.1) gave a partial classification theorem for span-symmetric generalized quadrangles by proving that for a span-symmetric generalized quadrangle of order (s, t), $s \neq 1 \neq t$, necessarily $t = s$ or $t = s^2$.

In Chapter 8, we completely classified the span-symmetric generalized quadrangles of order s, $s > 1$, by proving that every span-symmetric generalized quadrangle of order s is classical, i.e. isomorphic to $\mathcal{Q}(4, s)$. For span-symmetric generalized quadrangles of order (s, t) with $s \neq t$, $s \neq 1 \neq t$, however, a similar result cannot hold. For, let $\mathcal{K}$ be the quadratic cone with equation $X_0X_1 = X_2^2$ of $\mathbf{PG}(3, q)$, q odd. Then the q planes π_t with equation

$$tX_0 - mt^\sigma X_1 + X_3 = 0,$$

where $t \in \mathbf{GF}(q)$, m a given non-square in $\mathbf{GF}(q)$ and σ a given field automorphism of $\mathbf{GF}(q)$, define a flock $\mathcal{F}$ of $\mathcal{K}$, and the GQ which arises from $\mathcal{F}$ is the Kantor (flock) generalized quadrangle. If σ is the identity, the flock $\mathcal{F}$ is a linear flock and then the GQ is classical. Recently, S. E. Payne noted to us that the dual Kantor flock generalized quadrangles are span-symmetric, and this infinite class of generalized quadrangles contains non-classical examples. Moreover, every non-classical dual Kantor flock GQ even contains a line L for which every line which meets L is an axis of symmetry! Hence there is some line each point of which is a translation point. In order to initialize a theory for thick span-symmetric generalized quadrangles of order (s, t) with $s \neq t$, we therefore will first study the converse of the observation of S. E. Payne:

Classify all thick generalized quadrangles with distinct translation points.

Often in the existing literature, one speaks of 'the translation point' of a given TGQ $\mathcal{S}$. In the preceding chapter, we have shown that each SPGQ of order

$s = t > 1$ is classical, thus:

Each non-classical TGQ of order $s > 1$ has a unique, and whence well-defined, translation point.

Rather surprisingly, the (more involved) case of TGQ's of order (s, t) with $1 < s \neq t$ will yield non-classical examples with distinct translation points, thus TGQ's for which the notion of 'the translation point' is not well-defined! Moreover, in Chapter 10, we will completely determine those TGQ's.

Another goal of this chapter is to state elementary combinatorial and group theoretical conditions for a GQ $\mathcal{S}$ such that $\mathcal{S}$ essentially arises from a flock (see also J. A. Thas [177], [181], [183], [182], [186] and Chapters 6 and 12).

The present chapter is based on *K. Thas, The classification of generalized quadrangles with two translation points* [210], which is to appear in *Beiträge zur Algebra und Geometrie (Contributions to Algebra and Geometry).*

9.1 The Main Results

The following theorem will appear to be an essential tool in the theory of SPGQ's.

Theorem 9.1.1 *Suppose $\mathcal{S}$ is a span-symmetric generalized quadrangle of order (s, t), $s \neq 1 \neq t$, where $s \neq t$ and s is odd. Then $\mathcal{S}$ contains at least $s + 1$ subquadrangles isomorphic to the classical GQ $\mathcal{Q}(4, s)$.*

The main result reads as follows:

Theorem 9.1.2 *Suppose $\mathcal{S}$ is a generalized quadrangle of order (s, t), $s \neq 1 \neq t$, with two distinct collinear translation points. Then we have the following:*

(*i*) *$s = t$, s is a prime power and $\mathcal{S} \cong \mathcal{Q}(4, s)$;*

(*ii*) *$t = s^2$, s is even, s is a prime power and $\mathcal{S} \cong \mathcal{Q}(5, s)$;*

(*iii*) *$t = s^2$, $s = q^n$ with q odd, where $\mathbf{GF}(q)$ is the kernel of the TGQ $\mathcal{S} = \mathcal{S}^{(\infty)}$ with (∞) an arbitrary translation point of $\mathcal{S}$, $q \geq 4n^2 - 8n + 2$ and $\mathcal{S}$ is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ where $\mathcal{F}$ is a Kantor flock;*

(*iv*) $t = s^2$, $s = q^n$ *with* q *odd, where* $\mathbf{GF}(q)$ *is the kernel of the TGQ* $\mathcal{S} = \mathcal{S}^{(\infty)}$ *with* (∞) *an arbitrary translation point of* $\mathcal{S}$, $q < 4n^2 - 8n + 2$ *and* $\mathcal{S}$ *is the translation dual of the point-line dual of a flock GQ* $\mathcal{S}(\mathcal{F})$ *for some flock* $\mathcal{F}$.

If a thick GQ $\mathcal{S}$ *has two non-collinear translation points, then* $\mathcal{S}$ *is always of classical type, i.e. isomorphic to one of* $\mathcal{Q}(4,s)$, $\mathcal{Q}(5,s)$.

We give some variations and weakenings of the hypotheses of the main result in Section 9.12.

For s even the classification theorem is complete.
Whence

Corollary 9.1.3 *A generalized quadrangle of order* (s,t), $s \neq 1 \neq t$ *and* s *even, has two distinct collinear translation points if and only if* $\mathcal{S}$ *is isomorphic to* $\mathcal{Q}(4,s)$ *or* $\mathcal{Q}(5,s)$.

Recall that as s is even, $\mathcal{Q}(4,s) \cong W(s)$. In fact, for s even, one could restate Corollary 9.1.3 in the following more unified way.

Corollary 9.1.4 *A generalized quadrangle* $\mathcal{S}$ *of order* (s,t), $s \neq 1 \neq t$ *and* s *even, is isomorphic to one of* $H(3,s)$, $\mathcal{Q}(4,s) \cong W(s)$ *or* $\mathcal{Q}(5,s)$ *if and only if* $\mathcal{S}$ *is a TGQ for distinct elements of the same type.*

9.2 Setting of Notation

Let $\mathcal{S}$ be an SPGQ of order (s,t), $s,t \neq 1$, with base-span $\{L,M\}^{\perp\perp}$. In the sequel of this chapter, we will continuously use the following notation without further notice unless otherwise mentioned.
As in Chapter 8, the base-span is denoted by $\mathcal{L}$, and G denotes the base-group, which is generated by all the symmetries about the lines of $\mathcal{L}$. The set of all the points which are on lines of $\{L,M\}^{\perp\perp}$ is denoted by Ω, and the base-grid Γ is

$$\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^{\perp}, I'),$$

with I' being the restriction of I to $(\Omega \times (\mathcal{L} \cup \mathcal{L}^{\perp})) \cup ((\mathcal{L} \cup \mathcal{L}^{\perp}) \times \Omega)$. Put $\mathcal{L} = \{U_0, U_1, \ldots, U_s\}$ and suppose G_i is the group of symmetries about U_i for all $i = 0, 1, \ldots, s$. Finally, the elementwise stabilizer of $\mathcal{L}$ in G is denoted by N.

9.3 SPGQ's and Split BN-pairs of Rank 1

We start with recalling Theorem 8.6.1 from the preceding chapter, as that observation will be one of the crucial tools for the present chapter.

Theorem 9.3.1 *Suppose $\mathcal{S}$ is a span-symmetric generalized quadrangle of order (s,t), $s,t \neq 1$, with base-span $\mathcal{L}$ and base-group G. Then G/N either acts as a sharply 2-transitive group on $\mathcal{L}$, or is isomorphic, as a permutation group, to one of the following:*

(*i*) $\mathbf{PSL}(2,s)$;

(*ii*) *the Ree group* $\mathbf{R}(\sqrt[3]{s})$;

(*iii*) *the Suzuki group* $\mathbf{Sz}(\sqrt{s})$;

(*iv*) *the unitary group* $\mathbf{PSU}(3,\sqrt[3]{s^2})$,

each with its natural action of degree $s+1$.

■

9.4 A Lemma Concerning the Order of the Base-Group

It appears that the base-group of an SPGQ is always quite 'big'. It is the main objective of this section to illustrate this. Recall the notation of Chapter 8.

Lemma 9.4.1 *Let $\mathcal{S}$ be an SPGQ of order (s,t), $s \neq 1 \neq t$, with base-grid $\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^{\perp}, I')$ and base-group G, and let U_i, G_j, etc. be as in Section 9.2. If p is a point which is not an element of Ω, and U is a line through p which meets Ω in a certain point qIU_k of Γ, then every point on U which is different from q is a point of the G-orbit which contains p.*

Proof. The group G_k acts transitively on the points of U different from q. ■

Hence

Lemma 9.4.2 *Suppose $\mathcal{S}$ is an SPGQ of order (s,t), $s \neq 1 \neq t$, with base-grid Γ and base-group G. Then G has size at least $s^3 - s$.*

Proof. If $s = t$, then we already know by Section 8.2 (Theorem 8.2.3) that G has order $s^3 - s$, so suppose $s \neq t$.
Set $\mathcal{L} = \{U_0, U_1, \ldots, U_s\}$ and suppose G_i is the group of symmetries about U_i for all feasible i. Suppose p is a point of $\mathcal{S}$ not incident with a line of $\mathcal{L}$, and consider the following $s+1$ lines $M_i := [p, U_i]$. If Λ is the G-orbit which contains p, then by Lemma 9.4.1 every point of M_i not in Ω is also a point of Λ. Now consider an arbitrary point $q \neq p$ on M_0 which is not on a line of $\mathcal{L}$. Then again every point of $[q, U_i]$ not in Ω is a point of Λ. Hence we have the following inequality:

$$|\Lambda| \geq 1 + (s+1)(s-1) + (s-1)^2 s, \tag{9.1}$$

from which it follows that $|\Lambda| \geq s^3 - s^2 + s$. Now fix a line U of $\mathcal{L}^\perp$. Every line of $\mathcal{S}$ which meets this line and which contains a point of Λ is completely contained in $\Lambda \cup \Omega$ by Lemma 9.4.1. Also, G acts transitively on the points of U. Suppose k is the number of lines through a (= every) point of U which are completely contained in $\Lambda \cup \Omega$ (as point sets). If we count in two ways the number of point-line pairs (u, M) for which $u \in \Lambda, M \sim U$ and uIM, then it follows that

$$\begin{array}{rcl} k(s+1)s & = & |\Lambda| \geq s(s^2 - s + 1) \\ & \Longrightarrow & k \geq \dfrac{s^2 - s + 1}{s+1} = s - 2 + \dfrac{3}{s+1}, \end{array}$$

and hence, since $k \in \mathbb{N}$, we have that $k \geq s - 1$. Thus $|\Lambda| \geq s^3 - s$ and so also $|G|$. ∎

9.5 The Cases $s = 2$, $s = 3$ and $s = 4$

Suppose $\mathcal{S}$ is a GQ of order $(2, t)$, $t \geq 2$. Then by Section 1.3 we have that $t \in \{2, 4\}$, and $\mathcal{S}$ is isomorphic to $\mathcal{Q}(4, 2)$, respectively $\mathcal{Q}(5, 2)$. Next, suppose that $\mathcal{S}$ is of order $(3, t)$, $t \neq 1$, and suppose $\mathcal{S}$ is an SPGQ for some base-span $\mathcal{L}$. Then by Theorem 8.2.1, $t \in \{3, 9\}$, and as there is a unique GQ (up to duality) of order 3, respectively $(3, 9)$, namely the classical $\mathcal{Q}(4, 3)$, respectively $\mathcal{Q}(5, 3)$ (see Section 1.3), and since the GQ $W(3)$ has no regular lines, we have that

Any SPGQ of order (s, t) with $t \neq 1$ and $s \in \{2, 3\}$ is classical.

The following theorem classifies all TGQ's of order $(4, 16)$.

Theorem 9.5.1 (M. Lavrauw and T. Penttila [100]) *Any translation generalized quadrangle of order* $(4, 16)$ *is isomorphic to* $\mathcal{Q}(5, 4)$.

Finally, each GQ of order 4 is isomorphic to $\mathcal{Q}(4, 4) \cong W(4)$, see Section 1.3. In the following we will sometimes suppose that $s \neq 2, 3, 4$ if this seems convenient.

9.6 The Non-Semiregular Case

Suppose that $\mathcal{S}$ is a span-symmetric generalized quadrangle of order (s, t), $s, t \neq 1$, with base-span $\{L, M\}^{\perp\perp} = \mathcal{L}$, base-group G and base-grid $\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^{\perp}, I')$. Furthermore, put $\mathcal{L} = \{U_0, U_1, \ldots, U_s\}$ and assume G_i to be the group of symmetries about U_i for all $i = 0, 1, \ldots, s$. Since the case $s = t$ is completely settled by Chapter 8 (cf. Theorem 8.8.1), we suppose that $s \neq t$ for convenience. Thus, by Theorem 8.2.1, we have that $t = s^2$.

In this section, it is our aim to exclude the possibility that G does not act semiregularly on the points of $\mathcal{S} \setminus \Omega$ if s is odd. In the even case, we will start from a slightly different situation.

First, we recall an interesting structure theorem for SPGQ's.

Theorem 9.6.1 (FGQ, 10.7.1) *Let* $\mathcal{S}$ *be an SPGQ of order* (s, t), $s \neq 1 \neq t$, *with base-span* $\mathcal{L}$ *and base-group* G. *If* $\theta \neq \mathbf{1}$ *is an element of* G, *then the substructure* $\mathcal{S}_\theta = (P_\theta, B_\theta, I_\theta)$ *of elements fixed by* θ *must be given by one of the following.*

(*i*) $P_\theta = \emptyset$ *and* B_θ *is a partial spread containing* $\mathcal{L}^{\perp}$.

(*ii*) *There is a line* $L \in \mathcal{L}$ *for which* P_θ *is the set of points incident with* L, *and* $M \sim L$ *for each* $M \in B_\theta$ $(\mathcal{L}^{\perp} \subseteq B_\theta)$.

(*iii*) B_θ *consists of* $\mathcal{L}^{\perp}$ *together with a subset* B' *of* $\mathcal{L}$; P_θ *consists of those points incident with lines of* B'.

(*iv*) $\mathcal{S}_\theta$ *is a subGQ of order* (s, t') *with* $s \leq t' < t$. *This forces* $t' = s$ *and* $t = s^2$.

Suppose $\theta \neq \mathbf{1}$ is an element of G which fixes a point q of $\mathcal{S} \setminus \Omega$. Then by Theorem 9.6.1 the fixed element structure of θ is a subGQ $\mathcal{S}_\theta$ of order s. It is clear that $\mathcal{S}_\theta$ is also span-symmetric with respect to the same base-span. Hence $G_\theta = G/N_\theta$, with N_θ the kernel of the action of G on $\mathcal{S}_\theta$ (we can speak of "an action" since G fixes $\mathcal{S}_\theta$), has order $s^3 - s$ by Chapter 8; G_θ is *precisely* the base-group corresponding to $\mathcal{L}$ seen as a base-span of $\mathcal{S}_\theta$. Since by Theorem 8.8.1, $\mathcal{S}_\theta \cong \mathcal{Q}(4, s)$, we obtain

$$G_\theta \cong \mathbf{SL}(2, s).$$

Next, let x be an arbitrary point of $\mathcal{S} \setminus \mathcal{S}_\theta$, and consider the set of points $V = x^\perp \cap \mathcal{S}_\theta$ (and note that $|V| = t + 1 = s^2 + 1$, as $\mathcal{S}_\theta$ is a GQ of order s). Then x cannot be fixed by θ by Theorem 1.6.4, and $\{x, x^\theta\} \subseteq V^\perp$. As $\mathcal{S}$ has order (s, s^2), and hence $|\{x, x^\theta\}^{\perp\perp}| = 2$, and as N_θ acts semiregularly on the points of $\mathcal{S}$ outside $\mathcal{S}_\theta$, N_θ has size 2 if N_θ is not trivial. So, N_θ is a normal subgroup of G of order 2 and N_θ is thus contained in the center of G.
We obtain the following pathological situation:

- $G_\theta = G/N_\theta \cong \mathbf{SL}(2, s)$;
- $|N_\theta| = 2$ and, as a normal subgroup of order 2 of G, N_θ is contained in the center $Z(G)$ of G;
- $|G| = 2(s^3 - s)$.

Lemma 9.6.2 *Suppose $\mathcal{S}$ is an SPGQ of order (s, t), $s \neq 1 \neq t$, with base-span $\mathcal{L}$ and base-group G, and suppose s is a power of the prime p. If s is the largest power of p which divides $|G|$, then the groups of symmetries about the lines of $\mathcal{L}$ are precisely the Sylow p-subgroups of G, and hence G is generated by its Sylow p-subgroups.*

Proof. Put $\mathcal{L} = \{U_0, U_1, \ldots, U_s\}$, and suppose G_i is the full group of symmetries about U_i, $i = 0, 1, \ldots, s$. Since the groups G_j all have order s, all these groups are Sylow p-subgroups in G. One easily observes that the set $\mathcal{T} = \{G_0, G_1, \ldots, G_s\}$ is a complete conjugacy class in G, implying that every Sylow p-subgroup of G is contained in $\mathcal{T}$. Whence G has *exactly* $s + 1$ Sylow p-subgroups ∎

Lemma 9.6.3 *If G does not act semiregularly on $\mathcal{S} \setminus \Omega$, then G is perfect if $s > 3$ and s is odd.*

Proof. Suppose G does not act semiregularly on $\mathcal{S} \setminus \Omega$, and suppose $G \neq G'$. Then $(G/N_\theta)' = G'N_\theta/N_\theta = G/N_\theta$ (the group $\mathbf{SL}(2,s)$ is perfect if $s \neq 2,3$, see [65]), and hence $G'N_\theta = G$, and G' is a subgroup of G of index 2. Since s is odd, G and G' have exactly the same Sylow p-subgroups, where s is a power of the odd prime p. The group G is generated by its subgroups G_i, and since these are precisely the Sylow p-subgroups by Lemma 9.6.2, $G = G'$, a contradiction. Hence G is perfect. ∎

Lemma 9.6.4 *G acts semiregularly on $\mathcal{S} \setminus \Omega$ if s is odd.*

Proof. First suppose that $s = 3$. Then $\mathcal{S} \cong \mathcal{Q}(5,3)$ by Section 9.5, and it is well-known that $G \cong \mathbf{SL}(2,3)$. Hence G acts semiregularly on $\mathcal{S} \setminus \Omega$ by the proof of Lemma 9.4.2 and the fact that $|\mathbf{SL}(2,q)| = q^3 - q$ for arbitrary q.
Next suppose that G does not act semiregularly on the points of $\mathcal{S} \setminus \Omega$, and suppose that $\mathcal{S}$ is of order (s, s^2), $1 < s \neq 3, 9$ and s odd. We then know that $G/N_\theta \cong \mathbf{SL}(2,s)$ with s a power of an odd prime p and $s \neq 3, 9$, and where N_θ is a central subgroup of order 2. The group G is perfect by Lemma 9.6.3 and has size $2(s^3 - s)$, see above. The universal central extension of $\mathbf{SL}(2,s)$ coincides with $\mathbf{SL}(2,s)$ if $s \neq 4, 9$, see [65], and this contradicts the fact that $|G| = 2(s^3 - s)$. Hence G *does* act semiregularly on the points of $\mathcal{S} \setminus \Omega$.
Finally, suppose that $s = 9$. It is a well-known fact, see e.g. [65], that, if $\overline{\mathbf{PSL}(2,9)}$ is the universal central extension of $\mathbf{PSL}(2,9)$, four possibilities arise for the perfect central extension G of $\mathbf{PSL}(2,9)$:

(i) $G = \mathbf{PSL}(2,9)$;

(ii) $G \cong \overline{\mathbf{PSL}(2,9)}/C_2$;

(iii) $G \cong \mathbf{SL}(2,9) \cong \overline{\mathbf{PSL}(2,9)}/C_3$;

(iv) $G \cong \overline{\mathbf{PSL}(2,9)}$.

Suppose that G does not act semiregularly on $\mathcal{S} \setminus \Omega$. Then by Lemma 9.6.3 G is a perfect group, $G_\theta = G/N_\theta \cong \mathbf{SL}(2,9)$, and $N_\theta \leq Z(G)$ as a normal subgroup of G of size 2. Hence G is a perfect central extension of $\mathbf{SL}(2,9)$, and $|G| = 4|\mathbf{PSL}(2,9)| = 2|\mathbf{SL}(2,9)|$. This is clearly impossible. Hence G acts semiregularly on $\mathcal{S} \setminus \Omega$. ∎

For s even, we obtain the following less general result, which will be sufficient for our purposes for the time being. In Chapter 12, we will obtain the result in its full generality, however. As the main occupation of this chapter is to classify those GQ's with more than one translation point, we find it more convenient

to utilize a recent result by J. A. Thas on TGQ's $\mathcal{S}^{(x)}$ of order (s, s^2), $s > 1$ and s even, which have more than one classical subGQ of order s containing x.

Lemma 9.6.5 *Suppose $\mathcal{S}$ is a GQ of order (s,t), where $s \neq t$ and $s \neq 1 \neq t$, and suppose L is a line of which every point is a translation point. Furthermore, fix two non-concurrent lines U and V of $L^\perp$ and suppose that s is even. If G is the base-group which is defined by the base-span $\{U, V\}^{\perp\perp}$, and G does not act semiregularly on the points of $\mathcal{S} \setminus \Omega$, then $\mathcal{S}$ is classical, i.e. isomorphic to $\mathcal{Q}(5, s)$.*

Proof. If G does not act semiregularly on $\mathcal{S} \setminus \Omega$, then by the same argument as in the odd case, it follows that $\mathcal{S}_\theta$ is a classical subGQ of $\mathcal{S}$ of order s, with $\mathcal{S}_\theta$ as above. Since L contains a line of translation points, it follows easily that $\mathcal{S}$ contains at least two such classical subGQ's which contain a fixed translation point (∞) of $\mathcal{S}$, but which do not have the same line set on (∞). Hence by Theorem 1.7.7, we have that $\mathcal{S}$ is isomorphic to the classical $\mathcal{Q}(5, s)$.■

Note. It follows that this case does not occur since G acts semiregularly on $\mathcal{S} \setminus \Omega$ if $\mathcal{S} \cong \mathcal{Q}(5, s)$.

From now on we can hence suppose that G acts semiregularly on $\mathcal{S} \setminus \Omega$.

9.7 The Sharply 2-Transitive Case

In this section it is our goal to exclude the case where G/N acts sharply 2-transitively on the lines of $\mathcal{L}$ if $s > 3$. Recall that N is the kernel of the action of G on the lines of $\mathcal{L}$.

Lemma 9.7.1 *Let $\mathcal{S}$ be an SPGQ of order (s,t), $s \neq 1 \neq t \neq s$, with base-span $\mathcal{L}$, base-grid $\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^\perp, I')$ and base-group G. Then G/N cannot act as a sharply 2-transitive group on the lines of $\mathcal{L}$ if $s > 3$.*

Proof. Suppose G/N acts as a sharply two-transitive group on the lines of $\mathcal{L}$. Then $|G/N| = (s+1)s$. Hence, since $|G| \geq s^3 - s$, we have that $|N| \geq s-1$. Let q be an arbitrary point of $\mathcal{S} \setminus \Omega$, and define V as $V := q^\perp \cap \Omega$ (so $|V| = s+1$). Then since G acts semiregularly on the points of $\mathcal{S} \setminus \Omega$, $|N| = |q^N|$, and $q^N \subseteq V^\perp$. As $\mathcal{S}$ is a thick SPGQ of order (s,t) with $s \neq t$, Theorem 8.2.1 infers that $t = s^2$, hence every triad of points has exactly $s+1$ centers. So, we immediately have that $|N| = |q^N| \leq s+1$.
Now suppose that $|N| \neq s-1$, so that $|N| \in \{s, s+1\}$. First suppose $|N| = s$.

Then the order of G is $s^2(s+1)$, and by the semiregularity condition this must be a divisor of $|\mathcal{S} \setminus \Omega| = (s+1)(s^3-s)$, clearly a contradiction. Hence $|N| = s+1$. Assume that s is odd.
Since G/N is supposed to be a sharply 2-transitive group in its action on $\mathcal{L}$, $|G| = (s+1)^2 s$. Suppose that Λ is a G-orbit in $\mathcal{S} \setminus \Omega$; as G acts semiregularly on $\mathcal{S} \setminus \Omega$, $|\Lambda| = |G|$. Consider an arbitrary point p in Λ. Then every point of $\mathbf{X} = p^N$ is collinear with every point of $\mathbf{Y} = p^\perp \cap \Omega$, and we denote the set of points which are on a line of the form uv with $u \in \mathbf{X}$ and $v \in \mathbf{Y}$ by $\mathbf{XY}$. It is clear that $\mathbf{XY} \setminus \mathbf{Y}$ is completely contained in Λ, and that the order of this set is $(s+1)s^2$. Now take a point q of Λ outside $\mathbf{XY}$. The points of $\mathbf{X}$ and $\mathbf{Y}$ are the points of a dual grid with parameters $s+1, s+1$, and hence, if x, y and z are arbitrary distinct points of $\mathbf{Y}$ (or of $\mathbf{X}$), the triad $\{x, y, z\}$ is 3-regular. Put $q^N = \{q = q^0, q^1, q^2, \ldots, q^s\}$. If q^i is an arbitrary point of q^N, then $|\mathbf{Y} \cap (q^i)^\perp| =: k_{q_i} \leq 2$. One notes that $k_{q_i} = k_{q_j} =: k$ for some constant k, and that $\mathbf{Y} \cap (q^i)^\perp = \mathbf{Y} \cap (q^j)^\perp$, for all i and j, by the action of N. If W is an arbitrary line through q which intersects Ω, then W does not contain a point of $\mathbf{X}$ since this would imply that q is not outside $\mathbf{XY}$. Applying Theorem 1.1.2 (and recalling the fact that s is odd), we count the number of points which are collinear with a point of q^N and which are contained in Λ, together with the points of $\mathbf{XY} \cap \Lambda$. One notes that every point of q^N is collinear with every point of $q^\perp \cap \Omega$, and also that q^N is skew to $\mathbf{XY}$. We obtain the following.

$$|\Lambda| = (s+1)^2 s \geq (s+1)s^2 + s + 1 + k(s+1)(s-1) +$$

$$(s+1-k)(s+1)(s-3),$$

with $k \in \{0, 1, 2\}$, asserting that $s < 4$, a contradiction. Hence this case is excluded. Next, suppose that s is even.
From the fact that $|N| = s+1$, follows that $\mathcal{S}$ contains a 3-regular triad, and hence a subGQ $\mathcal{S}'$ of order s (by Section 1.6.3). Thus $|\mathbf{XY}| = |\mathcal{S}'|$. Take a point q of Λ outside $\mathbf{XY}$. Then every line which is incident with q and which intersects Ω has only that intersection point in common with $\mathcal{S}'$. Counting the number of points which are collinear with a point of q^N and which are contained in Λ, together with the points of $\mathbf{XY} \cap \Lambda$, we get the following inequality.

$$|\Lambda| = (s+1)^2 s \geq (s+1)s^2 + s + 1 + (s+1)^2(s-1),$$

and thus $s \geq s^3$, a contradiction if $s \geq 2$. Finally, suppose $|N| = s-1$ (so $|\Lambda| = (s+1)s(s-1)$). Define $\mathcal{S}' = (P', B', I')$ as in Section 9.8. Then by Section 9.8, $\mathcal{S}'$ is a subGQ of $\mathcal{S}$ of order s, and $\mathcal{S}'$ is isomorphic to $\mathcal{Q}(4, s)$. The

main theorem of Chapter 8 infers that $G \cong \mathbf{SL}(2,s)$, contradiction, as G/N then only acts sharply 2-transitively on $\mathcal{L}$ if $s \in \{2,3\}$. The proof is complete. ■

Corollary 9.7.2 *Let $\mathcal{S}$ be a span-symmetric generalized quadrangle of order (s,t), $s \neq 1 \neq t$. Then $s = t$ or $t = s^2$, and s and t are powers of the same prime.*

Proof. If $\mathcal{S}$ is an SPGQ of order (s,t), $s \neq 1 \neq t$, then $s = t$ or $t = s^2$ by Theorem 8.2.1. If $s = t$, then by Theorem 8.8.1, $\mathcal{S} \cong \mathcal{Q}(4,s)$, and hence s is a prime power. Now suppose $t = s^2$. Then by Theorem 9.7.1 and Theorem 9.3.1 the theorem follows. ■

9.8 Construction of Subquadrangles

Suppose $\mathcal{S}$ is an SPGQ of order (s,s^2), $s \neq 1$, with base-span $\mathcal{L} = \{U,V\}^{\perp\perp}$, base-group G and base-grid $\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^{\perp}, I')$. Then G acts semiregularly on the points of $\mathcal{S} \setminus \Omega$. Assume that G has order $(s+1)s(s-1)$. Let Λ be an arbitrary G-orbit in $\mathcal{S} \setminus \Omega$, and fix a line W of $\mathcal{L}^{\perp}$. By the semiregularity of G on the points of $\mathcal{S} \setminus \Omega$, the fact that $|G| = (s+1)s(s-1)$ and the fact that G acts transitively on the points incident with W, we have that any point on W is incident with exactly $s-1$ lines of $\mathcal{S}$ which are completely contained in Λ except for the point on W which is in Ω, and every point of Λ is incident with a line which meets W (recall that G is generated by groups of symmetries). Now define the following incidence structure $\mathcal{S}' = (P', B', I')$.

- LINES. The elements of B' are the lines of $\mathcal{S}'$ and they are essentially of two types:
  1. the lines of $\mathcal{L}^{\perp} \cup \mathcal{L}^{\perp\perp}$;
  2. the lines of $\mathcal{S}$ which contain a point of Λ and a point of Ω.
- POINTS. The elements of P' are the points of the incidence structure and they are just the points of $\Omega \cup \Lambda$.
- INCIDENCE. Incidence I' is the induced incidence.

Then by the previous lemmas and observations, any point of $\mathcal{S}'$ is incident with $s+1$ lines of $\mathcal{S}'$ and any line of $\mathcal{S}'$ is incident with $s+1$ points of the structure, and there are exactly $(s+1)(s^2+1)$ points and equally as many lines. Whence one can easily conclude that $\mathcal{S}'$ is a generalized quadrangle of order s (since it is an induced subgeometry of a GQ, it cannot contain triangles).

Remark 9.8.1 By Theorem 8.8.1, the GQ $\mathcal{S}'$ is isomorphic to the GQ $\mathcal{Q}(4,s)$, as $\mathcal{S}'$ is clearly span-symmetric for the base-span $\mathcal{L}$.

Hence

Theorem 9.8.2 *Suppose $\mathcal{S}$ is a span-symmetric generalized quadrangle with base-span $\mathcal{L}$ and of order (s,t), $s \neq 1 \neq t$ and $s \neq t$. Assume that G has size s^3-s. Then there exist $s+1$ subquadrangles of order s which are all isomorphic to $\mathcal{Q}(4,s)$, so that they mutually intersect in the points and lines of $\mathcal{L}^\perp \cup \mathcal{L}$.*

Proof. From each G-orbit in $\mathcal{S} \setminus \Omega$ there arises a subGQ of order s which is classical by Remark 9.8.1 (i.e. isomorphic to $\mathcal{Q}(4,s)$). There are exactly $s+1$ such distinct G-orbits, and G fixes each orbit. ∎

We now have

Lemma 9.8.3 *Suppose that $\mathcal{S}$ is a span-symmetric generalized quadrangle of order (s,s^2), $s>3$, with base-span $\mathcal{L}$, base-group G and base-grid $\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^\perp, I')$. Then G is perfect.*

Proof. Let N be the kernel of the action of G on $\mathcal{L}$. As G/N does not act as a sharply 2-transitive group on $\mathcal{L}$, Theorem 9.3.1 infers that G/N is isomorphic to one of the following list: (a) $\mathbf{PSL}(2,s)$, (b) $\mathbf{R}(\sqrt[3]{s})$, (c) $\mathbf{Sz}(\sqrt{s})$, or (d) $\mathbf{PSU}(3,\sqrt[3]{s^2})$, each with its natural permutation representation of degree $s+1$. All these groups are perfect groups, except when $s=27$ and $G/N \cong \mathbf{R}(3)$, see [65]. We may exclude the latter case by [94]. Hence, since G/N is a perfect group,

$$(G/N)' = (G/N) = G'N/N \cong G'/(G' \cap N),$$

and thus $G'N = G$.
Since G acts semiregularly on $\mathcal{S} \setminus \Omega$, the natural number

$$|G| = |G/N| \times |N| = \frac{(s^n-1)(s+1)s|N|}{r},$$

with $r \in \{1, 2, gcd(3, \sqrt[3]{s}+1)\}$ and $n \in \{1, 2/3, 1/2, 1/3\}$, is a divisor of $|\mathcal{S} \setminus \Omega| = (s+1)(s^3-s)$, where $r=2$ if and only if s is odd and $G/N \cong \mathbf{PSL}(2,s)$ and where $r = gcd(3, \sqrt[3]{s}+1)$ if and only if $G/N \cong \mathbf{PSU}(3,\sqrt[3]{s^2})$. Hence, we have that

$$r(s^2-1)/(s^n-1) \equiv 0 \mod |N|. \tag{9.2}$$

First suppose that s is odd and that $G/N \not\cong \mathbf{PSL}(2,s)$.
If $r = gcd(3, \sqrt[3]{s}+1)$ and $G/N \cong \mathbf{PSU}(3, \sqrt[3]{s^2})$, then s and $|N|$ have a non-trivial common divisor if and only if $r = 3$ and if 3 is a divisor of s, clearly in contradiction with $3 = gcd(3, \sqrt[3]{s}+1)$. It follows now immediately from (9.2) that $|N|$ and s are coprime if s is odd and $G/N \not\cong \mathbf{PSL}(2,s)$. Hence with $s = p^h$ for the odd prime p and $h \in \mathbb{N}_0$, s is the largest power of p which divides $|G|$. Thus the full groups of symmetries about the lines of $\mathcal{L}$ are exactly the Sylow p-subgroup of G by Lemma 9.6.2. Now suppose $G \neq G'$.
We know that $|N|$ and s are coprime, so since $|G'| = (|G| \times |G' \cap N|)/|N|$, there follows that $|G'| \equiv 0 \mod s$, whence G and G' have *exactly* the same Sylow p-subgroups. But $G' \leq G$ and G is generated by its Sylow p-subgroups, so $G = G'$, a contradiction. Hence G is perfect.

Next, suppose that s is odd and that $G/N \cong \mathbf{PSL}(2,s)$.
Then we know that $|G| = |G/N| \times |N| = |N| \times |\mathbf{PSL}(2,s)| = |N| \times \frac{(s^3-s)}{2}$ is a divisor of $(s+1)(s^3-s)$, and again s and $|N|$ are mutually coprime. In the same way as before, it follows now that G is a perfect group. ∎

Remark 9.8.4 For $s = 2$ or $s = 3$, $G/N \cong \mathbf{PSL}(2,2)$ or $\mathbf{PSL}(2,3)$, respectively, since $\mathcal{S}$ is then classical, and in both cases G/N acts sharply 2-transitively on $\mathcal{L}$.

Lemma 9.8.5 *Suppose $\mathcal{S}$ is an SPGQ of order (s,t), $s \neq 1 \neq t$, with base-span $\mathcal{L}$ and base-group G. If N is the kernel of the action of G on the lines of $\mathcal{L}$, then N is in the center of G.*

Proof. Fix non-concurrent lines U and U' of $\mathcal{L}$, and suppose that N is the kernel of the action of G on the lines of $\{U,U'\}^{\perp\perp}$ (so N fixes every point of Ω). Then N is a normal subgroup of G. Let H be the full group of symmetries about an arbitrary line M of $\{U,U'\}^{\perp\perp}$. Then N and H normalize each other, and hence they commute. As G is generated by all such groups H, the result follows. ∎

Lemma 9.8.6 *If $\mathcal{S}$ is an SPGQ of order (s,t), $s \neq 1 \neq t$ and $s \neq t$, with base-group G and base-span $\mathcal{L}$, then G/N acts as $\mathbf{PSL}(2,s)$ on $\mathcal{L}$.*

Proof. By Remark 9.8.4, we can suppose that $s > 3$. Assume that G/N does not act as $\mathbf{PSL}(2,s)$ on the lines of $\mathcal{L}$. First of all, by Lemma 9.8.3, G is a perfect group, and since N is contained in the center of G, we have that the group G is a perfect central extension of the group G/N which acts on $\mathcal{L}$. As G is a perfect group, there is a subgroup F of the center of the universal central extension $\overline{G/N}$ of G/N for which $\overline{G/N}/F \cong G$. Now take over the corresponding part of the proof of Lemma 8.6.5. ∎

Lemma 9.8.7 *Suppose $\mathcal{S}$ is an SPGQ of order (s,t), $s \neq 1 \neq t$ and $s \neq t$, with base-group G and base-span $\mathcal{L}$. If s is odd, then $G \cong \mathbf{SL}(2,s)$, thus G has size $s^3 - s$ and G acts semiregularly on the points of $\mathcal{S} \setminus \Omega$.*

Proof. For $s = 3$ the case is already settled, so we can suppose that $s \neq 3$, s odd.
Putting the results of the preceding sections together, we obtain the following properties if $s \neq 3$.

1. G acts semiregularly on $\mathcal{S} \setminus \Omega$.
2. G acts as $\mathbf{PSL}(2,s)$ on the lines of $\mathcal{L}$ (so $G/N \cong \mathbf{PSL}(2,s)$).
3. G is a perfect group.
4. $|G| \geq s^3 - s$.
5. N is a subgroup of $Z(G)$.

As $G/N \cong \mathbf{PSL}(2,s)$, and as G is a perfect group, the fact that N is contained in the center of G infers that G is a perfect central extension of $\mathbf{PSL}(2,s)$. By definition, this also asserts that G is a central quotient of the universal central extension of $\mathbf{PSL}(2,s)$, which we denote by $\overline{\mathbf{PSL}(2,s)}$. First suppose $s \neq 9$. Because of (4), G does not coincide with $\mathbf{PSL}(2,s)$, and if $s \neq 9$, s odd, then it is known that

$$\overline{\mathbf{PSL}(2,s)} = \mathbf{SL}(2,s),$$

and $|\mathbf{SL}(2,s)| = s^3 - s$. From (4) it follows that $G \cong \mathbf{SL}(2,s)$, and hence the order of G equals $s^3 - s$. Now put $s = 9$.
We already know that G is a perfect central extension of $\mathbf{PSL}(2,9)$. We recall from the proof of Lemma 9.8.3 that, since $G/N \cong \mathbf{PSL}(2,9)$, $|N|$ is a divisor of $2(s+1) = 20$, and by previous observations we can assume that $|N| < s+1$.

It should be noted at this point that $N \neq \{\mathbf{1}\}$, and whence $|N| \in \{2,4,5\}$. Recalling the possible perfect central extensions of $\mathbf{PSL}(2,9)$, it is clear that $|N| \notin \{4,5\}$, and hence $|N| = 2$. Thus $G \cong \mathbf{SL}(2,9)$, and we conclude that $|G| = 9^3 - 9$. ■

Lemma 9.8.8 *If s is even and if every line which meets some line $L \in \mathcal{L}^\perp$ is an axis of symmetry, then G is of order $s^3 - s$ and G acts semiregularly on the points of $\mathcal{S} \setminus \Omega$.*

Proof. In this case, we also have the Properties (1)–(4). And as in the proof of Lemma 9.8.7, G is a perfect central extension of $\mathbf{PSL}(2,s)$. Since

$$\overline{\mathbf{PSL}(2,s)} = \mathbf{SL}(2,s) \cong \mathbf{PSL}(2,s)$$

if s is even and $s \neq 4$, by (4) we have that $G \cong \mathbf{SL}(2,s)$, hence $|G| = s^3 - s$. Now put $s = 4$. Then by Theorem 9.5.1, $\mathcal{S} \cong \mathcal{Q}(5,4)$ and the assertion becomes trivial. ■

9.9 Proof of Theorem 9.1.1

The following theorem will be appear to be an essential tool for the theory of SPGQ's.

Theorem 9.9.1 *Suppose $\mathcal{S}$ is a span-symmetric generalized quadrangle of order (s,t), $s \neq 1 \neq t$, where $s \neq t$ and s is odd. Then $\mathcal{S}$ contains at least $s+1$ subquadrangles isomorphic to the classical GQ $\mathcal{Q}(4,s)$.*

Proof. Suppose $\mathcal{S}$ is an SPGQ with base-span $\mathcal{L}$ and of order (s,t), $s,t \neq 1$ and $s \neq t$, s odd. Then G acts semiregularly on $\mathcal{S} \setminus \Omega$ and G has order $s^3 - s$. Consequently, there are $s+1$ subquadrangles of order s, all isomorphic to $\mathcal{Q}(4,s)$, and so that they mutually intersect in the points and lines of $\mathcal{L}^\perp \cup \mathcal{L}$. ■

Note. In Chapter 12, we will establish the even case of Theorem 9.9.1.

9.10 On the Orders of SPGQ's

We are now ready to prove Theorem 8.2.1 of W. M. Kantor in a more combinatorial way; although we explicitly use a part of his proof, we will obtain the

remaining part without, e.g., "Maschke's Theorem" (see [94]). Suppose $\mathcal{S}$ is an SPGQ of order (s,t) with base-group G and base-grid Γ, where $s \neq 1 \neq t$ and $s < t < s^2$. Then by Theorem 8.2.3, G acts regularly on the $(s+1)s(t-1)$ points of $\mathcal{S}$ which are not in Γ. With the usual notation, G/N is one of the groups as listed in Theorem 9.3.1. We suppose that s is not 'small' to prevent degeneracy (see the present chapter for the details on the small cases). By W. M. Kantor [94], G is a perfect group if G/N does not act sharply 2-transitively[1], and hence G is one of the perfect central extensions of G/N as listed in the present chapter. Those cases are easily excluded as $t > s$ and as G acts regularly on the points of $\mathcal{S} \setminus \Gamma$, by comparing the sizes of the respective groups with $|\mathcal{S} \setminus \Omega| = (s+1)s(t-1)$. The sharply 2-transitive case cannot occur by Section 9.7.

9.11 An Observation of S. E. Payne and Proof of Theorem 9.1.2

9.11.1 Kantor flock generalized quadrangles as SPGQ's

It was a longstanding conjecture that every SPGQ of order s, $s > 1$, is isomorphic to the classical GQ $\mathcal{Q}(4,s)$ (and then s is a prime power). That conjecture was solved in Chapter 8. It is the goal of this paragraph to show that there is no such conjecture for the case $s \neq t$ $(s,t > 1)$; although it was conjectured in 1983 by S. E. Payne (amongst others) that every SPGQ of order (s,s^2), $s > 1$, is isomorphic to $\mathcal{Q}(5,s)$, see PROBLEM 26 of [122], we will indicate here that any member $\mathcal{S}$ of the class of dual Kantor flock generalized quadrangles has a line L so that

For any two non-concurrent lines $U, V \in L^\perp$, $\mathcal{S}$ is an SPGQ with base-span $\{U,V\}^{\perp\perp}$.

It was S. E. Payne who first noted this (in another context) in *A garden of generalized quadrangles, Alg., Groups, Geom.* **3**, *(1985), 323–354* [125].

For, suppose $\mathcal{S} = \mathcal{S}(G,\mathcal{J})$ is a 4-gonal representation of the Kantor flock GQ of order (q^2,q), q odd, with the convention that A_0 is the zero matrix (we use the (standard) notation of [125]). Then S. E. Payne shows that $\mathcal{S}$ is a TGQ for

[1] That proof could also be used for the case $t = s^2$. We choose our approach though, as the latter only uses — essentially — the orders of the groups, while W. M. Kantor his proof is more group theoretical (the intrinsic structure of those groups is utilized).

the line $[A(\infty)]$ (so each point of $[A(\infty)]$ is a center of symmetry). Next, for each $r \neq 0 \in \mathbf{GF}(q)$, the following automorphism of the group G is defined:

$$\theta_r : (\alpha, c, \beta) \longrightarrow (\alpha + \beta K_r^{-1}, c + \frac{1}{4}\beta A_r^{-1}\beta^T, \beta). \tag{9.3}$$

The automorphism θ_r has the properties that

(i) it maps $A(0)$ onto itself,

(ii) $A(\infty)$ onto $A(r)$, while $A(-r)$ is mapped onto $A(\infty)$, and, finally,

(iii) $A(t)$ is mapped onto $A(rt/(r+t))$ $(r \neq -t)$.

Also, he proves that the collineation of $\mathcal{S}$ induced by θ_r is a symmetry about $A^*(0)$, from which it follows that $A^*(0)$ is a center of symmetry. There is a major corollary.

Theorem 9.11.1 *Every point of* $(\infty)^\perp$ *is a center of symmetry.*

Proof. First of all, since (∞) is a center of symmetry, it follows that (∞) is also regular, and hence there is a net $\mathcal{N}_{(\infty)}$ associated to it. We know that $A^*(0)$ is a center of symmetry, and also every point of the line $[A(\infty)]$. By Theorem 3.2.1, the net which is generated (in the sense of Chapter 3) by all the centers of symmetry of $(\infty)^\perp \setminus \{(\infty)\}$, coincides with $\mathcal{N}_{(\infty)}$, and hence every point of $(\infty)^\perp$ is a center of symmetry of $\mathcal{S}$. ∎

Note that the dual of Theorem 9.11.1 is in fact the following.

Theorem 9.11.2 *Let* $\mathcal{S}(\mathcal{F})$ *be a Kantor flock GQ of order* (q^2, q), $q > 1$ *and* q *odd. Then the point-line dual* $\mathcal{S}(\mathcal{F})^D$ *contains a line* L *such that every line of* $L^\perp$ *is an axis of symmetry, i.e. every point of* L *is a translation point.*

∎

This observation is one of the main motivations of this chapter; our main result is a generalization of the converse of Theorem 9.11.2:

What are the GQ's which have two distinct translation points?

Note. We emphasize that if a flock GQ is not classical, then the automorphism group of the GQ fixes (∞) (cf. [140]).

9.11.2 Proof of Theorem 9.1.2

We are now ready to obtain Theorem 9.1.2. For the sake of completeness, we also repeat the statement of that theorem.

Theorem 9.11.3 *Suppose $\mathcal{S}$ is a generalized quadrangle of order (s,t), $s \neq 1 \neq t$, with two distinct collinear translation points. Then we have the following:*

(*i*) *$s = t$, s is a prime power and $\mathcal{S} \cong \mathcal{Q}(4,s)$;*

(*ii*) *$t = s^2$, s is even, s is a prime power and $\mathcal{S} \cong \mathcal{Q}(5,s)$;*

(*iii*) *$t = s^2$, $s = q^n$ with q odd, where $\mathbf{GF}(q)$ is the kernel of the TGQ $\mathcal{S} = \mathcal{S}^{(\infty)}$ with (∞) an arbitrary translation point of $\mathcal{S}$, $q \geq 4n^2 - 8n + 2$ and $\mathcal{S}$ is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ where $\mathcal{F}$ is a Kantor flock;*

(*iv*) *$t = s^2$, $s = q^n$ with q odd, where $\mathbf{GF}(q)$ is the kernel of the TGQ $\mathcal{S} = \mathcal{S}^{(\infty)}$ with (∞) an arbitrary translation point of $\mathcal{S}$, $q < 4n^2 - 8n + 2$ and $\mathcal{S}$ is the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ for some flock $\mathcal{F}$.*

If a thick GQ $\mathcal{S}$ has two non-collinear translation points, then $\mathcal{S}$ is always of classical type, i.e. isomorphic to one of $\mathcal{Q}(4,s)$, $\mathcal{Q}(5,s)$.
In particular, let $\mathcal{S}$ be a generalized quadrangle of order (s,t), $s \neq 1 \neq t$ and s even, with two distinct translation points. Then $\mathcal{S}$ is classical, i.e. isomorphic to $\mathcal{Q}(4,s)$ or $\mathcal{Q}(5,s)$.

Proof. Let $\mathcal{S}$ be as assumed. If $\mathcal{S}$ contains two non-collinear translation points, then it is clear that every point of $\mathcal{S}$ is a translation point, and hence that every line of $\mathcal{S}$ is an axis of symmetry. Hence every line is Moufang, and $\mathcal{S}$ is half Moufang. By Theorem 1.4.2, it follows that $\mathcal{S}$ is Moufang, and by Theorem 1.4.1, $\mathcal{S}$ is classical, i.e. $\mathcal{S}$ is isomorphic to one of $\mathcal{Q}(4,s)$,$\mathcal{Q}(5,s)$ (since these are the only classical GQ's with regular lines). For $s = t$, the statement follows immediately from Theorem 8.8.1. If $s \neq t$, then $t = s^2$ by Theorem 8.2.1.
Now suppose that s is even. We may assume that the translation points are collinear. Fix two non-concurrent lines U and V of $L^\perp$, where L is incident with the aforementioned translation points. If G is the base-group which is defined by the axes of symmetry U and V, and G does not act semiregularly on the points of $\mathcal{S} \setminus \Omega$ (we use the standard notation for SPGQ's), then $\mathcal{S}$ is classical, i.e. isomorphic to $\mathcal{Q}(5,s)$ by Lemma 9.6.5. Thus we can suppose that G acts semiregularly on the points of $\mathcal{S} \setminus \Omega$. By Lemma 9.8.8, $G \cong \mathbf{SL}(2,s)$, and hence $|G| = s^3 - s$. Hence Theorem 9.8.2 implies that $\mathcal{S}$ contains more

than 2 classical subGQ's of order s (containing L), and then by Theorem 1.7.7 the result follows.
Now let s be odd. Suppose that $\mathcal{S}$ contains two collinear distinct translation points u and v. Then clearly by transitivity, every point of $L := uv$ is a translation point, and hence every line of $L^\perp$ is an axis of symmetry. If we fix some base-span $\mathcal{L}$ for which $L \in \mathcal{L}^\perp$, then by Theorem 9.8.2 there are $s+1$ classical subGQ's of order s which mutually intersect precisely in the base-grid $(\Omega, \mathcal{L} \cup \mathcal{L}^\perp, I')$ (with the usual notation). Now fix an arbitrary translation point $(\infty)IL$, and consider the TGQ $\mathcal{S}^{(\infty)} = T(\mathcal{O})$ with base-point (∞). Then clearly $\mathcal{O}$ is good at its element π which corresponds to L. Hence, the translation dual $T(\mathcal{O}^*)$ of $T(\mathcal{O})$ satisfies Property (G) for the flag $((\infty)', L')$ by Theorem 1.11.3, where $(\infty)'$ corresponds to (∞), and L' to L. If we now apply Theorem 1.10.5, then it follows that $T(\mathcal{O}^*)$ is the point-line dual of a flock generalized quadrangle $\mathcal{S}(\mathcal{F})$. The theorem now follows from Theorem 6.5.12 and the fact that the dual Kantor flock quadrangles are isomorphic to their translation duals. ■

Remark 9.11.4 An SPGQ with base-span $\{U, U'\}^{\perp\perp}$ defines a split BN-pair of rank 1, where the root groups are the the full groups of symmetries about the lines of $\{U, U'\}^{\perp\perp}$. Suppose $\mathcal{S}$ is a thick GQ with two distinct collinear translation points p and q, and put $pq = L$. Suppose M and N are arbitrary non-concurrent lines of $L^\perp$. Since $L \cap M$ and $L \cap N$ are translation points of the GQ, the groups of symmetries about M and N are abelian groups (the translation groups corresponding to $L \cap M$ and $L \cap N$ are abelian), and hence every full group of symmetries about a line of $\{L, M\}^{\perp\perp}$ is abelian. As we mentioned before in Remark 8.3.2, the only transvection groups of a finite split BN-pair of rank 1 with abelian rootgroups of order s are isomorphic to $\mathbf{PSL}(2, s)$, or a sharply 2-transitive group. We did not use this fact since it was one of our main aims to obtain Theorem 9.1.1.

9.12 Generalizations of the Main Result

9.12.1 A useful lemma

In this section it is our aim to generalize Theorem 9.11.3 in various ways. We start with a lemma.

Lemma 9.12.1 *Suppose $\mathcal{S}$ is a GQ of order (s,t), $s \neq 1 \neq t$, and suppose L, p and q are so that L is a regular line, pIL is a point which is incident with at least $s+1$ axes of symmetry different from L, and qIL is a point different from*

p which is incident with at least one axis of symmetry which is not L. Then every point of L is a translation point.

Proof. Suppose $\mathcal{N}_L$ is the net which corresponds to the regular line L by the dual of Theorem 3.1.1, and suppose that $\mathcal{N}'_L$ is the subnet of $\mathcal{N}_L$ (of the same degree) which is generated (in the sense of Chapter 3) by the lines which correspond to the axes of symmetry meeting L (and different from it) in the quadrangle. Then by Theorem 3.2.1 there only are two possibilities:

(a) $\mathcal{N}'_L$ is an affine plane of order s;

(b) $\mathcal{N}'_L = \mathcal{N}_L$.

Since we supposed that there is a point on L which is incident with at least $s+1$ axes of symmetry different from L, the second possibility holds. It is clear that every line of $L^\perp \setminus \{L\}$ is an axis of symmetry, and by Theorem 6.6.9, L is also an axis of symmetry. Hence, $\mathcal{S}$ is a translation generalized quadrangle for every point on L. ■

9.12.2 Regularity revisited

The dual of Part (2) of the following lemma is a generalization of [139, 1.3.6 (iv)].

Lemma 9.12.2 *Suppose $\mathcal{S} = (P, B, I)$ is a GQ of order (s, t), $s \neq 1 \neq t$, and let L be a line of $\mathcal{S}$.*

1. *Assume that each line of $B \setminus L^\perp$ is regular. Then every line of $\mathcal{S}$ is regular.*

2. *If L is such that every line of $L^\perp \setminus \{L\}$ is regular, then L is also regular.*

Proof. Suppose we are in Case (1). Suppose N is an arbitrary line of $B \setminus L^\perp$. Then since N is a regular line, $\{L, N\}$ is a regular pair (of lines). Hence L is a regular line. Now suppose $L' \neq L$ is a line of $L^\perp$. Let M be a line of $L^\perp$, $L' \not\sim M$. Then since L is regular, $\{L', M\}$ is regular. Now suppose that M' is arbitrary in $B \setminus L^\perp$. Then M' is regular, and hence $\{L', M'\}$ is regular. Hence L' is regular, and then also every line of $\mathcal{S}$. This proves Part (1). Part (2) is easy. ■

9.12.3 Classifications for generalized quadrangles

Theorem 9.12.3 *Suppose $\mathcal{S}$ is a generalized quadrangle of order (s,t), $s \neq 1 \neq t$, and suppose that one of the following conditions holds.*

1. *There is a regular line L and there are points p and q so that pIL is a point which is incident with at least $s+1$ axes of symmetry different from L, and qIL is a point different from p which is incident with at least one axis of symmetry which is not L.*

2. *There is a regular line L and there are points p and q so that*

   (a) *pIL is a point which is incident with at least $s+1$ regular lines $L_0, L_1, \ldots, L_s$ different from L, for which there are lines $M_0, M_1, \ldots, M_s$ and points $p_0, p_1, \ldots, p_s$ such that $p_i I L_i \sim M_i \setminus p_i$ for all i, and such that there is a group of whorls about p_i which fixes M_i and which acts transitively on the points of M_i which are not on L_i, and*

   (b) *qIL is a point different from p which is incident with at least one regular U line which is not L, for which there is a line M and a point u such that $uIU \sim M \setminus u$, and such that there is a group of whorls about u which fixes M and which acts transitively on the points of M which are not incident with U.*

3. *$\mathcal{S}$ has two distinct centers of transitivity p and q and a regular line which is contained in $(pq)^\perp \setminus \{pq\}$ if p and q are collinear.*

4. *s is even and $\mathcal{S}$ contains a regular line and an elation point p which is not fixed by the group of automorphisms of $\mathcal{S}$.*

5. *s is odd and $\mathcal{S}$ contains two distinct regular lines and an elation point p which is not fixed by the group of automorphisms of $\mathcal{S}$.*

Then we have the following possibilities:

(*i*) *$s = t$, s is a prime power and $\mathcal{S} \cong \mathcal{Q}(4,s)$;*

(*ii*) *$t = s^2$, s is even, s is a prime power and $\mathcal{S} \cong \mathcal{Q}(5,s)$;*

(*iii*) *$t = s^2$, $s = q^n$, where $\mathbf{GF}(q)$ is the kernel of $\mathcal{S} = \mathcal{S}^{(\infty)}$ with (∞) an arbitrary translation point of $\mathcal{S}$ and q is odd, $q \geq 4n^2 - 8n + 2$ and $\mathcal{S}$ is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ where $\mathcal{F}$ is a Kantor flock;*

(*iv*) $t = s^2$, $s = q^n$, *where* $\mathbf{GF}(q)$ *is the kernel of* $\mathcal{S} = \mathcal{S}^{(\infty)}$ *with* (∞) *an arbitrary translation point of* $\mathcal{S}$ *with* q *odd,* $q < 4n^2 - 8n + 2$ *and* $\mathcal{S}$ *is the translation dual of the point-line dual of a flock GQ* $\mathcal{S}(\mathcal{F})$ *for some flock* $\mathcal{F}$.

Proof. If (1) is satisfied, then by Lemma 9.12.1 there follows that L is a line of translation points, and the statement follows by Theorem 9.11.3. Case (2) immediately follows from (1) and Theorem 5.2.7.
Next suppose that we are in Case (3). If p and q are non-collinear, then $\mathcal{S}$ is classical by Theorem 5.3.1. Suppose $p \sim q$; then clearly every point of pq is a center of transitivity. Since there is a regular line M in $(pq)^\perp \setminus \{pq\}$, every line of $(pq)^\perp \setminus \{pq\}$ is regular, and by Lemma 9.12.2, pq is also regular. Theorem 5.2.7 implies that every line of $(pq)^\perp \setminus \{pq\}$ is an axis of symmetry, and by Theorem 6.6.9 we can conclude that pq is also an axis of symmetry. The statement follows from Theorem 9.11.3 since pq is a line of translation points.
Now suppose we are in Case (4) of the theorem. If p is mapped onto a point not collinear with p by some automorphism of $\mathcal{S}$, then the statement follows from (3). Hence suppose that p is mapped onto a point $x \sim p \neq x$ by some automorphism of $\mathcal{S}$. Then px is a line each point of which is an elation point, and if the regular line is incident with one of these elation points, then by Theorem 5.5.2 every point incident with pq is a translation point. Now suppose that there is a regular line in $\mathcal{S} \setminus (px)^\perp$. Then by transitivity every line of $\mathcal{S} \setminus (px)^\perp$ is a regular line, and hence Lemma 9.12.2 implies that every line of $\mathcal{S}$ is regular. Now Theorem 5.5.2 applies.
The proof of Case (5) of the theorem is similar by using Theorem 5.5.1. ∎

As a direct corollary of the theorem, the following theorem is a characterization of the classical GQ's $\mathcal{Q}(4, s)$ and $\mathcal{Q}(5, s)$ with s even.

Theorem 9.12.4 *Suppose* $\mathcal{S}$ *is a generalized quadrangle of order* (s, t), *where* $s \neq 1 \neq t$ *and* s *even. Then* $\mathcal{S}$ *is isomorphic to one of* $\mathcal{Q}(4, s)$,$\mathcal{Q}(5, s)$ *if and only if* $\mathcal{S}$ *contains a regular line and an elation point* p *which is not fixed by the group of automorphisms of* $\mathcal{S}$.

∎

Let us end this chapter with a rather amusing observation in view of Lemma 9.12.2 and the proof of Theorem 9.12.3.

Suppose $\mathcal{S}$ is a GQ of order (s,t), $s \neq 1 \neq t$, with a line of elation points $[\infty]$, and assume that there is some regular line $L \not\sim [\infty]$.
As each point of $[\infty]$ is an elation point, each line of $\mathcal{S} \setminus [\infty]^{\perp}$ is regular. Whence Lemma 9.12.2 asserts that each line of $\mathcal{S}$ is regular. For $s = t$, it follows that $\mathcal{S}$ is isomorphic to $\mathcal{Q}(4,s)$.
Assume that $t > s$. By Chapter 5, each point incident with $[\infty]$ is a translation point. Hence by Theorem 9.11.3

(i) $\mathcal{S} \cong \mathcal{Q}(5,s)$ if s is even, or

(ii) $\mathcal{S}^{(\infty)}$ is the translation dual of the point-line dual of a flock GQ, where $(\infty)I[\infty]$ is arbitrary, and each line of $\mathcal{S}$ is regular.

We conjecture that, in the last case, $\mathcal{S} \cong \mathcal{Q}(5,s)$.

Appendix: Spreads in Span-Symmetric Generalized Quadrangles and Dual Kantor Flock Generalized Quadrangles

In this final part of the chapter, we describe a concrete construction of spreads in thick SPGQ's of order (s, s^2) (which is a direct corollary of some of the considerations of this chapter), which admit $\mathbf{SL}(2, s)$ as an automorphism group, and the construction applies to the dual Kantor GQ's. In fact, as we will show in Chapter 10, it will also work for each translation generalized quadrangle of order (q, q^2), $q > 1$ and q odd, which is the translation dual of the point-line dual of a flock GQ (or, equivalently, for which the corresponding generalized ovoid is good at some element). We note that for the latter class of TGQ's, other constructions are known, see e.g. J. A. Thas and S. E. Payne [190].

Spreads in Span-Symmetric Generalized Quadrangles

Theorem 9.12.5 *Suppose $\mathcal{S}$ is a span-symmetric generalized quadrangle of order (s,t), $s \neq 1 \neq t$ and $t \neq s$, with base-span $\mathcal{L}$ and base-group G. If $|G| = s^3 - s$, then $\mathcal{S}$ contains at least $2(s^2 - s)$ distinct spreads.*

Proof. Since $s \neq t$, we know by Theorem 8.2.1 that $t = s^2$. Suppose Λ is an arbitrary G orbit in $\mathcal{S} \setminus \Omega$, where Ω is the set of points incident with the lines of $\mathcal{L}$. Then by Section 9.8, $\Lambda \cup \Omega$ is the set of points of a subGQ $\mathcal{S}'$ of order s, which is classical by the main result of Chapter 8, and there are $s + 1$ such subGQ's which arise in this way, see Section 9.8. Let L be an arbitrary line of $\mathcal{S}$ which is not contained in $\mathcal{S}'$, and which does not intersect Ω. Then L intersects $\mathcal{S}'$ in exactly one point. Since $|\Lambda| = s^3 - s$, the action of G on the points of $\mathcal{S} \setminus \Omega$ is semiregular, and any of the $s + 1$ subGQ's of order s which contain Ω, is fixed by G. Now consider the line set L^G. Then $|L^G| = s^3 - s$ and no two distinct lines of L^G intersect since the action of G on Λ is regular and since there are $s + 1$ such G-orbits in $\mathcal{S} \setminus \Omega$. Now put $\mathbf{T} = \mathcal{L} \cup L^G$ and $\mathbf{T}' = \mathcal{L}^\perp \cup L^G$. Then $\mathbf{T}$ and $\mathbf{T}'$ clearly are spreads of $\mathcal{S}$. The theorem now follows from the fact that through a fixed point of Λ there are $s^2 - s$ choices

for L. ∎

Note. It may be clear to the reader that exactly the same $2(s^2 - s)$ spreads are obtained if one considers another G-orbit in $\mathcal{S} \setminus \Omega$.

As a direct consequence of Theorem 9.11.3 and Theorem 9.12.5, we obtain the following theorem which is valid for any SPGQ of order (s, t), $s \neq 1 \neq t$, with $s \neq t$ and s odd.

Theorem 9.12.6 *Suppose $\mathcal{S}$ is an SPGQ of order (s, t), $s \neq 1 \neq t$, with $s \neq t$ and s odd. Then $\mathcal{S}$ contains at least $2(s^2 - s)$ distinct spreads.*

Proof. Immediate. ∎

Remark 9.12.7 (i) If $\mathbf{T}$ is such a spread of $\mathcal{S}$, then the group of automorphisms of $\mathcal{S}$ which fix $\mathbf{T}$ contains a group isomorphic to $\mathbf{SL}(2, s)$.

(ii) In Chapter 12, we will show that Theorems 9.12.5 and 9.12.6 also hold for the even case.

Spreads in the Dual Kantor Flock Generalized Quadrangles

In the remainder of this section, we use the following notations: $\mathcal{S}$ is the point-line dual of a flock generalized quadrangle $\mathcal{S}^D = \mathcal{S}(\mathcal{F})$ with $\mathcal{F}$ a Kantor flock (so $\mathcal{S}^D$ is a GQ of order (q^2, q), q an odd prime power). Also, if $\mathcal{S}$ is not classical, i.e. not isomorphic to $\mathcal{Q}(5, q)$, then L will be the (unique) line of translation points of $\mathcal{S}$, see Theorem 9.11.3. If $\mathcal{S} \cong \mathcal{Q}(5, q)$, then L is arbitrary.

For *every* pair (M, N) of non-concurrent lines in $L^\perp$ (which are axes of symmetry), we know by now that for the dual Kantor flock GQ's, the base-group G corresponding to the base-span $\{M, N\}^{\perp\perp}$ is isomorphic to $\mathbf{SL}(2, q)$, and hence, since $|\mathbf{SL}(2, q)| = q^3 - q$, by Theorem 9.12.5 there are $2(q^2 - q)$ distinct spreads, all containing $\{M, N\}^\perp$ or $\{M, N\}^{\perp\perp}$, and such that G stabilizes all these spreads and fixes L. The class of spreads which arise with this construction method and containing a span of the form $\{U, V\}^{\perp\perp}$ with $U \neq V$ non-concurrent lines of $L^\perp$, will be denoted by $\mathfrak{T}$. The class of spreads which arise with this construction method but not containing such a line span is denoted by $\mathfrak{T}'$. Since an element of $\mathfrak{T} \cup \mathfrak{T}'$ is uniquely defined by some line span $\mathcal{L} = \{M, N\}^{\perp\perp}$, $M \not\sim N$ and $M, N \in L^\perp$, respectively $\mathcal{L} = \{M, N\}^\perp$,

$M \not\sim N$ and $M, N \in L^\perp$, and a line V which is not contained in each sub$\mathcal{Q}(4,q)$ of $\mathcal{S}$ which contains $\mathcal{L}$, we will sometimes denote a spread $\mathbf{T}$ of $\mathfrak{T} \cup \mathfrak{T}'$ by $\mathbf{T} = \mathbf{T}(\mathcal{L}, V)$. Hence,

$$\mathbf{T} = \mathbf{T}(\mathcal{L}, V) = \mathcal{L} \cup \{V^\theta \parallel \theta \in G\},$$

where G is the base-group corresponding to $\mathcal{L}$, respectively $\mathcal{L}^\perp$.

Observation 9.12.8 *If $\mathcal{S}$ is non-classical, then no spread of $\mathfrak{T}$ is isomorphic to a spread of $\mathfrak{T}'$.*

Proof. Suppose $\mathbf{T} \in \mathfrak{T}$ and $\mathbf{T}' \in \mathfrak{T}'$. If $\mathcal{S}$ is non-classical, then it is clear from Theorem 9.11.3 that no line of $\mathcal{S} \setminus L^\perp$ is an axis of symmetry, hence $\mathbf{T}$ and $\mathbf{T}'$ can never be isomorphic. ∎

The following lemma is easy:

Lemma 9.12.9 *Suppose $\mathbf{T}$ is a locally Hermitian spread w.r.t. a line M of a GQ $\mathcal{S}$ of order (s,t), $s \neq 1 \neq t$. For any $N \in \mathbf{T} \setminus \{M\}$, the set $(\mathbf{T} \setminus \{M,N\}^{\perp\perp}) \cup \{M,N\}^\perp$ is also a spread of $\mathcal{S}$ which is not locally Hermitian w.r.t. any line.*

Proof. First observe that $t > s$ as $\mathcal{S}$ contains a regular pair of lines and as GQ's of order s cannot have locally Hermitian spreads (clearly). The fact that $\mathbf{T}' = (\mathbf{T} \setminus \{M,N\}^{\perp\perp}) \cup \{M,N\}^\perp$ is a spread is well-known and easy to prove. Suppose that $U \in \{M,N\}^\perp$ and $U' \in \mathbf{T}' \setminus \{M,N\}^\perp = \mathbf{T} \setminus \{M,N\}^{\perp\perp}$ are arbitrary. If $\{U,U'\}^{\perp\perp} \not\subseteq \mathbf{T}'$ for every such U, U', then clearly $\mathbf{T}'$ is not locally Hermitian w.r.t. any line. Suppose $\{U,U'\}^{\perp\perp} \subseteq \mathbf{T}'$. It is clear that $\{U,U'\}^\perp \cap \{M,U'\}^\perp \neq \emptyset$, a contradiction since this implies that $\mathbf{T}$ has distinct concurrent lines. ∎

Observation 9.12.10 *Any spread of $\mathfrak{T}$ is locally Hermitian (w.r.t. some line). No spread of $\mathfrak{T}'$ is locally Hermitian (w.r.t. any line).*

Proof. Suppose $\mathbf{T} = \mathbf{T}(\mathcal{L}, U)$ is an element of $\mathfrak{T}$. Recall that $L \in \mathcal{L}^\perp$. Let M be an arbitrary line of $\mathcal{L}$. If $N \in \mathcal{L}$, $M \neq N$, then $\{M,N\}^{\perp\perp} = \mathcal{L} \subseteq \mathbf{T}$. Next suppose that $N \in \mathbf{T} \setminus \mathcal{L}$. If G is the base-group corresponding to the base-span $\mathcal{L}$, then the group G_M of symmetries about M is contained in G, implying that any line of $\{M,N\}^{\perp\perp}$ is also a line of $\mathbf{T}$. Hence $\mathbf{T}$ is locally Hermitian.
The fact that no spread of $\mathfrak{T}'$ is locally Hermitian follows from Lemma 9.12.9 and the definitions of $\mathfrak{T}$ and $\mathfrak{T}'$. ∎

Remark 9.12.11 If $\mathcal{S} \cong \mathcal{Q}(5,q)$, then every two elements of $\mathfrak{T}$, respectively $\mathfrak{T}'$, are isomorphic, essentially because the automorphism group of $\mathcal{Q}(5,q)$ acts transitively on its Hermitian spreads.

Observation 9.12.12 *For each* $\mathbf{T} \in \mathfrak{T} \cup \mathfrak{T}'$ *the group of automorphisms of* $\mathcal{S}$ *which stabilize* $\mathbf{T}$ *contains a subgroup isomorphic to* $\mathbf{SL}(2,q)$.

Proof. Immediate by the definition of $\mathfrak{T}$ and $\mathfrak{T}'$. ∎

Observation 9.12.13 $|\mathfrak{T}'| = q^6 - q^5 = |\mathfrak{T}|$. *Also,* $\mathfrak{T} \cap \mathfrak{T}' = \emptyset$.

