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ON THE LANDAU-LIFSHITZ EQUATION OF FERROMAGNETISM

Thesis submitted to Ghent University
in candidature for the degree
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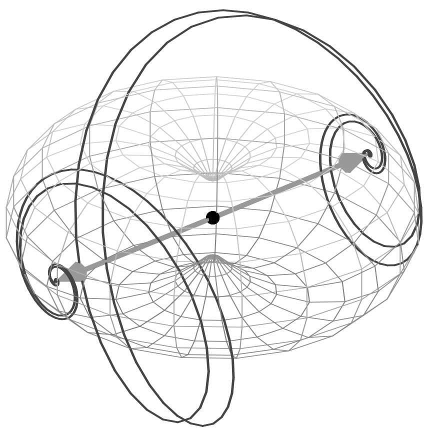
in mathematics



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Ghent University
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To Mirka, my parents and all I like.

Preface

Some time ago I read a book on ancient and contemporary coding. About how the ciphers are developed by encrypters and how decrypters try to break them. Through the ages as soon as a new cipher was invented, somebody always decrypted it in a short time. Both decryption and encryption was almost on the same level. Until the invasion of computers.

Since it is possible to perform millions mathematical operations in a short while, a new phenomena arose. Using simple algorithms it is easy to encrypt any text such that decryption becomes almost impossible. I'm speaking about the encryption methods based on the public and the private key using factorization of big numbers. Now, the relation between encryption and decryption is not balanced anymore. The difficulty begins when you do not have the private key. It would take too much time to find it. Millions of years.

But. Suppose that the decrypter doesn't play absolutely fair. There are several ways how to get the private key. For encryption you need a computer. Then, of course, viruses and Trojan horses in encryption programs can play a key role to find the private key. But for this you need at least a bug in the computers. However, there exists another way.

The decrypter, although this name is not appropriate anymore, comes next to the house of the encrypter, or better: user of an encryption program, with special equipment hidden in his van. With this equipment he is able to see the text of the message to be encrypted while the user enters his text via the keyboard. So actually, no decryption is necessary.

How? The keyboard, the monitor and in fact all electronic devices radiate electromagnetic waves. To trace them and to decode what was entered via the keyboard or what is on the screen, is not an easy task. But nowadays special devices are capable to do this. The construction of such devices would not be possible without a deep understanding and knowledge of the involved phenomena.

The scientific domain enabling all this magic is called Electromagnetism.

Acknowledgements

First of all I would like to thank Prof. Jozef Kačur for opening the world of numerical functional analysis to me and for motivating me to study this very rich branch of mathematics by his "unbounded" enthusiasm, concerning not only mathematics and physics. Next, I would like to express my thanks to Prof. Roger Van Keer for his co-promotorship, for his continuous interest, help and stimulation and for making perfect conditions for my work as a young researcher. I am also very grateful to my co-promotor Prof. Marián Slodička for his wisdom and good advises anytime, when mathematical difficulties seemed to be impossible to overcome. Also my teachers and professors deserve my thanks for giving me the basis of my knowledge.

My work was financially supported by an IWT/STWW and an IUAP project, with engineers from UGent, ULg and KULeuven from whom I learned a lot about real life applications of micromagnetism, especially from Prof. Luc Dupré.

Besides of the persons directly involved in writing this thesis I would like to thank also other people: my parents for giving me the the right education and care, my two sisters for their support and understanding, and Mirka, my wife and best friend, for her love and smiles whenever it was necessary.

I would like to thank also my slovak friends, which have persuaded me that physical distances are not always so important. Many thanks also to my Belgian friends Sofie and Joost for making the stay in Belgium so enjoyable and nice.

Finally, I would like to thank all my colleagues of the Department of Mathematical Analysis for creating a really nice atmosphere during the work.

Contents

I

Generalities

1 Samenvatting

(As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality. Albert Einstein)

De nadruk van dit werk ligt op de numerieke analyse van de Landau-Lifshitz vergelijking (LL-vergelijking). We ontwikkelen verscheidene numerieke schema's gebaseerd op tijdsdiscretisatie. De randwaardeproblemen worden niet volledig gediscretiseerd. De ruimtediscretisatie van alle opgestelde schema's zal het onderwerp zijn van toekomstig onderzoek. Studie van de ruimtediscretisatie lijkt echter niet zo uitdagend als de studie van tijdsdiscretisatie.

Ons werk bestaat uit drie delen. Eerst, in Hoofdstukken 1 t.e.m. 3 geven we o.a. een inleiding en bespreken we bondig Whitney-elementen. In het tweede deel (Hoofdstukken 4 t.e.m. 6) wordt het efective veld $\mathbf{H}_{\text{eff}}$ beschouwd zonder uitwisselingsenergieterm. In het derde deel (Hoofdstukken 7 t.e.m. 10) wordt wel met deze term rekening gehouden en bestuderen we de LL-vergelijking met uitwisselingsterm.

Tweede deel

Het geval zonder de uitwisselingsterm beschrijft uniform gemagnetiseerde media of andere configuraties waarbij de uitwisselingseffecten zeer klein zijn in vergelijking met andere bijdragen tot $\mathbf{H}_{\text{eff}}$.

In Hoofdstuk 4 starten we met het beschrijven van de natuurlijke aanpak van tijdsdiscretisatie van de LL-vergelijking die gebruikt werd door Joly, Vacus, Monk, Bertotti en anderen. De gebruikte idee in hun werk kreeg de naam “mid-point rule”.

De LL-vergelijking in de continue vorm heeft de eigenschap dat de modulus van de magnetisatie constant blijft in de tijd. Uiteraard is er nood aan numerieke schema's die deze belangrijke fysische eigenschap bewaren. De implementatie van de “mid-point rule” laat schema's toe deze voorwaarde te vervullen.

In Hoofdstuk 5, zie Figuur 2.7, benaderen we dit probleem op een andere manier. We discretiseren de tijdsafgeleide niet. We laten deze continu veranderen over het generieke interval (t_{i-1}, t_i). We kiezen de andere vectoren op zo'n manier dat de LL-vergelijking exact kan opgelost worden en dus verkrijgen we een continue approximatie van de magnetisatie $\mathbf{m}$.

We stellen foutenschattingen op voor de vermelde schema's, wat nog niet eerder gedaan werd voor op "mid-point rule" gebaseerde schema's.

In Hoofdstuk 6 passen we onze schema's aan met als doel het verhogen van de convergentiesnelheid, zie Figuur 2.7. Deze aanpassing is gebaseerd op iteraties die gebeuren bij elke tijdsstap. Deze iteraties convergeren dankzij de samentrekkingseigenschappen. De limiet, of anders gezegd, het vast punt van deze iteraties benadert de exacte oplossing met hogere nauwkeurigheid. De convergentiesnelheid ligt hierdoor hoger.

Derde deel

In Hoofdstukken 7 t.e.m. 10 behandelen we de LL-vergelijking waarbij het uitwisselingsveld een deel is van het efective veld $\mathbf{H}_{\text{eff}}$. Deze configuratie vormt een uitdaging omdat de uitwisselingsterm aanleiding geeft tot een partiële differentiële LL-vergelijking. Zonder $\mathbf{H}_{\text{eff}}$ krijgen we een gewone LL-vergelijking.

Om de convergentieresultaten van de numerieke schema's te bewijzen is het noodzakelijk om een gedetailleerde analyse van de LL-vergelijking te maken. In Sectie 2.4 geven we een overzicht van theoretische en numerieke resultaten voor de LL-vergelijking. We gebruiken deze gekende resultaten in numerieke analyse, maar soms is het noodzakelijk ze uit te breiden.

De studie van de LL-vergelijking maakt gebruik van de theorie van harmonische afbeeldingen. De vorm lijkt sterk op de vergelijking van de harmonisch warmtestroming. Struwe heeft een inleiding van de theorie van harmonische afbeeldingen geschreven in [76]. In [75] verkrijgt hij regulariteitsresultaten die direct kunnen gebruikt worden in het geval van de LL-vergelijking. Het gebruik van deze resultaten is besproken door Guo en Hong in [36].

Er is een grote leemte in de theorie van harmonische afbeeldingen tussen gekende resultaten van het 2D- en het 3D-geval. De aard van het probleem verandert drastisch wanneer men meer dan twee dimensies beschouwt. Dit fenomeen heeft als gevolg dat veel minder gekend is over regulariteitsresultaten van de exacte oplossing van de LL-vergelijking in 3D dan bij lagere dimensie.

In ons werk bestuderen we eerst de enkelvoudige LL-vergelijking in 3D, zonder de Maxwellvergelijkingen te beschouwen. In Hoofdstuk 7, zie Figuur 2.7, leiden we regulariteitsresultaten af, zoals

$$\max_{t\in(0,T_0)} \left\{ \kappa^p \|\partial_t^{p+1}\mathbf{m}\|_2 + \kappa^{\frac{2p+1}{2}} \|\nabla\partial_t^{p+1}\mathbf{m}\|_2 + \kappa^{p+1}\|\Delta\partial_t^{p+1}\mathbf{m}\|_2 \right\} \le C,$$

met p een niet negatief geheel getal en κ het tijdsgewicht gedefinieerd als

$$\kappa(s) = \begin{cases} 0, & \text{voor } s < 0, \\ \min\{1, s\}, & \text{voor } s \geq 0. \end{cases}$$

De sleutelafschatting om dit resultaat te bewijzen is

$$\max_{t \in (0,T_0)} \|\mathbf{m}\|_{W^{2,2}} \leq C,$$

wat bekomen werd door Carbou en Fabrie in [18].

In het volgend hoofstuk ligt de nadruk op een volledig Maxwell-LL systeem. Om gelijkaardige resultaten te bekomen als in Hoofdstuk 7 hebben we een afschatting nodig vergelijkbaar met het hulpresultaat van Carbou en Fabrie. We tonen aan dat

$$\max_{t \in (0,T_0)} \{\|\mathbf{m}\|_{W^{2,2}} + \|\mathbf{E}\|_{W^{1,2}} + \|\mathbf{H}\|_{W^{1,2}}\} \leq C,$$

wat een nieuw resultaat is het 3D-geval. Een bewijs wordt gegeven in Sectie 8.5, zie Figuur 2.7. We bekomen een rij van eindigdimensionale ruimtes die de vectorruimtes benaderen waartoe de verwachte oplossingen behoren. Vertrekkend van deze eindigdimensionale ruimtes construeren we een rij van benaderingen die convergeren naar de oplossing van de LL-vergelijking.

We bewijzen regulariteitsresultaten voor benaderingen die robuust genoeg zijn om op de oplossing van de LL-vergelijking getransfereerd te worden. Bovendien bewijzen we dat de oplossing van de LL-vergelijking lokaal uniek is. Met andere woorden, dat er een positieve T_0 bestaat zodat de oplossing van de LL-vergelijking uniek is op het interval $\langle 0, T_0 \rangle$. Op deze manier verkrijgen we de bovenvermelde originele afschatting.

Uiteindelijk gebruiken we de resultaten van Hoofdstukken 7 en 8. We introduceren een semi-impliciet schema in 3D. Convergentieresultaten worden bewezen in Hoofdstuk 9. Dit schema bewaart niet de lengte van de magnetisatie. We weten enkel dat deze lengte weinig varieert en bovendien afneemt bij dalende discretisatiestap. Voor meer details verwijzen we naar Stelling 9.2.

In de literatuur zijn andere schema's beschreven die enkelvoudige LL-vergelijkingen of volledige M-LL-systemen behandelen, waarbij de uitwisselingsterm in acht wordt genomen. Deze schema's zijn gebaseerd op penalisatietermen, die het verschil tussen $|\mathbf{m}^i|$ en $|\mathbf{m}^0|$ penaliseren. Dergelijk schema leidt tot resultaten die beter zijn in de zin van het bewaren van de modulus.

Deze methode werd gebruikt door Prohl in [61] voor het 2D-geval. Het is mogelijk om deze resultaten uit te breiden naar drie dimensies; dit wordt kort beschreven in Hoofdstuk 10. Daar deze extensie zeer technisch is, laten we hier de details achterwege.

Berekeningen

Om de theoretische resultaten te controleren voeren we enkele numerieke berekeningen uit. Hierbij ligt de nadruk eerder op de analyse. We beschouwen vooral academische voorbeelden. Echt praktische toepassingen worden niet behandeld. Hiertoe zou speciale, stabiele numerieke software dienen ontwikkeld te worden, wat zeer tijdsintensief is en buiten de opzet van deze thesis valt.

De software ALBERT werd aangepast aan onze doeleinden. Voor een overzicht van de mogelijkheden van ALBERT verwijzen we naar [63, 64]. Samen met L'. Baňas implementeerden we Whitney-elementen om magnetische en elektrische velden te benaderen. Voor een overzicht van Whitney-elementen verwijzen we naar Hoofdstuk 3.

2 Introduction

(Not everything can be described by PDE's)

Nowadays the study of magnetic materials and its behavior on very small scales is of huge interest for several technological devices. Applications such as data recording ask to understand the dynamics of the magnetization at nanometric scales. The micro-magnetic simulation allow us to predict this behavior with high accuracy.

Next major challenge for the tape recording industry is to move the limits of the information density stored on the magnetic medium. The amount of information stored on one workpiece doubles approximately every 18 months. The techniques of the recording, which were satisfactory 2 years ago, are not sufficient any more. To be able to improve these techniques one must understand what happens on very small scales. Micromagnetics suggest a physical model describing dynamics of sub-micron magnetic systems.

2.1 Maxwell's equations

In the theory of electromagnetism several vector fields describe the behavior of the medium and its properties. We consider the case of a time-varying field when electric and magnetic fields exist simultaneously. Then the following vector fields

H magnetic field intensity ($A\ m^{-1}$);

E electric flux intensity ($V\ m^{-1}$);

B magnetic flux density (magnetic induction) ($Wb\ m^{-2}$);

D electric flux density ($C\ m^{-2}$);

$\mathbf{J}$ electric current density ($A\ m^{-2}$);

and the function

ρ electric charge density ($C\ m^{-3}$),

have to be considered to describe the whole process. The properties of the media involved are described by

ε permitivity of medium ($F\ m^{-1}$);

μ permeability of medium $H\ m^{-1}$;

σ conductivity of medium ($S\ m^{-1}$).

Of course, there are special cases, when the material properties allow us to consider only some of these vector fields. For a more detailed review on the physical origins of these vector fields and constants describing material properties we refer to [15, 58, 70, 79].

The equations describing electromagnetics are called *Maxwell's equations*. They can be written in differential form as

$$\begin{aligned} \nabla \times \mathbf{H} &= \mathbf{J} + \partial_t \mathbf{D}, \\ \nabla \times \mathbf{E} &= -\partial_t \mathbf{B}, \\ \nabla \cdot \mathbf{D} &= \rho, \\ \nabla \cdot \mathbf{B} &= 0, \end{aligned}$$

for general time-varying fields. The fields are linked by so called constitutive laws

$$\begin{aligned} \mathbf{D} &= \epsilon \mathbf{E} \\ \mathbf{J} &= \mathbf{J}_0 + \mathbf{J}_c = \mathbf{J}_0 + \sigma \mathbf{E} \end{aligned}$$

describing macroscopic properties of the medium. The electric current density $\mathbf{J}$ can be split into a field dependent part $\mathbf{J}_c$ and a given value $\mathbf{J}_0$.

There is one more relation which we did not mention above: the relation between the magnetic field $\mathbf{H}$ and the magnetic flux density $\mathbf{B}$. In several situations one assumes that

$$\mathbf{B} = \mu \mathbf{H},$$

where μ is permeability of the medium. This approach is of course idealistic and is valid only in a few very special cases. In general the dependence is nonlinear. For ferromagnetic media the so called *Preisach model* is widely accepted, for a detailed description see [6, 55, 80]. However, the model describes the relation between $\mathbf{B}$ and $\mathbf{H}$ from a macroscopic point of view. Thus this model is not accurate for the applications demanding a microscopic description of the phenomena. We introduce another model based on the microscopic analysis of electromagnetic problems. For an overview of existing models and their numerical implementations we refer to [46].

2.2 Micromagnetism and free energy

For the purpose of a better understanding of the problem another vector field is introduced. We consider a magnetic body, made of a ferromagnetic material, which temperature is below a critical value, the so called Curie's temperature depending on the material. This ensures that thermal effects are negligible, see [49, 85]. The body is divided in elementary physical volumes ΔV being large enough to contain many atomic moments. The *magnetization vector* $\mathbf{M}$ is defined as the vector sum of the dipole moments in a unit volume ΔV. In such a way we can suppose that the magnetization vector $\mathbf{M}$, computed as a sum of individual atomic moments, has the same length for every volume ΔV to which $\mathbf{M}$ is associated so that

$$|\mathbf{M}(\mathbf{r})| = \mathrm{M}(\mathbf{r}), \quad \text{for all } \mathbf{r},$$

where $\mathbf{r}$ is a position vector. In general $\mathrm{M}(\mathbf{r})$ depends on the position $\mathbf{r}$ but for every $\mathbf{r}$ remains constant throughout the time evolution. We consider the case when $\mathrm{M}(\mathbf{r}) = \mathrm{M}$ is constant in the space, otherwise, we would gain only extra terms, that are constant in time. Thus this simplification is reasonable. Moreover, we suppose the elementary volumes ΔV to be small enough that the magnetization can be considered to be continuous in space. For simplicity we define the normalized vector with modulus 1 by

$$\mathbf{m} = \frac{\mathbf{M}}{\mathrm{M}}. \tag{2.1}$$

In a paramagnetic or diamagnetic medium $\mathbf{M}$ will be proportional to $\mathbf{B}$, the constant of proportionality being negative for diamagnetism, positive for paramagnetism. For a ferromagnetic medium the relationship between $\mathbf{M}$ and $\mathbf{B}$ is given by

$$\mathbf{B} = \mu(\mathbf{H} + \mathbf{M}).$$

Since a new vector field was introduced, we have to add an equation to have a fully determined system. We will discuss the derivation of this equation in Section 2.3.

Let us have a look on the energy of the whole electromagnetic system. There are several contributions to the total energy. Each contribution represents a different property of the material or different phenomena in the whole process.

Magnetostatic energy

To describe clearly the origins of magnetostatic energy let us focus only on a very simplified and idealistic model. Suppose the magnetic body to be represented as an assembly of elementary magnetic moments. It is assumed that magnetostatic interactions are the only relevant mechanism.

The *magnetostatic energy* represents the mechanical work spent to build up the body by bringing its magnetic moments, one after the other, from infinity to their final position, as depicted in Figure 2.1. Of course, this is an idealistic view and no one can even imagine building up a piece of iron in this way. But this approach is very useful as demonstrated in more detail in [6].

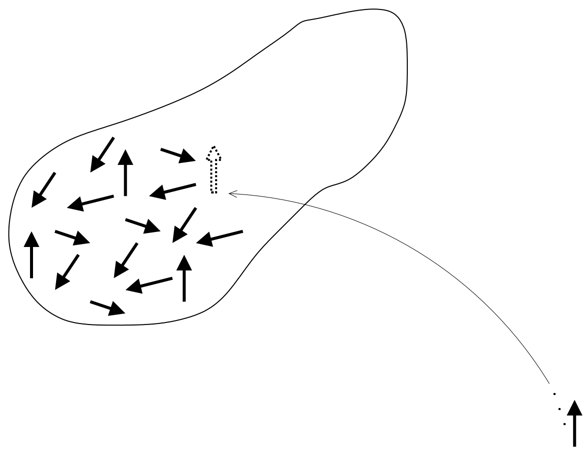


Figure 2.1: Locating elementary moments in a macroscopic body.

The magnetostatic energy denoted by E_{dem} can be written as

$$E_{\mathrm{dem}} = \int_V \langle \mathbf{M}, \mathbf{H}_{\mathrm{dem}} \rangle \mathrm{d}V,$$

where $\mathbf{H}_{\mathrm{dem}}$ denotes the demagnetizing field. For actual computations of $\mathbf{H}_{\mathrm{dem}}$ it is necessary to take into account the shape of the magnetic workpiece. We again refer to [6] for a detailed description of origins of demagnetizing field and of ways how to compute it for different settings. For example, for uniformly saturated samples, the field may be given in terms of a demagnetizing constant

$$\mathbf{H}_{\mathrm{dem}} = -\boldsymbol{N} \cdot \mathbf{M}.$$

In particular, for symmetric bodies with symmetry axis coincident with the coordinate axes, $\boldsymbol{N}$ is simply a diagonal matrix.

Anisotropy energy

The properties of a magnetic material are in general dependent on the directions in which they are measured. In the absence of all external forces, the magnetization

M would align itself in one or more specific directions in the crystal lattice. We call these directions *easy axes* of the material.

To rotate the magnetization away from the easy axis involves energy, namely *anisotropy energy* denoted by E_{ani}. The energy density of E_{ani} denoted by f_{ani} depends only on the direction in which the magnetization points out. If $\mathbf{M}$ is aligned with one of the easy directions, the value of f_{ani} will be small; otherwise it will be bigger. To depict this energy density we can draw a surface around the origin. The value $f_{\mathrm{ani}}(\mathbf{M})$ is then the distance from the origin to the point on the surface lying along the direction $\mathbf{M}$, see Figure 2.2. Thus we see in one picture the whole anisotropy energy for all directions $\mathbf{M}$.

Knowing the energy density f_{ani}, we compute the anisotropy energy as

$$E_{\mathrm{ani}} = \int_V f_{\mathrm{ani}} \mathrm{d}V.$$

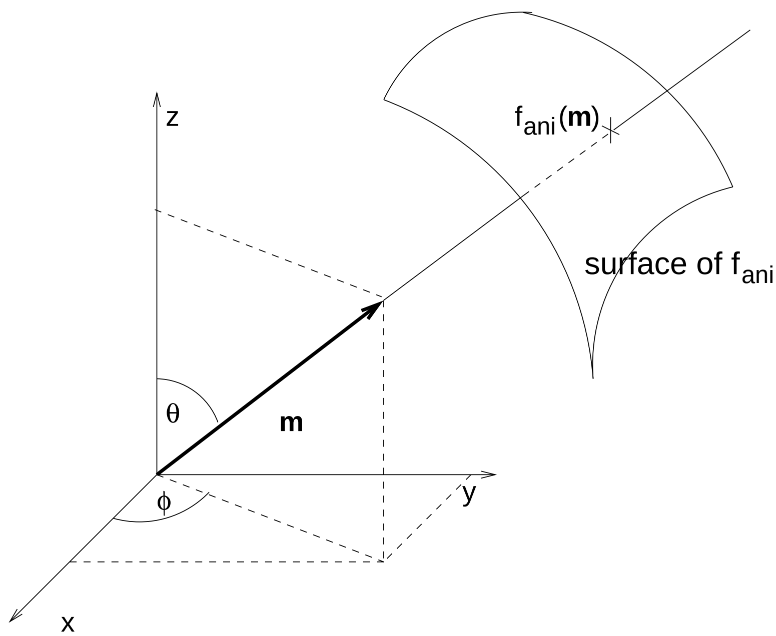


Figure 2.2: Graphical representation of anisotropy density

Uniaxial anisotropy

Let us describe the case when only one preferred direction controls the anisotropy energy. Suppose this direction is along the z-axis. Denote by θ the angle between $\mathbf{M}$ and the positive part of the z axis. Then the anisotropy energy depends only on the relative orientation of $\mathbf{M}$ with respect to the z axis. Under these conditions, the anisotropy energy is an even function of the magnetization component along z axis, $m_3 = \cos\theta$. It is common to use $m_1^2 + m_2^2 = 1 - m_3^2 = 1 - \cos^2\theta = \sin^2\theta$, instead of $\cos^2\theta$, as the expansion variable. Thus the energy density f_{ani} will have the general expansion

$$f_{\text{ani}} = K_0 + K_1 \sin^2(\theta) + K_2 \sin^4(\theta) + \sin^6(\theta) + \dots \tag{2.2}$$

where $K_1, K_2, K_3, \dots$ are the *anisotropy constants.* For the moment, let us limit our considerations to the case, where the expansion is truncated after the $\sin^2\theta$ term.

For one kind of uniaxial anisotropy, when $\mathbf{M}$ is aligned with the z-axis, f_{ani} is minimal. If $\mathbf{M}$ is perpendicular to the z-axis, the anisotropy energy density is maximal, see Figure 2.3. Here the anisotropy constants are $K_0 = 0.1, K_1 = 1$ and $K_i = 0$ for $i = 2, 3, \dots$

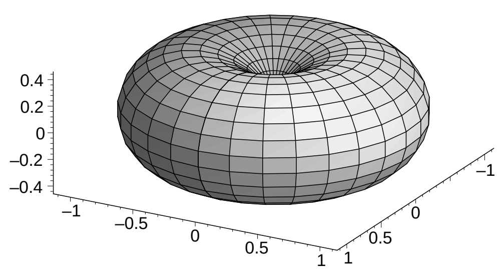


Figure 2.3: Uniaxial anisotropy for $K_0 = 0.1$ and $K_1 = 1$

It is possible also to consider an inverse setting. Some materials behave in such a way that an easy axis doesn't attract but repels the magnetization. In this case the anisotropy energy density is minimal when $\mathbf{M}$ is lying in the plane xy and maximal when $\mathbf{M}$ aligns the z-axis, see Figure 2.4.

Let us consider the case where $K_1 > 0$ and $\mathbf{M}$ lies along the easy axis. Let us take the energy of this state as zero energy level. For small deviations of the magnetization vector from the equilibrium position, the anisotropy energy density can be approximated, up to second order in θ, as

$$\begin{aligned} f_{\text{ani}} \cong K_1 &\cong 2K_1 - 2K_1\cos\theta \\ &= 2K_1 - \mu_0 \mathrm{M}\frac{2K_1}{\mu_0\mathrm{M}}\cos\theta = 2K_1 - \langle \mathbf{M}, \mathbf{H}_{\text{ani}}\rangle. \end{aligned} \tag{2.3}$$

The dependency of the energy is the same as if there was a field $\mathbf{H}_{\text{ani}}$ of strength $2K_1/(\mu_0 \text{M})$ acting along the easy axis. The *anisotropy field* $\mathbf{H}_{\text{ani}}$ gives a natural measure of the strength of the anisotropy effect and of the torque to take the magnetization away from the easy axis. $\mathbf{H}_{\text{ani}}$ will often appear in treatments of magnetic free energy.

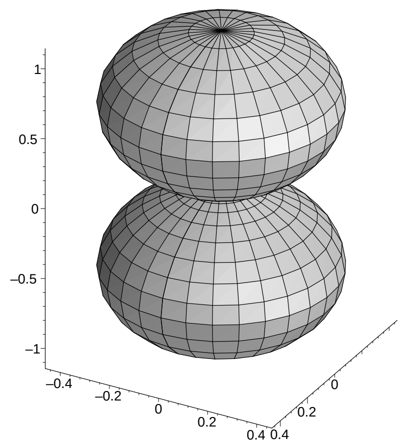


Figure 2.4: Uniaxial anisotropy for $K_0 = 1.1$ and $K_1 = -1$

For other types of anisotropy see [6].

Exchange energy

In the following we use the normalized vector field $\mathbf{m}$; for its definition see (2.1).

Although the modulus of $\mathbf{m}$ is constant, its orientation can vary from point to point, see Figure 2.5. Because of the interaction between neighboring volumes, it costs additional energy to change the direction of the magnetization. We call this the *exchange energy*. This energy can be measured by the gradient of $\mathbf{m}$ and the simplest approximation of exchange energy can be written as

$$\begin{aligned} E_{\text{exc}} &= \int_V A_{\text{exc}} \left[(\nabla \mathbf{m}_{x_1})^2 + (\nabla \mathbf{m}_{x_2})^2 + (\nabla \mathbf{m}_{x_3})^2 \right] \mathrm{d}V \\ &= \int_V A_{\text{exc}} |\nabla \mathbf{m}|^2 \mathrm{d}V \end{aligned}$$

where A_{exc} is a material constant, see [6].

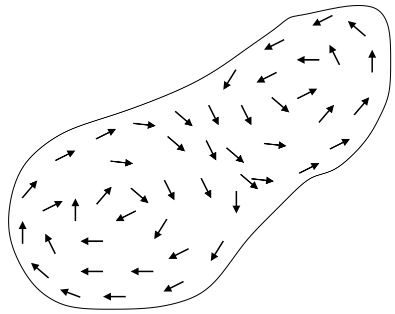


Figure 2.5: Different orientations of the magnetization vectors

Zeeman's energy

The magnetic system can be influenced by an externally applied field. The same kind of influence can have also a field caused by electric currents which can be described by Maxwell's equations. This type of field is called *applied field* and we denote it by $\mathbf{H}_{\text{app}}$. It interacts with the magnetization and creates an energy called *Zeeman's* or *applied field energy*. This energy contribution is simply given by

$$E_{\text{app}} = \int_v \langle \mathbf{H}_{\text{app}}, \mathbf{M} \rangle \mathrm{d}V.$$

Magnetostrictive energy

The mechanisms responsible for crystalline anisotropy also give rise to energy variations when the relative positions of the magnetic ions in the lattice are modified, that is, when the lattice is distorted. Due to the presence of magneto-elastic coupling to the lattice, the system will spontaneously deform in order to minimize its total free energy, and the ensuing deformation will be function of the magnetic state of the system. This phenomenon, in which the magnetic system gets deformed when it is magnetized, is called *magnetostriction* and the energy involved is *magnetostrictive energy*.

A complementary role is played by the so called inverse magnetostrictive effect, in which, again through magneto-elastic coupling, the deformation produced in the system by an externally applied stress makes certain magnetization directions energetically favored, so that the system will tend to align its magnetization to those directions.

We do not consider this energy term in the next chapters any more. Magnetostriction was studied for example in [81, 82] and in [14, 62]. In particular, the scheme introduced in Chapter 5, considering also the magnetostrictive energy, was discussed by Baňas in his PhD thesis [4] and together with Slodička in [5].

Above we have described important energy contributions that have to be taken into account for the total energy of the micromagnetic system. All these energies have totally different origins and their effect on the system is really dependent on the setting. We are interested mostly in the exchange, anisotropy, magnetostatic and applied field energy. To summarize all contributions we get

$$\begin{aligned} E_{\text{tot}} &= E_{\text{dem}} + E_{\text{ani}} + E_{\text{exc}} + E_{\text{app}} \\ &= -\frac{1}{2}\int_V \langle \mathbf{M}, \mathbf{H}_{\text{dem}} \rangle \mathrm{d}V + \int_V 2K_1 - \langle \mathbf{M}, \mathbf{H}_{\text{ani}} \rangle \mathrm{d}V \\ &\quad + \int_V A_{\text{exc}} |\nabla \mathbf{m}|^2 \mathrm{d}V + \int_v \langle \mathbf{H}_{\text{app}}, \mathbf{M} \rangle \mathrm{d}V. \end{aligned}$$

For a better understanding of situations when particular energies are large and when they are minimal, see Table 2.1. The exchange energy is large when the magnetization rapidly changes in space. When magnetization vectors are aligned to each other, then the exchange energy is minimal. For anisotropy effects to appear it is important that the direction of $\mathbf{M}$ is close to one of the easy axes. When a magnetic body is symmetrical and the direction of $\mathbf{M}$ is opposite in every two symmetrical points, then the magnetostatic energy vanishes. Finally, if the magnetization aligns the direction of the applied field, Zeeman's energy is minimal. When $\mathbf{M}$ becomes perpendicular or even opposite to the applied field then applied field energy rises.

2.3 Landau-Lifshitz equation

In the beginning of Section 2.2 we have mentioned that for a fully determined system of equations describing the electromagnetic phenomena, we need a new equation for the magnetization. Let us find out how does this equation looks like.

First, we have to understand that any system can only feel the magnetic field that results from an energy change. So we have to derive a total *effective field*, denoted by $\mathbf{H}_{\text{eff}}$, acting on the magnetization as derivative of the total energy E_{tot}

$$\mathbf{H}_{\text{eff}} = \partial_{\mathbf{M}} E_{\text{tot}}.$$

Denoting the *exchange field* by $\mathbf{H}_{\text{exc}}$, we define

$$\mathbf{H}_{\text{exc}} = 2A_{\text{exc}} \Delta \mathbf{m}.$$

Together with the definitions of the anisotropy, applied, and demagnetizing fields, the derivation of E_{tot} with respect $\mathbf{M}$ leads to

$$\mathbf{H}_{\text{eff}} = \mathbf{H}_{\text{dem}} + \mathbf{H}_{\text{ani}} + \mathbf{H}_{\text{exc}} + \mathbf{H}_{\text{app}}.$$

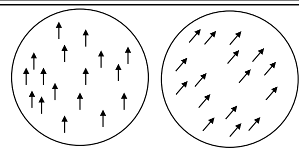
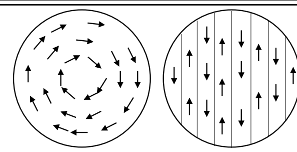
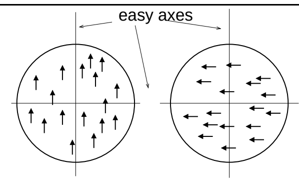
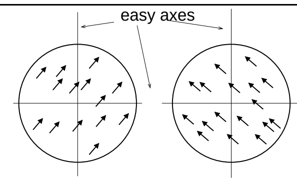
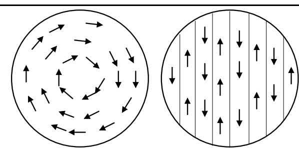
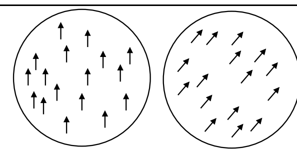
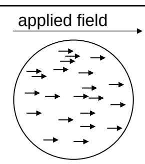
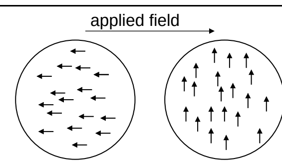
Type of energy	Minimal values	Large values
Exchange energy		
Anisotropy energy		
Magnetostatic energy		
Applied field energy		

Table 2.1: Competing energies

Having an expression for the field acting on the magnetization we are ready to study the dynamics of $\mathbf{m}$.

Precession of the magnetization

If the field $\mathbf{H}$ is acting on an elementary moment $\mathbf{m}_i$, it produces a motion of $\mathbf{m}_i$ described by

$$\partial_t \mathbf{m}_i = -\gamma \mathbf{m}_i \times \mathbf{H}, \tag{2.4}$$

where γ is the gyromagnetic factor. For more details we refer to [6, 49, 74]. We consider the magnetization to be a sum of magnetic moments over the volume ΔV. Therefore a similar equation as (2.4) will be valid for magnetization and the effective field

$$\partial_t \mathbf{m} = -\gamma \mathbf{m} \times \mathbf{H}_{\text{eff}}. \tag{2.5}$$

Let us have a closer look at the previous equation. Denote by $\mathbf{a}$ the vector product $\gamma\mathbf{m}\times\mathbf{H}_{\text{eff}}$. From the properties of a vector products it is clear that $\mathbf{a}$ is perpendicular to $\mathbf{m}$ and thus the length of $\mathbf{m}$ does not change in time. This corresponds to the fact that the modulus of magnetization has to remain constant. Furthermore, the vector $\mathbf{a}$ is also perpendicular to $\mathbf{H}_{\text{eff}}$ and thus $\mathbf{a}$ causes a circular movement of $\mathbf{m}$ around $\mathbf{H}_{\text{eff}}$, see Figure 2.6. This is not acceptable, because it would mean that $\mathbf{m}$ could never align $\mathbf{H}_{\text{eff}}$ and the energy could never reach its minimal values. A

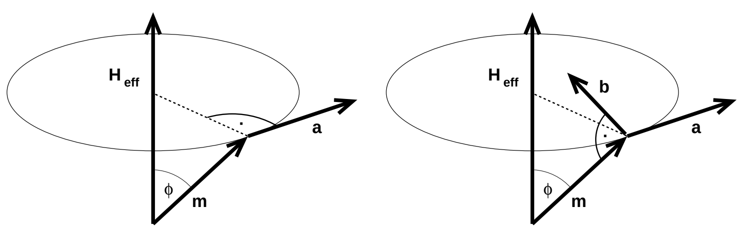


Figure 2.6: Precession of the magnetization vector around the magnetic field vector.

phenomenological way how to involve the dissipation of the energy is to add a new term to the equation (2.5)

$$\partial_t \mathbf{m} = -\gamma \mathbf{m} \times \mathbf{H}_{\text{eff}} + \mathbf{b}.$$

Immediately, we know that $\mathbf{b}$ has to be perpendicular to $\mathbf{m}$ to preserve the modulus of $\mathbf{m}$. Thus we get $\mathbf{b} = \mathbf{m} \times \mathbf{g}$. Since $\mathbf{d}$ should push $\mathbf{m}$ towards $\mathbf{H}_{\text{eff}}$, we choose

$\mathbf{g} = \alpha\mathbf{a}$, see Figure 2.6. Thus we get the expression

$$\partial_t \mathbf{m} = -\gamma \mathbf{m} \times \mathbf{H}_{\text{eff}} - \gamma\alpha \mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{\text{eff}}). \tag{2.6}$$

This equation was introduced for the first time by Landau and Lifshitz in [48]. Therefore it is known as the Landau-Lifshitz (LL) equation.

The constant γ is called the gyromagnetic factor and its value is very high, it can reach 1.67×10^7. The constant α depends on the material and denotes the damping parameter. Its values typically range between 0 and 1.

Gilbert's approach

A different approach to this problem was proposed by Gilbert in [34]. His equation takes the form of the standard precession equation (2.5) with the field term $\mathbf{H}_{\text{eff}}$ augmented by a damping term which is proportional to the rate of change of magnetization

$$\partial_t \mathbf{m} = -\gamma_G \mathbf{m} \times (\mathbf{H}_{\text{eff}} - \alpha_G \partial_t \mathbf{m}).$$

The magnitude of the field components in the cross product is then reduced. Thus, damping is incorporated implicitly as the precession direction is no longer perpendicular to $\mathbf{H}_{\text{eff}}$. We write the Gilbert equation in its more familiar form

$$\partial_t \mathbf{m} = -\gamma_G \mathbf{m} \times \mathbf{H}_{\text{eff}} + \gamma_G \alpha_G \mathbf{m} \times \partial_t \mathbf{m}. \tag{2.7}$$

Although these two approaches seem to be different, it can be proven that they are mathematically equivalent. To see this, compute the cross product of both sides of (2.6) with $\mathbf{m}$ to get

$$\mathbf{m} \times \partial_t \mathbf{m} = -\gamma \mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{\text{eff}}) - \gamma\alpha \mathbf{m} \times (\mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{\text{eff}})).$$

Using the identity

$$\mathbf{m} \times \mathbf{H}_{\text{eff}} = -\mathbf{m} \times (\mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{\text{eff}}))$$

which holds because of $|\mathbf{m}| = 1$, we get

$$-\alpha \mathbf{m} \times \partial_t \mathbf{m} = \alpha\gamma \mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{\text{eff}}) - \alpha^2 \gamma \mathbf{m} \times \mathbf{H}_{\text{eff}}.$$

Summing up the previous equation with (2.6) leads to

$$\partial_t \mathbf{m} = -\gamma(1 + \alpha^2) \mathbf{m} \times \mathbf{H}_{\text{eff}} + \alpha \mathbf{m} \times \partial_t \mathbf{m}.$$

Comparing the last relation with (2.7) and setting

$$\gamma(1 + \alpha^2) = \gamma_G \quad \text{and} \quad \alpha_G = \gamma^{-1} \frac{\alpha}{1 + \alpha^2},$$

we get the equivalence, between the Gilbert and the LL equation up to the constants.

2.4 Theory of the LL equation

The paper of Visintin [83] contains the first theoretical results concerning the LL equation. He studied the system of Maxwell's equations coupled with the LL equation considering both the exchange field and the anisotropy field. The case without Maxwell's equations was studied simultaneously in the papers [1] and [36]. They showed the existence of a global weak solution for the LL equation. Moreover, Alouges and Soyeur gave an example, in which the LL equation does not have an unique solution. However, the initial condition in this example doesn't belong to the $W^{2,2}(\Omega)-$ space. Therefore the question, if the LL equation has unique solution when we consider smooth initial condition, is still open.

Later, Guo, Ding and Su in [35, 37] proved the global existence of a weak solution for the Landau-Lifshitz-Maxwell equations with Neumann boundary conditions in two and three space dimensions.

In his PhD thesis [65] Seo intensively studies other types of regularity of weak solutions as well as a priori gradient estimates for weak solutions and well-posedness of the LL equation.

Guo and Hong discussed the question of uniqueness in two space dimensions in [36]. The same subject was studied in more detail in [21] and [24] by Chen and Guo. They proved that any weak solution with finite energy is unique and smooth with the exception of at most finitely many points. Chen devoted his paper [23] to the study and localization of these singularity points.

All the above authors have considered the LL equation with an exchange field.

Because of the high nonlinearity of the LL equation it is difficult to establish rigorous numerical analysis of the methods used for the computations. Prohl and Kružík have written a survey paper about numerical methods for micromagnetics, see [46]. The same authors together with Carstensen in [19, 45] proposed methods, which deal with the nonconvexity of a minimizing functional using Young measures. A different approach was used by E, Wang and Garcia-Cervera in [31, 84]. They introduced a numerical scheme for the LL equation and compared this scheme with other known schemes.

Joly, Komech, Vacus in [41] studied long-time convergence of solutions to the Maxwell-LL equations. The same authors together with Monk, see [43, 59] are interested in the numerical modeling of absorbing ferromagnetic materials. They proposed a numerical scheme which conserves the magnitude of magnetization, but they did not prove any error estimates in time. For more details we refer also to [78].

A similar problem was studied by Slodička, Baňas and the author of this thesis in [71, 72]. They suggested a new numerical scheme conserving the magnitude of magnetization and they also proved error estimates in time. They considered the LL equation in a simplified form considering a demagnetizing and anisotropy field but without an exchange field. More results on this scheme can be found also in [26, 27, 28, 73].

We mention also strong numerical analysis of Monk and Vacus, see [60], considering the full Maxwell-LL system, where the LL equation was considered with an exchange field. They proved the existence of a new class of Liapunov functions for the continuous problem, and then also for a variational formulation of the continuous problem. The authors also showed a special result on continuous dependence.

Next, numerical analysis for the two dimensional LL equation with an exchange field was done by Prohl in his monograph [61]. The author has proved some new regularity results for an exact solution of the LL equation. In this thesis we extend these results to three dimensions. He has also suggested a specific numerical scheme and he has proved error estimates in 2D. We adapt the scheme so that it becomes computationally cheaper and moreover we prove error estimates in time in three dimensions.

Carbou and Fabrie in [18] proved the local existence and uniqueness of regular solutions to the LL equation in 3D. So in three dimensions, solutions to the LL equation can blow up in a finite time. We study this case, when the existence and uniqueness of the solutions to the LL equation is guaranteed by the theory only on a finite interval.

Symmetrical solutions of LL equation

The single LL equation was intensively studied by Mayergoyz, Bertotti, Serpico and Magni. We provide a brief overview of their work, with references included in the text below.

Consider a special setting for the LL equation. Take the exchange field $\mathbf{H}_{\text{eff}}$ of the form

$$\mathbf{H}_{\text{eff}} = \mathbf{H}_{\text{app}} + \mathbf{H}_{\text{dem}} + \mathbf{H}_{\text{ani}}.$$

Since we do not consider the exchange term, we work only with uniformly magnetized sample. Nevertheless, it is shown that in this simple case the system exhibits complicated nonlinear phenomena such as symmetry breaking, bifurcations, quasi-periodicity and chaotic behavior. It is then desirable to understand this simple case before going to the complicated ones.

Moreover, suppose that the domain Ω has a spheroidal shape with the axis of the symmetry parallel to the z-axis. Furthermore, suppose that the material has an uniaxial anisotropy with the axis again parallel to the z-axis. Next we use the symbol $\perp$ ($\|$ respectively.) to express correspondence with the plane xy (axis z, respectively.) The sample will be subject to the applied field $\mathbf{H}_{\text{app}}$, which will be constantly rotated around the z-axis with constant magnitude. More specifically we have

$$\mathbf{H}_{\text{app}} = \mathbf{H}_{a\perp} + H_{\|}\mathbf{e}_z,$$

where $\mathbf{H}_{a\perp}$ has a constant amplitude and is rotated with angular frequency ω with initial angles θ_x, θ_y so that

$$\mathbf{H}_{a\perp} = \mathbf{e}_x H_{a_x} \cos(\omega t + \theta_x) + \mathbf{e}_y H_{a_z} \cos(\omega t + \theta_y).$$

Next, the demagnetizing field $\mathbf{H}_{\text{dem}}$ is of the form

$$\mathbf{H}_{\text{dem}} = -N_\perp \mathbf{m}_\perp - N_\parallel m_\parallel \mathbf{e}_z,$$

where $\mathbf{m}_\perp$ denotes component of $\mathbf{m}$ perpendicular to the z-axis and $m_\parallel$ denotes the length of the component of $\mathbf{m}$ parallel to the z-axis. Finally, the anisotropy field will be given by

$$\mathbf{H}_{\text{ani}} = K_\parallel \langle \mathbf{m}, \mathbf{e}_z \rangle \mathbf{e}_z,$$

where $K_\parallel$ is an anisotropy constant. Summarizing all components we arrive at

$$\mathbf{H}_{\text{eff}} = \mathbf{H}_{a\perp}(t) - N_\perp \mathbf{m}_\perp + \mathbf{e}_z [H_\parallel - (N_\parallel - K_\parallel) \mathbf{m}_\parallel].$$

With this setting the authors are able to derive analytical solutions when the applied field is circular, which is guaranteed when $H_{a_x} = H_{a_y}$, see [12, 13]. An analytical expression for the solutions of the LL equation is very useful for testing numerical schemes, as can be seen in Chapter 6. If the field $\mathbf{H}_{\text{app}}$ is considered to be constant and applied in the plane perpendicular to the anisotropy axis, the problems seem to be simpler, but this is not the case. In [11] the authors perform a rigorous analysis of the precessional magnetization and they again derive analytical solutions. The closed formulas in terms of Jacobi elliptic functions are based on the exact integration of the LL equation.

As soon as the values of H_{a_x} and H_{a_y} are different, the mathematical formulation is not rotationally symmetric in the xy plane. However, it can be easily verified that this formulation is invariant with respect to reflection around the origin of the plane xy. The study of this case was done in [67].

We refer to [7, 52, 54, 56, 66] for further studies of the LL equation.

When the field $\mathbf{H}_{a\perp}$ consists not only from one circularly rotated field, but consists of two circularly rotated fields in opposite directions, we speak about a radio-frequency field. To analyze this case a perturbation technique was developed in [8]. The study of spin-wave instabilities is provided in [9, 10]. The authors study in a analytical way the stability of large magnetization motions in systems with uniaxial symmetry under a circularly polarized radio-frequency field. They derive instability conditions valid for arbitrary values of the amplitude and frequency of the driving field. Moreover, they show that the input powers capable of inducting spin-wave instabilities are bounded both from below and above. It means that sufficiently large motions are always stable.

In [53] eddy-current effects were included in the previous model. The coupling between the LL equation and eddy currents was discussed in [57]. The system is a metallic thin disk lying in the xy-plane and therefore the total in-plane contribution will include also the $\mathbf{H}_{\text{eddy}}$ field computed from the general eddy-current law. The exact analytical solutions were derived under the assumption of small thickness of the material.

Concerning numerical techniques an interesting scheme was introduced in [68]. The scheme takes also into account the exchange field and is similar to those described in Chapter 4. However, the authors do not mention any convergence results or error estimates. They indeed prove quadratic accuracy of the scheme and provide a number of numerical experiments.

2.5 Overview of the thesis

In this work we focus on the numerical analysis of the LL equation. We develop several numerical schemes based on time stepping. The problem is not fully discretized. The space discretization of all schemes mentioned here can be considered as work for the future. However, we think, that the space discretization of the LL equation is not an equally challenging part of the problem as the time discretization.

Our work can be split in three parts. First, in Chapters 1–3 we summarize the generalities. In the next part (Chapters 4 to 6) we are interested in the case when the effective field $\mathbf{H}_{\text{eff}}$ is considered without the exchange energy term. In the third part (Chapters 7 to 10) we include this term and we study the LL equation with the exchange field.

Second part

The case without the exchange term describes a uniformly magnetized medium or other settings, when the exchange effects are very small in comparison to the other contributions of $\mathbf{H}_{\text{eff}}$.

In Chapter 4 we first describe the natural approach to the time discretization of the LL equation which was followed by Joly, Vacus, Monk, Bertotti and others. The idea used in their work is the called mid-point rule.

The LL equation in the continuous form has a property that the modulus of the magnetization remains constant in time. Then, of course, natural demand on numerical schemes is to preserve this important physical feature. The implementation of mid-point rule enables the schemes to fulfill this requirement.

The idea of the mid-point rule is based on an the well-known fact that if the scalar product $\langle \mathbf{a} - \mathbf{b}, \mathbf{a} + \mathbf{b} \rangle$ vanishes then the modulus of $\mathbf{a}$ is the same as the modulus of $\mathbf{b}$.

Let us discretize the LL equation in time in the following way. Imagine that $\mathbf{a}$ corresponds to $\mathbf{m}$ in time t_i; denote this value by $\mathbf{m}^i$, and let $\mathbf{b}$ correspond to $\mathbf{m}^{i-1}$

Then, the expression $(\mathbf{a}-\mathbf{b})/\tau$ corresponds to the time derivative of $\mathbf{m}$. Next, of course, the expression $(\mathbf{a}+\mathbf{b})/2$ corresponds to $\mathbf{m}$ in time $t_{i+1/2}$. We show that it is possible to implement the previous expressions into the LL equation in such a way that we obtain

$$\big\langle (\mathbf{m}^i - \mathbf{m}^{i-1})/\tau, (\mathbf{m}^i + \mathbf{m}^{i-1})/2 \big\rangle = 0,$$

which gives the desired relation $|\mathbf{m}^i| = |\mathbf{m}^{i-1}|$. For more details on this topic see Chapter 4.

In Chapter 5, see Figure 2.7, we use a different approach to this problem. We do not discretize the time derivative. We let the time vary continuously through the interval (t_{i-1}, t_i). We fix the other vectors in such a way that the LL equation is of the form

$$\partial_t \mathbf{u} = \mathbf{d} \times \mathbf{u} \qquad \text{in} \qquad (t_{i-1}, t_i),$$

where $\mathbf{d}$ is a constant vector on (t_{i-1}, t_i). This ordinary differential equation can be solved exactly. Thus we get a continuous approximation of $\mathbf{m}$. We also introduce a modification of this scheme where the LL equation has a quadratic form

$$\partial_t \mathbf{u} = \mathbf{d} \times \mathbf{u} + (\mathbf{g} \times \mathbf{u}) \times \mathbf{u} \qquad \text{in} \qquad (t_{i-1}, t_i),$$

which can be solved exactly on (t_{i-1}, t_i) too. For more details see Chapter 5.

For both schemes we derive error estimates, which is not the case for the schemes based on the mid-point rule.

In Chapter 6 we modify our schemes to obtain a better rate of convergence, see Figure 2.7. The modification is based on the iterations that are made on every time step. These iterations converge thanks to the contraction properties. The limit, or better, the fixed point of these iterations, approximates the exact solution with higher accuracy and therefore the rate of convergence is higher.

Third part

In Chapters 7 to 10 we deal with the LL equation when the exchange field is a part of the effective field $\mathbf{H}_{\text{eff}}$. This setting is quite challenging because the exchange term makes the LL equation to be a partial differential equation. Without $\mathbf{H}_{\text{eff}}$ the LL equation was an ordinary differential equation.

For the proofs of convergence results for the numerical schemes it is necessary to have a detailed analysis of the LL equation. In Section 2.4 we have mentioned the overview of theoretical and numerical results on the LL equation. We use these results in numerical analysis, but in some cases it is necessary to extend them.

The study of the LL equation can benefit from the theory of harmonic mappings. Its form is very close to the equation of the harmonic heat flow. Struwe introduced the theory of harmonic mappings in [76]. In [75] he derives a regularity result that can be directly used in the case of the LL equation. The application of these results was summarized by Huo and Hong in [36].

There is a big gap in the theory of harmonic mappings between known results for the 2D case and for the 3D case. The nature of the problem changes when going from two to higher dimensions. This phenomena causes also that much less is known for regularity results of the exact solution to the LL equation in 3D than in the lower dimensional case.

In our work we first study the single LL equation in 3D without considering Maxwell's equations. In Chapter 7, see Figure 2.7, we derive regularity results which can be written using standard symbols and notations, see List of symbols, page 170, as

$$\max_{t\in(0,T_0)} \left\{ \kappa^p \|\partial_t^{p+1}\mathbf{m}\|_2 + \kappa^{\frac{2p+1}{2}} \|\nabla\partial_t^{p+1}\mathbf{m}\|_2 + \kappa^{p+1}\|\Delta\partial_t^{p+1}\mathbf{m}\|_2 \right\} \le C, \tag{2.8}$$

where p is a nonnegative integer and κ is the time weight defined by

$$\kappa(s) = \begin{cases} 0, & \text{for } s < 0, \\ \min\{1, s\}, & \text{for } s \ge 0. \end{cases}$$

The key estimate for proving (2.8) was found to be

$$\max_{t\in(0,T_0)} \|\mathbf{m}\|_{W^{2,2}} \le C, \tag{2.9}$$

which was obtained by Carbou and Fabrie in [18].

In Chapter 8 we focus on the full Maxwell-LL system. To establish a similar results as in Chapter 7 we need an estimate similar to (2.9)

$$\max_{t\in(0,T_0)} \{\|\mathbf{m}\|_{W^{2,2}} + \|\mathbf{E}\|_{W^{1,2}} + \|\mathbf{H}\|_{W^{1,2}}\} \le C. \tag{2.10}$$

This result for the 3D case was not known before. In Chapter 8 we prove it, see Figure 2.7. We establish a sequence of finite-dimensional spaces approximating the vector spaces to which the solutions are expected to belong. Then, we construct a sequence of approximations from these finite-dimensional spaces, converging to the solution of the LL equation.

We prove regularity results for the approximations, which are robust enough to be transfered to the solution of the LL equation itself. Moreover, we prove that the solution is locally unique, in other words we prove that there exist a positive T_0 such that the solution of the LL equation is unique in the interval $\langle 0, T_0\rangle$. In such a way we obtain the estimate (2.10).

Finally, we use the results from Chapters 7 and 8. We introduce a semi-implicit scheme in 3D. We prove convergence results in Chapter 9. This scheme, however, does not preserve the length of magnetization. The only result we know is that, in some sense, this length changes very little and that with decreasing discretization step also the change of the length becomes smaller. For more details see Theorem 9.2.

In the literature exist other schemes dealing with the single LL equation or the full M-LL system, considering the nonzero exchange term. These schemes are based on penalty terms, which penalize the difference between $|\mathbf{m}^i|$ and $|\mathbf{m}^0|$. Such a scheme gives better results with respect to the conservation of the modulus.

This approach was used by Prohl in [61] for the 2D case. It is possible to extend his results also in three dimensions. We suggest how to do this in Chapter 10. However, this extension is more-less straightforward and we don't go into details.

Computations

For the verification of the theoretical results it is desirable to perform real computations. However, this thesis is oriented more on the analysis. We provide most calculations on academic examples. We do not focus on highly practical applications. For that aim it would be necessary to invest much more time in developing of robust numerical software.

For our purposes we adapted the software ALBERT. For the overview of ALBERT's features we refer to [63, 64]. Together with Baňas we implemented Whitney elements in order to approximate magnetic and electric field. For an overview of Whitney elements see Chapter 3.

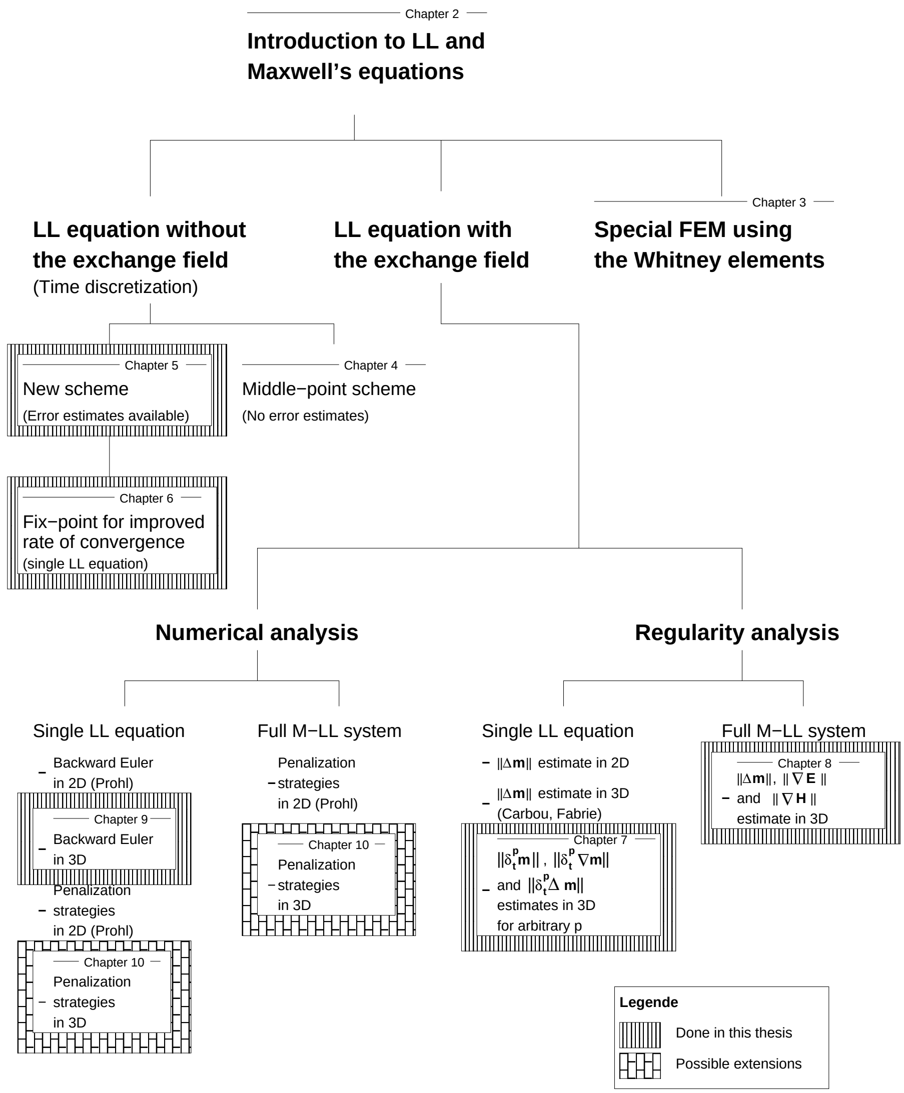


Figure 2.7: Overview of the thesis

3 Whitney elements

(Although this may seem a paradox, all exact science is dominated by the idea of approximation. Bertrand Russell)

This chapter can be considered to be independent from the other chapters. Therefore it can be skipped. We summarize basis of the theory of Whitney elements. For a more exhaustive introduction to Whitney elements we refer to [15].

In Section 3.1 we introduce some notations and define the orientation of tetrahedral simplices. In Sections 3.2–3.5 we build up four classes of Whitney elements and we point out some interesting metric properties which help to develop a computer code.

Next, in Section 3.6 we approve the use of Whitney elements for approximating the fields **E** and **B**. We confirm that this kind of elements satisfy continuity conditions when going from one element to the other, which is an important physical feature of the fields **E** and **B**.

Finally, we mention the embedding properties of the spaces build on the Whitney elements in Section 3.7. These embedding properties are equivalent to those for spaces $\mathbf{H}^1(\Omega), \mathbf{H}(\mathrm{curl};\Omega), \mathbf{H}(\mathrm{div};\Omega)$ and $\mathbf{L}^2(\Omega)$.

3.1 Notations

Each tetrahedra will be denoted by T. (E.g. T_1, T_3, ...) We use numbered symbols n for vertices of any tetrahedra. (E.g. n_2, n_5, ...) We sometimes simply write 0, 1, 2 for vertices. For edges we use a numbered letter e, for faces a numbered letter f. Let us introduce $\mathcal{N}_h$ as a set of all vertices of the mesh, $\mathcal{E}_h$ a set of all edges of the mesh, $\mathcal{F}_h$ a set of all faces and finally $\mathcal{T}_h$ a set of all tetrahedra.

Orientation of the simplices

Every component (except of vertices) has its own orientation. Edge $e = \{01\}$ is oriented from the vertex 0 to the vertex 1. Thus, in a point of the edge one can say which one from two possible directions along the edge is the positive one.

In a point of a face one can say which one from two possible rotations is positive. If we have for example the face $f = \{012\}$, as is depicted in Figure 3.1 (left), the positive rotation is from the vector **01** to the vector **02**. It means that the specified direction in the figure is negative.

One can define the orientation of the face $f = \{012\}$ as following: If you make a vector product $\boldsymbol{\nu} = \mathbf{01} \times \mathbf{02}$, the vector $\boldsymbol{\nu}$ defines the orientation of the face. Then if we have a positively oriented tetrahedra $T = \{0123\}$, its boundary is oriented outside from the tetrahedra and one can say if the orientation of the face f matches with the orientation of boundary of T. In the case depicted in Figure 3.1 (right), they don't match.

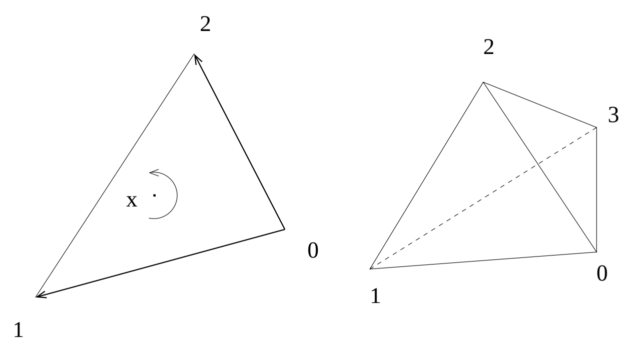


Figure 3.1: Orientation of the face

The orientation of the tetrahedra $T = \{0123\}$ follows from the rule of the right hand. We place vertex 0 to the palm of right hand. If it is possible to set the hand in such a way that vector **01** (**02**, **03**, respectively) follows the direction of the thumb (forefinger, middle finger, respectively), then tetrahedra T is oriented positively. Otherwise it has a negative orientation.

If $e_1 = \{01\}$ belongs to $\mathcal{E}_h$, then edge $\{10\}$ doesn't belong to $\mathcal{E}_h$. The same holds with faces and tetrahedra.

Tangential and normal part of the vector field

We will introduce the term *tangential part* and *normal part* of a vector field. Consider vector field $\mathbf{h}$ and some surface S. In point x we make a tangential plane T_x to the surface S. The normal part of $\mathbf{h}$ is the projection of $\mathbf{h}$ to the normal direction of the plane T_x. The tangential part of the vector $\mathbf{h}$ is projection of $\mathbf{h}$ to the plane T_x, see Figure 3.2.

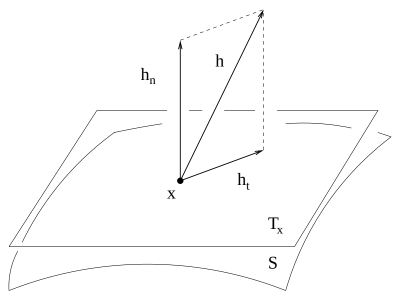


Figure 3.2: Tangential and normal part of the vector field

3.2 Nodal Whitney elements (WE)

Whitney elements are special kind of finite elements. We will study the 3D-case on a tetrahedra mesh. We will discuss finite elements which are linear on one element. The purpose of this study is to describe electromagnetic fields. The notation comes from the dimension of simplex to which WE belong. We consider four kinds of components of tetrahedra: vertex (node), edge, face and tetrahedra itself. Thus Whitney elements will be denoted as 0-elements (nodal elements), 1-elements (edge elements), 2-elements (facial elements) and 3-elements (tetrahedra elements).

In Galerkin approximations we have one basis function for each vertex n of the mesh. It is the only one continuous, linear function which has value 1 in the vertex n and 0 in other vertices. The equipotential levels are depicted in Figure 3.3. We call this element *nodal Whitney element* and we denote this function by w_n. It is worth to notice that

$$w_{n_i}(n_j) = \delta_{ij}, \quad \forall i, j.$$

The latter implies that $w_n(x) = 0$ on a tetrahedra that does not contain vertex n. This feature causes the sparseness of the system matrices.

Let us study the gradient of w_0. The gradient of a linear function is a constant vector. The direction of the gradient is perpendicular to the plane $\{123\}$, see Figure

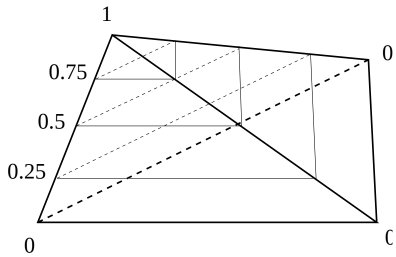


Figure 3.3: Equipotential levels of the nodal basis function

3.4. We compute the circulation of ∇w_0 along the height $\mathbf{A0}$ of the tetrahedra by two different ways. Let us denote the unit directional vector of $\mathbf{A0}$ by $\mathbf{t}$. Then

$$\int_A^0 \langle \nabla w_0, \mathbf{t} \rangle = |\nabla w_0| \langle \mathbf{t}, \mathbf{t} \rangle = \int_A^0 |\nabla w_0| |\mathbf{t}|^2 = |A0| |\nabla w_0|.$$

Using the fact that we are integrating the derivative of the function w_0 along the curve $\mathbf{A0}$ we can write

$$\int_A^0 \langle \nabla w_0, \mathbf{t} \rangle = w_0(0) - w_0(A) = 1 - 0 = 1.$$

Subsequently, the equation for the length of the gradient w_0 reads as

$$|\nabla w_0| h_0 = 1, \tag{3.1}$$

where h_0 is the height of the tetrahedra from the vertex 0. The unit normal vector $\mathbf{t}$ of the plane $\{123\}$ can be rewritten as

$$\tau = \frac{1}{|\mathbf{13} \times \mathbf{12}|} \mathbf{13} \times \mathbf{12}.$$

The doubled area $2\,|f|$ of the face $f = \{123\}$ is equal to $|\mathbf{13} \times \mathbf{12}|$ and the volume $|T|$ of the tetrahedra $T = \{0123\}$ can be computed as $\frac{1}{3}\,|f|\,.h_0$. Therefore we have

$$\nabla w_0 = |\nabla w_0| \mathbf{t} = \frac{1}{h_0} \frac{1}{|\mathbf{13} \times \mathbf{12}|} \mathbf{13} \times \mathbf{12}$$
$$= \frac{|f|}{3\,|T|} \frac{1}{2\,|f|} \mathbf{13} \times \mathbf{12} = \frac{1}{6\,|T|} \mathbf{13} \times \mathbf{12}.$$

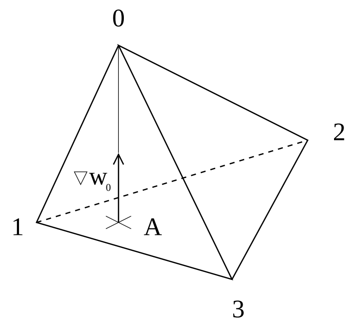


Figure 3.4: Gradient of the nodal function

Some identities

The following identities hold in tetrahedra $T = \{0123\}$ and face $f = \{123\}$:

$$
\begin{aligned}
|T| &= \int_T w_i, \quad i = 0,1,2,3, && (3.2)\\
|f| &= 3\,|T|\,|\nabla w_0|, \quad f = \{123\}, && (3.3)\\
|\mathbf{01}| &= 6\,|T|\,|\nabla w_2 \times \nabla w_3|, && (3.4)\\
1 &= 6\,|T| \det(\nabla w_1, \nabla w_2, \nabla w_3), && (3.5)\\
\int_T (w_1)^i (w_2)^j (w_3)^k &= 6\,|T| \frac{i!j!k!}{(i+j+k+3)!}, && (3.6)\\
\int_f (w_0)^i (w_1)^j &= 2\,|f| \frac{i!j!}{(i+j+2)!}, && (3.7)\\
\nabla \times (w_m \nabla w_n) &= \nabla w_m \times \nabla w_n, && (3.8)\\
\nabla w_2 \times \nabla w_3 &= \frac{\mathbf{01}}{6\,|T|}. && (3.9)
\end{aligned}
$$

From the above equations we can also derive identities for other edges and faces by cyclic replacement. For tetrahedra $T = \{0123\}$ we have

$$
\begin{aligned}
\nabla w_1 \times \nabla w_0 &= \frac{\mathbf{32}}{6\,|T|},\\
\nabla w_2 \times \nabla w_0 &= \frac{\mathbf{13}}{6\,|T|},\\
\nabla w_3 \times \nabla w_0 &= \frac{\mathbf{21}}{6\,|T|}.
\end{aligned}
$$

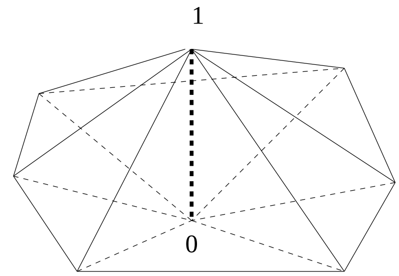


Figure 3.5: Tetrahedra with nonzero w_e

3.3 Edge Whitney elements

Using these elements we approximate the electric field. The reason will be given later. We define a vector field w_e associated with the edge $e = \{01\}$ on tetrahedra $T = \{0123\}$ by the following relation

$$w_e = w_0 \nabla w_1 - w_1 \nabla w_0. \tag{3.10}$$

On all tetrahedra that do not contain vertex 0, both w_0 and ∇w_0 are equal to 0. Therefore, w_e is zero. This is true for the vertex 1, too. It means that w_e is nonzero only on a tetrahedra that contain the edge $e = \{01\}$, see Figure 3.5. Thus we have again the property, which causes the spareness of the system matrix.

