



Faculteit Wetenschappen
Vakgroep Wiskundige Natuurkunde en Sterrenkunde

Vacuum structure and condensates in quantum chromodynamics

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Proefschrift ingediend tot het behalen van de graad van Doctor in de Wetenschappen: Natuurkunde

Academiejaar 2004-2005

Promotor : Prof. Dr. Henri Verschelde

UNDERSTANDING RESEARCH PAPERS

what is written

It has long been known that...

...of great theoretical and practical importance

While it has not been possible to provide definite answers to these questions...

High purity...

Extremely high purity...

Spectroscopically pure...

A fiducial reference line

Three samples were chosen for detailed study

...handled with extreme care during experiments

Typical results are shown...

Presumably at longer times...

The agreement with the predicted curve is excellent

good

satisfactory

fair

as good as it could be expected

These results will be reported at a later date

The most reliable values are those of Jones

It is believed that...

It is clear that much additional work will be required before a complete understanding...

Unfortunately, a quantitative theory to account for these effects has not been formulated

Correct within an order of magnitude

It is hoped that this work will stimulate further work in the field

Thanks to Jones for assistance with the experiments and to Smith for valuable discussions

A little algebra results in...

After a straightforward calculation...

what is meant

I haven't bothered to look up the reference

...interesting to me

The experiments didn't work out, but I figured I could at least get a publication out of it...

Composition unknown except for exaggerated claims of the supplier

A scratch

The others didn't make sense and were ignored

...not dropped on the floor

The best results are shown...

I didn't take the time to find out

fair

poor

doubtful

imaginary

non-existent

I might get around to this some time

He was a student of mine

I think...

I don't understand it

Neither does anybody else

Wrong

This paper isn't very good but neither are any of the other on this miserable subject

Jones did the work and Smith explained to me what it meant

After several pages of tedious calculations you might retrieve the result

Preface

It is a custom that the preface contains a word of gratitude towards the people that played a major role during the time that the research leading to this thesis took place. That role could have been scientifically important, mentally important, and, in some cases, important for God knows what reason.

Accommodating for this tradition, let us therefore start with thanking the human beings. To avoid that people might feel offended, the order is irrelevant.

Henri, the chief who knows almost anything, for his supporting criticisms and helpful insights. It was always a pleasure discussing, and I hope we can continue to do so.

Jos, for spending endless days and sometimes nights in my enlightening presence, for enduring my lack of football talents in the lavabo soccer games and other enjoyable office activities. Karel and Dirk, also for spending endless days in my presence and coping with Jos and me.

My parents and my little sister Lynn, I cannot say for their interest, as, except for my dad, they probably do not care at all about theoretical physics, QCD, and all those other pointless things, but just for being there.

O Silvio. Era sempre divertido estar no Brasil, o país caótico mas maravilhoso . Sem você, não era possível fazer toda esta pesquisa. Também graças ao Rodrigo e aos outros na UERJ, obrigado!

John, the wizard of loop calculations, without whom theory would have remained theory.

The great computer shaman G., for his patience when Jabba was hurt.

All the other colleagues of the magnificent-restaured-by-John Saey-building *S9*, the mechanics people from the first floor under the guidance of our big boss Willy (and my deepest sympathy for a desperate Antwerp fan, Frans), the astronomers and astrologists of the second floor and of course Anny and Hugo, the administrative (fifth) force of nature.

My friends and family, some mathematics and/or physics lovers and fortunately, also some ordinary humans, and everyone that might consider me as a friend or family.

Some great music bands that flourished up the long hours sitting on a chair behind a small computer screen staring at things that did not always do what one wanted them to do in room 1.08, often mistakenly called temple of wisdom: Sepultura, AC/DC, Rage Against the Machine, Soulfly, R.E.M., Counting Crows, Pantera, Metallica, Pearl Jam, Korn, Manic Street Preachers, ...

Club Brugge, for probably winning the 2005 championship.

John Stith Pemberton, a great inventor, for giving mankind Coca Cola, later perfected to Light and Light Lemon.

Het Laatste Nieuws, for keeping us informed about what is going on in the world.

The FWO, for giving me a Ph.D. grant, as from physics alone, one cannot eat or drink.

And almost last but not least, the two pussies of my life: Fluffy and Sloeber.

And no, I've not forgotten her, a special word of thanks for my nice, little, vegetarian (the universe is not perfect...) girlfriend Isabel!

And, finally, I thank and wish him/her great encouragement, the reader which has to plough through my perhaps not Nobel prizewinning (I mean for literature of course...) writings.

Greetings,

David

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Part I

Introduction

Chapter 1

Introduction

In this first part, we shall describe the content of our research. We shall explain the motivation behind it, how we came to investigate these particular topics and we give an overview of the obtained results. In the second part, we have collected the publications to which we shall refer during the first part for all the necessary details.

The third part contains a general conclusion.

In the Appendix, we present a Dutch summary of our work.

1.1 Explaining the title of this thesis.

1.1.1 Some basic notions: QCD, Yang-Mills gauge theories, gauge fixing, path integral, BRST symmetry.

As the title suggests, we will mainly focus our attention on quantum chromodynamics, commonly known as QCD, the gauge theory of the strong interactions. As we do not intend to give a complete pedagogical introduction to these topics, we refer to the vast amount of literature (see e.g. [1, 2, 3]) for all the historical and mathematical details. We suffice by stating the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a} + \bar{\psi}^{iI} i\gamma^\mu D_\mu^{IJ} \psi^{iJ}, \quad (1.1)$$

where the field strength $F_{\mu\nu}^a$ is defined by

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c. \quad (1.2)$$

$A_\mu = A_\mu^a T^a$ is the Lie algebra valued connection for the gauge group $SU(N)$, which is the group of unitary matrices ($U^\dagger = U^{-1}$) with $\det U = +1$. Its generators T^a , $[T^a, T^b] = f^{abc}T^c$, are chosen to be anti-Hermitian and to obey the orthonormality condition $\text{Tr}(T^a T^b) = \delta^{ab}$, with $a, b, c = 1, \dots, (N^2 - 1)$. The adjoint covariant derivative is given by

$$D_\mu^{ab} \equiv \partial_\mu \delta^{ab} - gf^{abc}A_\mu^c. \quad (1.3)$$

D_μ^{IJ} is the fundamental covariant derivative,

$$D_\mu^{IJ} = \partial_\mu \delta^{IJ} - igA_\mu^a T^{aIJ}. \quad (1.4)$$

The T^{aIJ} are the generators of the fundamental representation of $SU(N)$. The index i labels the number of flavours ($1 \leq i \leq N_f$). g is the coupling constant and serves as a measure for the strength of the interaction.

QCD is an example of a Yang-Mills gauge theory [4]. The Lagrangian given in eq.(1.1) exhibits a local invariance under $SU(N)$ transformations. More precisely, let S be a special unitary matrix which may depend on the space time coordinate x . Then, under the transformations

$$A_\mu \rightarrow S^+ \partial_\mu S + S^+ A_\mu S \quad (\text{adjoint transformation}) , \quad (1.5)$$

$$\psi \rightarrow S\psi \quad (\text{fundamental transformation}) , \quad (1.6)$$

the Lagrangian (1.1) is left unchanged.

The particles corresponding to the A_μ^a -fields are called gluons or gauge bosons. They are the mediators of the strong (color) interaction. The matter particles, described by the $\psi/\bar{\psi}$ fields, are called quarks/antiquarks.

In the case of QCD, there are 3 colors (thus $SU(3)$), known as red, green and blue, while there are six flavours ($N_f = 6$): the up, down, charm, strange, top and bottom quark.

Frequently, we shall omit the quark content of QCD (this corresponds to pure Yang-Mills gauge theory or, or as it is sometimes called, gluodynamics), and work with general $SU(N)$ or even $SU(2)$ instead of $SU(3)$.

In this thesis, we shall only rely on the path integral formulation of quantum field theory. Having found the Lagrangian, one defines the action, in four-dimensional Minkowski space, as

$$S = \int d^4x \mathcal{L} , \quad (1.7)$$

and the path integral by

$$\int [d\Phi] e^{iS} . \quad (1.8)$$

The path integral formulation can be compared with the distribution function in a statistical way, as the expectation value of any operator $\mathcal{O}(x)$ is determined by

$$\langle \mathcal{O}(x) \rangle = \int [d\Phi] \mathcal{O}(x) e^{iS} . \quad (1.9)$$

The integration goes over all possible field configurations, denoted by Φ . The exponential factor e^{iS} can thus be seen as the weight factor of any configuration, which contributes to the expectation value of $\mathcal{O}(x)$.

When Yang-Mills theories are quantized, the local gauge invariance gives rise to troubles for a proper implementation of the quantization. This local invariance is in a sense “too big” and should be restricted. The path integral, defined over all the possible gauge field configurations, should be reduced to one only over gauge inequivalent fields. Analogously as for the classical Maxwell theory of electromagnetism, this is achieved by fixing the gauge freedom, i.e. a certain condition is imposed on the gauge fields A_μ^a . Usually, a gauge fixing can be implemented in the path integral by adding extra terms to the action. These extra terms shall contain novel fields. Let us illustrate this when the Landau gauge is imposed, i.e. we demand that $\partial_\mu A^\mu = 0$. The corresponding Lagrangian turns out to be

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + b^a \partial_\mu A^{a\mu} + \bar{c}^a \partial^\mu D_\mu^{ab} c^b , \quad (1.10)$$

The fields $c^a/\bar{c}^a$ are called the ghost/antighost fields. They are not physical particles, as they are Grassmann (anticommuting) scalars. They are needed to localize a certain functional determinant showing up during the gauge fixing procedure, which is due to Faddeev and Popov [5]. Although the ghosts are unphysical, they are still important. They assure that only the physical, transversal polarizations of the massless gauge bosons are observable. The field b^a is a Lagrangian multiplier, ensuring the Landau gauge condition. For practical purposes, this auxiliary field is integrated out by adding $\alpha b^a b^a$ to (1.10). α is the gauge parameter where $\alpha = 0$ corresponds to the Landau gauge.

The Lagrangian (1.10) is no longer locally $SU(N)$ invariant, but some kind of "replacement" symmetry has come in the place, namely the BRST (Becchi-Rouet-Stora-Tyutin [6, 7]) symmetry given by

$$\begin{aligned} s\mathcal{A}_\mu^A &= -D_\mu^{AB}c^B, \\ s c^A &= \frac{g}{2}f^{ABC}c^B c^C, \\ s\bar{c}^A &= b^A, \\ s b^A &= 0, \\ s\psi^{iI} &= -igc^a T^{aIJ}\psi^{iJ}. \end{aligned} \tag{1.11}$$

As we shall see, all gauge fixings we shall consider possess this BRST symmetry. This invariance shall play an important role throughout this work.

As the original theory was locally gauge invariant, the observed physical quantities should not depend on the choice of gauge. Such statements can e.g. be proven starting from the BRST invariance.

1.1.2 Some further basic notions: perturbation theory, quantum corrections, renormalization, regularization, renormalization group, asymptotic freedom.

Once a classical theory is quantized, calculations are possible at the quantum level.

Starting from the path integral, a perturbation theory is typically achieved by considering an exactly calculable integral - i.e. one quadratic in the fields- and consequently a perturbative expansion in the coupling constant g^2 is performed for the nonquadratic terms. This results in a Taylor like series in g^2 . Each of these terms at a certain order $(g^2)^n$ consists of a number of Feynman diagrams, namely those having n loops. These Feynman diagrams are generated with a number of basic "graphical" elements, which are determined by the terms present in the starting theory. These basic elements are the propagators, describing the free propagation of the particles associated to the fields, and the vertices, describing the interactions between the fields (particles). The Feynman rules state the correspondence between the loop diagrams and the analytical expression behind them. The perturbative expansion is also known as the loop expansion.

In a typical four-dimensional theory, one shall find that these Feynman diagrams will contain expressions like

$$\int d^4q \frac{1}{q^2 + m^2} \propto \int_0^{+\infty} \frac{q^3}{q^2 + m^2} dq, \tag{1.12}$$

which are divergent, due to the bad behaviour at large momentum q . Hence, these infinities are called ultraviolet (UV) divergences. The presence of divergences in quantum field theory is a common phenomenon and can sometimes be cured for by *renormalization*. Essentially, renormalization means that all the divergences, appearing upon calculating Feynman diagrams, are reabsorbed in the "parameters"¹

¹Like masses, coupling constants, the fields themselves,...

of the original theory, which is called the bare theory. Assume that g_0^2 is the bare coupling constant. This g_0^2 represents the coupling constant without the incorporation of the quantum (loop) effects. Thus, g_0^2 can be supposed to be infinite in such a way that the infinite loop corrections to g_0^2 are canceled and a finite, meaningful quantity g^2 remains. The philosophy of renormalization is based on the fact that what we observe are not the bare quantities of the original theory, but the fully “dressed” quantities, i.e. with all the quantum corrections included. As long as these are finite, one can be satisfied.

The divergences can be kept track of by *regularizing* the considered theory. Regularizing means that one makes the divergent integrals well defined. A straightforward method would be cutting off the integration at a certain scale Λ . However, this cut-off regularization is incompatible with gauge invariance. Besides this, the integrations are quite complicated with a cut-off. *Dimensional regularization* [8] respects gauge invariance. The trick consists of defining the theory in $4 - \varepsilon$ dimensions, in which case it is finite. The divergences are recovered in the physical limit $\varepsilon \rightarrow 0$.

We have explained renormalization here in such a way that it might seem a quite trivial task to actually renormalize a theory. However, the reality is far more complicated than this. A priori, it is absolutely not straightforward all divergences that might appear could be reabsorbed. Indeed, if a divergence is generated which corresponds to a certain interaction not present in the original theory, then obviously it cannot be reabsorbed as there is nothing to reabsorb in. Problems could also occur when the same quantities appear in different combinations. Each combination should evidently lead to the same renormalization of those quantities. This is not necessarily true.

Theories with the aforementioned problems are called nonrenormalizable and are in principle bad quantum theories. The good quantum theories are the renormalizable ones. As an illustration, think about a theory having a coupling constant g^2 with negative mass dimension. At each order order in the series expansion, the negative dimensionality coming from powers of g^2 should be neutralized each time with novel operators having sufficiently high positive mass dimension. Clearly, this makes renormalization impossible.

It is one of the great achievements of theoretical physics that the Yang-Mills gauge theories and thus QCD, are indeed renormalizable, see e.g. [8, 6].

Returning to the concept of perturbation theory, it might have already been clear that, in order to be so that perturbation theory would be a useful tool, the expansion parameter in which one is perturbing, should be sufficiently small. As we shall see in this thesis, the explicit evaluation of Feynman diagrams is not that easy, so often only the first few orders are accessible.

Due to renormalization effects, one must introduce a mass scale $\bar{\mu}$ into the theory, while the parameters of the theory become a function of this in principle arbitrary mass scale. The functional behaviour w.r.t. $\bar{\mu}$ is usually called the running or the anomalous dimension of the considered quantity. For example, the running of the coupling constant g^2 is governed by the β -function

$$\bar{\mu} \frac{\partial}{\partial \bar{\mu}} g^2 \equiv \beta(g^2) = -2(\beta_0 g^2 + \beta_1 g^4 + \dots) . \quad (1.13)$$

Such equations are also frequently called renormalization group (RG) equations. Physical quantities $\mathcal{P}$, as well as bare quantities $\mathcal{B}_0$, which are inherent to the theory, should not depend on this arbitrary scale, and the renormalization group equations become in this case

$$\begin{cases} \bar{\mu} \frac{d}{d\bar{\mu}} \mathcal{P} = 0 , \\ \bar{\mu} \frac{d}{d\bar{\mu}} \mathcal{B}_0 = 0 . \end{cases} \quad (1.14)$$

At lowest order, the equation (1.13) can be solved by

$$g^2(\bar{\mu}) = \frac{1}{\beta_0 \ln \frac{\bar{\mu}^2}{\Lambda_{\text{MS}}^2}} . \quad (1.15)$$

The quantity $\Lambda_{\overline{\text{MS}}}$ is a reference scale, depending on the choice of renormalization scheme. We recall that renormalization means subtracting infinities, an operation which is clearly only well-defined up to finite subtractions. The freedom in the finite part is fixed by a (arbitrary) renormalization scheme. Expression (1.15) allows to trade the coupling constant for the ratio of the dimensionful parameters $\bar{\mu}$ and $\Lambda_{\overline{\text{MS}}}$. Such a replacement is sometimes called “dimensional transmutation”, as a dimensionless quantity is exchanged for dimensionful quantities.

Considering eq.(1.15), we recognize different cases:

- i. $\beta_0 < 0$: the theory is perturbatively only well-defined for scales $\bar{\mu} < \Lambda_{\overline{\text{MS}}}$, as the coupling should be positive. g^2 runs to zero for $\bar{\mu}$ going to zero, i.e. the theory is infrared well-defined perturbatively (IR stable). For $\bar{\mu} \uparrow \Lambda_{\overline{\text{MS}}}$, the coupling grows and perturbation theory will fail.
- ii. $\beta_0 = 0$: there is no running. We are considering a fixed point.
- iii. $\beta_0 > 0$: the theory is perturbatively only well-defined for scales $\bar{\mu} > \Lambda_{\overline{\text{MS}}}$, g^2 runs to zero for growing $\bar{\mu}$, i.e. the theory is ultraviolet well-defined perturbatively (UV stable). For $\bar{\mu} \downarrow \Lambda_{\overline{\text{MS}}}$, the coupling grows and perturbation theory will fail. Such theories are called *asymptotically free* [9, 10].

A similar analysis can be made when higher order corrections are included. Yang-Mills gauge theories are the most famous example of asymptotically free theories. Such theories are in some sense a bit counterintuitive, as one might expect that at low energies, not “much” would happen, while at high energies (small distance), the interaction could be expected to be strong.

In a typical experiment at a certain energy scale, one shall encounter the coupling at that scale. For sufficiently high scale, the coupling will be small and one can predict via perturbation theory the outcome of the experiment. However, at lower energies, nonperturbative effects enter the game, as the expansion parameter grows and make a series expansion useless.

1.1.3 Nonperturbative effects.

We have already mentioned the word “nonperturbative”. Logically, it indicates effects that are not accessible by perturbation theory. As a simple example, assume that we have an asymptotically free field theory with massless particles. Assume that the particles obtain by some dynamical mechanism a mass m . If this mass is supposed to be RG invariant at lowest order, then

$$\bar{\mu} \frac{d}{d\bar{\mu}} m = \left(\bar{\mu} \frac{\partial}{\partial \bar{\mu}} + \beta(g^2) \frac{\partial}{\partial g^2} \right) m = 0, \quad (1.16)$$

an equation that can be solved by

$$m \propto \bar{\mu} e^{-\frac{1}{2\beta_0 g^2}} \equiv \Lambda_{\overline{\text{MS}}}. \quad (1.17)$$

An expression like this can never be obtained in perturbation theory, as $e^{-\frac{1}{2\beta_0 g^2}}$ has no series expansion as it shows an essential singularity.

Probably, the most famous problem that cannot be solved in perturbation theory in QCD is the absence of separate gluons and quarks in the observable spectrum. The excitations that we do observe consist of mesons, baryons,.... This is called *confinement*: colored states like quarks are permanently “locked” into colorless states like baryons.

We recall that QCD² itself does not possess a mass scale at the classical level. Nevertheless, renormalization introduces a scale at the quantum level, and one might guess dimensionfull quantities might arise

²In the chiral limit of vanishing quark masses.

at the nonperturbative level in QCD. In practice, such a scale is generated through the *condensation* of a certain operator. Consider for example the operator $F_{\mu\nu}^2$ in QCD. This operator has mass dimension four. If we would like it to condense, then it is clearly not possible in perturbation theory due to the lack of an explicit mass scale to which it could be proportional. At the nonperturbative level, it could however happen that

$$\langle F_{\mu\nu}^2 \rangle \propto \Lambda_{\overline{MS}}^4. \quad (1.18)$$

Condensates are assumed to be the same everywhere in space time, i.e. $\langle F_{\mu\nu}^2(x) \rangle = \langle F_{\mu\nu}^2(0) \rangle = (VT)^{-1} \int d^4x \langle F_{\mu\nu}^2(x) \rangle$ where VT is the space time volume.

Condensates are thus the vacuum expectation value of certain operators. It is the expectation value in the vacuum, as no external particles are taken into account. Condensates can thus be thought of as characterizing the *vacuum structure* of a field theory. One might expect that the vacuum is an empty “thing” without energy. Nonetheless, condensates are nonzero in the vacuum and do even influence the energy of the vacuum. The vacuum is thus a much more exciting topic to investigate than might be thought. $\langle F_{\mu\nu}^2 \rangle$ for example can be related to the vacuum energy E , in fact for pure Yang-Mills theories, $E \propto \langle F_{\mu\nu}^2 \rangle$. This result can be obtained from the trace anomaly. An *anomaly* is another important subject in the quantum context. A theory can possess a certain symmetry at the classical level, however it is not always possible to retain all the symmetries of a theory when it is properly quantized. Such symmetries are called anomalous, since they are not real symmetries as the theory cannot maintain the classical invariance at the quantum level. The scale anomaly is due to the absence of a mass scale in the field theory at the classical level while the presence of the renormalization scale $\bar{\mu}$ does introduce a mass scale at the quantum level.

Let us clarify the possible importance of condensates with an example. Assume that we have determined experimentally a cross section σ at a certain scale Q . A cross section σ could be calculated approximately using a perturbative approach, but if nonperturbative condensates are existing, power corrections can arise, since a dimensionless quantity can be constructed with these condensates and the external momentum scale Q . Since σ is a physical and hence gauge invariant quantity, it can only receive contributions from gauge invariant quantities, like $\langle F_{\mu\nu}^2 \rangle$, which can give rise to power corrections proportional to $\frac{\langle F_{\mu\nu}^2 \rangle}{Q^4}$. Hence, we see that the vacuum structure of the theory has influence beyond the vacuum itself.

Having stated the importance of the nonperturbative aspects of vacuum condensates, the question arises how one may obtain estimates for these condensates. As already outlined above, a possible way is by matching experimentally determined quantities with theoretical predictions and extract values for the condensates. Such an approach is made possible by using the Operator Product Expansion (OPE) [11, 12] and QCD sum rules [13, 14]. Although very important and useful, these approaches are phenomenological in nature as an experimental input is necessary to obtain estimates for the condensates. It remains unclear what is the mechanism behind these condensates. A possible candidate is the instanton contribution [15, 16, 17, 18]. Instantons are solutions of the classical equations of motion of QCD carrying a topological winding number. Instantons describe the tunneling between topologically distinct vacua and lower the energy by “ $e^{-\frac{1}{g^2}}$ ” effects. As such, they can be the cause of nonperturbative effects in QCD, for example by inducing a condensate $\langle F_{\mu\nu}^2 \rangle$. However, calculations with instantons are not straightforward. Next to instantons, also other topological quantities might arise in QCD, like merons, magnetic monopoles, dyons, vortices,..., each with a possible influence on the vacuum structure.

Another powerful, nonperturbative technique is the numerical simulation of QCD on a discretized space time, i.e. on a (finite) lattice [12]. Putting a gauge theory on a lattice allows for a nonperturbative regularization in a gauge invariant way. Since lattice gauge theory is supposed to capture *all* effects as

the complete theory is simulated from the IR to UV region, a fortiori *all* nonperturbative information is also captured. In principle, one can calculate everything using lattice simulations. However, obtaining the numerical value does not always entail knowing the mechanism giving that value. Furthermore, in many cases the quantity is not calculated directly, but, e.g., extracted from a matching between the numerical values employing the OPE or even a simple fitting formula. Another point of care consists the extrapolation of the discrete lattice results to the continuum, i.e. the limit of vanishing lattice spacing and infinite volume.

1.1.4 Aim of our research.

This brings us to the main topic in this thesis: constructing analytical tools to calculate some nonperturbative effects in Yang-Mills gauge theories. Evidently, we are not the first ones using analytical tools to probe quantum field theories beyond the perturbative level. We already mentioned the instanton calculus [18]. Renowned examples are the CJT formalism [19], the Schwinger-Dyson equations [20], the $\frac{1}{N}$ expansion [21], the exact renormalization group approach,...

The CJT formalism relies on resumming certain types of diagrams, which leads to gap equations for consistency. However, these gap equations can become nonlocal and hence difficult to solve. The Schwinger-Dyson equations are the quantum equation of motion for the Greens functions of the theory, which are solved approximately by using certain ansatzes. In a $\frac{1}{N}$ expansion, one only retains those subclasses of diagrams that contribute at a certain order in $\frac{1}{N}$, where N , being the number of colors or flavours, is supposed to be large.

We shall focus our attention on techniques which are in essence perturbative, but still does allow to obtain some information on nonperturbative effects. Although this might sound as a contradictio in terminis, it is not, as it will become clear in the following sections.

1.2 Investigating a condensation by constructing the effective potential.

The first question coming to mind after having read the introduction, is how one can calculate the value of a certain condensate.

To get an idea, we return to classical mechanics. There one knows the concept of a potential, describing the energy of the model. The most stable state is the one which has minimal energy. This potential V is written in terms of some quantities (fields ϕ) and the extrema of V are found through solving

$$\frac{dV}{d\phi} = 0 . \quad (1.19)$$

This can be generalized to quantum field theory. The configuration ϕ_{vac} at which V obtains its absolute minimum describes the vacuum. Excitations $\tilde{\phi}$ above the vacuum are obtained from $\phi \equiv \phi_{\text{vac}} + \tilde{\phi}$. One should determine the quantum effective potential, which is nothing else than the classical potential supplemented with all quantum corrections. Evidently, things are slightly more complicated at the quantum level. It should be proven that any technique for constructing an effective potential maintains renormalizability. Since we will be working in a perturbative expansion³, reliability can only be obtained when the relevant expansion parameter is sufficiently small. Another pertinent question is which operator might be the one that condenses.

³We shall only calculate the potential in a perturbative series expansion, in principle there can be nonperturbative corrections to the potential itself, e.g. induced by instantons.

In the following section, we shall consider some of these questions in more detail using a toy model possessing interesting properties.

1.3 The Gross-Neveu model.

The Gross-Neveu model is a two-dimensional pure fermionic field theory, described by [21]

$$\mathcal{L} = \bar{\psi}\not{\partial}\psi - \frac{1}{2}g^2 (\bar{\psi}\psi)^2 . \quad (1.20)$$

This Lagrangian has a global $U(N)$ invariance and a discrete chiral symmetry $\psi \rightarrow \gamma_5 \psi$ which imposes $\langle \bar{\psi}\psi \rangle = 0$ perturbatively⁴. The model is asymptotically free and has a dynamical chiral symmetry breaking. This breaking cannot be realized perturbatively as an order parameter for this symmetry is provided by $\langle \bar{\psi}\psi \rangle$. The fermions acquire a nonperturbative dynamical mass $m_F \propto \langle \bar{\psi}\psi \rangle$.

One should thus construct an effective potential for the operator $\bar{\psi}\psi$ and find out whether a nonvanishing value for $\langle \bar{\psi}\psi \rangle$ is favoured if it lowers the vacuum energy. Here, we encounter another importance of condensates: they can sometimes break symmetries of the model at the nonperturbative level. This is another way of breaking symmetries, next to an anomalous breaking.

Gross and Neveu managed to construct the effective potential by decomposing the quartic interaction in the Lagrangian (1.21) by the introduction of an auxiliary field

$$\mathcal{L} = \bar{\psi}\not{\partial}\psi + \frac{1}{2}\sigma^2 - g\bar{\psi}\psi\sigma , \quad (1.21)$$

whereby $\langle \sigma \rangle = g \langle \bar{\psi}\psi \rangle$. Hence, one can obtain the effective potential for σ in a loop expansion by integrating out the fermionic fields, and one shall find, using the rules and techniques of perturbation theory,

$$V(\sigma) = a_1 \sigma^2 + g^2(\bar{\mu}) \left(b_1 \ln \frac{g\sigma}{\bar{\mu}} + b_2 \right) + g^4(\bar{\mu}) \left(c_1 \ln^2 \frac{g\sigma}{\bar{\mu}} + c_2 \ln \frac{g\sigma}{\bar{\mu}} + c_3 \right) + \dots \quad (1.22)$$

This potential will obey a RG equation $\bar{\mu} \frac{d}{d\bar{\mu}} V = 0$, allowing to choose freely at any order the scale⁵ $\bar{\mu}$. Usually, one chooses the scale $\bar{\mu}$ so that the logarithms vanish in the gap equation derived from $\frac{dV}{d\sigma} = 0$, and one obtains an effective potential in a $g^2|_{\bar{\mu}^2=g\sigma}$ series. The gap equation will give rise to

$$\frac{g^2(\bar{\mu})N}{4\pi} \bigg|_{\bar{\mu}^2=g\sigma} = c , \quad (1.23)$$

with c some constant determinable from the gap equation. Using the one-loop result (1.15), one consequently finds a value for the condensate $\langle g\sigma \rangle$. Here, the concept of “dimensional transmutation” comes clearly into the picture. If the condensate is large enough, the expansion parameter (1.23) will be sufficiently small⁶ and we can speak about reliable results. We see thus that, in principle, it is possible to find nonperturbative results using essentially nothing more than perturbation theory.

⁴Since there is no mass scale in the model, evidently $\langle \bar{\psi}\psi \rangle = 0$ in perturbation theory.

⁵The error will be one order higher compared to the considered order.

⁶In fact, the value of the coupling constant, found via eq.(1.23), is obtained first. It should be sufficiently small to find a reliable estimate of the condensate.

1.3.1 The local composite operator formalism.

A few things deserves attention considering the Gross-Neveu model. First of all, apart from being interesting in its own right, it can serve as a powerful testing model because the *exact* results for the mass gap and vacuum energy are known, due to some special techniques sometimes available in two-dimensional field theories [25, 26, 27]. Henceforth, the (numerical) reliability of analytical tools to investigate the dynamical generation of the fermion mass can be tested.

We notice that $\bar{\psi}\psi$ is a local, but composite operator. In general, local composite operators (LCO) bring along extra troubles at the quantum level. The operator should be renormalizable, and one should be able to construct a renormalizable effective potential. As we have already mentioned, renormalizability is not a trivial property. At one-loop or in the $\frac{1}{N}$ expansion, a decomposition of a quartic interaction like in eq.(1.21) is sensible. However, problems will appear concerning the renormalizability and/or the renormalization group at higher orders. Ad hoc counterterms have to be added. We refer to [22, 23, 24] for more details. The conclusion was that one must construct a method to effectively obtain an effective potential for local composite operators, whereby certain requirements have to be fulfilled. Such a method was found in [23], allowing one to calculate an effective potential exhibiting all the desired properties of renormalizability and renormalization group behaviour, and this for any N at any order of perturbation theory. Also some other old objections concerning the use of composite operators were neatly solved in [23]. The calculated values of the fermion mass were only differing a few percent from the exact results.

We shall not explain here in full detail the by now called LCO method, as it will sufficiently be discussed in the articles in part II of the thesis. We shall only outline the main idea. Given a certain LCO $\mathcal{O}$ of say dimension two in a four-dimensional field theory, the standard way of coupling this operator $\mathcal{O}$ to the theory is by using a source J , i.e. a term $J\mathcal{O}$ is added to the Lagrangian. This gives rise to a functional $W(J)$, whose Legendre transform is nothing else than the effective potential $V(\sigma)$, where σ is an auxiliary field describing the (composite) operator $\mathcal{O}$. However, novel infinities $\propto J^2$ shall arise due to the divergences in the correlator $\langle \mathcal{O}(x)\mathcal{O}(y) \rangle$ for $x \approx y$. These correspond to vacuum energy divergences. As a consequence, a counterterm $\propto J^2$ is needed in the Lagrangian, thus there should be a term ζJ^2 present in the starting Lagrangian, where the new LCO parameter ζ has to be introduced in order to make the reabsorption of the counterterm $\delta\zeta J^2$ possible. The value of ζ can be fixed by employing the RG in an intelligent fashion [23].

Concerning the choice of the operator, we already mentioned that $\langle \bar{\psi}\psi \rangle$ is an order parameter for the chiral γ_5 -symmetry, so it was quite clear that $\bar{\psi}\psi$ was an ideal candidate as LCO. However, in a more complex theory, the choice of a suitable operator might be less transparent, e.g. when the condensate is not directly linked to an order parameter of a (discrete) symmetry. However, a more general argument is available why the formation of $\langle \bar{\psi}\psi \rangle$ can be expected at a nonperturbative level. It is perhaps most easy visible employing the high N -expansion, although the argument also stands beyond that approximation. Considering the quartic interaction (the $\bar{\psi}\psi$ - $\bar{\psi}\psi$ channel) of the Gross-Neveu model, one shall find that all diagrams of leading order in N , can be summed in an explicit way. In this expression, poles at momenta that are negative and $\propto e^{-\frac{1}{g^2}}$ shall occur. Said otherwise, one encounters the problem of a tachyon pole in this channel. This kind of nonperturbative effects are called (infrared) renormalons. Tachyons are in general an indication of instabilities. We can interpret them as that the Gross-Neveu model is being considered around an instable vacuum. This is signalled by the problems in the $\bar{\psi}\psi$ - $\bar{\psi}\psi$ channel, which might trigger a condensation of $\bar{\psi}\psi$ at a nonperturbative level. If a mass would be generated, an infrared cutoff emerges “moving” away the IR instabilities.

1.4 The gluon condensate $\langle A_\mu^2 \rangle$.

The interesting the Gross-Neveu model might be as a testing ground, one should also try to probe physically more interesting theories like Yang-Mills gauge theories with the developed tools.

The question jumping to the readers mind should be: is there something analogous to the Gross-Neveu mass generating condensate $\langle \bar{\psi}\psi \rangle$ in the case of pure Yang-Mills theories? Here we are already confronted with the fact that there is no discrete symmetry shedding some light on a possible candidate. Furthermore, one is also faced with the fact of gauge invariance. A local gauge invariant operator of mass dimension two does not exist.

In [28, 29, 30, 31, 32], it was discussed that there does exist an instability in Yang-Mills theories in the IR. We could have guessed from our Gross-Neveu knowledge that similar problems with infrared renormalons and tachyonic instabilities might exist in gauge theories too: the Yang-Mills action contains also a quartic interaction $\sim A^4$ term. These instabilities might invoke the condensation of a dimension two operator like A_μ^2 .

Despite some early attempts, not much attention was paid to such a condensate $\langle A_\mu^2 \rangle$ and certainly no effective potential was available for it. A few years ago, interest in it was renewed due to the work of Zakharov et al [33, 34]. They attracted the attention to this condensate in the case of compact three-dimensional QED₃, where by numerical simulations, $\langle A_\mu^2 \rangle$ was shown to be an order parameter for the condensation of monopoles. It was already proven some time ago that monopoles are condensing in QED₃, giving rise to confinement [35, 36].

The speculation was that $\langle A_\mu^2 \rangle$ might also be relevant for four-dimensional QCD. It was argued in [34] that the condensate $\langle A_\mu^2 \rangle$ might receive “soft” nonperturbative contributions from the IR region as well as “hard” nonperturbative contributions from the UV. It was speculated that the “hard” part can contribute to $\frac{1}{Q^2}$ corrections to gauge invariant (physical) quantities.

The soft part should be accessible by an OPE analysis, a study explicitly performed in [37, 38, 39], resulting in a nonvanishing condensate $\langle A_\mu^2 \rangle_{\text{OPE}}$. This lattice study was based on the observed discrepancy existing between the calculated perturbative behaviour and the obtained lattice behaviour of the gluon two- and three-point function up to an energy region of 10 GeV. Results could be quite well fitted using the OPE condensate $\langle A_\mu^2 \rangle_{\text{OPE}}$ giving rise to $\frac{1}{Q^2}$ power corrections. The $\frac{1}{Q^4}$ power correction coming from $\langle F_{\mu\nu}^2 \rangle$ is too weak at such energies to be the cause of the discrepancy. Quadratic power corrections were already predicted in [40]. In [41], there was found lattice evidence for a linear part in the static quark-antiquark potential at short distances. This can be understood intuitively as follows. In one gluon exchange approximation, the static (Coulombic) potential $V(r)$ is determined by

$$V(r) \sim \int d^3k \alpha_s \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2} \sim \frac{1}{r}. \quad (1.24)$$

Slightly generalizing this to $\alpha_s \rightarrow \alpha_s + \frac{c}{k^2}$, an extra contribution to $V(r)$ is found, namely

$$V(r)_{\text{linear}} \sim \int d^3k \frac{c}{k^2} \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2} \sim cr. \quad (1.25)$$

This linear piece should not be confused with the confinement scenario of gauge theories. A linear rising potential between color charges is a very strong indication that they cannot be separated due to the enormous energies this would demand. Hence, the particles are confined to stay together. However, the foregoing (rude) approximations with the power corrections etc. are *not* applicable at low energies, exactly where confinement should be proven.

In a consequent work [38], it was argued, once more with lattice simulations, that instantons might give the main contribution to $\langle A_\mu^2 \rangle_{\text{OPE}}$.

All available estimates for $\langle A_\mu^2 \rangle$ were thus obtained in an indirect way via lattice simulations and an OPE analysis. A first independent *calculation*, using the LCO method, was presented in [42]. There, the two-loop effective potential was constructed and it was found that the operator A_μ^2 is renormalizable up to three-loop order by explicit calculation. We mention here a somewhat unfortunate feature of the LCO method, being that in order to be able to construct the n -loop potential, $(n + 1)$ -loop knowledge of the anomalous dimension of the operator as well as other RG functions is needed. Hence, the mentioned three-loop calculations are quite burdensome and were performed with (self-developed) computer packages.