Proof. It is trivial that $\mathfrak{T} \cap \mathfrak{T}' = \emptyset$ since every element of $\mathfrak{T}$ contains lines which intersect L. Now consider two not necessarily distinct elements of $\mathfrak{T}$, say $\mathbf{T} = \mathbf{T}(\mathcal{L}, U)$ and $\mathbf{T}' = \mathbf{T}(\mathcal{L}', U')$ (so $L \in \mathcal{L}^\perp, (\mathcal{L}')^\perp$). If $\mathcal{L} \neq \mathcal{L}'$, then clearly $\mathbf{T} \neq \mathbf{T}'$. If we now count the number k of pairs $(\mathcal{L}'', \mathbf{T}'')$, where $\mathcal{L}''$ is a line span for which $L \in (\mathcal{L}'')^\perp$ and where $\mathbf{T}''$ is of the form $\mathbf{T}(\mathcal{L}'', U'')$ for some line U'' (so $\mathbf{T}'' \in \mathfrak{T}$), then from Theorem 9.12.5 follows that $k = q^4(q^2 - q)$, and k is exactly the number of elements of $\mathfrak{T}$. The fact that $|\mathfrak{T}'| = |\mathfrak{T}|$ follows from the definition of $\mathfrak{T}$ and $\mathfrak{T}'$, and the proof of Observation 9.12.10 (no element of $\mathfrak{T}'$ contains two distinct regular line spans which both contain L). ∎

Chapter 10

Translation Generalized Quadrangles of which the Translation Dual Arises from a Flock

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Here, we will make the observation that each finite translation generalized quadrangle $\mathcal{S}$ which is the translation dual of the point-line dual of a flock generalized quadrangle, has a line $[\infty]$ each point of which is a translation point. This leads to the fact that the full group of automorphisms of $\mathcal{S}$ acts 2-transitively on the points of $[\infty]$, and the observation applies to the point-line duals of the Kantor flock generalized quadrangles, the Roman generalized quadrangles and the recently discovered Penttila-Williams generalized quadrangle. Moreover, by Chapter 9 the generalized quadrangles which have two distinct translation points are *precisely* the TGQ's of which the translation dual is the point-line dual of a flock GQ (that is, those TGQ's $\mathcal{S} = T(\mathcal{O})$ for which $\mathcal{O}$ is good at some element).

We emphasize that for a long time, it has been thought of that every non-

classical TGQ which is the translation dual of the point-line dual of a flock GQ has only one translation point. There are important consequences for the theory of generalized ovoids in $\mathbf{PG}(4n-1,q)$, the study of span-symmetric generalized quadrangles, derivation of flocks of the quadratic cone in $\mathbf{PG}(3,q)$, subtended ovoids in generalized quadrangles, and the understanding of automorphism groups of certain generalized quadrangles. Several well-known problems on these topics will be solved completely. We will also 'calculate' the size of the full automorphism group of the Roman GQ's of order (q,q^2), $q>9$ (as a corollary of a more general observation). Finally, as corollaries of some of the obtained results, we will prove that a TGQ $T(\mathcal{O})$ of order (q,q^2), q odd, where $\mathcal{O}$ is good at some element, is isomorphic to its translation dual $T(\mathcal{O}^*)$ if and only if it is the point-line dual of a Kantor flock GQ; we will also show that the Penttila-Williams TGQ and its translation dual, the Penttila-Williams flock and its derived (isomorphic) flocks, are all 'new' objects (those corollaries are all contained in [4], but we have doubts about some parts of the proofs).

The results of this chapter are taken from *K. Thas, On translation generalized quadrangles of which the translation dual arises from a flock* [213], which was submitted for publication to *Mathematische Zeitschrift.*

10.1 Proof of the Main Theorem

Suppose $\mathcal{F}$ is a flock of the quadratic cone $\mathcal{K}$ in $\mathbf{PG}(3,q)$. Suppose that $\mathcal{S}=\mathcal{S}(\mathcal{F})^D$ is the GQ of order (q,q^2) which arises from the flock $\mathcal{F}$, and suppose that $\mathcal{S}(\mathcal{F})^D$ is a TGQ (that is, suppose that one of the derived flocks of $\mathcal{F}$ is a semifield flock), with translation point x. Let $(\mathcal{S}(\mathcal{F})^D)^*$ be the TGQ which is the translation dual of $\mathcal{S}(\mathcal{F})^D$. Then there is some 'special' line $[\infty]Ix$ which is fixed by the full automorphism group of $(\mathcal{S}(\mathcal{F})^D)^*$ if the GQ is not classical, see [140, 3.3]. If q is even, then it was proved by N. L. Johnson [86] that the corresponding TGQ's $\mathcal{S}(\mathcal{F})^D$ and $(\mathcal{S}(\mathcal{F})^D)^*$ are classical. Hence it is no real restriction to ask that q be odd. That is what we will constantly assume in this chapter.

It is the main objective to prove that for q odd the point x of the TGQ $(\mathcal{S}(\mathcal{F})^D)^*$ is never fixed by the automorphism group of the GQ; this observation contradicts a well-established conviction that in the non-classical case the flag $(x,[\infty])$ is fixed by the full automorphism group of $(\mathcal{S}(\mathcal{F})^D)^*$. Our result implies that the line $[\infty]$ is a line of translation points (so, if $yI[\infty]$ is arbitrary, then $\mathcal{S}^{(y)}$ is a

TGQ with base-point y), and hence that the automorphism group of $(\mathcal{S}(\mathcal{F})^D)^*$ acts 2-transitively on the points of $[\infty]$.

Suppose that $\hat{f}$ is a biadditive and symmetric map of $\mathbb{F}^2 \times \mathbb{F}^2$ to $\mathbb{F}$, where $\mathbb{F} = \mathbf{GF}(q)$, q odd. Suppose $\hat{g}_t$ is a map of $\mathbb{F}^2$ to $\mathbb{F}$ so that, for all $d, t, u \in \mathbb{F}$ and $\alpha, \gamma \in \mathbb{F}^2$, the following conditions are satisfied:

1. $\hat{g}_t(\alpha+\gamma) - \hat{g}_t(\alpha) - \hat{g}_t(\gamma) = \hat{f}(t\alpha, \gamma) = \hat{f}(t\gamma, \alpha)$;
2. $\hat{g}_{t+u}(\alpha) = \hat{g}_t(\alpha) + \hat{g}_u(\alpha)$;
3. $\hat{g}_t(\gamma) = 0$, where t is nontrivial, implies that $\gamma = (0,0)$;
4. $\hat{g}_t(\gamma(t-d)^{-1}) - \hat{g}_d(\gamma(t-d)^{-1}) + \hat{g}_d(-\gamma(d-u)^{-1}) - \hat{g}_u(-\gamma(d-u)^{-1}) = 0$ implies $\gamma = 0$ if t, d, u are distinct (this is Condition (10) of [139, Section 10.4]).

Put $G = \mathbb{F}^4 = \{(r, c, b, d) \parallel r, c, b, d \in \mathbb{F}\}$ with coordinatewise addition. Define subgroups in the following way: $B(\infty) = \{(r, 0, 0, 0) \in G \parallel r \in \mathbb{F}\}$; $B^*(\infty) = \{(r, 0, \gamma) \in G \parallel r \in \mathbb{F}, \gamma \in \mathbb{F}^2\}$. For $\gamma \in \mathbb{F}^2$, write $B(\gamma) = \{(-\hat{g}_c(\gamma), c, -c\gamma) \in G \parallel c \in \mathbb{F}\}$. Put $\gamma = (g_1, g_2) \in \mathbb{F}^2$. Define $B^*(\gamma)$ in the usual way (so $B^*(\gamma) = B(\gamma) \cup \{g \in G \parallel B(\gamma)g \cap B(\gamma') = \emptyset \quad \forall \gamma' \in \mathbb{F}^2\}$; this will be inessential for the sequel). Then $(\mathcal{J}, \mathcal{J}^*)$ is a 4-gonal family for G [139], with $\mathcal{J} = \{B(\gamma) \parallel \gamma \in \mathbb{F}^2 \cup \{\infty\}\}$ and $\mathcal{J}^* = \{B^*(\gamma) \parallel \gamma \in \mathbb{F}^2 \cup \{\infty\}\}$. As G is abelian, we then have a TGQ $\mathcal{S} = (\mathcal{S}^{(\infty)}, G) = \mathcal{S}(G, \mathcal{J})$ (which satisfies some additional properties, see further) of order (q, q^2). Moreover, any TGQ which is the translation dual of the point-line dual of a flock GQ can be represented in this way, see [128], and also M. Lavrauw and T. Penttila [100]. Actually, we will show that the TGQ's as defined above are *precisely* the TGQ's for which the translation dual is the point-line dual of a flock GQ, see Theorem 10.1.2.

By [128] the point-line dual $\mathcal{S}^D$ of $\mathcal{S}$ can be represented by the 4-gonal family $\mathcal{J} = \{A(t) \parallel t \in \mathbb{F} \cup \{\infty\}\}$ in the group $H = \{(\alpha, c, \beta) \parallel \alpha, \beta \in \mathbb{F}^2, c \in \mathbb{F}\}$, where $A(t) = \{(\alpha, \hat{g}_t(\alpha), t\alpha) \parallel \alpha \in \mathbb{F}^2\}$, $A(\infty) = \{(0, 0, \beta) \parallel \beta \in \mathbb{F}^2\}$, and where the group operation of H is defined by

$$(\alpha, c, \beta)(\alpha', c', \beta') = (\alpha + \alpha', c + c' + \hat{f}(\beta, \alpha'), \beta + \beta').$$

The corresponding groups $A^*(t)$, with $t \in \mathbb{F} \cup \{\infty\}$, are defined by $A^*(t) = \{(\alpha, c, \alpha t) \parallel \alpha \in \mathbb{F}^2, c \in \mathbb{F}\}$ and $A^*(\infty) = \{(o, c, \beta) \parallel \beta \in \mathbb{F}^2, c \in \mathbb{F}\}$. With this representation, $\mathcal{S}^D$ is a TGQ with base-line $[A(\infty)]$ [128].

Theorem 10.1.1 *Suppose that $\mathcal{S} = \mathcal{S}^{(\infty)}$ is a TGQ with base-point (∞), which is the translation dual of the point-line dual of a flock GQ of order (q^2, q), q odd. Then there is a line $LI(\infty)$ so that every point on L is a translation point. In particular, the group of automorphisms of $\mathcal{S}$ acts 2-transitively on the points of L.*

Proof. As noted before, any TGQ $\mathcal{S}$ of order (q, q^2), q odd, which is the translation dual of the point-line dual of a flock GQ, can be represented in the aforementioned way. Dualize to obtain $\mathcal{S}^D$, and use the 4-gonal family $\mathcal{J} = \{A(t) \parallel t \in \mathbb{F} \cup \{\infty\}\}$ in the group $H = \{(\alpha, c, \beta) \parallel \alpha, \beta \in \mathbb{F}^2, c \in \mathbb{F}\}$, as above. For arbitrary $v \in \mathbb{F}$, $v \neq 0$, define a collination θ_v of $\mathcal{S}^D$, as follows:

$$(\alpha, c, \beta) \longrightarrow (\alpha + v^{-1}\beta, c + \hat{g}_{v^{-1}}(\beta), \beta).$$

It is easily checked that θ_v is indeed a nontrivial collineation of $\mathcal{S}^D$ (first note that θ_v induces a nontrivial automorphism of H, and observe that $A(t)$ is mapped onto $A(\frac{t}{1+t/v})$ if $t \neq -v$, that $A(-v)$ is mapped onto $A(\infty)$, and that $A(\infty)$ is mapped onto $A(v)$), and θ_v fixes any point of $(A^*(0))^\perp$. Hence θ_v is a nontrivial symmetry about $A^*(0)$, and $A^*(0)$ is a center of symmetry. Since S^D is a TGQ with base-line $[A(\infty)]$ (so every point on $[A(\infty)]$ is a center of symmetry), there easily follows that any point of $(\infty)^\perp$ is a center of symmetry, and hence $\mathcal{S}^D$ is a TGQ for every base-line through (∞). There follows that there is some line L through (∞) in $\mathcal{S}^{(\infty)}$ so that each point on L is a translation point, and the theorem easily follows. ∎

As a direct corollary, we obtain the following representation method for TGQ's of which the translation dual arises from a flock.

Theorem 10.1.2 *Suppose that $\hat{f}$ is a biadditive and symmetric map of $\mathbb{F}^2 \times \mathbb{F}^2$ to $\mathbb{F}$, where $\mathbb{F} = \mathbf{GF}(q)$, q odd. Suppose $\hat{g}_t$ is a map of $\mathbb{F}^2$ to $\mathbb{F}$ so that, for all $d, t, u \in \mathbb{F}$ and $\alpha, \gamma \in \mathbb{F}^2$, the Conditions (1)–(4) are satisfied. If the GQ $\mathcal{S}$ arises from the 4-gonal family $(\mathcal{J}, \mathcal{J}^*)$ as above, then the TGQ $\mathcal{S}$ of order (q, q^2) is the translation dual of the point-line dual of a flock GQ of order (q^2, q), and, conversely, any TGQ of which the translation dual arises from a flock arises in this way.*

Proof. By Theorem 10.1.1, we know that $\mathcal{S}$ has a line of translation points. The theorem now follows from the main theorem of Chapter 9. ∎

Remark 10.1.3 (i) If a GQ has non-collinear translation points, then it is classical, see, e.g., Chapter 5, Theorem 5.3.

(ii) In Theorem 10.1.1, we could also have stated that $\mathcal{S}^{(\infty)} = T(\mathcal{O})$ is a TGQ which is good at some element $\pi \in \mathcal{O}$ (which corresponds to the line $LI(\infty)$). Since q is odd, Theorem 1.10.5 infers that $\mathcal{S}^{(\infty)}$ is then the translation dual of the point-line dual of a flock GQ of order (q^2, q). The converse is true by J. A. Thas [177]. In the sequel, we will therefore make no distinction between TGQ's in odd characteristic which are good at some line containing the translation point (∞), and TGQ's which are the translation dual of a TGQ of order (q, q^2) arising from a flock, q odd.

(iii) There is also a purely geometrical proof without the use of coordinates, as was noted to us by J. A. Thas, which uses recent developments in the study of nets and GQ's. We give a sketch of that proof. Suppose $\mathcal{S}^{(\infty)}$ is a TGQ of order (q, q^2), $q \neq 1$ and q odd, which is good at its line $LI(\infty)$. Then by J. A. Thas [177], cf. Theorem 6.5.10, there are $q^3 + q^2$ subGQ's of order q, all isomorphic to $\mathcal{Q}(4, q)$, which contain the flag $((\infty), L)$. It follows immediately that L is regular, since any pair of lines in $\mathcal{S}$ of the form $\{L, M\}$ with $L \not\sim M$, is contained in such a classical subGQ of order q (and any line of $\mathcal{Q}(4, q)$ is regular). Now suppose $M \sim L \neq MImIL$. It is clear that if $N \sim L \neq N$ and $N \not\sim M$, then $\{M, N\}$ is a regular pair of lines (M and N are contained in one of the classical subGQ's). Now suppose $U \not\sim M$ is not a line of $L^\perp$. Consider an arbitrary point u of L different from m, and let V be the unique line of $\mathcal{S}$ for which $uIV \sim U$. Since there are $q^3 + q^2$ classical subGQ's of $\mathcal{S}$ of order q which contain L, it follows that there is a necessarily unique such classical subGQ of $\mathcal{S}$ of order q which contains L, M, V and U (this is also immediate by representing $\mathcal{S}^{(\infty)}$ as $T(\mathcal{O})$, with $T(\mathcal{O})$ good at L). Hence the pair $\{M, V\}$ is regular, and so M is a regular line. It follows that every point on L is coregular. Consider any such coregular point pIL. From the regular line L there arises a net $\mathcal{N}_L$, and $\mathcal{N}_L$ is a $\mathcal{P}$-net, with $\mathcal{P}$ the parallel class of $\mathcal{N}_L$ defined by p, since the dual of $\mathcal{N}_L$ satisfies the Axiom of Veblen ($\mathcal{N}_L$ is the dual of a H_s^3 by Theorem 3.1.2). Hence by Theorem 3.1.7, every point on L is a translation point, and so every line of $L^\perp$ is an axis of symmetry.

(iv) In some sense, Theorem 10.1.1 explains the intrinsic difference between a TGQ which arises from a flock and its translation dual, if the flock is not a Kantor flock.

(v) A proof of Theorem 10.1.1 is also implicitly contained in K. Thas [218].

(vi) Suppose that $\mathcal{S} = \mathcal{S}^{(\infty)}$ is a TGQ with base-point (∞), which is the translation dual of the point-line dual of a flock GQ of order (q^2, q), q odd. Then by Theorem 10.1.1, there is a line $[\infty]I(\infty)$ so that for each pair $\{U, V\}$ of non-concurrent lines of $[\infty]^\perp$, $\mathcal{S}$ is an SPGQ with base-span $\{U, V\}^{\perp\perp}$. Hence the appendix of Chapter 9 applies for each such TGQ, in particular for the Roman TGQ's and for the Penttila-Williams TGQ.

10.2 TGQ's of which the Translation Dual Arises from a Flock and Span-Symmetric Generalized Quadrangles

Theorem 10.1.1 implies that if $\mathcal{S}$ and L are as in Theorem 10.1.1, then every line of $L^\perp$ is an axis of symmetry, and then for every two non-concurrent lines U and V in $L^\perp$, the GQ $\mathcal{S}$ is span-symmetric with base-span $\{U, V\}^\perp$. Fix two such lines U and V in $L^\perp$, and assume that $\mathcal{S}$ is of order (s, s^2). Then by Chapter 9, we known that if H is the group generated by the symmetries about U and V, then $H \cong \mathbf{SL}(2, s)$, and H acts semiregularly on the points which are not incident with the lines of $\{U, V\}^{\perp\perp}$. Note that if N is the kernel of the action of H on $\mathcal{L} = \{U, V\}^{\perp\perp}$, then $H/N \cong \mathbf{PSL}(2, s)$ (cf. Chapter 9).

As a direct corollary of Theorem 10.1.1 and Theorem 9.11.3, we obtain the following result, which states that the (non-classical) generalized quadrangles with two distinct translation points are *exactly* the TGQ's of which the translation dual is the point-line dual of a flock GQ.

Theorem 10.2.1 *A generalized quadrangle $\mathcal{S}$ of order (s, t), $s \neq 1 \neq t \neq s$, has two distinct collinear translation points if and only if $\mathcal{S}$ is a TGQ which is the translation dual of the point-line dual of a flock GQ. In particular, if s is even, then $\mathcal{S} \cong \mathcal{Q}(5, s)$. If $\mathcal{S}$ has non-collinear translation points, then $\mathcal{S}$ is always of classical type.*

■

Theorem 10.2.2 *The automorphism groups of the non-classical dual Kantor GQ's, the non-classical Roman GQ's and the Penttila-Williams GQ act 2-transitively on the points incident with the line of translation points.*

Proof. Immediately by Theorem 10.1.1. ∎

Note. For the dual Kantor GQ's, this observation was already made by S. E. Payne in [125, 129], see Section 9.11.

10.3 Subtended Ovoids in the TGQ's $(\mathcal{S}(\mathcal{F})^D)^*$, $\mathcal{F}$ a Semifield Flock

Theorem 10.3.1 (M. Lavrauw [98, 99]) *Let $\mathcal{O}$ be an egg of $\mathbf{PG}(4n-1,q)$ which is good at the element π, q odd, and consider the TGQ $\mathcal{S}^{(\infty)} = T(\mathcal{O})$. Then all the ovoids of a fixed (arbitrary) subGQ $\mathcal{Q}(4,q^n)$ through the line $[\infty]$, where $[\infty]$ corresponds to π, which are subtended by a point of Type (2) of $T(\mathcal{O})$ are isomorphic translation ovoids. Moreover, these ovoids are isomorphic to the ovoid of $\mathcal{Q}(4,q^n)$ arising from the semifield flock which corresponds to the egg $\mathcal{O}$.*

Consider the TGQ $\mathcal{S}^{(\infty)} = T(\mathcal{O})$ as in Theorem 10.3.1. Then $[\infty]$ is a line of translation points by Theorem 10.1.1. Fix the subGQ $\mathcal{S}' = \mathcal{Q}(4,q^n)$ as above. Suppose that x is a point of Type (3) not in $\mathcal{S}'$, in the TGQ $T(\mathcal{O})$ (that is, $x \not\sim (\infty)$). Consider two non-concurrent lines U, V in $[\infty]^\perp \cap \mathcal{S}'$. Then U and V are axes of symmetry of $\mathcal{S}$, and the group G generated by all the symmetries about U and V fixes $\mathcal{S}'$ and $[\infty]$, and acts 2-transitively on the points of $[\infty]$. Since there is a collineation in G which maps (∞) onto the unique point on L which is collinear with x, there readily follows that x subtends an ovoid of $\mathcal{S}'$ which is isomorphic to the translation ovoid subtended by the points of Type (2). We obtain the following result, which completely solves the isomorphism problem for subtended ovoids in classical subGQ's of order q^n of TGQ's of order (q^n, q^{2n}) with a good element, q odd, which contain the flag $((\infty),[\infty])$. So we have the following theorem.

Theorem 10.3.2 *Let $\mathcal{O}$ be an egg of $\mathbf{PG}(4n-1,q)$ which is good at the element π, q odd, and consider the TGQ $\mathcal{S}^{(\infty)} = T(\mathcal{O})$. Then all subtended ovoids of a fixed (arbitrary) subGQ $\mathcal{Q}(4,q^n)$ through the line $[\infty]$, where $[\infty]$ corresponds to π, are isomorphic translation ovoids. Moreover, these ovoids are isomorphic to the ovoid of $\mathcal{Q}(4,q^n)$ arising from the semifield flock which corresponds to the egg $\mathcal{O}$.*

∎

As a direct corollary, Theorem 5.4 of M. R. Brown [21] follows (this is the 'isomorphism part' of Theorem 10.3.2 for the Kantor GQ's).

As the subGQ $\mathcal{Q}(4,q^n)$ through $[\infty]$ in Theorem 10.3.2, is arbitrary, we have the following result.

Theorem 10.3.3 *Two TGQ's $\mathcal{S}^{(\infty)} = T(\mathcal{O})$ and $(\mathcal{S}')^{(\infty)'} = T(\mathcal{O}')$ of order (q^n, q^{2n}), q odd and $\mathcal{O}$, respectively $\mathcal{O}'$, good at its element π, respectively π', are isomorphic if and only if the subtended ovoids of a fixed subGQ $\mathcal{Q}(4,q^n)$ of $\mathcal{S}$ through $[\infty]$, where $[\infty]$ corresponds to π, and the subtended ovoids of a fixed subGQ $\mathcal{Q}(4,q^n)$ of $\mathcal{S}'$ through $[\infty]'$, where $[\infty]'$ corresponds to π', are isomorphic translation ovoids of $\mathcal{Q}(4,q^n)$.*

■

Corollary 10.3.4 *The Penttila-Williams TGQ $\mathcal{S} = T(\mathcal{O}_{PW})$ is new.*

Proof. For the Penttila-Williams TGQ $\mathcal{S} = T(\mathcal{O}_{PW})$, we have that $\mathcal{O}_{PW}$ is good at some element. But the Penttila-Williams ovoid, which is isomorphic to the ovoid of $\mathcal{Q}(4,3^5)$ arising from the Penttila-Williams flock which corresponds to the egg $\mathcal{O}$, is not isomorphic to any other known ovoid of $\mathcal{Q}(4,3^5)$ [145], so by Theorem 10.3.3 the result follows. ■

Whence

Corollary 10.3.5 *The Penttila-Williams flock $\mathcal{F}_{PW}$ is new.*

■

10.4 Translation Generalized Quadrangles with Isomorphic Translation Duals

Theorem 10.4.1 *Suppose that $\mathcal{S}^{(x)}$ is a non-classical TGQ which is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ of order (q^2,q). So q is odd. Then the full automorphism group of $\mathcal{S}^{(x)}$ does not fix x if and only if $\mathcal{S}^{(x)}$ is the point-line dual of a non-classical Kantor flock GQ.*

Proof. Suppose that the translation point x of $\mathcal{S}$ is not fixed by $Aut(\mathcal{S})$. Then as $\mathcal{S}$ is non-classical, all the translation points of $\mathcal{S}$ are incident with the same line $[\infty]Ix$. By Chapter 9, there are q^3+q^2 classical subGQ's of $\mathcal{S}$ of order q which contain $[\infty]$. That line $[\infty]$ is thus fixed by the full automorphism group of $\mathcal{S}$ (and it is the only line with that property), and hence it follows that $[\infty]$ corresponds to the special point (∞) of $\mathcal{S}(\mathcal{F})$, as (∞) is fixed by each automorphism of $\mathcal{S}(\mathcal{F})$, see [140]. But by Theorem 3.1.5, it then follows that $\mathcal{F}$ is a Kantor flock, as the dual net $\mathcal{N}^*_{(\infty)}$ satisfies the Axiom of Veblen, contradiction. ∎

The following corollary characterizes the Kantor flock GQ's.

Theorem 10.4.2 *Suppose that $\hat{f}$ is a biadditive and symmetric map of $\mathbb{F}^2 \times \mathbb{F}^2$ to $\mathbb{F}$, where $\mathbb{F} = \mathbf{GF}(q)$, q odd. Suppose $\hat{g}_t$ is a map of $\mathbb{F}^2$ to $\mathbb{F}$ so that, for all $d,t,u \in \mathbb{F}$ and $\alpha,\gamma \in \mathbb{F}^2$, the following conditions are satisfied:*

1. $\hat{g}_t(\alpha+\gamma) - \hat{g}_t(\alpha) - \hat{g}_t(\gamma) = \hat{f}(t\alpha,\gamma) = \hat{f}(t\gamma,\alpha)$;
2. $\hat{g}_{t+u}(\alpha) = \hat{g}_t(\alpha) + \hat{g}_u(\alpha)$;
3. $\hat{g}_t(\gamma) = 0$, *where t is nontrivial, implies that* $\gamma = (0,0)$;
4. $\hat{g}_t(\gamma(t-d)^{-1}) - \hat{g}_d(\gamma(t-d)^{-1}) + \hat{g}_d(-\gamma(d-u)^{-1}) - \hat{g}_u(-\gamma(d-u)^{-1}) = 0$ *implies* $\gamma = 0$ *if t,d,u are distinct (this is Condition (10) of [139, Section 10.4]).*

Assume also the following additional condition:

(5) *If for $\alpha_1,\alpha_2,\beta_1,\beta_2 \in \mathbb{F}^2$ we have that $0 = \hat{f}(\alpha_1(t-u),\beta_1) = \hat{f}(\alpha_1(t-u),\beta_2) = \hat{f}(\alpha_2(t-u),\beta_1)$, then $\hat{f}(\alpha_2(t-u),\beta_2) = 0$ (this is Condition V.2 of [128]).*

Let $\mathcal{S}$ be the GQ which arises from the 4-gonal family $(\mathcal{J},\mathcal{J}^*)$ as before. Then $\mathcal{S}^D \cong \mathcal{S}(\mathcal{F})$ with $\mathcal{F}$ a Kantor flock (and conversely).

Proof. By Section 10.1 and Theorem 10.1.1, $\mathcal{S}$ is a TGQ for which $(\mathcal{S}^*)^D$ is a flock GQ, say $\mathcal{S}(\mathcal{F})$. Condition (5) is exactly the condition for $\mathcal{S}^D$ to be a flock GQ, see [128] (in fact, Condition (5) infers that $\mathcal{S}$ satisfies Property (G) at its point (∞), and then the main theorem of [183] implies that $\mathcal{S}^D$ is a flock GQ). The theorem now follows from Theorems 10.1.1 and 10.4.1. ∎

Another corollary of Theorem 10.4.1 is

Theorem 10.4.3 *Suppose $T(\mathcal{O})$ is a TGQ of order (q, q^2), q odd, where $\mathcal{O}$ is good at some element. Then $T(\mathcal{O})$ is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ if and only if $\mathcal{F}$ is a Kantor flock.*

Proof. Immediately by Theorem 10.1.1, Theorem 10.4.1 and the fact that as $\mathcal{O}$ is good at some element, $T(\mathcal{O})^*$ is the point-line dual of a flock GQ. ∎

There are some immediate corollaries, which we state as theorems.

Theorem 10.4.4 *(i) Suppose $\mathcal{S} = T(\mathcal{O})$ is a TGQ of order (q, q^2), q odd, where $\mathcal{O}$ is good at some element. Then $T(\mathcal{O}) \cong T(\mathcal{O}^*)$ if and only if $\mathcal{S}$ is the point-line dual of a Kantor flock GQ.*

(ii) Suppose $\mathcal{S} = T(\mathcal{O})$ is a TGQ of order (q, q^2), q odd, which is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$. Then $\mathcal{S}$ is isomorphic to its translation dual if and only if $\mathcal{F}$ is a Kantor flock.

Proof. Immediate. ∎

Finally, we have

Theorem 10.4.5 *The point-line dual of $\mathcal{S}(\mathcal{F}_{PW})$, which is the translation dual of $T(\mathcal{O}_{PW})$, is new.*

Proof. This follows from the fact that $T(\mathcal{O}_{PW})$ is new, that for the point-line dual $T(\mathcal{O})$ of $\mathcal{S}(\mathcal{F}_{PW})$, $\mathcal{O}$ is good at no element (by Theorem 10.4.3), and that for two TGQ's $T(\mathcal{O}_1)$ and $T(\mathcal{O}_2)$, we have that $T(\mathcal{O}_1) \cong T(\mathcal{O}_2)$ if and only if $T(\mathcal{O}_1^*) \cong T(\mathcal{O}_2^*)$ by the proof of Theorem 11.3.1 of the next chapter. ∎

10.5 The Automorphism Group of the Roman GQ's

The following theorem will be obtained in the next chapter, see Theorem 11.3.3.

Theorem 10.5.1 *Suppose $\mathcal{S} = T(\mathcal{O})$ is a TGQ of order (q^n, q^m) with base-point (∞), where q is odd if $n = m$, and suppose that $G_{(\infty)}$ is the stabilizer of (∞) in the automorphism group G of $\mathcal{S}$. Furthermore, suppose $(\infty)'$ is the base-point of $T(\mathcal{O}^*) = \mathcal{S}^*$, and let $G'_{(\infty)'}$ be the stabilizer of $(\infty)'$ in the automorphism group G' of $\mathcal{S}^*$. Then $|G_{(\infty)}| = |G'_{(\infty)'}|$.*

Proof. See Chapter 11, Theorem 11.3.3. ∎

As a corollary, we obtain

Corollary 10.5.2 *1. Suppose $\mathcal{S} = T(\mathcal{O})$ is a TGQ of order (q^n, q^{2n}), where q is even and where $\mathcal{O}$ is good at some element, and suppose that $Aut(\mathcal{S})$ is the automorphism group of $\mathcal{S}$. Suppose (∞) is the base-point. Let $\mathcal{S}^* = T(\mathcal{O}^*)$ be the translation dual of $\mathcal{S} = T(\mathcal{O})$ with base-point $(\infty)'$, and let $Aut(\mathcal{S}^*)$ be the automorphism group of $\mathcal{S}^*$. Then $|Aut(\mathcal{S})| = |Aut(\mathcal{S}^*)|$.*

2. Suppose $\mathcal{S} = T(\mathcal{O})$ is a TGQ of order (q^n, q^{2n}), q odd and $\mathcal{O}$ good at some element, which is not the point-line dual of a Kantor flock GQ, and suppose that $Aut(\mathcal{S})$ is the automorphism group of $\mathcal{S}$. Suppose (∞) is the base-point. Let $\mathcal{S}^* = T(\mathcal{O}^*)$ be the translation dual of $\mathcal{S} = T(\mathcal{O})$ with base-point $(\infty)'$, and let $Aut(\mathcal{S}^*)$ be the automorphism group of $\mathcal{S}^*$. Then $|Aut(\mathcal{S})| = (q^n + 1)|Aut(\mathcal{S}^*)|$.

Proof. Suppose we are in Case (1). If $\mathcal{S}$ is classical, then the result is obvious, hence suppose this is not the case. As q is even, the main result of Chapter 9 asserts that the translation points of both $\mathcal{S}$ and $\mathcal{S}^*$ are fixed by their respective automorphism group. The result then follows from Theorem 10.5.1.
Suppose we are in Case (2). As $\mathcal{S} = T(\mathcal{O})$ is not a dual Kantor flock GQ, there follows that the translation point of $\mathcal{S}^*$ is fixed by $Aut(\mathcal{S}^*)$ (see Theorem 10.4.1). The assertion follows from Theorem 10.1.1 and Theorem 10.5.1. ∎

We emphasize at this point that Corollary 10.5.2 implies that the size of the automorphism group of a TGQ in the even characteristic case always equals that of the automorphism group of its translation dual. There is another interesting corollary.

Corollary 10.5.3 (Automorphisms of the non-classical Roman GQ's) *Suppose $\mathcal{S}$ is the Roman GQ of order (q, q^2), $q = 3^h$, $h > 2$ (so $\mathcal{S}$ is the translation dual of the point-line dual of the flock GQ $\mathcal{S}(\mathcal{F})$ with $\mathcal{F}$ a Ganley flock). If $Aut(\mathcal{S})$ is the full automorphism group of $\mathcal{S}$, then*

$$|Aut(\mathcal{S})| = q^6(q+1)(q-1)2h.$$

Proof. Immediately by Corollary 10.5.2 and the fact that the full automorphism group of $\mathcal{S}(\mathcal{F})$, with $\mathcal{F}$ a Ganley flock, has size $q^6(q-1)2h$, where $q > 9$ (cf. Chapter 2). ∎

Remark 10.5.4 Corollary 10.5.2 will be considerably generalized in Chapter 11, see Theorem 11.3.4.

10.6 Automorphism Groups Revisited

In this section, it is our objective to obtain a lower bound for the size of the full automorphism group of a non-classical TGQ which is the translation dual of a TGQ that arises from a flock.

Suppose that $\mathcal{S}$ is a non-classical TGQ of order (q, q^2), q odd, which is the translation dual of a TGQ of order (q, q^2) arising from a flock. By Theorem 10.1.1 $\mathcal{S}$ has a line $[\infty]$ of translation points, and there are $q^3 + q^2$ classical subGQ's of order q all containing $[\infty]$, see Theorem 6.5.10 (or, e.g., Chapter 9). Consider a fixed subGQ $\mathcal{S}' \cong \mathcal{Q}(4, q)$ of order q through the line $[\infty]$, and note that each symmetry of $\mathcal{S}$ about a line of $\mathcal{S}' \cap [\infty]^\perp$ fixes $\mathcal{S}'$. It is then clear that $Aut(\mathcal{S})_{\mathcal{S}'} =: H$ (that is, the stabilizer of $\mathcal{S}'$ in the automorphism group of $\mathcal{S}$) acts transitively on the ordered pairs (x, y) of distinct points in $\mathcal{S}' \setminus [\infty]$ for which $x \sim y$, xy not intersecting $[\infty]$. Hence we obtain that

$$q^4(q+1) \mid |H|.$$

Note that H fixes $[\infty]$. Fix an ordinary quadrangle $\mathcal{A}$ in $\mathcal{S}'$ which contains $[\infty]$ as a side, and suppose $[\infty], U, V, W$ are the lines of $\mathcal{A}$, so that $U \not\sim [\infty]$. Consider the action of the elementwise stabilizer $H(\mathcal{A})$ of $\mathcal{A}$ in H on the lines of $X := \{U, [\infty]\}^{\perp\perp} \setminus \{U, [\infty]\}$ (so $H(\mathcal{A})$ fixes the two distinct collinear points $U \cap V$ and $U \cap W$ of $\mathcal{S} \setminus [\infty]$, $U \not\sim [\infty]$). By Theorem 10.1.1, V and W are axes of symmetry of $\mathcal{S}$ (and $\mathcal{S}'$), and the group G generated by the symmetries about V and W fixes $\mathcal{S}'$ and every line of $\{V, W\}^\perp$, and has been proved to be isomorphic to $\mathbf{SL}(2, q)$. Hence the kernel of the action of $H(\mathcal{A})$ on the lines of X has a subgroup of order $q - 1$ (recall that the action of G on $\mathcal{S}'$ is faithful). Hence

$$q^4(q+1)(q-1) \mid |H|.$$

Let $v = V \cap [\infty]$ and $w = W \cap U$. Then the group $\mathbf{W}(v, w)$ of whorls about v and w has size $|\mathbb{K}| - 1$, where $\mathbb{K}$ is the kernel of the TGQ. This group clearly fixes $\mathcal{S}'$, and acts semiregularly on X (by Theorem 1.6.5). Thus the following result:

$$(|\mathbb{K}| - 1)q^4(q+1)(q-1) \mid |H|.$$

It is also clear that each subGQ of order q of $\mathcal{S}$ which contains $[\infty]$ has an $Aut(\mathcal{S})$-orbit of size at least q^2, since $Aut(\mathcal{S})$ acts transitively on the pairs of non-concurrent lines in $[\infty]^\perp$, and hence

$$|Aut(\mathcal{S})| \geq q^6(q+1)(q-1)(|\mathbb{K}| - 1).$$

Recall at this point that, if x_1, x_2, x_3, x_4 are four collinear points of $\mathbf{PG}(n,q)$, with $|\{x_1, x_2, x_3, x_4\}| \geq 3$, then by $(x_1, x_2; x_3, x_4)$ we denote the usual *cross-ratio* given by

$$\frac{r_3 - r_1}{r_3 - r_2} : \frac{r_4 - r_1}{r_4 - r_2},$$

where the r_i, $i = 1, 2, 3, 4$, are non-homogeneous coordinates of the x_i on the line through x_1, x_2, x_3, x_4.
We also recall that, if a semilinear automorphism θ of $\mathbf{PG}(n,q)$ (i.e. $\theta \in \mathbf{P\Gamma L}(n+1,q)$) preserves the cross-ratio of all 4-tuples of points on at least one line, then θ is a linear automorphism of $\mathbf{PG}(n,q)$ (i.e. $\theta \in \mathbf{PGL}(n+1,q)$) [198, Hoofdstuk 5]. Note that if a semilinear automorphism θ of $\mathbf{PG}(n,q)$ fixes some $\mathbf{PG}(k,q)$ in $\mathbf{PG}(n,q)$, $k > 0$, elementwise, then clearly θ preserves the cross-ratio, and hence $\theta \in \mathbf{PGL}(n+1,q)$.

By Theorem 11.2.1 of the next chapter, we can now consider $Aut(\mathcal{S})_{(\infty)}$, for any fixed point $(\infty)I[\infty]$, as a group of automorphisms of $\mathbf{PG}(4n,q)$ which fixes the corresponding egg $\mathcal{O} \subseteq \mathbf{PG}(4n-1,q)$, and moreover, as the groups from the previous arguments, if restricted to $\mathbf{P\Gamma L}(4n+1,q)_{\mathcal{O}}$, all fix at least one line of $\mathbf{PG}(4n-1,q)$ pointwise, it is clear that

$$|Aut(\mathcal{S})_{(\infty)} \cap \mathbf{PGL}(4n+1,q)| \geq q^6(q-1)(|\mathbb{K}| - 1).$$

We now show that the latter bound is sharp. Suppose that $(\mathcal{S}(\mathcal{F})^D)^*$, with $\mathcal{F}$ a Ganley flock, is a Roman GQ of order (q, q^2), $q > 9$. Then

$$|Aut((\mathcal{S}(\mathcal{F})^D)^*)_{(\infty)} \cap \mathbf{PGL}(4n+1,q)| = q^6(q-1)2,$$

by the previous section and the preceding arguments, [129] and Theorem 11.3.1.

Remark 10.6.1 (On a special involution) Let $\mathcal{S}$ be a span-symmetric generalized quadrangle of order (s, s^2), $s > 1$ and s odd, with base-span $\mathcal{L}$ and base-group G. Then there is an involution θ in G which acts trivially on the points of the lines of $\mathcal{L}$ and which acts semiregularly on the other points (as the kernel of $G \cong \mathbf{SL}(2, s)$ on $\mathcal{L}$). Now suppose $\mathcal{O}$ is an egg of $\mathbf{PG}(4n-1, q) \subseteq \mathbf{PG}(4n, q)$, q odd, which is good at its element π. Then by Theorem 10.1.1, $T(\mathcal{O})$ is an SPGQ for every span $\{L, M\}^{\perp\perp}$ in $\pi^\perp$ (with the obvious notation), $L \not\sim M$. Hence Theorem 10.5.1 implies that for arbitrary $\pi' \in \mathcal{O} \setminus \{\pi\}$ and $p \in \mathbf{PG}(4n, q)$, there is an involution of $\mathbf{PG}(4n, q)$ (which is an element of $\mathbf{PGL}(4n+1, q)$) which fixes $\pi\pi' p$ pointwise and $\mathcal{O}$ as a set, and which acts faithfully as an involution on the elements of $\mathcal{O} \setminus \{\pi, \pi'\}$.

10.7 Derivation of Semifield Flocks, BLT-Sets and Automorphisms

We directly obtain the following theorem, which solves the isomorphism problem of derivation for the flocks of a BLT-set in the semifield case.

Theorem 10.7.1 *Suppose that $\mathcal{S}^{(\infty)}$ is a TGQ which is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ of order (q^2, q). Suppose $\{\mathcal{F}_0 = \mathcal{F}, \mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_q\}$ is the BLT-set of $q+1$ flocks which is derived from $\mathcal{F}$. Then all these flocks are isomorphic if and only if $\mathcal{F}$ is a Kantor flock. If $\mathcal{F}$ is not a Kantor flock, then $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_q$ are all isomorphic, but non-isomorphic to $\mathcal{F}$.*

Proof. If all these flocks are isomorphic, then this implies that $Aut(\mathcal{S}(\mathcal{F})^D)$ acts transitively on the points of the line $[\infty]$ which corresponds to the point (∞) of $\mathcal{S}(\mathcal{F})$. Hence $[\infty]$ is a line of translation points, and so $\mathcal{F}$ is a Kantor flock by Theorem 10.4.1. The theorem easily follows. ∎

Corollary 10.7.2 *Suppose that $\mathcal{S}^{(\infty)}$ is the point-line dual of the flock GQ $\mathcal{S}(\mathcal{F})$ of order (q^2, q), $\mathcal{F}$ the Penttila-Williams flock. Suppose $\{\mathcal{F}_0 = \mathcal{F}, \mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_q\}$ is the BLT-set of $q+1$ flocks which is derived from $\mathcal{F}$. Then $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_q$ are all isomorphic, but non-isomorphic to $\mathcal{F}$. Hence $\mathcal{F}_1$ is a new flock.*

Proof. The fact that $\mathcal{F}_1$ is new follows from the fact that $\mathcal{F}_1$ is non-isomorphic to $\mathcal{F}$, that both flocks give rise to the same generalized quadrangle, and that the latter is new by Theorem 10.4.5. ∎

Remark 10.7.3 Some of the results just obtained are contained in L. Bader, G. Lunardon and I. Pinneri [4]. As we have doubts about some parts of their proofs (especially of the essential Lemma 1 of that paper), we preferred to give new proofs of these results.

Chapter 11

Automorphisms and Determination of Translation Generalized Quadrangles

In this chapter, we will prove the well-known conjecture that every automorphism of a translation generalized quadrangle $\mathcal{S}^{(\infty)}$ of order (q^n, q^m) which fixes the translation point (∞) is induced by an automorphism of the Galois geometry $\mathbf{PG}(2n+m, q)$ in which $\mathcal{S}^{(\infty)}$ is represented. We will then explain the connection between the automorphism group of a TGQ and the automorphism group of its translation dual. As the most notable application, a classification result for translation generalized quadrangles will be obtained.

The results which we will obtain below stem from a collaboration with J. A. Thas, and were submitted to *Mathematical Proceedings of the Cambridge Philosophical Society* under the title 'Translation Generalized Quadrangles and Translation Duals, Part I' [191].

11.1 The Classification of Translation Generalized Quadrangles

The main examples of finite generalized quadrangles essentially belong to five types:

(i) they are fully embedded in a projective space $\mathbf{PG}(n,q)$ over the Galois field $\mathbf{GF}(q)$, and these are the classical examples;

(ii) they are the point-line duals of the classical examples (the dual $H(4,q^2)^D$ of the classical GQ $H(4,q^2)$ is never embedded (in the usual sense) in a projective space);

(iii) they are of order $(q-1,q+1)$ for some q or, dually, of order $(q+1,q-1)$, and the examples of this type all are in some way connected to ovals or hyperovals of $\mathbf{PG}(2,q)$;

(iv) they arise as translation generalized quadrangles (arising from generalized ovals or generalized ovoids);

(v) they arise from flocks of the quadratic cone in $\mathbf{PG}(3,q)$ for some q.

The examples which got the most attention in the past fifteen years are of Type (iv) and (v), and in particular, many classification results were obtained for the amalgamations of (iv) and (v), namely the translation generalized quadrangles which arise from a flock or of which the translation dual arises from a flock. A first major breakthrough came when J. A. Thas showed in the sequence of papers [177], [181] and especially [183], that each generalized quadrangle of order (q,q^2), q odd and $q \neq 1$, satisfying Property (G) at some flag is the point-line dual of a flock generalized quadrangle, i.e. is of Type (v). For translation generalized quadrangles, this asserts that a translation generalized quadrangle $\mathcal{S} = T(\mathcal{O})$ of order (q,q^2), $q \neq 1$, with q odd and with $\mathcal{O}$ good, is the translation dual of the point-line dual of a flock generalized quadrangle.

As any translation generalized quadrangle $\mathcal{S}$ can be represented in a projective space $\mathbf{PG}(n,q)$ for suitable n and q, a natural question is whether each automorphism of $\mathcal{S}$ which fixes a translation point is induced by an element of $\mathbf{P\Gamma L}(n+1,q)$. This question was open since the introduction of the notion of translation generalized quadrangles in 1974, and was recently (very) partially

solved by C. M. O'Keefe and T. Penttila in [108], namely only for the $T_d(\mathcal{O})$ of Tits, $d = 2, 3$. In this chapter, we will solve this problem completely. The following theorem will then be obtained:

Theorem 11.1.1 *Suppose $\mathcal{S} = T(\mathcal{O})$ is a TGQ of order (q^n, q^m) with translation point (∞) and kernel $\mathbf{GF}(q)$, where $\mathcal{O}$ is either a generalized ovoid ($n \neq m$) or a generalized oval ($n = m$) in $\mathbf{PG}(2n + m - 1, q) \subseteq \mathbf{PG}(2n + m, q)$. Then every automorphism of $\mathcal{S}$ which fixes (∞) is induced by an automorphism of $\mathbf{PG}(2n + m, q)$ which fixes $\mathcal{O}$, and conversely.*

To each TGQ $\mathcal{S}$ of order (q^n, q^m) (where $n \neq m$ if q is even), there can be associated another TGQ with the same parameters, namely its translation dual $\mathcal{S}^*$. Essential for the classification of translation generalized quadrangles is the isomorphism problem between $\mathcal{S}$ and $\mathcal{S}^*$. Only two (infinite) classes of translation generalized quadrangles $\mathcal{S}$ are known for which indeed $\mathcal{S} \cong \mathcal{S}^*$, namely the $T_d(\mathcal{O})$ of Tits, where $d \in \{2, 3\}$ and the characteristic is not even if $d = 2$, and the dual Kantor flock generalized quadrangles; probably these are the only examples. Using Theorem 11.1.1, we will start to investigate this problem by completely explaining the connection between the full automorphism groups of $\mathcal{S}$ and $\mathcal{S}^*$ in the following way:

Theorem 11.1.2 *Suppose $\mathcal{S} = T(\mathcal{O})$ is a TGQ of order (q^n, q^m) with base-point (∞), where q is odd if $n = m$, and suppose that $G_{(\infty)}$ is the stabilizer of (∞) in the automorphism group G of $\mathcal{S}$. Furthermore, suppose $(\infty)'$ is the base-point of $T(\mathcal{O}^*) = \mathcal{S}^*$, and let $G'_{(\infty)'}$ be the stabilizer of $(\infty)'$ in the automorphism group G' of $\mathcal{S}^*$. If u, respectively u', is a point of $T(\mathcal{O})$, respectively $T(\mathcal{O}^*)$, which is not collinear with (∞), respectively $(\infty)'$, then $[G_{(\infty)}]_u \cong [G'_{(\infty)'}]_{u'}$. Also, $|G_{(\infty)}| = |G'_{(\infty)'}|$.*

The only classes of translation generalized quadrangles where the sizes of the full automorphism groups (G and G') are different are *precisely* the non-classical translation generalized quadrangles of which the translation dual is the point-line dual of a flock generalized quadrangle $\mathcal{S}(\mathcal{F})$, $\mathcal{F}$ not a Kantor flock. This observation stems from [208] (see Chapter 8), the extensive paper [210] (see Chapter 9), and [213] (see Chapter 10), where it was first shown that these examples indeed have different automorphism groups than their translation dual. Before proceeding, recall that:

Remark 11.1.3 Every known TGQ $\mathcal{S}$ of order (s, t), $s \neq 1 \neq t$, is of one of the following two types:

1. $\mathcal{S} = T(\mathcal{O})$ with $\mathcal{O}$ an oval, respectively ovoid, in $\mathbf{PG}(2, s)$, respectively $\mathbf{PG}(3, s)$;

2. $\mathcal{S} = T(\mathcal{O})$ has order (s, s^2), s odd, and $\mathcal{S}$ or $\mathcal{S}^*$ contains a good line (that is, $\mathcal{S}$ is the point-line dual or the translation dual of the point-line dual of a flock GQ).

We end the chapter with a classification result for translation generalized quadrangles.

11.2 Automorphisms of Translation Generalized Quadrangles

Suppose that Π is a hyperplane of $\mathbf{PG}(n,q)$, $n \geq 2$. A *perspectivity of* $\mathbf{PG}(n,q)$ *with axis* Π is an automorphism of $\mathbf{PG}(n,q)$ fixing Π pointwise. Each nontrivial perspectivity ϕ of $\mathbf{PG}(n,q)$ (with axis Π) has a unique *center*; this is a point x of $\mathbf{PG}(n,q)$ so that each hyperplane of $\mathbf{PG}(n,q)$ through x is fixed by ϕ. If $\phi = \mathbf{1}$, then each point of $\mathbf{PG}(n,q)$ is a *center* by definition. If $x \in \Pi$, then we also call ϕ an *elation* (with *axis* Π and *center* x). If $x \notin \Pi$, then ϕ is sometimes called a *homology* (with *axis* Π and *center* x). By definition, the identical element of $\mathbf{PG}(n,q)$ is also an *elation* and a *homology* of $\mathbf{PG}(n,q)$. If ϕ is an elation of $\mathbf{PG}(n,q)$ with axis Π, then ϕ induces, by definition, a *translation* on $\mathbf{AG}(n,q) = \mathbf{PG}(n,q) \setminus \Pi$. Of course, for the case $n = 2$, the notions above can be generalized to arbitrary projective planes.

The following theorem solves a longstanding conjecture.

Theorem 11.2.1 *Suppose $\mathcal{S} = T(\mathcal{O})$ is a TGQ of order (q^n, q^m) with translation point (∞) and kernel $\mathbf{GF}(q)$, where $\mathcal{O}$ either is a generalized ovoid ($n \neq m$) or a generalized oval ($n = m$) in $\mathbf{PG}(2n+m-1,q) \subseteq \mathbf{PG}(2n+m,q)$. Then every automorphism of $\mathcal{S}$ which fixes (∞) is induced by an automorphism of $\mathbf{PG}(2n+m,q)$ which fixes $\mathcal{O}$, and conversely.*

Proof. Suppose $\mathcal{S}$ and $\mathcal{O}$, etc., are as above. First of all, note that any automorphism of $\mathbf{PG}(2n+m,q)$ which fixes $\mathcal{O}$ clearly induces an automorphism of $T(\mathcal{O})$ which fixes (∞). Hence, to prove the theorem, it suffices to show that any automorphism of $T(\mathcal{O})$ which fixes (∞), induces an automorphism of $\mathbf{PG}(2n+m,q)$ which fixes $\mathcal{O}$.

(1) First let G be the translation group of a TGQ $\mathcal{S}^{(x)}$. Let $\phi \in G \setminus \{\mathbf{1}\}$ and let θ be an automorphism of $\mathcal{S}$ which fixes x. Then $\theta^{-1}\phi\theta$ fixes x linewise and has no fixpoint in $P \setminus x^\perp$, with P the point set of $\mathcal{S}$. Hence $\theta^{-1}\phi\theta$ is an elation

with base-point x. As G is the group of all elations with base-point x, we have $\theta^{-1}\phi\theta \in G$. So G is a normal subgroup of the group of all automorphisms of $\mathcal{S}$ fixing x.[1]

(2) First assume that $q \neq 2$. Consider a point x in $\mathbf{PG}(2n+m,q) \setminus \mathbf{PG}(2n+m-1,q)$, and suppose that ϕ is an automorphism of $T(\mathcal{O})$ which fixes the points (∞) and x. Denote by $\mathcal{C} = \mathcal{C}((\infty),x)$ the group of automorphisms of $T(\mathcal{O})$ which fix (∞) and x linewise. Then by Chapter 8 of FGQ, we know that $\mathcal{C}$ is isomorphic to the multiplicative group of $\mathbf{GF}(q)$, and that $\mathcal{C}$ is induced by the perspectivities of $\mathbf{PG}(2n+m,q)$ with axis $\mathbf{PG}(2n+m-1,q)$ and center x. Take a nontrivial $\sigma \in \mathcal{C}$. Then $\phi^{-1}\sigma\phi$ is an element of $Aut(\mathcal{S})$ which fixes (∞) and x linewise, hence is also an element of $\mathcal{C}$. Put $\phi\sigma\phi^{-1} = \sigma'$. Consider a line L through x in $\mathbf{PG}(2n+m,q)$, and suppose y and z are points of L, both not in $\mathbf{PG}(2n+m-1,q)$, where x,y and z are distinct points, and so that $z = y^{\sigma'} = y^{\phi\sigma\phi^{-1}}$. As $z^{\phi} = (y^{\phi})^{\sigma}$ and σ fixes affine lines of $\mathbf{AG}(2n+m,q) = \mathbf{PG}(2n+m,q) \setminus \mathbf{PG}(2n+m-1,q)$ through x, the points x, y^{ϕ} and z^{ϕ} are on the same line of $\mathbf{AG}(2n+m,q)$, and hence ϕ maps affine lines of $\mathbf{AG}(2n+m,q)$ through x onto affine lines (through x). Now suppose θ' is an arbitrary nontrivial element of the translation group G of $T(\mathcal{O})$, and note that by Chapter 8 of FGQ (more specifically, the proof of [139, 8.7.1]), every element of G is induced by an elation of $\mathbf{PG}(2n+m,q)$ with axis $\mathbf{PG}(2n+m-1,q)$. Then $\phi^{-1}\theta'\phi = \theta$ is also an element of the translation group; see (1). Consider an arbitrary affine line M of $\mathbf{AG}(2n+m,q)$ not through x, and suppose that $\theta' \in G$ is so that M is mapped onto some affine line of $\mathbf{AG}(2n+m,q)$ through x. Then $M^{\phi} = M^{\theta'\phi\theta^{-1}}$, and hence M^{ϕ} is also an affine line of $\mathbf{AG}(2n+m,q)$. Thus, any element ϕ of $Aut(\mathcal{S})$ which fixes (∞) and some point x of $\mathbf{PG}(2n+m,q) \setminus \mathbf{PG}(2n+m-1,q)$ induces an element of the stabilizer of $\mathbf{PG}(2n+m-1,q)$ in $\mathbf{P\Gamma L}(2n+m+1,q)$ which fixes $\mathcal{O}$. Since G is a normal subgroup of $Aut(\mathcal{S})_{(\infty)}$, we have $Aut(\mathcal{S})_{(\infty)} = GH = HG$, where H is the stabilizer in $Aut(\mathcal{S})_{(\infty)}$ of an arbitrary point in $P \setminus (\infty)^{\perp}$. Since each element of G, respectively of H, maps affine lines of $\mathbf{AG}(2n+m,q)$ onto affine lines of $\mathbf{AG}(2n+m,q)$, the theorem follows.

Now assume $q = 2$. Suppose that ϕ is an automorphism of $T(\mathcal{O})$ which fixes (∞). Let L and M be lines of $\mathbf{PG}(2n+m,2)$ not contained in $\mathbf{PG}(2n+m-1,2)$, but containing a common point $u \in \mathbf{PG}(2n+m-1,2)$. Let θ be the translation of $T(\mathcal{O})$ defined by $l_1^{\theta} = l_2$, with l_1, l_2 the points of L not in $\mathbf{PG}(2n+m-1,2)$.

[1] Implicitly, this observation was already made in Chapter 8 of FGQ, see also Theorem 1.7.4.

Then θ is induced by a translation of $\mathbf{AG}(2n+m,2) = \mathbf{PG}(2n+m,2) \setminus \mathbf{PG}(2n+m-1,2)$. If m_1, m_2 are the points of M not in $\mathbf{PG}(2n+m-1,2)$, then clearly $m_1^\theta = m_2$. By (1), the automorphism $\phi^{-1}\theta\phi = \theta'$ is a translation of $T(\mathcal{O})$. We have $(l_1^\phi)^{\theta'} = l_2^\phi$ and $(m_1^\phi)^{\theta'} = m_2^\phi$. Hence the lines $l_1^\phi l_2^\phi$ and $m_1^\phi m_2^\phi$ of $\mathbf{PG}(2n+m,2)$ contain a common point u' of $\mathbf{PG}(2n+m-1,2)$. If we put $u' = u^\phi$, then ϕ defines a bijection of $\mathbf{PG}(2n+m,2)$ onto itself, such that lines of $\mathbf{PG}(2n+m,2)$ not contained in $\mathbf{PG}(2n+m-1,2)$ are mapped onto lines. It easily follows that also the lines of $\mathbf{PG}(2n+m-1,2)$ are mapped onto lines. So ϕ is induced by an automorphism of $\mathbf{PG}(2n+m,2)$ which fixes $\mathcal{O}$. ∎

Corollary 11.2.2 *Suppose $\mathcal{S} = T(\mathcal{O})$ is a TGQ of order (q^n, q^m) with translation point (∞) and $\mathbf{GF}(q)$ a proper subfield of the kernel, where $\mathcal{O}$ is either a generalized ovoid $(n \neq m)$ or a generalized oval $(n = m)$ in $\mathbf{PG}(2n+m-1,q) \subseteq \mathbf{PG}(2n+m,q)$. Then every automorphism of $\mathcal{S}$ which fixes (∞) is induced by an automorphism of $\mathbf{PG}(2n+m,q)$ which fixes $\mathcal{O}$, and conversely.*

Proof. Suppose $\mathcal{S}$ and $\mathcal{O}$, etc. are as above. First of all, note that any automorphism of $\mathbf{PG}(2n+m,q)$ which fixes $\mathcal{O}$ clearly induces an automorphism of $T(\mathcal{O})$ which fixes (∞). Hence to prove the theorem, it suffices to show that any automorphism of $T(\mathcal{O})$ which fixes (∞), induces an automorphism of $\mathbf{PG}(2n+m,q)$ which fixes $\mathcal{O}$.

Suppose that $\mathbf{GF}(q')$ is the kernel of the TGQ. Then $q' \neq 2$. Let $q' = q^r$. The GQ $\mathcal{S}$ is isomorphic to the TGQ $T(\mathcal{O}')$, where $\mathcal{O}'$ either is a generalized ovoid $(n' \neq m')$ or a generalized oval $(n' = m')$ in $\mathbf{PG}(2n'+m'-1,q') \subseteq \mathbf{PG}(2n'+m',q')$ with $n = rn'$ and $m = rm'$. Let $\{u_1, u_2, \ldots, u_r\}$ be a basis of $\mathbf{GF}(q')$ over $\mathbf{GF}(q)$. If $(x_1, x_2, \ldots, x_{2n'+m'})$ is any point of $\mathbf{AG}(2n'+m',q') = \mathbf{PG}(2n'+m',q') \setminus \mathbf{PG}(2n'+m'-1,q')$, then let $x_i = x_{i,1}u_1 + x_{i,2}u_2 + \cdots + x_{i,r}u_r$ with $x_{i,j} \in \mathbf{GF}(q)$, $j = 1, 2, \ldots, r$ and $i = 1, 2, \ldots, 2n'+m'$. Then the mapping

$$\xi : (x_1, x_2, \ldots, x_{2n'+m'}) \mapsto (x_{1,1}, x_{1,2}, \ldots, x_{1,r}, x_{2,1}, \ldots, x_{2n'+m',r})$$

defines an isomorphism of $T(\mathcal{O}')$ onto $T(\mathcal{O})$; see 8.7.1 of FGQ. Let γ be an automorphism of $T(\mathcal{O}')$ which fixes (∞). By Theorem 11.2.1, γ is induced by an automorphism γ' of $\mathbf{PG}(2n'+m',q')$ which fixes $\mathcal{O}'$. Applying ξ, with γ' there corresponds an automorphism γ'' of $\mathbf{PG}(2n+m,q)$ which fixes $\mathcal{O}$. This proves the corollary. ∎

There is another way to obtain Corollary 11.2.2.
Suppose $\mathcal{S}$ and $\mathcal{O}$, etc., are as in Corollary 11.2.2. Any automorphism of $\mathbf{PG}(2n+m,q)$ which fixes $\mathcal{O}$ induces an automorphism of $T(\mathcal{O})$ which fixes (∞). It suffices to show that any automorphism of $T(\mathcal{O})$ which fixes (∞), induces an automorphism of $\mathbf{PG}(2n+m,q)$ which fixes $\mathcal{O}$.
Consider a point x in $\mathbf{PG}(2n+m,q) \setminus \mathbf{PG}(2n+m-1,q)$, and suppose that ϕ is an automorphism of $T(\mathcal{O})$ which fixes the points (∞) and x. Denote by $\mathcal{C} = \mathcal{C}((\infty),x)$ the group of automorphisms of $\mathcal{S}$ which fix (∞) and x linewise. Then $\mathcal{C}$ is isomorphic to the multiplicative group of the kernel of $T(\mathcal{O})$. Let $\mathcal{C}'$ be the subgroup of $\mathcal{C}$ induced by the perspectivities of $\mathbf{PG}(2n+m,q)$ with axis $\mathbf{PG}(2n+m-1,q)$ and center x. Take a nontrivial $\sigma \in \mathcal{C}'$. Then $\phi^{-1}\sigma\phi$ is an element of $Aut(\mathcal{S})$ which fixes (∞) and x linewise, so belongs to $\mathcal{C}$. As $\mathcal{C}'$ is a subgroup of the cyclic group $\mathcal{C}$ and $\sigma \in \mathcal{C}'$, we have that $\sigma' = \phi^{-1}\sigma\phi \in \mathcal{C}'$. Now take over the rest of the proof of Theorem 11.2.1. ∎

In a completely similar way one proves the following theorem.

Theorem 11.2.3 *Suppose $\mathcal{S}_i = T(\mathcal{O}_i)$ is a TGQ of order (q^n, q^m) with translation point $(\infty)_i$ and $\mathbf{GF}(q)$ a subfield of the kernel, where $\mathcal{O}_i$ either is a generalized ovoid $(n \neq m)$ or a generalized oval $(n = m)$ in $\mathbf{PG}^{(i)}(2n+m-1,q) \subseteq \mathbf{PG}^{(i)}(2n+m,q)$, with $i = 1,2$. Then every isomorphism of $\mathcal{S}_1$ onto $\mathcal{S}_2$ which maps $(\infty)_1$ onto $(\infty)_2$ is induced by an isomorphism of $\mathbf{PG}^{(1)}(2n+m,q)$ onto $\mathbf{PG}^{(2)}(2n+m,q)$ which maps $\mathcal{O}_1$ onto $\mathcal{O}_2$, and conversely.*

∎

Remark 11.2.4 For the $\mathcal{S} = T_d(\mathcal{O})$ of Tits, $d = 2,3$, the case $q = 2$ does not have to be treated separately (as $\mathcal{S}$ is then classical).

H. Van Maldeghem pointed out to us that the following general theorem has a proof completely similar to the proof of Theorem 11.2.1. We first need a definition. Let $\Gamma = (P,B,I)$ be a point-line incidence geometry. A *generalized linear representation* of Γ in $\mathbf{AG}(n,q)$ is defined similarly as the usual linear representation of Γ in $\mathbf{AG}(n,q)$ (cf. Chapter 7), but where we allow subspaces of $\mathbf{AG}(n,q)$ instead of lines as in a usual linear representation. Thus, it is a monomorphism θ of Γ into the geometry of points and subspaces of the affine space $\mathbf{AG}(n,q)$, in such a way that P^θ is the set of all points of $\mathbf{AG}(n,q)$, that B^θ is a union of parallel classes of subspaces (not necessarily of the same dimension) of $\mathbf{AG}(n,q)$, and that each point of L^θ is the image of some point of L for any line L in B. We usually identify Γ with its image Γ^θ.

Theorem 11.2.5 *Let Γ be a point-line incidence geometry having a generalized linear representation θ in $\mathbf{AG}(n,q)$, $n>1$. Assume that G is a subgroup of $Aut(\Gamma)$ so that the group of translations of $\mathbf{AG}(n,q)$ is a normal subgroup of G, and so that the group of homologies of the projective completion $\mathbf{PG}(n,q)$ of $\mathbf{AG}(n,q)$ with axis $\mathbf{PG}(n-1,q)=\mathbf{PG}(n,q)\setminus\mathbf{AG}(n,q)$ and center $x\in\mathbf{AG}(n,q)$ is a normal subgroup of G_x. Then G is induced by a subgroup of $\mathbf{A\Gamma L}(n+1,q)$ fixing the set of all spaces at infinity of the elements of Γ^θ, and vice versa.*

■

In particular, Theorem 11.2.5 applies to translation planes, see Section 1 of M. Kallaher [88] (Chapter 5 of [28]).

11.3 The Stabilizer of the Base-Point of a Translation Generalized Quadrangle

We now have the following result which is an important step towards the determination of all translation generalized quadrangles.

Theorem 11.3.1 *Suppose $\mathcal{S}=T(\mathcal{O})$ is a TGQ of order (q^n,q^m) with base-point (∞), where q is odd if $n=m$, and suppose that $G_{(\infty)}$ is the stabilizer of (∞) in the automorphism group G of $\mathcal{S}$. Furthermore, suppose $(\infty)'$ is the base-point of $T(\mathcal{O}^*)=\mathcal{S}^*$, and let $G'_{(\infty)'}$ be the stabilizer of $(\infty)'$ in the automorphism group G' of $\mathcal{S}^*$. Suppose u and v are arbitrary points of $T(\mathcal{O})$ and $T(\mathcal{O}^*)$, not collinear with (∞) and $(\infty)'$, respectively. Then $[G_{(\infty)}]_u\cong[G'_{(\infty)'}]_v$. We also have that*

$$|G_{(\infty)}|=|G'_{(\infty)'}|.$$

Proof. Suppose $\mathcal{O}=\{\pi^0,\pi^1,\ldots,\pi^{q^m}\}$ is contained in $\Pi=\mathbf{PG}(2n+m-1,q)$, and embed Π in the $(2n+m)$-space $\Pi'=\mathbf{PG}(2n+m,q)$. Let $\zeta^0,\zeta^1,\ldots,\zeta^{q^m}$ be the tangent spaces at respectively $\pi^0,\pi^1,\ldots,\pi^m$. Now consider the dual space $(\Pi')^*$ of Π'; then the set of tangent spaces interpreted in $(\Pi')^*$, say $\mathcal{U}^*=\{\zeta^0_*,\zeta^1_*,\ldots,\zeta^{q^m}_*\}$, forms a set of q^m+1 spaces of dimension n which satisfy the following properties:

1. they intersect two by two in the same fixed point z (which corresponds to Π);

2. for distinct i, j and k, $\zeta_*^i \zeta_*^j \zeta_*^k$ has dimension $3n$;

3. for each i, there is an $(n+m)$-space π_*^i which contains z and ζ_*^i, and so that $\pi_*^i \cap \zeta_*^j = \{z\}$ if $i \neq j$.

Now consider an arbitrary point $x \in \Pi' \setminus \Pi$. Then the corresponding hyperplane Π_x in the dual space $(\Pi')^*$ of Π' is so that $\Pi_x \cap \mathcal{U}^*$ is an egg $\mathcal{O}^*$ in Π_x, which is clearly isomorphic to the dual egg of $\mathcal{O}$ (hence the notation). Now consider an arbitrary collineation θ of $T(\mathcal{O})$ which fixes (∞) and the point x. Then by Theorem 11.2.1, θ induces a collineation of $\Pi' = \mathbf{PG}(2n+m, q)$ which fixes $\mathcal{O}$ and x. Now interpret θ 'naturally' as a collineation of $(\Pi')^*$ (that is, if η is an r-dimensional space in $(\Pi')^*$ and if η^* is the corresponding $(2n+m-r-1)$-dimensional space in the dual space Π, then η^θ is the r-dimensional space of $(\Pi')^*$ which corresponds to $(\eta^*)^\theta$). Then θ fixes z, Π_x, and hence $\mathcal{O}^*$, and thus θ induces a collineation of $T(\mathcal{O}^*)$ which fixes $(\infty)'$ and z (where $(\infty)'$ is the translation point of $T(\mathcal{O}^*)$). Thus

$$[G_{(\infty)}]_u \cong [G'_{(\infty)'}]_v,$$

where u and v are arbitrary points of $T(\mathcal{O})$ and $T(\mathcal{O}^*)$, not collinear with (∞) and $(\infty)'$, respectively. For any TGQ $\mathcal{S}^{(y)} = (P, B, I)$ with translation point y, translation group H and full automorphism group G, there holds (by (1) of the proof of Theorem 11.2.1), that $G_y = (G_y)_z H = H(G_y)_z$, where z is arbitrary in $P \setminus y^\perp$. As the translation groups of $T(\mathcal{O})$ and $T(\mathcal{O}^*)$ are isomorphic, the theorem now readily follows. ∎

Remark 11.3.2 Motivated by the proof of Theorem 11.3.1, it may be useful for other purposes to define the following object $\mathcal{U}$ in $\mathbf{PG}(2n+m, q)$: $\mathcal{U} = \{\zeta^0, \zeta^1, \ldots, \zeta^{q^m}\}$ is a set of $q^m + 1$ spaces of dimension n which satisfy the following properties:

1. the elements of $\mathcal{U}$ intersect two by two in the same fixed point x;

2. for distinct i, j and k, $\zeta^i \zeta^j \zeta^k$ has dimension $3n$;

3. for each i, there is an $(n+m)$-space π^i which contains ζ^i, and so that $\pi^i \cap \zeta^j = \{x\}$ if $i \neq j$.

We call $\mathcal{U}$ a *generalized ovoid cone*, respectively *generalized oval cone*, with *vertex* x if $n \neq m$, respectively $n = m$.