There is a geometric interpretation of w_e, depicted in Figure 3.6. We can see that in the vertex 0 the value of w_1 is zero and the value of w_0 is one. Thus $w_e(0)$ is in fact the vector ∇w_1, similarly $w_e(1)$ is the vector $-\nabla w_0$. Moreover it is a linear function on the tetrahedra.

Why this definition of the basis function? Let us compute the circulation of w_e along the edge e. Realizing that the directional vector of the edge $e = \{01\}$ is the vector $\mathbf{t} = \frac{\mathbf{01}}{|\mathbf{01}|}$, we write the first part of the integral

$$\int_e \langle \mathbf{t}, w_0 \nabla w_1 \rangle = \int_e \langle \frac{\mathbf{01}}{|\mathbf{01}|}, w_0 \nabla w_1 \rangle = \frac{1}{|\mathbf{01}|} \langle \mathbf{01}, \nabla w_1 \rangle \int_e w_0. \tag{3.11}$$

The scalar product $\langle \mathbf{01}, \nabla w_1 \rangle$ is equal to $|\mathbf{01}||\nabla w_1| \cos\alpha$, where α is the angle between the vectors $\mathbf{01}$ and ∇w_1, see Figure 3.7. Because of the identity $|\mathbf{01}| \cos\alpha = h_1$, we can write

$$\langle \mathbf{01}, \nabla w_1 \rangle = |\mathbf{01}||\nabla w_1| \cos\alpha = h_1 |\nabla w_1| = 1, \tag{3.12}$$

due to equation (3.1). Thus, we can complete the computation of the first part of the circulation of w_e along the edge e with

$$\int_e \langle \mathbf{t}, w_0 \nabla w_1 \rangle = \frac{1}{|\mathbf{01}|} \langle \mathbf{01}, \nabla w_1 \rangle \int_e w_0 = \frac{1}{|\mathbf{01}|} \int_e w_0. \tag{3.13}$$

This together with the second part of w_e gives

$$\int_e \langle \mathbf{t}, w_e \rangle = \int_e \langle \mathbf{t}, w_0 \nabla w_1 \rangle - \int_e \langle \mathbf{t}, w_1 \nabla w_0 \rangle = \tag{3.14}$$

$$\int_e \langle \mathbf{t}, w_0 \nabla w_1 \rangle + \int_{-e} \langle \mathbf{t}, w_1 \nabla w_0 \rangle = \tag{3.15}$$

$$= \frac{1}{|\mathbf{01}|} \left(\int_e w_0 + \int_e w_1 \right) = 1, \tag{3.16}$$

because $w_0 + w_1 = 1$ on the edge e.

We see that if we compute the circulation of w_e along the edge e', it is zero if $e' \neq e$ and it is one if $e' = e$. Thus we have an analogue to the similar property of nodal Whitney elements.

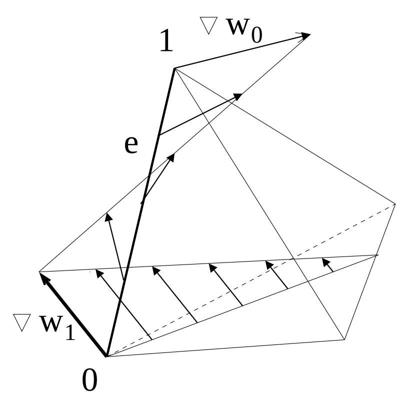


Figure 3.6: Geometric interpretation of w_e

3.4 Facial Whitney elements

To approximate the electric induction $\mathbf{B}$, we define facial elements. One element is linked to the face $f = \{123\}$ of the tetrahedra $T = \{0123\}$ by two equivalent

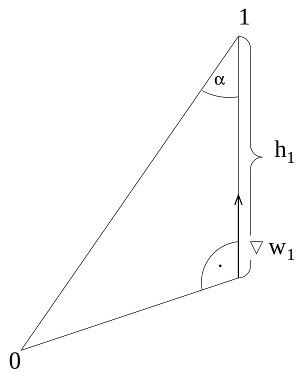


Figure 3.7: Computation of the scalar product

relations

$$w_f = 2(w_1\nabla w_2 \times \nabla w_3 + w_2\nabla w_3 \times \nabla w_1 + w_3\nabla w_1 \times \nabla w_2), \qquad (3.17)$$

$$w_f = \frac{1}{3\,|T|}(w_1\mathbf{01} + w_2\mathbf{02} + w_3\mathbf{03}). \qquad (3.18)$$

Notice, that the facial element is nonzero only on two tetrahedra with a common face. The latter definition of w_f is valid only if the orientation of the face f matches with the orientation of boundary of T. In the latter case they match. The definition of w_f on the tetrahedra $T' = \{4321\}$ will be different. T' is positively oriented, his boundary is oriented outside from T' and doesn't match with the orientation of the face $f = \{123\}$. Thus, the definition of w_f on tetrahedra T' is

$$w_f = -2(w_1\nabla w_2 \times \nabla w_3 + w_2\nabla w_3 \times \nabla w_1 + w_3\nabla w_1 \times \nabla w_2), \qquad (3.19)$$

$$w_f = -\frac{1}{3\,|T|}(w_1\mathbf{41} + w_2\mathbf{42} + w_3\mathbf{43}). \qquad (3.20)$$

The equivalence of the definitions comes from the vectorial identities

$$\nabla w_2 \times \nabla w_3 = (\mathbf{03} \times \mathbf{01}) \times \frac{\nabla w_3}{6\,|T|} = \frac{\langle \mathbf{03}, \nabla w_3\rangle \mathbf{01} - \langle \mathbf{01}, \nabla w_3\rangle \mathbf{03}}{6\,|T|} = \frac{\mathbf{01}}{6\,|T|},$$

because vectors **01** and w_3 are perpendicular, thus $\langle \mathbf{01}, \nabla w_3\rangle = 0$, and because $\langle \mathbf{03}, \nabla w_3\rangle = 1$, which comes from (3.12). The geometric interpretation is depicted in Figure 3.8.

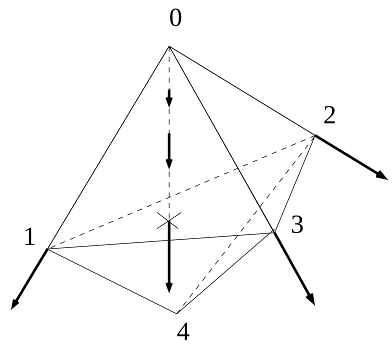


Figure 3.8: Facial WE

We can easily see that the flux of this vector field through the faces $\{012\}$, $\{023\}$ and $\{031\}$ is zero, because the normal part of this vector field on these faces is zero. Let us compute the flux through the face $f = \{123\}$. The normal vector $\mathbf{n}_f$ of the face f can be written as $-\nabla w_0 / |\nabla w_0|$. We can easily verify that

$$\int_f w_i = \frac{1}{3} |f| .$$

Then

$$\begin{aligned}\int_f \langle w_f, n_f \rangle &= -\frac{1}{3|T|} \frac{1}{|\nabla w_0|} \Big(\int_f \langle w_1 \mathbf{01}, \nabla w_0 \rangle + \int_f \langle w_2 \mathbf{02}, \nabla w_0 \rangle \\ &\quad + \int_f \langle w_3 \mathbf{03}, \nabla w_0 \rangle \Big).\end{aligned}$$

The equation (3.12) gives $\langle \mathbf{0i}, \nabla w_i \rangle = -1$, for i=1,2,3. If we notice that $3|T| |\nabla w_0| = |f|$, see equation (3.3), we can write

$$\int_f \langle w_f, n_f \rangle = \frac{1}{3|T|} \frac{1}{|\nabla w_0|} (\frac{1}{3} |f| + \frac{1}{3} |f| + \frac{1}{3} |f|) = 1.$$

3.5 Tetrahedral Whitney elements

We use constant functions for approximating $L^2(\Omega)$. The basis function w_T will be such a constant on tetrahedra T that after integrating this constant over the tetrahedra T, the result is 1. This leads to

$$w_T = \frac{1}{|T|}.$$

3.6 Continuity properties

For a good approximation of physical fields it is necessary to demand some kinds of continuity properties on approximating elements. We will use edge Whitney elements for the approximation of the magnetic intensity field **H**. This field has the property of continuity of the tangential part. It is important that our approximating element also have a tangential part continuous while moving from one tetrahedra to another one.

Tangential part of the edge elements

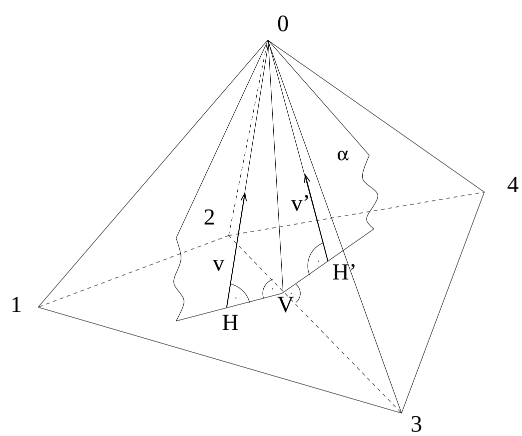


Figure 3.9: Tangential part of edge WE

If we have an edge element $w_e = w_0 \nabla w_3 - w_3 \nabla w_0$, it is sufficient to show that the tangential part of ∇w_0 is continuous through the faces, because of the continuity of w_i. Let us have two tetrahedra with common face. The vector ∇w_0 is constant on one tetrahedra. Denote by $\mathbf{v}$ the vector ∇w_0 on the first tetrahedra and by $\mathbf{v}'$ on the second. We will make projections of the vectors $\mathbf{v}$ and $\mathbf{v}'$ on the plane $\{023\}$.

In tetrahedra $T = \{0124\}$ and $T' = \{0243\}$ we consider the heights $0H$ and $0H'$, see Figure 3.9. Denote by α the plane containing the vectors **0H** and **0H'**. This plane is also perpendicular to the edge $\{23\}$ because both directional vectors **0H** and **0H'** of the plane are perpendicular to this edge. We cut the tetrahedron with the plane α and we denote by V the intersection of α and the edge $\{23\}$.

The projection $\mathbf{v}_p$ of $\mathbf{v}$ now appears on the line $0V$. The length $|\mathbf{v}_p|$ of this projection will be $|\mathbf{v}|.\cos H0V$. Because $|\mathbf{v}|.|\mathbf{0H}| = 1$, we have $|\mathbf{v}_p| = \cos H0V/|\mathbf{0H}| = 1/|\mathbf{0V}|$. We can see that the length of the projection $\mathbf{v}_p$ depends only on the value of $|\mathbf{0V}|$, which is same for both $\mathbf{v}$ and $\mathbf{v}'$. Thus, the tangential part of ∇w_0 is continuous through the face $\{023\}$. Also the same holds for the face $\{031\}$. The tangential part of w_e on the faces that do not contain edge e is zero. We have now proved that the tangential part of the edge element w_e is continuous through any face.

Normal part of the facial elements

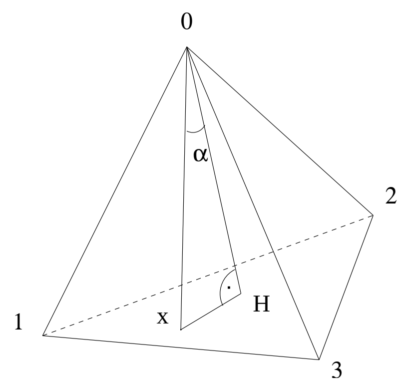


Figure 3.10: Normal part of facial WE

Consider the tetrahedra $T = \{0123\}$ and the face $f = \{123\}$. From the definition of the facial element w_f, it is clear that the normal part of w_f on the faces $\{012\}$, $\{023\}$ and $\{031\}$ is zero. Let us look at the face $f = \{123\}$. From (3.18) it is easy to see that

$$w_f(x) = \frac{1}{3\,|T|}\mathbf{0x},$$

because $w_f(i) = \frac{1}{3|T|}\mathbf{0i}$, for $i = 1, 2, 3$ and because w_f is linear.

For an arbitrary point x from the face f we make the projection $P(\vec{0x})$ to the normal vector of the face (123). The length of this projection will be exactly the height h of the tetrahedra. Thus, the length of projection of the vector $w_f(x)$ will be $\frac{1}{3|T|}.h = \frac{1}{|f|}$. We can see that it depends only on the area of the face f. Therefore the normal part of w_f is continuous through the face.

3.7 Embedding properties

When working with Maxwell's equations one is confronted with the following function spaces

$$\mathbf{H}^1(\Omega),\ \mathbf{H}(\mathrm{curl};\Omega),\ \mathbf{H}(\mathrm{div};\Omega),\ \mathbf{L}^2(\Omega),$$

or with appropriate subspaces when considering boundary conditions. The entities involved in Maxwell's equations such as the electric field $\mathbf{E}$, the magnetic field $\mathbf{H}$, the magnetic induction $\mathbf{B}$, or the magnetization $\mathbf{M}$, belong to these spaces or its subspaces. When approximating some function spaces by finite dimensional spaces it is desirable that the approximating spaces have similar properties as the approximated spaces. In particular this is the case when the properties are derived from physical phenomena.

Let us look at these function spaces in more detail. It is well known that curls of gradients vanish. But is it reciprocal? Is every curl-free field a gradient of some other function? The same question can be asked about divergence. We know that curls are divergence-free. But is every divergence-free field also a curl of some other field? A classical result of Poincaré, see for example [15, Appendix A], asserts that, in a contractible domain Ω, a smooth curl-free (divergence-free, respectively) field is a gradient (curl, respectively). A contractible domain is a simply connected domain with a connected boundary.

For further discussion we introduce some notations. A family of vector spaces is denoted by $X^0, X^1, \cdots, X^d$. By A^p we denote a linear map from X^{p-1} to X_p, $p = 1, \cdots, d$. We say that the sequence $\{X^p\}_{p=0}^d$ is an exact sequence at the level of X^p if $\mathrm{img}(A^p) = \ker(A^{p+1})$ in case $1 \le p \le d-1$, if A^1 is injective in case $p = 0$, and if A^d is surjective in case $p = d$. An exact sequence is one which is exact at all levels. We use the following diagram

$$X^0 \ \overset{A^1}{\rightarrow}\ X^1 \ \overset{A^2}{\rightarrow}\ \cdots \ \rightarrow\ X^{d-1} \ \overset{A^d}{\rightarrow}\ X^d$$

for a exact sequence. Using this diagram we can state the following result coming from Poincaré lemma, see [15, Chapter 5].

For a contractible domain Ω, the sequence

$$\mathbf{H}^1(\Omega) \ \overset{\mathrm{grad}}{\rightarrow}\ \mathbf{L}^2(\mathrm{curl};\Omega) \ \overset{\mathrm{curl}}{\rightarrow}\ \mathbf{L}^2(\mathrm{div};\Omega) \ \overset{\mathrm{div}}{\rightarrow}\ L^2(\Omega), \tag{3.21}$$

is exact at levels 1 and 2.

Let us go back to Whitney elements. The question is if finite dimensional spaces build from Whitney elements will satisfy a diagram similar to that above. For a given mesh, we can construct four classes of Whitney elements for the sets $\mathcal{N}_h, \mathcal{E}_h, \mathcal{F}_h, \mathcal{T}_h$. We then obtain basis functions denoted by w_n, w_e, w_f, w_T. Let us denote by W^0, W^1, W^2, W^3 approximation spaces build from each class of Whitney basis functions. The embeddings

$$W^0 \subset \mathbf{H}^1(\Omega), \quad W^1 \subset \mathbf{L}^2(\mathrm{curl};\Omega), \;\; W^2 \subset \mathbf{L}^2(\mathrm{div};\Omega), \;\; W^3 \subset L^2(\Omega), \tag{3.22}$$

are clear from the way how we have obtained W^0, W^1, W^2, W^3.

Next, we can prove that

$$\begin{aligned} \nabla W^0 &\subset W^1, \\ \nabla \times W^1 &\subset W^2, \\ \nabla \cdot W^2 &\subset W^3. \end{aligned}$$

The first statement follows from the fact that ∇w_n is a linear (constant) vector function on T, thus can be expressed by a linear combination of edge Whitney elements.

The second statement follows from the identity (we take tetrahedra $T = \{0123\}$ and edge $e = \{01\}$):

$$\nabla \times w_e = 2\nabla w_0 \times \nabla w_1 = \frac{\mathbf{23}}{3\,|T|},$$

which can be verified by direct computation using the equation (3.4). The continuity of $\nabla \times w_e$ can be shown using the same technique as in Section 3.6

Direct computation of $\nabla \cdot w_f$ shows the third statement

$$\nabla \cdot w_f = \frac{1}{3\,|T|}\Big(\langle \nabla w_1, \mathbf{01}\rangle + \langle \nabla w_2, \mathbf{02}\rangle + \langle \nabla w_3, \mathbf{03}\rangle\Big) = \frac{1}{|T|},$$

because by equation (3.1) we have

$$\langle \nabla w_i, \mathbf{0i}\rangle = 1.$$

Finally, we have shown that the spaces satisfy the diagram

$$W^0 \overset{\mathrm{grad}}{\rightarrow} W^1 \overset{\mathrm{curl}}{\rightarrow} W^2 \overset{\mathrm{div}}{\rightarrow} W^3 \tag{3.23}$$

To complete this section we combine the diagrams (3.21), (3.22) and (3.23) to get a so-called *Rham diagram*

$$
\begin{array}{ccccccc}
W^0 & \stackrel{\text{grad}}{\rightarrow} & W^1 & \stackrel{\text{curl}}{\rightarrow} & W^2 & \stackrel{\text{div}}{\rightarrow} & W^3 \\
\cap & & \cap & & \cap & & \cap \\
\mathbf{H}^1(\Omega) & \stackrel{\text{grad}}{\rightarrow} & \mathbf{L}^2(\text{curl};\Omega) & \stackrel{\text{curl}}{\rightarrow} & \mathbf{L}^2(\text{div};\Omega) & \stackrel{\text{div}}{\rightarrow} & L^2(\Omega),
\end{array}
$$

where also the embeddings (3.22) are included.

II

Effective field without exchange

4 Maxwell-LL system

(Since the mathematicians have invaded the theory of relativity, I do not understand it myself anymore. Albert Einstein)

In the LL equation plays a key role the term describing the total effective field $\mathbf{H}_{\text{eff}}$. In general it consists of several contributions and moreover every contribution features different behavior, as we have described already in Section 2.2. The case when the effective field consists of static applied field $\mathbf{H}_{\text{app}}$, anisotropy field $\mathbf{H}_{\text{ani}}$ and field coming from Maxwell's equations $\mathbf{H}$ was intensively studied by Joly, Métivier, Rauch, Komech, Vacus and Monk in [39, 40, 41, 42, 43, 59, 60].

They consider the case when

$$\mathbf{H}_{\text{eff}} = \mathbf{H} + \mathbf{H}_{\text{app}} - KP(\mathbf{M}),$$

when K is a constant characterizing anisotropy of the medium and $P(\mathbf{M}) = \langle \mathbf{p}, \mathbf{M} \rangle \mathbf{p}$ denotes the projection of the vector $\mathbf{M}$ into one specified direction represented by a fixed vector $\mathbf{p}$, see Figure 4.1. The anisotropy is thus uniaxial.

They neglect the exchange field $\mathbf{H}_{\text{exc}}$ having in mind applications such as modeling radar absorbing materials for stealth applications when this contribution is not important. For more detailed description of the model see [16, 50].

Joly and Vacus in [42, 43] have established existence and uniqueness of the solution for the system

$$\begin{aligned} \partial_t \mathbf{E} - \nabla \times \mathbf{H} &= -\mathbf{J}_0, \\ \partial_t \mathbf{H} + \nabla \times \mathbf{E} &= -\partial_t \mathbf{M}, \\ \partial_t \mathbf{M} &= \frac{\gamma}{1+\alpha^2} (\mathbf{H}_{\text{eff}} \times \mathbf{M} + \alpha \frac{\mathbf{M}}{|\mathbf{M}|} \times (\mathbf{H}_{\text{eff}} \times \mathbf{M})). \end{aligned} \tag{4.1}$$

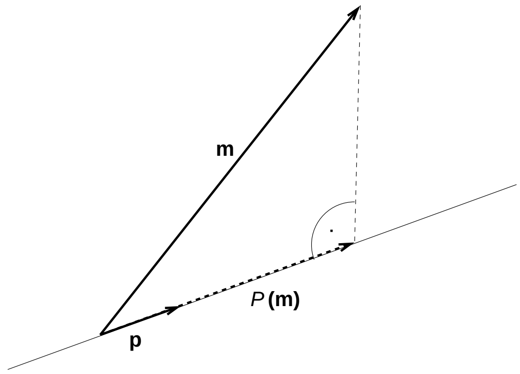


Figure 4.1: Anisotropy of the material

They proposed a finite difference scheme in 2D and 3D for the computation a solution of the M-LL system in one dimension and prove the convergence of this scheme. In [59] the authors construct three-dimensional finite element methods for the full system. They show how a certain class of finite element can be used to approximate the M-LL system while preserving energy decay and the norm $\mathbf{M}$. They prove error estimates.

After providing the space discretization, full space-time discretization of the problem is studied, too. When discretizing the LL equation in time, several problems arise. High nonlinearity of the LL equation causes troubles. There are two main conditions that should be fulfilled by any approximation scheme.

1. It should be possible to prove error estimates, if not, at least convergence to the solution.

2. Time stepping should conserve the magnitude of the magnetization $\mathbf{M}$.

In the sequel we present a time-discretization scheme used in the papers mentioned above. However, the authors do not prove convergence results for this scheme. In Chapter 5 we propose a new scheme discretizing the M-LL system, which fulfills both conditions.

4.1 Numerical scheme

We provide a standard equidistant time discretization with the step τ. The number of discretizing points is denoted by N. Let the triple $\mathbf{M}_n, \mathbf{E}_n, \mathbf{H}_n$ be the approximating solution on every time level. Denote by V_M, V_E, V_H a finite-dimensional spaces in which we look for the solution $\mathbf{M}_n, \mathbf{E}_n, \mathbf{H}_n$. These spaces are derived from a particular finite element method which we are using. Since we do not focus on the spatial discretization, we will not specify the spaces V_M, V_H, V_E closer anymore.

We denote the right-hand side of the LL equation by $f(\mathbf{M}, \mathbf{H})$ to make the text simpler.

We will compute the approximations of fields $\mathbf{H}$ and $\mathbf{M}$ in half steps such that $\mathbf{H}_{n+1/2}$ is the approximation of $\mathbf{H}(t_{n+1/2})$ and $\mathbf{M}_{n+1/2}$ is the approximation of $\mathbf{M}(t_{n+1/2})$ where $t_n = n\tau$ and $t_{n+1/2} = (n+1/2)\tau$. From the initial date we know $\mathbf{E}_0 = \mathbf{E}(0)$ and using for example explicit Runge-Kutta method we find $\mathbf{H}_{1/2}$ and $\mathbf{M}_{1/2}$. From then on the triple $\mathbf{M}_{n+3/2}, \mathbf{E}_{n+1}, \mathbf{H}_{n+3/2}$ is determined by

$$\frac{\mathbf{E}_{n+1} - \mathbf{E}_n}{\tau} - \nabla \times \mathbf{H}_{n+1/2} = \mathbf{J}(t_{n+1/2}), \tag{4.2}$$

$$\frac{\mathbf{H}_{n+3/2} - \mathbf{H}_{n+1/2}}{\tau} + \nabla \times \mathbf{E}_{n+1} = -\frac{\mathbf{M}_{n+3/2} - \mathbf{M}_{n+1/2}}{\tau}, \tag{4.3}$$

$$\frac{\mathbf{M}_{n+3/2} - \mathbf{M}_{n+1/2}}{\tau} = f(\mathbf{H}_{n+1}, \mathbf{M}_{n+1}), \tag{4.4}$$

where

$$\mathbf{H}_{n+1} = \frac{1}{2}(\mathbf{H}_{n+1/2} + \mathbf{H}_{n+3/2}) \quad \text{and} \quad \mathbf{M}_{n+1} = \frac{1}{2}(\mathbf{M}_{n+1/2} + \mathbf{M}_{n+3/2}).$$

The equalities (4.2)–(4.4) are considered in a weak sense in function spaces V_M, V_H, V_E. This scheme seems to be explicit-implicit. The field $\mathbf{E}_{n+1}$ is computed explicitly and other two fields $\mathbf{M}_{n+3/2}$ and $\mathbf{H}_{n+3/2}$ are computed implicitly. However, the authors point out that due to the special features of this scheme it is possible to write down an explicit form for $\mathbf{M}_{n+3/2}$ and $\mathbf{H}_{n+3/2}$. Then it is not necessary to use, for example, Newton's method on every time level. This scheme in combination with finite differences or finite elements becomes very fast.

The question if this scheme conserves the magnitude of $\mathbf{M}_n$ can be answered positive. If we rewrite (4.4) we get up to constants

$$\begin{aligned}
&\frac{\mathbf{M}_{n+3/2} - \mathbf{M}_{n+1/2}}{\tau} \\
&\quad = \mathbf{H}_{\text{eff}} \times \frac{\mathbf{M}_{n+3/2} + \mathbf{M}_{n+1/2}}{2} \\
&\qquad - \alpha \frac{(\mathbf{M}_{n+3/2} + \mathbf{M}_{n+1/2})/2}{\left|(\mathbf{M}_{n+3/2} + \mathbf{M}_{n+1/2})/2\right|} \times \left(\mathbf{H}_{\text{eff}} \times \frac{\mathbf{M}_{n+3/2} + \mathbf{M}_{n+1/2}}{2} \right).
\end{aligned}$$

Multiplying the previous equation by $(\mathbf{M}_{n+3/2}+\mathbf{M}_{n+1/2})/2$, the right-hand side vanishes and on the left-hand side we get

$$\frac{|\mathbf{M}_{n+3/2}|^2 - |\mathbf{M}_{n+1/2}|^2}{\tau} = 0,$$

which guarantees the conservation of the norm $\mathbf{M}_{n+1/2}$.

However, it is not known if it is possible to derive error estimates. In the next chapter we introduce scheme which is comparable in computational efficiency and we prove error estimates for it.

5 Numerical analysis of the M-LL system

(For those who want some proof that physicists are human, the proof is in the idiocy of all the different units which they use for measuring energy. Richard Feynman)

A substantial part of this chapter was already published by author of this thesis and Slodička in [28, 72].

We consider quasi-static Maxwell equations of the form

$$\begin{aligned} \nabla \times \mathbf{H} &= \sigma \mathbf{E} + \mathbf{J}_0, \\ \partial_t \mathbf{H} + \nabla \times \mathbf{E} &= -\partial_t \mathbf{M}, \end{aligned} \tag{5.1}$$

where $\mathbf{J}_0$ is the current density and $\sigma > 0$ denotes the conductivity of a medium. The case of full M-LL system was studied by Slodička and Baňas in [71] and in the PhD thesis of Baňas [4].

The coupling between $\mathbf{M}$ and $\mathbf{H}$ is given by the LL equation of the form

$$\partial_t \mathbf{M} = \frac{\gamma}{1+\alpha^2} (\mathbf{H}_{\text{eff}} \times \mathbf{M} + \alpha \mathbf{M} \times (\mathbf{H}_{\text{eff}} \times \mathbf{M})) =: f(\mathbf{H}, \mathbf{M}) \tag{5.2}$$

The vector $\mathbf{H}_{\text{eff}}$ represents the total magnetic field in the ferromagnet

$$\mathbf{H}_{\text{eff}} = \mathbf{H} + \mathbf{H}_{\text{app}} + KP(\mathbf{M}), \tag{5.3}$$

where $\mathbf{H}_{\text{app}}$ is a given static applied field. The constant K is a constant characterizing the material. We discuss the case of a ferromagnetic crystal with uniaxial anisotropy represented by a unit vector $\mathbf{p}$, $|\mathbf{p}| = 1$. The symbol $P(\mathbf{M})$ was defined in the previous chapter.

Let us note that we have neglected the exchange magnetic field in (5.3). For a more complete discussion of the model see, e.g., [2, 35, 37, 77, 83].

We know already that the LL equation conserves the modulus of $\mathbf{M}$, thus for any time $t > 0$ we have

$$|\mathbf{M}(t)| = |\mathbf{M}(0)|. \tag{5.4}$$

Eliminating $\mathbf{B}$ and $\mathbf{E}$ in (5.1) and assuming $\sigma = const$, we get the following M-LL system

$$\begin{aligned} \mu_0\partial_t\mathbf{H} + \sigma^{-1}\nabla\times\nabla\times\mathbf{H} &= \sigma^{-1}\nabla\times\mathbf{J}_0 - \mu_0\partial_t\mathbf{M}, \\ \partial_t\mathbf{M} &= f(\mathbf{H},\mathbf{M}). \end{aligned} \tag{5.5}$$

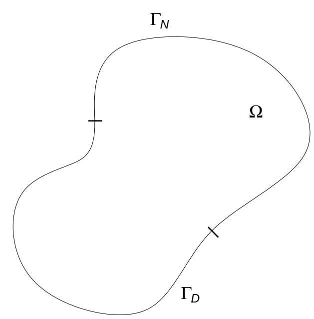


Figure 5.1: Definition of the domain Ω

The boundary Γ of Ω is split into two non-overlapping parts Γ_D and Γ_N, see Figure 5.1. For simplicity, we consider homogeneous boundary conditions for $\mathbf{H}$ of the type

$$\begin{aligned} \boldsymbol{\nu}\times\boldsymbol{\nu}\times\mathbf{H} &= \mathbf{0} && \text{on } \Gamma_D, \\ \boldsymbol{\nu}\times\mathbf{H} &= \mathbf{0} && \text{on } \Gamma_N, \end{aligned} \tag{5.6}$$

where $\boldsymbol{\nu}$ stands for the outward unit normal vector on the boundary.

We assume that the fields $\mathbf{H}$ and $\mathbf{M}$ are specified at time $t = 0$, i.e., $\mathbf{H}(0) = \mathbf{H}_0$ and $\mathbf{M}(0) = \mathbf{M}_0$. We tacitly presume that they are sufficiently smooth for our purposes. For physical reasons we suppose that

$$\nabla\cdot(\mathbf{H}_0 + \mathbf{M}_0) = 0 \qquad \text{in } \Omega,$$

which ensures that $\mathbf{B}$ is divergence free.

A single LL equation has been intensively studied by many authors, e.g., [22, 23, 31, 87]. One dimensional M-LL problem has been considered for example in [38, 41, 42].

The main purpose of this chapter is to present two new approximation schemes for the time discretization of a quasi-static M-LL system. We design a linear (see Section 5.2) and a nonlinear (see Section 5.4) numerical algorithm for the computation of the vector field $\mathbf{M}$. Both schemes conserve the length of $\mathbf{M}$. We derive exact formulas for the approximation of $\mathbf{M}$, see Lemmas 5.1 and 5.4. We use Rothe's method for the time discretization for $\mathbf{H}$. Assuming that the exact solution of a M-LL system is bounded, we derive the error estimates for the proposed numerical schemes, cf. Theorems 5.1 and 5.2. In Section 5.6 we present a numerical example in order to demonstrate suggested algorithms.

5.1 Preliminaries

For ease of exposition we put $\sigma = \mu_0 = \alpha = K = 1,\ \gamma = 2$ and $\mathbf{J}_0 = \mathbf{H}_{\text{app}} = \mathbf{0}$ in the theoretical part of the chapter, but not in the numerical one.

Let us introduce the following space of test vector-functions

$$\mathbf{V} \quad = \{\boldsymbol{\phi} \in \mathbf{H}(\text{curl};\Omega);\ \boldsymbol{\phi} = \mathbf{0} \text{ on } \Gamma_D, \boldsymbol{\nu} \times \boldsymbol{\phi} = \mathbf{0} \text{ on } \Gamma_N\}.$$

Throughout the rest of the paper we assume the following regularity of an exact solution to the boundary value problem (5.5), (5.6)

$$\begin{aligned} \partial_t \mathbf{H} &\in L_2\left((0,T), \mathbf{L}^2(\Omega)\right), \\ \mathbf{H} &\in L_\infty\left((0,T), \mathbf{H}(\text{curl};\Omega)\right) \cap L_\infty\left((0,T)\times\Omega\right), \\ \partial_t \mathbf{M} &\in L_\infty\left((0,T), \mathbf{L}^2(\Omega)\right), \\ \mathbf{M} &\in L_\infty\left((0,T)\times\Omega\right). \end{aligned} \tag{5.7}$$

Let us note that the assumption $\mathbf{H} \in L_\infty\left((0,T)\times\Omega\right)$ together with (5.4) guarantee the Lipschitz continuity of the right-hand side f of (5.2) (see [59, Lemma 2.2], thus also the uniqueness of a solution.

The variational formulation of (5.5) reads as

$$\begin{aligned} (\partial_t \mathbf{H}, \boldsymbol{\varphi}) + (\nabla \times \mathbf{H}, \nabla \times \boldsymbol{\varphi}) &= -\left(\partial_t \mathbf{m}, \boldsymbol{\varphi}\right), \\ (\partial_t \mathbf{M}, \boldsymbol{\psi}) &= (f(\mathbf{H},\mathbf{M}), \boldsymbol{\psi}) \end{aligned} \tag{5.8}$$

for any $\boldsymbol{\varphi} \in \mathbf{V}$ and any $\boldsymbol{\psi} \in \mathbf{L}^2(\Omega)$.

5.2 Linear approximation scheme

We divide the time interval $[0,T]$ into n equidistant subintervals $[t_{i-1}, t_i]$ for $t_i = i\tau$, where $\tau = \frac{T}{n}$ for any $n \in \mathbb{N}$.

We suggest the following recurrent linear approximation scheme for $i = 1, \ldots, n$, see Figure 5.2. In this chapter the notation $\mathbf{m}$ has no link with the normalized magnetization $\mathbf{M}/\mathrm{M}$.

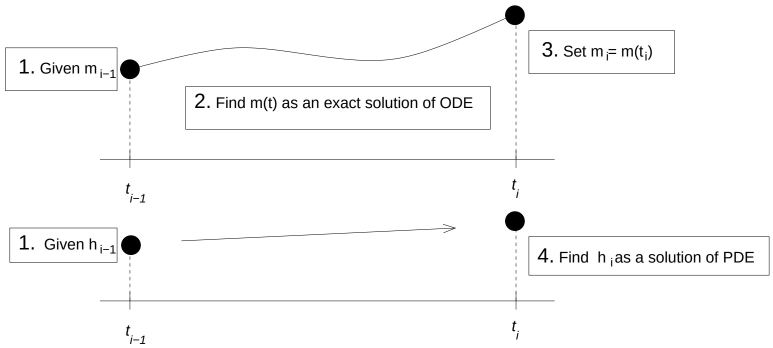


Figure 5.2: Algorithm 1 (linear)

Algorithm 1 (linear)

1. *We start from* $\mathbf{h}_{i-1}$ *and* $\mathbf{m}_{i-1}$ *taking into account* $\mathbf{h}_0 = \mathbf{H}_0$ *and* $\mathbf{m}_0 = \mathbf{M}_0$.

2. *We solve the linear ordinary differential equation (ODE) with an unknown* $\mathbf{m}(t)$ *on the subinterval* $[t_{i-1}, t_i]$

$$\begin{aligned} \partial_t \mathbf{m} \quad &= [\mathbf{h}_{i-1} + P(\mathbf{m}_{i-1})] \times \mathbf{m} \\ &+ \frac{\mathbf{m}}{|\mathbf{m}_{i-1}|} \times ([\mathbf{h}_{i-1} + P(\mathbf{m}_{i-1})] \times \mathbf{m}_{i-1}) \,. \end{aligned} \tag{5.9}$$

3. *We set* $\mathbf{m}_i := \mathbf{m}(t_i)$.

4. *We solve the partial differential equation (PDE) for* $\mathbf{h}_i$

$$(\delta \mathbf{h}_i, \boldsymbol{\varphi}) + (\nabla \times \mathbf{h}_i, \nabla \times \boldsymbol{\varphi}) \quad = - (\partial_t \mathbf{m}(t_i), \boldsymbol{\varphi}) \tag{5.10}$$

for $\boldsymbol{\varphi} \in \mathbf{V}$.

Suppose that $\mathbf{h}_{i-1}$ and $\mathbf{m}_{i-1}$ are given. A scalar multiplication of (5.9) by $\mathbf{m}$ implies

$$\langle \partial_t \mathbf{m}, \mathbf{m} \rangle = \frac{1}{2} \partial_t |\mathbf{m}|^2 = 0.$$

The time integration over $[t_{i-1}, t_i]$ immediately gives

$$|\mathbf{m}_{i-1}| = |\mathbf{m}(t)| \qquad \text{for } t \in [t_{i-1}, t_i].$$

Thus $\mathbf{m}$ preserves its modulus. Further, the equation (5.9) admits a unique solution for any $\mathbf{x} \in \Omega$ which is given by the following lemma for $\mathbf{u}_0 = \mathbf{m}_{i-1}$ and $\mathbf{a} = \mathbf{h}_{i-1} + P(\mathbf{m}_{i-1}) - [\mathbf{h}_{i-1} + P(\mathbf{m}_{i-1})] \times \frac{\mathbf{m}_{i-1}}{|\mathbf{m}_{i-1}|}$.

Lemma 5.1 *Let* $\mathbf{a}$ *and* $\mathbf{u}_0$ *be any vectors in* $\mathbb{R}^3$. *Then the unique solution of*

$$\begin{aligned} \partial_t \mathbf{u}(t) &= \mathbf{a} \times \mathbf{u}(t) \qquad t > 0, \\ \mathbf{u}(0) &= \mathbf{u}_0 \end{aligned} \tag{5.11}$$

is given by

$$\mathbf{u}(t) = \mathbf{u}_0^{\|} + \mathbf{u}_0^{\perp} \cos(|\mathbf{a}|t) + \frac{\mathbf{a}}{|\mathbf{a}|} \times \mathbf{u}_0^{\perp} \sin(|\mathbf{a}|t),$$

where $\mathbf{u}_0 = \mathbf{u}_0^{\perp} + \mathbf{u}_0^{\|}$, $\mathbf{u}_0^{\|}$ *is parallel to* $\mathbf{a}$, *and* $\mathbf{u}_0^{\perp}$ *is perpendicular to* $\mathbf{a}$. *Moreover, the vector field* $\mathbf{u}(t)$ *preserves its modulus, i.e.,* $|\mathbf{u}(t)| = |\mathbf{u}_0|$ *for any time* $t > 0$.

PROOF:
The assertion of Lemma 5.1 for $\mathbf{a} = \mathbf{0}$ is trivial. Now, we suppose that $\mathbf{a} \neq \mathbf{0}$. Let us introduce the notation for any $k \in \mathbb{N}$

$$\begin{aligned} \mathbf{a}^0 \times \mathbf{u}_0 &:= \mathbf{u}_0, \\ \mathbf{a}^k \times \mathbf{u}_0 &:= \mathbf{a} \times (\mathbf{a}^{k-1} \times \mathbf{u}_0). \end{aligned}$$

A simple calculation gives (see Figure 5.3)

$$\begin{aligned} \mathbf{a}^{2k} \times \mathbf{u}_0 &= (-1)^k |\mathbf{a}|^{2k} \mathbf{u}_0^{\perp}, \\ \mathbf{a}^{2k+1} \times \mathbf{u}_0 &:= (-1)^k |\mathbf{a}|^{2k} \mathbf{a} \times \mathbf{u}_0^{\perp}. \end{aligned}$$

The solution of (5.11) is given by

$$\begin{aligned} \mathbf{u}(t) &= e^{\mathbf{a}t} \times \mathbf{u}_0 \\ &= \sum_{k=0}^{\infty} \frac{\mathbf{a}^k \times \mathbf{u}_0}{k!} t^k \\ &= \mathbf{u}_0^{\|} + \sum_{k=0}^{\infty} \mathbf{a}^{2k} \times \mathbf{u}_0 \frac{t^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \mathbf{a}^{2k+1} \times \mathbf{u}_0 \frac{t^{2k+1}}{(2k+1)!} \\ &= \mathbf{u}_0^{\|} + \mathbf{u}_0^{\perp} \sum_{k=0}^{\infty} (-1)^k \frac{(|\mathbf{a}|t)^{2k}}{(2k)!} + \frac{\mathbf{a}}{|\mathbf{a}|} \times \mathbf{u}_0^{\perp} \sum_{k=0}^{\infty} (-1)^k \frac{(|\mathbf{a}|t)^{2k+1}}{(2k+1)!} \\ &= \mathbf{u}_0^{\|} + \mathbf{u}_0^{\perp} \cos(|\mathbf{a}|t) + \frac{\mathbf{a}}{|\mathbf{a}|} \times \mathbf{u}_0^{\perp} \sin(|\mathbf{a}|t). \end{aligned}$$

The vector field $\mathbf{u}(t)$ conserves its norm. One can easily see that

$$|\mathbf{u}(t)| = \sqrt{|\mathbf{u}_0^{\|}|^2 + |\mathbf{u}_0^{\perp}|^2 \cos^2(|\mathbf{a}|t) + |\mathbf{u}_0^{\perp}|^2 \sin^2(|\mathbf{a}|t)} = |\mathbf{u}_0|.$$

The uniqueness of a solution follows from the linearity of (5.11). □

From (5.9) we easily deduce that

$$|\partial_t \mathbf{m}| \leq C |\mathbf{m}_{i-1}| \left(|\mathbf{h}_{i-1}| + |\mathbf{m}_{i-1}| \right) \leq C \left(|\mathbf{h}_{i-1}| + 1 \right). \tag{5.12}$$

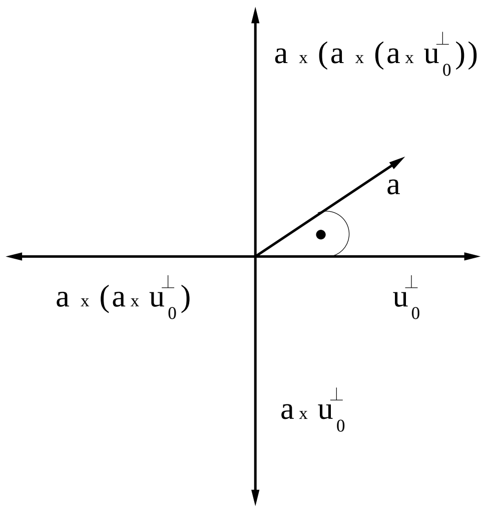


Figure 5.3: Rotations of $u_0^{\perp}$

Therefore, if $\mathbf{h}_{i-1} \in \mathbf{L}^2(\Omega)$, then (5.10) admits a unique solution $\mathbf{h}_i \in \mathbf{V}$. This follows from Lax-Milgram lemma (see [20]). In this way we successively obtain $\mathbf{m}_i$ and $\mathbf{h}_i$ for all $i = 1, \ldots, n$. The next step is to prove suitable a priori estimates. Here, we use standard Rothe's technique to get uniform energy estimates for the approximations $\mathbf{h}_i$ with respect to the index i.

Lemma 5.2 *Let $j \in \{1, \ldots, n\}$. Then there exists a positive constant C such that*

$$\|\mathbf{h}_j\|^2 + \sum_{i=1}^{j} \|\mathbf{h}_i - \mathbf{h}_{i-1}\|^2 + \sum_{i=1}^{j} \|\nabla \times \mathbf{h}_i\|^2 \tau \leq C.$$

Proof:
We set $\boldsymbol{\varphi} = \mathbf{h}_i \tau$ in (5.10) and sum the equation for $i = 1, \ldots, j$. We have

$$\sum_{i=1}^{j} (\mathbf{h}_i - \mathbf{h}_{i-1}, \mathbf{h}_i) + \sum_{i=1}^{j} \|\nabla \times \mathbf{h}_i\|^2 \tau = -\sum_{i=1}^{j} (\partial_t \mathbf{m}(t_i), \mathbf{h}_i)\, \tau. \tag{5.13}$$

The first term on the left can be written as

$$\sum_{i=1}^{j} (\mathbf{h}_i - \mathbf{h}_{i-1}, \mathbf{h}_i) = \tfrac{1}{2} \left(\|\mathbf{h}_j\|^2 - \|\mathbf{h}_0\|^2 + \sum_{i=1}^{j} \|\mathbf{h}_i - \mathbf{h}_{i-1}\|^2 \right).$$

For the right-hand side of (5.13) we use Cauchy's and Young's inequalities and (5.12). We get

$$\begin{aligned}\left|\sum_{i=1}^{j}(\partial_t\mathbf{m}(t_i),\mathbf{h}_i)\,\tau\right| &\le C\sum_{i=1}^{j}(1+\|\mathbf{h}_{i-1}\|)\,\|\mathbf{h}_i\|\,\tau\\ &\le C\left(1+\sum_{i=1}^{j}\|\mathbf{h}_i\|^2\,\tau\right).\end{aligned}$$

Summarizing all estimates we get

$$\|\mathbf{h}_j\|^2+\sum_{i=1}^{j}\|\mathbf{h}_i-\mathbf{h}_{i-1}\|^2+\sum_{i=1}^{j}\|\nabla\times\mathbf{h}_i\|^2\,\tau\le C\left(1+\sum_{i=1}^{j}\|\mathbf{h}_i\|^2\,\tau\right).$$

The desired result follows from Gronwall's lemma. □

We have needed $\mathbf{H}_0\in\mathbf{L}^2(\Omega)$ in Lemma 5.2. When $\mathbf{H}_0\in\mathbf{V}$, then we are capable to get better a priori estimates.

Lemma 5.3 *Let $j\in\{1,\dots,n\}$ and $\mathbf{H}_0\in\mathbf{V}$. Then there exists a positive constant C such that*

$$\sum_{i=1}^{j}\|\delta\mathbf{h}_i\|^2\,\tau+\|\nabla\times\mathbf{h}_j\|^2+\sum_{i=1}^{j}\|\nabla\times[\mathbf{h}_i-\mathbf{h}_{i-1}]\|^2\le C.$$

Proof:
Setting $\boldsymbol{\varphi}=\mathbf{h}_i-\mathbf{h}_{i-1}$ in (5.10) and summing up for $i=1,\dots,j$ we get

$$\sum_{i=1}^{j}\|\delta\mathbf{h}_i\|^2\,\tau+\sum_{i=1}^{j}(\nabla\times\mathbf{h}_i,\nabla\times[\mathbf{h}_i-\mathbf{h}_{i-1}])=-\sum_{i=1}^{j}(\partial_t\mathbf{m}(t_i),\delta\mathbf{h}_i)\,\tau. \qquad (5.14)$$

The second term on the left can be written as

$$\begin{aligned}\sum_{i=1}^{j}(\nabla\times\mathbf{h}_i,\nabla\times[\mathbf{h}_i-\mathbf{h}_{i-1}]) &= \tfrac{1}{2}\left(\|\nabla\times\mathbf{h}_j\|^2-\|\nabla\times\mathbf{h}_0\|^2\right)\\ &\quad+\tfrac{1}{2}\sum_{i=1}^{j}\|\nabla\times[\mathbf{h}_i-\mathbf{h}_{i-1}]\|^2.\end{aligned}$$

For the right-hand side of (5.14) we use Cauchy's inequality, (5.12), Lemma 5.2 and

Young's inequality. We obtain

$$\left|\sum_{i=1}^{j}(\partial_t \mathbf{m}(t_i), \delta \mathbf{h}_i)\,\tau\right| \le C\sum_{i=1}^{j}(1+\|\mathbf{h}_{i-1}\|)\,\|\delta \mathbf{h}_i\|\,\tau$$
$$\le C\sum_{i=1}^{j}\|\delta \mathbf{h}_i\|\,\tau$$
$$\le C_\epsilon + \epsilon\sum_{i=1}^{j}\|\delta \mathbf{h}_i\|^2\,\tau.$$

Choosing a sufficiently small positive ϵ and collecting all estimates, we arrive at

$$\sum_{i=1}^{j}\|\delta \mathbf{h}_i\|^2\,\tau + \|\nabla\times\mathbf{h}_j\|^2 + \sum_{i=1}^{j}\|\nabla\times[\mathbf{h}_i-\mathbf{h}_{i-1}]\|^2 \le C,$$

which concludes the proof. □

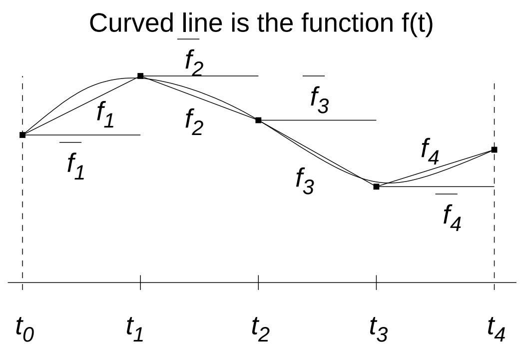


Figure 5.4: Definition of step functions related to a general function $f(t)$

5.3 Sub-linear convergence

Now, let us introduce the following piecewise linear in time vector field $\mathbf{h}_n$ ($i = 1,\dots,n$) see Figure 5.4

$$\begin{aligned}\mathbf{h}_n(0) &= \mathbf{H}_0,\\ \mathbf{h}_n(t) &= \mathbf{h}_{i-1} + (t-t_{i-1})\delta\mathbf{h}_i \qquad \text{for } t\in(t_{i-1},t_i],\end{aligned}$$

and the step vector fields $\overline{\mathbf{h}}_n, \overline{\mathbf{m}}_n$ and $\overline{\partial_t \mathbf{m}}_n$

$$\begin{array}{llll} \overline{\mathbf{h}}_n(0) & = \mathbf{H}_0, & \overline{\mathbf{h}}_n(t) & = \mathbf{h}_i, \\ \overline{\mathbf{m}}_n(0) & = \mathbf{M}_0, & \overline{\mathbf{m}}_n(t) & = \mathbf{m}_i, \\ \overline{\partial_t \mathbf{m}}_n(0) & = \partial_t \mathbf{M}(0), & \overline{\partial_t \mathbf{m}}_n(t) & = \partial_t \mathbf{m}(t_i), \qquad \text{for } t \in (t_{i-1}, t_i]. \end{array}$$

Further we define the vector field $\mathbf{m}_n$ as follows

$$\mathbf{m}_n(t) = \mathbf{m}(t) \qquad \text{for } t \in [t_{i-1}, t_i]$$

and for all $i = 1, \ldots, n$.

Using the new notation we rewrite (5.10) into the following form, which is more convenient for our purposes

$$(\partial_t \mathbf{h}_n, \boldsymbol{\varphi}) + \left(\nabla \times \overline{\mathbf{h}}_n, \nabla \times \boldsymbol{\varphi}\right) = -\left(\overline{\partial_t \mathbf{m}}_n, \boldsymbol{\varphi}\right) \tag{5.15}$$

for any $\boldsymbol{\varphi} \in \mathbf{V}$.

Now, we are in a position to derive the error estimates for the linear approximation scheme (5.9), (5.10). We use the standard proof-technique for parabolic equations. The only difficulty will be the handling of the right-hand side.

Theorem 5.1 *There exist positive constants C and τ_0 such that*

(i) $\displaystyle \max_{t\in[0,T]} \|\mathbf{H}(t) - \mathbf{h}_n(t)\|^2 + \int_0^T \left\|\nabla \times \left[\mathbf{H} - \overline{\mathbf{h}}_n\right]\right\|^2 \leq C\tau,$

(ii) $\displaystyle \max_{t\in[0,T]} \|\mathbf{M}(t) - \mathbf{m}_n(t)\|^2 + \int_0^T \|\partial_t \mathbf{M} - \partial_t \mathbf{m}_n\|^2 \leq C\tau,$

hold for any $0 < \tau < \tau_0$.

Proof:
(i) Using the definitions of the vector fields $\mathbf{M}$ and $\mathbf{m}_n$ we can write for any time t

$$\begin{aligned} \partial_t \mathbf{M}(t) \quad & -\partial_t \mathbf{m}_n(t) = [\mathbf{H}(t) + P(\mathbf{M}(t))] \times \mathbf{M}(t) \\ & + \frac{\mathbf{M}(t)}{|\mathbf{M}(t)|} \times ([\mathbf{H}(t) + P(\mathbf{M}(t))] \times \mathbf{M}(t)) \\ & - \left[\overline{\mathbf{h}}_n(t-\tau) + P(\overline{\mathbf{m}}_n(t-\tau))\right] \times \mathbf{m}_n(t) \\ & - \frac{\mathbf{m}_n(t)}{|\mathbf{m}_n(t)|} \times \left(\left[\overline{\mathbf{h}}_n(t-\tau) + P(\overline{\mathbf{m}}_n(t-\tau))\right] \times \overline{\mathbf{m}}_n(t-\tau)\right) \\ & = R_1 + R_2 + R_3 + R_4 + R_5, \end{aligned} \tag{5.16}$$

where

$$\begin{aligned}
R_1 &= [\mathbf{H}(t) + P(\mathbf{M}(t))] \times (\mathbf{M}(t) - \mathbf{m}_n(t)) \\
R_2 &= \left([\mathbf{H}(t) + P(\mathbf{M}(t))] - \left[\overline{\mathbf{h}}_n(t-\tau) + P(\overline{\mathbf{m}}_n(t-\tau))\right]\right) \times \mathbf{m}_n(t) \\
R_3 &= \frac{\mathbf{M}(t)}{|\mathbf{M}(t)|} \times ([\mathbf{H}(t) + P(\mathbf{M}(t))] \times [\mathbf{M}(t) - \overline{\mathbf{m}}_n(t-\tau)]) \\
R_4 &= \frac{\mathbf{M}(t) - \mathbf{m}_n(t)}{|\mathbf{M}(t)|} \times ([\mathbf{H}(t) + P(\mathbf{M}(t))] \times \overline{\mathbf{m}}_n(t-\tau)) \\
R_5 &= \frac{\mathbf{m}_n(t)}{|\mathbf{M}(t)|} \times ([\mathbf{H}(t) + P(\mathbf{M}(t))] \times \overline{\mathbf{m}}_n(t-\tau)) \\
&\quad - \frac{\mathbf{m}_n(t)}{|\mathbf{M}(t)|} \times \left(\left[\overline{\mathbf{h}}_n(t-\tau) + P(\overline{\mathbf{m}}_n(t-\tau))\right] \times \overline{\mathbf{m}}_n(t-\tau)\right).
\end{aligned}$$

Taking into account the fact that both $\mathbf{H}$ and $\mathbf{M}$ are bounded in $L_\infty\left((0,T)\times\Omega\right)$, we get in a straightforward way

$$\begin{aligned}
|\partial_t \mathbf{M}(t) - \partial_t \mathbf{m}_n(t)| &\leq C\left(|\mathbf{H}(t) - \overline{\mathbf{h}}_n(t-\tau)| + |\mathbf{M}(t) - \mathbf{m}_n(t)|\right) \\
&\quad + C|\mathbf{M}(t) - \overline{\mathbf{m}}_n(t-\tau)| \\
&\leq C\left(|\mathbf{H}(t) - \mathbf{h}_n(t)| + |\mathbf{h}_n(t) - \overline{\mathbf{h}}_n(t-\tau)|\right) \\
&\quad + C\left(|\mathbf{M}(t) - \mathbf{m}_n(t)| + |\mathbf{m}_n(t) - \overline{\mathbf{m}}_n(t-\tau)|\right).
\end{aligned} \tag{5.17}$$

The fields $\mathbf{m}_n$ and $\mathbf{M}$ are continuous in time and they start from the same initial datum $\mathbf{M}_0$. Therefore

$$\mathbf{M}(t) - \mathbf{m}_n(t) = \int_0^t \partial_t \mathbf{M} - \partial_t \mathbf{m}_n.$$

In virtue of (5.17) and Lemma 5.3 we can write

$$\begin{aligned}
\|\mathbf{M}(t) - \mathbf{m}_n(t)\| &\leq \int_0^t \|\partial_t \mathbf{M} - \partial_t \mathbf{m}_n\| \\
&\leq C \int_0^t \left(\tau + \|\mathbf{H} - \mathbf{h}_n\| + \tau\|\partial_t \mathbf{h}_n\| + \|\mathbf{M} - \mathbf{m}_n\|\right) \\
&\leq C\left(\tau + \int_0^t \left(\|\mathbf{H} - \mathbf{h}_n\| + \|\mathbf{M} - \mathbf{m}_n\|\right)\right),
\end{aligned}$$

because of the inequality

$$\|\mathbf{m}_n(t) - \overline{\mathbf{m}}_n(t-\tau)\| \leq \int_{t-\tau}^t \|\partial_t \mathbf{m}_n\| \leq C\tau.$$

Gronwall's argument gives

$$\|\mathbf{M}(t) - \mathbf{m}_n(t)\| \leq C\left(\tau + \int_0^t \|\mathbf{H} - \mathbf{h}_n\|\right). \tag{5.18}$$

This together with (5.17) give

$$\int_0^t \|\partial_t \mathbf{M} - \partial_t \mathbf{m}_n\|^2 \le C\left(\tau^2 + \int_0^t \|\mathbf{H} - \mathbf{h}_n\|^2\right). \tag{5.19}$$

Now, we subtract (5.15) from (5.8a), set $\boldsymbol{\varphi} = \mathbf{H} - \mathbf{h}_n$ and integrate the equation over the time interval $(0,t)$. We get

$$\begin{aligned}
&\tfrac{1}{2}\|\mathbf{H}(t) - \mathbf{h}_n(t)\|^2 + \int_0^t \left\|\nabla\times\left(\mathbf{H} - \overline{\mathbf{h}}_n\right)\right\|^2 \\
&= \int_0^t \left(\nabla\times\left(\mathbf{H} - \overline{\mathbf{h}}_n\right), \nabla\times\left(\mathbf{h}_n - \overline{\mathbf{h}}_n\right)\right) + \int_0^t \left(\overline{\partial_t \mathbf{m}}_n - \partial_t \mathbf{M}, \mathbf{H} - \mathbf{h}_n\right).
\end{aligned} \tag{5.20}$$

We estimate the first term on the right-hand side using Cauchy's and Young's inequalities and Lemma 5.3 as follows

$$\begin{aligned}
\int_0^t & \left(\nabla\times\left(\mathbf{H} - \overline{\mathbf{h}}_n\right), \nabla\times\left(\mathbf{h}_n - \overline{\mathbf{h}}_n\right)\right) \\
&\le \int_0^t \left\|\nabla\times\left(\mathbf{H} - \overline{\mathbf{h}}_n\right)\right\| \left\|\nabla\times\left(\mathbf{h}_n - \overline{\mathbf{h}}_n\right)\right\| \\
&\le \epsilon \int_0^t \left\|\nabla\times\left(\mathbf{H} - \overline{\mathbf{h}}_n\right)\right\|^2 + C_\epsilon \int_0^t \left\|\nabla\times\left(\mathbf{h}_n - \overline{\mathbf{h}}_n\right)\right\|^2 \\
&\le \epsilon \int_0^t \left\|\nabla\times\left(\mathbf{H} - \overline{\mathbf{h}}_n\right)\right\|^2 + C_\epsilon \tau.
\end{aligned} \tag{5.21}$$

For any $t \in [t_{i-1}, t_i]$ we can write

$$\left\|\overline{\partial_t \mathbf{m}}_n(t) - \partial_t \mathbf{M}(t)\right\| \le \|\partial_t \mathbf{m}_n(t_i) - \partial_t \mathbf{M}(t_i)\| + \|\partial_t \mathbf{M}(t_i) - \partial_t \mathbf{M}(t)\|. \tag{5.22}$$

The difference $\partial_t \mathbf{M}(t_i) - \partial_t \mathbf{M}(t)$ can be estimated using [59, Lemma 2.2] due to the assumption $\mathbf{H} \in L_\infty((0,T)\times\Omega)$, namely,

$$\begin{aligned}
\|\partial_t \mathbf{M}(t_i) - \partial_t \mathbf{M}(t)\| &\le C\left(\|\mathbf{M}(t_i) - \mathbf{M}(t)\| + \|\mathbf{H}(t_i) - \mathbf{H}(t)\|\right) \\
&\le C\left(\tau + \int_{t_{i-1}}^{t_i} \|\partial_t \mathbf{H}\|\right).
\end{aligned}$$

For the term $\partial_t \mathbf{m}_n(t_i) - \partial_t \mathbf{M}(t_i)$ we apply (5.17), triangle inequality and (5.18)

$$\begin{aligned}\|\partial_t \mathbf{m}_n(t_i) - \partial_t \mathbf{M}(t_i)\| &\le C\left(\|\mathbf{H}(t_i) - \mathbf{h}_n(t_{i-1})\| + \|\mathbf{M}(t_i) - \mathbf{m}_n(t_i)\|\right) \\ &+C\,\|\mathbf{m}(t_i) - \mathbf{m}_n(t_{i-1})\| \\ &\le C\left(\|\mathbf{H}(t) - \mathbf{h}_n(t)\| + \|\mathbf{m}(t_i) - \mathbf{m}_n(t_i)\|\right) \\ &+C\left(\tau + \int_{t_{i-1}}^{t_i} \|\partial_t \mathbf{H}\| + \int_{t_{i-1}}^{t_i} \|\partial_t \mathbf{h}_n\|\right) \\ &\le C\left(\|\mathbf{H}(t) - \mathbf{h}_n(t)\| + \int_0^t \|\mathbf{H} - \mathbf{h}_n\|\right) \\ &+C\left(\tau + \int_{t_{i-1}}^{t_i} \|\partial_t \mathbf{H}\| + \int_{t_{i-1}}^{t_i} \|\partial_t \mathbf{h}_n\|\right).\end{aligned}$$

Taking the second power in (5.22) and integrating over the time, we deduce

$$\int_0^t \left\|\overline{\partial_t \mathbf{m}}_n - \partial_t \mathbf{M}\right\|^2 \quad \le C\left(\tau^2 + \int_0^t \|\mathbf{H} - \mathbf{h}_n\|^2\right).$$

For the last term on the right in (5.20) we deduce using Cauchy's and Young's inequalities

$$\begin{aligned}\left|\int_0^t \left(\overline{\partial_t \mathbf{m}}_n - \partial_t \mathbf{M}, \mathbf{H} - \mathbf{h}_n\right)\right| &\le \int_0^t \left\|\overline{\partial_t \mathbf{m}}_n - \partial_t \mathbf{M}\right\|^2 + \int_0^t \|\mathbf{H} - \mathbf{h}_n\|^2 \\ &\le C\left(\tau^2 + \int_0^t \|\mathbf{H} - \mathbf{h}_n\|^2\right).\end{aligned} \tag{5.23}$$

Summarizing (5.20), (5.21), (5.23) and choosing a sufficiently small positive ϵ, we arrive at ($\tau < \tau_0 \le 1$)

$$\|\mathbf{H}(t) - \mathbf{h}_n(t)\|^2 + \int_0^t \left\|\nabla \times \left(\mathbf{H} - \overline{\mathbf{h}}_n\right)\right\|^2 \le C\left(\tau + \int_0^t \|\mathbf{H} - \mathbf{h}_n\|^2\right).$$

Applying the Gronwall's lemma, we conclude the proof.

(ii) The assertion immediately follows from the just proved part (i) and the relations (5.18), (5.19). □

5.4 Nonlinear approximation scheme

As a second possibility, we design the following recurrent nonlinear approximation scheme for $i = 1, \dots, n$

Algorithm 2 (nonlinear)

1. *We start from* $\mathbf{h}_{i-1}$ *and* $\mathbf{m}_{i-1}$ *taking into account* $\mathbf{h}_0 = \mathbf{H}_0$ *and* $\mathbf{m}_0 = \mathbf{M}_0$.

2. *We solve the quadratic ODE with an unknown* $\mathbf{m}(t)$ *on the subinterval* $[t_{i-1}, t_i]$

$$\begin{aligned} \partial_t \mathbf{m} &= [\mathbf{h}_{i-1} + P(\mathbf{m}_{i-1})] \times \mathbf{m} \\ &+ \frac{\mathbf{m}}{|\mathbf{m}|} \times ([\mathbf{h}_{i-1} + P(\mathbf{m}_{i-1})] \times \mathbf{m}) \,. \end{aligned} \tag{5.24}$$

3. *We set* $\mathbf{m}_i := \mathbf{m}(t_i)$.

4. *We solve the PDE for* $\mathbf{h}_i$

$$(\delta \mathbf{h}_i, \boldsymbol{\varphi}) + (\nabla \times \mathbf{h}_i, \nabla \times \boldsymbol{\varphi}) \quad = - (\partial_t \mathbf{m}(t_i), \boldsymbol{\varphi}) \tag{5.25}$$

for $\boldsymbol{\varphi} \in \mathbf{V}$.

The approximation of $\mathbf{M}$ is now nonlinear (compare with (5.9)). The conservation of the modulus for $\mathbf{m}$ can be proved exactly in the same way as it has been done for the linear algorithm.