The LCO method provided a nonvanishing value for the mass dimension two gluon condensate as it lowers in a nonperturbative way the vacuum energy. A consequence of the condensate is the appearance of a nonvanishing mass parameter in the lowest order gluon propagator. The occurrence of massive parameters in the gluon propagator has received many confirmations from the lattice community, see e.g. [43, 44, 45, 46, 47, 48, 49, 50]. Also some recent work using the Schwinger-Dyson equations revealed that massive parameters can occur [51, 52]. Even from the phenomenological side, evidence has been presented that massive gluons give better predictions for some processes, see e.g. [53, 54]. Dynamically massive gluons were also used in other contexts, see e.g. [55, 56, 57, 58].

One should not confuse $\langle A_\mu^2 \rangle_{\text{LCO}}$ with $\langle A_\mu^2 \rangle_{\text{OPE}}$. The LCO value is obtained with perturbative techniques that are a fortiori valid in a region where perturbation theory applies. It might be said that $\langle A_\mu^2 \rangle_{\text{LCO}}$ contains (part of) the “hard” content of $\langle A_\mu^2 \rangle$. The OPE value should find its existence in the IR, from topological quantities like instantons,.... Thus, there is no direct connection between estimates obtained from the OPE or LCO method. One might state that both are complementary in the sense that they are giving strong evidence for the existence of a gluon condensate of mass dimension two. In the case of the Gross-Neveu model, the success in comparing the LCO estimate with the exact mass gap is partially due to the absence of any topological content in that model.

The attention of the mentioned works was mainly focused on the Landau gauge. It is worth explaining the preferred role of this gauge. Consider the quantity

$$\langle A_\mu^2 \rangle_{\min} \equiv (VT)^{-1} \min_{U \in SU(N)} \left\langle \int d^4x (A_\mu^U)^2 \right\rangle. \quad (1.26)$$

This quantity is gauge invariant albeit highly nonlocal. Although, in the Landau gauge it reduces to the vacuum expectation value of the square of the gauge potential, i.e. $\langle A_\mu^2 \rangle$. As a matter of fact, the Landau gauge can be defined as the gauge minimizing the functional

$$\mathcal{R}_{\text{Landau}}[A] = \int d^4x (A_\mu^U)^2, \quad U \in SU(N). \quad (1.27)$$

The minimization of this particular functional is something that, in principle, can be arranged for in an algorithmic form, i.e. the Landau gauge is very well suited for lattice simulations.

1.5 Algebraic renormalization and the role of symmetries in establishing renormalizability.

Before turning to an overview of our own accomplishments, we shall spend a few more words on the issue of renormalization. More precisely, we shall say something more on the *algebraic renormalization* formalism, as this has turned out to be a tool of major importance during our research. Evidently, this thesis is not the place to give a complete introduction to this formalism. We kindly refer to the literature [59, 60] for a thorough introduction. We intend only to give a simplified idea of the concept.

Consider a quantum field theory with a certain symmetry of the action Σ , expressed in a functional way by $\mathcal{F}(\Sigma) = 0$. Such a constraint is usually called a *Ward identity*. A Ward identity controls more than just the action, it also put constraints on the Greens functions of the theory, but this will be of lesser importance in this thesis.

As the quantum theory enjoys a symmetry, the quantum corrections to it, giving the renormalization, should be invariant under this symmetry⁷. As such, the symmetry is putting a constraint on the form of the most general quantum correction, Σ' . If we can solve for the most general quantum correction which obeys $\mathcal{F}(\Sigma') = 0$, it is a straightforward exercise to determine whether Σ' can be reabsorbed in the original action by a suitable renormalization of the available parameters. The advantage of such an approach is that renormalizability might be obtained to *all* orders of perturbation theory without the necessity of performing complicated Feynman diagram calculations, only allowing to establish renormalizability order by order.

Of course, the difficulty remains in finding the most general solution of $\mathcal{F}(\Sigma') = 0$. For simple models, this might be not much of a problem. Although, for e.g. Yang-Mills gauge theories with many interactions, this task could become quite complicated. However, there is a beautiful way of reducing the work. We remind the reader here of the existence of the BRST symmetry in gauge theories, describing the “relics” of the gauge invariance at the level of the gauge fixed theory. This symmetry enjoys the property of being nilpotent, i.e. the charge Q_{BRST} , corresponding to it vanishes upon squaring, $Q_{\text{BRST}}^2 = 0$. The Ward identity arisen from the BRST invariance is known as the Slavnov-Taylor identity, $\mathcal{S}(\Sigma) = 0$. The most general counterterm is thus restricted by $\mathcal{B}_\Sigma(\Sigma') = 0$, where $\mathcal{B}_\Sigma$ is the nilpotent linearized version of $\mathcal{S}$. Due to the nilpotency of $\mathcal{B}_\Sigma$, one surely has

$$\mathcal{B}_\Sigma \mathcal{B}_\Sigma(\text{anything}) = 0 . \quad (1.28)$$

If

$$\text{something} = \mathcal{B}_\Sigma(\text{something else}) , \quad (1.29)$$

then “something” is called $(\mathcal{B}_\Sigma)$ -exact. If

$$\mathcal{B}_\Sigma(\text{something}) = 0 , \quad (1.30)$$

then “something” is called $(\mathcal{B}_\Sigma)$ -closed. For a nilpotent transformation $\mathcal{B}_\Sigma$, exactness clearly induces closedness. The set of quantities that are closed but not exact are said to belong to the $(\mathcal{B}_\Sigma)$ -cohomology.

From the general results on Yang-Mills cohomology [60, 59], it can be derived that the most general solution can be written as the sum of a gauge invariant part ($\sim F_{\mu\nu}^2$), which belongs to the cohomology, and an exact part. This provides one with a powerful tool for discussing the renormalizability of Yang-Mills gauge theories in certain classes of gauges. In particular, it is the Slavnov-Taylor identity which assures that the coupling constant g^2 can be renormalized, despite the fact that the same coupling appears in several interaction vertices.

Let us also mention that there can exist several more Ward identities next to the Slavnov-Taylor identity. Consequently, these put further restrictions on the most general quantum correction.

The essence of the algebraic renormalization is incorporated in the *quantum action principle*, stating precisely what constraints can be put on a quantum theory. Although we have invoked the algebraic renormalization in the present work for the purpose of proving renormalizability to all orders of perturbation theory for the LCO method when a certain composite operator is coupled to the theory, much more information can be drawn from it: nonrenormalization theorems, finding out whether symmetries

⁷As far as the symmetry is not anomalous.

can a priori be maintained at the quantum level, discussing the (gauge) anomalies, the general form of these anomalies, proving the finiteness of some theories, ... Sometimes we shall also be able in our work to draw more information than just the renormalizability of e.g. a composite operator from the algebraic renormalization formalism.

Before turning to an overview of the publications bundled in part II, let us mention that the concept of (BRST)-cohomology plays an important role in Yang-Mills gauge theories. Kugo and Ojima have proven the unitarity⁸ of gauge theories defining the physical states as BRST closed but not exact states. Then it can be proven that, due to some mechanism, all unphysical degrees of freedom (longitudinal and scalar gluon polarizations together with the ghosts and auxiliary fields) decouple from the physical spectrum, only retaining the transversal gluon modes, thereby having unitarity of the scattering matrix. Said in other language, the physical states belong to the gauge cohomology. This is also the main reason why physical amplitudes should not depend on the gauge parameter. In the context of BRST gauge fixing, the terms that are added to the Lagrangian are BRST exact, and as such are trivial in the cohomology, thus variations in the gauge parameter should not change gauge invariant quantities. For example, the Landau gauge fixing in (1.10) can be rewritten as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + s(\bar{c}^a \partial_\mu A^{\mu a}) . \quad (1.31)$$

1.6 Summary of own research.

1.6.1 The $2PPI$ expansion.

As we have already mentioned, a first calculation of the condensate $\langle A_\mu^2 \rangle$ was performed using the LCO method.

Another approach that can reveal some information on the vacuum expectation value of local composite operators was developed in [61, 62, 63, 64] in the case of the $\lambda\varphi^4$ theory. The $2PPI$ expansion contains all Feynman diagrams which remain connected when two lines meeting at the same point are cut and therefore sums systematically the bubble graphs. Diagrams falling apart after such a cutting operation are called 2-point-particle-reducible ($2PPR$). One only keeps the 2-point-particle-irreducible ($2PPI$) diagrams, hence the name $2PPI$ expansion. In Chapter 2, we give an alternative derivation of the $2PPI$ expansion using the Gross-Neveu model. The essence of the $2PPI$ resummation lies in the fact that one can remove the $2PPR$ diagrams from the sum of Feynman diagram building up the vacuum energy E by replacing them with an effective mass scale m . In the case of the Gross-Neveu model, $m \propto \langle \bar{\psi}\psi \rangle$. The selfconsistency of the $2PPI$ resummation is guaranteed by an algebraic gap equation for E , namely $\frac{dE}{dm} = 0$. We have also paid attention to the renormalizability of the $2PPI$ expansion. The explicit evaluation of the gap equation was worked out at two-loop order, as well as the diagrams leading to the pole mass in the fermion propagator, in order to give an estimate of the Gross-Neveu mass gap. The final numerical results for the mass gap and vacuum energy were in a relatively fine agreement with the exact results: we found a disagreement of a few percent for the mass gap. The results for the energy were a bit worse, although this can be understood as the error on the vacuum energy is already “doubled” when compared with the error on the mass gap, as the energy is proportional to the square of the mass gap.

Having tested the $2PPI$ expansion on the Gross-Neveu model, we consequently applied the method to Yang-Mills gauge theory, when the Landau is imposed, in Chapter 3. In this case, the $2PPI$ mass parameter m^2 is proportional to the gauge condensate $\langle A_\mu^2 \rangle$. As such, we presented some further analytical evidence for the mass dimension two condensate $\langle A_\mu^2 \rangle$ in the Landau gauge. The results

⁸At the perturbative level where gluons and quarks are the elementary excitations.

were compatible with the ones obtained earlier with the LCO method in [42]: a lower vacuum energy and a dynamical gluon mass parameter of a few hundred MeV.

Although the $2PPI$ expansion has its attractiveness as it allows to resum a certain class of diagrams without the need of perturbing the action with anything, its usefulness might also be a bit limited. We believe that for e.g. other gauges than the Landau gauge, the renormalizability might be more difficult to prove. Moreover, for other dimension two condensates, we do not immediately see how to use the $2PPI$ expansion. It might be possible, but for the rest of the thesis, we shall rely on the LCO formalism. Before turning to this, let us first explain another concept that initiated some of our research.

1.6.2 The dual superconductor and the maximal Abelian gauge.

As already mentioned, an unresolved problem of $SU(N)$ Yang-Mills theory is color confinement. A physical picture that might explain confinement is based on the mechanism of the dual superconductivity [65, 66, 67, 68], according to which the low energy regime of QCD should be described by an effective Abelian theory in the presence of magnetic monopoles.

Let us provide a very short overview of the concept of Abelian gauges, which are useful in the search for magnetic monopoles, a crucial ingredient in the dual superconductivity picture.

Abelian gauges.

We recall that $SU(N)$ has a $U(1)^{N-1}$ Abelian subgroup, consisting of the diagonal generators. In [68], 't Hooft proposed the idea of the Abelian gauges. Consider a quantity $X(x)$, transforming in the adjoint representation of $SU(N)$,

$$X(x) \rightarrow U(x)X(x)U^\dagger(x) \text{ with } U(x) \in SU(N). \quad (1.32)$$

The transformation $U(x)$ which diagonalizes $X(x)$ is the one that defines the gauge. If $X(x)$ is already diagonal, then clearly $X(x)$ remains diagonal under the action of the $U(1)^{N-1}$ subgroup. Hence, the gauge is only partially fixed because there is a residual Abelian gauge freedom.

In certain space time points x_i , the eigenvalues of $X(x)$ can coincide, so that $U(x_i)$ becomes singular. These possible singularities give rise to the concept of (Abelian) magnetic monopoles. They have a topological meaning since $\pi_2(SU(N)/U(1)^{N-1}) \neq 0$. We kindly refer to [69, 70] for all the necessary details.

The dual superconductor as a possible mechanism behind confinement.

We give a simplified picture of the dual superconductor to explain the importance of the idea. If the QCD vacuum contains monopoles and if these monopoles condense, there will be a dual Meissner effect which squeezes the chromoelectric field into a thin flux tube. This results in a linearly rising potential, $V(r) = \sigma r$, between static charges, as can be guessed from Gauss' law, $\int E dS = q$ or, since the main contribution is coming from the flux tube, one finds $E \Delta S \approx q$, hence $V = -\int E dr \propto r$. In fact, it is not difficult to imagine the longer the flux tube (string) gets, the more energy it will carry.

An example of an Abelian gauge: the maximal Abelian gauge (MAG).

Let A_μ be the Lie algebra valued connection for the gauge group $SU(N)$. We decompose the gauge field into its off-diagonal and diagonal parts, namely

$$A_\mu = A_\mu^A T^A = A_\mu^a T^a + A_\mu^i T^i, \quad (1.33)$$

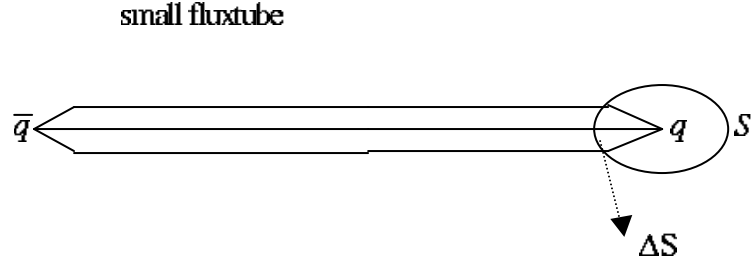


Figure 1.1: A chromoelectric flux tube between a static quark-antiquark pair.

where the indices $i, j, \dots$ label the $N-1$ generators of the Cartan subalgebra. The remaining $N(N-1)$ off-diagonal generators will be labeled by the indices $a, b, \dots$. The field strength decomposes as

$$F_{\mu\nu} = F_{\mu\nu}^A T^A = F_{\mu\nu}^a T^a + F_{\mu\nu}^i T^i, \quad (1.34)$$

with the off-diagonal and diagonal parts given respectively by

$$\begin{aligned} F_{\mu\nu}^a &= D_\mu^{ab} A_\nu^b - D_\nu^{ab} A_\mu^b + g f^{abc} A_\mu^b A_\nu^c, \\ F_{\mu\nu}^i &= \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g f^{abi} A_\mu^a A_\nu^b, \end{aligned} \quad (1.35)$$

where the covariant derivative D_μ^{ab} is defined with respect to the diagonal components A_μ^i

$$D_\mu^{ab} \equiv \partial_\mu \delta^{ab} - g f^{abi} A_\mu^i. \quad (1.36)$$

For the Yang-Mills action one obtains

$$S_{\text{YM}} = -\frac{1}{4} \int d^4x \left(F_{\mu\nu}^a F^{\mu\nu a} + F_{\mu\nu}^i F^{\mu\nu i} \right). \quad (1.37)$$

The maximal Abelian gauge (MAG), introduced in [68, 69, 70], corresponds to minimizing the functional

$$\mathcal{R}[A] = \int d^4x \left[A_\mu^a A^{\mu a} \right]. \quad (1.38)$$

One checks that $\mathcal{R}[A]$ does exhibit a residual $U(1)^{N-1}$ invariance. As the norm of the off-diagonal gluons is minimized, the remaining theory is a sense maximally Abelian, hence the name.

The MAG can be recast into a differential form

$$D_\mu^{ab} A^{\mu b} = 0. \quad (1.39)$$

Although we have introduced the MAG here in a functional way, it is worth mentioning that the MAG does correspond to the diagonalization of a certain adjoint operator, see e.g. [49]. The renormalizability in the continuum of the MAG was proven in [71, 72], at the cost of introducing a quartic ghost interaction.

In order to have a complete quantization of the theory, one has to fix the residual Abelian gauge freedom by means of a suitable further gauge condition on the diagonal components A_μ^i of the gauge field. A common choice for the Abelian gauge fixing, also adopted in the lattice papers [49, 50], is the Landau gauge, $\partial_\mu A^{\mu i} = 0$.

Abelian dominance.

According to the concept of Abelian dominance, the low energy regime of QCD can be expressed solely in terms of Abelian degrees of freedom [73]. Lattice confirmations of the Abelian dominance can be found in [74, 75, 76]. To our knowledge, there is no analytic proof of the Abelian dominance. Nevertheless, an argument that can be interpreted in favour of it, is the fact that the off-diagonal gluons would attain a large, dynamical mass. At energies below the scale set by this mass, the off-diagonal gluons should decouple, and in this way one should end up with an Abelian theory at low energies.

A lattice study of the MAG reported an off-diagonal gluon mass of approximately 1.2GeV [49], while the diagonal gluons behaved massless. Another, more recent, study reported a similar result [50].

Analytical evidence for the off-diagonal mass.

There have been efforts to give an analytical description of a mechanism responsible for the dynamical generation of the off-diagonal gluon mass. In [77, 78, 79, 80], a certain ghost condensate, namely $\langle f^{abc} \bar{c}^a c^b \rangle$, was used to construct an effective, off-diagonal mass. A key ingredient in this construction was the presence of the quartic ghost interaction. Analogously as for the Gross-Neveu model, the interaction was decomposed using an auxiliary field, and an one-loop effective potential for the ghost condensate was calculated, showing that a condensation lowers the vacuum energy. Due to the one-loop effects of the ghost condensation, the off-diagonal gluons required a dynamical mass, while the diagonal ones remained massless.

1.6.3 Ghost condensation and $SL(2, \mathbb{R})$ symmetry.

However, in Chapter 3, we show that the mass obtained from the ghost condensate is a tachyonic one, a fact confirmed later in [82]. We also raise some questions concerning the renormalizability of this approach, making the comparison with the Gross-Neveu model. We propose the LCO method as more suitable for investigating the possible condensation of the ghosts.

In [81], a different decomposition of the quartic interaction gave rise to different ghost condensates, namely the Faddeev-Popov charged ones $\langle f^{abc} c^a c^b \rangle$ and $\langle f^{abc} \bar{c}^a \bar{c}^b \rangle$. This should not be a big surprise, as the ghost condensation is an order parameter for a continuous $SL(2, \mathbb{R})$ symmetry present in the MAG. The $SL(2, \mathbb{R})$ rotations interchange the different channels in which the ghost condensation could occur. It should be investigated if a certain channel might be preferred. By a simple decomposition, this does not seem to be possible, as all the relevant ghost operators should be considered simultaneously.

Another operator, namely the mixed gluon-ghost operator⁹ $\frac{1}{2} A_\mu^a A^{\mu a} + \alpha \bar{c}^a c^a$, of which the condensation could be responsible for the off-diagonal mass, was proposed by Kondo in [83]. That this operator should condense can be expected on the basis of a close analogy existing between the MAG and the renormalizable nonlinear Curci-Ferrari gauge [84, 85], which is the massless limit of a renormalizable massive $SU(N)$ gauge model. The mass term of the Curci-Ferrari model is exactly given by $m^2 (\frac{1}{2} A_\mu^a A^{\mu a} + \alpha \bar{c}^a c^a)$. The Landau gauge is a special case of the Curci-Ferrari gauge.

In Chapter 5, we discuss in some more detail the $SL(2, \mathbb{R})$ symmetry. We recover the already known results that the Landau and the Curci-Ferrari gauge exhibit this symmetry, and we show that it is possible to extend the full symmetry to the MAG in a renormalizable way by choosing a suitable diagonal gauge fixing¹⁰. The algebra built out of the $SL(2, \mathbb{R})$ and (anti)-BRST transformation is known as the Nakanishi-Ojima (NO) algebra [86]. The ghost condensation signals a (partial) breakdown of this NO symmetry.

⁹The index a runs only over the $N(N-1)$ off-diagonal generators in case of the MAG.

¹⁰If the diagonal Landau gauge is imposed, the “diagonal” part of the $SL(2, \mathbb{R})$ symmetry is lost from the beginning.

1.6.4 The anomalous dimension of the composite operator A_μ^2 in the Landau gauge.

Despite the fact that the composite operator A_μ^2 was already studied using the LCO method in the Landau gauge, it was only by explicit calculations that it became clear that the operator is renormalizable, at least up to the considered three-loop order. In Chapter 6, we employed the algebraic renormalization to prove that A_μ^2 is renormalizable to all orders of perturbation theory. We were also able to provide the all order proof of a conjectured relation existing between the anomalous dimension of A_μ^2 and other, more elementary RG functions. From the explicit evaluation in [87], Gracey found that the anomalous dimension of A_μ^2 is a linear combination of the anomalous dimension of the gluon field A_μ and of the β -function, values of which were already known some time to three-loop order [88]. Our proof relied on a restrictive Ward identity present in the Landau gauge, the *ghost Ward identity* [89].

1.6.5 The anomalous dimension of the gluon-ghost mass operator in Yang-Mills theory and gluon-ghost condensate of mass dimension two in the Curci-Ferrari gauge.

In Chapter 7, we set a first step towards extending the LCO formalism to other gauges. Until then, A_μ^2 was restricted to the Landau gauge. The question arises if a renormalizable mass dimension two operator may exist in other gauges. We already mentioned the Landau gauge is a limiting case of the Curci-Ferrari gauge, namely $\alpha = 0$. We have shown that the operator $\mathcal{O} = \frac{1}{2} A_\mu^a A^{\mu a} + \alpha \bar{c}^a c^a$ is indeed renormalizable to all orders of perturbation theory in the Curci-Ferrari gauge. We also briefly discussed the generalization of this operator to the MAG supplemented with the Abelian Landau gauge fixing.

We made use of a set of external sources for the relevant operators so that the BRST transformation s was no longer nilpotent, s^2 being related to the $SL(2, \mathbb{R})$ symmetry. However, the fact that $s^2 \neq 0$ does not prevent to prove the renormalizability. In Chapter 12, we employed an alternative set of sources, capable of maintaining a nilpotent BRST transformation.

We also found relations between the anomalous dimension of $\mathcal{O}$ in case of the Curci-Ferrari gauge and MAG. In case of the Curci-Ferrari gauge, the relation is however not that useful as the relation involves the anomalous dimension of another composite operator, being $gf^{abc}c^a c^b$. We have numerically verified the obtained relation up to three-loop order. In the case of the MAG, the relation is of more practical use as it expresses the anomalous dimension of the operator in terms of that of the diagonal ghost and the β -function. This further simplification is due to an additional Ward identity present in the MAG, the *diagonal ghost Ward identity* [72]. The details of the MAG can be found in Chapter 12.

Perhaps, it is worth drawing the attention to the fact that the operator $\mathcal{O}$ is no longer identifiable with a gauge invariant quantity as it was the case for $\langle A_\mu^2 \rangle$ and $\langle A_\mu^2 \rangle_{\min}$ in the Landau gauge. So, although we are able to generalize the operator A_μ^2 , we cannot establish a direct link with a gauge invariant operator. Nevertheless, it is a truly remarkable property that such a nontrivial operator is renormalizable to any order of perturbation theory.

Let us also mention a property the operator $\mathcal{O}$ in the Curci-Ferrari gauge and MAG does share with A_μ^2 in the Landau gauge, namely the on-shell BRST invariance¹¹. This on-shell invariance can be translated into a functional way, giving rise to a Ward identity, see Chapter 12. We repeat here the importance of having a sufficient number of Ward identities to constrain the most general form of the counterterms. Chapter 8 is devoted to the extension of the LCO formalism itself to the Curci-Ferrari gauge to investigate the condensation of the operator $\mathcal{O}$. We did not push the calculations to an effective determination

¹¹On-shell means that the equations of motion of the auxiliary b -field may be used.

of the potential, as the Curci-Ferrari gauge was mainly studied preparative for the MAG. The condensation of the dimension two operator in the MAG would give a dynamical off-diagonal gluon mass, thereby serving as an indication of Abelian dominance. However, the Curci-Ferrari gauge did learn us some things. In the Landau gauge, there is obviously no gauge parameter present. As the considered LCO is not gauge invariant, one might wonder what will happen with physical, thus gauge invariant quantities. For our research, the physical quantity we are immediately faced with is the minimum of the effective potential, i.e. the vacuum energy E . We have provided a argument that the vacuum energy should not change formally if the gauge parameter α is varied amongst the Curci-Ferrari gauge.

The following issue is related to the presence of the gauge parameter. Considering the Landau gauge, the equation determining the LCO parameter ζ is a differential equation with g^2 as variable, which can be simply solved using the Frobenius method and an unique ζ guaranteeing multiplicative renormalizability is found. Though, when a gauge parameter α is present, the equation for ζ reduces to differential equations in α for all the coefficients ζ_i of the powers of g^2 in the series expansion of ζ , as such arbitrary integration constants can enter ζ , which in turn shall influence the effective potential. In Chapter 8, we have set these integration constants equal to zero simply on the ground that the α -dependence should not influence the eventual vacuum energy. A more profound explanation why this setting to zero is justified, can be found in Chapter 12.

1.6.6 Return of the ghost condensation.

In Chapter 8, we elaborate more on the ghost condensation. In the MAG, the presence of a quartic ghost interaction hinted at the possible condensation of ghosts, inducing the breakdown of a continuous $SL(2, \mathbb{R})$ symmetry. As the quartic interaction and the $SL(2, \mathbb{R})$ symmetry are also present in the Curci-Ferrari gauge, the ghost condensation could also be expected [90].

As there is no quartic ghost interaction in the Landau gauge, there is absolutely no reason to believe in a ghost condensation in this gauge too, despite the $SL(2, \mathbb{R})$ symmetry. Surprisingly, in [91] it was shown that the LCO method allows to study the ghost condensation even in the Landau gauge.

We took a closer look at the Landau gauge in Chapter 8. This study was performed using the LCO method, which allows to deal simultaneously with the different channels in which the condensation could take place. These channels are linked by the $SL(2, \mathbb{R})$ rotations. The condensation $\langle f^{abc} \bar{c}^a c^b \rangle$ corresponds to the Overhauser channel, while $\langle f^{abc} c^a c^b \rangle$ and $\langle f^{abc} \bar{c}^a \bar{c}^b \rangle$ are corresponding to the BCS channel. We choose this names because of the analogy existing with the Overhauser and BCS effect in the theory of ordinary superconductivity [92, 93, 94, 95].

It becomes apparent that the simultaneous analysis of the ghost operators demands the introduction of two LCO parameters, which however turned out to be proportional to each other, due to the symmetries of the theory. We also provided a diagrammatic explanation of the relation between these two LCO parameters.

We constructed the one-loop effective potential, which is written in terms of two $SL(2, \mathbb{R})$ invariants. As the explicit calculation turned out to be rather cumbersome, we managed to reduce the effort by showing that one of these two invariants necessarily equals zero in the vacuum, at least at one-loop. Consequently, we found a lower vacuum energy due to a non-vanishing ghost condensate.

Choosing a vacuum along a certain $SL(2, \mathbb{R})$ -direction dynamically breaks this symmetry, but any $SL(2, \mathbb{R})$ rotated configuration of the vacuum has the same energy, thus there is no distinction made between e.g. the BCS or Overhauser effect.

As the BRST and ghost number symmetry are a key ingredient in perturbative gauge field theory [177], we paid a little more attention to vacua other than the Overhauser one. To situate the problem, in the BCS vacuum one has, for the ghost charged operator $f^{abc} c^a c^b$, $\langle Q_{\text{BRST}}(\dots) \rangle \propto \langle f^{abc} c^a c^b \rangle \neq 0$, thus the BRST, as well as the ghost number symmetry is broken, while in the Overhauser vacuum these

are preserved. We motivated that even in such vacua the concept of a ghost number symmetry and nilpotent BRST charge exist, being the “rotated” version of the common ghost number and BRST charge.

We then concentrated on the Overhauser vacuum. As the continuous $SL(2, \mathbb{R})$ symmetry is broken, the Goldstone theorem predicts the presence of massless bosons. It can however be argued that these will belong to the unphysical sector of the theory, using a BRST argument due to Kugo and Ojima [177].

A consequence of the ghost condensation is the different behaviour of the ghosts in function of the color, due to the absence of the global $SU(N)$ symmetry, which is indeed broken due to a condensate like $\langle f^{abc} \bar{c}^a c^b \rangle$. However, if we consider the expression for the global color charge Q_c in the unbroken case, it reduces, under certain conditions¹², to a BRST exact form, i.e. $Q_{\text{BRST}}(\dots)$. Thus, although Q_c no longer generates a global symmetry of the action, the action of Q_c on physical states¹³ will vanish. Said otherwise, the breaking of $SU(N)$ is located in an unphysical sector.

1.6.7 Mass dimension two gluon condensate in linear covariant gauges.

Until now, we investigated the condensation of A_μ^2 in the Landau gauge and its generalization in the Curci-Ferrari gauge, while we proved already the renormalizability in the MAG. The operator turned out to be on-shell BRST invariant every time. One could imagine that this invariance would be a *conditio sine qua non*.

There is however another, perhaps the most familiar one, class of covariant gauge fixings: the linear covariant gauges, including the Landau and Feynman gauge. Would a mass dimension two condensation occur in these gauges too? The answer is yes. We proposed the operator A_μ^2 , which is then *not* even on-shell BRST invariant. Nevertheless, we are still able to prove the renormalizability to all orders of perturbation theory, we calculated the two-loop anomalous dimension¹⁴ of A_μ^2 and the one-loop effective potential, once more finding that $\langle A_\mu^2 \rangle \neq 0$.

Although we are able to repeat our proof of gauge parameter independence of the vacuum energy for the linear covariant gauges, the explicit results for various choices of the gauge parameter do not stroke with it. To our understanding, this mismatch between theory and practice is due to a mixing of different orders of perturbation theory at finite order, and should not appear at infinite order precision. We propose a way to reduce the problem at finite order approximation, and solve for the tree level mass and vacuum energy.

The aforementioned issues can be found in Chapters 10 and 11.

1.6.8 Off-diagonal mass generation for Yang-Mills theories in the MAG.

With our knowledge gained from the investigation of relatively simple gauges, we are finally equipped to tackle the MAG, supplemented with a Landau gauge fixing for the Abelian degrees of freedom. The results are presented in Chapter 12.

We examined the renormalizability, we determined the one-loop effective potential and did find a non-trivial value for the off-diagonal gluon mass, while the diagonal gluons remains massless. These results are in qualitative agreement with the lattice simulations of the MAG [49, 50]. Let us also point out that some authors invoked the off-diagonal mass to construct effective low energy actions for Yang-Mills theories in the MAG, see e.g. [96, 97, 98, 99, 100, 101, 102, 103].

¹²Conditions that are in fact the same in the absence of any ghost condensate.

¹³We remember the reader that these are defined as BRST closed.

¹⁴For general gauge parameter, there is no relation between this anomalous dimension and other, more elementary RG functions.

Another issue we were faced with is the following: all the previously investigated gauges, possess the Landau gauge as a limiting case. Therefore, the vacuum energy due to dimension two condensate is related in any of these gauges. Unfortunately, the MAG clearly does not possess the Landau gauge as a special case and as such, there does not seem to be a connection between the respective dimension two condensates in these two gauges. We managed to construct a gauge that does not only interpolates between the MAG and Landau, but does allow to introduce an interpolating local composite operator as well. This construction involved the introduction of an extra gauge parameter. We showed the renormalizability to all orders of perturbation theory of this interpolating gauge, and it turned out that the extra gauge parameter does not renormalize independently from the other fields/parameters of the model. Henceforth, we made a connection between the MAG, Landau and as such also between the linear covariant and Curci-Ferrari gauges.

1.6.9 Small intermezzo on renormalization schemes.

Let us pause here a little while by spending some words on the issue of the choice of renormalization scheme. During most of our work, we relied on the $\overline{\text{MS}}$ -the modified minimal subtraction- scheme. This is a popular scheme as it is also a very efficient scheme for performing actual computations.

Most of the time, we have found that the relevant expansion parameter, being $\frac{g^2 N}{16\pi^2}$ for gluodynamics, is sufficiently small¹⁵ to speak about qualitatively reliable results, by which we mean we do not expect spectacularly different results at higher order or in other renormalization schemes.

When studying the Gross-Neveu model, as a nonperturbative technique like the LCO or $2PPI$ method can capture all the relevant nonperturbative information regarding the mass gap formation, it is worth reducing the dependence on the renormalization scheme in order to get better numerics. There are several techniques at hand to do such an optimization, see [104, 105, 24, 106] for additional details. In essence, these are based on replacing the relevant quantities by their scheme and scale invariant counterparts¹⁶. We shall not explain such an approach here, we refer to part II.

In the case of gauge theories, performing such an optimization is time consuming and perhaps not very productive, as there are many other sources of nonperturbative effects. Our efforts were meant to get an idea of the order of magnitude of some effects, and this by using an inconclusive perturbative approach. For example, in all the investigated gauges, for the $\overline{\text{MS}}$ estimate of the tree level gluon mass stemming from the condensation of the mass dimension two operator, we found values in the range of a few hundred MeV, with the $\overline{\text{MS}}$ expansion parameter smaller than one half. In the case that the $\overline{\text{MS}}$ expansion parameter would turn out to be too large, it might be worthwhile to go beyond the $\overline{\text{MS}}$ scheme.

1.6.10 The Gribov problem: gauge copies.

We recall that, in order to quantize a gauge theory using the path integral approach, a gauge fixing condition has to be imposed, in order to ensure that only one representative of each gauge equivalence class would contribute to the path integral. So far, it was always tacitly assumed that the gauge condition did select a unique representant.

Gribov illustrated that the Landau gauge does not entirely fix the gauge freedom [107]: there exist several equivalent gauge field configurations and each fulfills the Landau gauge condition $\partial_\mu A_\mu = 0$. The Gribov ambiguity is not restricted to the Landau gauge, but a common feature of non-Abelian gauge theories [108]. As a consequence of the existence of these copies, the domain of integration of

¹⁵In general, the expansion parameter should certainly be (much) smaller than 1.

¹⁶This is achieved order by order.

the path integral should be restricted further. The problem is whether this could be translated into a practical framework. It turned out that this is more or less possible in the Landau gauge, and to some extent in the noncovariant Coulomb gauge [107], but beyond these gauges, not much is known about solving the Gribov ambiguity.

At the infinitesimal level, Gribov showed that the existence of gauge copies is equivalent with the existence of zero modes of the Faddeev-Popov operator, $-\partial_\mu (\partial_\mu \delta^{ab} + g f^{acb} A_\mu^c)$. Hence, he proposed in [107] to restrict the integration to the so-called Gribov region, where the eigenvalues of the Faddeev-Popov operator are positive¹⁷. On the border of this region, the (first) Gribov horizon, the first vanishing eigenvalue appears.

Evidently, each gauge configuration should have a representant inside the Gribov region. This was shown in [107] by Gribov for configurations near to the outer border of the Gribov region, while in general, the statement was proven in [109].

Furthermore, nothing guarantees the absence of gauge copies inside the Gribov region, and in fact it turns out copies are existing in this region [110, 111, 112]. The smaller region free of copies is called the fundamental modular region (FMR). It has been argued in [113] that expectation values evaluated within the FMR do coincide with those within the Gribov region.

Returning to problem of the restriction, Gribov performed a heuristic first order approximation in [107], nevertheless all the essential features of this restriction were already present: a massive parameter is introduced into the theory, the value of which is determined by a gap equation. The main consequence is an infrared enhancement of the ghost propagator and a suppression of the gluon propagator. Such a behaviour is in qualitative agreement with the findings on the lattice [45, 48, 114, 115, 116, 117, 118, 119, 120] or with solutions of the Schwinger-Dyson equations [121, 122, 123, 20, 124, 125, 126]. Let us also mention here that a sufficient condition in the Landau gauge for a certain confinement criterion, due to Kugo and Ojima [177], to be fulfilled, is a stronger than quadratic divergence of the ghost propagator. This feeds the belief that the Gribov ambiguity, and more particularly a solution to it, might be important for the IR dynamics of gauge theories.

The possibility of improving the definition of the path integral in the Landau gauge using the Gribov approach initiated another part of our research. We became interested in what the influence might be on the condensation of the operator A_μ^2 .

In Chapter 13, we followed closely Gribov's approach in case the possible existence of a condensate $\langle A_\mu^2 \rangle$ is taken into consideration. We came to the same conclusion: in the infrared one observes a suppression of the gluon propagator and an enhancement of the ghost propagator.

In [127, 128], Zwanziger constructed a local Lagrangian allowing to implement the restriction to the Gribov region order by order in a renormalizable fashion. This provides an answer to the yet unsolved issue of renormalizability of the Gribov approach. The restriction is explicitly imposed through the horizon condition, which is a gap equation for the massive Gribov parameter. This gap equation is derived from the effective action, calculable with the Zwanziger Lagrangian.

Chapter 13 briefly discusses the algebraic setup to prove to all orders of perturbation theory of the, à la Zwanziger, localized version of Gribov's original approximation.

In Chapter 14, we have proven the Zwanziger Lagrangian maintains its renormalizability when the operator A_μ^2 is coupled to the action following the LCO method. An interesting property of the action is that no new renormalization factors are needed due to the rich symmetry structure of the model. In particular, what could be called the LCO parameter for the massive Gribov parameter is exactly "1", a property that shows to be important in order to find the enhancement of the ghost propagator in the infrared sector. We have also given a few examples of the possible importance of the presence of the Gribov parameter.

¹⁷The Faddeev-Popov operator is Hermitian in the Landau gauge, thus the eigenvalues are real.