Corollary 11.3.3 *Suppose $\mathcal{S} = T(\mathcal{O})$ is a TGQ of order (q^n, q^m), where q is odd if $n = m$, and suppose that $Aut(\mathcal{S})$ is the automorphism group of $\mathcal{S}$. Suppose (∞) is the base-point of the TGQ. Let $\mathcal{S}^* = T(\mathcal{O}^*)$ be its translation dual with base-point $(\infty)'$, and let $Aut(\mathcal{S}^*)$ be the full group of automorphisms of $\mathcal{S}^*$. Then we have one of the following possibilities:*

(*i*) $|Aut(\mathcal{S})| = |Aut(\mathcal{S}^*)|$;

(*ii*) $|Aut(\mathcal{S})| = (q^n + 1)|Aut(\mathcal{S}^*)|$;

(*iii*) $|Aut(\mathcal{S}^*)| = (q^n + 1)|Aut(\mathcal{S})|$;

and all possibilities occur.

Proof. The $Aut(\mathcal{S})$-orbit of (∞), with $\mathcal{S} = (P, B, I)$, is either P, a line, or $\{(\infty)\}$. If every point of $\mathcal{S}$ is a translation point, then $\mathcal{S}$ is classical[2], $\mathcal{S} \cong \mathcal{S}^*$, and so $Aut(\mathcal{S}) \cong Aut(\mathcal{S}^*)$. In all other cases we have one of (i),(ii),(iii) by Theorem 11.3.1. By the next corollary all possibilities occur. ∎

The following corollary was already deduced in Chapter 10 in the context of TGQ's arising from flocks.

Corollary 11.3.4 (See Chapter 10) *1. Suppose $\mathcal{S} = T(\mathcal{O})$ is a TGQ of order (s, s^2), s even, and suppose that $Aut(\mathcal{S})$ is the automorphism group of $\mathcal{S}$. Suppose (∞) is the base-point. Let $\mathcal{S}^* = T(\mathcal{O}^*)$ be the translation dual with base-point $(\infty)'$, and let $Aut(\mathcal{S}^*)$ be the automorphism group of $\mathcal{S}^*$. Then $|Aut(\mathcal{S})| = |Aut(\mathcal{S}^*)|$.*

2. Suppose $\mathcal{S} = T(\mathcal{O})$ is a TGQ of order (s, s^2), s odd, which arises from a flock but which is not the point-line dual of a Kantor flock GQ, and suppose that $Aut(\mathcal{S})$ is the full group of automorphisms of $\mathcal{S}$. Suppose (∞) is the base-point of $T(\mathcal{O})$. Let $\mathcal{S}^* = T(\mathcal{O}^*)$ be the translation dual of $T(\mathcal{O})$ with base-point $(\infty)'$, and let $Aut(\mathcal{S}^*)$ be the full group of automorphisms of $\mathcal{S}^*$. Then $|Aut(\mathcal{S}^*)| = (s + 1)|Aut(\mathcal{S})|$.

∎

[2]See, e.g., Chapter 5.

11.4 Structure of the Automorphism Group of a TGQ

Let $\mathcal{S} = T(n,m,q) = T(\mathcal{O})$ be a TGQ of order (q^n, q^m) for the generalized ovoid $\mathcal{O}$ $(n \neq m)$, respectively generalized oval $\mathcal{O}$ $(n = m)$, with full group of automorphisms $Aut(\mathcal{S})$ and base-point (∞). By Theorem 11.2.1, we know that $Aut(\mathcal{S})_{(\infty)} \cong \mathbf{P\Gamma L}(2n+m+1,q)_{\mathcal{O}}$, where $\mathcal{O}$ is contained in $\mathbf{PG}(2n+m-1,q) \subseteq \mathbf{PG}(2n+m,q)$. The following theorem analyses more closely the structure of $Aut(\mathcal{S})_{(\infty)}$.

Theorem 11.4.1 *Let $\mathcal{S} = T(n,m,q) = T(\mathcal{O})$ be a TGQ of order (q^n, q^m) for the generalized ovoid $\mathcal{O}$ $(n \neq m)$, respectively generalized oval $\mathcal{O}$ $(n = m)$, with full group of automorphisms $Aut(\mathcal{S})$ and base-point (∞). Let T be the translation group of $\mathcal{S}$, and suppose x is an arbitrary point of $P \setminus (\infty)^{\perp}$, where P is the point set of $\mathcal{S}$. Let $H = Aut(\mathcal{S})_{(\infty)}$ be the stabilizer of (∞) in $Aut(\mathcal{S})$. Then we have that*

$$H \cong T \rtimes H_x,$$

where '$\rtimes$' denotes the natural semidirect product.

Proof. The theorem follows from the fact that $H_x \cap T = \{\mathbf{1}\}$, that T is a normal subgroup of $H = Aut(\mathcal{S})_{(\infty)}$, and that $H = TH_x = H_xT$. ∎

11.5 On the Classification of TGQ's

We now can easily deduce the following.

Theorem 11.5.1 *Let $\mathcal{S} = T(\mathcal{O})$ be a TGQ of order q^n, q odd, with translation point (∞). Then we have that:*

(*i*) *either $\mathcal{S}$ is classical, or*

(*ii*) *if u and v are arbitrary points of $T(\mathcal{O})$ and $T(\mathcal{O}^*)$, not collinear with (∞) and $(\infty)'$ (where $(\infty)'$ is the translation point of $\mathcal{S}^* = T(\mathcal{O}^*)$), respectively, then $Aut(\mathcal{S})_u = [Aut(\mathcal{S})_{(\infty)}]_u \cong Aut(\mathcal{S}^*)_v = [Aut(\mathcal{S}^*)_{(\infty)'}]_v$. Also, $|Aut(\mathcal{S})| = |Aut(\mathcal{S}^*)|$.*

Proof. Suppose we are not in Case (i). Then also $\mathcal{S}^*$ is not classical. Then $\mathcal{S}$ and $\mathcal{S}^*$ have precisely one translation point by Theorem 9.11.3, and hence $Aut(\mathcal{S}) = Aut(\mathcal{S})_{(\infty)}$ and $Aut(\mathcal{S}^*) = Aut(\mathcal{S}^*)_{(\infty)'}$. The result now follows from Theorem 11.3.1. ■

Combining the obtained results, we arrive at the following theorem.

Theorem 11.5.2 *Let $\mathcal{S} = T(\mathcal{O})$ be a TGQ of order (q^n, q^m) with translation point (∞). Then we have the following possibilities.*

(*i*) *$n = m$, and either $\mathcal{S} \cong \mathcal{Q}(4, q^n)$, or*

- (*a*) *$\mathcal{S}$ has one translation point, q is even, and $Aut(\mathcal{S}) = Aut(\mathcal{S})_{(\infty)}$;*
- (*b*) *$\mathcal{S}$ has one translation point, q is odd, and if u and v are arbitrary points of $T(\mathcal{O})$ and $T(\mathcal{O}^*)$, not collinear with (∞) and $(\infty)'$ (where $(\infty)'$ is the translation point of $\mathcal{S}^* = T(\mathcal{O}^*)$), respectively, then*

$$Aut(\mathcal{S})_u = [Aut(\mathcal{S})_{(\infty)}]_u \cong Aut(\mathcal{S}^*)_v = [Aut(\mathcal{S}^*)_{(\infty)'}]_v.$$

Also, $|Aut(\mathcal{S})| = |Aut(\mathcal{S}^*)|$.

(*ii*) *$n < m$, and then we have one of the following cases.*

- (*a*) *$\mathcal{S} \cong \mathcal{Q}(5, q^n)$.*
- (*b*) *$\mathcal{S}$ and $\mathcal{S}^*$ are not classical, have two distinct collinear translation points, and then q is odd, $m = 2n$, and $\mathcal{S} \cong \mathcal{S}^* \cong \mathcal{S}(\mathcal{F})^D$, where $\mathcal{F}$ is a Kantor flock.*
- (*c*) *$\mathcal{S}$ is not classical, is not the point-line dual of a Kantor flock GQ, has two distinct collinear translation points (and then also a line of translation points), and then $m = 2n$, q is odd, and $\mathcal{S}$ is the translation dual of the point-line dual of a flock GQ. In this case we have that $|Aut(\mathcal{S})| = (q^n + 1)|Aut(\mathcal{S}^*)|$.*
  Moreover, we have that

$$[Aut(\mathcal{S})_{(\infty)}]_u \cong Aut(\mathcal{S}^*)_v = [Aut(\mathcal{S}^*)_{(\infty)'}]_v,$$

where (∞) is an arbitrary translation point on the line of translation points $[\infty]$ of $\mathcal{S}$, where $(\infty)'$ is the translation point of $\mathcal{S}^*$, and where u and v are arbitrary points of $T(\mathcal{O})$ and $T(\mathcal{O}^*)$, not collinear with (∞) and $(\infty)'$, respectively.

(*d*) *$\mathcal{S}^*$ is not classical, is not the point-line dual of a Kantor flock GQ, has two distinct collinear translation points, and then $m = 2n$, q is odd, and $\mathcal{S}^*$ is the translation dual of the point-line dual of a flock GQ. Similarly as in Case (ii) (c), we have that $|Aut(\mathcal{S}^*)| = (q^n+1)|Aut(\mathcal{S})|$. Also,*

$$[Aut(\mathcal{S}^*)_{(\infty)'}]_v \cong Aut(\mathcal{S})_u = [Aut(\mathcal{S})_{(\infty)}]_u,$$

where $(\infty)'$ is an arbitrary translation point on the line of translation points $[\infty]'$ of $\mathcal{S}^*$, where (∞) is the translation point of $\mathcal{S}$, and where u and v are arbitrary points of $T(\mathcal{O})$ and $T(\mathcal{O}^*)$, not collinear with (∞) and $(\infty)'$, respectively.

(*e*) *$\mathcal{S}$ and $\mathcal{S}^*$ both have precisely one translation point, and then $Aut(\mathcal{S}) = Aut(\mathcal{S})_{(\infty)}$ and $Aut(\mathcal{S}^*) = Aut(\mathcal{S}^*)_{(\infty)'}$, where the notation is obvious. Furthermore, $|Aut(\mathcal{S})| = |Aut(\mathcal{S}^*)|$, and*

$$Aut(\mathcal{S})_u \cong Aut(\mathcal{S}^*)_v,$$

where u and v are arbitrary points of $T(\mathcal{O})$ and $T(\mathcal{O}^*)$, not collinear with (∞) and $(\infty)'$, respectively.

If $H = Aut(\mathcal{S})_{(\infty)}$ is the stabilizer of (∞) in $Aut(\mathcal{S})$, then we have that $H \cong T \rtimes H_x$, with T the translation group of $\mathcal{S}^{(\infty)}$.

Proof. First suppose that $n = m$. Then by Theorems 8.8.1 and 9.11.3, $\mathcal{S}$ has one translation point if $\mathcal{S}$ is not classical. Part (i) then follows from Theorem 11.5.1.
Next suppose that $n < m$, and that $\mathcal{S}$ is not classical (i.e. not isomorphic to $\mathcal{Q}(5, q^n)$). If $\mathcal{S} = T(\mathcal{O})$ has two distinct translation points, then they are necessarily collinear, and from Theorem 9.11.3 it follows that q is odd and that $(\mathcal{S}^*)^D$ is a flock GQ $\mathcal{S}(\mathcal{F})$, or, equivalently, $\mathcal{O}$ is good at some element π. If $\mathcal{S}^*$ also contains distinct translation points, then Theorem 9.11.3 and Theorem 10.4.1 imply that $\mathcal{F}$ is a Kantor flock, and hence $\mathcal{S}^* \cong \mathcal{S}$ by Theorem 2.4.1. Suppose $\mathcal{S} \ncong \mathcal{S}^*$. Then $\mathcal{S}^*$ has one translation point and Part (c) follows from Theorem 11.3.1 and Corollary 11.3.3. Part (d) is obtained from Part (c) by interchanging the role of $\mathcal{S}$ and $\mathcal{S}^*$. If we are not in one of the previous cases, then we are in Case (e) by Theorem 11.3.1. The final part of the statement

now follows from Theorem 11.4.1. ■

Final Remark

The paper [191] is the first in a series of papers. It is the eventual goal to determine all translation generalized quadrangles arising from flocks, to analyse more closely the relation between $\mathcal{S}$ and $\mathcal{S}^*$, and to refine the classification of all TGQ's.

Chapter 12

A Lenz-Barlotti Classification for Finite Generalized Quadrangles

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In *Finite Geometries* [48], P. Dembowski wrote that an alternative approach to the study of projective planes began with the paper *Homogeneity of projective planes, R. Baer (1942)* [6] in which the close relationship between Desargues' theorem and the existence of central collineations was pointed out. Baer's notion of (p, L)-transitivity, corresponding to this relationship, proved to be extremely fruitful; it provided a better understanding of coordinate structures and it led eventually to the only coordinate-free (and hence geometrically satisfactory) classification of projective planes existing today, namely the classification by H. Lenz in *Kleiner Desarguesscher Satz und Dualität in projektiven Ebenen (1954)* [104] and A. Barlotti in *Sulle possibili configurazioni del sistema delle coppie punto-retta* (A, a) *per cui un piano grafico risulta* (A, a)*-transitivo (1958)* [9], see also [161]. The brilliant idea of H. Lenz was to consider all possible subconfigurations of point-line pairs

$$\{\{p, L\} \parallel pIL\},$$

of projective planes Π so that Π is (p, L)-transitive. A. Barlotti then considered all such point-line pairs. Due to deep discoveries in finite group theory, the analysis of this classification has been particularly penetrating for finite projective planes in recent years.
For generalized quadrangles, J. A. Thas and H. Van Maldeghem [194] gave a (first) definition of Desargues configurations and proved a result analogous to the theorem of R. Baer for projective planes, i.e. it asserts that a local configurational condition holds if and only if a certain locally defined collineation group is as large as possible. In the theory of projective planes, the configuration involves two triangles in perspective from a point (or a line); in the theory of generalized quadrangles the configuration involves two quadrilaterals in perspective from a panel (or root).

In H. Van Maldeghem, J. A. Thas and S. E. Payne [234], there was a second approach to the problem, as follows. Let (p, L) be a flag of a generalized quadrangle $\mathcal{S}$. A collineation θ of $\mathcal{S}$ is called a (p, L)-*collineation* if θ fixes each point on L and each line through p. The group $G(p, L)$ of all (p, L)-collineations acts semiregularly on the set of points of $\mathcal{S}$ collinear with u but not on L, where $u \neq p$ is a point on L, and dually on the set of lines of $\mathcal{S}$ concurrent with N but not passing through p, where $N \neq L$ is a line through p. If $G(p, L)$ acts regularly on these sets, then $\mathcal{S}$ is said to be (p, L)-*transitive*. It is easy to see that each Moufang generalized quadrangle is (p, L)-transitive for all flags (p, L). The authors proved that for finite $\mathcal{S}$ the converse also holds; a finite generalized quadrangle $\mathcal{S}$ is (p, L)-transitive for all flags (p, L) if and only if $\mathcal{S}$ is Moufang and hence classical or dual classical by Theorem 1.4.1. As a geometric counterpart to (p, L)-transitivity, they introduced the notion of a (p, L)-*Desarguesian generalized quadrangle*, and they proved that a finite generalized quadrangle is (p, L)-Desarguesian if and only if it is (p, L)-transitive.

The most natural generalization of the notion of (p, L)-transitivity, pIL, for projective planes to generalized quadrangles is not that of (p, L)-transitivity however; it is the notion of '(p, L, q)-transitivity':

A GQ is (p, L, q)-transitive *for its panel* (p, L, q) *if* (p, L, q) *is Moufang.*

The natural analogue of (p, L)-transitivity, $p\not|L$, for projective planes in the theory of generalized quadrangles is that of '(x, y)-transitivity':

A GQ is (x,y)-transitive*, where x and y are non-collinear points, if there is a group H of whorls about x and y (such automorphisms are in general called* generalized homologies*) which acts transitively (and then regularly) on the points incident with at least one line M through x, not contained in $x \cup \{x,y\}^{\perp}$.*

If there is such a group H for x and y, then the choice of M is arbitrary, and the notion appears to be symmetric for both x and y. It should be noted that the notion of (p,L)-transitivity for projective planes is self-dual; the notions of (p,L,q)-transitivity and (x,y)-transitivity are *not.*
Some configurational results were already obtained by several authors:

- Theorem 1.4.2 of J. A. Thas, S. E. Payne and H. Van Maldeghem, which asserts that every half Moufang GQ is automatically Moufang;
- the result of P. Fong and G. M. Seitz for generalized quadrangles, see Theorem 1.4.1, which infers that each Moufang GQ is classical or dual classical[1];
- work of J. A. Thas [170, 171] implying that each thick GQ $\mathcal{S}$ which is (x,y)-transitive for each $x \not\sim y$ in $\mathcal{S}$ (where x and y are points), is classical[2].

However, despite these promising results, a 'good' classification based on subconfigurations of flags (p,L), respectively panels (p,L,q), for which the generalized quadrangle is (p,L)-transitive, respectively (p,L,q)-transitive, seems (quite) far away and would yield many open classes very hard to deal with. We present the following alternative. From Chapter 5 we recall that:

A line L of a generalized quadrangle $\mathcal{S}$ is an axis of symmetry if it is regular, and if there is a pair of distinct points (p,q) both incident with L for which the generalized quadrangle is (p,L,q)-transitive.

A classification based on subconfigurations of axes of symmetry will appear to be a good choice[3]. Note that this is a 'Lenz-type' classification.

[1] We also refer to the work of J. Tits and R. Weiss [226].

[2] We note that the proof of this result is almost completely combinatorial; only Theorem 1.4.1 is utilized in the case which is, in fact, the same open case as in Chapter 9 of FGQ.

[3] Some classical examples will be in the 'lowest' class, namely those which have no regular lines (which must be the case if $t < s$). In particular, $H(4,q^2)$ is the only classical example in Class **I** whose point-line dual is also in Class **I**.

The chapter is organized as follows. in § 12.1, we prove a first version of the classification theorem, which will define 'symmetry-classes' **I**-**VI**. The next section contains a short discussion concerning TGQ's arising from flocks. Then, in § 12.3–12.4, the classes **I**-**II** are investigated. In § 12.5, necessary theorems on SPGQ's of order (s, s^2) with s even are proved, and in § 12.6, we digress to make an observation on spreads in SPGQ's which will later be used. In Section 12.7, two useful results are obtained. Then, in § 12.8–12.11, we investigate the classes **III**-**VI**. The next section contains a classification theorem for span-symmetric generalized quadrangles. In a last section, we give a table which overviews a finalized version of the classification.
In an appendix, we obtain a strong characterization theorem of the classical GQ $\mathcal{Q}(5, q)$, which solves a recent conjecture of W. M. Kantor (which we posed independently).

The preliminary work done in Chapters 3, 4, 5, 6, 8, 9, 10 and 11 will be essential for the present chapter.

This chapter is based on *K. Thas, A Lenz-Barlotti Classification of Finite Generalized Quadrangles* [218]. We also refer the interested reader to K. Thas [217].

12.1 The Classification Theorem

Our first lemma is well-known.

Lemma 12.1.1 *Suppose $\mathcal{S}$ is a GQ of order s, $s > 1$, which contains a regular pair of distinct non-concurrent lines $\{U, V\}$. Then every line of $\mathcal{S} \setminus \{U, V\}^{\perp\perp}$ intersects at least one line of $\{U, V\}^{\perp\perp}$.*

Proof. Easy counting. ∎

Lemma 12.1.2 *Suppose $\mathcal{S}$ is an SPGQ of order (s, t), $s \neq 1 \neq t$, with base-span $\mathcal{L}$, and suppose that every point of $\mathcal{S}$ is incident with a constant number, say $k + 1$ $(k > 0)$, of axes of symmetry. Moreover, suppose that every axis of symmetry hits the base-grid. Then $(t, k) = (s, s)$, i.e. $\mathcal{S} \cong \mathcal{Q}(4, s)$.*

Proof. If $s = t$, then $\mathcal{S} \cong \mathcal{Q}(4, s)$ by Theorem 8.8.1, and $k = s$, see Remark 1.7.1. Suppose by way of contradiction that $s \neq t$, i.e. that $t = s^2$ by Theorem 8.2.1. We count the number of axes of symmetry of $\mathcal{S} \setminus (\mathcal{L} \cup \mathcal{L}^{\perp})$ in two ways,

and distinguish three cases. We obtain the following three equalities, according as $\mathcal{L}^\perp$ contains no, respectively one, respectively $s+1$ axes of symmetry:

(1) $(s+1)^2k = (s^3-s)(k+1)$;

(2) $(s+1)(k-1)+(s+1)sk = (s^3-s)(k+1)$;

(3) $(s+1)^2(k-1) = (s^3-s)(k+1)$.

None of these equalities is possible if $s > 2$ (by Section 1.3, we do not consider the case $s = 2$). The proof is thus complete. ∎

Lemma 12.1.3 *Suppose $\mathcal{S}$ is a thick GQ of order (s,t) which contains an axis of symmetry L, and suppose $\mathcal{S}'$ is a thick proper subGQ of $\mathcal{S}$ of order (s,t') which contains L. Then L is also an axis of symmetry in $\mathcal{S}$.*

Proof. It is clear that any nontrivial symmetry about L fixes $\mathcal{S}'$ and acts nontrivially on $\mathcal{S}'$. ∎

The following result, which we present as a lemma, is taken from Chapter 5.

Lemma 12.1.4 *Suppose $\mathcal{S}$ is a GQ of order (s,t), $s,t \neq 1$. Then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(4,s)$ or $\mathcal{Q}(5,s)$ if and only if $\mathcal{S}$ contains a center of transitivity p, a collineation θ of $\mathcal{S}$ for which $p^\theta \not\sim p$, and a regular line.*

∎

Note that the proof of Lemma 12.1.4 uses Theorem 1.4.1. There is an easy and well-known corollary, which we already met several times, but it will be convenient to state it here as a lemma.

Lemma 12.1.5 *Let $\mathcal{S}$ be a thick GQ of order (s,t) every line of which is an axis of symmetry. Then $\mathcal{S}$ is isomorphic to one of $\mathcal{Q}(4,s)$, $\mathcal{Q}(5,s)$.*

Proof. If $\mathcal{S}$ is as above, then every point of $\mathcal{S}$ is a translation point, and then the corollary follows from the fact that an axis of symmetry is regular, and Lemma 12.1.4. ∎

The following theorem describes the structure of the possible subconfigurations of axes of symmetry of generalized quadrangles. In the next sections every class will be investigated in detail.

Theorem 12.1.6 *Suppose $\mathcal{S} = (P, B, I)$ is a generalized quadrangle of order (s, t), $s \neq 1 \neq t$. Then we have one of the following possibilities.*

I *$\mathcal{S}$ contains no axis of symmetry.*

II *Every axis of symmetry is incident with some fixed point $p \in P$.*

III *There is a line $L \in B$ which is not an axis of symmetry such that every point qIL is incident with exactly $k+1$ axes of symmetry, $k \in \{0, s-1, s^2-1\}$, and there are no other axes of symmetry.*

IV *There is an axis of symmetry $L \in B$ such that every point qIL is incident with exactly $k+1$ axes of symmetry, $k \in \{1, s, s^2\}$, and there are no other axes of symmetry.*

V *Suppose none of the previous cases holds. Define an incidence structure $\mathcal{S}' = (P', B', I')$ as follows.*

- LINES *are of two types:*
  (1) *the axes of symmetry of $\mathcal{S}$;*
  (2) *the lines of $\mathcal{S}$ such that each point of such a line is incident with a line of Type (1).*
- *The* POINTS *of $\mathcal{S}'$ are the points lying on the lines of Type (1), and*
- INCIDENCE *is the restriction of I to $(P' \times B') \cup (B' \times P')$.*

Then $\mathcal{S}'$ is a subGQ of $\mathcal{S}$, every point of $\mathcal{S}'$ is incident with a constant number $k+1$ of axes of symmetry, and one of the following possibilities holds.

(*i*) *$k = 0$ and $\mathcal{S}$ has a regular spread $\mathbf{T}_N$ of which any line is an axis of symmetry of $\mathcal{S}$. Furthermore, the group $Aut(\mathcal{S})$ acts transitively on the lines of $\mathbf{T}_N$, $\mathcal{S}' = \mathcal{S}$ and $t = s^2$.*

(*ii*) *We have that $k = 1$, and there are two possibilities.*
  (*a*) *$\mathcal{S} = \mathcal{S}'$ and $\mathcal{S}$ is of order (s, s^2).*
  (*b*) *$\mathcal{S}'$ is a grid with parameters $s+1, s+1$, and hence each line of $\mathcal{S}'$ is an axis of symmetry of $\mathcal{S}$. Moreover, $\mathcal{S}$ is a GQ of order (s, s^2).*

(*iii*) *$2 \leq k < t$ and one of the following possibilities holds.*
  (*a*) *$\mathcal{S}' = \mathcal{S}$ and $\mathcal{S}$ is of order (s, s^2).*
  (*b*) *$k = s$, $\mathcal{S}' \neq \mathcal{S}$, $\mathcal{S}' \cong \mathcal{Q}(4, s)$ and $t = s^2$.*

VI *Every line of $\mathcal{S}$ is an axis of symmetry and then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(4,s)$ or $\mathcal{Q}(5,s)$.*

Proof. First suppose that $\mathcal{S}$ is not contained in **I**, and suppose p is a point which is incident with exactly $k+1$ axes of symmetry, $k \geq 0$, and let L be an axis of symmetry not incident with p. Then $Aut(\mathcal{S})$ acts transitively on the points of $proj_pL$. Whence *every* point of the line $proj_pL$ is incident with **exactly** $k+1$ axis of symmetry. Define k in this way for the rest of the proof. Suppose $\mathcal{S}$ is not an element of **I**, **II**, **III**, **IV** and **VI** in STEP 1-4. Then we have the following.

STEP 1: The structure $\mathcal{S}'$ is a (possibly thin) generalized quadrangle

The definition of $\mathcal{S}'$ implies that every line of $\mathcal{S}'$ is incident with $s+1$ points of $\mathcal{S}'$. Let L be an arbitrary line of $\mathcal{S}$ which contains at least two different points q and r of $\mathcal{S}'$. Then there is an axis of symmetry through q and also through r, and it follows that there exist axes of symmetry through each point of L. Thus L is completely contained in $\mathcal{S}'$. Since $\mathcal{S}'$ has at least one axis of symmetry, and since all the axes of symmetry of $\mathcal{S}$ do not contain a common point, we can conclude by Theorem 1.6.3 that $\mathcal{S}'$ is a subquadrangle of order (s,t') with $1 \leq t' \leq t$. It is important to observe that every point of $\mathcal{S}'$ is incident with $k+1$ axes of symmetry of $\mathcal{S}$ with k as above, $0 \leq k \leq t'$; we only show this property for the case $\mathcal{S} = \mathcal{S}'$, with $s \neq t$. The other cases are left to the reader as an easy exercise (and they follow essentially from the rest of the proof).
We start with observing the following easy property:

(O) *If $x \sim y \neq x$ and if there are axes of symmetry LIx and MIy so that $L \not\sim M$, then x and y are incident with the same number of axes of symmetry.*

Suppose that p and q are points of $\mathcal{S}'$ which are both incident with the same line U which is not an axis of symmetry. Then there are axes of symmetry V and W of $\mathcal{S}$ so that VIp and WIq. The result then follows from (O). Suppose $p \not\sim q$, and distinguish the following two cases:

- THERE ARE AXES OF SYMMETRY LIp AND MIq SO THAT $L \not\sim M$. Suppose that x is a point not on a line of $\{L,M\}^{\perp\perp} = \{L_0 = L, L_1, \ldots, L_s = M\}$, which is incident with an axis of symmetry skew to $\{L,M\}^{\perp\perp}$. Define the points $x_i := proj_{L_i}x$ for $i = 0,1,\ldots,s$. Then by (O), each x_i is incident with the same number of axes of symmetry of $\mathcal{S}$. Since

each line of $\{L, M\}^{\perp}$ contains a point of $\{x_0, x_1, \ldots, x_s\}$, the statement now follows from (O). The case where each axis of symmetry through x intersects $\{L, M\}^{\perp\perp}$ is handled similarly.

- WE ARE NOT IN THE PREVIOUS CASE. Then LIp and MIq are the only axes of symmetry through, respectively, p and q, and L and M are concurrent. Suppose w is an arbitrary point of $\{p, q\}^{\perp} \setminus \{L \cap M\}$. Then there is an axis of symmetry WIw. It is now obvious that there is an automorphism of $\mathcal{S}$ mapping p onto q.

Finally, if $pILIq \neq p$ with L an axis of symmetry, choose a point u in $p^{\perp}$ or $q^{\perp}$ such that up or uq is not an axis of symmetry, and apply the preceding observations to obtain that u is incident with the same number of axes of symmetry as p and q. If no such point exists, then every line of $L^{\perp}$ is an axis of symmetry, and (O) applies.

STEP 2: The case $k = 0$

As $k = 0$ and as we are not in Case **III**, $\mathcal{S}'$ is a thick GQ which contains non-concurrent axes of symmetry (by Lemma 12.1.3), $\mathcal{S}'$ is an SPGQ of order (s, t'), $t' > 1$, and hence $t' \in \{s, s^2\}$ by Theorem 8.2.1. Since $k = 0$, there is exactly one axis of symmetry (of $\mathcal{S}$) through each point of $\mathcal{S}'$, so $\mathcal{S}'$ has a spread $\mathbf{T}_N$ each line of which is an axis of symmetry. Observe that $\mathbf{T}_N$ is a Hermitian spread, since every axis of symmetry is regular and since it is clear that $\{L, M\}^{\perp\perp} \subseteq \mathbf{T}_N$ if $L, M \in \mathbf{T}_N$, $L \neq M$. However, a GQ of order s cannot have regular spreads by Lemma 12.1.1 , and so $\mathcal{S} = \mathcal{S}'$ and $t = s^2$.

STEP 3: The case $k = 1$

Again, we conclude that $t \in \{s, s^2\}$. Moreover, if $\mathcal{S}$ is of order s, then $\mathcal{S} \cong \mathcal{Q}(4, s)$ by Theorem 8.8.1, contradiction, since in that case every line is an axis of symmetry (we assumed not to be in **VI**). Hence $t = s^2$. Suppose that $\mathcal{S}' \neq \mathcal{S}$. Then we have two possibilities (by Theorem 1.6.2):

- $\mathcal{S}'$ is a grid with parameters $s+1, s+1$ (in which case any of the $2(s+1)$ lines of $\mathcal{S}'$ is an axis of symmetry of $\mathcal{S}$);

- $\mathcal{S}'$ is a GQ of order s (in which case Lemma 12.1.3 infers that $\mathcal{S}'$ is an SPGQ of order s, and then Theorem 8.8.1 implies that $\mathcal{S}' \cong \mathcal{Q}(4, s)$).

Suppose we are in the second case, and suppose U and V are arbitrary but non-concurrent axes of symmetry of $\mathcal{S}$. Then $U, V \in \mathcal{S}'$, and every line of

$\mathcal{L} = \{U, V\}^{\perp\perp}$ is also an axis of symmetry of $\mathcal{S}$. It is clear that at most one line of $\{U, V\}^{\perp}$ can be an axis of symmetry; otherwise every line of $\{U, V\}^{\perp}$ is, and then by Lemma 12.1.1 no point of $\mathcal{S}' \setminus \{U, V\}^{\perp}$ can be incident with an axis of symmetry, contradiction. Now count the number of axes of symmetry of $\mathcal{S}' \setminus (\mathcal{L} \cup \mathcal{L}^{\perp})$ in two ways, and we obtain that

$$(s+1)^2 = \frac{(s^3 - s)2}{s},$$

if there is no axis of symmetry in $\{U, V\}^{\perp}$, and then $s = 3$, or that

$$(s+1)s = \frac{(s^3 - s)2}{s},$$

if there is one axis of symmetry in $\{U, V\}^{\perp}$, and then $s = 2$. If $s = 3$, then by Section 1.3 $\mathcal{S} \cong \mathcal{Q}(5, 3)$, a contradiction since every line of $\mathcal{Q}(5, 3)$ is an axis of symmetry; if $s = 2$, then $\mathcal{S}$ is isomorphic to the unique GQ of order $(2, 4)$, namely $\mathcal{Q}(5, 2)$, contradiction.

STEP 4: The case $k \geq 2$

Since $\mathcal{S}$ is an SPGQ, $t \in \{s, s^2\}$. The case $t = s$ is clearly only possible when $s = t = k$ by Theorem 8.8.1, contradiction.
Suppose that $\mathcal{S}' \neq \mathcal{S}$. By Theorem 1.6.2, $\mathcal{S}'$ is a GQ of order s which is isomorphic to $\mathcal{Q}(4, s)$ by similar arguments as before, and $t = s^2$. Consider a fixed axis of symmetry L of $\mathcal{S}$; since L is regular, we can associate a projective plane Π_L of order s to L by Theorem 3.1.1. Each point of L is incident with k axes of symmetry of $\mathcal{S}'$ which are different from L, and these lines are points of Π_L. Consider the set $\mathcal{B}$ of line-spans which contain two distinct axes of symmetry $M, N \sim L$ of $\mathcal{S}$, $M \not\sim N$; then every line of $\{M, N\}^{\perp\perp}$ is an axis of symmetry meeting L, and every span of this form is a line of Π_L.
It is easily seen that the set of all axes of symmetry meeting L and L itself, together with $\mathcal{B}$ and the points of L, with the induced incidence of Π_L, form a subplane Π' of Π_L. Since $k \geq 2$, the plane Π' is not degenerate. However, every line of Π' is incident with exactly $s + 1$ points, and so, as Π_L is a projective plane of order s, we have that $\Pi' = \Pi_L$. We can conclude that every line of $\mathcal{S}'$ meeting L is an axis of symmetry of $\mathcal{S}$.

STEP 5: The classes III and IV

Every point of the line L is incident with exactly $k+1$ axis of symmetry, $k \in \{0, 1, \ldots, t\}$. Also, since $\mathcal{S}$ is an SPGQ, $t \in \{s, s^2\}$, and we can suppose that $t = s^2$ as before. By the proof of STEP 1, it is clear that the incidence structure $\mathcal{S}'$ as defined above is a (possibly thin) GQ in this case. Suppose $k > 0$ if $\mathcal{S} \in$ **III**, and that $k > 1$ if $\mathcal{S} \in$ **IV**. Then $\mathcal{S}'$ is thick, and $\mathcal{S}'$ is a GQ of order s if $\mathcal{S}' \neq \mathcal{S}$. If we are in the latter case, then it is clear that $k = s - 1$, respectively $k = s$, by Chapter 3. If $\mathcal{S}' = \mathcal{S}$, then $k = s^2 - 1$, respectively $k = s^2$, by that same chapter.

By Lemma 12.1.5, the proof of the theorem now follows. ∎

Remark 12.1.7 (*i*) We will call the classes of Theorem 12.1.6 *symmetry-classes*, since each class reflects in a certain way the 'amount of symmetry' of its members. For convenience, we will also call their subclasses (see further) *symmetry-classes*.

(*ii*) In the determination of the symmetry-classes, we will only consider thick generalized quadrangles 'with symmetry'.

12.2 Some Observations of TGQ's which Arise from Flocks

In this section, we state two results concerning configurations of translation points in the known TGQ's. They are direct corollaries (or restatements) of the results obtained in Chapter 9 and Chapter 10.

Theorem 12.2.1 *Suppose $\mathcal{S} = T(\mathcal{O})$ is a TGQ of order (q, q^2), $q > 1$ and q odd, which is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$. Then $T(\mathcal{O})$ has two distinct translation points if and only if $\mathcal{F}$ is a Kantor flock.*

∎

Theorem 12.2.2 (*i*) *Assume that $\mathcal{S}$ is a known non-classical TGQ which is the translation dual of a TGQ which arises from a flock, i.e. suppose that $\mathcal{S}$ is, respectively, a (non-classical) Roman GQ, the Penttila-Williams GQ or a non-classical Kantor GQ. Then $\mathcal{S}$ contains a line L of translation points.*

(*ii*) *Let $\mathcal{F}_G$, respectively $\mathcal{F}_{PW}$, be a non-linear Ganley flock, respectively Penttila-Williams flock. Then $\mathcal{S}(\mathcal{F}_G)^D$ (which is the translation dual of the Roman GQ), respectively $\mathcal{S}(\mathcal{F}_{PW})^D$ (which is the translation dual of the Penttila-Williams GQ), has exactly one translation point.*

(*iii*) *Suppose $\mathcal{S}$ is GQ of order (s,t), $s \neq 1 \neq t \neq s$, which contains two distinct translation points. If s is even, then $\mathcal{S} \cong \mathcal{S}^* \cong \mathcal{Q}(5,s)$.*

■

12.3 The Symmetry-Class I

We first note that a GQ of order (s,t), $s \neq 1 \neq t$, with $t < s$ cannot have axes of symmetry since an axis of symmetry is a regular line.

12.3.1 The classical and dual classical examples

The following classical or dual classical GQ's contain no axes of symmetry: $H(3,q^2)$, $H(4,q^2)^D$, $W(q)$ with q odd, and $H(4,q^2)$.

12.3.2 The GQ's $\mathcal{P}(\mathcal{S},x)$ of S. E. Payne

Rather than referring constantly to Chapter 2, we recall the construction of the GQ's $\mathcal{P}(\mathcal{S},x)$ to set the notation for this section. Assume s to be a regular point of the GQ $\mathcal{S} = (P,B,I)$ of order s, $s > 1$. Define a GQ $\mathcal{P}(\mathcal{S},x) = \mathcal{S}' = (P',B',I')$ of order $(s-1,s+1)$. as follows.

- The POINT SET P' is the set $P \setminus x^{\perp}$.
- The LINES of B' are of two types:
  (a) the elements of Type (a) are the lines of B which are not incident with x;
  (b) the elements of Type (b) are the hyperbolic lines $\{x,y\}^{\perp\perp}$ where $y \not\sim x$.
- INCIDENCE I' is containment (if one regards a line of $\mathcal{S}$ as a set of points).

Suppose $\mathcal{S}' = \mathcal{P}(\mathcal{S},x)$ is as above, and suppose θ is a nontrivial symmetry about some line in $\mathcal{S}'$. Then by Theorem 6.3.1, we have that

$$(s+1)(s-1)s \equiv 0 \mod 2s,$$

and hence s must be odd[4]. Let us first have a look at the known GQ's $\mathcal{P}(\mathcal{S},x)$ of order $(s-1,s+1)$ (i.e. those coming from known $\mathcal{S}$), s odd.
The only known GQ's of order s (without restriction on the parity) with a regular point are:

(i) the dual of the $T_2(\mathcal{O})$ of Tits of order s;

(ii) the $T_2(\mathcal{O})$ of Tits of order s with s even.

If we are in Case (i) and s is odd, then $T_2(\mathcal{O})^D \cong W(q)$, $s=q$ an odd prime power. The following is due to S. E. Payne and J. A. Thas:

Theorem 12.3.1 (FGQ, 3.3.5) *Suppose L and M are distinct lines of the GQ $\mathcal{P}(W(q),x)$, where $\mathcal{S}$ is of order $q>3$, q odd. Then $\{L,M\}$ is a regular pair if and only if one of the following holds:*

(i) L and M are concurrent lines in $\mathcal{S}$;

(ii) L and M are hyperbolic lines of $\mathcal{S}$ which contain x.

Hence, this result yields the fact that the GQ's $\mathcal{P}(W(q),x)$, $q>3$ and q odd, x any regular point of $W(q)$, have no regular lines. Hence no axes of symmetry.

Theorem 12.3.2 *The GQ $\mathcal{P}(\mathcal{S},x)$ is an element of the symmetry-class* **I** *for all the known GQ's $\mathcal{S}$ of order s with regular point x if $s>3$ and s is odd. If s is even and $s>2$, then $\mathcal{P}(\mathcal{S},x)$ is always a member of* **I**. *If $s=3$, then any $\mathcal{P}(\mathcal{S},x)$ is isomorphic to $\mathcal{Q}(5,2)$, and then every line is an axis of symmetry (and so $\mathcal{P}(\mathcal{S},x)$ is not a member of* **I**). *If $s=2$, then $\mathcal{S}\cong\mathcal{Q}(4,2)$, and $\mathcal{P}(\mathcal{S},x)$ is just a plain dual grid with parameters $4,4$.*

■

The GQ's $\mathcal{P}(\mathcal{S},x)^D$ are also members of **I** since there are more points on a line than lines through a point (a $\mathcal{P}(\mathcal{S},x)^D$ cannot have regular lines). In fact, suppose $\mathcal{S}$ is any thick GQ of order $(s+1,s-1)$ which admits a nontrivial symmetry about some line. Then

$$(s+1)(s-1)(s+2) \equiv 0 \mod 2s,$$

forcing $s=2$.

[4]Note that this works with each thick GQ of order $(s-1,s+1)$.

12.3.3 Regularity (for lines) in the GQ's $\mathcal{P}(\mathcal{S}, x)$

In this section, we push our results a little further in a more general context, and we have a brief look at regularity (for lines) in the GQ's $\mathcal{P}(\mathcal{S}, x)$. As will be shown, this will be very useful in the context of the classification.
We will come to a rather remarkable observation which links several combinatorial problems to each other.

The following observation is an easy generalization of Theorem 12.3.1, but the proof is (necessarily) completely different.

Observation 12.3.3 *Suppose L and M are two distinct non-concurrent lines of the GQ $\mathcal{P}(\mathcal{S}, x)$, where $\mathcal{S}$ is of order $s > 3$. Suppose that one of the following holds:*

(*i*) *L and M are distinct concurrent lines in $\mathcal{S}$;*

(*ii*) *L and M are distinct hyperbolic lines of $\mathcal{S}$ which contain x.*

Then $\{L, M\}$ is a regular pair of lines.

Proof. If we are in Case (i), then it is an easy exercise to show that $\{L, M\}$ is a regular pair of lines (the lines of Type (a) of $\mathcal{P}(\mathcal{S}, x)$ through $L \cap M$ form $\{L, M\}^{\perp\perp}$). Now suppose we are in Case (ii), and put $L = P_L$ and $M = P_M$, where P_L and P_M are the corresponding point sets in $\mathcal{S}$. By Theorem 3.1.1, we know that $P_L^\perp \cap P_M^\perp$ is a point r in $x^\perp \setminus \{x\}$, and clearly every line in $\mathcal{S}$ through r and different from rx forms a line of $\{L, M\}^\perp$ in $\mathcal{P}(\mathcal{S}, x)$. By (i), we can now conclude that $\{L, M\}$ is regular. ∎

Now suppose $\{L, M\}$ is a non-concurrent regular pair of lines which is not of one of the Types (i) and (ii) of Observation 12.3.3.
By the previous observations, we know that if H and H' are distinct hyperbolic lines through x in $\mathcal{S}$, then

(a) they form a regular pair of lines in $\mathcal{P}(\mathcal{S}, x)$;

(b) $\{H, H'\}^\perp$ (where '$\perp$' is taken in $\mathcal{P}(\mathcal{S}, x)$) consists only of lines of Type (a);

(c) $\{H, H'\}^{\perp\perp}$ only consists of lines of Type (b).

Hence we can suppose that L is of Type (a), that no line of Type (a) of $\{L, M\}^{\perp\perp} \setminus \{L\}$ (in $\mathcal{P}(\mathcal{S}, x)$) intersects L in $\mathcal{S}$, and that $\{L, M\}^{\perp\perp}$ contains

at most one line of Type (b). Thus, $\{L,M\}^{\perp\perp}$ has at least $s-1$ lines of Type (a) which are mutually non-concurrent in $\mathcal{S}$. The same can be concluded for $\{L,M\}^{\perp}$. Hence the lines of Type (a) in $\{L,M\}^{\perp}\cup\{L,M\}^{\perp\perp}$ form ('at least') a grid with parameters $s-1,s-1$ in $\mathcal{S}$. It is also clear that $\{L,M\}^{\perp}$ and $\{L,M\}^{\perp\perp}$ cannot contain a line of Type (b) at the same time since such lines are disjoint in $\mathcal{P}(\mathcal{S},x)$. Now suppose that the 'missing lines' in $\{L,M\}^{\perp}$ and $\{L,M\}^{\perp\perp}$ are both of Type (a). From Chapter 4 we know that

A grid with parameters k,s, with $k>2$, in a thick GQ of order (s,t), is always contained in a grid with parameters $k,s+1$.

Hence, there then would follow that $\{L,M\}^{\perp}\cup\{L,M\}^{\perp\perp}$ is contained in a grid with parameters $s+1,s+1$ of $\mathcal{S}$, that is, $\mathcal{S}$ contains a regular pair of lines. By a previous observation, we thus know that at least one of the missing lines in $\{L,M\}^{\perp}$ and $\{L,M\}^{\perp\perp}$ is of Type (a). Suppose now the other is of Type (b). Again from Chapter 4 we conclude that $\{L,M\}^{\perp}\cup\{L,M\}^{\perp\perp}$ is contained in a grid with parameters $s-1,s+1$ of $\mathcal{S}$, and one observes that this grid is complete. So by Theorem 4.4.3, we have that

$\mathcal{S}$ contains a dual complete (s^2-1)-arc (and a spread).

We now have the following.

Observation 12.3.4 *Consider the GQ $\mathcal{P}(\mathcal{S},x)$ which arises from the GQ $\mathcal{S}$ of order s, $s>1$, with regular point x. Suppose L is a regular line in $\mathcal{P}(\mathcal{S},x)$. Then we have that $\mathcal{S}$ contains a dual complete (s^2-1)-arc. Hence if $\mathcal{S}$ is a known GQ of order $s>1$, then we have one of the following possibilities:*

- *(i)* $\mathcal{S}\cong\mathcal{Q}(4,2)$ *and up to isomorphism there is a unique example;*
- *(ii)* $\mathcal{S}\cong W(s)$ *with s odd.*

Proof. First suppose that L is of Type (a) and that L is a regular line in $\mathcal{S}$. Let $M\not\sim L$ be a line of Type (b). Then by preceding considerations one easily constructs a dual complete (s^2-1)-arc from $\{L,M\}^{\perp\perp}$ (considered in $\mathcal{P}(\mathcal{S},x)$).
Suppose that L is of Type (a) and that L is not a regular line in $\mathcal{S}$. Then by the preceding considerations, we have that $\{L,M\}^{\perp}\cup\{L,M\}^{\perp\perp}$ is contained in a complete grid Γ of $\mathcal{S}$ with parameters $s-1,s+1$, if M is a line of Type (a). Now consider the lines of $\mathcal{S}$ which do not intersect lines of this grid. Then together with the $s-1$ mutually non-concurrent lines of Γ of one type, they

form a dual complete (s^2-1)-arc $\mathcal{U}$ of $\mathcal{S}$, and the dual of Theorem 4.3.1 applies. The case where L is of Type (b) follows in the same way. ∎

Since in a $\mathcal{P}(\mathcal{S},x)$ of order $(s-1,s+1)$, axes of symmetry only can exist for s odd, Observation 12.3.4 yields the fact that existence of axes of symmetry in $\mathcal{P}(\mathcal{S},x)$ implies existence of 'nontrivial' dual complete (s^2-1)-arcs if $s>3$.

Remark 12.3.5 (i) In [44], B. De Bruyn and S. E. Payne show that for a GQ $\mathcal{S}$ of order $s>1$ with regular point x which is also coregular (and then by [139, 1.5.1(ii)] s is even), $\mathcal{P}(\mathcal{S},x)$ has the property that any two distinct concurrent lines are contained in a unique grid of $\mathcal{S}$ with parameters s,s. Hence in that case, there are no regular lines.

(ii) It is not known when a collineation θ of $\mathcal{P}(\mathcal{S},x)$ induces a collineation θ' of $\mathcal{S}$ (which fixes x) and vice versa. Of course, a necessary and sufficient condition is that θ preserves the types of the lines (and clearly, any collineation θ' of $\mathcal{S}$ which fixes x induces an automorphism of $\mathcal{P}(\mathcal{S},x)$). A recent reference on this problem is S. E. Payne and M. Miller [135], see also [13, 51, 52]. (The references [13, 51, 52], for instance, easily yield the information that the collineation groups of the GQ's which arise from hyperovals in $\mathbf{PG}(2,q)$, q even, are completely determined by the stabilizer in $\mathbf{P\Gamma L}(4,q)$, of the associated hyperoval.)

(iii) Suppose L is a line of Type (a) of $\mathcal{P}(\mathcal{S},x)$; if a nontrivial symmetry about L in $\mathcal{P}(\mathcal{S},x)$ would induce an automorphism of $\mathcal{S}$, then this would be a nontrivial symmetry about L in $\mathcal{S}$ which fixes x, clearly a contradiction. Next, suppose that L is a line of Type (b) in $\mathcal{P}(\mathcal{S},x)$ which is an axis of symmetry in $\mathcal{P}(\mathcal{S},x)$, with corresponding (full) group of symmetries G_L which extends to a group of automorphisms of $\mathcal{S}$. Then clearly G_L induces a group of collineations G_L' of $\mathcal{S}$ of size $s-1$ which fixes $\{x,p\}^\perp \cup \{x,p\}^{\perp\perp}$ pointwise, where $\{x,p\}^{\perp\perp}$ corresponds to L in $\mathcal{S}$. Moreover, by Theorem 1.6.4, G_L' acts regularly on the $s-1$, points of any line N of $\mathcal{S}$ incident with some point of $\{x,p\}^\perp \cup \{x,p\}^{\perp\perp}$, and which are not contained in $\{x,p\}^\perp \cup \{x,p\}^{\perp\perp}$. Let M be an arbitrary line of $\mathcal{S}$ which is not incident with some point of $\{x,p\}^\perp \cup \{x,p\}^{\perp\perp}$. Then $M^{G_L'}$ together with all the lines of $\mathcal{S}$ which are incident with a point of $\{x,p\}^{\perp\perp}$ and which hit M, form a complete grid $\mathcal{G}$ with parameters $s-1,s+1$. Now consider the lines of $\mathcal{S}$ which do not intersect a line of $(M^{G_L'})^\perp$. Then together with $M^{G_L'}$ they form a dual complete (s^2-1)-arc $\mathcal{U}$ of $\mathcal{S}$.

12.3.4 Other known (non-classical) GQ's of order (s,t) with $t < s$, and flock generalized quadrangles

For any flock $\mathcal{F}$ of the quadratic cone of $\mathbf{PG}(3,q)$, it holds that $\mathcal{S}(\mathcal{F})$ is an element of **I** since $\mathcal{S}(\mathcal{F})$ is of order (q^2,q).
Also, for any ovoid $\mathcal{O}$ of $\mathbf{PG}(3,q)$, the GQ $T_3(\mathcal{O})^D$ is contained in **I**.

The following GQ's all have **at least one** axis of symmetry (see further for more details):

(i) a $T_2(\mathcal{O})$ and $T_3(\mathcal{O})$, where $\mathcal{O}$ is an oval, respectively ovoid, of $\mathbf{PG}(2,q)$, respectively $\mathbf{PG}(3,q)$ (note that $T_3(\mathcal{O})^* \cong T_3(\mathcal{O})$);

(ii) the dual $T_2(\mathcal{O})^D$ of the $T_2(\mathcal{O})$ of Tits of order s, s even;

(iii) any GQ $\mathcal{S}(\mathcal{F})^D$ with $\mathcal{F}$ a flock of the quadratic cone;

(iv) the translation duals $(\mathcal{S}(\mathcal{F})^D)^*$ of the point-line duals of the flock GQ's $\mathcal{S}(\mathcal{F})$, where $\mathcal{F}$ is a Ganley flock, a Penttila-Williams flock or a Kantor flock (note that in the last case $(\mathcal{S}(\mathcal{F})^D)^* \cong \mathcal{S}(\mathcal{F})^D$).

12.4 The Symmetry-Class II

This class is a very large class of GQ's, which in fact contains most of the known examples of GQ's (up to duality). We first observe that if x is the point of an arbitrary element $\mathcal{S}$ of **II** which is incident with the $k+1$ axes of symmetry $L_0, L_1, \ldots, L_k$ of $\mathcal{S}$, then any automorphism of $\mathcal{S}$ must fix x if $k > 0$. The set

$$\{L_0, L_1, \ldots, L_k\}$$

is always fixed by any automorphism of $\mathcal{S}$.

12.4.1 Symmetry-class II.1

The symmetry class II.1. *Each element of* **II.1** *is defined to have exactly one axis of symmetry.*

Suppose $\mathcal{S}$ is a (thick) element of **II.1** of order (s,t) with axis of symmetry L. Then by Theorem 5.5.2, p is not an elation point if pIL and if s is even. By Theorem 6.3.1 we also have that

$$st(s+1) \equiv 0 \mod s+t.$$

As was remarked in Chapter 1, a flock GQ $\mathcal{S}(\mathcal{F})$ always contains a center of symmetry (namely the point (∞)), hence for every flock $\mathcal{F}$ of the quadratic cone of $\mathbf{PG}(3,q)$, the GQ $\mathcal{S}(\mathcal{F})^D$ of order (q,q^2) contains at least one axis of symmetry. If there is a center of symmetry of $\mathcal{S}(\mathcal{F}) = (P,B,I)$ in $P \setminus (\infty)^\perp$, then every point of $\mathcal{S}(\mathcal{F})$ is a center of symmetry (the point (∞) is an elation point), and $\mathcal{S}(\mathcal{F})$ (and then also $\mathcal{S}(\mathcal{F})^D$) is classical by Lemma 12.1.5, i.e. $\mathcal{F}$ is linear. If $\mathcal{F}$ is a Kantor flock, a Penttila-Williams flock or a Ganley flock, then by Theorem 10.1.1 $(\mathcal{S}(\mathcal{F})^D)^*$ contains a line $[\infty]$ which is a line of translation points, i.e. each line of $[\infty]^\perp$ is an axis of symmetry. If $\mathcal{F}$ is a Kantor flock, then $\mathcal{S}(\mathcal{F})^D \cong (\mathcal{S}(\mathcal{F})^D)^*$ by Theorem 2.4.1. By Theorem 12.2.1, $\mathcal{S}(\mathcal{F})^D$ contains exactly one translation point if $\mathcal{F}$ is a non-classical (i.e. non-linear) Ganley flock (and then $s > 9$), or a Penttila-Williams flock. More generally,

If $\mathcal{S}(\mathcal{F})^D$ is TGQ (for some point), $\mathcal{F}$ not a Kantor flock, then $\mathcal{S}(\mathcal{F})^D$ always has precisely one translation point.

Observation 12.4.1 *If $\mathcal{F}$ is a known flock of the quadratic cone of $\mathbf{PG}(3,q)$ and $\mathcal{F}$ is not one of the previous flocks (or, more generally, if $\mathcal{S}(\mathcal{F})^D$ is not a TGQ for some point), then $\mathcal{S}(\mathcal{F})^D$ is an element of* **II.1**.

■

The following theorem considers the possible subconfigurations of axes of symmetry for the point-line dual of the $T_2(\mathcal{O})$ of Tits.

Theorem 12.4.2 *Suppose $\mathcal{S}$ is isomorphic to a $T_2(\mathcal{O})^D$ of order $q > 1$, $\mathcal{O}$ an oval of $\mathbf{PG}(2,q)$. If q is odd, then no line of $\mathcal{S}$ is an axis of symmetry. Suppose q is even. Then $\mathcal{S} = \mathcal{S}^{(x)}$ is a TGQ for some point x if and only if $\mathcal{O}$ is a translation oval of $\mathbf{PG}(2,q)$, and then $\mathcal{S}^{(x)} \cong T_2(\mathcal{O})$. In particular, if $\mathcal{O}$ is a conic, then every line of $\mathcal{S}$ is an axis of symmetry. If $\mathcal{O}$ is not a translation oval of $\mathbf{PG}(2,q)$, then $\mathcal{S} \cong T_2(\mathcal{O})^D$ has exactly one axis of symmetry.*

Proof. If q is odd, then by the result of Segre, see Theorem 1.12.4, $\mathcal{O}$ is a conic and then $T_2(\mathcal{O}) \cong \mathcal{Q}(4,q)$, and hence by Section 1.2.1, $\mathcal{S} \cong W(q)$. Since no line of $W(q)$ is regular if q is odd, $\mathcal{S}$ cannot contain axes of symmetry. Suppose q is even, and consider $T_2(\mathcal{O})$. Then (∞) is an elation point (since it is a translation point), and as a coregular point, (∞) is regular since q is even (by [139, 1.5.2(iv)]). By Corollary 3.3.6 (∞) is a center of symmetry. Suppose

$T_2(\mathcal{O})$ contains another center of symmetry u. If $u \not\sim (\infty)$, then $\mathcal{S} \cong W(q)$ by Theorem 1.2.2. If $u \sim (\infty)$, clearly $u(\infty)$ is a line each point of which is a center of symmetry, and hence $\mathcal{S}$ is a TGQ with base-line $u(\infty)$. Also, if $T_2(\mathcal{O})$ is non-classical, there are no other centers of symmetry in this case by Theorem 8.8.1.

Now suppose that $\mathcal{S}^{(x)}$ is a TGQ for the point x. Suppose $[\infty]$ corresponds to the translation point (∞) of $T_2(\mathcal{O})$ under some duality δ.[5] First suppose $x \not I [\infty]$. As there is an elation about $[\infty]$ which maps some axis of symmetry through x onto a non-concurrent axis of symmetry, $\mathcal{S}$ is an SPGQ, and then $\mathcal{S} \cong \mathcal{Q}(4,q)$ by Theorem 8.8.1. So suppose $xI[\infty]$. Apply δ to obtain, in the obvious notation, $[X]I(\infty)$, where $[X]$ is a translation line. We now work in $\mathbf{PG}(3,q)$, where $\mathcal{O}$ lies in some fixed $\mathbf{PG}(2,q)$. Recall from Chapter 11 that an automorphism of $T_2(\mathcal{O})$ which fixes (∞) is induced by an automorphism of $\mathbf{PG}(3,q)$ which fixes $\mathcal{O}$ (and $\mathbf{PG}(2,q)$). Every point on $[X]$ is a center of symmetry, and hence, if $p_{[X]}$ is the point of $\mathcal{O}$ which corresponds to $[X]$, then for each plane π of $\mathbf{PG}(3,q)$ through $p_{[X]}$ which intersects $\mathcal{O}$ in only $p_{[X]}$, there is a group H_π of colllineations of $\mathbf{PG}(3,q)$ which stabilizes $\mathcal{O}$, which fixes π pointwise, and which acts regularly on $\mathcal{O} \setminus p_{[X]}$. Hence $\mathcal{O}$ is a translation oval w.r.t. $p_{[X]}$, and so

$$\mathcal{S}^{(x)} = T_2(\mathcal{O})^D \cong T_2(\mathcal{O})$$

by Section 1.12.3. Finally, if $\mathcal{O}$ is a conic, then $T_2(\mathcal{O}) \cong \mathcal{Q}(4,q)$, and $\mathcal{Q}(4,q)$ is isomorphic to $W(q)$ if and only if q is even, see Section 1.2.1. ■

There is a nice corollary which characterizes translation ovals $\mathcal{O}$ of $\mathbf{PG}(2,q)$, q even, in terms of the $T_2(\mathcal{O})$'s which arise from it.

Corollary 12.4.3 *Suppose $\mathcal{O} \subseteq \mathbf{PG}(2,q)$ is an oval of $\mathbf{PG}(2,q)$, where q is even. Then $\mathcal{O}$ is a translation oval if and only if $T_2(\mathcal{O})^D$ is a TGQ for some translation point x, in which case $T_2(\mathcal{O})^D \cong T_2(\mathcal{O})$.*

■

12.4.2 Symmetry-class II.2

The symmetry-class II.2. *Each element of* **II.2** *has exactly $k+1$ concurrent axes of symmetry (through the point p), with $1 \leq k \leq t-1$.*

[5] The only reason why we mention this duality is formal; in the classical case the choice of $[\infty]$ is not well-defined (it is arbitrary).

Since we define the classes to be disjoint, we have the following restrictions on any element $\mathcal{S}$ of **II.2** of order (s,t), $s \neq 1 \neq t$. More such restrictions are contained in Chapter 6.

(R1) By Theorem 1.7.5, we know that if $s = t$, that $\mathcal{S}$ has a translation point if $k > 1$, and hence $t > s$ if $k > 1$.

(R2) By Theorem 6.3.1, we have that $st(s+1) \equiv 0 \mod s+t$. Moreover, if $k > 1$, then by Theorem 6.3.2, $s|t$ and $\frac{t}{s} + 1$ divides $(s+1)t$.

(R3) By Theorem 6.6.9, we have that $k \leq t - s + 1$.

(R4) By Chapter 6, we have that if $t > s^2/2$, then $k \geq s+1$ implies that $k = t$, that is, $\mathcal{S}$ is a TGQ. Hence if $t > s^2/2$, we have that $k < s+1$. If for each point $x \not\sim p$ of $\mathcal{S}$, $|\{p,x\}^{\perp\perp}| = 2$, then the same remarks could be made.

(R5) Suppose that $k > 2$ (so $s \neq t$), and let $L_0, L_1, \ldots, L_k$ be the axes of symmetry of $\mathcal{S}$. Then there is no $i \in \{0,1,\ldots,k\}$ so that Property (T) (recall Chapter 6) is satisfied for (L_i, p) w.r.t. some other three distinct elements of $\{L_0, L_1, \ldots, L_k\}$ (cf. Theorem 6.1.3).

(R6) Assume that k and the L_j are as in (R5). Then $\mathcal{S}$ has no subGQ of order s which contains three distinct elements of $\{L_0, L_1, \ldots, L_k\}$ but not all (cf. Theorem 6.1.4).

(R7) By Chapter 5, p is not an elation point.

We do not have examples of the class **II.2** *at present.*

CONJECTURE. **II.2** *is empty.*

This seems like a reasonable conjecture to make, as the only known thick GQ's which are not EGQ's have order $(s-1, s+1)$ or $(s+1, s-1)$ for some s; see (R7) and recall the discussion of Section 12.3.

12.4.3 Symmetry-class II.3

The symmetry-class II.3. *The elements of* **II.3** *are precisely the TGQ's which have one translation point.*

For classification of TGQ's, see, e.g., A. Blokhuis et al. [16], Chapter 8 of FGQ, J. A. Thas [177], [181], [183], [182] and [186], J. A. Thas and K. Thas [191] (cf. Chapter 11), and also K. Thas [211] (cf. Chapter 6), [210] (cf. Chapter 9) and

[213] (cf. Chapter 10).

In particular, we refer the reader to Chapter 15.

Here, we only summarize the known GQ's with one translation point (as a direct result of previous observations).

(i) For $\mathcal{O}$ a non-classical oval, respectively ovoid, in $\mathbf{PG}(2,q)$, respectively $\mathbf{PG}(3,q)$, the GQ $T_2(\mathcal{O})$, respectively $T_3(\mathcal{O})$ ($\cong T_3(\mathcal{O})^*$), has precisely one translation point.

(ii) A $T_2(\mathcal{O})^D$ of order q has one translation point if and only if $\mathcal{O}$ is a translation oval in $\mathbf{PG}(2,q)$ which is not a conic (and note that q is therefore even). In that case, $T_2(\mathcal{O})^D \cong T_2(\mathcal{O})$.

(iii) Suppose $\mathcal{F}$ is a Ganley flock or a Penttila-Williams flock. Then $\mathcal{S}(\mathcal{F})^D$, which is a GQ of order (q,q^2), has exactly one translation point (where $q>9$ in the case $\mathcal{F}$ is a Ganley flock).
More generally, if $\mathcal{F}$ is derived from a non-linear semifield flock which is not a Kantor flock, the same property holds for $\mathcal{S}(\mathcal{F})^D$.

Remark 12.4.4 Each non-classical TGQ of order (s,t), $s \neq 1 \neq t$ and s even, has one and only one translation point.

12.5 SPGQ's of Order (s, s^2) with s Even

In Chapter 9, we proved that if $\mathcal{S}$ is a span-symmetric generalized quadrangle of order (s,t), $s \neq 1 \neq t$, with base-span $\mathcal{L}$, where $s \neq t$ and s is odd, then $\mathcal{S}$ contains $s+1$ subquadrangles, all isomorphic to the classical GQ $\mathcal{Q}(4,s)$, which mutually intersect in the base-grid $\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^\perp, I')$. Also, the base-group G acts semiregularly on the points of $\mathcal{S} \setminus \Omega$ and $G \cong \mathbf{SL}(2,s)$. Note that $|G| = |\mathbf{SL}(2,s)| = s^3 - s$. In this section, it is our goal to obtain the same result for s even. This result will be crucial for the determination of several symmetry-classes.

Remark 12.5.1 Already since the very beginning of the study of SPGQ's, it was thought of that SPGQ's of order (s,s^2), $s>1$, always have ('many') classical subGQ's of order s, all passing through the base-grid. In Chapter 9, we solved that conjecture for the odd case. In this section, the problem is completely solved for the general case. We emphasize that this is a very strong result, which will be illustrated many times in the sequel.

Suppose $\mathcal{S}$ is an SPGQ of order (s, s^2), $s > 1$ and s even, with base-grid $\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^\perp, I')$ and base-group G ($\mathcal{L}$ is like before). For convenience, we will suppose that $s \neq 2$; if $s = 2$, then $\mathcal{S} \cong \mathcal{Q}(5, 2)$ by Section 1.3, $G \cong \mathbf{SL}(2, 2)$ and G acts semiregularly on the points of $\mathcal{S} \setminus \Gamma$.

First suppose that G does not act semiregularly on $\mathcal{S} \setminus \Omega$. Suppose that $\theta \neq \mathbf{1}$ is an element of G which fixes a point q of $\mathcal{S} \setminus \Omega$. Then by Theorem 9.6.1, the fixed element structure of θ is a subGQ $\mathcal{S}_\theta$ of order s. It is clear that $\mathcal{S}_\theta$ is also span-symmetric with respect to the same base-span. Hence $G_\theta := G/N_\theta$ with N_θ the kernel of the action of G on $\mathcal{S}_\theta$ — we can speak of an action since G stabilizes $\mathcal{S}_\theta$ — has order $s^3 - s$; G_θ is exactly the base-group corresponding to $\mathcal{L}$ seen as a base-span of $\mathcal{S}_\theta$. Also, by Chapter 8, we have that $\mathcal{S}_\theta \cong \mathcal{Q}(4, s)$, and

$$G_\theta \cong \mathbf{SL}(2, s).$$

Next, let x be an arbitrary point of $\mathcal{S} \setminus \mathcal{S}_\theta$, and consider the set of points $V = x^\perp \cap \mathcal{S}_\theta$. Note that $|V| = t + 1 = s^2 + 1$ because $\mathcal{S}_\theta$ is a GQ of order s. Then x cannot be fixed by θ — as otherwise $x \in \mathcal{S}_\theta$ — and $\{x, x^\theta\} \subseteq V^\perp$. Since $\mathcal{S}$ has order (s, s^2), we know that

$$|\{x, x^\theta\}^{\perp\perp}| = 2,$$

and thus, as N_θ acts semiregularly on the points of $\mathcal{S}$ outside $\mathcal{S}_\theta$, N_θ has size 2 (N_θ is not trivial). So, N_θ is a normal subgroup of G of order 2 and N_θ is thus contained in the center of G.
It is important to note that $|G| = 2(s^3 - s)$. Recall the following result of Chapter 9 which worked for any $s > 1$.

Lemma 12.5.2 *Suppose $\mathcal{S}$ is an SPGQ of order (s, t), $s \neq 1 \neq t$, with base-grid Γ and base-group G. Then G has size at least $s^3 - s$. Moreover, for each G-orbit Λ in $\mathcal{S} \setminus \Omega$, there holds that $|\Lambda| \geq s^3 - s$.*

■

Observe that

$$(G/N_\theta)' = G'N_\theta/N_\theta = G/N_\theta;$$

the group $\mathbf{SL}(2,s)$, s even, is its own derived group if $s \neq 2$ [65], and hence $G'N_\theta = G$. It follows that

$$G/N_\theta \cong \mathbf{SL}(2,s).$$

(Easy exercise — recall that $|N_\theta| = 2$.)
We now use a recent lemma of W. M. Kantor which uses the internal structure of $\mathbf{PSL}(2,s)$.

Lemma 12.5.3 (W. M. Kantor [94]) *G is a perfect group.*

■

Observe

Lemma 12.5.4 *G acts semiregularly on $\mathcal{S} \setminus \Omega$ if s is even.*

Proof. As before, we can suppose that $s > 2$. Suppose by way of contradiction that G does not act semiregularly on the points of $\mathcal{S} \setminus \Omega$. We then know that $G/N_\theta \cong \mathbf{SL}(2,s)$ with s a power of 2, and where N_θ is a central subgroup of order 2. The group G is perfect by Lemma 12.5.3 and has size $2(s^3 - s)$. The universal central extension of $\mathbf{SL}(2,s)$, s even, coincides with $\mathbf{SL}(2,s)$ if $s \neq 4$, see [65], and this contradicts the fact that $|G| = 2(s^3 - s)$ ($|\mathbf{SL}(2,s)| = s^3 - s$). Hence G **does** act semiregularly on the points of $\mathcal{S} \setminus \Omega$ if $s \neq 4$. Now put $s = 4$. Then $\mathcal{S}$ is a GQ of order $(4,16)$ which contains a subGQ isomorphic to $\mathcal{Q}(4,4)$, hence by [169] $\mathcal{S} \cong \mathcal{Q}(5,4)$. The result follows. ■

We are ready to prove the main result of this section. The corresponding part of Chapter 9 could now be taken over, but we include a sketch of the proof for the convenience of the reader.

Theorem 12.5.5 *Suppose $\mathcal{S}$ is a span-symmetric generalized quadrangle of order (s,t), $s \neq 1 \neq t$ and $s \neq t$. Then $\mathcal{S}$ contains $s+1$ subquadrangles, all isomorphic to the classical GQ $\mathcal{Q}(4,s)$, which mutually intersect in the base-grid Γ. Also, the base-group G acts semiregularly on $\mathcal{S} \setminus \Omega$, $|G| = s^3 - s$ and $G \cong \mathbf{SL}(2,s)$.*

Proof. For s odd, see Chapter 9. Suppose that s is even. Suppose that $\mathcal{S}$ is an SPGQ of order (s,s^2), $s \neq 1$, with base-span $\mathcal{L} = \{U,V\}^{\perp\perp}$, base-group G and base-grid $\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^\perp, I')$. Furthermore, let N be the kernel of the

action of G on $\mathcal{L}$. As before, we assume w.l.o.g. that $s > 2$. If G acts semiregularly on $\mathcal{S} \setminus \Omega$, then G/N cannot act as a sharply 2-transitive group on the lines of $\mathcal{L}$ if $s > 3$ by Section 9.7.