Let $\mathbf{u}(t)$ be the solution of

$$\begin{aligned} \partial_t \mathbf{u} &= \mathbf{a} \times \mathbf{u} + c\mathbf{u} \times (\mathbf{a} \times \mathbf{u}) \qquad t > 0, \\ \mathbf{u}(0) &= \mathbf{u}_0. \end{aligned} \tag{5.26}$$

Due to the properties of any rotation $\mathcal{R}$ we have the identity

$$\mathcal{R}(\mathbf{x} \times \mathbf{y}) = \mathcal{R}\mathbf{x} \times \mathcal{R}\mathbf{y},$$

which is valid for any vectors $\mathbf{x}$ and $\mathbf{y}$. Therefore, we can write

$$\partial_t \mathcal{R}\mathbf{u} = \mathcal{R}\mathbf{a} \times \mathcal{R}\mathbf{u} + c\mathcal{R}\mathbf{u} \times (\mathcal{R}\mathbf{a} \times \mathcal{R}\mathbf{u}) \qquad t > 0$$

along with $\mathcal{R}\mathbf{u}(0) = \mathcal{R}\mathbf{u}_0$. Hence we see that it is enough to study the solvability of (5.26) for a vector $\mathbf{a}$ being parallel to $(1,0,0)^T$.

Let us fix any $\mathcal{X} \in \Omega$. The existence of $\mathbf{m}$ on $[t_{i-1}, t_i]$ follows from the next lemma. First, we set $\tilde{\mathbf{u}}_0 = \mathbf{m}_{i-1}$, $\tilde{\mathbf{a}} = \mathbf{h}_{i-1} + P(\mathbf{m}_{i-1})$ and $c = \frac{1}{|\mathbf{m}|}$, then we perform a rotation $\mathcal{R}$ of the coordinate system in such a way that $\mathbf{a} := \mathcal{R}(\tilde{\mathbf{a}}) = |\tilde{\mathbf{a}}|(1,0,0)$. Then we denote $\mathbf{u}_0 := \mathcal{R}(\tilde{\mathbf{u}}_0)$ and $a = |\tilde{\mathbf{a}}|$.

Lemma 5.4 *Let* $a, c \in \mathbb{R}$, $\mathbf{u}_0 = (x_0, y_0, z_0)^T$ *be any vector in* $\mathbb{R}^3$. *Then the solution* $\mathbf{u}(t) = (x(t), y(t), z(t))^T$ *of (5.26) for* $\mathbf{a} = a(1,0,0)^T$ *is given by*

$$\begin{aligned} x(t) &= |\mathbf{u}_0| \frac{e^{act|\mathbf{u}_0|}(|\mathbf{u}_0| + x_0) - e^{-act|\mathbf{u}_0|}(|\mathbf{u}_0| - x_0)}{e^{act|\mathbf{u}_0|}(|\mathbf{u}_0| + x_0) + e^{-act|\mathbf{u}_0|}(|\mathbf{u}_0| - x_0)}, \\ y(t) &= 2|\mathbf{u}_0| \frac{y_0 \cos(at) - z_0 \sin(at)}{e^{act|\mathbf{u}_0|}(|\mathbf{u}_0| + x_0) + e^{-act|\mathbf{u}_0|}(|\mathbf{u}_0| - x_0)}, \\ z(t) &= 2|\mathbf{u}_0| \frac{y_0 \sin(at) + z_0 \cos(at)}{e^{act|\mathbf{u}_0|}(|\mathbf{u}_0| + x_0) + e^{-act|\mathbf{u}_0|}(|\mathbf{u}_0| - x_0)}. \end{aligned}$$

Proof:
Let us denote by $R = |\mathbf{u}_0| = \sqrt{x_0^2 + y_0^2 + z_0^2}$. A scalar multiplication of (5.26) by $\mathbf{u}(t)$ gives

$$\langle \partial_t \mathbf{u}(t), \mathbf{u}(t) \rangle = \tfrac{1}{2}\partial_t |\mathbf{u}(t)|^2 = 0,$$

which after the time integration yields $|\mathbf{u}(t)| = R$ for all $t > 0$.

Further, a simple computation implies

$$\mathbf{a} \times \mathbf{u} = (a, 0, 0)^T \times (x, y, z)^T = a(0, -z, y)^T$$

and

$$\mathbf{u} \times (\mathbf{a} \times \mathbf{u}) = a(y^2 + z^2, -xy, -xz)^T.$$

Therefore, (5.26) for the x-coordinate reads as

$$\begin{aligned} \partial_t x &= ac(y^2 + z^2) = ac(R^2 - x^2), \\ x(0) &= x_0. \end{aligned}$$

This ordinary differential equation can be explicitly solved. We demonstrate it for $|x| < R$. The case $|x| = R$ is trivial. Thus, we can write

$$\partial_t x \quad \left(\frac{1}{R - x} + \frac{1}{R + x} \right) = 2acR.$$

We integrate this equation over $(0, t)$ and get

$$\ln \frac{R + x(t)}{R - x(t)} = \ln \frac{R + x_0}{R - x_0} + 2acRt,$$

or an equivalent form

$$\frac{R + x(t)}{R - x(t)} = \frac{R + x_0}{R - x_0} \; e^{2acRt}.$$

The solution of this algebraic equation is

$$x(t) = R \frac{e^{acRt}(R + x_0) - e^{-acRt}(R - x_0)}{e^{acRt}(R + x_0) + e^{-acRt}(R - x_0)}.$$

Once we have the formula for the x-coordinate, we have to solve the system of ordinary differential equations for the y- and z-coordinate, which has the form

$$\begin{aligned} \partial_t y &= -az - acxy, \\ \partial_t z &= ay - acxz, \end{aligned}$$

along with the starting data (y_0, z_0). This system can also be explicitly solved, e.g., by MAPLE, which gives the solution of the form

$$\begin{aligned} y(t) &= 2R\frac{y_0\cos(at) - z_0\sin(at)}{e^{acRt}(R+x_0) + e^{-acRt}(R-x_0)}, \\ z(t) &= 2R\frac{y_0\sin(at) + z_0\cos(at)}{e^{acRt}(R+x_0) + e^{-acRt}(R-x_0)}. \end{aligned}$$

□

Further we follow the same way as we did for the linear algorithm. We can show the existence of all $\mathbf{h}_i$ for $i = 1, \ldots, n$ and we can get the same a priori estimates as in Lemmas 5.2 and 5.3. The following theorem derives the error estimates for the nonlinear algorithm.

Theorem 5.2 *There exist positive constants C and τ_0 such that*

(i) $$\max_{t\in[0,T]} \|\mathbf{H}(t) - \mathbf{h}_n(t)\|^2 + \int_0^T \left\|\nabla \times \left[\mathbf{H} - \overline{\mathbf{h}}_n\right]\right\|^2 \le C\tau,$$

(ii) $$\max_{t\in[0,T]} \|\mathbf{M}(t) - \mathbf{m}_n(t)\|^2 + \int_0^T \|\partial_t\mathbf{M} - \partial_t\mathbf{m}_n\|^2 \le C\tau,$$

hold for any $0 < \tau < \tau_0$.

PROOF:
We use the definitions of the vector fields $\mathbf{m}$ and $\mathbf{m}_n$ and we write for any time t

$$\begin{aligned} \partial_t\mathbf{M}(t) \quad & -\partial_t\mathbf{m}_n(t) = [\mathbf{H}(t) + P(\mathbf{M}(t))] \times \mathbf{M}(t) \\ & + \frac{\mathbf{M}(t)}{|\mathbf{M}(t)|} \times ([\mathbf{H}(t) + P(\mathbf{M}(t))] \times \mathbf{M}(t)) \\ & - \left[\overline{\mathbf{h}}_n(t-\tau) + P(\overline{\mathbf{m}}_n(t-\tau))\right] \times \mathbf{m}_n(t) \\ & - \frac{\mathbf{m}_n(t)}{|\mathbf{m}_n(t)|} \times \left(\left[\overline{\mathbf{h}}_n(t-\tau) + P(\overline{\mathbf{m}}_n(t-\tau))\right] \times \mathbf{m}_n(t)\right) \\ & = \tilde{R}_1 + \tilde{R}_2 + \tilde{R}_3 + \tilde{R}_4 + \tilde{R}_5, \end{aligned} \tag{5.27}$$

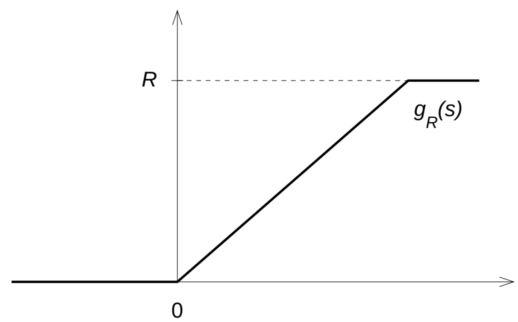


Figure 5.5: Definition of a cut-off function g_R

where

$$
\begin{aligned}
\tilde{R}_1 &= [\mathbf{H}(t) + P(\mathbf{M}(t))] \times (\mathbf{M}(t) - \mathbf{m}_n(t)) \\
\tilde{R}_2 &= \left([\mathbf{H}(t) + P(\mathbf{M}(t))] - \left[\overline{\mathbf{h}}_n(t-\tau) + P(\overline{\mathbf{m}}_n(t-\tau))\right]\right) \times \mathbf{m}_n(t) \\
\tilde{R}_3 &= \frac{\mathbf{M}(t)}{|\mathbf{M}(t)|} \times ([\mathbf{H}(t) + P(\mathbf{M}(t))] \times [\mathbf{M}(t) - \mathbf{m}_n(t)]) \\
\tilde{R}_4 &= \frac{\mathbf{M}(t) - \mathbf{m}_n(t)}{|\mathbf{M}(t)|} \times ([\mathbf{H}(t) + P(\mathbf{M}(t))] \times \mathbf{m}_n(t)) \\
\tilde{R}_5 &= \frac{\mathbf{m}_n(t)}{|\mathbf{M}(t)|} \times ([\mathbf{H}(t) + P(\mathbf{M}(t))] \times \mathbf{m}_n(t)) \\
&\quad - \frac{\mathbf{m}_n(t)}{|\mathbf{M}(t)|} \times \left(\left[\overline{\mathbf{h}}_n(t-\tau) + P(\overline{\mathbf{m}}_n(t-\tau))\right] \times \mathbf{m}_n(t)\right).
\end{aligned}
$$

Further we follow exactly the same line as in the proof of Theorem 5.1, therefore we omit the rest. □

5.5 Linear convergence

Up to now, we have proved in Theorems 5.1 and 5.2 a sub-linear convergence of the algorithms, see [72]. As for the linear algorithm also for the nonlinear one we are able to improve error estimates and obtain linear convergence. This result was published in [28]. We denote by R upper bound for $\mathbf{H}$ satisfying

$$|\mathbf{H}(t, \mathbf{x})| \leq R \qquad \text{a.e. in } Q_T.$$

We introduce a real function g_R, see Figure 5.5, as

$$g_R(s) = \begin{cases} 0 & \text{for } s < 0, \\ \min(s, R) & \text{else,} \end{cases}$$

and the vector function

$$\mathbf{w}(\mathbf{H}) = \frac{\mathbf{H}}{|\mathbf{H}|} g_R(|\mathbf{H}|).$$

Using this cut-off function we can slightly change the problem in weak formulation such that its solution thanks its boundedness does not change. First we modify right hand side

$$f_R(\mathbf{H}, \mathbf{M}) = \frac{|\gamma|}{1+\alpha^2} \left(\mathbf{w}(\mathbf{H}_{\mathrm{eff}}) \times \mathbf{M} + \alpha \frac{\mathbf{M}}{|\mathbf{M}|} \times (\mathbf{w}(\mathbf{H}_{\mathrm{eff}}) \times \mathbf{M}) \right). \tag{5.28}$$

And now the system:

$$\begin{aligned} (\partial_t \mathbf{H}, \boldsymbol{\varphi}) + (\nabla \times \mathbf{H}, \nabla \times \boldsymbol{\varphi}) &= -(\partial_t \mathbf{M}, \boldsymbol{\varphi}), & (5.29)\\ (\partial_t \mathbf{M}, \boldsymbol{\psi}) &= (f_R(\mathbf{H}, \mathbf{M}), \boldsymbol{\psi}), & (5.30) \end{aligned}$$

for any $\boldsymbol{\varphi} \in \mathbf{V}$ and any $\boldsymbol{\psi} \in \mathbf{W}$.

Both algorithms will change only in the step 2. So in Algorithm 1 will be the equation (5.9) replaced by

$$\begin{aligned} \partial_t \mathbf{m} &= \mathbf{w}\left(\mathbf{h}_{i-1} + P(\mathbf{m}_{i-1})\right) \times \mathbf{m} \\ &\quad + \frac{\mathbf{m}}{|\mathbf{m}_{i-1}|} \times \left(\mathbf{w}\left(\mathbf{h}_{i-1} + P(\mathbf{m}_{i-1})\right) \times \mathbf{m}_{i-1}\right). \end{aligned} \tag{5.31}$$

In Algorithm 2 will be the equation (5.24) replaced by

$$\begin{aligned} \partial_t \mathbf{m} &= \mathbf{w}\left(\mathbf{h}_{i-1} + P(\mathbf{m}_{i-1})\right) \times \mathbf{m} \\ &\quad + \frac{\mathbf{m}}{|\mathbf{m}_{i-1}|} \times \left(\mathbf{w}\left(\mathbf{h}_{i-1} + P(\mathbf{m}_{i-1})\right) \times \mathbf{m}\right). \end{aligned}$$

We will prove the following lemmas.

Lemma 5.5

$$\begin{aligned} |\partial_t \mathbf{m}(t)| &\leq C & (5.32)\\ |\partial_t (\mathbf{m}(t_i) - \mathbf{m}(t_{i-1}))| &\leq |\mathbf{h}_{i-1} - \mathbf{h}_{i-2}| + \tau C. & (5.33) \end{aligned}$$

PROOF:
First statement can be directly verified from (5.31) taking into account the boundedness of $\mathbf{m}$ and $\mathbf{w}(\mathbf{h}_i)$. Then

$$\begin{aligned} \left|\partial_t \left(\mathbf{m}(t_i) - \mathbf{m}(t_{i-1})\right)\right| \leq\ & \big|\mathbf{w}(\mathbf{h}_{i-1} + P(\mathbf{m}_{i-1})) \times \mathbf{m}(t_i) \\ & - \mathbf{w}(\mathbf{h}_{i-2} + P(\mathbf{m}_{i-2})) \times \mathbf{m}(t_{i-1})\big| \\ & + \left| \frac{\mathbf{m}(t_i)}{|\mathbf{m}(t_{i-1})|} \times (\mathbf{w}(\mathbf{h}_{i-1} + P(\mathbf{m}_{i-1})) \times \mathbf{m}(t_{i-1})) \right. \\ & \left. - \frac{\mathbf{m}(t_i - 1)}{|\mathbf{m}(t_{i-2})|} \times (\mathbf{w}(\mathbf{h}_{i-2} + P(\mathbf{m}_{i-2})) \times \mathbf{m}(t_{i-2})) \right|. \end{aligned}$$

We have boundedness of $\mathbf{w}(\mathbf{h}_i + P(\mathbf{m}_{i-1}))$. Together with the fact that $\mathbf{m}_i$ is bounded too we have

$$\left|\partial_t \left(\mathbf{m}(t_i) - \mathbf{m}(t_{i-1})\right)\right| \leq \left|\mathbf{h}_{i-1} - \mathbf{h}_{i-2}\right| + \left|\mathbf{m}_i - \mathbf{m}_{i-1}\right| + \left|\mathbf{m}_{i-1} - \mathbf{m}_{i-2}\right|. \quad (5.34)$$

We rewrite two last terms for $k \in \{i, i-1\}$

$$\mathbf{m}_k - \mathbf{m}_{k-1} = \int_{t_{k-1}}^{t_k} \partial_t \mathbf{m} \leq \tau C,$$

because of the first statement of the lemma. It concludes the proof. □

In the next we need the following compatibility condition:

$$(\partial_t \mathbf{H}(0), \boldsymbol{\varphi}) + (\nabla \times \mathbf{H}_0, \nabla \times \boldsymbol{\varphi}) = -(\partial_t \mathbf{M}(0), \boldsymbol{\varphi}),$$

for any $\varphi \in \mathbf{V}$. It means that Maxwell's equations are satisfied in time $t = 0$.

Lemma 5.6 *If the compatibility condition is satisfied and $\nabla \times \mathbf{H}_0 \in \mathbf{V}$ then the following estimate holds*

$$||\partial_t \mathbf{H}(0)|| \leq C$$

for some positive C.

Proof:
Take (5.29) for $t = 0$ and set $\varphi = \nabla \times \nabla \times \mathbf{H}(0)$. Then we get

$$\partial_t ||\nabla \times \mathbf{H}_0||^2 + ||\nabla \times \nabla \times \mathbf{H}_0||^2 = -(\partial_t \mathbf{M}(0), \nabla \times \nabla \times \mathbf{H}_0),$$

and subsequently

$$\begin{aligned} \left|\partial_t ||\nabla \times \mathbf{H}_0||^2\right| &\leq ||\nabla \times \nabla \times \mathbf{H}_0||^2 + ||\partial_t \mathbf{M}(0)||\ ||\nabla \times \nabla \times \mathbf{H}_0|| \\ &\leq \frac{3}{2} ||\nabla \times \nabla \times \mathbf{H}_0||^2 + C. \end{aligned}$$

We have used Young's inequality. Now we set $\varphi = \partial_t \mathbf{H}_0$ in (5.29) and use again Young's inequality and previous to obtain

$$\begin{aligned} ||\partial_t \mathbf{H}_0||^2 + \partial_t ||\nabla \times \mathbf{H}_0||^2 &= -(\partial_t \mathbf{M}(0), \partial_t \mathbf{H}_0), \\ ||\partial_t \mathbf{H}_0||^2 &\leq \left|\partial_t ||\nabla \times \mathbf{H}_0||^2\right| + \frac{1}{2} ||\partial_t \mathbf{M}(0)||^2 + \frac{1}{2} ||\partial_t \mathbf{H}_0||^2 \\ &\leq 3 ||\nabla \times \nabla \times \mathbf{H}_0||^2 + C. \end{aligned}$$

The last statement concludes the proof of lemma. □

Lemma 5.7 *Let $j \in \{1, \dots, n\}$. Then there exists a positive constant C such that*

$$||\delta \mathbf{h}_j|| + \sum_{i=1}^{j} ||\delta \mathbf{h}_i - \delta \mathbf{h}_{i-1}|| + \sum_{i=1}^{j} ||\nabla \times \delta \mathbf{h}_i||^2 \tau \leq C$$

PROOF:
If we define $\delta \mathbf{h}_0 = \partial_t \mathbf{H}(0)$, the equation (5.10) is valid also for $i = 0$ because of the compatibility conditions. We take (5.10) for i and $i-1$ and we subtract both equations. We set $\varphi = \delta \mathbf{h}_i \tau$ and then we get

$$\begin{aligned} &(\delta \mathbf{h}_i - \delta \mathbf{h}_{i-1}, \delta \mathbf{h}_i)\tau + (\nabla \times (\mathbf{h}_i - \mathbf{h}_{i-1}), \nabla \times (\mathbf{h}_i - \mathbf{h}_{i-1})) \\ &\quad = -(\partial_t(\mathbf{m}(t_i) - \mathbf{m}(t_{i-1})), \mathbf{h}_i - \mathbf{h}_{i-1}) \end{aligned}$$

Applying

$$(\delta \mathbf{h}_i - \delta \mathbf{h}_{i-1}, \delta \mathbf{h}_i) = 1/2\big(||\delta \mathbf{h}_i||^2 - ||\delta \mathbf{h}_{i-1}||^2 + ||\delta \mathbf{h}_i - \delta \mathbf{h}_{i-1}||^2\big)$$

we get

$$\begin{aligned} &\frac{\tau}{2}\left[||\delta \mathbf{h}_i||^2 - ||\delta \mathbf{h}_{i-1}||^2\right] + \frac{\tau}{2}||\delta \mathbf{h}_i - \delta \mathbf{h}_{i-1}||^2 + ||\nabla \times (\mathbf{h}_i - \mathbf{h}_{i-1})||^2 \\ &\quad \leq ||\partial_t(\mathbf{m}(t_i) - \mathbf{m}(t_{i-1}))|| \, ||\mathbf{h}_i - \mathbf{h}_{i-1})||. \end{aligned} \tag{5.35}$$

We sum up (5.35) for $i = 1, \dots, j$, then we apply Lemma 5.5, (5.33) and Young's inequality to obtain

$$\begin{aligned} &\sum_{i=1}^{j} ||\partial_t(\mathbf{m}(t_i) - \mathbf{m}(t_{i-1}))|| \, ||\mathbf{h}_i - \mathbf{h}_{i-1}|| \\ &\quad \leq \sum_{i=1}^{j} \big(\tau C + ||\mathbf{h}_{i-1} - \mathbf{h}_{i-2}||\big)||\mathbf{h}_i - \mathbf{h}_{i-1}|| \\ &\quad \leq \sum_{i=1}^{j} \tau C||\mathbf{h}_i - \mathbf{h}_{i-1}|| + \sum_{i=1}^{j} ||\mathbf{h}_{i-1} - \mathbf{h}_{i-2}|| \, ||\mathbf{h}_i - \mathbf{h}_{i-1}|| \\ &\quad \leq \frac{1}{2}\sum_{i=1}^{j} \tau^2 C^2 + \frac{1}{2}\sum_{i=1}^{j} ||\mathbf{h}_i - \mathbf{h}_{i-1}||^2 + \sum_{i=1}^{j} ||\mathbf{h}_i - \mathbf{h}_{i-1}||^2 \\ &\quad \leq \tau C^2 + \frac{3}{2}\sum_{i=1}^{j} ||\mathbf{h}_i - \mathbf{h}_{i-1}||^2 = \tau C^2 + \frac{3}{2}\tau^2 \sum_{i=1}^{j} ||\delta \mathbf{h}_i||^2. \end{aligned}$$

Together with (5.35) we have

$$\frac{\tau}{2}\sum_{i=1}^{j}\left[||\delta \mathbf{h}_i||^2 - ||\delta \mathbf{h}_{i-1}||^2\right] + \sum_{i=1}^{j}\frac{\tau}{2}\left[||\delta \mathbf{h}_i - \delta \mathbf{h}_{i-1}||^2\right] + \tau^2 \sum_{i=1}^{j} ||\nabla \times \delta \mathbf{h}_i||^2$$

$$\leq \tau C^2 + \frac{3}{2}\tau^2 \sum_{i=1}^{j} ||\delta \mathbf{h}_i||^2.$$

After division by $\tau/2$ we arrive to

$$||\delta \mathbf{h}_j||^2 + \sum_{i=1}^{j} ||\delta \mathbf{h}_i - \delta \mathbf{h}_{i-1}||^2 + 2\tau \sum_{i=1}^{j} ||\nabla \times \delta \mathbf{h}_i||^2 \leq 2C^2 + 4\tau \sum_{i=1}^{j} ||\delta \mathbf{h}_i||^2.$$

The desired result comes from Gronwall's lemma. □

Lemma 5.8 *Let $j \in \{1, \ldots, n\}$. Then there exists a positive constant C such that*

$$\sum_{i=1}^{j} ||\delta \mathbf{h}_i - \delta \mathbf{h}_{i-1}||^2 + \frac{\tau}{2}||\nabla \times \delta \mathbf{h}_j||^2 + \sum_{i=1}^{j} \frac{\tau}{2}||\nabla \times (\delta \mathbf{h}_i - \delta \mathbf{h}_{i-1})||^2$$
$$\leq \tau C + \frac{\tau}{2}||\nabla \times \delta \mathbf{h}_1||^2$$

PROOF:
We take (5.10) for i and $i-1$ and we subtract both equations. We set $\varphi = \delta \mathbf{h}_i - \delta \mathbf{h}_{i-1}$ and then we get

$$||\delta \mathbf{h}_i - \delta \mathbf{h}_{i-1}||^2 + \tau(\nabla \times \delta \mathbf{h}_i, \nabla \times (\delta \mathbf{h}_i - \delta \mathbf{h}_{i-1}))$$
$$= -(\partial_t(\mathbf{m}(t_i) - \mathbf{m}(t_{i-1})), \delta \mathbf{h}_i - \delta \mathbf{h}_{i-1})$$

Using similar technique for the second term on the left side as in the proof of Lemma 5.5 and applying results from Lemma 5.5, (5.33) and Lemma 5.7 we arrive at

$$||\delta \mathbf{h}_i - \delta \mathbf{h}_{i-1}||^2 + \frac{\tau}{2}(||\nabla \times \delta \mathbf{h}_i||^2 - ||\nabla \times \delta \mathbf{h}_{i-1}||^2)$$
$$+\frac{\tau}{2}||\nabla \times (\delta \mathbf{h}_i - \delta \mathbf{h}_{i-1})||^2 \leq \big(\tau C + \tau||\delta \mathbf{h}_{i-1}||\big)||\delta \mathbf{h}_i - \delta \mathbf{h}_{i-1}||$$
$$\leq \tau C||\delta \mathbf{h}_i - \delta \mathbf{h}_{i-1}||$$

Summing up for $i = 1, \ldots, j$ and applying result from Lemma 5.7 we get

$$\sum_{i=1}^{j} ||\delta \mathbf{h}_i - \delta \mathbf{h}_{i-1}||^2 + \frac{\tau}{2}||\nabla \times \delta \mathbf{h}_j||^2 + \sum_{i=1}^{j} \frac{\tau}{2}||\nabla \times (\delta \mathbf{h}_i - \delta \mathbf{h}_{i-1})||^2$$
$$\leq \tau C \sum_{i=1}^{j} ||\delta \mathbf{h}_i - \delta \mathbf{h}_{i-1}|| + \frac{\tau}{2}||\nabla \times \delta \mathbf{h}_1||^2 \leq \tau C + \frac{\tau}{2}||\nabla \times \delta \mathbf{h}_1||^2$$

It concludes the proof of lemma. □

Theorem 5.3 *There exist positive constants C and τ_0 such that*

$$\begin{aligned}\max_{t\in[0,T]}||\mathbf{H}(t)-\mathbf{h}_n(t)||^2 &\leq C\tau^2,\\ \max_{t\in[0,T]}||\mathbf{M}(t)-\mathbf{m}_n(t)||^2 &\leq C\tau^2.\end{aligned}$$

PROOF:
We use the same definitions for $\mathbf{m}_n(t), \overline{\mathbf{m}}_n(t), \mathbf{h}_n(t)$ and $\overline{\mathbf{h}}_n(t)$.

For any time we can write

$$\begin{aligned}|\partial_t\mathbf{M}(t)-\partial_t\mathbf{m}_n(t)| \leq\ & C\big(|\mathbf{H}(t)-\mathbf{h}_n(t)|+|\mathbf{h}_n(t)-\overline{\mathbf{h}}(t-\tau)|\\ &+|\mathbf{M}(t)-\mathbf{m}_n(t)|+|\mathbf{m}_n(t)-\overline{\mathbf{m}}(t-\tau)|\big).\end{aligned}$$

Following the steps in [72] we get

$$||\mathbf{M}(t)-\mathbf{m}_n(t)|| \leq C\left(\tau+\int_0^t||\mathbf{H}-\mathbf{h}_n||\right), \tag{5.36}$$

and

$$\int_0^t||\partial_t\mathbf{M}-\partial_t\mathbf{m}_n||^2 \leq C\left(\tau^2+\int_0^t||\mathbf{H}-\mathbf{h}_n||^2\right). \tag{5.37}$$

Next we subtract equations (5.29) and (5.10). We set $\varphi=\mathbf{H}-\mathbf{h}_n$ to obtain

$$\begin{aligned}&\frac{1}{2}||\mathbf{H}(t)-\mathbf{h}_n(t)||^2+\int_0^t||\nabla\times(\mathbf{H}-\overline{\mathbf{h}}_n)||^2\\ &\leq \left|\int_0^t(\nabla\times(\mathbf{H}-\overline{\mathbf{h}}_n),\nabla\times(\mathbf{h}_n-\overline{\mathbf{h}}_n))\right|\\ &\quad+\left|\int_0^t(\partial_t\overline{\mathbf{m}}_n-\partial_t\mathbf{M},\mathbf{H}-\mathbf{h}_n)\right|.\end{aligned} \tag{5.38}$$

We estimate the first term on the right hand side using Young's inequality:

$$\begin{aligned}&\left|\int_0^t(\nabla\times(\mathbf{H}-\overline{\mathbf{h}}_n),\nabla\times(\mathbf{h}_n-\overline{\mathbf{h}}_n))\right|\\ &\leq \int_0^t||\nabla\times(\mathbf{H}-\overline{\mathbf{h}}_n)||\;||\nabla\times(\mathbf{h}_n-\overline{\mathbf{h}}_n)||\\ &\leq \varepsilon\int_0^t||\nabla\times(\mathbf{H}-\overline{\mathbf{h}}_n)||^2+C_\varepsilon\int_0^t||\nabla\times(\mathbf{h}_n-\overline{\mathbf{h}}_n)||^2\\ &\leq \varepsilon\int_0^t||\nabla\times(\mathbf{H}-\overline{\mathbf{h}}_n)||^2+C_\varepsilon\tau^2.\end{aligned}$$

We have used the result of Lemma 5.7. We have to estimate the last term in (5.38):

$$\begin{aligned} ||\partial_t \overline{\mathbf{m}}_n - \partial_t \mathbf{M}|| &= ||\partial_t \mathbf{m}_n(t_i) - \partial_t \mathbf{M}|| \\ \leq \quad & ||\partial_t \mathbf{m}_n(t_i) - \partial_t \mathbf{m}_n(t)|| + ||\partial_t \mathbf{m}_n(t) - \partial_t \mathbf{M}(t)|| \end{aligned} \tag{5.39}$$

The first term on right-hand side includes two functions. They are both solutions of the same ODE, but taken in different times. That's why we have

$$||\partial_t \mathbf{m}_n(t_i) - \partial_t \mathbf{m}_n(t)|| \leq C||\mathbf{m}_n(t_i) - \mathbf{m}_n(t)|| \leq C\tau.$$

Now take the second power of (5.39) and integrate the result in time, then using (5.37) we get

$$\begin{aligned} \int_0^t ||\partial_t \overline{\mathbf{m}}_n - \partial_t \mathbf{M}||^2 \quad &\leq \quad \int_0^t C\tau^2 + \int_0^t ||\partial_t \mathbf{m}_n(s) - \partial_t \mathbf{M}(s)||^2 \\ &\leq \quad C\tau^2 + \int_0^t ||\mathbf{H} - \mathbf{h}_n||^2. \end{aligned}$$

Finally we are able to estimate the last term in (5.38)

$$\begin{aligned} \left| \int_0^t (\partial_t \overline{\mathbf{m}}_n - \partial_t \mathbf{M}, \mathbf{H} - \mathbf{h}_n) \right| \quad &\leq \quad \int_0^t ||\partial_t \overline{\mathbf{m}}_n - \partial_t \mathbf{M}||^2 + \int_0^t ||\mathbf{H} - \mathbf{h}_n||^2 \\ &\leq \quad C\tau^2 + C \int_0^t ||\mathbf{H} - \mathbf{h}_n||^2. \end{aligned}$$

We turn back to (5.38) by writing

$$\frac{1}{2}||\mathbf{H}(t) - \mathbf{h}_n(t)||^2 + \int_0^t ||\nabla \times (\mathbf{H} - \overline{\mathbf{h}}_n)||^2 \leq C_\varepsilon \tau^2 + c\tau^2 + C \int_0^t ||\mathbf{H} - \mathbf{h}_n||^2.$$

Gronwall's lemma concludes the proof of the first inequality from the theorem. The second inequality follows directly from the first one and (5.36). □

5.6 Numerical experiments

In this section we present two numerical examples. The first one simulates an applied situation, but here we do not have any exact solution. The second example with a prescribed solution demonstrates the convergence rates.

Example 1

Consider a ferromagnetic sample occupying a rectangular cuboid with the length $dl = 4$, the width $dw = 0.5$ and the height $dh = 0.5$. There is an electrical wire wrapped around it, see Figure 5.6. When the electrical current starts to flow through the wire, the induced electromagnetic field influences the magnetization $\mathbf{M}$. We apply the nonlinear algorithm in order to simulate the evolution of $\mathbf{M}$ and $\mathbf{H}$.

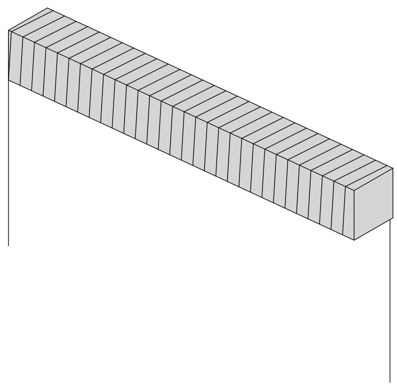


Figure 5.6: Model situation

First, we split the domain into small blocks with $\Delta x = 0.25$ and $\Delta y = \Delta z = \frac{0.5}{6}$. Then we divide each block into 6 tetrahedra. For the approximation of the magnetic field $\mathbf{H}$ we use Whitney edge elements, cf. Bossavit [15], Cessenat [20]. Recall that the approximation of $\mathbf{M}$ can be settled down at any point $\mathcal{X}$ from the approximation of the LLG equation, see (5.24). For computations we use the time step $\tau = 0.02$.

The material constants appearing in the problem setting are $\mu = \sigma = 1, \alpha = 0.3, \gamma = 1.5, K = 10$ and the easy magnetization axis is given by $\mathbf{p} = (1, 0, 0)^T$. The static magnetic field $\mathbf{H}_s$ vanishes and $\mathbf{J}_0 = \mathbf{0}$. Further we set $f = 0.5$ and $H_{amp} = 50$.

We consider the following boundary conditions for $\mathbf{H}$

Γ_D**:** $\mathbf{H}(t) = (H(t), 0, 0)^T$ for $H(t) = H_{amp}\cos(2\pi f t)$ on the long boundary parts (with the size $dl \times dw$ and $dl \times dh$). This boundary condition can be interpreted as $\boldsymbol{\nu} \times \mathbf{H} \times \boldsymbol{\nu} = 0$ because of $(\boldsymbol{\nu} \times \boldsymbol{\phi}) \cdot (\nabla \times \mathbf{H}) = [\boldsymbol{\nu} \times (\boldsymbol{\nu} \times \boldsymbol{\phi})] \cdot [\boldsymbol{\nu} \times (\nabla \times \mathbf{H})] = \boldsymbol{\phi} \cdot [(\nabla \times \mathbf{H}) \times \boldsymbol{\nu}]$.

Γ_N**:** $\boldsymbol{\nu} \times \mathbf{H} = \mathbf{0}$ on the small boundary parts (with the size $dh \times dw$).

Thus, the boundary conditions are periodical with the period $T_{per} = 2$. Initial data are $\mathbf{H}_0 = (H_{amp}, 0, 0)^T$ and $\mathbf{m}_0 = (0, 1, 0)^T$.

Consider a vertical cross-section S of the magnet through its barycenter. We define

$$\hat{M}(t) \quad = average\ M(t) \quad := \frac{1}{|S|}\int_S \mathbf{M}(t)\cdot\boldsymbol{\nu},$$
$$H_{int}(t) \quad = integral\ H(t) \quad := \int_0^t H(s)\ \mathrm{d}s,$$

where $|S|$ is the 2D-measure of S and $\boldsymbol{\nu}$ stands for the unit normal vector on S. As physically relevant curves characterizing the ferromagnetic material, we can take $\big(\hat{M}(t), H(t)\big)$- and $\big(\hat{M}(t), H_{int}(t)\big)$-loops, where $H(t)$ describes the x-coordinate of $\mathbf{H}$ on Γ_D. Figure 5.7 describes point-wise $\mathbf{H}$–$\mathbf{m}$ dependence and depicts $\big(\hat{M}(t), H(t)\big)$-loop. In Figure 5.8 you can see integral $\mathbf{H}$–$\mathbf{m}$ dependence represented by $\big(\hat{M}(t), H_{int}(t)\big)$-loop.

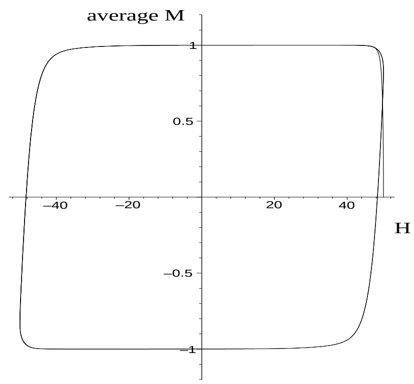


Figure 5.7: Point-wise $\mathbf{H}$–$\mathbf{m}$ dependence

Let us note that the area of such a loop describes the energy losses in the ferromagnet due to the hysteresis effects. In Figure 5.7 we arrive in a short time at a *stable regime*, which is represented by the closed curve.

Figure 5.9 shows the movement of $\mathbf{M}(t)$ at the barycenter of the ferromagnet in the stable regime. The length of $\mathbf{M}(t)$ remains constant, thus we associated the starting point of $\mathbf{m}(t)$ with the origin and the end point travels on the unit sphere. Gray arrows point out the end points of the trajectory, which is denoted by the bold curve.

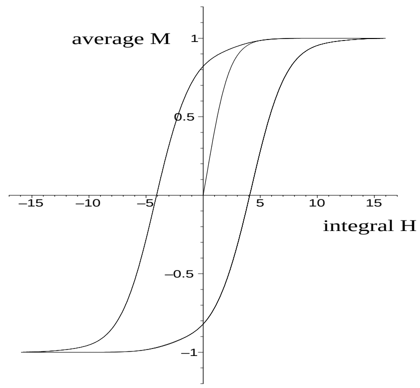


Figure 5.8: Integral **H**–**m** dependence

Example 2

We performed two computations on different domains. First, consider a ferromagnet of a cubic shape such that $\Omega_1 = [0,1]^3$. Second shape of the domain was taken $\Omega_2 = [0,1]^2 \times [0,3]$. We have computed following problems

$$\begin{aligned} \mu_0 \partial_t \mathbf{H} + \sigma^{-1} \nabla \times \nabla \times \mathbf{H} &= \mathbf{R}_i - \mu_0 \partial_t \mathbf{M}, \\ \partial_t \mathbf{M} &= f(\mathbf{H}, \mathbf{M}) + \mathbf{S}_i, \end{aligned} \tag{5.40}$$

for $i = 1, 2$ on the interval $(0, T)$ in the domain Ω_i, where the vector fields $\mathbf{R}_i$ and $\mathbf{S}_i$ are chosen in such a way that the exact solution takes the form

$$\begin{aligned} \mathbf{H}_{exact_1} &= 0.1 \sin(t) \begin{pmatrix} \sin(x) + 0.5\cos(y) + \sin(z) + \cos(z) \\ \sin(x) + \cos(x) + 2\cos(y) + 0.5\sin(z) \\ 0.5\cos(x) + \sin(y) + \cos(y) + 2\sin(z) \end{pmatrix}, \\ \mathbf{M}_{exact_1} &= \begin{pmatrix} \sin(|x|t)\cos(t) \\ \cos(|x|t)\cos(t) \\ \sin(t) \end{pmatrix}. \\ \mathbf{H}_{exact_2} &= 0.1 \sin(t) \begin{pmatrix} 0.5\cos(x_1) + 1.5\cos(x_2) + \sin(x_3) \\ \sin(x_1) + 2\cos(x_1) + 0.5\sin(x_3) \\ \sin(x_2) + 0.5\cos(x_3) \end{pmatrix}, \\ \mathbf{M}_{exact_2} &= \begin{pmatrix} \sin(|\mathbf{x}|t)\sin(t) \\ \cos(|\mathbf{x}|t)\sin(t) \\ \cos(t) \end{pmatrix}. \end{aligned}$$

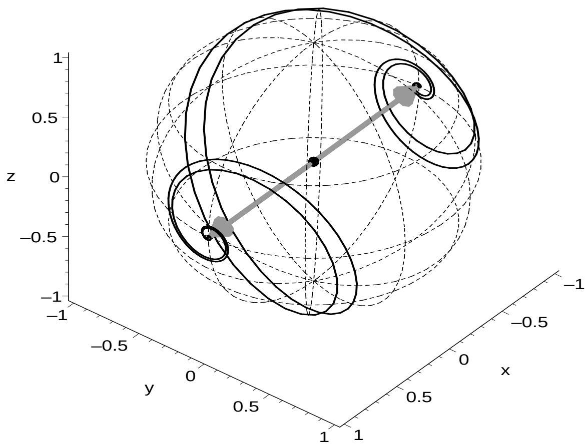


Figure 5.9: Time evolution of $\mathbf{m}$

We consider Neumann boundary conditions and use the following constants:

$$\alpha = \mu_0 = \sigma = \gamma = 1, K = 0, T = 0.5.$$

The results in Table 5.1 for problem considering $\mathbf{R}_1, \mathbf{S}_1, \Omega_1$ were obtained on a uniform grid with 16 939 unknowns associated with the Whitney edge elements. The results in Table 5.2 for problem considering $\mathbf{R}_2, \mathbf{S}_2, \Omega_2$ were obtained on a uniform grid with 11 457 unknowns. We have computed the absolute and the relative errors

$$\begin{aligned} e^{H,\tau}_{abs_i} &= \max_{[0,T]} |\mathbf{H}_{exact_i} - \mathbf{H}_{computed_i}| \\ e^{M,\tau}_{abs_i} &= \max_{[0,T]} |\mathbf{M}_{exact_i} - \mathbf{M}_{computed_i}| \\ e^{H,\tau}_{rel_i} &= \max_{[0,T]} \frac{|\mathbf{H}_{exact_i} - \mathbf{H}_{computed_i}|}{|\mathbf{H}_{exact_i}|} \\ e^{M,\tau}_{rel_i} &= \max_{[0,T]} \frac{|\mathbf{M}_{exact_i} - \mathbf{M}_{computed_i}|}{|\mathbf{M}_{exact_i}|} \end{aligned}$$

as well as the convergence rates

$$\omega^{H,\tau}_i = \frac{\log \left[\frac{e^{H,2\tau}_{abs_i}}{e^{H,\tau}_{abs_i}} \right]}{\log 2}, \qquad \omega^{M,\tau}_i = \frac{\log \left[\frac{e^{M,2\tau}_{abs_i}}{e^{M,\tau}_{abs_i}} \right]}{\log 2}.$$

We have used a modified linear Algorithm 1 in computations. The modification was caused due to the presence of the vector field $\mathbf{S}_i$ in the LL equation. This straightforward change was necessary, because we have prescribed an exact solution.

τ	$e^{H,\tau}_{rel_1}$ [%]	$e^{H,\tau}_{abs_1}$	$\omega^{H,\tau}_1$	$e^{M,\tau}_{rel_1}$ [%]	$e^{M,\tau}_{abs_1}$	$\omega^{M,\tau}_1$
0.1	28.53	0.135	-	2.48	0.061	-
0.05	16.21	0.0763	0.82	1.42	0.0348	0.81
0.025	7.22	0.0341	1.16	0.698	0.0171	1.02
0.0125	3.72	0.0176	0.95	0.363	0.0089	0.94
0.00625	2.07	0.0098	0.84	0.200	0.0049	0.86

Table 5.1: Absolute and relative errors for $\mathbf{R}_1, \mathbf{S}_1, \Omega_1$

τ	$e^{H,\tau}_{rel_2}$ [%]	$\omega^{H,\tau}_1$	$e^{M,\tau}_{rel_2}$ [%]	$\omega^{M,\tau}_1$
0.2	20.93		1.27	
0.1	5.41		0.36	
0.05	1.96		0.098	
0.025	0.72		0.032	
0.0125	0.35		0.0117	

Table 5.2: Relative errors for $\mathbf{R}_2, \mathbf{S}_2, \Omega_2$

We are not aware of any known example of (5.5) with a given exact solution, i.e., for $\mathbf{S}_i = \mathbf{0}$.

Inspecting Tables 5.1 and 5.2 we see that the actual convergence rates for the approximations of $\mathbf{H}$ and $\mathbf{M}$ correspond to the theoretical results obtained in Theorem 5.3.

We have not tested the nonlinear Algorithm 2 on an example with a prescribed solution. The reason is that Lemma 5.4 is valid for a homogeneous equation (5.26) and a generalization to a non-homogeneous case is not an easy matter.

6 Fixed point technique for higher convergence rate

(This paper gives wrong solutions to trivial problems. The basic error,however, is not new. Clifford Truesdell)

We suggest a new numerical algorithm (6.5) for computations of the single LL equation. In fact, we use relaxation iterations at each time point of a time partitioning. The basis for the development of (6.5) was a non-iterative algorithm (5.9) from Chapter 5, which was introduced in [71, 72]. We prove the convergence of iterations at each time step using a fixed point argument, see Lemma 6.1. In practice, the relaxation process stops when a given tolerance is achieved. We derive error estimates for our scheme in Theorem 6.2 taking into account the stopping criterion. At the end we present a numerical example with a known solution, where we confirm the theoretical results from Theorem 6.2, i.e., the second-order of convergence. Let us note that our algorithm conserves the modulus of $\mathbf{M}$, which is a very important feature from the physical point of view.

These results were published by author of this thesis and Slodička in [27].

6.1 Approximation scheme

For ease of exposition we again put $\alpha = K = 1$ and $\gamma = 2$ in the theoretical part of the chapter, but not in the numerical one.

The following error estimate has been proved for algorithm (5.9)

$$\max_{t\in[0,T]} \|\mathbf{M}(t) - \mathbf{m}_n(t)\|^2 + \int_0^T \|\partial_t \mathbf{M} - \partial_t \mathbf{m}_n\|^2 \leq C\tau^2, \tag{6.1}$$

for the vector field $\mathbf{m}_n$ defined as

$$\mathbf{m}_n(t) = \mathbf{m}(t) \qquad \text{for } t \in [t_{i-1}, t_i]$$

and for all $i = 1, \ldots, n$. The crucial assumption was $\mathbf{H} \in L_\infty((0,T) \times \Omega)$, which implies the global Lipschitz continuity of the right-hand side in (5.2) (see [59, Lemma 2.2]).

One can easily see that (5.9) preserves the modulus of $\mathbf{m}$. We recall that (5.9) admits a unique solution, which follows from Lemma 5.1 for

$$\mathbf{u}_0 = \mathbf{m}_{i-1} \quad \text{and} \quad \mathbf{a} = \mathbf{h}_{i-1} + P\mathbf{m}_{i-1} - [\mathbf{h}_{i-1} + P\mathbf{m}_{i-1}] \times \frac{\mathbf{m}_{i-1}}{|\mathbf{m}_{i-1}|}.$$

Iteration scheme

Throughout the rest of this chapter we assume that

$$\begin{array}{ccccc} \mathbf{H} & \in & C^2([0,T]), & & \\ 0 & < & c_0 \;<\; |\mathbf{M}_0| & < C. \end{array} \tag{6.2}$$

One can easily deduce from (5.2) and (5.3) that

$$\begin{aligned} |\mathbf{M}(t)| &\leq C, \\ |\partial_t \mathbf{M}| &\leq C\left(1 + \max_{[0,T]} |\mathbf{H}|\right) \leq C, \\ |\partial_{tt} \mathbf{M}| &\leq C\left(1 + \max_{[0,T]} |\mathbf{H}| + |\partial_t \mathbf{H}|\right) \leq C. \end{aligned} \tag{6.3}$$

Consider any sufficiently smooth function f on $[a,b]$. Let Qf be defined as a quadrature operator on $[a,b]$ satisfying

$$\int_a^b f = (b-a)\; Qf + \mathcal{O}\left((b-a)^3\right).$$

Simplest examples are (which are also considered in the proofs)

- $Qf = f\left(\frac{a+b}{2}\right)$
- $Qf = \frac{f(a)+f(b)}{2}$.

Thus, in both cases we have for any $i = 1, \ldots, n$

$$\begin{aligned} |f - Q_i f| &\leq C\tau, \\ \left|\int_{t_{i-1}}^{t_i} (f - Q_i f)\right| &\leq C\tau^3, \end{aligned} \tag{6.4}$$

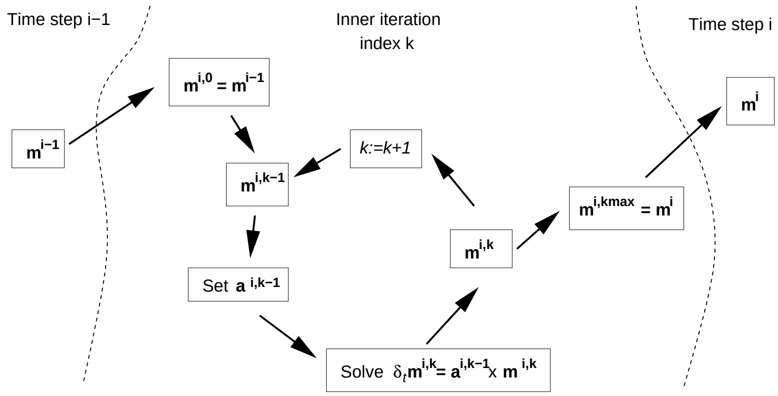


Figure 6.1: Inner iterations

where $Q_i f$ is the quadrature operator on $[t_{i-1}, t_i]$ and $f \in C^2([0,T])$.

We will modify the algorithm (5.9) in the following way, see Figure 6.1. The approximate solution $\mathbf{m}^{i,k} \approx \mathbf{M}$ on $[t_{i-1}, t_i]$ is obtained in an iteration process with respect to the relaxation parameter k. The linearized scheme for a fixed $i \in \{1, \dots, n\}$ and running $k = 1, \dots, k_{i,\max}$ reads as

$$\begin{aligned} \partial_t \mathbf{m}^{i,k} &= \left(Q_i \mathbf{H} + Q_i P \mathbf{m}^{i,k-1}\right) \times \mathbf{m}^{i,k} \\ &+ \frac{\mathbf{m}^{i,k}}{|\mathbf{m}^{i,k}|} \times \left[\left(Q_i \mathbf{H} + Q_i P \mathbf{m}^{i,k-1}\right) \times Q_i \mathbf{m}^{i,k-1}\right] \end{aligned} \tag{6.5}$$

for

$$\begin{aligned} \mathbf{m}^{i,0} &= \mathbf{m}^{i-1,k_{i-1,\max}}, \\ \mathbf{m}^{0,k} &= \mathbf{M}_0, \\ \mathbf{m}^{i,k}(t_{i-1}) &= \mathbf{m}^{i-1,k_{i-1,\max}}(t_{i-1}). \end{aligned} \tag{6.6}$$

The iteration process stops when the following condition is satisfied

$$|\mathbf{m}^{i,k}(t_i) - \mathbf{m}^{i,k-1}(t_i)| \le \tau^\beta \tag{6.7}$$

for a given β, which will be specified later.

A short inspection of (6.5) and (6.6) gives

$$\begin{aligned} |\mathbf{m}^{i,k}(t)| &= |\mathbf{m}^{i,k}(t_{i-1})| = |\mathbf{m}^{i-1,k_{i-1,\max}}(t_{i-1})| = \ldots = |\mathbf{M}_0| \le C, \\ |\partial_t \mathbf{m}^{i,k}| &\le C \left(1 + \max_{[0,T]} |\mathbf{H}|\right) \le C, \\ |\partial_{tt} \mathbf{m}^{i,k}| &\le C \left(1 + \max_{[0,T]} |\mathbf{H}|\right) |\partial_t \mathbf{m}^{i,k}| \le C. \end{aligned} \tag{6.8}$$

Auxiliary problem

Let us consider the following temporary nonlinear problem for $t \in [t_{i-1}, t_i]$

$$\begin{aligned} \partial_t \mathbf{u}^i &= \left(Q_i\mathbf{H} + Q_i P\mathbf{u}^i\right) \times \mathbf{u}^i + \frac{\mathbf{u}^i}{|\mathbf{u}^i|} \times \left[\left(Q_i\mathbf{H} + Q_i P\mathbf{u}^i\right) \times Q_i\mathbf{u}^i\right], \\ \mathbf{u}^i(t_{i-1}) &= \mathbf{m}^{i-1,k_{i-1,\max}}(t_{i-1}). \end{aligned} \tag{6.9}$$

One can easily deduce that

$$\begin{aligned} |\mathbf{u}^i(t)| &= |\mathbf{u}^i(t_{i-1})| = |\mathbf{m}^{i-1,k_{i-1,\max}}(t_{i-1})| = |\mathbf{M}_0| \le C, \\ |\partial_t \mathbf{u}^i| &\le C\left(1 + \max_{[0,T]}|\mathbf{H}|\right) \le C, \\ |\partial_{tt}\mathbf{u}^i| &\le C\left(1 + \max_{[0,T]}|\mathbf{H}|\right)|\partial_t\mathbf{u}^i| \le C. \end{aligned} \tag{6.10}$$

We will show that $\lim_{k\to\infty}\mathbf{m}^{i,k} = \mathbf{u}^i (= \mathbf{m}^{i,\infty})$, more exactly we prove the following lemma.

Lemma 6.1 (contraction) *There exist $\tau_0 > 0$ and $0 < q = q(\tau_0) < 1$ such that*

$$\max_{[t_{i-1},t_i]} |\mathbf{m}^{i,k} - \mathbf{u}^i| \le q \max_{[t_{i-1},t_i]} |\mathbf{m}^{i,k-1} - \mathbf{u}^i|$$

holds for any k, any i and any $\tau < \tau_0$.

Proof:
First, we denote

$$\begin{aligned} \mathbf{a}^{i,k} &= Q_i\mathbf{H} + Q_iP\mathbf{m}^{i,k} - \left(Q_i\mathbf{H} + Q_iP\mathbf{m}^{i,k}\right) \times \frac{Q_i\mathbf{m}^{i,k}}{|\mathbf{M}_0|}, \\ \mathbf{a}^i &= Q_i\mathbf{H} + Q_iP\mathbf{u}^i - \left(Q_i\mathbf{H} + Q_iP\mathbf{u}^i\right) \times \frac{Q_i\mathbf{u}^i}{|\mathbf{M}_0|}. \end{aligned}$$

Therefore, we have for the difference

$$\begin{aligned} \mathbf{a}^{i,k-1} - \mathbf{a}^i &= Q_iP(\mathbf{m}^{i,k-1} - \mathbf{u}^i) + \left(Q_i\mathbf{H} + Q_iP\mathbf{u}^i\right) \times \frac{Q_i(\mathbf{u}^i - \mathbf{m}^{i,k-1})}{|\mathbf{M}_0|} \\ &\quad - Q_iP(\mathbf{m}^{i,k-1} - \mathbf{u}^i) \times \frac{Q_i\mathbf{m}^{i,k-1}}{|\mathbf{M}_0|}. \end{aligned}$$

Using the triangle inequality and the definition of Q_i we deduce

$$\begin{aligned} |\mathbf{a}^{i,k-1} - \mathbf{a}^i| &\le |Q_iP(\mathbf{u}^i - \mathbf{m}^{i,k-1})| + \left|Q_i\left(\mathbf{H} + P\mathbf{u}^i\right)\right| \frac{|Q_i(\mathbf{u}^i - \mathbf{m}^{i,k-1})|}{|\mathbf{M}_0|} \\ &\quad + \left|Q_iP\left(\mathbf{m}^{i,k-1} - \mathbf{u}^i\right)\right| \\ &\le C \max_{[t_{i-1},t_i]} |\mathbf{u}^i - \mathbf{m}^{i,k-1}|, \end{aligned}$$

which implies

$$\max_{[t_{i-1},t_i]} |\mathbf{a}^{i,k-1} - \mathbf{a}^i| \leq C \max_{[t_{i-1},t_i]} |\mathbf{u}^i - \mathbf{m}^{i,k-1}|. \tag{6.11}$$

Using (6.5), (6.9) and applying the new notation we can write for $t \in [t_{i-1}, t_i]$

$$\begin{aligned} \partial_t(\mathbf{m}^{i,k} - \mathbf{u}^i) &= \mathbf{a}^{i,k-1} \times (\mathbf{m}^{i,k} - \mathbf{u}^i) + (\mathbf{a}^{i,k-1} - \mathbf{a}^i) \times \mathbf{u}^i, \\ (\mathbf{m}^{i,k} - \mathbf{u}^i)(t_{i-1}) &= \mathbf{0}. \end{aligned}$$

The semi-group theory gives

$$(\mathbf{m}^{i,k} - \mathbf{u}^i)(t) = \int_{t_{i-1}}^{t} e^{\mathbf{a}^{i,k-1}(t-s)} \times \left[(\mathbf{a}^{i,k-1} - \mathbf{a}^i(s)) \times \mathbf{u}^i(s)\right] \, \mathrm{d}s.$$

Hence, for the absolute value we deduce

$$\begin{aligned} |(\mathbf{m}^{i,k} - \mathbf{u}^i)(t)| &\leq \int_{t_{i-1}}^{t} \left| e^{\mathbf{a}^{i,k-1}(t-s)} \times \left[(\mathbf{a}^{i,k-1} - \mathbf{a}^i(s)) \times \mathbf{u}^i(s)\right] \right| \, \mathrm{d}s \\ &= \int_{t_{i-1}}^{t} \left| (\mathbf{a}^{i,k-1} - \mathbf{a}^i(s)) \times \mathbf{u}^i(s) \right| \, \mathrm{d}s \\ &\leq C\tau \max_{[t_{i-1},t_i]} |\mathbf{a}^{i,k-1} - \mathbf{a}^i|. \end{aligned}$$

This together with (6.11) yields

$$|(\mathbf{m}^{i,k} - \mathbf{u}^i)(t)| \leq C\tau \max_{[t_{i-1},t_i]} |\mathbf{u}^i - \mathbf{m}^{i,k-1}|$$

and

$$\max_{[t_{i-1},t_i]} |\mathbf{m}^{i,k} - \mathbf{u}^i| \leq C\tau \max_{[t_{i-1},t_i]} |\mathbf{u}^i - \mathbf{m}^{i,k-1}|.$$

Hence, for $\tau < \tau_0$ we conclude the proof. □

The vector field $\mathbf{u}^i$ is continuous in $[t_{i-1}, t_i]$, but there are discontinuities between $\mathbf{u}^i$ and $\mathbf{u}^{i+1}$

$$\begin{array}{ccc} \mathbf{u}^i(t_i) & \neq & \mathbf{u}^{i+1}(t_i) \\ \| & & \| \\ \mathbf{m}^{i,\infty}(t_i) & \neq & \mathbf{m}^{i,k_{i,\max}}(t_i). \end{array}$$

According to Lemma 6.1 and (6.7) we successively deduce

$$\begin{aligned} |\mathbf{m}^{i,k_{i,\max}}(t_i) - \mathbf{u}^i(t_i)| &\leq |\mathbf{m}^{i,k_{i,\max}}(t_i) - \mathbf{m}^{i,k_{i,\max}+1}(t_i)| \\ &\quad + |\mathbf{m}^{i,k_{i,\max}+1}(t_i) - \mathbf{u}^i(t_i)| \\ &\leq \tau^\beta + q|\mathbf{m}^{i,k_{i,\max}}(t_i) - \mathbf{u}^i(t_i)|, \end{aligned}$$

which implies

$$|\mathbf{u}^{i+1}(t_i) - \mathbf{u}^i(t_i)| = |\mathbf{m}^{i,k_{i,\max}}(t_i) - \mathbf{u}^i(t_i)| \leq \frac{\tau^\beta}{1-q} \leq C\tau^\beta. \tag{6.12}$$

This estimate will keep the discontinuity between $\mathbf{u}^{i+1}(t_i)$ and $\mathbf{u}^i(t_i)$ under control.

6.2 Error estimates

Let us denote by $\mathbf{a}$ the following expression

$$\mathbf{a} = \mathbf{H} + P\mathbf{M} - (\mathbf{H} + P\mathbf{M}) \times \frac{\mathbf{M}}{|\mathbf{M}|}.$$

Then, using (5.2), (6.9) and the new notation we obtain on $[t_{i-1}, t_i]$

$$\partial_t(\mathbf{M} - \mathbf{u}^i) = \mathbf{a}^i \times (\mathbf{M} - \mathbf{u}^i) + (\mathbf{a} - \mathbf{a}^i) \times \mathbf{M}, \tag{6.13}$$

where $\mathbf{a}^i$ was introduced in the proof of Lemma 6.1. We apply the semi-group theory and get

$$\begin{aligned}(\mathbf{M} - \mathbf{u}^i)(t) \quad &= e^{\mathbf{a}^i(t-t_{i-1})} \times (\mathbf{M} - \mathbf{u}^i)(t_{i-1}) \\ &+ \int_{t_{i-1}}^{t} e^{\mathbf{a}^i(t-s)} \times [(\mathbf{a} - \mathbf{a}^i) \times \mathbf{M}](s)\ \mathrm{d}s.\end{aligned} \tag{6.14}$$

Now, we are in a position to derive the error estimates for $\mathbf{M} - \mathbf{u}^i$.

Theorem 6.1 *Assume (6.2) and $\beta = 3$. Then there exist positive constants C and τ_0 such that*

$$|(\mathbf{M} - \mathbf{u}^i)(t_i)| \le C\tau^2$$

holds for any $1 \le i \le n$ and any $0 < \tau < \tau_0$.

PROOF:
We use (6.14) for $t = t_i$ and we apply the integration by parts formula to the integral term. We get

$$\begin{aligned}(\mathbf{M} - \mathbf{u}^i)(t_i) \quad &= e^{\mathbf{a}^i\tau} \times (\mathbf{M} - \mathbf{u}^i)(t_{i-1}) + \int_{t_{i-1}}^{t_i} (\mathbf{a} - \mathbf{a}^i) \times \mathbf{M} \\ &+ \int_{t_{i-1}}^{t_i} \mathbf{a}^i \times \left[e^{\mathbf{a}^i(t_i - s)} \times \int_{t_{i-1}}^{s} (\mathbf{a}(\xi) - \mathbf{a}^i) \times \mathbf{M}(\xi)\ \mathrm{d}\xi \right]\ \mathrm{d}s \\ &= A_1 + A_2 + A_3.\end{aligned} \tag{6.15}$$

For the first term we have

$$|A_1| = |(\mathbf{M} - \mathbf{u}^i)(t_{i-1})|. \tag{6.16}$$

The second term can be written as

$$\begin{aligned}A_2 \quad &= \int_{t_{i-1}}^{t_i} (\mathbf{a} - \mathbf{a}^i) \times \mathbf{M} \\ &= \int_{t_{i-1}}^{t_i} [\mathbf{a} \times \mathbf{M} - Q_i(\mathbf{a} \times \mathbf{M})] + \int_{t_{i-1}}^{t_i} (Q_i\mathbf{a} - \mathbf{a}^i) \times Q_i\mathbf{M} \\ &+ \int_{t_{i-1}}^{t_i} \mathbf{a}^i \times (Q_i\mathbf{M} - \mathbf{M}) + \int_{t_{i-1}}^{t_i} [Q_i(\mathbf{a} \times \mathbf{M}) - Q_i\mathbf{a} \times Q_i\mathbf{M}] \\ &= A_{21} + A_{22} + A_{23} + A_{24}.\end{aligned} \tag{6.17}$$

According to the properties of the quadrature operator Q_i - see (6.4) - we deduce

$$\begin{aligned} |A_{21}| &\leq C\tau^3, \\ |A_{23}| &= \left| \mathbf{a}^i \times \int_{t_{i-1}}^{t_i} (Q_i\mathbf{M} - \mathbf{M}) \right| \leq C\tau^3. \end{aligned} \tag{6.18}$$

In the case when $Q_i f = f\left(\frac{t_{i-1}+t_i}{2}\right)$, we have $A_{24} = 0$. For the second event $Q_i f = \frac{f(t_{i-1})+f(t_i)}{2}$, we have

$$Q_i\left(\mathbf{a} \times \mathbf{M}\right) = Q_i\mathbf{a} \times Q_i\mathbf{M} + \frac{\mathbf{a}(t_{i-1}) - \mathbf{a}(t_i)}{2} \times \frac{\mathbf{M}(t_{i-1}) - \mathbf{M}(t_i)}{2}.$$

Hence, we deduce

$$\begin{aligned} |A_{24}| &\leq \int_{t_{i-1}}^{t_i} |Q_i\left(\mathbf{a} \times \mathbf{M}\right) - Q_i\mathbf{a} \times Q_i\mathbf{M}| \\ &\leq C \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} |\partial_t \mathbf{a}| \int_{t_{i-1}}^{t_i} |\partial_t \mathbf{M}| \\ &\leq C\tau^3. \end{aligned} \tag{6.19}$$

We rewrite $\mathbf{a} - \mathbf{a}^i$ into a different form

$$\begin{aligned} \mathbf{a} - \mathbf{a}^i &= \mathbf{H} - Q_i\mathbf{H} - Q_iP\mathbf{u}^i + P\mathbf{M} \\ &\quad -(\mathbf{H} - Q_i\mathbf{H} - Q_iP\mathbf{u}^i + P\mathbf{M}) \times \frac{\mathbf{M}}{|\mathbf{M}_0|} \\ &\quad -(Q_i\mathbf{H} + Q_iP\mathbf{u}^i) \times \frac{\mathbf{M} - Q_i\mathbf{u}^i}{|\mathbf{M}_0|}. \end{aligned}$$

For the absolute value we successively deduce

$$\begin{aligned} |\mathbf{a} - \mathbf{a}^i| &\leq C\left(|\mathbf{H} - Q_i\mathbf{H}| + |P\mathbf{M} - Q_iP\mathbf{M}| + |Q_iP\mathbf{M} - Q_iP\mathbf{u}^i| \right. \\ &\quad \left. + |\mathbf{M} - Q_i\mathbf{M}| + |Q_i\mathbf{M} - Q_i\mathbf{u}^i|\right) \\ &\leq C\left(\tau + |Q_i\mathbf{M} - Q_i\mathbf{u}^i|\right) \\ &\leq C\left(\tau + |(\mathbf{M} - \mathbf{u}^i)(t_{i-1})| + \int_{t_{i-1}}^{t_i} |\partial_t(\mathbf{M} - \mathbf{u}^i)|\right) \\ &\leq C\left(\tau + |(\mathbf{M} - \mathbf{u}^i)(t_{i-1})|\right). \end{aligned} \tag{6.20}$$

Using (6.13) and (6.20) we obtain

$$\begin{aligned} |\partial_t(\mathbf{M} - \mathbf{u}^i)| &\leq C\left(\tau + |\mathbf{M} - \mathbf{u}^i| + |(\mathbf{M} - \mathbf{u}^i)(t_{i-1})|\right) \\ &\leq C\left(\tau + |(\mathbf{M} - \mathbf{u}^i)(t_{i-1})|\right). \end{aligned} \tag{6.21}$$

If $\mathbf{w}(t)$ and $\mathbf{v}(t)$ are any vector fields with bounded derivatives with respect to the time variable, then

$$Q_i(\mathbf{w} \times \mathbf{v}) - Q_i\mathbf{w} \times Q_i\mathbf{v} = \mathcal{O}\left(\tau^2\right).$$

Therefore, in virtue of the definition of $\mathbf{a}^i$ we can write

$$\begin{aligned} \mathbf{a}^i - Q_i\mathbf{a} \quad &= Q_i(P\mathbf{u}^i - P\mathbf{M}) \\ &+ Q_i\left[(P\mathbf{M} - P\mathbf{u}^i) \times \frac{\mathbf{M}}{|\mathbf{M}_0|} + (\mathbf{H} + P\mathbf{u}^i) \times \frac{\mathbf{M} - \mathbf{u}^i}{|\mathbf{M}_0|}\right] + \mathcal{O}\left(\tau^2\right) \end{aligned}$$

and for the absolute value we get

$$|\mathbf{a}^i - Q_i\mathbf{a}| \leq C|Q_i(\mathbf{M} - \mathbf{u}^i)| + C\tau^2.$$

According to this inequality and (6.21) we deduce for A_{22} the following

$$\begin{aligned} |A_{22}| \quad &\leq C \int_{t_{i-1}}^{t_i} |Q_i\mathbf{a} - \mathbf{a}^i)| \\ &\leq C \int_{t_{i-1}}^{t_i} |Q_i(\mathbf{M} - \mathbf{u}^i)| + C\tau^3 \\ &\leq C\tau \left(|(\mathbf{M} - \mathbf{u}^i)(t_{i-1})| + \int_{t_{i-1}}^{t_i} |\partial_t(\mathbf{M} - \mathbf{u}^i)| \right) + C\tau^3 \\ &\leq C\left(\tau^3 + \tau|(\mathbf{M} - \mathbf{u}^i)(t_{i-1})|\right). \end{aligned} \tag{6.22}$$

Finally, we have to estimate the term A_3. We use the relation (6.20) and obtain

$$|A_3| \quad \leq C \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{s} |\mathbf{a} - \mathbf{a}^i| \quad \leq C\left(\tau^3 + \tau^2|(\mathbf{M} - \mathbf{u}^i)(t_{i-1})|\right). \tag{6.23}$$

Collecting (6.15)-(6.19), (6.22), (6.23) and applying (6.12) we arrive at the recursion formula

$$\begin{aligned} |(\mathbf{M} - \mathbf{u}^i)(t_i)| \quad &\leq C\tau^3 + (1 + C\tau)|(\mathbf{M} - \mathbf{u}^i)(t_{i-1})| \\ &\leq C\tau^3 + (1 + C\tau)|(\mathbf{M} - \mathbf{u}^{i-1})(t_{i-1})| + C|(\mathbf{u}^i - \mathbf{u}^{i-1})(t_{i-1})| \\ &\leq C\tau^3 + (1 + C\tau)|(\mathbf{M} - \mathbf{u}^{i-1})(t_{i-1})|. \end{aligned}$$

This implies

$$|(\mathbf{M} - \mathbf{u}^i)(t_i)| \leq C\tau^3 \sum_{j=0}^{i-1} (1 + C\tau)^j \leq C\tau^2,$$

which concludes the proof. □

The following theorem shows the main result of this chapter.