Using the effective action, one consequently obtains two gap equations, for the Gribov parameter as well as for the mass parameter associated to $\langle A_\mu^2 \rangle$. This allows to study the effects of the condensation of A_μ^2 on the Gribov parameter and vice versa. We obtained explicit values in the $\overline{\text{MS}}$ scheme for the Gribov parameter and for the mass parameter due to $\langle A_\mu^2 \rangle$, but the expansion parameter turned out to be larger than one. We performed an optimization of the perturbative expansion in order to reduce the dependence on the renormalization scheme. The properties of the vacuum energy, with or without the inclusion of the condensate $\langle A_\mu^2 \rangle$, are investigated. In particular, it is shown that in the original Gribov-Zwanziger formulation, i.e. without the inclusion of the operator A_μ^2 , the resulting vacuum energy is always positive at one-loop order, independently from the choice of the renormalization scheme and scale. We prove that in the $\overline{\text{MS}}$ scheme, and at one-loop order, the solution of the gap equations is necessarily one with $\langle A_\mu^2 \rangle > 0$. Without the restriction to the Gribov region, the value found for $\langle A_\mu^2 \rangle$ using the LCO formalism is negative, see e.g. [42] or Chapter 11.

It is an unfortunate finding that the vacuum energy is positive, as, through the trace anomaly¹⁸, the vacuum energy can be traced back to the value of the gluon condensate $\langle F_{\mu\nu}^2 \rangle$. In fact, a positive vacuum energy implies a negative value for the condensate $\langle \frac{g^2}{4\pi^2} F_{\mu\nu}^2 \rangle$. This is in contradiction with what is found. In “real life” QCD, with quarks present, one can extract phenomenological values for $\langle \frac{g^2}{4\pi^2} F_{\mu\nu}^2 \rangle$ via the sum rules [13], obtaining positive values for this condensate. For gluodynamics, estimates were obtained using lattice techniques, also leading to a positive value [129]. From this viewpoint, it seems to us that it would be an asset that the vacuum energy obtained from any kind of calculation is at least negative. Adding the operator A_μ^2 , opens the possibility to have a negative vacuum energy, although we are unable to come to a definite conclusion at the order considered, as the dependence on the renormalization scheme is pathologically strong.

1.6.11 Three-dimensional gauge theories.

In Chapter 16, we have considered three-dimensional gauge theories. These are physically relevant as the high temperature limit of their four-dimensional counterpart [130]. Also for lattice computations these are interesting due to the reduced simulation time [131, 115]. We have taken a first look at three-dimensional Yang-Mills theories in order to find out whether (some of) our results could also be generalized to three dimensions, e.g. the condensation of A_μ^2 .

Unfortunately, the situation is asking for more than a simple adaptation of our existing research. In three dimensions, Yang-Mills theories are *superrenormalizable* as the coupling constant g^2 itself has a mass dimension. Superrenormalizable theories are more than renormalizable in the sense that only a few “basic” UV divergent diagrams emerge. However, massless superrenormalizable theories are plagued by severe infrared problems [132], which can be easily understood. Imagine that one calculates a cross section at a certain external scale Q . As g^2 is massive, the series will necessarily be one in $\frac{g^2}{Q}$, and clearly the IR limit $Q \rightarrow 0$ causes difficulties. If a dynamical mass would be generated in some way, the IR limit could be protected, hence the interest in finding such a mechanism, see e.g. [133, 134, 135, 136].

We have set a first step towards extending the LCO formalism to three dimensions by showing that, using the Landau gauge, the insertion of the operator A_μ^2 via a mass term $\frac{1}{2}m^2 A_\mu^2$ gives a renormalizable theory to all orders of perturbation theory, while we, surprisingly, recovered that the relation for the anomalous dimension in four dimensions remains valid in three dimensions. This relation was explicitly verified using the large N_f approach, where N_f represents the number of quarks. We also briefly

¹⁸The Gribov-Zwanziger approach introduces the massive Gribov parameter γ in the Yang-Mills action, but the trace anomaly remains valid. This was discussed with great rigor in [128], but is not difficult to imagine that the extra contributions are something like $\frac{\partial \Gamma}{\partial \gamma}$, which equals zero as this is precisely the horizon condition. A similar thing can be said about the contributions from the condensation of A_μ^2 .

touched the three-dimensional Curci-Ferrari gauge. Let us end by mentioning that the theory turns out to be finite at one-loop order in dimensional regularization. We do not have yet any information beyond this order, or on a possible extension of the LCO formalism.

1.7 Conclusion of the introduction.

We hope the reader is now more or less prepared to jump to the detailed articles in part II, completing the sketchy account given above of our efforts. Needless to say, our investigations are incomplete. We postpone this discussion to the final conclusion of the thesis.

Part II

Articles

Chapter 2

The mass gap and vacuum energy of the Gross-Neveu model via the $2PPI$ expansion

*D. Dudal and H. Verschelde (UGent),
published in Physical Review D **67** (2003) 025011.*

We introduce the $2PPI$ (2-point-particle-irreducible) expansion, which sums bubble graphs to all orders. We prove the renormalizability of this summation. We use it on the Gross-Neveu model to calculate the mass gap and vacuum energy. After an optimization of the expansion, the final results are qualitatively good.

2.1 Introduction.

The Gross-Neveu (GN) model [21] is plagued by infrared renormalons. The origin of this problem lies in the fact that we perturb around an instable (zero) vacuum. A remedy would be the mass generation of the particles, connected to a non-perturbative, lower value of the vacuum energy. Such a dynamical mass must be of a non-perturbative nature, since the GN Lagrangian possesses a discrete chiral symmetry. A dynamical mass is closely related to a nonzero vacuum expectation value (VEV) for a local composite operator (i.e. $\bar{\psi}\psi$). This condensate introduces a mass scale into the model. We consider GN because the exact mass gap [25] and vacuum energy [26] are known. This allows a test for the reliability of approximative frameworks before attention is paid to dynamical mass generation in more complex theories like $SU(N)$ Yang-Mills [42]. The last few years, several methods have been proposed to solve this problem and get non-perturbative information out of the model [27, 24, 106].

In this paper, we address another approach, the so-called $2PPI$ expansion. Its first appearance and use for analytical finite temperature research can be found in [61, 62, 63, 64, 137]. In section 2.2, we give a new derivation of the expansion. Section 2.3 is devoted to the renormalization of the $2PPI$ technique. Preliminary numerical results, using the $\overline{MS}$ scheme, are presented in section 2.4. We recover the $N \rightarrow \infty$ approximation, but we encounter the problem that the coupling is infinite. In section 2.5 we optimize the $2PPI$ technique. We rewrite the expansion in terms of a scheme and scale independent mass parameter M . The freedom in coupling constant renormalization is reduced to a single parameter b_0 by a reorganization of the series. We discuss how to fix b_0 . Numerical results can be found in section

2.6. We also give some evidence to motivate why results are acceptable. We end with conclusions in section 2.7.

2.2 The 2PPI expansion.

We start from the (unrenormalized) GN Lagrangian in two-dimensional Euclidean space time.

$$\mathcal{L} = \bar{\psi}\not{\partial}\psi - \frac{1}{2}g^2 (\bar{\psi}\psi)^2 \quad (2.1)$$

This Lagrangian has a global $U(N)$ invariance and a discrete chiral symmetry $\psi \rightarrow \gamma_5 \psi$ which imposes $\langle \bar{\psi}\psi \rangle = 0$ perturbatively. This model is asymptotically free and has spontaneous chiral symmetry breaking. As such, it is a toy model which mimics QCD in some ways.

First of all, we focus on the topology of vacuum diagrams. We can divide them in 2 disjoint classes:

- Those diagrams falling apart in 2 separate pieces when 2 lines meeting at the same point x are cut. We call those *2-point-particle-reducible* or *2PPR*. x is named the *2PPR insertion point*. Figure 2.1 depicts the most simple 2PPR vacuum bubble.

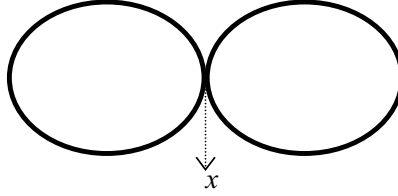


Figure 2.1: A 2PPR vacuum bubble. x is the 2PPR insertion point.

- The other type is the complement of the 2PPR class, we baptize such diagrams *2-point-particle-irreducible* (2PPI) diagrams. Figure 2.2 shows a 2PPI bubble.

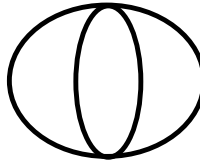


Figure 2.2: A 2PPI vacuum bubble.

We could now remove all 2PPR bubbles from the diagrammatic sum building up the vacuum energy by summing them in an effective mass. To proceed, we must use a little trick. Let us define

$$\Delta = \langle \bar{\psi}\psi \rangle = \langle \bar{\psi}_i\psi_i \rangle \quad (2.2)$$

where the index $i = 1 \dots 2N$ goes over space as well as internal values. Obviously, we have

$$\Delta_{ij} \equiv \langle \bar{\psi}_i\psi_j \rangle = \delta_{ij} \frac{\Delta}{2N} \quad (2.3)$$

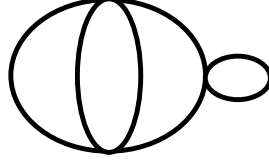


Figure 2.3: Generic vacuum bubble.

Figure 2.4: Diagrammatic depiction of $\frac{d}{dg^2}$ (fat dot) applied on the bubble of Figure 2.4.

We now calculate $\frac{dE}{dg^2}$ where E is the vacuum energy. The g^2 derivative can hit a 2PPR vertex or a 2PPI vertex. (see Figure 2.3 and Figure 2.4) In the first case, we have diagrammatically the contribution

$$-\frac{1}{2}\Delta_{ij}(\delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk})\Delta_{kl} = -\frac{1}{2}\left(1 - \frac{1}{2}\right)\Delta^2 \quad (2.4)$$

In the second case, we can unambiguously subdivide the vacuum diagram in one maximal 2PPI part, which contains the vertex hit by $\frac{d}{dg^2}$, and one or several 2PPR parts which can be deleted and replaced by an effective mass $\bar{m}$. A simple diagrammatical argument gives

$$\bar{m}\delta_{ij} = -g^2(\delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk})\Delta_{kl} \quad (2.5)$$

or

$$\bar{m} = -g^2\Delta\left(1 - \frac{1}{2N}\right) \quad (2.6)$$

Summarizing, we have

$$\frac{dE}{dg^2} = -\frac{1}{2}\Delta^2\left(1 - \frac{1}{2N}\right) + \frac{\partial E_{2PPI}}{\partial g^2}(\bar{m}, g^2) \quad (2.7)$$

The g^2 dependence in E_{2PPI} comes from the 2PPI vertices. To integrate (2.7), we use the Ansatz

$$E(g^2) = E_{2PPI}(\bar{m}, g^2) + cg^2\Delta^2 \quad (2.8)$$

with c a constant to be determined. Using (2.6), we find from (2.8)

$$\frac{dE}{dg^2} = \frac{\partial E_{2PPI}}{\partial g^2}(\bar{m}, g^2) + \frac{\partial E_{2PPI}}{\partial \bar{m}}\left(-\Delta\left(1 - \frac{1}{2N}\right) - g^2\frac{d\Delta}{dg^2}\left(1 - \frac{1}{2N}\right)\right) + c\Delta^2 + 2cg^2\Delta\frac{d\Delta}{dg^2} \quad (2.9)$$

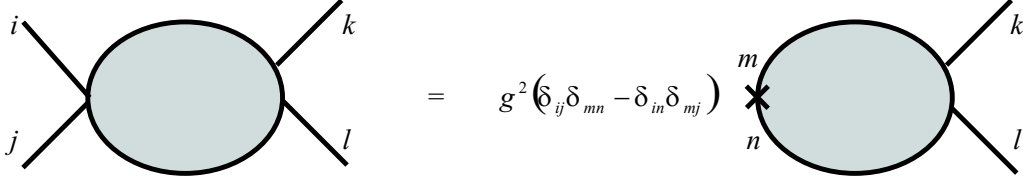


Figure 2.5: A diagrammatic identity.

A simple diagrammatic argument gives

$$\frac{\partial E_{2PPI}}{\partial \bar{m}}(\bar{m}, g^2) = \Delta \quad (2.10)$$

This is a (local) gap equation, summing the bubble graphs into $\bar{m}$. Using (2.10) and comparing (2.9) with (2.7), we find $c = \frac{1}{2} \left(1 - \frac{1}{2N}\right)$, so that we finally have that

$$E(g^2) = \frac{1}{2} g^2 \left(1 - \frac{1}{2N}\right) \Delta^2 + E_{2PPI}(\bar{m}, g^2) \quad (2.11)$$

It is easy to show that the following equivalence hold.

$$\frac{\partial E_{2PPI}}{\partial \bar{m}} = \Delta \Leftrightarrow \frac{\partial E}{\partial \bar{m}} = 0 \quad (2.12)$$

One should not confuse (2.12) with the usual procedure of minimizing an effective potential $V(\varphi)$ with respect to the field variable φ . First of all, $\bar{m}$ is not a field variable. Secondly, the expression for E in terms of the 2PPI expansion is only correct if the gap equation is fulfilled.

2.3 Renormalization of the 2PPI expansion.

Up to now, we have not paid any attention to divergences. We will now show that an equation such as (2.11) is valid for the vacuum energy E with fully renormalized and finite quantities. Since in the original Lagrangian there is no mass counterterm, one could naively expect problems with the non-perturbative mass $\bar{m}$, which generates mass renormalization in E_{2PPI} . Another possible problem is vacuum energy renormalization. Perturbatively, the vacuum energy is zero and hence no vacuum energy renormalization is needed. Non-perturbatively, we expect logarithmic divergences proportional to $\bar{m}^2$ for E_{2PPI} . As we will show, both these problems are solved with coupling constant renormalization.

The trick is to separate the contribution of the coupling constant renormalization counterterm $-\frac{1}{2} \delta Z_4 (\bar{\psi} \psi)^2$ into 2PPR and 2PPI parts, corresponding with the topology of the original divergent subgraphs. Let i and j be the indices carried by the lines meeting at the 2PPR vertex, then we have

$$\delta Z_4 (\delta_{ij} \delta_{kl} - \delta_{il} \delta_{kj}) = \delta Z_{4;ij,kl}^{2PPI} + \delta Z_{4;ij,kl}^{2PPR} \quad (2.13)$$

Note that crossing will change a 2PPR part into a 2PPI part.

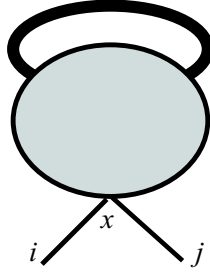


Figure 2.6: Divergent subgraph containing the 2PPR vertex x . Fat lines denote full propagators.

Because of the diagrammatical identity shown in Figure 2.5, with ${}^m\text{---}x\text{---}^n$ a $\bar{\psi}_m\psi_n$ insertion, we have a relation between the 2PPR part of coupling constant and mass renormalization.

$$\delta Z_{4;ij,kl}^{2PPR} = (\delta_{ij}\delta_{mn} - \delta_{in}\delta_{mj}) \delta Z_{2;mn,kl} \quad (2.14)$$

This identity can be used to show that the divergent effective mass $\bar{m}$, given by (2.6), gets replaced by a finite renormalized mass $\bar{m}_R = Z_2\bar{m} = -g^2 \left(1 - \frac{1}{2N}\right) \Delta_R$, where $\Delta_R = Z_2\Delta$ is the finite, renormalized expectation value of the composite operator $\bar{\psi}\psi$. Indeed, let us consider a generic 2PPR subgraph or bubble graph with a 2PPR vertex x . The divergent subgraphs of this bubble graph, which do not contain x , can be made finite by the usual counterterms for wavefunction and coupling constant renormalization. The resulting effective mass will be given by (2.6), but now with $\Delta = \langle \bar{\psi}\psi \rangle$ evaluated with the full Lagrangian, i.e. including counterterms. We still have to consider the subgraphs of the bubble graph which do contain the 2PPR vertex x . They can be made finite by coupling constant renormalization, but because the subgraph is 2PPR at x , only the 2PPR part of the counterterm has to be inserted and we get the contribution (see Figure 2.6)

$$-g^2 \delta Z_{4;ij,kl}^{2PPR} \Delta_{kl} = -g^2 (\delta_{ij}\delta Z_{2;mm,kk} - \delta Z_{2;ij,kk}) \frac{\Delta}{2N} \quad (2.15)$$

where use was made of (2.3) and (2.14). Since for a diagonal mass matrix, we can define δZ_2 by

$$\delta Z_2 \delta_{kl} = \delta Z_{2;mm,kl} \quad (2.16)$$

we have

$$2N\delta Z_2 = \delta Z_{2;mm,kk} \quad (2.17)$$

and

$$\delta Z_{2;ij,kk} = \delta Z_2 \delta_{ij} \quad (2.18)$$

After substitution of (2.17) and (2.18) into (2.15), we find that the contribution of the 2PPR counterterm insertion gives

$$-\delta_{ij} g^2 \left(1 - \frac{1}{2N}\right) \delta Z_2 \Delta \quad (2.19)$$

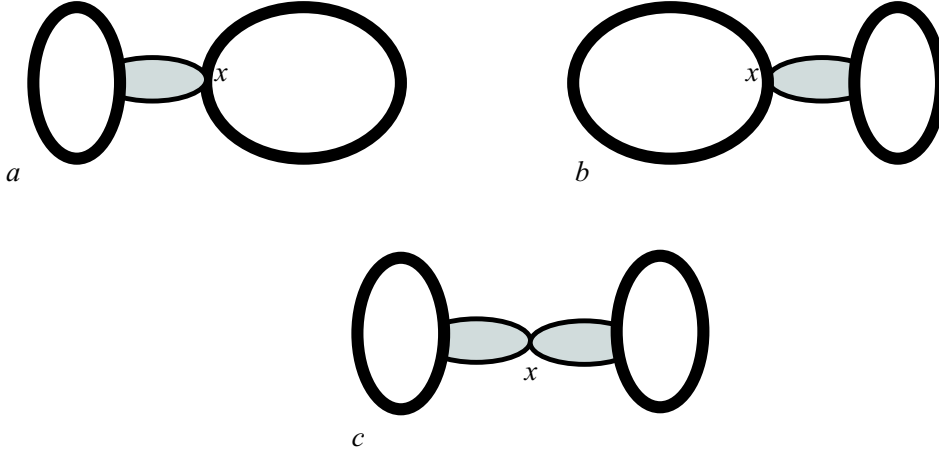


Figure 2.7: Divergent subgraphs of $\frac{dE}{dg^2}$ containing the 2PPR vertex x .

and hence a mass renormalization

$$\delta\bar{m} = -g^2 \left(1 - \frac{1}{2N}\right) \delta Z_2 \Delta \quad (2.20)$$

so that we obtain a finite, effective renormalized mass

$$\bar{m}_R = Z_2 \bar{m} = -g^2 \left(1 - \frac{1}{2N}\right) \Delta_R \quad (2.21)$$

with $\Delta_R = Z_2 \Delta = Z_2 \langle \bar{\psi}\psi \rangle$ the finite, renormalized VEV of the composite operator $\bar{\psi}\psi$.

To obtain a finite, renormalized expression for the vacuum energy as a function of Δ_R or $\bar{m}_R$, we have to use the same trick as in the unrenormalized case and consider the renormalization of $\frac{dE}{dg^2}$. Let us first consider the case when the vertex x hit by $\frac{d}{dg^2}$ is a 2PPR vertex and restrict ourselves to divergent subgraphs which contain x (the ones not containing x pose no problem and simply replace the original Δ evaluated without counterterms by Δ with counterterms included). The divergent subgraph can just end at x from the left or the right (Figure 2.7a and 2.7b) or the 2PPR vertex x can be embedded in it (Figure 2.7c). Graphs 2.7a and 2.7b can be made finite by the 2PPR part of the coupling constant counterterm and making use of (2.14), their renormalization contributes

$$(7a) + (7b) = 2 \left(-\frac{1}{2}\right) \Delta_{kl} \delta Z_{2;ij,kl} (\delta_{ij} \delta_{pq} - \delta_{iq} \delta_{pj}) \Delta_{pq} = -\left(1 - \frac{1}{2N}\right) \delta Z_2 \Delta^2 \quad (2.22)$$

where we have used (2.3), (2.14) and (2.17). Graph 2.7c can be made finite with that part of coupling constant renormalization that factorizes at the 2PPR vertex x . Its renormalization therefore contributes

$$(7c) = -\frac{1}{2} \Delta_{kl} (\delta Z_{2;ii,kl} \delta Z_{2;jj,pq} - \delta Z_{2;ij,kl} \delta Z_{2;ij,pq}) \Delta_{pq} = -\frac{1}{2} \left(1 - \frac{1}{2N}\right) \delta Z_2^2 \Delta^2 \quad (2.23)$$

where we made use of (2.17) and (2.18). Adding the counterterm contributions (2.22) and (2.23) to the original unrenormalized expression (2.4), we obtain $-\frac{1}{2} \left(1 - \frac{1}{2N}\right) (Z_2 \Delta)^2 = -\frac{1}{2} \left(1 - \frac{1}{2N}\right) \Delta_R^2$, which is finite. When $\frac{d}{dg^2}$ hits a 2PPI vertex, we can unambiguously subdivide the vacuum diagrams in a

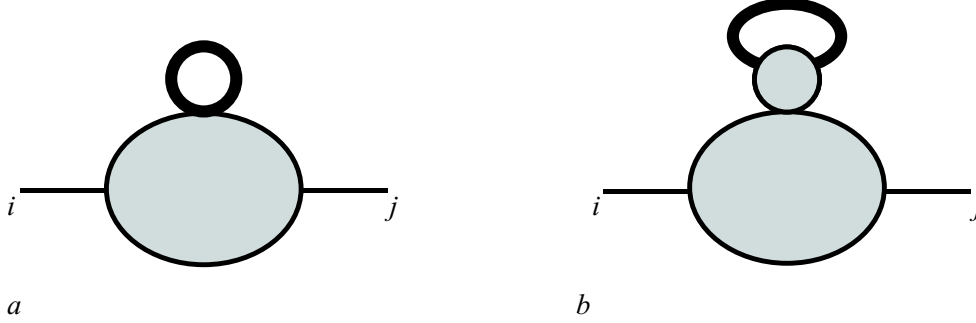


Figure 2.8: Selfenergy subgraphs needing mass renormalization.

maximal 2PPI part, which contains the vertex hit by $\frac{d}{dg^2}$, and one or more 2PPR bubble insertions which, after renormalization, can be replaced by the effective renormalized mass $\bar{m}_R$. We therefore have

$$\frac{dE}{dg^2} = -\frac{1}{2} \left(1 - \frac{1}{2N}\right) \Delta_R^2 + \frac{\partial E_{2PPI}}{\partial g^2} (\bar{m}_R, g^2) \quad (2.24)$$

We still have to show that the usual counterterms make $\frac{\partial E_{2PPI}}{\partial g^2} (\bar{m}_R, g^2)$ finite. The non-perturbative mass $\bar{m}_R$, running in the propagatorlines, will now generate selfenergies which require mass renormalization, which is not present in the original Lagrangian. Again coupling constant renormalization will solve the problem. Let us consider a generic selfenergy subgraph which needs mass renormalization. Since the divergence is linear in $\bar{m}_R$, we can restrict ourselves to 2PPI diagrams with only one 2PPR bubble insertion (Figure 2.8). The divergent part of this subgraph, that one wants to renormalize, can end at the 2PPR vertex (Figure 2.8a) or can continue throughout the 2PPR bubble (Figure 2.8b). In the first case, one needs the 2PPR part of coupling constant renormalization which contains only one 2PPR vertex (because the divergent part considered belongs to the 2PPI part of the diagram). We obviously have

$$\delta Z_{4;ij,kl}^{2PPR,1} = (\delta_{ij}\delta_{mn} - \delta_{in}\delta_{mj}) \delta Z_{2;mn,kl}^{2PPI} \quad (2.25)$$

so that the counterterm contribution is

$$(8a) = -g^2 \Delta_{kl} \delta Z_{4;kl,ij}^{2PPR,1} = -g^2 \Delta \left(1 - \frac{1}{2N}\right) \delta Z_2^{2PPI} \delta_{ij} \quad (2.26)$$

where use was made of (2.16) and (2.25).

In the second case, the divergence factorizes into a 2PPR coupling constant renormalization part (the bubble graph part) and a 2PPI mass renormalization part, so that the counterterm contribution is

$$(8b) = -g^2 \Delta_{kl} \delta Z_{4;mn,kl}^{2PPR} \delta Z_{2;mn,ij}^{2PPI} = -g^2 \Delta \left(1 - \frac{1}{2N}\right) \delta Z_2 \delta Z_2^{2PPI} \delta_{ij} \quad (2.27)$$

Adding both contributions, the relevant parts of the coupling constant counterterms give

$$(8a) + (8b) = -g^2 Z_2 \Delta \left(1 - \frac{1}{2N}\right) \delta Z_2^{2PPI} \delta_{ij} = \bar{m}_R \delta Z_2^{2PPI} \delta_{ij} \quad (2.28)$$

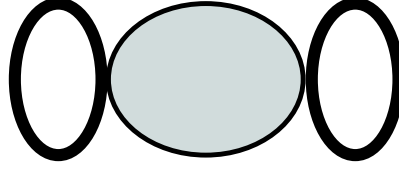


Figure 2.9: Coupling constant renormalization graph with 2 2PPR vertices.

which is exactly what we need for mass renormalization in E_{2PPI} .

In an analogous way, we can consider the logarithmic overall divergences of the vacuum diagrams which are quadratic in $\overline{m}_R$. We now consider 2PPI vacuum diagrams with two bubble insertions. One type of coupling constant renormalization subgraphs end at both 2PPR vertices (Figure 2.9). They can be renormalized by the corresponding 2PPR part of the coupling constant renormalization counterterm.

$$\delta Z_{4;ij,kl}^{2PPR,2} = g^2 (\delta_{ij}\delta_{mn} - \delta_{in}\delta_{mj}) \delta \zeta_{mn,rs}^{2PPI} (\delta_{rs}\delta_{kl} - \delta_{rl}\delta_{ks}) \quad (2.29)$$

where $\delta \zeta_{mn,rs}$ is the overall divergent part of $\langle \overline{\psi}_m \psi_n \overline{\psi}_r \psi_s \rangle$. Adding the contributions from coupling constant renormalization graphs which also go through the bubble parts, we find

$$\delta E_{2PPI} = \frac{1}{2} \delta Z_{4;ij,kl}^{2PPR,2} \Delta_{ij}^R \Delta_{kl}^R = \frac{1}{2} \overline{m}_R^2 \delta \zeta^{2PPI} \quad (2.30)$$

with

$$\delta \zeta^{2PPI} = \delta \zeta_{mm,nn}^{2PPI} \quad (2.31)$$

and use was made of (2.29). Again coupling constant renormalization provides us with the necessary additive renormalization of the 2PPI vacuum energy. Furthermore, completely analogous arguments can be used to show that the unrenormalized gap equation (2.10) gets renormalized to

$$\frac{\partial E_{2PPI}}{\partial \overline{m}_R} (\overline{m}_R, g) = \Delta_R \quad (2.32)$$

It is clear that the 2PPI coupling constant and wave function renormalization subgraphs can be renormalized with the original counterterms. We therefore conclude that $\frac{\partial E_{2PPI}}{\partial g^2} (\overline{m}_R, g)$ is finite and hence (2.24) is finite and can be integrated. Making use of the gap equation (2.32), we find

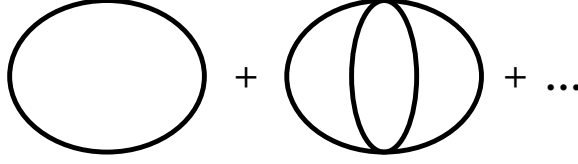
$$E(g^2) = \frac{1}{2} g^2 \left(1 - \frac{1}{2N} \right) \Delta_R^2 + E_{2PPI}(\overline{m}_R, g^2) \quad (2.33)$$

Of course, we also have the equivalence (2.12) in the renormalized case.

For the rest of the paper, it is implicitly understood we are working with renormalized quantities, so that we can drop the R -subscripts.

2.4 Preliminary results for the mass gap and vacuum energy.

Figure 2.10 shows the first terms in the loop expansion for E_{2PPI} . Restricting ourselves to the one-loop

**Figure 2.10:** E_{2PPI} .

vacuum bubble, we have in dimensional regularization with $d = 2 - \varepsilon$,

$$E_{2PPI} = -2N\bar{m}^2 \frac{1}{2-\varepsilon} \mu^\varepsilon \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + \bar{m}^2} \quad (2.34)$$

Using the $\overline{\text{MS}}$ scheme, we arrive at

$$E = \frac{1}{2}g^2 \left(1 - \frac{1}{2N}\right) \Delta^2 + \frac{N}{4\pi} \bar{m}^2 \left(\ln \frac{\bar{m}^2}{\bar{\mu}^2} - 1\right) \quad (2.35)$$

The gap equation (2.32) gives

$$\frac{N\bar{m}}{2\pi} \ln \frac{\bar{m}^2}{\bar{\mu}^2} = \bar{\Delta} \quad (2.36)$$

Consequently, the vacuum energy is expressed by

$$E = -\frac{N}{4\pi} \bar{m}^2 \quad (2.37)$$

At one-loop order, we have

$$\bar{g}^2(\bar{\mu}) = \frac{1}{\beta_0 \ln \frac{\bar{\mu}^2}{\Lambda_{\overline{\text{MS}}}^2}} \quad (2.38)$$

where β_0 is the leading order coefficient of the $\bar{\beta}$ -function

$$\bar{\mu} \frac{\partial \bar{g}^2}{\partial \bar{\mu}} = \bar{\beta}(\bar{g}^2) = -2 (\beta_0 \bar{g}^4 + \beta_1 \bar{g}^6 + \bar{\beta}_2 \bar{g}^8 + \dots) \quad (2.39)$$

The values of the coefficients can be found in [138, 139, 140, 22]

$$\beta_0 = \frac{N-1}{2\pi}, \quad \beta_1 = -\frac{N-1}{4\pi^2}, \quad \bar{\beta}_2 = -\frac{(N-1)(N-\frac{7}{2})}{16\pi^3} \quad (2.40)$$

To get a numerical value for the mass gap¹, we have to choose the subtraction scale $\bar{\mu}$. The choice immediately coming to mind is setting $\bar{\mu} = \bar{m}$, which eliminates the potentially large logarithm present in (2.35). Doing so, we find, next to the perturbative solution $\bar{m} = 0$,

$$m = \bar{m} = \Lambda_{\overline{\text{MS}}} \quad (2.41)$$

¹At one-loop $2PPI$ order, there is no mass renormalization, hence $\bar{m}$ is the physical mass.

N	P	Q	$P_{1/N}$	$Q_{1/N}$
2	-46.3%	-	-21.9%	-
3	-32.5%	-6.7%	-12.2%	5.8%
4	-24.2%	-8.0%	-7.0%	1.3%
5	-19.1%	-7.2%	-4.5%	0.4%
6	-15.8%	-6.2%	-3.1%	0.1%
7	-13.5%	-5.5%	-2.3%	0.007%
8	-11.7%	-4.8%	-1.8%	-0.03%
9	-10.4%	-4.3%	-1.4%	-0.04%
10	-9.3%	-3.9%	-1.1%	-0.04%
20	-4.6%	-2.0%	-0.3%	-0.02%

Table 2.1: one-loop results for mass gap and vacuum energy.

while

$$E = -\frac{N}{4\pi} \Lambda_{\overline{\text{MS}}}^2 \quad (2.42)$$

The exact mass gap is given by [25]

$$m_{\text{exact}} = (4e)^{\frac{1}{2N-2}} \frac{1}{\Gamma\left(1 - \frac{1}{2N-2}\right)} \Lambda_{\overline{\text{MS}}} \quad (2.43)$$

while the exact vacuum energy is [26]

$$E_{\text{exact}} = -\frac{1}{8} m_{\text{exact}}^2 \cot\left(\frac{\pi}{2(N-1)}\right) \quad (2.44)$$

We expect that the error on E consists of the error on the mass *squared* and the error on the function multiplying that mass squared. Therefore we will consider the quantity $\sqrt{-E}$ to test the reliability of our results. We define the deviations in terms of percentage P and Q , i.e.

$$P = 100 \frac{m_{\text{eff}} - m_{\text{exact}}}{m_{\text{exact}}} \quad (2.45)$$

$$Q = 100 \frac{\sqrt{-E} - \sqrt{-E_{\text{exact}}}}{\sqrt{-E_{\text{exact}}}} \quad (2.46)$$

Looking at Table 2.1, we notice that our results² are quite acceptable. We notice there is convergence ($P \rightarrow 0$ and $Q \rightarrow 0$) to the exact result in case of $N \rightarrow \infty$. In fact, we recovered the $N \rightarrow \infty$ approximation. For comparison, we also displayed the next to leading results³, given by expanding (2.43) and (2.44) in powers of $1/N$.

$$m_{1/N} = \left(1 + \frac{1 - \gamma_E + \ln 4}{2N} + \mathcal{O}\left(\frac{1}{N^2}\right)\right) \Lambda_{\overline{\text{MS}}} \quad (2.47)$$

$$E_{1/N} = \left(-\frac{N}{4\pi} + \frac{\gamma_E - \ln 4}{4\pi} + \mathcal{O}\left(\frac{1}{N}\right)\right) \Lambda_{\overline{\text{MS}}}^2 \quad (2.48)$$

² $Q(2)$ is not defined since $E_{\text{exact}}(2) = 0$. We have $E(2) = -0.16\Lambda_{\overline{\text{MS}}}^2$.

³ $E_{1/N}(2) = -0.22\Lambda_{\overline{\text{MS}}}^2$.

where $\gamma_E \approx 0.577216$ is the Euler-Mascheroni constant.

However, the choice $\bar{\mu} = \bar{m} = \Lambda_{\overline{\text{MS}}}$ cannot satisfy us, since we are expanding in $\bar{g}^2(\bar{m}) = \infty$. We may have qualitatively good results, but for a field theory where the exact results are unknown, $\bar{g}^2 = \infty$ gives by no means an indication about how trustworthy our approximations are. It is clear we must find a better method to achieve results with the $2PPI$ expansion.

2.5 Optimization and two-loop corrections.

2.5.1 Renormalization group equation for E .

A standard approach to get better results is the usage of the renormalization group equation (RGE). We already mentioned that E cannot be treated on equal footing with an effective potential due to the demand that $\frac{\partial E}{\partial \bar{m}} = 0$ must hold.

Since E is the vacuum energy, it is a physical quantity and therefore, it should not depend on the subtraction scale $\bar{\mu}$. This is expressed in a formal way by means of the RGE

$$\bar{\mu} \frac{dE}{d\bar{\mu}} = 0 \quad (2.49)$$

In a perturbative series expansion, this means the differential equation (2.49) must be fulfilled order by order, when all quantities obey their running w.r.t. $\bar{\mu}$. Out of (2.16), we extract the running of $\bar{m}$, namely

$$\bar{\mu} \frac{\partial \bar{m}}{\partial \bar{\mu}} = \left(\frac{\bar{\beta}(\bar{g}^2)}{\bar{g}^2} + \bar{\gamma}(\bar{g}^2) \right) \bar{m} \equiv \bar{\kappa}(\bar{g}^2) \bar{m} \quad (2.50)$$

where $\bar{\gamma}(\bar{g}^2)$ governs the scaling behaviour of $\bar{\Delta}$

$$\bar{\mu} \frac{\partial \bar{\Delta}}{\partial \bar{\mu}} = \bar{\gamma}(\bar{g}^2) \bar{\Delta} \quad (2.51)$$

with

$$\bar{\gamma}(\bar{g}^2) = \gamma_0 \bar{g}^2 + \gamma_1 \bar{g}^4 + \gamma_2 \bar{g}^6 + \dots \quad (2.52)$$

The coefficients are given by [138, 139, 140, 22]

$$\gamma_0 = \frac{N - \frac{1}{2}}{\pi}, \quad \bar{\gamma}_1 = -\frac{N - \frac{1}{2}}{4\pi^2}, \quad \bar{\gamma}_2 = -\frac{(N - \frac{1}{2})(N - \frac{3}{4})}{4\pi^3} \quad (2.53)$$

After some calculation, we find

$$\begin{aligned} \bar{\mu} \frac{dE}{d\bar{\mu}} &= \left(\bar{\mu} \frac{\partial}{\partial \bar{\mu}} + \bar{\beta}(\bar{g}^2) \frac{\partial}{\partial \bar{g}^2} + \bar{\kappa}(\bar{g}^2) \bar{m} \frac{\partial}{\partial \bar{m}} \right) E \\ &= \frac{1}{4\pi} \frac{\bar{m}^2}{1 - \frac{1}{2N}} + \mathcal{O}(\bar{g}^2) \end{aligned} \quad (2.54)$$

It seems that E does not obey its RGE. Perturbatively, it of course fullfills the RGE up to $\mathcal{O}(g^2)$ since $\bar{m} = 0$ to all orders in perturbation theory. We must not be tempted to interpret this failure as the need to introduce some non-perturbative running coupling constant, as can be found in literature sometimes.

The nature of the *apparent* problem lies in the fact that we forgot about the gap equation $\frac{\partial E}{\partial \bar{m}} = 0$, because only then our 2PPI expression for E is meaningful. (2.36) gives that

$$\ln \frac{\bar{m}^2}{\bar{\mu}^2} \propto \frac{1}{\bar{g}^2} \quad (2.55)$$

It is easy to check that (2.55) means that all leading log terms in the expansion of E are of the order unity. Consequently, we cannot simply show order by order that $\bar{\mu} \frac{dE}{d\bar{\mu}} = 0$. The problem extends to higher orders: when we would calculate E up to a certain order n , we would need knowledge of *all* leading, subleading, ..., n^{th} leading log terms.

The above discussion reveals a possible strategy : we could do a (leading) log expansion for E_{2PPI} , with a source J coupled to $\bar{\psi}\psi$. Then we could use the RGE for E to sum all (leading) logs in E_{2PPI} . We leave this idea, because the RGE for E itself is non-linear when⁴ $J \neq 0$. This is accompanied with its own problems. A thorough discussion of this subject can be consulted in [24].