G IS A PERFECT GROUP. As G/N does not act sharply 2-transitively on $\mathcal{L}$ and as G acts semiregularly on $\mathcal{S} \setminus \Omega$, we have that G is its own derived group. Clearly, G/N is isomorphic to one of the following list: (a) $\mathbf{PSL}(2, s)$, (b) $\mathbf{Sz}(\sqrt{s})$, (c) $\mathbf{PSU}(3, \sqrt[3]{s^2})$ (the Ree group cannot arise since then s is a power of 3). All these groups are perfect groups [65], hence $(G/N)' = (G/N) = G'N/N$, and so $G'N = G$. Since G acts semiregularly on $\mathcal{S} \setminus \Omega$ by Lemma 12.5.4, it follows that $|G| = |G/N| \times |N| = (s^n - 1)(s + 1)s/r \times |N|$, with $r \in \{1, (3, \sqrt[3]{s}+1)\}$ and $n \in \{1, 2/3, 1/2\}$, is a divisor of $|\mathcal{S} \setminus \Omega| = (s+1)(s^3 - s)$, where $r = (3, \sqrt[3]{s} + 1)$ if $G/N \cong \mathbf{PSU}(3, \sqrt[3]{s^2})$. Hence $r(s^2 - 1)/(s^n - 1) \equiv 0 \mod |N|$. If $r = (3, \sqrt[3]{s} + 1)$ and $G/N \cong \mathbf{PSU}(3, \sqrt[3]{s^2})$, then s and N have a nontrivial common divisor if and only if $r = 3$ and if 3 is a divisor of s, in contradiction with the fact that s is an even prime power. It follows that $|N|$ and s are coprime. Hence s is the largest power of 2 which divides $|G|$. Thus the full groups of symmetries about the lines of $\mathcal{L}$ are exactly the Sylow 2-subgroup of G. The other cases are similar. Whence we know that $|N|$ and s are coprime, so G and G' have precisely the same Sylow 2-subgroups. As $G' \leq G$ and as G is generated by its Sylow 2-subgroups, we have that G is indeed a perfect group.

G/N IS ISOMORPHIC TO $\mathbf{PSL}(2, s)$. This follows exactly as in Chapter 9.

PROOF OF THE THEOREM. The semiregularity of G on $\mathcal{S} \setminus \Omega$ implies that G is a perfect group, and also that $G/N \cong \mathbf{PSL}(2, s)$ if N is the kernel of the action of G on $\mathcal{L}$. Moreover, N is contained in the center of G. Thus, G is a perfect central extension of $\mathbf{PSL}(2, s)$, and this leads to the fact that $G \cong \mathbf{SL}(2, s)$ (and $|G| = s^3 - s$). Recalling that G acts semiregularly on $\mathcal{S} \setminus \Omega$, it follows by earlier observations made in this section, that there are $s + 1$ subGQ's of $\mathcal{S}$ of order s which are all ismorphic to $\mathcal{Q}(4, s)$, and which mutually intersect in the base-grid Γ. The result follows. ∎

Corollary 12.5.6 *The construction method for spreads of SPGQ's of order (s, s^2), $s > 1$ and s odd, and its implications — see the appendix of Chapter 9 — also applies to SPGQ's of order (s, s^2), $s > 1$ and s even.*

∎

Whence

Corollary 12.5.7 *Suppose $\mathcal{S}$ is an SPGQ of order (s,t), $s \neq 1 \neq t$ and $t \neq s$. Then $\mathcal{S}$ contains at least $2(s^2 - s)$ distinct spreads.*

■

12.6 Digression: Spreads in SPGQ's

We digress with the following easy theorem.

Theorem 12.6.1 *An SPGQ of order (s,t), $s \neq 1 \neq t$, contains spreads if and only if $s = t$ with s even or if $s \neq t$.*

Proof. If $s = t$, then $\mathcal{S} \cong \mathcal{Q}(4,s)$ by Theorem 8.8.1, and then by Theorem 1.12.2 the statement follows. If $s \neq t$, then $t = s^2$ by Theorem 8.2.1. Corollary 12.5.6 yields the result. ■

12.7 A Characterization Theorem and a Classification Theorem

The two main theorems of this section will appear to be very valuable for the sequel.

Theorem 12.7.1 *Suppose L is a line of the GQ $\mathcal{S}$ of order (s,t), $s \neq 1 \neq t$ and $s \neq t$, such that there are $s^3 + s^2$ distinct subGQ's of $\mathcal{S}$ of order s, all isomorphic to $\mathcal{Q}(4,s)$ and all containing L.*

- *If s is even, then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5,s)$.*
- *If s is odd, then $\mathcal{S}$ is the translation dual (w.r.t. some translation point) of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ of order (s^2,s), s an odd prime power, and $\mathcal{S}$ contains a line of translation points.*

Proof. Suppose that L is so that there are s^3+s^2 such subGQ's of $\mathcal{S}$ of order s all containing L. Then it follows immediately that L is regular, since any pair of lines in $\mathcal{S}$ of the form $\{L,U\}$, $L \not\sim U$, is contained in such a classical subGQ of order s (and any line of $\mathcal{Q}(4,s)$ is regular). Now suppose $M \sim L \neq MImIL$. It is clear that if $N \sim L \neq N$ and $N \not\sim M$, then $\{M,N\}$ is a regular pair of lines. Let $U \not\sim M$ be a line not contained in $L^\perp$. Consider an arbitrary point u of L different from m, and let V be the unique line of $\mathcal{S}$ for which $uIV \sim U$. Since there are $s^3 + s^2$ subGQ's which are isomorphic to $\mathcal{Q}(4,s)$ and which

contain L, there is a (necessarily) unique such subGQ of $\mathcal{S}$ which contains L, M, V and U (cf. Theorems 1.6.2 and 1.6.3). Hence the pair $\{M, V\}$ is regular, and M thus is a regular line. It follows that every point on L is coregular. Consider any such coregular point pIL. From the regular line L there arises a net $\mathcal{N}_L$, and $\mathcal{N}_L$ is a $\mathcal{P}$-net with $\mathcal{P}$ the parallel class of $\mathcal{N}_L$ defined by p, as the dual of $\mathcal{N}_L$ clearly satisfies the Axiom of Veblen (and so $\mathcal{N}_L$ is isomorphic to the dual of H_s^3 by Theorem 3.1.2). Hence by Theorem 3.1.7, every point on L is a translation point, and every line of $L^\perp$ is an axis of symmetry. The complete result now follows, for example, from Theorem 9.11.3. ∎

Recall

Theorem 12.7.2 (FGQ, 5.2.6) *A GQ of order $s > 1$ is isomorphic to $W(s)$ if and only if it has a regular pair of non-concurrent lines $\{L, M\}$ with the property that any triad of points lying on lines of $\{L, M\}^\perp$ is centric.*

The following theorem is very strong.

Theorem 12.7.3 *Let $\mathcal{S}$ be a GQ of order (s, t), $s \neq 1 \neq t$, and suppose $L \sim M \neq L$ are axes of symmetry. Moreover, suppose that N is an axis of symmetry of $\mathcal{S}$ which does not meet L or M. Then we have that one of the following holds:*

(*i*) *$\mathcal{S}$ is isomorphic to $\mathcal{Q}(4, s)$;*

(*ii*) *$\mathcal{S}$ is isomorphic to $\mathcal{Q}(5, s)$;*

(*iii*) *$\mathcal{S}$ is the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ of order (s^2, s), s an odd prime power, and $\mathcal{S}$ contains a line of translation points.*

Proof. Let L, M and N be as above, and define U as the line through $M \cap L$ which intersects N. We can suppose that $s \neq t$, since otherwise $\mathcal{S} \cong \mathcal{Q}(4, s)$ by Theorem 8.8.1. By Theorem 12.1.6, we also know that each point of U is incident with $k + 1$ axes of symmetry, where

$$k \in \{s - 1, s, s^2 - 1, s^2\}.$$

If $k = s^2$, then every point on U is a translation point, and Theorem 9.11.3 applies. Now put $k = s^2 - 1$. Then by Theorem 6.6.9, every point on U is a translation point, and again Theorem 12.7.1 applies. Next suppose that

$k \in \{s-1, s\}$. Then by Step 5 of the proof of Theorem 12.1.6, the set of points on the axes of symmetry of $\mathcal{S}$ form a classical subGQ $\mathcal{S}'$ of $\mathcal{S}$ of order s (isomorphic to $\mathcal{Q}(4,s)$). Fix an arbitrary point u on U. In $\mathcal{S}'$, the symmetries about any three distinct lines through u different from U generate a group of elations about u of size s^3 which extends to a group of automorphisms of $\mathcal{S}$ of size s^3 (cf. Chapter 6). In $\mathcal{S}'$, U is also an axis of symmetry, see the appendix of Chapter 6. Let θ be a symmetry about U in $\mathcal{S}'$, and let L_1, L_2, L_3 be three distinct lines through u, all different from U and contained in $\mathcal{S}'$. Then we can write θ as

$$\theta = \theta_1\theta_2\theta_3,$$

where θ_i is a symmetry about L_i. Now consider an extended automorphism θ' of $\mathcal{S}$ of θ. Then θ' fixes all lines through u in $\mathcal{S}$, and all lines of $U^\perp \cap \mathcal{S}'$. Hence, by Chapter 3 (e.g. Theorem 3.3.2), θ' is a symmetry about U. It now readily follows that U is also an axis of symmetry of $\mathcal{S}$, and we can suppose that $k = s$. Also, through any two non-concurrent axes of symmetry V, W in $U^\perp$ there are $s+1$ classical subGQ's of order s (one of which is $\mathcal{S}'$), mutually intersecting in the points and lines $\{V,W\}^{\perp\perp} \cup \{V,W\}^\perp \subseteq \mathcal{S}'$, see Theorem 12.5.5. Counting all the classical subGQ's of order s which arise in this way, we easily obtain s^3+1 such GQ's (including $\mathcal{S}'$).
Suppose L_1, L_2, L_3, u are as above, and suppose G is the group of size s^3 of automorphisms of $\mathcal{S}$ which is generated by the symmetries about L_1, L_2, L_3. Suppose G_* is an arbitrary G-orbit of the permutation group $(P \setminus u^\perp, G)$, where P is the point set of $\mathcal{S}$. Now define the incidence structure $\mathcal{S}(G_*) = \mathcal{S}^* = (P^*, B^*, I^*)$ as follows.

- The POINTS of P^* are of three types:
  (1) the point u;
  (2) the points of G_*;
  (3) any point which is incident with a line of $\mathcal{S}'$ through u.
- We have two types of LINES:
  (a) the s + 1 lines through u in $\mathcal{S}'$;
  (b) the lines of $\mathcal{S}$ which intersect a line of the first type and contain at least one point of G_*.
- The INCIDENCE RELATION $I^* \subseteq I$ is the restriction of I to $(P^* \times B^*) \cup (B^* \times P^*)$.

Then one easily observes that $\mathcal{S}^*$ has $(s+1)(s^2+1)$ points and the same number of lines, and that any line has $s+1$ points. Hence $\mathcal{S}^*$ is a subGQ of $\mathcal{S}$ of order s. Furthermore, it clearly is a TGQ with translation point u. By considering all the G-orbits in $P \setminus u^\perp$, s such GQ's arise (including $\mathcal{S}'$) for a fixed u. In total, s^2-1 distinct such subGQ's of order s arise, all distinct from $\mathcal{S}'$, and which intersect $\mathcal{S}'$ in the lines of $\mathcal{S}'$ through some point $xIU \in \mathcal{S}'$, together with the points incident with those lines (we will call such a subGQ, 'a subGQ of Type (2)'). Hence U is contained in s^3+s^2 distinct subGQ's of $\mathcal{S}$ of order s.
Consider an arbitrary subGQ $\mathcal{S}''$ of $\mathcal{S}$ of Type (2). Then $\mathcal{S}''$ is a TGQ with respect to the point zIU for which $\mathcal{S}'' \cap \mathcal{S}'$ is the induced subgeometry of $\mathcal{S}$ on the lines of $\mathcal{S}'$ through u.
We distinguish two cases.

- s IS ODD. Consider two non-concurrent lines U' and U'' in $U^\perp$; then there are $s+1$ distinct subGQ's of order s containing U, U' and U'', and thus $\{U', U''\}$ is a regular pair of lines by Theorem 1.6.3. It follows easily that U is a regular line. As U is contained in s^3+s^2 distinct subGQ's of $\mathcal{S}$ of order s, the dual net $\mathcal{N}_U^*$ satisfies the Axiom of Veblen, and hence $\mathcal{N}_U^* \cong H_s^3$ by Theorem 3.1.2. Hence, if $\mathcal{S}''$ is an arbitrary subGQ of $\mathcal{S}$ of order s containing the line U, then $(\Pi_U)_{\mathcal{S}''}$ (the projective plane of order s which arises from the line U, which is regular in $\mathcal{S}''$), is Desarguesian. As s is odd, the result follows from Corollary 3.1.9 and Theorem 12.7.1.

- s IS EVEN. Let $\mathcal{S}''$ be a subGQ of $\mathcal{S}$ of Type (2), and suppose that z is the point on U so that $\mathcal{S}'' \cap \mathcal{S}'$ is the induced subgeometry of $\mathcal{S}$ defined by $z^\perp$ in $\mathcal{S}'$. Then each line of $\mathcal{S}''$ through z is regular (in $\mathcal{S}''$), as $\mathcal{S}''$ is a TGQ with translation point z. Fix such a line ZIz which is different from U. Let Γ be an arbitrary grid with parameters $s+1, 2$ which is contained in $\mathcal{S}''$ and which contains Z and U. Then Γ is contained in $s+1$ subGQ's of $\mathcal{S}$ of order s (which contain U), s of which are not of Type (2), hence which are isomorphic to $\mathcal{Q}(4,s) \cong W(s)$ (recall that s is even). Denote these subGQ's by $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_s$, and suppose $\{p_1, p_2, p_3\}$ is an arbitrary triad of points which are contained in Γ. Then as $\mathcal{S}_i \cong W(s)$ for all $i = 1, 2, \ldots, s$, $\{p_1, p_2, p_3\}$ (clearly) is unicentric in $\mathcal{S}_i$. As $\mathcal{S}$ has order (s, s^2), we also know that $|\{p_1, p_2, p_3\}^\perp| = s+1$, and so we infer that $\{p_1, p_2, p_3\}$ is also (uni)centric in $\mathcal{S}''$. By Theorem 12.7.2, we conclude that $\mathcal{S}'' \cong W(s)$, and thus U is contained in s^3+s^2 subGQ's of $\mathcal{S}$ of order s which are isomorphic to $\mathcal{Q}(4,s) \cong W(s)$, and Theorem 12.7.1 applies.

Corollary 12.7.4 *Let $\mathcal{S}$ be a GQ of order (s,t), $s \neq 1 \neq t$, and suppose $L \sim M \neq L$ are axes of symmetry. Moreover, suppose that N is an axis of*

symmetry of $\mathcal{S}$ which does not meet L or M, and let s be even. Then $\mathcal{S}$ is ismorphic to $\mathcal{Q}(4,s)$ or $\mathcal{Q}(5,s)$.

■

12.8 The Symmetry-Class III

The symmetry-class III. *There is a line $L \in B$ which is not an axis of symmetry so that every point qIL is incident with exactly $k+1$ axes of symmetry, $k \in \{0, s-1, s^2-1\}$, and there are no other axes of symmetry. Respectively, we call those classes* **III.1**, **III.2** *and* **III.3**.

12.8.1 Symmetry-class III.1

First suppose that $\mathcal{S} \in$ **III.1**. Then we have the following restrictions for $\mathcal{S}$ (which are valid for any member of **III**).

(R1) By Theorem 8.2.1 and Theorem 8.8.1, $\mathcal{S}$ is of order (s, s^2); if $s = t$, then $\mathcal{S} \cong \mathcal{Q}(4,s)$ and all lines are thus axes of symmetry. We also note that s is a prime power (which follows, e.g., from (R2)).

(R2) By Theorem 12.5.5, $\mathcal{S}$ contains $s+1$ subGQ's of order s, all isomorphic to $\mathcal{Q}(4,s)$, which mutually intersect in the grid Γ with parameters $s+1, s+1$ which is defined by the axes of symmetry of $\mathcal{S}$.

Needless to say, many other restrictions could be stated, such as the fact that any automorphism of $\mathcal{S}$ fixes the grid Γ and stabilizes any of its two reguli of lines.

We do not know examples of the class **III.1**.

Remark 12.8.1 I personally think that examples of **III.1** need some extra conditions to be determined in a sensible way. Restriction (R2) is already an indication what conditions should be the most natural. We refer to Section 12.10 for more details to that end.

12.8.2 Determination of III.2 and III.3

By Section 12.7, we can assert that

Theorem 12.8.2 *The symmetry-classes* **III.2** *and* **III.3** *are empty.*

Proof. Since an element $\mathcal{S}$ of **III.2** or **III.3** contains axes of symmetry L, M and N so that $L \sim M \neq L$ and such that N does not meet L or M, we can apply Theorem 12.7.3 to obtain that $\mathcal{S}$ is the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ of order (s^2, s), s an odd prime power, and that $\mathcal{S}$ contains a line of translation points, contradiction. ∎

12.9 The Symmetry-Class IV

The symmetry-class IV. *Each element of* **IV** *has an axis of symmetry $L \in B$ so that every point qIL is incident with exactly $k+1$ axes of symmetry, $k \in \{1, s, s^2\}$, and there are no other axes of symmetry.*

Every element of **IV** has order (s, s^2) by Theorem 8.2.1 and Theorem 8.8.1. For $k+1 = 2, s+1, s^2+1$, we will denote the corresponding subclasses of **IV** by **IV.1**, **IV.2** and **IV.3**, respectively.

12.9.1 IV.1

Assume that $\mathcal{S} \in$ **IV.1**. Then we have the same restrictions for $\mathcal{S}$ as in the case of class **III.1**. Also, any automorphism of $\mathcal{S}$ fixes the line L and the grid Γ as defined by the axes of symmetry of $\mathcal{S}$ (the reguli of lines of Γ are automatically fixed in this case).

We do not know examples of **IV.1**.

12.9.2 IV.2 and IV.3

Regarding the class **IV.2**, we can immediately proceed as in Theorem 12.8.2, and thus we have the following.

Theorem 12.9.1 *The class* **IV.2** *is empty.*

Proof. See the proof of Theorem 12.8.2. ∎

For **IV.3** we have the following by Chapter 9.

Theorem 12.9.2 *Suppose $\mathcal{S}$ is a GQ of order (s, t), $s \neq 1 \neq t$, which has a line L for which each element of $L^\perp$ is an axis of symmetry, and which contains no other axes of symmetry. Then $\mathcal{S}$ is the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ of order (q^2, q) with q an odd prime power, and we have the following possibilities.*

(*i*) *If $t = s^2$, $s = q^n$ and q odd, where* $\mathbf{GF}(q)$ *is the kernel of the TGQ $\mathcal{S} = \mathcal{S}^{(\infty)}$ with (∞) an arbitrary translation point of $\mathcal{S}$, and if $q \geq 4n^2 - 8n + 2$, then $\mathcal{S}$ is the point-line dual of the flock GQ $\mathcal{S}(\mathcal{F})$, $\mathcal{F}$ a non-linear Kantor flock.*

(*ii*) *If $t = s^2$, $s = q^n$ and q odd, where* $\mathbf{GF}(q)$ *is as in (i), and if $q < 4n^2 - 8n + 2$, then $\mathcal{S}$ is the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ for some non-linear semifield flock.*

Conversely, if $\mathcal{S} = T(\mathcal{O})$ is a TGQ of order (q, q^2), q odd, which is the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ (or, equivalently, if $\mathcal{O}$ is good at some element), then $\mathcal{S}$ is an element of **IV.3**.

Proof. Immediately by Theorem 9.11.3 and Theorem 10.1.1. ∎

So the elements of **IV.3** are precisely the non-classical TGQ's $T(\mathcal{O})$ which are the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$.

The only known examples of **IV.3** *besides the dual Kantor flock GQ's are the (non-classical) Roman GQ's and the Penttila-Williams GQ. Both examples satisfy the properties of Theorem 12.9.2 (ii) (recall that the kernel of both is isomorphic to* $\mathbf{GF}(3)$*).*

12.10 The Symmetry-Class V

The symmetry-class V. *We define the subclasses of* **V** *as follows:*

(*i*) **V.1** *is the class* **V***.1 of Theorem 12.1.6;*

(*ii*) **V.2** *is the class* **V***.2.(a);*

(*iii*) **V.3** *is* **V***.2.(b);*

(*iv*) **V.4** *corresponds to* **V***.3.(a) and, finally,*

(*v*) **V.5** *corresponds to* **V***.3.(b).*

Whence if $\mathcal{S}'$ is the (possibly thin) subGQ of order (s, t') which is 'generated' by the axes of symmetry as described in Theorem 12.1.6, and if k is the constant so that every point of $\mathcal{S}'$ is incident with $k + 1$ axes of symmetry (of $\mathcal{S}$), then we have

V.1 $k = 0$ and $\mathcal{S}$ has a Hermitian spread $\mathbf{T}_N$ of which any line is an axis of symmetry;

V.2 $k = 1$, and $\mathcal{S} = \mathcal{S}'$, where $\mathcal{S}$ is of order (s, s^2), $s > 1$;

V.3 $k = 1$, and $\mathcal{S}'$ is a grid with parameters $s+1, s+1$, $s > 1$. Moreover, $\mathcal{S}$ is a GQ of order (s, s^2);

V.4 $2 \leq k < t$, $\mathcal{S}' = \mathcal{S}$ and $\mathcal{S}$ is of order (s, s^2), $s > 1$;

V.5 $k = s$, $\mathcal{S}' \neq \mathcal{S}$, $\mathcal{S}' \cong \mathcal{Q}(4, s)$ and $t = s^2$, $s > 1$.

12.10.1 The symmetry-classes V.1, V.2 and V.4

The complete determination of **V.2** and **V.4** follows from Section 12.7. However, in this section we will give a totally different proof which is valuable for any element of **V.1** $\cup$ **V.2** $\cup$ **V.4**, thus providing more insight in the structure of the generalized quadrangle *for which the axes of symmetry cover the quadrangle*. We will show that **V.1**, **V.2** and **V.4** are empty, that is, every element of these classes is automatically classical. The essential ingredient is that each element of **V.1** $\cup$ **V.2** $\cup$ **V.4** has a 'special' spread, forcing the GQ to have many classical subGQ's.

Lemma 12.10.1 *Any thick element of* $\mathcal{V} :=$ **V.1** $\cup$ **V.2** $\cup$ **V.4**, *contains a spread each line of which is an axis of symmetry.*

Proof. Suppose $\mathcal{S}$ is an arbitrary thick element of $\mathcal{V}$ of order (s, t). Then $s \neq t$ by Theorem 8.8.1, and every point of $\mathcal{S}$ is incident with at least one axis of symmetry. Fix a base-span $\mathcal{L}$ for the SPGQ $\mathcal{S}$. Then there is a classical subGQ $\mathcal{S}'$ of order s containing the base-grid by Theorem 12.5.5. Now fix an axis of symmetry M of $\mathcal{S}$ not in $\mathcal{S}'$. Then $\mathbf{T}(\mathcal{L}, M)$, see the Appendix of Chapter 9 and Section 12.6, is the desired spread. ∎

We hence have

Theorem 12.10.2 $\mathcal{V} =$ **V.1** $\cup$ **V.2** $\cup$ **V.4** *is empty.*

Proof. Suppose $\mathcal{S}$ is an element of $\mathcal{V}$ of order (s, s^2), $s > 1$, fix an axis of symmetry L which is contained in a spread $\mathbf{T}_L$ each line of which is an axis of symmetry (and which is locally Hermitian w.r.t. L). By Section 12.5 every two non-concurrent lines U and V in $\mathbf{T}_L$ are contained in $s+1$ classical subGQ's of order s which mutually intersect in the geometry of $\{U, V\}^{\perp\perp} \cup \{U, V\}^{\perp}$. If we now count the number ν of classical subGQ's of order s which arise from

all the base-spans of the SPGQ $\mathcal{S}$ which are contained in $\mathbf{T}_L$, then by Lemma 12.1.1 we obtain

$$\nu = \frac{s^3(s^3+1)(s+1)}{(s+1)s} = (s^3+1)s^2.$$

It now easily follows that every centric triad of lines of $\mathcal{S}$ is contained in a proper subGQ of order s, and by Theorem 1.6.6 (i), the theorem follows[6]. ■

Corollary 12.10.3 *Suppose $\mathcal{S}$ is a GQ of order (s,t), $s \neq 1 \neq t$, which admits a spread each line of which is an axis of symmetry. Then $\mathcal{S} \cong \mathcal{Q}(5,s)$.*

■

12.10.2 The symmetry-class V.3

For an element $\mathcal{S}$ of **V.3**, the axes of symmetry of $\mathcal{S}$ form a grid Γ with parameters $s+1, s+1$, and $\mathcal{S}$ is a GQ of order (s,s^2), $s>1$.

(R1) By the same arguments as usual, $\mathcal{S}$ is of order (s,s^2), s is a prime power.

(R2) By Theorem 12.5.5, $\mathcal{S}$ contains $s+1$ subGQ's of order s, all isomorphic to $\mathcal{Q}(4,s)$, which mutually intersect in the grid Γ. For each regulus in Γ, there is an automorphism group of $\mathcal{S}$ which

- is isomorphic to $\mathbf{SL}(2,s)$;
- which acts semiregularly on the points of $\mathcal{S} \setminus \Gamma$;
- which fixes the regulus elementwise.

We do not know examples of **V.3**.

CONJECTURE. *We conjecture that every element of* **V.3** *is of classical type.*

Remark 12.10.4 Many other restrictions hold using (R2).

[6]Note that Theorem 1.6.6 (i) does not require the subGQ's to be classical.

12.10.3 The symmetry-class V.5

If $\mathcal{S}$ is an element of **V.5**, then the axes of symmetry of $\mathcal{S}$ form a proper subGQ $\mathcal{S}' \cong \mathcal{Q}(4,s)$ of order s.

Each element of **V.5** is thus of order (s,s^2), s is a prime power. By Section 12.7, we can conclude that $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5,s)$, but in this section we want to give a more direct approach without the explicit construction of subGQ's as we did in the proof of Theorem 12.7.3, and which will also work in a more general setting, see Theorem 12.10.8.

The following result is due to J. A. Thas and S. E. Payne [190] for the even case, M. R. Brown [20] for the odd case, and L. Brouns, J. A. Thas and H. Van Maldeghem [18] for a uniform proof of both cases.

Theorem 12.10.5 ([190],[20],[18]) *Suppose $\mathcal{S}$ is a GQ of order (s,t), $s \neq 1 \neq t$, which contains a subGQ $\mathcal{S}'$ isomorphic to $\mathcal{Q}(4,s)$, so that every subtended ovoid is an elliptic quadric. Then $\mathcal{S} \cong \mathcal{Q}(5,s)$.*

Let us now obtain

Theorem 12.10.6 *Suppose $\mathcal{S}$ is a GQ of order (s,t), $s \neq 1 \neq t$, which contains a subGQ $\mathcal{S}'$ isomorphic to $\mathcal{Q}(4,s)$, such that every symmetry about a line of $\mathcal{S}'$ can be extended to a symmetry of $\mathcal{S}$. Then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5,s)$.*

Proof. By preceding considerations, we know that if L and M are non-concurrent lines of $\mathcal{S}'$, then there are $s+1$ classical subGQ's ($\cong \mathcal{Q}(4,s)$) of order s which mutually intersect in the points and lines of $\{L,M\}^{\perp\perp} \cup \{L,M\}^{\perp}$. Fix a point $x \in \mathcal{S} \setminus \mathcal{S}'$, and consider the ovoid $\mathcal{O}_x$ of $\mathcal{S}'$ which is subtended by x. Consider three arbitrary but distinct points u,v,w on $\mathcal{O}_x$. Let U,V,W and U',V',W' be lines in $\mathcal{S}'$ such that $UIuIU' \neq U$, $VIvIV' \neq V$ and $WIwIW' \neq W$, and such that $U \not\sim V$ and $W \in \{U,V\}^{\perp}$, and $V' \not\sim W'$ and $U' \in \{V',W'\}^{\perp}$. Define $\mathcal{S}^1$, respectively $\mathcal{S}^2$, as the unique (classical) GQ of order s through x and U,V,W, respectively x and U',V',W', and note that these GQ's are different. (Note also that, for example, $\mathcal{S}^1$ contains the grid with parameters $s+1,s+1$ which contains U,V and W.) The GQ's $\mathcal{S}^1$ and $\mathcal{S}^2$ intersect in the geometry defined by the points and lines of $x^{\perp} \cap \mathcal{S}^1 = x^{\perp} \cap \mathcal{S}^2$, see Theorem 1.6.3. But

$$\{U,V\}^{\perp} \cap \{V',W'\}^{\perp} \subseteq \mathcal{S}^1 \cap \mathcal{S}^2 \cap \mathcal{S}' = \mathcal{O}_x \cap \mathcal{S}^1 = \mathcal{O}_x \cap \mathcal{S}^2$$

(the latter objects viewed as point sets), implying that the conic $\{U,V\}^\perp \cap \{V',W'\}^\perp$ of $\mathcal{S}'$ is completely contained in $\mathcal{O}_x$.
Hence, through any three distinct points of $\mathcal{O}_x$ there goes a conic of $\mathcal{S}'$ which lies on $\mathcal{O}_x$, and hence $\mathcal{O}_x$ is an elliptic quadric (see, e.g., [190]). Thus, every subtended ovoid of $\mathcal{S}'$ is an elliptic quadric, and by Theorem 12.10.5, $\mathcal{S} \cong \mathcal{Q}(5,s)$.■

We now remind the reader of the following recent result due to L. Brouns, J. A. Thas and H. Van Maldeghem [18].

Theorem 12.10.7 ([18]) *Suppose $\mathcal{S}$ is a GQ of order (s,t), $s \neq 1 \neq t$, which contains a subGQ $\mathcal{S}'$ isomorphic to $\mathcal{Q}(4,s)$, such that every linear automorphism of $\mathcal{S}'$ can be extended to an automorphism of $\mathcal{S}$. Then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5,s)$.*

Using the results of this section, we will try to obtain the following generalization of this theorem.

Theorem 12.10.8 *Suppose $\mathcal{S}$ is a GQ of order (s,t), $s \neq 1 \neq t$, which contains a subGQ $\mathcal{S}'$ isomorphic to $\mathcal{Q}(4,s)$, such that every symmetry about a line of $\mathcal{S}'$ can be extended to an automorphism of $\mathcal{S}$. Then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5,s)$.*

Proof. By Section 1.3, we can suppose that $s > 3$ in the rest of the proof.
Assume that $Aut^+(\mathcal{S}')$ is the group of automorphisms of $\mathcal{S}'$ generated by the symmetries about all lines of $\mathcal{S}'$. As $\mathcal{S}$ is a GQ of order (s,s^2), each span of non-collinear points of $\mathcal{S}$ has size 2. Now fix a point $x \in \mathcal{S} \setminus \mathcal{S}'$, and consider the ovoid $\mathcal{O}_x$ of $\mathcal{S}'$ which is subtended by x. Then there is at most one other point y which also subtends $\mathcal{O}_x$. Hence the group extension H^+ of $Aut^+(\mathcal{S}')$ in $Aut(\mathcal{S})$ can have size at most $2|Aut^+(\mathcal{S}')|$ (by Theorem 1.6.4). We distinguish two cases (according whether the extension is proper or not).

(a) $|H^+| = |Aut^+(\mathcal{S}')|$. So H^+ acts faithfully on $\mathcal{S}'$. Fix two non-concurrent lines L and M of $\mathcal{S}'$. As we know, the group G generated by the symmetries about L and M has size s^3-s (in its action on $\mathcal{S}'$), and is isomorphic to $\mathbf{SL}(2,s)$ (in its action on $\mathcal{S}'$). Suppose that H is the subgroup of H^+ which induces G on $\mathcal{S}'$.
We have that

$$H = G \cong \mathbf{SL}(2,s).$$

Note that H, respectively G, acts semiregularly on the points of $\mathcal{S}' \setminus \{L, M\}^{\perp\perp}$.

THE SEMIREGULAR CASE. We first suppose that H acts semiregularly on (the points of) $\mathcal{S} \setminus \{L, M\}^{\perp\perp}$. If G' is the base-group of $\mathcal{S}'$ corresponding to the base-span $\{L, M\}^{\perp}$, and H' is the subgroup of H^+ which induces G' on $\mathcal{S}'$, then we may thus suppose that H' acts semiregularly on $\mathcal{S} \setminus \{L, M\}^{\perp}$ (note that $Aut^+(\mathcal{S}')$ acts transitively on the pairs of non-concurrent lines of $\mathcal{S}'$). The following property is now clear:

(N) *H and H' normalize each other.*

By (N), the fact that H and H' act semiregularly on $\mathcal{S} \setminus \{L, M\}^{\perp\perp}$ and the fact that $H \cong H'$, we can thus conclude that H and H' have exactly the same orbits in $\mathcal{S} \setminus \{L, M\}^{\perp\perp}$. Fix such an orbit Λ in $\mathcal{S} \setminus \mathcal{S}'$. Then $|\Lambda| = s^3 - s$, and the group

$$\Phi = \langle H, H' \rangle$$

acts on Λ. It is clear that

$$|\Phi| = \frac{(s+1)^2 s^2 (s-1)^2}{gcd(s-1, 2)},$$

by (N), and the fact that $H \cong \mathbf{SL}(2, s) \cong H'$ (and their respective action on $\mathcal{S}' \cong \mathcal{Q}(4, s)$). It is straightforward to see that there is a line U of $\mathcal{S}$ which is concurrent with a line of $\{L, M\}^{\perp\perp}$ and which is incident with at least two distinct points of Λ (by, e.g., considering the action of Φ on all such lines, or the rest of the proof). Suppose that one of these points is x. Then $|\Phi_{x,U,y}| \geq \frac{s-1}{gcd(s-1,2)}$, where y is incident with some line of $\{L, M\}^{\perp\perp}$ but not with U. Suppose z is the point of U which is incident with some line of $\{L, M\}^{\perp\perp}$. If s is even, then either $\Phi_{x,U,y}$ acts transitively on the points of $U \setminus \{x, z\}$, or there is a nontrivial $\theta \in \Phi_{x,U,y}$ which fixes each point of $\{L, M\}^{\perp\perp}$ (recall that H, respectively H', acts as $\mathbf{PSL}(2, s)$ on the lines of $\{L, M\}^{\perp\perp}$, respectively $\{L, M\}^{\perp}$). Clearly (cf. Theorems 1.6.2 and 1.6.4), θ fixes a subGQ $\mathcal{S}_\theta$ of $\mathcal{S}$ of order s elementwise. Suppose that this is not the case. As $\Phi_{x,U,y}$ acts transitively on $U \setminus \{x, z\}$, the line U is completely contained in the point set of $\Lambda \cup \{L, M\}^{\perp\perp}$ (recall

that $|U \cap \Lambda| \geq 2$). Define an incidence structure $\mathcal{S}' = (P', B', I')$ as follows.

- LINES. The elements of B' are the lines of $\mathcal{S}'$ and they are of two types:
  1. the lines of $\{L, M\}^{\perp} \cup \{L, M\}^{\perp\perp}$;
  2. the lines of $\mathcal{S}$ which contain a point of Λ and a point of a line of $\{L, M\}^{\perp\perp}$.
- POINTS. The elements of P' are the points of $\mathcal{S}'$ and they are the points of $\{L, M\}^{\perp\perp} \cup \Lambda$, where $\{L, M\}^{\perp\perp}$ is viewed as a point set.
- INCIDENCE. Incidence I' is the induced incidence.

Then it is easy to see that $\mathcal{S}'$ is a subGQ of $\mathcal{S}$ of order s, as there is a line completely contained in the point set of $\Lambda \cup \{L, M\}^{\perp\perp}$, and using the transitivity of Φ on Λ (and the fact that $|\Lambda| = s^3 - s$). Hence we can conclude the following property:

(C) *If Z and Z' are arbitrary non-concurrent lines of $\mathcal{S}'$, and z is a point of $\mathcal{S} \setminus \mathcal{S}'$, then there is a subGQ of $\mathcal{S}$ of order s containing $\{Z, Z'\}^{\perp\perp}$ (as a grid), and the point z.*

Now take over the corresponding part of Theorem 12.10.6. The case where s is odd is completely similar; recall that if U would not be contained in $\Lambda \cup \{L, M\}^{\perp\perp}$ (as a point set), then, considering the action of $\Phi_{x,U,y}$ on $U \setminus (U \cap [\Lambda \cup \{L, M\}^{\perp\perp}])$, we obtain one of the following:

- there is a subGQ of $\mathcal{S}$ of order s containing U, L and M which is fixed elementwise by some element of $\Phi_{x,U,y}$;
- a contradiction, as $|U \setminus (U \cap [\Lambda \cup \{L, M\}^{\perp\perp}])| = \frac{s-1}{2}$.

Hence (C) is satisfied, and the corresponding part of Theorem 12.10.6 can be taken over.

THE NON-SEMIREGULAR CASE. Suppose that H does not act semiregularly on $\mathcal{S} \setminus \{L, M\}^{\perp\perp}$. Since $H \cong \mathbf{SL}(2, s)$, we know (cf. Chapter 8) that H contains precisely $s + 1$ subgroups of order s (which are Sylow p-subgroups, $s = p^h$), and we denote these groups by $H_0, H_1, \ldots, H_s$. It is important to note that for any H_i, there is a line $N \in \{L, M\}^{\perp\perp}$ which is pointwise fixed by each element of H_i, and moreover, H_i induces

a group of symmetries about N in $\mathcal{S}'$. For convenience, suppose that this line is L_i for H_i, $i = 0, 1, \ldots, s$.

Consider an H-orbit O in $\mathcal{S} \setminus \mathcal{S}'$, and suppose $x \in O$ is fixed by some non-identical element $\theta \in H$. Note that θ does not act trivially on $\mathcal{S}'$. By Theorems 1.6.2 and 1.6.4, the fixed element structure $\mathcal{S}_\theta$ of θ is a subGQ of order s. Clearly, the corresponding part of the proof of the semiregular case can be taken over.

(b) $|H^+| = 2|Aut^+(\mathcal{S}')|$. This case can essentially be treated similarly as the case $|H^+| = |Aut^+(\mathcal{S}')|$.

The result follows. ■

There is another way to obtain Theorem 12.10.8.
Suppose $L, M, \mathcal{S}', H$, etc. are as above. We again work with the case where $|Aut^+\mathcal{S}'| = |H^+|$ (but the method works in general, see, e.g., [218] for more details; if $2|Aut^+\mathcal{S}'| = |H^+|$, then the proof essentially boils down to replacing H by its derived group). We also suppose first that H acts semiregularly on $\mathcal{S} \setminus \{L, M\}^{\perp\perp}$.

THE SEMIREGULAR CASE. Suppose O is an H-orbit in $\mathcal{S} \setminus \mathcal{S}'$. Define the following incidence structure $\mathcal{S}'' = \mathcal{S}''(O) = (P'', B'', I'')$.

- LINES. The elements of B'' are the lines of $\mathcal{S}''$ and they are of two types:
  1. the lines of $\{L, M\}^\perp \cup \{L, M\}^{\perp\perp}$;
  2. sets $X \cup proj_{L_j} y$, where X is an H_j-orbit in O, and where y is arbitrary in X, $j \in \{0, 1, \ldots, s\}$.
- POINTS. The elements of P'' are the points of the incidence structure and they are just the points of $\{L, M\}^{\perp\perp} \cup O$.
- INCIDENCE. Incidence I'' is the induced incidence.

First note that the lines are well-defined: suppose A is an H_j-orbit in O, and let y and z be distinct points of A. Then $proj_{L_j} y = proj_{L_j} z$, since H_j fixes L_j pointwise.

Any point of $\mathcal{S}''$ is incident with $s+1$ lines of $\mathcal{S}''$ and any line of $\mathcal{S}''$ is incident with $s+1$ points, and there are exactly $(s+1)(s^2+1)$ points and equally as many lines. Suppose $\mathcal{S}''$ contains a triangle Δ. Then it is clear that the sides of Δ must be lines of Type (2). This implies that there are $i, j, k \in \{0, 1, \ldots, s\}$

and a point $x \in \Delta$, for which there are nontrivial distinct collineations $\theta_i \in H_i$, $\theta_j \in H_j$ and $\theta_k \in H_k$, so that $\theta_i\theta_j\theta_k$ fixes x. The semiregularity of H on $\mathcal{S} \setminus \{L, M\}^{\perp\perp}$ implies that $\theta_i\theta_j\theta_k = \mathbf{1}$. But, then $\mathcal{S}'$ also would contain triangles, a contradiction (note that this is the geometrical interpretation of one of the main properties of $\mathbf{SL}(2, s)$ as a group with a 4-gonal basis $H_0, H_1, \ldots, H_s$; for distinct i, j, k, $H_iH_j \cap H_k = \{\mathbf{1}\}$). Thus $\mathcal{S}''$ has no triangles, and $\mathcal{S}''$ is a generalized quadrangle of order s. Also, $\mathcal{S}'' \cong \mathcal{Q}(4, s)$ since $\mathcal{S}$ is an SPGQ for the base-span $\{L, M\}^{\perp\perp}$ (the groups H_j are groups of symmetries of $\mathcal{S}''$).
So, we know that if L' and M' are arbitrary non-concurrent lines of $\mathcal{S}'$, then there arise $s+1$ classical GQ's of order s which mutually intersect in the points and lines of $\{L', M'\}^{\perp\perp} \cup \{L', M'\}^{\perp}$. The point sets of all these GQ's are also point sets of $\mathcal{S}$. We will now repeat the same kind of argument as in the proof of Theorem 12.10.6.
Fix a point $x \in \mathcal{S} \setminus \mathcal{S}'$, and consider the ovoid $\mathcal{O}_x$ of $\mathcal{S}'$ which is subtended by x. Consider three arbitrary but distinct points u, v, w on $\mathcal{O}_x$. Let U, V, W and U', V', W' be lines in $\mathcal{S}'$ such that $UIuIU' \neq U$, $VIvIV' \neq V$ and $WIwIW' \neq W$, and such that $U \not\sim V$ and $W \in \{U, V\}^{\perp}$, and $V' \not\sim W'$ and $U' \in \{V', W'\}^{\perp}$. Define $\mathcal{S}^1$, respectively $\mathcal{S}^2$, as the (classical) GQ of order s through x and U, V, W, respectively x and U', V', W' which arises as above. Observe the following properties:

- for any line $Z \in \{U, V\}^{\perp\perp}$, $(proj_Z x)_{\mathcal{S}^1} = (proj_Z x)_{\mathcal{S}}$, where the notation is obvious, and the same property holds for $\mathcal{S}^2$; whence
- $x^{\perp} \cap \mathcal{S}^1 \cap \mathcal{S}' \subseteq \mathcal{O}_x$, where '$\perp$' is taken in $\mathcal{S}^1$, and the same property holds for $\mathcal{S}^2$;
- $\mathcal{S}' \cap \mathcal{S}^1 \cap \mathcal{S}^2 = \{U, V\}^{\perp} \cap \{V', W'\}^{\perp}$.
- $\mathcal{S}^1 \cap x^{\perp} = \mathcal{S}^2 \cap x^{\perp}$, where '$\perp$' is taken in $\mathcal{S}^1$ and $\mathcal{S}^2$, respectively.

The last property follows implicitly from the proof of Theorem 12.10.8. Hence the conic $\{U, V\}^{\perp} \cap \{V', W'\}^{\perp}$ of $\mathcal{S}'$ is completely contained in $\mathcal{O}_x$. Hence, through any three distinct points of $\mathcal{O}_x$ there goes a conic of $\mathcal{S}'$ which lies on $\mathcal{O}_x$, and hence $\mathcal{O}_x$ is an elliptic quadric. Thus, every subtended ovoid of $\mathcal{S}'$ is an elliptic quadric, and Theorem 12.10.5 implies that $\mathcal{S} \cong \mathcal{Q}(5, s)$.

THE NON-SEMIREGULAR CASE. Suppose that H does not act semiregularly on $\mathcal{S} \setminus \{L, M\}^{\perp\perp}$. As $H \cong \mathbf{SL}(2, s)$, H contains precisely $s+1$ subgroups of order s, denoted by $H_0, H_1, \ldots, H_s$. For any H_i, there is a line $N \in \{L, M\}^{\perp\perp}$ which is pointwise fixed by each element of H_i, and H_i induces a group of symmetries about N in $\mathcal{S}'$. Suppose that this line is L_i for H_i, $i = 0, 1, \ldots, s$. Consider an

H-orbit O in $\mathcal{S} \setminus \mathcal{S}'$, and suppose $x \in O$ is fixed by some non-identical element $\theta \in H$. Then the fixed element structure $\mathcal{S}_\theta$ of θ is a subGQ of order s.
Now note the following properties:

- H acts on $\mathcal{S}_\theta$;
- there is a central subgroup N^* of H of size 2 so that $H/N^* = H^*$ acts faithfully and semiregularly on $\mathcal{S}_\theta \setminus \{L, M\}^{\perp\perp}$;
- H^* has precisely $s+1$ subgroups of order s, say $H_0^*, H_1^*, \ldots, H_s^*$ (which correspond to the groups $H_0, H_1, \ldots, H_s$), and which respectively fix the lines $L_0, L_1, \ldots, L_s$ pointwise (use Theorems 1.6.2 and 1.6.4);
- for distinct i, j, k we have that $H_i^* H_j^* \cap H_k^* = \{\mathbf{1}\}$.

Consider an arbitrary H^*-orbit Λ in $\mathcal{S}_\theta \setminus \{L, M\}^\perp$. Then $|\Lambda| = \frac{s^3-s}{2}$, and there are two such orbits. The counting argument of the proof of Lemma 9.4.2 can now be repeated (in Λ); let p be a point of Λ, and consider the $s+1$ H_i^*-orbits O_i which contain p. Then every point of O_i is also a point of Λ. By the fact that $H_i^* H_j^* \cap H_k^* = \{\mathbf{1}\}$ for distinct i, j, k (and the semiregular action of H^* on $\mathcal{S}_\theta \setminus \{L, M\}^{\perp\perp}$), we can now take over the counting arguments of the proof of Lemma 9.4.2 in the obvious way (see also Chapter 14 for more details), to obtain that

$$|\Lambda| \geq s^3 - s,$$

contradiction. ∎

12.11 The Symmetry-Class VI

Since every line of an element of **VI** is an axis of symmetry, there follows by Lemma 12.1.5 that the only (thick) elements of **VI** are $\mathcal{Q}(4, s)$ and $\mathcal{Q}(5, s)$, s an arbitrary prime power.

12.12 Classification of Span-Symmetric Generalized Quadrangles

As a corollary of the complete classification theorem, we obtain the following result on span-symmetric generalized quadrangles:

Theorem 12.12.1 *Suppose $\mathcal{S}$ is a span-symmetric generalized quadrangle of order (s,t), $s \neq 1 \neq t$, with base-span $\mathcal{L}$ and base-grid Γ. Then we have one of the following possibilities.*

(*i*) *We have that $s = t$ and $\mathcal{S}$ is isomorphic to $\mathcal{Q}(4,s)$, s a prime power;*

(*ii*) *$t = s^2$ with s a prime power, and the only axes of symmetry of $\mathcal{S}$ are contained in $\mathcal{L}$. Also, there are $s+1$ distinct subGQ's of order s, all isomorphic to $\mathcal{Q}(4,s)$ and mutually intersecting in Γ;*

(*iii*) *$t = s^2$ with s a prime power, and there is exactly one axis of symmetry which is not contained in $\mathcal{L}$, and which is a line of $\mathcal{L}^\perp$. As in the preceding case, there are $s+1$ distinct subGQ's of order s, all isomorphic to $\mathcal{Q}(4,s)$ and mutually intersecting in Γ;*

(*iv*) *$t = s^2$ with s a prime power, and the set of axes of symmetry of $\mathcal{S}$ is precisely $\mathcal{L} \cup \mathcal{L}^\perp$. There are $s+1$ distinct subGQ's of order s, all isomorphic to $\mathcal{Q}(4,s)$ and mutually intersecting in Γ;*

(*v*) *we are not in one of the preceding cases, s is even and $\mathcal{S} \cong \mathcal{Q}(5,s)$ with s a power of 2;*

(*vi*) *s is odd and $\mathcal{S}$ is the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$, and there are at least two infinite classes of non-classical examples and one sporadic non-classical example which can occur. Also, if $\mathcal{S}$ is not classical, then $\mathcal{S}$ contains a line of translation points (so there is some line L so that each line of $L^\perp$ is an axis of symmetry) and s is a prime power.*

■

12.13 Recapitulation of the Classification Theorem

This section contains a table which states a short description of each of the symmetry-classes. All the known generalized quadrangles — sometimes slightly more general — are in the appropriate class. Still, we emphasize that for the classification in its full generality, the table is not complete; the semifield flock GQ's and their translation duals, for example, are completely classified w.r.t. the symmetry-classes by the earlier considerations (so the table is focused more

on 'concrete' examples).

Other information will be clear from the table. The notation of Theorem 12.1.6 is used.

Symmetry-class I. (D) *$\mathcal{S}$ contains no axis of symmetry.*

(M) $H(3,q^2)$, $H(4,q^2)^D$, $W(q)$ with q odd, $H(4,q^2)$, $T_3(\mathcal{O})^D$ with $\mathcal{O}$ an ovoid of $\mathbf{PG}(3,q)$, $\mathcal{S}(\mathcal{F})$ with $\mathcal{F}$ a flock of the quadratic cone in $\mathbf{PG}(3,q)$, $\mathcal{P}(\mathcal{S},x)$, where $\mathcal{S}$ is a known GQ of order $s>3$ with regular point x, and $\mathcal{P}(\mathcal{S},x)^D$ with $\mathcal{S}$ arbitrary.

We also have a classification of **I** based on the possible subconfigurations of centers of symmetry. A classification of those GQ's having both axes and centers of symmetry will be done separately, see below.
We do not mention each time that there are no axes of symmetry.

I.A.1. (D) *There is precisely one center of symmetry.*

(M) $\mathcal{S}(\mathcal{F})$, with $\mathcal{F}$ a known flock of the quadratic cone of $\mathbf{PG}(3,q)$, $\mathcal{F}$ not a linear flock, a Kantor flock, a Ganley flock or a Penttila-Williams flock.

I.A.2. (D) *There are $k+1$ distinct collinear centers of symmetry, $1 \leq k < s$.*

(M) No examples known.

I.A.3. (D) *TGQ's which have one translation line.*

(M) A $T_3(\mathcal{O})^D$ with $\mathcal{O}$ a non-classical ovoid, $\mathcal{S}(\mathcal{F})$ with $\mathcal{F}$ a non-classical Ganley flock or a Penttila-Williams flock.

I.B.1. (D) *p is not a center of symmetry and each line incident with p contains one center of symmetry.*

(M) No examples known.

I.B.2. (D) *p is not a center of symmetry and each line incident with p is incident with t centers of symmetry.*

(M) The class **I.B.2** is empty.

I.B.3. (D) *p is not a center of symmetry and each line incident with p contains t^2 centers of symmetry.*

(M) The class **I.B.3** is empty.

I.C.1. (D) *p is a center of symmetry such that each line through p is incident with 2 centers of symmetry.*

(M) No examples known.

I.C.2. (D) *p is a center of symmetry such that each line through p is incident with $t+1$ centers of symmetry.*

(M) The class **I.C.2** is empty.

I.C.3. (D) *p is a center of symmetry such that each line through p is incident with t^2+1 centers of symmetry.*

(M) $(\mathcal{S}(\mathcal{F}))^*$ with $\mathcal{F}$ a non-classical Kantor flock, Ganley flock or Penttila-Williams flock (where the translation dual is taken w.r.t. an arbitrary translation line of $\mathcal{S}(\mathcal{F})$).

I.D.1. (D) *$\mathcal{S}$ has a normal (regular) ovoid $\mathbf{O}_N$ of which element is a center of symmetry of $\mathcal{S}$.*

(M) The class **I.D.1** is empty.

I.D.2. (D) *The centers of symmetry of $\mathcal{S}$ form a subGQ $\mathcal{S}'$ of order t and $\mathcal{S} \neq \mathcal{S}'$.*

(M) The class **I.D.2** is empty.

I.D.3. (D) *The centers of symmetry of $\mathcal{S}$ form a dual grid $\mathcal{G}$ with parameters $t+1, t+1$, and $\mathcal{S}$ is a GQ of order (t^2, t).*

(M) No examples known.

I.D.4. (D) *Each line of $\mathcal{S}$ intersects the set of centers of symmetry of $\mathcal{S}$, $\mathcal{S}$ is of order (t^2, t), and each line of $\mathcal{S}$ is incident with $k+1$ centers of symmetry, $t^2 > k \geq 1$.*

(M) The class **I.D.4** is empty.

I.D.5. (D) *The centers of symmetry of $\mathcal{S}$ form a proper subGQ $\mathcal{S}' \cong W(t)$ of order t.*

(M) The class **I.D.5** is empty.

I.E. (D) *Every point of $\mathcal{S}$ is a center of symmetry.*

(M) $W(s)$ with s odd, $H(3, s)$.

Symmetry-class II.1. (D) *There is precisely one axis of symmetry.*

(M) $\mathcal{S}(\mathcal{F})^D$, with $\mathcal{F}$ a known flock of the quadratic cone of $\mathbf{PG}(3,q)$, $\mathcal{F}$ not a linear flock, a Kantor flock, a Ganley flock or a Penttila-Williams flock, $T_2(\mathcal{O})^D$ with $\mathcal{O}$ not a translation oval of $\mathbf{PG}(2,q)$.

Symmetry-class II.2. (D) *There are $k+1$ distinct concurrent axes of symmetry, $1 \leq k < t$.*

(M) No examples known.

Symmetry-class II.3. (D) *TGQ's which have one translation point.*

(M) A $T_2(\mathcal{O})$ with $\mathcal{O}$ a non-classical translation oval, a $T_3(\mathcal{O})$ with $\mathcal{O}$ a non-classical ovoid, $\mathcal{S}(\mathcal{F})^D$ with $\mathcal{F}$ a non-classical Ganley flock or a Penttila-Williams flock.

Symmetry-class III.1. (D) *L is not an axis of symmetry and each point on L is incident with one axis of symmetry.*

(M) No examples known.

Symmetry-class III.2. (D) *L is not an axis of symmetry and each point on L is incident with s axes of symmetry.*

(M) The class **III.2** is empty.

Symmetry-class III.3. (D) *L is not an axis of symmetry and each point on L is incident with s^2 axes of symmetry.*

(M) The class **III.3** is empty.

Symmetry-class IV.1. (D) *L is an axis of symmetry such that each point on L is incident with 2 axes of symmetry.*

(M) No examples known.

Symmetry-class IV.2. (D) *L is an axis of symmetry such that each point on L is incident with $s+1$ axes of symmetry.*

(M) The class **IV.2** is empty.

Symmetry-class IV.3. (D) *L is an axis of symmetry such that each point on L is incident with s^2+1 axes of symmetry.*

(M) $(\mathcal{S}(\mathcal{F})^D)^*$ with $\mathcal{F}$ a non-classical Kantor flock, Ganley flock or Penttila-Williams flock.

Symmetry-class V.1. (D) *$\mathcal{S}$ has a normal (regular) spread $\mathbf{T}_N$ of which any line is an axis of symmetry of $\mathcal{S}$.*

(M) The class **V.1** is empty.

Symmetry-class V.2. (D) *The axes of symmetry of $\mathcal{S}$ form a subGQ $\mathcal{S}'$ of order s and $\mathcal{S} \neq \mathcal{S}'$.*

(M) The class **V.2** is empty.

Symmetry-class V.3. (D) *The axes of symmetry of $\mathcal{S}$ form a grid $\mathcal{G}$ with parameters $s+1, s+1$, and $\mathcal{S}$ is a GQ of order (s, s^2).*

(M) No examples known.

Symmetry-class V.4. (D) *The axes of symmetry of $\mathcal{S}$ cover $\mathcal{S}$, $\mathcal{S}$ is of order (s, s^2), and each point of $\mathcal{S}$ is incident with $k+1$ axes of symmetry, $s^2 > k \geq 1$.*

(M) The class **V.4** is empty.

Symmetry-class V.5. (D) *The axes of symmetry of $\mathcal{S}$ form a proper subGQ $\mathcal{S}' \cong \mathcal{Q}(4,s)$ of order s.*

(M) The class **V.5** is empty.

Symmetry-class VI. (D) *Every line of $\mathcal{S}$ is an axis of symmetry.*

(M) $\mathcal{Q}(4,s)$, $\mathcal{Q}(5,s)$.

There also easily follows a classification for those GQ's having both centers and axes of symmetry.

Symmetry-class A.1 (D) *There is a flag (p, L) so that p is a center of symmetry and L is an axis of symmetry.*

(M) No examples known.

Symmetry-class A.2 (D) *There is an axis of symmetry L each point of which is a center of symmetry (so L is a translation line).*

(M) $T_2(\mathcal{O})^D$ with $\mathcal{O}$ not a translation oval of $\mathbf{PG}(2, q)$.

Symmetry-class A.3 (D) *There is a center of symmetry p each line through which is an axis of symmetry (so p is a translation point).*

(M) $T_2(\mathcal{O})$ with $\mathcal{O}$ not a translation oval of $\mathbf{PG}(2, q)$.

Symmetry-class A.4 (D) *There is point p each line through which is an axis of symmetry, and there is one line MIp each point of which is a center of symmetry (so p is a translation point and M is a translation line).*

(M) $T_2(\mathcal{O}) \cong T_2(\mathcal{O})^D$ with $\mathcal{O}$ a translation oval of $\mathbf{PG}(2, q)$.

Symmetry-class A.5 (D) *Each point is a center of symmetry and each line is an axis of symmetry.*

(M) $W(q)$, q even.

Appendix: Solution of a Conjecture of W. M. Kantor

The following is an immediate corollary of the classification proposed in this chapter, and is in fact the solution of the natural analogue of the 'SPGQ-conjecture' for GQ's of order (s, s^2), $s > 1$. It is also the solution of a recent conjecture of W. M. Kantor[7] (*Some Conjectures Concerning Generalized Quadrangles*, W. M. Kantor, January 2001, Private communication).

Theorem 12.13.1 *A generalized quadrangle of order (s,t), $s \neq 1 \neq t$, is isomorphic to $\mathcal{Q}(5,s)$ if and only if there are three distinct axes of symmetry U, V and W for which $U \cap \{V,W\}^{\perp\perp} = \emptyset$ (where we view U and $\{V,W\}^{\perp\perp}$ as point sets).*

■

Remark 12.13.2 In *Some Conjectures Concerning Generalized Quadrangles*, W. M. Kantor also describes an 'SPGQ-conjecture' for generalized quadrangles of order (s^2, s^3), $s > 1$.

[7] Which I posed independently.

Chapter 13

Half Pseudo Moufang Generalized Quadrangles are Classical or Dual Classical

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Finite Moufang generalized quadrangles were classified in 1974 as a corollary to the classification of finite groups with a split BN-pair of rank 2 by P. Fong and G. M. Seitz [60, 61], see also Theorem 1.4.1. Later on, work of S. E. Payne and J. A. Thas culminated in an almost complete, elementary proof of that classification, see Chapter 9 of FGQ. Using slightly more group theory, first W. M. Kantor [93], and then the author of this text [206], see Appendix A, completed this geometric approach. Recently, J. Tits and R. Weiss classified all Moufang polygons [226], and this provides a third independent proof for the classification of finite Moufang quadrangles. In 1991, J. A. Thas, S. E. Payne and H. Van Maldeghem introduced the notion of 'half Moufang quadrangle', and showed that every half Moufang quadrangle is a Moufang quadrangle [201], cf. Theorem 1.4.2. In the present chapter, we introduce the notions of *pseudo*

Moufang quadrangle and *half pseudo Moufang quadrangle* as a further generalization of Moufang quadrangle and half Moufang quadrangle, respectively. We show that every finite half pseudo Moufang quadrangle is a Moufang quadrangle, or, equivalently, that every finite half pseudo Moufang quadrangle is classical or dual classical. This provides a new global characterization of all classical and dual classical finite generalized quadrangles.

The present chapter is joint work with H. Van Maldeghem, and is based on *Half Pseudo Moufang Generalized Quadrangles are Classical or Dual Classical* [220], which is submitted for publication to *Journal für die reine und angewandte Mathematik*.

13.1 Half Pseudo Moufang Quadrangles and the Main Results

We recall the notion of 'Moufang generalized quadrangle' for the convenience of the reader. A *Moufang generalized quadrangle* is a GQ $\mathcal{S} = (P, B, I)$ in which the following conditions hold:

(M) for any panel (p, L, q) of $\mathcal{S}$, the group of all automorphisms of $\mathcal{S}$ fixing all lines incident with p, all lines incident with q, and all points incident with L (sometimes called *root-elations* in the sequel), acts transitively on the set of points that are incident with a given line MIp, $M \neq L$, and different from p;

(M′) for any dual panel (L, p, M) of $\mathcal{S}$, the group of all automorphisms of $\mathcal{S}$ fixing all points incident with L, all those incident with M, and all lines incident with p (also called *root-elations*), acts transitively on the set of lines that are incident with a given point xIL, $x \neq p$, and different from L.

Note. Suppose (p, L, q) is a panel of a thick GQ $\mathcal{S}$, and assume that the group H of all automorphisms of $\mathcal{S}$ fixing all lines incident with p, all lines incident with q, and all points incident with L, acts transitively on the set of points incident with the line MIp, $M \neq L$, and different from p. If $M' \neq L$ is an arbitrary line through p, respectively q, then H also acts transitively on the points incident with M', different from p, respectively q. The same remark holds for dual panels.

Let $\mathcal{S}$ be a thick Moufang GQ. The group generated by all root-elations will be called the *little projective group.*
Moufang quadrangles were introduced in the appendix of [222] (as a special case of *Moufang spherical buildings*). It was noted by J. Tits in [223] that the classification of finite split BN-pairs of rank 2 by P. Fong and G. M. Seitz [60, 61] (or rather a corollary of their main result, see Section 5.7 of [229]), implies that a finite generalized quadrangle is Moufang if and only if it is a classical or dual classical GQ. In [201], J. A. Thas, S. E. Payne and H. Van Maldeghem proved that the Properties (M) and (M′) are equivalent, recall Theorem 1.4.2 (in this context, the reader also recalls the notion of 'half Moufang GQ' from the introduction of this work).

After 1974, it was a challenge to prove the classification of finite Moufang GQ's in a purely geometric way or, at least, in an elementary way without using the results of P. Fong and G. M. Seitz. We refer the reader to Appendix A for more details.

Now let $\mathcal{S} = (P, B, I)$ be a thick GQ of order (s, t) and let H be an automorphism group of $\mathcal{S}$. For points p, q of $\mathcal{S}$, we denote by $H_{p,q}$ the stabilizer of both p and q in H. Similarly for lines. Then $\mathcal{S}$ is called *half pseudo Moufang (with respect to H)* if either *Property (PM)* or *Property (PM′)* (below) is satisfied.

(PM) For every panel (p, L, q), with $p, q \in P$ and $L \in B$, there is a normal subgroup $H(p, q)$ of $H_{p,q}$ of elations about both p and q which acts regularly on the set of points that are incident with any line MIp, respectively $M'Iq$, $M \neq L \neq M'$, and different from p, respectively q. The group $H(p, q)$ will be sometimes referred to as a *pseudo elation group.*

(PM′) For every dual panel (L, p, M), with $L, M \in B$ and $p \in P$, there is a normal subgroup $H(L, M)$ of $H_{L,M}$ of elations about both L and M which acts regularly on the set of lines that are incident with an arbitrary point xIL, respectively yIM, $x \neq p \neq y$, and different from L, respectively M.

A GQ is called *pseudo Moufang (with respect to H)* if both Properties (PM) and (PM′) hold. In the sequel, we will sometimes write 'HPMGQ' instead of 'half pseudo Moufang generalized quadrangle' for the sake of convenience, and we always assume that the corresponding group is H. Hence we will often forget to mention this group explicitly.
Let (p, L, q) be a panel of the thick generalized quadrangle $\mathcal{S} = (P, B, I)$ of order (s, t) and suppose that $s < t$. Let $S(p, q)$ be the set of elations about both p and q, and suppose that $|S(p, q)| \geq s$. Suppose moreover that $S(p, q)$ is

not a group. Then clearly there is some automorphism $\phi \neq \mathbf{1}$ of $\mathcal{S}$ generated by $S(p,q)$ which fixes all lines incident with p and with q, and which fixes some point of $\mathcal{S}$ not incident with L. By Theorem 1.6.4, the set of fixed points and lines of ϕ forms a subGQ of $\mathcal{S}$ of order (s',t), $s' < s$, contradicting the fact that $s < t$ (cf. Theorem 1.6.2). Hence if $s < t$, then Property (PM) is equivalent with requiring that for every panel (p,L,q), there is a set of s elations about both p and q all contained in H.

It is easy to see that in a pseudo Moufang GQ every point is a point of transitivity and every line is a line of transitivity. Hence by [230], the pseudo Moufang GQ is a Moufang GQ. It is now our aim to classify the thick half pseudo Moufang GQ's.

Main Result. *Every finite thick half pseudo Moufang generalized quadrangle is a classical or dual classical quadrangle. Conversely, every (finite) classical or dual classical generalized quadrangle is a half pseudo Moufang quadrangle with respect to any group H containing the little projective group, and the pseudo elation groups are independent of H. In particular, every pseudo elation group is a group of root-elations.*

Equivalently, we have the following.

Main Result, Second Version. *Every finite thick half pseudo Moufang generalized quadrangle is a Moufang generalized quadrangle. Conversely, every (finite) Moufang generalized quadrangle is a half pseudo Moufang quadrangle with respect to any group H containing the little projective group, and the pseudo elation groups are independent of H. In particular, every pseudo elation group is a group of root-elations.*

The first part of the Main Result can be proved using the classification of finite simple groups (using the main result of [31]). Here, we present a proof which does not depend on that classification. In fact, since by the foregoing remarks Moufang quadrangles can be classified without using the results of P. Fong and G. M. Seitz mentioned above, our proof will also be independent of their proof. But we will use the classification of split BN-pairs of rank 1.

In the next section, we prove the second part of the Main Result, namely, that every classical GQ and every dual classical GQ is an HPMGQ "in a unique way". The rest of the chapter is devoted to the proof of the first part of the Main Result.

We start with some general observations. We then invoke the classification of finite split BN-pairs of rank 1 and consider the case of a sharply 2-transitive

group first. The other cases are treated afterwards and we make a distinction between the cases $s \leq t$ and $s > t$. Whenever we can, we state lemmas and propositions in a more general context.
For the sake of simplicity, we will from now on use the notion of a half pseudo Moufang GQ only for GQ's satisfying Property (PM).

13.2 Finite Moufang Quadrangles Are Pseudo Moufang Quadrangles

We first show a general lemma.

Lemma 13.2.1 *If $\mathcal{S}$ is an HPMGQ with respect to the group H, then H acts transitively on the set of ordered quadruples (p_1, p_2, p_3, p_4) of points with $p_1 \sim p_2 \sim p_3 \sim p_4 \sim p_1$ and $p_1 \not\sim p_3$; $p_2 \not\sim p_4$.*

Proof. Let L, M, M' be three lines of $\mathcal{S}$ with $M \not\sim L \not\sim M' \sim M$. Let $p = proj_L(M \cap M')$ and let q be an arbitrary point on L distinct from p. Select a nontrivial $\theta \in H(p,q)$ and denote $M'' = M^\theta$. Choose any line M_0 meeting both M' and M'', but not incident with p, and put $L' = proj_p M_0$. Set $q' = M_0 \cap L'$. Then there exists a $\theta \in H(p,q')$ mapping M'' to M'. Hence we found $\sigma = \theta\theta' \in H$ fixing L and mapping M to M'. As M and M' are not concurrent, then either $L \in \{M, M'\}^{\perp\perp}$ — in which case there exists $\sigma \in H(p,q)$ mapping M to M' — or there exists a line M^* meeting both M and M' and not L. Applying twice the foregoing argument, we conclude that H acts transitively on the set of ordered non-concurrent pairs of lines. Hence, for two quadruples (p_1, p_2, p_3, p_4) and (p_1', p_2', p_3', p_4') as in the statement of the lemma, we may assume $p_1p_2 = p_1'p_2'$ and $p_3p_4 = p_3'p_4'$. Applying elements of $H(x,y)$, with $p_1p_2Ix \sim yIp_3p_4$, the result follows easily. ∎

A GQ is called *flag-transitive GQ* if it admits an automorphism group A acting transitively on its flags. The group A is then also called *flag-transitive*.

Proposition 13.2.2 *Let $\mathcal{S}$ be a finite Moufang quadrangle with little projective group H. Then $\mathcal{S}$ is an HPMGQ with respect to H if one defines the pseudo elation groups as the corresponding root-elation groups. Moreover, if $\mathcal{S}$ is half pseudo Moufang with respect to some other group H^*, then $H \leq H^*$ and the pseudo elation groups are necessarily the root-elation groups.*

Proof. The first statement is obvious and follows from the semiregular action of any root-elation. Suppose now that the classical or dual classical GQ $\mathcal{S}$ is

half pseudo Moufang with respect to the group H^*. By Lemma 13.2.1, H^* is flag-transitive and hence it contains all root-elations, except if $\mathcal{S}$ is isomorphic to $W(2)$, $W(3)$, $\mathcal{Q}(4,2)$, $\mathcal{Q}(4,3)$ or $H(3,9)$ (see [154]; see also Theorem 4.8.7 of [229]). But it is easily seen that in the first four cases the elements of $H^*(p,q)$ must be root-elations. If $\mathcal{S}$ is isomorphic to $H(3,9)$ and if H^* does not contain all root-elations of $\mathcal{S}$, then it follows from [154] that H^* has order at most $4|\mathbf{PSL}(3,4)| = 2^8.3^2.5.7$, which is much less than the number of ordered quadruples of points (as in the statement of Lemma 13.2.1) in $H(3,9)$ (that number is equal to $2^5.3^6.5.7$), contradicting the fact that H^* acts transitively on the set of such ordered quadruples of $\mathcal{S}$. Hence we have shown that H^* contains all root-elations of $\mathcal{S}$ and consequently that $H \leq H^*$.
Now let the order of $\mathcal{S}$ be equal to (s,t) and let $\theta \in H^*(p,q)$, for some collinear distinct points p,q. Note that there exists a root-elation σ fixing all lines through p and through q, and such that $x^\theta = x^\sigma$ for at least one point x not incident with pq. The automorphism $\theta\sigma^{-1}$ fixes pointwise a subquadrangle of order (s',t), with $1 < s' \leq s$. Indeed, $1 < s'$ because θ is an elation and hence its order divides the prime power s, which implies that it fixes at least three points of the line pq. If $s \leq t$, then necessarily $s = s'$ by Theorem 1.6.2, and so $\theta = \sigma$. That leaves two cases.