Theorem 6.2 *Let the assumptions of Theorem 6.1 be fulfilled. Then there exist positive constants C and τ_0 such that*

(i)

$$|(\mathbf{M} - \mathbf{m}^{i,k_{i,\max}})(t_i)| \leq C\tau^2,$$

(ii)

$$|(\mathbf{M} - \mathbf{m}^{i,k_{i,\max}})(t)| \leq C\tau^2, \qquad t \in [0,T]$$

hold for any $1 \leq i \leq n$ and any $0 < \tau < \tau_0$.

PROOF:
(i) The assertion is a consequence of the triangle inequality, Theorem 6.1 and (6.12).
(ii) We subtract (6.5) from (5.2) and get

$$\partial_t(\mathbf{M} - \mathbf{m}^{i,k_{i,\max}}) = \mathbf{a}^{i,k_{i,\max}} \times (\mathbf{M} - \mathbf{m}^{i,k_{i,\max}}) + (\mathbf{a} - \mathbf{a}^{i,k_{i,\max}}) \times \mathbf{M}, \qquad (6.24)$$

where $\mathbf{a}^{i,k_{i,\max}}$ was introduced in the proof of Lemma 6.1. We rewrite $\mathbf{a} - \mathbf{a}^{i,k}$ into a different form

$$\begin{aligned} \mathbf{a} - \mathbf{a}^{i,k_{i,\max}} \quad &= \mathbf{H} - Q_i\mathbf{H} - Q_iP\mathbf{m}^{i,k_{i,\max}} + P\mathbf{M} \\ &-(\mathbf{H} - Q_i\mathbf{H} - Q_iP\mathbf{m}^{i,k_{i,\max}} + P\mathbf{M}) \times \frac{\mathbf{M}}{|\mathbf{M}_0|} \\ &-(Q_i\mathbf{H} + Q_iP\mathbf{m}^{i,k_{i,\max}}) \times \frac{\mathbf{M} - Q_i\mathbf{m}^{i,k_{i,\max}}}{|\mathbf{M}_0|}. \end{aligned}$$

For the absolute value we successively deduce

$$\begin{aligned} |\mathbf{a} - \mathbf{a}^{i,k_{i,\max}}| \quad &\leq C\,(|\mathbf{H} - Q_i\mathbf{H}| + |P\mathbf{M} - Q_iP\mathbf{M}| \\ &+|Q_iP\mathbf{M} - Q_iP\mathbf{m}^{i,k_{i,\max}}| \\ &+|\mathbf{M} - Q_i\mathbf{M}| + |Q_i\mathbf{M} - Q_i\mathbf{m}^{i,k_{i,\max}}|\big) \\ &\leq C\left(\tau + |Q_i\mathbf{M} - Q_i\mathbf{m}^{i,k_{i,\max}}|\right) \\ &\leq C\left(\tau + |(\mathbf{M} - \mathbf{m}^{i,k_{i,\max}})(t_{i-1})|\right) \\ &+C\int_{t_{i-1}}^{t_i} |\partial_t(\mathbf{M} - \mathbf{m}^{i,k_{i,\max}})| \\ &\leq C\left(\tau + |(\mathbf{M} - \mathbf{m}^{i,k_{i,\max}})(t_{i-1})|\right). \end{aligned} \qquad (6.25)$$

Using (6.24) and (6.25) we obtain

$$\begin{aligned} |\partial_t(\mathbf{M} - \mathbf{m}^{i,k_{i,\max}})| \quad &\leq C|\mathbf{M} - \mathbf{m}^{i,k_{i,\max}}| \\ &+ C\left(\tau + |(\mathbf{M} - \mathbf{m}^{i,k_{i,\max}})(t_{i-1})|\right) \\ &\leq C\left(\tau + |(\mathbf{M} - \mathbf{m}^{i,k_{i,\max}})(t_{i-1})|\right). \end{aligned} \qquad (6.26)$$

Finally, we apply the triangle inequality, (6.26), Theorem 6.2 (i) and we have for any $t \in [t_{i-1}, t_i]$

$$\begin{aligned}\left|(\mathbf{M} - \mathbf{m}^{i,k_{i,\max}})(t)\right| &= \left|(\mathbf{M} - \mathbf{m}^{i,k_{i,\max}})(t_{i-1}) + \int_{t_{i-1}}^{t} \partial_t \left(\mathbf{M} - \mathbf{m}^{i,k_{i,\max}}\right)\right| \\ &\leq \left|(\mathbf{M} - \mathbf{m}^{i,k_{i,\max}})(t_{i-1})\right| + \int_{t_{i-1}}^{t_i} \left|\partial_t \left(\mathbf{M} - \mathbf{m}^{i,k_{i,\max}}\right)\right| \\ &\leq C\tau^2,\end{aligned}$$

which concludes the proof. □

6.3 Numerical experiment

In this section we demonstrate on an example with an exact solution that the iteration scheme (6.5) really has a second order of convergence. The scheme (5.9) without iterations (which gave an theoretical background for the development of (6.5)) has only the first order of convergence.

Exact solution of the LL equation

We consider the following equation

$$\partial_t \mathbf{M} = -\mathbf{M} \times \mathbf{H}_{\text{eff}} - \alpha \mathbf{M} \times (\mathbf{M} \times \mathbf{H}_{\text{eff}}), \tag{6.27}$$

where $\mathbf{H}_{\text{eff}} = \mathbf{H}_{\text{app}} + \mathbf{H}_{\text{dem}} + \mathbf{H}_{\text{ani}}$. The applied field $\mathbf{H}_{\text{app}}$ is a given spatially uniform function in time and other terms represent the magneto-static and anisotropy fields, respectively. For numerical tests we have used an exact analytical solution introduced in [13], which was derived for non-conducting ferromagnetic bodies with a symmetry axis. Rotational symmetry, see Figure 6.2, leads to the following conditions

(i) The shape of a body is spheroidal with a symmetry axis along z.

(ii) The dissipative parameter α is a positively defined function of $\mathbf{H}_{\text{eff}}$, in our case identically equal to 1, and $\mathbf{M} = (m_x, m_y, m_z)$ invariant with respect to rotations of the reference frame around the z axis.

(iii) Crystal anisotropy is uniaxial with respect to the z axis, i.e., $\mathbf{H}_{\text{ani}} = (2K_1/\mu_0 \mathrm{M}^2) m_z \mathbf{e}_z$ ($\mathbf{e}_z$ is the unit vector along z).

(iv) The external field is of the form $\mathbf{H}_{\text{app}} = \mathbf{H}_{a\perp}(t) + h_{az}\mathbf{e}_z$. The component $h_{az}\mathbf{e}_z$ is constant in time, whereas $\mathbf{H}_{a\perp}(t)$ is a circularly rotated component with an angular frequency ω and a constant amplitude $h_{a\perp}$ perpendicular to the z axis.

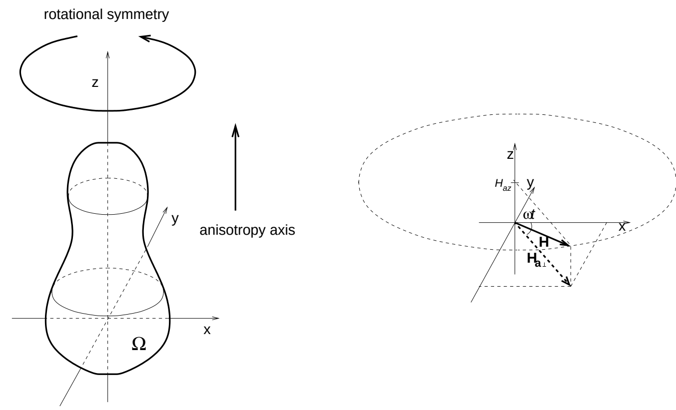


Figure 6.2: The setting for the numerical experiment

We consider the case with an demagnetizing term of the form $\mathbf{H}_{\text{dem}} = N_\perp \mathbf{M}_\perp - N_z m_z \mathbf{e}_z$, where N_z and $N_\perp$ represent the z axis and its perpendicular demagnetizing factors, respectively.

Therefore, the effective field takes the form

$$\mathbf{H}_{\text{eff}} = \mathbf{H}_{a\perp}(t) + (h_{az} + \kappa_{\text{eff}} m_z)\mathbf{e}_z,$$

where $\kappa_{\text{eff}} = 2K_1/\mu_0 M_s^2 + N_\perp - N_z$.

Because of the rotational symmetry it is much simpler to rewrite the equations in the terms of spherical coordinates. We introduce spherical coordinates with respect to rotated applied field $\mathbf{H}_{\text{app}}$ in such a way that this field remains constant. Let us denote by ϕ the lag of $\mathbf{M}_\perp$ with respect to $\mathbf{H}_{a\perp}$ and by θ the angle between $\mathbf{M}$ and the z axis. Thus, we are looking for $\mathbf{M}$ in the form $m_x = \sin\theta\cos(\omega t - \phi)$, $m_y = \sin\theta\sin(\omega t - \phi)$, $m_z = \cos\theta$. The equation (6.27) becomes the following form in terms of (θ, ϕ)

$$\begin{aligned} \partial_t\theta - \alpha\sin\theta\partial_t\phi &= \kappa_{\text{eff}}\left[b_\perp \sin\phi - B\sin\theta\right], \\ \alpha\partial_t\theta + \sin\theta\partial_t\phi &= \kappa_{\text{eff}}\left[b_\perp\cos\theta\cos\phi - (b_z + \cos\theta)\sin\theta\right], \end{aligned} \tag{6.28}$$

where $b_z = (h_{az} - \omega)/\kappa_{\text{eff}}$, $b_\perp = h_{a\perp}/\kappa_{\text{eff}}$ and $B = \alpha\omega/\kappa_{\text{eff}}$. We set the values in such a way that

$$\begin{aligned} b_z &= m_z(v-1), \\ |b_\perp| &= B\frac{\sin\theta}{\sin\phi}, \end{aligned}$$

where $v = B\cot\phi$.

Following the motivation in [13] it is easy to see, that the right-hand sides of (6.28) are equal to zero and functions ϕ and θ are constant. Thus, also the left-hand sides of (6.28) vanish and the equations are fulfilled.

τ	$k_{\max}=1$	$k_{\max}=2$	$k_{\max}=3$	$k_{\max}=4$
10^{-1}	3.66E−01	6.46E−02	2.15E−02	2.83E−02
10^{-2}	3.20E−03	9.82E−05	1.77E−04	1.76E−04
10^{-3}	3.01E−05	1.69E−06	1.76E−06	1.76E−06
10^{-4}	2.99E−07	1.75E−08	1.76E−08	1.76E−08
10^{-5}	2.99E−09	1.76E−10	1.76E−10	1.76E−10
10^{-6}	2.99E−11	1.77E−12	1.77E−12	1.77E−12

Table 6.1: Iteration scheme (6.5). Discrete error $\|\mathbf{M} - \mathbf{m}^{i,k_{\max}}\|_{L_1([0,T])}$.

τ	$\|\mathbf{M} - \mathbf{m}_n\|_{L_1([0,T])}$
10^{-1}	6.23E−01
10^{-2}	1.24E−01
10^{-3}	1.42E−02
10^{-4}	1.43E−03
10^{-5}	1.44E−04
10^{-6}	1.44E−05

Table 6.2: Scheme (5.9). Discrete error $\|\mathbf{M} - \mathbf{m}_n\|_{L_1([0,T])}$.

Numerical implementation

We have chosen parameters in the following order

1. arbitrary values of angles θ, ϕ and variables B, κ_{eff} and ω,
2. $b_z = (B\cot\phi - 1)\cos\theta$ and $b_\perp = |B\sin\theta\ (\sin\phi)^{-1}|$,
3. $h_{az} = b_z\kappa_{\text{eff}} + \omega$, $h_{a\perp} = \kappa_{\text{eff}} b_\perp$ and $\alpha = B\kappa_{\text{eff}}\omega^{-1}$.

Thus, we arrived at (6.27), where $\mathbf{H}_{\text{eff}}$ takes the form

$$\mathbf{H}_{\text{eff}} = h_{a\perp}\cos(\omega t)\mathbf{e}_x + h_{a\perp}\sin(\omega t)\mathbf{e}_y + (h_{az} + \kappa_{\text{eff}} m_z)\mathbf{e}_z,$$

with the exact analytical solution

$$\mathbf{M} = \sin\theta\ \cos(\omega t - \phi)\mathbf{e}_x + \sin\theta\ \sin(\omega t - \phi)\mathbf{e}_y + \cos\theta\ \mathbf{e}_z.$$

In the calculations we have used the values: $T = 2\pi$, $\theta = \frac{\pi}{3}$, $\phi = \frac{\pi}{4}$, $B = 1$, $\omega = 2$, $\kappa_{\text{eff}} = 1$. The quadrature operator Q was chosen as $Qf = \frac{f(a)+f(b)}{2}$ on any interval $[a, b]$.

We have performed computations for scheme (6.5) for a given number of iterations $k_{\max}$ at each time step t_i. The results for $\|\mathbf{M} - \mathbf{m}^{i,k_{\max}}\|_{L_1([0,T])}$ are shown in Table 6.1. Then, we have used the algorithm (5.9) for a comparison, see Table 6.2. We see that (5.9) has the first order of convergence, while (6.5) shows the second-order. Let us note that the length of $\mathbf{M}$ is conserved by computations, which is an important feature especially for engineers.

Further, we have performed computational tests with the stopping criterion (6.7) for $\beta = 3$. They showed that this condition was fulfilled for $k_{i,\max} = 2$ for all time points of the time partitioning and we also got the second-order of convergence.

For better visual comparison see Figure 6.3.

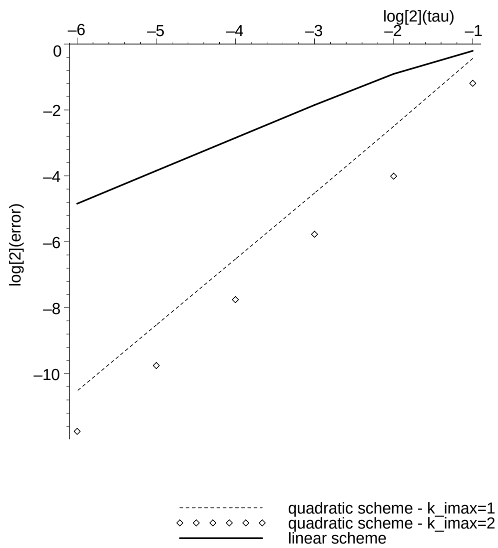


Figure 6.3: Comparison of quadratic and linear algorithm

III

Effective field with exchange

7 Regularity results for single LL equation

("Obvious" is the most dangerous word in mathematics. Eric Temple Bell)

This chapter is theoretical. Therefore we work with normalized magnetization

$$\mathbf{m} = \frac{\mathbf{M}}{|\mathrm{M}|}.$$

We consider single LL equation

$$\partial_t \mathbf{m} = -\gamma \mathbf{m} \times \mathbf{H}_{\text{eff}} - \alpha\gamma \mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{\text{eff}}), \quad \mathbf{m}|_{t=0} = \mathbf{m}_0, \tag{7.1}$$

where α is a positive constant, called Gilbert damping constant, and γ denotes the gyromagnetic factor. The unknown $\mathbf{m}$ stands for a spin vector of magnetization and $\mathbf{H}_{\text{eff}}$ denotes the effective field.

For the purposes of mathematical analysis we can skip the gyromagnetic factor γ. It disappears after a time rescaling of the LL equation.

For the sake of simplicity we consider the effective field $\mathbf{H}_{\text{eff}}$ of the form that corresponds to the pure isotropic case without any external field. Then, $\mathbf{H}_{\text{eff}} = \Delta\mathbf{m}$. However it would be no serious obstacle to add also terms describing anisotropy, magneto-static energy or applied magnetic field.

After these assumptions the problem we are interested in reads as

$$\partial_t \mathbf{m} = -\mathbf{m} \times \Delta\mathbf{m} - \alpha \mathbf{m} \times (\mathbf{m} \times \Delta\mathbf{m}) \qquad \text{in } \mathbb{R}^+ \times \Omega, \tag{7.2}$$

$$\frac{\partial \mathbf{m}}{\partial \boldsymbol{\nu}} = 0 \qquad \text{on } \mathbb{R}^+ \times \partial\Omega, \tag{7.3}$$

$$\mathbf{m}(0, .) = \mathbf{m}_0 \qquad \text{in } \Omega. \tag{7.4}$$

Simple multiplication of (7.2) gives us directly that the modulus of $\mathbf{m}$ remains constant in time. Thus, we can use throughout the text the inequality

$$\|\mathbf{m}\|_{L^\infty} \leq C. \tag{7.5}$$

Then, (7.2) according to [18] becomes equivalent to

$$\partial_t \mathbf{m} - \alpha \Delta \mathbf{m} = -\mathbf{m} \times \Delta \mathbf{m} + \alpha |\nabla \mathbf{m}|^2 \mathbf{m}. \tag{7.6}$$

Our aim in this section is to derive estimates for the time derivatives of the solution $\mathbf{m}$.

7.1 Regularity results

G. Carbou and P. Fabrie in [18] have proved the following result for the solution to the LL equation:

Theorem 7.1 *Supposing* $\mathbf{m}_0 \in W^{2,2}(\Omega)$, *there exists a positive* T_0 *such that for the solution* $\mathbf{m}$ *to the problem (7.2)-(7.4) the following estimate is valid:*

$$\max_{t \in (0,T_0)} \|\mathbf{m}(t)\|_{W^{2,2}(\Omega)} \leq C. \tag{7.7}$$

Remark 7.1 *Using (10.12) we can get*

$$\max_{t \in (0,T_0)} \|\nabla \mathbf{m}(t)\|_4 \quad \leq \quad C, \tag{7.8}$$

since $\mathbf{m}$ *is bounded in the* $W^{2,2}(\Omega)$ *norm.*

In [61] the author has derived and proved new-type regularity results for the exact solution of the LL equation introducing a time weight $\kappa(s) = \min\{1, s\}$. This weight helps to get the highly nonlinear terms under control at the beginning of the time interval on which we solve the problem (7.2)-(7.4). This reduces requirements on the regularity of initial data. The estimates were proved for the first and the second time derivatives of the exact solution, as well as for the gradient and the second space derivatives of the exact solution. The author obtained these regularity results in two dimensions.

We extend these results in the following way. We consider the problem in three space dimensions and we prove estimates for time derivatives $\partial_t^p \mathbf{m}, \nabla \partial_t^p \mathbf{m}$ and $\Delta \partial_t^p \mathbf{m}$ for arbitrary value of p, not only for $p = 1, 2$ as was considered for 2D in [61]. The author of this thesis and Van Keer deal with this problem in [29]. These results constitute a first step in higher order analysis of the LL equation in three dimensions.

We prove the following theorem.

Theorem 7.2 (Regularity theorem) *The solution* $\mathbf{m}$ *to the problem (7.2)-(7.4), taking* T_0 *from Theorem 7.1, satisfies the following estimates for any positive* $p \in \mathbb{Z}$

$$\max_{t\in(0,T_0)} \left\{\kappa^p \|\partial_t^{p+1}\mathbf{m}\|_2\right\} + \left(\int_0^{T_0} \kappa^{2p}\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2 \mathrm{d}s\right)^{\frac{1}{2}} \leq C(p,\alpha,\Omega), \tag{7.9}$$

$$\max_{t\in(0,T_0)} \left\{\kappa^p \|\Delta\partial_t^{p}\mathbf{m}\|_2\right\} \leq C(p,\alpha,\Omega), \tag{7.10}$$

$$\max_{t\in(0,T_0)} \left\{\kappa^{\frac{2p+1}{2}} \|\nabla\partial_t^{p+1}\mathbf{m}\|_2\right\} + \left(\int_0^{T_0} \kappa^{2p+1}\|\Delta\partial_t^{p+1}\mathbf{m}\|_2^2 \mathrm{d}s\right)^{\frac{1}{2}} \leq C(p,\alpha,\Omega), \tag{7.11}$$

$$\left(\int_0^{T_0} \kappa^{2p+1}\|\partial_t^{p+2}\mathbf{m}\|_2^2\right)^{\frac{1}{2}} \leq C(p,\alpha,\Omega), \tag{7.12}$$

where κ *is a time weight equal to* $\kappa(s) = \min\{1,s\}$ *and* $C(p,\alpha,\Omega)$ *is a constant depending only on* p,α *and* Ω.

From now on we do not explicitly write the dependence of the constant $C(p,\alpha,\Omega)$ on p,α and Ω.

We introduce similar theorem, which includes the statement of Theorem 7.2 for $p = 0$ and the inequality (7.15). It helps to keep the text more readable.

Theorem 7.3 *For the solution* $\mathbf{m}$ *to the problem (7.2)-(7.4), taking* T_0 *from Theorem 7.1, the following estimates are valid:*

$$\max_{t\in(0,T_0)} \|\partial_t\mathbf{m}\|_2 + \left(\int_0^{T_0} \|\nabla\partial_t\mathbf{m}\|_2^2 \mathrm{d}s\right)^{\frac{1}{2}} \leq C, \tag{7.13}$$

$$\max_{t\in(0,T_0)} \sqrt{\kappa}\|\nabla\partial_t\mathbf{m}\|_2 + \left(\int_0^{T_0} \kappa\left\{\|\partial_t^2\mathbf{m}\|_2^2 + \|\Delta\partial_t\mathbf{m}\|_2^2\right\} \mathrm{d}s\right)^{\frac{1}{2}} \leq C, \tag{7.14}$$

$$\int_0^{T_0} \|\partial_t^2\mathbf{m}\|_{W^{-1,2}} \mathrm{d}s \leq C, \tag{7.15}$$

where κ *is a time weight equal to* $\kappa(s) = \min\{1,s\}$.

Proof of inequality (7.13):
The time derivation of (7.6) leads to

$$\partial_t^2\mathbf{m} - \alpha\Delta\partial_t\mathbf{m} = 2\alpha\langle\nabla\mathbf{m},\nabla\partial_t\mathbf{m}\rangle_{\mathbb{R}^9}\mathbf{m} + \alpha|\nabla\mathbf{m}|^2\partial_t\mathbf{m} - \partial_t\mathbf{m}\times\Delta\mathbf{m} - \mathbf{m}\times\Delta\partial_t\mathbf{m}. \tag{7.16}$$

Now, we test this with the function $\partial_t \mathbf{m}$ to find

$$\frac{1}{2}\partial_t\|\partial_t\mathbf{m}\|_2^2+\alpha\|\nabla\partial_t\mathbf{m}\|_2^2=C(A_1+A_2+A_3+A_4), \tag{7.17}$$

where the definitions of $A_1,\dots,A_4$ are in the text below.

The integrals over the boundary always vanish thanks to the homogeneous Neumann boundary conditions.

We make use of Lemma 10.4 to estimate the terms A_1, A_2, A_3 and A_4. At the beginning of every estimate we use some of the inequalities stated in Lemma 10.3 and then we apply the estimate (7.7) or (7.8). Thus we have

$$\begin{aligned}|A_1|:=2|(\langle\nabla\mathbf{m},\nabla\partial_t\mathbf{m}\rangle_{\mathbb{R}^9}\mathbf{m},\partial_t\mathbf{m})| &\le C\|\nabla\mathbf{m}\|_4\|\nabla\partial_t\mathbf{m}\|_2\|\partial_t\mathbf{m}\|_4\|\mathbf{m}\|_{L^\infty}\\ &\le C\|\partial_t\mathbf{m}\|_{W^{1,2}}\|\partial_t\mathbf{m}\|_4. \end{aligned}\tag{7.18}$$

We apply the inequality $(a^2+b^2)^{\frac{1}{2}}\le a+b$ to the first factor. To estimate the L^4 norm of $\partial_t\mathbf{m}$ we use inequality (10.11) and then we separate the terms by the Young inequality with exponents 4 and 4/3 and appropriate weights ε and C_ε:

$$\|\partial_t\mathbf{m}\|_4\le C\|\partial_t\mathbf{m}\|_2+C\|\partial_t\mathbf{m}\|_2^{\frac{1}{4}}\|\nabla\partial_t\mathbf{m}\|_2^{\frac{3}{4}}\le C_\varepsilon\|\partial_t\mathbf{m}\|_2+\varepsilon\|\nabla\partial_t\mathbf{m}\|_2. \tag{7.19}$$

Together we get

$$|A_1|\le C(\|\partial_t\mathbf{m}\|_2+\|\nabla\partial_t\mathbf{m}\|_2)(C_\varepsilon\|\partial_t\mathbf{m}\|_2+\varepsilon\|\nabla\partial_t\mathbf{m}\|_2)\le C_\varepsilon\|\partial_t\mathbf{m}\|_2^2+\varepsilon\|\nabla\partial_t\mathbf{m}\|_2^2,$$

where ε is a generic small constant, which could be changed if needed. To estimate the term $\|\partial_t\mathbf{m}\|_4^2$ in A_2 we again use the same technique as we have used in (7.19). Then we get

$$\begin{aligned}|A_2|:=\alpha(|\nabla\mathbf{m}|^2,|\partial_t\mathbf{m}|^2) &\le C\|\nabla\mathbf{m}\|_4^2\|\partial_t\mathbf{m}\|_4^2\le C(C_\varepsilon\|\partial_t\mathbf{m}\|_2+\varepsilon\|\nabla\partial_t\mathbf{m}\|_2)^2\\ &\le C_\varepsilon\|\partial_t\mathbf{m}\|_2^2+\varepsilon\|\nabla\partial_t\mathbf{m}\|_2^2.\end{aligned}$$

For the term A_3, it is simply

$$|A_3|:=|(\partial_t\mathbf{m}\times\Delta\mathbf{m},\partial_t\mathbf{m})|=0.$$

In the following estimates we perform integration by parts. In virtue of homogeneous Neumann boundary conditions we can write

$$|A_4|:=|(\mathbf{m}\times\Delta\partial_t\mathbf{m},\partial_t\mathbf{m})|=|(\nabla(\mathbf{m}\times\partial_t\mathbf{m}),\nabla\partial_t\mathbf{m})|=|(\nabla\mathbf{m}\times\partial_t\mathbf{m},\nabla\partial_t\mathbf{m})|.$$

We make use of inequality (10.8) and we arrive to the following:

$$|A_4|\le\|\nabla\mathbf{m}\|_4\|\partial_t\mathbf{m}\|_4\|\nabla\partial_t\mathbf{m}\|_2\le C\|\partial_t\mathbf{m}\|_{W^{1,2}}\|\partial_t\mathbf{m}\|_4.$$

On the right-hand side of the inequality appears the same expression as in (7.18). Therefore we can directly end up at

$$|A_4| \le \|\nabla \mathbf{m}\|_4 \|\partial_t \mathbf{m}\|_4 \|\nabla \partial_t \mathbf{m}\|_2 \le C_\varepsilon \|\partial_t \mathbf{m}\|_2^2 + \varepsilon \|\nabla \partial_t \mathbf{m}\|_2^2.$$

Since we have successfully bounded the terms A_1, A_2, A_3 and A_4, we can now continue in (7.17) setting $3\varepsilon = \alpha/2$ to get

$$\begin{aligned} \frac{1}{2}\partial_t \|\partial_t \mathbf{m}\|_2^2 + \alpha \|\nabla \partial_t \mathbf{m}\|_2^2 &\le \frac{\alpha}{2}\|\nabla \partial_t \mathbf{m}\|_2^2 + 3C\|\partial_t \mathbf{m}\|_2^2, \\ \frac{1}{2}\partial_t \|\partial_t \mathbf{m}\|_2^2 + \frac{\alpha}{2} \|\nabla \partial_t \mathbf{m}\|_2^2 &\le 3C\|\partial_t \mathbf{m}\|_2^2. \end{aligned}$$

Using Gronwall's lemma in the previous relation we obtain the following result

$$\|\partial_t \mathbf{m}\|_2 + \left(\int_0^T \|\nabla \partial_t \mathbf{m}\|_2^2 \mathrm{d}s \right)^{\frac{1}{2}} \le C,$$

which completes the proof of inequality (7.13). □

Proof of inequality (7.14):

Now we begin the proof starting again with (7.16). We make a formal step by testing the relation (7.16) with function $-\Delta \partial_t \mathbf{m}$. This can be done rigorous by difference quotient method. We make similar steps also further. Because of vanishing Neumann boundary conditions we get

$$\frac{1}{2}\partial_t \|\nabla \partial_t \mathbf{m}\|_2^2 + \alpha \|\Delta \partial_t \mathbf{m}\|_2^2 \le C(B_1 + B_2 + B_3 + B_4). \tag{7.20}$$

We deal with terms B_1, B_2, B_3 and B_4 independently. First we have

$$\begin{aligned} B_1 &:= |(\langle \nabla \mathbf{m}, \nabla \partial_t \mathbf{m} \rangle_{\mathbb{R}^9} \mathbf{m}, -\Delta \partial_t \mathbf{m})| \\ &\le \|\nabla \mathbf{m}\|_4 \|\nabla \partial_t \mathbf{m}\|_4 \|\mathbf{m}\|_{L^\infty} \|\Delta \partial_t \mathbf{m}\|_2 \\ &\le (\|\partial_t \mathbf{m}\|_4 + \|\nabla \partial_t \mathbf{m}\|_4) \|\Delta \partial_t \mathbf{m}\|_2. \end{aligned} \tag{7.21}$$

To estimate L^4 norms, we apply inequalities (10.11) and (10.12) and then we separate the individual L^2 norms by applying the Young inequality with exponents 4 and 4/3 and appropriate weights ε and C_ε:

$$\begin{aligned} \|\partial_t \mathbf{m}\|_4 + \|\nabla \partial_t \mathbf{m}\|_4 &\le C(\|\partial_t \mathbf{m}\|_2 + \|\partial_t \mathbf{m}\|_2^{\frac{1}{4}} \|\nabla \partial_t \mathbf{m}\|_2^{\frac{3}{4}} \\ &\quad + \|\nabla \partial_t \mathbf{m}\|_2 + \|\nabla \partial_t \mathbf{m}\|_2^{\frac{1}{4}} \|\Delta \partial_t \mathbf{m}\|_2^{\frac{3}{4}}) \\ &\le C_\varepsilon \|\partial_t \mathbf{m}\|_2 + C_\varepsilon \|\nabla \partial_t \mathbf{m}\|_2 + \varepsilon \|\Delta \partial_t \mathbf{m}\|_2. \end{aligned} \tag{7.22}$$

We can now continue in (7.21) using two times the weighted Young inequality to conclude

$$\begin{aligned} B_1 &\le (C_\varepsilon\|\partial_t\mathbf{m}\|_2 + C_\varepsilon\|\nabla\partial_t\mathbf{m}\|_2 + \varepsilon\|\Delta\partial_t\mathbf{m}\|_2)\|\Delta\partial_t\mathbf{m}\|_2 \\ &\le C_\varepsilon\|\partial_t\mathbf{m}\|_2^2 + C_\varepsilon\|\nabla\partial_t\mathbf{m}\|_2^2 + \varepsilon\|\Delta\partial_t\mathbf{m}\|_2^2. \end{aligned}$$

We proceed with estimate for B_2:

$$\begin{aligned} B_2 := \alpha|(|\nabla\mathbf{m}|^2\partial_t\mathbf{m}, -\Delta\partial_t\mathbf{m})| &\le C\|\nabla\mathbf{m}\|_4^2\|\partial_t\mathbf{m}\|_{L^\infty}\|\Delta\partial_t\mathbf{m}\|_2 \\ &\le C\|\partial_t\mathbf{m}\|_{L^\infty}\|\Delta\partial_t\mathbf{m}\|_2. \end{aligned}$$

Now we make use of inequality (10.14) to arrive at the same situation as in (7.21):

$$B_2 \le C(|\partial_t\mathbf{m}\|_4 + \|\nabla\partial_t\mathbf{m}\|_4)\|\Delta\partial_t\mathbf{m}\|_2 \le C_\varepsilon\|\partial_t\mathbf{m}\|_2^2 + C_\varepsilon\|\nabla\partial_t\mathbf{m}\|_2^2 + \varepsilon\|\Delta\partial_t\mathbf{m}\|_2^2.$$

To estimate B_3, we use the same technique as for B_2:

$$\begin{aligned} B_3 := |(\partial_t\mathbf{m}\times\Delta\mathbf{m}, -\Delta\partial_t\mathbf{m})| &\le \|\partial_t\mathbf{m}\|_{L^\infty}\|\Delta\mathbf{m}\|_2\|\Delta\partial_t\mathbf{m}\|_2 \le \|\partial_t\mathbf{m}\|_{L^\infty}\|\Delta\partial_t\mathbf{m}\|_2 \\ &\le C(|\partial_t\mathbf{m}\|_4 + \|\nabla\partial_t\mathbf{m}\|_4)\|\Delta\partial_t\mathbf{m}\|_2 \\ &\le C_\varepsilon\|\partial_t\mathbf{m}\|_2^2 + C_\varepsilon\|\nabla\partial_t\mathbf{m}\|_2^2 + \varepsilon\|\Delta\partial_t\mathbf{m}\|_2^2. \end{aligned}$$

Finally, for B_4 we have simply

$$B_4 := |(\mathbf{m}\times\Delta\partial_t\mathbf{m}, -\Delta\partial_t\mathbf{m})| = 0.$$

After applying these estimates, we continue in (7.20) setting $3\varepsilon = \alpha/2$:

$$\frac{1}{2}\partial_t\|\nabla\partial_t\mathbf{m}\|_2^2 + \alpha\|\Delta\partial_t\mathbf{m}\|_2^2 \le \frac{\alpha}{2}\|\Delta\partial_t\mathbf{m}\|_2^2 + C\|\nabla\partial_t\mathbf{m}\|_2^2 + C\|\partial_t\mathbf{m}\|_2^2.$$

We can get rid of the coefficients $1/2$ and α by dividing the equation by $\min\{1/2, \alpha\}$. Applying the inequality (7.13) and multiplying by the time weight $\kappa(s) = \min(1, s)$ leads to

$$\kappa(s)\partial_t\|\nabla\partial_t\mathbf{m}\|_2^2 + \alpha\kappa(s)\|\Delta\partial_t\mathbf{m}\|_2^2 \le C_\alpha\kappa(s) + C_\alpha\kappa(s)\|\nabla\partial_t\mathbf{m}\|_2^2.$$

We integrate both sides of the previous equation. Integration by parts gives

$$\begin{aligned} &\Big[\kappa(s)\|\nabla\partial_t\mathbf{m}(s)\|_2^2\Big]_0^t - \int_0^t(\kappa'(s))\|\nabla\partial_t\mathbf{m}\|_2^2 + \int_0^t\kappa(s)\|\Delta\partial_t\mathbf{m}\|_2^2 \\ &\le C_\varepsilon\int_0^t\kappa(s) + C_\varepsilon\int_0^t\kappa(s)\|\nabla\partial_t\mathbf{m}\|_2^2. \end{aligned}$$

Using $\kappa'(s) \le 1$ we have

$$\begin{aligned}&\kappa(t)\|\nabla\partial_t \mathbf{m}(t)\|_2^2 + \int_0^t \kappa(s)\|\Delta\partial_t\mathbf{m}\|_2^2\\ &\le \int_0^t \|\nabla\partial_t\mathbf{m}\|_2^2 + C_\alpha + C_\alpha\int_0^t \kappa(s)\|\nabla\partial_t\mathbf{m}\|_2^2.\end{aligned}$$

Now we use (7.13) and we arrive at the formula

$$\kappa(t)\|\nabla\partial_t \mathbf{m}(t)\|_2^2 + \int_0^t \kappa(s)\|\Delta\partial_t\mathbf{m}\|_2^2 \quad\le\quad C_\alpha + C_\alpha\int_0^t \kappa(s)\|\nabla\partial_t\mathbf{m}\|_2^2.$$

We are ready to use Gronwall's lemma to obtain

$$\sqrt{\kappa}\|\nabla\partial_t\mathbf{m}\|_2 + \left(\int_0^T \kappa\|\Delta\partial_t\mathbf{m}\|_2^2\right)^{\frac{1}{2}} \le C. \tag{7.23}$$

Next we take (7.16) and test it with the function $\partial_t^2\mathbf{m}$ to get

$$\|\partial_t^2\mathbf{m}\|_2^2 + \frac{\alpha}{2}\partial_t\|\nabla\partial_t\mathbf{m}\|_2^2 \le C(D_1 + D_2 + D_3 + D_4). \tag{7.24}$$

The boundary terms vanish. We estimate D_1, D_2, D_3, D_4 as follows. First

$$\begin{aligned}D_1 &:= |(\langle\nabla\mathbf{m}, \nabla\partial_t\mathbf{m}\rangle_{\mathbb{R}^9}\mathbf{m}, \partial_t^2\mathbf{m})|\\ &\le \|\nabla\mathbf{m}\|_4\|\nabla\partial_t\mathbf{m}\|_4\|\mathbf{m}\|_{L^\infty}\|\partial_t^2\mathbf{m}\|_2\\ &\le C(\|\partial_t\mathbf{m}\|_4 + \|\nabla\partial_t\mathbf{m}\|_4)\|\partial_t^2\mathbf{m}\|_2.\end{aligned} \tag{7.25}$$

Now we use (7.22) together with the weighted Young inequalities to get

$$\begin{aligned}D_1 &\le C_\varepsilon(\|\partial_t\mathbf{m}\|_2 + \|\nabla\partial_t\mathbf{m}\|_2 + \|\Delta\partial_t\mathbf{m}\|_2)\|\partial_t^2\mathbf{m}\|_2\\ &\le C_\varepsilon\|\partial_t\mathbf{m}\|_2^2 + C_\varepsilon\|\nabla\partial_t\mathbf{m}\|_2^2 + C_\varepsilon\|\Delta\partial_t\mathbf{m}\|_2^2 + \varepsilon\|\partial_t^2\mathbf{m}\|_2^2.\end{aligned}$$

Using inequality (10.14) we estimate the terms D_2 and D_3

$$\begin{aligned}D_2 := |(|\nabla\mathbf{m}|^2\partial_t\mathbf{m}, \partial_t^2\mathbf{m})| &\le \|\nabla\mathbf{m}\|_4^2\|\partial_t\mathbf{m}\|_{L^\infty}\|\partial_t^2\mathbf{m}\|_2\\ &\le (\|\partial_t\mathbf{m}\|_4 + \|\nabla\partial_t\mathbf{m}\|_4)\|\partial_t^2\mathbf{m}\|_2,\\ D_3 := |(\partial_t\mathbf{m}\times\Delta\mathbf{m}, \partial_t^2\mathbf{m})| &\le \|\partial_t\mathbf{m}\|_{L^\infty}\|\Delta\mathbf{m}\|_2\|\partial_t^2\mathbf{m}\|_2\\ &\le (\|\partial_t\mathbf{m}\|_4 + \|\nabla\partial_t\mathbf{m}\|_4)\|\partial_t^2\mathbf{m}\|_2.\end{aligned}$$

Notice that the expressions at the ends of the previous two inequalities are the same as the expression at the end of (7.25). Therefore we use the upper bounds obtained for D_1 also for the terms D_2 and D_3.

For D_4 we have the following result using the weighted Young inequality:

$$D_4 := |(\mathbf{m} \times \Delta\partial_t \mathbf{m}, \partial_t^2 \mathbf{m})| \leq \|\mathbf{m}\|_{L^\infty} \|\Delta\partial_t \mathbf{m}\|_2 \|\partial_t^2 \mathbf{m}\|_2 \leq C_\varepsilon \|\Delta\partial_t \mathbf{m}\|_2^2 + \varepsilon \|\partial_t^2 \mathbf{m}\|_2.$$

Now using the previous estimates we can continue in (7.24) setting $4\varepsilon = \alpha/4$:

$$\|\partial_t^2 \mathbf{m}\|_2^2 + \frac{\alpha}{2}\partial_t \|\nabla\partial_t \mathbf{m}\|_2^2 \leq \frac{\alpha}{4}\|\partial_t^2 \mathbf{m}\|_2^2 + C\|\Delta\partial_t \mathbf{m}\|_2^2 + C\|\nabla\partial_t \mathbf{m}\|_2^2 + C\|\partial_t \mathbf{m}\|_2^2.$$

Coefficients $\alpha/2, \alpha/4$ can be absorbed on the left-hand side. Multiplication by $\kappa(s)$ followed by time integration now leads to

$$\begin{aligned} \int_0^t \kappa(s)\|\partial_t^2 \mathbf{m}\|_2^2 + \kappa(t)\|\nabla\partial_t \mathbf{m}\|_2^2 \quad &\leq \quad C\int_0^t \kappa(s)\|\Delta\partial_t \mathbf{m}\|_2^2 + C\int_0^t \kappa(s)\|\nabla\partial_t \mathbf{m}\|_2^2 \\ &\quad + C\int_0^t \kappa(s)\|\partial_t \mathbf{m}\|_2^2. \end{aligned}$$

Using (7.23) and (7.13) leads us to the desired result

$$\int_0^t \kappa(s)\|\partial_t^2 \mathbf{m}\|^2 + \kappa(t)\|\nabla\partial_t \mathbf{m}\|_2^2 \leq C.$$

This completes the proof of inequality (7.14). □

PROOF OF INEQUALITY (7.15):
In order to show (7.15) we employ the previous results. First, we make use of (7.16) and estimate

$$\begin{aligned} \int_0^T \|\partial_t^2 \mathbf{m}\|_{W^{-1,2}}^2 &= \|\partial_t^2 \mathbf{m}\|_{L^2(I,W^{-1,2})} \\ &\leq \sup_{\substack{\boldsymbol{\varphi} \in L^2(I,W^{1,2}), \\ \|\boldsymbol{\varphi}\|_{L^2(I,W^{1,2})} \leq 1}} \int_0^T |(\alpha\nabla\partial_t \mathbf{m}, \nabla\boldsymbol{\varphi}) + E_1 + \cdots + E_4|\, \mathrm{d}s. \end{aligned} \tag{7.26}$$

The first term on the right-hand side of (7.26) can be split using the Young inequality:

$$\int_0^T |(\alpha\nabla\partial_t \mathbf{m}, \nabla\boldsymbol{\varphi})| \mathrm{d}s \leq C\int_0^T \|\nabla\partial_t \mathbf{m}\|_2^2 + \|\nabla\boldsymbol{\varphi}\|_2^2 \mathrm{d}s \leq C. \tag{7.27}$$

The boundedness of the first part comes from (7.13). The definition of φ in the supremum guarantees the boundedness of the second part.

The terms $E_1, \ldots, E_4$ can be bounded separately. For the term E_1, we write

$$\left|\int_0^T E_1\right| \quad \leq \quad C\int_0^T |(\langle\nabla\mathbf{m}, \nabla\partial_t \mathbf{m}\rangle_{\mathbb{R}^9} \mathbf{m}, \boldsymbol{\varphi})|$$

$$\begin{aligned} &\leq C\int_0^T \|\nabla \mathbf{m}\|_4 \|\nabla \partial_t \mathbf{m}\|_2 \|\mathbf{m}\|_{L^\infty} \|\boldsymbol{\varphi}\|_4 \\ &\leq C\int_0^T \|\nabla \partial_t \mathbf{m}\|_2^2 + \|\boldsymbol{\varphi}\|_{W^{1,2}}^2. \end{aligned}$$

We see similar terms as in (7.27). This allows us to use the same arguments to estimate also the term $\int_0^T E_1$.

Applying the embedding $W^{1,2} \hookrightarrow L^4$, we get the following estimate for E_2:

$$\begin{aligned} \left|\int_0^T E_2\right| &\leq C\int_0^T |(|\nabla \mathbf{m}|^2 \partial_t \mathbf{m}, \boldsymbol{\varphi})| \leq C\int_0^T \|\nabla \mathbf{m}\|_4^2 \|\partial_t \mathbf{m}\|_4 \|\boldsymbol{\varphi}\|_4 \\ &\leq C\int_0^T \|\partial_t \mathbf{m}\|_2^2 + \|\nabla \partial_t \mathbf{m}\|_2^2 + \|\boldsymbol{\varphi}\|_{W^{1,2}}^2. \end{aligned} \tag{7.28}$$

We make use of (7.13) to estimate the first two terms. The last term is bounded from the definition. Thus we get boundedness of the term $\left|\int_0^T E_2\right|$. We continue with E_3:

$$\begin{aligned} \left|\int_0^T E_3\right| &= \int_0^T |(\partial_t \mathbf{m} \times \Delta \mathbf{m}, \boldsymbol{\varphi})| = \int_0^T |(\nabla(\boldsymbol{\varphi} \times \partial_t \mathbf{m}), \nabla \mathbf{m})| \\ &\leq \int_0^T \|\nabla \boldsymbol{\varphi}\|_2 \|\partial_t \mathbf{m}\|_4 \|\nabla \mathbf{m}\|_4 + \int_0^T \|\boldsymbol{\varphi}\|_4 \|\nabla \partial_t \mathbf{m}\|_2 \|\nabla \mathbf{m}\|_4. \end{aligned}$$

We get rid of the term $\|\nabla \mathbf{m}\|_4$ thanks to (7.8). The embedding $W^{1,2} \hookrightarrow L^4$ used for the L^4 norms in both terms followed by application of the Young inequality gives us

$$\begin{aligned} \left|\int_0^T E_3\right| &\leq \int_0^T \|\boldsymbol{\varphi}\|_{W^{1,2}} \|\partial_t \mathbf{m}\|_{W^{1,2}} + \int_0^T \|\boldsymbol{\varphi}\|_{W^{1,2}} \|\partial_t \mathbf{m}\|_{W^{1,2}} \\ &\leq \int_0^T \|\boldsymbol{\varphi}\|_{W^{1,2}}^2 + \|\partial_t \mathbf{m}\|_2^2 + \|\nabla \partial_t \mathbf{m}\|_2^2, \end{aligned}$$

which is the same upper bound as in (7.28) and confirms the boundedness of the term $\left|\int_0^T E_3\right|$. For the term E_4, we write

$$\begin{aligned} \left|\int_0^T E_4\right| &= \int_0^T |(\mathbf{m} \times \Delta \partial_t \mathbf{m}, \boldsymbol{\varphi})| = \int_0^T |(\nabla(\boldsymbol{\varphi} \times \mathbf{m}), \nabla \partial_t \mathbf{m})| \\ &\leq \int_0^T \|\nabla \boldsymbol{\varphi}\|_2 \|\mathbf{m}\|_{L^\infty} \|\nabla \partial_t \mathbf{m}\|_2 + \int_0^T \|\boldsymbol{\varphi}\|_4 \|\nabla \mathbf{m}\|_4 \|\nabla \partial_t \mathbf{m}\|_2. \end{aligned}$$

Now we can do the same tricks as we did above for the term $\int_0^T E_3$ to get

$$\begin{aligned}\left|\int_0^T E_4\right| &\leq \int_0^T \|\varphi\|_{W^{1,2}}\|\partial_t \mathbf{m}\|_{W^{1,2}} + \int_0^T \|\varphi\|_{W^{1,2}}\|\partial_t \mathbf{m}\|_{W^{1,2}} \\ &\leq \int_0^T \|\varphi\|^2_{W^{1,2}} + \|\partial_t \mathbf{m}\|_2^2 + \|\nabla \partial_t \mathbf{m}\|_2^2 \leq C,\end{aligned}$$

which finally concludes the proof of boundedness of $\int_0^T \|\partial_t^2 \mathbf{m}\|^2_{W^{-1,2}}$ and completes the proof of inequality (7.15). □

7.2 Notations

Before we prove Theorem 7.2 we summarize necessary inequalities needed in the proofs.

By X_p and Y_p we denote the following sets:

$$\begin{aligned} X_p &= \big\{(i,j,k) : i,j,k \in \mathbb{Z}, \\ &\qquad 0 \leq i \leq p,\ 0 \leq j \leq p,\ 0 \leq k \leq p,\ i+j+k = p\big\}, \\ Y_p &= \{(i,j) : i,j \in \mathbb{Z},\ 0 \leq i \leq p,\ 0 \leq j \leq p,\ i+j = p\}. \end{aligned}$$

Denote $\mathcal{A} = \langle \nabla \mathbf{m}, \nabla \mathbf{m} \rangle \mathbf{m}$ and $\mathcal{B} = \mathbf{m} \times \Delta \mathbf{m}$. For the time derivatives of $\mathcal{A}$ and $\mathcal{B}$ we can write

$$\partial_t^p \mathcal{A} = \sum_{X_p} \langle \nabla \partial_t^i \mathbf{m}, \nabla \partial_t^j \mathbf{m} \rangle \partial_t^k \mathbf{m}, \tag{7.29}$$

$$\partial_t^p \mathcal{B} = \sum_{Y_p} \partial_t^i \mathbf{m} \times \Delta \partial_t^j \mathbf{m}. \tag{7.30}$$

We can formally take the r-th derivative in time of the LL equation to get

$$\partial_t^{r+1} \mathbf{m} - \alpha \Delta \partial_t^r \mathbf{m} = \alpha \partial_t^r \mathcal{A} - \partial_t^r \mathcal{B}. \tag{7.31}$$

7.3 Proof of regularity theorem

We use the notation $\mathcal{P}(p)$ for the statement of Theorem 7.2 for specific value of p. We will prove the theorem by mathematical induction. First we prove $\mathcal{P}(0)$. We deal with this also in [25]. Then we prove $\mathcal{P}(p)$ using $\mathcal{P}(0), \ldots, \mathcal{P}(p-1)$.

First step in the proof by mathematical induction

This step is in fact included in the proof of Theorem 7.3.

Second step in the proof of mathematical induction

We prove all four inequalities (7.9)–(7.12) sequentially. It means that in the proof of (7.9) we use $\mathcal{P}(m)$ for $m \le p-1$ only, whereas in the proof of (7.10) we can use (7.9) from $\mathcal{P}(p)$, too. Finally, in the proof of (7.12) we will use (7.9)–(7.11) from $\mathcal{P}(p)$.

PROOF OF INEQUALITY (7.9):
We take the $(p+1)$-th derivative of the LL equation (7.31). Then we multiply the result by $\kappa^{2p}\partial_t^{p+1}\mathbf{m}$ and finally we integrate it over Ω

$$\begin{aligned}&\kappa^{2p}(\partial_t^{p+2}\mathbf{m},\partial_t^{p+1}\mathbf{m}) - \alpha\kappa^{2p}(\Delta\partial_t^{p+1}\mathbf{m},\partial_t^{p+1}\mathbf{m})\\ &\quad = \alpha\kappa^{2p}(\partial_t^{p+1}\mathcal{A},\partial_t^{p+1}\mathbf{m}) - \kappa^{2p}(\partial_t^{p+1}\mathcal{B},\partial_t^{p+1}\mathbf{m}).\end{aligned}$$

Because of the following identity

$$\kappa^{2p}(\partial_t^{p+2}\mathbf{m},\partial_t^{p+1}\mathbf{m}) = \frac{1}{2}\partial_t(\kappa^{2p}\|\partial_t^{p+1}\mathbf{m}\|_2^2) - p\kappa^{2p-1}\|\partial_t^{p+1}\mathbf{m}\|_2^2, \tag{7.32}$$

together with integration by parts in the term $\alpha\kappa^{2p}(\Delta\partial_t^{p+1}\mathbf{m},\partial_t^{p+1}\mathbf{m})$, we can deduce

$$\begin{aligned}&\frac{1}{2}\partial_t\left(\kappa^{2p}\|\partial_t^{p+1}\mathbf{m}\|_2^2\right) + \alpha\kappa^{2p}\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2\\ &\quad = p\kappa^{2p-1}\|\partial_t^{p+1}\mathbf{m}\|_2^2 + \alpha\kappa^{2p}(\partial_t^{p+1}\mathcal{A},\partial_t^{p+1}\mathbf{m})\\ &\qquad -\kappa^{2p}(\partial_t^{p+1}\mathcal{B},\partial_t^{p+1}\mathbf{m}).\end{aligned} \tag{7.33}$$

In Lemmas 7.1 and 7.2 we estimate absolute value of terms $\kappa^{2p}(\partial_t^{p+1}\mathcal{A},\partial_t^{p+1}\mathbf{m})$ and $\kappa^{2p}(\partial_t^{p+1}\mathcal{B},\partial_t^{p+1}\mathbf{m})$.

Lemma 7.1 *For any fixed ε the following estimate holds*

$$\int_0^{T_0}\left|\kappa^{2p}(\partial_t^{p+1}\mathcal{A},\partial_t^{p+1}\mathbf{m})\right| \le C + \varepsilon\int_0^{T_0}\kappa^{2p}\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2 + C_\varepsilon\int_0^{T_0}\kappa^{2p}\|\partial_t^{p+1}\mathbf{m}\|_2^2.$$

PROOF:
From the definition of $\mathcal{A}$ we have

$$|\kappa^{2p}(\partial_t^{p+1}\mathcal{A},\partial_t^{p+1}\mathbf{m})| \quad \le \quad \kappa^{2p}\sum_{X_{p+1}}|(\langle\nabla\partial_t^i\mathbf{m},\nabla\partial_t^j\mathbf{m}\rangle\partial_t^k\mathbf{m},\partial_t^{p+1}\mathbf{m})|. \tag{7.34}$$

Class	Value of k	i	j
1a	$p+1$	0	0
1b	0	$p+1$ (or 0)	0 (or $p+1$)
1c	p	0 (or 1)	1 (or 0)
1d	$\leq p-1$	$\leq p$	$\leq p$

Table 7.1: Classes of indices i, j, k when $2i-1+2j-1+2k = 2p$.

We divide the proof of lemma in several steps. All possible choices of values k, i, j, such that $i+j+k = p+1$, can be put in four classes, see Table 7.1. We study these cases separately.

Class 1a

When $k = p+1$ in (7.34), we have the following estimate using (7.7)

$$|(\langle\nabla\mathbf{m},\nabla\mathbf{m}\rangle\partial_t^{p+1}\mathbf{m},\partial_t^{p+1}\mathbf{m})| \leq \|\nabla\mathbf{m}\|_4^2\|\partial_t^{p+1}\mathbf{m}\|_4^2 \leq C\|\partial_t^{p+1}\mathbf{m}\|_4^2. \quad (7.35)$$

Now, we first apply (10.11) and then the Young inequality with exponents 4/3 and 4 to get

$$\begin{aligned}|(\langle\nabla\mathbf{m},\nabla\mathbf{m}\rangle\partial_t^{p+1}\mathbf{m},\partial_t^{p+1}\mathbf{m})| &\leq \|\partial_t^{p+1}\mathbf{m}\|_2^2 + \|\partial_t^{p+1}\mathbf{m}\|_2^{\frac{1}{2}}\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^{\frac{3}{2}} \\ &\leq C_\varepsilon\|\partial_t^{p+1}\mathbf{m}\|_2^2 + \varepsilon\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2.\end{aligned}$$

Class 1b

When $i = p+1$ or $j = p+1$, we can estimate the corresponding terms in (7.34) as follows, using (7.8) and the Young inequality,

$$\begin{aligned}|(\langle\nabla\partial_t^{p+1}\mathbf{m},\nabla\mathbf{m}\rangle\mathbf{m},\partial_t^{p+1}\mathbf{m})| &\leq \|\nabla\partial_t^{p+1}\mathbf{m}\|_2\|\nabla\mathbf{m}\|_4\|\mathbf{m}\|_{L^\infty}\|\partial_t^{p+1}\mathbf{m}\|_4 \\ &\leq C\|\nabla\partial_t^{p+1}\mathbf{m}\|_2\|\partial_t^{p+1}\mathbf{m}\|_4 \\ &\leq \varepsilon\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2 + C_\varepsilon\|\partial_t^{p+1}\mathbf{m}\|_4^2. \quad (7.36)\end{aligned}$$

For the term $C_\varepsilon\|\partial_t^{p+1}\mathbf{m}\|_4^2$ we can use the same technique as in (7.35). Then we get

$$|(\langle\nabla\partial_t^{p+1}\mathbf{m},\nabla\mathbf{m}\rangle\mathbf{m},\partial_t^{p+1}\mathbf{m})| \leq \varepsilon\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2 + C_\varepsilon\|\partial_t^{p+1}\mathbf{m}\|_2^2.$$

Class 1c

In this case we have $k = p$ and $i + j = 1$. Without loss of generality we assume $i = 0$. Then we have

$$\begin{aligned}\kappa^{2p}|(\langle\nabla\partial_t^i\mathbf{m},\nabla\partial_t^j\mathbf{m}\rangle\partial_t^k\mathbf{m},\partial_t^{p+1}\mathbf{m})| &\leq \kappa^{2p}\|\nabla\mathbf{m}\|_4\|\nabla\partial_t\mathbf{m}\|_4\|\partial_t^p\mathbf{m}\|_4\|\partial_t^{p+1}\mathbf{m}\|_4 \\ &\leq \kappa^{2p}\|\nabla\partial_t\mathbf{m}\|_4^2\|\partial_t^p\mathbf{m}\|_4^2 + \|\partial_t^{p+1}\mathbf{m}\|_4^2,\end{aligned}$$

where we used the embedding $W^{2,2} \hookrightarrow W^{1,4}$, the boundedness of $\mathbf{m}$ in $W^{2,2}$ norm and the Young inequality. The term $\|\partial_t^{p+1}\mathbf{m}\|_4^2$ was already successfully estimated in (7.35) and (7.36). We now apply the embeddings $W^{2,2} \hookrightarrow W^{1,4}$ and $W^{1,2} \hookrightarrow L^4$ to get

$$\begin{aligned} &\kappa^{2p}|(\langle\nabla\partial_t^i\mathbf{m},\nabla\partial_t^j\mathbf{m}\rangle\partial_t^k\mathbf{m},\partial_t^{p+1}\mathbf{m})| \\ &\quad \le C\kappa(\|\Delta\partial_t\mathbf{m}\|_2^2+\|\nabla\partial_t\mathbf{m}\|_2^2)\kappa^{2p-1}(\|\nabla\partial_t^p\mathbf{m}\|_2^2+\|\partial_t^p\mathbf{m}\|_2^2) \\ &\qquad +C_\varepsilon\|\partial_t^{p+1}\mathbf{m}\|_2^2+\varepsilon\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2. \end{aligned} \tag{7.37}$$

We use $\mathcal{P}(p-1)$ and $\mathcal{P}(0)$ to get

$$\begin{aligned} \kappa^{2p-1}\|\nabla\partial_t^p\mathbf{m}\|_2^2 &\le C, \\ \kappa^{2p-1}\|\partial_t^p\mathbf{m}\|_2^2 &\le \kappa^{2p-2}\|\partial_t^p\mathbf{m}\|_2^2 \le C, \\ \kappa\|\nabla\partial_t\mathbf{m}\|_2^2 &\le C, \end{aligned}$$

which can be applied in (7.37) to arrive at

$$\begin{aligned} &\kappa^{2p}|(\langle\nabla\partial_t^i\mathbf{m},\nabla\partial_t^j\mathbf{m}\rangle\partial_t^k\mathbf{m},\partial_t^{p+1}\mathbf{m})| \\ &\quad \le C+C\kappa\|\Delta\partial_t\mathbf{m}\|_2^2+C_\varepsilon\|\partial_t^{p+1}\mathbf{m}\|_2^2+\varepsilon\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2. \end{aligned}$$

After the time integration we use (7.11) for $\mathcal{P}(0)$, which completes the case that $k = p$.

<u>Class 1d</u>

We have $k \le p-1$. For the terms on the right-hand side of (7.34) we have

$$\begin{aligned} F_1 &:= \kappa^{2p}|(\langle\nabla\partial_t^i\mathbf{m},\nabla\partial_t^j\mathbf{m}\rangle\partial_t^k\mathbf{m},\partial_t^{p+1}\mathbf{m})| \\ &\quad \le \kappa^{2p}\|\nabla\partial_t^i\mathbf{m}\|_2\|\nabla\partial_t^j\mathbf{m}\|_4\|\partial_t^k\mathbf{m}\|_{L^\infty}\|\partial_t^{p+1}\mathbf{m}\|_4 \end{aligned}$$

Further, we have to estimate the term $\|\partial_t^k\mathbf{m}\|_{L^\infty}$. We use the embedding $W^{1,4} \hookrightarrow L^\infty$ and then we obtain $\|\partial_t^k\mathbf{m}\|_{L^\infty} \le C(\|\nabla\partial_t^j\mathbf{m}\|_4+\|\partial_t^j\mathbf{m}\|_4)$. To be exact we should go on with the previous inequality but we proceed only with estimating worse term, namely with the term $\|\nabla\partial_t^j\mathbf{m}\|_4$. It is clear that if we successfully estimate this term, then better term $\|\partial_t^j\mathbf{m}\|_4$ would not cause any problems. And in such a way we rapidly decrease the length of the text. Similar technique we use more times on different places. Thus we get

$$\begin{aligned} F_1 &\le \kappa^{2p}\|\nabla\partial_t^i\mathbf{m}\|_2^2\|\nabla\partial_t^j\mathbf{m}\|_4^2\|\nabla\partial_t^k\mathbf{m}\|_4^2 \\ &\quad +\kappa^{2p}\|\partial_t^{p+1}\mathbf{m}\|_4^2, \end{aligned}$$

where we have used the Young inequality and the embedding $W^{1,4} \hookrightarrow L^\infty$. We use estimates, which we have already proved in (7.35) for the term $\kappa^{2p}\|\partial_t^{p+1}\mathbf{m}\|_4^2$. Note that $2i-1+2j-1+2k=2p$. Now we use the embedding $W^{2,2}\hookrightarrow W^{1,4}$, (10.11) and (10.12) to get

$$\begin{aligned} F_1 \quad \leq \quad & \kappa^{2i-1}\|\nabla\partial_t^i\mathbf{m}\|_2^2\kappa^{2j-1}(\|\nabla\partial_t^j\mathbf{m}\|_2^2+\|\Delta\partial_t^j\mathbf{m}\|_2^2) \\ & \kappa^{2k}(\|\nabla\partial_t^k\mathbf{m}\|_2^2+\|\Delta\partial_t^k\mathbf{m}\|_2^2) \\ & +C_\varepsilon\kappa^{2p}\|\partial_t^{p+1}\mathbf{m}\|_2^2+\varepsilon\kappa^{2p}\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2. \end{aligned} \tag{7.38}$$

Since $k\leq p-1, i\leq p, j\leq p$, using $\mathcal{P}(k),\mathcal{P}(i-1)$ and $\mathcal{P}(j-1)$, we have

$$\begin{aligned} \kappa^{2i-1}\|\nabla\partial_t^i\mathbf{m}\|_2^2 &\leq C, \\ \kappa^{2j-1}\|\nabla\partial_t^j\mathbf{m}\|_2^2 &\leq C, \\ \kappa^{2k}\|\nabla\partial_t^k\mathbf{m}\|_2^2\leq\kappa^{2k-1}\|\nabla\partial_t^k\mathbf{m}\|_2^2 &\leq C, \\ \kappa^{2k}\|\Delta\partial_t^k\mathbf{m}\|_2^2 &\leq C, \end{aligned}$$

which can be applied in (7.38) to get

$$F_1\leq C+C\kappa^{2j-1}\|\Delta\partial_t^j\mathbf{m}\|_2^2+C_\varepsilon\kappa^{2p}\|\partial_t^{p+1}\mathbf{m}\|_2^2+\varepsilon\kappa^{2p}\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2.$$

After time integration we use (7.11) from $\mathcal{P}(j-1)$ for the term $\kappa^{2j-1}\|\Delta\partial_t^j\mathbf{m}\|_2^2$, which completes the proof of the lemma. □

Lemma 7.2 *For any fixed ε the following estimate holds*

$$\int_0^{T_0}|\kappa^{2p}(\partial_t^{p+1}\mathcal{B},\partial_t^{p+1}\mathbf{m})|\leq C+\varepsilon\int_0^{T_0}\kappa^{2p}\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2+C_\varepsilon\int_0^{T_0}\kappa^{2p}\|\partial_t^{p+1}\mathbf{m}\|_2^2.$$

Proof:
The definition of $\mathcal{B}$ gives

$$|\kappa^{2p}(\partial_t^{p+1}\mathcal{B},\partial_t^{p+1}\mathbf{m})| \quad\leq\quad \kappa^{2p}\sum_{Y_{p+1}}|(\partial_t^i\mathbf{m}\times\Delta\partial_t^j\mathbf{m},\partial_t^{p+1}\mathbf{m})|. \tag{7.39}$$

Now we make another distribution of possibilities for the values of i,j such that $i+j=p+1$. See Table 7.2.