2.5.2 Optimization.

We have seen that the $\overline{\text{MS}}$ scheme is not optimal for the 2PPI expansion used on GN. We could have renormalized the coupling constant in another way and hope that this gives better results. It is easily verified that going to a scheme with coupling $\bar{g}^2$, determined at lowest order by $\bar{g}^2 = g^2 (1 + b_0 g^2)$, gives the same results as in (2.41) and (2.42), but now with $g^2 = b_0^{-1}$. This means results are as good as before, but for a sufficiently large b_0 , g^2 is small. Again, we put $\bar{\mu} = \bar{m}$ to cancel logarithms.

Till now, we kept $\bar{m}$ as the mass parameter, however we should go to another scheme for this quantity too. The results are then no longer independent of the renormalization prescriptions, i.e. if $\bar{m} = m (1 + a_0 g^2)$ at lowest order, then a_0 enters the final results, and a_0 is completely free to choose. We tackle the problem of freedom of renormalization of the coupling constant and mass parameter in 4 consecutive steps.

Step 1

First of all, we remove the freedom how the mass parameter is renormalized. We can replace $\bar{m}$ by an unique⁵ M such that M is renormalization scale and scheme independent (RSSI) [106]. Out of (2.50), we immediately deduce that

$$M = \bar{f}(\bar{g}^2) \bar{m} \quad (2.56)$$

where $\bar{f}(\bar{g}^2)$ is the solution of

$$\bar{\mu} \frac{\partial \bar{f}}{\partial \bar{\mu}} = -\bar{\kappa}(\bar{g}^2) \bar{f} \quad (2.57)$$

When we change our MRS, we have relations of the form

$$\bar{g}^2 = g^2 (1 + b_0 g^2 + b_1 g^4 + \dots) \quad (2.58)$$

$$\bar{m} = m (1 + m_0 g^2 + m_1 g^4 + \dots) \quad (2.59)$$

$$\bar{f}(\bar{g}^2) = f(g^2) (1 + f_0 g^2 + f_1 g^4 + \dots) \quad (2.60)$$

⁴We must add a source term, otherwise E cannot be treated as an effective potential in the usual sense.

⁵Up to an irrelevant (integration) constant that can be dropped.

Whenever a quantity is barred, it is understood we are considering $\overline{MS}$, otherwise we are considering an arbitrary MRS⁶. Using the foregoing relations, it is easy to show the scheme independence of M .

The explicit solution, up to the order we will need it, is given by

$$\begin{aligned} \bar{f}(\bar{g}^2) &= (\bar{g}^2)^{-1+\frac{\gamma_0}{2\beta_0}} \left\{ 1 + \frac{\bar{g}^2}{2} \left(-\frac{\beta_1\gamma_0}{\beta_0^2} + \frac{\bar{\gamma}_1}{\beta_0} \right) \right. \\ &\quad \left. + \frac{\bar{g}^4}{4} \left[\frac{1}{2} \left(-\frac{\beta_1\gamma_0}{\beta_0^2} + \frac{\bar{\gamma}_1}{\beta_0} \right)^2 + \frac{\gamma_0 \left(\frac{\beta_1^2}{\beta_0^2} - \frac{\bar{\beta}_2}{\beta_0} \right)}{\beta_0} - \frac{\beta_1\bar{\gamma}_1}{\beta_0^2} + \frac{\bar{\gamma}_2}{\beta_0} \right] \right\} \end{aligned} \quad (2.61)$$

Next, we rewrite $\bar{m}$ in terms of M by inverting (2.56)

$$\bar{m} = M(\bar{g}^2)^{1-\frac{\gamma_0}{2\beta_0}} (1 + c_1\bar{g}^2 + c_2\bar{g}^4) \quad (2.62)$$

where

$$c_1 = \frac{1}{2} \left(\frac{\beta_1\gamma_0}{\beta_0^2} - \frac{\bar{\gamma}_1}{\beta_0} \right) \quad (2.63)$$

$$c_2 = \frac{1}{8} \left(-\frac{\beta_1\gamma_0}{\beta_0^2} + \frac{\bar{\gamma}_1}{\beta_0} \right)^2 - \frac{1}{4} \left(\frac{\gamma_0 \left(\frac{\beta_1^2}{\beta_0^2} - \frac{\bar{\beta}_2}{\beta_0} \right)}{\beta_0} \right) + \frac{1}{4} \left(\frac{\beta_1\bar{\gamma}_1}{\beta_0^2} - \frac{\bar{\gamma}_2}{\beta_0} \right) \quad (2.64)$$

Step 2

Transformation (2.62) allows to rewrite E in terms of M . Since the next contribution to (2.35) is proportional to $\bar{g}^4\bar{m}^2$ (see Figure 2.10), we can rewrite E up to order $\bar{g}^2$ when (2.62) is applied. Explicitly,

$$\begin{aligned} E &= M^2(\bar{g}^2)^{2-\frac{\gamma_0}{\beta_0}} \left[\frac{N}{4\pi} (1 + 2c_1\bar{g}^2) \left(\ln \frac{M^2}{\mu^2} + \left(2 - \frac{\gamma_0}{\beta_0} \right) \ln \bar{g}^2 \right) - \frac{N}{4\pi} \right. \\ &\quad \left. + \frac{1}{2 \left(1 - \frac{1}{2N} \right)} \left(\frac{1}{\bar{g}^2} + 2c_1 + (2c_2 + c_1^2) \bar{g}^2 \right) \right] \end{aligned} \quad (2.65)$$

It is important to notice that the demand $\frac{\partial E}{\partial \bar{m}} = 0$ is translated into $\frac{\partial E}{\partial M} = 0$, because M and $\bar{m}$ differ only by an overall factor $\bar{f}$ which depends solely on $\bar{g}^2(\bar{\mu})$.

Step 3

(2.65) is still written in terms of $\bar{g}^2$. Using (2.58), we exchange $\bar{g}^2$ for g^2 , where the b_i parametrize the coupling constant renormalization. We find

$$E = M^2(g^2)^{2-\frac{\gamma_0}{\beta_0}} \left(\frac{e_{-1}}{g^2} + e_0 + e_1 g^2 \right) \quad (2.66)$$

with

$$e_{-1} = \frac{1}{2 \left(1 - \frac{1}{2N} \right)} \quad (2.67)$$

$$e_0 = -\frac{N}{4\pi} + \frac{-b_0 + 2c_1}{2 \left(1 - \frac{1}{2N} \right)} + \frac{b_0 U}{2 \left(1 - \frac{1}{2N} \right)} + \frac{N}{4\pi} V \quad (2.68)$$

⁶We notice that β_0 , β_1 and γ_0 are the same for each MRS.

$$\begin{aligned}
e_1 &= \frac{b_0^2 - b_1 + c_1^2 + 2c_2}{2\left(1 - \frac{1}{2N}\right)} + \frac{1}{2\left(1 - \frac{1}{2N}\right)} \left(b_1 U + \frac{b_0^2}{2} U(U-1) \right) \\
&+ b_0 U \left(-\frac{N}{4\pi} + \frac{-b_0 + 2c_1}{2\left(1 - \frac{1}{2N}\right)} + \frac{N}{4\pi} V \right) + \frac{N}{4\pi} (b_0 U + 2c_1 V)
\end{aligned} \tag{2.69}$$

$$U = 2 - \frac{\gamma_0}{\beta_0} \tag{2.70}$$

$$V = \ln \frac{M^2}{\bar{\mu}^2} + \left(2 - \frac{\gamma_0}{\beta_0} \right) \ln g^2 \tag{2.71}$$

Step 4

Consider (2.66). We notice that the degrees of freedom, concerning the scheme, are settled in the b_i . When we rewrite the expansion in terms of $g_{\text{one-loop}}^2$ instead of g^2 , all scheme dependence is reduced to *one* parameter, namely b_0 . This was also recognized in [106]. It is in a way more “natural” to rewrite a perturbative series in terms of $g_{\text{one-loop}}^2$, because g^2 itself is changed whenever we include the next loop order, while $g_{\text{one-loop}}^2$ of course remains the same.

The necessary formulas are given by

$$g^2(\bar{\mu}) = \frac{1}{x} - \frac{\beta_1}{\beta_0} \frac{\ln \frac{x}{\beta_0}}{x^2} + \frac{\left(\frac{\beta_1}{\beta_0}\right)^2 \left(\left(\ln \frac{x}{\beta_0} \right)^2 - \ln \frac{x}{\beta_0} \right) + \left(\frac{\beta_2}{\beta_0} - \left(\frac{\beta_1}{\beta_0} \right)^2 \right)}{x^3} + \mathcal{O}\left(\frac{1}{x^4}\right) \tag{2.72}$$

where

$$x = \frac{1}{g_{\text{one-loop}}^2} = \beta_0 \ln \frac{\bar{\mu}^2}{\Lambda^2} \tag{2.73}$$

Λ is the scale parameter of the corresponding MRS. In [141], it was shown that

$$\Lambda = \Lambda_{\overline{\text{MS}}} e^{-\frac{b_0}{2\beta_0}} \tag{2.74}$$

For β_2 , we have [105]

$$\beta_2 = (b_0^2 - b_1)\beta_0 + \beta_1 b_0 + \bar{\beta}_2 \tag{2.75}$$

Since (2.66) is correct up to order $g^2 M^2$, we can expand up to order $x^{-1} M^2$. Using (2.72), (2.74) and (2.75), the vacuum energy becomes

$$E = M^2 \left(\frac{1}{x} \right)^{2 - \frac{\gamma_0}{\beta_0}} \left(E_{-1} x + E_0 + \frac{E_1}{x} \right) \tag{2.76}$$

with

$$E_{-1} = \frac{1}{2\left(1 - \frac{1}{2N}\right)} \tag{2.77}$$

$$E_0 = -\frac{N}{4\pi} + \frac{-b_0 + 2c_1}{2\left(1 - \frac{1}{2N}\right)} + \frac{b_0 U}{2\left(1 - \frac{1}{2N}\right)} - \frac{\beta_1(U-1)L}{2\beta_0\left(1 - \frac{1}{2N}\right)} + \frac{N}{4\pi} W \tag{2.78}$$

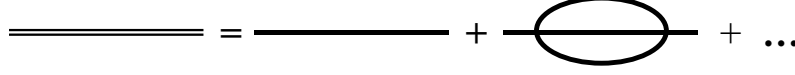


Figure 2.11: Diagrams needed to calculate m_{eff} in function of $\overline{m}$.

$$\begin{aligned}
 E_1 &= \frac{c_1^2 + 2c_2}{2\left(1 - \frac{1}{2N}\right)} + \frac{\frac{\beta_1^2}{\beta_0^2}(1+L) - \frac{b_0\beta_1 + \overline{\beta}_2}{\beta_0}}{2\left(1 - \frac{1}{2N}\right)} - \frac{N}{4\pi} \frac{\beta_1}{\beta_0} UL + \frac{b_0^2 U(U-1)}{4\left(1 - \frac{1}{2N}\right)} \\
 &+ \frac{1}{2\left(1 - \frac{1}{2N}\right)} \left[\left(-\frac{\beta_1^2}{\beta_0^2} (1+L-L^2) + \frac{b_0^2\beta_0 + b_0\beta_1 + \overline{\beta}_2}{\beta_0} \right) U + \frac{\beta_1^2 L^2 U(U-1)}{2\beta_0^2} \right] \\
 &+ b_0 U \left[\frac{N}{4\pi} (W-1) + \frac{2c_1 - b_0}{2\left(1 - \frac{1}{2N}\right)} \right] - \frac{\beta_1}{\beta_0} LU \left[\frac{N}{4\pi} (W-1) \right. \\
 &\left. + \frac{2c_1 - b_0}{2\left(1 - \frac{1}{2N}\right)} + \frac{\beta_1 L}{2\beta_0\left(1 - \frac{1}{2N}\right)} + \frac{b_0 U}{2\left(1 - \frac{1}{2N}\right)} \right] + \frac{N}{4\pi} (b_0 U + 2c_1 W) \quad (2.79)
 \end{aligned}$$

$$L = \ln \frac{x}{\beta_0} \quad (2.80)$$

$$U = 2 - \frac{\gamma_0}{\beta_0} \quad (2.81)$$

$$W = \ln \frac{M^2}{\mu^2} + \left(2 - \frac{\gamma_0}{\beta_0} \right) \ln \frac{1}{x} \quad (2.82)$$

2.5.3 Two-loop corrections.

The next order corrections are two-loop for the mass (the setting sun diagram of Figure 2.11) and three-loop for the vacuum energy (the basket ball diagram of Figure 2.2). We will restrict ourselves to two-loop corrections. The diagram displayed in Figure 2.11 gives a mass renormalization. The double line is the full propagator $S_{\text{full}}(p)$. We first employ the $\overline{\text{MS}}$ scheme again for the calculation. Let $\mathcal{P}$ be the value of the (amputated) setting sun diagram. Since

$$S(p) = \frac{1}{i\not{p} + \overline{m}} \quad (2.83)$$

we have

$$S_{\text{two-loop}}(p) = \frac{1}{i\not{p} + \overline{m} - \mathcal{P}} \quad (2.84)$$

The effective mass m_{eff} is the pole of $S_{\text{two-loop}}(p)$. From [24], we obtain

$$\mathcal{P} = \left(N - \frac{1}{2} \right) \overline{g}^4 \left(-\overline{m} I^2 + i\not{p} \frac{\varepsilon}{2 - \varepsilon} I^2 + \frac{1}{16\pi^2} (\overline{m} F_1 + i\not{p} F_2) \right) \quad (2.85)$$

where

$$I = \frac{1}{4\pi} \left[\frac{2}{\varepsilon} - \ln \frac{\bar{m}^2}{\bar{\mu}^2} + \frac{\varepsilon}{4} \left(\frac{\pi^2}{6} + \ln^2 \frac{\bar{m}^2}{\bar{\mu}^2} \right) + \mathcal{O}(\varepsilon^2) \right] \quad (2.86)$$

$$F_1 = -\frac{2\pi^2}{9} + 12q_1 - 24q_2 \quad (2.87)$$

$$F_2 = 2 - \frac{2\pi^2}{3} \quad (2.88)$$

$$q_1 = \int_0^1 dt \frac{\ln t}{t^2 - t + 1} \approx -1.17195 \quad (2.89)$$

$$q_2 = \int_0^1 dt \frac{\ln t}{t^3 + 1} \approx -0.951518 \quad (2.90)$$

Working up to order $\bar{g}^4$, we find for the inverse propagator

$$\begin{aligned} S_{\text{two-loop}}^{-1} &= i\not{p} \left[1 - \frac{(N - \frac{1}{2}) \bar{g}^4}{16\pi^2} \left(1 - 2 \ln \frac{\bar{m}^2}{\bar{\mu}^2} + F_2 \right) \right] \\ &+ \bar{m} \left[1 + \frac{(N - \frac{1}{2}) \bar{g}^4}{16\pi^2} \left(2 \ln^2 \frac{\bar{m}^2}{\bar{\mu}^2} + \frac{\pi^2}{6} - F_1 \right) \right] \end{aligned} \quad (2.91)$$

Solving for the pole gives

$$m_{\text{eff}} = \bar{m} \left[1 + \frac{(N - \frac{1}{2}) \bar{g}^4}{16\pi^2} \left(2 \ln^2 \frac{\bar{m}^2}{\bar{\mu}^2} + \frac{\pi^2}{6} - F_1 + 1 - 2 \ln \frac{\bar{m}^2}{\bar{\mu}^2} + F_2 \right) \right] \quad (2.92)$$

With $\bar{\mu} = \bar{m} = \Lambda_{\overline{\text{MS}}}$, the above equation has no sense.

Next, we follow the same steps as executed for E to reexpress m_{eff} in terms of M and x . A little algebra results in

$$\begin{aligned} m_{\text{eff}} &= M \left(\frac{1}{x} \right)^{1 - \frac{\gamma_0}{2\beta_0}} \left\{ 1 + \frac{1}{x} \left[c_1 + \frac{b_0 U}{2} - \frac{\beta_1 U L}{2\beta_0} \right] + \frac{1}{x^2} \left[b_0 c_1 \left(1 + \frac{U}{2} \right) + c_2 + \frac{U}{2} \times \right. \right. \\ &\quad \left. \left(\frac{\beta_1^2}{\beta_0^2} (L^2 - L - 1) + \frac{b_0^2 \beta_0 + b_0 \beta_1 + \beta_2}{\beta_0} \right) - \frac{b_0^2 \gamma_0 U}{8\beta_0} - \frac{L^2 \beta_1^2 \gamma_0 U}{8\beta_0^3} - \frac{L \beta_1 (1 + \frac{U}{2}) (c_1 + b_0 \frac{U}{2})}{\beta_0} \right. \\ &\quad \left. \left. + \frac{N - \frac{1}{2}}{16\pi^2} \left(\frac{\pi^2}{6} + 1 - F_1 + F_2 - 2W + 2W^2 \right) \right] \right\} \end{aligned} \quad (2.93)$$

The quantities c_1 , c_2 , U , W and L are the same as defined before. Again, only b_0 is left over as scheme parameter.

2.6 Second numerical results for the mass gap and vacuum energy.

We first discuss how we can fix the parameter b_0 in a reasonable, *self-consistent* way. A frequently used method is the *principle of minimal sensitivity* (PMS)[104]. This is based on the concept that *physical*

N	P	Q	$\frac{N}{4\pi x}$
2	?	?	?
3	-22.5%	43.9%	0.60
4	-19.4%	25.9%	0.56
5	-16.8%	17.9%	0.55
6	-14.6%	13.8%	0.54
7	-12.7%	11.3%	0.53
8	-11.2%	9.5%	0.53
9	-10.1%	8.2%	0.53
10	-9.1%	7.2%	0.52
20	-4.5%	3.4%	0.50

Table 2.2: Optimized first order results for mass gap and vacuum energy (Choice I)

quantities should not depend on the renormalization prescriptions. In our case, the vacuum energy E as well as the mass gap m_{eff} are physical, so we could apply PMS. However, PMS does not always work out. Sometimes there is no minimum, then an alternative is picking that b_0 for which the derivative of the considered quantity is minimal ($\rightarrow$ as near as possible to a minimum). Also *fastest apparent convergence criteria* (FACC) can be practiced.

But maybe the biggest barrier to a fruitful use of PMS (or FACC) arises from the same origin why E did not seem to obey its RGE. Just as the scale dependence of E is not canceled order by order, the scheme dependence of E will not cancel order by order, so we may find no optimal b_0 , and even if we would have such b_0 , it would not be certain that the corresponding E really is a good approximation to E_{exact} . The same obstacle will arise for the mass gap m_{eff} .

Apparently, we are not any further. We may have a way out through. M , as defined in (2.56), is RSSI, independent of the fact that it satisfies its gap equation or not. The $2PPI$ formalism provides us with an equation to calculate M approximately. This equation, $\frac{\partial E}{\partial M} = 0$, is correct up to a certain order and M is RSSI up to that order by construction. Hence, we can ask that the (non-zero) solution M has minimal dependence on b_0 . This also gives a value for b_0 to calculate the vacuum energy, because the b_0 for E and M must be equal, again because E is only correct when the gap equation is fulfilled. Also the mass gap m_{eff} can be calculated with this b_0 .

2.6.1 First order results.

We start from the expression (2.76), but we first restrict ourselves to the lowest order correction.

$$E = M^2 \left(\frac{1}{x} \right)^{2 - \frac{\gamma_0}{\beta_0}} (E_{-1}x + E_0) \quad (2.94)$$

Until now, we have not said anything about the freedom in scale $\bar{\mu}$. Analogously as we fixed b_0 , we can ask $\frac{\partial M}{\partial \bar{\mu}} = 0$ due to the scale independence of M . For the sake of simplicity, we will however make a reasonable *choice* for $\bar{\mu}$. In order to cancel logarithms, we could set $\bar{\mu} = M$. We refer to this as Choice I. We observe that $\ln \frac{M^2}{\bar{\mu}^2}$ always appears in the form $W \equiv \ln \frac{M^2}{\bar{\mu}^2} + \left(2 - \frac{\gamma_0}{\beta_0} \right) \ln \frac{1}{x}$; we could determine $\bar{\mu}$ such that $W = 0$, then the danger of exploding logarithms is also averted. We refer to this as Choice II.

Tables 2.2 and 2.3 summarize the corresponding results.

Some remarks must be made.

N	P	Q	$\frac{N}{4\pi x}$
2	?	?	?
3	19.9%	120.7%	0.30
4	4.5%	57.2%	0.31
5	0.3%	36.8%	0.32
6	-1.2%	27.0%	0.33
7	-1.9%	21.3%	0.33
8	-2.1%	17.5%	0.34
9	-2.2%	14.9%	0.34
10	-2.2%	12.9%	0.34
20	-1.6%	5.5%	0.35

Table 2.3: Optimized first order results for mass gap and vacuum energy (Choice II).

- We have determined the parameter b_0 by requiring that $\left|\frac{\partial M}{\partial b_0}\right|$ is minimal⁷. In Figure 2.12, $M(b_0)$ is plotted for the case $N = 5$ and Choice I. Figure 2.13 shows $\frac{\partial M}{\partial b_0}$, again for $N = 5$ and Choice I. The plots for Choice II are completely similar. Notice that $\left|\frac{\partial M}{\partial b_0}\right|$ is relatively small. For both choices, it tended to zero for growing N , f.i. $\left|\frac{\partial M}{\partial b_0}\right| \approx 0.045$ for $N = 10$, Choice I.

Results for the mass gap agree very well with the exact values for Choice II, this is quite remarkable since we used a lowest order approximation. Choice I gives almost the same results as the $N \rightarrow \infty$ approximation.

For the vacuum energy, the results are somewhat less good than those obtained with a straightforward $\overline{\text{MS}}$ calculation. Nevertheless, the mass gap as well as the vacuum energy are converging, and we retrieve the correct $N \rightarrow \infty$ limit. Moreover, the relevant expansion parameter $\frac{N}{4\pi x}$ is relatively small, and behaves more or less as a constant.

- For $N = 2$ we did not find an optimal b_0 . In the light of the exact results (2.43) and (2.44), it is not unexpected that $N = 2$ causes trouble. $N = 2$ is a maximum of m_{exact} , and close to $N = \frac{3}{2}$, which is a root of m_{exact} . There is a sharp drop between 2 and $\frac{3}{2}$, and somewhat lower than $\frac{3}{2}$, oscillating behaviour begins. What is more, $N = 2$ and $N = \frac{3}{2}$ are both roots of E_{exact} , while $E_{\text{exact}} > 0$ between them. Again there is a sharp drop at $\frac{3}{2}$ with oscillation somewhat before $\frac{3}{2}$. Problems with $N = 2$ persist at second order too, as will be seen shortly.

2.6.2 Second order results.

In Table 2.4, we present second order results for Choice I, while Table 2.5 displays those for Choice II. Just as for the first order approximation, we plotted $M(b_0)$ in Figure 2.14, and $\left|\frac{\partial M}{\partial b_0}\right|$ in Figure 2.15 for the case $N = 5$, Choice I. Notice that $\left|\frac{\partial M}{\partial b_0}\right|$ is smaller at second order. For $N = 10$, $\left|\frac{\partial M}{\partial b_0}\right| \approx 0.022$. Again it reaches zero for infinite N . Again, we were unable to extract a value for m_{eff} or E for $N = 2$.

⁷No b_0 satisfying $\frac{\partial M}{\partial b_0} = 0$ was found.

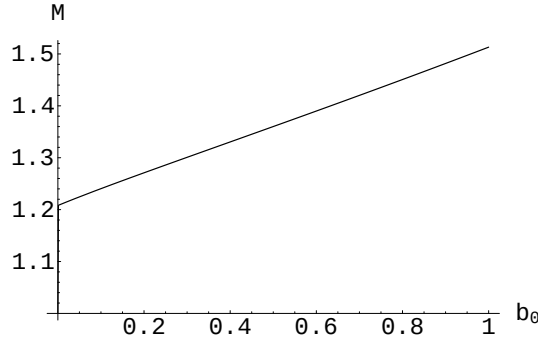


Figure 2.12: $M(b_0)$ in units of $\Lambda_{\overline{MS}}$ for $N = 5$ (Choice I, 1st order).

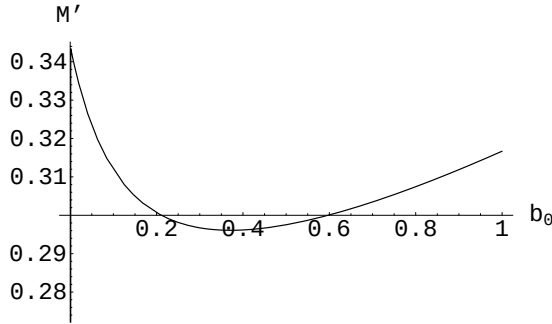


Figure 2.13: $\frac{\partial M}{\partial b_0}$ in units of $\Lambda_{\overline{MS}}$ for $N = 5$ (Choice I, 1st order).

2.6.3 Interpretation of the results.

When we compare the second with the first order results, a strange feature immediately catches our eyes. For Choice I, the mass gap results are better at second order, while the energy results are worse. For Choice II, the energy results are better, while the mass gap performs worse (except for $N = 3$). To make the comparison more transparent, we plotted the different mass gap results in Figure 2.16 and energy results in Figure 2.17. One should not be alarmed that second order results are “worse”. We see that the difference between the Choice I and II results at first order are relatively large, for m_{eff} as well as for E . But at second order, the results are almost the same for both choices, whereas $\frac{N}{4\pi x}$ is the same. This pleases us, because these results indicate that the choice of $\bar{\mu}$ is getting less relevant in the final results at second order. The fact that both (reasonable) choices for the scale $\bar{\mu}$ give results that are close to each other and are converging to the same $N \rightarrow \infty$ limit, convinces us that our method is consistent and should give trustable results.

Yet, there is another way to check reliability. We already said FACC could be used as an alternative to PMS to fix b_0 . More precisely, we could use a FACC on both the energy E as the mass gap equation $\frac{\partial E}{\partial M} = 0$. Explicitly, define

$$\delta_E = \left| \frac{E_1 x^{-1} - E_0}{E_0} \right| \quad (2.95)$$

measuring the relative correction of the second order on the first order contribution. The closer δ_E is to 1, the better it is, as an indication that the series expansion is under control. The quantity δ_M is

N	P	Q	$\frac{N}{4\pi x}$
2	?	?	?
3	-0.2%	54.8%	0.16
4	-2.6%	33.5%	0.16
5	-3.3%	23.8%	0.16
6	-3.7%	18.1%	0.16
7	-3.8%	14.5%	0.17
8	-3.9%	11.9%	0.17
9	-3.9%	10.0%	0.17
10	-4.0%	8.5%	0.17
20	-3.7%	2.8%	0.19

Table 2.4: Optimized second order results for mass gap and vacuum energy (Choice I).

N	P	Q	$\frac{N}{4\pi x}$
2	?	?	?
3	-4.5%	47.7%	0.17
4	-6.5%	27.9%	0.17
5	-6.1%	19.9%	0.17
6	-5.4%	15.6%	0.17
7	-4.8%	12.8%	0.17
8	-4.3%	10.9%	0.17
9	-3.9%	9.5%	0.17
10	-3.5%	8.4%	0.17
20	-1.8%	3.9%	0.17

Table 2.5: Optimized second order results for mass gap and vacuum energy (Choice II)

defined in a similar fashion. Unfortunately, no b_0 exists such that $\left|\frac{\partial\delta_E}{\partial b_0}\right|$ or $\left|\frac{\partial\delta_M}{\partial b_0}\right|$ are zero or minimal. However, we can substitute our PMS results in δ_E and δ_M and find out what these give.

Consulting Figure 2.18 and Figure 2.19, we are able to understand why we should have ended up with qualitatively good results, since δ_E as well as δ_M are close to 1, even for small N . We also see that both choices for $\bar{\mu}$ should give comparable results, since δ_E and δ_M fit with each other.

We also fixed b_0 by demanding that $\left|\frac{\partial E}{\partial b_0}\right|$ was minimal⁸, and we found that results were less good than those obtained by fixing b_0 by means of M , except for small N values⁹. However, the convergence to the exact results for growing N was very slow. For example with Choice I, $Q(5) = 19.4\%$, $Q(10) = 19.1\%$, $Q(20) = 14.1\%$. An analogous story held true for m_{eff} , where b_0 was determined by demanding that $\left|\frac{\partial m_{\text{eff}}}{\partial b_0}\right|$ was minimal. There, the deviation from the exact results was always bigger¹⁰, and the convergence was again rather slow. For example, with Choice I, $P(5) = 30.1\%$, $P(10) = 25.7\%$, $P(20) = 18.5\%$. All this corroborates our conjecture that M is indeed the best quantity to fix b_0 .

⁸Again, no solution for $\frac{\partial E}{\partial b_0} = 0$.

⁹To be more precise, $Q(3) = 2.1\%$ and $Q(4) = 14.4\%$. The fact that the error grows fast between $N = 3$ and $N = 4$, and goes slowly to 0 for $N > 5$, makes us believe it is a rather lucky shot that the energy values are better for small N .

¹⁰Also for m_{eff} , the error grows between $N = 3$ and $N = 5$ ($P(3) = 16.8\%$, $P(4) = 27.8\%$), and drops slowly to 0 for $N > 5$.

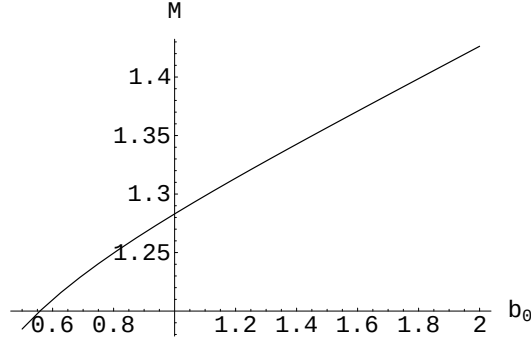


Figure 2.14: $M(b_0)$ in units of $\Lambda_{\overline{MS}}$ for $N = 5$ (Choice I, 2nd order).

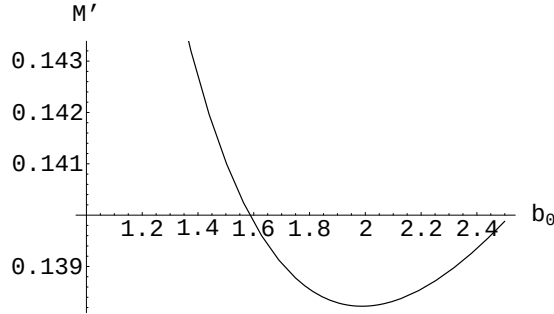


Figure 2.15: $\frac{\partial M}{\partial b_0}$ in units of $\Lambda_{\overline{MS}}$ for $N = 5$ (Choice I, 2nd order).

Before we formulate our conclusions, we just like to mention that also in case of $N = 2$ there exist a mass gap and a non-perturbative vacuum energy. We already pointed out why we probably did not find an optimal b_0 with our method. The best we can do with this special N value, is just choosing a (physical) renormalization scheme, but we must realize we can easily obtain highly over- or underestimated values in this case and that this is not a self-consistent way to obtain results.

2.7 Conclusion.

This paper, which had the purpose to investigate the dynamical mass generation and non-perturbative vacuum energy of the two-dimensional Gross-Neveu field theory, consisted of two main parts. In the first part, we proved how all bubble Feynman diagrams can be consistently resummed up to all orders in an effective mass $\overline{m}$. We showed that this $\overline{m}$ can be calculated from the gap equation $\frac{\partial E}{\partial \overline{m}} = 0$, whereby E is the vacuum energy. E is given by the sum of the $2PPI$ vacuum bubbles, calculated with the $2PPI$ massive propagator (i.e. with mass $\overline{m}$), plus an extra term, accounting for a double counting ambiguity.

We showed that the $2PPI$ expansion can be renormalized with the original counterterms of the model.

A very important fact is that the $2PPI$ expansion for E is only correct if the gap equation $\frac{\partial E}{\partial \overline{m}} = 0$ is fulfilled. In this context, we discussed the renormalization group equation for E , and showed why E does not obey its RGE *order by order*, because the requirement of the gap equation turns terms of different orders into the same order. We stress that this does not mean E does not obey its RGE, or

ask for the introduction of a “non-perturbative” β -function.

To get actual values for m_{eff} and E , we employed the $\overline{\text{MS}}$ scheme, and after the classical choice $\bar{\mu} = \bar{m}$ to cancel logarithms, we recovered the $N \rightarrow \infty$ results. However, the corresponding coupling constant was infinite, so we could not say anything about validity of the results, without the foreknowledge of exact values. This compelled us to search for a more sophisticated way to improve the 2PPI technique.

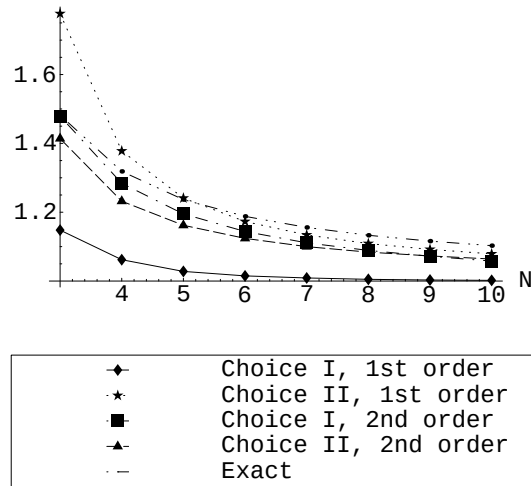
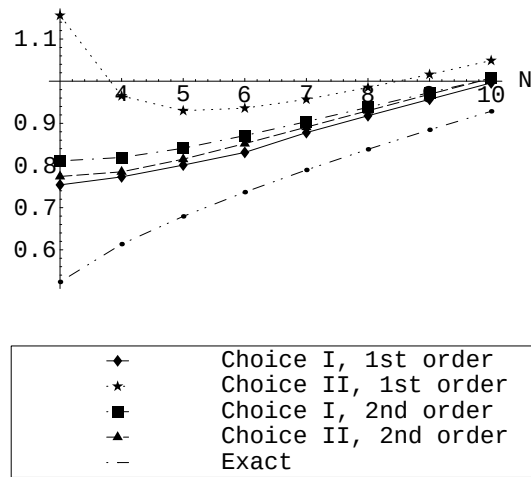
In the second part, we first eliminated the freedom in the renormalization of the 2PPI mass parameter, by transforming $\bar{m}$ to a renormalization scheme and scale independent M . The consistency relation $\frac{\partial E}{\partial \bar{m}} = 0$ was completely equivalent to $\frac{\partial E}{\partial M} = 0$. Secondly, we parametrized the coupling constant renormalization. After a reorganization of the series, all scheme dependence was reduced to a single parameter b_0 , equivalent to the choice of a certain scale parameter Λ . We fixed this b_0 by means of the principle of minimal sensitivity (PMS). Originally, PMS was founded on the logical requirement that observable physics cannot depend on how one chooses to renormalize. Translated to our case, E and m_{eff} should not depend on the arbitrary parameter b_0 . But we showed on theoretical grounds why applying PMS on neither m_{eff} nor E would be valid, because analogously as $E(m_{\text{eff}})$ does not lose its scale dependence order by order, it does not lose its scheme dependence order by order.

Nevertheless, we gave an outcome to the problem of PMS. By construction, M is scheme and scale independent, so we can apply PMS on this mass parameter. This provides us with an optimal b_0 to calculate M , and consequently E and m_{eff} . For the scale $\bar{\mu}$, we made 2 reasonable choices. These 2 choices gave acceptable results at first order, yet there was quite a big difference between them. The second order results were comparable and qualitatively good, converging to the exact values for growing N .

The relevant expansion parameter was relatively small. We gave extra evidence why results were good, by using a fastest apparent convergence argument.

We explicitly checked that using PMS on E and m_{eff} to fix b_0 gave worse results, and the convergence was very slow.

Summarizing, we have constructed a self consistent method to calculate the mass gap and non-perturbative vacuum energy.

Figure 2.16: Different results for m_{eff} .Figure 2.17: Different results for $\sqrt{-E}$.

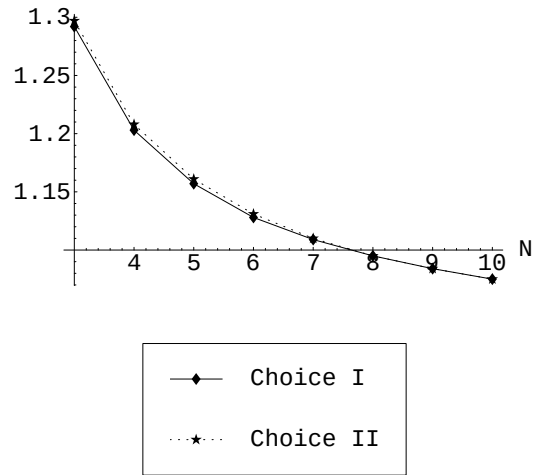


Figure 2.18: δ_E as a function of N .

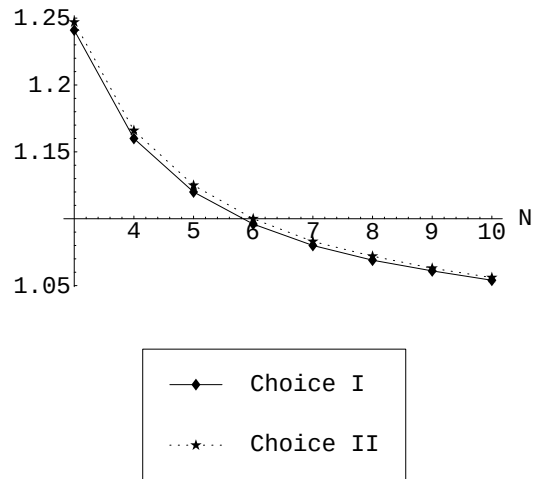


Figure 2.19: δ_M as a function of N .