1. $\mathcal{S}$ IS DUAL TO $H(4,t)$. We have $H(p,q) \trianglelefteq H^*_{p,q}$, so every Sylow r-subgroup S of H^*p,q (where r is the unique prime number dividing s and t) contains $H(p,q)$ (which, we recall, consists of root-elations). We choose S such that it contains θ. Write $(s,t) = (r^3, r^2)$. If $\theta \notin H(p,q)$, then $\theta\sigma^{-1}$ fixes pointwise a subquadrangle $\mathcal{S}'$ of order (s', r^2). By Theorem 1.6.1, $s'r^2 \leq r^3$, hence $s' \leq r$. So $s' = r$ by the inequality of Higman, and $\mathcal{S}'$ is dual to $H(3,t)$. Now $\theta\sigma^{-1}$ must act trivially on the set of lines through p, so interpreted in $\mathbf{PG}(4,t)$, the collineation $\theta\sigma^{-1}$ is linear. If we now interpret the set of points incident with pq as a Hermitian curve in a projective plane, then $\theta\sigma^{-1}$ is a linear automorphism of that curve fixing a line of the plane pointwise. Hence the order of $\theta\sigma^{-1}$ divides $r+1$ and therefore cannot be contained in S.

2. $\mathcal{S}$ IS ISOMORPHIC TO $H(3,s)$. In this case, it is easily seen that $H^*(p,q)$ must contain at least one root-elation σ (otherwise a Sylow r-subgroup (r as before) of $H^*_{p,q}$ would have order at least s^2, a contradiction). Let L be any line through p distinct from pq. Then H_L acts on L as $\mathbf{PSL}(2,s)$, see e.g. [229, Table 8.1]. It is not so hard to see that $H_{L,p}$ has the same action on the points incident with L as $H_{L,p,q}$. So we may conjugate σ by $H_{L,p,q}$ and obtain, if s is even, all root-elations of $H(p,q)$, or, if s is odd, $\frac{s-1}{2}$ elements of $H(p,q)$, which generate $H(p,q)$. So since $H^*(p,q)$

is normal in $H^*_{p,q}$, we conclude $H^*(p,q) = H(p,q)$. ∎

We now prove the first part of the Main Result in the next sections.

13.3 Some General Observations

13.3.1 Property (H) and HPMGQ's

The first properties only rely on the fact that in a half pseudo Moufang GQ every point is a point of transitivity, up to duality:

Lemma 13.3.1 *Suppose the thick GQ $\mathcal{S} = (P, B, I)$ of order (s,t) is half pseudo Moufang. Then every point of $\mathcal{S}$ is a center of transitivity.*

Proof. Suppose p is an arbitrary point of $\mathcal{S}$. Let $H(p)$ be the group defined by $\langle H(p,q) \parallel q \sim p, q \neq p \rangle$, then it is clear that every element of $H(p)$ fixes every line incident with p. It is also clear, by the definition of $H(p)$, that if q and r are distinct collinear points in $P \setminus p^\perp$, there is an element of $H(p)$ mapping q onto r. Now suppose q and r are non-collinear points in $P \setminus p^\perp$. Then there is an element of $H(p)$ mapping q onto r by the previous observation, since by a result of A. Brouwer [19] the set $P \setminus x^\perp$ is connected in $\mathcal{S}$. ∎

Lemma 13.3.2 *Suppose $\mathcal{S}$ is a thick GQ of order (s,t) such that some point p of $\mathcal{S}$ is a center of transitivity. Then p has Property (H).*

Proof. Let $\{q, r, s\}$ be a triad in $p^\perp$ with the property that $q \in cl(r,s)$. Suppose o is a point such that $o \sim q$ and $o \in \{r,s\}^{\perp\perp}$. Consider a whorl ϕ about p which maps o onto q, and which fixes r (such a whorl exists by the transitivity assumption). Then $s^\phi \sim s$ and $q \in \{r, s^\phi, q\}^{\perp\perp}$, and hence $s \in cl(r,q)$. It now easily follows that the point p has Property (H). ∎

From FGQ we recall

Theorem 13.3.3 (FGQ, 5.6.2) *Suppose every point of the thick GQ $\mathcal{S}$ of order (s,t) satisfies Property (H). Then we have one of the following possibilities:*

(a) *every span of non-collinear points has size 2;*

(b) *every point is regular;*

(c) $\mathcal{S} \cong H(4,s)$.

Whence

Proposition 13.3.4 *Suppose every point of the thick GQ $\mathcal{S}$ of order (s,t) is a center of transitivity. Then we have one of the following possibilities:*

(a) *every span of non-collinear points has size 2;*

(b) *every point is regular;*

(c) $\mathcal{S} \cong H(4,s)$.

In particular, this applies to half pseudo Moufang generalized quadrangles.

Proof. Immediately from Theorem 13.3.3. ∎

Since a thick GQ of order (s,t) with $s \leq t$ and all points regular is necessarily isomorphic to $W(s)$ by Theorem 1.2.2, we may already conclude:

Corollary 13.3.5 *Let $\mathcal{S}$ be an HPMGQ of order (s,t). Then either*

- *$\mathcal{S}$ is classical (and isomorphic to $W(s)$ or $H(4,s)$), or*
- *every span of non-collinear points has size 2, or*
- *$s > t$ and every point is regular.*

∎

13.3.2 A nonexistence theorem for certain elation generalized quadrangles

If $s \leq t$, then there is a further restriction on the structure of an HPMGQ of order (s,t), which can be stated in the slightly more general context of EGQ's.

Proposition 13.3.6 *Let $\mathcal{S} = (P,B,I)$ be a thick GQ of order (s,t) and suppose $p \in P$ is an elation point with elation group G. Suppose that for every $q \in P \setminus p^\perp$ the span $\{p,q\}^{\perp\perp}$ has exactly 2 elements, and that for every LIp and every $M \in B \setminus L^\perp$ the span $\{L,M\}^{\perp\perp}$ has exactly 2 elements. Then the center of G is trivial. In particular, $s > t$.*

Proof. Denote the lines through p by L_i, $i \in \{0,1,\ldots,t\}$, and let q be an arbitrary point of $P \setminus p^\perp$. Put $M_i := proj_q L_i$, for every i, and let G_i be the stabilizer of M_i in G (so G_i acts regularly on the points of M_i distinct from $L_i \cap M_i$). Furthermore, let G_i^* be the stabilizer of $p_i := M_i \cap L_i$ in G. Suppose

by way of contradiction that the center $Z(G)$ of G is nontrivial and let g be a nontrivial element of $Z(G)$. If g is a symmetry about p, then, assuming without loss of generality that $q^g \neq q$, we have $\{q, q^g, p\} \subseteq \{p, q\}^{\perp\perp}$, a contradiction. Hence, again without loss of generality, we may assume that g does not belong to G_0^* (if $g \in \cap_{i=0}^{s} G_i^*$, then g is a symmetry about p). Since G_0 centralizes g, the group G_0 fixes M_0^g. The latter line does not meet M_0. Let M_j be the unique line through q which meets $M_0' = M_0^g$. Then we conclude that every element of the orbit of M_j under the action of G_0 meets every element of the triad $\{M_0, M_0', L_j\}$. Since that orbit has size s and also L_0 is a center of the triad, we see that $\{M_0, M_0', L_j\} \subseteq \{M_0, L_j\}^{\perp\perp}$, again a contradiction. Hence $Z(G)$ is trivial.
Suppose now that $s \leq t$. Then by D. Frohardt [63], s and t are powers of the same prime number n, and hence G is an n-group. But every such group has a nontrivial center.
The proposition is proved. ∎

The connection of HPMGQ's with GQ's having an elation point is given by the following lemma.

Lemma 13.3.7 *Let $\mathcal{S}$ be a thick HPMGQ of order (s,t) with $s \leq t$, so that each span of non-collinear points has size 2. Then each point x is an elation point with corresponding elation group $H(x)$ (see above for this notation). Moreover, $H(x)$ is the complete set of elations of $\mathcal{S}$ about x.*

Proof. Immediate from Theorem 1.7.2 (v) and the fact that the automorphism group of $\mathcal{S}$ acts transitively on the points of $\mathcal{S}$. ∎

These results have for our situation the following interesting consequence.

Corollary 13.3.8 *If $\mathcal{S}$ is a thick HPMGQ of order (s,t) with $s \leq t$, so that each span of non-collinear points has size 2, then each span of non-concurrent lines has at least size 3.*

∎

13.3.3 Subquadrangles and HPMGQ's

We now investigate the role of thick subquadrangles of an HPMGQ which have 'a lot of points on a line' or 'a lot of lines through a point'.

Proposition 13.3.9 *Let $\mathcal{S}$ be a thick HPMGQ of order (s,t) and assume that $\mathcal{S}'$ is a thick subGQ of $\mathcal{S}$ of order (s,t'). Then $\mathcal{S}'$ is also an HPMGQ (with*

respect to the induced group). If moreover $\mathcal{S}'$ is classical or dual classical, then $\mathcal{S}$ is also classical or dual classical.

Proof. Consider an arbitrary panel (p, L, q) in $\mathcal{S}'$ and consider this as a panel of $\mathcal{S}$. Let $\theta \in H(p,q)$ be arbitrary. Then $\mathcal{S}' \cap (\mathcal{S}')^\theta$ is a subGQ of $\mathcal{S}'$ of order (s, t'), hence it coincides with $\mathcal{S}'$ and so θ is an elation of $\mathcal{S}'$ about both p and q. Let H' be the (global) stabilizer in H of $\mathcal{S}'$. Then we may put $H'(p,q) = H(p,q)$ (with obvious notation), and clearly $H'(p,q) \trianglelefteq H'_{p,q}$. Hence $\mathcal{S}'$ is an HPMGQ with respect to H'.
Now suppose that $\mathcal{S}'$ is classical or dual classical. We may assume that $t' < t$, otherwise the result is trivial. By Proposition 13.2.2, $H'(p,q)$ consists of root-elations and so every element of $H(p,q)$ fixes L pointwise. Hence $\mathcal{S}$ is half Moufang and the result follows. ∎

Proposition 13.3.10 *Suppose that $\mathcal{S} = (P, B, I)$ is a thick HPMGQ of order (s,t) and let $\mathcal{S}' = (P', B', I')$ be a thick subGQ of $\mathcal{S}$ of order (s', t). Then $\mathcal{S}'$ is also an HPMGQ (with respect to the induced group). If moreover $\mathcal{S}$ contains another subquadrangle $\mathcal{S}^*$ of order (s', t), $\mathcal{S}' \neq \mathcal{S}^*$, such that $\mathcal{S}'$ and $\mathcal{S}^*$ have two non-collinear points in common, then $\mathcal{S} \cong H(3, s)$ (and consequently $\mathcal{S}' \cong \mathcal{S}^* \cong W(t)$).*

Proof. Let (p, L, q) be an arbitrary panel of $\mathcal{S}'$ and regard it as a panel of $\mathcal{S}$. Let UIp and VIq be so that $U \neq L \neq V$, and let M and M' be distinct lines, $M \neq L \neq M'$, so that $M, M' \in \{U, V\}^\perp \cap B'$. Let θ be the element of $H(p,q)$ that maps M onto M'. It is clear that, if for each such $(p, L, q), U, V$, M, M', we have that θ stabilizes $\mathcal{S}'$ (globally), then $\mathcal{S}'$ is an HPMGQ with respect to the stabilizer H' in H of $\mathcal{S}'$. Suppose now either that θ does not fix $\mathcal{S}'$, or that there is a second subGQ $\mathcal{S}^*$ of order (s', t) containing two non-collinear points of $\mathcal{S}'$, $\mathcal{S}' \neq \mathcal{S}^*$. Then by Theorem 1.6.2 and Theorem 1.6.3, $\mathcal{S}' \cap (\mathcal{S}')^\theta$, respectively $\mathcal{S}' \cap \mathcal{S}^*$, is a subGQ of $\mathcal{S}$ of order $(1, t)$; moreover $s' = t$ and $s = t^2$. Hence $\mathcal{S}$ has a regular pair of points. Since H acts transitively on the set of pairs of non-collinear points, each point of $\mathcal{S}$ is regular. This implies that each point of $\mathcal{S}'$ is regular and so $\mathcal{S}'$ is isomorphic to $W(t)$ by Theorem 1.2.2.
Fix some point $x \in \mathcal{S}$, and let y and z be non-collinear points in $x^\perp$. Let $H_0(x)$ be the subgroup of the group $H(x)$ of whorls about x which fixes both y and z. We know that t is a prime power. Let n be the unique prime dividing t. Since $H_0(x)$ acts transitively on $\{y, z\}^\perp \setminus \{x\}$, there is a nontrivial Sylow n-subgroup K of $H_0(x)$ of order at least t. Since t and $s - 1 = t^2 - 1$ are relatively prime, the group K fixes some point u on the line xy different from x and y. Since K also fixes the set $\{y, z\}^{\perp\perp}$ pointwise, it fixes all points of $\{u, u'\}^{\perp\perp}$, with $u' \in \{y, z\}^{\perp\perp}$. It follows easily that K fixes at least $t + 1$ points incident with

xy, and if it fixes at least $t+2$ of such points, then it must fix $x^\perp$ pointwise (see Chapter 3). We consider two possibilities.

1. K DOES NOT ACT SEMIREGULARLY ON $\{y,z\}^\perp \setminus \{x\}$. In this case, let $\theta \in K$ be such that θ fixes some point $v \in \{y,z\}^\perp \setminus \{x\}$, $\theta \neq \mathbf{1}$. Then θ fixes a subGQ $\mathcal{S}'' \cong W(t)$ of order t pointwise, and hence θ is an involution (by, e.g., Chapter 3). Also, θ fixes $\{y,z\}^\perp$ pointwise, $n=2$ (as K is a Sylow n-subgroup), and the order of K is at least $2t$. Also, every element of K fixes every point of $\mathcal{S}''$ on xy.

   By the first part of this proof, respectively by assumption, we may assume that there is a second subquadrangle $\mathcal{S}''' \cong W(t)$ containing y,z and meeting $\mathcal{S}''$ in the dual grid with parameters $t+1, t+1$ determined by x,y,z. If an element of K preserves $\mathcal{S}'''$, then it must fix all points of $\mathcal{S}'''$ collinear with x (otherwise there arises a subGQ of $W(t)$ of order (t',t), with $1<t'<t$, a contradiction), and hence it must fix $x^\perp$ pointwise in $\mathcal{S}$. If no element of K preserves $\mathcal{S}'''$, then there arises a set of at least $2t(t-1)$ different points on xy (corresponding with $2t$ subGQ's containing x,y,z which are images of $\mathcal{S}'''$ under K), a contradiction. So K contains some nontrivial symmetry σ about x. Clearly, the group of symmetries about x has even order. By C. E. Ealy, Jr. [56], we now have that $\mathcal{S} \cong H(3,s)$, as the automorphism group of $\mathcal{S}$ acts transitively on the points of $\mathcal{S}$, and the proposition follows.

2. K ACTS SEMIREGULARLY ON $\{y,z\}^\perp \setminus \{x\}$. Since $|K| \geq t$, this implies that $|K| = t$, and K acts transitively on $\{y,z\}^\perp \setminus \{x\}$. If K does not contain any symmetry about x, then as in the previous case, we obtain t subGQ's containing x,y,z, and the union of their point set contains all points of the line xy of $\mathcal{S}$ not contained in $\mathcal{S}'$. Hence it is now easy to see that every centric triad of lines is contained in a subGQ of order t, and so $\mathcal{S}$ is isomorphic to $H(3,s)$ by the dual of Theorem 1.6.6. The proposition again follows in this case. So we may assume that K contains a nontrivial symmetry σ about x. By conjugation, there is a nontrivial symmetry about every point $w \in \{y,z\}^\perp$, and these symmetries preserve $\mathcal{S}'$. The group K^* generated by K and these symmetries acts on $\{y,z\}^\perp$ as $\mathbf{PSL}(2,t)$, and so we see that $\frac{t-1}{2}$ elements of K are conjugate to σ by elements of K^*. So at least $\frac{t-1}{2}$ nontrivial elements of K are symmetries about x, which implies that all elements of K must be symmetries about x since $\frac{t-1}{2}+1$ distinct elements of K generate K. Consequently $\mathcal{S}$ is half Moufang, hence Moufang by Theorem 1.4.2, and isomorphic to $H(3,s)$.

The proposition is proved. ∎

13.4 HPMGQ's and Groups with a Split BN-Pair of Rank 1

Let $\mathcal{S}$ be a thick HPMGQ, and let U and W be non-concurrent lines of $\mathcal{S}$. Put $X = \{U, W\}^{\perp} = \{L_0, L_1, \ldots, L_s\}$, and set $u_i = L_i \cap U$, respectively $w_i = L_i \cap W$, for $i = 0, 1, \ldots, s$. Define $G = \langle H(u_j, w_j) \parallel j \in \{0, 1, \ldots, s\}\rangle$. Then (X, G) is a (not necessarily faithful) permutation group.

Lemma 13.4.1 *Let (X, G) be as defined above. Then (X, G) satisfies the following two properties.*

(BN1) *G acts 2-transitively on X, and $|X| > 2$;*

(BN2) *for every $L_i \in X$, the stabilizer of L_i in H has a normal subgroup $H(u_i, w_i)$ which acts regularly on $X \setminus \{L_i\}$.*

Proof. This follows immediately from the definition of an HPMGQ. ∎

Hence (X, G) is a finite group with a split BN-pair of rank 1. We now invoke the classification of the latter groups to obtain:

Proposition 13.4.2 *Let $\mathcal{S}$ be a thick HPMGQ of order (s, t) and let (X, G) be as above. Denote by N the kernel of the action of G on X. Then the permutation group $(X, G/N)$ is either a sharply 2-transitive group, or one of natural modules of degree $s + 1$ of the groups $\mathbf{PSL}(2, s)$, $\mathbf{R}(\sqrt[3]{s})$ (a Ree group in characteristic 3), $\mathbf{Sz}(\sqrt{s})$ (a Suzuki group; $\sqrt{s}$ is an odd exponent of 2), $\mathbf{PSU}(3, \sqrt[3]{s^2})$.*

Proof. This follows immediately from Lemma 13.4.1 and Theorem 8.3.1. ∎

We will denote the group $H(u_i, w_i)$ also by $G(L_i)$ and call it a 'root group' of $(X, G/N)$, or of G/N.
Proposition 13.4.2 allows us to divide the rest of the proof into different cases according to the isomorphism class of G/N. We will first treat the sharply 2-transitive case, and then treat the other cases separately for $s \leq t$ and $s > t$. From now on, we use the notation introduced in the beginning of this section.

13.5 The Sharply 2-Transitive Case

In this section, we assume that $(X, G/N)$ is a sharply 2-transitive permutation group. We will denote by $\Gamma(X)$ the set of points on the lines of X. Also, we assume that $\mathcal{S}$ is non-classical, in particular, $s > 3$, see Section 1.3.

Before proceeding, we recall the following.

Theorem 13.5.1 (FGQ, 9.4.1) *Let $\mathcal{S}$ be a GQ of order (s,t), $s \neq 1 \neq t$. Suppose that Ω and Δ are disjoint sets of points of $\mathcal{S}$. Suppose that K is a group of collineations of $\mathcal{S}$ which acts on Ω, but not transitively. Suppose the following conditions are satisfied:*

1. *$|\Delta| > 2$;*
2. *$|K_y|$ is independent of y for $y \in \Omega$;*
3. *each element of Ω is collinear with a constant number of points of Δ;*
4. *if x and y are points of Δ, then there is a sequence of points $x = z_1, z_2, \ldots, z_n = y$ so that $z_1, z_2, \ldots, z_n \in \Delta$, and $z_i \not\sim z_{i+1}$ for $i = 1, 2, \ldots, n-1$;*
5. *if v is a point of Ω and v' is a point of Δ for which $v \sim v'$, then each point of vv' which is different from v' is a point of v^K.*

Then $|\Delta| \leq t + 1 + \overline{b}$, where $\overline{b}$ is the average number of points of Δ different from d which are collinear with a given point $d \in \Delta$.

Lemma 13.5.2 *If (s,t) is the order of $\mathcal{S}$, then $s \leq t$.*

Proof. Suppose by way of contradiction that $s > t$. Let Ω be the set of points of $\mathcal{S}$ which are not on a line of X. Also, define Δ as the set of points which are incident with U and W. Now suppose that G does not act transitively on Ω. Then all conditions of Theorem 13.5.1 above are clearly satisfied ($|G_y|$ is independent of $y \in \Omega$ since G_y fixes a subquadrangle of order (s',t) pointwise, but then by Theorem 1.6.2, $s' = s$, hence G_y is always trivial), and with $\overline{b} = s + 1$, we obtain $2(s+1) \leq t + 1 + (s+1)$, and hence $s \leq t$, a contradiction. So G acts transitively on Ω. Suppose that some nontrivial element θ of G fixes a point v of Ω. Then θ fixes $proj_U v$ and $proj_W v$; hence θ fixes two elements of X and so $\theta \in N$. This implies that θ fixes a subGQ of order (s,t'), contradicting $s > t$ and Theorem 1.6.2. So G acts regularly on Ω and $|G| = (1+s)(1+st) - (1+s)^2 = (t-1)(s+1)s$. This implies $|N| = t-1$, and hence N is a group of generalized homologies with axes U and W which acts regularly on the lines through each point of U and W which are not contained in $\{U, W\} \cup X$ (indeed, if some element of N would fix such a line, then again a subGQ of order (s,t') with $1 < t' < t$ arises). Hence by J. A. Thas [170, 171], $\mathcal{S}$ is classical, a contradiction to our assumptions. ∎

As an immediate corollary of the first part of the proof of the previous lemma, we obtain, with the same notation:

Corollary 13.5.3 *If $\mathcal{S}$ is a thick HPMGQ of order (s,t) with $s > t$, then G acts transitively on Ω.*

■

Lemma 13.5.4 *If (s,t) is the order of $\mathcal{S}$, then $s \neq t$.*

Proof. Suppose by way of contradiction that $s = t$. By Corollary 13.3.5, the span of any pair of non-collinear points has size 2 and by Corollary 13.3.8, the span of every pair of non-concurrent lines has (constant) size $r+1 \geq 3$. Put $\{U,W\}^{\perp\perp} = \{L'_0, L'_1, \ldots, L'_r\}$, where $r \geq 2$. Define p_{ij}, $i \in \{0,1,\ldots,s\}$ and $j \in \{0,1,\ldots,r\}$, by $p_{ij} = u_i w_i \cap L'_j$. Next define the following group Φ:

$$\Phi = \langle H(p_{ij}, p_{ik}) \parallel i \neq j \neq k \neq i, 0 \leq i \leq s, 0 \leq j,k \leq r \rangle.$$

Then note the following properties:

(i) Φ acts on the lines $\{u_0w_0, u_1w_1, \ldots, u_sw_s\} = \{U,W\}^{\perp}$;

(ii) Φ acts on the lines $\{L'_0, L'_1, \ldots, L'_r\}$.

Moreover

(ST) *The permutation group $(\{U,W\}^{\perp}, \Phi)$ is sharply 2-transitive.*

This follows inductively from the following direct property:

(I) *The action of $H(p_{i0}, p_{i1})$ and $H(p_{i1}, p_{i2})$, $0 \leq i \leq s$, is exactly the same (that is, if $\theta \in H(p_{i0}, p_{i1})$, respectively $\theta' \in H(p_{i1}, p_{i2})$, maps u_jw_j onto u_kw_k, then $\theta = \theta'$ in their action on $\{U,W\}^{\perp}$).*[1]

Let Ω be the set of points of $\mathcal{S}$ which are not on a line of X. Define Δ as the set of points which are incident with an element of $\{U,W\}^{\perp\perp}$. Suppose that G does not act transitively on Ω.
Clearly all conditions of Theorem 13.5.1 are satisfied, and hence, with $\overline{b} = s+r$, we obtain $(r+1)(s+1) \leq s+1+(s+r)$, and hence $rs \leq s$, a contradiction. So G acts transitively on Ω. But, as G acts sharply 2-transitively on X, as in the proof of Lemma 13.5.2, we obtain a group N of $s-1$ generalized homologies with axes U and W. As before, this leads to a contradiction. ■

We can now finish the sharply 2-transitive case.

[1] Note that θ and θ' both are elements of the full group of elations of $\mathcal{S}$ about p_{i1}.

Proposition 13.5.5 *If G/N acts sharply 2-transitively on X, then $\mathcal{S}$ must be classical or dual classical.*

Proof. We assume that $\mathcal{S}$ is not classical or dual classical. By the previous results, we may assume $s < t$. Let i, j, k be distinct elements of $\{0, 1, \ldots, s\}$. We claim first that $H(u_i, w_i)H(u_j, w_j) \cap H(u_k, w_k)$ is trivial. Indeed, suppose $\theta_i\theta_j = \theta_k$, with $\theta_\ell \in H(u_\ell, w_\ell)$, $\ell = i, j, k$. Clearly, we may assume that none of $\theta_i, \theta_j, \theta_k$ is trivial. Let Y be the set of lines through u_k different from U and L_k. Then for every $M \in Y$, there is a line M_iIw_i which meets both M and M^{θ_i}, and there is a line M_jIw_j which meets M^{θ_i} and $M^{\theta_i\theta_j} = M^{\theta_k} = M$. Hence, there exists a $\sigma_M \in H(w_i, w_j)$ mapping M onto M^{θ_i}. Since $|H(w_i, w_j)| - 1 = s - 1 < t - 1 = |Y|$, there exist $M, M' \in Y$ so that $\sigma_M = \sigma_{M'}$. Hence σ_M maps $M \cap M' = u_k$ onto $M^{\theta_i} \cap (M')^{\theta_i} = u_k^{\theta_i}$. Hence σ_M maps $proj_{u_k} L_i = U$ onto $proj_{u_k^{\theta_i}} L_i = U$, contradicting the regular action of $H(w_i, w_j)$ on $\{L_i, L_j\}^\perp \setminus \{W\}$. The claim is proved.

Next, we claim that, if $\theta := \theta_i\theta_j$ fixes L_k, then θ acts semiregularly on Y (same notation as above). Indeed, suppose by way of contradiction that θ fixes a line $L \in Y$. Let $\sigma \in H(u_k, w_k)$ be such that $\sigma N = \theta N$. Then $\sigma\theta^{-1} \in N$ and fixes L, but it is not trivial by the first part of the proof. Hence $\sigma\theta^{-1}$ fixes a subquadrangle $\mathcal{S}'$ of order (s, t'), with $t' \leq s$. By Proposition 13.3.9, $\mathcal{S}'$ is an HPMGQ and clearly the corresponding group G'/N' acts sharply 2-transitively on X (because $G = G'$ and $N = N'$). By the foregoing results, we conclude that $\mathcal{S}'$ is classical, and so using the second statement of Proposition 13.3.9, we conclude that $\mathcal{S}$ is classical, a contradiction. The claim follows.

Now notice that for every $\ell \in \{0, 1, \ldots, s\} \setminus \{i, j, k\}$, where i, j, k are arbitrary but mutually different and where $L_{k'} = L_k^{\theta_i}$, there exists θ_ℓ such that $\sigma_\ell := \theta_i\theta_\ell$ fixes L_k. Suppose there exists $L \in Y$ such that for two such ℓ, ℓ' the automorphisms σ_ℓ and $\sigma_{\ell'}$ have the same action on L, i.e. $L^{\sigma_\ell} = L^{\sigma_{\ell'}}$. Then L^{θ_i} is fixed by $\theta_\ell\theta_{\ell'}^{-1}$, contradicting our previous claim. Hence N has orbits of length at least $s - 1$ on Y.

By Corollary 13.3.5, the span of any pair of non-collinear points has size 2 and by Corollary 13.3.8, the span $\{U, W\}^{\perp\perp}$ has size at least 3. If $|\{U, W\}^{\perp\perp}| = s + 1$, then all lines are regular and $\mathcal{S}$ is half Moufang by Chapter 5, hence classical. So there is a line M meeting at least two elements of $X \setminus \{L_k\}$, and not meeting L_k. Let $L \in Y$ be concurrent with M. Clearly, if $\theta \in N$ and $L^\theta \neq L$, then $M^\theta \neq M$ (otherwise there arises a triangle). So the orbit of M under N has size at least $s - 1$, which easily implies that $\{U, W\}^{\perp\perp} = \{U, W\}$, a final contradiction.

We conclude that $\mathcal{S}$ must be classical. Note that only the values $s = 2, 3$ turn up here. ∎

We give a slightly different proof of the sharply 2-transitive case with the explicit use of 4-gonal bases.

By the preceding observations, we can assume that $s < t$.

G ACTS SEMIREGULARLY ON THE POINTS OF $\mathcal{S} \setminus \Gamma(X)$. Suppose that $\theta \in G$ is a non-identical element which fixes some point $z \in \mathcal{S} \setminus \Gamma(X)$. Then $proj_U z$ and $proj_W z$, and hence also the (distinct) lines $ZIproj_U z$ and $Z'Iproj_W z$ which are contained in $X = \{U, W\}^\perp$ are fixed by θ. As $(\{U, W\}^\perp, G/N)$ is a sharply 2-transitive permutation group, there follows that θ fixes each line of $\{U, W\}^\perp$. By Theorems 1.6.2 and 2.4.1, the fixed element structure $\mathcal{S}_\theta$ of θ is a thick subGQ of order (s, t'), $t' \leq s < t$. But G induces a sharply 2-transitive group on $\{U, W\}^\perp$ in $\mathcal{S}_\theta$, which leads to $s \leq 3$, as $t' \leq s$.

$|G| \geq s^3 - s$. Suppose x is a point of $\mathcal{S} \setminus \Gamma(X)$, and consider the G-orbit Λ which contains x. Consider $x^{H(u_0,w_0)}$ — and note that this orbit has s points — and put $x^{H(u_0,w_0)} = \{z_1 = x, z_2, \ldots, z_s\}$. Then note that all the orbits $z_i^{H(u_j,w_j)}$, where $i \in \{1, 2, \ldots, s\}$ and $j \in \{1, 2, \ldots, s\}$, are completely contained in Λ. Suppose there is a point y in $\Lambda \setminus x^{H(u_0,w_0)}$ so that $y^\sigma = y^{\theta\phi}$, where σ, θ and ϕ are nontrivial elements of respectively $H(u_0, w_0), H(u_j, w_j)$ and $H(u_k, w_k)$, where $0, j$ and k are mutually distinct. Then the fact that G acts semiregularly on $\mathcal{S} \setminus \Gamma(X)$ implies that $\sigma = \theta\phi$, a contradiction. Hence none of these orbits mutually intersect. Hence counting the points of the orbits $z_i^{H(u_j,w_j)}$, we obtain

$$|\Lambda| \geq s + s^2(s-1) = s^3 - s^2 + s.$$

Suppose n is the number of lines through every point of W which are completely contained in Λ. If we count in two ways the number of point-line pairs (y, M) for which $y \in \Lambda, M \sim W$ and yIM, then we have that $n(s+1)s = |\Lambda| \geq s(s^2 - s + 1) \Longrightarrow n \geq \frac{s^2-s+1}{s+1} = s - 2 + \frac{3}{s+1}$, and hence, since $n \in \mathbb{N}$, we have that $n \geq s - 1$. Thus $|\Lambda| \geq s^3 - s$ and so also $|G|$.

EACH SPAN OF NON-COLLINEAR POINTS OF $\mathcal{S}$ HAS SIZE 2. Immediate by Corollary 13.3.5 and the fact that $s < t$.

EACH POINT x IS AN ELATION POINT. Immediate by the preceding point and Theorem 1.7.2 (v).

$|G| = s^3 - s$ OR $s \in 2,3$. Suppose $|N| > s-1$, and fix some line $L = L_i$ of $\{U,W\}^\perp$, $i \in \{0,1,\ldots,s\}$ (each line of the latter set is fixed by each element of N). As $|N| > s-1$, we clearly have two possibilitties:

(i) $\{U,W\}$ (and then also $\{L,M\}$) is a regular pair of lines;

(ii) there is a point $l \in L_i \setminus \{u_i, w_i\}$ so that no line of $\{U,W\}^{\perp\perp}$ is incident with l, and there must be some nontrivial element $\theta \in N$ which fixes l.

Consider Case (i). Then by Chapter 5, each line of $\mathcal{S}$ is an axis of symmetry (recall that each point of $\mathcal{S}$ is an elation point), hence $\mathcal{S}$ is half Moufang and thus Moufang by Theorem 1.4.2. The result easily follows. Suppose that we are in Case (ii). As there is no line of $\{U,W\}^{\perp\perp}$ incident with l, we have that the fixed element structure of θ is a (proper) subGQ of $\mathcal{S}$ order (s,t') (use again Theorems 1.6.2 and 2.4.1), with $t' \leq s$ and $t' > 1$ (the latter because otherwise we are in Case (i)). Theorem 13.3.9 (i) asserts that $\mathcal{S}'$ also is an HPMGQ, and hence $s = 2,3$. Hence $|N| = s-1$. There thus follows that $|G|$ indeed equals $s^3 - s$.

WE HAVE THAT $s \in \{2,3\}$. Now suppose that $|G| = s^3 - s$. Then G and $H(u_0,w_0), H(u_1,w_1), \ldots, H(u_s,w_s)$ have the following properties:

1. $|G| = s^3 - s$;
2. the groups $H(u_0,w_0), H(u_1,w_1), \ldots, H(u_s,w_s)$ form a complete conjugacy class in G and $|H(u_i,w_i)| = s$ for all feasible i;
3. $N_G(H(u_i,w_i)) \cap H(u_j,w_j) = \{\mathbf{1}\}$ for distinct i and j in $\{0,1,\ldots,s\}$;
4. $H(u_i,w_i)H(u_j,w_j) \cap H(u_k,w_k) = \{\mathbf{1}\}$ for distinct i,j and k.

Hence G is a group with a 4-gonal basis, and by Chapter 8, we have that $G \cong \mathbf{SL}(2,s)$, and $\mathbf{SL}(2,s)$ only acts sharply 2-transitively on X if $s = 2,3$. ■

13.6 The Case $s \leq t$

We use the same notation as in the previous section. In particular, $\mathcal{S}$ is an HPMGQ of order (s,t) with respect to the group H, and G is the group generated by all $H(u_i,w_i)$, $0 \leq i \leq s$. The kernel of the action of G on the set of lines $X = \{U,W\}^\perp$ is denoted by N. By Lemma 13.3.8, we may assume without loss of generality that each span of non-concurrent lines has at least

size 3. Also, by Section 13.5, we can assume that G/N does not act sharply 2-transitively on X. We denote G/N by K.
We will need the following lemma, which is essentially about finite rank 2 Chevalley groups.

Lemma 13.6.1 *Suppose that (X, K) is permutation equivalent to the action of $\mathbf{PSL}(2,q)$, $\mathbf{PSU}(3,q^2)$, $\mathbf{Sz}(q)$ or $\mathbf{R}(q)$, q an arbitrary prime power (with q not even in the case of a Ree group and q not odd in the case of a Suzuki group), on the natural module (respectively of size $q+1$, q^3+1, q^2+1 or q^3+1). Let $Y \subseteq X$, with $|Y| \geq 3$, be such that, for any $g \in K$, $|Y \cap Y^g| \geq 2$ implies $Y = Y^g$. Then either $Y = X$, or we have one of the following:*

(*i*) *$K \cong \mathbf{PSU}(3,q^2)$, $|Y| = q+1$ and the stabilizer of Y in K contains a group isomorphic to $\mathbf{PGL}(2,q)$ with its natural action on Y;*

(*ii*) *$K \cong \mathbf{R}(q)$, $|Y| = q+1$ and the stabilizer of Y in K is isomorphic to $2 \times \mathbf{PSL}(2,q)$ with its natural action on Y.*

Proof. Let $y_0, y_1 \in Y$, $y_0 \neq y_1$. The assumption implies that $Y \setminus \{y_0\}$ is the image of y_1 under a subgroup of the root group $K(y_0)$ (with obvious notation). Also, if $y_2 \in Y \setminus \{y_0, y_1\}$, then Y contains the image of y_2 under the stabilizer in K of $\{y_0, y_1\}$. If $K \cong \mathbf{PSL}(2,q)$, the second observation readily implies that $|Y| \geq \frac{q+1}{2}$, and the first one then yields $Y = X$. If $K \cong \mathbf{PSU}(3,q^2)$ or $K \cong \mathbf{R}(q)$, then X carries the extra structure of a unital[2], see [229], and if Y is contained in a block of that unital, then since the stabilizer in K of a block contains a group isomorphic to $\mathbf{PSL}(2,q)$ in its natural action, we deduce from the previous case that Y coincides with that block. If Y is not contained in a block, then we now show that Y contains all elements of the block determined by any two elements of Y. By the previous observation, it suffices to show that Y contains three points of a common block. So we may assume that y_2 does not belong to the block β defined by y_0 and y_1. We may also assume by way of contradiction that no block meets Y in more than 2 points. If $K \cong \mathbf{PSU}(3,q^2)$, then the stabilizer of y_0 and y_1 — which has size q^2-1 — acts semiregularly on the points off β, and hence Y contains at least q^2+1 points. Considering the set of q^2 blocks through any point contained in Y, we see that Y contains exactly q^2+1 points. By the 2-transitivity of K, we obtain a 2-transitive subgroup of K acting on Y. Hence K contains a subgroup whose order is divisible by q^2+1, but this is a contradiction, as q^2+1 can never divide the order of K, which is $q^3(q^3+1)(q^2-1)$.

[2] A *unital of order q* is just a $2-(q^3+1, q+1, 1)$-design.

Now suppose that $K \cong \mathbf{R}(q)$. In this case, we have to make an explicit computation. Therefore, we take the notation of Section 7.7.7 of [229]. We put y_0 equal to the point (∞), and y_1 is the point with coordinates $(0,0,0)$. Every other point has coordinates $(x,x',x'') \in \mathbf{GF}(q) \times \mathbf{GF}(q) \times \mathbf{GF}(q)$ and can be identified with the unique root element with respect to (∞) mapping $(0,0,0)$ to (x,x',x''). This way, we obtain a group with action

$$(x,x',x'').(y,y',y'') = (x+y, x'+y'+xy^\theta, x''+y''+xy'-x'y-xy^{\theta+1}),$$

where θ is the square root of the Frobenius endomorphism $x \mapsto x^3$. We now look at the possibilities for y_2. Note that, if $y_2 = (0,x',0)$, with $x' \in \mathbf{GF}(q) \setminus \{0\}$, then y_2 belongs to the block determined by y_0 and y_1 and the result follows. Also remark that, if two elements (x,x',x'') and (y,y',y'') belong to Y, then, since $(0,0,0) \in Y$, the product $(x,x',x'').(y,y',y'')$ belongs to Y. We will frequently use this observation without further notice. Also, for any $z \in \mathbf{GF}(q) \setminus \{0\}$ and any $(x,x',x'') \in Y$, the element $(zx, z^{\theta+1}x', z^{\theta+2}x'')$ belongs to Y (since the corresponding mapping belongs to K and fixes both y_0 and y_1). Suppose now that $y_2 = (0,x',x'')$ with $x' \neq 0$. Putting $z = -1$ in the previous remark, we obtain $(0,x',-x'') \in Y$, and hence also $(0,x',x'').(0,x',-x'') = (0,-x',0)$ belongs to Y, thus three points of the block through y_0 and y_1 belong to Y, and the result follows. Suppose next that $y_2 = (x,x',x'')$ with $x \neq 0$. Performing the same trick as before, we deduce that $(x,x',x'').(-x,x',-x'') = (0,-x'-x^{\theta+1},-xx'-x^{\theta+2})$ belongs to Y. If $x' \neq -x^{\theta+1}$, then the result follows from the previous case. Suppose now $x' = -x^{\theta+1}$. If $q \neq 3$, then we can select $z \in \mathbf{GF}(q)$ such that $z \neq z^\theta$. One calculates that, replacing (x,x',x'') by $(x,x',x'').(zx, z^{\theta+1}x', z^{\theta+2}x'')$, the sum $-x'-x^{\theta+1}$ is not zero, and hence the result follows. If $q = 3$, then $(1,-1,x'') \in Y$ easily implies (by consecutively combining with itself) that $|Y| \geq 10 = 3^2+1$ (since $(1,-1,x'')$ has order 9 in K). Similarly as above in the case of $\mathbf{PSU}(3,q^2)$, this implies $|Y| = 10$ and 27.28.2 must be divisible by 10.9, a contradiction. Finally, if $y_2 = (0,0,x'')$, then we first perform a transformation that interchanges (∞) and $(0,0,0)$. One can check that the image of an element $(0,0,x'')$, with $x'' \neq 0$, under that transformation has the form (y,y',y'') with $(y,y') \neq (0,0)$. The result now follows from th previous cases.

Hence we obtain a subdesign Y implying that $|Y|(|Y|-1)$ is divisible by $(q+1)q$ (counting the number of blocks in the subdesign) and q divides $|Y|-1$ (counting the blocks through a fixed point). Our assumption also implies that the orbit of Y under K is the block set of a linear space with point set X, implying that $|Y|(|Y|-1)$ divides $(q^3+1)q^3$ (again counting all lines of the linear space), and $|Y|-1$ divides q^3 (counting lines through a fixed point). We easily deduce

that, putting $q = p^n$, for some prime p, $|Y| - 1$ must be equal to p^h, for some positive integer h, with $n < h \leq 3n$, and $n|h$ and $h|3n$. This yields $h = 3n$, hence $Y = X$. If $H \cong \mathbf{Sz}(q)$, then let Y_0 be the orbit of y_1 under H^{y_0}. The orbits of y_0 under H^y, with $y \in Y_0$, meet pairwise in Y_0 (this follows for instance from [228], where these orbits correspond to the circles with corner y), and hence Y contains at least $q^2 + 1$ elements, implying $Y = X$.
The lemma is proved. ∎

The next proposition takes care of the case $s \leq t$. But it is valid under slightly more general hypotheses, which we will need in the case $s > t$. Therefore we state it in this general context.

Proposition 13.6.2 *Let $\mathcal{S}$ be a thick HPMGQ of order (s,t) such that the span of every pair of non-concurrent lines has size at least 3 (which we in particular may assume if $s \leq t$). Then $\mathcal{S}$ is classical or dual classical.*

Proof. Let us denote G/N briefly by K. Note that K acts on X as one of the groups in the previous lemma. Let $L, M \in X$, $L \neq M$. Put $Y = \{L, M\}^{\perp\perp}$. Then it is clear that Y satisfies the assumptions of Lemma 13.6.1, because each collineation g of $\mathcal{S}$ which stabilizes two elements of Y must stabilize the perp of these two elements, which coincides with $\{L, M\}^{\perp}$. If $Y = X$, then all lines of $\mathcal{S}$ are regular, and the result follows from Chapter 5. Hence we may assume from now on that $Y \neq X$. So K is isomorphic to either $\mathbf{PSU}(3, q^2)$ or $\mathbf{R}(q)$, with $q^3 = s$, and every span of non-concurrent lines has size $q+1$. The group G acts on the $q-1$ elements of $\{U, W\}^{\perp\perp} \setminus \{U, W\}$. Hence, this set contains an orbit of length d coprime to q. Such an orbit consequently contains an element Z fixed by some Sylow p-subgroup of G, where q is a power of p. This in turn implies that there is some line in X, which we may without loss of generality assume to be L, such that the group $H(L)$ fixes U, W and Z. Lemma 13.3.7 implies that every element of $H(L)$ is an elation about $Z \cap L$. Let $Y_\perp \subseteq \{U, W\}^{\perp\perp}$ be the set of lines fixed by every element of $H(L)$. Consider the group $G_\perp$ generated by the groups $H(M \cap U, L \cap U)$ and $H(M \cap W, L \cap W)$, and let $N_\perp$ be the kernel of the action of $G_\perp$ on $\{L, M\}^{\perp}$. Then by the foregoing, $G_\perp / N_\perp$ contains a subgroup $K_\perp^*$ isomorphic to $\mathbf{PSL}(2, q)$ acting on $\{U, W\}^{\perp\perp}$ in its natural action, and it is not so hard to see that the set $Y_\perp$ satisfies the assumptions of Lemma 13.6.1. This implies that $Y_\perp$ equals $\{U, W\}^{\perp\perp}$. Hence every element of $H(L)$ fixes every line meeting L in a point of the set $A := \{L \cap Z \parallel Z \in \{U, W\}^{\perp\perp}\}$.
Now let M' be an arbitrary line not concurrent with L in $\mathcal{S}$. By considering the images of M' under the elements of $H(L)$, we see that every line meeting every element of $\{L, M'\}^{\perp}$ contains an element of A. Now Property (H) for

the line L follows. By transitivity, every line has Property (H), and so, since no line is regular and all line spans have size $q+1 \geq 3$, we conclude that $\mathcal{S}$ is isomorphic to the dual of $H(4,t)$. ■

13.7 The Case $s > t$

We continue with the same notation as before. The assumptions now are that $\mathcal{S}$ has order (s,t) with $s > t$ and that the group $K = G/N$ does not act sharply 2-transitively on X.
We distinguish two different cases.

Proposition 13.7.1 *If every point of $\mathcal{S}$ is regular, then $\mathcal{S}$ is classical.*

Proof. First we assume that there is a subquadrangle $\mathcal{S}'$ of order (s',t), with $1 < s' < s$. Then by Theorem 1.6.2, $s = t^2$ and $s' = t$ (as $t' \leq s$, since $\mathcal{S}'$ has regular points). Also, $\mathcal{S}'$ is classical and isomorphic to $W(t)$ by Theorem 1.2.2. Without loss of generality we may assume that $\mathcal{S}'$ contains U, W, L_0, L_1. Since the group G'/N induced by G/N on $\mathcal{S}'$ contains a group isomorphic to $\mathbf{PSL}(2,t)$, the order of the latter must divide $|G/N|$; after a quick and elementary inspection, this leaves only the possibility $G/N \cong \mathbf{PSL}(2,s)$. The action of the latter now implies the existence of at least one other subquadrangle $\mathcal{S}''$ isomorphic to $W(t)$ and containing U, W, L_0, L_1. By Proposition 13.3.10, $\mathcal{S}$ is classical.
Now suppose that $\mathcal{S}$ does not admit any subquadrangle of order (s',t), with $1 < s' < s$. The group $H(u_0)$ of whorls about u_0 generated by all groups $H(u_0,x)$, with $x \sim u_0$, $x \neq u_0$, acts transitively on the set of points of $\mathcal{S}$ not collinear with u_0. If an element of $H(u_0)$ fixes at least two distinct traces in $u_0^\perp$, then by Chapter 3, either it fixes a subquadrangle of order (t,t) — impossible by our assumption — or it fixes $u_0^\perp$ pointwise. Hence the action of $H(u_0)$ on the set of traces in $u_0^\perp$ has a Frobenius kernel and so we obtain a group F, which is a normal subgroup of $H(u_0)$, containing the normal subgroup $N' \trianglelefteq H(u_0)$ which fixes $u_0^\perp$ pointwise and, modulo N', acting regularly on the set of traces in $u_0^\perp$ (clearly, $H(u_0)$ acts transitively on those traces). It is easy to see that the group $H(u_0,w_0)$ must fix some point v_0 on L_0 with $u_0 \neq v_0 \neq w_0$. Also, clearly $H(u_0,w_0) \leq F$ (by the definition of Frobenius kernel), and so if some element $\theta \in H(u_0,w_0)$ fixes a line V through v_0, then the element θ' of $H(u_0,v_0)$ mapping u_1 onto u_1^θ has the same action on $u_0^\perp$ and $\theta'\theta^{-1}$ is a symmetry about u_0 fixing a line V not through u_0. Consequently $\theta'\theta^{-1}$ is the identity and θ fixes all lines through v_0. It follows immediately that s and t have a nontrivial common divisor.
We now distinguish two cases.

- s IS EVEN. As each point of $\mathcal{S}$ is regular, $t+1$ divides $s^2(s^2-1)$ by 1.5.1 (iv) of FGQ. Using the inequality of Higman, the fact that s and t have a nontrivial common divisor (say g) and the fact that $t < s$, we obtain that t cannot divide $s-1$. Hence by Chapter 3, there are nontrivial symmetries about u_0, and hence about each point. Also, as s is an even prime power, g is also a power of 2, and thus the group of symmetries about each point has even order. The result follows from C. E. Ealy, Jr. [56].

- s IS ODD. We know that $H(u_0, w_0)$ fixes the point v_0. Suppose M is an arbitrary line of $\mathcal{S}$ containing v_0 and different from L_0. Then by the dual of 1.5.1 (iii) of FGQ, there is at least one other line of $\{U, W\}^\perp$ which hits M (the triad $\{U, M, W\}$ has at least two centers). Hence $H(u_0, w_0)$ acts transitively on the lines of $\mathcal{S}$ through v_0 different from L_0. Hence t divides s, and it follows easily, in view of Chapter 3, that the group of symmetries about u_0 has order t.

Hence $\mathcal{S}$ is classical. ∎

Proposition 13.7.2 *If the span of every pair of non-collinear points has size 2, then $\mathcal{S}$ is classical or dual classical.*

Proof. First suppose that $\mathcal{S}$ does not admit any subquadrangle of order (s', t), with $1 < s' < s$. This immediately implies that, with the notation of the previous proof, and using Theorem 1.6.5, the group $H(u_0)$ acts as a Frobenius group on the set of points of $\mathcal{S}$ not collinear with u_0. Hence there is a Frobenius kernel F. Now suppose that $H(u_0)$ does not act regularly on the points non-collinear with u_0. Then by Hauptsatz 8.7 of [82], $H(u_0)$ is nilpotent, and hence has a nontrivial center. But then some pair of non-concurrent lines must have a span of size at least 3 by Proposition 13.3.6. Consequently $\mathcal{S}$ is classical or dual classical by Proposition 13.6.2. Now suppose that $H(u_0)$ does act regularly on the points non-collinear with u_0. It is clear by the preceding argument that we may assume that the full set of whorls about u_0 is a group of elations about u_0, otherwise $\mathcal{S}$ is classical. Suppose $z = w_j$ is an arbitrary point on W not collinear with u_0 (so $j \neq 0$), and put $\{u_0, z\}^\perp = \{z_0, z_1, \ldots, z_t\}$. Let G_i be the full group of elations about z_i, $i = 0, 1, \ldots, t$, and suppose G_i' is the subgroup of G_i fixing u_0 and z. Then clearly, $(\{u_0, z\}^\perp, \langle G_i' \parallel i \in \{0, 1, \ldots, t\}\rangle)$ is a group with a split BN-pair of rank 1 (the root groups are the G_i''s), which induces a sharply 2-transitive group T on the lines through u_0. But this case is ruled out easily after inspection of the possible automorphism groups which G_{L_0, L_j} (where $j \neq 0$), induces in the automorphism group of the sharply 2-transitive group T. Now suppose that $\mathcal{S}$ admits a subGQ $\mathcal{S}'$ of order (s', t),

with $1 < s' < s$, which is half pseudo Moufang by Proposition 13.3.10. By Section 13.5 and Proposition 13.6.2, $\mathcal{S}'$ is classical (as $s' \leq t$). So we have the following three possibilities:

(a) $\mathcal{S}' \cong \mathcal{Q}(4,t)$;

(b) $\mathcal{S}' \cong H(4, \sqrt[3]{t^2})$;

(c) $\mathcal{S} \cong \mathcal{Q}(5, \sqrt{t})$.

In Case (a), any line not belonging to $\mathcal{S}'$ subtends some spread of $\mathcal{S}'$ implying by Theorem 1.12.2 that t is even; but then each span of non-collinear points has size t, a contradiction. Case (b) is not possible as each span of non-collinear points of $H(4, \sqrt[3]{t^2})$ has size $\sqrt[3]{t} + 1$, see Section 1.2.3. Finally in Case (c), putting $q^2 = t$, the stabilizer K_0 of L_0 in the group G/N must have orbits whose union has size $q + 1$ (otherwise $\mathcal{S}'$, which can be assumed to contain U, W, L_0, L_1 has an image containing L_0, L_1, U, W distinct from $\mathcal{S}'$, and the result follows by Proposition 13.3.10). Also, clearly q divides $|H(u_0, w_0)|$. By inspection, this immediately implies that G/N is isomorphic to either $\mathbf{PSU}(3, q^2)$ or $\mathbf{R}(q)$. Hence $(s,t) = (q^3, q^2)$ and $\mathcal{S}$ is dual to $H(4,s)$ by the fact that H acts transitively on the pairs of non-collinear points and applying the dual of Appendix B of [200].
This completes the proof of the proposition. ∎

The proof of the Main Result is thus complete.

To end this chapter, we give an alternative proof of the case $t < s$, but with the additional assumption that G acts transitively on the points of $\mathcal{S} \setminus \Gamma(X)$. This point of view could be useful in the general context of Moufang conditions for finite GQ's. Note that the additional assumption is not so restrictive in view of Theorem 13.5.1. We continue with the same notation as before. The assumptions are that $\mathcal{S}$ has order (s,t) with $s > t$ and that the group $K = G/N$ does not act sharply 2-transitively on X.

We consider two cases.

Lemma 13.7.3 *If $|N| > 1$, then $\mathcal{S}$ is classical or dual classical.*

Proof. Let $i \in \{0, 1, \ldots, s\}$ be arbitrary. The stabilizer $G_{u_i,W} = G_{u_i}$ in G of u_i and W acts transitively on the set of lines through u_i different from U and from L_i. By switching the roles of U and an arbitrary line through u_i different from U and from L_i, we easily see that there is a subgroup $\overline{G}$ of H that fixes W

and u_i and acts doubly transitively on the lines through u_i different from L_i. Consider two distinct arbitrary lines L and L' through u, so that $L \notin \{L_i, U\}$ and $L' \notin \{L_i, L\}$. Suppose θ is an arbitrary nontrivial element of N, and assume that $L^\theta \neq L'$. Note that θ does not fix L, otherwise there arises a thick subGQ of order (s, t') with $1 < t' < t$, a contradiction to the assumption $s > t$. Let $\phi \in \overline{G}$ be such that $L^\phi = L$ and $L^{\theta\phi} = L'$. Then $L^{\phi^{-1}\theta\phi} = L'$ and $\phi^{-1}\theta\phi$ fixes W pointwise. It readily follows that there is a group of automorphisms of $\mathcal{S}$ which fixes W pointwise and which acts transitively on the lines through u different from L_i. By letting u vary on U, and U in $B \setminus W^\perp$ (where B is the line set of $\mathcal{S}$), we obtain that, using the result of A. Brouwer [19] that states that the geometry of points of $\mathcal{S} \setminus W$ and lines of $B \setminus W^\perp$ is connected, there is an automorphism group of $\mathcal{S}$ fixing W pointwise and acting transitively on the lines of $B \setminus W^\perp$. Hence W is a line of transitivity. So each line of $\mathcal{S}$ is a line of transitivity, and as we already knew that each point of $\mathcal{S}$ is a center of transitivity, we conclude by the first main result of [230] that $\mathcal{S}$ is a Moufang GQ, and hence $\mathcal{S}$ is classical or dual classical.

Lemma 13.7.4 *If $|N| = 1$, then $\mathcal{S}$ is classical or dual classical.*

Proof. Since N is trivial, the group G is isomorphic to either $\mathbf{PSL}(2, s)$, or to $\mathbf{PSU}(3, \sqrt[3]{s^2})$, or to $\mathbf{Sz}(\sqrt{s})$, or to $\mathbf{R}(\sqrt[3]{s})$.
Now, consider two points u_i and w_j, $0 \leq i < j \leq s$. Since G_{L_i,L_j} acts transitively on the set $\{u_i, w_j\}^\perp \setminus \{w_i, u_j\}$, and since any element of G stabilizing this set must fix L_i and L_j, we infer that $t - 1$ divides $|G_{L_i,L_j}|$. For $G \cong \mathbf{R}(\sqrt[3]{s})$, this implies that $t^3 \leq s$, contradicting the inequality of Higman. Also, if $G \cong \mathbf{Sz}(\sqrt{s})$, then $t^2 = s$, by the same inequality of Higman. Hence, putting $t = q$, $\mathcal{S}$ is a GQ of order (q^2, q), with q an even prime power. Suppose first that each point of $\mathcal{S}$ is regular. Fix a point x and two non-collinear points y, z in $x^\perp$. As in the proof of Proposition 13.3.10, there is a Sylow 2-subgroup S of the group of all whorls about x fixing y, z, and S acts transitively on $\{y, z\}^\perp \setminus \{x\}$. Then by Chapter 3, either x is a center of symmetry, and then $\mathcal{S}$ is half Moufang and hence classical, or there is some subGQ $\mathcal{S}'$ of order (q, q), which must necessarily be isomorphic to $W(q)$. Since $\mathcal{S}'$ is an HPMGQ with respect to the induced group, we infer that $\mathbf{PSL}(2, q) \leq \mathbf{Sz}(q)$, which implies that $q(q^2 - 1)$ divides $q^2(q^2 + 1)(q - 1)$, a contradiction. Hence we may assume, by Corollary 13.3.5, that the span of every pair of non-collinear points has size 2. Fix again a point x. If x is an elation point, then by Proposition 13.3.6, each span of non-concurrent lines has at least size 3. But this is in contradiction with the fact that $\mathcal{S}$ has order (q^2, q). Suppose some whorl about x fixes two points u, u' not collinear with x. Them since $u' \notin \{u, x\}^{\perp\perp}$, this whorl fixes a thick subquadrangle. Hence either the elations about x generate a Frobenius

group on the set of points not collinear with x, but then the Frobenius kernel is an elation group and x is an elation point, or there is some thick subGQ $\mathcal{S}'$ of $\mathcal{S}$ of order (s', q), $s' \leq q$. In this case Proposition 13.3.10 asserts that $\mathcal{S}'$ is an HPMGQ with respect to the induced group. Hence by Proposition 13.6.2, $\mathcal{S}'$ is classical or dual classical. So we have the following three possibilities (recall that q is even):

(a) $\mathcal{S}' \cong W(q)$;

(b) $\mathcal{S}' \cong H(4, \sqrt[3]{q^2})$;

(c) $\mathcal{S}' \cong \mathcal{Q}(5, \sqrt{q})$.

Case (a) is not possible, as each point of $W(q)$ is regular, and we assume that each span of non-collinear points of $\mathcal{S}$ has size 2. Case (b) is not possible for the same reason, as each span of non-collinear points of $H(4, \sqrt[3]{q^2})$ has size $\sqrt[3]{q} + 1$, see Section 1.2.3. Finally, Case (3) is impossible because q is an odd power of 2.
So we may assume that $G \cong \mathbf{PSL}(2, s)$, or $G \cong \mathbf{PSU}(3, \sqrt[3]{s^2})$. Fix two distinct lines L_i, L_j of X and put $K = G_{L_i, L_j}$. Then K is a cyclic group. Let N_K be the subgroup of K which fixes every line through u_i. The action of K/N_K on the set of lines $\mathcal{L}$ of $\mathcal{S}$ through u_i which are different from U and L_i is regular.

1. N_K IS NONTRIVIAL. Then by the uniqueness of N_K as a subgroup of K of order $|K|/(t-1)$, each element of N_K also fixes the lines through u_j. So each nontrivial element θ of N_K fixes a subGQ $\mathcal{S}'$ of $\mathcal{S}$ of order (s', t), $1 \leq s' \leq t$. First assume that $s' = 1$, for every element of N_K. Note that, if there is some symmetry about some point, then we may argue as in the previous lemma to prove that each point is a center of transitivity, and so $\mathcal{S}$ is Moufang. If there is some subquadrangle $\mathcal{S}'$ of order (s', t), then using the action of K, we see that there are at least two such subquadrangles intersecting in a dual grid, and we may argue as in the proof of Proposition 13.3.10 to obtain $\mathcal{S} \cong H(3, s)$.

   Hence, by Chapter 3, we may assume that the group H_0 of whorls about some fixed point x fixing two non-collinear points $y, z \in x^\perp$, acts semiregularly on the set of points of the line xy distinct from x and y. Since H_0 acts transitively on $\{y, z\}^\perp \setminus \{x\}$, and the corresponding stabilizer contains N_K, the order of H_0 is divisible by $t|N_K|$, hence $t|N_K|$ divides $s-1$. Also, clearly $t-1$ divides $|K|$. If $G \cong \mathbf{PSL}(2, s)$, then this implies that $t-1$ divides $s-1$. Hence in this case $t(t-1)$ divides $s-1$, so $t = \sqrt{s}$ by the inequality of Higman. But $\sqrt{s}(\sqrt{s}-1)$ never divides $s-1$. So we

may assume that $G \cong \mathbf{PSU}(3, \sqrt[3]{s^2})$. Put $r = \sqrt[3]{s}$. Here, the condition that $t|H_K|$ divides $s-1$ implies easily that $t|K|$ divides $(r^3-1)(t-1)$, or $t(r+1)$ divides $3(r^2+1+1)(t-1)$ which, in view of the fact that t and $t-1$ are relatively prime, means that t divides r^2+r+1 (noting that the factor 3 appears precisely when $r+1$ is divisible by it). We conclude that t and r are relatively prime, so the condition that $s+t$ divides $st(s+1)(t+1)$ (see Chapter 1) simplifies to $s+t$ divides $(s+1)(t+1)$, which is equivalent to $s+t$ divides t^2-1. Put $t^2-1 = k(r^3+t)$ and $\ell t = r^2+r+1$, with k, ℓ positive integers. Then $t^2 = (k(r-1)\ell+1)t + k + 1$, and so t divides $k+1$. This implies $t^2-1 \geq (t-1)(r^3+t)$, or $t+1 \geq r^3+t$, clearly a contradiction.

So we may assume that some $\theta \in N_K$ fixes a thick subGQ $\mathcal{S}'$ of order (s', t), $s' \leq t$. We then necessarily have $G \cong \mathbf{PSU}(3, \sqrt[3]{s^2})$ and θ fixes $\sqrt[3]{s}+1$ points on U. Hence $s' = \sqrt[3]{s}$. Clearly no point of $\mathcal{S}$ is regular as otherwise $s' = t = \sqrt[3]{s}$, contradicting the inequality of Higman. Consequently the span of every pair of non-collinear points has size 2. By Corollary 13.3.10, $\mathcal{S}'$ is also an HPMGQ with respect to the induced group. Also, by Theorem 1.6.2, $t \geq s' = \sqrt[3]{s}$. Whence Proposition 13.6.2 yields that $\mathcal{S}'$ is classical. As each span of non-collinear points of $\mathcal{S}'$ clearly also has size 2, and as $\sqrt{s} \leq t < s$, the only possibility for $\mathcal{S}'$ is $\mathcal{Q}(5, \sqrt[3]{s})$, Section 1.2.3. Hence $\mathcal{S} \cong H(4,t)^D$ by the fact that H acts transitively on the pairs of non-collinear points and applying the dual of Appendix B of [200].

2. N_K IS TRIVIAL. Then we have the following possibilities for t:

   - either $t = s$ (and $G \cong \mathbf{PSL}(2, s)$ with s even), or
   - $t = \frac{s+1}{2}$ (and $G \cong \mathbf{PSL}(2, s)$ with s odd), or
   - $t = \sqrt[3]{s^2}$ (and $G \cong \mathbf{PSU}(3, \sqrt[3]{s^2})$ with $s \not\equiv 2 \mod 3$), or
   - $t = \frac{\sqrt[3]{s^2}+2}{3}$ (and $G \cong \mathbf{PSU}(3, \sqrt[3]{s^2})$ with $s \equiv 2 \mod 3$).

   Of course $t = s$ contradicts our main assumption $s > t$ of this section. Suppose now first $t^3 = s^2$. If every point of $\mathcal{S}$ is regular, then we consider an arbitrary point x and two non-collinear points $y, z \in x^\perp$. As in the proof of Proposition 13.3.10, there is a Sylow 2-subgroup S of the group of all whorls about x fixing y, z, and S acts transitively on $\{y,z\}^\perp \setminus \{x\}$. Then by Chapter 3, either x is a center of symmetry, and then $\mathcal{S}$ is half Moufang and hence classical, or there is some thick subGQ $\mathcal{S}'$ of order (s', t), with $s't \leq s$ (by Theorem 1.6.1), so the inequality of Higman

implies $s' = \sqrt{t}$, contradicting the fact that $\mathcal{S}'$ has regular points. Hence every span of two non-collinear points has size 2, and if there is some elation point, then by Proposition 13.3.6, each span of two non-concurrent lines has size at least 3 (noting that s^2t is a prime power and hence every group of that order has a nontrivial center), and consequently $\mathcal{S}$ is classical or dual classical by Proposition 13.6.2. Hence, arguing as in the first main paragraph of the proof, there is some subquadrangle $\mathcal{S}'$ of order (s', t) with $1 < s' < s$. As above, we have $s' = \sqrt[3]{s}$. Also, $\mathcal{S}'$ is classical (by Corollary 13.3.10 and Proposition 13.6.2), and hence isomorphic to $\mathcal{Q}(5, \sqrt[3]{s})$. We again use Appendix B of [200] to conclude that $\mathcal{S}$ is classical.

Next, suppose $t = \frac{s+1}{2}$. Recall that $s+t$ must divide $st(s+1)(t+1)$. Since the greatest common divisor of $3s + 1$ and s, $s + 1$, $s + 3$, respectively, is equal to at most 1, 2 and 8, respectively, we infer that, since $4st(s + 1)(t + 1) = s(s + 1)^2(s + 3)$, $2st$ is at most equal to $2 \times 2 \times 8$, and it is a power of 2. But as $t = \frac{s+1}{2}$, s is odd, contradiction.

Finally, suppose $t = \frac{\sqrt[3]{s^2}+2}{3}$. Set $r^3 = s$, then $t = \frac{r^2+2}{3}$. As in the previous paragraph, we deduce that $3r^3+r^2+2$ must divide $r^3(r^2+2)(r^3+1)(r^2+5)$, hence $3r^2 - 2r + 2$ must divide $R := r^3(r^2 + 2)(r^2 - r + 1)(r^2 + 5)$. One calculates that $3r^2 - 2r + 2$ and each of the factors r^3, $r^2 + 2$, $r^2 - r + 1$, $r^2 + 5$, of R has as greatest common divisor a divisor of 2, 2, 1 and 189, respectively. Hence $3r^2 - 2r + 2$ divides 756, which implies $r = 4$ (remember that r is a prime power not divisible by 3). But if $r = 4$, then $(s, t) = (64, 6)$, contradicting the inequality of Higman.

The lemma is proved. ∎

This completely finishes the case $s > t$.

Chapter 14

An Application: Generalized Quadrangles with a Spread of Symmetry

Let $\mathcal{S}$ be a finite generalized quadrangle of order (s,t), $s \neq 1 \neq t$. A spread $\mathbf{T}$ of $\mathcal{S}$ is called a spread of symmetry if there is a group of automorphisms of $\mathcal{S}$ which fixes $\mathbf{T}$ elementwise and which acts transitively on the points of at least one line of $\mathbf{T}$. From spreads of symmetry of generalized quadrangles can be constructed near polygons, and new spreads of symmetry would yield new near polygons. In this chapter, we focus on spreads of symmetry in generalized quadrangles of order (s,s^2). The main tools will be the results and techniques obtained in, especially, Chapter 9. Many new characterizations of the classical generalized quadrangle $\mathcal{Q}(5,q)$ will be obtained. We point out that part of this chapter (up to Section 14.2) is mainly expository; it is our intention to describe in detail the interrelation between the study of spreads of symmetry of generalized quadrangles and that of near polygons[1], as it puts the results of

[1]Some of the elementary results presented in Sections 14.2 and 14.3.1 are also contained in [41], but the proofs are slightly different.

the present chapter in the right perspective.
One of the main purposes of this chapter is to apply some of the ideas obtained in the previous chapters.

The results of this chapter up to Section 14.7 stem from *B. De Bruyn and K. Thas, Generalized quadrangles with a spread of symmetry and near polygons* [45], which is accepted for publication in *Illinois Journal of Mathematics*.
From Section 14.7 on, the results are taken from *K. Thas, A Characterization of the Classical Generalized Quadrangle $\mathcal{Q}(5,q)$ and the Nonexistence of Certain Near Polygons* [216], which was submitted for publication to *Journal of Combinatorial Theory, Series A*.

REMARK

A "proof" of the main result of Section 14.4 already was written down in B. De Bruyn [41]. This proof appears to be wrong, as that author used a theorem of S. E. Payne of [125], which is stated incorrectly.

14.1 The Relation Between Spreads of Symmetry, Admissible Triples and Near Polygons

14.1.1 Some definitions

A *near polygon* is a partial linear space with the property that for every point p and every line L there exists a unique point on L nearest to p (with respect to the distance in the point graph G). If d is the diameter of G, then the near polygon is called a *near $2d$-gon*. A near 0-gon is a point, a near 2-gon is a line, and the class of the near 4-gons coincides with the class of the generalized quadrangles. Near 6-gons and near 8-gons are also called, respectively, *near hexagons* and *near octagons*. Also, generalized $2d$-gons and dual polar spaces are examples of near polygons. Near polygons were introduced by E. E. Shult and A. Yanushka in [160] because of the relationship with the so-called 'tetrahedrally closed systems of lines' in Euclidean spaces. For a survey on near polygons, see B. De Bruyn [43] and F. De Clerck and H. Van Maldeghem [46].

Two lines L and M of a near polygon are called *parallel* if $\mathrm{d}(L,m)$ is independent of the chosen point mIM. Clearly, every two disjoint lines of a generalized

quadrangle are parallel. A *spread* of a near polygon is a set of lines which partitions the point set. A *spread of symmetry* of a near polygon Γ is a spread $\mathbf{T}$ such that for every line $L \in \mathbf{T}$ and for every two points x and y of L, there exists an automorphism of Γ fixing each line of $\mathbf{T}$ and mapping x onto y. Clearly, every two lines of a spread of symmetry are parallel. If Γ is a GQ, then dually one defines *ovoids of symmetry*.

14.1.2 Spreads of symmetry and admissible triples

By [39], every spread of symmetry of a GQ can be derived from a so-called *admissible triple* in the way we will describe now.

A triple $T = (\mathcal{D}, H, \Delta)$ is called *admissible* if the following conditions are satisfied.

(1) $\mathcal{D}$ is a linear space of order $(s, t-1)$ with s and t nonzero positive integers. Let $\mathcal{P}$ denote the point set of $\mathcal{D}$.

(2) H is a (multiplicative) group of order $s+1$. Let $\mathbf{1}$ denote its identity element.

(3) Δ is a map from $\mathcal{P} \times \mathcal{P}$ to H such that x, y and z are collinear if and only if $\Delta(x,y)\,\Delta(y,z) = \Delta(x,z)$.

Let Γ be the graph on the vertex set $H \times \mathcal{P}$; two vertices (h_1, x) and (h_2, y) are adjacent if and only if:

(i) $x = y$ and $h_1 \neq h_2$, or

(ii) $x \neq y$ and $h_2 = h_1\,\Delta(x,y)$.