Class 2a

When $i=p+1$ we have

$$(\partial_t^{p+1}\mathbf{m}\times\Delta\mathbf{m},\partial_t^{p+1}\mathbf{m})=0.$$

Class 2b

When $j=p+1$ we can estimate

$$\begin{aligned} |(\mathbf{m}\times\Delta\partial_t^{p+1}\mathbf{m},\partial_t^{p+1}\mathbf{m})| \quad &= \quad |(\partial_t^{p+1}\mathbf{m}\times\mathbf{m},\Delta\partial_t^{p+1}\mathbf{m})| \\ &\leq \quad |(\nabla\partial_t^{p+1}\mathbf{m}\times\mathbf{m},\nabla\partial_t^{p+1}\mathbf{m})| \\ & \quad\quad +|(\partial_t^{p+1}\mathbf{m}\times\nabla\mathbf{m},\nabla\partial_t^{p+1}\mathbf{m})|, \end{aligned}$$

Class	Value of i	Value of j
2a	0	$p+1$
2b	$p+1$	0
2c	$\leq p$	$\leq p$

Table 7.2: Classes of indices i, j when $2i-1+2j-1=2p$.

where we have applied integration by parts. The boundary terms vanish. In the next step we use (7.8) and (10.11)

$$\begin{aligned}
&|(\mathbf{m}\times\Delta\partial_t^{p+1}\mathbf{m},\partial_t^{p+1}\mathbf{m})|\\
&\quad\leq |(\partial_t^{p+1}\mathbf{m}\times\nabla\mathbf{m},\nabla\partial_t^{p+1}\mathbf{m})|\\
&\quad\leq \|\partial_t^{p+1}\mathbf{m}\|_4\|\nabla\mathbf{m}\|_4\|\nabla\partial_t^{p+1}\mathbf{m}\|_2\\
&\quad\leq C(\|\partial_t^{p+1}\mathbf{m}\|_2+\|\partial_t^{p+1}\mathbf{m}\|_2^{\frac14}\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^{\frac34})\|\nabla\partial_t^{p+1}\mathbf{m}\|_2.
\end{aligned}$$

Now we apply the Young inequality first with both exponents equal to 2 and next with the exponents taking the value 4/3 and 4, respectively. We get

$$|(\mathbf{m}\times\Delta\partial_t^{p+1}\mathbf{m},\partial_t^{p+1}\mathbf{m})| \quad\leq\quad C_\varepsilon\|\partial_t^{p+1}\mathbf{m}\|_2^2+\varepsilon\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2.$$

Class 2c

When $i\leq p$ and $j\leq p$ we have

$$\begin{aligned}
&|(\partial_t^i\mathbf{m}\times\Delta\partial_t^j\mathbf{m},\partial_t^{p+1}\mathbf{m})|\\
&\quad= |(\partial_t^{p+1}\mathbf{m}\times\partial_t^i\mathbf{m},\Delta\partial_t^j\mathbf{m})|\\
&\quad\leq |(\nabla\partial_t^{p+1}\mathbf{m}\times\partial_t^i\mathbf{m},\nabla\partial_t^j\mathbf{m})|+|(\partial_t^{p+1}\mathbf{m}\times\nabla\partial_t^i\mathbf{m},\nabla\partial_t^j\mathbf{m})|\\
&\quad\leq \|\nabla\partial_t^{p+1}\mathbf{m}\|_2\|\partial_t^i\mathbf{m}\|_4\|\nabla\partial_t^j\mathbf{m}\|_4+\|\partial_t^{p+1}\mathbf{m}\|_4\|\nabla\partial_t^i\mathbf{m}\|_2\|\nabla\partial_t^j\mathbf{m}\|_4\\
&\quad\leq \|\nabla\partial_t^{p+1}\mathbf{m}\|_2(\|\partial_t^i\mathbf{m}\|_2^2+\|\nabla\partial_t^i\mathbf{m}\|_2^2)^{\frac12}(\|\nabla\partial_t^j\mathbf{m}\|_2^2+\|\Delta\partial_t^j\mathbf{m}\|_2^2)^{\frac12}\\
&\qquad+(\|\partial_t^{p+1}\mathbf{m}\|_2^2+\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2)^{\frac12}\|\nabla\partial_t^i\mathbf{m}\|_2(\|\nabla\partial_t^j\mathbf{m}\|_2^2+\|\Delta\partial_t^j\mathbf{m}\|_2^2)^{\frac12},
\end{aligned}$$

where we have used the embeddings $W^{1,2}\hookrightarrow L^4, W^{2,2}\hookrightarrow W^{1,4}$ and (10.11). Now, from Young's inequality we get

$$\begin{aligned}
&\kappa^{2p}|(\partial_t^i\mathbf{m}\times\Delta\partial_t^j\mathbf{m},\partial_t^{p+1}\mathbf{m})|\\
&\quad\leq \varepsilon\kappa^{2p}\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2\\
&\qquad+C_\varepsilon\kappa^{2p}(\|\partial_t^i\mathbf{m}\|_2^2+\|\nabla\partial_t^i\mathbf{m}\|_2^2)(\|\nabla\partial_t^j\mathbf{m}\|_2^2+\|\Delta\partial_t^j\mathbf{m}\|_2^2)\\
&\qquad+\varepsilon\kappa^{2p}\|\partial_t^{p+1}\mathbf{m}\|_2^2+C_\varepsilon\kappa^{2p}\|\nabla\partial_t^i\mathbf{m}\|_2^2(\|\nabla\partial_t^j\mathbf{m}\|_2^2+\|\Delta\partial_t^j\mathbf{m}\|_2^2)\\
&\quad\leq \varepsilon\kappa^{2p}\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2+\varepsilon\kappa^{2p}\|\partial_t^{p+1}\mathbf{m}\|_2^2\\
&\qquad+C_\varepsilon\kappa^{2i-1}(\|\partial_t^i\mathbf{m}\|_2^2+\|\nabla\partial_t^i\mathbf{m}\|_2^2)\kappa^{2j-1}(\|\nabla\partial_t^j\mathbf{m}\|_2^2+\|\Delta\partial_t^j\mathbf{m}\|_2^2).
\end{aligned}$$

Note that $2i-1+2j-1 = 2(i+j)-2 = 2p$. Since $j \le p$ and $i \le p$, we have from $\mathcal{P}(j-1)$ and $\mathcal{P}(i-1)$ that

$$\begin{aligned} \kappa^{2i-1}\|\partial_t^i \mathbf{m}\|_2^2 &\le \kappa^{2i-2}\|\partial_t^i \mathbf{m}\|_2^2 \le C \\ \kappa^{2i-1}\|\nabla\partial_t^i \mathbf{m}\|_2^2 &\le C, \\ \kappa^{2j-1}\|\nabla\partial_t^j \mathbf{m}\|_2^2 &\le C. \end{aligned}$$

Thus, we get

$$\begin{aligned} &\kappa^{2p}|(\partial_t^i \mathbf{m}\times\Delta\partial_t^j \mathbf{m}, \partial_t^{p+1}\mathbf{m})| \\ &\quad \le \varepsilon\kappa^{2p}\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2 + \varepsilon\kappa^{2p}\|\partial_t^{p+1}\mathbf{m}\|_2^2 + C + C_\varepsilon\kappa^{2j-1}\|\Delta\partial_t^j \mathbf{m}\|_2^2. \end{aligned}$$

We integrate the previous inequality in time. Then we use (7.11) from $\mathcal{P}(j-1)$ to estimate the term $\kappa^{2j-1}\|\Delta\partial_t^j \mathbf{m}\|_2^2$. This completes the proof of the lemma. □

Proof of Inequality (7.9) (Continuation):
Now, we can pass to the proof of inequality (7.9). From Lemma 7.1 and Lemma 7.2 we can estimate the terms $|\alpha\kappa^{2p}(\partial_t^{p+1}\mathcal{A}, \partial_t^{p+1}\mathbf{m})|$ and $|\kappa^{2p}(\partial_t^{p+1}\mathcal{B}, \partial_t^{p+1}\mathbf{m})|$ in (7.33). After time integration we find

$$\begin{aligned} &\kappa^{2p}\|\partial_t^{p+1}\mathbf{m}(T)\|_2^2 + \alpha\int_0^T \kappa^{2p}\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2 \\ &\quad\le p\int_0^T \kappa^{2p-1}\|\partial_t^{p+1}\mathbf{m}\|_2^2 \\ &\qquad + \alpha\int_0^T |\kappa^{2p}(\partial_t^{p+1}\mathcal{A}, \partial_t^{p+1}\mathbf{m})| + \int_0^T |\kappa^{2p}(\partial_t^{p+1}\mathcal{B}, \partial_t^{p+1}\mathbf{m})| \\ &\quad\le p\int_0^T \kappa^{2p-1}\|\partial_t^{p+1}\mathbf{m}\|_2^2 \\ &\qquad + (\alpha+1)\left[T.C + C_\varepsilon\int_0^T \kappa^{2p}\|\partial_t^{p+1}\mathbf{m}\|_2^2 + \varepsilon\int_0^T \kappa^{2p}\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2\right]. \end{aligned}$$

Taking ε sufficiently small and using $\mathcal{P}(p-1)$ for the term $\int_0^T \kappa^{2p-1}\|\partial_t^{p+1}\mathbf{m}\|_2^2$, we get

$$\kappa^{2p}\|\partial_t^{p+1}\mathbf{m}(T)\|_2^2 + \int_0^T \kappa^{2p}\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2 \le C(\alpha,p) + C(\alpha,p)\int_0^T \kappa^{2p}\|\partial_t^{p+1}\mathbf{m}\|_2^2.$$

After applying Gronwall's lemma we complete the proof of inequality (7.9) for $\mathcal{P}(p)$. □

Proof of Inequality (7.10):
Take the p-th derivative of the LL equation (7.31) and multiply it by $-\kappa^{2p}\Delta\partial_t^p\mathbf{m}$.

Class	Value of k	i	j
3a	p	0	0
3b	0	p (or 0)	0 (or p)
3c	$\leq p-1$	$\leq p-1$	$\leq p-1$

Table 7.3: Classes of indices i, j, k when $2i+2j+2k=2p$.

We find

$$\begin{aligned}&-\kappa^{2p}(\partial_t^{p+1}\mathbf{m}, \Delta\partial_t^p\mathbf{m}) + \alpha\kappa^{2p}\|\Delta\partial_t^p\mathbf{m}\|_2^2\\ &\quad = -\alpha\kappa^{2p}(\partial_t^p\mathcal{A}, \Delta\partial_t^p\mathbf{m}) + \kappa^{2p}(\partial_t^p\mathcal{B}, \Delta\partial_t^p\mathbf{m}).\end{aligned}$$

Using Young's inequality we get

$$\begin{aligned}\alpha\kappa^{2p}\|\Delta\partial_t^p\mathbf{m}\|_2^2 \quad &\leq \quad \alpha\kappa^{2p}|(\partial_t^p\mathcal{A}, \Delta\partial_t^p\mathbf{m})| + \kappa^{2p}|(\partial_t^p\mathcal{B}, \Delta\partial_t^p\mathbf{m})|\\ &\quad + C_\varepsilon\kappa^{2p}\|\partial_t^{p+1}\mathbf{m}\|_2^2 + \varepsilon\kappa^{2p}\|\Delta\partial_t^p\mathbf{m}\|_2^2.\end{aligned}$$

We take ε small. Since we have proved (7.9) for $\mathcal{P}(p)$, we can estimate the term $\kappa^{2p}\|\partial_t^{p+1}\mathbf{m}\|_2^2$ by C to get

$$\alpha\kappa^{2p}\|\Delta\partial_t^p\mathbf{m}\|_2^2 \leq \alpha\kappa^{2p}|(\partial_t^p\mathcal{A}, \Delta\partial_t^p\mathbf{m})| + \kappa^{2p}|(\partial_t^p\mathcal{B}, \Delta\partial_t^p\mathbf{m})| + C_\varepsilon. \tag{7.40}$$

The terms $\kappa^{2p}|(\partial_t^p\mathcal{A}, \Delta\partial_t^p\mathbf{m})|$ and $\kappa^{2p}|(\partial_t^p\mathcal{B}, \Delta\partial_t^p\mathbf{m})|$ are estimated in Lemmas 7.3 and 7.4.

Lemma 7.3 *For any fixed ε the following estimate holds*

$$\kappa^{2p}|(\partial_t^p\mathcal{A}, \Delta\partial_t^p\mathbf{m})| \leq C_\varepsilon + \varepsilon\kappa^{2p}\|\Delta\partial_t^p\mathbf{m}\|_2^2.$$

PROOF:
From the definition of $\mathcal{A}$ we have the following

$$|\kappa^{2p}(\partial_t^p\mathcal{A}, \Delta\partial_t^p\mathbf{m})| \quad \leq \quad \kappa^{2p}\sum_{X_p}|(\langle\nabla\partial_t^i\mathbf{m}, \nabla\partial_t^j\mathbf{m}\rangle\partial_t^k\mathbf{m}, \Delta\partial_t^p\mathbf{m})|. \tag{7.41}$$

We again split all possible combinations of values i, j, k into several classes, see Table 7.3.

<u>Class 3a</u>
When $k=p$ in (7.41) we have the following estimate:

$$\begin{aligned}|(\langle\nabla\mathbf{m}, \nabla\mathbf{m}\rangle\partial_t^p\mathbf{m}, \Delta\partial_t^p\mathbf{m})| \quad &\leq \quad \|\nabla\mathbf{m}\|_4^2\|\partial_t^p\mathbf{m}\|_{L^\infty}\|\Delta\partial_t^p\mathbf{m}\|_2\\ &\leq \quad C_\varepsilon\|\nabla\partial_t^p\mathbf{m}\|_4^2 + \varepsilon\|\Delta\partial_t^p\mathbf{m}\|_2^2,\end{aligned} \tag{7.42}$$

where we have used the boundedness of $\mathbf{m}$ in the $W^{2,2}$ norm, the embedding $W^{1,4} \hookrightarrow L^\infty$ and Young's inequality. We have estimated again only the worst terms, namely the terms with highest space derivative. Using first (10.12) and next Young's inequality with coefficients 4 and 3/4 we find

$$\begin{aligned}
&\kappa^{2p}|(\langle \nabla \mathbf{m}, \nabla \mathbf{m}\rangle \partial_t^p \mathbf{m}, \Delta \partial_t^p \mathbf{m})| \\
&\le C_\varepsilon \kappa^{2p} \|\nabla \partial_t^p \mathbf{m}\|_2^2 + C_\varepsilon \kappa^{2p} \|\nabla \partial_t^p \mathbf{m}\|_2^{\frac{1}{2}} \|\Delta \partial_t^p \mathbf{m}\|_2^{\frac{3}{2}} + \varepsilon \kappa^{2p} \|\Delta \partial_t^p \mathbf{m}\|_2^2 \\
&\le C_\varepsilon \kappa^{2p} \|\nabla \partial_t^p \mathbf{m}\|_2^2 + \varepsilon \kappa^{2p} \|\Delta \partial_t^p \mathbf{m}\|_2^2. \qquad (7.43)
\end{aligned}$$

From $\mathcal{P}(p-1)$ we have

$$\kappa^{2p} \|\nabla \partial_t^p \mathbf{m}\|_2^2 \le \kappa^{2p-1} \|\nabla \partial_t^p \mathbf{m}\|_2^2 \le C.$$

Then we arrive at

$$\kappa^{2p}|(\langle \nabla \mathbf{m}, \nabla \mathbf{m}\rangle \partial_t^p \mathbf{m}, \Delta \partial_t^p \mathbf{m})| \quad \le \quad C_\varepsilon + \varepsilon \kappa^{2p} \|\Delta \partial_t^p \mathbf{m}\|_2^2.$$

Class 3b

When $i = p$ or $j = p$ we can estimate the corresponding terms in (7.41) as follows

$$\begin{aligned}
|(\langle \nabla \partial_t^p \mathbf{m}, \nabla \mathbf{m}\rangle \mathbf{m}, \Delta \partial_t^p \mathbf{m})| &\le \|\nabla \partial_t^p \mathbf{m}\|_4 \|\nabla \mathbf{m}\|_4 \|\mathbf{m}\|_{L^\infty} \|\Delta \partial_t^p \mathbf{m}\|_2 \\
&\le C\|\nabla \partial_t^p \mathbf{m}\|_4 \|\Delta \partial_t^p \mathbf{m}\|_2 \\
&\le C_\varepsilon \|\nabla \partial_t^p \mathbf{m}\|_4^2 + \varepsilon \|\Delta \partial_t^p \mathbf{m}\|_2^2.
\end{aligned}$$

We end up with the same terms as on the right-hand side of (7.42).

Class 3c

When $i \le p-1$, $j \le p-1$ and $k \le p-1$, on the right-hand side of (7.41) the following argument can be used

$$\begin{aligned}
G_1 \quad := \quad & \kappa^{2p}|(\langle \nabla \partial_t^i \mathbf{m}, \nabla \partial_t^i \mathbf{m}\rangle \partial_t^k \mathbf{m}, \Delta \partial_t^p \mathbf{m})| \\
& \le \kappa^{2p} \|\nabla \partial_t^i \mathbf{m}\|_4 \|\nabla \partial_t^j \mathbf{m}\|_4 \|\partial_t^k \mathbf{m}\|_{L^\infty} \|\Delta \partial_t^p \mathbf{m}\|_2 \\
& \le C_\varepsilon \kappa^{2p} \|\nabla \partial_t^i \mathbf{m}\|_4^2 \|\nabla \partial_t^j \mathbf{m}\|_4^2 \|\nabla \partial_t^k \mathbf{m}\|_4^2 \\
& \quad + \varepsilon \kappa^{2p} \|\Delta \partial_t^p \mathbf{m}\|_2^2,
\end{aligned}$$

where we invoked the embedding $W^{1,4} \hookrightarrow L^\infty$ and Young's inequality estimating only the terms with the highest space derivative.

Note that $2i + 2j + 2k = 2p$. Now, we use the embedding $W^{1,2} \hookrightarrow L^4$ to get

$$\begin{aligned}
G_1 \quad \le \quad & C_\varepsilon \kappa^{2i} (\|\nabla \partial_t^i \mathbf{m}\|_2^2 + \|\Delta \partial_t^i \mathbf{m}\|_2^2) \kappa^{2j} (\|\nabla \partial_t^j \mathbf{m}\|_2^2 + \|\Delta \partial_t^j \mathbf{m}\|_2^2) \\
& \kappa^{2k} (\|\nabla \partial_t^k \mathbf{m}\|_2^2 + \|\Delta \partial_t^k \mathbf{m}\|_2^2) \\
& + \varepsilon \kappa^{2p} \|\Delta \partial_t^p \mathbf{m}\|_2^2. \qquad (7.44)
\end{aligned}$$

Class	Value of i	j
4a	0	p
4b	p	0
4c	$\leq p-1$	$\leq p-1$

Table 7.4: Classes of indices i, j when $2i + 2j = 2p$.

Since $i \leq p-1, j \leq p-1$ and $k \leq p-1$ using $\mathcal{P}(i), \mathcal{P}(j), \mathcal{P}(k)$ we have

$$\begin{aligned} \kappa^{2i}\|\Delta\partial_t^i \mathbf{m}\|_2^2 &\leq C, \\ \kappa^{2i}\|\nabla\partial_t^i \mathbf{m}\|_2^2 \leq \kappa^{2i-1}\|\nabla\partial_t^i \mathbf{m}\|_2^2 &\leq C, \\ \kappa^{2j}\|\Delta\partial_t^j \mathbf{m}\|_2^2 &\leq C, \\ \kappa^{2j}\|\nabla\partial_t^j \mathbf{m}\|_2^2 \leq \kappa^{2j-1}\|\nabla\partial_t^j \mathbf{m}\|_2^2 &\leq C, \\ \kappa^{2k}\|\Delta\partial_t^k \mathbf{m}\|_2^2 &\leq C, \\ \kappa^{2k}\|\nabla\partial_t^k \mathbf{m}\|_2^2 \leq \kappa^{2k-1}\|\nabla\partial_t^k \mathbf{m}\|_2^2 &\leq C, \end{aligned}$$

which can be applied in (7.44) to get

$$G_1 \leq C_\varepsilon + \varepsilon\kappa^{2p}\|\Delta\partial_t^{p+1}\mathbf{m}\|_2^2.$$

This completes the proof of the lemma. □

Lemma 7.4 *For any fixed ε the following estimate holds*

$$\kappa^{2p}|(\partial_t^p \mathcal{B}, \Delta\partial_t^p \mathbf{m})| \leq C_\varepsilon + \varepsilon\kappa^{2p}\|\Delta\partial_t^p \mathbf{m}\|_2^2.$$

Proof:
From the definition of $\mathcal{B}$ we have

$$|\kappa^{2p}(\partial_t^p \mathcal{B}, \Delta\partial_t^p \mathbf{m})| = \kappa^{2p}\sum_{Y_p} |(\partial_t^i \mathbf{m} \times \Delta\partial_t^j \mathbf{m}, \Delta\partial_t^p \mathbf{m})|. \tag{7.45}$$

We use Table 7.4 in order to proceed in several steps.
Class 4a
When $j = p$ in (7.45) we have

$$(\mathbf{m} \times \Delta\partial_t^p \mathbf{m}, \Delta\partial_t^p \mathbf{m}) = 0.$$

Class 4b
When $i = p$ we can estimate

$$\begin{aligned} |(\partial_t^p \mathbf{m} \times \Delta\mathbf{m}, \Delta\partial_t^p \mathbf{m})| &\leq \|\partial_t^p \mathbf{m}\|_{L^\infty}\|\Delta\mathbf{m}\|_2\|\Delta\partial_t^p \mathbf{m}\|_2 \\ &\leq C_\varepsilon\|\nabla\partial_t^p \mathbf{m}\|_4^2 + \varepsilon\|\Delta\partial_t^p \mathbf{m}\|_2^2. \end{aligned}$$

where we invoked the boundedness of $\mathbf{m}$ in $W^{2,2}$, the embedding $W^{1,4} \hookrightarrow L^\infty$, and Young's inequality. Using (10.12) and then Young's inequality with exponents 4 and 4/3, we arrive at

$$\begin{aligned}
&\kappa^{2p}|(\partial_t^p \mathbf{m} \times \Delta \mathbf{m}, \Delta \partial_t^p \mathbf{m})| \\
&\quad \le C_\varepsilon \kappa^{2p}\|\nabla \partial_t^p \mathbf{m}\|_2^2 + C_\varepsilon \kappa^{2p}\|\nabla \partial_t^p \mathbf{m}\|_2^{\frac{1}{2}}\|\Delta \partial_t^p \mathbf{m}\|_2^{\frac{3}{2}} + \varepsilon \kappa^{2p}\|\Delta \partial_t^p \mathbf{m}\|_2^2 \\
&\quad \le C_\varepsilon \kappa^{2p}\|\nabla \partial_t^p \mathbf{m}\|_2^2 + \varepsilon \kappa^{2p}\|\Delta \partial_t^p \mathbf{m}\|_2^2.
\end{aligned}$$

Since $\mathcal{P}(p-1)$ we have

$$\kappa^{2p}\|\nabla \partial_t^p \mathbf{m}\|_2^2 \le \kappa^{2p-1}\|\nabla \partial_t^p \mathbf{m}\|_2^2 \le C.$$

Thus we find

$$\kappa^{2p}|(\partial_t^p \mathbf{m} \times \Delta \mathbf{m}, \Delta \partial_t^p \mathbf{m})| \quad \le \quad C_\varepsilon + \varepsilon \kappa^{2p}\|\Delta \partial_t^p \mathbf{m}\|_2^2.$$

<u>Class 4c</u>
When $i \le p-1$ and $j \le p-1$, one has

$$\begin{aligned}
|(\partial_t^i \mathbf{m} \times \Delta \partial_t^j \mathbf{m}, \Delta \partial_t^p \mathbf{m})| \quad &\le \quad \|\partial_t^i \mathbf{m}\|_{L^\infty}\|\Delta \partial_t^j \mathbf{m}\|_2\|\Delta \partial_t^p \mathbf{m}\|_2 \\
&\le \quad C_\varepsilon \|\nabla \partial_t^i \mathbf{m}\|_4^2\|\Delta \partial_t^j \mathbf{m}\|_2^2 + \varepsilon\|\Delta \partial_t^p \mathbf{m}\|_2^2,
\end{aligned}$$

where we have used the embedding $W^{1,4} \hookrightarrow L^\infty$ and Young's inequality. Now, we apply the embedding $W^{2,2} \hookrightarrow W^{1,4}$ to get

$$\begin{aligned}
&|(\partial_t^i \mathbf{m} \times \Delta \partial_t^j \mathbf{m}, \Delta \partial_t^p \mathbf{m})| \\
&\quad \le C_\varepsilon(\|\nabla \partial_t^i \mathbf{m}\|_2^2 + \|\Delta \partial_t^i \mathbf{m}\|_2^2)\|\Delta \partial_t^j \mathbf{m}\|_2^2 + \varepsilon\|\Delta \partial_t^p \mathbf{m}\|_2^2 \\
&\quad \le C_\varepsilon\|\nabla \partial_t^i \mathbf{m}\|_2^2\|\Delta \partial_t^j \mathbf{m}\|_2^2 + C_\varepsilon\|\Delta \partial_t^i \mathbf{m}\|_2^2\|\Delta \partial_t^j \mathbf{m}\|_2^2 + \varepsilon\|\Delta \partial_t^p \mathbf{m}\|_2^2.
\end{aligned}$$

Since $i \le p-1$ and $j \le p-1$, we can deduce from $\mathcal{P}(i)$ and $\mathcal{P}(j)$ that

$$\begin{aligned}
\kappa^{2i}\|\nabla \partial_t^i \mathbf{m}\|_2^2 \le \kappa^{2i-1}\|\nabla \partial_t^i \mathbf{m}\|_2^2 \quad &\le \quad C, \\
\kappa^{2i}\|\Delta \partial_t^i \mathbf{m}\|_2^2 \quad &\le \quad C, \\
\kappa^{2j}\|\Delta \partial_t^j \mathbf{m}\|_2^2 \quad &\le \quad C.
\end{aligned}$$

Then we find

$$\begin{aligned}
&\kappa^{2p}|(\partial_t^i \mathbf{m} \times \Delta \partial_t^j \mathbf{m}, \Delta \partial_t^p \mathbf{m})| \\
&\quad \le \varepsilon \kappa^{2p}\|\Delta \partial_t^p \mathbf{m}\|_2^2 + C_\varepsilon \kappa^{2i}\|\nabla \partial_t^i \mathbf{m}\|_2^2 \kappa^{2j}\|\Delta \partial_t^j \mathbf{m}\|_2^2 \\
&\qquad + C_\varepsilon \kappa^{2i}\|\Delta \partial_t^i \mathbf{m}\|_2^2 \kappa^{2j}\|\Delta \partial_t^j \mathbf{m}\|_2^2 \\
&\quad \le \varepsilon \kappa^{2p}\|\Delta \partial_t^p \mathbf{m}\|_2^2 + C_\varepsilon,
\end{aligned}$$

which completes the proof of the lemma. □

PROOF OF INEQUALITY (7.10) (Continuation):
To finish the proof of inequality (7.10) we simply apply Lemma 7.3 and Lemma 7.4 in the relation (7.40) and we take ε sufficiently small. □

PROOF OF INEQUALITY (7.11):
Take the $(p+1)$-th derivative of the LL equation (7.31). Then, we multiply the result by $-\kappa^{2p+1}\Delta\partial_t^{p+1}\mathbf{m}$ and get

$$\begin{aligned}&-\kappa^{2p+1}(\partial_t^{p+2}\mathbf{m},\Delta\partial_t^{p+1}\mathbf{m})+\alpha\kappa^{2p+1}\|\Delta\partial_t^{p+1}\mathbf{m}\|_2^2\\&\quad=-\alpha\kappa^{2p+1}(\partial_t^{p+1}\mathcal{A},\Delta\partial_t^{p+1}\mathbf{m})+\kappa^{2p+1}(\partial_t^{p+1}\mathcal{B},\Delta\partial_t^{p+1}\mathbf{m}).\end{aligned}$$

In the first term on the left-hand side we apply integration by parts. On account of the analogy with (7.32) we get

$$\begin{aligned}&\frac{1}{2}\partial_t\left(\kappa^{2p+1}\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2\right)+\alpha\kappa^{2p+1}\|\Delta\partial_t^{p+1}\mathbf{m}\|_2^2\\&\quad=-\kappa^{2p+1}\alpha(\partial_t^{p+1}\mathcal{A},\Delta\partial_t^{p+1}\mathbf{m})+\kappa^{2p+1}(\partial_t^{p+1}\mathcal{B},\Delta\partial_t^{p+1}\mathbf{m})\\&\qquad+\frac{2p+1}{2}\kappa^{2p}\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2.\end{aligned}\tag{7.46}$$

We again estimate the terms containing $\partial_t^{p+1}\mathcal{A}$ and $\partial_t^{p+1}\mathcal{B}$ in separate lemmas.

Lemma 7.5 *For any fixed ε the following estimate holds*

$$\begin{aligned}&\int_0^{T_0}|\kappa^{2p+1}(\partial_t^{p+1}\mathcal{A},\Delta\partial_t^{p+1}\mathbf{m})|\\&\quad\le C_\varepsilon+\varepsilon\int_0^{T_0}\kappa^{2p+1}\|\Delta\partial_t^{p+1}\mathbf{m}\|_2^2+C_\varepsilon\int_0^{T_0}\kappa^{2p+1}\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2.\end{aligned}$$

PROOF:
Starting from the definition of $\mathcal{A}$ we have

$$\begin{aligned}&|\kappa^{2p+1}(\partial_t^{p+1}\mathcal{A},\Delta\partial_t^{p+1}\mathbf{m})|\\&\quad\le\kappa^{2p+1}\sum_{X_{p+1}}|(\langle\nabla\partial_t^i\mathbf{m},\nabla\partial_t^j\mathbf{m}\rangle\partial_t^k\mathbf{m},\Delta\partial_t^{p+1}\mathbf{m})|.\end{aligned}\tag{7.47}$$

For the overview of classes of indices i,j,k this time, see Table 7.5.

Class 5a

When $k=p+1$ in (7.47) we have the estimate

$$\begin{aligned}|(\langle\nabla\mathbf{m},\nabla\mathbf{m}\rangle\partial_t^{p+1}\mathbf{m},\Delta\partial_t^{p+1}\mathbf{m})|&\le\|\nabla\mathbf{m}\|_4^2\|\partial_t^{p+1}\mathbf{m}\|_{L^\infty}\|\Delta\partial_t^{p+1}\mathbf{m}\|_2\\&\le C_\varepsilon\|\nabla\partial_t^{p+1}\mathbf{m}\|_4^2+\varepsilon\|\Delta\partial_t^{p+1}\mathbf{m}\|_2^2,\end{aligned}\tag{7.48}$$

Class	Value of k	i	j
5a	$p+1$	0	0
5b	0	$p+1$ (or 0)	0 (or $p+1$)
5c	$\leq p$	$\leq p$	$\leq p$

Table 7.5: Classes of indices i, j, k when $2i + 2j - 1 + 2k = 2p + 1$.

where we have used the embedding $W^{1,4} \hookrightarrow L^\infty$. We continue, using first (10.12) and next Young's inequality with coefficients 4 and 3/4 to get

$$\begin{aligned}
&|(\langle \nabla \mathbf{m}, \nabla \mathbf{m}\rangle \partial_t^{p+1}\mathbf{m}, \Delta\partial_t^{p+1}\mathbf{m})| \\
&\leq C_\varepsilon \|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2 + C_\varepsilon \|\nabla\partial_t^{p+1}\mathbf{m}\|_2^{\frac{1}{2}} \|\Delta\partial_t^{p+1}\mathbf{m}\|_2^{\frac{3}{2}} + \varepsilon\|\Delta\partial_t^{p+1}\mathbf{m}\|_2^2 \\
&\leq C_\varepsilon \|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2 + \varepsilon\|\Delta\partial_t^{p+1}\mathbf{m}\|_2^2.
\end{aligned}$$

Class 5b

When $i = p + 1$ or $j = p + 1$, using (7.8) we can estimate the corresponding terms in (7.47) as follows

$$\begin{aligned}
|(\langle \nabla\partial_t^{p+1}\mathbf{m}, \nabla\mathbf{m}\rangle\mathbf{m}, \Delta\partial_t^{p+1}\mathbf{m})| &\leq \|\nabla\partial_t^{p+1}\mathbf{m}\|_4 \|\nabla\mathbf{m}\|_4 \|\mathbf{m}\|_{L^\infty} \|\Delta\partial_t^{p+1}\mathbf{m}\|_2 \\
&\leq C\|\nabla\partial_t^{p+1}\mathbf{m}\|_4 \|\Delta\partial_t^{p+1}\mathbf{m}\|_2 \\
&\leq C_\varepsilon \|\nabla\partial_t^{p+1}\mathbf{m}\|_4^2 + \varepsilon\|\Delta\partial_t^{p+1}\mathbf{m}\|_2^2.
\end{aligned}$$

We end up with the same terms as on the right-hand side of (7.48). Thus we obtain the same estimates.

Class 5c

When $i \leq p$, $j \leq p$ and $k \leq p$, we can use the following argument on the right-hand side of (7.34)

$$\begin{aligned}
H_1 &:= \kappa^{2p+1} |(\langle \nabla\partial_t^i\mathbf{m}, \nabla\partial_t^i\mathbf{m}\rangle \partial_t^k\mathbf{m}, \Delta\partial_t^{p+1}\mathbf{m})| \\
&\leq \kappa^{2p+1} \|\nabla\partial_t^i\mathbf{m}\|_4 \|\nabla\partial_t^j\mathbf{m}\|_4 \|\partial_t^k\mathbf{m}\|_{L^\infty} \|\Delta\partial_t^{p+1}\mathbf{m}\|_2 \\
&\leq C_\varepsilon \kappa^{2p+1} \|\nabla\partial_t^i\mathbf{m}\|_4^2 \|\nabla\partial_t^j\mathbf{m}\|_4^2 \|\nabla\partial_t^k\mathbf{m}\|_4^2 \\
&\quad + \varepsilon\kappa^{2p+1}\|\Delta\partial_t^{p+1}\mathbf{m}\|_2^2,
\end{aligned}$$

where we have used the embedding $W^{1,4} \hookrightarrow L^\infty$ and Young's inequality. Note that $2i + 2j - 1 + 2k = 2p + 1$. Now from the embedding $W^{1,2} \hookrightarrow L^4$ we get

$$\begin{aligned}
H_1 \leq\ & C_\varepsilon \kappa^{2i} (\|\nabla\partial_t^i\mathbf{m}\|_2^2 + \|\Delta\partial_t^i\mathbf{m}\|_2^2) \kappa^{2j-1} (\|\nabla\partial_t^j\mathbf{m}\|_2^2 + \|\Delta\partial_t^j\mathbf{m}\|_2^2) \\
& \kappa^{2k} (\|\nabla\partial_t^k\mathbf{m}\|_2^2 + \|\Delta\partial_t^k\mathbf{m}\|_2^2) \\
& + \varepsilon\kappa^{2p+1}\|\Delta\partial_t^{p+1}\mathbf{m}\|_2^2. \qquad (7.49)
\end{aligned}$$

Since $i \leq p, j \leq p$ and $k \leq p$, using $\mathcal{P}(i-1), \mathcal{P}(j-1), \mathcal{P}(k-1)$, we have

$$\begin{aligned}\kappa^{2i}\|\nabla\partial_t^i\mathbf{m}\|_2^2 \leq \kappa^{2i-1}\|\nabla\partial_t^i\mathbf{m}\|_2^2 &\leq C,\\ \kappa^{2k}\|\nabla\partial_t^k\mathbf{m}\|_2^2 \leq \kappa^{2k-1}\|\nabla\partial_t^k\mathbf{m}\|_2^2 &\leq C,\\ \kappa^{2j-1}\|\nabla\partial_t^j\mathbf{m}\|_2^2 &\leq C.\end{aligned}$$

Inequality (7.10) was already proved for $\mathcal{P}(p)$, too. Thus we get

$$\begin{aligned}\kappa^{2i}\|\Delta\partial_t^i\mathbf{m}\|_2^2 &\leq C,\\ \kappa^{2k}\|\Delta\partial_t^k\mathbf{m}\|_2^2 &\leq C.\end{aligned} \tag{7.50}$$

We can proceed in (7.49) to get

$$H_1 \leq C_\varepsilon + C_\varepsilon\kappa^{2j-1}\|\Delta\partial_t^j\mathbf{m}\|_2^2 + \varepsilon\kappa^{2p+1}\|\Delta\partial_t^{p+1}\mathbf{m}\|_2^2.$$

We integrate the previous inequality in time. Then using (7.11) from $\mathcal{P}(j-1)$ we estimate the term $\kappa^{2j-1}\|\Delta\partial_t^j\mathbf{m}\|_2^2$. This completes the proof of the lemma. □

Lemma 7.6 *For any fixed ε the following estimate holds*

$$\begin{aligned}&\int_0^{T_0} |\kappa^{2p+1}(\partial_t^{p+1}\mathcal{B}, \Delta\partial_t^{p+1}\mathbf{m})|\\ &\leq C_\varepsilon + \varepsilon\int_0^{T_0}\kappa^{2p+1}\|\Delta\partial_t^{p+1}\mathbf{m}\|_2^2 + C_\varepsilon\int_0^{T_0}\kappa^{2p+1}\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2.\end{aligned}$$

Proof:
From the definition of $\mathcal{B}$ we have the following

$$|\kappa^{2p+1}(\partial_t^{p+1}\mathcal{B}, \Delta\partial_t^{p+1}\mathbf{m})| \leq \kappa^{2p+1}\sum_{Y_{p+1}}|(\partial_t^i\mathbf{m}\times\Delta\partial_t^j\mathbf{m}, \Delta\partial_t^{p+1}\mathbf{m})|. \tag{7.51}$$

The Classes 6a–6c are described in Table 7.6.

Class 7a

When $j = p+1$ in (7.51) we have

$$(\mathbf{m}\times\Delta\partial_t^{p+1}\mathbf{m}, \Delta\partial_t^{p+1}\mathbf{m}) = 0.$$

Class 7b

When $i = p+1$ we can estimate

$$\begin{aligned}|(\partial_t^{p+1}\mathbf{m}\times\Delta\mathbf{m}, \Delta\partial_t^{p+1}\mathbf{m})| &\leq \|\partial_t^{p+1}\mathbf{m}\|_{L^\infty}\|\Delta\mathbf{m}\|_2\|\Delta\partial_t^{p+1}\mathbf{m}\|_2\\ &\leq C_\varepsilon\|\nabla\partial_t^{p+1}\mathbf{m}\|_4^2 + \varepsilon\|\Delta\partial_t^{p+1}\mathbf{m}\|_2^2,\end{aligned}$$

	Value of	
Class	i	j
6a	0	$p+1$
6b	$p+1$	0
6c	$\leq p$	$\leq p$

Table 7.6: Classes of indices i, j when $2i+2j-1 = 2p+1$.

where we have invoked the boundedness of $\mathbf{m}$ in the $W^{2,2}$ norm, the embedding $W^{1,4} \hookrightarrow L^\infty$, and Young's inequality. Using (10.12) and Young's inequality with exponents 4 and 4/3, we arrive at

$$\begin{aligned}
&|(\partial_t^{p+1}\mathbf{m} \times \Delta\mathbf{m}, \Delta\partial_t^{p+1}\mathbf{m})| \\
&\leq C_\varepsilon \|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2 + C_\varepsilon \|\nabla\partial_t^{p+1}\mathbf{m}\|_2^{\frac{1}{2}} \|\Delta\partial_t^{p+1}\mathbf{m}\|_2^{\frac{3}{2}} + \varepsilon\|\Delta\partial_t^{p+1}\mathbf{m}\|_2^2 \\
&\leq C_\varepsilon \|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2 + \varepsilon\|\Delta\partial_t^{p+1}\mathbf{m}\|_2^2.
\end{aligned}$$

Class 7c

When $i \leq p$ and $j \leq p$ we get

$$\begin{aligned}
|(\partial_t^i\mathbf{m} \times \Delta\partial_t^j\mathbf{m}, \Delta\partial_t^{p+1}\mathbf{m})| &\leq \|\partial_t^i\mathbf{m}\|_{L^\infty} \|\Delta\partial_t^j\mathbf{m}\|_2 \|\Delta\partial_t^{p+1}\mathbf{m}\|_2 \\
&\leq C_\varepsilon \|\nabla\partial_t^i\mathbf{m}\|_4^2 \|\Delta\partial_t^j\mathbf{m}\|_2^2 + \varepsilon\|\Delta\partial_t^{p+1}\mathbf{m}\|_2^2,
\end{aligned}$$

where we have invoked the embedding $W^{1,4} \hookrightarrow L^\infty$ and Young's inequality. Now, we apply the embedding $W^{2,2} \hookrightarrow W^{1,4}$

$$\begin{aligned}
&|(\partial_t^i\mathbf{m} \times \Delta\partial_t^j\mathbf{m}, \Delta\partial_t^{p+1}\mathbf{m})| \\
&\leq C_\varepsilon (\|\nabla\partial_t^i\mathbf{m}\|_2^2 + \|\Delta\partial_t^i\mathbf{m}\|_2^2) \|\Delta\partial_t^j\mathbf{m}\|_2^2 + \varepsilon\|\Delta\partial_t^{p+1}\mathbf{m}\|_2^2 \\
&\leq C_\varepsilon \|\nabla\partial_t^i\mathbf{m}\|_2^2 \|\Delta\partial_t^j\mathbf{m}\|_2^2 + C_\varepsilon \|\Delta\partial_t^i\mathbf{m}\|_2^2 \|\Delta\partial_t^j\mathbf{m}\|_2^2 + \varepsilon\|\Delta\partial_t^{p+1}\mathbf{m}\|_2^2.
\end{aligned}$$

Since $i \leq p$ and $j \leq p$ we can deduce from $\mathcal{P}(i-1)$ and from (7.10) for $\mathcal{P}(i)$ and $\mathcal{P}(j)$ that

$$\begin{aligned}
\kappa^{2i}\|\nabla\partial_t^i\mathbf{m}\|_2^2 \leq \kappa^{2i-1}\|\nabla\partial_t^i\mathbf{m}\|_2^2 &\leq C, \\
\kappa^{2i}\|\Delta\partial_t^i\mathbf{m}\|_2^2 &\leq C, \\
\kappa^{2j}\|\Delta\partial_t^j\mathbf{m}\|_2^2 &\leq C.
\end{aligned}$$

Again notice that $2i + 2j - 1 = 2p + 1$. Then we get

$$\begin{aligned}
&\kappa^{2p+1}|(\partial_t^i\mathbf{m} \times \Delta\partial_t^j\mathbf{m}, \Delta\partial_t^{p+1}\mathbf{m})| \\
&\leq \varepsilon\kappa^{2p+1}\|\Delta\partial_t^{p+1}\mathbf{m}\|_2^2 + C_\varepsilon\kappa^{2i}\|\nabla\partial_t^i\mathbf{m}\|_2^2 \kappa^{2j-1}\|\Delta\partial_t^j\mathbf{m}\|_2^2 \\
&\quad + C_\varepsilon\kappa^{2i}\|\Delta\partial_t^i\mathbf{m}\|_2^2 \kappa^{2j-1}\|\Delta\partial_t^j\mathbf{m}\|_2^2 \\
&\leq \varepsilon\kappa^{2p+1}\|\Delta\partial_t^{p+1}\mathbf{m}\|_2^2 + C_\varepsilon\kappa^{2j-1}\|\Delta\partial_t^j\mathbf{m}\|_2^2.
\end{aligned}$$

We integrate the previous inequality in time and using (7.11) from $\mathcal{P}(j-1)$ we estimate the term $\kappa^{2j-1}\|\Delta\partial_t^j \mathbf{m}\|_2^2$. This completes the proof of the lemma. □

Proof of Inequality (7.11) (Continuation):
We apply the results from Lemma 7.5 and Lemma 7.6 in (7.46) to estimate the terms $|\alpha\kappa^{2p+1}(\partial_t^{p+1}\mathcal{A},\Delta\partial_t^{p+1}\mathbf{m})|$ and $|\kappa^{2p+1}(\partial_t^{p+1}\mathcal{B},\Delta\partial_t^{p+1}\mathbf{m})|$. After time integration we find

$$\begin{aligned}
&\kappa^{2p+1}\|\nabla\partial_t^{p+1}\mathbf{m}(T)\|_2^2+\alpha\int_0^T\kappa^{2p+1}\|\Delta\partial_t^{p+1}\mathbf{m}\|_2^2\\
&\quad\leq\frac{2p+1}{2}\int_0^T\kappa^{2p}\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2\\
&\qquad+\alpha\int_0^T|\kappa^{2p+1}(\partial_t^{p+1}\mathcal{A},\Delta\partial_t^{p+1}\mathbf{m})|+\int_0^T|\kappa^{2p+1}(\partial_t^{p+1}\mathcal{B},\Delta\partial_t^{p+1}\mathbf{m})|\\
&\quad\leq\frac{2p+1}{2}\int_0^T\kappa^{2p}\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2\\
&\qquad+(\alpha+1)\Big[T.C_\varepsilon+C_\varepsilon\int_0^T\kappa^{2p+1}\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2+\varepsilon\int_0^T\kappa^{2p+1}\|\Delta\partial_t^{p+1}\mathbf{m}\|_2^2\Big].
\end{aligned}$$

Taking ε sufficiently small and using $\mathcal{P}(p)$ for the term $\kappa^{2p}\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2$, we get

$$\begin{aligned}
&\kappa^{2p+1}\|\nabla\partial_t^{p+1}\mathbf{m}(T)\|_2^2+\int_0^T\kappa^{2p+1}\|\Delta\partial_t^{p+1}\mathbf{m}\|_2^2\\
&\quad\leq C(\alpha,p)+C(\alpha,p)\int_0^T\kappa^{2p+1}\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2.
\end{aligned}$$

After applying Gronwall's lemma we complete the proof of inequality (7.11) for $\mathcal{P}(p)$. □

Proof of Inequality (7.12):
Take the $(p+1)$-th derivative of the LL equation. Then multiply the result by $\kappa^{2p+1}\partial_t^{p+2}\mathbf{m}$. Integration by parts gives

$$\begin{aligned}
&\kappa^{2p+1}\|\partial_t^{p+2}\mathbf{m}\|_2^2+\alpha\kappa^{2p+1}(\nabla\partial_t^{p+1}\mathbf{m},\nabla\partial_t^{p+2}\mathbf{m})\\
&\quad=\alpha\kappa^{2p+1}(\partial_t^{p+1}\mathcal{A},\partial_t^{p+2}\mathbf{m})-\kappa^{2p+1}(\partial_t^{p+1}\mathcal{B},\partial_t^{p+2}\mathbf{m}).
\end{aligned}$$

On account of the analogy with (7.32) we get

$$\begin{aligned}
&\kappa^{2p+1}\|\partial_t^{p+2}\mathbf{m}\|_2^2+\frac{\alpha}{2}\partial_t(\kappa^{2p+1}\|\nabla\partial_t^{p+1}\mathbf{m}\|^2)\\
&\quad=\kappa^{2p+1}\Big(\alpha(\partial_t^{p+1}\mathcal{A},\partial_t^{p+2}\mathbf{m})-(\partial_t^{p+1}\mathcal{B},\partial_t^{p+2}\mathbf{m})\Big)\\
&\qquad+\alpha\frac{2p+1}{2}\kappa^{2p}\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2.
\end{aligned}\tag{7.52}$$

We will estimate the terms containing $\partial_t^{p+1}\mathcal{A}$ and $\partial_t^{p+1}\mathcal{B}$ separately in Lemmas 7.7 and 7.8.

Lemma 7.7 *For any fixed ε the following estimate holds*

$$\int_0^{T_0} |\kappa^{2p+1}(\partial_t^{p+1}\mathcal{A}, \partial_t^{p+2}\mathbf{m})| \le C_\varepsilon + \varepsilon \int_0^{T_0} \kappa^{2p+1}\|\partial_t^{p+2}\mathbf{m}\|_2^2.$$

Proof:
From the definition of $\mathcal{A}$ we have the following

$$|\kappa^{2p+1}(\partial_t^{p+1}\mathcal{A}, \partial_t^{p+2}\mathbf{m})| \le \kappa^{2p+1} \sum_{X_{p+1}} |(\langle \nabla\partial_t^i\mathbf{m}, \nabla\partial_t^j\mathbf{m}\rangle \partial_t^k\mathbf{m}, \partial_t^{p+2}\mathbf{m})|. \quad (7.53)$$

For all terms in (7.53) we estimate:

$$\begin{aligned} &|(\langle \nabla\partial_t^i\mathbf{m}, \nabla\partial_t^j\mathbf{m}\rangle \partial_t^k\mathbf{m}, \partial_t^{p+2}\mathbf{m})| \\ &\quad \le \|\nabla\partial_t^i\mathbf{m}\|_4 \|\nabla\partial_t^j\mathbf{m}\|_4 \|\partial_t^k\mathbf{m}\|_{L^\infty} \|\partial_t^{p+2}\mathbf{m}\|_2 \\ &\quad \le C_\varepsilon \|\nabla\partial_t^i\mathbf{m}\|_4^2 \|\nabla\partial_t^j\mathbf{m}\|_4^2 \|\nabla\partial_t^k\mathbf{m}\|_4^2 + \varepsilon\|\partial_t^{p+2}\mathbf{m}\|_2^2, \end{aligned}$$

where we have used the embedding $W^{1,4} \hookrightarrow L^\infty$ and Young's inequality. Further, we apply the embedding $W^{2,2} \hookrightarrow W^{1,4}$ to find

$$\begin{aligned} &|(\langle \nabla\partial_t^i\mathbf{m}, \nabla\partial_t^j\mathbf{m}\rangle \partial_t^k\mathbf{m}, \partial_t^{p+2}\mathbf{m})| \\ &\quad \le C_\varepsilon (\|\nabla\partial_t^i\mathbf{m}\|_2^2 + \|\Delta\partial_t^i\mathbf{m}\|_2^2)(\|\nabla\partial_t^j\mathbf{m}\|_2^2 + \|\Delta\partial_t^j\mathbf{m}\|_2^2) \\ &\qquad (\|\nabla\partial_t^k\mathbf{m}\|_2^2 + \|\Delta\partial_t^k\mathbf{m}\|_2^2) + \varepsilon\|\partial_t^{p+2}\mathbf{m}\|_2^2. \end{aligned}$$

Without loss of generality we can assume that $i \le j \le k$. Then $i \le p$ and $j \le p$. From (7.9), (7.10) and (7.11), which is already proved for $\mathcal{P}(p)$ too, we have

$$\begin{aligned} \kappa^{2i}\|\Delta\partial_t^i\mathbf{m}\|_2^2 &\le C, \\ \kappa^{2i}\|\nabla\partial_t^i\mathbf{m}\|_2^2 &\le \kappa^{2i-1}\|\nabla\partial_t^i\mathbf{m}\|_2^2 \le C, \\ \kappa^{2j}\|\Delta\partial_t^j\mathbf{m}\|_2^2 &\le C, \\ \kappa^{2j}\|\nabla\partial_t^j\mathbf{m}\|_2^2 &\le \kappa^{2j-1}\|\nabla\partial_t^j\mathbf{m}\|_2^2 \le C, \\ \kappa^{2k-1}\|\nabla\partial_t^k\mathbf{m}\|_2^2 &\le C. \end{aligned}$$

Then, because $2i + 2j + 2k - 1 = 2p + 1$, we have

$$\begin{aligned} &\kappa^{2p+1}|(\langle \nabla\partial_t^i\mathbf{m}, \nabla\partial_t^j\mathbf{m}\rangle \partial_t^k\mathbf{m}, \partial_t^{p+2}\mathbf{m})| \\ &\quad \le C_\varepsilon \kappa^{2i}(\|\nabla\partial_t^i\mathbf{m}\|_2^2 + \|\Delta\partial_t^i\mathbf{m}\|_2^2)\kappa^{2j}(\|\nabla\partial_t^j\mathbf{m}\|_2^2 + \|\Delta\partial_t^j\mathbf{m}\|_2^2) \\ &\qquad \kappa^{2k-1}(\|\nabla\partial_t^k\mathbf{m}\|_2^2 + \|\Delta\partial_t^k\mathbf{m}\|_2^2) + \varepsilon\kappa^{2p+1}\|\partial_t^{p+2}\mathbf{m}\|_2^2, \\ &\quad \le C_\varepsilon + C_\varepsilon\kappa^{2k-1}\|\Delta\partial_t^k\mathbf{m}\|_2^2 + \varepsilon\|\partial_t^{p+2}\mathbf{m}\|_2^2. \end{aligned}$$

We integrate the previous relation in time and use (7.11) to estimate the term $\kappa^{2k-1}\|\Delta\partial_t^k\mathbf{m}\|_2^2$, which is already proved for $\mathcal{P}(p)$, too. This completes the proof of the lemma. $\square$

Lemma 7.8 *For any fixed ε the following estimate holds*

$$\int_0^{T_0} |\kappa^{2p+1}(\partial_t^{p+1}\mathcal{B}, \partial_t^{p+2}\mathbf{m})| \le C_\varepsilon + \varepsilon \int_0^{T_0} \kappa^{2p+1}\|\partial_t^{p+2}\mathbf{m}\|_2^2.$$

PROOF:
From the definition of $\mathcal{B}$ we get

$$|\kappa^{2p+1}(\partial_t^{p+1}\mathcal{B}, \partial_t^{p+2}\mathbf{m})| \quad \le \quad \kappa^{2p+1}\sum_{Y_{p+1}} |(\partial_t^i\mathbf{m} \times \Delta\partial_t^j\mathbf{m}, \partial_t^{p+2}\mathbf{m})|. \tag{7.54}$$

For all terms in (7.54) we have

$$\begin{aligned} |(\partial_t^i\mathbf{m} \times \Delta\partial_t^j\mathbf{m}, \partial_t^{p+2}\mathbf{m})| &\le \|\partial_t^i\mathbf{m}\|_{L^\infty}\|\Delta\partial_t^j\mathbf{m}\|_2\|\partial_t^{p+2}\mathbf{m}\|_2 \\ &\le C_\varepsilon\|\nabla\partial_t^i\mathbf{m}\|_4^2\|\Delta\partial_t^j\mathbf{m}\|_2^2 + \varepsilon\|\partial_t^{p+2}\mathbf{m}\|_2^2, \end{aligned}$$

where we have used the embedding $W^{1,4} \hookrightarrow L^\infty$ and Young's inequality. Now, we apply the embedding $W^{2,2} \hookrightarrow W^{1,4}$

$$\begin{aligned} &|(\partial_t^i\mathbf{m} \times \Delta\partial_t^j\mathbf{m}, \partial_t^{p+2}\mathbf{m})| \\ &\quad \le C_\varepsilon(\|\nabla\partial_t^i\mathbf{m}\|_2^2 + \|\Delta\partial_t^i\mathbf{m}\|_2^2)\|\Delta\partial_t^j\mathbf{m}\|_2^2 + \varepsilon\|\partial_t^{p+2}\mathbf{m}\|_2^2 \\ &\quad \le C_\varepsilon(\|\nabla\partial_t^i\mathbf{m}\|_2^2 + \|\Delta\partial_t^i\mathbf{m}\|_2^2)(\|\nabla\partial_t^j\mathbf{m}\|_2^2 + \|\Delta\partial_t^j\mathbf{m}\|_2^2) + \varepsilon\|\partial_t^{p+2}\mathbf{m}\|_2^2. \end{aligned}$$

Without loss of generality we may assume that $i \le j$. Then $i \le p$ and from (7.10) for $\mathcal{P}(p)$ and from $\mathcal{P}(p-1)$ we have

$$\begin{aligned} \kappa^{2i}\|\nabla\partial_t^i\mathbf{m}\|_2^2 \le \kappa^{2i-1}\|\nabla\partial_t^i\mathbf{m}\|_2^2 &\le C, \\ \kappa^{2i}\|\nabla\partial_t^i\mathbf{m}\|_2^2 \le \kappa^{2i-1}\|\nabla\partial_t^i\mathbf{m}\|_2^2 &\le C, \\ \kappa^{2i}\|\Delta\partial_t^i\mathbf{m}\|_2^2 &\le C. \end{aligned}$$

Then, we get

$$\begin{aligned} &\kappa^{2p+1}|(\partial_t^i\mathbf{m} \times \Delta\partial_t^j\mathbf{m}, \partial_t^{p+2}\mathbf{m})| \\ &\quad \le \varepsilon\kappa^{2p+1}\|\partial_t^{p+2}\mathbf{m}\|_2^2 \\ &\qquad + C_\varepsilon\kappa^{2i}(\|\nabla\partial_t^i\mathbf{m}\|_2^2 + \|\Delta\partial_t^i\mathbf{m}\|_2^2)\kappa^{2j-1}(\|\nabla\partial_t^j\mathbf{m}\|_2^2 + \|\Delta\partial_t^j\mathbf{m}\|_2^2) \\ &\quad \le C_\varepsilon + \varepsilon\kappa^{2p+1}\|\partial_t^{p+2}\mathbf{m}\|_2^2 + C_\varepsilon\kappa^{2j-1}\|\Delta\partial_t^j\mathbf{m}\|_2^2. \end{aligned}$$

After the time integration we use (7.11) from $\mathcal{P}(j-1)$ to estimate the terms $\kappa^{2j-1}\|\Delta\partial_t^j \mathbf{m}\|_2^2$. This completes the proof of the lemma. □

Proof of Inequality (7.12) (Continuation):
We apply the results from Lemma 7.7 and Lemma 7.8 into (7.52) and estimate the terms $|\alpha\kappa^{2p+1}(\partial_t^{p+1}\mathcal{A}, \partial_t^{p+2}\mathbf{m})|$ and $|\kappa^{2p+1}(\partial_t^{p+1}\mathcal{B}, \partial_t^{p+2}\mathbf{m})|$. Then, after the time integration we obtain

$$\begin{aligned}
&\int_0^T \kappa^{2p+1}\|\partial_t^{p+2}\mathbf{m}\|_2^2 + \alpha\kappa^{2p+1}\|\nabla\partial_t^{p+1}\mathbf{m}(T)\|_2^2 \\
&\quad \leq \alpha\frac{2p+1}{2}\int_0^T \kappa^{2p}\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2 \\
&\qquad +\alpha\int_0^T |\kappa^{2p+1}(\partial_t^{p+1}\mathcal{A}, \partial_t^{p+2}\mathbf{m})| + \int_0^T |\kappa^{2p+1}(\partial_t^{p+1}\mathcal{B}, \partial_t^{p+2}\mathbf{m})| \\
&\quad \leq \alpha\frac{2p+1}{2}\int_0^T \kappa^{2p}\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2 + (\alpha+1)\left[T.C_\varepsilon + \varepsilon\int_0^T \kappa^{2p+1}\|\partial_t^{p+2}\mathbf{m}\|_2^2\right].
\end{aligned}$$

Taking ε sufficiently small and using $\mathcal{P}(p)$ for the term $\kappa^{2p}\|\nabla\partial_t^{p+1}\mathbf{m}\|_2^2$, we get

$$\int_0^T \kappa^{2p+1}\|\partial_t^{p+2}\mathbf{m}\|_2^2 \leq C(\alpha, p).$$

This completes the proof of inequality (7.12) for $\mathcal{P}(p)$. Consequently, the proof of Theorem 7.2 is now completed. □

8 Regularity results for the M-LL system

(Carpe Diem, you live just once)

In this chapter we discuss the existence, uniqueness and regularity of a regular solutions to the coupled full M-LL system. The system reads as

$$\begin{aligned} \partial_t \mathbf{m} &= -\mathbf{m} \times (\Delta \mathbf{m} + \mathbf{H}) \\ & \quad -\alpha \mathbf{m} \times (\mathbf{m} \times (\Delta \mathbf{m} + \mathbf{H})), && (8.1) \\ \partial_t \mathbf{E} + \sigma \mathbf{E} - \nabla \times \mathbf{H} &= \mathbf{0}, && (8.2) \\ \partial_t \mathbf{H} + \nabla \times \mathbf{E} &= -\beta \partial_t \mathbf{m}, && (8.3) \\ \nabla \cdot \mathbf{H} + \beta \nabla \cdot \mathbf{m} &= 0, && (8.4) \\ \nabla \cdot \mathbf{E} &= 0, && (8.5) \end{aligned}$$

where α, β and σ are constants, $\alpha > 0,\ \sigma \geq 0$. In practical applications is σ a "nice" function of the space describing the conductivity of the medium. Nevertheless, non-constant σ would not change the mathematical analysis and therefore we can consider it as a constant.

We consider the following boundary conditions

$$\begin{aligned} \frac{\partial \mathbf{m}}{\partial \boldsymbol{\nu}}\Big|_{\partial\Omega} &= 0, && (8.6) \\ \mathbf{E} \times \boldsymbol{\nu}\big|_{\partial\Omega} &= 0, && (8.7) \\ (\mathbf{H} + \beta \mathbf{m}) \cdot \boldsymbol{\nu}\big|_{\partial\Omega} &= 0, && (8.8) \end{aligned}$$

where $\boldsymbol{\nu}$ is the unit outward normal vector to $\partial\Omega$.

The initial conditions read as

$$\mathbf{m}(x,0)=\mathbf{m}_0(x),\quad \mathbf{H}(x,0)=\mathbf{H}_0(x),\quad \mathbf{E}(x,0)=\mathbf{E}_0(x)\quad (x\in\Omega\subset\mathbb{R}^3).$$

A crucial observation is, that $|\mathbf{m}|=1$, for almost all $t\in\langle 0,\infty)$ provided that the solution to (8.1)–(8.5) is sufficiently smooth. This comes from a scalar multiplication of (8.1) with $\mathbf{m}$. Then the equation (8.1) is equivalent to:

$$\partial_t\mathbf{m}-\alpha\Delta\mathbf{m}-\alpha|\nabla\mathbf{m}|^2\mathbf{m}+\mathbf{m}\times\Delta\mathbf{m}=-\mathbf{m}\times\mathbf{H}-\alpha\mathbf{m}\times(\mathbf{m}\times\mathbf{H}). \tag{8.9}$$

The transformation of the equation (8.1) to the equation (8.9) is a classical approach used for example in [18, 35, 61].

In Section 8.1 we define a weak solution to the problem (8.9), along with (8.2)–(8.5).

In Section 8.2 we look at the appropriate function spaces, in which we seek the solutions. We establish the theory of the approximation of this spaces by finite-dimensional spaces. Thus, we will be able to define a finite dimensional problem, which has an unique solution.

In the next section we derive estimates for the approximate solution in various function spaces. These estimates are crucial when passing to the limit for $n\to\infty$. The main results are mentioned in Lemma 8.7.

Section 8.4 is devoted to the limit process when $n\to\infty$. First, we prove the existence of functions $\mathbf{m},\mathbf{E},\mathbf{H}$, which are the limits of subsequences of $\{\mathbf{m}_n\},\{\mathbf{E}_n\},\{\mathbf{H}_n\}$ in various function spaces. Then we compute limits of all terms appearing in the definition of a weak solution and finally we conclude that the functions $\mathbf{m},\mathbf{E},\mathbf{H}$ are really weak solutions.

Finally, in Section 8.5 we prove the regularity results for the weak solution.

8.1 Weak solution to the M-LL system

Definition 8.1 *The triple* $(\mathbf{m},\mathbf{E},\mathbf{H})$*, where*

$$\begin{aligned}\mathbf{m} &\in L^\infty(I,\mathbf{H}^1_{\nabla 0}(\Omega))\cap H^1(I,\mathbf{L}^2(\Omega)),\\ \mathbf{E} &\in L^\infty(I,\mathbf{H}_{t0}(\mathrm{curl},\mathrm{div},\Omega)),\\ \mathbf{H}+\beta\mathbf{m} &\in L^\infty(I,\mathbf{H}_{no}(\mathrm{curl},\mathrm{div},\Omega)),\end{aligned}$$

is called a weak solution of problem stated in (8.9), (8.2)–(8.5) if $|\mathbf{m}|=1$*, almost everywhere in* $I\times\Omega$*,* $\mathbf{m}(\mathbf{x},0)=\mathbf{m}_0$ *holds in the sense of traces, and for all test functions*

$$\boldsymbol{\varphi}\ \in\ L^\infty(I,\mathbf{H}^1(\Omega)),$$

$$\begin{aligned}
\boldsymbol{\psi} &\in L^\infty(I, \mathbf{H}_{t0}(\mathrm{curl}, \Omega)) \cap H^1(I, \mathbf{L}^2(\Omega)), \quad \boldsymbol{\psi}(\mathbf{x}, T) = 0,\\
\boldsymbol{\phi} &\in L^\infty(I, \mathbf{H}(\mathrm{curl}, \Omega)) \cap H^1(I, \mathbf{L}^2(\Omega)), \quad \boldsymbol{\phi}(\mathbf{x}, T) = 0,\\
\xi &\in L^\infty(I, \mathbf{H}^1(\Omega)),\\
\zeta &\in L^\infty(I, \mathbf{H}^1_0(\Omega)),
\end{aligned}$$

the following identities hold,

$$\begin{aligned}
&\int_I (\partial_t \mathbf{m}, \boldsymbol{\varphi}) \mathrm{d}s + \alpha \int_I (\nabla \mathbf{m}, \nabla \boldsymbol{\varphi}) \mathrm{d}s = \alpha \int_I (|\nabla \mathbf{m}|^2 \mathbf{m}, \boldsymbol{\varphi}) \mathrm{d}s\\
&\quad \int_I (\mathbf{m} \times \nabla \mathbf{m}, \nabla \boldsymbol{\varphi}) \mathrm{d}s - \int_I (\mathbf{m} \times \mathbf{H}, \boldsymbol{\varphi}) \mathrm{d}s\\
&\quad +\alpha \int_I (\mathbf{m} \times \mathbf{H}, \mathbf{m} \times \boldsymbol{\varphi}) \mathrm{d}s, \qquad (8.10)
\end{aligned}$$

$$\int_I e^{\sigma s} (\nabla \times \mathbf{H}, \boldsymbol{\psi}) \mathrm{d}s + \int_I (\mathbf{E} e^{\sigma s}, \partial_t \boldsymbol{\psi}) \mathrm{d}s + (\mathbf{E}_0, \boldsymbol{\psi}(\mathbf{x}, 0)) = 0, \qquad (8.11)$$

$$\int_I (\mathbf{H} + \beta \mathbf{m}, \partial_t \boldsymbol{\phi}) \mathrm{d}s + (\mathbf{H}_0 + \beta \mathbf{m}_0, \boldsymbol{\phi}(\mathbf{x}, 0)) - \int_I (\nabla \times \mathbf{E}, \boldsymbol{\phi}) \mathrm{d}s = 0, \qquad (8.12)$$

$$\int_I (\nabla \cdot \mathbf{H} + \beta \nabla \cdot \mathbf{m}, \xi) \mathrm{d}s = 0, \qquad (8.13)$$

$$\int_I (\nabla \cdot \mathbf{E}, \zeta) \mathrm{d}s = 0. \qquad (8.14)$$

8.2 Finite approximation

Looking at the problem given by (8.9), (8.2)–(8.5) and the boundary conditions (8.6)–(8.8), we demand that the solution has to belong to the following functional spaces:

$$\begin{aligned}
\mathbf{m} &\in \mathbf{H}^1_{\nabla 0}(\Omega),\\
\mathbf{E} &\in \mathbf{H}_{t0}(\mathrm{curl}, \mathrm{div}0, \Omega),\\
\mathbf{H} + \beta \mathbf{m} &\in \mathbf{H}_{n0}(\mathrm{curl}, \mathrm{div}0, \Omega).
\end{aligned}$$

Now, we define approximation spaces for these spaces. It is a natural choice to approximate the space $\mathbf{H}^1_{\nabla 0}(\Omega)$ by finite-dimensional spaces $\mathbf{V}_n$ build on the first n eigenvectors of the "weak" operator $I - \Delta$ with domain $\mathbf{H}^1_{\nabla 0}(\Omega)$. We denote by P_n orthogonal projection of $\mathbf{L}^2(\Omega)$ on $\mathbf{V}_n$.

Next, we define the following eigenvalue problems.