Chapter 3

A determination of $\langle A_\mu^2 \rangle$ and the non-perturbative vacuum energy of Yang-Mills theory in the Landau gauge

*D. Dudal , H. Verschelde (UGent), R. E. Browne and J. A. Gracey (Liverpool University), published in Physics Letters B **562** (2003) 87.*

We discuss the 2-point-particle-irreducible (2PPI) expansion, which sums bubble graphs to all orders, in the context of $SU(N)$ Yang-Mills theory in the Landau gauge. Using the method we investigate the possible existence of a gluon condensate of mass dimension two, $\langle A_\mu^a A_\mu^a \rangle$, and the corresponding non-zero vacuum energy. This condensate gives rise to a dynamically generated mass for the gluon.

3.1 Introduction.

Recently there has been growing evidence for the existence of a condensate of mass dimension two in $SU(N)$ Yang-Mills theory with N colours. An obvious candidate for such a condensate is $\langle A_\mu^a A_\mu^a \rangle$. The phenomenological background of this type of condensate can be found in [142, 33, 34]. However, if one first considers simpler models such as massless $\lambda\phi^4$ theory or the Gross-Neveu model [21] and the role played by their quartic interaction in the formation of a (local) composite quadratic condensate and the consequent *dynamical mass generation* for the originally massless fields [21, 62, 64], it is clear that the *possibility* exists that the quartic gluon interaction gives rise to a quadratic composite operator condensate in Yang Mills theory and hence a dynamical mass parameter for the gluons. The formation of such a dynamical mass is strongly correlated to a lower value of the vacuum energy. In other words the casual perturbative Yang-Mills vacuum is unstable. From this viewpoint, mass generation in connection with gluon pairing has already been discussed a long time ago in, for example, [28, 29, 30, 31]. There the analogy with the BCS superconductor and its gap equation was examined. It was shown that the zero vacuum is tachyonic in nature and the gluons achieve a mass due to a non-trivial solution of the gap equation. Moreover, recent work using lattice regularized Yang Mills theory has indicated the existence of a non-zero condensate, $\langle A_\mu^a A_\mu^a \rangle$, [37]. There the authors invoked the operator product expansion, (OPE), on the gluon propagator as well as on the effective coupling α_s in the Landau gauge. Their work

was based on the perception that, even in the relatively high energy region ($\sim 10\text{GeV}$), a discrepancy existed between the expected perturbative behaviour and the lattice results. It was shown that, within the momentum range accessible to the OPE, that this discrepancy could be solved with a $1/q^2$ power correction¹. They concluded that a non-vanishing dimension two condensate must exist. Further, the results of [38] give some evidence that instantons might be the mechanism behind the low-momentum contribution to condensate. As has been argued in [34], only the low-momentum content of the squared vector potential is accessible with the OPE. Moreover, they argue that there are also *short-distance* non-perturbative contributions to $\langle A_\mu^2 \rangle$.

It is no coincidence the Landau gauge is used for the search for a dimension two condensate. Naively, the operator A_μ^2 is not gauge invariant. Although this does not prevent the condensate $\langle A_\mu^2 \rangle$ showing up in gauge variant quantities like the gluon propagator, we should instead consider the gauge-invariant operator $(VT)^{-1} \min_U \int d^4x (A_\mu^U)^2$, where VT is the space-time volume and U is an arbitrary gauge transformation in order to assign some physical meaning to the operator. Clearly from its structure this operator is non-local and thus is difficult to handle. However, when we impose the Landau gauge, it reduces² to the local operator A_μ^2 . Moreover, it has been shown that $\langle A_\mu^2 \rangle$ is (on-shell) BRST invariant [84, 85, 83, 144]. Another motivation for this study is that the gluon propagator seems to exhibit an infrared suppression, as has been reported in many lattice simulations, [44, 45, 46] and using the Schwinger-Dyson approach, [146, 20, 126]. A dynamical gluon mass might serve as an indication for such a suppression. An attempt has already been made to explain confinement by a dual Ginzburg-Landau model or an effective string theory, in the Landau gauge, with the help of $\langle A_\mu^2 \rangle$ [147]. The fact that $\langle A_\mu^2 \rangle$ might be central to confinement, is supported by the observation that it undergoes a phase transition due to the monopole condensation in three-dimensional compact QED [33, 34].

From these various analyses the importance of $\langle A_\mu^2 \rangle$ must have become clear. Therefore, the aim of this article is to provide some *analytical* evidence that gluons do condense. To our knowledge, [42] is the only paper which effectively calculates $\langle A_\mu^2 \rangle$, without referring to lattice regularization. In [42] the standard way of calculating the effective potential for a particular quantity was followed, and all the problems concerning the fact that the considered quantity was a composite operator were elegantly solved.

In a previous paper [148], we have discussed the $2PPI$ expansion for the Gross-Neveu model and found results close to the exact values for the Gross-Neveu mass gap and the vacuum energy. The $2PPI$ expansion does not rely on the effective action formalism of [42]. Instead it is directly based on the path integral and the topology of its Feynman diagrammatical expansion. In this paper we will discuss how to apply it to $SU(N)$ Yang-Mills theories in the Landau gauge. Of course, it is not our aim to provide a complete picture of $\langle A_\mu^2 \rangle$ but rather give further evidence for its existence as it lowers the vacuum energy also in the $2PPI$ approach.

3.2 The $2PPI$ expansion.

The $SU(N)$ Yang Mills Lagrangian in d -dimensional Euclidean space time is given by

$$\mathcal{L}(A_\mu, c, \bar{c}) = \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \mathcal{L}_{\text{G.F.}+\text{F.P.}} \quad (3.1)$$

¹The $\frac{1}{q^4}$ power correction due to the $\langle F_{\mu\nu}^2 \rangle$ condensate is too weak at such energies to be the cause of the discrepancy.

²Although this equality is somewhat disturbed by Gribov ambiguities [143]. In this paper Gribov copies are neglected since we will work in the perturbative Landau gauge and sum a certain class of bubble diagrams in this particular gauge. It is a pleasant feature of the Landau gauge that A_μ^2 can be given some gauge invariant meaning. In another gauge, the bubbles will no longer correspond to A_μ^2 and the correspondence with $A_\mu^2_{\text{min}}$ is more of academic interest.

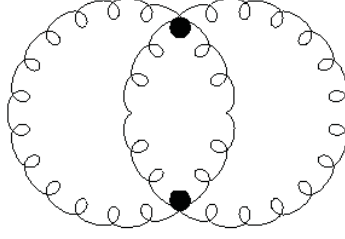
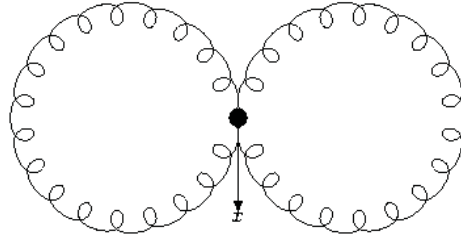


Figure 3.1: A 2PPI vacuum bubble.

Figure 3.2: A 2PPR vacuum bubble. x is the 2PPR insertion point.

where $F_{\mu\nu}^a$ is the gluon field strength, $1 \leq a \leq N^2 - 1$, $\mathcal{L}_{\text{G.F.}+\text{F.P.}}$ implements the Landau gauge and its corresponding Faddeev-Popov part and ξ and $\bar{\xi}$ denote the ghosts and anti-ghost fields respectively. Issues concerning the counterterm part of (3.1) will be discussed later. First, we consider the diagrammatical expansion for the vacuum energy which we denote by E . As is well known, this is a series consisting of one particle irreducible, (1PI), diagrams. These 1PI diagrams can be divided into two disjoint classes:

- those diagrams not falling apart into two separate pieces when two lines meeting at the same point x are cut, which we call 2-point-particle-irreducible, (2PPI); (an example is given in Figure 3.1)
- those diagrams falling apart into two separate pieces when two lines meeting at the same point x are cut which we call 2-point-particle-reducible, (2PPR), while x is called the 2PPR insertion point; (an example is given in Figure 3.2).

We may now resum this series of 2PPR and 2PPI graphs, where the propagators are the usual massless ones, by retaining only the 2PPI graphs, whereby the 2PPR insertions, or bubbles, are resummed into an effective (mass)² $m_{2\text{PPI}}^2 \equiv \bar{m}^2$. The bubble graph gluon polarization is then given by

$$\begin{aligned} \Pi_{\mu\nu}^{ab} = & -\frac{g^2}{2} [f_{eab}f_{ecd} (\langle A_\mu^c A_\nu^d \rangle - \langle A_\mu^d A_\nu^c \rangle) + f_{eac}f_{edb} (\langle A_\mu^d A_\nu^c \rangle - \langle A_\rho^c A_\rho^d \rangle \delta_{\mu\nu}) \\ & + f_{ead}f_{ebc} (\delta_{\mu\nu} \langle A_\rho^c A_\rho^d \rangle - \langle A_\mu^c A_\nu^d \rangle)] \end{aligned} \quad (3.2)$$

where f_{abc} are the $SU(N)$ structure constants. We define the vacuum expectation value of A_μ^2 as

$$\Delta = \langle A_\mu^a A_\mu^a \rangle. \quad (3.3)$$

The global $SU(N)$ symmetry can then be used to show that

$$\langle A_\mu^a A_\nu^b \rangle = \frac{\delta^{ab} \delta_{\mu\nu}}{d(N^2 - 1)} \Delta \quad (3.4)$$

Substitution of (3.4) in (3.2) yields

$$\Pi_{\mu\nu}^{ab} = -g^2 \frac{N}{N^2-1} \frac{d-1}{d} \delta^{ab} \delta_{\mu\nu} \Delta \quad (3.5)$$

which results in an effective mass, $\bar{m}$, running in the $2PPI$ propagators, given by

$$\bar{m}^2 = g^2 \frac{N}{N^2-1} \frac{d-1}{d} \Delta. \quad (3.6)$$

If we let E_{2PPI} be the sum of the $2PPI$ vacuum bubbles, calculated with the effective $2PPI$ propagator, then this E_{2PPI} is *not* equal to the vacuum energy E , because simply removing all $2PPR$ insertions is too naive. For instance, there is a double counting problem which is already visible in the $2PPR$ diagram of Figure 3.2 where each bubble can be seen as a $2PPR$ insertion on the other one. However, we can resolve this ambiguity. A dimensional argument results in

$$E = E_{2PPI} + c g^2 \Delta^2 \quad (3.7)$$

where $c \neq 0$ will accomodate the double counting. To determine the appropriate value of c , we use the path integral which gives

$$\begin{aligned} \frac{\partial E}{\partial g^2} &= -\frac{1}{4g} f_{abc} \langle (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) A_\mu^b A_\nu^c \rangle - \frac{1}{2g} f_{abc} \langle \partial_\mu \bar{\xi}^a \xi^c A_\mu^b \rangle \\ &\quad + \frac{1}{4} f_{abc} f_{ade} \langle A_\mu^b A_\nu^c A_\mu^d A_\nu^e \rangle. \end{aligned} \quad (3.8)$$

The first two terms contribute unambiguously to the $2PPI$ part. For the last term, we rewrite

$$\begin{aligned} \langle A_\mu^b A_\nu^c A_\mu^d A_\nu^e \rangle &= \langle A_\mu^b A_\nu^c \rangle \langle A_\mu^d A_\nu^e \rangle + \langle A_\mu^b A_\mu^d \rangle \langle A_\nu^c A_\nu^e \rangle + \langle A_\mu^b A_\nu^e \rangle \langle A_\nu^c A_\mu^d \rangle \\ &\quad + \langle A_\mu^b A_\nu^c A_\mu^d A_\nu^e \rangle_{2PPI}. \end{aligned} \quad (3.9)$$

Using (3.4) and the properties of the structure constants f_{abc} , we obtain

$$\begin{aligned} \frac{\partial E}{\partial g^2} &= -\frac{1}{4g} f_{abc} \langle (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) A_\mu^b A_\nu^c \rangle_{2PPI} - \frac{1}{2g} f_{abc} \langle \partial_\mu \bar{\xi}^a \xi^c A_\mu^b \rangle_{2PPI} \\ &\quad + \frac{1}{4} f_{abc} f_{ade} \langle A_\mu^b A_\nu^c A_\mu^d A_\nu^e \rangle_{2PPI} + \frac{1}{4} \frac{N}{N^2-1} \frac{d-1}{d} \Delta^2 \\ &= \frac{\partial E_{2PPI}}{\partial g^2} + \frac{1}{4} \frac{N}{N^2-1} \frac{d-1}{d} \Delta^2. \end{aligned} \quad (3.10)$$

From (3.7), we derive

$$\frac{\partial E}{\partial g^2} = \frac{\partial E_{2PPI}}{\partial g^2} + \frac{\partial E_{2PPI}}{\partial \bar{m}^2} \frac{\partial \bar{m}^2}{\partial g^2} + c \Delta^2 + 2c g^2 \Delta \frac{\partial \Delta}{\partial g^2}. \quad (3.11)$$

Combining (3.6), (3.10) and (3.11) gives

$$\frac{\partial E_{2PPI}}{\partial \bar{m}^2} \left(\frac{N}{N^2-1} \frac{d-1}{d} \Delta + g^2 \frac{N}{N^2-1} \frac{d-1}{d} \frac{\partial \Delta}{\partial g^2} \right) = \frac{1}{4} \frac{N}{N^2-1} \frac{d-1}{d} \Delta^2. \quad (3.12)$$

Then a simple diagrammatical argument gives

$$\frac{\partial E_{2PPI}}{\partial \bar{m}^2} = \frac{\Delta}{2}. \quad (3.13)$$

which is a local gap equation, summing the bubble graphs into $\bar{m}^2$. Using this together with (3.12) finally gives

$$c = -\frac{1}{4} \frac{N}{N^2-1} \frac{d-1}{d}. \quad (3.14)$$

It is easy to show that the following equivalence holds

$$\frac{\partial E_{2PPI}}{\partial \bar{m}^2} = \frac{\Delta}{2} \Leftrightarrow \frac{\partial E}{\partial \bar{m}^2} = 0. \quad (3.15)$$

To summarize, we have summed the bubble insertions into an effective (mass)², $\bar{m}^2$. The vacuum energy is expressed by

$$E = E_{2PPI} - \frac{g^2}{4} \frac{N}{N^2-1} \frac{d-1}{d} \Delta^2. \quad (3.16)$$

We stress the fact that (3.16) is *only meaningful* if the gap equation (3.15) is satisfied. This means we cannot consider $\bar{m}$ or Δ as a real variable on which E depends. It is a quantity which has to obey its gap equation, otherwise the 2PPI expansion loses its validity. This also means that $E(\bar{m})$, or equivalently $E(\Delta)$, is not a function depending on $\bar{m}$ (Δ), in contrast³ with the usual concept of an effective potential $V(\varphi)$ which is a function of the field φ .

In order to ensure the usefulness of the 2PPI formalism for actual calculations, we should prove it can be fully renormalized with the counterterms available from the original (bare) Lagrangian, (3.1). However, it is sufficient to say that all our derived formulae remain valid and are finite when the counterterms are included. This also implies the 2PPI mass $\bar{m}$ is renormalized. Furthermore, no new counterterms are needed to remove the vacuum energy divergences. The renormalizability of the 2PPI expansion has been discussed in detail in [148] in the case of the Gross-Neveu model. Since the arguments for Yang Mills theory are completely analogous, we refer to [148] for technical details concerning the renormalization.

Another point worth emphasizing here, is the renormalization group equation, (RGE), for E . The first diagram of E_{2PPI} is given by the O-bubble. Using the MS renormalization scheme in dimensional regularization in d dimensions, we find

$$E = \frac{3}{4} \frac{N^2-1}{16\pi^2} \bar{m}^4 \left(\ln \frac{\bar{m}^2}{\bar{\mu}^2} - \frac{5}{6} \right) - \frac{1}{4\bar{g}^2} \frac{d}{d-1} \frac{N^2-1}{N} \bar{m}^4. \quad (3.17)$$

Since E is a physical quantity, it should not depend on the subtraction scale $\bar{\mu}$. This is expressed by the RGE

$$\bar{\mu} \frac{dE}{d\bar{\mu}} = \left(\bar{\mu} \frac{\partial}{\partial \bar{\mu}} + \bar{\beta}(\bar{g}^2) \frac{\partial}{\partial \bar{g}^2} + \bar{\kappa}(\bar{g}^2) \bar{m}^2 \frac{\partial}{\partial \bar{m}^2} \right) E = 0 \quad (3.18)$$

where $\bar{\beta}(\bar{g}^2)$ governs the scaling behaviour of the coupling constant

$$\bar{\beta}(\bar{g}^2) = \bar{\mu} \frac{\partial \bar{g}^2}{\partial \bar{\mu}} = -2(\beta_0 \bar{g}^4 + \beta_1 \bar{g}^6 + \beta_2 \bar{g}^8 + \dots) \quad (3.19)$$

and $\bar{\kappa}(\bar{g}^2)$ is the anomalous dimension of $\bar{m}^2$

$$\bar{\mu} \frac{\partial \bar{m}^2}{\partial \bar{\mu}} = \left(\frac{\bar{\beta}(\bar{g}^2)}{\bar{g}^2} + \bar{\gamma}_{A_\mu^2}(\bar{g}^2) \right) \bar{m}^2 \equiv \bar{\kappa}(\bar{g}^2) \bar{m}^2 \quad (3.20)$$

$$\bar{\gamma}_{A_\mu^2}(\bar{g}^2) = \frac{\bar{\mu}}{\Delta} \frac{\partial \Delta}{\partial \bar{\mu}} = \gamma_0 \bar{g}^2 + \bar{\gamma}_1 \bar{g}^4 + \bar{\gamma}_2 \bar{g}^6 + \dots \quad (3.21)$$

³ $V(\varphi)$ also makes sense if $\frac{dV}{d\varphi} \neq 0$.

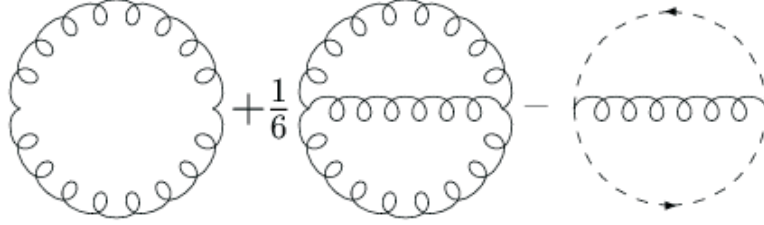


Figure 3.3: The first diagrams contributing to E_{2PPI}

The coefficients can be found in [9, 10, 149, 150, 151, 152, 88, 87] for $\bar{\beta}$ and in [42, 87, 153] for $\bar{\gamma}_{A_\mu^2}$,

$$\beta_0 = \frac{11}{3} \left(\frac{N}{16\pi^2} \right) \quad \beta_1 = \frac{34}{3} \left(\frac{N}{16\pi^2} \right)^2 \quad \bar{\beta}_2 = \frac{2857}{54} \left(\frac{N}{16\pi^2} \right)^3 \quad (3.22)$$

$$\gamma_0 = \frac{35}{6} \left(\frac{N}{16\pi^2} \right) \quad \bar{\gamma}_1 = \frac{449}{24} \left(\frac{N}{16\pi^2} \right)^2 \quad \bar{\gamma}_2 = \left[\frac{75607}{864} - \frac{9\zeta(3)}{16} \right] \left(\frac{N}{16\pi^2} \right)^3. \quad (3.23)$$

When we combine all this information and determine $\bar{\mu} \frac{dE}{d\bar{\mu}}$ up to lowest order in $\bar{g}^2$, we find

$$\bar{\mu} \frac{dE}{d\bar{\mu}} \neq 0. \quad (3.24)$$

Apparently, it seems that E does not obey its RGE. However, this is not a contradiction because of the demand that the gap equation (3.15) *must* be satisfied. The gap equation implies that $\ln \frac{\bar{m}^2}{\bar{\mu}^2} \propto \frac{1}{\bar{g}^2} + \text{constants}$. The consequence is that all leading logarithms contain terms of order unity. Hence, we cannot show that the RGE for E is obeyed order by order. The same phenomenon extends to higher orders. In other words, knowledge of $\bar{\mu} \frac{dE}{d\bar{\mu}}$ up to a certain order n , would require knowledge of *all* leading and subleading log terms to order n , to show explicitly that $\bar{\mu} \frac{dE}{d\bar{\mu}} = 0$. We must therefore be careful not to interpret the non-vanishing of the RGE as a reason to introduce a “non-perturbative” β -function, as is sometimes done, [154].

3.3 Results.

Up to two-loop order in the $2PPI$ expansion (see Figure 3.3), we find in the $\overline{\text{MS}}$ scheme

$$\begin{aligned} E(\bar{\Delta}) = & -\frac{3}{16} \frac{\bar{g}^2 N}{N^2 - 1} \bar{\Delta}^2 + \frac{27}{64} \frac{\bar{g}^4 N^2}{16\pi^2} \frac{\bar{\Delta}^2}{N^2 - 1} \left(\ln \frac{3}{4} \frac{\bar{g}^2 N}{N^2 - 1} \frac{\bar{\Delta}}{\bar{\mu}^2} - \frac{5}{6} \right) + \frac{9}{16} \frac{\bar{g}^6 N^3}{(16\pi^2)^2} \frac{\bar{\Delta}^2}{N^2 - 1} \\ & \times \left[-\frac{31}{2} \left(\ln \frac{3}{4} \frac{\bar{g}^2 N}{N^2 - 1} \frac{\bar{\Delta}}{\bar{\mu}^2} \right)^2 + \frac{259}{8} \ln \frac{3}{4} \frac{\bar{g}^2 N}{N^2 - 1} \frac{\bar{\Delta}}{\bar{\mu}^2} - \frac{1043}{32} + \frac{891}{16} s_2 - \frac{63}{8} \zeta(2) \right] \end{aligned} \quad (3.25)$$

where $\zeta(n)$ is the Riemann zeta function,

$$s_2 = \frac{4}{9\sqrt{3}} \mathcal{C}\ell_2\left(\frac{\pi}{3}\right) \approx 0.2604341 \quad (3.26)$$

and $\mathcal{C}\ell_2(x)$ is the Clausen function. We have computed the relevant two-loop vacuum bubble diagrams to the finite part using the massive gluon and massless ghost propagators which are respectively

$$-\frac{1}{p^2 + \overline{m}^2} \left[\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right] \quad \text{and} \quad \frac{1}{p^2} \quad (3.27)$$

in the Landau gauge. The expressions for the general massive and massless two-loop bubble integrals were derived from the results of [155] and implemented in the symbolic manipulation language FORM, [156]. It is easy to check that solving the gap equation $\frac{\partial E}{\partial \Delta} = 0$, with $\overline{\mu}^2$ set equal to $\frac{3}{4} \frac{\overline{g}^2 N}{N^2 - 1} \overline{\Delta}$ to kill potentially large logarithms, gives no solution at one-loop or two-loops, apart from the trivial one $\overline{\Delta} = 0$. This does not imply $\langle A_\mu^2 \rangle$ does not exist but that the $\overline{\text{MS}}$ scheme might not be the best choice for the $2PPI$ expansion. To address this we first remove the freedom existing in how the mass parameter Δ is renormalized by replacing Δ by a renormalization scheme and scale independent quantity $\tilde{\Delta}$. This can be accommodated by⁴

$$\tilde{\Delta} = \bar{f}(\overline{g}^2) \overline{\Delta} \quad (3.28)$$

with

$$\overline{\mu} \frac{\partial \bar{f}}{\partial \overline{\mu}} = - \bar{\gamma}_{A_\mu^2}(\overline{g}^2) \bar{f}. \quad (3.29)$$

A change in massless renormalization scheme corresponds to relations of the form

$$\overline{g}^2 = g^2 (1 + b_0 g^2 + b_1 g^4 + \dots) \quad (3.30)$$

$$\overline{\Delta} = \Delta (1 + d_0 g^2 + d_1 g^4 + \dots) \quad (3.31)$$

$$\bar{f}(\overline{g}^2) = f(g^2) (1 + f_0 g^2 + f_1 g^4 + \dots). \quad (3.32)$$

With these, it is easily checked that (3.28) is renormalization scheme and scale independent. The explicit solution of (3.29) reads

$$\begin{aligned} \bar{f}(\overline{g}^2) = (\overline{g}^2)^{\frac{\gamma_0}{2\beta_0}} & \left\{ 1 + \frac{\overline{g}^2}{2} \left(-\frac{\beta_1 \gamma_0}{\beta_0^2} + \frac{\bar{\gamma}_1}{\beta_0} \right) + \frac{\overline{g}^4}{4} \left[\frac{1}{2} \left(-\frac{\beta_1 \gamma_0}{\beta_0^2} + \frac{\bar{\gamma}_1}{\beta_0} \right)^2 \right. \right. \\ & \left. \left. + \frac{\gamma_0 \left(\frac{\beta_1^2}{\beta_0^2} - \frac{\bar{\beta}_2}{\beta_0} \right)}{\beta_0} - \frac{\beta_1 \bar{\gamma}_1}{\beta_0^2} + \frac{\bar{\gamma}_2}{\beta_0} \right] + \mathcal{O}(\overline{g}^6) \right\}. \end{aligned} \quad (3.33)$$

Since the gap equation is still a series expansion in $\frac{g^2 N}{16\pi^2}$ and we hope to find (at least qualitatively) acceptable results, $\frac{g^2 N}{16\pi^2}$ should be small. We will therefore choose to renormalize the coupling constant in such a scheme so that E is of the form

$$E = \frac{3}{16} (g^2)^{1 - \frac{\gamma_0}{\beta_0}} \frac{N}{N^2 - 1} \tilde{\Delta}^2 \left(-1 + \frac{g^2 N}{16\pi^2} E_1^1 L + \left(\frac{g^2 N}{16\pi^2} \right)^2 (E_2^1 L + E_2^2 L^2) + \dots \right) \quad (3.34)$$

⁴Barred quantities refer to the $\overline{\text{MS}}$ scheme, otherwise any other massless renormalization scheme is meant.

where

$$L = \ln \frac{\frac{3}{4} (g^2)^{1-\frac{\gamma_0}{2\beta_0}} \frac{N}{N^2-1} \tilde{\Delta}}{\bar{\mu}^2}. \quad (3.35)$$

Otherwise, we remove all the terms of the form $\left(\frac{g^2 N}{16\pi^2}\right)^n \times \text{constant}$, and only keep the terms that contain a power of the logarithm L . This is always possible by calculating the $\overline{\text{MS}}$ value of E as in (3.25) and using (3.30) to change the coupling constant renormalization by a suitable choice for the coefficients b_i . In other words the one-loop $\overline{\text{MS}}$ contribution to E allows one to determine b_0 . Once b_0 is fixed, the two-loop $\overline{\text{MS}}$ contribution to E can be used to fix b_1 , and so on for the higher order contributions. We note that the gap equation (3.15) is translated into $\frac{\partial E}{\partial \tilde{\Delta}} = 0$ since $\bar{m}^2 \propto \bar{\Delta} \propto \tilde{\Delta}$. In this gap equation, we will set $\bar{\mu}^2 = \frac{3}{4} (g^2)^{1-\frac{\gamma_0}{2\beta_0}} \frac{N}{N^2-1} \tilde{\Delta}$ so that all logarithms vanish. In other words $L = 0$. Once a solution $\tilde{\Delta}_*$ of the gap equation is found, then we will always have

$$E_{\text{vac}} = - \frac{3}{16} (g^2)^{1-\frac{\gamma_0}{2\beta_0}} \frac{N}{N^2-1} \tilde{\Delta}_*^2. \quad (3.36)$$

If the constructed value for $\frac{g^2 N}{16\pi^2}$ is small enough, then we can trust, at least qualitatively, the results we will obtain. Now we are ready to rewrite (3.25) in terms of $\tilde{\Delta}$. After a little algebra, one finds

$$\begin{aligned} E = & \frac{3}{16} (g^2)^{1-\frac{\gamma_0}{2\beta_0}} \frac{N}{N^2-1} \tilde{\Delta}^2 \left\{ -1 + \left[\frac{9N}{64\pi^2} \left(-\frac{5}{6} + L \right) - 2c_1 - c_4 \right] g^2 \right. \\ & + \left[\frac{3N^2}{256\pi^4} \left(c_3 + \frac{259}{8} L - \frac{31}{2} L^2 \right) - 2b_0 c_1 - c_1^2 - 2c_2 + \left(\frac{9N}{64\pi^2} \left(-\frac{5}{6} + L \right) - 2c_1 \right) c_4 \\ & \left. - c_5 + \frac{9N}{64\pi^2} \left(\left(-\frac{5}{6} + L \right) b_0 + c_1 + 2 \left(-\frac{5}{6} + L \right) c_1 + b_0 \left(1 - \frac{\gamma_0}{2\beta_0} \right) \right) \right] g^4 + \mathcal{O}(g^6) \right\} \end{aligned} \quad (3.37)$$

with

$$c_1 = \frac{1}{2} \left(\frac{\beta_1 \gamma_0}{\beta_0^2} - \frac{\bar{\gamma}_1}{\beta_0} \right) \quad (3.38)$$

$$c_2 = \frac{1}{8} \left(-\frac{\beta_1 \gamma_0}{\beta_0^2} + \frac{\bar{\gamma}_1}{\beta_0} \right)^2 - \frac{1}{4} \left(\frac{\gamma_0 \left(\frac{\beta_1^2}{\beta_0^2} - \frac{\bar{\beta}_2}{\beta_0} \right)}{\beta_0} \right) + \frac{1}{4} \left(\frac{\beta_1 \bar{\gamma}_1}{\beta_0^2} - \frac{\bar{\gamma}_2}{\beta_0} \right) \quad (3.39)$$

$$c_3 = -\frac{1043}{32} - \frac{63}{8} \zeta(2) + \frac{891}{16} s_2 \quad (3.40)$$

$$c_4 = b_0 \left(1 - \frac{\gamma_0}{\beta_0} \right) \quad (3.41)$$

$$c_5 = b_1 \left(1 - \frac{\gamma_0}{\beta_0} \right) - b_0^2 \frac{\gamma_0}{2\beta_0} \left(1 - \frac{\gamma_0}{\beta_0} \right). \quad (3.42)$$

Next, we determine b_0 and b_1 so that (3.37) reduces to

$$\begin{aligned} E = & \frac{3}{16} (g^2)^{1-\frac{\gamma_0}{2\beta_0}} \tilde{\Delta}^2 \frac{N}{N^2-1} \left\{ -1 + g^2 \frac{9N}{64\pi^2} L + g^4 \left[\frac{3N^2}{256\pi^4} \left(\frac{259}{8} L - \frac{31}{2} L^2 \right) \right. \right. \\ & \left. \left. + \frac{9N}{64\pi^2} (b_0 + c_4 + 2c_1) L \right] + \mathcal{O}(g^6) \right\}. \end{aligned} \quad (3.43)$$

We find that b_0 is

$$b_0 = \frac{409}{2288} \frac{N}{\pi^2} . \quad (3.44)$$

We do not list the value for b_1 since it is no longer required. From the β -function we find the two-loop expression for the coupling constant is

$$g^2(\bar{\mu}) = \frac{1}{\beta_0 \ln \frac{\bar{\mu}^2}{\Lambda^2}} - \frac{\beta_1}{\beta_0} \frac{\ln \ln \frac{\bar{\mu}^2}{\Lambda^2}}{\beta_0^2 \ln^2 \frac{\bar{\mu}^2}{\Lambda^2}} \quad (3.45)$$

where Λ is the scale parameter of the corresponding massless renormalization scheme. We will express everything in terms of the $\overline{\text{MS}}$ scale parameter $\Lambda_{\overline{\text{MS}}}$. In [141], it was shown that

$$\Lambda = \Lambda_{\overline{\text{MS}}} e^{-\frac{b_0}{2\beta_0}} . \quad (3.46)$$

We will also derive a value for the $\langle \frac{\alpha_s}{\pi} F_{\mu\nu}^2 \rangle$ condensate from the trace anomaly

$$\Theta_{\mu\mu} = \frac{\beta(g)}{2g} (F_{\rho\sigma}^a)^2 . \quad (3.47)$$

This anomaly allows us to deduce for $N = 3$ the following relation between the vacuum energy and the gluon condensate

$$\left\langle \frac{\alpha_s}{\pi} F_{\mu\nu}^2 \right\rangle = - \frac{32}{11} E_{\text{vac}} . \quad (3.48)$$

At one-loop order, the results for $N = 3$ are

$$\left. \frac{g^2 N}{16\pi^2} \right|_{\text{one-loop}} = \frac{8}{9} \quad (3.49)$$

$$\left. \sqrt{\widetilde{\Delta}} \right|_{\text{one-loop}} \approx 1.004 \Lambda_{\overline{\text{MS}}} \approx 233 \text{MeV} \quad (3.50)$$

$$\left. E_{\text{vac}} \right|_{\text{one-loop}} \approx -0.0074 \Lambda_{\overline{\text{MS}}}^4 \approx -0.00002 \text{GeV}^4 \quad (3.51)$$

$$\left\langle \frac{\alpha_s}{\pi} F_{\mu\nu}^2 \right\rangle \Big|_{\text{one-loop}} \approx 0.02 \Lambda_{\overline{\text{MS}}}^4 \approx 0.00007 \text{GeV}^4 \quad (3.52)$$

while the scale parameter $\bar{\mu}^2 \approx (184 \text{MeV})^2$. We have used $\Lambda_{\overline{\text{MS}}} \approx 233 \text{MeV}$ which was the value reported in [37]. We see that the one-loop expansion parameter is quite large and we conclude that we should go to the next order where the situation is improved. We find

$$\left. \frac{g^2 N}{16\pi^2} \right|_{\text{two-loop}} \approx 0.131 \quad (3.53)$$

$$\left. \sqrt{\widetilde{\Delta}} \right|_{\text{two-loop}} \approx 2.3 \Lambda_{\overline{\text{MS}}} \approx 536 \text{MeV} \quad (3.54)$$

$$\left. E_{\text{vac}} \right|_{\text{two-loop}} \approx -0.63 \Lambda_{\overline{\text{MS}}}^4 \approx -0.002 \text{GeV}^4 \quad (3.55)$$

$$\left\langle \frac{\alpha_s}{\pi} F_{\mu\nu}^2 \right\rangle \Big|_{\text{two-loop}} \approx 1.84 \Lambda_{\overline{\text{MS}}}^4 \approx 0.005 \text{GeV}^4 . \quad (3.56)$$

with $\bar{\mu}^2 \approx (347 \text{MeV})^2$. Although there is a sizeable difference between one-loop and two-loop results, the relative smallness of the two-loop expansion parameter, indicates that the two-loop values are

qualitatively trustworthy. It is well known that in order to find reliable perturbative results, one must go beyond one-loop, and even beyond two-loop approximations. Therefore, one should not attach a firm quantitative meaning to the numerical values. Let us compare our results with what was found elsewhere with different methods. A combined lattice fit resulted in $\sqrt{\langle A_\mu^2 \rangle} \approx 1.64 \text{ GeV}$, [37]. We

cannot really compare this with our result for $\sqrt{\widetilde{\Delta}}$, since the lattice value was obtained with the OPE at a scale $\mu = 10 \text{ GeV}$ in a specific renormalization scheme (MOM). However, it is satisfactory that (3.54) is at least of the same order of magnitude. More interesting is the comparison with what was found in [42] with the local composite operator formalism for $\langle A_\mu^2 \rangle$. In the $\overline{\text{MS}}$ scheme at two-loop order, it was found that $\frac{\bar{g}^2 N}{16\pi^2} \approx 0.141247$ while $E_{\text{vac}} \approx -0.789 \Lambda_{\overline{\text{MS}}}^4$ which is in quite good agreement with our results. An estimate of the tree level gluon mass of $\sim 500 \text{ MeV}$ was also given in [42] which compares well with the lattice value of $\sim 600 \text{ MeV}$ of [47, 48]. With the $2PPI$ method, one does not really have the concept of a tree level mass. Instead, one would need the calculation of the highly non-trivial two-loop $2PPI$ mass renormalization graphs which is beyond the scope of this article.

In conclusion we note that the perturbative Yang-Mills vacuum is unstable and lowers its value through a non-perturbative mass dimension two gluon condensate $\langle A_\mu^2 \rangle$. We have omitted quark contributions in our analysis but it is straightforward to extend the $2PPI$ expansion to QCD with quarks included. Indeed an idea of the effect they have could be gained by an extension of [42].

Chapter 4

On ghost condensation, mass generation and Abelian dominance in the maximal Abelian gauge

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published in Journal of Physics A: Math.Gen. 36 (2003) 8507.*

Recent work claimed that the off-diagonal gluons (and ghosts) in pure Yang-Mills theories, with maximal Abelian gauge fixing (MAG), attain a dynamical mass through an off-diagonal ghost condensate. This condensation takes place due to a quartic ghost interaction, unavoidably present in MAG for renormalizability purposes. The off-diagonal mass can be seen as evidence for Abelian dominance. We discuss why ghost condensation of the type discussed in those works cannot be the reason for the off-diagonal mass and Abelian dominance, since it results in a tachyonic mass. We also point out what the full mechanism behind the generation of a real mass might look like.