It is proved in B. De Bruyn [39] that Γ is the point graph of a generalized quadrangle $\mathcal{S}$ of order (s,t). The set $\mathbf{T} = \{L_x \parallel x \in \mathcal{P}\}$ with $L_x = \{(h,x) \parallel h \in H\}$ is a spread of $\mathcal{S}$ and we put $\Omega(T) := (\mathcal{S}, \mathbf{T})$. For every $h \in H$, the map $\theta_h : (g,x) \mapsto (h\,g, x)$, $g \in H$ and $x \in \mathcal{P}$, defines an automorphism of $\mathcal{S}$ that fixes each line of $\mathbf{T}$, proving that $\mathbf{T}$ is a spread of symmetry.

The following properties must be satisfied for an admissible triple $T = (\mathcal{D}, H, \Delta)$:

(A) $\Delta(x,x) = \mathbf{1}$ for all $x \in \mathcal{P}$;

(B) $\Delta(y,x) = [\Delta(x,y)]^{-1}$ for all $x, y \in \mathcal{P}$;

(C) if $\mathcal{S}$ is not a grid, then $H = \{\Delta(a,x)\,\Delta(x,y)\,\Delta(y,a) \parallel x,y \in \mathcal{P}\}$ for every $a \in \mathcal{P}$.

Proof. Property (A) is obtained by putting $x = y = z$ in (3). Property (B) is obtained by putting $z = x$ in (3). Let h be an arbitrary element of $H \setminus \{\mathbf{1}\}$, and let $a, x \in \mathcal{P}$, $a \neq x$. Since $\mathcal{S}$ is not a grid, $(\Delta(a,x),x)$ and (h,a) have a common neighbour (g,y) with $a \neq y \neq x$. We have $g = \Delta(a,x)\,\Delta(x,y)$ and $h = g\,\Delta(y,a)$, proving Property (C). ∎

Admissible triples (*AT's* for short) yield spreads of symmetry. It is possible however that two admissible triples, although different, yield 'equivalent spreads', see below. In the following paragraph, we examine why this happens and introduce the notion of *equivalent AT's*.

14.1.3 Equivalence of admissible triples

Let $\mathbf{T}_i$, $i \in \{1,2\}$, be a spread of a generalized quadrangle $\mathcal{S}_i$. Then we say that $(\mathcal{S}_1, \mathbf{T}_1)$ and $(\mathcal{S}_2, \mathbf{T}_2)$ are *equivalent* if and only if there exists an isomorphism from $\mathcal{S}_1$ to $\mathcal{S}_2$ mapping $\mathbf{T}_1$ onto $\mathbf{T}_2$.

Let $T_1 = (\mathcal{D}_1, H_1, \Delta_1)$ and $T_2 = (\mathcal{D}_2, H_2, \Delta_2)$ be two admissible triples. If $\mathcal{D}_1$ and $\mathcal{D}_2$ are two lines of the same length[2], then T_1 and T_2 are said to be *equivalent*. Otherwise, T_1 and T_2 are called *equivalent* if there exist:

1. an isomorphism from $\mathcal{D}_1$ to $\mathcal{D}_2$ determined by $\alpha : \mathcal{P}_1 \to \mathcal{P}_2$;
2. an isomorphism β from H_1 to H_2;
3. a map γ from $\mathcal{P}_1$ to H_1 such that $\Delta_2(\alpha(x),\alpha(y)) = \beta(\gamma^{-1}(x)\,\Delta_1(x,y)\,\gamma(y))$ holds for all $x, y \in \mathcal{P}_1$.

We now have

Theorem 14.1.1 *Two admissible triples T_1 and T_2 are equivalent if and only if $\Omega(T_1)$ and $\Omega(T_2)$ are equivalent.*

Proof. Put $T_i = (\mathcal{D}_i, H_i, \Delta_i)$ and $\Omega(T_i) = (\mathcal{S}_i, \mathbf{T}_i)$ for every $i \in \{1,2\}$. We may suppose that that $\mathcal{S}_1$ and $\mathcal{S}_2$ are not grids, or, equivalently, that $\mathcal{D}_1$ and $\mathcal{D}_2$ are not lines. If T_1 and T_2 are equivalent, let α, β and γ be as above. Then the map $(h,x) \mapsto (\beta(h\,\gamma(x)),\alpha(x))$ defines an isomorphism from $\mathcal{S}_1$ to

[2]That is, two sets of size $s+1$ without any further structure.

$\mathcal{S}_2$ mapping $\mathbf{T}_1$ onto $\mathbf{T}_2$.

Conversely, suppose that $(\mathcal{S}_1, \mathbf{T}_1)$ and $(\mathcal{S}_2, \mathbf{T}_2)$ are equivalent. We will freely use Properties (A), (B) and (C) of the previous section without further notice. Let θ be an isomorphism from $\mathcal{S}_1$ to $\mathcal{S}_2$ mapping $\mathbf{T}_1$ to $\mathbf{T}_2$. For a point x_i of $\mathcal{P}_i$, $i \in \{1,2\}$, let $L_{x_i}^{(i)} = \{(h, x_i) \parallel h \in H_i\} \in \mathbf{T}_i$. There exists a bijection $\alpha : \mathcal{P}_1 \to \mathcal{P}_2$ such that $\theta\left(L_{x_1}^{(1)}\right) = L_{\alpha(x_1)}^{(2)}$ for all $x_1 \in \mathcal{P}_1$. Since sets of the form $\{L, M\}^{\perp\perp}$ with $L, M \in \mathbf{T}_1$, are mapped by θ onto sets of the form $\{L', M'\}^{\perp\perp}$ with $L', M' \in \mathbf{T}_2$, α induces an isomorphism from $\mathcal{D}_1$ to $\mathcal{D}_2$. For every $x \in \mathcal{P}_1$, let β_x be the bijection from H_1 to H_2 such that $\theta[(h, x)] = (\beta_x(h), \alpha(x))$ for all $x \in \mathcal{P}_1$ and all $h \in H_1$. Consider now the adjacent points (h, x) and $(h\Delta(x, y), y)$ of $\mathcal{S}_1$; then $(\beta_x(h), \alpha(x))$ and $(\beta_y(h\Delta_1(x, y)), \alpha(y))$ are two adjacent points of $\mathcal{S}_2$. Hence $\beta_y(h\Delta_1(x, y)) = \beta_x(h)\Delta_2(\alpha(x), \alpha(y))$. Let $a \in \mathcal{P}_1$ be fixed and put $\bar{\beta} := \beta_a$, then $\beta_y(k) = \bar{\beta}(k\Delta_1(y, a))\Delta_2(\alpha(a), \alpha(y))$. Hence

$$\bar{\beta}(h\Delta_1(x, y)\Delta_1(y, a))\Delta_2(\alpha(a), \alpha(y)) = \bar{\beta}(h\Delta_1(x, a))\Delta_2(\alpha(a), \alpha(x))\Delta_2(\alpha(x), \alpha(y))$$

for all $h \in H$ and all $x, y \in \mathcal{P}_1$. Putting $h = \Delta_1(a, x)$, we find

$$\bar{\beta}(\Delta_1(a, x)\Delta_1(x, y)\Delta_1(y, a)) = \bar{\beta}(\mathbf{1})\Delta_2(\alpha(a), \alpha(x))\Delta_2(\alpha(x), \alpha(y))\Delta_2(\alpha(y), \alpha(a)).$$

Hence

$$\bar{\beta}(h\Delta_1(x, a)\Delta_1(a, x)\Delta_1(x, y)\Delta_1(y, a)) =$$

$$\bar{\beta}(h\Delta_1(x, a))[\bar{\beta}(\mathbf{1})]^{-1}\bar{\beta}(\Delta_1(a, x)\Delta_1(x, y)\Delta_1(y, a)).$$

Now, let h_1, h_1' be arbitrary elements of H_1. Since $\mathcal{S}_1$ is not a grid, we can choose the points x and y such that $\Delta_1(a, x)\Delta_1(x, y)\Delta_1(y, a) = h_1'$. Choose now h such that $h\Delta_1(x, a) = h_1$. Hence $\bar{\beta}(h_1h_1') = \bar{\beta}(h_1)[\bar{\beta}(\mathbf{1})]^{-1}\bar{\beta}(h_1')$ for all $h_1, h_1' \in H_1$. As a consequence the map $\beta : H_1 \to H_2, h \mapsto \bar{\beta}(h_1)[\bar{\beta}(\mathbf{1})]^{-1}$ is an isomorphism from H_1 to H_2. Hence $\Delta_2(\alpha(x), \alpha(y))$ is equal to

$$\left(\Delta_2(\alpha(x), \alpha(a))[\bar{\beta}(\mathbf{1})]^{-1}\beta(\Delta_1(a, x))\right)\beta(\Delta_1(x, y))\left(\beta(\Delta_1(y, a)\bar{\beta}(\mathbf{1})\Delta_2(\alpha(a), \alpha(y)))\right).$$

The theorem follows now if we put

$$\gamma : \mathcal{P}_1 \to H_1; x \mapsto \Delta_1(x,a)\beta^{-1}(\bar{\beta}(\mathbf{1})\Delta_2(\alpha(a),\alpha(x))).$$

■

14.1.4 The known admissible triples

Every known admissible triple is equivalent to one of the following examples (see [41]).

(1) Let $\mathcal{D}$ be the line of length $s+1$ and let H be the cyclic group of order $s+1$. Put $\Delta(x,y)$ equal to $\mathbf{1}$ for all points x and y of $\mathcal{D}$.

(2) Let $\mathcal{D}$ be the complete graph on $t+1$ vertices and let H be the group of order 2. Put $\Delta(x,y)$ equal to $\mathbf{1}$ if and only if $x = y$.

(3) Consider a nonsingular nondegenerate Hermitian form $(\cdot,\cdot)$ in the vector space $V(3,q^2)$ and let $\mathcal{U}$ be the corresponding Hermitian unital in $\mathbf{PG}(2,q^2)$. With this unital there is associated the following linear space $\mathcal{D}$.

- The POINTS of $\mathcal{D}$ are the points of $\mathcal{U}$;
- the LINES of $\mathcal{D}$ are all the sets of order $q+1$ arising as an intersection of $\mathcal{U}$ with lines of the projective plane $\mathbf{PG}(2,q^2)$.

Put $H = \{x \in \mathbf{GF}(q^2) \parallel x^{q+1} = \mathbf{1}\}$. Let $\alpha = \langle\bar{a}\rangle$ be a fixed point of $\mathcal{U}$. For every two points $\beta = \langle\bar{b}\rangle$ and $\gamma = \langle\bar{c}\rangle$ of $\mathcal{U}$, we define $\Delta(\beta,\gamma) = -(\bar{a},\bar{b})^{q-1}(\bar{b},\bar{c})^{q-1}(\bar{c},\bar{a})^{q-1}$ if $\alpha \neq \beta \neq \gamma \neq \alpha$; $\Delta(\beta,\gamma) = \mathbf{1}$ otherwise.

In the Examples (4), (5) and (6), the linear space $\mathcal{D}$ is the Desarguesian affine plane $\mathbf{AG}(2,q)$ and H is the additive group of $\mathbf{GF}(q)$.
In Examples (5) and (6) a function $f : \mathbf{GF}(q) \to \mathbf{GF}(q)$ occurs which satisfies one of the following two equivalent statements:

(I) the set $\mathcal{H} := \{(1,0,0),(0,1,0)\} \cup \{(f(\lambda),\lambda,1) \parallel \lambda \in \mathbf{GF}(q)\}$ is a hyperoval in $\mathbf{PG}(2,q)$ (hence q is even);

(II) $$\begin{vmatrix} f(\lambda_1) & \lambda_1 & 1 \\ f(\lambda_2) & \lambda_2 & 1 \\ f(\lambda_3) & \lambda_3 & 1 \end{vmatrix} \neq 0 \Leftrightarrow \lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1.$$

Now, let (α_1, β_1) and (α_2, β_2) be two arbitrary points of $\mathbf{AG}(2, q)$. Examples (4), (5) and (6) are then given by:

(4) we put $\Delta((\alpha_1, \beta_1), (\alpha_2, \beta_2)) = \alpha_1\beta_2 - \alpha_2\beta_1$;

(5) we put $\Delta((\alpha_1, \beta_1), (\alpha_2, \beta_2)) = (\alpha_1 - \alpha_2) f\left(\frac{\beta_1 - \beta_2}{\alpha_1 - \alpha_2}\right)$ if $\alpha_1 \neq \alpha_2$ and 0 otherwise;

(6) we put $\Delta((\alpha_1, \beta_1), (\alpha_2, \beta_2)) = (f(\alpha_1) - f(\alpha_2))\frac{\beta_1 - \beta_2}{\alpha_1 - \alpha_2}$ if $\alpha_1 \neq \alpha_2$ and 0 otherwise.

14.1.5 Spreads of symmetry in the known GQ's

If a generalized quadrangle $\mathcal{S}$ of order (s, t) has a spread of symmetry, then we have the following restrictions on the parameters, see [39]:

(R1) $s + 1 \mid t(t - 1)$;

(R2) $s + t \mid s(s + 1)(t + 1)$;

(R3) $s + 2 \leq t \leq s^2$ if $s \neq 1 \neq t$.

If $\mathcal{S}$ is one of the known GQ's, then (R1), (R2) and (R3) imply that either $s = 1$, $t = 1$, $(s, t) = (q - 1, q + 1)$ or $(s, t) = (q, q^2)$. Here q denotes an arbitrary prime power. If $t = 1$, then $\mathcal{S}$ is a grid and the corresponding AT is given in (1) of the previous section. If $s = 1$, then $\mathcal{S}$ is a dual grid and the corresponding AT is given in (2). Every known GQ of order $(q-1, q+1)$ has a spread of symmetry and the corresponding admissible triples are given in (4), (5) and (6); they respectively correspond to the GQ $\mathcal{S} = \mathcal{P}(W(q), x)$ (where $\mathcal{S} \cong AS(q)$ if q is odd, and $\mathcal{S} \cong T_2^*(\mathcal{O})$, $\mathcal{O}$ a regular hyperoval of $\mathbf{PG}(2, q)$, if q is even), $T_2^*(\mathcal{O})$ with $\mathcal{O}$ an arbitrary hyperoval in $\mathbf{PG}(2, q)$, q even, and the GQ $(S_{xy}^-)^D$ arising from a hyperoval in $\mathbf{PG}(2, q)$, q even. For more details, see B. De Bruyn [39]. Two AT's T_1 and T_2 of Type (4), (5) or (6) are equivalent if and only if $\Omega(T_1)$ and $\Omega(T_2)$ are equivalent, and by a paper of S. E. Payne ([126]) we precisely know when this happens. All spreads of symmetry in the classical GQ's of order (q, q^2) (i.e. the GQ's $\mathcal{Q}(5, q)$) were determined in [39]. The corresponding admissible triples are given in (3).

The problem that is still open today is whether there are known non-classical GQ's of order (q, q^2) with a spread of symmetry. This is the main concern of this chapter.

14.1.6 Glued near polygons and spreads of symmetry

Let k and s be nonzero integers and let X be a set of size $s+1$. For every $i \in \{1, 2, \ldots, k\}$ consider the following objects:

(A) a near polygon Γ_i;

(B) a spread $\mathbf{T}_i = \{L_1^{(i)}, L_2^{(i)}, \ldots, L_{n_i}^{(i)}\}$ of Γ_i, consisting of lines which are two by two parallel;

(C) a bijection $\theta_i : X \mapsto L_1^{(i)}$[3].

Conditions (B) and (C) assert that all lines $L_j^{(i)}$, $i \in \{1, 2, \ldots, k\}$ and $j \in \{1, 2, \ldots, n_i\}$, have the same length $s+1$. If x is a point of Γ_l and $L_m^{(l)} \in \mathbf{T}_l$, then $p_m^{(l)}(x)$ denotes the unique point of $L_m^{(l)}$ nearest to x. The following graph G can now be defined. The vertices of G are the elements of $X \times \mathbf{T}_1 \times \ldots \times \mathbf{T}_k$. Two vertices $(x, L_{i_1}^{(1)}, L_{i_2}^{(2)}, \ldots, L_{i_k}^{(k)})$ and $(y, L_{j_1}^{(1)}, L_{j_2}^{(2)}, \ldots, L_{j_k}^{(k)})$ are adjacent if and only if

(i) there exists an $l \in \{1, 2, \ldots, k\}$ such that $i_m = j_m$ for all $m \in \{1, 2, \ldots, k\} \setminus \{l\}$, and

(ii) for every l like in (i), $p_{i_l}^{(l)} \circ \theta_l(x)$ and $p_{j_l}^{(l)} \circ \theta_l(y)$ are collinear points in Γ_l.

The following incidence structure Γ can then be defined:

- the POINTS of Γ are the vertices of G, and
- the LINES of Γ are the maximal cliques of G.

Theorem 14.1.2 (B. De Bruyn [43]) *The incidence structure Γ is a near polygon if and only if the permutations $\theta_i^{-1} \circ p_1^{(i)} \circ p_\alpha^{(i)} \circ p_\beta^{(i)} \circ \theta_i$ and $\theta_j^{-1} \circ p_1^{(j)} \circ p_\gamma^{(j)} \circ p_\delta^{(j)} \circ \theta_j$ commute for all possible α, β, γ, δ, i and j with $i \neq j$.*

The group $G_i := \langle p_1^{(i)} \circ p_\alpha^{(i)} \circ p_\beta^{(i)} \parallel \alpha, \beta \in \{1, 2, \ldots, n_i\} \rangle$, $i \in \{1, 2, \ldots, k\}$, is called the *group of projectivities of $L_1^{(i)}$ with respect to $\mathbf{T}_i$*. If Γ is a near polygon, then it is called a *glued near polygon*. In this case the following conditions necessarily are satisfied:

[3]Of course, it will be clear that the construction method presented in this section is intrinsically independent of the fact that here, the underscore '1' is fixed; this is merely a (convenient) choice of notation.

(i) $\mathbf{T}_i$ is a spread of symmetry of Γ_i;

(ii) there exists a group G such that G_i, $i \in \{1, 2, \ldots, k\}$, is either trivial or isomorphic to G;

(iii) G is abelian if there exists at least three elements $i \in \{1, 2, \ldots, k\}$ for which G_i is not trivial.

Conversely, if for fixed X, Γ_i, $\mathbf{T}_i$, $L_1^{(i)}$, $i \in \{1, 2, \ldots, k\}$, Conditions (i), (ii) and (iii) are satisfied, then there always exist maps θ_i, $i \in \{1, 2, \ldots, k\}$, such that Γ is a glued near polygon, see [41].

In this way, generalized quadrangles with a spread $\mathbf{T}$ of symmetry always yield glued near hexagons [40]. If the group of projecivities of a line $L \in \mathbf{T}$ with respect to $\mathbf{T}$ is commutative, then also near $2d$-gons with $d \geq 4$ can be derived [40]. This is another motiviation for a study of spreads of symmetry in generalized quadrangles.

14.2 Spreads of Symmetry in Generalized Quadrangles: Basic Observations

Let $\mathbf{T} = \{L_1, L_2, \ldots, L_{st+1}\}$ be a spread of a generalized quadrangle $\mathcal{S}$ of order (s, t) and let $H_{\mathbf{T}}$ be the group of automorphisms of $\mathcal{S}$ fixing each line of $\mathbf{T}$. If $\mathcal{S}$ is an $(s+1) \times (s+1)$-grid, then $|H_{\mathbf{T}}| = (s+1)!$ for both spreads of $\mathcal{S}$.

Theorem 14.2.1 *If $\mathcal{S}$ is not a grid, then each nontrivial element of $H_{\mathbf{T}}$ has no fixed points; hence $|H_{\mathbf{T}}| = \frac{s+1}{n}$ with n some nonzero integer. In particular, $\mathbf{T}$ is a spread of symmetry if and only if $|H_{\mathbf{T}}| = s + 1$.*

Proof. The fact that each nontrivial element of $H_{\mathbf{T}}$ has no fixed point readily follows from Theorem 1.6.4. This implies that $|H_{\mathbf{T}}| = \frac{s+1}{n}$ since $H_{\mathbf{T}}$ acts semiregularly on the set of points of any line of $\mathbf{T}$. ∎

Theorem 14.2.2 *If a generalized quadrangle $\mathcal{S}$ of order $(s, t) = (s, s^\alpha)$, $s \neq 1$ and $\alpha \in \mathbb{Q} \setminus \{0\}$, contains a Hermitian spread, then $\alpha = 2$.*

Proof. Put $s = q^n$ and $t = q^m$ where q, n and m are strictly positive integers. Since $s, t > 1$ and $t \leq s^2$, we have that $m \leq 2n$. For every Hermitian spread $\mathbf{T}$ of $\mathcal{S}$, one can define the following linear space $\mathcal{L}(\mathbf{T})$. The points of $\mathcal{L}(\mathbf{T})$ are the elements of $\mathbf{T}$, the lines of $\mathcal{L}(\mathbf{T})$ are the sets $\{L, M\}^{\perp\perp}$ with L and M distinct lines of $\mathbf{T}$, and incidence is containment. Counting the number of lines

of $\mathcal{L}(\mathbf{T})$, one finds that $s(s+1) \mid st(st+1)$ or $s+1 \mid t(t-1)$. From $s = q^n$ and $t = q^m$ with $0 < n, m \leq 2n$, it readily follows that $m = 2n$. ∎

14.3 Ovoids of Symmetry and Dual Nets

We recall some relevant facts:

1. the point (∞) of a flock GQ $\mathcal{S}(\mathcal{F})$ is a regular point;
2. if $\mathcal{S}(\mathcal{F})$ is not isomorphic to $H(3, q^2)$, then
   (i) the point (∞) is fixed by each automorphism of $\mathcal{S}(\mathcal{F})$, see S. E. Payne and J. A. Thas [140];
   (ii) every ovoid of symmetry $\mathbf{O}$ contains the point (∞) (for, let θ be a nontrivial automorphism of $\mathcal{S}(\mathcal{F})$ which fixes each point of $\mathbf{O}$, then the only fixed points of θ are the elements of $\mathbf{O}$).

By (i), there arises a net $\mathcal{N}_{(\infty)}$ from the point (∞) of each flock GQ. As flock GQ's are the most widely investigated type of generalized quadrangles, we therefore will study the interaction between nets which arise from GQ's with a regular point and ovoids of symmetry.

14.3.1 Ovoids of symmetry through a regular point

Let $\mathcal{S}$ be a GQ of order (s,t), $s \neq 1 \neq t$, with a regular point (∞). A *spread* $\mathbf{T}$ of $\mathcal{N}^*_{(\infty)}$ (or, more generally, of any dual net) is a set of lines partitioning the point set of $\mathcal{N}^*_{(\infty)}$. Note that $|\mathbf{T}| = s$. The following observation was first made by J. A. Thas in [182], but we include a proof for the sake of completeness.

Theorem 14.3.1 *Let $\mathcal{S} = (P, B, I)$ be a GQ of order (s,t), $s,t > 1$, with a regular point (∞). Let $\mathbf{T} = \{L_1, L_2, \ldots, L_s\}$ be a spread of $\mathcal{N}^*_{(\infty)}$, then $O_{\mathbf{T}} = \{(\infty)\} \cup \{x \parallel x \in P \setminus (\infty)^\perp$ and $\{x, (\infty)\}^\perp \in \mathbf{T}\}$ is an ovoid of $\mathcal{S}$.*

Proof. Let x_1 and x_2 be two collinear points of $O_{\mathbf{T}}$, then $\{x_1, (\infty)\}^\perp$ and $\{x_2, (\infty)\}^\perp$ have at least one point in common. Hence $\{x_1, (\infty)\}^\perp = \{x_2, (\infty)\}^\perp$ and $x_2 \in \{x_1, (\infty)\}^{\perp\perp}$. This implies that $x_1 = x_2$. Since $O_{\mathbf{T}}$ is a set of $st+1$ two by two non-collinear points, the theorem follows. ∎

A spread $\mathbf{T}$ of $\mathcal{N}^*_{(\infty)}$ is called *regular* [41, 45] if for every line M_1 of $\mathcal{N}^*_{(\infty)}$ not belonging to $\mathbf{T}$ (then there exist $t+1$ lines $L_1, L_2, \ldots, L_{t+1}$ of $\mathbf{T}$ intersecting

M_1) there are t other lines $M_2, M_3, \ldots, M_{t+1}$ such that M_i is disjoint from M_j for all $i, j \in \{1, 2, \ldots, t+1\}$ with $i \neq j$, and so that M_i meets L_j in a point for all $i, j \in \{1, 2, \ldots, t+1\}$. The notion of 'regular spread' for arbitrary dual nets is defined in the same way.

Theorem 14.3.2 *Let $\mathcal{S}$ be a GQ of order (s,t), $s,t > 1$, with a regular point (∞) which is contained in an ovoid of symmetry $\mathbf{O}$. Then there exists a regular spread $\mathbf{T}$ of $\mathcal{N}^*_{(\infty)}$ such that $\mathbf{O} = O_{\mathbf{T}}$.*

Proof. We first prove that $\{o_1, (\infty)\}^\perp$ and $\{o_2, (\infty)\}^\perp$ are equal or disjoint for every two distinct points o_1, o_2 of $\mathbf{O} \setminus \{(\infty)\}$. Suppose $x \in \{o_1, (\infty)\}^\perp \cap \{o_2, (\infty)\}^\perp$ and let θ be a nontrivial automorphism of $\mathcal{S}$ which fixes each point of $\mathbf{O}$, then $x, x^\theta \in \{o_1, (\infty)\}^\perp \cap \{o_2, (\infty)\}^\perp$ and hence $\{o_1, (\infty)\}^\perp = \{o_2, (\infty)\}^\perp = \{x, x^\theta\}^{\perp\perp}$. Hence $\mathbf{T} = \{\{o, (\infty)\}^\perp \parallel o \in \mathbf{O} \setminus \{(\infty)\}\}$ is a spread of $\mathcal{N}^*_{(\infty)}$. We now show that $\mathbf{T}$ is regular. Let M_1 be an arbitrary line of $\mathcal{N}^*_{(\infty)}$ not belonging to $\mathbf{T}$ and let $M_1 = \{(\infty), x_1\}^\perp$. Let $\{x_1, x_2, \ldots, x_{t+1}\}$ denote the orbit of x_1 determined by the group of automorphisms fixing each point of $\mathbf{O}$ and put $M_i = \{(\infty), x_i\}^\perp$ for all $i \in \{1, 2, \ldots, t+1\}$. The above conditions are then satisfied for $M_1, M_2, \ldots, M_{t+1}$. ∎

Every automorphism θ of $\mathcal{S}$ which fixes (∞) induces an automorphism $\bar{\theta}$ of $\mathcal{N}^*_{(\infty)}$. If θ fixes every point of $O_{\mathbf{T}}$, then $\bar{\theta}$ fixes every line of $\mathbf{T}$.

Theorem 14.3.3 *Let ϕ be an automorphism of $\mathcal{N}^*_{(\infty)}$ fixing each line of a spread $\mathbf{T}$ of $\mathcal{N}^*_{(\infty)}$. Then there exists at most one automorphism θ of $\mathcal{S}$, with $\bar{\theta} = \phi$, which fixes each point of $O_{\mathbf{T}}$.*

Proof. If θ and θ' are two such automorphisms, then by Theorem 1.6.2, $\theta(\theta')^{-1}$ is the identity on $\mathcal{S}$. ∎

If θ exists, then it must be equal to the following map. Put $(\infty)^\theta = (\infty)$ and put $z^\theta = z^\phi$ for every $z \in (\infty)^\perp \setminus \{(\infty)\}$. For $z \notin (\infty)^\perp$, let L_1 and L_2 denote two lines through z. The line L_i, $i \in \{1, 2\}$, intersects $(\infty)^\perp$ in a point z_i and the ovoid $O_{\mathbf{T}}$ in a point o_i. Now, z^θ is the intersection of the lines $z_1^\theta o_1$ and $z_2^\theta o_2$. This is another proof for Theorem 14.3.3.

To determine all ovoids of symmetry of $\mathcal{S}$ through the regular point (∞), one could proceed as follows.

(i) Determine all regular spreads of $\mathcal{N}^*_{(\infty)}$.

(ii) For each regular spread $\mathbf{T}$ of $\mathcal{N}^*_{(\infty)}$, calculate the ovoid $O_{\mathbf{T}}$ of $\mathcal{S}$.

(iii) Determine all the automorphisms of $\mathcal{N}^*_{(\infty)}$ fixing each line of $\mathbf{T}$. There have to be at least $t+1$ such automorphisms (including the identity), otherwise $O_{\mathbf{T}}$ cannot be an ovoid of symmetry.

(iv) Check for each of these automorphisms whether it corresponds to an automorphism of $\mathcal{S}$ which fixes each point of $O_{\mathbf{T}}$. If we find $t+1$ such automorphisms, then $O_{\mathbf{T}}$ is an ovoid of symmetry.

This method allows us to determine the group of automorphisms that fixes each point of an ovoid of symmetry $O_{\mathbf{T}}$ if the dual net $\mathcal{N}^*_{(\infty)}$ satisfies the Axiom of Veblen.
Suppose that $\mathcal{N}^*_{(\infty)}$ satisfies the Axiom of Veblen. Since $s \neq t$, it follows by Theorem 3.1.2 that $\mathcal{N}^*_{(\infty)} \cong H^n_q$, $n > 2$. From $s = q^{n-1}$, $t = q$ and $s \leq t^2$, it then follows that $n = 3$. Adding $L = \mathbf{PG}(n-2, q)$ to the spread $\mathbf{T}$ of $\mathcal{N}^*_{(\infty)}$, we then obtain a spread $\bar{\mathbf{T}}$ of $\mathbf{PG}(3, q)$.

In this section, we restricted our search to those ovoids of symmetry through a fixed regular point (∞), as these restrictions are justified if the GQ $\mathcal{S}$ is a flock GQ. Also, the case where the dual net arising from (∞) is isomorphic to H^3_q occurs, e.g. in the GQ's arising from the Kantor flocks. These GQ's will be treated in Section 14.4.

Remark 14.3.4 (i) Let m be an arbitrary non-square of $\mathbf{GF}(q)$, q odd. For $(\lambda, \mu) \in \mathbf{GF}(q) \times \mathbf{GF}(q)$, define

$$L_{\lambda,\mu} = \langle (\lambda, \mu, 1, 0), (\mu, m\lambda, 0, 1) \rangle,$$

and set

$$\mathbf{T} = \{\langle (1,0,0,0), (0,1,0,0) \rangle\} \cup \{L_{\lambda,\mu} \parallel \lambda, \mu \in \mathbf{GF}(q)\}.$$

Then $\mathbf{T}$ is easily seen to be a spread of $\mathbf{PG}(3, q)$, see B. De Bruyn [41]. For $\alpha, \beta \in \mathbf{GF}(q)$, let $A(\alpha, \beta)$ be the following matrix:

$$A(\alpha, \beta) = \begin{pmatrix} \alpha & \beta & 0 & 0 \\ m\beta & \alpha & 0 & 0 \\ 0 & 0 & \alpha & m\beta \\ 0 & 0 & \beta & \alpha \end{pmatrix}.$$

If $(\alpha,\beta) \neq (0,0)$, then the linear automorphism $\theta : \overline{v}' = A(\alpha,\beta)\overline{v}$ of $\mathbf{PG}(3,q)$ fixes each line of $\mathbf{T}$; in this way, all $q+1$ such automorphisms are obtained. In [41] B. De Bruyn also observes that the group of automorphisms of $\mathbf{PG}(3,q)$ fixing $\mathbf{T}$ linewise — which is a subgroup of $\mathbf{PGL}(4,q)$ — is isomorphic to the cyclic group C_{q+1}.

(ii) Let n be an element of $\mathbf{GF}(q)$, q even, so that $tr(n) = 1$. For $(\lambda,\mu) \in \mathbf{GF}(q) \times \mathbf{GF}(q)$, put

$$L_{\lambda,\mu} = \langle (\lambda,\mu,1,0), (\lambda + n\mu, \lambda, 0, 1) \rangle,$$

and set

$$\mathbf{T} = \{\langle (1,0,0,0), (0,1,0,0) \rangle\} \cup \{L_{\lambda,\mu} \parallel \lambda,\mu \in \mathbf{GF}(q)\}.$$

Then $\mathbf{T}$ is a spread of $\mathbf{PG}(3,q)$, see [41]. For $\alpha,\beta \in \mathbf{GF}(q)$, define $A(\alpha,\beta)$ as being the following matrix:

$$A(\alpha,\beta) = \begin{pmatrix} \alpha+\beta & n\beta & 0 & 0 \\ \beta & \alpha & 0 & 0 \\ 0 & 0 & \alpha & n\beta \\ 0 & 0 & \beta & \alpha+\beta \end{pmatrix}.$$

With $(\alpha,\beta) \neq (0,0)$, the linear automorphism $\theta : \overline{v}' = A(\alpha,\beta)\overline{v}$ of $\mathbf{PG}(3,q)$ fixes each line of $\mathbf{T}$; all $q+1$ such automorphisms are thus obtained. Also, the group of automorphisms of $\mathbf{PG}(3,q)$ which fix $\mathbf{T}$ linewise — which is a subgroup of $\mathbf{PGL}(4,q)$ — is isomorphic to the cyclic group C_{q+1} [41].

(iii) As there is precisely one $\mathbf{P\Gamma L}(4,q)$-orbit of regular spreads of $\mathbf{PG}(3,q)$, from (i) and (ii) one can conclude that the group of all automorphisms of an arbitrary regular spread of $\mathbf{PG}(3,q)$ is a cyclic subgroup of $\mathbf{PGL}(4,q)$.

(iv) We will also obtain (iii) in an alternative way, see the proof of Theorem 14.5.2.

Theorem 14.3.5 *The spread $\bar{\mathbf{T}}$ is a regular spread of $\mathbf{PG}(3,q)$ and the group of automorphisms of $\mathcal{S}$ fixing each point of $O_{\mathbf{T}}$ is isomorphic to the cyclic group C_{q+1}.*

Proof. Let $\theta_i, i \in \{1, 2, \ldots, q+1\}$, denote the $q+1$ automorphisms of $\mathcal{S}$ fixing each point of $O_{\mathbf{T}}$. The automorphism θ_i corresponds to an automorphism $\bar{\theta}_i$ of $\mathcal{N}^*_{(\infty)}$ which can be extended to an automorphism $\tilde{\theta}_i$ of $\mathbf{PG}(3,q)$, see e.g. Theorem 1.4.3 of [41]. The automorphism $\tilde{\theta}_i$ fixes each line of $\bar{\mathbf{T}}$. Now, let M be any line of $\mathbf{PG}(3,q)$ not belonging to $\bar{\mathbf{T}}$, then M meets the lines $L_1, L_2, \ldots, L_{q+1}$ of $\bar{\mathbf{T}}$. Put $\{M_1, M_2, \ldots, M_{q+1}\} = \{\tilde{\theta}_i(M) \parallel i \in \{1, 2, \ldots, q+1\}\}$. Clearly $L_1, L_2, \ldots, L_{q+1}$ is a regulus of $\mathbf{PG}(3,q)$; whence $\bar{\mathbf{T}}$ is regular. The group $\{\tilde{\theta}_i \parallel i \in \{1, 2, \ldots, q+1\}\}$ is a subgroup of the full group of automorphisms of $\mathbf{PG}(3,q)$ which fix each element of $\bar{\mathbf{T}}$ and this latter group is isomorphic to C_{q+1}, see Remark 14.3.4. The theorem follows now from Theorem 14.2.1. ■

14.4 The Nonexistence of Ovoids of Symmetry in Non-Classical Kantor Flock Quadrangles

In this section we will show that each Kantor flock generalized quadrangle of order (q^2, q) with an ovoid of symmetry is classical (i.e. isomorphic to $H(3, q^2)$). This will be a crucial result for the next section.

Remark 14.4.1 (i) Recall from Chapter 9 that if $\mathcal{S}(\mathcal{F})$ is a Kantor flock GQ of order (q^2, q), $q > 1$, with special point (∞), then there are (precisely) $q^3 + q^2$ subGQ's of order q which contain the point (∞). These subGQ's are all isomorphic to $W(q)$. We will use this property (or its dual) in the sequel without further notice.

(ii) By Chapter 10 each TGQ $\mathcal{S}$ of order (q, q^2), $q > 1$, of which the translation dual is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$, has a line $[\infty]$ of translation points. In particular, if $\mathcal{F}$ is a Kantor flock, the property holds, and as $\mathcal{S} \cong \mathcal{S}^*$ by Theorem 2.4.1, also $\mathcal{S}^*$ has a line of translation points.

14.4.1 Nonexistence of ovoids of symmetry in non-classical Kantor flock GQ's

We now come to the main result of this section; for the sake of convenience, we will work with the dual of $\mathcal{S}(\mathcal{F})^D$, $\mathcal{F}$ a Kantor flock.

Theorem 14.4.2 *Suppose $\mathcal{S} = \mathcal{S}(\mathcal{F})^D$ is the dual of the flock GQ $\mathcal{S}(\mathcal{F})$ of order (q^2, q), where $\mathcal{F}$ is a Kantor flock. If $\mathcal{S}$ allows a spread of symmetry $\mathbf{T}$, then $\mathcal{F}$ is linear, that is, $\mathcal{S} \cong \mathcal{Q}(5, q)$.*

Proof. Let $[\infty]$ be the line of $\mathcal{S}(\mathcal{F})^D$ which corresponds to the point (∞) in $\mathcal{S}(\mathcal{F})$. Consider a (classical) subGQ $\mathcal{S}'$ of order q through $[\infty]$. As $\mathcal{S}$ is the dual of a flock GQ, $\mathbf{T}$ must contain the line $[\infty]$ if $\mathcal{S}$ is not classical. Since $\mathcal{S}'$ is of order q, there are lines U and V of $\mathbf{T}$ so that $\{U, V\}^{\perp\perp} \subseteq \mathbf{T} \cap \mathcal{S}'$ (this is an easy counting argument). If $H_{\mathbf{T}}$ is the group of automorphisms of $\mathcal{S}$ which fix $\mathbf{T}$ linewise, then $|[\mathcal{S}']^{H_{\mathbf{T}}}| = q + 1$ (as each line of $\mathbf{T}$ hits $\mathcal{S}'$ in 1 or $q + 1$ points). This implies that the $q^3 + q^2$ subGQ's of order q through $[\infty]$ are all in the same $Aut(\mathcal{S})$-orbit as, since $[\infty]$ is a line of translation points, $Aut(\mathcal{S})_{[\infty]}$ acts transitively on the pairs of non-concurrent lines in $[\infty]^{\perp}$. This yields a contradiction since non-classical Kantor GQ's have two such $Aut(\mathcal{S})$-orbits of subGQ's of order q [188]. ∎

14.5 TGQ's and EGQ's with a Spread of Symmetry

14.5.1 Generalized quadrangles with translation points or elation points which have a spread of symmetry

We now eliminate the possibility that non-classical TGQ's of order (s, t), $s \neq 1 \neq t$, which contain a spread of symmetry exist.

We will need the following recent result:

Theorem 14.5.1 (M. R. Brown and J. A. Thas [24]) *Suppose $\mathcal{S} = \mathcal{S}(\mathcal{F})$ is a flock GQ of order (q^2, q), $q > 1$ and q odd, and assume that the point (∞) of $\mathcal{S}(\mathcal{F})$ is contained in at least one subGQ of $\mathcal{S}$ of order q. Then $\mathcal{F}$ is a Kantor flock.*

Theorem 14.5.2 *Let $\mathcal{S} = T(\mathcal{O})$ be a TGQ of order (s, t), $s \neq 1 \neq t$, with base-point (∞), and which contains a spread of symmetry $\mathbf{T}$. Then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5, s)$.*

Proof. Suppose L is the line of $\mathbf{T}$ which is incident with the base-point (∞) of $\mathcal{S}$. Since there is a group $H_{\mathbf{T}}$ of $s + 1$ automorphisms of $\mathcal{S}$ which fixes $\mathbf{T}$ elementwise and which acts transitively on the points incident with any line

of $\mathbf{T}$, each point on L is a translation point (so every line of $L^\perp$ is an axis of symmetry, and hence $\mathcal{S}$ is an SPGQ for every two non-concurrent lines of $L^\perp$). By the main result of Chapter 9, one of the following holds:

(i) s is even and $\mathcal{S} \cong \mathcal{Q}(5, s)$;

(ii) s is odd and $\mathcal{S}^{(\infty)}$ is the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$, that is, $\mathcal{O}$ is good at the element π which corresponds to L.

(Recall that $t > s$ by the assumptions.) Suppose we are in Case (ii) for the sequel of the proof. Suppose $M \neq L$ is a line of $\mathbf{T}$, and put $\mathcal{L} = \{L, M\}^\perp$. Then every line of $\mathcal{L}$ is an axis of symmetry. Let G be the group which is generated by the symmetries about the lines of $\mathcal{L}$, and define H by $H = \langle G, H_{\mathbf{T}} \rangle$. First of all, note that any element of H fixes $\mathcal{L}^\perp$ linewise. Also, by Theorem 12.5.5, G acts semiregularly on $\mathcal{S} \setminus \Omega$ (where Ω is the set of all points on the lines of $\{L, M\}^\perp$), and $G \cong \mathbf{SL}(2, s)$. Since s is odd, the kernel of the action of G on the lines of $\mathcal{L}$ has size 2. We now show that $H_{\mathbf{T}} \cap G = \{\mathbf{1}\}$.
As $G \cong \mathbf{SL}(2, s)$, G has order $(s+1)s(s-1)$. Let Λ be an arbitrary G-orbit in $\mathcal{S} \setminus \Omega$, and fix an arbitrary line W of $\mathcal{L}^\perp$. By the semiregularity of G on the set of points of $\mathcal{S} \setminus \Omega$, the fact that $|G| = (s+1)s(s-1)$ and the fact that G acts transitively on the points of W, any point on W is incident with exactly $s-1$ lines of $\mathcal{S}$ which are completely contained in Λ except for the point which is in Ω, and every point of Λ is incident with a line which meets W (recall that G is generated by the groups of symmetries about the lines of $\mathcal{L}$). Now define the following incidence structure $\mathcal{S}' = (P', B', I')$;

- LINES. The elements of B' are the lines of $\mathcal{S}'$ and they are essentially of two types:
  1. the lines of Γ (where Γ is the induced subgeometry of $\mathcal{S}$ defined by $\mathcal{L}$);
  2. the lines of $\mathcal{S}$ which contain a point of Λ and a point of Ω.
- POINTS. The elements of P' are the points of $\mathcal{S}'$ and they are just the points of $\Omega \cup \Lambda$.
- INCIDENCE. Incidence I' is the natural one.

Then $\mathcal{S}'$ is a generalized quadrangle of order s, see also Chapter 9, and hence any line of $\mathcal{S}$ intersects $\mathcal{S}'$ in 1 or $s+1$ points. Now suppose that $\theta \in H_{\mathbf{T}} \cap G$, $\theta \neq \mathbf{1}$. Then by the semiregularity of G on $\mathcal{S} \setminus \Omega$, each line of $\mathbf{T} \setminus \mathcal{L}^\perp$ intersects

$\mathcal{S}' \setminus \Gamma$ in at least two points (and hence in $s+1$ points) clearly a contradiction. Thus $H_{\mathbf{T}} \cap G = \{\mathbf{1}\}$. Since $H_{\mathbf{T}} \cap G = \{\mathbf{1}\}$, we have $|H| \geq (s^3 - s)(s+1)$. Actually, since it is now clear that H acts transitively on the points of $\mathcal{S} \setminus \Gamma$, the number $(s+1)(s^3-s)$ divides $|H|$. Consider the dual net $\mathcal{N}_L^*$ which arises from the regular line L. Since $\mathcal{O}$ is good at π, Theorem 3.1.4 implies that $\mathcal{N}_L^*$ satisfies the Axiom of Veblen. Hence by Theorem 3.1.2, $\mathcal{N}_L^* \cong H_s^3$. The points of H_s^3 are the points of $\mathbf{PG}(3,s)$ not on a given line Z of $\mathbf{PG}(3,s)$; the lines are the lines of $\mathbf{PG}(3,s)$ which have no point in common with Z. With $\{L,M\}^\perp$ corresponds a line $Z' \not\sim Z$ of $\mathbf{PG}(3,s)$ and with each line L_i, $i = 0, 1, \ldots, s$, of $\{L,M\}^\perp$, corresponds a point z_i' on Z'. Now we interpret the group $H = \langle G, H_{\mathbf{T}} \rangle$ as a group of collineations of H_s^3. First of all, the subgroup of G of symmetries about L_j, $j = 0, 1, \ldots, s$, clearly induces the group of all elations of $\mathbf{PG}(3,s)$ with axis $\langle z_j', Z \rangle$ and center z_j'. Hence if G induces G' on $\mathbf{PG}(3,s)$, then G' is a subgroup of $\mathbf{PGL}(4,s)$. Also, $H_{\mathbf{T}}$ induces the full group $H'_{\overline{\mathbf{T}}}$ of automorphisms of $\mathbf{PG}(3,s)$ which fix the spread $\overline{\mathbf{T}}$ of $\mathbf{PG}(3,s)$, recall Section 14.3.1, linewise. As $H'_{\overline{\mathbf{T}}}$ preserves the cross-ratio of $\mathbf{PG}(3,s)$, the group $H'_{\overline{\mathbf{T}}}$ is also a subgroup of $\mathbf{PGL}(4,s)$, see Remark 14.3.4 (i)-(iii) for an alternative proof of the latter observation. Hence H induces a subgroup H' of $\mathbf{PGL}(4,s)$ on $\mathbf{PG}(3,s)$ (which fixes Z and Z'). So we have the following property:

(E) *If $\theta \in H$ fixes three lines of $\{L,M\}^\perp$, then θ fixes every line of $\{L,M\}^\perp$.*

Now fix some point u in $\mathcal{S} \setminus \Gamma$, and consider u^N. There is at most one nontrivial $\theta \in N$ which fixes u since the fixed element structure of such an element is a subGQ $\mathcal{S}_\theta$ of $\mathcal{S}$ of order s (by Theorems 1.6.2 and 1.6.4), and such a subGQ can be fixed pointwise by at most one nontrivial collineation (which is then an involution), as in a GQ of order (s, s^2), $s > 1$, for each two distinct non-collinear points u and v, $\{u,v\}^{\perp\perp} = \{u,v\}$. Hence $|H| \in \{(s+1)(s^3-s), 2(s+1)(s^3-s)\}$, and so, as H acts transitively on the points of $\mathcal{S} \setminus \Gamma$, we have in both cases that $|u^N| = s+1$ by Property (E)[4]. Now also suppose that $u \in (\infty)^\perp$. Recall that u is not a point of Γ. Suppose $\mathcal{S}^{(\infty)} = T(\mathcal{O})$ for the generalized ovoid $\mathcal{O}$ in $\mathbf{PG}(4n-1,q) \subseteq \mathbf{PG}(4n,q)$, where $q^n = s$. Then u^N is a set of q^n+1 points of Type (2) which — as subspaces of $\mathbf{PG}(4n,q)$ — all contain the same q^n points in $\mathbf{PG}(4n,q) \setminus \mathbf{PG}(4n-1,q)$. Note that in the GQ, (∞) is also a point of $(u^N)^\perp$. Interpreted in the translation dual $T(\mathcal{O}^*)$ of $T(\mathcal{O})$, the q^n+1 tangent spaces of $\mathcal{O}$ as defined by u^N become q^n+1 elements $\pi_0, \pi_1, \ldots, \pi_{q^n}$ of $\mathcal{O}^*$ which are contained in a $\mathbf{PG}(3n-1,q)$. Hence $\{\pi_0, \pi_1, \ldots, \pi_{q^n}\}$ is a generalized oval which lies on $\mathcal{O}^*$, and so induces a subGQ of $T(\mathcal{O}^*)$. Since $T(\mathcal{O}^*)$

[4] Observe that H/N is a subgroup of $\mathbf{PGL}(2,s)$.

is the point-line dual of a flock GQ and since q is odd, we can now conclude by Theorem 14.5.1 that $T(\mathcal{O}^*)$ is the point-line dual of a Kantor flock GQ $\mathcal{S}(\mathcal{F})$. Since the dual Kantor flock GQ's are isomorphic to their translation duals (see Theorem 2.4.1), we can conclude that $\mathcal{S}^{(\infty)}$ is the point-line dual of a Kantor flock GQ. Now Theorem 14.4.2 applies. ∎

The following theorem is a strong generalization of Theorem 14.5.2, and relies on Chapter 12.

Theorem 14.5.3 *Let $\mathcal{S}$ be a GQ of order (s,t), $s \neq 1 \neq t$, with a point x which is incident with at least three axes of symmetry. Moreover, suppose that $\mathcal{S}$ contains a spread of symmetry* $\mathbf{T}$. *Then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5,s)$.*

Proof. Suppose L is the line of $\mathbf{T}$ which is incident with x. Then, as before, each point on L is incident with three axes of symmetry. By Chapter 12, each point on L is a translation point. The theorem now follows from the proof of Theorem 14.5.2. ∎

Remark 14.5.4 By Chapter 12, it is in fact sufficient to ask that $\mathcal{S}$ be a GQ of order (s,t), $s \neq 1 \neq t$, with x a point incident with at least two axes of symmetry U and V, and $\mathcal{S}$ having a spread of symmetry $\mathbf{T}$ which does not contain U or V, to conclude that $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5,s)$.

The following very general theorem eliminates almost all known possible classes of GQ's with a spread of symmetry in the 'even characteristic' case.

Theorem 14.5.5 *Let $\mathcal{S} = (P, B, I)$ be a GQ of order (s, s^2), s even, with a center of transitivity (∞), and which contains a spread of symmetry* $\mathbf{T}$. *Then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5,s)$.*

Proof. As before, there is some line L incident with (∞) of which each point is a center of transitivity, and which is a line of the spread of symmetry $\mathbf{T}$. Suppose U and V are two distinct concurrent lines in $B \setminus L^\perp$, and let $u = U \cap V$. Suppose u' is the unique point on L which is collinear with u. Then since u' is a center of transitivity, there is a collineation of $\mathcal{S}$ which fixes u' linewise and which maps U onto V. Using this observation, one easily derives that the group of automorphisms of $\mathcal{S}$ which fixes L, say H_L, acts transitively on the lines of $B \setminus L^\perp$. The spread $\mathbf{T}$ is semiregular with respect to L. By the transitivity of H_L on $B \setminus L^\perp$, we can hence conclude that L is a regular line of $\mathcal{S}$. Also, by Corollary 6.6.2, the fact that $\mathcal{S}$ is of order (s, s^2) implies that $\mathcal{S}^{(\infty)}$ is an EGQ with base-point (∞) for some elation group G. By Theorem 5.5.2, we

then know that, since s is even, $\mathcal{S}$ is a TGQ for every point incident with L. The result now follows from Theorem 14.5.2. ∎

From the proof of Theorem 14.5.5, we immediately have:

Corollary 14.5.6 *Let $\mathcal{S}$ be an EGQ (for some elation point) of order (s,t), $s \neq 1 \neq t$ and s even, which contains a spread of symmetry $\mathbf{T}$. Then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5,s)$.*

∎

The case where s is odd will be considered later, as a completely different (and slightly more involved) approach will be needed.

14.5.2 Generalized quadrangles with translation lines or elation lines which have a spread of symmetry

Theorem 14.5.7 *Suppose $\mathcal{S}$ is an EGQ of order (s,t), $s \neq 1 \neq t$, with baseline L. Assume that $\mathbf{T}$ is a spread of symmetry of $\mathcal{S}$ which does not contain the line L. Then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5,s)$.*

Proof. Since L is not contained in $\mathbf{T}$, one easily observes that each line of $\mathcal{S}$ is an elation line, and hence, as $Aut(\mathcal{S})$ clearly acts transitively on the pairs of non-concurrent lines of $\mathcal{S}$ and as $\mathbf{T}$ is a Hermitian spread, every line of $\mathcal{S}$ is regular. Now fix an arbitrary line M of $\mathcal{S}$. Then by Chapter 3 (cf. Theorem 3.3.2), we have two possibilities:

(i) M is an axis of symmetry, or

(ii) there is a subGQ $\mathcal{S}'$ of $\mathcal{S}$ of order s which contains M.

Suppose we are in Case (i). Then each line is an axis of symmetry, hence $\mathcal{S}$ is half Moufang as each line is Moufang. Thus, by Theorem 1.4.2, $\mathcal{S}$ is Moufang and then by Theorem 1.4.1, it follows that $\mathcal{S}$ is classical. Since $\mathcal{S}$ cannot be of order s and since $\mathcal{S}$ contains regular lines, we now obtain that $\mathcal{S} \cong \mathcal{Q}(5,s)$.
Next suppose we are in Case (ii). As all lines of $\mathcal{S}$ are regular, there easily follows that $\mathcal{S}' \cong \mathcal{Q}(4,s)$. Consider two lines U and V of $\mathbf{T}$. Then $\{U,V\}^{\perp\perp} \subseteq \mathbf{T}$, and there is a subGQ $\mathcal{S}'' \cong \mathcal{Q}(4,s)$ which contains U and V (and hence also $\{U,V\}^{\perp\perp}$), since $Aut(\mathcal{S})$ acts transitively on the pairs of non-concurrent lines of $\mathcal{S}$. It is clear that $\mathbf{T} \cap \mathcal{S}'' = \{U,V\}^{\perp\perp}$ (both $\mathbf{T}$ and $\mathcal{S}''$ viewed as line sets). Hence, if $H_{\mathbf{T}}$ is the group of automorphisms of $\mathcal{S}$ which fix $\mathbf{T}$ linewise, then $|[\mathcal{S}'']^{H_{\mathbf{T}}}| = s+1$, and there are $s+1$ (classical) subGQ's of order s which

mutually intersect in the induced subgeometry of $\mathcal{S}$ defined by $\{U, V\}^{\perp\perp}$. By the transitivity of $Aut(\mathcal{S})$ on the pairs of non-concurrent lines of $\mathcal{S}$, the result now follows from Theorem 1.6.6. ■

Remark 14.5.8 A generalized quadrangle $\mathcal{S}$ of order (s,t), $s,t > 1$, with a translation line L cannot have a spread of symmetry. For, suppose that this is the case. As L contains centers of symmetry, and hence regular points, $t \leq s$, a contradiction by (iii) of Section 14.1.5.

14.6 Remaining Open Cases

We end this part of the chapter with mentioning the most important remaining open cases.

PROBLEM A. *Classify all generalized quadrangles of order (s,t), $s,t > 1$ and s odd, with an elation point p which have a spread of symmetry* $\mathbf{T}$.

PROBLEM B. *Classify all generalized quadrangles of order (s,t), $s,t > 1$, with an elation line L which allow a spread of symmetry* $\mathbf{T}$ *for which* $L \in \mathbf{T}$.

We consider Problem B as the harder but also the less interesting of both problems. By Corollary 14.5.5, the solution of Problem A would yield a complete classification of the elation generalized quadrangles (w.r.t. a point) which admit a spread of symmetry! Since almost all known GQ's are EGQ's for some base-point, this would be a very interesting result.

Note. For more on Problem A, see the next section.

14.7 Solution of Problem A

In this section, we completely solve Problem A. No new near polygons arise. Our result also contributes to the classification of those generalized quadrangles having a line of elation points.

14.7.1 Setting of notation and introductory results

Suppose $(\mathcal{S}^{(x)}, G)$ is an EGQ of order (s, s^2), $s \neq 1$ and s odd, and suppose that $\mathbf{T}$ is a spread of symmetry of $\mathcal{S}$ with corresponding group $H_{\mathbf{T}}$. We denote

the unique line of $\mathbf{T}$ through x by $[\infty]$. Clearly we have that

Every point on $[\infty]$ *is an elation point.*

Fix an arbitrary line $U \not\sim [\infty]$ of $\mathcal{S}$. Denote by u_i, $i = 0, 1, \ldots, s$, the points on U, and write x_j for the unique point on $[\infty]$ incident with u_j for each j. For each i, define H_i as the group of elations about x_i which fixes U. That we can speak of 'the group' is shown by the followng three lemmas, especially Lemma 14.7.3.

Lemma 14.7.1 $[\infty]$ *is a regular line.*

Proof. We know that the spread $\mathbf{T}$, which contains the line $[\infty]$, is a Hermitian spread, so for each line O of $\mathbf{T} \setminus \{[\infty]\}$, $\{O, [\infty]\}$ is a regular pair of lines. Take an arbitrary line $V \notin [\infty]^\perp$ which is not contained in $\mathbf{T}$. If v is an arbitrary point on V and O' is the line of $\mathbf{T}$ through v, then there is an elation θ about $proj_{[\infty]}v$ which maps O' onto V, and hence, since θ fixes $[\infty]$, also $\{V, [\infty]\}$ is a regular pair of lines. Whence the result. ∎

Lemma 14.7.2 $\mathcal{S}$ *has order* (s, s^2), s *a prime power.*

Proof. By Lemma 14.7.1, $\mathcal{S}$ contains a regular line, and hence $s \leq t$. By D. Frohardt [63], the fact that $\mathcal{S}$ is an EGQ implies that s and t are powers of the same prime p. By Theorem 14.2.2, $t = s^2$. ∎

Lemma 14.7.3 *The set of all elations about* x_i *is the full group of elations about* x_i.

Proof. By Lemma 14.7.2, we have that $t = s^2$. So for each two non-collinear points u and v of $\mathcal{S}$, $|\{u, v\}^{\perp\perp}| = 2$, and then Theorem 1.7.2 (iv) implies that the full set of all elations about x_i forms an elation group with elation point x_i. ∎

Note that H_i fixes U, the point u_i, and each line through x_i. Define $H = H(U, [\infty])$ as the group generated by all the groups H_i, $i = 0, 1, \ldots, s$, and let N be the kernel of the action of H on $\{U, [\infty]\}^\perp$. Sometimes we will write X for $\{U, [\infty]\}^\perp$.

Lemma 14.7.4 $(X, H/N)$ *is a split BN-pair of rank* 1.

Proof. It is clear that H/N acts 2-transitively on the set $X = \{L_0, L_1, \ldots, L_s\}$. Suppose $L = L_i \in X$ is arbitrary, $i \in \{0, 1, \ldots, s\}$, and consider $g^{-1}hg$, where

h is a nontrivial element of H_i, and where g is a nontrivial element of H_L. Then clearly, $g^{-1}hg$ fixes x_i linewise, and also $U^{g^{-1}hg} = U$. As h is an elation about x_i, we have that $g^{-1}hg$ is also an elation about x_i. By Lemma 14.7.3, $g^{-1}hg \in H_i$. Hence the lemma. ∎

Lemma 14.7.5 *H/N is one of the following:*

1. *a sharply 2-transitive group;*
2. $\mathbf{PSL}(2, s)$*;*
3. $\mathbf{R}(\sqrt[3]{s})$*;*
4. $\mathbf{PSU}(3, \sqrt[3]{s^2})$.

Proof. Immediately by Theorem 8.3.1, Lemma 14.7.4, and the fact that s is odd. ∎

Denote by $\Gamma(X)$ the induced subgeometry of $\mathcal{S}$ which is defined by the lines of X (we will also use that notation for each such set X). If a dual net $\mathcal{N}^*$ satisfies the Axiom of Veblen, then we also say that $\mathcal{N}$ satisifies the Axiom of Veblen.

Lemma 14.7.6 *N acts semiregularly on $\mathcal{S} \setminus \Gamma(X)$ or $\mathcal{S} \cong \mathcal{Q}(5, q)$.*

Proof. Suppose that N does not act semiregularly on $\mathcal{S} \setminus \Gamma(X)$; let p be a fixed point of $\theta \in N$, θ not the identity, $p \notin \Gamma(X)$. Then by Theorems 1.6.2 and 1.6.4, θ fixes a subGQ of $\mathcal{S}$ of order s elementwise. Note that:

(i) $Aut(\mathcal{S})$ acts transitively on the pairs of non-concurrent lines of $[\infty]^\perp$;

(ii) if $\mathcal{S}'$ is a subGQ of $\mathcal{S}$ of order s which completely contains the $(s+1) \times (s+1)$-grid $\Gamma(\{U', V'\}^{\perp\perp})$, where $[\infty] \in \{U', V'\}^\perp$ and $U' \not\sim V'$, then, since $|\mathcal{S}' \cap \mathbf{T}| = s + 1$ by an easy counting argument and considering the action of $H_{\mathbf{T}}$ on $\mathcal{S}$, (i) leads to the fact that $[\infty]$ is contained in at least (and then precisely; this is just easy counting) $s^3 + s^2$ subGQ's of order s (it is clear that no nontrivial element of $H_{\mathbf{T}}$ stabilizes $\mathcal{S}'$).

By Property (ii), the net $\mathcal{N}_{[\infty]}$ arising from the regular line $[\infty]$ satisfies the Axiom of Veblen. Hence by Theorem 3.1.2, $\mathcal{N}^*_{[\infty]}$, which is the point-line dual of $\mathcal{N}_{[\infty]}$, is isomorphic to H^3_s. Fix such a subGQ $\mathcal{S}'$ of order s through $[\infty]$, and suppose that L and M are non-concurrent lines of $[\infty]^\perp \cap \mathcal{S}'$. For the sake of convenience, assume that $U \in \{L, M\}^\perp$. Put $H = H(U, [\infty])$. Then clearly

H and each H_i, $i = 0, 1, \ldots, s$, fixes the affine plane $\Pi_{\mathcal{S}'}$ in $\mathcal{N}_{[\infty]}$ as defined by $\mathcal{S}'$. If we interpret H_i in the projective completion Π of $\Pi_{\mathcal{S}'}$, then H_i fixes all the lines through (the point which corresponds to) x_i in Π, and the point zIx_iu_i, where z corresponds to $\{L, M\}^{\perp\perp}$. Hence each point of x_iu_i is fixed by each element of H_i as interpreted as a collineation of the automorphism group $Aut(\Pi)$ of Π. As $\mathcal{N}^*_{[\infty]} \cong H^3_s$, we also know that Π is Desarguesian. Thus $H/N' \cong \mathbf{SL}(2, s)$, where N' is the kernel of the action of H on $\mathcal{S}'$. The result now follows by taking over the final part of the proof of Theorem 14.7.15 (starting from "Property (F)"). ∎

As a corollary of Lemma 14.7.6, we obtain:

Lemma 14.7.7 *If $\mathcal{S}$ is not classical, then $|N|$ divides $s^2 - 1$.*

Proof. By Lemma 14.7.6, N acts semiregularly on the points of $\mathcal{S} \setminus \Gamma(X)$. Let $i \neq j$ be elements of $\{0, 1, \ldots, s\}$, and consider x_i and u_j. Then $x_i \not\sim u_j$ and $|\{x_i, u_j\}^\perp \cap [\mathcal{S} \setminus \Gamma(X)]| = s^2 - 1$. As N fixes both x_i and u_j, the lemma follows. ∎

Lemma 14.7.8 *$H_iH_j \cap H_k = \{\mathbf{1}\}$ for distinct i, j and k.*

Proof. Suppose that θ, ϕ and σ are nontrivial elements of respectively H_i, H_j and H_k so that $\theta\phi = \sigma$. As s is a prime power (and as $t = s^2$), σ fixes some line M through u_k different from U and x_ku_k. Then $M = M^\sigma = M^{\theta\phi}$, and hence $proj_{x_i}M^\theta$ and $proj_{x_j}M^\theta$ both intersect M. But since $[\infty]$ is a regular line by Lemma 14.7.1, this implies that U intersects $[\infty]$, clearly a contradiction. ∎

Lemma 14.7.9 *$|H| \geq s^3 - s$.*

Proof. Let $U, [\infty], u_i, x_j$, etc. be as before. Let Λ be an arbitrary H-orbit in $\mathcal{S} \setminus \Gamma(X)$. Each H_i, $i = 0, 1, \ldots, s$, fixes at least one line O through u_i different from U and $proj_{[\infty]}u_i$, and which is, as a point set, contained in Λ (by considering the possible orders of the H_i-orbits in the set of lines through u_i distinct from U and $proj_{[\infty]}u_i = x_iu_i$ — recall that s is a prime power). Each point of O is a point of $\Lambda \cup \Gamma(X)$, and hence the points on the lines of $\{O, [\infty]\}^\perp$ are completely contained in $\mathcal{S}'$. Let W be a line of $\{O, [\infty]\}^\perp$ which is not contained in X (note that $\{O, [\infty]\}^\perp = \{W, x_iu_i\}^{\perp\perp}$). The stabilizer H_W of W in H acts transitively on $X \setminus Z$, where $Z \sim W$ and $Z \in X$. Hence each point of $[\infty] \setminus \{Z \cap W\}$ is incident with at least s lines different from $[\infty]$, of which the point sets are completely contained in $\Lambda \cup \Gamma(X)$. As H acts transitively on the points of $[\infty]$, the lemma follows. ∎

Lemma 14.7.10 *H/N cannot act as a sharply 2-transitive group on X unless $s = 3$ (recall that s is odd).*

Proof. Let $H_{\mathbf{T}}$ be the group of $s+1$ collineations of $\mathcal{S}$ which all fix $\mathbf{T}$ elementwise, and note the following property (stated in terms of the line L_0):

(F) *Suppose that $h \in H_{\mathbf{T}}$ is nontrivial, and suppose that $L_0^h = L_i$ for some $i \in \{1, 2, \ldots, s\}$. Let $g \in H_0$ be arbitrary but nontrivial. Then $h^{-1}gh = g^h$ is an element of H_i with the property that, if L is an arbitrary line of $\{[\infty], U\}^{\perp\perp}$, then $L^h = L^{h^g}$.*

Hence using Lemma 14.7.8 (see the proof of Theorem 14.7.15 for more details), the group Φ defined by

$$\Phi = \langle g(g^h)^{-1} \parallel g \in H_0, h \in H_{\mathbf{T}} \rangle,$$

which has the property that each of its elements fixes $\{[\infty], U\}^{\perp\perp}$ elementwise, has at least size s^2 (in its action on X). Hence H/N fixes $X^\perp$ elementwise. Consider an arbitrary H_i, $i \in \{0, 1, \ldots, s\}$. Then it follows readily that H_i fixes $X^\perp$ elementwise. As H is generated by all such H_i, it follows that also N fixes $X^\perp$ elementwise. Consider an arbitrary point $x \in \mathcal{S} \setminus \Gamma(X)$. Then $|x^N| = |N|$ by Lemma 14.7.6. As N fixes $\Gamma(X)$ elementwise, we also obtain that each point of x^N is collinear with each point of $x^\perp \cap \Gamma(X)$, and hence, since $\mathcal{S}$ is a GQ of order (s, s^2), we have that $|N| \leq s+1$. Since $|N|$ divides $s^2 - 1$ and since $|H| \geq s^3 - s$, we obtain that $|N| \in \{s-1, s+1\}$. The case where $|N| = s+1$ is not possible by Section 9.7 of Chapter 9. Assume that $|N| = s-1$. Then H and $H_0, H_1, \ldots, H_s$ have the following properties:

1. $|H| = s^3 - s$;
2. the groups $H_0, H_1, \ldots, H_s$ form a complete conjugacy class in H and $|H_i| = s$ for all feasible i;
3. $N_H(H_i) \cap H_j = \{\mathbf{1}\}$ for distinct i and j in $\{0, 1, \ldots, s\}$;
4. $H_i H_j \cap H_k = \{\mathbf{1}\}$ for distinct i, j and k.

Hence H is a group with a 4-gonal basis, and by Chapter 8, $H \cong \mathbf{SL}(2, s)$. But $\mathbf{SL}(2, s)$ only acts sharply 2-transitively on X if $s = 3$ when s is odd. ∎

Lemma 14.7.11 *$\{H_0, H_1, \ldots, H_s\}$ is the set of Sylow p-subgroups of H, where $s = p^h$, $h \in \mathbb{N}$.*

Proof. We know that $|H| = |H/N| \times |N| = (s^n - 1)(s+1)s/r \times |N|$, with $r \in \{1, 2, gcd(3, \sqrt[3]{s}+1)\}$ and $n \in \{1, 2/3, 1/2, 1/3\}$, where $r = 2$ if and only if $H/N \cong \mathbf{PSL}(2,s)$ and where $r = gcd(3, \sqrt[3]{s}+1)$ if and only if $H/N \cong \mathbf{PSU}(3, \sqrt[3]{s^2})$. By Lemma 14.7.7, we have that $|N|$ divides $s^2 - 1$, and hence, as $gcd(|N|, s) = 1$, $s = p^h$ is the greatest power of p which divides $|H|$. Hence the H_i, $i = 0, 1, \ldots, s$, are Sylow p-subgroups of H. As the H_i form a complete conjugacy class in H, and as all Sylow p-subgroups are mutually conjugated, the result follows. ∎

Lemma 14.7.12 *H is a perfect central extension of H/N.*

Proof. To obtain this result, we prove:

(i) H is a perfect group, and that

(ii) $N \leq Z(H)$.

(1) As H/N cannot act sharply 2-transitively on X and as s is odd, we have that H/N is isomorphic to one of $\mathbf{PSL}(2,s)$, $\mathbf{PSU}(3, \sqrt[3]{s^2})$, $\mathbf{R}(\sqrt[3]{s})$, and each of these groups is perfect, except when $s = 27$ and $H/N \cong \mathbf{R}(3)$. But that case does not occur by [94]. Hence $(H/N)' = H/N = H'N/N$, and so $H = H'N$. Note that $H = H'N = NH'$, and $|H| = \frac{|H'| \times |N|}{|H' \cap N|}$. As noted before, $H_0, H_1, \ldots, H_s$ is the complete set of Sylow p-subgroups of H, $s = p^h$. But s and $|N|$ are coprime, and so s must divide $|H'|$. As an immediate corollary, H' and H have the same Sylow p-subgroups. As $H' \leq H$ and as H is generated by its Sylow p-subgroups (recall that $H = \langle H_0, H_1, \ldots, H_s \rangle$), $H' = H$, and thus H is a perfect group.