Problem 8.1 *Find* $(\mathbf{u}, \omega) \in \mathbf{H}_{t0}(\mathrm{curl}, \mathrm{div}0, \Omega) \times \mathbb{R}$ *such that*

$$(\nabla \times \mathbf{u}, \nabla \times \mathbf{v}) = \omega^2 (\mathbf{u}, \mathbf{v}), \qquad \forall \mathbf{v} \in \mathbf{H}_{t0}(\mathrm{curl}, \mathrm{div}0, \Omega).$$

Problem 8.2 *Find* $(\mathbf{u},\omega) \in \mathbf{H}_{n0}(\mathrm{curl}, \mathrm{div}0, \Omega) \times \mathbb{R}$ *such that*

$$(\nabla \times \mathbf{u}, \nabla \times \mathbf{v}) = \omega^2(\mathbf{u},\mathbf{v}), \qquad \forall \mathbf{v} \in \mathbf{H}_{n0}(\mathrm{curl}, \mathrm{div}0, \Omega).$$

Both problems are equivalent to an eigenvalue problem for a self-adjoint unbounded operator with its compact inverse. The spectrum is a denumerable set of real isolated eigenvalues with final multiplicities. Moreover, the direct Hilbert sum of the eigenspaces related to Problem 8.1 actually covers $\mathbf{H}_{t0}(\mathrm{curl}, \mathrm{div}0, \Omega)$. Therefore, we can build finite-dimensional spaces $\mathbf{V}_n^{t0}$ from the first n eigenvectors related to Problem 8.1. The spaces $\mathbf{V}_n^{t0}$ are a good approximation of the space $\mathbf{H}_{t0}(\mathrm{curl}, \mathrm{div}0, \Omega)$. We denote by Q_n the orthogonal projection from $\mathbf{L}^2(\Omega)$ on $\mathbf{V}_n^{t0}$.

Similarly we can build finite-dimensional spaces $\mathbf{V}_n^{n0}$ from the first n eigenvectors related to Problem 8.2. The spaces $\mathbf{V}_n^{n0}$ are a good approximation of the space $\mathbf{H}_{n0}(\mathrm{curl}, \mathrm{div}0, \Omega)$. We denote by R_n the orthogonal projection from $\mathbf{L}^2(\Omega)$ on $\mathbf{V}_n^{n0}$.

For more details see [17, 30].

We are ready to define an approximated solution to the problem defined by (8.9), (8.2)–(8.5) along with (8.6)–(8.8). We seek for a triple $(\mathbf{m}_n, \mathbf{E}_n, \mathbf{H}_n)$ such that

$$\mathbf{m}_n \in \mathbf{V}_n, \quad \mathbf{E}_n \in \mathbf{V}_n^{t0}, \quad \text{and} \quad \mathbf{H}_n + \beta\mathbf{m}_n \in \mathbf{V}_n^{n0}.$$

satisfying the discretized M-LL system

$$\begin{aligned}
&\partial_t \mathbf{m}_n - \alpha\Delta\mathbf{m}_n - P_n\left[\alpha|\nabla\mathbf{m}_n|^2\mathbf{m}_n - \mathbf{m}_n \times \Delta\mathbf{m}_n\right] \\
&\quad = P_n\left[-\mathbf{m}_n \times \mathbf{H}_n - \alpha\mathbf{m}_n \times (\mathbf{m}_n \times \mathbf{H}_n)\right], && (8.15)\\
&\partial_t \mathbf{E}_n + \sigma\mathbf{E}_n - Q_n\left[\nabla \times \mathbf{H}_n\right] = 0, && (8.16)\\
&\partial_t \mathbf{H}_n + R_n\left[\nabla \times \mathbf{E}_n\right] = -\beta\partial_t\mathbf{m}_n, && (8.17)\\
&\nabla \cdot \mathbf{H}_n + \beta\nabla \cdot \mathbf{m}_n = 0, && (8.18)\\
&\nabla \cdot \mathbf{E}_n = 0, && (8.19)
\end{aligned}$$

8.3 Estimates for approximating solution

Next we do some analysis using techniques such as multiplication of the equations (8.15)–(8.19) with some vector function and then integrating over Ω to obtain estimates in $L^2(\Omega)$ norm. The projection operators P_n, Q_n and R_n seem to cause troubles. For example, the vectors $\mathbf{m}_n \times \mathbf{H}_n$ and $\mathbf{m}_n$ are orthogonal in $\mathbb{R}^3$ space, therefore after a multiplication of (8.15) by $\mathbf{m}_n$ would the term $(\mathbf{m}_n \times \mathbf{H}_n, \mathbf{m}_n)$ normally disappear. Question is, if the term $(P_n(\mathbf{m}_n \times \mathbf{H}_n), \mathbf{m}_n)$ will be zero. The answer on this question is simple. There is a little bit different situation depicted in Figure 8.1. Here the bigger space is $\mathbb{R}^3$ and not $L^2(\Omega)$ and the smaller space is the plane $\mathbb{R}^2$ given by axes x and y. This corresponds to the space V_n.

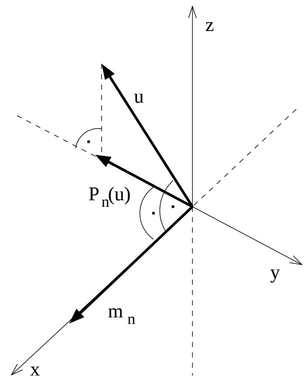


Figure 8.1: Orthogonality of the projection

The vector $\mathbf{u} = \mathbf{m}_n \times \mathbf{H}_n$ is orthogonal to vector $\mathbf{m}_n$ and his projection $P_n(\mathbf{u})$ lies in the plane xy. We simply see that if $(\mathbf{u}, \mathbf{m}_n) = 0$ then also $(P(\mathbf{u}), \mathbf{m}_n) = 0$. This can be written exactly in the following way. Since $P_n(\mathbf{u})$ is a orthogonal projection into space V_n we know that

$$(\mathbf{u} - P_n(\mathbf{u}), \mathbf{w}) = 0,$$

for all $\mathbf{w} \in V_n$. Taking $\mathbf{w} = \mathbf{m}_n$ and supposing $(\mathbf{u}, \mathbf{m}_n) = 0$ we get

$$(P_n(\mathbf{u}), \mathbf{m}_n) = -(\mathbf{u} - P_n(\mathbf{u}), \mathbf{m}_n) + (\mathbf{u}, \mathbf{m}_n) = 0.$$

Moreover, we use the property

$$(P_n(\mathbf{u}), \mathbf{v}) = (\mathbf{u}, P_n(\mathbf{v})).$$

In the following lemmas we use the previous remark and the fact that orthogonal projections satisfy

$$\|P_n(\mathbf{u})\| \leq \|\mathbf{u}\|.$$

Lemma 8.1 *For every real positive T the following estimates are valid for the solution $\mathbf{m}_n$ to (8.15)–(8.19):*

$$\partial_t \|\mathbf{m}_n\|_2^2 + \alpha \|\nabla \mathbf{m}_n\|_2^2 \leq C(1 + \|\mathbf{m}_n\|_{W^{2,2}}^{18}). \tag{8.20}$$

Proof:
Take (8.15) and multiply by $\mathbf{m}_n$ to get

$$\frac{1}{2}\partial_t\|\mathbf{m}_n\|_2^2 + \alpha\|\nabla\mathbf{m}_n\|_2^2 \le \alpha|(|\nabla\mathbf{m}_n|^2\mathbf{m}_n, \mathbf{m}_n)| \le \alpha\|\nabla\mathbf{m}_n\|_2^2\|\mathbf{m}_n\|_{L^\infty}^2.$$

We use the embedding $W^{2,2}(\Omega) \hookrightarrow L^\infty(\Omega)$ to conclude

$$\partial_t\|\mathbf{m}_n\|_2^2 + \alpha\|\nabla\mathbf{m}_n\|_2^2 \le C\|\mathbf{m}_n\|_{W^{2,2}}^4 \le C(1 + \|\mathbf{m}_n\|_{W^{2,2}}^{18}),$$

where we have used the Young inequality. We are able to prove better result with lower exponent than 18. Such high value is however necessary in the next. □

Lemma 8.2 *For every real positive T the following estimates are valid for the solution $\mathbf{m}_n$ to (8.15)–(8.19):*

$$\partial_t\|\nabla\mathbf{m}_n\|_2^2 + \alpha\|\Delta\mathbf{m}_n\|_2^2 \le C(1 + \|\mathbf{m}_n\|_{W^{2,2}}^{18} + \|\mathbf{H}_n\|_2^4) \tag{8.21}$$

Proof:
Take (8.15) and multiply by $-\Delta\mathbf{m}_n$ to get

$$\begin{aligned}\frac{1}{2}\partial_t\|\nabla\mathbf{m}_n\|_2^2 + \alpha\|\Delta\mathbf{m}_n\|_2^2 \quad \le \quad & \alpha\|\nabla\mathbf{m}_n\|_4\|\nabla\mathbf{m}_n\|_4\|\mathbf{m}_n\|_{L^\infty}\|\Delta\mathbf{m}_n\|_2 \\ & +|P_n(\mathbf{m}_n \times \Delta\mathbf{m}_n), \Delta\mathbf{m}_n)| \\ & +\|\mathbf{m}_n\|_{L^\infty}\|\mathbf{H}_n\|_2\|\Delta\mathbf{m}_n\|_2 \\ & +\|\mathbf{m}\|_{L^\infty}^2\|\mathbf{H}_n\|_2\|\Delta\mathbf{m}_n\|_2\end{aligned}$$

The term $P_n(\mathbf{m}_n \times \Delta\mathbf{m}_n), \Delta\mathbf{m}_n)$ vanishes due to the properties of the projection. We use the embeddings $W^{2,2}(\Omega) \hookrightarrow W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$ and Young's inequality to proceed and get

$$\partial_t\|\nabla\mathbf{m}_n\|_2^2 + \alpha\|\Delta\mathbf{m}_n\|_2^2 \quad \le \quad C(\|\mathbf{m}_n\|_{W^{2,2}}^6 + \|\mathbf{m}_n\|_{W^{2,2}}^4 + \|\mathbf{H}_n\|_2^2).$$

□

Lemma 8.3 *For every real positive T the following estimates are valid for the solution $\mathbf{m}_n$ to (8.15)–(8.19):*

$$\partial_t\|\Delta\mathbf{m}_n\|_2^2 + \alpha\|\nabla\Delta\mathbf{m}_n\|_2^2 \le C(1 + \|\mathbf{m}_n\|_{W^{2,2}}^{18} + \|\mathbf{H}_n\|_{W^{1,2}}^4). \tag{8.22}$$

Proof:
We apply operator Δ on (8.15) to get

$$\begin{aligned}\partial_t\Delta\mathbf{m}_n - \alpha\Delta^2\mathbf{m}_n \quad = \quad & \Delta P_n\alpha|\nabla\mathbf{m}_n|^2\mathbf{m}_n - \Delta P_n\mathbf{m}_n \times \Delta\mathbf{m}_n \\ & +\Delta P_n\mathbf{m}_n \times \mathbf{H}_n - \alpha\Delta P_n\mathbf{m}_n \times (\mathbf{m}_n \times \mathbf{H}_n).\end{aligned}$$

Then we multiply the previous identity by $\Delta\mathbf{m}_n$ and we perform integration by parts. Thanks to the properties of P_n mentioned in the beginning of this section we get

$$\begin{aligned}(\partial_t\Delta\mathbf{m}_n,\Delta\mathbf{m}_n)+\alpha\|\nabla\Delta\mathbf{m}_n\|_2^2 \quad\le\quad & \alpha|(\nabla(|\nabla\mathbf{m}_n|^2\mathbf{m}_n),\nabla\Delta\mathbf{m}_n)| \\ & +|(\nabla(\mathbf{m}_n\times\Delta\mathbf{m}_n),\nabla\Delta\mathbf{m}_n)| \\ & +|(\nabla(\mathbf{m}_n\times\mathbf{H}_n),\nabla\Delta\mathbf{m}_n)| \\ & +\alpha|(\nabla(\mathbf{m}_n\times(\mathbf{m}_n\times\mathbf{H}_n)),\nabla\Delta\mathbf{m}_n)|.\end{aligned}$$

Next we derive the terms on the right-hand side. Then we use appropriate integral inequalities:

$$\begin{aligned}&\partial_t\|\Delta\mathbf{m}_n\|_2^2+\alpha\|\nabla\Delta\mathbf{m}_n\|_2^2 \\ &\quad\le 2\|\nabla\mathbf{m}_n\|_{W^{1,4}}\|\nabla\mathbf{m}_n\|_4\|\mathbf{m}_n\|_{L^\infty}\|\nabla\Delta\mathbf{m}_n\|_2 \\ &\qquad+\|\nabla\mathbf{m}_n\|_4^2\|\nabla\mathbf{m}_n\|_{L^\infty}\|\nabla\Delta\mathbf{m}_n\|_2+\|\nabla\mathbf{m}_n\|_4\|\Delta\mathbf{m}_n\|_4\|\nabla\Delta\mathbf{m}_n\|_2 \\ &\qquad+|(\mathbf{m}_n\times\nabla\Delta\mathbf{m}_n,\nabla\Delta\mathbf{m}_n)|+\|\nabla\mathbf{m}_n\|_4\|\mathbf{H}_n\|_4\|\nabla\Delta\mathbf{m}_n\|_2 \\ &\qquad+\|\mathbf{m}_n\|_{L^\infty}\|\nabla\mathbf{H}_n\|_2\|\nabla\Delta\mathbf{m}_n\|_2 \\ &\qquad+2\|\nabla\mathbf{m}_n\|_4\|\mathbf{m}_n\|_{L^\infty}\|\mathbf{H}_n\|_4\|\nabla\Delta\mathbf{m}_n\|_2+\|\mathbf{m}_n\|_{L^\infty}^2\|\nabla\mathbf{H}_n\|_2\|\nabla\Delta\mathbf{m}_n\|_2.\end{aligned}$$

Notice that the term $(\mathbf{m}_n\times\nabla\Delta\mathbf{m}_n,\nabla\Delta\mathbf{m}_n)=0$. To proceed we use embeddings $W^{2,2}(\Omega)\hookrightarrow L^\infty(\Omega)$ and $W^{1,4}(\Omega)\hookrightarrow L^\infty(\Omega)$:

$$\begin{aligned}&\partial_t\|\Delta\mathbf{m}_n\|_2^2+\alpha\|\nabla\Delta\mathbf{m}_n\|_2^2 \\ &\quad\le 2\|\nabla\mathbf{m}_n\|_{W^{1,4}}\|\nabla\mathbf{m}_n\|_4\|\mathbf{m}_n\|_{W^{2,2}}\|\nabla\Delta\mathbf{m}_n\|_2 \\ &\qquad+\|\nabla\mathbf{m}_n\|_4^2\|\nabla\mathbf{m}_n\|_{W^{1,4}}\|\nabla\Delta\mathbf{m}_n\|_2+\|\nabla\mathbf{m}_n\|_4\|\Delta\mathbf{m}_n\|_4\|\nabla\Delta\mathbf{m}_n\|_2 \\ &\qquad+\|\nabla\mathbf{m}_n\|_4\|\mathbf{H}_n\|_4\|\nabla\Delta\mathbf{m}_n\|_2+\|\mathbf{m}_n\|_{W^{2,2}}\|\nabla\mathbf{H}_n\|_2\|\nabla\Delta\mathbf{m}_n\|_2 \\ &\qquad+2\|\nabla\mathbf{m}_n\|_4\|\mathbf{m}_n\|_{W^{2,2}}\|\mathbf{H}_n\|_4\|\nabla\Delta\mathbf{m}_n\|_2 \\ &\qquad+\|\mathbf{m}_n\|_{W^{2,2}}^2\|\nabla\mathbf{H}_n\|_2\|\nabla\Delta\mathbf{m}_n\|_2\end{aligned}$$

and then we get rid of the term $\|\nabla\Delta\mathbf{m}_n\|_2$ using the weighted Young inequality:

$$\begin{aligned}&\partial_t\|\Delta\mathbf{m}_n\|_2^2+\alpha\|\nabla\Delta\mathbf{m}_n\|_2^2 \\ &\quad\le\varepsilon\|\nabla\Delta\mathbf{m}_n\|_2^2+C_\varepsilon\|\nabla\mathbf{m}_n\|_{W^{1,4}}^2\|\nabla\mathbf{m}_n\|_4^2\|\mathbf{m}_n\|_{W^{2,2}}^2 \\ &\qquad+C_\varepsilon\|\nabla\mathbf{m}_n\|_4^4\|\nabla\mathbf{m}_n\|_{W^{1,4}}^2+C_\varepsilon\|\nabla\mathbf{m}_n\|_4^2\|\Delta\mathbf{m}_n\|_4^2 \\ &\qquad+C_\varepsilon\|\nabla\mathbf{m}_n\|_4^2\|\mathbf{H}_n\|_4^2+C_\varepsilon\|\nabla\mathbf{H}_n\|_2^2\|\mathbf{m}_n\|_{W^{2,2}}^2 \\ &\qquad+C_\varepsilon\|\nabla\mathbf{m}_n\|_4^2\|\mathbf{H}_n\|_4^2\|\mathbf{m}_n\|_{W^{2,2}}^2+C_\varepsilon\|\nabla\mathbf{H}_n\|_2^2\|\mathbf{m}_n\|_{W^{2,2}}^4.\end{aligned}\tag{8.23}$$

We make use of (10.16) (10.12) and (10.13) to continue in (8.23)

$$\partial_t\|\Delta\mathbf{m}_n\|_2^2+\alpha\|\nabla\Delta\mathbf{m}_n\|_2^2$$

$$
\begin{aligned}
&\leq \varepsilon\|\nabla\Delta\mathbf{m}_n\|_2^2 + C_\varepsilon\|\Delta\mathbf{m}_n\|_2^{\frac{1}{2}}\|\nabla\Delta\mathbf{m}_n\|_2^{\frac{3}{2}}\|\nabla\mathbf{m}_n\|_2^{\frac{1}{2}}\|\Delta\mathbf{m}_n\|_2^{\frac{3}{2}}\|\mathbf{m}_n\|_{W^{2,2}}^2 \\
&\quad + C_\varepsilon\|\nabla\mathbf{m}_n\|_2\|\Delta\mathbf{m}_n\|_2^3\|\Delta\mathbf{m}_n\|_2^{\frac{1}{2}}\|\nabla\Delta\mathbf{m}_n\|_2^{\frac{3}{2}} \\
&\quad + C_\varepsilon\|\nabla\mathbf{m}_n\|_2^{\frac{1}{2}}\|\Delta\mathbf{m}_n\|_2^{\frac{3}{2}}\|\Delta\mathbf{m}_n\|_2^{\frac{1}{2}}\|\nabla\Delta\mathbf{m}_n\|_2^{\frac{3}{2}} \\
&\quad + C_\varepsilon\|\nabla\mathbf{m}_n\|_2^{\frac{1}{2}}\|\Delta\mathbf{m}_n\|_2^{\frac{3}{2}}\|\mathbf{H}_n\|_2^{\frac{1}{2}}\|\nabla\mathbf{H}_n\|_2^{\frac{3}{2}} + C_\varepsilon\|\nabla\mathbf{H}_n\|_2^4 + \|\mathbf{m}_n\|_{W^{2,2}}^4 \\
&\quad + C_\varepsilon\|\nabla\mathbf{m}_n\|_2^{\frac{1}{2}}\|\Delta\mathbf{m}_n\|_2^{\frac{3}{2}}\|\mathbf{H}_n\|_2^{\frac{1}{2}}\|\nabla\mathbf{H}_n\|_2^{\frac{3}{2}}\|\mathbf{m}_n\|_{W^{2,2}}^2 \\
&\quad + C_\varepsilon\|\nabla\mathbf{H}_n\|_2^4 + \|\mathbf{m}_n\|_{W^{2,2}}^8. \qquad (8.24)
\end{aligned}
$$

In the previous expression we skipped some terms. We considered only terms including the highest space derivation as the "worst" case. The terms with lowest space derivative would not cause any problems.

After using the weighted Young inequality and setting ε small enough we get

$$\partial_t\|\Delta\mathbf{m}_n\|_2^2 + \|\nabla\Delta\mathbf{m}_n\|_2^2 \quad \leq \quad C_\varepsilon(1 + \|\mathbf{H}_n\|_2^4 + \|\nabla\mathbf{H}_n\|_2^4 + \|\mathbf{m}_n\|_{W^{2,2}}^{18}),$$

which verifies the result of the lemma. □

Lemma 8.4 *For every real positive T the following estimates are valid for the solution $\mathbf{m}_n$ to (8.15)–(8.19):*

$$\partial_t\|\mathbf{E}_n\|_2^2 + \partial_t\|\mathbf{H}_n\|_2^2 \leq C(1 + \|\mathbf{m}_n\|_{W^{2,2}}^{18} + \|\mathbf{H}_n\|_2^4). \qquad (8.25)$$

PROOF:
Take (8.16) and (8.17) and multiply by $\mathbf{E}_n$, $\mathbf{H}_n$ respectively, to get

$$
\begin{aligned}
(\partial_t\mathbf{E}_n, \mathbf{E}_n) + \sigma\|\mathbf{E}_n\|_2^2 - (\nabla\times\mathbf{H}_n, \mathbf{E}_n) &= 0, \qquad (8.26)\\
(\partial_t\mathbf{H}_n, \mathbf{H}_n) + (\nabla\times\mathbf{E}_n, \mathbf{H}_n) &= (\partial_t\mathbf{m}_n, \mathbf{H}_n). \qquad (8.27)
\end{aligned}
$$

Because of the boundary conditions (8.7)–(8.8) we have

$$(\nabla\times\mathbf{H}_n, \mathbf{E}_n) - (\nabla\times\mathbf{E}_n, \mathbf{H}_n) = 0.$$

Then after summing up the equations (8.26)–(8.27) we get

$$\partial_t\|\mathbf{E}_n\|_2^2 + \partial_t\|\mathbf{H}_n\|_2^2 \leq |(\partial_t\mathbf{m}_n, \mathbf{H}_n)|. \qquad (8.28)$$

Using (8.15) we estimate the term $|(\partial\mathbf{m}_n, \mathbf{H}_n)|$ as follows

$$
\begin{aligned}
|(\partial_t\mathbf{m}_n, \mathbf{H}_n)| \quad \leq \quad & \alpha|(\Delta\mathbf{m}_n, \mathbf{H}_n)| + \alpha|(|\nabla\mathbf{m}_n|^2\mathbf{m}_n, \mathbf{H}_n)| + |(\mathbf{m}_n\times\Delta\mathbf{m}_n, \mathbf{H}_n)| \\
& + |(\mathbf{m}_n\times\mathbf{H}_n, \mathbf{H}_n)| + \alpha|(\mathbf{m}_n\times(\mathbf{m}_n\times\mathbf{H}_n), \mathbf{H}_n)|.
\end{aligned}
$$

We use appropriate integral inequalities to get

$$\begin{aligned}|(\partial_t \mathbf{m}_n, \mathbf{H}_n)| \quad &\le \quad \alpha\|\Delta\mathbf{m}_n\|_2\|\mathbf{H}_n\|_2 + \alpha\|\nabla\mathbf{m}_n\|_4^2\|\mathbf{m}_n\|_{L^\infty}\|\mathbf{H}_n\|_2 \\ &\quad + \|\mathbf{m}_n\|_{L^\infty}\|\Delta\mathbf{m}_n\|_2\|\mathbf{H}_n\|_2 + \alpha\|\mathbf{m}_n\|_{L^\infty}^2\|\mathbf{H}_n\|_2^2.\end{aligned}$$

We use the embeddings $W^{2,2}(\Omega) \hookrightarrow W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$ and Young's inequality to proceed and get

$$|(\partial_t \mathbf{m}_n, \mathbf{H}_n)| \le C\Big[\|\mathbf{m}_n\|_{W^{2,2}}^2 + \|\mathbf{H}_n\|_2^2 + \|\mathbf{m}_n\|_{W^{2,2}}^6 + \|\mathbf{m}_n\|_{W^{2,2}}^4 + \|\mathbf{H}_n\|_2^4\Big].$$

Then we again apply the Young inequality to verify that

$$|(\partial_t \mathbf{m}_n, \mathbf{H}_n)| \le C(\|\mathbf{m}_n\|_{W^{2,2}}^{18} + \|\mathbf{H}_n\|_2^4).$$

The previous result together with (8.28) concludes the proof. □

Lemma 8.5 *For every real positive T and ε the following estimates are valid for the solution $\mathbf{m}_n$ to (8.15)–(8.19):*

$$\partial_t\|\nabla\mathbf{E}_n\|_2^2 + \partial_t\|\nabla\mathbf{H}_n\|_2^2 \le C_\varepsilon(1 + \|\mathbf{m}_n\|_{W^{2,2}}^{18} + \|\nabla\mathbf{H}_n\|_2^4) + \varepsilon\|\nabla\Delta\mathbf{m}_n\|_2^2. \tag{8.29}$$

PROOF:
We multiply (8.16) and (8.17) with $-\Delta\mathbf{E}_n$ and $-\Delta\mathbf{H}_n$, respectively. Because of the boundary conditions (8.7)–(8.8) we get

$$\partial_t\|\nabla\mathbf{E}_n\|_2^2 + \partial_t\|\nabla\mathbf{H}_n\|_2^2 \le |(\partial_t\nabla\mathbf{m}_n, \nabla\mathbf{H}_n)|. \tag{8.30}$$

We multiply (8.15) with $-\Delta\mathbf{H}_n$ to estimate the term $|(\partial_t\nabla\mathbf{m}_n, \nabla\mathbf{H}_n)|$. Thus we get

$$\begin{aligned}&|(\partial_t\nabla\mathbf{m}_n, \nabla\mathbf{H}_n)| \\ &\quad \le \alpha|(\nabla\Delta\mathbf{m}_n, \nabla\mathbf{H}_n)| + \alpha|(\nabla(|\nabla\mathbf{m}_n|^2\mathbf{m}_n), \nabla\mathbf{H}_n)| \\ &\qquad + |(\nabla(\mathbf{m}_n \times \mathbf{H}_n), \nabla\mathbf{H}_n)| + |(\nabla(\mathbf{m}_n \times \Delta\mathbf{m}_n), \nabla\mathbf{H}_n)| \\ &\qquad + \alpha|(\nabla(\mathbf{m}_n \times (\mathbf{m}_n \times \mathbf{H}_n)), \nabla\mathbf{H}_n)| \\ &\quad \le \varepsilon\|\nabla\Delta\mathbf{m}_n\|_2^2 + C_\varepsilon\|\nabla\mathbf{H}_n\|_2^2 \\ &\qquad + 2\|\nabla\mathbf{m}_n\|_{W^{1,4}}\|\nabla\mathbf{m}_n\|_4\|\mathbf{m}_n\|_{L^\infty}\|\nabla\mathbf{H}_n\|_2 \\ &\qquad + \|\nabla\mathbf{m}_n\|_4^2\|\nabla\mathbf{m}_n\|_{L^\infty}\|\nabla\mathbf{H}_n\|_2 + \|\nabla\mathbf{m}_n\|_4\|\Delta\mathbf{m}_n\|_4\|\nabla\mathbf{H}_n\|_2 \\ &\qquad + \|\mathbf{m}_n\|_{L^\infty}\|\nabla\Delta\mathbf{m}_n\|_2\|\nabla\mathbf{H}_n\|_2 + \|\nabla\mathbf{m}_n\|_4\|\mathbf{H}_n\|_4\|\nabla\mathbf{H}_n\|_2 \\ &\qquad + \|\mathbf{m}_n\|_{L^\infty}\|\nabla\mathbf{H}_n\|_2^2 + 2\|\nabla\mathbf{m}_n\|_4\|\mathbf{m}_n\|_{L^\infty}\|\mathbf{H}_n\|_4\|\nabla\mathbf{H}_n\|_2 \\ &\qquad + \|\mathbf{m}_n\|_{L^\infty}^2\|\nabla\mathbf{H}_n\|_2^2.\end{aligned}$$

Next we use (10.16), the embeddings $W^{2,2}(\Omega) \hookrightarrow W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$, and the Young inequality for several times to arrive at

$$\begin{aligned} |(\partial_t \nabla \mathbf{m}_n, \nabla \mathbf{H}_n)| \quad \le \quad & \varepsilon \|\nabla \Delta \mathbf{m}_n\|_2^2 + C_\varepsilon \big[\|\nabla \mathbf{H}_n\|_2^2 + \|\Delta \mathbf{m}_n\|_4^2 \\ & + \|\mathbf{m}_n\|_{W^{2,2}}^4 \|\nabla \mathbf{H}_n\|_2^2 + \|\mathbf{m}_n\|_{W^{2,2}}^2 \|\nabla \mathbf{H}_n\|_2^2 \\ & + \|\mathbf{m}_n\|_{W^{2,2}}^2 + \|\mathbf{H}_n\|_{W^{1,2}}^4 + \|\nabla \mathbf{H}_n\|_2^4 + \|\mathbf{m}_n\|_{W^{2,2}}^4 \big]. \end{aligned}$$

Using (10.13) and again the Young inequality gives us

$$|(\partial_t \nabla \mathbf{m}_n, \nabla \mathbf{H}_n)| \le \varepsilon \|\nabla \Delta \mathbf{m}_n\|_2^2 + C_\varepsilon \big[1 + \|\mathbf{m}_n\|_{W^{2,2}}^8 + \|\mathbf{H}_n\|_{W^{1,2}}^4 \big],$$

which confirms the result of the lemma. □

If we summarize the results from Lemmas 8.1–8.5, we get

$$\begin{aligned} & \partial_t \|\mathbf{m}_n\|_2^2 + \partial_t \|\Delta \mathbf{m}_n\|_2^2 + \alpha \|\nabla \Delta \mathbf{m}_n\|_2^2 \\ & + \partial_t \|\mathbf{E}_n\|_2^2 + \partial_t \|\mathbf{H}_n\|_2^2 + \partial_t \|\nabla \mathbf{E}_n\|_2^2 + \partial_t \|\nabla \mathbf{H}_n\|_2^2 \\ & \quad \le C_\varepsilon (1 + \|\mathbf{m}_n\|_{W^{2,2}}^{18} + \|\mathbf{E}_n\|_2^4 + \|\mathbf{H}_n\|_2^4 + \|\nabla \mathbf{E}_n\|_2^4 + \|\nabla \mathbf{H}_n\|_2^4) \\ & \qquad + \varepsilon \|\nabla \Delta \mathbf{m}_n\|_2^2. \end{aligned} \tag{8.31}$$

As ε was arbitrary positive real number suppose $\varepsilon = \alpha/2$. Then, we easily get rid of the term $\varepsilon \|\nabla \Delta \mathbf{m}_n\|_2^2$.

As Ω is a bounded regular open set we have the equivalence of the norms $\|u\|_{W^{2,2}}$ and $\left(\|u\|_2^2 + \|\Delta u\|_2^2 \right)^{\frac{1}{2}}$, see Appendix, remark after Lemma 10.4. The proof of this equivalence is based on the regularity result for the operator $I - \Delta$ with domain

$$\left\{ u \in W^{2,2}(\Omega), \frac{\partial u}{\partial \boldsymbol{\nu}}\Big|_{\partial \Omega} = 0. \right\}$$

We refer to [18, 32]. Then we use this result in (8.31) to replace $\|\mathbf{m}_n\|_{W^{2,2}}$ with $\|\mathbf{m}\|_2^2 + \|\Delta \mathbf{m}\|_2^2$. to get

$$\begin{aligned} & \partial_t \|\mathbf{m}_n\|_2^2 + \partial_t \|\Delta \mathbf{m}_n\|_2^2 + \tfrac{\alpha}{2} \|\nabla \Delta \mathbf{m}_n\|_2^2 \\ & + \partial_t \|\mathbf{E}_n\|_2^2 + \partial_t \|\mathbf{H}_n\|_2^2 + \partial_t \|\nabla \mathbf{E}_n\|_2^2 + \partial_t \|\nabla \mathbf{H}_n\|_2^2 \\ & \quad \le C(1 + \|\mathbf{m}_n\|_2^{18} + \|\Delta \mathbf{m}_n\|_2^{18} + \|\mathbf{E}_n\|_2^4 + \|\mathbf{H}_n\|_2^4 \\ & \qquad + \|\nabla \mathbf{E}_n\|_2^4 + \|\nabla \mathbf{H}_n\|_2^4). \end{aligned} \tag{8.32}$$

Let us define the following functions

$$\begin{aligned} f(t) &= \partial_t \|\mathbf{m}_n\|_2^2 + \partial_t \|\Delta \mathbf{m}_n\|_2^2, \\ g(t) &= \partial_t \|\mathbf{E}_n\|_2^2 + \partial_t \|\mathbf{H}_n\|_2^2 + \partial_t \|\nabla \mathbf{E}_n\|_2^2 + \partial_t \|\nabla \mathbf{H}_n\|_2^2. \end{aligned}$$

From (8.32) we can then conclude that

$$f'(t) + g'(t) \le C(1 + f^9(t) + g^2(t)). \tag{8.33}$$

In the next lemma we mention a Bihary-type inequality, see for example [3].

Lemma 8.6 *Let u, a, b and k be nonnegative continuous functions in $J = [\alpha_1, \beta_1]$, and let $p > 1$ be a constant. Suppose a/b is nondecreasing in J and*

$$u(t) \leq a(t) + b(t) \int_{\alpha_1}^{\beta_1} k(s) u^p(s) \mathrm{d}s, \quad t \in J.$$

Then

$$u(t) \leq a(t) \left\{ 1 - (p-1) \int_{\alpha_1}^{\beta_1} k(s) b(s) a^{p-1}(s) \mathrm{d}s \right\}^{\frac{1}{1-p}}, \quad \alpha \leq t \leq \beta_p,$$

where $\beta_p = \sup\{t \in J : (p-1) \int_{\alpha_1}^{t} k(s) b(s) a^{p-1}(s) \mathrm{d}s < 1\}$.

The following lemma summarize the in-time local regularity results for the solution $(\mathbf{m}_n, \mathbf{E}_n, \mathbf{H}_n)$ to the system (8.15)–(8.19).

Lemma 8.7 *There exist a positive T_0 and a constant C both depending only on the domain Ω, α, and on the size of initial data in $W^{2,2}(\Omega)$ such that for every positive $T < T_0$*

$$\sup_{0<t<T} \left[\|\mathbf{m}_n\|^2_{W^{2,2}} + \|\mathbf{E}_n\|^2_{W^{1,2}} + \|\mathbf{H}_n\|^2_{W^{1,2}} \right] \leq C, \tag{8.34}$$

$$\sup_{0<t<T} \|\mathbf{m}_n\|^2_{L^\infty} \leq C, \tag{8.35}$$

$$\int_0^T \|\mathbf{m}_n\|^2_{W^{3,2}} \mathrm{d}\tau \leq C. \tag{8.36}$$

PROOF:
The first result comes directly using Lemma 8.6 for the functions f and g. The second results is a simple consequence of the first one and the embedding $L^\infty(\Omega) \hookrightarrow W^{2,2}(\Omega)$. The third result can be verified by integrating of (8.32) in time and using (8.34). □

Lemma 8.8 *There exist a positive T_0 and a constant C both depending only on the domain Ω, α and on the size of initial data in $W^{2,2}(\Omega)$ such that for every positive $T < T_0$*

$$\int_0^T \|\partial_t \mathbf{m}_n\|^2_{W^{1,2}} \leq C. \tag{8.37}$$

PROOF:
Multiply (8.15) first with $\partial_t \mathbf{m}_n$ and then with $-\partial_t \Delta \mathbf{m}_n$ to verify the result of lemma. □

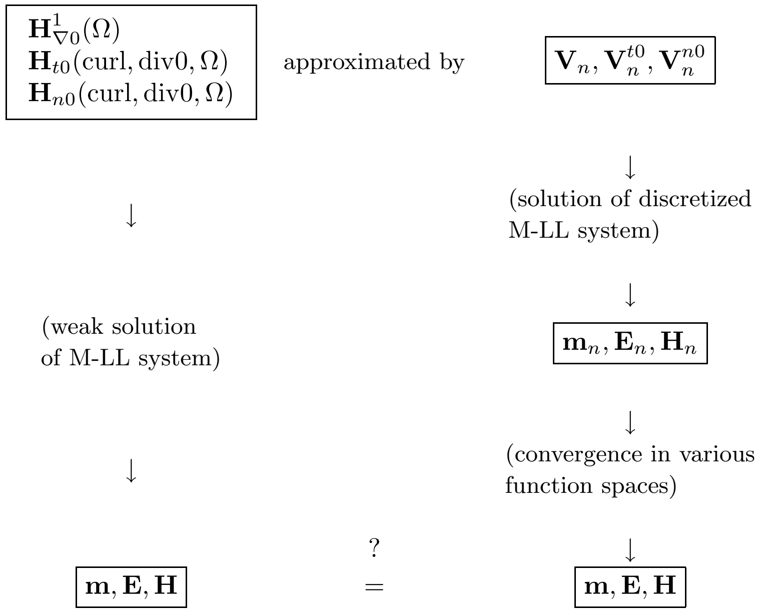


Figure 8.2: Convergence diagram

8.4 The convergence of finite approximation

Now we would like to prove that the sequences $\{\mathbf{m}_n\}$, $\{\mathbf{E}_n\}$, and $\{\mathbf{H}_n\}$ converge in some sense and that the limits are weak solutions to (8.1)–(8.5). By this we complete the convergence diagram in Figure 8.2.

Lemma 8.9 *For T_0 from Lemma 8.7 there exists a triple* $(\mathbf{m}, \mathbf{E}, \mathbf{H})$, *where*

$$\begin{aligned}
&\mathbf{m} \in L^2(I, W^{3,2}(\Omega)) \cap L^\infty(I, W^{1,2}(\Omega)),\\
&\mathbf{E}, \mathbf{H} \in L^2(I, W^{1,2}(\Omega)) \cap L^\infty(I, L^2(\Omega)),\\
&\partial_t \mathbf{m}, \partial_t \mathbf{E}, \partial_t \mathbf{H} \in L^2(I, L^2(\Omega)),
\end{aligned}$$

for $I = \langle 0, T_0 \rangle$ and subsequences still denoted $\{\mathbf{m}_n\}_n$, $\{\mathbf{E}_n\}_n$, $\{\mathbf{H}_n\}_n$ such that for any $1 < p < \infty$

$$\mathbf{m}_n \rightharpoonup \mathbf{m} \text{ in } L^2(I, W^{3,2}(\Omega)), \tag{8.38}$$

$$\partial_t \mathbf{m}_n \rightharpoonup \partial_t \mathbf{m} \text{ in } L^2(I, W^{1,2}(\Omega)), \tag{8.39}$$

$$\mathbf{m}_n \to \mathbf{m} \text{ in } L^2(I, W^{2,2}(\Omega)), \tag{8.40}$$
$$\mathbf{m}_n \to \mathbf{m} \text{ in } L^p(I, W^{1,2}(\Omega)), \tag{8.41}$$
$$\mathbf{H}_n \rightharpoonup \mathbf{H} \text{ in } L^2(I, W^{1,2}(\Omega)), \tag{8.42}$$
$$\mathbf{E}_n \rightharpoonup \mathbf{E} \text{ in } L^2(I, W^{1,2}(\Omega)), \tag{8.43}$$
$$\partial_t \mathbf{H}_n \rightharpoonup \partial_t \mathbf{H} \text{ in } L^2(I, L^2(\Omega)), \tag{8.44}$$
$$\partial_t \mathbf{E}_n \rightharpoonup \partial_t \mathbf{E} \text{ in } L^2(I, L^2(\Omega)), \tag{8.45}$$
$$\mathbf{H}_n \to \mathbf{H} \text{ in } L^p(I, L^2(\Omega)), \tag{8.46}$$
$$\mathbf{E}_n \to \mathbf{E} \text{ in } L^p(I, L^2(\Omega)). \tag{8.47}$$

PROOF:
From (8.36) we have boundedness of $\{\mathbf{m}_n\}$ in $L^2(I, W^{3,2}(\Omega))$. Since the space $L^2(I, W^{3,2}(\Omega))$ is a reflexive Banach space we can conclude that there exists a subsequence of $\{\mathbf{m}_n\}$ still denoted by $\{\mathbf{m}_n\}$ such that (8.38) is valid.

Similarly we get weak convergence of $\{\partial_t \mathbf{m}_n\}$ in $L^2(I, W^{1,2}(\Omega))$, because of (8.37). We denote the convergent subsequence of $\{\partial_t \mathbf{m}_n\}$ again by $\{\partial_t \mathbf{m}_n\}$:

$$\partial_t \mathbf{m}_n \rightharpoonup \mathbf{w} \text{ in } L^2(I, W^{1,2}(\Omega)).$$

We use Theorem 10.8. This theorem says if $u_n \rightharpoonup u$ and $\partial_t u_n \rightharpoonup v$ in $L^p(I, V)$, where V is a reflexive Banach space then $\partial_t u = v$ in sense of $L^p(I, V)$. So, we confirm (8.39).

Now, we show that $\{\mathbf{m}_n\}$ is relatively compact in $L^2(I, W^{2,2}(\Omega))$. Set $X = W^{3,2}(\Omega)$, $B = W^{2,2}(\Omega)$ and $Y = L^2(\Omega)$. Then the embedding schema $X \hookrightarrow\hookrightarrow B \hookrightarrow Y$ is valid. We have also the estimates

$$\|\mathbf{m}_n\|_{L^2(I,X)} \le C \text{ and } \|\partial_t \mathbf{m}_n\|_{L^2(I,Y)} \le C.$$

Then according to Theorem 10.9 we deduce that $\{\mathbf{m}_n\}$ is relatively compact in $L^2(I, W^{2,2}(\Omega))$.

Since $\{\mathbf{m}_n\}$ is relatively compact in $L^2(I, W^{2,2}(\Omega))$ we can choose a subsequence still denoted by $\{\mathbf{m}_n\}$ such that (8.40) is valid.

Similarly we show that $\{\mathbf{m}_n\}$ is relatively compact in $L^p(I, W^{1,2}(\Omega))$. Set $X = W^{2,2}(\Omega)$, $B = W^{1,2}(\Omega)$ and $Y = L^2(\Omega)$. Then $X \hookrightarrow\hookrightarrow B \hookrightarrow Y$. Using the estimates $\|\mathbf{m}_n\|_{L^p(I,X)} \le C\|\mathbf{m}_n\|_{L^\infty(I,X)} \le C$ and $\|\partial_t \mathbf{m}_n\|_{L^2(I,Y)} \le C$ according to Theorem 5.1 from [51] we deduce that $\{\mathbf{m}_n\}$ is relatively compact in $L^p(I, W^{1,2}(\Omega))$. Therefore

$$\mathbf{m}_n \to \mathbf{m} \text{ in } L^p(I, W^{1,2}(\Omega)).$$

Using the same argumentation as for $\{\mathbf{m}_n\}$ while deriving (8.38)–(8.39) we can prove also the results (8.42)–(8.45).

For sequences $\{\mathbf{H}_n\}$ and $\{\mathbf{E}_n\}$ we can prove that they are relatively compact in $L^p(I, L^2(\Omega))$. Set $X = W^{1,2}(\Omega)$ and $B = Y = L^2(\Omega)$. Then $X \hookrightarrow\hookrightarrow B \hookrightarrow Y$. We have also

$$\|\mathbf{H}_n\|_{L^p(I,X)} \leq C\|\mathbf{H}_n\|_{L^\infty(I,X)} \leq C \text{ and } \|\partial_t \mathbf{H}_n\|_{L^2(I,Y)} \leq C.$$

Therefore, according to Theorem 5.1 from [51] we deduce that $\{\mathbf{H}_n\}$ is relatively compact in $L^p(I, L^2(\Omega))$. The same can be done also for $\{\mathbf{E}_n\}$. Thus

$$\begin{aligned} \mathbf{H}_n &\to \mathbf{H} \text{ in } L^p(I, L^2(\Omega)), \\ \mathbf{E}_n &\to \mathbf{E} \text{ in } L^p(I, L^2(\Omega)). \end{aligned}$$

□

From (8.38)–(8.47) we can compute the following limits:

$$\lim_{n\to\infty} \int_0^t (\nabla\Delta\mathbf{m}_n - \nabla\Delta\mathbf{m}, \phi) = 0, \quad (8.48)$$

$$\lim_{n\to\infty} \int_0^t (\partial_t\mathbf{m}_n - \partial_t\mathbf{m}, \phi) = 0, \quad (8.49)$$

$$\lim_{n\to\infty} \int_0^t (\nabla\partial_t\mathbf{m}_n - \nabla\partial_t\mathbf{m}, \phi) = 0, \quad (8.50)$$

$$\lim_{n\to\infty} \int_0^t \|\mathbf{m}_n - \mathbf{m}\|^2_{W^{2,2}(\Omega)} = 0, \quad (8.51)$$

$$\lim_{n\to\infty} \int_0^t \|\mathbf{m}_n - \mathbf{m}\|^p_{W^{1,2}(\Omega)} = 0, \quad (8.52)$$

for $1 < p < \infty$ and for all admissible ϕ.

For functions $\mathbf{H}_n$ and $\mathbf{E}_n$ we have

$$\lim_{n\to\infty} \int_0^t (\nabla\mathbf{H}_n - \nabla\mathbf{H}, \phi) = 0, \quad (8.53)$$

$$\lim_{n\to\infty} \int_0^t (\nabla\mathbf{E}_n - \nabla\mathbf{E}, \phi) = 0, \quad (8.54)$$

$$\lim_{n\to\infty} \int_0^t (\partial_t\mathbf{H}_n - \partial_t\mathbf{H}, \phi) = 0, \quad (8.55)$$

$$\lim_{n\to\infty} \int_0^t (\partial_t\mathbf{E}_n - \partial_t\mathbf{E}, \phi) = 0, \quad (8.56)$$

$$\lim_{n\to\infty} \int_0^t \|\mathbf{H}_n - \mathbf{H}\|^p_{L^2(\Omega)} = 0, \quad (8.57)$$

$$\lim_{n\to\infty} \int_0^t \|\mathbf{E}_n - \mathbf{E}\|^p_{L^2(\Omega)} = 0, \quad (8.58)$$

for $1 < p < \infty$ and for all admissible function ϕ.

We would like to pass $n \to \infty$ in the weak formulation of the equations (8.16)–(8.17), (8.15). To do so, the following lemma will be useful.

Lemma 8.10 *For triples* $(\mathbf{m}_n, \mathbf{E}_n, \mathbf{H}_n)$ *and* $(\mathbf{m}, \mathbf{E}, \mathbf{H})$ *from Lemma 8.9 the following equalities hold:*

$$\begin{aligned}
L_1 &:= \lim_{n\to\infty} \int_0^t (P_n(\mathbf{m}_n \times \nabla \mathbf{m}_n), \nabla\phi) - \int_0^t (\mathbf{m} \times \nabla \mathbf{m}, \nabla \phi) = 0,\\
L_2 &:= \lim_{n\to\infty} \int_0^t (P_n(|\nabla \mathbf{m}_n|^2 \mathbf{m}_n), \phi) - \int_0^t (|\nabla \mathbf{m}|^2 \mathbf{m}, \phi) = 0,\\
L_3 &:= \lim_{n\to\infty} \int_0^t (P_n(\mathbf{m}_n \times \mathbf{H}_n), \phi) - \int_0^t (\mathbf{m} \times \mathbf{H}, \phi) = 0,\\
L_4 &:= \lim_{n\to\infty} \int_0^t (P_n((\mathbf{m}_n \times \mathbf{H}_n) \times \mathbf{m}_n), \phi) - \int_0^t (\mathbf{m} \times \mathbf{H}, \mathbf{m} \times \phi) = 0,\\
L_5 &:= \lim_{n\to\infty} \int_0^t e^{\sigma s}(Q_n(\nabla \times \mathbf{H}_n), \phi) - \int_0^t e^{\sigma s}(\nabla \times \mathbf{H}, \phi) = 0,\\
L_6 &:= \lim_{n\to\infty} \int_0^t (R_n(\nabla \times \mathbf{E}_n), \phi) - \int_0^t (\nabla \times \mathbf{E}, \phi) = 0.
\end{aligned}$$

Proof:
Let us compute the first limit:

$$\begin{aligned}
L_1 \quad &:= \quad \lim_{n\to\infty} \int_0^t (P_n(\mathbf{m}_n \times \nabla \mathbf{m}_n), \nabla\phi) - \int_0^t (\mathbf{m} \times \nabla \mathbf{m}, \nabla \phi),\\
&\leq \quad \lim_{n\to\infty} \int_0^t (\mathbf{m}_n \times \nabla \mathbf{m}_n, P_n(\nabla\phi)) - \int_0^t (\mathbf{m} \times \nabla \mathbf{m}, \nabla \phi),\\
&\leq \quad \lim_{n\to\infty} \int_0^t (\mathbf{m}_n \times \nabla \mathbf{m}_n, \nabla\phi) + \int_0^t (\mathbf{m}_n \times \nabla \mathbf{m}_n, P_n(\nabla\phi) - \nabla\phi)\\
&\qquad - \int_0^t (\mathbf{m} \times \nabla \mathbf{m}, \nabla \phi),
\end{aligned}$$

The projection operator P_n converges strongly to identity so one of the terms vanish and we get

$$\begin{aligned}
L_1 \quad &\leq \quad \lim_{n\to\infty} \int_0^t (\mathbf{m}_n \times \nabla \mathbf{m}_n, \nabla\phi) - \int_0^t (\mathbf{m} \times \nabla \mathbf{m}, \nabla \phi)\\
&\leq \quad \lim_{n\to\infty} \left| \int_0^t ((\mathbf{m}_n - \mathbf{m}) \times \nabla \mathbf{m}_n, \nabla\phi) \right|
\end{aligned}$$

$$
\begin{aligned}
&+ \lim_{n\to\infty}\left|\int_0^t (\mathbf{m}\times(\nabla\mathbf{m}_n-\nabla\mathbf{m}),\nabla\phi)\right| \\
\leq\ & \lim_{n\to\infty}\int_0^t \|\mathbf{m}_n-\mathbf{m}\|_4\|\nabla\mathbf{m}_n\|_4\|\nabla\phi\|_2 \\
&+ \lim_{n\to\infty}\int_0^t \|\mathbf{m}\|_4\|\nabla\mathbf{m}_n-\nabla\mathbf{m}\|_4\|\nabla\phi\|_2,
\end{aligned}
$$

Next we use embedding $W^{1,2}(\Omega)\hookrightarrow L^4(\Omega)$ and since the admissible function ϕ belong to $W^{1,2}(\Omega)$ we get

$$
\begin{aligned}
L_1 \quad\leq\quad & \lim_{n\to\infty}\int_0^t \|\mathbf{m}_n-\mathbf{m}\|_{W^{1,2}(\Omega)}\|\mathbf{m}_n\|_{W^{2,2}} \\
&+ \lim_{n\to\infty}\int_0^t \|\mathbf{m}\|_{W^{1,2}(\Omega)}\|\mathbf{m}_n-\mathbf{m}\|_{W^{2,2}},
\end{aligned}
$$

As a simple consequence of the Cauchy inequality and (8.51) we get

$$
\begin{aligned}
L_1 \quad\leq\quad & C\lim_{n\to\infty}\left(\int_0^t \|\mathbf{m}_n-\mathbf{m}\|^2_{W^{1,2}(\Omega)}\right)^{\frac{1}{2}} \\
&+ \lim_{n\to\infty}\left(\int_0^t \|\mathbf{m}\|^2_{W^{1,2}(\Omega)}\right)^{\frac{1}{2}}\left(\int_0^t \|\mathbf{m}_n-\mathbf{m}\|^2_{W^{2,2}}\right)^{\frac{1}{2}}.
\end{aligned}
$$

According to (8.51) the term between the second brackets is bounded. Thus, from (8.51) we get

$$
L_1 \quad\leq\quad C\lim_{n\to\infty}\left(\int_0^t \|\mathbf{m}_n-\mathbf{m}\|^2_{W^{2,2}(\Omega)}\right)^{\frac{1}{2}} = 0.
$$

To compute $L_2, L_3, \ldots, L_6$ we can always perform the same computations as when computing L_1 and get rid of projection operators P_n, Q_n and R_n. Then we get

$$
\begin{aligned}
L_2 \quad\leq\quad & \lim_{n\to\infty}\left|\int_0^t (\langle\nabla\mathbf{m}_n-\nabla\mathbf{m},\nabla\mathbf{m}_n\rangle\mathbf{m}_n,\phi)\right| \\
&+ \lim_{n\to\infty}\left|\int_0^t (\langle\nabla\mathbf{m},\nabla\mathbf{m}_n-\nabla\mathbf{m}\rangle\mathbf{m}_n,\phi)\right| \\
&+ \lim_{n\to\infty}\left|\int_0^t (|\nabla\mathbf{m}|^2(\mathbf{m}_n-\mathbf{m}),\phi)\right| \\
\leq\quad & \lim_{n\to\infty}\int_0^t \|\nabla\mathbf{m}_n-\nabla\mathbf{m}\|_2\|\nabla\mathbf{m}_n\|_4\|\mathbf{m}_n\|_{L^\infty}\|\phi\|_4
\end{aligned}
$$

$$+\lim_{n\to\infty}\int_0^t \|\nabla\mathbf{m}\|_2\|\nabla\mathbf{m}_n-\nabla\mathbf{m}\|_4\|\mathbf{m}_n\|_{L^\infty}\|\phi\|_4$$
$$+\lim_{n\to\infty}\int_0^t \|\nabla\mathbf{m}\|_2\|\nabla\mathbf{m}\|_4\|\mathbf{m}_n-\mathbf{m}\|_{L^\infty}\|\phi\|_4.$$

Since $\phi\in W^{1,2}(\Omega)\hookrightarrow L^4(\Omega)$ we have $\|\phi\|_4\le C$. We use (8.34) and (8.35) to obtain

$$\begin{aligned}L_2 \quad \le \quad & \lim_{n\to\infty}\int_0^t \|\nabla\mathbf{m}_n-\nabla\mathbf{m}\|_2\\ & +\lim_{n\to\infty}\int_0^t \|\nabla\mathbf{m}\|_2\|\nabla\mathbf{m}_n-\nabla\mathbf{m}\|_4\\ & +\lim_{n\to\infty}\int_0^t \|\nabla\mathbf{m}\|_2\|\nabla\mathbf{m}\|_4\|\mathbf{m}_n-\mathbf{m}\|_{L^\infty(\Omega)}.\end{aligned}$$

We make use of embedding $W^{1,2}(\Omega)\hookrightarrow L^4(\Omega)$, $W^{2,2}(\Omega)\hookrightarrow L^\infty(\Omega)$, the Cauchy inequality, (10.12), and (10.14) to get

$$\begin{aligned}L_2 \quad \le \quad & C\lim_{n\to\infty}\left(\int_0^t \|\nabla\mathbf{m}_n-\nabla\mathbf{m}\|_2^2\right)^{\frac{1}{2}}\\ & +\lim_{n\to\infty}\left(\int_0^t \|\nabla\mathbf{m}\|_2^2\right)^{\frac{1}{2}}\left(\int_0^t \|\nabla\mathbf{m}_n-\nabla\mathbf{m}\|_{W^{1,2}(\Omega)}^2\right)^{\frac{1}{2}}\\ & +\lim_{n\to\infty}\left(\int_0^t \left(\|\nabla\mathbf{m}\|_2\|\nabla\mathbf{m}\|_2^{\frac{1}{4}}\|\Delta\mathbf{m}\|_2^{\frac{3}{4}}\right)^2\right)^{\frac{1}{2}}\left(\int_0^t \|\mathbf{m}_n-\mathbf{m}\|_{W^{2,2}(\Omega)}^2\right)^{\frac{1}{2}}.\end{aligned}$$

According to (8.51) we have boundedness of $\int_0^t \|\nabla\mathbf{m}\|_2^2$. Thus after using the Cauchy inequality with exponents 4/3 and 4 we get

$$\begin{aligned}L_2 \quad \le \quad & C\lim_{n\to\infty}\left(\int_0^t \|\mathbf{m}_n-\mathbf{m}\|_{W^{2,2}(\Omega)}^2\right)^{\frac{1}{2}}\\ & +\lim_{n\to\infty}\left(\int_0^t \|\nabla\mathbf{m}\|_2^{10}\right)^{\frac{1}{4}}\left(\int_0^t \|\Delta\mathbf{m}\|_2^2\right)^{\frac{1}{4}}\left(\int_0^t \|\mathbf{m}_n-\mathbf{m}\|_{W^{2,2}(\Omega)}^2\right)^{\frac{1}{2}}.\end{aligned}$$

Now according to (8.51) and (8.52) we can successfully bound another two terms:

$$L_2\le C\lim_{n\to\infty}\left(\int_0^t \|\mathbf{m}_n-\mathbf{m}\|_{W^{2,2}(\Omega)}^2\right)^{\frac{1}{2}}=0,$$

where we have used (8.51).

Next we compute L_3 using embedding $W^{1,2}(\Omega) \hookrightarrow L^4(\Omega)$ and the Cauchy inequality

$$
\begin{aligned}
L_3 &\leq \lim_{n\to\infty} \left| \int_0^t ((\mathbf{m}_n - \mathbf{m}) \times \mathbf{H}_n, \phi) \right| + \lim_{n\to\infty} \left| \int_0^t (\mathbf{m} \times (\mathbf{H}_n - \mathbf{H}), \phi) \right| \\
&\leq \lim_{n\to\infty} \int_0^t \|\mathbf{m}_n - \mathbf{m}\|_2 \|\mathbf{H}_n\|_4 \|\phi\|_4 + \lim_{n\to\infty} \int_0^t \|\mathbf{m}\|_4 \|\mathbf{H}_n - \mathbf{H}\|_2 \|\phi\|_4 \\
&\leq \lim_{n\to\infty} \int_0^t \|\mathbf{m}_n - \mathbf{m}\|_2 \|\mathbf{H}_n\|_{W^{1,2}(\Omega)} \|\phi\|_4 \\
&\quad + \lim_{n\to\infty} \int_0^t \|\mathbf{m}\|_{W^{1,2}(\Omega)} \|\mathbf{H}_n - \mathbf{H}\|_2 \|\phi\|_4 \\
&\leq \lim_{n\to\infty} \left(\int_0^t \|\mathbf{m}_n - \mathbf{m}\|_2^2 \right)^{\frac{1}{2}} \left(\int_0^t \|\mathbf{H}_n\|_{W^{1,2}(\Omega)}^2 \right)^{\frac{1}{2}} \\
&\quad + \lim_{n\to\infty} \left(\int_0^t \|\mathbf{m}\|_{W^{1,2}(\Omega)}^2 \right)^{\frac{1}{2}} \left(\int_0^t \|\mathbf{H}_n - \mathbf{H}\|_2^2 \right)^{\frac{1}{2}},
\end{aligned}
$$

where we have already used that $\|\phi\|_4 \leq C$. Using the Young inequality, (8.34), and (8.51) we have

$$
L_3 \leq \lim_{n\to\infty} \left(\int_0^t \|\mathbf{m}_n - \mathbf{m}\|_2^2 \right)^{\frac{1}{2}} + \lim_{n\to\infty} \left(\int_0^t \|\mathbf{H}_n - \mathbf{H}\|_2^2 \right)^{\frac{1}{2}} = 0,
$$

which comes from (8.51) and (8.57).

Next we compute L_4 using embeddings $W^{1,2}(\Omega) \hookrightarrow L^4(\Omega)$ and $W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$:

$$
\begin{aligned}
L_4 &\leq \lim_{n\to\infty} \left| \int_0^t ((\mathbf{m}_n - \mathbf{m}) \times \mathbf{H}_n, \mathbf{m}_n \times \phi) \right| \\
&\quad + \lim_{n\to\infty} \left| \int_0^t (\mathbf{m} \times \mathbf{H}_n, (\mathbf{m}_n - \mathbf{m}) \times \phi) \right| \\
&\quad + \lim_{n\to\infty} \left| \int_0^t (\mathbf{m} \times (\mathbf{H}_n - \mathbf{H}), \mathbf{m} \times \phi) \right| \\
&\leq \lim_{n\to\infty} \int_0^t \|\mathbf{m}_n - \mathbf{m}\|_4 \|\mathbf{H}_n\|_4 \|\mathbf{m}_n\|_4 \|\phi\|_4 \\
&\quad + \lim_{n\to\infty} \int_0^t \|\mathbf{m}\|_4 \|\mathbf{H}_n\|_4 \|\mathbf{m}_n - \mathbf{m}\|_4 \|\phi\|_4 \\
&\quad + \lim_{n\to\infty} \int_0^t \|\mathbf{m}\|_4 \|\mathbf{H}_n - \mathbf{H}\|_2 \|\mathbf{m}\|_{L^\infty} \|\phi\|_4
\end{aligned}
$$

$$\begin{aligned}
&\leq \lim_{n\to\infty}\int_0^t \|\mathbf{m}_n-\mathbf{m}\|_{W^{1,2}}\|\mathbf{H}_n\|_{W^{1,2}}\|\mathbf{m}_n\|_{W^{1,2}} \\
&\quad + \lim_{n\to\infty}\int_0^t \|\mathbf{m}\|_{W^{1,2}}\|\mathbf{H}_n\|_{W^{1,2}}\|\mathbf{m}_n-\mathbf{m}\|_{W^{1,2}} \\
&\quad + \lim_{n\to\infty}\int_0^t \|\mathbf{m}\|_{W^{1,2}}\|\mathbf{H}_n-\mathbf{H}\|_2\|\mathbf{m}\|_{W^{1,4}},
\end{aligned}$$

where we have used $\|\phi\|_4 \leq C$. The boundedness of $\mathbf{m}_n$ in $W^{2,2}$ and boundedness of $\mathbf{H}_n$ in $W^{1,2}$ together with (10.12) and the Cauchy inequality give us

$$\begin{aligned}
L_4 &\leq \lim_{n\to\infty}\left(\int_0^t \|\mathbf{m}_n-\mathbf{m}\|_{W^{1,2}}^2\right)^{\frac{1}{2}} \\
&\quad + \lim_{n\to\infty}\left(\int_0^t \|\mathbf{m}\|_{W^{1,2}}^2\right)^{\frac{1}{2}}\left(\int_0^t \|\mathbf{m}_n-\mathbf{m}\|_{W^{1,2}}^2\right)^{\frac{1}{2}} \\
&\quad + \lim_{n\to\infty}\left(\int_0^t \left(\|\mathbf{m}\|_{W^{1,2}}^{\frac{5}{4}}\|\mathbf{m}\|_{W^{2,2}}^{\frac{3}{4}}\right)^2\right)^{\frac{1}{2}}\left(\int_0^t \|\mathbf{H}_n-\mathbf{H}\|_2^2\right)^{\frac{1}{2}} \\
&\leq \lim_{n\to\infty}\left(\int_0^t \|\mathbf{m}_n-\mathbf{m}\|_{W^{1,2}}^2\right)^{\frac{1}{2}} \\
&\quad + \lim_{n\to\infty}\left(\int_0^t \|\mathbf{m}\|_{W^{1,2}}^2\right)^{\frac{1}{2}}\left(\int_0^t \|\mathbf{m}_n-\mathbf{m}\|_{W^{1,2}}^2\right)^{\frac{1}{2}} \\
&\quad + \lim_{n\to\infty}\left(\int_0^t \|\mathbf{m}\|_{W^{1,2}}^{10}\right)^{\frac{1}{4}}\left(\int_0^t \|\mathbf{m}\|_{W^{2,2}}^2\right)^{\frac{1}{4}}\left(\int_0^t \|\mathbf{H}_n-\mathbf{H}\|_2^2\right)^{\frac{1}{2}}
\end{aligned}$$

Using (8.51) and (8.52) we get

$$L_4 \leq C\lim_{n\to\infty}\left(\int_0^t \|\mathbf{m}_n-\mathbf{m}\|_{W^{1,2}}^2\right)^{\frac{1}{2}} + \lim_{n\to\infty}\left(\int_0^t \|\mathbf{H}_n-\mathbf{H}\|_2^2\right)^{\frac{1}{2}} = 0,$$

where we have used (8.51) and (8.57).

To compute L_5 we use (8.53) and $\|\phi\|_{W^{1,2}} \leq C$ to conclude

$$\begin{aligned}
L_5 &\leq \left|\lim_{n\to\infty}\int_0^t e^{\sigma s}(\nabla\times\mathbf{H}_n-\nabla\times\mathbf{H},\phi)\right| \leq C\lim_{n\to\infty}\int_0^t \|\nabla\mathbf{H}_n-\nabla\mathbf{H}\|_2\|\phi\|_2 \\
&\leq C\lim_{n\to\infty}\int_0^t \|\nabla\mathbf{H}_n-\nabla\mathbf{H}\|_2^2 = 0.
\end{aligned}$$

Similarly we can also prove that

$$L_6 = 0.$$

□

Remark 8.1 *From Lemma 8.9 and 8.10 it follows, that the limit triple* $(\mathbf{m}, \mathbf{E}, \mathbf{H})$ *defined in Lemma 8.9 satisfies the equations (8.10)–(8.12). The fact that* $\|\mathbf{m}\|_2 < C$ *and* $\|\mathbf{H}\|_2 < C$ *together with the following lemma ensures that equations (8.13) and (8.14) are also satisfied.*

Lemma 8.11 *If the initial value vector functions* $\mathbf{E}_0(\mathbf{x}), \mathbf{H}_0(\mathbf{x}), \mathbf{m}_0(\mathbf{x})$ *satisfy for all* ϑ_1 *and* ϑ_2 *the condition*

$$(\vartheta_1, \nabla\cdot\mathbf{E}_0) = 0, \qquad (\vartheta_2, \nabla\cdot(\mathbf{H}_0 + \beta\mathbf{m}_0)) = 0, \tag{8.59}$$

where $\vartheta_1(\mathbf{x}) \in \mathbf{H}^1_0(\Omega)$ *and* $\vartheta_2(\mathbf{x}) \in \mathbf{H}^1(\Omega)$ *and if* $\|\mathbf{H}+\beta\mathbf{m}\|_2$ *and* $\|\mathbf{E}\|_2$ *are bounded, then from (8.11), (8.12) it follows that*

$$\int_I (\nabla\cdot\mathbf{E}, \zeta)\mathrm{d}s = 0, \qquad \int_I (\nabla\cdot\mathbf{H} + \beta\nabla\cdot\mathbf{m}, \xi)\mathrm{d}s = 0,$$

where ξ *and* ζ *are from the definition of weak solution.*

Proof:
Let us take arbitrary $\theta \in C^\infty(I, C^\infty(\Omega))$. Then we can set $\boldsymbol{\phi} = \nabla\theta$ in (8.12) to get

$$\int_I (\nabla\times\mathbf{E}, \nabla\theta) = \int_I (\mathbf{H} + \beta\mathbf{m}, \nabla\partial_t\theta) + (\mathbf{H}_0 + \beta\mathbf{m}_0, \nabla\theta(0))$$

and consequently

$$\begin{aligned}\int_I (\mathbf{E}, \nabla\times\nabla\theta) - \int_I (\mathbf{E}\times\boldsymbol{\nu}, \nabla\theta)_{\partial\Omega} \;=\; & -\int_I (\nabla\cdot(\mathbf{H}+\beta\mathbf{m}), \partial_t\theta) \\ & + \int_I ((\mathbf{H}+\beta\mathbf{m})\cdot\boldsymbol{\nu}, \theta_t)_{\partial\Omega} \\ & -(\nabla\cdot(\mathbf{H}_0+\beta\mathbf{m}_0), \theta(0)) \\ & +((\mathbf{H}_0+\beta\mathbf{m}_0)\cdot\boldsymbol{\nu}, \theta(0))_{\partial\Omega}.\end{aligned}$$

Left-hand side is zero since $\nabla\times\nabla\theta = 0$ and the values of $\mathbf{E}\cdot\boldsymbol{\nu}$ are zero on the boundary. The boundary terms on the right-hand side vanish because $(\mathbf{H}+\beta\mathbf{m})\cdot\boldsymbol{\nu}$ is zero on the boundary. Since we suppose (8.59) we get

$$0 = \int_I ((\nabla\cdot\mathbf{H} + \beta\nabla\cdot\mathbf{m}), \partial_t\theta).$$

Finally we get

$$\int_I (\nabla\cdot(\mathbf{H}+\beta\mathbf{m}), \theta) = 0, \qquad \forall\theta \in C^\infty(I, C^\infty(\Omega)). \tag{8.60}$$

From the density of $C^\infty(I, C^\infty(\Omega))$ functions in $\mathbf{L}^\infty(I, \mathbf{H}^1(\Omega))$ we have

$$\forall \xi \in \mathbf{L}^\infty(I, \mathbf{H}^1(\Omega))\ \forall \varepsilon > 0\ \exists \theta \in C^\infty(I, C^\infty(\Omega)) :\ \max_{t \in I} \|\xi - \theta\|_{H^1} < \varepsilon. \tag{8.61}$$

Then we have

$$\begin{aligned}
\int_I (\nabla \cdot (\mathbf{H} + \beta \mathbf{m}), \xi) &= \int_I (\nabla \cdot (\mathbf{H} + \beta \mathbf{m}), \theta) \\
&\quad + \int_I (\nabla \cdot (\mathbf{H} + \beta \mathbf{m}), \xi - \theta) \\
&= -\int_I (\mathbf{H} + \beta \mathbf{m}, \nabla(\xi - \theta)) \\
&\quad + \int_I ((\mathbf{H} + \beta \mathbf{m}) \cdot \boldsymbol{\nu}, \xi - \theta)_{\partial\Omega}.
\end{aligned}$$

The boundary term on the right-hand side vanish. We obtain

$$\int_I (\nabla \cdot (\mathbf{H} + \beta \mathbf{m}), \xi) \le \int_I \|\mathbf{H} + \beta \mathbf{m}\|_2 \|\nabla(\xi - \theta)\|_2 \le C\varepsilon.$$

Since ε was arbitrary small we can conclude (8.13).