4.1 Introduction.

As everybody knows, quarks are confined: nature as well as lattice simulations of nature are telling us that. Still, there is no rigorous proof of confinement. One proposal for the explanation of confinement is the idea of the dual superconductor: magnetic monopoles condense and induce a dual Meissner effect: color-electric flux between charges is squeezed and a string is created in between. The original work on this topic can be found in [65, 66, 67]. Abelian projection [68] is a way to reveal the relevant degrees of freedom (the monopoles). In a loose way of speaking, at points where the projection is ill-defined, singularities invoke (Abelian) monopoles. Abelian dominance means that low energy QCD is dominated by Abelian degrees of freedom. Some early work on this is presented in [73]. Numerical evidence can be found in e.g. [76, 75, 74] and more recently [49].

Can this Abelian dominance be founded on more theoretical grounds? In the light of renormalization à la Wilson, and assuming that the *off-diagonal* gluons (ghosts) attain a mass M while the *diagonal* ones remain massless, an effective theory in terms of the massless diagonal fields could be achieved at low energy ($\ll M$), thereby realizing a kind of Abelian dominance. In the context of low energy theories, we like to refer to the Appelquist-Carazzone decoupling theorem [166], which states that heavy particle modes decouple at low energy. Notice that this decoupling does not mean “heavy terms” are simply

removed by hand from the Lagrangian, their influence is still present through renormalization effects. As an illustration of this: a low energy, *Abelian* theory for Yang-Mills was derived in [99], but the corresponding β -function was shown to be the same as the full Yang-Mills one.

The aforementioned pathway has been followed in a series of papers by Kondo et al [99, 98, 80, 167, 168, 96, 165, 97] and more recently the technique of the exact renormalization group has been employed by Freire [100, 101] to construct effective low energy descriptions of Yang-Mills theory. The results have been used in order to construct a linearly rising potential between static quarks, a criterion for confinement. Their efforts were based on the dual superconductor picture, realized with MAG. Also the monopole condensation was discussed in their framework. An essential ingredient of their work is the mass scale of the off-diagonal fields. The monopole condensate is proportional to this mass squared [99]. The lattice reported a value of approximately 1.2 GeV for the off-diagonal gluon mass in MAG Yang-Mills [49]. Next to these numerical results, analytical information is needed how this mass raises. A few papers have been written on this issue [77, 78, 79, 80]. All these authors came to the same conclusion: *a dimension two ghost condensation gives an off-diagonal mass M* . We already mentioned (but did not show explicitly) in a previous paper that we found the ghost condensation gives a *tachyonic* off-diagonal gluon mass [169]. In this paper, we will perform the calculations explicitly step by step. To make it self-contained, we will start from the beginning and in order to make comparison as transparent as possible, we will follow the (notational) conventions of [80]. For the sake of simplicity, we will restrict ourselves to the $SU(2)$ case. We discuss the (in)completeness of presented work. We end with the path we intend to follow in the future to investigate dynamical mass generation in MAG.

4.2 Ghost condensation in the maximal Abelian gauge.

Consider the Yang-Mills Lagrangian in four-dimensional Minkowski space time

$$\mathcal{L} = -\frac{1}{4}\mathcal{F}_{\mu\nu}^A\mathcal{F}^{A\mu\nu} + \mathcal{L}_{GF+FP} \quad (4.1)$$

where $\mathcal{L}_{GF+FP}$ is the gauge fixing and Faddeev-Popov part.

We decompose the gauge field as

$$\mathcal{A}_\mu = \mathcal{A}_\mu^A T^A = a_\mu T^3 + A_\mu^a T^a \quad (4.2)$$

$$\mathcal{F}_{\mu\nu} = \mathcal{F}_{\mu\nu}^A T^A = \mathcal{D}_\mu \mathcal{A}_\nu - \mathcal{D}_\nu \mathcal{A}_\mu = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - ig [\mathcal{A}_\mu, \mathcal{A}_\nu] \quad (4.3)$$

The T^A 's are the Hermitian generators of $SU(2)$ and obey the commutation relations $[T^A, T^B] = if^{ABC}T^C$. T^3 is the diagonal generator. The capital index A runs from 1 to 3. Small indices like $a, b, \dots$ run from 1 to 2 and label the off-diagonal components. We will drop the index 3 later on.

As a gauge fixing procedure, we use MAG. Introducing the functional

$$\mathcal{R}[A] = (VT)^{-1} \int d^4x \left(\frac{1}{2} A_\mu^a A^{\mu a} \right) \quad (4.4)$$

with VT the space time volume, MAG is defined as that gauge which minimizes $\mathcal{R}$ under local gauge transformations. Since (4.4) is invariant under $U(1)$ transformations w.r.t. the “photon” a_μ , MAG is only a partial gauge fixing. We do not fix the residual $U(1)$ gauge freedom, since it plays no role for what we are discussing here.

To implement the gauge fixing in the Lagrangian (4.1), we use the so-called modified MAG. This gauge is slightly different from the ordinary MAG, it possesses for instance some more symmetry (see [165])

and references therein). Moreover, it generates the four-point ghost interaction, indispensable for the renormalizability of MAG, as was proven in [71]. Explicitly, we get

$$\mathcal{L}_{GF+FP} = i\delta_B \bar{\delta}_B \left(\frac{1}{2} A_\mu^a A^{\mu a} - \frac{\alpha}{2} i C^a \bar{C}^a \right) \quad (4.5)$$

where α is a gauge parameter, C and $\bar{C}$ denote the (off-diagonal) ghosts and anti-ghosts, δ_B and $\bar{\delta}_B$ are the BRST and anti-BRST transformation respectively, defined by¹

$$\begin{aligned} \delta_B A_\mu &= \mathcal{D}_\mu C = \partial_\mu C - ig [A_\mu, C] \\ \delta_B C &= \frac{ig}{2} [C, C] \\ \delta_B \bar{C} &= i\mathcal{B} \\ \delta_B \mathcal{B} &= 0 \end{aligned} \quad (4.6)$$

$$\begin{aligned} \bar{\delta}_B A_\mu &= \mathcal{D}_\mu \bar{C} = \partial_\mu \bar{C} - ig [A_\mu, \bar{C}] \\ \bar{\delta}_B \bar{C} &= \frac{ig}{2} [\bar{C}, \bar{C}] \\ \bar{\delta}_B C &= i\bar{\mathcal{B}} \\ \bar{\delta}_B \bar{\mathcal{B}} &= 0 \\ \mathcal{B} + \bar{\mathcal{B}} &= g [C, \bar{C}] \end{aligned} \quad (4.7)$$

with the following properties

$$\begin{aligned} \delta_B^2 &= \bar{\delta}_B^2 = \{\delta_B, \bar{\delta}_B\} = 0 \\ \delta_B (XY) &= \delta_B (X) Y \pm X \delta_B (Y) \\ \bar{\delta}_B (XY) &= \bar{\delta}_B (X) Y \pm X \bar{\delta}_B (Y) \end{aligned} \quad (4.8)$$

where the upper sign is taken for bosonic X , and the lower sign for fermionic X .

Performing the BRST and anti-BRST transformations, yields

$$\begin{aligned} \mathcal{L}_{GF+FP} &= B^a D_\mu^{ab} A^{\mu b} + \frac{\alpha}{2} B^a B^a + i\bar{C}^a D_\mu^{ac} D^{\mu cb} C^b - ig^2 \epsilon^{ad} \epsilon^{cb} \bar{C}^a C^b A^{\mu c} A_\mu^d \\ &+ i\bar{C}^a g \epsilon^{ab} C^3 D_\mu^{bc} A^{\mu c} - i\alpha g \epsilon^{ab} B^a \bar{C}^b C^3 + \frac{\alpha}{4} g^2 \epsilon^{ab} \epsilon^{cd} \bar{C}^a \bar{C}^b C^c C^d \end{aligned} \quad (4.9)$$

where

$$D_\mu^{ab} \equiv D_\mu^{ab}[a] = \partial_\mu \delta^{ab} - g \epsilon^{ab} a_\mu \quad (4.10)$$

is the covariant derivative w.r.t. the $U(1)$ symmetry and

$$\epsilon^{12} = -\epsilon^{21} = 1 \quad (4.11)$$

$$\epsilon^{11} = \epsilon^{22} = 0 \quad (4.12)$$

When we integrate the multipliers $\mathcal{B}$ out, we finally obtain

$$\mathcal{L}_{GF+FP} = -\frac{1}{2\alpha} (D_\mu^{ab} A^{\mu b})^2 + i\bar{C}^a D_\mu^{ac} D^{\mu cb} C^b - ig^2 \epsilon^{ad} \epsilon^{cb} \bar{C}^a C^b A^{\mu c} A_\mu^d + \frac{\alpha}{4} g^2 \epsilon^{ab} \epsilon^{cd} \bar{C}^a \bar{C}^b C^c C^d \quad (4.13)$$

¹ $C = (C^a, C^3)$ with C^3 the diagonal ghost. Analogously for $\bar{C}$.

Notice that the diagonal ghost C^3 has dropped out of (4.13).

For the (singular) choice $\alpha = 0$, the 4-ghost interaction cancels from the Lagrangian. However, radiative corrections due to the other, non-vanishing quartic interactions, reintroduce this term. We further assume that $\alpha \neq 0$. Some more details concerning the properties for $\alpha = 0$ can be found in [77].

To discuss the ghost condensation mechanism, we “Gaussianize” the 4-ghost interaction in the Lagrangian by means of the ($U(1)$ invariant) auxiliary field ϕ

$$\frac{\alpha}{4} g^2 \epsilon^{ab} \epsilon^{cd} \bar{C}^a \bar{C}^b C^c C^d \rightarrow -\frac{1}{2\alpha g^2} \phi^2 - i\phi \epsilon^{ab} \bar{C}^a C^b \quad (4.14)$$

A useful identity to prove (4.14), reads

$$\epsilon^{ab} \epsilon^{cd} \bar{C}^a \bar{C}^b C^c C^d = 2 \left(i\epsilon^{ab} \bar{C}^a C^b \right)^2 \quad (4.15)$$

The part of the Lagrangian which concerns us for the moment is

$$\hat{\mathcal{L}} = i\bar{C}^a \partial_\mu \partial^\mu C^a - \frac{1}{2\alpha g^2} \phi^2 - i\phi \epsilon^{ab} \bar{C}^a C^b \quad (4.16)$$

Assuming constant ϕ , we use the Coleman-Weinberg construction [170] of the effective potential $V(\phi)$. This means we are summing all one-loop (off-diagonal) ghost bubbles with any number of ϕ -insertions. This yields

$$(VT)V(\phi) = \int d^4x \frac{\phi^2}{2\alpha g^2} + i \ln \det (\partial_\mu \partial^\mu \delta^{ab} - \phi \epsilon^{ab}) \quad (4.17)$$

or

$$V(\phi) = \frac{\phi^2}{2\alpha g^2} - \frac{1}{i} \int \frac{d^4k}{(2\pi)^4} \ln(k^4 + \phi^2) \quad (4.18)$$

Employing the Wick rotation² $k_0 \rightarrow ik_0$, and performing the integration in dimensional regularization within the $\overline{\text{MS}}$ scheme, we arrive at

$$V(\phi) = \frac{\phi^2}{2\alpha g^2} + \frac{\phi^2}{32\pi^2} \left(\ln \frac{\phi^2}{\mu^4} - 3 \right) \quad (4.19)$$

This potential possesses a local maximum at $\phi = 0$ (the usual vacuum), but has global minima at

$$\phi = \pm v = \pm \bar{\mu}^2 e^{1 - \frac{8\pi^2}{\alpha \bar{g}^2 (\bar{\mu}^2)}} \quad (4.20)$$

We take $\alpha > 0$ since v diverges for $\bar{g}^2 \rightarrow 0$ if $\alpha < 0$.

Up to now, we find complete agreement with [80]. We proceed by calculating the ghost propagator in the non-zero vacuum ($V(v) < 0$). Substituting $\phi = v$ in (4.16), it is straightforward to determine the Feynman propagator

$$\langle C^a(x) \bar{C}^b(y) \rangle = \int \frac{d^4k}{(2\pi)^4} \frac{-k^2 \delta^{ab} + v \epsilon^{ab}}{k^4 + v^2} e^{-ik(x-y)} \quad (4.21)$$

With the above propagator, we are ready to determine the one-loop off-diagonal gauge boson polarization. Now, there exists a non-trivial contribution coming from the ghost bubble, originating in the

²If one would like to avoid Wick rotations, one could start immediately from the Euclidean version of Yang-Mills.

interaction term $-ig^2\epsilon^{ad}\epsilon^{cb}\bar{C}^a C^b A^{\mu c} A_\mu^d$, resulting in a mass M for the off-diagonal gluons. Again Wick rotating $k_0 \rightarrow ik_0$ to get an integral over Euclidean space time, one easily obtains

$$M^2 = g^2 \int \frac{d^4 k}{(2\pi)^4} \frac{2k^2}{k^4 + v^2} \quad (4.22)$$

There is one remaining step, we still have to calculate the integral of (4.22). Using dimensional regularization, we find the finite result

$$M^2 = \frac{-g^2 v}{16\pi} < 0 \quad (4.23)$$

where we have used that $v > 0$. Here we find a *different* result in comparison with the other references [77, 78, 79, 80]. To be more precise, we find the opposite sign. This sign difference is not meaningless, since the negative sign we find means that the off-diagonal fields have a *tachyonic* mass.

Hence, we state that a ghost condensation à la $\langle \epsilon^{ab}\bar{C}^a C^b \rangle$ is *not* the mechanism behind the off-diagonal mass generation in MAG, and consequently does not give evidence for Abelian dominance.

Another important point is what happens with the diagonal gluon. Consider the term $i\bar{C}^a D_\mu^{ac} D^{\mu cb} C^b$ of (4.13), it contains a part proportional to $i\bar{C}^a C^a a_\mu a^\mu$. Doing the same as for the off-diagonal gluons, the diagonal gluon a_μ seems to get a (real) mass too, which is of the same order as the off-diagonal one (up to the sign). However, there are other one-loop contributions coming from the terms proportional to $\epsilon^{ab}(\partial_\mu \bar{C}^a) C^b a^\mu$ and $\epsilon^{ab}\bar{C}^a (\partial_\mu C^b) a^\mu$. These contributions cancel the one coming from the term proportional to $i\bar{C}^a C^a a_\mu a^\mu$. Consequently, the “photon” a_μ remains massless, as could be expected by the residual $U(1)$ invariance.

Another point of concern is the renormalizability of the “Gaussianized” Lagrangian. A completely analogous approach can be done in case of the two-dimensional Gross-Neveu model [21], where the quartic fermion interaction can also be made Gaussian by the introduction of an auxiliary field σ . This works well at one-loop order, but from two loops on, *ad hoc* counterterms have to be added in order to end up with finite results [22]. A successful formalism to deal with local composite operators in case of the Gross-Neveu model was developed in [24]. A similar approach should be used to investigate the ghost condensates.

One could wonder what the mechanism behind the mass generation might be, since the previous paragraphs showed that we did not find a dynamically generated real mass for the (off-diagonal) particles. In order to find an answer to this question, we first give a very short overview of recent results in the Landau gauge, giving us a hint in which direction we should look for the mass generation.

4.3 Gluon condensation via $\mathcal{A}^2$ in the Landau gauge and its nephew A^2 in the maximal Abelian gauge.

A well known condensate in QCD (or Yang-Mills) is the dimension four gluon condensate $\langle \mathcal{F}_{\mu\nu}^A \mathcal{F}^{\mu\nu A} \rangle$. This is the lowest dimensional gluonic condensate that can exist, since no local, gauge-invariant condensates with dimension lower than 4 exist. However, recently interest arised concerning a dimension 2 gluon condensate in Yang-Mills theory in the Landau gauge. One way it came to attention was the conclusion that there exists a non-negligible discrepancy between the lattice strong coupling constant α_s (determined via the 3-point gluon interaction) and the perturbative one, into a relatively high energy region where this would not be expected (up to 10 GeV). Also the propagator showed a similar discrepancy. The $\frac{1}{p^4}$ power correction due to $\langle \mathcal{F}^2 \rangle$ is far to small to explain this. It was shown that a $\frac{1}{p^2}$ power

correction could solve the discrepancy. More precisely, the Operator Product Expansion (OPE) used in combination with the $\langle \mathcal{A}^2 \rangle$ condensate was able to fit both predictions [37, 39]. An important question that naturally arises, sounds: has $\langle \mathcal{A}^2 \rangle$ any physical meaning, or is it merely a gauge artefact? The point is that $\mathcal{A}^2$ equals $(VT)^{-1} \min_U \int d^4x \mathcal{A}_\mu^U \mathcal{A}^{\mu U}$ in the Landau gauge, and this latter operator is, although non-local, gauge-invariant. Hence, $\mathcal{A}^2$ can be given some physical sense in the Landau gauge. Moreover, [33] discussed the relevance of $\mathcal{A}^2$ in connection with topological structure (monopoles) of compact QED. The physical relevance of the Landau gauge, in the framework of geometrical monopoles, is explained in [34]. The authors of that paper also stress that the values found with an OPE calculation, only describe the soft (infrared) content of $\langle \mathcal{A}^2 \rangle$, while they argue that also hard (short range) contributions, inaccessible for OPE, may occur. In this context, we cite [42], where a formalism was constructed for the calculation of the vacuum expectation value of (local) *composite* operators. Since this is based on the effective action, it should in principle, give the “full” value of $\langle \mathcal{A}^2 \rangle$, i.e. soft and hard part. For example, one could assume an instanton background as a possible source of long range contributions. In fact, there is some preliminary evidence that instantons can explain the OPE values [38]. The conclusion that one can draw from all this is that the dimension 2 condensate $\langle \mathcal{A}^2 \rangle$ may have some physical relevance in the Landau gauge.

Let us go back to MAG³ now. In this particular gauge, $(VT)^{-1} \min_U \int d^4x \mathcal{A}_\mu^U \mathcal{A}^{\mu U}$ no longer reduces to a local operator. It would be interesting to repeat e.g. the OPE calculations of [37] for the coupling constant and propagators in MAG, but which dimension 2 condensate(s) could take over the role of $\langle \mathcal{A}^2 \rangle$ in the Landau gauge? To solve this, we draw attention to the striking similarity existing between the Landau gauge and MAG. The former one can be seen as that gauge minimizing $(VT)^{-1} \int d^4x (\mathcal{A}_\mu^A \mathcal{A}^{\mu A})^U$, while the latter one minimizes $(VT)^{-1} \int d^4x (A_\mu^a A^{\mu a})^U$. This operator reduces to the local one A^2 in MAG and can be seen as the MAG version of $\mathcal{A}^2$. Other dimension 2 condensates can exist (the ghost condensates). Notice that, in the case of the MAG, all these ghost condensates are $U(1)$ invariants, hence the $U(1)$ symmetry will be preserved.

The physics we see behind all these condensates is that they might have a common, deeper reason for existence. In this context, we quote [28, 29], where it was shown that the zero vacuum is instable (tachyonic) and a vacuum with lower energy is achieved through gluon pairing, and an accompanying gluon mass. The vacuum energy itself is a physical object. After choosing a certain gauge, the different types of dimension 2 condensates are just an expression of the fact that $E = 0$ is a wrong vacuum state. In this sense, all these different condensates in different gauges are equivalent in a way, since they lower the vacuum energy to a stable $E < 0$ vacuum. In [34], discussion can be found on the appearance of the soft part of $\langle \mathcal{A}^2 \rangle$ in gauge-variant quantities like in an OPE improvement of the gluon propagator, while the hard part enters physical quantities. It is imaginable that the mechanism behind this hard part (see [34] for more details) is the same in different gauges, but reveals its importance with different condensates, depending on the specific gauge. This might justify its possible appearance in gauge-invariant quantities.

4.4 Further discussion on the ghost condensation and mass generation in the modified MAG.

In [171], it was shown that it is possible to fix the residual Abelian gauge freedom of MAG in such a way that the ghost condensate $\langle \epsilon^{ab} \bar{C}^a C^b \rangle$ does not give rise to any mass term, at least at the one-loop level. This Abelian gauge fixing (needed for a complete quantization of the theory) was based on the requirement that the fully gauge fixed Lagrangian has a $SL(2, \mathbb{R})$ and anti-BRST invariance.

³Here, with MAG we mean the gauge minimizing the functional (4.4), and not the modified MAG.

A restricted⁴ version of this $SL(2, \mathbb{R})$ symmetry was originally observed in $SU(2)$ MAG in [77], and later generalized to $SU(N)$ MAG [164]. In [171], the symmetry was defined on all the fields (diagonal and off-diagonal). In fact, that $SL(2, \mathbb{R})$ symmetry together with the (anti-) BRST symmetry form a larger algebra, the Nakanishi-Ojima (NO) algebra. This NO algebra is known to generate a symmetry of the Landau gauge and a certain class of generalized covariant gauges, more precisely the Curci-Ferrari gauges, given by the gauge fixing Lagrangian

$$\mathcal{L}_{GF+FP} = i\delta_B \bar{\delta}_B \left(\frac{1}{2} \mathcal{A}_\mu^A \mathcal{A}^{\mu A} - \frac{\alpha}{2} i \mathcal{C}^A \bar{\mathcal{C}}^A \right) \quad (4.24)$$

The Landau gauge corresponds to the gauge parameter choice $\alpha = 0$. For more details, see [86, 158, 159, 160, 161, 162, 163].

Yang-Mills theory with the gauge fixing (4.24) possesses a generalization to a massive $SU(N)$ gauge model, the so-called Curci-Ferrari model [84, 85]. Although this model is non-unitary, it is known to be (anti-)BRST invariant and renormalizable, whereby the mass term is of the form

$$\mathcal{L}_{mass} = M^2 \left(\frac{1}{2} \mathcal{A}_\mu^A \mathcal{A}^{\mu A} - i\alpha \mathcal{C}^A \bar{\mathcal{C}}^A \right) \quad (4.25)$$

Keeping this in mind and recalling that in [42], a dynamically generated mass was found in case of the Landau gauge by coupling a source J to the operator A^2 , it becomes clear that in case of the Curci-Ferrari gauge, the same technique could be employed by coupling a source J to the composite operator

$$\mathcal{L}_{source} = J \left(\frac{1}{2} \mathcal{A}_\mu^A \mathcal{A}^{\mu A} - i\alpha \mathcal{C}^A \bar{\mathcal{C}}^A \right) \quad (4.26)$$

Returning to the case of MAG and comparing the gauge fixing Lagrangians (4.5) and (4.24), the equivalent of (4.26) reads

$$\mathcal{L}_{source} = J \left(\frac{1}{2} A_\mu^a A^{\mu a} - i\alpha C^a \bar{C}^a \right) \quad (4.27)$$

This idea to arrive at a dynamically generated mass in case of the Curci-Ferrari and maximal Abelian gauge was already proposed in [83, 144]. There, it was explicitly shown that the operator coupled to the source J in the expressions (4.26) or (4.27), is on-shell BRST invariant.

We reserve the actual discussion of the aforementioned framework to get a dynamical mass for future publications, since it is quite involved and a clean treatment of it needs a combination of the local composite operator formalism [42] and the algebraic renormalization technique [59, 60].

Before turning to conclusions, we want to draw attention to the following. We decomposed the 4-ghost interaction with a real auxiliary field ϕ whereby $\phi \sim \epsilon^{ab} \bar{C}^a C^b$. Let us make a small comparison with ordinary superconductivity. Usually, there is talked about BCS pairing, i.e. particle-particle and hole-hole pairing. The analogy of this in the ghost condensation case would be ghost-ghost pairing and antighost-antighost pairing. This can be achieved by an alternative decomposition of the 4-ghost interaction via a pair of auxiliary fields σ and $\bar{\sigma}$ such that $\sigma \sim \epsilon^{ab} C^a C^b$ and $\bar{\sigma} \sim \epsilon^{ab} \bar{C}^a \bar{C}^b$. This kind of pairing⁵ was considered in [81]. A less known effect is the particle-hole pairing, the so-called Overhauser pairing [92]. This corresponds to the kind of condensation we and the papers [77, 78, 79, 80] considered. From the viewpoint of the $SL(2, \mathbb{R})$ symmetry, the existence of different channels where

⁴By restricted, we mean that the symmetry only acts non-trivially on the off-diagonal fields.

⁵Our conclusion about the tachyonic mass is unaltered by this alternative decomposition of the 4-ghost interaction.

the ghost condensation can take place should not be surprising. The different composite ghost operators are mutually changed into each other under the action of the symmetry. Here and in the other papers the *choice* was made to work with the Overhauser channel, but a complete treatment would need an analysis of all channels at once, and with the local composite operator technique. This analysis of the BCS versus Overhauser effect is nicely intertwined with the existence of the NO algebra and its (partial) breakdown, and it is very much alike for the MAG, Landau [172] and Curci-Ferrari gauge, just as in case of the mass generation mechanism. As an indication, it has been found recently that, although no 4-ghost interaction is present in the Landau gauge, the condensation à la $f^{ABC}\bar{C}^A C^B$ etc. also occurs [91].

4.5 Conclusion.

We considered Yang-Mills theory in the maximal Abelian gauge. With this non-linear gauge choice, a 4-ghost interaction enters the Lagrangian. Such an interaction could allow a non-zero vacuum expectation value for (off-diagonal) dimension 2 ghost condensates. Consequently, it was expected that a mass generating mechanism for the off-diagonal gluons *and* the diagonal gluons due to quartic interaction terms of the form gluon-gluon-ghost-anti-ghost was found.

We explained why this particular type of ghost condensation is not sufficient to construct a (off-diagonal) dynamical mass in $SU(2)$ Yang-Mills theory in the maximal Abelian gauge, an indicator for Abelian dominance. We have restricted ourselves to the $SU(2)$ case, but a similar conclusion will exist for general $SU(N)$. Explicit calculations showed that we ended up with a tachyonic off-diagonal mass M ($M^2 < 0$). This result indicate something is missing. A comparison with Yang-Mills theory in the Landau gauge and the role played by the mass dimension 2 gluon condensate $\langle A^2 \rangle$, shed some light on the route that should be followed.

We revealed certain shortcomings of the present available studies on the ghost condensation (renormalizability, existence of more than one condensation channel).

The actual study of the mass generation and the ghost condensation with its symmetry breaking pattern will be discussed elsewhere. We will follow the local composite operator formalism of [42], where a source is coupled to each operator and the effective action can be treated consistently. This effective potential formalism allows a clean treatment of the role played by the dimension 2 operators. We remark that with essentially perturbative techniques one can obtain at least qualitatively trustworthy results⁶ on the stability of the condensates and their relevance for e.g. mass generation and symmetry breakdown, without making it directly necessary to go to (or extrapolating to) strong coupling.

We conclude by mentioning that the dimension 2 condensates and the accompanying mass generation in Yang-Mills are not only of theoretical importance (the role of $\langle A^2 \rangle$ for OPE corrections [39, 37], monopoles [33, 34], short range linear correction to the Coulomb-like potential [39], low energy effective theories [147],...) but also have their importance for automated Feynmandiagram calculations [87, 173, 174] where a gluon mass serves as a infrared regulator. If this mass is generated in massless Yang-Mills, it does not have to be implemented by hand.

⁶In case of the Gross-Neveu model, very accurate results were obtained [24]. In case of the A^2 condensate in the Landau gauge, the relevant coupling constant was quite small, making the expansion acceptable [42].

Chapter 5

On the $SL(2, \mathbb{R})$ symmetry in Yang-Mills Theories in the Landau, Curci-Ferrari and maximal Abelian gauge

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The existence of a $SL(2, \mathbb{R})$ symmetry is discussed in $SU(N)$ Yang-Mills in the maximal Abelian gauge. This symmetry, also present in the Landau and Curci-Ferrari gauge, ensures the absence of tachyons in the maximal Abelian gauge at the one-loop level in the presence of ghost condensates. In all these gauges, $SL(2, \mathbb{R})$ turns out to be dynamically broken by these ghost condensates.

5.1 Introduction.

It is widely believed that the dual superconductivity mechanism [65, 66, 67, 68] can be at the origin of color confinement. The key ingredients of this mechanism are the Abelian dominance and the monopoles condensation. According to the dual superconductivity picture, the low energy behavior of QCD should be described by an effective Abelian theory in the presence of monopoles. The condensation of the monopoles gives rise to the formation of Abrikosov-Nielsen-Olesen flux tubes which confine all chromoelectric charges. This mechanism has received many confirmations from lattice simulations in Abelian gauges, which are very useful in order to characterize the effective relevant degrees of freedom at low energies.

Among the Abelian gauges, the so called maximal Abelian gauge (MAG) plays an important role. This gauge, introduced in [68, 70, 69], has given evidences for the Abelian dominance and for the monopoles condensation, while providing a renormalizable gauge in the continuum. Here, the Abelian degrees of freedom are identified with the components of the gauge field belonging to the Cartan subgroup of the gauge group $SU(N)$. The other components correspond to the $(N^2 - N)$ off-diagonal generators of $SU(N)$ and, being no longer protected by gauge invariance, are expected to acquire a mass, thus

decoupling at low energies. The understanding of the mechanism for the dynamical mass generation of the off-diagonal components is fundamental for the Abelian dominance.

A feature to be underlined is that the MAG is a nonlinear gauge. As a consequence, a quartic self-interaction term in the Faddeev-Popov ghosts is necessarily required for renormalizability [71, 72]. Furthermore, as discussed in [77, 79, 79, 80] and later on in [81], the four ghost interaction gives rise to an effective potential whose vacuum configuration favors the formation of off-diagonal ghost condensates $\langle c\bar{c} \rangle$, $\langle cc \rangle$ and $\langle \bar{c}\bar{c} \rangle$ ¹. However, these ghost condensates were proven [157] to originate an unwanted effective tachyon mass for the off-diagonal gluons, due to the presence in the MAG of an off-diagonal interaction term of the type $AAc\bar{c}$.

Meanwhile, the ghost condensation has been observed in others gauges, namely in the Curci-Ferrari gauge [90] and in the Landau gauge [91]. The existence of these condensates turns out to be related to the dynamical breaking of a $SL(2, \mathbb{R})$ symmetry which is known to be present in both Curci-Ferrari and Landau gauge since long time [86, 158, 159, 160, 161, 162]. It is worth noticing that in the Curci-Ferrari gauge an on-shell BRST invariant mass term $(\frac{1}{2}\mathcal{A}^2 - \xi c\bar{c}) \equiv (\frac{1}{2}\mathcal{A}^A \mathcal{A}^A - \xi c^A \bar{c}^A)$, with $A = 1, \dots, N^2 - 1$ and ξ being the gauge parameter, can be introduced without spoiling the renormalizability of the model [84, 85, 163].

Recent investigations [83, 144] suggested that the existence of a nonvanishing condensate, given by $(\frac{1}{2}\langle \mathcal{A}^2 \rangle - \xi \langle c\bar{c} \rangle)$ could yield an on-shell BRST invariant dynamical mass for both gluons and ghosts. We observe that in the limit $\xi \rightarrow 0$, this condensate reduces to the pure gauge condensate $\frac{1}{2}\langle \mathcal{A}^2 \rangle$ whose existence is well established in the Landau gauge [42, 37, 38], providing indeed an effective mass for the gluons.

The aim of this paper is to show that the $SL(2, \mathbb{R})$ symmetry is also present in the MAG for $SU(N)$ Yang-Mills, with any value of N . This result generalizes that of [77, 79, 79, 80, 164], where the $SL(2, \mathbb{R})$ symmetry has been established only for a partially gauge-fixed version of the action. In particular, the requirement of an exact $SL(2, \mathbb{R})$ invariance for the complete quantized action, including the diagonal part of the gauge fixing, will have very welcome consequences. In fact, as we shall see, this requirement introduces new interaction terms in the action, which precisely cancel the term $AAc\bar{c}$ responsible for the generation of the tachyon mass. In other words, no tachyons are present in the MAG at one-loop order if the $SL(2, \mathbb{R})$ symmetry is required from the beginning as an exact invariance of the fully gauge-fixed action.

This observation allows us to make an interesting *trait d'union* between the Landau gauge, the Curci-Ferrari gauge and the MAG, providing a more consistent and general understanding of the ghost condensation and of the mechanism for the dynamical generation of the effective gluon masses. The whole framework can be summarized as follows. The ghost condensates signal the dynamical breaking of the $SL(2, \mathbb{R})$ symmetry, present in all these gauges. Also, the condensed vacuum has the interesting property of leaving unbroken the Cartan subgroup of the gauge group. As a consequence of the ghost condensation, the off-diagonal ghost propagator is modified in the infrared region [77, 79, 79, 80, 81, 157, 90]. This feature might be relevant for the analysis of the infrared behavior of the gluon propagator, through the ghost-gluon mixed Schwinger-Dyson equations. Moreover, the ghost condensates contribute to the dimension four condensate $\langle \frac{\alpha}{\pi} F^2 \rangle$ through the trace anomaly.

On the other hand, the dynamical mass generation for the gluons is expected to be related to the on-shell BRST invariant condensate $(\frac{1}{2}\langle \mathcal{A}^2 \rangle - \xi \langle c\bar{c} \rangle)$. It is remarkable that this condensate can be defined in a (on-shell) BRST invariant way also in the MAG [83, 144, 157], where it can give masses for all off-diagonal fields, thus playing a pivotal role for the Abelian dominance.

The paper is organized as follows. Section 5.2 is devoted to the analysis of the $SL(2, \mathbb{R})$ symmetry

¹The notation for the ghost condensates $\langle c\bar{c} \rangle$, $\langle cc \rangle$ and $\langle \bar{c}\bar{c} \rangle$ stands for $f^{iab}c^a\bar{c}^b$, $f^{iab}c^ac^b$ and $f^{iab}\bar{c}^a\bar{c}^b$, where f^{iab} are the structure constants of the gauge group. The index i runs over the Cartan generators, while the indices a, b correspond to the off-diagonal generators.

in the Landau, Curci-Ferrari and MAG gauge. In section 5.3 we prove that the requirement of the $SL(2, \mathbb{R})$ symmetry for the MAG in $SU(N)$ Yang-Mills yields a renormalizable theory. In section 5.4 we discuss the issue of the ghost condensation. In section 5.5 we present the conclusions.

5.2 Yang-Mills theories and the $SL(2, \mathbb{R})$ symmetry.

Let $\mathcal{A}_\mu$ be the Lie algebra valued connection for the gauge group $SU(N)$, whose generators T^A , $[T^A, T^B] = f^{ABC}T^C$, are chosen to be anti-Hermitian and to obey the orthonormality condition $\text{Tr}(T^A T^B) = \delta^{AB}$, with $A, B, C = 1, \dots, (N^2 - 1)$. The covariant derivative is given by

$$\mathcal{D}_\mu^{AB} \equiv \partial_\mu \delta^{AB} - g f^{ABC} \mathcal{A}_\mu^C. \quad (5.1)$$

Let s and $\bar{s}$ be the nilpotent BRST and anti-BRST transformations respectively, acting on the fields as

$$\begin{aligned} s\mathcal{A}_\mu^A &= -\mathcal{D}_\mu^{AB} c^B \\ s c^A &= \frac{g}{2} f^{ABC} c^B c^C \\ s\bar{c}^A &= b^A \\ s b^A &= 0, \end{aligned} \quad (5.2)$$

$$\begin{aligned} \bar{s}\mathcal{A}_\mu^A &= -\mathcal{D}_\mu^{AB} \bar{c}^B \\ \bar{s} c^A &= -b^A + g f^{ABC} c^B \bar{c}^C \\ \bar{s} \bar{c}^A &= \frac{g}{2} f^{ABC} \bar{c}^B \bar{c}^C \\ \bar{s} b^A &= -g f^{ABC} b^B \bar{c}^C. \end{aligned} \quad (5.3)$$

Here c^A and $\bar{c}^A$ generally denote the Faddeev-Popov ghosts and anti-ghosts, while b^A denote the Lagrange multipliers.

Furthermore, we define the operators δ and $\bar{\delta}$ by

$$\begin{aligned} \delta \bar{c}^A &= c^A \\ \delta b^A &= \frac{g}{2} f^{ABC} c^B c^C \\ \delta \mathcal{A}_\mu^A &= \delta c^A = 0, \end{aligned} \quad (5.4)$$

$$\begin{aligned} \bar{\delta} c^A &= \bar{c}^A \\ \bar{\delta} b^A &= \frac{g}{2} f^{ABC} \bar{c}^B \bar{c}^C \\ \bar{\delta} \mathcal{A}_\mu^A &= \bar{\delta} \bar{c}^A = 0. \end{aligned} \quad (5.5)$$

Together with the Faddeev-Popov ghost number operator δ_{FP} , δ and $\bar{\delta}$ generate a $SL(2, \mathbb{R})$ algebra. This algebra is a subalgebra of the algebra generated by δ_{FP} , δ , $\bar{\delta}$ and the BRST and anti-BRST

operators s and $\bar{s}$. The algebra

$$\begin{aligned}
s^2 &= 0, & \bar{s}^2 &= 0, \\
\{s, \bar{s}\} &= 0, & [\delta, \bar{\delta}] &= \delta_{FP}, \\
[\delta, \delta_{FP}] &= -2\delta, & [\bar{\delta}, \delta_{FP}] &= 2\bar{\delta}, \\
[s, \delta_{FP}] &= -s, & [\bar{s}, \delta_{FP}] &= \bar{s}, \\
[s, \delta] &= 0, & [\bar{s}, \bar{\delta}] &= 0, \\
[s, \bar{\delta}] &= -\bar{s}, & [\bar{s}, \delta] &= -s,
\end{aligned} \tag{5.6}$$

is known as the Nakanishi-Ojima (NO) algebra [86].

5.2.1 Landau gauge.

In the Landau gauge, we have

$$S = S_{YM} + S_{GF+FP} = -\frac{1}{4} \int d^4x \mathcal{F}_{\mu\nu}^A \mathcal{F}^{A\mu\nu} + s \int d^4x \bar{c}^A \partial_\mu \mathcal{A}^{A\mu}. \tag{5.7}$$

The BRST invariance is immediate, just as the δ invariance since

$$\delta S_{GF+FP} = s \int d^4x \bar{c}^A \partial_\mu \mathcal{A}^{A\mu} = \frac{s^2}{2} \int d^4x \mathcal{A}^2 = 0. \tag{5.8}$$

It is easy checked that also $\bar{s}$ and $\bar{\delta}$ leave the action (5.7) invariant. Hence, the NO algebra is a global symmetry of Yang-Mills theories in the Landau gauge, a fact already longer known [86, 158, 159].