(2) As N is the kernel of the action of H on X, N is a normal subgroup of H. Consider H_i, where $i \in \{0, 1, \ldots, s\}$ is arbitrary. Then N normalizes H_i, and so, as H is generated by the H_j, $j = 0, 1, \ldots, s$, $[N, H] = \{\mathbf{1}\}$ since $N \cap H = \{\mathbf{1}\}$. ∎

Lemma 14.7.13 *$H \cong \mathbf{SL}(2,s)$ and $H/N \cong \mathbf{PSL}(2,s)$.*

Proof. The proof of this lemma is essentially the same of that of Lemma 8.6.5. ∎

14.7.2 Solution of Problem A

Before proceeding with the main result, recall the following result of I. Bloemen, J. A. Thas and H. Van Maldeghem [14]:

Theorem 14.7.14 ([14]) *Each EGQ of order (p,t), $p,t>1$, with p a prime, is isomorphic to $W(p)$, $\mathcal{Q}(4,p)$ or $\mathcal{Q}(5,p)$.*

We arrive at

Theorem 14.7.15 *Suppose $\mathcal{S}$ is an elation generalized quadrangle of order (s,t), $s,t>1$ and s odd, which has a spread of symmetry $\mathbf{T}$. Then $\mathcal{S}\cong\mathcal{Q}(5,s)$.*

Proof. Define $U,[\infty],u_i,x_j,H=H(X)$, etc. as before. Let Λ be an arbitrary H-orbit in $\mathcal{S}\setminus\Gamma(X)$. We already know that $|H|=s^3-s$ and that $H\cong\mathbf{SL}(2,s)$. Note that if $x\in\Lambda$, and if MIx and $M\sim[\infty]$, that $M\setminus proj_{[\infty]}x$ is completely contained in Λ. Also, as $|\Lambda|=s^3-s$, H acts semiregularly on $\mathcal{S}\setminus\Gamma(X)$. Since Λ has size s^3-s, each point of $[\infty]$ is incident with precisely $s-1$ lines which are completely contained in $\Lambda\cup\Gamma(X)$ (as point sets). Define the following point-line incidence structure $\mathcal{S}'=(P',B',I')$;

- LINES. The elements of B' are the lines of $\mathcal{S}'$ and they are of two types:
  1. the lines of $\{[\infty],U\}^\perp\cup\{[\infty],U\}^{\perp\perp}$;
  2. the lines of $\mathcal{S}$ which contain a point of Λ and a point of $\Gamma(X)$.
- POINTS. The elements of P' are the points of the incidence structure and they are just the points of $\Gamma(X)\cup\Lambda$.
- INCIDENCE. Incidence I' is the 'induced incidence'.

Observe that H_i, where $i=0,1,\ldots,s$ is arbitrary, fixes at least one line O through u_i different from U and $proj_{[\infty]}u_i$, and which is, as a point set, contained in Λ (as in Lemma 14.7.9). We now repeat the same kind of argument as in the proof of Lemma 14.7.9. As each point of O is a point of $\mathcal{S}'$, it follows easily that the points on the lines of $\{O,[\infty]\}^\perp$ are completely contained in $\mathcal{S}'$. Now note that H acts transitively on the lines of $\mathcal{S}'$ which are incident with an arbitrary but fixed point z of $[\infty]$ and which are not contained in X. Take such a line WIz. Then clearly the stabilizer H_W of W in H acts transitively on $X\setminus Z$, where ZIz and $Z\in X$. As the point set and the line set of $\{O,[\infty]\}^\perp$ are completely contained in $\mathcal{S}'$, then, as a line set,

$$|(\{O,[\infty]\}^{\perp})^H| \geq (s-1)s + (s-1).$$

This implies immediately that for each pair $\{W, W'\}$ of non-concurrent lines in $[\infty]^{\perp} \cap \mathcal{S}'$, the regular span of lines $\{W, W'\}^{\perp\perp}$ is completely contained in $\mathcal{S}'$ (since there are $s^2(s^2+s)$ such ordered pairs $\{W, W'\}$ in $\mathcal{S}'$, and noting that $Aut(\mathcal{S})_{\mathcal{S}'}$ acts transitively on the lines of $(\mathcal{S}' \cap [\infty]^{\perp}) \setminus X$). By Theorem 3.2.1, $\mathcal{S}'$ is a subGQ of $\mathcal{S}$ of order s, as $\mathcal{S}'$ clearly induces a proper subnet of $\mathcal{N}_{[\infty]}$ of the same degree. Now observe the following two facts.

(i) $Aut(\mathcal{S})$ acts transitively on the pairs of non-concurrent lines in $[\infty]^{\perp}$.

(ii) Fix such a pair $\{W, W'\}$ of lines. Then $\{W, W'\}$ is regular, and since there are $s+1$ $H(\{W, W'\})$-orbits in $\mathcal{S} \setminus \Gamma(\{W, W'\}^{\perp\perp})$, $s+1$ subGQ's of $\mathcal{S}$ arise of order s which mutually intersect in $\Gamma(\{W, W'\}^{\perp\perp})$.

From (i) and (ii) and an easy counting argument, it follows that $\mathcal{S}$ contains s^3+s^2 subGQ's of order s which all contain $[\infty]$. Now consider the net $\mathcal{N}_{[\infty]}$. Then $\mathcal{N}_{[\infty]}$ contains at least s^3+s^2 (and then precisely) distinct affine planes of order s, and hence, by Theorem 3.1.2, $N_{[\infty]}$ is isomorphic to $(H_s^3)^D$ (as a corollary, the aforementioned s^3+s^2 affine planes are Desarguesian). Fix W and W', and suppose that $\mathcal{S}'$ is a subGQ of order s containing $\Gamma(\{W, W'\}^{\perp\perp})$. Put $X = \{W, W'\}^{\perp\perp}$, and adopt the notations U, x_i, u_j, H_k, H, etc. from above. Put $L_i = x_i u_i$, $i = 0, 1, \ldots, s$. The group H_i, where $i \in \{0, 1, \ldots, s\}$ is arbitrary, fixes $\mathcal{S}'$, all lines through x_i, the line U and $\{W, W'\}^{\perp\perp}$. Now we interpret H_i (with the same notation) in the projective plane Π which is the projective completion of the affine plane $\mathcal{N}_{\mathcal{S}'}$ which is induced by $\mathcal{S}'$ in $\mathcal{N}_{[\infty]}$. Then each element of H_i fixes the point $\{W, W'\}^{\perp\perp}$, and all lines through the point x_i. Suppose that Z is the line of $\{W, W'\}^{\perp}$ which is incident with x_i. As a central collineation of a finite projective plane is also axial (and conversely), each element of H_i also fixes each point of Z (in Π). Since H_i acts faithfully on Π, and since the same reasoning can be made for each of the $s+1$ subGQ's of order s through $\Gamma(\{W, W'\}^{\perp\perp})$, each $(s+1) \times (s+1)$-grid which contains Z and $[\infty]$ is fixed by each element of H_i. Hence, in $\mathcal{S}$, each line through u_i is fixed by each element of H_i. Let $H_{\mathbf{T}}$ be the group of $s+1$ collineations of $\mathcal{S}$ which all fix $\mathbf{T}$ elementwise, and note the following important property (which we state, for the sake of convenience, in terms of the line L_0):

(F) *Suppose that $h \in H_{\mathbf{T}}$ is nontrivial, and suppose that $L_0^h = L_i$ for some $i \in \{1, 2, \ldots, s\}$. Let $g \in H_0$ be arbitrary but nontrivial. Then $h^{-1}gh = g^h$ is an element of H_i with the property that, if L is an arbitrary line of $\{[\infty], U\}^{\perp\perp}$, then $L^h = L^{h^g}$.*

Property (F) readily follows from the fact that each element of $H_{\mathbf{T}}$ fixes each line of $\{[\infty], U\}^{\perp\perp}$. Whence, for each $g \in H_0$ and each $h \in H_{\mathbf{T}}$, we have that

(F′) *The element* $g(g^h)^{-1}$ *of* H *fixes each line of* $\{[\infty], U\}^{\perp\perp}$.

Now suppose that $g(g^h)^{-1} = g'((g')^{h'})^{-1}$ for nontrivial $h, h' \in H_{\mathbf{T}}$ and $g, g' \in H_0$. Then $(g')^{-1}g = ((g')^{h'})^{-1}g^h$. First note that the left hand side is an element of H_0. If $((g')^{h'})^{-1}$ and g^h both are elements of the same H_j for some $j \in \{1, 2, \ldots, s\}$, then clearly $g = g'$ and $h = h'$. If $((g')^{h'})^{-1}$ and g^h are not elements of the same H_j for all $j \in \{1, 2, \ldots, s\}$, then we have a contradiction against the fact that $H_m H_n \cap H_k = \{\mathbf{1}\}$ for distinct m, n and k (as $\{H_0, H_1, \ldots, H_s\}$ is a 4-gonal basis of $H \cong \mathbf{SL}(2, s)$). Hence the group Φ which is defined by

$$\Phi = \langle g(g^h)^{-1} \parallel g \in H_0, h \in H_{\mathbf{T}} \rangle,$$

and which has the property that each of its elements fixes $\{[\infty], U\}^{\perp\perp}$ elementwise, has at least size s^2. Now recall Dickson's classification of the subgroups of $\mathbf{PSL}(2, q)$, with $q = p^h$, p a prime (see [82, Hauptsatz 8.27, p. 213]); we list the possible subgroups $H \leq \mathbf{PSL}(2, q)$, as follows:

(i) H is an elementary abelian p-group;

(ii) H is a cyclic group of order k, where k divides $\frac{q \pm 1}{r}$, where $r = gcd(q-1, 2)$;

(iii) H is a dihedral group of order $2k$, where k is as in (ii);

(iv) H is the alternating group A_4, where $p > 2$ or $p = 2$ and $h \equiv 0 \mod 2$;

(v) H is the symmetric group S_4, where $p^{2h} - 1 \equiv 0 \mod 16$;

(vi) H is the alternating group A_5, where $p = 5$ or $p^{2h} - 1 \equiv 0 \mod 5$;

(vii) H is a semidirect product of an elementary abelian group of order p^m with a cyclic group of order k, where k divides $p^m - 1$ and $p^h - 1$;

(viii) H is a $\mathbf{PSL}(2, p^m)$, where m divides h, or a $\mathbf{PGL}(2, p^n)$, where $2n$ divides h.

First, one observes that Φ interpreted as a permutation group on $\{[\infty], U\}^{\perp}$ induces a subgroup Φ' of $\mathbf{PSL}(2, s)$ which also has size at least s^2. Considering Dickson's classification, we clearly can only have the following three possibilities for Φ' if $\Phi' \neq \mathbf{PSL}(2, s)$:

1. $\Phi' \cong A_4$, then $|\Phi'| = 12$, and so $s = 3$;
2. $\Phi' \cong A_5$, then $|\Phi'| = 60$, and so $s \leq 7$;
3. $\Phi' \cong S_4$, then $|\Phi'| = 24$, and so $s \leq 4$.

As s is odd, we hence have that $s \in \{3, 5, 7\}$ if $\Phi' \neq \mathbf{PSL}(2, s)$. But as $\mathcal{S}$ is an EGQ (for each point on $[\infty]$), $\mathcal{S}$ is classical by Theorem 14.7.14, so $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5, 3)$, $\mathcal{Q}(5, 5)$ or $\mathcal{Q}(5, 7)$, respectively. Suppose we are not in one of these cases. Then Φ' coincides with $H/N \cong \mathbf{PSL}(2, s)$, and hence s divides $|\Phi'|$, and so also $|\Phi|$. As H is generated by its (Sylow) subgroups of order s, we hence can conclude that $\Phi = H \cong \mathbf{SL}(2, s)$. So each element of H fixes each line of $\{[\infty], U\}^{\perp\perp}$. Now consider H_i, $i \in \{0, 1, \ldots, s\}$ arbitrary. As was noted above, each element θ of H_i has the property that it fixes each line through every point on L_i which is fixed by θ. Hence H_i is a group of symmetries. It readily follows that each line of $[\infty]^\perp$ is an axis of symmetry, and hence each point on $[\infty]$ is a translation point. The theorem now follows from Theorem 14.5.2. ∎

We obtain a complete classification of those EGQ's of order (s, s^2), $s > 1$, admitting a spread of symmetry.

Theorem 14.7.16 *Suppose $\mathcal{S}$ is an elation generalized quadrangle of order (s, t), $s, t > 1$, which has a spread of symmetry $\mathbf{T}$. Then $\mathcal{S} \cong \mathcal{Q}(5, s)$.*

Proof. For s even, the result follows from Theorem 14.5.5. For s odd, Theorem 14.7.15 yields the result. ∎

Chapter 15

Blueprint for the Classification of Translation Generalized Quadrangles

In the present chapter, we will describe a classification program for all translation generalized quadrangles, which is suggested directly by the main results of Chapter 8, Chapter 9, Chapter 10 and certain results of Chapter 12. Other (possible) indications of the correctness of that program were obtained in Chapter 11. The blueprint thus obtained is so that some large parts of that program are completely solved in the aforementioned chapters.

The ideas presented here stem from some conjectures which we wrote down in the extensive (partial) survey *Automorphisms and Characterizations of Finite Generalized Quadrangles*, which appeared in the book *Generalized Polygons* (Proceedings of the Academy Contact Forum 'Generalized Polygons', 20 October 2000, Palace of the Academies, Brussels) [206].

15.1 The Classification of Translation Generalized Quadrangles

Essential in the classification of all translation generalized quadrangles is, clearly, the determination of those TGQ's $\mathcal{S} = T(\mathcal{O})$ of order (q^n, q^m), where q is odd if $n = m$, for which $\mathcal{S} = T(\mathcal{O}) \cong \mathcal{S}^* = T(\mathcal{O}^*)$. The only known examples with that property are:

(i) FOR $n = m$. The $T_2(\mathcal{O})$ of Tits of order q^n, q odd (i.e. the classical GQ $\mathcal{Q}(4, q^n)$, as in that case $\mathcal{O}$ is a conic of $\mathbf{PG}(2, q^n)$);

(ii) FOR $n \neq m$.

(a) The $T_3(\mathcal{O})$ of Tits of order (q^n, q^{2n}).

(b) The TGQ's $\mathcal{S}(\mathcal{F})^D$, $\mathcal{F}$ a Kantor flock.

Note that a first step in a classification of the TGQ's with the aforementioned property was already obtained in Chapter 11, where we indicated that, if $\mathcal{S} = T(\mathcal{O})$ is as above, and if (∞), respectively $(\infty)'$, is a translation point of $\mathcal{S}$, respectively $\mathcal{S}^* = T(\mathcal{O}^*)$, then

$$[Aut(\mathcal{S})_{(\infty)}]_u \cong [Aut(\mathcal{S}^*)_{(\infty)'}]_v,$$

where u and v are arbitrary points of $T(\mathcal{O})$ and $T(\mathcal{O}^*)$, not collinear with (∞) and $(\infty)'$, respectively, and to that end generalized oval cones, respectively generalized ovoid cones, were introduced. Also, from Chapter 8 and Chapter 9, we can derive that if a TGQ $\mathcal{S}$ of order (q^n, q^m) (without the restriction on q if $n = m$) has distinct translation points, then one of the following necessarily holds:

(i) $\mathcal{S} \cong \mathcal{Q}(4, q^n)$;

(ii) $\mathcal{S} \cong \mathcal{Q}(5, q^n)$;

(iii) $\mathcal{S}^* \cong \mathcal{S}(\mathcal{F})^D$ for some flock $\mathcal{F}$ (and hence $2n = m$), q odd,

and if, in the last case, $\mathcal{S}^*$ also has distinct translation points, then

(iv) $\mathcal{S} \cong \mathcal{S}^* \cong \mathcal{S}(\mathcal{F})^D$, where $\mathcal{F}$ is a Kantor flock.

In fact, the main result of Chapter 10 asserts that, conversely, each TGQ $\mathcal{S} = T(\mathcal{O})$ for which $\mathcal{S}^* \cong \mathcal{S}(\mathcal{F})^D$ for some flock $\mathcal{F}$ and q is odd, always has distinct translation points. If $\mathcal{S}$ and $\mathcal{S}^*$ both have one translation point, where q is odd if $n = m$, then

$$Aut(\mathcal{S})_u \cong Aut(\mathcal{S}^*)_v,$$

where u and v are arbitrary points of $T(\mathcal{O})$ and $T(\mathcal{O}^*)$, not collinear with (∞) and $(\infty)'$, respectively.
Now suppose that

(TGQ1) *the TGQ's $\mathcal{S} = T(\mathcal{O})$ of order (q^n, q^m), where q is odd if $n = m$, for which $\mathcal{S} = T(\mathcal{O}) \cong \mathcal{S}^* = T(\mathcal{O}^*)$, precisely are the classical GQ $\mathcal{Q}(4, q^n)$, q odd, a $T_3(\mathcal{O})$ of Tits of order (q^n, q^{2n}), and the TGQ's $\mathcal{S}(\mathcal{F})^D$ with $\mathcal{F}$ a Kantor flock;*

(TGQ2) *if $\mathcal{S}$ and $\mathcal{S}^*$ (in the case where the latter is defined) both have one translation point, then*

$$Aut(\mathcal{S}) \cong Aut(\mathcal{S}^*);$$

(TGQ3) *if $\mathcal{S}$ and $\mathcal{S}^*$ (in the case where the latter is defined) both have one translation point, then (TGQ2) suggests that $\mathcal{S} \cong \mathcal{S}^*$.*

Then we immediately would obtain that each TGQ $\mathcal{S} = T(\mathcal{O})$ of order (q^n, q^m) is of one of the following types:

(C1) $\mathcal{S}$ is a $T_2(\mathcal{O})$ of Tits of order q^n, q even;

(C2) $\mathcal{S} \cong \mathcal{Q}(4, q^n)$, q odd;

(C3) $\mathcal{S}$ is a $T_3(\mathcal{O})$ of Tits of order (q^n, q^{2n}), q even;

(C4) $\mathcal{S} \cong \mathcal{Q}(5, q^n)$, q odd;

(C5) $\mathcal{S}$ is the translation dual of the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ of order (q^{2n}, q^n) (so $\mathcal{O}$ is good at some element π), q odd;

(C6) $\mathcal{S}$ is the point-line dual of a flock GQ $\mathcal{S}(\mathcal{F})$ of order (q^{2n}, q^n) (so $\mathcal{O}^*$ is good at some element π', where $\mathcal{S}^* = T(\mathcal{O}^*)$), q odd.

15.2 Corollaries

There would be several 'direct' corollaries. The high degree of difficulty of the parts of the blueprint which remain to be solved is read from the importance of those corollaries.

1. EACH TGQ HAS CLASSICAL ORDER, I.E., IS OF ORDER s OR OF ORDER (s, s^2) FOR SOME s. This is one of the most intriguing conjectures in the theory of finite (translation) generalized quadrangles, and was already verified to be true for s even, see Chapter 1. It is also an essential part of the "classical order conjecture" for finite thick generalized quadrangles, stating that for each finite GQ of order (s,t), $s \neq 1 \neq t$, s and t have a 'classical' form, namely, $t \in \{\sqrt{s}, \sqrt[3]{s^2}, s-2, s, s+2, \sqrt{s^3}, s^2\}$, where s is a prime power.

2. THERE ARE NO NON-CLASSICAL TGQ'S OF ORDER s WITH s ODD. Here it should certainly be noted that there are no non-classical GQ's known of order $s > 1$, s odd. To that end, it would thus be a very interesting observation.

3. THE ONLY TGQ'S WITH AN ODD NUMBER OF POINTS ON A LINE ARISE FROM THE $T_d(\mathcal{O})$ CONSTRUCTION OF TITS, $d = 2, 3$. This would be an impressive result as it already is a famous open problem to classify those TGQ's $\mathcal{S} = T(\mathcal{O})$ of order (s, s^2), s even, which have a good element.

4. CLASSIFICATION OF GENERALIZED OVOIDS (AND GENERALIZED OVALS). If the predicted classification is true, then the only possible generalized ovoids $\mathcal{O}$ in $\mathbf{PG}(2n+m-1, q)$ would be so that:

   (a) $m = 2n$ and $\mathcal{O}$ can be interpreted as an ovoid in $\mathbf{PG}(3, q^n)$ (in the obvious sense), and in that case, at least two (and then each) element of $\mathcal{O}$ is good; also, in the even characteristic case, each generalized ovoid is obtained in this way;

   (b) $\mathcal{O}$ is good at some element π;

   (c) $\mathcal{O}^*$ is good at some element π'.

   For generalized ovals in $\mathbf{PG}(3n-1, q)$, there would follow that if q is odd, each such $\mathcal{O}$ arises as in (a) from a plain conic of $\mathbf{PG}(2, q^n)$; if q is even, $\mathcal{O}$ arises as in (a) from some oval in $\mathbf{PG}(2, q^n)$.

5. NONEXISTENCE OF m-SYSTEMS. Suppose $\mathcal{M}$ is a collection of totally singular subspaces of a finite classical non-degenerate polar space $\triangle$. Then $\mathcal{M}$ is called an *m-system* if the elements of $\mathcal{M}$ are pairwise opposite $\mathbf{PG}(m,q)$'s of $\triangle$, and if a theoretical maximal bound μ_m is attained for $|\mathcal{M}|$, see [158] (this number μ_m is just the number of elements of a partition of $\triangle$ by generators).
In *Constructions of polygons from buildings* [159], E. E. Shult and J. A. Thas (who also introduced the notion of m-systems), developed a construction method of finite geometries (with special emphasis on generalized quadrangles) using a strange subset $\mathcal{C}$ of flags of a known finite geometry and then defining new objects by a process of sequentially taking collections of new flags in various residues according to certain rules (which can be thought of as a game played on a Dynkin diagram). The 'new' geometry is denoted by $\Gamma(\mathcal{C})$. The construction of the point-line geometry $\Gamma(\mathcal{C})$ starts with a 'point' $p = F_0$, where F_0 is a flag of a spherical building $\triangle$. The stabilizer of F_0 in the automorphism group $Aut(\triangle)$ of $\triangle$ (which is a Chevalley group) is a parabolic subgroup of $Aut(\triangle)$ which induces an action on the residue $Res_{\triangle}(F_0)$. The kernel of this action is a semidirect product of a normal unipotent group U, possibly extended by a certain group[1]. That unipotent group U induces a group of automorphisms of $\Gamma(\mathcal{C})$ which fixes the 'point' $p = F_0$ and all 'lines' incident with it. Also, U acts regularly on the set of objects of $\Gamma(\mathcal{C})$ farthest from p. In the special case of generalized quadrangles, this means that $\Gamma(\mathcal{C})$ is an EGQ with elation point p. If the unipotent group U is abelian, then it follows that $\Gamma(\mathcal{C})^{(p)}$ is a TGQ with translation point p. This is the case when $\triangle$ is a polar space defined by a quadratic form [159], and F_0 is a point of the quadric. Let us list three distinct possible cases for TGQ's which arise from the latter observation, in the notation of [159]. Recall from [158] that an m-sytem $\mathcal{M}$ of a polar space $\triangle$ satisfies the *BLT-Property* if each line which lies on $\triangle$ intersects no more than two distinct elements of $\mathcal{M}$ nontrivially.

   (a) $\triangle = \mathcal{Q}^+(7,q)$, $\overline{\triangle} = \mathcal{Q}^+(5,q)$, $\overline{\mathcal{C}}$ IS A 1-SYSTEM OF q^2+1 LINES IN $\mathcal{Q}^+(5,q)$, q ODD, WITH THE BLT-PROPERTY. The corresponding TGQ has order (q^2,q^2). By [158], only one such $\Gamma(\mathcal{C})$ exists; it is isomorphic to $\mathcal{Q}(4,q^2)$. Thus it is an extra reason for the blueprint to be true in the $(s=t)$-case.

   (b) $\triangle = \mathcal{Q}^+(11,q)$, $\overline{\triangle} = \mathcal{Q}^+(9,q)$, $\overline{\mathcal{C}}$ IS A 2-SYSTEM OF q^4+1 PLANES IN

[1] A scalar action centralized by the 'Levi factor', see [159].

$\mathcal{Q}^+(9,q)$, q ODD, WITH THE BLT-PROPERTY. The corresponding TGQ would have order (q^3,q^4). No such TGQ, and whence no such 2-sytem $\overline{\mathcal{C}}$, exists if the blueprint appears to be true.

(c) $\triangle = \mathcal{Q}^-(4r-3,q)$, $\overline{\triangle} = \mathcal{Q}^-(4r-5,q)$, $\overline{\mathcal{C}}$ IS AN $(r-2)$-SYSTEM OF $q^{2r-2}+1$ $(r-2)$-DIMENSIONAL SPACES IN $\mathcal{Q}^-(4r-5,q)$, $r \geq 3$, WITH THE BLT-PROPERTY. The corresponding TGQ has order (q^{r-1},q^{2r-2}). For each value of r, one such $\Gamma(\mathcal{C})$ is known; it is isomorphic to $\mathcal{Q}(5,q^{r-1})$. If it appears that two such TGQ's $\Gamma(\mathcal{C})$ and $\Gamma(\mathcal{C}')$ are isomorphic if and only if the corresponding $(r-2)$-systems $\overline{\mathcal{C}}$ and $\overline{\mathcal{C}'}$ are isomorphic (as $(r-2)$-systems of $\mathcal{Q}^-(4r-5,q)$), and if the blueprint is true, then as $\mathcal{Q}(5,q^{r-1})$ is the only GQ arising from this construction, there can be only one such an $(r-2)$-system.

It is the major aim of the papers [191], [193] (etc.)[2] to complete the classification program as described in this chapter. We refer the reader to the extensive paper [192] for a very thorough synopsis on the subject of this chapter.

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[2]Which are all co-authored by J. A. Thas.

Appendix A

A Proof for the Hole in the Moufang Theorem

A.1 The Hole in the Moufang Theorem

In 1978, S. E. Payne and J. A. Thas started the program to prove the Moufang theorem of P. Fong and G. M. Seitz [60, 61], cf. Theorem 1.4.1, for finite generalized quadrangles, without the use of 'deep group theory' (and hence geometrically more satisfactory), see Chapter 9 of FGQ, or Chapter 13. They came very close to obtaining a proof; their only obstacle was (essentially) the following problem:

PROBLEM. *Suppose that $\mathcal{S}$ is a thick GQ of order (s, s^2), all lines of which are axes of symmetry. Then $\mathcal{S} \cong \mathcal{Q}(5, s)$.*

W. M. Kantor gave a proof of this theorem in [93], where he used the classification of the split BN-pairs of rank 1, and 4B,C of P. Fong and G. M. Seitz

[60], but the proof is still not elementary in the sense of S. E. Payne and J. A. Thas.

Theorem A.1.1 ([60, 61]; [93]) *If $\mathcal{S}$ is a GQ of order (s, s^2), $s > 1$, each line of which is an axis of symmetry, then $\mathcal{S}$ is isomorphic to $\mathcal{Q}(5, s)$.*

We do not finish the aforementioned program here, but we give a proof of Theorem A.1.1 without the use of any result of [60, 61]; we still have to invoke the classification of the split BN-pairs of rank 1 [156, 72], however. As will be pointed out, we do not need that result 'intrinsically'; we will at some stage only need the order of some group (it is there that our geometrical proof fails). The proof given in the next section is taken from Section 8.4 of *K. Thas, Automorphisms and Characterizations of Finite Generalized Quadrangles*, which was published in *Proceedings of the Academy Contact Forum 'Generalized Polygons', 20 October 2000, Palace of the Academies, Brussels* [206].

A.2 Sketch of the Proof of the Theorem

In Chapter 9, we showed geometrically that given an SPGQ $\mathcal{S}$ of order (s, t), $s \neq 1 \neq t$ and $s \neq t$ (and hence $t = s^2$), with base-group G and base-grid Γ, we have $|G| \geq s^3 - s$. Moreover, if $|G| = s^3 - s$, then there are $s + 1$ subGQ's of order s, all isomorphic to $\mathcal{Q}(4, s)$ and mutually intersecting in Γ. Although we used Theorem 8.8.1 to show that these subGQ's are classical, this result is not needed here since all lines of each such subGQ are regular as each line of $\mathcal{S}$ is regular; by the dual of Theorem 1.2.2, we can then conclude that each such subGQ indeed is isomorphic to $\mathcal{Q}(4, s)$. It is at this point that we need to prove that $|G|$ equals $s^3 - s$, so that we can apply Section 9.8 of Chapter 9. Relying on [156, 72], it is possible to show this, see Chapters 9 and 12, and since $\mathcal{S}$ is an SPGQ for any two non-concurrent lines, there easily follows by e.g. Theorem 1.6.6 that $\mathcal{S} \cong \mathcal{Q}(5, s)$. ■

Appendix B

A Generalization of the Theorem of Higman-Bose-Shrikhande-Cameron

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In the present appendix, we will prove Theorem 7.4.2 (Theorem B.1.1 below) of Chapter 7. As a corollary, we will obtain the inequality of D. G. Higman [74, 75] for finite generalized quadrangles, and the inequality of P. J. Cameron [33] for finite partial quadrangles. Also, the proof will yield some valuable information for those semi quadrangles having triads with a constant number of centers, and that information generalizes corresponding results of C. C. Bose and S. S. Shrikhande [17] (for finite generalized quadrangles) and P. J. Cameron [33] (for finite partial quadrangles).

Theorem B.1.1 and its proof are taken from [212]. The method is essentially that of P. J. Cameron [33].

B.1 A Generalization of the Theorem of Higman-Bose-Shrikhande-Cameron

Theorem B.1.1 *Suppose $\mathcal{S}$ is a semi quadrangle with extremal order $(s; t_1, t_n)$ and with extremal μ-parameters (μ_1, μ_m). Then we have the following inequality:*

$$[(t_1-1)s\mu_1]^2 \leq \mu_m[(\mu_m-1)(\mu_m-2)+(t_n-1)s](\lfloor\frac{(t_n+1)t_n s^2}{\mu_1}\rfloor$$

$$-s(t_1+1)+\mu_m-1). \tag{B.1}$$

If equality holds, then there is a constant $x_0 = \frac{(t_1-1)s\mu_1}{\lfloor\frac{(t_n+1)t_n s^2}{\mu_1}\rfloor - s(t_1+1)+\mu_m-1}$ such that each triad of points has exactly x_0 centers.
Also, if each triad of points has a constant number of centers, then

$$[(t_n-1)s\mu_m]^2 \geq \mu_1[(t_1-1)s+(\mu_1-1)(\mu_1-2)](\lceil\frac{(t_1+1)t_1 s^2}{\mu_m}\rceil$$

$$-s(t_n+1)+\mu_1-1). \tag{B.2}$$

Proof. Suppose p is a point of degree t_1 of $\mathcal{S}$. We repeat our general assumption that $t_1 \leq t_n$ and $\mu_1 \leq \mu_m$. There are $(t_1+1)s$ points collinear with, and different from, p and, if p' is such a point, then there are at least $t_1 s$ points collinear with, and different from, p' and not collinear with p. Since $\mu_1 \leq \mu_m$, there are at least $\frac{(t_1+1)t_1 s^2}{\mu_m}$ points not collinear with p.
If q is a point not collinear with p, then there are at most $s(t_n+1)-\mu_1$ points collinear with q and not with p, but different from q. So, there are at least $a = \lceil\frac{(t_1+1)t_1 s^2}{\mu_m}\rceil - s(t_n+1)+\mu_1-1$ points not collinear with p and q. Analogously there are at most $a'' = \lfloor\frac{(t_n+1)t_n s^2}{\mu_1}\rfloor - s(t_1+1)+\mu_m-1$ points not collinear with both p and q.
We suppose a' is the *precise* number of points not collinear with p and q. Suppose $p_1, p_2, \ldots, p_{a'}$ are these a' points, and suppose x_i is the number of points collinear with p, q and p_i $(1 \leq i \leq a')$. Then we have the following inequalities:

$$b = (t_1-1)s\mu_1 \leq \sum_{i=1}^{i=a'} x_i \leq (t_n-1)s\mu_m = b' \tag{B.3}$$

(since for each of the at most μ_m points collinear with p and q, there are at most $(t_n - 1)s$ choices for p_i and for each of the at least μ_1 points collinear with p and q there are at least $(t_1 - 1)s$ choices for p_i);

$$\mu_1(\mu_1 - 1)(\mu_1 - 2) \leq \sum_{i=1}^{i=a'} x_i(x_i - 1) \leq \mu_m(\mu_m - 1)(\mu_m - 2) \tag{B.4}$$

(since for each pair of distinct points collinear with p and q there are at most $\mu_m - 2$ choices for p_i and at least $\mu_1 - 2$ choices for p_i).
It follows that

$$c = \mu_1(\mu_1 - 1)(\mu_1 - 2) + (t_1 - 1)s\mu_1 \leq \sum_{i=1}^{i=a'} x_i^2 \tag{B.5}$$

and

$$\sum_{i=1}^{i=a'} x_i^2 \leq \mu_m(\mu_m - 1)(\mu_m - 2) + (t_n - 1)s\mu_m = c'. \tag{B.6}$$

So, first of all,

$$\sum_{i=1}^{i=a'} (x_i - x)^2 = \sum_{i=1}^{i=a'} x^2 - 2\sum_{i=1}^{i=a'} xx_i + \sum_{i=1}^{i=a'} x_i^2 \leq a'x^2 - 2bx + c'. \tag{B.7}$$

Since the left-hand side is a positive semi-definite quadratic form, we can conclude that

$$b^2 \leq a'c' \leq a''c', \tag{B.8}$$

and $b^2 \leq a''c'$ gives the first inequality of the theorem.
If equality holds, then with $a' = a''$ and $x_0 = \frac{b}{a'} = \frac{c'}{b} = \frac{b}{a''}$, we have that

$$\sum_{i=1}^{i=a'} (x_i - x_0)^2 = 0,$$

and so $x_i = x_0$ for $i = 1, 2, \ldots, a'$.
Now we clearly also have the following inequality:

$$ax^2 - 2b'x + c \leq \sum_{i=1}^{i=a'} (x_i - x)^2. \tag{B.9}$$

Since (B.2) is trivial if $a \leq 0$, we suppose that $a > 0$. If we suppose that $x_i = x_0$ for a certain constant x_0 and every i, then we have that

$$ax_0^2 - 2b'x_0 + c \leq 0, \tag{B.10}$$

and since $a > 0$, we now know that this quadratic form has at least one real root. Hence $(b')^2 > ac$, which yields the complete proof of the theorem. ∎

Note. If equality holds in (B.1), we have the divisibility conditions $a''|b$ and $b|c'$, with $a'' = a'$, b and c' defined as above.

Remark B.1.2 One notes that we used (SQ3) implicitly in the proof of Theorem B.1.1, since we divided by μ_1. Also, we did not 'completely' use (SQ4), and hence the preceding theorem holds for some other incidence geometries.

For the sake of completeness, we recall the following immediate corollaries from Chapter 7.

Corollary B.1.3 (P. J. Cameron [33]) *Suppose $\mathcal{S}$ is a partial quadrangle with parameters (s, t, μ). Then*

$$\mu(t-1)^2 s^2 \leq [s(t-1) + (\mu-1)(\mu-2)][\frac{(t+1)ts^2}{\mu} - (t+1)s + \mu - 1]. \tag{B.11}$$

Equality holds if and only if the number of points collinear with each of any three pairwise non-collinear points is a constant; if this occurs, the constant is $1 + \frac{(\mu-1)(\mu-2)}{s(t-1)}$.

∎

Corollary B.1.4 (D. G. Higman [74, 75]) *Suppose $\mathcal{S}$ is a generalized quadrangle with parameters (s,t), $s \neq 1 \neq t$. Then $t \leq s^2$ and, dually, $s \leq t^2$.*

■

Corollary B.1.5 (C. C. Bose and S. S. Shrikhande [17]) *Let $\mathcal{S}$ be a generalized quadrangle with parameters (s,t), $s \neq 1 \neq t$. Then $t = s^2$ if and only if the number of points collinear with every three pairwise non-collinear points is a constant, and if this occurs, the constant is $s+1$. Dually, $s = t^2$ if and only if the number of lines concurrent with every three pairwise non-concurrent lines is a constant, and if this occurs, the constant is $t+1$.*

■

Appendix C

Some Open Problems

In this appendix, we describe some open problems and conjectures which are relevant for this thesis. We do not repeat the problems which were discussed in detail at the beginning of Chapters 9 and 12, and in Chapter 15, and we refer the interested reader to those chapters. This appendix is mainly based on J. A. Thas [179], K. Thas [206], *K. Thas, Finite Flag-Transitive Projective Planes: A Survey and Some Remarks* [214], which is submitted for publication to *Discrete Mathematics*, and *K. Thas, Finite Flag-Transitive Projective Planes and Fermat Curves* [215], which is submitted for publication to *Journal of Algebraic Combinatorics*.

C.1 Generalized Quadrangles with Symmetry

The following problems are directly inspired by Chapter 6.

PROBLEM. *What is — in general — the minimal number N of distinct axes of symmetry through a point p of a thick GQ $\mathcal{S}$ of order (q^n, q^m), $n \neq m$, forcing $\mathcal{S}^{(p)}$ to be a TGQ?*

By Chapter 6, we know that $N \leq q^m - q^n + 3$.

PROBLEM. *Given a thick TGQ $\mathcal{S}^{(p)}$ of order (q^n, q^m), $n \neq m$, what is the minimal number of lines N' through p such that the translation group is generated by the symmetries about these lines?*

If $n = n'k$ and $m = n'(k+1)$, k odd, and if $\mathbf{GF}(q)$ is the kernel of the TGQ, then

$$N' - 3 \leq n'.$$

PROBLEM. *Are there GQ's of order (s,t), $s \neq 1 \neq t$, for which there is a point incident with precisely $k+1$ axes of symmetry with $k = 1$ if $s = t$, and with $1 \leq k \leq t - s + 1$ if $s \neq t$?*

PROBLEM. *State minimal conditions for (thick) EGQ's $(\mathcal{S}^{(p)}, G)$ such that G is the set of all elations about p.*

If $\mathcal{S}^{(p)}$ is a TGQ with translation point p, then the problem is completely solved, see Chapter 8 of FGQ. Also, if $\mathcal{S}^{(p)}$ is a thick EGQ of order (s,t), then G is always the complete set of elations about p if $t > s^2/2$, see Chapter 6 for more details.

C.2 Some Problems Concerning The Orders of Generalized Quadrangles

PROBLEM. *Assume that $\mathcal{S}$ is a GQ of order (s,t), $s \neq 1 \neq t$. Define minimal hypotheses for $\mathcal{S}$ such that s and t have the same parity.*

There are no examples known for which

$$t - s \not\equiv 0 \mod 2.$$

It seems tempting to conjecture that no such examples exist.

Note. Recall from Chapter 6 that if a thick GQ $\mathcal{S} = (P, B, I)$ of order (s,t) has a point p incident with some axes of symmetry, so that the group generated by the symmetries about these lines acts transitively on $P \setminus p^\perp$, then s and t

have the same parity.

PROBLEM. *Does there exist a GQ of order (s,t), $s < t < s^2$, for which all lines are regular?*

J. A. Thas and H. Van Maldeghem showed that $t \neq s+2$ if $s > 2$, see [200].

PROBLEM. *Suppose $\mathcal{S}$ is an EGQ of order (s,t), $s \neq 1 \neq t$. Show that s and t are powers of the same prime.*

D. Frohardt solved the problem completely if $s \leq t$, and if $\mathcal{S}$ is an EGQ w.r.t. some base-point [63].

PROBLEM. *Does there exist a TGQ of order (q^a, q^{a+1}), q odd and $a > 1$?*

For more details on the latter problem, see Chapter 15.

We end this section with an old and well-known problem, first stated by J. Tits. The problem can be stated for generalized polygons in general, but we only do it for GQ's. In the next problem, we allow the parameters of a GQ to be infinite.

PROBLEM. *Suppose $\mathcal{S}$ is a GQ of order (s,t), $s,t > 1$, and suppose that either $s \in \mathbb{N}$ or $t \in \mathbb{N}$. Then $s,t \in \mathbb{N}$.*

An interesting account on that problem can be found in K. Tent [163], J. A. Thas [179] and H. Van Maldeghem [229].

C.3 Generalized Quadrangles with Small Parameters

PROBLEM. *Is there a unique GQ of order $(4,t)$, $t \in \{6,8,16\}$?*

PROBLEM. *Does there exist a GQ of order 6?*

C.4 Regularity in Finite Generalized Quadrangles

PROBLEM. *Give a proof of the following: 'Suppose $\mathcal{S}^{(x)}$ is a TGQ of order (q, q^2), $q > 1$, with translation point x and all lines regular. Then $\mathcal{S}^{(x)}$ is isomorphic to $\mathcal{Q}(5, q)$'.*

J. A. Thas has some strong unpublished results on this subject. Note that the solution of this problem would almost certainly lead to the solution of the 'Hole in the Moufang Theorem', see Appendix A.

CONJECTURE. *Suppose $\mathcal{S}$ is a thick GQ of order (s, t) with all lines regular. Then $\mathcal{S}$ is of classical type.*

For $s = t$, the conjecture was answered affirmatively by C. T. Benson [10], see Theorem 1.2.2, and this is probably the oldest combinatorial characterization of a class of GQ's. For $s \neq t$, to my knowledge the only known result without additional hypotheses is the result of J. A. Thas and H. Van Maldeghem, saying that $t \neq s + 2$ if $s > 2$ [200]. The situation in the general case seems hopeless at present.

A more reasonable conjecture is the following:

CONJECTURE. *Let $\mathcal{S}$ be a GQ of order (s, s^2), $s > 1$, so that every two nonconcurrent lines of $\mathcal{S}$ are contained in a subGQ of $\mathcal{S}$ of order s which is isomorphic to $\mathcal{Q}(4, s)$. Then $\mathcal{S} \cong \mathcal{Q}(5, s)$.*

There is an analogue for GQ's of order (s^2, s^3), $s > 1$, which is solved completely, see J. A. Thas and H. Van Maldeghem [200].

C.5 Construction and Nonexistence of Finite Generalized Quadrangles

PROBLEM. *Are $\mathcal{Q}(4, q)$ and $W(q)$ the only GQ's of order q, q odd and $q \geq 3$?*

PROBLEM. *Is $AS(q)$ the only GQ of order $(q - 1, q + 1)$, q odd?*

PROBLEM. *Is $H(4, q^2)$ the only GQ of order (q^2, q^3), $q \geq 3$?*

Regarding the last problem, there is not even an 'abstract' construction method which 'could' yield new examples, besides the one of E. E. Shult and J. A. Thas, cf. Chapter 15. This should be a first step in the study of the last problem.

C.6 Ovoids, Spreads and k-Arcs

PROBLEM. *Are there thick GQ's of order (s,t) which have complete $(st-t/s)$-arcs, $s>3$?*

See Chapter 4 for a detailed account on that problem.

PROBLEM. *Classify all ovoids of $\mathcal{Q}(4,q)$.*

PROBLEM. *Classify all spreads of $\mathcal{Q}(5,q)$.*

PROBLEM. *Does $H(4,q^2)$, $q>2$, have a spread?*

Concerning the last problem, we refer the reader to K. Thas [202].

C.7 Other Combinatorial Characterizations of the Finite Generalized Quadrangles

PROBLEM. *Is every GQ of order s, s odd and $s>1$, for which each point is antiregular, isomorphic to $\mathcal{Q}(4,s)$?*

Let $\mathcal{S}=(P,B,I)$ be a GQ of order $s>1$. Recall the following facts from FGQ (see Theorem 3.1.1 and [139, 1.3.2]).

- For a regular point x, the incidence structure π_x with point set $x^\perp$, with line set the set of spans $\{y,z\}^{\perp\perp}$, where $y,z\in x^\perp$ with $y\neq z$, and with the natural incidence, is a projective plane of order s.
- For an antiregular point x and a point y in $x^\perp\setminus\{x\}$, the incidence structure $\pi(x,y)$ with point set $x^\perp\setminus\{x,y\}^\perp$, with lines the sets $\{x,u\}^\perp\setminus\{y\}$ with $y\sim u\not\sim x$, and with the natural incidence, is an affine plane of order s.

PROBLEM. *Are the planes as described here always Desarguesian?*

C.8 Distance-Transitivity in Generalized Polygons

Using the classification of the finite simple groups, F. Buekenhout and H. Van Maldeghem [31] classified all finite generalized polygons having an automorphism group which acts *distance-transitively* [31] on the set of points. This proved an old conjecture of J. Tits saying that every group with an irreducible BN-pair of rank 2 arises from a group of Lie type.

PROBLEM. *Obtain the results of F. Buekenhout and H. Van Maldeghem without the classification of the finite simple groups.*

H. Van Maldeghem recently showed, in the spirit of the latter problem, that a distance-transitive GQ of order p, p a prime, is isomorphic to $W(p)$ or $\mathcal{Q}(4,p)$ [231].

C.9 A Final Problem: Flag-Transitivity of Generalized Polygons

A *flag* of a generalized polygon is just an incident point-line pair. A generalized polygon is *flag-transitive* if it has an automorphism group acting transitively on its flags. The following conjecture was first made by W. M. Kantor in [93], and is regarded as one of the most important conjectures in the theory of collineations in finite generalized polygons:

CONJECTURE. *A finite thick flag-transitive generalized n-gon, $n \geq 3$, is either a classical polygon, or isomorphic to one of the following:*

- *the unique generalized quadrangle of order $(3,5)$ or its point-line dual;*
- *the generalized quadrangle of order $(15,17)$ arising from the Lunelli-Sce oval in $\mathbf{PG}(2,16)$ [106] or its dual.*

For $n = 3$, the problem is much older, and is in fact one of the oldest problems in the theory of projective planes. It was first mentioned in D. G. Higman and J. E. McLaughlin [76], and, by closely following ideas of [110], they showed that

A finite flag-transitive projective plane is Desarguesian if the order is suitably restricted.

Recently, relying on results of W. M. Kantor [92] and W. Feit [57], we have obtained

Theorem C.9.1 (K. Thas [215]) *A non-Desarguesian flag-transitive projective plane of order n, where n necessarily is even, $n > 8$ and $p = n^2 + n + 1$ is a prime, is equivalent to a set of Fermat curves*

$$\mathcal{F} : X^n - Y^n = \eta Z^n,$$

(where $\eta \in \mathbf{GF}(p)^* = \mathbf{GF}(p) \setminus \{0\}$), with the properties that if η or $-\eta \in \mathcal{D}$, where $\mathcal{D}$ be the set of n-th powers of $\mathbf{GF}(p)^*$, then $\mathcal{F}$ has $n^2 + 2n$ $\mathbf{GF}(p)$-rational points, and if $\eta, -\eta \in \mathbf{GF}(p)^* \setminus \mathcal{D}$, then $\mathcal{F}$ has $n^2 + n$ $\mathbf{GF}(p)$-rational points.

Final Remark

It is a pleasing fact that a substantial part of the open problems as presented in Chapter 5 by J. A. Thas [179] in 'Handbook of Incidence Geometry' [28], which appeared in 1995, is completely solved by now[1]. I think that this emphasizes the rate at which the theory is emerging.

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[1]The same could be observed concerning the problems described by S. E. Payne in [122].

Appendix D

Nederlandstalige Samenvatting

In dit proefschrift stellen we een meetkundige studie voor van *eindige veralgemeende vierhoeken*.

- Enerzijds bestuderen we veralgemeende vierhoeken die zekere automorfismen toelaten.
- Anderzijds onderzoeken we zuiver combinatorische problemen (en structuren) in veralgemeende vierhoeken.

Op een natuurlijke manier komen we tot karakteriseringen van de klassieke en andere voorbeelden.

D.1 Een Voorbeschouwing

Een *veralgemeende vierhoek* van de orde (s,t), $s,t>0$, is een 1–$(v,s+1,t+1)$ design van welke de incidentiegraaf omvang (girth) 8 en diameter 4 heeft. De klassieke voorbeelden van veralgemeende vierhoeken ontstaan uit groepen met

een BN-paar van rang 2, voor dewelke de Weyl groep een dihedrale groep is. Veralgemeende vierhoeken werden ingevoerd door Jacques Tits [221] in 1959, als werktuig ter studie van halfenkelvoudige algebraïsche groepen (alsook de groepen van Lie-type en de Chevalley-groepen) van relatieve rang 2, als een bijzondere deelklasse van de *veralgemeende veelhoeken*. Veralgemeende veelhoeken vormen tevens de basisstructuren voor de meer algemene *Tits-gebouwen*; dit zijn de natuurlijke modellen waarop de halfenkelvoudige algebraïsche groepen van willekeurige rang werken. Veralgemeende veelhoeken zijn ook natuurlijke veralgemeningen van *projectieve vlakken*, en toepassingen van de theorie van de veralgemeende veelhoeken kunnen bijna overal gesitueerd worden in de *Incidentiemeetkunde*, maar ook in de studie van de *Designs* en in de *Informatietechnologie* (met in het bijzonder de *Codeertheorie* en de *Cryptologie*, zie bijvoorbeeld [53, 49, 50]).

In 1954 beschreef H. Lenz [104] een classificatie van projectieve vlakken op basis van de deelmeetkunden die gevormd worden door de incidente punt-rechte paren (p, L) waarvoor de vlakken (p, L)-transitief zijn. In 1957 beschouwde A. Barlotti [9] alle punt-rechte paren met deze eigenschap. Op die manier ontstond de bekende 'Lenz-Barlotti classificatie voor projectieve vlakken'. Een van de hoofdpunten van deze thesis is het doorvoeren van gelijkaardige classificatie(s) voor eindige veralgemeende vierhoeken. Dankzij de structuur van veralgemeende vierhoeken zijn er hier meerdere opties. Meer algemeen kunnen we het probleem als volgt stellen:

'Indien we een deelmeetkunde $\mathcal{S}$ beschouwen van een veralgemeende vierhoek, gevormd door objecten van een bepaald type A, wat is dan de grootste deelmeetkunde van de veralgemeende vierhoek die $\mathcal{S}$ bevat en tevens enkel bestaat uit objecten van hetzelfde type A ?'.

Een mogelijkheid is de deelstructuren te onderzoeken die gevormd worden door de *assen van symmetrie* van de veralgemeende vierhoek.

Een dergelijke classificatie werd reeds gestart in onze Licentiaatsthesis [203], en met de technieken die we ontwikkeld hebben in [209, 205, 211, 208, 210, 213, 191], hebben we deze classificatie verder uitgewerkt, wat resulteerde in [218]. Essentieel betreft het

(i) de ontwikkeling van een uitgewerkte theorie voor *span-symmetrische veralgemeende vierhoeken*, een probleem dat reeds meer dan 20 jaar open is (en dat we in [208] volledig opgelost hebben voor veralgemeende vierhoeken van de orde (s, s));

(ii) de classificatie van *translatie veralgemeende vierhoeken* (en meer algemeen, de studie van veralgemeende vierhoeken met een aantal snijdende assen van symmetrie);

(iii) de beschrijving van een theorie die (i) en (ii) unificeert.

Een ander hoofdpunt in de probleemstelling is het vinden van nieuwe karakteriseringen van

- de *klassieke veralgemeende vierhoeken* enerzijds, en van, voornamelijk,
- *flock veralgemeende translatie vierhoeken* anderzijds (die meer en meer een basisplaats innemen in de moderne studie van de veralgemeende vierhoeken).

Uiteindelijk zal ook de studie van combinatorische structuren in veralgemeende vierhoeken, zoals *deelvierhoeken*, *ovoïden*, *spreads* en *complete bogen*, maar ook de interactie tussen *netten* en veralgemeende vierhoeken met een *regulier punt*, een cruciale rol spelen in de behandeling van (i), (ii) en (iii). [1] Deelvierhoeken duiken in bijna alle gebieden van de veralgemeende vierhoeken op een natuurlijke manier op. In het bijzonder is het classificeren van deelvierhoeken in veralgemeende vierhoeken die welbepaalde eigenschappen hebben van fundamenteel belang, omdat er veel karakteriseringen van o.a. klassieke -en flock veralgemeende vierhoeken zijn die (impliciet) gebruik maken van de aanwezigheid van deelvierhoeken. In [210, 218] bewezen we onder andere:

Elke span-symmetrische veralgemeende vierhoek van de orde (s, s^2), $s > 1$, *heeft* $s + 1$ *deelvierhoeken van de orde* s *(in een welbepaalde ligging), die allen isomorf zijn met* $\mathcal{Q}(4, s)$.

Dit resultaat zal een bepalende observatie blijken voor het proefschrift.

D.2 Veralgemeende Vierhoeken

D.2.1 Eindige veralgemeende vierhoeken

Een (eindige) *veralgemeende vierhoek van de orde* (s, t) is een incidentiestructuur $\mathcal{S} = (P, B, I)$, waarbij P en B niet-ledige, disjuncte verzamelingen zijn

[1] In de appendix van Hoofdstuk 9 geven we, bijvoorbeeld, een eenvoudige nieuwe constructie van grote klassen van spreads in span-symmetrische veralgemeende vierhoeken van de orde (s, s^2), $s > 1$. Deze spreads worden vastgehouden door een automorfismengroep van de veralgemeende vierhoek die $\mathbf{SL}(2, s)$ bevat.

van objecten die we respectievelijk 'punten' en 'rechten' noemen, en waarvoor I een symmetrische punt-rechte incidentierelatie is die voldoet aan de volgende drie axioma's.

(VV1) Elk punt is incident met $t+1$ rechten, $t \geq 1$, en twee verschillende punten zijn incident met ten hoogste één rechte ($t \in \mathbb{N}$).

(VV2) Elke rechte is incident met $s+1$ punten, $s \geq 1$, en twee verschillende rechten zijn incident met ten hoogste één punt ($s \in \mathbb{N}$).

(VV3) Als p een punt is en L is een rechte niet incident met p, dan is er een uniek paar $(q, M) \in P \times B$ waarvoor $pIMIqIL$.

Als $s = t$, dan zeggen we ook dat $\mathcal{S}$ van *de orde* s is.

Het basiswerk betreffende eindige veralgemeende vierhoeken is de monograaf *Finite Generalized Quadrangles* van S. E. Payne en J. A. Thas [139] (in het vervolg vaak afgekort door **FGQ**). We schrijven in het vervolg soms 'GQ' i.p.v. 'veralgemeende vierhoek', naar de Engelse term 'generalized quadrangle'. Ook zullen we soms i.p.v. 'veralgemeende vierhoek' gewoon de term 'vierhoek' gebruiken.

Onderstel nu dat $\mathcal{S} = (P, B, I)$ een veralgemeende vierhoek van de orde (s, t) is, $s, t > 1$. Dan bedoelen we met $\mathcal{S}^D$ de *punt-rechte duaal* van $\mathcal{S}$; dit is de veralgemeende vierhoek van de orde (t, s) die we uit $\mathcal{S}$ verkrijgen door de rol van punten en rechten om te wisselen.
Als twee (niet noodzakelijk verschillende) punten p en q op eenzelfde rechte liggen, dan zeggen we dat ze *collineair* zijn, en we schrijven $p \sim q$. Voor rechten gebruiken we dezelfde notatie, en in dat geval spreken we van *concurrentie* (van rechten). Voor $A \subseteq P$ stellen we

$$A^\perp = \{x \in P \parallel x \sim y \text{ voor elke } y \in A\}.$$

Tevens stellen we $A^{\perp\perp} = (A^\perp)^\perp$, en dit is de *span* van A. Als $p \not\sim q$ punten zijn van $\mathcal{S}$, dan is $|\{p, q\}^\perp| = t+1$. Het is duidelijk dat voor $p \not\sim q$, $|\{p, q\}^{\perp\perp}| \leq t+1$, en als gelijkheid optreedt, dan zeggen we dat $\{p, q\}$ een *regulier* puntenpaar is. Het punt p is *regulier* als $\{p, q\}$ regulier is voor elke $q \in P \setminus p^\perp$. Een *triade* van punten is een drietal twee aan twee niet-collineaire punten, en x is een *centrum* van de triade T als $x \in T^\perp$. *Triades van rechten* en hun *centra* worden op dezelfde manier gedefinieerd. Een *vlag* van een GQ is een incident punt-rechte

paar.

Nota. We gebruiken dezelfde begrippen en notaties als hierboven vaak voor algemenere meetkunden, zonder daarbij verwarring te scheppen.

Voor notaties en noties die we hier niet uitleggen, verwijzen we naar FGQ.

D.2.2 De klassieke en duaal klassieke veralgemeende vierhoeken

We beschrijven nu drie families van veralgemeende vierhoeken die beter bekend staan als de *klassieke veralgemeende vierhoeken.* Ze werden eerst (als vierhoek) ontdekt door J. Tits (in [48]), en zijn allen associeerbaar met klassieke groepen.

Beschouw een niet-singuliere kwadriek $\mathcal{Q}$ van projectieve index 1 in de projectieve ruimte $\mathbf{PG}(d,q)$ (d.w.z. dat de maximale projectieve dimensie van de projectieve ruimten die volledig op $\mathcal{Q}$ liggen, gegeven wordt door 1), waarbij $d = 3, 4$ of 5. De punten en rechten van $\mathcal{Q}$ vormen dan een GQ, die we noteren met $\mathcal{Q}(d,q)$, met de volgende parameters:

1. $(s,t) = (q,1)$ als $d = 3$;
2. $(s,t) = (q,q)$ wanneer $d = 4$;
3. $(s,t) = (q,q^2)$ als $d = 5$.

De structuur van $\mathcal{Q}(3,q)$ is triviaal, daar het een *grid* is; dit is een veralgemeende vierhoek van de orde $(s,1)$ voor een bepaalde s. Duaal definiëren we *duale grids.* Als een veralgemeende vierhoek $\mathcal{S}$ een grid of een duale grid is, dan zeggen we dat $\mathcal{S}$ *dun* is; anders zeggen we dat $\mathcal{S}$ *dik* is.
We herinneren er nog aan dat de kwadriek $\mathcal{Q}$ de volgende canonische vergelijking heeft:

1. $X_0X_1 + X_2X_3 = 0$ wanneer $d = 3$;
2. $X_0^2 + X_1X_2 + X_3X_4 = 0$ met $d = 4$;
3. $f(X_0, X_1) + X_2X_3 + X_4X_5 = 0$ als $d = 5$. (Hierbij is f een irreduciebele binaire kwadratische vorm.)

Onderstel dat H een niet-singuliere Hermitische variëteit is van de projectieve ruimte $\mathbf{PG}(d,q^2)$, $d = 3$ of 4. De punten en de rechten van H vormen dan een GQ $H(d,q^2)$ met volgende parameters:

1. $(s,t) = (q^2, q)$ als $d = 3$;
2. $(s,t) = (q^2, q^3)$ als $d = 4$.

We merken nog op dat H de volgende canonische vergelijking heeft:

$$X_0^{q+1} + X_1^{q+1} + \ldots + X_d^{q+1} = 0.$$

De punten van $\mathbf{PG}(3,q)$, samen met de totaal isotrope rechten van een symplectische polariteit, zijn de punten en rechten van een GQ die we met $W(q)$ noteren, en die als orde $(s,t) = (q,q)$ heeft.

Een symplectische polariteit van $\mathbf{PG}(3,q)$ wordt gedefinieerd door de volgende canonische bilineaire vorm:

$$X_0Y_1 - X_1Y_0 + X_2Y_3 - X_3Y_2.$$

De GQ's die we in deze sectie ingevoerd hebben, noemt men de *klassieke veralgemeende vierhoeken*. Een veralgemeende vierhoek is *duaal klassiek* als hij de punt-rechte duaal is van een klassieke veralgemeende vierhoek. Er geldt dat

(i) $\mathcal{Q}(4,q)^D \cong W(q)$;

(ii) $\mathcal{Q}(4,q) \cong \mathcal{Q}(4,q)^D$ als en slechts als q even is;

(iii) $\mathcal{Q}(5,q)^D \cong H(3,q^2)$.

Opmerking D.2.1 De klassieke voorbeelden spelen een belangrijke rol in de theorie van de GQ's, en een cruciale probleemstelling in deze theorie is het karakteriseren van deze klassieke voorbeelden en hun dualen. We komen hier later meerdere malen op terug.

D.3 Collineaties in Veralgemeende Vierhoeken, EGQ's en TGQ's

Onderstel nu dat $\mathcal{S} = (P, B, I)$ een GQ is van de orde (s,t), $s \neq 1$ en $t \neq 1$, en dat p een punt is van $\mathcal{S}$. Een *collineatie* van $\mathcal{S}$ is een afbeelding van $P \cup B$ op zichzelf die aan de volgende voorwaarden voldoet:

- ze induceert een bijectie van de puntenverzameling op zichzelf;

- ze induceert een bijectie van de rechtenverzameling op zichzelf;

- ze behoudt de incidentie.

Een collineatie θ van $\mathcal{S}$ is een *whorl met centrum* p als θ elke rechte incident met het punt p vasthoudt. Onderstel nu dat θ een whorl is met centrum p van een vierhoek $\mathcal{S} = (P, B, I)$. Als θ identiek is, of als geen enkel punt van $P \setminus p^\perp$ wordt vastgehouden door θ, dan zeggen we dat θ een *elatie met centrum* p is. Fixeert θ elk punt van $p^\perp$, dan is θ een *symmetrie met centrum* p. Als er een groep G is van elaties met centrum p die regulier werkt op de punten van $P \setminus p^\perp$, dan zeggen we dat $\mathcal{S}$ een *elatie veralgemeende vierhoek* (EGQ) is met *elatiegroep* G en *basispunt* of *elatiepunt* p. We schrijven ook dat $(\mathcal{S}^{(p)}, G)$ of $\mathcal{S}^{(p)}$ een EGQ is, en soms spreken we gewoon van 'elatievierhoek'.

De meeste onder de gekende veralgemeende vierhoeken zijn EGQ's; de opmerkelijke uitzonderingen zijn de vierhoeken van de orde $(s-1, s+1)$, $s > 3$, *en hun dualen.*

Zij $(\mathcal{S}^{(p)}, G)$ een EGQ van de orde (s, t), $s, t > 1$, en onderstel dat $L_0, L_1, \ldots, L_t$ de rechten zijn door p. Als G voor elke L_i een deelgroep heeft van s symmetrieën om L_i, dan noemen we $\mathcal{S}$ een *translatie veralgemeende vierhoek (TGQ)* met *basispunt* p en *translatiegroep* G. Men spreekt meestal van een 'translatievierhoek'. Deze veralgemeende vierhoeken werden ingevoerd door J. A. Thas in [165] voor enkele bijzondere gevallen, en veralgemeend door S. E. Payne en J. A. Thas in FGQ. Er geldt (cf. Hoofdstuk 8 van FGQ):

Een EGQ $(\mathcal{S}^{(p)}, G)$ *is een TGQ als en slechts* G *abels is.*

Een rechte met de eigenschap dat de groep van symmetrieën om deze rechte de (maximale) orde s heeft (in een dikke GQ van de orde (s, t)), noemt men een *symmetrie-as*, of een *as van symmetrie*. Een punt waardoor elke rechte een as van symmetrie is, is een *translatiepunt*.

D.4 Veralgemeende Vierhoeken met een Aantal Snijdende Symmetrie-Assen

Het basispunt p van een TGQ $(\mathcal{S}^{(p)}, G)$ is een translatiepunt. Omgekeerd hebben we dat, als p een translatiepunt is van een dikke GQ $\mathcal{S} = (P, B, I)$, de groep G die voortgebracht wordt door de symmetrieën om de rechten door p

abels is, en regulier werkt op $P \setminus p^\perp$. Dus

De translatievierhoeken zijn precies de veralgemeende vierhoeken met een translatiepunt.

In [203, 211] bewezen we dat het voldoende is om minstens $t-s+3$ symmetrie-assen door een punt te hebben om 'hetzelfde' te besluiten. Een belangrijk probleem in deze context is het optimaliseren van deze grens, als uitgangspunt om de verzamelingen van symmetrie-assen door een punt te classificeren. De studie van dit probleem is een van de hoofdmotivaties van Hoofdstuk 6, dat uitgewerkt is naar ideeën van [203], en van het ariktel [211], dat aanvaard is ter publicatie in *Bulletin of the Belgian Mathematical Society — Simon Stevin.* Daarin hebben we ook naar minimale eigenschappen voor TGQ's gezocht opdat ze 'klassieke' ordes zouden hebben (namelijk (s,s) of (s,s^2)) indien dergelijke eigenschap geldt. Een eerste eigenschap luidt als volgt.

Onderstel dat $\mathcal{S}$ een GQ is van de orde (s,t), $s \neq 1 \neq t$, en onderstel dat p een punt is van $\mathcal{S}$.

Eigenschap (T). *Een geordende vlag (L,p) heeft* Eigenschap (T) *met betrekking tot de rechten L_1, L_2, L_3, waar L_1, L_2, L_3 drie verschillende rechten door p zijn, verschillend van L, als het volgende geldt: als (i,j,k) een permutatie is van $(1,2,3)$, en $M \sim L$ en $M \not\sim p$, en als $N \sim L_i$ en $N \not\sim p$ met $M \not\sim N$, dan zijn de triaden $\{M,N,L_j\}$ en $\{M,N,L_k\}$ niet beide centrisch. Als de geordende vlag (L,p) Eigenschap (T) heeft met betrekking tot de rechten L_1, L_2, L_3, dan zeggen we dat $\mathcal{S}$* Eigenschap (T) *heeft voor de geordende vlag (L,p) m.b.t. L_1, L_2, L_3.*

Stelling D.4.1 *Onderstel dat $\mathcal{S}$ een GQ is van de orde (s,t), $s \neq 1 \neq t$, en onderstel dat p een punt is van $\mathcal{S}$ dat incident is met vier verschillende assen van symmetrie L_1, L_2, L_3, L_4. Als Eigenschap (T) voldaan is voor (L_4,p) m.b.t. L_1, L_2, L_3, dan is $t = s^2$. Definieert men G als zijnde de groep voortgebracht door alle symmetrieën om L_1, L_2, L_3, L_4, dan is $(\mathcal{S}^{(p)}, G)$ een TGQ.*

Onder de voorwaarden van Stelling D.4.1 hebben we dus dat

p een translatiepunt is.

We introduceren nu *Eigenschap (T').* Onderstel dat $\mathcal{S}$ een GQ is van de orde (s,t), $s \neq 1 \neq t$, en onderstel dat p een punt is van $\mathcal{S}$.

Eigenschap (T'). *Een geordende vlag (L,p) heeft* Eigenschap (T') *met betrekking tot de rechten L_1, L_2, L_3, waar L_1, L_2, L_3 drie verschillende rechten door p zijn, verschillend van L, als de volgende eigenschap voldaan is: als $M \not\mid p$ en $M \sim L$, en als q en q' verschillende punten zijn op M die niet incident zijn met L, dan is er een permutatie (i,j,k) van $(1,2,3)$, zodat er rechten M_i, M_j, M_k bestaan, waar $M_r \sim L_r$ en $r \in \{i,j,k\}$, waarvoor $M \in \{M_i, M_k, L\}^\perp$, $M_j \in \{M_i, M_k, L_j\}^\perp$, met qIM_i en $q'IM_k$.*
Als de geordende vlag (L,p) Eigenschap (T') heeft met betrekking tot de rechten L_1, L_2, L_3, dan zeggen we ook dat $\mathcal{S}$ Eigenschap (T') *heeft voor de geordende vlag (L,p) m.b.t. L_1, L_2, L_3.*

We komen verder terug op Eigenschap (T'), maar eerst hebben we enkele definities nodig.

D.4.1 TGQ's, veralgemeende ovoïden, flocks en enkele resultaten

Beschouw de $(2n+m-1)$-dimensionale projectieve ruimte $\mathbf{PG}(2n+m-1,q)$ over $\mathbf{GF}(q)$. Een *veralgemeende ovoïde* $\mathcal{O}$ is een verzameling van q^m+1 deelruimten van dimensie $n-1$, zodat

(i) elke drie verschillende elementen van $\mathcal{O}$ een $\mathbf{PG}(3n-1,q)$ opspannen;

(ii) en zodat er q^m+1 deelruimten bestaan van dimensie $n+m-1$, die elk precies één element van $\mathcal{O}$ bevatten en disjunct zijn van de rest.

Als $n=m$ spreekt men ook van *veralgemeende ovaal*. Die q^m+1 deelruimten van dimensie $n+m-1$ zijn de *raakruimten* aan de veralgemeende ovoïde. In FGQ tonen S. E. Payne en J. A. Thas aan dat

Veralgemeende ovoïden en TGQ's equivalente objecten zijn.

Indien men de raakruimten interpreteert in de duale ruimte van $\mathbf{PG}(2n+m-1,q)$, waar q oneven is als $n=m$, verkrijgen we opnieuw een veralgemeende ovoïde die niet noodzakelijk isomorf is met de oude, en de corresponderende TGQ noemt men de *translatieduaal* van de oorspronkelijke TGQ (die correspondeert met de oorspronkelijke veralgemeende ovoïde). De *kern* van een TGQ $\mathcal{S}^{(p)}$ (zie FGQ voor meer precieze informatie), is het veld $\mathbb{K}$ dat de eigenschap heeft dat zijn multiplicatieve groep isomorf is met de groep van whorls met centrum p die een willekeurig (maar vast) punt $x \not\sim p$ fixeren. Als $\mathbf{GF}(q)$ de kern is van de TGQ $\mathcal{S}$ van de orde (s,t), dan is elke veralgemeende ovoïde

die correspondeert met $\mathcal{S}$ beschreven in een $\mathbf{PG}(2n'+m'-1,q')$, waar $\mathbf{GF}(q')$ een deelveld is van $\mathbf{GF}(q)$, en s en t zijn machten van q. Als $s \neq t$, dan is er een natuurlijke n en een oneven a waarvoor $s = q^{na}$ en $t = q^{n(a+1)}$. [2]

Een TGQ $\mathcal{S}^{(p)} = T(\mathcal{O})$ van de orde (q^n, q^{2n}) is *goed* voor de rechte *LIp* indien het corresponderend element π van de veralgemeende ovoïde $\mathcal{O}$ de eigenschap heeft dat elke $(3n-1)$-dimensionale deelruimte van $\mathbf{PG}(4n-1,q)$ die voortgebracht wordt door π en twee andere elementen van $\mathcal{O}$, precies q^n+1 elementen van $\mathcal{O}$ volledig bevat. We zeggen ook dat $\mathcal{O}$ 'goed' is in het element π.

Een *flock* van de kwadratische kegel in $\mathbf{PG}(3,q)$ is een partitie van de kegel zonder top in disjuncte irreduciebele kegelsneden. In 1987 ontdekte J. A. Thas [172] dat

Uit elke flock van de kwadratische kegel in $\mathbf{PG}(3,q)$ *een veralgemeende vierhoek van de orde* (q^2,q) *kan geconstrueerd worden.*

We noemen deze vierhoeken *flock (veralgemeende) vierhoeken.*

Opmerking D.4.2 Door de observatie van J. A. Thas werden snel nieuwe veralgemeende vierhoeken (maar ook veel andere meetkunden) gevonden. Flocks worden tevens gebruikt om hyperovalen, ovalen, spreads en translatievlakken te construeren.

Probleem D.4.3 *Wat zijn minimale voorwaarden voor een veralgemeende vierhoek om een flock veralgemeende vierhoek te zijn?*

In Hoofdstuk 6 bekwamen we de volgende classificatiestelling van TGQ's die 'aan Eigenschap (T) en Eigenschap (T') voldoen'.

Stelling D.4.4 *Onderstel dat* $\mathcal{S}$ *een TGQ is van de orde* (s,t), $s \neq 1 \neq t$, *en onderstel dat* p *een punt is van* $\mathcal{S}$, *en* pIL *een rechte door* p, *zodat Eigenschap (T) of Eigenschap (T')* *voldaan is voor de geordende vlag* (L,p) *m.b.t. elke drie verschillende rechten door* p *die verschillend zijn van* L. *Dan hebben we de volgende mogelijkheden.*

(*i*) $\mathcal{S}$ *is van de orde* s, *en* $\mathcal{S}^{(p)}$ *is een TGQ met geen verdere restricties.*

(*ii*) $t = s^2$, s *is even en* $\mathcal{S}^{(p)}$ *is goed; er zijn tevens* s^3+s^2 *deelvierhoeken van de orde* s *die* L *bevatten, en als minstens één van die deelvierhoeken isomorf is met* $\mathcal{Q}(4,s)$, *dan is* $\mathcal{S} \cong \mathcal{Q}(5,s)$.