Similar procedure we can apply to (8.11). Take arbitrary $\theta \in C_0^\infty(I, C^\infty(\Omega))$. Then we can set $\boldsymbol{\psi} = \nabla\theta$ in (8.11) to get

$$\begin{aligned}
\int_I e^{\sigma s}(\nabla \times \mathbf{H}, \nabla\theta) + \int_I (\mathbf{E} e^{\sigma s}, \nabla \partial_t \theta) + (\mathbf{E}_0, \nabla\theta(0)) &= 0, \\
\int_I e^{\sigma s}(\mathbf{H}, \nabla \times \nabla\theta) - \int_I e^{\sigma s}(\mathbf{H} \times \boldsymbol{\nu}, \nabla\theta)_{\partial\Omega} - \int_I (\nabla \cdot (\mathbf{E} e^{\sigma s}), \partial_t \theta) & \\
+ \int_I (e^{\sigma s} \mathbf{E} \cdot \boldsymbol{\nu}, \partial_t \theta)_{\partial\Omega} - (\nabla \cdot \mathbf{E}_0, \theta(0)) + (\mathbf{E}_0 \cdot \boldsymbol{\nu}, \theta(0))_{\partial\Omega} &= 0.
\end{aligned}$$

Realizing that θ vanish on the boundary and using (8.59) we get

$$0 = \int_I (\nabla \cdot (\mathbf{E} e^{\sigma s}), \partial_t \theta).$$

Note that the function $e^{\sigma t} > 1$ is constant in space. Then we get

$$\int_I (\nabla \cdot \mathbf{E}, \theta) = 0, \qquad \forall \theta \in C^\infty(I, C_0^\infty(\Omega)). \tag{8.62}$$

From the density of $C^\infty(I, C_0^\infty(\Omega))$ functions in $\mathbf{L}^\infty(I, \mathbf{H}_0^1(\Omega))$ we have

$$\forall \zeta \in \mathbf{L}^\infty(I, \mathbf{H}_0^1(\Omega))\ \forall \varepsilon > 0\ \exists \theta \in C^\infty(I, C_0^\infty(\Omega)) :\ \max_{t \in I} \|\xi - \theta\|_{H^1} < \varepsilon. \tag{8.63}$$

Then we have

$$\begin{aligned}\int_I (\nabla\cdot\mathbf{E},\zeta) &= \int_I (\nabla\cdot\mathbf{E},\theta) + \int_I (\nabla\cdot\mathbf{E},\zeta-\theta)\\ &= -\int_I (\mathbf{E},\nabla(\zeta-\theta)) + \int_I (\mathbf{E}\cdot\boldsymbol{\nu},\zeta-\theta)_{\partial\Omega}.\end{aligned}$$

The boundary term on the right-hand side vanish. We obtain

$$\int_I (\nabla\cdot\mathbf{E},\zeta) \le \int_I \|\mathbf{E}\|_2\|\nabla(\zeta-\theta)\|_2 \le C\varepsilon.$$

Since ε was arbitrary small we can conclude (8.14). □

8.5 Regularity results for weak solution

The final aim of this chapter is to derive regularity results for the weak solution and prove its uniqueness. Estimates are summarized in the following theorem.

Theorem 8.1 *Suppose that the initial conditions satisfy (8.59), and moreover*

$$\mathbf{m}_0 \in W^{2,2}(\Omega) \text{ and } \mathbf{H}_0,\mathbf{E}_0 \in W^{1,2}(\Omega).$$

Then there exists a positive $T^* > 0$ such that on the interval $(0,T^*)$ there exists a unique weak solution of the system (8.9), (8.2)–(8.5) defined by Definition 8.1 satisfying

$$\sup_{t\in I}\left\{\|\mathbf{m}\|_{W^{2,2}(\Omega)} + \|\mathbf{E}\|_{W^{1,2}(\Omega)} + \|\mathbf{H}\|_{W^{1,2}(\Omega)}\right\} \le C, \tag{8.64}$$

$$\int_0^{T^*} \|\mathbf{m}\|^2_{W^{3,2}} \le C. \tag{8.65}$$

Proof:
The existence of a weak solution was already mentioned in Remark 8.1.

Coming from (8.34) and embedding $L^\infty \hookrightarrow L^p$, $1<p<\infty$, we know that there exists one constant C such that

$$\|\mathbf{m}_n\|_{L^p(I,W^{2,2}(\Omega))} \le \|\mathbf{m}_n\|_{L^\infty(I,W^{2,2}(\Omega))} \le C.$$

Notice, that C does not depend on p. Since $L^p(I,W^{2,2}(\Omega))$ is a reflexive Banach space we conclude from Theorem 10.6 that for all p

$$\mathbf{m}_n \rightharpoonup \mathbf{m} \text{ in } L^p(I,W^{2,2}(\Omega)).$$

From Theorem 10.5 we get uniform bound for all p

$$\|\mathbf{m}\|_{L^p(I,W^{2,2}(\Omega))} \leq \liminf_{n\to\infty} \|\mathbf{m}_n\|_{L^p(I,W^{2,2}(\Omega))} \leq C. \tag{8.66}$$

Realizing that

$$\mathbf{m} \in \bigcap_{p\in\mathbb{N}} L^p(I, W^{2,2}(\Omega))$$

we verify together with (8.66) the assumptions of Theorem 10.7 and thus $\mathbf{m}$ belongs to $L^\infty(I, W^{2,2}(\Omega))$.

Similarly we can prove that the functions $\mathbf{E}$ and $\mathbf{H}$ belong to $L^\infty(I, W^{1,2}(\Omega))$ which completes the proof of (8.64).

The proof of (8.65) is a consequence of (8.38) and Theorem 10.5.

Now we prove uniqueness of the weak solution on the interval $\langle 0, T_0\rangle$. Suppose that $(\mathbf{m}_1, \mathbf{E}_1, \mathbf{H}_1)$ and $(\mathbf{m}_2, \mathbf{E}_2, \mathbf{H}_2)$ are two solutions of the system (8.9), (8.2)–(8.5) with the same initial conditions. Denote

$$\bar{\mathbf{m}} = \mathbf{m}_1 - \mathbf{m}_2,\ \bar{\mathbf{E}} = \mathbf{E}_1 - \mathbf{E}_2,\ \bar{\mathbf{H}} = \mathbf{H}_1 - \mathbf{H}_2.$$

The solutions $\mathbf{m}_1, \mathbf{m}_2, \mathbf{H}_1, \mathbf{H}_2$ satisfy equation (8.9). The differences $\bar{\mathbf{m}}, \bar{\mathbf{H}}$ then satisfy

$$\begin{aligned}\partial_t \bar{\mathbf{m}} - \alpha\Delta\bar{\mathbf{m}} \quad = \quad & \alpha|\nabla\mathbf{m}_1|^2\bar{\mathbf{m}} + (|\nabla\mathbf{m}_1|^2 - |\nabla\mathbf{m}_2|^2)\mathbf{m}_2 - \bar{\mathbf{m}} \times \Delta\mathbf{m}_1 \\ & -\mathbf{m}_2 \times \Delta\bar{\mathbf{m}} - \bar{\mathbf{m}} \times \mathbf{H}_1 - \mathbf{m}_2 \times \bar{\mathbf{H}} \\ & +\bar{\mathbf{m}} \times (\mathbf{m}_1 \times \mathbf{H}_1) + \mathbf{m}_2 \times (\bar{\mathbf{m}} \times \mathbf{H}_1) \\ & +\mathbf{m}_2 \times (\mathbf{m}_2 \times \bar{\mathbf{H}})\end{aligned} \tag{8.67}$$

in a weak sense. Taking the previous equation, multiplying by $\bar{\mathbf{m}}$ and integrating over Ω we directly get rid of the terms $\bar{\mathbf{m}} \times \mathbf{H}_1$, $\bar{\mathbf{m}} \times \Delta\mathbf{m}_1$ and $\bar{\mathbf{m}} \times (\mathbf{m}_1 \times \mathbf{H}_1)$. Remaining equation reads as

$$\begin{aligned}\frac{1}{2}\partial_t\|\bar{\mathbf{m}}\|_2^2 + \alpha\|\nabla\bar{\mathbf{m}}\|_2^2 \;=\; & \alpha\int_\Omega |\nabla\mathbf{m}_1|^2\langle\bar{\mathbf{m}}, \bar{\mathbf{m}}\rangle + \int_\Omega (\langle\nabla\bar{\mathbf{m}}, \nabla\mathbf{m}_1 + \nabla\mathbf{m}_2\rangle\langle\mathbf{m}_2, \bar{\mathbf{m}}\rangle \\ & - \int_\Omega \langle\mathbf{m}_2 \times \Delta\bar{\mathbf{m}}, \bar{\mathbf{m}}\rangle - \int_\Omega \langle\mathbf{m}_2 \times \bar{\mathbf{H}}, \bar{\mathbf{m}}\rangle \\ & + \int_\Omega \langle\mathbf{m}_2 \times (\bar{\mathbf{m}} \times \mathbf{H}_1), \bar{\mathbf{m}}\rangle + \int_\Omega \langle\mathbf{m}_2 \times (\mathbf{m}_2 \times \bar{\mathbf{H}}, \bar{\mathbf{m}})\rangle.\end{aligned}$$

Next we perform integration by parts in the term including $\mathbf{m}_2 \times \Delta\bar{\mathbf{m}}$ and then we use integral inequalities to estimate

$$\frac{1}{2}\partial_t\|\bar{\mathbf{m}}\|_2^2 + \alpha\|\nabla\bar{\mathbf{m}}\|_2^2 \;\leq\; \alpha\|\nabla\mathbf{m}_1\|_4^2\|\bar{\mathbf{m}}\|_4^2$$

$$
\begin{aligned}
&+\|\nabla\bar{\mathbf{m}}\|_2(\|\nabla\mathbf{m}_1\|_4+\|\nabla\mathbf{m}_2\|_4)\|\mathbf{m}_2\|_{L^\infty}\|\bar{\mathbf{m}}\|_4\\
&+\|\nabla\mathbf{m}_2\|_4\|\nabla\bar{\mathbf{m}}\|_2\|\bar{\mathbf{m}}\|_4+\|\mathbf{m}_2\|_4\|\bar{\mathbf{H}}\|_2\|\bar{\mathbf{m}}\|_4\\
&+\|\mathbf{m}_2\|_{L^\infty}\|\bar{\mathbf{m}}\|_4^2\|\mathbf{H}_1\|_2+\|\mathbf{m}_2\|_{L^\infty}^2\|\bar{\mathbf{H}}\|_2\|\bar{\mathbf{m}}\|_2.
\end{aligned}
$$

From the embeddings $W^{2,2}(\Omega)\hookrightarrow L^\infty(\Omega)$ and $W^{2,2}(\Omega)\hookrightarrow W^{1,4}(\Omega)$ we see for $i=1,2$ that

$$
\begin{aligned}
\|\nabla\mathbf{m}_i\|_4 &\leq C\|\mathbf{m}_i\|_{W^{2,2}}\\
\|\mathbf{m}_i\|_{L^\infty} &\leq C\|\mathbf{m}_i\|_{W^{2,2}}
\end{aligned}
$$

Further, using (8.64), (10.11), (10.12) and the Young inequality we arrive at

$$
\partial_t\|\bar{\mathbf{m}}\|_2^2+\|\nabla\bar{\mathbf{m}}\|_2^2\leq\varepsilon\|\nabla\bar{\mathbf{m}}\|_2^2+C_\varepsilon\|\bar{\mathbf{m}}\|_2^2+C_\varepsilon\|\bar{\mathbf{H}}\|_2^2.
$$

Setting ε small enough and integrating the equation in time we get

$$
\|\bar{\mathbf{m}}(t)\|_2^2-\|\bar{\mathbf{m}}(0)\|_2^2+\int_0^t\|\nabla\bar{\mathbf{m}}(s)\|_2^2\mathrm{d}s\leq C\int_0^t\|\bar{\mathbf{m}}(s)\|_2^2\mathrm{d}s+C\int_0^t\|\bar{\mathbf{H}}(s)\|_2^2\mathrm{d}s.
$$

Since the initial conditions for $\mathbf{m}_1$ and $\mathbf{m}_2$ are the same we have $\|\bar{\mathbf{m}}(0)\|_2=0$. Note that the time variable t is not bigger that T_0. Next we use Gronwall's lemma, see Appendix, and change the constant C if necessary to obtain

$$
\begin{aligned}
\|\bar{\mathbf{m}}(t)\|_2^2 &\leq C\int_0^t\|\bar{\mathbf{H}}(s)\|_2^2\mathrm{d}s\\
&\quad+\int_0^t\left(C\int_0^s\|\bar{\mathbf{H}}(\tau)\|_2^2\mathrm{d}\tau\right)Ce^{(t-s)C}\mathrm{d}s\\
&\leq C\int_0^t\|\bar{\mathbf{H}}(s)\|_2^2\mathrm{d}s\\
&\quad+\int_0^t\left(C\int_0^t\|\bar{\mathbf{H}}(\tau)\|_2^2\mathrm{d}\tau\right)Ce^{T_0C}\mathrm{d}s\\
&\leq C\int_0^t\|\bar{\mathbf{H}}(s)\|_2^2\mathrm{d}s+T_0C\int_0^t\|\bar{\mathbf{H}}(\tau)\|_2^2\mathrm{d}\tau\\
&\leq C\int_0^t\|\bar{\mathbf{H}}(s)\|_2^2\mathrm{d}s. \qquad (8.68)
\end{aligned}
$$

Direct use of Maxwell's equations to obtain estimate on the term $\|\bar{\mathbf{H}}\|_2$ would not be successful at this stage, because on the right-hand side of Maxwell's equations the term $\|\partial_t\bar{\mathbf{m}}\|_2$ arises and up to now we do not have any estimate on it. Therefore we need to obtain first estimate on $\|\Delta\bar{\mathbf{m}}\|_2$, which will be used to estimate $\|\partial_t\bar{\mathbf{m}}.\|_2$

Multiplying (8.67) by $-\Delta\bar{\mathbf{m}}$ and integrating over Ω gives

$$\begin{aligned}
&\frac{1}{2}\partial_t\|\nabla\bar{\mathbf{m}}\|_2^2 + \alpha\|\Delta\bar{\mathbf{m}}\|_2^2 \\
&\quad \le \alpha\|\nabla\mathbf{m}_1\|_4^2\|\bar{\mathbf{m}}\|_{L^\infty}\|\Delta\bar{\mathbf{m}}\|_2 \\
&\qquad +\|\nabla\bar{\mathbf{m}}\|_4(\|\nabla\mathbf{m}_1\|_4 + \|\nabla\mathbf{m}_2\|_4)\|\mathbf{m}_2\|_{L^\infty}\|\Delta\bar{\mathbf{m}}\|_2 \\
&\qquad +\|\bar{\mathbf{m}}\|_{L^\infty}\|\Delta\mathbf{m}_1\|_2\|\Delta\bar{\mathbf{m}}\|_2 + \|\bar{\mathbf{m}}\|_{L^\infty}\|\mathbf{H}_1\|_2\|\Delta\bar{\mathbf{m}}\|_2 \\
&\qquad +\|\mathbf{m}_2\|_{L^\infty}\|\bar{\mathbf{H}}\|_2\|\Delta\bar{\mathbf{m}}\|_2 + \|\bar{\mathbf{m}}\|_{L^\infty}\|\mathbf{m}_1\|_{L^\infty}\|\mathbf{H}_1\|_2\|\Delta\bar{\mathbf{m}}\|_2 \\
&\qquad +\|\mathbf{m}_2\|_{L^\infty}\|\bar{\mathbf{m}}\|_{L^\infty}\|\mathbf{H}_1\|_2\|\Delta\bar{\mathbf{m}}\|_2 + \|\mathbf{m}_2\|_{L^\infty}^2\|\bar{\mathbf{H}}\|_2\|\Delta\bar{\mathbf{m}}\|_2.
\end{aligned}$$

From the embedding $W^{2,2}(\Omega) \hookrightarrow L^\infty(\Omega)$ and (8.64) we get rid of terms

$$\|\mathbf{m}_i\|_{L^\infty},\ \|\Delta\mathbf{m}_i\|_2,\ \|\nabla\mathbf{m}_i\|_4,\ \|\mathbf{H}_i\|_2.$$

In what remains we use the embedding $W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$, and the Young inequality to obtain

$$\partial_t\|\nabla\bar{\mathbf{m}}\|_2^2 + \|\Delta\bar{\mathbf{m}}\|_2^2 \le C\|\nabla\bar{\mathbf{m}}\|_4\|\Delta\bar{\mathbf{m}}\|_2 + C_\varepsilon\|\bar{\mathbf{H}}\|_2^2 + \varepsilon\|\Delta\bar{\mathbf{m}}\|_2^2.$$

Next we apply (10.12) and the Young inequality with exponents 8/7 and 8 to obtain

$$\begin{aligned}
\partial_t\|\nabla\bar{\mathbf{m}}\|_2^2 + \|\Delta\bar{\mathbf{m}}\|_2^2 &\le C\|\nabla\bar{\mathbf{m}}\|_2^{\frac{1}{4}}\|\Delta\bar{\mathbf{m}}\|_2^{\frac{7}{8}} + C_\varepsilon\|\bar{\mathbf{H}}\|_2^2 + \varepsilon\|\Delta\bar{\mathbf{m}}\|_2^2 \\
&\le C_\varepsilon\|\nabla\bar{\mathbf{m}}\|_2^2 + C_\varepsilon\|\bar{\mathbf{H}}\|_2^2 + \varepsilon\|\Delta\bar{\mathbf{m}}\|_2^2.
\end{aligned}$$

Integrating over time interval $\langle 0,t\rangle$, realizing that $\|\nabla\bar{\mathbf{m}}(0)\|_2 = 0$, setting ε small enough, and using Gronwall's lemma we conclude

$$\|\nabla\bar{\mathbf{m}}(t)\|_2^2 + \int_0^t \|\Delta\bar{\mathbf{m}}(s)\|_2^2 \mathrm{d}s \le C\int_0^t \|\bar{\mathbf{H}}(s)\|_2^2\mathrm{d}s. \tag{8.69}$$

Further we continue with estimating $\|\partial_t\bar{\mathbf{m}}\|_2$. Take (8.67), multiply it by $\partial_t\bar{\mathbf{m}}$ and integrate over Ω to get

$$\begin{aligned}
&\|\partial_t\bar{\mathbf{m}}\|_2^2 + \frac{1}{2}\partial_t\|\nabla\bar{\mathbf{m}}\|_2^2 \\
&\quad \le \|\nabla\mathbf{m}_1\|_4^2\|\bar{\mathbf{m}}\|_{L^\infty}\|\partial_t\bar{\mathbf{m}}\|_2 + \|\nabla\bar{\mathbf{m}}\|_4(\|\nabla\mathbf{m}_1\|_4 + \|\nabla\mathbf{m}_2\|_4)\|\mathbf{m}_2\|_{L^\infty}\|\partial_t\bar{\mathbf{m}}\|_2 \\
&\qquad +\|\bar{\mathbf{m}}\|_{L^\infty}\|\Delta\mathbf{m}_1\|_2\|\partial_t\bar{\mathbf{m}}\|_2 + \|\mathbf{m}_2\|_{L^\infty}\|\Delta\bar{\mathbf{m}}\|_2\|\partial_t\bar{\mathbf{m}}\|_2 \\
&\qquad +\|\bar{\mathbf{m}}\|_{L^\infty}\|\mathbf{H}_1\|_2\|\partial_t\bar{\mathbf{m}}\|_2 + \|\mathbf{m}_2\|_{L^\infty}\|\bar{\mathbf{H}}\|_2\|\partial_t\bar{\mathbf{m}}\|_2 \\
&\qquad +\|\bar{\mathbf{m}}\|_{L^\infty}\|\mathbf{m}_1\|_{L^\infty}\|\mathbf{H}_1\|_2\|\partial_t\bar{\mathbf{m}}\|_2 + \|\mathbf{m}_2\|_{L^\infty}^2\|\bar{\mathbf{H}}\|_2\|\partial_t\bar{\mathbf{m}}\|_2.
\end{aligned}$$

We can again get rid of terms containing $\mathbf{m}_i$ and $\mathbf{H}_i$. Then we use the embedding $W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$, and thus

$$\|\partial_t\bar{\mathbf{m}}\|_2^2 + \frac{1}{2}\partial_t\|\nabla\bar{\mathbf{m}}\|_2^2 \le C_\varepsilon\|\nabla\bar{\mathbf{m}}\|_4^2 + C_\varepsilon\|\Delta\bar{\mathbf{m}}\|_2^2 + C_\varepsilon\|\bar{\mathbf{H}}\|_2^2 + \varepsilon\|\partial_t\bar{\mathbf{m}}\|_2^2.$$

The embedding $W^{2,2}(\Omega) \hookrightarrow W^{1,4}(\Omega)$ and setting ε small enough give

$$\|\partial_t \bar{\mathbf{m}}\|_2^2 + \partial_t \|\nabla \bar{\mathbf{m}}\|_2^2 \leq C_\varepsilon \|\nabla \bar{\mathbf{m}}\|_2^2 + C_\varepsilon \|\Delta \bar{\mathbf{m}}\|_2^2 + C_\varepsilon \|\bar{\mathbf{H}}\|_2^2.$$

Integrating over time interval, realizing that $\|\nabla \bar{\mathbf{m}}(t)\|_2 = 0$, and using (8.69) leads to

$$\int_0^t \|\partial_t \bar{\mathbf{m}}(s)\|_2^2 \mathrm{d}s + \|\nabla \bar{\mathbf{m}}(t)\|_2^2 \leq C_\varepsilon \int_0^t \|\nabla \bar{\mathbf{m}}(s)\|_2^2 \mathrm{d}s + C_\varepsilon \int_0^t \|\bar{\mathbf{H}}\|_2^2.$$

The use of Gronwall's lemma gives us finally

$$\int_0^t \|\partial_t \bar{\mathbf{m}}(s)\|_2^2 \mathrm{d}s + \|\nabla \bar{\mathbf{m}}(t)\|_2^2 \leq C_\varepsilon \int_0^t \|\bar{\mathbf{H}}\|_2^2. \tag{8.70}$$

At this stage we are ready to incorporate Maxwell's equations. Since (8.2) and (8.3) are linear we can directly consider both equations valid also for the triple $(\bar{\mathbf{m}}, \bar{\mathbf{E}}, \bar{\mathbf{H}})$

$$\begin{aligned} \partial_t \bar{\mathbf{E}} + \sigma \bar{\mathbf{E}} - \nabla \times \bar{\mathbf{H}} &= 0 \\ \partial_t \bar{\mathbf{H}} + \nabla \times \bar{\mathbf{E}} &= -\beta \partial_t . \bar{\mathbf{m}} \end{aligned}$$

Next we multiply the equations by $\bar{\mathbf{E}}, \bar{\mathbf{H}}$, integrate it over Ω and after summation we get using the Young inequality

$$\frac{1}{2} \partial_t \|\bar{\mathbf{E}}\|_2^2 + \frac{1}{2} \partial_t \|\bar{\mathbf{H}}\|_2^2 + \sigma \|\bar{\mathbf{E}}\|_2^2 \leq C \|\partial_t \bar{\mathbf{m}}\|_2^2 + C \|\bar{\mathbf{H}}\|_2^2.$$

After integration in time, realizing that $\|\bar{\mathbf{H}}(0)\|_2 = \|\bar{\mathbf{E}}(0)\|_2 = 0$ and using (8.70) we get

$$\|\bar{\mathbf{E}}(t)\|_2^2 + \|\bar{\mathbf{H}}(t)\|_2^2 \leq C_\varepsilon \int_0^t \|\bar{\mathbf{H}}\|_2^2,$$

which after using Gronwall's lemma gives

$$\|\bar{\mathbf{E}}\|_2^2 \leq 0, \quad \|\bar{\mathbf{H}}\|_2^2 \leq 0.$$

Going back to the relations (8.68),(8.69) and (8.70) we conclude

$$\sup_{0<t<T_0} \|\bar{\mathbf{m}}(t)\|_{W^{1,2}} + \int_0^{T_0} \left(\|\Delta \bar{\mathbf{m}}(s)\|_2^2 + \|\partial_t \bar{\mathbf{m}}(s)\|_2^2 \mathrm{d}s \right) = 0,$$

which concludes the proof of the uniqueness. □

9 Numerical schemes for the single LL equation

(In the fall of 1972 President Nixon announced that the rate of increase of inflation was decreasing. This was the first time a sitting president used the third derivative to advance his case for reelection. Hugo Rossi)

In Chapter 7 we stated the results concerning the regularity of an exact solution to the LL equation in 3D. We study the system (7.2)–(7.4).

These results will be used in this chapter to prove the convergence results of the numerical scheme introduced later in the chapter. In Section 9.1 we introduce a numerical scheme and we state Theorems 9.1 and 9.2 concerning the error estimates in time for the numerical scheme. We prove these theorems in Sections 9.2 and 9.3.

Finally, we confirm the theoretical results with a numerical example in Section 9.4.

9.1 Numerical scheme and error estimates

In the following, any number placed to the upper right of a function represents an index, not a power. We use the symbol $\delta \mathbf{f}^i$ for the backward Euler approximation of the time derivation, so $\delta \mathbf{f}^i = \frac{1}{\tau}(\mathbf{f}^{i+1} - \mathbf{f}^i)$.

We provide a standard equidistant discretization of the time interval $(0, T_0)$ with J time steps of a size $\tau = T_0/J$ and we denote $t_j = j\tau$ for $j = 0, \dots, J$. The author in [61, Section 4.2.1] considers the following semi-implicit scheme and proves the error estimates, all in 2D.

$$\delta \mathbf{m}^{j+1} - \alpha\Delta \mathbf{m}^{j+1} = \alpha|\nabla \mathbf{m}^j|^2\mathbf{m}^{j+1} - \mathbf{m}^{j+1} \times \Delta\mathbf{m}^{j+1}, \quad \mathbf{m}^0 = \mathbf{m}_0. \tag{9.1}$$

Note that the previous scheme is nonlinear. The term $\mathbf{m}^{j+1} \times \Delta\mathbf{m}^{j+1}$ makes the scheme quadratic on each time level.

We change the scheme so that it becomes linear on each time level and we consider the case of 3D

$$\delta \mathbf{m}^{j+1} - \alpha\Delta \mathbf{m}^{j+1} = \alpha|\nabla \mathbf{m}^j|^2\mathbf{m}^{j+1} - \mathbf{m}^{j} \times \Delta\mathbf{m}^{j+1}, \quad \mathbf{m}^0 = \mathbf{m}_0. \tag{9.2}$$

The difference is in the curl term. We consider $\mathbf{m}$ in this term taken from the previous time level.

The existence and uniqueness of $\mathbf{m}^{j+1}$ on every time step is guaranteed by the Lax-Milgram theorem, see Appendix, as soon as we verify the V-ellipticity of the bilinear form

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) \quad := \quad & (\mathbf{u},\mathbf{v}) + \alpha\tau(\nabla\mathbf{u},\nabla\mathbf{v}) - \alpha\tau(|\nabla\mathbf{m}^j|^2\mathbf{u},\mathbf{v}) \\ & -\tau(\nabla\mathbf{v}\times\mathbf{m}^j,\nabla\mathbf{u}) - \tau(\mathbf{v}\times\nabla\mathbf{m}^j,\nabla\mathbf{u}). \end{aligned}$$

Let us compute:

$$\begin{aligned} a(\mathbf{u},\mathbf{u}) \quad \geq \quad & \|\mathbf{u}\|_2^2 + \alpha\tau\|\nabla\mathbf{u}\|_2^2 - \alpha\tau|(|\nabla\mathbf{m}^j|^2\mathbf{u},\mathbf{u})| - \tau|(\mathbf{u}\times\nabla\mathbf{m}^j,\nabla\mathbf{u})| \\ \geq \quad & \|\mathbf{u}\|_2^2 + \alpha\tau\|\nabla\mathbf{u}\|_2^2 - \alpha\tau\|\nabla\mathbf{m}^j\|_4^2\|\mathbf{u}\|_4^2 - \tau\|\mathbf{u}\|_4\|\nabla\mathbf{m}^j\|_4\|\nabla\mathbf{u}\|_2 \\ \geq \quad & \|\mathbf{u}\|_2^2 + \alpha\tau\|\nabla\mathbf{u}\|_2^2 - C\alpha\tau\|\mathbf{u}\|_4^2 - C\tau\|\mathbf{u}\|_4\|\nabla\mathbf{u}\|_2, \end{aligned}$$

where we have already used Remark 9.1 below. Now, we use inequality (10.11) and the Young inequality to finish verifying the V-ellipticity of $a(\mathbf{u},\mathbf{v})$

$$\begin{aligned} a(\mathbf{u},\mathbf{u}) \quad \geq \quad & \|\mathbf{u}\|_2^2 + \alpha\tau\|\nabla\mathbf{u}\|_2^2 - C_\epsilon\alpha\tau\|\mathbf{u}\|_4^2 - \epsilon\tau\|\nabla\mathbf{u}\|_2^2 \\ \geq \quad & \|\mathbf{u}\|_2^2 + \alpha\tau\|\nabla\mathbf{u}\|_2^2 - C_\epsilon\alpha\tau\|\mathbf{u}\|_2^2 - C_\epsilon\alpha\tau\|\mathbf{u}\|_2^{\frac{1}{2}}\|\nabla\mathbf{u}\|_2^{\frac{3}{2}} - \epsilon\tau\|\nabla\mathbf{u}\|_2^2 \\ \geq \quad & \|\mathbf{u}\|_2^2(1 - C_\epsilon\alpha\tau) + \|\nabla\mathbf{u}\|_2^2(\alpha\tau - 2\epsilon\alpha\tau) \geq \frac{\alpha\tau}{2}\|\mathbf{u}\|_{W^{1,2}}^2, \end{aligned}$$

setting $\epsilon = 1/4$ and considering $\tau \leq (2C_\epsilon\alpha)^{-1}$.

To verify the boundedness of $a(\mathbf{u},\mathbf{v})$ in $W^{1,2}(\Omega)$ norm we simply write

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) \quad \leq \quad & \|\mathbf{u}\|_2\|\mathbf{v}\|_2 + \alpha\tau\|\nabla\mathbf{u}\|_2\|\nabla\mathbf{v}\|_2 + \alpha\tau\|\nabla\mathbf{m}^j\|_4^2\|\mathbf{u}\|_4\|\mathbf{v}\|_4 \\ & +\tau\|\nabla\mathbf{v}\|_2\|\mathbf{m}^j\|_{L^\infty}\|\nabla\mathbf{u}\|_2 + \tau\|\mathbf{v}\|_4\|\nabla\mathbf{m}^j\|_4\|\nabla\mathbf{u}\|_2. \end{aligned}$$

Now, we use the embedding $W^{2,2} \hookrightarrow W^{1,4} \hookrightarrow L^\infty$ and Remark 9.1 below to get

$$a(\mathbf{u},\mathbf{v}) \leq C\|\mathbf{u}\|_{W^{1,2}}\|\mathbf{v}\|_{W^{1,2}}.$$

We demonstrate the usefulness of the scheme (9.2) by the following theorem which guarantees the convergence of the scheme in time. This theorem is the main result of this chapter.

Theorem 9.1 (Convergence theorem) *Let* $0 < t_J < T_0(\mathbf{m})$. *Let* $\{\mathbf{m}^j\}_{j=0}^J$ *be the solution to (9.2) and* $\mathbf{m}$ *solves (7.2)–(7.4) for* $\mathbf{m}_0 \in W^{2,2}(\Omega)$, $|\mathbf{m}_0| = 1$ *on* Ω. *Let* $\tau \leq \tau_0$ *for* τ_0 *being sufficiently small. Then we have*

$$\max_{0\leq j\leq J} \|\mathbf{m}(t_j) - \mathbf{m}^j\|_2 + \left(\tau \sum_{j=0}^J \|\nabla\{\mathbf{m}(t_j) - \mathbf{m}^j\}\|_2^2\right)^{\frac{1}{2}} \leq C\tau, \tag{9.3}$$

$$\max_{0\leq j\leq J} \|\nabla\mathbf{m}(t_j) - \nabla\mathbf{m}^j\|_2 + \left(\tau \sum_{j=0}^J \|\Delta\{\mathbf{m}(t_j) - \mathbf{m}^j\}\|_2^2\right)^{\frac{1}{2}} \leq C\sqrt{\tau}. \tag{9.4}$$

Remark 9.1 *As a simple consequence of Theorem 7.3 and Theorem 9.1 we have that the solution* $\{\mathbf{m}^j\}_{j=0}^J$ *to (9.2) enjoys*

$$\max_{0\leq j\leq J} \left\|\mathbf{m}^j\right\|_{W^{2,2}} \leq C.$$

Notice that we do not enforce $|\mathbf{m}^j| = 1$ and moreover the scheme (9.2) does not guarantee this feature. However, we are able to prove some kind of convergence of the modulus $|\mathbf{m}^j|$ to 1 as indicated by the next theorem.

Theorem 9.2 (Modulus theorem) *Let the time discretization step* τ *be the same as in the previous theorem. Then the solution* $\{\mathbf{m}^j\}_{j=0}^J$ *satisfies the following condition*

$$\max_{0\leq j\leq J} \|1 - |\mathbf{m}^j|^2\|_2 \leq C\tau.$$

9.2 Proof of the convergence theorem

Let us denote by $\mathbf{e}^{j+1} := \mathbf{m}(t_{j+1}) - \mathbf{m}^{j+1}$ the error of the approximation. Consider (9.2) and (7.2) in a weak form. Then for all $\boldsymbol{\phi} \in W^{1,2}(\Omega)$ we have

$$\begin{aligned}(\delta\mathbf{e}^{j+1}, \boldsymbol{\phi}) + \alpha(\nabla\mathbf{e}^{j+1}, \nabla\boldsymbol{\phi}) &= (\mathcal{F}^{j+1}(\mathbf{m}), \boldsymbol{\phi}) \\ &+\alpha\Big[(|\nabla\mathbf{m}(t_{j+1})|^2\mathbf{m}(t_{j+1}), \boldsymbol{\phi}) - (|\nabla\mathbf{m}^j|^2\mathbf{m}^{j+1}, \boldsymbol{\phi})\Big] \\ &-\Big[(\mathbf{m}(t_{j+1}) \times \Delta\mathbf{m}(t_{j+1}), \boldsymbol{\phi}) - (\mathbf{m}^j \times \Delta\mathbf{m}^{j+1}, \boldsymbol{\phi})\Big],\end{aligned} \tag{9.5}$$

where

$$\mathcal{F}^{j+1}(\mathbf{m}) := -\frac{1}{\tau}\int_{t_j}^{t_{j+1}} (s - t_j)\partial_t^2\mathbf{m}(s)\mathrm{d}s.$$

Let us compute

$$
\begin{aligned}
&\tau \sum_{j=0}^{J} \|\mathcal{F}^{j+1}(\mathbf{m})\|_2^2 \\
&\quad = \tau \sum_{j=0}^{J} \tau^{-2} \int_\Omega \left(\int_{t_j}^{t_{j+1}} (s-t_j)\partial_t^2 \mathbf{m}(s)\mathrm{d}s \right)^2 \\
&\quad = \tau^{-1} \sum_{j=0}^{J} \int_\Omega \left(\int_{t_j}^{t_{j+1}} \underbrace{(s-t_j)^{\frac{1}{2}}}_{f} \underbrace{(s-t_j)^{\frac{1}{2}} \partial_t^2 \mathbf{m}(s)}_{g} \mathrm{d}s \right)^2
\end{aligned}
$$

We use Cauchy inequality for functions f and g to get

$$
\begin{aligned}
&\tau \sum_{j=0}^{J} \|\mathcal{F}^{j+1}(\mathbf{m})\|_2^2 \\
&\leq \tau^{-1} \sum_{j=0}^{J} \int_\Omega \left(\int_{t_j}^{t_{j+1}} (s-t_j)\mathrm{d}s \right) \left(\int_{t_j}^{t_{j+1}} (s-t_j)|\partial_t^2 \mathbf{m}(s)|^2 \mathrm{d}s \right) \\
&\leq \tau^{-1} \sum_{j=0}^{J} \frac{\tau^2}{2} \left(\int_{t_j}^{t_{j+1}} s\|\partial_t^2 \mathbf{m}(s)\|_2^2 \mathrm{d}s \right) \\
&\leq C\tau
\end{aligned}
$$

where we have used (7.14) at the end.

Summarizing previous we get

$$
\tau \sum_{j=0}^{J} \|\mathcal{F}^{j+1}(\mathbf{m})\|_2^2 \quad \leq \quad C\tau, \tag{9.6}
$$

Let us estimate $\mathcal{F}^{j+1}(\mathbf{m})$ in the norm of $W^{-1,2}$

$$
\begin{aligned}
\|\mathcal{F}^{j+1}(\mathbf{m})\|_{W^{-1,2}} \quad &= \quad \sup_{u\in W^{1,2}} \frac{|\int_\Omega \mathcal{F}^{j+1}(\mathbf{m}).\mathbf{u}\ \mathrm{d}x|}{\|\mathbf{u}\|_{W^{1,2}}} \\
&= \quad \sup_{u\in W^{1,2}} \frac{\left|\int_\Omega \left[\frac{1}{\tau}\int_{t_j}^{t_{j+1}} (s-t_j)\partial_t^2 \mathbf{m}(s)\mathrm{d}s\right].\mathbf{u}\ \mathrm{d}x\right|}{\|\mathbf{u}\|_{W^{1,2}}} \\
&= \quad \frac{1}{\tau} \sup_{u\in W^{1,2}} \frac{\left|\int_\Omega \int_{t_j}^{t_{j+1}} (s-t_j)\partial_t^2 \mathbf{m}(s).\mathbf{u}\ \mathrm{d}s\mathrm{d}x\right|}{\|\mathbf{u}\|_{W^{1,2}}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\tau}\sup_{u\in W^{1,2}} \frac{\left|\int_{t_j}^{t_{j+1}}(s-t_j)\int_\Omega \partial_t^2\mathbf{m}(s).\mathbf{u}\ \mathrm{d}x\mathrm{d}s\right|}{\|\mathbf{u}\|_{W^{1,2}}} \\
&= \frac{1}{\tau}\int_{t_j}^{t_{j+1}}(s-t_j)\sup_{u\in W^{1,2}}\frac{\left|\int_\Omega \partial_t^2\mathbf{m}(s).\mathbf{u}\ \mathrm{d}x\right|}{\|\mathbf{u}\|_{W^{1,2}}}\mathrm{d}s \\
&= \frac{1}{\tau}\int_{t_j}^{t_{j+1}}(s-t_j)\|\partial_t^2\mathbf{m}(s)\|_{W^{-1,2}}\mathrm{d}s.
\end{aligned}
$$

Further we get

$$
\|\mathcal{F}^{j+1}(\mathbf{m})\|^2_{W^{-1,2}} \quad = \quad \frac{1}{\tau^2}\left(\int_{t_j}^{t_{j+1}}\underbrace{(s-t_j)}_{f}\underbrace{\|\partial_t^2\mathbf{m}(s)\|_{W^{-1,2}}}_{g}\mathrm{d}s\right)^2.
$$

We use again Cauchy inequality for functions f and g to obtain

$$
\begin{aligned}
\|\mathcal{F}^{j+1}(\mathbf{m})\|^2_{W^{-1,2}} &= \frac{1}{\tau^2}\left(\int_{t_j}^{t_{j+1}}(s-t_j)^2\mathrm{d}s\right)\left(\int_{t_j}^{t_{j+1}}\|\partial_t^2\mathbf{m}(s)\|^2_{W^{-1,2}}\mathrm{d}s\right) \\
&= \frac{1}{\tau^2}\frac{\tau^3}{3}\int_{t_j}^{t_{j+1}}\|\partial_t^2\mathbf{m}(s)\|^2_{W^{-1,2}}\mathrm{d}s \\
&= \frac{\tau}{3}\int_{t_j}^{t_{j+1}}\|\partial_t^2\mathbf{m}(s)\|^2_{W^{-1,2}}\mathrm{d}s.
\end{aligned}
$$

Finally, using (7.15) we derive an estimate, which we use in the proof of Statement 1 (see below)

$$
\tau\sum_{j=0}^{J}\|\mathcal{F}^{j+1}(\mathbf{m})\|^2_{W^{-1,2}} \quad \leq \quad \tau\frac{\tau}{3}\int_0^T\|\partial_t^2\mathbf{m}(s)\|^2_{W^{-1,2}}\mathrm{d}s \leq C\tau^2. \tag{9.7}
$$

Let us do some technical steps in (9.5). We add and subtract some terms to obtain

$$
\begin{aligned}
&(|\nabla\mathbf{m}(t_{j+1})|^2\mathbf{m}(t_{j+1}),\boldsymbol{\phi}) - (|\nabla\mathbf{m}^j|^2\mathbf{m}^{j+1},\boldsymbol{\phi}) \\
&\quad = (\langle\nabla\mathbf{m}(t_{j+1})-\nabla\mathbf{m}(t_j),\nabla\mathbf{m}(t_{j+1})+\nabla\mathbf{m}(t_j)\rangle_{\mathbb{R}^9}\mathbf{m}(t_{j+1}),\boldsymbol{\phi}) \\
&\qquad +2(\langle\nabla\mathbf{m}(t_j),\nabla\mathbf{e}^j\rangle_{\mathbb{R}^9}\mathbf{m}(t_{j+1}),\boldsymbol{\phi}) - (|\nabla\mathbf{e}^j|^2\mathbf{m}(t_{j+1}),\boldsymbol{\phi}) \\
&\qquad +(|\nabla\mathbf{m}(t_j)|^2\mathbf{e}^{j+1},\boldsymbol{\phi}) - 2(\langle\nabla\mathbf{m}(t_j),\nabla\mathbf{e}^j\rangle_{\mathbb{R}^9}\mathbf{e}^{j+1},\boldsymbol{\phi}) \\
&\qquad +(|\nabla\mathbf{e}^j|^2\mathbf{e}^{j+1},\boldsymbol{\phi}).
\end{aligned}
$$

The last term in (9.5) is

$$
\begin{aligned}
&(\mathbf{m}(t_{j+1}) \times \Delta\mathbf{m}(t_{j+1}), \boldsymbol{\phi}) - (\mathbf{m}^j \times \Delta\mathbf{m}^{j+1}, \boldsymbol{\phi}) \\
&\quad = ([\mathbf{m}(t_{j+1}) - \mathbf{m}(t_j)] \times \Delta\mathbf{m}(t_{j+1}), \boldsymbol{\phi}) + (\mathbf{m}(t_j) \times \Delta\mathbf{e}^{j+1}, \boldsymbol{\phi}) \\
&\qquad + (\mathbf{e}^j \times \Delta\mathbf{m}(t_{j+1}), \boldsymbol{\phi}) - (\mathbf{e}^j \times \Delta\mathbf{e}^{j+1}, \boldsymbol{\phi})\big].
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&(\delta\mathbf{e}^{j+1}, \boldsymbol{\phi})) + \alpha(\nabla\mathbf{e}^{j+1}, \nabla\boldsymbol{\phi}) \\
&\quad = (\mathcal{F}^{j+1}(\mathbf{m}), \boldsymbol{\phi}) \\
&\quad + (\langle\nabla\mathbf{m}(t_{j+1}) - \nabla\mathbf{m}(t_j), \nabla\mathbf{m}(t_{j+1}) + \nabla\mathbf{m}(t_j)\rangle_{\mathbb{R}^9}\mathbf{m}(t_{j+1}), \boldsymbol{\phi}) \\
&\quad + 2(\langle\nabla\mathbf{m}(t_j), \nabla\mathbf{e}^j\rangle_{\mathbb{R}^9}\mathbf{m}(t_{j+1}), \boldsymbol{\phi}) - (|\nabla\mathbf{e}^j|^2\mathbf{m}(t_{j+1}), \boldsymbol{\phi}) \\
&\quad + (|\nabla\mathbf{m}(t_j)|^2\mathbf{e}^{j+1}, \boldsymbol{\phi}) - 2(\langle\nabla\mathbf{m}(t_j), \nabla\mathbf{e}^j\rangle_{\mathbb{R}^9}\mathbf{e}^{j+1}, \boldsymbol{\phi}) + (|\nabla\mathbf{e}^j|^2\mathbf{e}^{j+1}, \boldsymbol{\phi}) \\
&\quad - ([\mathbf{m}(t_{j+1}) - \mathbf{m}(t_j)] \times \Delta\mathbf{m}(t_{j+1}), \boldsymbol{\phi}) \\
&\quad - (\mathbf{m}(t_j) \times \Delta\mathbf{e}^{j+1}, \boldsymbol{\phi}) - (\mathbf{e}^j \times \Delta\mathbf{m}(t_{j+1}), \boldsymbol{\phi}) + (\mathbf{e}^j \times \Delta\mathbf{e}^{j+1}, \boldsymbol{\phi}) \;=:\; \mathcal{Y},
\end{aligned}
\tag{9.8}
$$

where we have denoted the right-hand side of (9.8) by $\mathcal{Y}$.

Now, we derive two statements, one by testing (9.8) by $\boldsymbol{\phi} = \mathbf{e}^{j+1}$ and the second one by $\boldsymbol{\phi} = -\Delta\mathbf{e}^{j+1}$.

STATEMENT 1

Take $\boldsymbol{\phi} = \mathbf{e}^{i+1}$ in (9.8). We denote by $\mathcal{Y}_1, \ldots, \mathcal{Y}_{11}$, the terms arising in $\mathcal{Y}$ when $\boldsymbol{\phi} = \mathbf{e}^{i+1}$. Our goal in the following part will be to arrive at the inequality

$$
\begin{aligned}
&(\delta\mathbf{e}^{j+1}, \mathbf{e}^{j+1}) + \alpha\|\nabla\mathbf{e}^{j+1}\|_2^2 \\
&\quad \le |(\mathcal{F}^{j+1}(\mathbf{m}), \mathbf{e}^{j+1})| + \epsilon\|\nabla\mathbf{e}^j\|_2^2 + C\|\nabla\mathbf{e}^j\|_4^2\|\nabla\mathbf{e}^j\|_2^2 \\
&\qquad + C_\epsilon\|\mathbf{e}^{j+1}\|_4^2 + C\|\mathbf{e}^{j+1}\|_4^2\|\nabla\mathbf{e}^j\|_4^2 \\
&\qquad + C\|\nabla\mathbf{e}^{j+1}\|_2\|\mathbf{e}^{j+1}\|_4[\|\nabla\mathbf{e}^j\|_4 + 1] + C\|\nabla\mathbf{e}^{j+1}\|_2\|\mathbf{e}^j\|_4 \\
&\qquad + \|\mathbf{m}(t_{j+1}) - \mathbf{m}(t_j)\|_2^2 + \|\nabla\mathbf{m}(t_{j+1}) - \nabla\mathbf{m}(t_j)\|_2^2,
\end{aligned}
\tag{9.9}
$$

estimating each term in (9.8) separately for $\boldsymbol{\phi} = \mathbf{e}^{i+1}$. We leave the term $\mathcal{Y}_1$ without any change. The term $\mathcal{Y}_2$ will be estimated using the inequalities (7.8), (10.7), $\|\mathbf{m}\|_{L^\infty} = 1$ and the Young inequality:

$$
\begin{aligned}
\mathcal{Y}_2 &= (\langle\nabla\mathbf{m}(t_{j+1}) - \nabla\mathbf{m}(t_j), \nabla\mathbf{m}(t_{j+1}) + \nabla\mathbf{m}(t_j)\rangle_{\mathbb{R}^9}\mathbf{m}(t_{j+1}), \mathbf{e}^{j+1}) \\
&\le \|\nabla\mathbf{m}(t_{j+1}) - \nabla\mathbf{m}(t_j)\|_2\|\nabla\mathbf{m}(t_{j+1}) + \nabla\mathbf{m}(t_j)\|_4\|\mathbf{m}(t_{j+1})\|_{L^\infty}\|\mathbf{e}^{j+1}\|_4 \\
&\le C\|\nabla\mathbf{m}(t_{j+1}) - \nabla\mathbf{m}(t_j)\|_2^2 + C\|\mathbf{e}^{j+1}\|_4^2.
\end{aligned}
$$

In a similar way, also the terms $\mathcal{Y}_3$ to $\mathcal{Y}_7$ can be bounded:

$$
\begin{aligned}
\mathcal{Y}_3 &= (\langle\nabla\mathbf{e}^j, \nabla\mathbf{m}(t_j)\rangle_{\mathbb{R}^9}\mathbf{m}(t_{j+1}), \mathbf{e}^{j+1}) \\
&\le \|\nabla\mathbf{e}^j\|_2\|\nabla\mathbf{m}(t_j)\|_4\|\mathbf{m}\|_{L^\infty}\|\mathbf{e}^{j+1}\|_4 \\
&\le \epsilon\|\nabla\mathbf{e}^j\|_2^2 + C_\epsilon\|\mathbf{e}^{j+1}\|_4^2,
\end{aligned}
$$

$$\begin{aligned}
\mathcal{Y}_4 &= (|\nabla \mathbf{e}^j|^2 \mathbf{m}(t_{j+1}), \mathbf{e}^{j+1}) \\
&\le \|\nabla \mathbf{e}^j\|_4 \|\nabla \mathbf{e}^j\|_2 \|\mathbf{m}\|_{L^\infty} \|\mathbf{e}^{j+1}\|_4 \\
&\le C\|\nabla \mathbf{e}^j\|_4^2 \|\nabla \mathbf{e}^j\|_2^2 + C\|\mathbf{e}^{j+1}\|_4^2,
\end{aligned}$$

$$\begin{aligned}
\mathcal{Y}_5 &= (|\nabla \mathbf{m}(t_j)|^2 \mathbf{e}^{j+1}, \mathbf{e}^{j+1}) \\
&\le \|\nabla \mathbf{m}\|_4^2 \|\mathbf{e}_{j+1}\|_4^2 \le C\|\mathbf{e}^{j+1}\|_4^2,
\end{aligned}$$

$$\begin{aligned}
\mathcal{Y}_6 &= (\langle \nabla \mathbf{e}^j, \nabla \mathbf{m}(t_j)\rangle_{\mathbb{R}^9}, |\mathbf{e}^{j+1}|^2) \\
&\le \|\nabla \mathbf{e}^j\|_4 \|\nabla \mathbf{m}(t_{j+1})\|_4 \|\mathbf{e}^{j+1}\|_4^2 \\
&\le C\|\nabla \mathbf{e}^j\|_4 \|\mathbf{e}^{j+1}\|_4^2 \le C(1+\|\nabla \mathbf{e}^j\|_4^2)\|\mathbf{e}^{j+1}\|_4^2,
\end{aligned}$$

$$\mathcal{Y}_7 = (|\nabla \mathbf{e}^j|^2, |\mathbf{e}^{j+1}|^2) \quad \le \quad \|\nabla \mathbf{e}^j\|_4^2 \|\mathbf{e}^{j+1}\|_4^2.$$

We estimate the term $\mathcal{Y}_8$ using the embedding $W^{1,2} \hookrightarrow L^4$ and the Young inequality:

$$\begin{aligned}
\mathcal{Y}_8 &= ([\mathbf{m}(t_{j+1}) - \mathbf{m}(t_j)] \times \Delta \mathbf{m}(t_{j+1}), \mathbf{e}^{j+1}) \\
&\le \|\Delta \mathbf{m}(t_{j+1})\|_2 \|\mathbf{e}^{j+1}\|_4 \|\mathbf{m}(t_{j+1}) - \mathbf{m}(t_j)\|_4 \\
&\le C\|\mathbf{e}^{j+1}\|_4 \|\mathbf{m}(t_{j+1}) - \mathbf{m}(t_j)\|_2 \\
&\quad + C\|\mathbf{e}^{j+1}\|_4 \|\nabla \mathbf{m}(t_{j+1}) - \nabla \mathbf{m}(t_j)\|_2 \\
&\le C\|\mathbf{e}^{j+1}\|_4^2 + C\|\mathbf{m}(t_{j+1}) - \mathbf{m}(t_j)\|_2^2 \\
&\quad + C\|\nabla \mathbf{m}(t_{j+1}) - \nabla \mathbf{m}(t_j)\|_2^2.
\end{aligned}$$

Next, we use integration by parts to get

$$\begin{aligned}
\mathcal{Y}_9 &= |(\mathbf{m}(t_j) \times \Delta \mathbf{e}^{j+1}, \mathbf{e}^{j+1})| \\
&= |(\nabla(\mathbf{e}^{j+1} \times \mathbf{m}(t_j)), \nabla \mathbf{e}^{j+1})| \\
&\le \|\mathbf{e}^{j+1}\|_4 \|\nabla \mathbf{m}\|_4 \|\nabla \mathbf{e}^{j+1}\|_2 \le C\|\mathbf{e}^{j+1}\|_4 \|\nabla \mathbf{e}^{j+1}\|_2.
\end{aligned}$$

We estimate also the terms $\mathcal{Y}_{10}$ and $\mathcal{Y}_{11}$ in a similar way:

$$\begin{aligned}
\mathcal{Y}_{10} &= (\mathbf{e}^j \times \Delta \mathbf{m}(t_{j+1}), \mathbf{e}^{j+1}) \\
&\le |(\nabla(\mathbf{e}^{j+1} \times \mathbf{e}^j), \nabla \mathbf{m}(t_{j+1}))| \\
&\le \|\nabla \mathbf{e}^{j+1}\|_2 \|\mathbf{e}^j\|_4 \|\nabla \mathbf{m}\|_4 + \|\mathbf{e}^{j+1}\|_4 \|\nabla \mathbf{e}^j\|_2 \|\nabla \mathbf{m}\|_4 \\
&\le C\|\nabla \mathbf{e}^{j+1}\|_2 \|\mathbf{e}^j\|_4 + C_\epsilon \|\mathbf{e}^{j+1}\|_4^2 + \epsilon \|\nabla \mathbf{e}^j\|_2^2,
\end{aligned}$$

$$\begin{aligned}
\mathcal{Y}_{11} &= (\mathbf{e}^j \times \Delta \mathbf{e}^{j+1}, \mathbf{e}^{j+1}) \\
&\le |\nabla(\mathbf{e}^{j+1} \times \mathbf{e}^j), \nabla \mathbf{e}^{j+1})| = \|\mathbf{e}^{j+1}\|_4 \|\nabla \mathbf{e}^{j+1}\|_2 \|\nabla \mathbf{e}^j\|_4.
\end{aligned}$$

We can now proceed with estimating the right-hand side of (9.9). The letters $\mathcal{X}_1$ and $\mathcal{X}_2$ denote positive real numbers. We first do some calculations, using (10.11) and then the Young inequality with exponents 4/3 and 4:

$$\begin{aligned}\|\mathbf{e}^{j+1}\|_4^2\mathcal{X}_1 &\leq C(\|\mathbf{e}^{j+1}\|_2^2\mathcal{X}_1 + \|\mathbf{e}^{j+1}\|_2^{\frac{1}{2}}\|\nabla\mathbf{e}^{j+1}\|_2^{\frac{3}{2}}\mathcal{X}_1)\\ &\leq C_\epsilon\mathcal{X}_1\|\mathbf{e}^{j+1}\|_2^2 + C_\epsilon\mathcal{X}_1^4\|\mathbf{e}^{j+1}\|_2^2 + \epsilon\|\nabla\mathbf{e}^{j+1}\|_2^2.\end{aligned}$$

Analogously, using first the Young inequality with exponents 2 and then the Young inequality with exponents 8/7 and 8:

$$\begin{aligned}\|\nabla\mathbf{e}^{j+1}\|_2\|\mathbf{e}^{j+1}\|_4\mathcal{X}_2 &\leq C\mathcal{X}_2\|\nabla\mathbf{e}^{j+1}\|_2(\|\mathbf{e}^{j+1}\|_2 + \|\mathbf{e}^{j+1}\|_2^{\frac{1}{4}}\|\nabla\mathbf{e}^{j+1}\|_2^{\frac{3}{4}})\\ &\leq C_\epsilon\mathcal{X}_2^2\|\mathbf{e}^{j+1}\|_2^2 + C_\epsilon\mathcal{X}_2^8\|\mathbf{e}^{j+1}\|_2^2 + \epsilon\|\nabla\mathbf{e}^{j+1}\|_2^2.\end{aligned}$$

We apply the previous two inequalities to (9.9). First we set $\mathcal{X}_1 = C_\epsilon$ then $\mathcal{X}_1 = C\|\nabla\mathbf{e}^j\|_4^2$. Then we set $\mathcal{X}_2 = \|\nabla\mathbf{e}^j\|_4$ and $\mathcal{X}_2 = C$. Thus supposing $\alpha > \epsilon/2$ we can write

$$\begin{aligned}&(\delta\mathbf{e}^{j+1},\mathbf{e}^{j+1}) + (\alpha-\epsilon)\|\nabla\mathbf{e}^{j+1}\|_2^2 \qquad (9.10)\\ &\quad\leq |(\mathcal{F}^{j+1}(\mathbf{m}),\mathbf{e}^{j+1})| + C\|\nabla\mathbf{e}^j\|_2^2\|\nabla\mathbf{e}^j\|_4^2 + C_\epsilon\|\nabla\mathbf{e}^j\|_4^2 + \epsilon\|\nabla\mathbf{e}^j\|_2^2\\ &\qquad + C_\epsilon\|\mathbf{e}^{j+1}\|_2^2 + C_\epsilon\|\mathbf{e}^{j+1}\|_2^2\big(\|\nabla\mathbf{e}^j\|_4^2 + \|\nabla\mathbf{e}^j\|_4^4 + \|\nabla\mathbf{e}^j\|_4^8\big)\\ &\qquad + \|\mathbf{m}(t_{j+1}) - \mathbf{m}(t_j)\|_2^2 + \|\nabla\mathbf{m}(t_{j+1}) - \nabla\mathbf{m}(t_j)\|_2^2.\end{aligned}$$

The next step is to sum up all equations (9.10) for $j = 0,\ldots,l$. To get rid of the last two terms in (9.10), we introduce the following lemma.