5.2.2 Curci-Ferrari gauge.

Next, we discuss Yang-Mills theories in a class of generalized covariant non-linear gauges proposed in [160, 161]. The action is given by

$$\begin{aligned}
S &= S_{YM} + S_{GF+FP} \\
&= -\frac{1}{4} \int d^4x \mathcal{F}_{\mu\nu}^A \mathcal{F}^{A\mu\nu} + s\bar{s} \int d^4x \left(\frac{1}{2} \mathcal{A}_\mu^A \mathcal{A}^{A\mu} - \frac{\xi}{2} c^A \bar{c}^A \right) \\
&= -\frac{1}{4} \int d^4x \mathcal{F}_{\mu\nu}^A \mathcal{F}^{A\mu\nu} + \int d^4x \left(b^A \partial_\mu \mathcal{A}^{A\mu} + \frac{\xi}{2} b^A b^A + \bar{c}^A \partial^\mu \mathcal{D}_\mu^{AB} c^B \right. \\
&\quad \left. - \frac{\xi}{2} g f^{ABC} b^A \bar{c}^B c^C - \frac{\xi}{8} g^2 f^{ABC} f^{CDE} \bar{c}^A \bar{c}^B c^D c^E \right),
\end{aligned} \tag{5.9}$$

where ξ is the gauge parameter. A gauge fixing as in (5.9) is sometimes called the Curci-Ferrari (CF) gauge, since its gauge fixing part resembles the gauge fixing part of the massive, $SU(N)$ gauge model introduced in [84, 85].

In addition to the BRST and anti-BRST symmetries, the action (5.9) is also invariant under the global $SL(2, \mathbb{R})$ symmetry generated by δ , $\bar{\delta}$ [163, 164] and δ_{FP} . We conclude that Yang-Mills theories in the CF gauge have the NO symmetry.

5.2.3 Maximal Abelian gauge.

We decompose the gauge field into its off-diagonal and diagonal parts, namely

$$\mathcal{A}_\mu = \mathcal{A}_\mu^A T^A = A_\mu^a T^a + A_\mu^i T^i, \quad (5.10)$$

where the index i labels the $N - 1$ generators T^i of the Cartan subalgebra. The remaining $N(N - 1)$ off-diagonal generators T^a will be labelled by the index a . Accordingly, the field strength decomposes as

$$\mathcal{F}_{\mu\nu} = \mathcal{F}_{\mu\nu}^A T^A = F_{\mu\nu}^a T^a + F_{\mu\nu}^i T^i, \quad (5.11)$$

with the off-diagonal and diagonal parts given respectively by

$$\begin{aligned} F_{\mu\nu}^a &= D_\mu^{ab} A_\nu^b - D_\nu^{ab} A_\mu^b + g f^{abc} A_\mu^b A_\nu^c, \\ F_{\mu\nu}^i &= \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g f^{abi} A_\mu^a A_\nu^b, \end{aligned} \quad (5.12)$$

where the covariant derivative D_μ^{ab} is defined with respect to the diagonal components A_μ^i

$$D_\mu^{ab} \equiv \partial_\mu \delta^{ab} - g f^{abi} A_\mu^i. \quad (5.13)$$

For the pure Yang-Mills action one obtains

$$S_{\text{YM}} = -\frac{1}{4} \int d^4x (F_{\mu\nu}^a F^{a\mu\nu} + F_{\mu\nu}^i F^{i\mu\nu}). \quad (5.14)$$

The so called MAG gauge condition amounts to fix the value of the covariant derivative ($D_\mu^{ab} A^{b\mu}$) of the off-diagonal components [68, 70, 69]. However, this condition being nonlinear, a quartic ghost self-interaction term is required for renormalizability [71, 72]. The corresponding gauge fixing term turns out to be [165]

$$S_{\text{MAG}} = s\bar{s} \int d^4x \left(\frac{1}{2} A_\mu^a A^{a\mu} - \frac{\xi}{2} c^a \bar{c}^a \right), \quad (5.15)$$

where s denotes the nilpotent BRST operator

$$\begin{aligned} sA_\mu^a &= -(D_\mu^{ab} c^b + g f^{abc} A_\mu^b c^c + g f^{abi} A_\mu^b c^i), & sA_\mu^i &= -(\partial_\mu c^i + g f^{iab} A_\mu^a c^b), \\ sc^a &= g f^{abi} c^b c^i + \frac{g}{2} f^{abc} c^b c^c, & sc^i &= \frac{g}{2} f^{iab} c^a c^b, \\ s\bar{c}^a &= b^a, & s\bar{c}^i &= b^i, \\ sb^a &= 0, & sb^i &= 0, \end{aligned} \quad (5.16)$$

and $\bar{s}$ the nilpotent anti-BRST operator, which acts as

$$\begin{aligned} \bar{s}A_\mu^a &= -(D_\mu^{ab} \bar{c}^b + g f^{abc} A_\mu^b \bar{c}^c + g f^{abi} A_\mu^b \bar{c}^i), & \bar{s}A_\mu^i &= -(\partial_\mu \bar{c}^i + g f^{iab} A_\mu^a \bar{c}^b), \\ \bar{s}c^a &= g f^{abi} \bar{c}^b c^i + \frac{g}{2} f^{abc} \bar{c}^b c^c, & \bar{s}c^i &= \frac{g}{2} f^{iab} \bar{c}^a c^b, \\ \bar{s}c^i &= -b^i + g f^{ibc} c^b \bar{c}^c, & \bar{s}b^i &= -g f^{ibc} b^b \bar{c}^c, \\ \bar{s}c^a &= -b^a + g f^{abc} c^b \bar{c}^c + g f^{abi} c^b \bar{c}^i + g f^{abi} \bar{c}^b c^i, \\ \bar{s}b^a &= -g f^{abc} b^b \bar{c}^c - g f^{abi} b^b \bar{c}^i + g f^{abi} \bar{c}^b b^i. \end{aligned} \quad (5.17)$$

Here c^a and c^i are the off-diagonal and the diagonal components of the Faddeev-Popov ghost field, $\bar{c}^a$ and $\bar{c}^i$ the off-diagonal and the diagonal anti-ghost fields and b^a and b^i are the off-diagonal and diagonal Lagrange multipliers. These transformation are nothing else than the projection on diagonal and off-diagonal fields of (5.2) and (5.3). Expression (5.15) is easily worked out and yields

$$\begin{aligned}
S_{\text{MAG}} &= s \int d^4x \left(\bar{c}^a \left(D_\mu^{ab} A^{b\mu} + \frac{\xi}{2} b^a \right) - \frac{\xi}{2} g f^{abi} \bar{c}^a \bar{c}^b c^i - \frac{\xi}{4} g f^{abc} c^a \bar{c}^b \bar{c}^c \right) \\
&= \int d^4x \left(b^a \left(D_\mu^{ab} A^{b\mu} + \frac{\xi}{2} b^a \right) + \bar{c}^a D_\mu^{ab} D^{\mu bc} c^c + g \bar{c}^a f^{abi} (D_\mu^{bc} A^{c\mu}) c^i \right. \\
&\quad + g \bar{c}^a D_\mu^{ab} (f^{bcd} A^{c\mu} c^d) - g^2 f^{abi} f^{cdi} \bar{c}^a c^d A_\mu^b A^{c\mu} - \xi g f^{abi} b^a \bar{c}^b c^i - \frac{\xi}{2} g f^{abc} b^a \bar{c}^b c^c \\
&\quad \left. - \frac{\xi}{4} g^2 f^{abi} f^{cdi} \bar{c}^a \bar{c}^b c^c c^d - \frac{\xi}{4} g^2 f^{abc} f^{adi} \bar{c}^b \bar{c}^c c^d c^i - \frac{\xi}{8} g^2 f^{abc} f^{ade} \bar{c}^b \bar{c}^c c^d c^e \right). \quad (5.18)
\end{aligned}$$

The MAG condition allows for a residual local $U(1)^{N-1}$ invariance with respect to the diagonal subgroup, which has to be fixed by means of a suitable further gauge condition on the diagonal components A_μ^i . We shall choose a diagonal gauge fixing term which is BRST and anti-BRST invariant. The diagonal gauge fixing is then given by

$$\begin{aligned}
S_{\text{diag}} &= s \bar{s} \int d^4x \left(\frac{1}{2} A_\mu^i A^{i\mu} \right) \\
&= s \int d^4x \left(\bar{c}^i \partial_\mu A^{i\mu} - g f^{iab} A_\mu^i A^{a\mu} \bar{c}^b \right) \\
&= \int d^4x \left(b^i \partial_\mu A^{i\mu} + \bar{c}^i \partial^2 c^i + g f^{iab} A_\mu^a (\partial_\mu c^i \bar{c}^b - \partial_\mu \bar{c}^i c^b) + g^2 f^{iab} f^{icd} \bar{c}^a c^d A_\mu^b A^{c\mu} \right. \\
&\quad \left. - g f^{iab} A_\mu^i A^{a\mu} (b^b - g f^{ibc} \bar{c}^c c^i) + g f^{iab} A^{i\mu} (D_\mu^{ac} c^c) \bar{c}^b + g^2 f^{abi} f^{acd} A_\mu^i A^{c\mu} c^d \bar{c}^b \right). \quad (5.19)
\end{aligned}$$

In addition to the BRST and the anti-BRST symmetry, the gauge-fixed action ($S_{\text{YM}} + S_{\text{MAG}} + S_{\text{diag}}$) is invariant under a global $SL(2, \mathbb{R})$ symmetry, which is generated by the operators δ , $\bar{\delta}$ and the ghost number operator δ_{FP} . For the δ transformations we have

$$\begin{aligned}
\delta \bar{c}^a &= c^a, \quad \delta \bar{c}^i = c^i, \\
\delta b^a &= g f^{abi} c^b c^i + \frac{g}{2} f^{abc} c^b c^c, \\
\delta b^i &= \frac{g}{2} f^{iab} c^a c^b, \\
\delta A_\mu^a &= \delta A_\mu^i = \delta c^a = \delta c^i = 0. \quad (5.20)
\end{aligned}$$

The operator $\bar{\delta}$ acts as

$$\begin{aligned}
\bar{\delta} c^a &= \bar{c}^a, \quad \bar{\delta} c^i = \bar{c}^i, \\
\bar{\delta} b^a &= g f^{abi} \bar{c}^b \bar{c}^i + \frac{g}{2} f^{abc} \bar{c}^b \bar{c}^c, \\
\bar{\delta} b^i &= \frac{g}{2} f^{iab} \bar{c}^a \bar{c}^b, \\
\bar{\delta} A_\mu^a &= \bar{\delta} A_\mu^i = \bar{\delta} \bar{c}^a = \bar{\delta} \bar{c}^i = 0. \quad (5.21)
\end{aligned}$$

The existence of the $SL(2, \mathbb{R})$ symmetry has been pointed out in [77, 78, 79] in the maximal Abelian gauge for the gauge group $SU(2)$. A generalization of it can be found in [164]. There are, however, important differences between [77, 78, 79, 164] and the present analysis.

The first point relies on the choice of the diagonal part of the gauge fixing S_{diag} , a necessary step towards a complete quantization of the model. We remark that with our choice of S_{diag} in eq.(5.19), the whole NO algebra becomes an exact symmetry of the gauge fixed action ($S_{\text{YM}} + S_{\text{MAG}} + S_{\text{diag}}$) with gauge group $SU(N)$, for any value of N . In particular, as one can see from expression (5.19), S_{diag} contains the interaction term $g^2 f^{iab} f^{icd} \bar{c}^a c^d A_\mu^b A^{c\mu}$, which precisely cancels the corresponding term appearing in eq.(5.18) for S_{MAG} . This is a welcome feature, implying that no tachyons are generated at one-loop order in the MAG if the $SL(2, \mathbb{R})$, and thus the NO algebra, is required as an exact invariance for the starting gauge-fixed action. We remark that a similar compensation holds also for the interaction terms of (5.19) and (5.18) containing two diagonal gluons and a pair of off-diagonal ghost-antighost, implying that the diagonal gauge fields remain massless.

A second difference concerns the way the fields are transformed. We observe that in the present case, the field transformations (5.16) – (5.17) and (5.20) – (5.21) are obtained from (5.2) – (5.3) and (5.4) – (5.5) upon projection of the group index $A = 1, \dots, (N^2 - 1)$ over the Cartan subgroup of $SU(N)$ and over the off-diagonal generators, thus preserving the whole NO structure. As it is apparent from eqs.(5.20), (5.21), the diagonal fields $c^i, \bar{c}^i, b^i$ transform nontrivially, a necessary feature for the NO algebra. These transformations were not taken into account in the original work [77, 78, 79]. Also, in [164], the NO structure is analysed only on the off-diagonal fields, the diagonal components $\bar{c}^i, b^i$ being set to zero.

In summary, it is possible to choose the diagonal part S_{diag} of the gauge fixing so that the whole NO structure is preserved. It remains now to prove that the choice of the diagonal gauge fixing (5.19) will lead to a renormalizable model. This will be the task of the next section.

5.3 Stability of the MAG under radiative corrections.

In order to discuss the renormalizability of the action ($S_{\text{YM}} + S_{\text{MAG}} + S_{\text{diag}}$) within the BRST framework, we have to write down the Ward identities corresponding to the symmetries of the classical action. The expression of the BRST invariance as a functional identity requires the introduction of invariant external sources

$$S_{\text{ext}} = \int d^4x \left(\Omega^{a\mu} sA_\mu^a + \Omega^{i\mu} sA_\mu^i + L^a s c^a + L^i s c^i \right), \quad (5.22)$$

with $s\Omega_\mu^a = s\Omega_\mu^i = sL^a = sL^i = 0$.

The δ transformations of the external fields can be fixed by imposing $\delta S_{\text{ext}} = 0$, which yields $\delta\Omega_\mu^a = \delta\Omega_\mu^i = \delta L^a = \delta L^i = 0$. Therefore the classical action

$$\Sigma = S_{\text{YM}} + S_{\text{MAG}} + S_{\text{diag}} + S_{\text{ext}}, \quad (5.23)$$

is invariant under BRST and δ transformations, obeying the following identities

- Slavnov-Taylor identity:

$$S(\Sigma) = \int d^4x \left(\frac{\delta\Sigma}{\delta A_\mu^a} \frac{\delta\Sigma}{\delta\Omega^{a\mu}} + \frac{\delta\Sigma}{\delta A_\mu^i} \frac{\delta\Sigma}{\delta\Omega^{i\mu}} + \frac{\delta\Sigma}{\delta c^a} \frac{\delta\Sigma}{\delta L^a} + \frac{\delta\Sigma}{\delta c^i} \frac{\delta\Sigma}{\delta L^i} + b^a \frac{\delta\Sigma}{\delta \bar{c}^a} + b^i \frac{\delta\Sigma}{\delta \bar{c}^i} \right) = 0 \quad (5.24)$$

- δ symmetry Ward identity:

$$D(\Sigma) = \int d^4x \left(c^a \frac{\delta\Sigma}{\delta \bar{c}^a} + c^i \frac{\delta\Sigma}{\delta \bar{c}^i} + \frac{\delta\Sigma}{\delta b^a} \frac{\delta\Sigma}{\delta L^a} + \frac{\delta\Sigma}{\delta b^i} \frac{\delta\Sigma}{\delta L^i} \right) = 0 \quad (5.25)$$

- Integrated diagonal ghost equation:

$$\mathcal{G}^i \Sigma = \Delta_{cl}^i, \quad (5.26)$$

where

$$\mathcal{G}^i = \int d^4x \left(\frac{\delta}{\delta c^i} + g f^{abi} \bar{c}^a \frac{\delta}{\delta b^b} \right) \quad (5.27)$$

and the classical breaking Δ_{cl}^i is given by

$$\Delta_{cl}^i = \int d^4x (g f^{abi} \Omega_\mu^a A^{b\mu} - g f^{abi} L^a c^b). \quad (5.28)$$

- Integrated diagonal antighost equation:

$$\int d^4x \frac{\delta \Sigma}{\delta \bar{c}^i} = 0. \quad (5.29)$$

Similarly, we could also impose the anti-BRST and $\bar{\delta}$ symmetries in a functional way by introducing an additional set of external sources. However, the Ward identities (5.24)-(5.29) are sufficient to ensure the stability of the classical action under quantum corrections. It is not difficult indeed, by using the algebraic renormalization procedure [59, 60], to prove that the model is renormalizable.

5.4 Ghost condensation and the breakdown of $SL(2, \mathbb{R})$ and NO symmetry.

In this section, we give a brief discussion of the existence of ghost condensates and of their relationship with $SL(2, \mathbb{R})$ and hence with NO symmetry.

The condensation of ghosts came to attention originally in the works of [77, 78, 79, 80] in the context of $SU(2)$ MAG. The decomposition of the 4-ghost interaction allowed to construct an effective potential with a nontrivial minimum for the off-diagonal condensate $\langle \varepsilon^{3ab} c^a \bar{c}^b \rangle$. This condensate implies a dynamical breaking of $SL(2, \mathbb{R})$. Order parameters for this breaking are given by

$$\langle \varepsilon^{3ab} c^a \bar{c}^b \rangle = \frac{1}{2} \langle \delta (\varepsilon^{3ab} \bar{c}^a c^b) \rangle = \frac{1}{2} \langle \bar{\delta} (\varepsilon^{3ab} c^a c^b) \rangle. \quad (5.30)$$

As a consequence the generators δ and $\bar{\delta}$ of $SL(2, \mathbb{R})$ are broken [77, 78, 79], while the Faddeev-Popov ghost number generator $\delta_{FP} = [\delta, \bar{\delta}]$ remains unbroken. The ghost condensate resulted in a mass for the off-diagonal gluons, whose origin can be traced back to the presence of the off-diagonal interaction term $\bar{c} c A_\mu A^\mu$ term in expression (5.18). However, as was shown in [157], this mass is a tachyonic one. Furthermore, the requirement of invariance of S_{diag} under anti-BRST and, as a consequence, under $SL(2, \mathbb{R})$ transformations, also gives rise to quartic terms of the kind $\bar{c} c A_\mu A^\mu$ (see (5.19)), which precisely cancel those of S_{MAG} in (5.18). Thus, if the $SL(2, \mathbb{R})$ symmetry (NO symmetry (5.6)) is required, the ghost condensates do not induce any unphysical mass for the off-diagonal gluons at one-loop order.

The 4-ghost interaction can be decomposed in a different way, so that the ghost condensation takes places in the Faddeev-Popov charged channels $\varepsilon^{3ab} c^a c^b$ and $\varepsilon^{3ab} \bar{c}^a \bar{c}^b$ instead of $\varepsilon^{3ab} c^a \bar{c}^b$. In this case, the ghost number symmetry is broken [81]. Consequently, the NO algebra is broken again. It is interesting to see that the existence of different ghost channels in which the ghost condensation can take place

has an analogy in ordinary superconductivity, known as BCS (particle-particle and hole-hole pairing) versus Overhauser (particle-hole pairing) [92, 95]. In the present case the BCS channel corresponds to the Faddeev-Popov charged condensates $\langle \varepsilon^{3ab} c^a c^b \rangle$ and $\langle \varepsilon^{3ab} \bar{c}^a \bar{c}^b \rangle$, while the Overhauser channel to $\langle \varepsilon^{3ab} c^a \bar{c}^b \rangle$.

The CF gauge (5.9) contains a 4-ghost interaction too, so it is expected that the ghost condensation can take place also in the CF gauge. This was confirmed in [90]. Notice that the CF and MAG gauges look very similar.

More surprising is the fact that also in the Landau gauge, the ghost condensation occurs [91]. Since there is no 4-ghost interaction to be decomposed, another technique was used to discuss this gauge. A combination of the algebraic renormalization technique [59, 60] and the local composite operator technique [42] allowed a clean treatment, with the result that also in case of the Landau gauge the NO symmetry is broken.

One could wonder what the mechanism behind the dynamical generation of gluon masses could be. It was proposed in [83] that the generation of a real mass for the gluons in case of the CF gauge is related to a non-vanishing vacuum expectation value for the two-dimensional local, composite operator $(\frac{1}{2} \mathcal{A}_\mu^A \mathcal{A}^{A\mu} - \xi c^A \bar{c}^A)$. It is interesting to notice that this is exactly the kind of mass term that is present in the massive Lagrangian of Curci and Ferrari [84, 85]. In the case of MAG, the relevant operator is believed to be $(\frac{1}{2} A_\mu^a A^{a\mu} - \xi c^a \bar{c}^a)$, and is expected to provide an effective mass for both off-diagonal gauge and off-diagonal ghost fields [83, 157].

5.5 Conclusion.

- In this paper the presence of the $SL(2, \mathbb{R})$ symmetry has been analysed in the Landau, Curci-Ferrari and maximal Abelian gauge for $SU(N)$ Yang-Mills. In all these gauges $SL(2, \mathbb{R})$ can be established as an exact invariance of the complete gauge fixed action. Together with the BRST and anti-BRST, the generators of $SL(2, \mathbb{R})$ are part of a larger algebra, known as the Nakanishi-Ojima algebra [86].
- In particular, we have been able to show that in the case of the maximal Abelian gauge, the requirement of $SL(2, \mathbb{R})$ for the complete action, including the diagonal gauge fixing term, ensures that no tachyons will be generated at one-loop order.
- In all these gauges, the $SL(2, \mathbb{R})$ symmetry turns out to be dynamically broken by the existence of off-diagonal ghost condensates $\langle cc \rangle$, $\langle \bar{c}\bar{c} \rangle$ and $\langle c\bar{c} \rangle$. As a consequence, the NO algebra is also broken. These condensates modify the infrared behavior of the off-diagonal ghost propagator, while contributing to the vacuum energy density and hence to the trace anomaly [77, 78, 79, 80, 81].
- Finally, let us spend a few words on future research. As already remarked, the ghost condensation can be observed in different channels, providing a close analogy with the BCS versus Overhauser effect of superconductivity. We have also pointed out that the existence of the condensate $(\frac{1}{2} \langle A_\mu^a A^{a\mu} \rangle - \xi \langle c^a \bar{c}^a \rangle)$ can be at the origin of the dynamical mass generation in the MAG for all off-diagonal gluons and ghosts [80, 157], a feature of great relevance for the Abelian dominance. Both aspects will be analysed by combining the algebraic renormalization [59, 60] with the local composite operator technique [42], as done in the case of the ghost condensation in the Landau gauge [91]. The combination of these two procedures results in a very powerful framework for discussing the ghost condensation in the various channels as well as for studying the condensate $(\frac{1}{2} \langle A_\mu^a A^{a\mu} \rangle - \xi \langle c^a \bar{c}^a \rangle)$ and its relationship with the dynamical mass generation.

Also, the detailed analysis of the decoupling at low energies of the diagonal ghosts and of the validity of the local $U(1)^{N-1}$ Ward identity in the MAG deserves careful attention.

Chapter 6

The anomalous dimension of the composite operator A^2 in the Landau gauge

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published in Physics Letters B **555** (2003) 126.*

The local composite operator A^2 is analysed in pure Yang-Mills theory in the Landau gauge within the algebraic renormalization. It is proven that the anomalous dimension of A^2 is not an independent parameter, being expressed as a linear combination of the gauge β function and of the anomalous dimension of the gauge fields.

6.1 Introduction

Nowadays an increasing evidence has been reported on the relevance of the local composite operator $A_\mu^a A^{a\mu}$ for the nonperturbative regime of Yang-Mills theories quantized in the Landau gauge. That this operator has a certain special meaning in the Landau gauge can be simply understood by observing that, due to the transversality condition $\partial_\mu A^{a\mu} = 0$, the integrated mass dimension two operator $(VT)^{-1} \int d^4x A_\mu^a A^{a\mu}$ is gauge invariant, VT being the space-time volume. Lattice simulations [37, 38, 39, 175] have indeed provided strong indications for the existence of the pure gluon condensate $\langle A_\mu^a A^{a\mu} \rangle$, confirming its relevance for the infrared dynamics of Yang-Mills. Also, the monopole condensation in compact QED turns out to be related to a phase transition for this condensate [33, 34]. Recently, a renormalizable effective potential for $\langle A_\mu^a A^{a\mu} \rangle$ has been obtained in [42] by using the local composite operator (LCO) technique [24, 23, 176]. This result shows that the vacuum of pure Yang-Mills theory favors a nonvanishing value of this condensate, which provides effective masses for the gluons while contributing to the dimension four condensate $\langle \alpha F^2 \rangle$ through the trace anomaly. It is worth remarking here that this mass has a pure dynamical origin and manifests itself without breaking the gauge group. Both features are indeed expected in a confining gauge theory, being in agreement with the Kugo-Ojima criterion for color confinement [177].

An important ingredient in the analysis of the effective potential for the gluon condensate is the anomalous dimension γ_{A^2} of the operator $A_\mu^a A^{a\mu}$. Till now, γ_{A^2} has been computed up to three

loops in the $\overline{\text{MS}}$ renormalization scheme [173, 174]. The available expression for γ_{A^2} shows rather interesting properties concerning the operator $A_\mu^a A^{a\mu}$ in the Landau gauge. It turns out indeed that, besides being multiplicative renormalizable, its anomalous dimension can be expressed as a combination of the gauge β function and of the anomalous dimension γ_A of the gauge fields, according to the relation

$$\gamma_{A^2} = - \left(\frac{\beta(a)}{a} + \gamma_A(a) \right), \quad a = \frac{g^2}{16\pi^2}, \quad (6.1)$$

which can be easily checked up to three loops [173, 174]. This feature strongly supports the existence of some underlying Ward identities which should be at the origin of eq.(6.1), meaning that γ_{A^2} is not an independent parameter of the theory.

The aim of this paper is to provide an affirmative answer concerning the possibility of giving a purely algebraic proof of the relation (6.1), which extends to all orders of perturbation theory. Our proof will rely only on the use of the Slavnov-Taylor identity and of an additional Ward identity, known as the ghost Ward identity, present in the Landau gauge [89]. Furthermore, according to [89], it turns out that also the anomalous dimension γ_c of the Faddeev-Popov ghosts can be written as a combination of β and γ_A , namely

$$2\gamma_c(a) = \frac{\beta(a)}{a} - \gamma_A(a). \quad (6.2)$$

Both relations (6.1) and (6.2) can be used as a very useful check for higher order computations in Yang-Mills theories quantized in gauges which reduce to the Landau gauge when the gauge parameters are set to zero, as in the case of the nonlinear Curci-Ferrari gauge [173, 174].

The work is organized as follows. In section 6.2 we give a brief account of eqs.(6.1) and (6.2) by making use of the available three loops expressions. Section 6.3 is devoted to their algebraic proof.

6.2 The anomalous dimension of the operator A_μ^2 in the Landau gauge.

In order to give a short account of the relations (6.1) and (6.2), let us recall the three-loop expressions of the gauge β function and of the gauge and ghost fields anomalous dimensions γ_A and γ_c , as given in [173, 174]. They read

$$\frac{\beta(a)}{a} = -\frac{11}{3}(Na) - \frac{34}{3}(Na)^2 - \frac{2857}{54}(Na)^3, \quad (6.3)$$

$$\gamma_A = -\frac{13}{6}(Na) - \frac{59}{8}(Na)^2 + \frac{(648\zeta(3) - 39860)}{1152}(Na)^3, \quad (6.4)$$

and

$$\gamma_c = -\frac{3}{4}(Na) - \frac{95}{48}(Na)^2 - \frac{(1944\zeta(3) + 63268)}{6912}(Na)^3, \quad (6.5)$$

where N is the number of colors corresponding to the gauge group $SU(N)$. Making use of the relation

$$\gamma_{A^2} = - \left(\frac{\beta(a)}{a} + \gamma_A(a) \right), \quad (6.6)$$

for the anomalous dimension of $A_\mu^a A^{a\mu}$ one gets, up to the third order,

- first order

$$\gamma_{A^2}^{(1)} = - \left(\frac{\beta^{(1)}}{a} + \gamma_A^{(1)} \right) = \frac{35}{6} (Na) . \quad (6.7)$$

- second order

$$\gamma_{A^2}^{(2)} = - \left(\frac{\beta^{(2)}}{a} + \gamma_A^{(2)} \right) = \frac{449}{24} (Na)^2 . \quad (6.8)$$

- third order

$$\gamma_{A^2}^{(3)} = - \left(\frac{\beta^{(3)}}{a} + \gamma_A^{(3)} \right) = \left(\frac{75607}{864} - \frac{9}{16} \zeta(3) \right) (Na)^3 . \quad (6.9)$$

Expressions (6.7), (6.8), (6.9) are in complete agreement with those given in¹ [173, 174]. Concerning now the ghost anomalous dimension γ_c in eq.(6.5), it is straightforward to verify in fact that the relation

$$2\gamma_c(a) = \frac{\beta(a)}{a} - \gamma_A(a) , \quad (6.10)$$

holds. This equation expresses the nonrenormalization properties of the ghost fields in the Landau gauge and, as shown in [89], follows from the ghost Ward identity. Although we are considering only pure Yang-Mills theory, it is worth mentioning that eqs.(6.6) and (6.10) remain valid also in the presence of matter fields, as one can verify from [173, 174].

6.3 Algebraic proof.

In this section we provide an algebraic proof of the relation (6.6). We shall make use of a suitable set of Ward identities which can be derived in the Landau gauge in order to characterize the local operator A^2 . Let us begin by reminding the expression of the pure Yang-Mills action in the Landau gauge

$$\begin{aligned} S &= S_{\text{YM}} + S_{\text{GF+FP}} \\ &= -\frac{1}{4} \int d^4x F_{\mu\nu}^a F^{a\mu\nu} + \int d^4x (b^a \partial_\mu A^{a\mu} + \bar{c}^a \partial^\mu D_\mu^{ab} c^b) , \end{aligned} \quad (6.11)$$

where

$$D_\mu^{ab} \equiv \partial_\mu \delta^{ab} + g f^{acb} A_\mu^c . \quad (6.12)$$

To study the operator $A_\mu^a A^{a\mu}$, we introduce it in the action by means of a set of external sources. It turns out that three external sources J , η^μ and τ^μ are needed, so that

$$\begin{aligned} S_J &= \int d^4x \left[J \frac{1}{2} A_\mu^a A^{a\mu} + \frac{\xi}{2} J^2 - \eta^\mu A_\mu^a c^a - \tau^\mu s(A_\mu^a c^a) \right] \\ &= \int d^4x \left[J \frac{1}{2} A_\mu^a A^{a\mu} + \frac{\xi}{2} J^2 - \eta^\mu A_\mu^a c^a + \tau^\mu \partial_\mu c^a c^a + \frac{g}{2} \tau^\mu f^{abc} A_\mu^a c^b c^c \right] , \end{aligned} \quad (6.13)$$

¹Notice that the anomalous dimension γ_O for A^2 given in [173, 174] is related to γ_{A^2} in eq.(6.6) by $\gamma_{A^2} = -4\gamma_O$.

where s denotes the BRST operator acting as

$$\begin{aligned}
sA_\mu^a &= -D_\mu^{ab}c^b \\
sc^a &= \frac{g}{2}f^{abc}c^bc^c \\
s\bar{c}^a &= b^a \\
sb^a &= -Jc^a \\
sJ &= 0 \\
s\eta^\mu &= \partial^\mu J \\
s\tau^\mu &= \eta^\mu .
\end{aligned} \tag{6.14}$$

It is easy to check that

$$s(S_{\text{YM}} + S_{\text{GF+FP}} + S_J) = 0 . \tag{6.15}$$

According to the LCO procedure [42, 24, 23, 176], the dimensionless parameter ξ is needed to account for the divergences present in the vacuum Green function $\langle A^2(x)A^2(y) \rangle$, which are proportional to J^2 .

6.3.1 Ward identities.

In order to translate the BRST invariance (6.15) into the corresponding Slavnov-Taylor identity [59], we introduce two additional external sources Ω_μ^a and L^a coupled to the nonlinear BRST variation of A_μ^a and c^a

$$S_{\text{ext}} = \int d^4x \left[-\Omega^{a\mu} D_\mu^{ab}c^b + L^a \frac{g}{2}f^{abc}c^bc^c \right] , \tag{6.16}$$

with

$$s\Omega_\mu^a = sL^a = 0 .$$

The complete action

$$\Sigma = S_{\text{YM}} + S_{\text{GF+FP}} + S_J + S_{\text{ext}} . \tag{6.17}$$

turns out thus to obey the following identities:

- The Slavnov-Taylor identity

$$\mathcal{S}(\Sigma) = 0 , \tag{6.18}$$

$$\mathcal{S}(\Sigma) = \int d^4x \left(\frac{\delta\Sigma}{\delta A_\mu^a} \frac{\delta\Sigma}{\delta\Omega^{a\mu}} + \frac{\delta\Sigma}{\delta L^a} \frac{\delta\Sigma}{\delta c^a} + b^a \frac{\delta\Sigma}{\delta\bar{c}^a} + \partial_\mu J \frac{\delta\Sigma}{\delta\eta_\mu} + \eta^\mu \frac{\delta\Sigma}{\delta\tau^\mu} - Jc^a \frac{\delta\Sigma}{\delta b^a} \right) \tag{6.19}$$

- The Landau gauge condition

$$\frac{\delta\Sigma}{\delta b^a} = \partial_\mu A^{\mu a} , \tag{6.20}$$

and the antighost Ward identity

$$\frac{\delta\Sigma}{\delta\bar{c}^a} + \partial_\mu \frac{\delta\Sigma}{\delta\Omega_\mu^a} = 0 , \tag{6.21}$$

- The ghost Ward identity [89, 59]

$$\mathcal{G}^a \Sigma = \Delta_{\text{cl}}^a, \quad (6.22)$$

where

$$\mathcal{G}^a \Sigma = \int d^4x \left(\frac{\delta \Sigma}{\delta c^a} + g f^{abc} \bar{c}^b \frac{\delta \Sigma}{\delta b^c} - \tau_\mu \frac{\delta \Sigma}{\delta \Omega_\mu^a} \right), \quad (6.23)$$

and

$$\Delta_{\text{cl}}^a = \int d^4x \left(g f^{abc} \Omega_\mu^b A^{c\mu} - g f^{abc} L^b c^c + \eta^\mu A_\mu^a \right). \quad (6.24)$$

Notice that expression (6.24), being purely linear in the quantum fields, is a classical breaking. It is remarkable that the ghost Ward identity can be established also in the presence of the external sources (J, η^μ, τ^μ) . As we shall see, this identity will play a fundamental role for the algebraic proof of the relation (6.6).