[2] Het vermoeden is dat $a = 1$ voor elke TGQ, en als s even is, dan is dit vermoeden reeds bewezen, zie FGQ.

(iii) $t = s^2$, *s is oneven, en de translatieduaal* $(\mathcal{S}^{(p)})^*$ *van* $\mathcal{S}^{(p)}$ *is de puntrechte duaal van een flock vierhoek* $\mathcal{S}(\mathcal{F})$.

In het bewijs van deze stelling gebruiken we als essentieel onderdeel het hoofdresultaat van J. A. Thas uit [183].

De volgende stelling toont een verband aan tussen de kern van een TGQ en het minimaal aantal assen van symmetrie door het translatiepunt die nodig zijn om de translatiegroep te 'genereren'.

Stelling D.4.5 *Onderstel dat* $(\mathcal{S}^{(p)}, G)$ *een TGQ is van de orde* (s,t), $s \neq 1 \neq t \neq s$. *Onderstel dat* $\mathbf{GF}(q)$ *de kern is van de TGQ. Als* $k+3$ *het minimum aantal rechten door* p *is zodat* G *voortgebracht wordt door de symmetrieën om deze rechten, dan is*

$$k \leq n,$$

waar $s = q^{na}$ *en* $t = q^{n(a+1)}$.

D.5 Symmetrie en TGQ's

In het artikel *K. Thas, On symmetries and translation generalized quadrangles* [205], hebben we verdere (minimale) voorwaarden gesteld op EGQ's om TGQ's te zijn. De essentie was om minimale (en meer combinatorische) voorwaarden te stellen op rechten om symmetrie-assen te zijn. Een belangrijke observatie was dat

Een rechte M van een (dikke) veralgemeende vierhoek $\mathcal{S}$ een as van symmetrie is als en slechts als M een reguliere rechte is, en als er een groep H bestaat van whorls met centrum p voor een bepaald punt p op M, die transitief werkt op de punten van een willekeurige rechte $L \sim M$ verschillend van $L \cap M$, waar $p \not\in L$.

Daardoor bekwamen we de volgende (meer) meetkundige definitie voor TGQ's:

Een EGQ met elatiepunt p is een TGQ met basispunt p als elke rechte door p regulier is.

Als gevolg van onze observaties gaven we in [205] de volgende verbeteringen (en veralgemeningen) van stellingen van X. Chen en D. Frohardt [35] en van D. Hachenberger [68].

Stelling D.5.1 *Een EGQ $\mathcal{S}^{(p)}$ van de orde (s,t), $s,t>1$, is een TGQ met basispunt p als en slechts als p incident is met tenminste twee verschillende reguliere rechten.*

Stelling D.5.2 *Een EGQ $\mathcal{S}^{(p)}$ van de orde (s,t), $s,t>1$ en s even, is een TGQ met basispunt p als en slechts als p incident is met tenminste één reguliere rechte.*

De resultaten van [205] worden uiteengezet in Hoofdstuk 5.

D.6 Netten en Veralgemeende Vierhoeken met een Regulier Punt

We beginnen met een essentiële observatie uit FGQ. Onderstel dat $\mathcal{S}$ een veralgemeende vierhoek is van de orde (s,t), $s\neq 1\neq t$, met een regulier punt x. Definieer een incidentiestructuur $\Pi=(P,B,I)$ als volgt: P is de verzameling van de punten van $x^\perp\setminus\{x\}$; B is de verzameling spans $\{q,r\}^{\perp\perp}$, waar q en r niet-collineaire punten zijn in $x^\perp$, en I is de natuurlijke incidentie. Dan is Π een *duaal net* (zie bijvoorbeeld Hoofdstuk 3) *van de orde s en graad $t+1$*. In het bijzonder, als $s=t$, dan is Π een *duaal affien vlak van de orde s* [81]. Definieer in het geval $s=t$ een incidentiestructuur $\Pi=(P,B,I)$ als volgt: P is de verzameling van de punten van $x^\perp$; B is de verzameling spans $\{q,r\}^{\perp\perp}$, waar q en r verschillende punten zijn in $x^\perp$; I is de natuurlijke incidentie. Dan is Π een *projectief vlak van de orde s* [81].

Het net dat het duale is van het duaal net dat we zojuist geconstrueerd hebben, noteren we met $\mathcal{N}_x$.

We vermelden in deze paragraaf enkel de hoofdstelling van Hoofdstuk 3, dat gebaseerd is op het artikel *A theorem concerning nets arising from generalized quadrangles with a regular point* [209], dat gepubliceerd wordt in *Designs, Codes and Cryptography*. Die stelling blijkt essentieel voor de herkenning van deelvierhoeken:

Stelling D.6.1 (Structuurstelling) *Onderstel dat $\mathcal{S}$ een veralgemeende vierhoek is van de orde (s,t), $s\neq 1\neq t$, met een regulier punt x. Als $\mathcal{N}'$ een deelnet is van het net $\mathcal{N}_x$ van dezelfde graad als $\mathcal{N}_x$, dan hebben we de volgende mogelijkheden.*

(*i*) *$\mathcal{N}'$ en $\mathcal{N}_x$ vallen samen.*

(*ii*) *$\mathcal{N}' \neq \mathcal{N}_x$, $s = t^2$ en $\mathcal{N}'$ is een affien vlak van de orde t. In dit geval heeft $\mathcal{S}$ een deelvierhoek van de orde t die x bevat (als regulier punt).*

Als omgekeerd $\mathcal{S}$ een eigenlijke deelvierhoek van de orde (s',t) heeft die x bevat, $s' \neq 1$, dan is $s' = t$, $s = t^2$ en heeft $\mathcal{N}_x$ een deelnet $\mathcal{N}'$ van de orde t en graad $t+1$ (dus $\mathcal{N}'$ is een affien vlak van de orde t).

Vervolgens passen we deze structuurstelling toe op de theorie van de symmetrieën in GQ's, op de theorie van de *gepunte translatie veralgemeende vierhoeken (STGQ's)*, en op de theorie van de *translatienetten* en *translatievlakken*. Voor meer details verwijzen we naar Hoofdstuk 3.

D.7 Complete $(st-t/s)$-Bogen in Veralgemeende Vierhoeken van de Orde (s,t)

Onderstel dat $\mathcal{S}$ een GQ is van de orde (s,t), $s \neq 1 \neq t$. Een *k-boog* $\mathcal{K}$ in $\mathcal{S}$ is een verzameling van k onderling niet-collineaire punten, $k \in \mathbb{N}$. Een k-boog noemt men *compleet* als er geen k'-boog $\mathcal{K}'$ van $\mathcal{S}$ bestaat met $k' > k$ die $\mathcal{K}$ bevat. Als $k = st+1$, dan noemt men $\mathcal{K}$ ook een *ovoïde* van $\mathcal{S}$. Een ovoïde $\mathcal{K}$ heeft de karakteristieke eigenschap dat elke rechte van de vierhoek $\mathcal{K}$ in precies één punt snijdt (vandaar dat $st+1$ ook een bovengrens is voor k). Duaal definieert men respectievelijk *(complete) duale k-bogen* en *spreads*. De volgende interessante observatie komt van FGQ.

Observatie D.7.1 (FGQ, 2.7.1) *Onderstel dat $\mathcal{K}$ een $(st-\rho)$-boog is met $\rho < t/s$ in de veralgemeende vierhoek $\mathcal{S}$ van de orde (s,t), $s \neq 1 \neq t$. Dan is $\mathcal{K}$ op een unieke manier uitbreidbaar tot een ovoïde van $\mathcal{S}$.*

Het is dus een natuurlijke vraag of complete $(st-t/s)$-bogen bestaan in veralgemeende vierhoeken van de orde (s,t), $s \neq 1 \neq t$.

Dit probleem is reeds open sinds 1983, zie FGQ (of Hoofdstuk 4) voor meer details. Het is de moeite waard om op te merken dat geen enkel voorbeeld van een complete $(st-t/s)$-boog expliciet in de literatuur vermeld is, alhoewel men in bepaalde 'kleine' gevallen gemakkelijk een voorbeeld kan construeren; bijvoorbeeld in het geval van de klassieke vierhoek $\mathcal{Q}(5,2)$.

Opmerking D.7.2 Het onderzoek naar het wel dan niet bestaan van ovoïden en spreads in veralgemeende vierhoeken, maar bijvoorbeeld ook in *projectieve ruimten* $\mathbf{PG}(n,q)$ (zie verder), of in *veralgemeene veelhoeken*, wordt beschouwd

als een van de belangrijkste problemen in de betrokken theorieën. Daarom is het onderzoek naar het wel dan niet bestaan van complete $(st - t/s)$-bogen in GQ's van de orde (s,t) ook belangrijk; door Observatie D.7.1 impliceert het bestaan van een $(st - \rho)$-boog, met $\rho < t/s$ het bestaan van een ovoïde.

In het artikel *Nonexistence of Complete $(st-t/s)$-Arcs in Generalized Quadrangles of Order (s,t), I* [207], dat gepubliceerd werd in *Journal of Combinatorial Theory, Series A*, hebben we aangetoond dat complete $(st-t/s)$-bogen in alle gekende GQ's van de orde (s,t), $s,t > 1$, behalve in het geval van de vierhoek $\mathcal{Q}(4,q)$ met q oneven (en we verwijzen hier direct ook naar [219]), niet bestaan op twee sporadische gevallen na. Hoofdstuk 4 is (onder andere) op [207] gebaseerd. De volgende stelling werd bewezen.

Stelling D.7.3 *Onderstel dat $\mathcal{K}$ een complete $(st-t/s)$-boog is in een gekende veralgemeende vierhoek $\mathcal{S}$ van de orde (s,t), $s \neq 1 \neq t$. Dan hebben we een van de volgende gevallen.*

1. *$\mathcal{S} \cong \mathcal{Q}(4,2)$ en $\mathcal{S}$ bevat een unieke complete 3-boog (op isomorfisme na).*
2. *$\mathcal{S} \cong \mathcal{Q}(5,2)$ en $\mathcal{S}$ bevat een unieke complete 6-boog (op isomorfisme na).*
3. *$\mathcal{S} \cong \mathcal{Q}(4,q)$ met q oneven.*

In het laatste geval is er tenminste één voorbeeld van een complete 8-boog als $\mathcal{S} \cong \mathcal{Q}(4,3)$. We conjecteren dat dit voorbeeld uniek is, en dat als $q > 3$ er geen andere voorbeelden zijn. Het onderzoeken van dit vermoeden is een van de doelstellingen van het artikel in voorbereiding [219].

In hetzelfde hoofdstuk voeren we *(partiële) spreads* en *(partiële) ovoïden* in van *affiene veralgemeende vierhoeken (AGQ's)* [147]. We hebben namelijk ontdekt dat het bestaan van ovoïden in affiene vierhoeken van een zeker type [3] tevens

het bestaan van complete $(st - t/s)$-bogen impliceert in de geassocieerde veralgemeende vierhoek van de orde (s,t).

Op die manier, door bijvoorbeeld Stelling D.7.3 te gebruiken, kunnen we direct het niet bestaan van veel van die objecten aantonen. De studie van een ander open probleem wordt tevens geïnitialiseerd, namelijk het wel dan niet bestaan van *complete grids met parameters $s-1, s+1$* in veralgemeende vierhoeken van de orde (s,t), $s \neq 1 \neq t$. We hebben bijvoorbeeld kunnen bewijzen dat

[3] Een AGQ is altijd van drie mogelijke types.

het bestaan van complete grids met parameters $s-1, s+1$ in een veralgemeende vierhoek $\mathcal{S}$ van de orde $s > 1$ het bestaan impliceert van een complete duale (s^2-1)-boog in $\mathcal{S}$. Ook hier is Stelling D.7.3 dan van toepassing. Het is trouwens die observatie die direct leidt tot een voorbeeld van een complete 8-boog in $\mathcal{Q}(4,3)$.

Eigenaardig genoeg duiken complete $(st-t/s)$-bogen ook op andere plaatsen in de theorie van de veralgemeende vierhoeken op; in Hoofdstuk 12 hebben we Stelling D.7.3 gebruikt in een op het eerste zicht volledig andere context. Daarom ook is Stelling D.7.3 essentieel voor dit werk.

D.8 Semi Vierhoeken

Een *semi vierhoek* $\mathcal{S} = (P, B, I)$ van de *orde* $(s; t_1, t_2, \ldots, t_n)$ met *μ-parameters* $(\mu_1, \mu_2, \ldots, \mu_m)$ (waarbij alle parameters 'natuurlijk' zijn), is een punt-rechte incidentiestructuur met de volgende eigenschappen:

(SV1) elke rechte is incident met $s+1$ punten;

(SV2) elk punt is incident met $t_1+1, t_2+1, \ldots$, of t_n+1 rechten (en elk van deze mogelijkheden doet zich voor);

(SV3) voor elke twee niet-collineaire punten zijn er $\mu_1, \mu_2, \ldots, \mu_m$ punten collineair met beide (en elk van de mogelijkheden doet zich voor), en elke μ_i is strikt positief;

(SV4) er zijn geen ('gewone') driehoeken maar wel ('gewone') vierhoeken en vijfhoeken.

Het axioma (SV3) impliceert dat er 'veel' vierhoeken zijn in een semi vierhoek.

We hebben semi vierhoeken ingevoerd in verband met zekere natuurlijke meetkunden die ontstaan uit bepaalde configuraties van assen van symmetrie in een GQ. In [203] noemden we een variant op deze structuren *(l,k)-partiële vierhoeken*. Hoofdstuk 7 is geïnspireerd op het artikel *On Semi Quadrangles*, dat zal gepubliceerd worden in *Ars Combinatoria*, en bevat delingsvoorwaarden, ongelijkheden, natuurlijke voorbeelden (die voornamelijk gehecht zijn aan veralgemeende vierhoeken), een beschouwing over de lineaire representaties, karakterisering van hun puntgrafen, enzoverder.

In deze sectie geven we enkel twee resultaten uit dat hoofdstuk. Het eerste resultaat is een ongelijkheid die deze van P. J. Cameron voor *partiële vierhoeken* (zie [33]) uitbreidt, en deze van D. G. Higman voor veralgemeende vierhoeken [74, 75]. Het bewijs van dit resultaat wordt gegeven in Appendix B.
Een tweede resultaat beschrijft de lineaire representaties van semi vierhoeken, en geeft een interessant verband tussen semi vierhoeken en *complete kappen* in $\mathbf{PG}(n,q)$.

Stelling D.8.1 *Onderstel dat $\mathcal{S}$ een semi vierhoek is van de orde $(s;t_1,t_2,\ldots,t_n)$, met μ-parameters $(\mu_1,\mu_2,\ldots,\mu_m)$. Dan hebben we de volgende ongelijkheid:*

$$[(t_1-1)s\mu_1]^2 \leq \mu_m[(\mu_m-1)(\mu_m-2)+(t_n-1)s](\lfloor\frac{(t_n+1)t_ns^2}{\mu_1}\rfloor)$$

$$-s(t_1+1)+\mu_m-1). \qquad \text{(D.1)}$$

Als de gelijkheid geldt, dan is er een constante $x_0 = \frac{(t_1-1)s\mu_1}{\lfloor\frac{(t_n+1)t_ns^2}{\mu_1}\rfloor-s(t_1+1)+\mu_m-1}$ zodat elke triade van punten x_0 centra heeft.

Indien elke triade van punten een constant aantal centra heeft, dan is

$$[(t_n-1)s\mu_m]^2 \geq \mu_1[(t_1-1)s+(\mu_1-1)(\mu_1-2)](\lceil\frac{(t_1+1)t_1s^2}{\mu_m}\rceil)$$

$$-s(t_n+1)+\mu_1-1). \qquad \text{(D.2)}$$

Gevolg D.8.2 (P. J. Cameron [33]) *Onderstel dat $\mathcal{S}$ een partiële vierhoek is met parameters (s,t,μ). Dan is*

$$\mu(t-1)^2s^2 \leq [s(t-1)+(\mu-1)(\mu-2)][\frac{(t+1)ts^2}{\mu}-(t+1)s+\mu-1]. \qquad \text{(D.3)}$$

Gelijkheid treedt op als en slechts als elke triade van punten een constant aantal centra heeft, en deze constante is $1+\frac{(\mu-1)(\mu-2)}{s(t-1)}$.

Een ander gevolg is de stelling van D. G. Higman [74, 75]/C. C. Bose en S. S. Shrikhande [17].

Gevolg D.8.3 ([74, 75];[17]) *Onderstel dat $\mathcal{S}$ een GQ is van de orde (s,t), $s \neq 1 \neq t$. Dan is $t \leq s^2$, en gelijkheid treedt op als en slechts als elke triade van punten een constant aantal centra heeft, en deze constante is $1+s$. Duaal hebben we dat $s \leq t^2$, en dan treedt gelijkheid op als en slechts als elke triade van rechten $t+1$ centra heeft.*

Een *lineaire representatie* van een semi vierhoek $\mathcal{S} = (P, B, I)$ is een monomorfisme θ van $\mathcal{S}$ in de meetkunde van punten en rechten van de affiene ruimte $\mathbf{AG}(n,q)$, $n > 1$, op zo'n manier dat P^θ de volledige puntenverzameling van $\mathbf{AG}(n,q)$ is, dat B^θ een unie van parallelklassen van rechten is van $\mathbf{AG}(n,q)$, en dat elk punt van L^θ het beeld is van een bepaald punt van L voor elke rechte L in B. Meestal identificeren we $\mathcal{S}$ met het beeld $\mathcal{S}^\theta$. Aangezien parallelklassen van rechten in $\mathbf{AG}(n,q)$ corresponderen met punten in $\mathbf{PG}(n-1,q)$, definieert een lineaire representatie $\mathcal{S}^\theta$ van $\mathcal{S}$ in $\mathbf{AG}(n,q)$ een puntenverzameling van $\mathbf{PG}(n-1,q)$.

Stelling D.8.4 *(i) Een deelverzameling $\mathcal{K}$ van de puntenverzameling van $\mathbf{PG}(n-1,q)$, $n \geq 3$, definieert een lineaire representatie van een semi vierhoek met μ-parameters $(\mu_1, \mu_2, \ldots, \mu_k)$ als en slechts als een van de volgende voorwaarden geldt:*

1. *het is een complete $(t+1)$-cap voor een bepaalde t, met de eigenschap dat elk punt van $\mathbf{PG}(n-1,q)$ dat niet in $\mathcal{K}$ ligt, op precies $t-\mu_j+1$ tangenten aan $\mathcal{K}$ ligt voor een $\mu_j \in \{\mu_1, \mu_2, \ldots, \mu_k\}$, en elk geval doet zich voor;*
2. *als $q = 2$, dan is $\mathcal{K}$ niet het complement van een hypervlak.*

(ii) Als een $(t+1)$-cap $\mathcal{K}$ van $\mathbf{PG}(n-1,q)$ een lineaire representatie definieert van de semi vierhoek $\mathcal{S}$, dan is elk punt van $\mathcal{S}$ incident met $t+1$ rechten.

(iii) Onderstel dat $\mathcal{S} = (P, B, I)$ een semi vierhoek is met μ-parameters $(\mu_1, \mu_2, \ldots, \mu_k)$ die een lineaire representatie heeft in $\mathbf{AG}(n,q)$, en definieer P_j als $P_j = \{\{x,y\} \in P \times P, x \not\sim y \parallel |\{x,y\}^\perp| = \mu_j\}$. Dan hebben we voor elke j, dat $|P_j| \equiv 0 \mod q(q-1)/2$.

Het verband tussen semi vierhoeken en complete kappen in $\mathbf{PG}(m,q)$ gaan we in de toekomst verder onderzoeken.

D.9 Span-Symmetrische Veralgemeende Vierhoeken

Zoals eerder gezegd is een *as van symmetrie* (of *symmetrie-as*) van een veralgemeende vierhoek van de orde (s,t), $s \neq 1 \neq t$, een rechte waarvoor er een (maximale) groep van s symmetrieën bestaat met deze rechte als as. Men bewijst gemakkelijk dat een as van symmetrie noodzakelijk regulier is, zie FGQ. Een GQ is *span-symmetrisch* indien er twee niet-snijdende assen van symmetrie (L en M) zijn. De verzameling $\{L,M\}^{\perp\perp}$ noemt men dan de *basis-span*. In het vervolg korten we de notie 'span-symmetrische veralgemeende vierhoek' soms af door 'SPGQ' (naar de Engelstalige term 'span-symmetric generalized quadrangle'). Reeds in 1980 werd het probleem gesteld (door S. E. Payne) om alle SPGQ's van de orde (s,t), $s \neq 1 \neq t$, te classificeren. Het oorspronkelijk vermoeden luidde als volgt:

Vermoeden D.9.1 *Alle SPGQ's van de orde (s,t), $s \neq 1 \neq t$, zijn klassiek, i.e. isomorf met $\mathcal{Q}(4,s)$ of $\mathcal{Q}(5,s)$.*

Een eerste (classificatie) resultaat werd bekomen door W. M. Kantor:

Stelling D.9.2 *Onderstel dat $\mathcal{S}$ een SPGQ is van de orde (s,t), $s \neq 1 \neq t$, met basis-span $\mathcal{L}$. Dan is $s = t$ of $t = s^2$.*

S. E. Payne bewees in [121] dat als $s = t$, dan automatisch elke rechte van $\mathcal{L}^{\perp}$ ook een as van symmetrie is. In datzelfde artikel staat ook een 'bewijs' van het SPGQ-vermoeden voor het geval $s = t$. Later bleek dit echter foutief te zijn. Tevens toont hij aan dat een SPGQ van de orde $s > 1$ equivalent is met een zogenaamde *groep met een 4-gonale basis*, zie [139, Hoofdstuk 10] of [121].

In *K. Thas, Classification of span-symmetric generalized quadrangles of order s* [208], zijn we erin geslaagd om het probleem voor het geval $s = t$ volledig op te lossen. Het bewijs van dit resultaat werd gepubliceerd in het tijdschrift *Advances in Geometry*, en staat beschreven in Hoofdstuk 8 van deze thesis:

Stelling D.9.3 *Een SPGQ van de orde s, $s > 1$, is altijd isomorf met $\mathcal{Q}(4,s)$.*

Een eerste essentiële observatie in het bewijs van Stelling D.9.3 is dat, als G de groep is, voortgebracht door de symmetrieën om de rechten van de basis-span $\mathcal{L}$, en als N de kern is van de actie van G op $\mathcal{L}$, dan is de permutatiegroep $(\mathcal{L}, G/N)$ een zogenaamd *gespleten BN-paar van rang 1* [149, 222][4]. Dit wil

[4] J. Tits noemt zulke permutatiegroepen ook 'Moufang sets', zie [225].

zeggen dat de permutatiegroep $(\mathcal{L}, G/N)$ aan de volgende twee eigenschappen voldoet:

(BN1) G/N werkt 2-transitief op de verzameling $\mathcal{L}$;

(BN2) voor elke $L \in \mathcal{L}$ bevat de stabilizator $(G/N)_L$ van L in G/N een normaaldeler die regulier werkt op $\mathcal{L} \setminus \{L\}$.

De eindige gespleten BN-paren van rang 1, genoteerd (X, H), werden onafhankelijk geclassificeerd door E. E. Shult [156] en C. Hering, W. M. Kantor en G. M. Seitz [72], en men heeft de volgende mogelijkheden:

1. H is een scherp 2-transitieve groep (op X);
2. $H \cong \mathbf{PSL}(2, s)$;
3. H is isomorf met de Ree groep $\mathbf{R}(\sqrt[3]{s})$ (soms ook door ${}^2\mathbf{G_2}(\sqrt[3]{s})$ genoteerd), waar $\sqrt[3]{s}$ een oneven macht is van 3;
4. H is een Suzuki groep $\mathbf{Sz}(\sqrt{s})$ (soms ook door ${}^2\mathbf{B_2}(\sqrt{s})$ genoteerd), waar $\sqrt{s}$ een oneven macht is van 2;
5. H is de unitaire groep $\mathbf{PSU}(3, \sqrt[3]{s^2})$,

elk in hun natuurlijke permutatiegroep van de graad $s + 1$.

Dus, met N de kern van de actie van G op $\mathcal{L}$, is G/N een van de vorige gevallen. De groep G/N kan niet scherp 2-transitief werken op $\mathcal{L}$; als $L \in \mathcal{L}$ namelijk een willekeurige rechte is van $\mathcal{L}$, dan induceren N en de groepen van alle symmetrieën om de rechten van $\mathcal{L}^\perp$ zekere groepen van collineaties in het projectief vlak Π_L van de orde s dat gehecht is aan de reguliere rechte L, zie eerder. We hebben dan bewezen in Hoofdstuk 8 dat dit leidt tot het feit dat Π_L een projectief vlak uit de *Lenz-Barlotti klasse* **III.2** is. Vervolgens gebruiken we een resultaat van J. C. S. D. Yaqub [238] (die de vlakken uit de Lenz-Barlotti klasse **III.2** bepaald heeft), om het scherp 2-transitieve geval uit te sluiten (tenzij in enkele 'kleine' gevallen, die echter geen problemen opleveren). Vervolgens tonen we aan, door gebruik te maken van de theorie van de universele centrale extensies van perfecte groepen [3, 96], dat enkel het geval $G/N \cong \mathbf{PSL}(2, s)$ zich voordoet, en dat dan $G \cong \mathbf{SL}(2, s)$. De stelling volgt dan, en we hebben direct ook het volgende groepentheoretisch analogon van Stelling D.9.3:

Stelling D.9.4 *Een eindige groep G heeft een 4-gonale basis als en slechts als hij isomorf is met $\mathbf{SL}(2, s)$ voor een bepaalde s.*

D.10 Veralgemeende Vierhoeken met Twee (Verschillende) Translatiepunten

We beginnen met een definitie. Onderstel dat $\mathcal{F}$ de flock is van de kwadratische kegel $\mathcal{K}$ in $\mathbf{PG}(3,q)$ met vergelijking $X_0X_1 = X_2^2$, q oneven, die gedefinieerd is door de q vlakken met de volgende vergelijkingen:

$$tX_0 - mt^\sigma X_1 + X_3 = 0,$$

waar $t \in \mathbf{GF}(q)$, m een gegeven niet-kwadraat in $\mathbf{GF}(q)$ is, en waar σ een automorfisme van $\mathbf{GF}(q)$ is. Deze flock werd eerst ontdekt door W. M. Kantor [90], en om die reden noemen we de geassocieerde veralgemeende vierhoek $\mathcal{S}(\mathcal{F})$ de *Kantor (semifield flock) veralgemeende vierhoek*.

In 1985 bewees S. E. Payne [125] dat de duale Kantor vierhoeken, die we met $\mathcal{S}(\mathcal{F})^D$ noteren, altijd een rechte L hebben zodat elke rechte van $L^\perp$ een symmetrie-as is. Of, anders geformuleerd: L is een rechte van translatiepunten (wat equivalent is met het feit dat L minstens twee verschillende translatiepunten bevat). Dit impliceert dat Vermoeden D.9.1 verkeerd is voor het (algemene) geval $t = s^2$, omdat een (duale) Kantor vierhoek $\mathcal{S}(\mathcal{F})^D$ klassiek is als en slechts als $\sigma = \mathbf{1}$, en $\mathcal{S}(\mathcal{F})^D$ is een SPGQ voor elk niet-concurrent rechtenpaar in $L^\perp$!!!

Deze observatie motiveerde ons om in Hoofdstuk 9 het volgende probleem te onderzoeken, met het oog op de classificatie van SPGQ's.

Probleem D.10.1 *Is een (dikke) veralgemeende vierhoek van de order (s, s^2), $s > 1$, met een even aantal punten op een rechte en met twee verschillende collineaire translatiepunten altijd isomorf met een duale Kantor vierhoek?*

Of, meer algemeen:

Probleem D.10.2 *Classificeer de veralgemeende vierhoeken van de orde (s,t), $s,t > 1$, met twee verschillende translatiepunten.*

Merk op dat door de stelling van W. M. Kantor, $t = s$ of $t = s^2$, en dat Stelling D.9.3 impliceert dat een veralgemeende vierhoek van de orde s, $s > 1$, met twee verschillende translatiepunten isomorf is met $\mathcal{Q}(4,s)$. Door gelijkaardige technieken (meer diepgaand) te gebruiken dan in het bewijs van Stelling D.9.3, slaagden we erin de volgende stelling te bewijzen.

Stelling D.10.3 *Onderstel dat $\mathcal{S}$ een SPGQ is van de orde (s, s^2), $s > 1$, met basis-span $\mathcal{L}$. Dan zijn er $s+1$ deelvierhoeken van de orde s in $\mathcal{S}$ die twee aan twee snijden in de grid (met parameters $s+1, s+1$) die bepaald is door $\mathcal{L}$, en die isomorf zijn met $\mathcal{Q}(4, s)$.*

Door vervolgens gebruik te maken van het hoofdresultaat van [177, 181, 183], dat de link legt tussen goede TGQ's, *Eigenschap (G)* [128] en flock veralgemeende vierhoeken, bekomen we het volgende.

Stelling D.10.4 *Onderstel dat $\mathcal{S}$ een GQ is van de orde (s, t), $s \neq 1 \neq t$, met twee verschillende translatiepunten. Dan hebben een van de volgende mogelijkheden:*

- *(i) $\mathcal{S}$ is van de orde s, en $\mathcal{S} \cong \mathcal{Q}(4, s)$;*
- *(ii) $t = s^2$, s is even en $\mathcal{S} \cong \mathcal{Q}(5, s)$;*
- *(iii) $t = s^2$, s is oneven, de translatiepunten zijn niet-collineair, en $\mathcal{S} \cong \mathcal{Q}(5, s)$;*
- *(iv) $t = s^2$, s is oneven, de translatiepunten zijn collineair, en als (∞) een willekeurig translatiepunt is (dit is, indien $\mathcal{S}$ niet klassiek is, een willekeurig punt op de unieke rechte van translatiepunten), dan is de translatieduaal $(\mathcal{S}^{(\infty)})^*$ van $\mathcal{S}^{(\infty)}$ de punt-rechte duaal van een flock veralgemeende vierhoek $\mathcal{S}(\mathcal{F})$.*

Het is duidelijk dat dit een krachtig resultaat is. Voor het even geval is de classificatie compleet.

Stelling D.10.5 *Onderstel dat $\mathcal{S}$ een GQ is van de orde (s, t), $s \neq 1 \neq t$ en s even, met twee verschillende translatiepunten. Dan hebben we een van de volgende mogelijkheden:*

- *(i) $\mathcal{S}$ is van de orde s, en $\mathcal{S} \cong \mathcal{Q}(4, s)$;*
- *(ii) $t = s^2$ en $\mathcal{S} \cong \mathcal{Q}(5, s)$.*

D.11 Translatie Vierhoeken waarvan de Translatieduaal de Punt-Rechte Duaal van een Flock Vierhoek is

De vraag bleef echter open of $\mathcal{F}$ nu wel degelijk een Kantor flock is als we in Geval (iv) zijn. In [210] slaagden we er echter niet in om dit aan te tonen.

De reden daarvan werd ons later duidelijk, en wel om de volgende (hoofd) observatie van Hoofdstuk 10, waarop we [213] baseerden:

(E) *Elke TGQ waarvan de translatieduaal de punt-rechte duaal is van een flock vierhoek heeft twee verschillende collineaire translatiepunten.*

Dus in combinatie met Stelling D.10.4 krijgen we:

(E′) *De niet-klassieke TGQ's waarvan de translatieduaal de punt-rechte duaal is van een flock veralgemeende vierhoek zijn precies de niet-klassieke veralgemeende vierhoeken van de orde (s,t), $s \neq 1 \neq t \neq s$ en s oneven, met twee translatiepunten.*

Dit impliceert dat ook de *Roman veralgemeende vierhoeken* van S. E. Payne [128] en de onlangs ontdekte *Penttila-Williams veralgemeende vierhoek* [145] een rechte $[\infty]$ van translatiepunten hebben, en de vermelde voorbeelden zijn dus SPGQ's voor elk paar niet-snijdende rechten in $[\infty]$! Dit is een op zijn minst eigenaardige observatie, omdat dit impliceert dat het 'bijzondere punt' (∞) van deze vierhoeken niet vastgehouden wordt door hun automorfismengroep, wat ook bewijst dat niet alleen het punt (∞) in feite niet zo 'bijzonder' is, maar ook dat de automorfismengroep $Aut(\mathcal{S})$ veel groter is dan oorspronkelijk gedacht (de actie van $Aut(\mathcal{S})$ op $[\infty]$ is 2-transitief).

Observatie D.11.1 Eigenschap (E′) toont aan dat de theorie van de SPGQ's van de orde (s, s^2), $s > 1$, veel ruimer is dan aanvankelijk gedacht, in het perspectief van Vermoeden D.9.1.

D.12 Een Lenz-Barlotti Classificatie voor Eindige Veralgemeende Vierhoeken

In 1954 beschreef H. Lenz [104] een classificatie van projectieve vlakken op basis van de deelmeetkunden die gevormd worden door de incidente punt-rechte paren (p, L) waarvoor de vlakken (p, L)-*transitief* zijn [81]. In 1957 beschouwde A. Barlotti [9] alle punt-rechte paren met deze eigenschap. Op die manier ontstond de bekende *Lenz-Barlotti classificatie voor projectieve vlakken.* De bedoeling van Hoofdstuk 12 is een gelijkaardige classificatie door te voeren voor eindige veralgemeende vierhoeken.

We zullen ons hier voornamelijk richten op een 'Lenz-type classificatie'.

Het natuurlijke analogon voor '(p, L)-transitiviteit', pIL, in veralgemeende vierhoeken is de notie van '(p, L, q)-transitiviteit'. Laat ons eerst vermelden dat een *paneel* in een veralgemeende vierhoek $\mathcal{S} = (P, B, I)$ een element (p, L, q) is in $P \times B \times P$ waarvoor $pILIq$ en $p \neq q$. Een veralgemeende vierhoek $\mathcal{S}$ van de orde (s, t), $s \neq 1 \neq t$, is (p, L, q)-*transitief*, met (p, L, q) een paneel van $\mathcal{S}$, als de groep $H(p, L, q)$ van whorls om p, q en L de orde s heeft. Als UIp en $U'Iq$, $U \neq L \neq U'$, dan werkt $H(p, L, q)$ transitief (en dan regulier) op $\{U, U'\}^{\perp} \setminus \{L\}$ als en slechts $|H(p, L, q)| = s$. We zeggen ook dat paneel (p, L, q) van een GQ $\mathcal{S}$ *Moufang* is als $\mathcal{S}$ (p, L, q)-transitief is. Beschouw nu de volgende eigenschap

(M) Elk paneel is Moufang.

Eigenschap $(\hat{M})$ definiëren we als de duale eigenschap van Eigenschap (M). Een GQ $\mathcal{S}$ is *half Moufang* indien Eigenschap (M) of Eigenschap $(\hat{M})$ voldaan is; zijn beide eigenschappen voldaan, dan zeggen we dat $\mathcal{S}$ *Moufang* is.
De volgende bekende stelling is een gevolg van resultaten van P. Fong en G. M. Seitz.

Stelling D.12.1 (P. Fong en G. M. Seitz [60, 61]) *Elke eindige Moufang veralgemeende vierhoek is klassiek of duaal klassiek.*

Een sterke veralgemening daarvan is de volgende stelling van J. A. Thas, S. E. Payne en H. Van Maldeghem [201].

Stelling D.12.2 ([201]) *Een eindige veralgemeende vierhoek die half Moufang is, is ook Moufang (en dus klassiek of duaal klassiek).*

Ondanks deze veelbelovende resultaten lijkt het bekomen van een 'goede classificatie' momenteel niet evident zonder extra voorwaarden. Daarom merken we eerst het volgende op:

Een rechte L in een dikke GQ is een as van symmetrie als en slechts als ze regulier is en als er een paar $\{p, q\}$ is van verschillende punten op L waarvoor de GQ (p, L, q)-transitief is.

De mogelijkheid die voorgesteld wordt in Hoofdstuk 12, is

De mogelijke deelstructuren te onderzoeken (in (eindige) veralgemeende vierhoeken) die gevormd worden door de assen van symmetrie van de vierhoek.

Een classificatie van dit type werd reeds begonnen in onze Licentiaatsthesis [203]. In het artikel in voorbereiding *K. Thas, A Lenz-Barlotti Classification*

of Finite Generalized Quadrangles [218], en waarop Hoofdstuk 12 gebaseerd is, diepten we die classificatie uit, door gebruik te maken van de ideeën en technieken die we ontwikkelden in [209, 207, 205, 211, 208, 210] en [213]. In deze paragraaf geven we een 'eerste' versie van de classificatiestelling, en vermelden we enkele hoofdresultaten van het artikel. Voor meer details verwijzen we naar Hoofdstuk 12.

Een *Hermitische spread* van een GQ is een spread $\mathbf{T}$ met de eigenschap dat als L en M twee verschillende rechten zijn van $\mathbf{T}$, dan $\{L, M\}$ een regulier rechtenpaar is waarvoor $\{L, M\}^{\perp\perp} \subseteq \mathbf{T}$.

Stelling D.12.3 (Lenz-type classificatie) *Onderstel dat $\mathcal{S} = (P, B, I)$ een eindige veralgemeende vierhoek is van de orde (s, t), $s \neq 1 \neq t$. Dan hebben we een van de volgende mogelijkheden.*

I *$\mathcal{S}$ heeft geen assen van symmetrie.*

II *Elke as van symmetrie is incident met een bepaald punt $p \in P$.*

III *Er is een rechte $L \in B$ die geen as van symmetrie is, zodat elk punt van L incident is met precies $k + 1$ assen van symmetrie, waarbij $k \in \{0, s - 1, s^2 - 1\}$, en er zijn geen andere symmetrie-assen.*

IV *Er is een symmetrie-as $L \in B$, zodat elk punt van L incident is met precies $k + 1$ assen van symmetrie, waarbij $k \in \{1, s, s^2\}$, en er zijn geen andere symmetrie-assen.*

V *Onderstel dat we geen van de voorgaande gevallen hebben. We definiëren een incidentiestructuur $\mathcal{S}'$ als volgt.*

- RECHTEN *zijn van twee types:*
  (1) *de symmetrie-assen van $\mathcal{S}$;*
  (2) *de rechten van $\mathcal{S}$ zodanig dat elk punt op zo een rechte ook gelegen is op een rechte van Type (1).*
- *De* PUNTEN *van $\mathcal{S}'$ zijn gewoon de punten die gelegen zijn op de rechten van Type (1), en*
- INCIDENTIE *is de restrictie van I tot $\mathcal{S}'$.*

Dan hebben we dat $\mathcal{S}'$ een veralgemeende vierhoek is waarvan elk punt incident is met $k + 1$ assen van symmetrie, en we hebben een van de volgende mogelijkheden.

1. *Er geldt dat $k = 0$, en $\mathcal{S}$ heeft een Hermitische spread $\mathbf{T}_N$ van symmetrie-assen van $(\mathcal{S})$. De groep $Aut(\mathcal{S})$ van alle automorfismen van $\mathcal{S}$ werkt transitief op de rechten van $\mathbf{T}_N$. Tevens hebben we dat $\mathcal{S}' = \mathcal{S}$, en dat $t = s^2$.*
2. *We hebben dat $k = 1$, en er zijn twee mogelijkheden:*
   (a) *$\mathcal{S} = \mathcal{S}'$ en $\mathcal{S}$ is van de orde (s, s^2);*
   (b) *$\mathcal{S}'$ is een grid met parameters $s+1, s+1$, en elke rechte van $\mathcal{S}'$ is een as van symmetrie van $\mathcal{S}$. We hebben tevens dat $\mathcal{S}$ de orde (s, s^2) heeft.*
3. *$2 \leq k < t$ en we hebben twee mogelijkheden:*
   (a) *$\mathcal{S}' = \mathcal{S}$ en $\mathcal{S}$ is van de orde (s, s^2);*
   (b) *$k = s$, $\mathcal{S}' \neq \mathcal{S}$, $\mathcal{S}' \cong \mathcal{Q}(4, s)$ en $t = s^2$.*

VI *Elke rechte van $\mathcal{S}$ is een symmetrie-as, en $\mathcal{S}$ is isomorf met $\mathcal{Q}(4, s)$ of $\mathcal{Q}(5, s)$.*

Opmerking D.12.4 (i) We noemen de (disjuncte) klassen **I**–**VI** en hun onderverdelingen *symmetrie-klassen*[5].

(ii) Tevens hebben we in Hoofdstuk 12 alle gekende eindige veralgemeende vierhoeken (vaak als lid van meer algemene 'abstracte' klassen van vierhoeken) in hun (unieke) symmetrie-klasse geplaatst.

(iii) Veel nieuwe karakteriseringen en eigenschappen (en niet-existentie stellingen) duiken op in Hoofdstuk 12 (als gevolg van (ii)). We hebben tevens veralgemeningen bewezen van recente stellingen van L. Brouns, J. A. Thas en H. Van Maldeghem, zie [18].

(iv) Een classificatie van span-symmetrische veralgemeende vierhoeken is bevat in het einde van Hoofdstuk 12.

We geven hier slechts twee voorbeelden van karakteriseringen. In beide gevallen onderstellen we dat $s \neq t$ wegens het hoofdresultaat van Hoofdstuk 8.

Stelling D.12.5 *Onderstel dat $\mathcal{S}$ een GQ is van de orde (s, t), $s \neq 1 \neq t \neq s$, en onderstel dat $L \neq M$ concurrente symmetrie-assen zijn. Onderstel tevens dat N een symmetrie-as is van $\mathcal{S}$ die $M \cup L$ niet snijdt (als puntenverzameling). Dan hebben we de volgende mogelijkheden.*

1. *s is even en $\mathcal{S} \cong \mathcal{Q}(5, s)$;*

[5] In [218] onderscheiden we er 36 in totaal.

2. *s is oneven, $t = s^2$, en de unieke rechte U die $M \cap L$ bevat en N snijdt, is een rechte van translatiepunten. Dus, als x een willekeurig punt is op U, dan is de translatieduaal $(\mathcal{S}^{(x)})^*$ van de TGQ $\mathcal{S}^{(x)}$ de punt-rechte duaal van een flock veralgemeende vierhoek $\mathcal{S}(\mathcal{F})$.*

Merk op dat Stelling D.12.5 een sterke karakterisering oplevert van de klassieke vierhoek $\mathcal{Q}(5, s)$, s even.

Stelling D.12.6 *Onderstel dat $\mathcal{S}$ een SPGQ is van de orde (s, s^2), $s > 1$, met basis-span $\mathcal{L}$, en onderstel dat N een symmetrie-as is van $\mathcal{S}$ die geen rechte van $\mathcal{L}$ snijdt. Dan is $\mathcal{S} \cong \mathcal{Q}(5, s)$.*

Opmerking D.12.7 (i) Stelling D.12.6 is eigenlijk de oplossing van het natuurlijk 'SPGQ-vermoeden' voor het geval $t = s^2$, dat onlangs door W. M. Kantor werd gesteld in [95].

(ii) Ook voor veralgemeende vierhoeken van de orde (s^2, s^3), $s > 1$, zijn er gelijkaardige 'SPGQ-vermoedens', zie [95]. W. M. Kantor heeft enkele interessante deelresultaten gevonden in deze context [95].

Nota. Naast de vermelde technieken die we voor Hoofdstuk 12 gebruikten, hebben we ook gesteund op [18], de artikelenreeks [177, 181, 183] van J. A. Thas, en de recente preprints [185] en [187].

We eindigen deze sectie met twee open gevallen van Hoofdstuk 12, en die we hier vermelden als vermoedens.

Vermoeden D.12.8 *Onderstel dat $\mathcal{S}$ een SPGQ is van de orde (s, s^2), $s > 1$, waarvan de symmetrie-assen de twee reguli vormen van een grid met parameters $s + 1, s + 1$. Dan is $\mathcal{S}$ isomorf met $\mathcal{Q}(5, s)$.*

Vermoeden D.12.9 *Onderstel dat $\mathcal{S}$ een SPGQ is van de orde (s, s^2), $s > 1$, waarvan de symmetrie-assen één regulus vormen van een grid met parameters $s + 1, s + 1$. Dan is $\mathcal{S}$ isomorf met $\mathcal{Q}(5, s)$ als s even is. Als s oneven is, dan is $\mathcal{S}$ een TGQ, en is $\mathcal{S}$ bijgevolg de translatieduaal van de punt-rechte duaal van een flock vierhoek.*

D.13 Automorfismen van TGQ's

Een fundamenteel probleem in de theorie van de TGQ's $T(\mathcal{O})$ van de orde (q^n, q^m) is of de groep van automorfismen die het translatiepunt invariant

houden, geïnduceerd wordt door de groep van automorfismen van de projectieve ruimte $\mathbf{PG}(2n+m,q)$, waarin $T(\mathcal{O})$ gerepresenteerd is, die $\mathcal{O}$ vasthouden. Dit probleem is reeds open sinds 1974, en werd volledig opgelost in *J. A. Thas and K. Thas, On Translation Generalized Quadrangles and Translation Duals, Part I* [191], waarop Hoofdstuk 11 gebaseerd is.

Stelling D.13.1 *Onderstel dat $\mathcal{S} = T(\mathcal{O})$ een TGQ is van de orde (q^n, q^m) met translatiepunt (∞), waarbij $\mathcal{O} \subseteq \mathbf{PG}(2n+m-1,q) \subseteq \mathbf{PG}(2n+m,q)$. Dan wordt $Aut(\mathcal{S})_{(\infty)}$ geïnduceerd door de deelgroep van $\mathbf{P\Gamma L}(2n+m+1,q)$ die $\mathcal{O}$ vasthoudt.*

Gebruik makende van Stelling D.13.1 werd de volgende stelling verkregen:

Stelling D.13.2 *Onderstel dat $\mathcal{S} = T(\mathcal{O})$ een TGQ is van de orde (q^n, q^m) met translatiepunt (∞), waarbij $\mathcal{O} \subseteq \mathbf{PG}(2n+m-1,q) \subseteq \mathbf{PG}(2n+m,q)$, en waar q oneven is als $n = m$. Als T de translatiegroep is van $\mathcal{S}^{(\infty)}$, en x is een willekeurig punt van $\mathcal{S} \setminus (\infty)^\perp$, dan is*

$$Aut(\mathcal{S})_{(\infty)} \cong T \rtimes (Aut(\mathcal{S})_{(\infty)})_x.$$

Onderstel dat $\mathcal{S}^* = T(\mathcal{O}^*)$ de translatieduaal is van $\mathcal{S}$, en onderstel dat $(\infty)'$ het translatiepunt is van $\mathcal{S}^*$. Als x, respectievelijk x', een willekeurig punt is van $\mathcal{S} \setminus (\infty)^\perp$, respectievelijk $\mathcal{S}^* \setminus [(\infty)']^\perp$, dan is

$$(Aut(\mathcal{S})_{(\infty)})_x \cong (Aut(\mathcal{S}^*)_{(\infty)'})_{x'}.$$

Vervolgens bewezen we een classificatieresultaat voor TGQ's.

D.14 Een Elementair Bewijs van de Moufang Stelling

Het is een berucht probleem om een 'elementair' bewijs te geven van de classificatie van de eindige Moufang veralgemeende vierhoeken. S. E. Payne en J. A. Thas slaagden bijna in deze opzet, zie Hoofdstuk 9 van FGQ, op één geval na dat ondertussen bijna 20 jaar open is in deze context:

Stelling D.14.1 (S. E. Payne en J. A. Thas [139]) *Onderstel dat $\mathcal{S}$ een eindige Moufang veralgemeende vierhoek is van de orde (s,t), $s,t > 1$. Dan hebben we een van de volgende mogelijkheden:*

(*i*) *$\mathcal{S}$ is klassiek of duaal klassiek;*

(*ii*) *$\mathcal{S}$ is van de orde (s, s^2), en elke rechte van $\mathcal{S}$ is een symmetrie-as.*

W. M. Kantor gaf een oplossing voor Geval (ii) in [91], maar naast de classificatie van de eindige split BN-paren van rang 1, maakt hij tevens gebruik van 4 B,C van P. Fong en G. M. Seitz [60, 61].

In Appendix A bewijzen we dan het volgende, met gebruik van de classificatie van de eindige split BN-paren van rang 1 [156, 72][6], en resultaten van Hoofdstuk 8 en Hoofdstuk 9:

Stelling D.14.2 *Onderstel dat $\mathcal{S}$ een GQ is van de orde (s, s^2), $s > 1$, en dat elke rechte van $\mathcal{S}$ een symmetrie-as is. Dan is $\mathcal{S}$ klassiek, i.e. isomorf met $\mathcal{Q}(5, s)$.*

Appendix A is gebaseerd op *K. Thas, Automorphisms and Characterizations of Finite Generalized Quadrangles* [206].

D.15 Half Pseudo Moufang Veralgemeende Vierhoeken

We noemen een paneel (p, L, q) in een GQ van de orde (s, t), $s, t > 1$, *pseudo Moufang* t.o.v. een automorfismengroep H van $\mathcal{S}$, als de volledige verzameling van elaties met centrum p en q een normale deelgroep is van $(H_p)_q$ van de orde s. We voeren nu *Voorwaarde (PM)* in.

(PM) *Elk paneel is pseudo Moufang.*

Een GQ is *half pseudo Moufang* als Voorwaarde (PM) of de duale voorwaarde voldaan is. Half pseudo Moufang veralgemeende vierhoeken zijn dus veel algemener gedefinieerd dan Moufang veralgemeende vierhoeken.
In samenwerking met H. Van Maldeghem slaagden we erin de half pseudo Moufang veralgemeende vierhoeken volledig te classificeren (zonder [60, 61] te gebruiken):

Stelling D.15.1 *Een half pseudo Moufang veralgemeende vierhoek is klassiek of duaal klassiek.*

[6] In ons bewijs is de groepentheorie van [156, 72] van weinig belang; eigenlijk hebben we enkel de orde van zekere groepen nodig.

Het bewijs van Stelling D.15.1 wordt uitvoerig beschreven in Hoofdstuk 13, en is ingediend ter publicatie in *Journal für die reine und angewandte Mathematik*, onder de naam *K. Thas and H. Van Maldeghem, Half Pseudo Moufang Generalized Quadrangles are Classical or Dual Classical* [220].

Opmerking D.15.2 De 'pseudo Moufang voorwaarde' hebben we (in een andere vorm) ingevoerd in [204] in verband met de studie van de *sterke elatie veralgemeende vierhoeken* (dit zijn de GQ's waarvan elk punt een elatiepunt is). In samenwerking met H. Van Maldeghem zijn we reeds grotendeels geslaagd in de opzet om ook deze objecten volledig te classificeren. Veel ideeën van [220] kunnen in deze context overgenomen worden.

D.16 Toepassing: Veralgemeende Vierhoeken met een Spread van Symmetrie

Onderstel dat $\mathcal{S}$ een GQ is van de orde (s,t), $s \neq 1 \neq t$. Een *spread* is een verzameling van $st+1$ rechten die twee aan twee niet snijden. Een spread heeft de karakteristieke eigenschap dat elk punt van de vierhoek op precies een rechte van de spread ligt. Een *spread van symmetrie* van $\mathcal{S}$ is een spread waarvoor er een groep H is van collineaties van $\mathcal{S}$ die de spread rechtegewijs vasthoudt, en die transitief (en dan regulier) werkt op de punten van tenminste één (en dan alle) rechte(n) van de spread.

B. De Bruyn [39, 41] heeft aangetoond dat het bestaan van een spread van symmetrie in een GQ kan gebruikt worden om *schier veelhoeken* te construeren. Een *schier $2n$-hoek van de orde (s,t)*, met $n > 1$ en $s, t > 1$, is een punt-rechte meetkunde $\mathcal{S} = (P, B, I)$ die aan de volgende vier voorwaarden voldoet.

(NP1) Er zijn $s+1$ punten op een rechte en twee verschillende punten zijn incident met ten hoogste één rechte.

(NP2) Er zijn $t+1$ rechten door een punt en twee verschillende rechten zijn incident met ten hoogste één punt.

(NP3) De diameter van de puntgraaf is n.

(NP4) Voor elk punt-rechte paar (p, L) is er een uniek punt op L dat het dichtst ligt bij p.

Schier veelhoeken werden ingevoerd door E. E. Shult en A. Yanushka in [160]. Het vinden van nieuwe spreads van symmetrie levert nieuwe *schier zeshoeken*

op, en indien H abels is, kunnen ook nieuwe *schier achthoeken* opgeleverd worden.

In samenwerking met B. De Bruyn hebben we in [45], dat aanvaard is ter publicatie in *Illinois Journal of Mathematics*, en waarop Hoofdstuk 14 gedeeltelijk gebaseerd is, aangetoond dat translatie veralgemeende vierhoeken met een spread van symmetrie altijd klassiek zijn, namelijk isomorf met $\mathcal{Q}(5,q)$ voor een bepaalde q. Meer zelfs: we zijn er tevens in geslaagd aan te tonen dat een elatie veralgemeende vierhoek (t.o.v. een punt) van de orde (s,t), waar $s \neq 1 \neq t$ en s even, met een spread van symmetrie altijd isomorf is met $\mathcal{Q}(5,s)$. De voornaamste technieken die we gebruiken, komen uit Hoofdstuk 8, Hoofdstuk 9 en Hoofdstuk 12. Essentieel is de observatie dat een GQ die aan deze voorwaarden voldoet

Twee translatiepunten heeft.

De vraag bleef open in [45] om ook aan te tonen dat

(NPO) *Een elatie veralgemeende vierhoek (t.o.v. een punt) van de orde (s,t), $s \neq 1 \neq t$ en s oneven, met een spread van symmetrie, isomorf is met $\mathcal{Q}(5,s)$.*

Dat probleem werd volledig opgelost in K. Thas [216], dat ingediend werd ter publicatie in *Journal of Combinatorial Theory, Series A*, en de oplossing vormt het tweede deel van Hoofdstuk 14.

Opmerking D.16.1 (i) Een moeilijker probleem is het classificeren van elatie veralgemeende vierhoeken *ten opzichte van een rechte* met een spread van symmetrie. Hier kunnen we (voorlopig) niet steunen op de resultaten van [208, 210, 218]. Duale flock veralgemeende vierhoeken, die altijd EGQ's zijn t.o.v. een rechte, zouden bijvoorbeeld een eerste objectief kunnen zijn in deze context.

(ii) De oplossing van (NPO) samen met het tweede hoofdresultaat van [45] leidt tot een volledige classificatie van dikke EGQ's (t.o.v. een punt) met een spread van symmetrie.

(iii) Merk op dat de vermelde resultaten belangrijk zijn omdat elke nieuwe spread van symmetrie nieuwe schier veelhoeken oplevert.

D.17 Blueprint voor de Classificatie van Alle Eindige TGQ's

De enige gekende TGQ's die isomorf zijn met hun translatieduaal, zijn:

- $\mathcal{Q}(4,q)$, q oneven, en $\mathcal{Q}(5,q)$;
- de $T_3(\mathcal{O})$ van Tits;
- de GQ $\mathcal{S}(\mathcal{F})^D$, $\mathcal{F}$ een Kantor flock.

Vermoeden D.17.1 *Dit zijn de enige TGQ's met die eigenschap.*

Onderstel dat $\mathcal{S}$ een TGQ is van de orde (q^n, q^m).

(i) Als $n = m$, en $\mathcal{S}$ heeft verschillende translatiepunten, dan is $\mathcal{S} \cong \mathcal{Q}(4,q^n)$.

(ii) Als $n = m$, q oneven, en $\mathcal{S}$ heeft één translatiepunt, dan is

$$Aut(\mathcal{S})_x \cong Aut(\mathcal{S}^*)_{x'},$$

waar x, respectievelijk x', een willekeurig punt niet collineair met het translatiepunt van $\mathcal{S}$, respectievelijk $\mathcal{S}^*$, is.

(iii) Als $n \neq m$, en $\mathcal{S}$ heeft twee verschillende translatiepunten, dan is $(\mathcal{S}^*)^D$ een flock GQ (en dus is $m = 2n$). We hebben dat

(a) als $\mathcal{S}^*$ ook verschillende translatiepunten heeft, dan is $\mathcal{S} \cong \mathcal{S}^*$, en $\mathcal{F}$ is een Kantor flock;

(b) als $\mathcal{S}^*$ één translatiepunt $(\infty)'$ heeft, en (∞) is een willekeurig translatiepunt van $\mathcal{S}$, dan is

$$(Aut(\mathcal{S})_{(\infty)})_x \cong (Aut(\mathcal{S}^*)_{(\infty)'})_{x'} = Aut(\mathcal{S}^*)_{x'},$$

waar x, respectievelijk x', een willekeurig punt niet collineair met (∞), respectievelijk $(\infty)'$, is.

(iv) Als $\mathcal{S}$ en $\mathcal{S}^*$ beide één translatiepunt hebben, dan is

$$Aut(\mathcal{S})_x \cong Aut(\mathcal{S}^*)_{x'},$$

waar x, respectievelijk x', een willekeurig punt niet collineair met het translatiepunt van $\mathcal{S}$, respectievelijk $\mathcal{S}^*$, is.

Vermoeden D.17.2 *Zijn we in Geval (iv), dan is* $\mathcal{S} \cong \mathcal{S}^*$.

Vermoeden D.17.3 *De enige TGQ's van de orde s, s even, zijn isomorf met een* $T_2(\mathcal{O})$ *van Tits.*

D.17.1 Classificatie van eindige translatie veralgemeende vierhoeken

Als Vermoeden D.17.1, Vermoeden D.17.2 en Vermoeden D.17.3 juist zijn, dan hebben we dat elke TGQ van de orde (q^n, q^m) een van de volgende is:

- $\mathcal{Q}(4, q^n)$ of $\mathcal{Q}(5, q^n)$;
- een $T_2(\mathcal{O})$ van Tits, $\mathcal{O}$ een ovaal van $\mathbf{PG}(2, q^n)$;
- een $T_3(\mathcal{O})$ van Tits, $\mathcal{O}$ een ovoïde is van $\mathbf{PG}(3, q^n)$;
- $\mathcal{S}(\mathcal{F})^D$, $\mathcal{F}$ een semifield flock van de kwadratische kegel in $\mathbf{PG}(3, q^n)$, q oneven;
- $(\mathcal{S}(\mathcal{F})^D)^*$, $\mathcal{F}$ een semifield flock van de kwadratische kegel in $\mathbf{PG}(3, q^n)$, q oneven.

Voor meer details en gevolgen verwijzen we naar Hoofdstuk 15.

Appendix E

A Table of the Sizes of Some Groups and a Table of Schur Multipliers

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E.1 A Table of the Sizes of Some Groups

In Table E.1 are listed the sizes of some groups of the form $X(r, q')$, respectively $G(q')$, $q' = p^h$ with p prime, which frequently arise in this work; X denotes one of **PΓL**, **PGL**, **PSL**, **PSU**, **SL**, **SU**; G denotes one of **R**, **Sz**. If $X \in \{\mathbf{PSU}, \mathbf{SU}\}$, then $q' = q^2$; in all the other cases $q' = q$.

E.2 A Table of Schur Multipliers

If G is a perfect group and $(\overline{G}, \eta)$ is its universal central extension, then $ker(\eta)$ is sometimes also called the *Schur multiplier* of G. In Table E.2, we list the

Schur multiplier of a group of the form $X(r,q)$, respectively $G(q)$, $q = p^h$, as $R \times P$, where R is a p'-group and P is a p-group with $p \neq p'$, see D. Gorenstein [65] (see also J. L. Alperin and D. Gorenstein [2], R. L. Griess, Jr. [67] and I. Schur [150]). Here, the groups the $X(r,q)$ and $G(q)$ are the Lie groups of which the (size of the) Schur multiplier was explicitly used in the body of this work (so $X \in \{\mathbf{PSL}, \mathbf{PSU}\}$; $G \in \{\mathbf{R}, \mathbf{Sz}\}$).

E.1. The Sizes of Some Groups.

Group	Order
$\mathbf{P\Gamma L}(r,q)$	$h(q-1)^{-1}q^{r(r-1)/2}\prod_{i=1}^{r}(q^i-1)$
$\mathbf{PGL}(r,q)$	$(q-1)^{-1}q^{r(r-1)/2}\prod_{i=1}^{r}(q^i-1)$
$\mathbf{PSL}(r,q)$	$(gcd(r,q-1))^{-1}(q-1)^{-1}q^{r(r-1)/2}\prod_{i=1}^{r}(q^i-1)$
$\mathbf{PSU}(r,q^2)$	$(gcd(r,q+1))^{-1}(q+1)^{-1}q^{r(r-1)/2}\prod_{i=1}^{r}(q^i-(-1)^i)$
$\mathbf{SL}(r,q)$	$(q-1)^{-1}q^{r(r-1)/2}\prod_{i=1}^{r}(q^i-1)$
$\mathbf{SU}(r,q^2)$	$(q+1)^{-1}q^{r(r-1)/2}\prod_{i=1}^{r}(q^i-(-1)^i)$
$\mathbf{Sz}(q) \cong {}^2\mathbf{B}_2(q)$	$(q^2+1)q^2(q-1)$
$\mathbf{R}(q) \cong {}^2\mathbf{G}_2(q)$	$(q^3+1)q^3(q-1)$

E.2. Table of Schur multipliers.

Group	R (if $P = \{\mathbf{1}\}$)	$R \times P$ (if $P \neq \{\mathbf{1}\}$)	$(r-1, q)$
$\mathbf{PSL}(r,q)$	$\mathbb{Z}_{gcd(r,q-1)}$	$\mathbb{Z}_2$ $\mathbb{Z}_2 \times \mathbb{Z}_3$ $\mathbb{Z}_2$ $\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ $\mathbb{Z}_2$	$(1,4)$ $(1,9)$ $(2,2)$ $(2,4)$ $(3,2)$
$\mathbf{PSU}(r,q^2)$	$\mathbb{Z}_{gcd(r,q+1)}$	$\mathbb{Z}_2$ $\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$(3,2)$ $(3,3)$ $(5,2)$
$\mathbf{Sz}(q) \cong {}^2\mathbf{B}_2(q)$	$\{\mathbf{1}\}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(2,8)$
$\mathbf{R}(q) \cong {}^2\mathbf{G}_2(q)$	$\{\mathbf{1}\}$		

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Index of Notations

We only list the notations and (mathematical) abbreviations which are frequently and systematically used.

$\mathbf{A\Gamma L}(n+1,q)$	The full semilinear automorphism group of $\mathbf{AG}(n,q)$
$\mathbf{AG}(n,q)$	The n-dimensional affine space over $\mathbf{GF}(q)$
AGQ	Affine generalized quadrangle
A_n	The alternating group of degree n
AT	Admissible triple
$Aut(\mathcal{S})$	The full automorphism group of the GQ $\mathcal{S}$
$cl(p,q)$	$\{z \in \mathcal{S} \parallel z^\perp \cap \{p,q\}^{\perp\perp} \neq \emptyset\}$
EGQ	Elation generalized quadrangle
$\mathcal{F}$	A flock of the quadratic cone in $\mathbf{PG}(3,q)$
$\mathbb{F}$	A (finite) field
$gcd(n,m)$	The greatest common divisor of the natural numbers n and m
$\mathbf{GF}(q)$	The finite Galois field with q elements
$\Gamma = (\Omega, \mathcal{L} \cup \mathcal{L}^\perp, I')$	The base-grid of an SPGQ
G'	The derived group of G
GQ	Generalized quadrangle
$H(4,q^2)$	The classical Hermitian quadrangle of order (q^2,q^3)
$H(4,q^2)^D$	The point-line dual of the classical Hermitian quadrangle of order (q^2,q^3)
$H \rtimes H'$	Semidirect product of H and H' (where $H \trianglelefteq H \rtimes H'$)

H_q^n	The dual net with $q+1$ points on any line and q^{n-1} lines through any point which satisfies the Axiom of Veblen
$(H_q^n)^D$	The point-line dual of H_q^n
HPMGQ	Half pseudo Moufang generalized quadrangle
$H_{\mathbf{T}}$	The group which fixes the spread of symmetry $\mathbf{T}$ elementwise
$H(3,q^2)$	The classical Hermitian quadrangle of order (q^2,q)
I	Incidence relation
$(\mathcal{J},\mathcal{J}^*)=\mathcal{J}$	A 4-gonal family
$\mathbb{K}$	A (general finite) field, or the kernel of some TGQ
$\mathcal{K}$	The quadratic cone of $\mathbf{PG}(3,q)$, or a complete $(st-t/s)$-arc in a GQ of order (s,t)
$\mathcal{L}$	The base-span of an SPGQ
$\mathbb{N}$	The set of natural numbers (including 0)
$N_H(H')$	The normalizer of H' in H (where $H' \leq H$)
$\mathcal{N}_L$	The net which arises from the regular line L of a GQ
$\mathcal{N}_p$	The net which arises from the regular point p of a GQ
$\mathcal{N}_L^*$	The point-line dual of $\mathcal{N}_L$
$\mathcal{N}_p^*$	The point-line dual of $\mathcal{N}_p$
$\mathbb{N}_0$	The set of strictly positive natural numbers
$\mathbf{O}$	A (partial) ovoid
$\mathcal{O}=\mathcal{O}(n,m,q)$	A generalized ovoid, respectively generalized oval, in $\mathbf{PG}(2n+m-1,q)$, where $n \neq m$, respectively $n=m$
$\mathcal{O}^*=\mathcal{O}^*(n,m,q)$	The 'dual egg' of $\mathcal{O}=\mathcal{O}(n,m,q)$
$\Omega(T)=(\mathcal{S},\mathbf{T})$	A GQ-spread pair
$\mathcal{O}_x$	The ovoid subtended by x
$\mathbf{PGL}(n+1,q)$	The full linear automorphism group of $\mathbf{PG}(n,q)$
$\mathbf{P\Gamma L}(n+1,q)$	The full semilinear automorphism group of $\mathbf{PG}(n,q)$
$\mathbf{PG}(n,q)$	The n-dimensional projective space over $\mathbf{GF}(q)$
$\mathbf{P\Gamma O}(2n+1,q)$	The projective orthogonal subgroup of $\mathbf{P\Gamma L}(2n+1,q)$
$[p,L]$	The unique line through p which is concurrent with L
$proj_L p$	The unique point on L which is collinear with p
$proj_p L$	The unique line through p which is concurrent with L
PQ	Partial quadrangle

$\mathbf{PSL}(n+1,q)$	The projective special linear subgroup of $\mathbf{PGL}(n+1,q)$
$\mathbf{PSU}(n+1,q)$	The projective special unitary subgroup of $\mathbf{PGL}(n+1,q)$
$\mathcal{P}(\mathcal{S},x)$	The GQ which arises from the GQ $\mathcal{S}$ of order s with regular point x
$\mathcal{Q}(5,q)$	The classical GQ of order (q,q^2) arising from a nonsingular quadric in $\mathbf{PG}(5,q)$
$\mathcal{Q}(4,q)$	The classical GQ of order (q,q) arising from a nonsingular quadric in $\mathbf{PG}(4,q)$
$\mathcal{Q}(3,q)$	The classical GQ of order $(q,1)$ arising from a nonsingular quadric in $\mathbf{PG}(3,q)$
$\mathbf{R}(q)$	The Ree group of degree q^3+1 in characteristic 3
$\mathcal{S}^D$	The point-line dual of $\mathcal{S}$
SEGQ	Strong elation generalized quadrangle
$\mathcal{S}(\mathcal{F})$	The flock GQ which corresponds to the flock $\mathcal{F}$
$\mathcal{S}(\Gamma)$	The GQ spanned by the AGQ Γ
$\mathcal{S}(G_*)$	The closure of G_*
$\mathcal{S}(\mathcal{K})$	The GQ which arises from the complete $(st-t/s)$-arc $\mathcal{K}$ in a GQ of order (s,t)
$\mathbf{SL}(n+1,q)$	The special linear subgroup of $\mathbf{GL}(n+1,q)$
S_n	The symmetric group of degree n
$\mathcal{S}=(P,B,I)$	The GQ with point set P, line set B, and incidence relation I
SPGQ	Span-symmetric generalized quadrangle
SQ	Semi quadrangle
$\mathcal{S}^*$	The translation dual of $\mathcal{S}$
$\mathcal{S}_\theta$	The subGQ fixed elementwise by θ
STGQ	Skew translation generalized quadrangle
$\mathbf{SU}(n+1,q)$	The special unitary subgroup of $\mathbf{GL}(n+1,q)$
$\mathcal{S}^{(x)}=(\mathcal{S}^{(x)},G)$	An EGQ or TGQ with base-point x (and base-group G)
$\mathbf{Sz}(q)$	The Suzuki group of degree q^2+1
$\mathbf{T}$	A (partial) spread
$T=(\mathcal{D},H,\Delta)$	The admissible triple defined by $\mathcal{D}$, H and Δ
TGQ	Translation generalized quadrangle
$\mathbf{T}(\mathcal{L},M)$	The spread determined by the base-span $\mathcal{L}$ and the line M
$T(n,m,q)$	A TGQ which can be represented in $\mathbf{PG}(2n+m-1,q)$
$T(\mathcal{O})$	The TGQ which arises from the generalized ovoid (respectively oval) $\mathcal{O}$

tr	Trace function
$T^*(\mathcal{O}) = T(\mathcal{O}^*)$	The translation dual of $T(\mathcal{O})$
$T_3(\mathcal{O})$	The GQ of Tits which arises from the ovoid $\mathcal{O}$ in $\mathbf{PG}(3,q)$
$T_2(\mathcal{O})$	The GQ of Tits which arises from the oval $\mathcal{O}$ in $\mathbf{PG}(2,q)$
$V^\perp$	The set of elements which are collinear, respectively concurrent, with each element of V, where $V \subseteq P$, respectively $V \subseteq B$
$V^{\perp\perp}$	The set of elements which are collinear, respectively concurrent, with each element of $V^\perp$, where $V \subseteq P$, respectively $V \subseteq B$
$W(q)$	The classical symplectic quadrangle of order q
(X, G)	A permutation group, where G acts on X
$\mathbb{Z}$	The set of integers
$Z(G)$	The center of the group G
$\mathbb{Z}_0$	The set of nonzero integers

Miscellaneous Notations

$\sim$	Collinear, or concurrent (or adjacent)
$\cong$	Isomorphic
$\lfloor x \rfloor$	The greatest natural number which is at most x
$\lceil x \rceil$	The smallest natural number which is at least x

* * *

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We only list the notions which are used frequently and systematically, or which are substantial for this work. Some miscellaneous terms are included.

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