Lemma 9.1 *Keeping all notations from the above discussion the following inequality is valid*

$$\sum_{j=0}^{l}\|\mathbf{m}(t_{j+1}) - \mathbf{m}(t_j)\|_2^2 + \sum_{j=0}^{l}\|\nabla\mathbf{m}(t_{j+1}) - \nabla\mathbf{m}(t_j)\|_2^2 \leq C\tau.$$

PROOF:
From the definition of L^2 norm and using the integral Young inequality with respect to time we get

$$\begin{aligned}&\sum_{j=0}^{l}\|\nabla\mathbf{m}(t_{j+1}) - \nabla\mathbf{m}(t_j)\|_2^2 = \sum_{j=0}^{j}\int_\Omega\left(\int_{t_j}^{t_{j+1}}\langle\mathbf{1},\nabla\partial_t\mathbf{m}(s)\rangle_{\mathbb{R}^9}\mathrm{d}s\right)^2\mathrm{d}x\\ &\leq C\sum_{j=0}^{j}\int_\Omega\tau\int_{t_j}^{t_{j+1}}|\nabla\partial_t\mathbf{m}(s)|^2\,\mathrm{d}s\mathrm{d}x \leq C\sum_{j=0}^{j}\tau\int_{t_j}^{t_{j+1}}\|\nabla\partial_t\mathbf{m}(s)\|_2^2\mathrm{d}s\\ &\leq C\tau\int_0^{t_{l+1}}\|\nabla\partial_t\mathbf{m}(s)\|_2^2\mathrm{d}s \leq C\tau,\end{aligned}$$

where we have used (7.13) at the end. The rest of Lemma 9.1 can be proved in the same way. □

To get rid of the first term on the right-hand side of (9.10) we have

$$|(\mathcal{F}^{j+1}(\mathbf{m}),\mathbf{e}^{j+1})| = \left|\int_\Omega \mathcal{F}^{j+1}(\mathbf{m}).\mathbf{e}^{i+1}\right| \le \|\mathcal{F}^{j+1}(\mathbf{m})\|_{W^{-1,2}}\|\mathbf{e}^{j+1}\|_{W^{1,2}}.$$

According to (9.7), after performing the Young inequality we get for the term $(\mathcal{F}^{j+1}(\mathbf{m}),\mathbf{e}^{j+1})$ the estimate

$$\tau\sum_{j=0}^{J}|(\mathcal{F}^{j+1}(\mathbf{m}),\mathbf{e}^{j+1})| \tag{9.11}$$

$$\le \tau\sum_{j=0}^{J}\|\mathcal{F}^{j+1}(\mathbf{m})\|_{W^{-1,2}}(\|\mathbf{e}^{j+1}\|_2^2+\|\nabla\mathbf{e}^{j+1}\|_2^2)^{\frac{1}{2}} \tag{9.12}$$

$$\le C_\epsilon\tau\sum_{j=0}^{J}\|\mathcal{F}^{j+1}(\mathbf{m})\|_{W^{-1,2}}^2+\tau\epsilon\sum_{j=0}^{J}\left(\|\nabla\mathbf{e}^{j+1}\|_2^2+\|\mathbf{e}^{j+1}\|_2^2\right)$$

$$\le C_\epsilon\tau^2+\tau\epsilon\sum_{j=0}^{J}\left(\|\nabla\mathbf{e}^{j+1}\|_2^2+\|\mathbf{e}^{j+1}\|_2^2\right).$$

We can now continue in (9.10). On the left-hand side we use the inequality

$$\tau\sum_{j=0}^{l}(\delta\mathbf{e}^{j+1},\mathbf{e}^{j+1}) \ge \frac{1}{2}\|\mathbf{e}^{l+1}\|_2^2$$

to find that

$$\|\mathbf{e}^{l+1}\|_2^2+\tau\sum_{j=0}^{l}\|\nabla\mathbf{e}^{j+1}\|_2^2$$

$$\le C_\epsilon\tau^2+C_\epsilon\tau\sum_{j=0}^{l}\|\nabla\mathbf{e}^j\|_4^2\sum_{j=0}^{l}\|\nabla\mathbf{e}^j\|_2^2+\epsilon\tau\sum_{j=0}^{l}(\|\nabla\mathbf{e}^j\|_2^2+\|\nabla\mathbf{e}^{j+1}\|_2^2)$$

$$+C\tau\sum_{j=0}^{l}\|\mathbf{e}^{j+1}\|_2^2+C\tau\sum_{j=0}^{l}(\|\nabla\mathbf{e}^j\|_4^2+\|\nabla\mathbf{e}^j\|_4^4+\|\nabla\mathbf{e}^j\|_4^8)\sum_{j=0}^{l}\|\mathbf{e}^{j+1}\|_2^2,$$

where for some terms we have already used the inequality

$$\sum_{j=0}^{l}a_jb_j \le \sum_{j=0}^{l}a_j\sum_{j=0}^{l}b_j, \tag{9.13}$$

which holds for any nonnegative a_j, b_j. Setting $\epsilon < 1/4$ we can now absorb the term $\epsilon\tau \sum_{j=0}^{l} \|\nabla \mathbf{e}^j\|_2^2$ on the right-hand side into the similar term on the left to get the desired STATEMENT 1:

$$\|\mathbf{e}^{l+1}\|^2 + \tau \sum_{j=0}^{l} \|\nabla \mathbf{e}^{j+1}\|_2^2 \tag{9.14}$$
$$\leq C\tau^2 + C\tau \sum_{j=0}^{l} \|\nabla \mathbf{e}^j\|_4^2 \sum_{j=0}^{l} \|\nabla \mathbf{e}^j\|_2^2 + C\tau \sum_{j=0}^{l} \|\mathbf{e}^{j+1}\|_2^2$$
$$+C\tau \sum_{j=0}^{l} (\|\nabla \mathbf{e}^j\|_4^2 + \|\nabla \mathbf{e}^j\|_4^4 + \|\nabla \mathbf{e}^j\|_4^8) \sum_{j=0}^{l} \|\mathbf{e}^{j+1}\|_2^2.$$

END OF STATEMENT 1

STATEMENT 2

In this statement we do mostly the same as in STATEMENT 1. We choose $\boldsymbol{\phi} = -\Delta \mathbf{e}^{i+1}$ in (9.8) and denote by $\mathcal{Y}_1', \mathcal{Y}_2', \ldots, \mathcal{Y}_{11}'$ the terms arising on the right-hand side of (9.8), respectively. Our goal will be to arrive at the inequality

$$(\delta \nabla \mathbf{e}^{j+1}, \nabla \mathbf{e}^{j+1}) + \alpha \|\Delta \mathbf{e}^{j+1}\|_2^2 \tag{9.15}$$
$$\leq C_\epsilon \|\mathcal{F}^{j+1}(\mathbf{m})\|_2^2 + \epsilon \|\Delta \mathbf{e}^{j+1}\|_2^2 + C_\epsilon \big(\|\nabla \mathbf{e}^j\|_4^2 + C\|\mathbf{e}^j\|_4^2 + \|\nabla \mathbf{e}^j\|_4^4\big)$$
$$+C_\epsilon \|\nabla \mathbf{e}^{j+1}\|_4^2 + C_\epsilon \|\nabla \mathbf{e}^{j+1}\|_4^2 \big(\|\nabla \mathbf{e}^j\|_4^2 + \|\nabla \mathbf{e}^j\|_4^4\big)$$
$$+C_\epsilon \|\mathbf{e}^{j+1}\|_4^2 + C_\epsilon \|\mathbf{e}^{j+1}\|_4^2 \big(\|\nabla \mathbf{e}^j\|_4^2 + \|\nabla \mathbf{e}^j\|_4^4\big)$$
$$+C_\epsilon \|\mathbf{m}(t_{j+1}) - \mathbf{m}(t_j)\|_4^2 + C_\epsilon \|\nabla \mathbf{m}(t_{j+1}) - \nabla \mathbf{m}(t_j)\|_4^2.$$

The term $\mathcal{Y}_1'$ causes no troubles. The term $\mathcal{Y}_2'$ can be estimated using inequalities (7.8) and (10.7), the equality $\|\mathbf{m}\|_{L^\infty} = 1$ and the Young inequality:

$$\begin{aligned}
\mathcal{Y}_2' &= |(\langle \nabla \mathbf{m}(t_{j+1}) - \nabla \mathbf{m}(t_j), \nabla \mathbf{m}(t_{j+1}) + \nabla \mathbf{m}(t_j)\rangle_{\mathbb{R}^9} \mathbf{m}(t_{j+1}), \Delta \mathbf{e}^{j+1})| \\
&\leq 2\|\nabla \mathbf{m}(t_{j+1}) - \nabla \mathbf{m}(t_j)\|_4 \|\nabla \mathbf{m}(t_{j+1})\|_4 \|\mathbf{m}(t_{j+1})\|_{L^\infty} \|\Delta \mathbf{e}^{j+1}\|_2 \\
&\leq C_\epsilon \|\nabla \mathbf{m}(t_{j+1}) - \nabla \mathbf{m}(t_j)\|_4^2 + \epsilon \|\Delta \mathbf{e}^{j+1}\|_2^2.
\end{aligned}$$

We handle the terms $\mathcal{Y}_3'$ and $\mathcal{Y}_4'$ in a similar way:

$$\begin{aligned}
\mathcal{Y}_3' &= |(\langle \nabla \mathbf{e}^j, \nabla \mathbf{m}(t_j)\rangle_{\mathbb{R}^9} \mathbf{m}(t_{j+1}), \Delta \mathbf{e}^{j+1})| \\
&\leq \|\nabla \mathbf{e}^j\|_4 \|\nabla \mathbf{m}(t_j)\|_4 \|\mathbf{m}(t_{j+1})\|_{L^\infty} \|\Delta \mathbf{e}^{j+1}\|_2 \\
&\leq C_\epsilon \|\nabla \mathbf{e}^j\|_4^2 + \epsilon \|\Delta \mathbf{e}^{j+1}\|_2^2,
\end{aligned}$$

$$\mathcal{Y}_4' = |(|\nabla \mathbf{e}^j|^2 \mathbf{m}(t_{j+1}), \Delta \mathbf{e}^{j+1})|$$

$$\begin{aligned} &\leq \|\nabla \mathbf{e}^j\|_4^2 \|\mathbf{m}(t_{j+1})\|_{L^\infty} \|\Delta \mathbf{e}^{j+1}\|_2 \\ &\leq C_\epsilon \|\nabla \mathbf{e}^j\|_4^4 + \epsilon \|\Delta \mathbf{e}^{j+1}\|_2^2. \end{aligned}$$

We estimate the terms $\mathcal{Y}_5'$, $\mathcal{Y}_6'$ and $\mathcal{Y}_7'$ using (10.14) and the Young inequality:

$$\begin{aligned} \mathcal{Y}_5' &= |(|\nabla \mathbf{m}(t_j)|^2 \mathbf{e}^{j+1}, \Delta \mathbf{e}^{j+1})| \\ &\leq \|\nabla \mathbf{m}(t_j)\|_4^2 \|\mathbf{e}^{j+1}\|_{L^\infty} \|\Delta \mathbf{e}^{j+1}\|_2 \\ &\leq C_\epsilon \|\mathbf{e}^{j+1}\|_{L^\infty}^2 + \epsilon \|\Delta \mathbf{e}^{j+1}\|_2^2 \\ &\leq C_\epsilon (\|\mathbf{e}^{j+1}\|_4^2 + \|\nabla \mathbf{e}^{j+1}\|_4^2) + \epsilon \|\Delta \mathbf{e}^{j+1}\|_2^2, \end{aligned}$$

$$\begin{aligned} \mathcal{Y}_6' &= |(\langle \nabla \mathbf{e}^j, \nabla \mathbf{m}(t_j) \rangle_{\mathbb{R}^9} \mathbf{e}^{j+1}, \Delta \mathbf{e}^{j+1})| \\ &\leq \|\nabla \mathbf{e}^j\|_4 \|\nabla \mathbf{m}(t_j)\|_4 \|\mathbf{e}^{j+1}\|_{L^\infty} \|\Delta \mathbf{e}^{j+1}\|_2 \\ &\leq C \|\nabla \mathbf{e}^j\|_4 (\|\mathbf{e}^{j+1}\|_4 + \|\nabla \mathbf{e}^{j+1}\|_4) \|\Delta \mathbf{e}^{j+1}\|_2 \\ &\leq C_\epsilon \|\nabla \mathbf{e}^j\|_4^2 \|\mathbf{e}^{j+1}\|_4^2 + C_\epsilon \|\nabla \mathbf{e}^j\|_4^2 \|\nabla \mathbf{e}^{j+1}\|_4^2 + \epsilon \|\Delta \mathbf{e}^{j+1}\|_2^2, \end{aligned}$$

$$\begin{aligned} \mathcal{Y}_7' &= |(\langle \nabla \mathbf{e}^j, \nabla \mathbf{e}^j \rangle_{\mathbb{R}^9} \mathbf{e}^{j+1}, \Delta \mathbf{e}^{j+1})| \\ &\leq \|\nabla \mathbf{e}^j\|_4^2 \|\mathbf{e}^{j+1}\|_{L^\infty} \|\Delta \mathbf{e}^{j+1}\|_2 \\ &\leq \|\nabla \mathbf{e}^j\|_4^2 (\|\mathbf{e}^{j+1}\|_4 + \|\nabla \mathbf{e}^{j+1}\|_4) \|\Delta \mathbf{e}^{j+1}\|_2 \\ &\leq C_\epsilon \|\nabla \mathbf{e}^j\|_4^4 \|\mathbf{e}^{j+1}\|_4^2 \\ &\quad + C_\epsilon \|\nabla \mathbf{e}^j\|_4^4 \|\nabla \mathbf{e}^{j+1}\|_4^2 + \epsilon \|\Delta \mathbf{e}^{j+1}\|_2^2. \end{aligned}$$

The term $\mathcal{Y}_8'$ satisfies

$$\begin{aligned} \mathcal{Y}_8' &= |([\mathbf{m}(t_{j+1}) - \mathbf{m}(t_j)] \times \Delta \mathbf{m}(t_{j+1}), \Delta \mathbf{e}^{j+1})| \\ &\leq \|\mathbf{m}(t_{j+1}) - \mathbf{m}(t_j)\|_{L^\infty} \|\Delta \mathbf{m}\|_2 \|\Delta \mathbf{e}^{j+1}\|_2 \\ &\leq (\|\mathbf{m}(t_{j+1}) - \mathbf{m}(t_j)\|_4 + \|\nabla \mathbf{m}(t_{j+1}) - \nabla \mathbf{m}(t_j)\|_4) \|\Delta \mathbf{e}^{j+1}\|_2 \\ &\leq C_\epsilon \|\mathbf{m}(t_{j+1}) - \mathbf{m}(t_j)\|_4^2 + C_\epsilon \|\nabla \mathbf{m}(t_{j+1}) - \nabla \mathbf{m}(t_j)\|_4^2 + \epsilon \|\Delta \mathbf{e}^{j+1}\|_2^2. \end{aligned}$$

The terms $\mathcal{Y}_9'$ and $\mathcal{Y}_{11}'$ are equal to 0 and the last term $\mathcal{Y}_{10}'$ can be bounded as follows:

$$\begin{aligned} \mathcal{Y}_{10}' = |(\mathbf{e}^j \times \Delta \mathbf{m}(t_{j+1}), \Delta \mathbf{e}^{j+1})| &\leq C \|\mathbf{e}^j\|_{L^\infty} \|\Delta \mathbf{m}\|_2 \|\Delta \mathbf{e}^{j+1}\|_2 \\ &\leq C(\|\nabla \mathbf{e}^j\|_4 + \|\mathbf{e}^j\|_4) \|\Delta \mathbf{e}^{j+1}\|_2 \\ &\leq C_\epsilon \|\nabla \mathbf{e}^j\|_4^2 + C_\epsilon \|\mathbf{e}^j\|_4^2 + \epsilon \|\Delta \mathbf{e}^{j+1}\|_2^2. \end{aligned}$$

We can now proceed with estimating the right-hand side of (9.15). We again denote by $\mathcal{X}_1, \mathcal{X}_2$ arbitrary positive real numbers. We first do some calculations using the embedding $W^{1,2} \hookrightarrow L^4$ with the Young inequality. We get

$$\|\mathbf{e}^{j+1}\|_4^2 \mathcal{X}_1 \le C(\|\mathbf{e}^{j+1}\|_2^2 \mathcal{X}_1 + \|\nabla \mathbf{e}^{j+1}\|_2^2 \mathcal{X}_1).$$

Similarly, using (10.12) and the Young inequality with exponents 4/3 and 4 we arrive at

$$\begin{aligned}\|\nabla \mathbf{e}^{j+1}\|_4^2 \mathcal{X}_2 &\le C\mathcal{X}_2(\|\nabla \mathbf{e}^{j+1}\|_2^2 + \|\nabla \mathbf{e}^{j+1}\|_2^{\frac{1}{2}} \|\Delta \mathbf{e}^{j+1}\|_2^{\frac{3}{2}}) \\ &\le C_\epsilon \mathcal{X}_2 \|\nabla \mathbf{e}^{j+1}\|_2^2 + C_\epsilon \mathcal{X}_2^4 \|\nabla \mathbf{e}^{j+1}\|_2^2 + \epsilon \|\Delta \mathbf{e}^{j+1}\|_2^2.\end{aligned}$$

We continue in estimating the right-hand side of (9.15). We use the previous two inequalities for several values of $\mathcal{X}_1$ and $\mathcal{X}_2$. First take $\mathcal{X}_2 = C_\epsilon$, second take $\mathcal{X}_2 = \|\nabla \mathbf{e}^j\|_4^2$ and finally put $\mathcal{X}_2 = \|\nabla \mathbf{e}^j\|_4^4$. Consider that $\mathcal{X}_1$ has the same values as $\mathcal{X}_2$. Then supposing $\alpha > \epsilon/2$, we get

$$\begin{aligned}&(\delta \nabla \mathbf{e}^{j+1}, \nabla \mathbf{e}^{j+1}) + (\alpha - \epsilon)\|\Delta \mathbf{e}^{j+1}\|_2^2 \\ &\le C_\epsilon \|\mathcal{F}^{j+1}(\mathbf{m})\|_2^2 + C_\epsilon\big(\|\nabla \mathbf{e}^j\|_4^2 + \|\mathbf{e}^j\|_4^2 + \|\nabla \mathbf{e}^j\|_4^4\big) \\ &\quad + C_\epsilon \|\mathbf{e}^{j+1}\|_2^2 + C_\epsilon \|\mathbf{e}^{j+1}\|_2^2\big(\|\nabla \mathbf{e}^j\|_4^2 + \|\nabla \mathbf{e}^j\|_4^4\big) + C_\epsilon \|\nabla \mathbf{e}^{j+1}\|_2^2 \\ &\quad + C_\epsilon \|\nabla \mathbf{e}^{j+1}\|_2^2\big(\|\nabla \mathbf{e}^j\|_4^2 + \|\nabla \mathbf{e}^j\|_4^4 + \|\nabla \mathbf{e}^j\|_4^8 + \|\nabla \mathbf{e}^j\|_4^{16}\big) \\ &\quad + C_\epsilon \|\mathbf{m}(t_{j+1}) - \mathbf{m}(t_j)\|_4^2 + C_\epsilon \|\nabla \mathbf{m}(t_{j+1}) - \nabla \mathbf{m}(t_j)\|_4^2.\end{aligned} \tag{9.16}$$

We would like to follow the strategy of summing up the equations (9.16) for $j = 1, \ldots, l$. First, we introduce a weakened version of Lemma 9.1.

Lemma 9.2 *Keeping all notations from the above discussion the following inequality holds:*

$$\sum_{j=1}^{l} \|\mathbf{m}(t_{j+1}) - \mathbf{m}(t_j)\|_4^2 + \sum_{j=1}^{l} \|\nabla \mathbf{m}(t_{j+1}) - \nabla \mathbf{m}(t_j)\|_4^2 \le C.$$

Proof:
From the embedding $W^{1,2} \hookrightarrow L^4$, we have

$$\|\nabla \mathbf{m}(t_{j+1}) - \nabla \mathbf{m}(t_j)\|_4^2 \le C\|\nabla \mathbf{m}(t_{j+1}) - \nabla \mathbf{m}(t_j)\|_2^2 + C\|\Delta \mathbf{m}(t_{j+1}) - \Delta \mathbf{m}(t_j)\|_2^2.$$

From the definition of L^2 norm and using the integral Young inequality with respect to time we get

$$\begin{aligned}\sum_{j=1}^{l} \|\Delta \mathbf{m}(t_{j+1}) - \Delta \mathbf{m}(t_j)\|_2^2 &= \sum_{j=1}^{l} \int_\Omega \left(\int_{t_j}^{t_{j+1}} \langle \mathbf{1}, \Delta \partial_t \mathbf{m}(s)\rangle_{\mathbb{R}^3} \mathrm{d}s \right)^2 \mathrm{d}x \\ &\le C \sum_{j=1}^{l} \int_\Omega \tau \int_{t_j}^{t_{j+1}} |\Delta \partial_t \mathbf{m}(s)|^2 \, \mathrm{d}s \mathrm{d}x.\end{aligned}$$

Since we sum up from the index $j=1$, we have $\tau \le \kappa(s)$ on the interval (t_j, t_{j+1}). Thus

$$\begin{aligned}\sum_{j=1}^{l} \|\Delta\mathbf{m}(t_{j+1}) - \Delta\mathbf{m}(t_j)\|_2^2 &\le C\sum_{j=1}^{l}\int_\Omega\int_{t_i}^{t_{i+1}} \kappa(s)\,|\Delta\partial_t\mathbf{m}(s)|^2\,\mathrm{d}s\mathrm{d}x \\ &\le C\sum_{j=1}^{l}\int_{t_j}^{t_{j+1}} \kappa(s)\|\Delta\partial_t\mathbf{m}(s)\|_2^2\mathrm{d}s \\ &\le C\int_0^{t_{l+1}} \kappa(s)\|\Delta\partial_t\mathbf{m}(s)\|_2^2\mathrm{d}s \le C,\end{aligned}$$

where we have used (7.14) at the end. The rest of the lemma follows directly from Lemma 9.1.

This completes the proof of Lemma 9.2. □

Now, we can write the STATEMENT 2 summing up the equations (9.16) for $j = 1, \ldots, l$ and applying Lemma 9.2 and (9.6). Directly after the summation we use the inequality (9.13) for some terms

$$\begin{aligned}&\|\nabla\mathbf{e}^{l+1}\|_2^2 + \tau\sum_{j=1}^{l}\|\Delta\mathbf{e}^{j+1}\|_2^2 \qquad (9.17)\\ &\le C\tau + C_\epsilon\tau\sum_{j=1}^{l}(\|\mathbf{e}^j\|_4^2 + \|\nabla\mathbf{e}^j\|_4^2 + \|\nabla\mathbf{e}^j\|_4^4)\\ &\quad + C_\epsilon\tau\sum_{j=1}^{l}\|\mathbf{e}^{j+1}\|_2^2\\ &\quad + C_\epsilon\tau\sum_{j=1}^{l}(\|\nabla\mathbf{e}^j\|_4^2 + \|\nabla\mathbf{e}^j\|_4^4)\sum_{j=1}^{l}\|\mathbf{e}^{j+1}\|_2^2 + C_\epsilon\tau\sum_{j=1}^{l}\|\nabla\mathbf{e}^{j+1}\|_2^2\\ &\quad + C_\epsilon\tau\sum_{j=1}^{l}(\|\nabla\mathbf{e}^j\|_4^2 + \|\nabla\mathbf{e}^j\|_4^4 + \|\nabla\mathbf{e}^j\|_4^8 + \|\nabla\mathbf{e}^j\|_4^{16})\sum_{j=1}^{l}\|\nabla\mathbf{e}^{j+1}\|_2^2.\end{aligned}$$

END OF STATEMENT 2

We proceed with the proof of Theorem 9.1. To prove it, we employ an inductive argument. There exists a constant $\mathcal{A}$ depending only on $\omega, \alpha, \mathbf{m}$ such that for a sufficiently small τ_0 we have for every $\tau \le \tau_0$

$$\|\mathbf{e}^{l+1}\|_2^2 + \tau\sum_{j=0}^{l}\|\nabla\mathbf{e}^{j+1}\|_2^2 \quad\le\quad \mathcal{A}\tau^2, \qquad (9.18)$$

$$\|\nabla \mathbf{e}^{l+1}\|_2^2 + \tau \sum_{j=0}^{l} \|\Delta \mathbf{e}^{j+1}\|_2^2 \quad \leq \quad \mathcal{A}\tau. \tag{9.19}$$

First, these statements are valid for $l = 0$. To see this, we employ (9.5) for $j = 0$.

Let us prove the statements for $l = r$ assuming that they are valid for $l = r-1$. Note that $\mathcal{A}$ is independent of l. At this stage of mathematical induction, the following lemma will be useful.

Lemma 9.3 *Assuming (9.18) and (9.19) hold true for* $l = r - 1$, *the following estimate holds for sufficiently small* τ*:*

$$\mathcal{B} := \sum_{j=0}^{r} \|\nabla \mathbf{e}^j\|_4^2 + \|\nabla \mathbf{e}^j\|_4^4 + \|\nabla \mathbf{e}^j\|_4^8 + \|\nabla \mathbf{e}^j\|_4^{16} \leq 1. \tag{9.20}$$

PROOF:
For $a = 2, 4, 8, 16$, it is valid that

$$(x + y)^a \leq a(x^a + y^a) \leq 16(x^a + y^a).$$

Thus we can use the inequality (10.12) in the form

$$\|\nabla \mathbf{u}\|_4^a \leq C\|\nabla \mathbf{u}\|_2^a + C\|\nabla \mathbf{u}\|_2^{\frac{a}{4}} \|\Delta \mathbf{u}\|_2^{\frac{3a}{4}}.$$

We apply the previous inequality for $a = 2, 4, 8, 16$ and sum it up for $j = 0, \ldots, r$ to get

$$\begin{aligned} \sum_{j=0}^{r} \|\nabla \mathbf{e}^j\|_4^a \quad &\leq \quad \sum_{j=0}^{r} \|\nabla \mathbf{e}^j\|_2^a + \sum_{j=0}^{r} \|\nabla \mathbf{e}^j\|_2^{\frac{a}{4}} \|\Delta \mathbf{e}^j\|_2^{\frac{3a}{4}} \\ &\leq \quad \sum_{j=0}^{r} \|\nabla \mathbf{e}^j\|_2^a + \left(\sum_{j=0}^{r} \|\nabla \mathbf{e}^j\|_2^a\right)^{\frac{1}{4}} \left(\sum_{j=0}^{r} \|\Delta \mathbf{e}^j\|_2^a\right)^{\frac{3}{4}}. \end{aligned}$$

At the end we have used the discrete Hölder inequality with exponents 4 and 4/3. Now we apply Lemma 10.2 on the following terms. Note that $2 \leq a$. We have

$$\begin{aligned} \left(\sum_{j=0}^{r} \|\nabla \mathbf{e}^j\|_2^a\right)^2 \quad &\leq \quad \left(\sum_{j=0}^{r} \|\nabla \mathbf{e}^j\|_2^2\right)^a, \\ \left(\sum_{j=0}^{r} \|\Delta \mathbf{e}^j\|_2^a\right)^2 \quad &\leq \quad \left(\sum_{j=0}^{r} \|\Delta \mathbf{e}^j\|_2^2\right)^a. \end{aligned}$$

This gives us

$$\sum_{j=0}^{r}\|\nabla\mathbf{e}^j\|_4^a \le \left(\sum_{j=0}^{r}\|\nabla\mathbf{e}^j\|_2^2\right)^{\frac{a}{2}} + \left(\sum_{j=0}^{r}\|\nabla\mathbf{e}^j\|_2^2\right)^{\frac{a}{8}}\left(\sum_{j=0}^{r}\|\Delta\mathbf{e}^j\|_2^2\right)^{\frac{3a}{8}}.$$

Notice that in the expression of $\mathcal{B}$ we have summed up to the index $j=r$ but the indices of the terms are shifted by one when compared with those that appeared in (9.18) and (9.19). Thus we are able to apply (9.18) and (9.19) for $l=r-1$. We combine it with the previous inequality to obtain

$$\mathcal{B} \le \sum_{a=2,4,8,16} (\mathcal{A}\tau)^{\frac{a}{2}} + (\mathcal{A}\tau)^{\frac{a}{8}}\mathcal{A}^{\frac{3a}{8}}.$$

Since τ appears in every term with exponent at least $1/4$, we can guarantee that $\mathcal{B}$ remains less than 1 by setting τ sufficiently small.

This completes the proof of Lemma 9.3. □

If we look at the STATEMENT 1, we are able to get rid of some problematic terms by applying the previous lemma. Thus

$$\begin{aligned}&\|\mathbf{e}^{l+1}\|^2 + \tau\sum_{j=0}^{l}\|\nabla\mathbf{e}^{j+1}\|_2^2\\ &\le C\tau^2 + C\tau\sum_{j=0}^{l}\|\nabla\mathbf{e}^j\|_4^2\sum_{j=0}^{l}\|\nabla\mathbf{e}^j\|_2^2 + 2C\tau\sum_{j=0}^{l}\|\mathbf{e}^{j+1}\|_2^2.\end{aligned}$$

In Lemma 9.3 we can easily gain also the estimate $\mathcal{B}\le\mathcal{A}^{-1}$. Then, if we apply the relations (9.18) and (9.19) with $l=r-1$ to the term $\sum_{j=0}^{l}\|\nabla\mathbf{e}^j\|_2^2$ and we apply the estimate $\mathcal{B}\le\mathcal{A}^{-1}$ to the term $\sum_{j=0}^{l}\|\nabla\mathbf{e}^j\|_4^2$, we gain

$$\begin{aligned}\|\mathbf{e}^{l+1}\|^2 + \tau\sum_{j=0}^{l}\|\nabla\mathbf{e}^{j+1}\|_2^2 &\le C\tau^2 + C\tau\mathcal{A}^{-1}\mathcal{A}\tau + 2C\tau\sum_{j=0}^{l}\|\mathbf{e}^{j+1}\|_2^2\\ &\le 2C\tau^2 + 2C\tau\sum_{j=0}^{l}\|\mathbf{e}^{j+1}\|_2^2.\end{aligned}$$

Notice that the previous inequality was obtained independently of the index l. Now we apply Gronwall's argument to verify (9.18) for $l=r$.

If we look at the STATEMENT 2, we can eliminate some terms by applying Lemma 9.3. Then we get

$$\|\nabla\mathbf{e}^{l+1}\|_2^2 + \tau\sum_{j=1}^{l}\|\Delta\mathbf{e}^{j+1}\|_2^2 \le C\tau + C_\epsilon\tau\sum_{j=1}^{l}\|\mathbf{e}^j\|_4^2$$

$$+2C_\epsilon\tau\sum_{j=1}^{l}\|\mathbf{e}^{j+1}\|_2^2+2C_\epsilon\tau\sum_{j=1}^{l}\|\nabla\mathbf{e}^{j+1}\|_2^2.$$

For the term $\sum_{j=1}^{l}\|\mathbf{e}^j\|_4^2$, we use the embedding $W^{1,2}\hookrightarrow L^4$ and (9.18), (9.19) for $l=r-1$. For the term $\sum_{j=1}^{l}\|\mathbf{e}^{j+1}\|_2^2$, we use (9.18) for $l=r$, which has already been proved. Thus

$$\begin{aligned}\|\nabla\mathbf{e}^{l+1}\|_2^2+\tau\sum_{j=1}^{l}\|\Delta\mathbf{e}^{j+1}\|_2^2 &\le C\tau+2C_\epsilon\tau\mathcal{A}\tau+2C_\epsilon\tau\sum_{j=1}^{l}\|\nabla\mathbf{e}^{j+1}\|_2^2\\ &\le 2C\tau+2C_\epsilon\tau\sum_{j=1}^{l}\|\nabla\mathbf{e}^{j+1}\|_2^2.\end{aligned}$$

Here we have the sum over the index j starting from 1 and ending with $j=r$. To make the sum start with $j=0$, we should know that $\|\Delta\mathbf{e}^1\|_2^2\le C$. This is given by the following lemma.

Lemma 9.4 *Keeping the notations from the above discussion we have*

$$\|\Delta\mathbf{e}^1\|_2^2\le C.$$

Proof:
We take (9.8) for $j=0$ and set $\boldsymbol{\phi}=-\Delta\mathbf{e}^1$. After considering $\mathbf{e}^0=\mathbf{0}$ and $\nabla\mathbf{e}^0=\mathbf{0}$ we get

$$\begin{aligned}&\left(\frac{\mathbf{e}^1}{\tau},-\Delta\mathbf{e}^1\right)+\alpha(\Delta\mathbf{e}^1,\Delta\mathbf{e}^1)\le|(\mathcal{F}^1(\mathbf{m}),\Delta\mathbf{e}^1)|\\ &\quad+|(\langle\nabla\mathbf{m}(t_1)-\nabla\mathbf{m}(t_0),\nabla\mathbf{m}(t_1)+\nabla\mathbf{m}(t_0)\rangle_{\mathbb{R}^9}\mathbf{m}(t_1),\Delta\mathbf{e}^1)|\\ &\quad+|(|\nabla\mathbf{m}(t_0)|^2\mathbf{e}^1,\Delta\mathbf{e}^1)|+|([\mathbf{m}(t_1)-\mathbf{m}(t_0)]\times\Delta\mathbf{m}(t_1),\Delta\mathbf{e}^1)|.\end{aligned}$$

Now we apply some inequalities from Lemma 10.3, the estimates (7.7) and (7.8) and the Young inequality:

$$\begin{aligned}&\frac{1}{\tau}\|\nabla\mathbf{e}^1\|_2^2+\alpha\|\Delta\mathbf{e}^1\|_2^2\le C\|\mathcal{F}^1(\mathbf{m})\|_2^2+\epsilon\|\Delta\mathbf{e}^1\|_2^2 \qquad (9.21)\\ &\quad+\|\nabla\mathbf{m}(t_1)-\nabla\mathbf{m}(t_0)\|_4\|\nabla\mathbf{m}(t_1)+\nabla\mathbf{m}(t_0)\|_4\|\mathbf{m}(t_1)\|_{L^\infty}\|\Delta\mathbf{e}^1\|_2\\ &\quad+\|\nabla\mathbf{m}(t_0)\|_4^2\|\mathbf{e}^1\|_{L^\infty}\|\Delta\mathbf{e}^1\|_2\\ &\quad+\|\mathbf{m}(t_1)-\mathbf{m}(t_0)\|_{L^\infty}\|\Delta\mathbf{m}(t_1)\|_2\|\Delta\mathbf{e}^1\|_2\\ &\le C\|\mathcal{F}^1(\mathbf{m})\|_2^2+\epsilon\|\Delta\mathbf{e}^1\|_2^2+\|\mathbf{e}^1\|_{L^\infty}^2.\end{aligned}$$

To estimate the term $\|\mathbf{e}^1\|_{L^\infty}^2$ we use the same techniques as we have already done several times using (10.14) and (10.12) and the Young inequality with exponents 4 and 4/3:

$$\begin{aligned}\|\mathbf{e}^1\|_{L^\infty} &\leq \|\mathbf{e}^1\|_4^2 + \|\nabla\mathbf{e}^1\|_4^2 \leq C_\epsilon\|\mathbf{e}^1\|_2^2 + C_\epsilon\|\nabla\mathbf{e}^1\|_2^2 + \|\nabla\mathbf{e}^1\|_2^{\frac{1}{2}}\|\Delta\mathbf{e}^1\|_2^{\frac{3}{2}} \\ &\leq C_\epsilon\|\mathbf{e}^1\|_2^2 + C_\epsilon\|\nabla\mathbf{e}^1\|_2^2 + \epsilon\|\Delta\mathbf{e}^1\|_2^2.\end{aligned}$$

We suppose that $\alpha > \epsilon$ and $1 - C_\epsilon\tau > 0$. Then, together with (9.21), since $\|\mathbf{e}^1\|_2^2 \leq C$ we have

$$\frac{1-\tau C_\epsilon}{\tau}\|\nabla\mathbf{e}^1\|_2^2 + (\alpha-\epsilon)\|\Delta\mathbf{e}^1\|_2^2 \quad\leq\quad C\|\mathcal{F}^1(\mathbf{m})\|_2^2 \leq C,$$

which completes the proof of Lemma 9.4. □

Thus we have arrived at

$$\begin{aligned}\|\nabla\mathbf{e}^{l+1}\|_2^2 + \tau\sum_{j=0}^{l}\|\Delta\mathbf{e}^{j+1}\|_2^2 &\leq C\tau + 2C\tau\mathcal{A}\tau + 2C\tau\sum_{j=0}^{l}\|\nabla\mathbf{e}^{j+1}\|_2^2 \\ &\leq 2C\tau + 2C\tau\sum_{j=0}^{l}\|\nabla\mathbf{e}^{j+1}\|_2^2.\end{aligned}$$

Notice that this inequality was also obtained independently of the index l. We now apply Gronwall's argument to verify (9.19) for $l = r$.

This completes the proof of Theorem 9.1. □

9.3 Proof of the modulus theorem

The result follows directly from inequality (10.12) and Theorem 9.1:

$$\begin{aligned}\||\mathbf{m}(t_j)|^2 - |\mathbf{m}^j|^2\|_2 &= \|\langle\mathbf{e}^j, \mathbf{e}^j + 2\mathbf{m}(t_j)\rangle_{\mathbb{R}^3}\|_2 \\ &\leq 2(\|\mathbf{e}^j\|_4^2 + \|\mathbf{e}^j\|_2\|\mathbf{m}(t_j)\|_{L^\infty}) \\ &\leq C(\|\mathbf{e}^j\|_2^2 + \|\mathbf{e}^j\|_2^{\frac{1}{2}}\|\nabla\mathbf{e}^j\|_2^{\frac{3}{2}} + \|\mathbf{e}^j\|_2) \\ &\leq C(\tau^2 + \tau^{\frac{1}{2}}\tau^{\frac{3}{4}} + \tau) \leq C\tau.\end{aligned}$$

□

9.4 Numerical tests

In order to verify the theoretical results, we solve the problem (7.2)–(7.4) with an artificial right-hand side and with a prescribed solution. Up to now there is no example of such a problem with a known analytical solution in the literature. We set

$$\mathbf{m}^{\mathrm{ex}}(t,x) = \begin{cases} \mathrm{m}_0^{\mathrm{ex}} &= \left(1 - \left(\frac{x_0 t}{2}\right)^2 - \left(\frac{(x_1+x_2)t}{4}\right)\right)^{\frac{1}{2}}, \\ \mathrm{m}_1^{\mathrm{ex}} &= \frac{(x_1+x_2)t}{4}, \\ \mathrm{m}_2^{\mathrm{ex}} &= \frac{x_0 t}{2}, \end{cases}$$

in order to ensure that the modulus of $\mathbf{m}^{\mathrm{ex}}$ remains constant equal to 1. Then we solve the following problem on a cube $\Omega = (0,1)^3$

$$\begin{aligned} \partial_t \mathbf{m} = \mathbf{m} \times \Delta \mathbf{m} - \alpha \mathbf{m} \times (\mathbf{m} \times \Delta \mathbf{m}) + \mathbf{f} \qquad & \text{in } \mathbb{R}^+ \times \Omega, \\ \nabla \mathbf{m} \cdot \boldsymbol{\nu} = \nabla \mathbf{f} \cdot \boldsymbol{\nu} \qquad & \text{on } \mathbb{R}^+ \times \partial\Omega, \\ \mathbf{m}(0,.) = \mathbf{m}^{\mathrm{ex}}(0,.) \qquad & \text{in } \Omega, \end{aligned}$$

where $\mathbf{f} = \partial_t \mathbf{m}^{\mathrm{ex}} - \mathbf{m}^{\mathrm{ex}} \times \Delta \mathbf{m}^{\mathrm{ex}} - \alpha \mathbf{m}^{\mathrm{ex}} \times (\mathbf{m}^{\mathrm{ex}} \times \Delta \mathbf{m}^{\mathrm{ex}})$. Note that the sign of the term $\mathbf{m} \times \Delta \mathbf{m}$ is negative, however, it has no impact on the results. The sign defines the direction of the rotation of $\mathbf{m}$ around $\Delta \mathbf{m}$. If the sign changes, the movement of $\mathbf{m}$ is in fact only mirrored. For the spatial discretization, we use the standard $W^{1,2}(\Omega)$-conforming finite element formulation. We establish a finite dimensional approximation space V_h consisting of piece-wise linear functions. The symbol h denotes the size of the space discretization step. The projection operator P_h associated with the space V_h has the necessary properties needed for further analysis of the full discretization in time and space.

τ^{-1}	$\|\mathbf{e}^j\|_2$	$\|\nabla \mathbf{e}^j\|_2$	$\|\mathbf{e}^j\|_{L^\infty}$	$\|1 - \|\mathbf{m}^j\|^2\|_2$
10	0.001669	0.013666	0.003988	0.003270
20	0.000863	0.009331	0.002053	0.001793
40	0.000444	0.007153	0.001100	0.000980
80	0.000221	0.006031	0.000603	0.000485
160	0.000139	0.005431	0.000403	0.000318

Table 9.1: Errors of the approximation

In order to confirm the theoretical results, we test the rate of convergence of the numerical scheme (9.2). We divide one edge of the cubic domain into 24 elements which gives us 82 944 tetrahedra and 15 625 vertices. The number of vertices determines also the number of degrees of freedom.

Due to the linearity of scheme (9.2), we have to solve only a linear elliptic problem on each time level. Compared with (9.1), our scheme is computationally

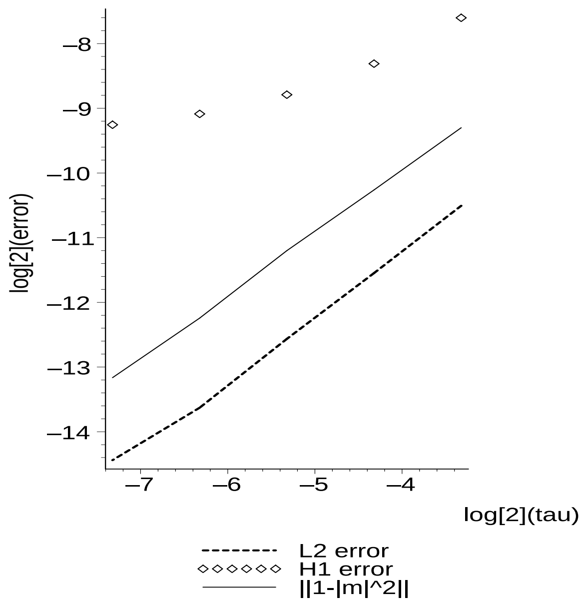


Figure 9.1: The graphs plotted with the logarithmic scale

cheaper. To compute the solution on a new time level in (9.1), it is necessary to solve a nonlinear elliptic problem whose solution involves the use of a nonlinear solver.

Table 9.1 shows the results for a time step running from 10^{-1} to 160^{-1}. The graphs in Figure 9.1 are plotted with a logarithmic scale so that we can easily see that the numerical results confirm the theoretical ones.

10 Penalization strategies

(An expert is someone who knows some of the worst mistakes that can be made in his subject, and how to avoid them. Werner Heisenberg)

10.1 Single LL equation

In Chapter 9 we have proposed a numerical scheme derived from 2D version introduced in [61]. The author analyses also another schemes.

We have already mentioned that the scheme (9.2) does not preserve the length of the magnetization. But Theorem 9.2 says that in some sense the modulus of approximating solutions $\mathbf{m}^j$ goes to 1 when time step τ goes to 0. There is also another approach to keep the modulus of the magnetization under control. This approach uses penalization strategies.

The idea of a penalization term artificially introduced to the equation which is solved is not new. Let us take a function $\Phi(\mathbf{m}^i, \mathbf{m}^{i+1})$ in such a way that its value is small when $|\mathbf{m}^i|$ is very close to $|\mathbf{m}^{i+1}|$. We will not write down particular expression for Φ yet. Then add a term $\Phi(\mathbf{m}^i, \mathbf{m}^{i+1})$ with the weight $1/\varepsilon$ to the LL equation

$$\delta\mathbf{m}^{j+1} - \alpha\Delta\mathbf{m}^{j+1} + \frac{1}{\varepsilon}\Phi(\mathbf{m}^i, \mathbf{m}^{i+1}) = \alpha|\nabla\mathbf{m}^j|^2\mathbf{m}^{j+1} + \mathbf{m}^{j+1} \times \Delta\mathbf{m}^{j+1}. \quad (10.1)$$

We see that if we go with ε to 0 then the weight of function Φ becomes larger. Then reciprocally the value of $\Phi(\mathbf{m}^i, \mathbf{m}^{i+1})$ tends to be small and therefore the modulus of $\mathbf{m}^i$ is close to the modulus of $\mathbf{m}^{i+1}$.

We introduce three possibilities for the function $\Phi = \Phi_i$ for $i = 1, 2, 3$

$$\Phi_1(\mathbf{m}^i, \mathbf{m}^{i+1}) = |\mathbf{m}^{i+1}|^2 - 1,$$

$$\Phi_2(\mathbf{m}^i, \mathbf{m}^{i+1}) = 1 - \frac{1}{|\mathbf{m}^i|^2},$$
$$\Phi_3(\mathbf{m}^i, \mathbf{m}^{i+1}) = 1 - \frac{1}{|\mathbf{m}^i|}.$$

Prohl in [61] has proved the following convergence results for the numerical schemes (10.1) for all three choices of Φ

$$\max_{0\le j\le J} \|\mathbf{m}(t_j) - \mathbf{m}^j\|_2 + \left(\tau \sum_{j=0}^{J} \|\nabla\{\mathbf{m}(t_j) - \mathbf{m}^j\}\|_2^2\right)^{\frac{1}{2}}$$
$$+ \frac{1}{\sqrt{\varepsilon}} \left(\tau \sum_{j=0}^{J} \left\{\|\mathbf{m}(t_j) - \mathbf{m}^j\|_4^4 + \|\langle\mathbf{m}(t_j) - \mathbf{m}^j\rangle\|_2^2\right\}\right)^{\frac{1}{2}} \le C\tau.$$

The differences between different choices of Φ were only in the dependence of ε on τ. For Φ_1 it is necessary that $\varepsilon^{-1} = o(\tau^{-1})$, for Φ_2 was the estimate sharpened to $\varepsilon > 1.9\tau$ and finally for Φ_3 it is $\varepsilon \ge \tau$.

The reason why we have introduced the penalization term was to get better control over the modulus of the magnetization. The estimates

$$\max_{0\le j\le J} \|1 - |\mathbf{m}^j|^2\|_2 \le \sqrt{\tau\epsilon}$$

are also proved and demonstrate the usefulness of the term $\Phi(\mathbf{m}^i, \mathbf{m}^{i+1})$ in (10.1).

What we have mentioned in this chapter was done in 2D. The use of techniques similar to those used in Chapter 9 opens the way to make the same analysis of the penalization techniques in 3D. We will not do this here since the proofs would be similar to the proofs in Chapter 9.

10.2 Full M-LL system

For more general case we would like to study numerical schemes dealing with the full Maxwell-Landau-Lifshitz system. In [61] the author has suggested a couple of such schemes and has derived also error estimates for these schemes. However, he considered all schemes only in two dimensional case. The key estimate enabling the use of these schemes was the upper bound of $\mathbf{m}$ in the space $W^{2,2}(\Omega)$. Prohl has this estimate only in 2D. In Chapter 8, Theorem 8.1, we have proved missing estimate also in 3D. This result suggests the extension of Prohl's results also in the case of three dimensions. However, the detailed analysis of these schemes exceeds the range of this work and we omit the proofs.

For the full M-LL system we provide one example of the scheme together with error estimates. Let us consider the implicit penalized Euler scheme

$$\begin{aligned}
&\delta\mathbf{m}^{j+1} - \alpha\Delta\mathbf{m}^{j+1} - \mathbf{l}_\epsilon(\mathbf{m}^{j+1})\mathbf{m}^{j+1} = \alpha|\nabla\mathbf{m}^j|^2\mathbf{m}^{j+1} \\
&\quad +\mathbf{m}^j \times (\Delta\mathbf{m}^{j+1} + \mathbf{H}^{j+1}) + \alpha(\mathbf{H}^{j+1} - \langle\mathbf{m}^j, \mathbf{H}^j\rangle\mathbf{m}^{j+1}), \\
&\nabla \times \mathbf{H}^{j+1} = \delta\mathbf{E}^{j+1} + \sigma\mathbf{E}^{j+1}, \\
&\nabla \times \mathbf{E}^{j+1} = -\delta\mathbf{H}^{j+1} - \beta(\delta\mathbf{m}^{j+1} - \mathbf{l}_\epsilon(\mathbf{m}^{j+1})\mathbf{m}^{j+1}), \\
&\nabla \cdot \mathbf{E}^{j+1} = 0,
\end{aligned}$$

where $\mathbf{l}_\epsilon(\phi) = \epsilon^{-1}(|\phi|^2 - 1)$.

Considering $\epsilon^{-1} = o(\tau^{-1})$, the error estimates for this scheme are

$$\begin{aligned}
&\max_{0\le j\le J}\big\{\|\mathbf{m}(t_j) - \mathbf{m}^j\|_2 + \|\mathbf{H}(t_j) - \mathbf{H}^j\|_{W^{-1,2}} + \|\mathbf{E}(t_j) - \mathbf{E}^j\|_{W^{-1,2}}\big\} \\
&\quad +\Big(\tau\sum_{j=0}^{J}\big\{\|\mathbf{m}(t_j) - \mathbf{m}^j\|^2_{\mathbf{W}^{1,2}} + \frac{\beta}{\epsilon}\big[\|\langle\mathbf{m}(t_j) - \mathbf{m}^j, \mathbf{m}(t_j)\rangle\|_2^2 \\
&\quad +\|\mathbf{m}(t_j) - \mathbf{m}^j\|_4^4\big]\big\}\Big)^{1/2} \le C\tau \\
&\max_{0\le j\le J}\Big\{\|\mathbf{H}(t_j) - \mathbf{H}^j\|_2 + \|\mathbf{E}(t_j) - \mathbf{E}^j\|_2\Big\} \le C\sqrt{\tau}, \\
&\Big(\tau\sum_{j=1}^{J}\|1 - |\mathbf{m}^j|^2\|_2^2\Big)^{1/2} \le C\sqrt{\epsilon\tau^2}, \\
&\max_{0\le j\le J}\Big\{\|\nabla\cdot(\mathbf{H}^j + \beta\mathbf{m}^j)\|_{W^{-1,2}} + \sqrt{\frac{\tau}{\epsilon}}\|\nabla\cdot(\mathbf{H}^j + \beta\mathbf{m}^j)\|_2\Big\} \le C\sqrt{\frac{\tau^2}{\epsilon}},
\end{aligned}$$

where $\mathbf{m}, \mathbf{E}, \mathbf{H}$ are exact solutions of the full M-LL system on the interval $(0, T_0)$. The constant T_0 comes from Lemma 8.7.

We can see that original constraint $\mathbf{H} + \beta\mathbf{m}$ from Maxwell's equations is not satisfied for its discrete version. But still, the last estimate gives control over the term $\mathbf{H}^j + \beta\mathbf{m}^j$.

Appendix

Vector identities

We recall the following vector identities

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{c}\cdot\mathbf{a})\ \mathbf{b}-(\mathbf{c}\cdot\mathbf{b})\ \mathbf{a}, \tag{10.2}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})\ (\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})\ (\mathbf{b}\cdot\mathbf{c}), \tag{10.3}$$

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}), \tag{10.4}$$

which can be verified by simple calculations.

Simple mathematical analysis

Lemma 10.1 *For* $a, b \geq 0$ *and* $1 \leq s \leq t$ *we have*

$$(a^t+b^t)^s \leq (a^s+b^s)^t.$$

Proof:
We follow the idea of Slodička. For any real x such that $0 \leq x \leq 1$, it is clear that $x^t \leq x^s$. Then we have

$$(1+x^t)^s \leq (1+x^s)^s.$$

Since $1+x^s \geq 1$, we get $(1+x^s)^s \leq (1+x^s)^t$. Therefore, we arrive at

$$(1+x^t)^s \leq (1+x^s)^t. \tag{10.5}$$

Suppose that $b \leq a$. By substituting $x = b/a$ into (10.5) we obtain

$$\left[1+\left(\frac{b}{a}\right)^t\right]^s \leq \left[1+\left(\frac{b}{a}\right)^s\right]^t,$$

which gives the desired result when multiplied by a^{ts}. □

Lemma 10.2 *For* $a_i > 0,\ i = 2, \ldots, n \in \mathbb{N}$ *and* $1 \le s \le t$ *it holds that*

$$\left(\sum_{i=1}^{n} a_i^t\right)^s \le \left(\sum_{i=1}^{n} a_i^s\right)^t.$$

Proof:
We prove this lemma by mathematical induction. The lemma is valid for $n = 2$, as it is the case of Lemma 10.1. Suppose this lemma is valid for $n = j$. We would like to prove that

$$\left(\sum_{i=1}^{j+1} a_i^t\right)^s \le \left(\sum_{i=1}^{j+1} a_i^s\right)^t.$$

Let $A := (\sum_{i=1}^{j} a_i^t)^{\frac{1}{t}}$ and $B := \sum_{i=1}^{j} a_i^s$. Then using Lemma 10.1 with $a = A$ and $b = a_{j+1}$ we compute

$$\left(\sum_{i=1}^{j+1} a_i^t\right)^s = \left(A^t + a_{j+1}^t\right)^s \le \left(A^s + a_{j+1}^s\right)^t. \tag{10.6}$$

Using the inductive hypothesis we get

$$A^s = \left(\sum_{i=1}^{j} a_i^t\right)^{\frac{s}{t}} \le \left(\sum_{i=1}^{j} a_i^s\right) = B.$$

Since the function $f(x) = x^t$ is monotonically increasing, we can put B instead of A^s in the last expression of (10.6) keeping the desired inequality, thus

$$\left(\sum_{i=1}^{j+1} a_i^t\right)^s \le \left(A^s + a_{j+1}^s\right)^t \le \left(B + a_{j+1}^s\right)^t \le \left(\sum_{i=1}^{j+1} a_i^s\right)^t,$$

which completes the proof of Lemma 10.2. □

Theorem 10.1 (Gronwall's lemma) *Let $r(t), h(t), y(t)$ are continuous real functions defined on the interval $[a, b]$ such that $r(t), h(t) \ge 0$. Suppose that*

$$y(t) \le h(t) + \int_a^t r(s)y(s)\mathrm{d}s \quad \text{for } a \le t \le b.$$

Then

$$y(t) \le h(t) + \int_a^t h(s)r(s)e^{\int_s^t r(\tau)\mathrm{d}\tau}\mathrm{d}s$$

is valid for all $t \in [a, b]$.

Theorem 10.2 (Gronwall's lemma - discrete version) *Let $\{A_i\}, \{a_i\}$ be the sequences of nonnegative real numbers and let $q \geq 0$. Suppose*

$$a_i \leq A_i + \sum_{j=1}^{i-1} a_j q$$

holds for $i \in \mathbb{N}$. Then

$$a_i \leq A_i + e^{qi} \sum_{j=1}^{i-1} A_j q.$$

Function spaces

Lemma 10.3 *Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{m}, \mathbf{n}$ be vector valued functions from L^2 (or L^4, L^∞ if necessary). Then*

$$\begin{aligned} (\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{R}^3} \mathbf{m}, \mathbf{n})_\Omega &= \int_\Omega \langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{R}^3} \langle \mathbf{m}, \mathbf{n} \rangle_{\mathbb{R}^3} \\ &\leq t \|\mathbf{u}\|_{L^4} \|\mathbf{v}\|_{L^4} \|\mathbf{m}\|_{L^\infty} \|\mathbf{n}\|_{L^2}, \qquad (10.7) \\ (\mathbf{u} \times \mathbf{v}, \mathbf{m})_\Omega &= \int_\Omega \langle \mathbf{u} \times \mathbf{v}, \mathbf{m} \rangle_{\mathbb{R}^3} \leq \|\mathbf{u}\|_{L^4} \|\mathbf{v}\|_{L^4} \|\mathbf{m}\|_{L^2}, \qquad (10.8) \\ (\mathbf{u} \times \mathbf{v}, \mathbf{m})_\Omega &= \int_\Omega \langle \mathbf{u} \times \mathbf{v}, \mathbf{m} \rangle_{\mathbb{R}^3} \leq \|\mathbf{u}\|_{L^\infty} \|\mathbf{v}\|_{L^2} \|\mathbf{m}\|_{L^2}. \qquad (10.9) \end{aligned}$$

Proof of this lemma is a straightforward computation using the integral Hölder inequality.

Theorem 10.3 (Extended Sobolev inequalities) *([33],Theorem 10.1) Let Ω be a bounded domain with $\partial\Omega$ in C^m, and let u be any function in $W^{m,r}(\Omega) \cap L^q(\Omega)$, $1 \leq r,\ q \leq \infty$. For any integer j, $0 \leq j < m$, and for any number a in the interval $j/m \leq a \leq 1$, set*

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q}.$$

If $m - j - n/r$ is not a nonnegative integer, then

$$||D^j u||_{L^p} \leq C \left(||u||_{W^{m,r}}\right)^a \left(||u||_{L^q}\right)^{1-a}. \qquad (10.10)$$

If $m - j - n/r$ is a nonnegative integer, then (10.10) holds for $a = j/m$. The constant C depends only on Ω, r, q, m, j, a.

Special cases of the previous theorem are summed up in the following theorem.

Lemma 10.4 (Special cases of Sobolev inequalities) *Let Ω be a bounded domain with $\partial\Omega$ in C^2, and let u be any function in $W^{2,2}(\Omega) \cap L^2(\Omega)$. Consider the case of spatial dimension $n = 3$. Then the following hold*

$$\|u\|_{L^4} \leq C\|u\|_{W^{1,2}}^{3/4}\|u\|_{L^2}^{1/4} \leq C\|u\|_{L^2} + C\|\nabla u\|_{L^2}^{3/4}\|u\|_{L^2}^{1/4}, \qquad (10.11)$$

$$\|\nabla u\|_{L^4} \leq C\|\nabla u\|_{W^{1,2}}^{3/4}\|\nabla u\|_{L^2}^{1/4} \leq C\|\nabla u\|_{L^2} + C\|\Delta u\|_{L^2}^{3/4}\|\nabla u\|_{L^2}^{1/4}, \qquad (10.12)$$

$$\|\Delta u\|_{L^4} \leq C\|\Delta u\|_{W^{1,2}}^{3/4}\|\Delta u\|_{L^2}^{1/4} \leq C\|\Delta u\|_{L^2} + C\|\nabla\Delta u\|_{L^2}^{3/4}\|\Delta u\|_{L^2}^{1/4}, \qquad (10.13)$$

$$\|u\|_{L^\infty} \leq C\|u\|_{W^{1,4}} \leq C\|u\|_{L^4} + C\|\nabla u\|_{L^4}, \qquad (10.14)$$

$$\|\nabla u\|_{L^\infty} \leq C\|\nabla u\|_{W^{1,4}} \leq C\|u\|_{W^{2,2}}^{\frac{1}{4}}\|u\|_{W^{3,2}}^{\frac{3}{4}}, \qquad (10.15)$$

$$\|\nabla u\|_{W^{1,4}} \leq C\Big[\|\nabla u\|_2 + \|\Delta u\|_2 + (\|\nabla u\|_2^{\frac{1}{4}} + \|\Delta u\|_2^{\frac{1}{4}})\|\nabla\Delta u\|_2^{\frac{3}{4}}\Big]. \qquad (10.16)$$

In the previous lemma it is possible to replace scalar functions u with vector functions $\mathbf{u}$. So that's why we can use all these inequalities for vector fields $\mathbf{m}, \mathbf{H}$ and $\mathbf{E}$.

Remark 10.1 *Equivalence of the norms $\|u\|_{W^{2,2}}$ and $\|\Delta u\|_2 + \|u\|_2$.*

To derive (10.12) we simply put ∇u instead of u in (10.11). With this approach, however, we would gain inequality

$$\|\nabla u\|_{L^4} \leq C\|u\|_{W^{2,2}}^{3/4}\|\nabla u\|_{L^2}^{1/4}$$

instead of (10.12). To verify (10.12) we have to show the equivalence of the norms $\|u\|_{W^{2,2}}$ and $\|\Delta u\|_2 + \|u\|_2$. In general it is not so, but in the case of zero Neumann boundary conditions we can use Theorem 2.50 from [32]. Similar remark can be applied for derivation of (10.16).

General functional analysis

Theorem 10.4 (Lax-Milgram lemma) *Let a be a bounded coercive bilinear functional on a Hilbert space H. Then for every bounded linear functional f on H, there exists a unique u such that*

$$a(u, v) = f(v)$$

for all $v \in H$.

Theorem 10.5 *(From [86, p.120]) Let X be a Banach space and let $\{x_n\}$ be weakly convergent to x_∞. Then $\{\|x_n\|\}$ is bounded and*

$$\|x_\infty\| \leq \liminf_{n\to\infty} \|x_n\|.$$

Theorem 10.6 *(From [86, p.126]) Let X be a reflexive Banach space and let $\{x_n\}$ be any sequence, which is in norm bounded. Then we choose a subsequence $\{x_{n_i}\}$, which converges weakly to an element of X.*

Theorem 10.7 *(From [47, Theorem 2.11.5]) Let $\mu(\Omega) < \infty$. Let $1 \leq p_1 \leq p_2 \leq \ldots$ and suppose $\lim_{k\to\infty} p_k = \infty$. Let*

$$f \in \bigcap_{k=1}^{\infty} L^{p_k}(\Omega)$$

and suppose $a = \sup_{k\in\mathbb{N}} \|f\|_{p_k} < \infty$. Then $f \in L^\infty(\Omega)$.

By $\mathcal{D}'(I, V)$ we denote the space of distributions on I with values in V.

Theorem 10.8 *(From [44])*

- **(i)** *Let $t_0 \in I$ and let $u : I \to V$ be integrable. Then $v(t) = \int_{t_0}^t u(s)\mathrm{d}s\ (t, t_0)$ is in $C(I, V) \subset \mathcal{D}'(I, V)$ and $\frac{\partial v}{\partial t} = u$ in the sense of $\mathcal{D}'(I, V)$.*
- **(ii)** *If $u \in \mathcal{D}'(I, V)$ and $\frac{\partial v}{\partial t} = 0$, then $u : I \to V$ is constant. i.e., $u \equiv u(t) : I \to V$ and $u(t) = x \in V$ for a.e. $t \in I$:*
- **(iii)** *If $u_n \to u$ in $\mathcal{D}'(I, V)$, then $\frac{\partial u_n}{\partial t} \to \frac{\partial u}{\partial t}$ in $\mathcal{D}'(I, V)$. In particular, if $u_n \rightharpoonup u$ in $L_p(I, V)$ and $\frac{\partial u_n}{\partial t} \rightharpoonup x$ in $L_p(I, V)$ then $\frac{\partial u_n}{\partial t} = x \in L_p(I, V)$.*

Theorem 10.9 *(From [51]) Let $\{u_n\}_{n=1}^\infty$ is a sequence of functions belonging to $L_{p_0}(I, B)$ where B is a Banach space and $1 < p_i < \infty$ for $i = 0, 2$. Let X and Y are Banach spaces such that $X \hookrightarrow\hookrightarrow B \hookrightarrow Y$. If u_n are uniformly bounded in $L_{p_0}(I, X)$ and $\frac{\partial u_n}{\partial t}$ are uniformly bounded in $L_{p_1}(I, Y)$ then the sequence $\{u_n\}_{n=1}^\infty$ is relatively compact in $L_{p_2}(I, B)$.*

More results on compactness in spaces $L^p(I, B)$ can be found in [69].

List of Figures

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Various symbols

f, u	scalar functions, denoted with normal font
$\mathbf{f}, \mathbf{u}, \mathbf{m}$	vector functions, denoted with bold font
$\boldsymbol{N}, \boldsymbol{A}$	matrix denoted with italic bold font
$(.,.)$	standard scalar product in the space $L^2(\Omega)$
$\lvert . \rvert$	the modulus
$\lVert \cdot \rVert_X$	the norm in the space X
$\lVert \cdot \rVert_p$	the norm in the space $L^p(\Omega)$
$\lVert \cdot \rVert$	the norm in the space $L^2(\Omega)$
$\langle .,. \rangle_m$	scalar product in the space $\mathbb{R}^m$
$\langle .,. \rangle$	scalar product in the space $\mathbb{R}^3$
$\partial_t^p f$	p-th time derivative of time-dependent function f
$\partial_{x_i} f$	partial derivative of the function f
C	generic real number, which may change if necessary
ε	generic real small number, which may change if necessary
$\delta \mathbf{f}^i$	$= (\mathbf{f}(t_i) - \mathbf{f}(t_{i-1}))/tau$, approximation of time derivative of the function $\mathbf{f}(t)$.
T_i	a tetrahedron from the mesh
n_i	a vertex from the mesh
e_i	an edge from the mesh
f_i	a face from the mesh

$\mathcal{N}_h$	a set of all vertexes in the mesh
$\mathcal{E}_h$	a set of all edges in the mesh
$\mathcal{F}_h$	a set of all faces in the mesh
$\mathcal{T}_h$	a set of all tetrahedra in the mesh
T_x	a tangential plane in the point x
δ_{ij}	Kronecker delta function
w_n	nodal basis Whitney functions
w_e	edge basis Whitney functions
w_T	tetrahedral basis Whitney functions
w_f	facial basis Whitney functions
$\rightarrow$	strong convergence
$\rightharpoonup$	weak convergence
$\hookrightarrow$	continuous embedding
$\hookrightarrow\hookrightarrow$	compact embedding
γ	gyromagnetic factor
α	damping constant
σ	conductivity of medium
ρ	electric charge density
ϵ	permitivity of medium
μ	permeability of medium
K	anisotropy constant
$E_{\text{exc}}, E_{\text{app}}, E_{\text{ani}}, E_{\text{dem}}$	energy contributions to the total energy E_{tot} of a micromagnetic system
Ω	domain, representing working piece
Γ_N	part of the boundary $\partial\Omega$
Γ_D	part of the boundary $\partial\Omega$

Vectors

$\mathbf{M}$	vector of the magnetization
$\mathbf{m}$	normalized vector of the magnetization
$\mathbf{H}$	vector of the magnetic field
$\mathbf{E}$	vector of the electric field
$\mathbf{B}$	vector of the magnetic induction
$\mathbf{D}$	vector of the electric flux density
$\mathbf{J}$	vector of the electric current density
$\mathbf{H}_{\text{exc}}, \mathbf{H}_{\text{app}}, \mathbf{H}_{\text{ani}}, \mathbf{H}_{\text{dem}}$	field contributions to the total field $\mathbf{H}_{\text{eff}}$ of a micromagnetic system
$\mathbf{e}_x$	vector $(1, 0, 0)^T$, mostly describing an x-axis

$\mathbf{e}_y$	vector $(0,1,0)^T$, mostly describing an y-axis
$\mathbf{e}_z$	vector $(0,0,1)^T$, mostly describing an z-axis
$\mathbf{01}, \mathbf{32}$	vectors beginning in the vertex 0 ending in the vertex 1, resp. beginning in the vertex 3 and ending in the vertex 2

Functions and operators

Φ_i	penalization terms for $i = 1, 2, 3$
$g_R(s)$	cut-off function
$P(\mathbf{m})$	projection of the vector $\mathbf{m}$ to the direction given by the vector $\mathbf{p}$
$P_n(\mathbf{m})$	orthogonal projection of $\mathbf{L}^2(\Omega)$ on $\mathbf{H}^1_{\nabla 0}(\Omega)$
$Q_n(\mathbf{m})$	orthogonal projection of $\mathbf{L}^2(\Omega)$ on $\mathbf{V}^{t0}_n$
$R_n(\mathbf{m})$	orthogonal projection of $\mathbf{L}^2(\Omega)$ on $\mathbf{V}^{n0}_n$
$\mathcal{R}$	a rotation in 3D space
$\kappa(s)$	weight function
Qf	approximates the function f on the whole interval $[a,b]$ by its value in the middle or by arithmetic mean value in the points a, b
$\nabla \times \mathbf{u}$	$= (\partial_{x_2} u_3 - \partial_{x_3} u_2, \partial_{x_3} u_1 - \partial_{x_1} u_3, \partial_{x_1} u_2 - \partial_{x_2} u_1)$
$\nabla \cdot \mathbf{u}$	$= \partial_{x_1} u_1 + \partial_{x_2} u_2 + \partial_{x_3} u_3$
∇f	$= (\partial_{x_1} f, \partial_{x_2} f, \partial_{x_3} f)$
$\nabla \mathbf{u}$	$= (\partial_{x_1} u_1, \partial_{x_2} u_1, \partial_{x_3} u_1, \partial_{x_1} u_2, \partial_{x_2} u_2, \partial_{x_3} u_2, \partial_{x_1} u_3, \partial_{x_2} u_3, \partial_{x_3} u_3)$
Δf	$= \partial_{x_1 x_1} f + \partial_{x_2 x_2} f + \partial_{x_3 x_3} f$
$\Delta \mathbf{u}$	$= (\partial_{x_1 x_1} u_1 + \partial_{x_2 x_2} u_1 + \partial_{x_3 x_3} u_1, \partial_{x_1 x_1} u_2 + \partial_{x_2 x_2} u_2 + \partial_{x_3 x_3} u_2, \partial_{x_1 x_1} u_3 + \partial_{x_2 x_2} u_3 + \partial_{x_3 x_3} u_3)$

Function spaces

$L^p(\Omega)$	space of p-th power integrable functions
$\mathbf{L}^2(\Omega)$	$= (L^2(\Omega))^3$
$\mathbf{H}^1(\Omega)$	$= (H^1(\Omega))^3$
$W^{k,p}(\Omega)$	Sobolev spaces
$L^p(I, V)$	spaces of Bochner-type integrable abstract functions

$\mathbf{H}^1_{\nabla 0}(\Omega)$	$=\left\{\mathbf{v}\in\mathbf{H}^1(\Omega), \frac{\partial\mathbf{v}}{\partial\boldsymbol{\nu}}\Big\|_{\partial\Omega}=\mathbf{0}\right\}$
$\mathbf{H}(\mathrm{curl},\Omega)$	$=\left\{\mathbf{v}\in\mathbf{L}^2(\Omega), \nabla\times\mathbf{v}\in\mathbf{L}^2(\Omega)\right\}$
$\mathbf{H}(\mathrm{div},\Omega)$	$=\left\{\mathbf{v}\in\mathbf{L}^2(\Omega), \nabla\cdot\mathbf{v}\in\mathbf{L}^2(\Omega)\right\}$
$\mathbf{H}_{t0}(\mathrm{curl},\Omega)$	$=\left\{\mathbf{v}\in\mathbf{L}^2(\Omega), \nabla\times\mathbf{v}\in\mathbf{L}^2(\Omega), \mathbf{v}\times\boldsymbol{\nu}\big\|_{\partial\Omega}=\mathbf{0}\right\}$
$\mathbf{H}_{t0,\Gamma^h}(\mathrm{curl},\Omega)$	$=\left\{\mathbf{v}\in\mathbf{L}^2(\Omega), \nabla\times\mathbf{v}\in\mathbf{L}^2(\Omega), \mathbf{v}\times\boldsymbol{\nu}\big\|_{\partial\Gamma^h}=\mathbf{0}\right\}$
$\mathbf{H}_{n0}(\mathrm{curl},\Omega)$	$=\left\{\mathbf{v}\in\mathbf{L}^2(\Omega), \nabla\times\mathbf{v}\in\mathbf{L}^2(\Omega), \mathbf{v}\cdot\boldsymbol{\nu}\big\|_{\partial\Omega}=0\right\}$
$\mathbf{H}_{n0,\Gamma^b}(\mathrm{curl},\Omega)$	$=\left\{\mathbf{v}\in\mathbf{L}^2(\Omega), \nabla\times\mathbf{v}\in\mathbf{L}^2(\Omega), \mathbf{v}\cdot\boldsymbol{\nu}\big\|_{\partial\Gamma^b}=0\right\}$
$\mathbf{H}_{t0}(\mathrm{div},\Omega)$	$=\left\{\mathbf{v}\in\mathbf{L}^2(\Omega), \nabla\cdot\mathbf{v}\in\mathbf{L}^2(\Omega), \mathbf{v}\times\boldsymbol{\nu}\big\|_{\partial\Omega}=\mathbf{0}\right\}$
$\mathbf{H}_{n0}(\mathrm{div},\Omega)$	$=\left\{\mathbf{v}\in\mathbf{L}^2(\Omega), \nabla\cdot\mathbf{v}\in\mathbf{L}^2(\Omega), \mathbf{v}\cdot\boldsymbol{\nu}\big\|_{\partial\Omega}=\mathbf{0}\right\}$
$\mathbf{H}(\mathrm{curl},\mathrm{div},\Omega)$	$=\mathbf{H}(\mathrm{curl},\Omega)\cap\mathbf{H}(\mathrm{div},\Omega)$
$\mathbf{H}(\mathrm{curl},\mathrm{div}0,\Omega)$	$=\{\mathbf{v}\in\mathbf{H}(\mathrm{curl},\mathrm{div},\Omega), \nabla\cdot\mathbf{v}=0 \text{ in } \Omega\}$
$\mathbf{H}_{t0}(\mathrm{curl},\mathrm{div}0,\Omega)$	$=\left\{\mathbf{v}\in\mathbf{H}(\mathrm{curl},\mathrm{div},\Omega), \nabla\cdot\mathbf{v}=0 \text{ in } \Omega, \mathbf{v}\times\boldsymbol{\nu}\big\|_{\partial\Omega}=\mathbf{0}\right\}$
$\mathbf{H}_{n0}(\mathrm{curl},\mathrm{div}0,\Omega)$	$=\left\{\mathbf{v}\in\mathbf{H}(\mathrm{curl},\mathrm{div},\Omega), \nabla\cdot\mathbf{v}=0 \text{ in } \Omega, \mathbf{v}\cdot\boldsymbol{\nu}\big\|_{\partial\Omega}=0\right\}$
$\mathbf{V}_n^{t0}$	finite-dimensional approximation space for $\mathbf{H}_{t0}(\mathrm{curl},\mathrm{div}0,\Omega)$
$\mathbf{V}_n^{n0}$	finite-dimensional approximation space for $\mathbf{H}_{n0}(\mathrm{curl},\mathrm{div}0,\Omega)$

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