6.3.2 Algebraic characterization of the general local counterterm.

We are now ready to analyse the structure of the most general local counterterm compatible with the identities (6.18) – (6.22). Let us begin by displaying the quantum numbers of all fields and sources, namely In order to characterize the most general invariant counterterm which can be freely added to all

	A_μ^a	c^a	$\bar{c}^a$	b^a	L^a	Ω_μ^a	J	η^μ	τ^μ
Gh. number	0	1	-1	0	-2	-1	0	-1	-2
Dimension	1	0	2	2	4	3	2	3	3

Table 6.1:

orders of perturbation theory, we perturb the classical action Σ by adding an arbitrary integrated local polynomial Σ^{count} in the fields and external sources of dimension bounded by four and with zero ghost number, and we require that the perturbed action $(\Sigma + \varepsilon \Sigma^{\text{count}})$ satisfies the same Ward identities and constraints as Σ to the first order in the perturbation parameter ε , *i.e.*

$$\begin{aligned} \mathcal{S}(\Sigma + \varepsilon \Sigma^{\text{count}}) &= 0 + O(\varepsilon^2), \\ \frac{\delta(\Sigma + \varepsilon \Sigma^{\text{count}})}{\delta b^a} &= \partial^\mu A_\mu^a + O(\varepsilon^2), \\ \left(\frac{\delta}{\delta \bar{c}^a} + \partial_\mu \frac{\delta}{\delta \Omega_\mu^a} \right) (\Sigma + \varepsilon \Sigma^{\text{count}}) &= 0 + O(\varepsilon^2), \\ \mathcal{G}^a(\Sigma + \varepsilon \Sigma^{\text{count}}) &= \Delta_{\text{cl}}^a + O(\varepsilon^2). \end{aligned} \quad (6.25)$$

This amounts to impose the following conditions on Σ^{count}

$$\begin{aligned} \mathcal{B}_\Sigma \Sigma^{\text{count}} &= 0, \\ \mathcal{B}_\Sigma &= \int d^4x \left(\frac{\delta \Sigma}{\delta A_\mu^a} \frac{\delta}{\delta \Omega^{a\mu}} + \frac{\delta \Sigma}{\delta \Omega^{a\mu}} \frac{\delta}{\delta A_\mu^a} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta}{\delta c^a} + \frac{\delta \Sigma}{\delta c^a} \frac{\delta}{\delta L^a} \right. \\ &\quad \left. + b^a \frac{\delta}{\delta \bar{c}^a} + \partial_\mu J \frac{\delta}{\delta \eta_\mu} + \eta^\mu \frac{\delta}{\delta \tau^\mu} - J c^a \frac{\delta}{\delta b^a} \right), \end{aligned} \quad (6.26)$$

and

$$\frac{\delta \Sigma^{\text{count}}}{\delta b^a} = 0, \quad (6.27)$$

$$\frac{\delta \Sigma^{\text{count}}}{\delta \bar{c}^a} + \partial_\mu \frac{\delta \Sigma^{\text{count}}}{\delta \Omega_\mu^a} = 0, \quad (6.28)$$

$$\mathcal{G}^a \Sigma^{\text{count}} = 0. \quad (6.29)$$

From equations (6.27) and (6.28) it follows that Σ^{count} does not depend on the Lagrange multiplier field b^a and that the antighost $\bar{c}^a$ enters only through the combination $\hat{\Omega}_\mu^a = (\Omega_\mu^a + \partial_\mu \bar{c}^a)$. As a consequence, Σ^{count} can be parametrized as follows

$$\begin{aligned} \Sigma^{\text{count}} &= S^{\text{count}}(A) \\ &+ \int d^4x \left(\frac{a_1}{2} f^{abc} L^a c^b c^c + a_2 \hat{\Omega}_\mu^a \partial^\mu c^a + a_3 f^{abc} \hat{\Omega}_\mu^a A_\mu^b c^c + \frac{a_4}{2} \xi J^2 \right) \\ &+ \int d^4x \left(\frac{a_5}{2} J A_\mu^a A^{a\mu} + a_6 \eta^\mu A_\mu^a c^a + \frac{a_7}{2} \tau^\mu f^{abc} A_\mu^a c^b c^c + a_8 \tau^\mu \partial_\mu c^a c^a \right), \end{aligned} \quad (6.30)$$

where a_i , $i = 1 \dots 8$ are free parameters and $S^{\text{count}}(A)$ depends only on the gauge fields A_μ^a . From the ghost Ward identity condition (6.29) it follows that

$$\begin{aligned} a_1 &= a_3 = a_6 = a_7 = 0, \\ a_8 &= -a_2. \end{aligned} \quad (6.31)$$

The vanishing of the coefficient a_1 expresses the absence of the counterterm $f^{abc} L^a c^b c^c$. Also, $a_3 = 0$ states the nonrenormalization of the ghost-antighost-gluon vertex, stemming from the transversality of the Landau propagator and from the factorization of the ghost momentum. As shown in [89], these features are related to a set of nonrenormalization theorems of the Landau gauge. Furthermore, the vanishing of a_6 implies the ultraviolet finiteness of the operator $A_\mu c$. These finiteness properties extend to all orders, due to the ghost Ward identity (6.22). It remains now to work out the condition (6.26). Making use of the well known results on the cohomology of Yang-Mills theory [59, 60], it turns out that the condition (6.26) implies that the coefficient a_5 is related to a_2 ,

$$a_5 = a_2, \quad (6.32)$$

and that $S^{\text{count}}(A)$ can be written as

$$\begin{aligned} S^{\text{count}}(A) &= \rho S_{YM}(A) + a_2 \int d^4x A_\mu^a \frac{\delta S_{YM}(A)}{\delta A_\mu^a}, \\ S_{YM}(A) &= -\frac{1}{4} \int d^4x F_{\mu\nu}^a F^{a\mu\nu}, \end{aligned} \quad (6.33)$$

where ρ is a free parameter. In summary, the most general local counterterm compatible with the Ward identities (6.18) – (6.22) contains three independent parameters ρ , a_2 , a_4 , and reads

$$\begin{aligned} \Sigma^{\text{count}} &= \rho S_{YM}(A) + a_2 \int d^4x A_\mu^a \frac{\delta S_{YM}(A)}{\delta A_\mu^a} \\ &+ \int d^4x \left(a_2 (\Omega_\mu^a + \partial_\mu \bar{c}^a) \partial^\mu c^a + \frac{a_4}{2} \xi J^2 + \frac{a_2}{2} J A_\mu^a A^{a\mu} - a_2 \tau^\mu \partial_\mu c^a c^a \right). \end{aligned} \quad (6.34)$$

It is apparent from the above expression that the parameters ρ and a_2 are related to the renormalization of the gauge coupling g and of the gauge fields A_μ^a , while the parameter a_4 corresponds to the renormalization of ξ . It should be remarked also that the coefficient of the counterterm $JA_\mu^a A^{a\mu}$ is given by a_2 . This means that the renormalization of the external source J , and thus of the composite operator $A_\mu^a A^{a\mu}$, can be expressed as a combination of the renormalization constants of the gauge coupling and of the gauge fields. As we shall see in the next section, this property will lead to the eq.(6.6).

6.3.3 Stability and renormalization constants.

Having found the most general local counterterm compatible with all Ward identities, it remains to discuss the stability [59] of the classical starting action, *i.e.* to check that Σ^{count} can be reabsorbed in the starting action Σ by means of a multiplicative renormalization of the coupling constant g , the parameter ξ , the fields $\{\phi = A, c, \bar{c}, b\}$ and the sources $\{\Phi = J, \eta, \tau, L, \Omega\}$, namely

$$\Sigma(g, \xi, \phi, \Phi) + \varepsilon \Sigma^{\text{count}} = \Sigma(g_0, \xi_0, \phi_0, \Phi_0) + O(\varepsilon^2), \quad (6.35)$$

where, adopting the same conventions of [173, 174]

$$\begin{aligned} g_0 &= Z_g g, \\ \xi_0 &= Z_\xi \xi, \\ \phi_0 &= Z_\phi^{1/2} \phi, \\ \Phi_0 &= Z_\Phi \Phi. \end{aligned} \quad (6.36)$$

As already mentioned, the parameters ρ and a_2 are related to the renormalization of g and A_μ^a , according to

$$\begin{aligned} Z_g &= 1 - \varepsilon \frac{\rho}{2}, \\ Z_A^{1/2} &= 1 + \varepsilon \left(a_2 + \frac{\rho}{2} \right). \end{aligned} \quad (6.37)$$

Concerning the other fields, it is almost immediate to verify that they are renormalized as follows

$$Z_b = Z_A^{-1}, \quad (6.38)$$

and

$$Z_{\bar{c}} = Z_c = Z_g^{-1} Z_A^{-1/2}. \quad (6.39)$$

Similar relations are easily found for the sources. In particular, for the source J and for the parameter ξ one has

$$Z_J = Z_{A^2} = Z_g Z_A^{-1/2}, \quad (6.40)$$

and

$$Z_\xi = 1 + \varepsilon (a_4 + 2a_2 + 2\rho) = (1 + \varepsilon a_4) Z_g^{-2} Z_A. \quad (6.41)$$

We see therefore that the relation

$$\gamma_{A^2} = - \left(\frac{\beta(a)}{a} + \gamma_A(a) \right), \quad (6.42)$$

follows from eq.(6.40). Concerning now the eq.(6.10) for the ghost anomalous dimension, it is a direct consequence of eq.(6.39).

Summarizing, we have been able to give a purely algebraic proof of the relationship (6.6). This means that the anomalous dimension γ_{A^2} of the composite operator $A_\mu^a A^{a\mu}$ is not an independent parameter for Yang-Mills theory in the Landau gauge.

Chapter 7

The anomalous dimension of the gluon-ghost mass operator in Yang-Mills theory

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The local composite gluon-ghost operator $(\frac{1}{2}A^{a\mu}A_\mu^a + \alpha\bar{c}^a c^a)$ is analysed in the framework of the algebraic renormalization in $SU(N)$ Yang-Mills theories in the Landau, Curci-Ferrari and maximal Abelian gauges. We show, to all orders of perturbation theory, that this operator is multiplicatively renormalizable. Furthermore, its anomalous dimension is not an independent parameter of the theory, being given by a general expression valid in all these gauges. We also verify the relations we obtain for the operator anomalous dimensions by explicit three-loop calculations in the $\overline{MS}$ scheme for the Curci-Ferrari gauge.

7.1 Introduction.

Vacuum condensates are believed to play an important role in the understanding of the nonperturbative dynamics of Yang-Mills theories. In particular, much effort has been devoted to the study of condensates of dimension two built up with gluons and ghosts. For instance, the relevance of the pure gluon condensate $\langle A^{a\mu}A_\mu^a \rangle$ in the Landau gauge has been discussed from lattice simulations [37, 38, 39, 175] as well as from a phenomenological point of view [33, 34]. That the operator A_μ^2 has a special meaning in the Landau gauge follows by observing that, due to the transversality condition $\partial_\mu A^{a\mu} = 0$, the integrated operator $(VT)^{-1} \int d^4x A_\mu^a A^{a\mu}$ is gauge invariant, with VT denoting the space-time volume. An effective potential for $\langle A^{a\mu}A_\mu^a \rangle$ has been constructed in [42], showing that the vacuum of Yang-Mills favors a nonvanishing value for this condensate, which gives rise to a dynamical mass generation for the gluons.

The operator A_μ^2 in the Landau gauge can be generalized to other gauges such as the Curci-Ferrari gauge and maximal Abelian gauge, (MAG). Indeed, as was shown in [83, 144], the mixed gluon-ghost operator¹

¹In the case of the maximal Abelian gauge, the color index a runs only over the $N(N-1)$ off-diagonal components.

$\mathcal{O} = (\frac{1}{2}A^{a\mu}A_\mu^a + \alpha\bar{c}^a c^a)$ turns out to be BRST invariant on-shell, where α is the gauge parameter. Also, the Curci-Ferrari gauge has the Landau gauge, $\alpha = 0$, as a special case. Thus, the gluon-ghost condensate $(\frac{1}{2}\langle A^{a\mu}A_\mu^a \rangle + \alpha\langle \bar{c}^a c^a \rangle)$ might be suitable for the description of dynamical mass generation in these gauges. Recently, the effective potential for this condensate in the Curci-Ferrari gauge has been constructed in [178] by combining the algebraic renormalization [59] with the local composite operators technique [42, 176], resulting in a dynamical mass generation. In this formalism, an essential step is the renormalizability of the local composite operator related to the condensate, which is fundamental to obtaining its anomalous dimension. It is worth mentioning that the anomalous dimension of the gluon-ghost operator $\mathcal{O} = (\frac{1}{2}A^{a\mu}A_\mu^a + \alpha\bar{c}^a c^a)$ in the Curci-Ferrari gauge, and thus of the gluon operator A_μ^2 in the Landau gauge, has been computed to three-loop in the $\overline{\text{MS}}$ renormalization scheme, [87]. In addition, it has been proven [153] by using BRST Ward identities that the anomalous dimension $\gamma_{A_\mu^2}(a)$ of the operator A_μ^2 in the Landau gauge is not an independent parameter, being expressed as a combination of the gauge beta function, $\beta(a)$, and of the anomalous dimension, $\gamma_A(a)$, of the gauge field, according to the relation

$$\gamma_{A_\mu^2}(a) = - \left(\frac{\beta(a)}{a} + \gamma_A(a) \right), \quad a = \frac{g^2}{16\pi^2}. \quad (7.1)$$

The aim of this paper is to extend the analysis of [153] to the Curci-Ferrari and maximal Abelian gauges. We shall prove that the operator $(\frac{1}{2}A^{a\mu}A_\mu^a + \alpha\bar{c}^a c^a)$ is multiplicatively renormalizable to all orders of perturbation theory. Furthermore, as in the case of the Landau gauge, its anomalous dimension $\gamma_{\mathcal{O}}(a)$ is not an independent parameter of the theory, being given in fact by a general relationship valid in the Landau, Curci-Ferrari and maximal Abelian gauges, which is

$$\gamma_{\mathcal{O}}(a) = - 2(\gamma_c(a) + \gamma_{gc^2}(a)), \quad (7.2)$$

where $\gamma_c(a)$, $\gamma_{gc^2}(a)$ are the anomalous dimensions of the Faddeev-Popov ghost c^a and of the composite operator $\frac{1}{2}gf^{abc}c^b c^c$ corresponding to the BRST variation of c^a . In other words $sc^a = \frac{1}{2}gf^{abc}c^b c^c$ where s is the BRST operator.

The paper is organized as follows. In section 7.2, the renormalization of the dimension two operator $(\frac{1}{2}A^{a\mu}A_\mu^a + \alpha\bar{c}^a c^a)$ is considered in detail, by taking the Curci-Ferrari gauge as an example and the relationship (7.2) is derived. In section 7.3, we verify the relation we obtain between the anomalous dimensions explicitly at three loops in the Curci-Ferrari gauge in the $\overline{\text{MS}}$ scheme. Section 7.4 is devoted to the Landau gauge, showing that the expression (7.2) reduces to that of (7.1). In section 7.5 we shall analyse the maximal Abelian gauge, where the results established in [179] for the case of $SU(2)$ will be recovered. Finally, in section 7.6 we present our conclusions.

7.2 The gluon-ghost operator in Yang-Mills theories in the Curci-Ferrari gauge.

7.2.1 The Curci-Ferrari action.

We begin by reviewing the quantization of pure $SU(N)$ Yang-Mills in the Curci-Ferrari gauge. The pure Yang-Mills action is given by

$$S_{\text{YM}} = - \frac{1}{4} \int d^4x F^{a\mu\nu} F_{\mu\nu}^a, \quad (7.3)$$

with $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c$. The so called Curci-Ferrari gauge, [162, 160, 161, 84, 85], is defined by the following gauge fixing term

$$S_{\text{GF+FP}} = \int d^4x \left(b^a \partial_\mu A^{\mu a} + \frac{\alpha}{2} b^a b^a + \bar{c}^a \partial^\mu (D_\mu c)^a - \frac{\alpha}{2} g f^{abc} b^a \bar{c}^b c^c - \frac{\alpha}{8} g^2 f^{abc} \bar{c}^a \bar{c}^b f^{cmn} c^m c^n \right), \quad (7.4)$$

with

$$(D_\mu c)^a = \partial_\mu c^a + g f^{abc} A_\mu^b c^c. \quad (7.5)$$

In order to analyse the renormalization of the operator $(\frac{1}{2} A^{a\mu} A_\mu^a + \alpha \bar{c}^a c^a)$, we have to introduce it in the action by means of a set of external sources. Following [153], it turns out that three external sources J , η^μ and τ^μ are required, so that

$$S_J = \int d^4x \left(J \left(\frac{1}{2} A^{a\mu} A_\mu^a + \alpha \bar{c}^a c^a \right) + \frac{\xi}{2} J^2 - \eta^\mu A_\mu^a c^a - \tau^\mu (\partial_\mu c^a) c^a - \frac{g}{2} \tau^\mu f^{abc} A_\mu^a c^b c^c \right) \quad (7.6)$$

where ξ is a dimensionless parameter, accounting for the divergences present in the vacuum Green function $\left\langle \left(\frac{1}{2} A^{a\mu} A_\mu^a + \alpha \bar{c}^a c^a \right)_x \left(\frac{1}{2} A^{a\mu} A_\mu^a + \alpha \bar{c}^a c^a \right)_y \right\rangle$, which are proportional to J^2 [178]. The action $(S_{\text{YM}} + S_{\text{GF+FP}} + S_J)$ is invariant under the BRST transformations, which read

$$\begin{aligned} s A_\mu^a &= - (D_\mu c)^a, & s c^a &= \frac{g}{2} f^{abc} c^b c^c, \\ s \bar{c}^a &= b^a, & s b^a &= - J c^a, \\ s \tau^\mu &= - \eta^\mu, & s \eta^\mu &= \partial^\mu J, \\ s J &= 0. \end{aligned} \quad (7.7)$$

Also, due to the introduction of the external sources J , η^μ and τ^μ it follows that the operator s is not strictly nilpotent, namely

$$\begin{aligned} s^2 \Phi &= 0, & (\Phi &= A^{a\mu}, c^a, J, \eta^\mu), \\ s^2 \bar{c}^a &= - J c^a, \\ s^2 b^a &= - J \frac{g}{2} f^{abc} c^b c^c, \\ s^2 \tau^\mu &= - \partial^\mu J. \end{aligned} \quad (7.8)$$

Therefore, setting

$$s^2 \equiv \delta, \quad (7.9)$$

we have $\delta(S_{\text{YM}} + S_{\text{GF+FP}} + S_J) = 0$. The operator δ is related to a global $SL(2, \mathbb{R})$ symmetry [163], which is known to be present in the Landau, Curci-Ferrari and maximal Abelian gauges, [171]. Finally, in order to express the BRST and δ invariances in a functional way, we introduce the external action

$$\begin{aligned} S_{\text{ext}} &= \int d^4x \left(\Omega^{a\mu} s A_\mu^a + L^a s c^a \right) \\ &= \int d^4x \left(- \Omega^{a\mu} (D_\mu c)^a + L^a \frac{g}{2} f^{abc} c^b c^c \right), \end{aligned} \quad (7.10)$$

where $\Omega^{a\mu}$ and L^a are external sources invariant under both BRST and δ transformations, coupled to the nonlinear variations of the fields A_μ^a and c^a . It is worth noting that the source L^a couples to the composite operator $\frac{g}{2}f^{abc}c^b c^c$, thus defining its renormalization properties. It is easy to check that the complete classical action

$$\Sigma = S_{\text{YM}} + S_{\text{GF+FP}} + S_J + S_{\text{ext}} \quad (7.11)$$

is invariant under BRST and δ transformations

$$s\Sigma = 0, \quad \delta\Sigma = 0. \quad (7.12)$$

When translated into functional form, the BRST and the δ invariances give rise to the following Ward identities for the complete action Σ , namely

- the Slavnov-Taylor identity

$$\mathcal{S}(\Sigma) = 0, \quad (7.13)$$

- with

$$\mathcal{S}(\Sigma) = \int d^4x \left(\frac{\delta\Sigma}{\delta\Omega^{a\mu}} \frac{\delta\Sigma}{\delta A_\mu^a} + \frac{\delta\Sigma}{\delta L^a} \frac{\delta\Sigma}{\delta c^a} + b^a \frac{\delta\Sigma}{\delta \bar{c}^a} - J c^a \frac{\delta\Sigma}{\delta b^a} + \partial^\mu J \frac{\delta\Sigma}{\delta \eta^\mu} - \eta^\mu \frac{\delta\Sigma}{\delta \tau^\mu} \right) \quad (7.14)$$

- The δ Ward identity

$$\mathcal{W}(\Sigma) = 0, \quad (7.15)$$

with

$$\mathcal{W}(\Sigma) = \int d^4x \left(-J c^a \frac{\delta\Sigma}{\delta \bar{c}^a} - J \frac{\delta\Sigma}{\delta L^a} \frac{\delta\Sigma}{\delta b^a} - \partial^\mu J \frac{\delta\Sigma}{\delta \tau^\mu} \right). \quad (7.16)$$

7.2.2 The invariant counterterm and the renormalization constants.

We are now ready to analyse the structure of the most general local counterterm compatible with the identities (7.13) and (7.15). Let us begin by displaying the quantum numbers of all fields and sources in

	A_μ^a	c^a	$\bar{c}^a$	b^a	L^a	Ω_μ^a	J	η^μ	τ^μ
Ghost number	0	1	-1	0	-2	-1	0	-1	-2
Dimension	1	0	2	2	4	3	2	3	3

Table 7.1:

order to characterize the most general invariant counterterm which can be freely added to all orders of perturbation theory, we perturb the classical action Σ by adding an arbitrary integrated local polynomial Σ^{count} in the fields and external sources of dimension bounded by four and with zero ghost number, and we require that the perturbed action $(\Sigma + \varepsilon \Sigma^{\text{count}})$ satisfies the same Ward identities and constraints as Σ to first order in the perturbation parameter ε , which are

$$\begin{aligned} \mathcal{S}(\Sigma + \varepsilon \Sigma^{\text{count}}) &= 0 + O(\varepsilon^2), \\ \mathcal{W}(\Sigma + \varepsilon \Sigma^{\text{count}}) &= 0 + O(\varepsilon^2), \end{aligned} \quad (7.17)$$

This amounts to imposing the following conditions on Σ^{count}

$$\mathcal{B}_\Sigma \Sigma^{\text{count}} = 0, \quad (7.18)$$

with

$$\begin{aligned} \mathcal{B}_\Sigma = \int d^4x \left(\frac{\delta \Sigma}{\delta A_\mu^a} \frac{\delta}{\delta \Omega^{a\mu}} + \frac{\delta \Sigma}{\delta \Omega^{a\mu}} \frac{\delta}{\delta A_\mu^a} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta}{\delta c^a} + \frac{\delta \Sigma}{\delta c^a} \frac{\delta}{\delta L^a} \right. \\ \left. + b^a \frac{\delta}{\delta \bar{c}^a} + \partial_\mu J \frac{\delta}{\delta \eta_\mu} + \eta^\mu \frac{\delta}{\delta \tau^\mu} - J c^a \frac{\delta}{\delta b^a} \right), \end{aligned} \quad (7.19)$$

and

$$\int d^4x \left(-J c^a \frac{\delta \Sigma^{\text{count}}}{\delta \bar{c}^a} - J \frac{\delta \Sigma}{\delta L^a} \frac{\delta \Sigma^{\text{count}}}{\delta b^a} - J \frac{\delta \Sigma}{\delta b^a} \frac{\delta \Sigma^{\text{count}}}{\delta L^a} - \partial^\mu J \frac{\delta \Sigma^{\text{count}}}{\delta \tau^\mu} \right) = 0. \quad (7.20)$$

Proceeding as in [153], it turns out that the most general local invariant counterterm compatible with the Ward identities (7.13) and (7.15) contains six independent parameters denoted by σ , a_1 , a_2 , a_3 , a_4 and a_5 , and is given by

$$\begin{aligned} \Sigma^{\text{count}} = \int d^4x \left(\frac{\sigma}{4} F^{a\mu\nu} F_{\mu\nu}^a + (a_3 - a_4) (D_\mu F^{\mu\nu})^a A_\nu^a + \frac{a_1}{2} b^a b^a + a_2 b^a \partial^\mu A_\mu^a \right. \\ + (a_2 - a_3) \bar{c}^a \partial^2 c^a + (a_4 - a_2) g f^{abc} \bar{c}^a \partial^\mu (c^b A_\mu^c) \\ + \frac{(\alpha a_4 - a_1)}{2} g f^{abc} b^a \bar{c}^b c^c + (\alpha a_4 - \frac{a_1}{2}) \frac{g^2}{4} f^{abc} \bar{c}^a \bar{c}^b f^{cmn} c^m c^n \\ + a_3 \Omega^{a\mu} \partial_\mu c^a + a_4 g f^{abc} \Omega^{a\mu} A_\mu^b c^c - \frac{a_4}{2} L^a g f^{abc} c^b c^c \\ + \frac{(a_2 + a_3)}{2} J A^{a\mu} A_\mu^a + a_1 J \bar{c}^a c^a + a_5 \frac{\xi}{2} J^2 - a_2 \eta^\mu A_\mu^a c^a \\ \left. + (a_2 - a_3) \tau^\mu c^a \partial_\mu c^a + (a_4 - a_2) \frac{g}{2} \tau^\mu f^{abc} A_\mu^a c^b c^c \right). \end{aligned} \quad (7.21)$$

It therefore remains to discuss the stability of the classical action. In other words to check that Σ^{count} can be reabsorbed in the classical action Σ by means of a multiplicative renormalization of the coupling constant g , the parameters α and ξ , the fields $\{\phi = A, c, \bar{c}, b\}$ and the sources $\{\Phi = J, \eta, \tau, L, \Omega\}$, namely

$$\Sigma(g, \xi, \alpha, \phi, \Phi) + \varepsilon \Sigma^{\text{count}} = \Sigma(g_0, \xi_0, \alpha_0, \phi_0, \Phi_0) + O(\varepsilon^2), \quad (7.22)$$

with the bare fields and parameters defined as

$$\begin{aligned} A_{0\mu}^a &= Z_A^{1/2} A_\mu^a, \quad \Omega_{0\mu}^a = Z_\Omega \Omega_\mu^a, \quad \tau_{0\mu} = Z_\tau \tau_\mu \\ c_0^a &= Z_c^{1/2} c^a, \quad L_0^a = Z_L L^a, \quad g_0 = Z_g g, \\ \bar{c}_0^a &= Z_{\bar{c}}^{1/2} \bar{c}^a, \quad J_0 = Z_J J, \quad \alpha_0 = Z_\alpha \alpha, \\ b_0^a &= Z_b^{1/2} b^a, \quad \eta_{0\mu} = Z_\eta \eta_\mu, \quad \xi_0 = Z_\xi \xi. \end{aligned} \quad (7.23)$$

The parameters σ , a_1 , a_2 , a_3 , a_4 and a_5 turn out to be related to the renormalization of the gauge coupling constant g , of the gauge parameter α , and of L^a , c^a , A_μ^a , and ξ respectively, according to

$$\begin{aligned}
Z_g &= 1 + \varepsilon \frac{\sigma}{2} , \\
Z_\alpha &= 1 + \varepsilon \left(\frac{a_1}{\alpha} - \sigma - 2a_2 + 2a_3 - 2a_4 \right) , \\
Z_L &= 1 + \varepsilon \left(-a_2 - \frac{\sigma}{2} + a_3 - a_4 \right) , \\
Z_c^{1/2} &= 1 + \varepsilon \left(\frac{-a_3 + a_2}{2} \right) , \\
Z_A^{1/2} &= 1 + \varepsilon \left(-\frac{\sigma}{2} + a_3 - a_4 \right) , \\
Z_\xi &= 1 + \varepsilon (a_5 - 2\sigma - 2a_2 + 2a_3 - 4a_4) .
\end{aligned} \tag{7.24}$$

Concerning the other fields and the sources Ω_μ^a , η_μ , and τ_μ , it can be verified that they are renormalized as

$$\begin{aligned}
Z_{\bar{c}} &= Z_c , & Z_b^{1/2} &= Z_L^{-1} , & Z_\Omega &= Z_L Z_A^{-1/2} Z_c^{1/2} \\
Z_\eta &= Z_L^{-1} Z_c^{-1/2} , & Z_\tau &= 1 .
\end{aligned} \tag{7.25}$$

Finally, for the source J , one has

$$Z_J = Z_L^{-2} Z_c^{-1} , \tag{7.26}$$

from which it follows that

$$\gamma_{\mathcal{O}}(a) = -2(\gamma_c(a) + \gamma_{gc^2}(a)) , \tag{7.27}$$

where $\gamma_c(a)$ and $\gamma_{gc^2}(a)$ are the anomalous dimensions of the Faddeev-Popov ghost c^a and of the composite operator $\frac{g}{2} f^{abc} c^b c^c$, defined as

$$\begin{aligned}
\gamma_c(a) &= \mu \partial_\mu \ln Z_c^{1/2} & \gamma_{gc^2}(a) &= \mu \partial_\mu \ln Z_L & \gamma_{\mathcal{O}}(a) &= \mu \partial_\mu \ln Z_J \\
\frac{\beta(a)}{a} &= \mu \partial_\mu \ln Z_g^{-1} & \gamma_\alpha(a) &= \mu \partial_\mu \ln Z_\alpha^{-1}
\end{aligned} \tag{7.28}$$

where μ is the renormalization scale.

Therefore, we have provided a purely algebraic proof of the multiplicative renormalizability of the gluon-ghost operator to all orders of perturbation theory. In particular, we have been able to show, as explicitly exhibited in (7.27), that the anomalous dimension of $(\frac{1}{2} A^{a\mu} A_\mu^a + \alpha \bar{c}^a c^a)$ is not an independent parameter of the theory, being given by a combination of the anomalous dimensions $\gamma_c(a)$ and $\gamma_{gc^2}(a)$. It is worth mentioning that it has also been proven, [172], for the Curci-Ferrari gauge that the anomalous dimension of the ghost operators $\frac{g}{2} f^{abc} c^b c^c$, $\frac{g}{2} f^{abc} \bar{c}^b c^c$ and $\frac{g}{2} f^{abc} \bar{c}^b \bar{c}^c$ are the same.

Although we did not consider matter fields in the previous analysis, it can be checked that the renormalizability of $\mathcal{O}$ and the relation (7.27) remain unchanged if matter fields are included.

7.3 Three-loop verification.

In this section, we will explicitly verify the relation (7.27) up to three-order in the Curci-Ferrari gauge in a particular renormalization scheme, $\overline{\text{MS}}$. The values for the β -function and the anomalous dimensions

of the gluon, ghost, the operator $\mathcal{O}$ and the gauge parameter α have already been calculated in the presence of matter fields in [87]. For completeness we note that for an arbitrary colour group these are

$$\begin{aligned}\beta(a) = & - \left[\frac{11}{3} C_A - \frac{4}{3} T_F N_f \right] a^2 - \left[\frac{34}{3} C_A^2 - 4 C_F T_F N_f - \frac{20}{3} C_A T_F N_f \right] a^3 \\ & + \left[2830 C_A^2 T_F N_f - 2857 C_A^3 + 1230 C_A C_F T_F N_f - 316 C_A T_F^2 N_f^2 \right. \\ & \quad \left. - 108 C_F^2 T_F N_f - 264 C_F T_F^2 N_f^2 \right] \frac{a^4}{54} + O(a^5) ,\end{aligned}\tag{7.29}$$

$$\begin{aligned}\gamma_A(a) = & [(3\alpha - 13) C_A + 8 T_F N_f] \frac{a}{6} \\ & + [(\alpha^2 + 11\alpha - 59) C_A^2 + 40 C_A T_F N_f + 32 C_F T_F N_f] \frac{a^2}{8} \\ & + [(54\alpha^3 + 909\alpha^2 + (6012 + 864\zeta(3))\alpha + 648\zeta(3) - 39860) C_A^3 \\ & \quad - (2304\alpha + 20736\zeta(3) - 58304) C_A^2 T_F N_f + (27648\zeta(3) + 320) C_A C_F T_F N_f \\ & \quad - 9728 C_A T_F^2 N_f^2 - 2304 C_F^2 T_F N_f - 5632 C_F T_F^2 N_f^2] \frac{a^3}{1152} + O(a^4)\end{aligned}\tag{7.30}$$

$$\begin{aligned}\gamma_c(a) = & (\alpha - 3) C_A \frac{a}{4} + [(3\alpha^2 - 3\alpha - 95) C_A^2 + 40 C_A T_F N_f] \frac{a^2}{48} \\ & + [(162\alpha^3 + 1485\alpha^2 + (3672 - 2592\zeta(3))\alpha - (1944\zeta(3) + 63268)) C_A^3 \\ & \quad - (6048\alpha - 62208\zeta(3) - 6208) C_A^2 T_F N_f - (82944\zeta(3) - 77760) C_A C_F T_F N_f \\ & \quad + 9216 C_A T_F^2 N_f^2] \frac{a^3}{6912} + O(a^4) ,\end{aligned}\tag{7.31}$$

$$\begin{aligned}\gamma_{\mathcal{O}}(a) = & - [16 T_F N_f + (3\alpha - 35) C_A] \frac{a}{6} \\ & - [280 C_A T_F N_f + (3\alpha^2 + 33\alpha - 449) C_A^2 + 192 C_F T_F N_f] \frac{a^2}{24} \\ & - [((2592\alpha + 1944)\zeta(3) + 162\alpha^3 + 2727\alpha^2 + 18036\alpha - 302428) C_A^3 \\ & \quad - (62208\zeta(3) + 6912\alpha - 356032) C_A^2 T_F N_f + (82944\zeta(3) + 79680) C_A C_F T_F N_f \\ & \quad - 49408 C_A T_F^2 N_f^2 - 13824 C_F^2 T_F N_f - 33792 C_F T_F^2 N_f^2] \frac{a^3}{3456} + O(a^4) .\end{aligned}\tag{7.32}$$

$$\begin{aligned}\gamma_{\alpha}(a) = & \alpha \left[\frac{a}{4} C_A + (\alpha + 5) C_A^2 \frac{a^2}{16} \right. \\ & \quad \left. + 3 C_A^2 [(\alpha^2 + 13\alpha + 67\alpha) C_A - 40 T_F N_f] \frac{a^3}{128} \right] + O(a^4)\end{aligned}\tag{7.33}$$

where the anomalous dimension of $\mathcal{O}$ in our conventions is given by (-4) times the result quoted in [87]. The group Casimirs are $\text{tr}(T^a T^b) = T_F \delta^{ab}$, $T^a T^a = C_F I$, $f^{acd} f^{bcd} = \delta^{ab} C_A$, N_f is the number of quark flavours and $\zeta(n)$ is the Riemann zeta function. Our definition here of $\gamma_{\alpha}(a)$, which denotes the running of α , differs from that of [87] due to a different definition of Z_{α} . For computational reasons, it turns out to be more convenient to consider the renormalization of the ghost operators $f^{abc} \bar{c}^b c^c$, $f^{abc} \bar{c}^b \bar{c}^c$ and $f^{abc} \bar{c}^b \bar{c}^c$ instead of $g f^{abc} \bar{c}^b c^c$, $g f^{abc} \bar{c}^b c^c$ and $g f^{abc} \bar{c}^b \bar{c}^c$ respectively. We note that we have first verified that to three loops the anomalous dimension of each of the three operators are in

fact equal, in agreement with [172]. Accordingly, we find

$$\begin{aligned}\gamma_{c^2}(a) = & \frac{3}{2}C_A a + [(18\alpha + 95)C_A^2 - 40C_A T_F N_f] \frac{a^2}{24} \\ & + [(621\alpha^2 + (7182 + 2592\zeta(3))\alpha + (1944\zeta(3) + 63268))C_A^3 \\ & - (432\alpha + 62208\zeta(3) + 6208)C_A^2 T_F N_f + (82944\zeta(3) - 77760)C_A C_F T_F N_f \\ & - 8960C_A T_F^2 N_f^2] \frac{a^3}{3456} + O(a^4) .\end{aligned}\quad (7.34)$$

We have deduced this result using the MINCER package, [180], written in FORM, [156], where the Feynman diagrams are generated in FORM input format by QGRAF, [181]. For instance, for $f^{abc}\bar{c}^b c^c$ there are 529 diagrams to determine at three loops and 376 for the operator $f^{abc}c^b c^c$ where each is inserted in the appropriate ghost two-point function. The same FORM converter functions of [87] were used here. Since the operator $f^{abc}\bar{c}^b c^c$ has the same ghost structure as the operator $\bar{c}^a c^a$, we were able merely to replace the Feynman rule for the operator insertion of $\bar{c}^a c^a$ in the ghost two-point function with the new operator and use the same routine which determined $\gamma_{\mathcal{O}}(a)$ in [87]. However, as $f^{abc}c^b c^c$ has a different structure we had to generate a new QGRAF set of diagrams to renormalize this operator. That the anomalous dimensions of both operators emerged as equivalent at three loops for all α provides a strong check on our programming as well as justifying the general result of section 7.2. Now, taking into account the extra factor g , the anomalous dimension $\gamma_{gc^2}(a)$ is found to be

$$\begin{aligned}\gamma_{gc^2}(a) = & [8T_F N_f - 13C_A] \frac{a}{6} + [(6\alpha - 59)C_A^2 + 40C_A T_F N_f + 32C_F T_F N_f] \frac{a^2}{8} \\ & + [(207\alpha^2 + (2394 + 864\zeta(3))\alpha + (648\zeta(3) - 39860))C_A^3 \\ & - (144\alpha + 20736\zeta(3) - 58304)C_A^2 T_F N_f + (27648\zeta(3) + 320)C_A C_F T_F N_f \\ & - 9728C_A T_F^2 N_f^2 - 2304C_F^2 T_F N_f - 5632C_F T_F^2 N_f^2] \frac{a^3}{1152} + O(a^4) .\end{aligned}\quad (7.35)$$

It is then easily checked from the expressions (7.32), (7.33) and (7.35) that, up to three-loop order,

$$\gamma_{\mathcal{O}}(a) = -2(\gamma_c(a) + \gamma_{gc^2}(a)) .\quad (7.36)$$

It is worth mentioning that the renormalizability of the operator $\mathcal{O}$ was already discussed in [182] from the viewpoint of the massive Curci-Ferrari model. Whilst the relation (7.27) was not explicitly given in [182], it is possible to obtain the relation from that analysis. Although the relation (7.27) has been established in the case of the Curci-Ferrari gauge, it expresses a general property of the gluon-ghost operator which remains valid also in the Landau and maximal Abelian gauges, as will be shown in the following sections.

7.4 The Landau gauge.

The Landau gauge is a particular case of the Curci-Ferrari gauge, corresponding to $\alpha = 0$. The Landau gauge is known to possess further additional Ward identities [59, 89], implying that the renormalization constants Z_L and Z_c can be expressed in terms of Z_g and Z_A , according to [153]

$$Z_L = Z_A^{1/2} , \quad Z_c = Z_g^{-1} Z_A^{-1/2} .\quad (7.37)$$

Therefore, it follows that (7.26) reduces to

$$Z_J = Z_g Z_A^{-1/2}\quad (7.38)$$

from which the expression (7.1) is recovered, providing a nontrivial check of the validity of the general relationship (7.2).

As another internal check of our computations, we note that we should also find in the Landau gauge that $\gamma_{gc^2}(a) = \gamma_A(a)$, as is obvious from (7.37). It can indeed be checked from (7.30) and (7.35) that

$$\gamma_{gc^2}(a)|_{\alpha=0} = \gamma_A(a)|_{\alpha=0} . \quad (7.39)$$

7.5 The maximal Abelian gauge.

As is well known, the maximal Abelian gauge is a nonlinear partial gauge fixing allowing for a residual $U(1)^{N-1}$ local invariance [71, 72, 179, 183]. In the following, a Landau type gauge fixing will be assumed for this local residual invariance. The Slavnov-Taylor and the δ Ward identities (7.13) and (7.15) can be straightforwardly generalized to this case. It is useful to recall that the gauge field is now decomposed into its off-diagonal and diagonal components

$$A_\mu^a T^a = A_\mu^i T^i + A_\mu^\alpha T^\alpha , \quad (7.40)$$

where the index i labels the $N - 1$ generators T^i of the Cartan subalgebra of $SU(N)$. The remaining $N(N-1)$ off-diagonal generators T^α will be labelled by the index α . Accordingly, for the Faddeev-Popov ghost c^a we have

$$c^a T^a = c^i T^i + c^\alpha T^\alpha , \quad (7.41)$$

with

$$\begin{aligned} sc^\alpha &= gf^{\alpha\beta i} c^\beta c^i + \frac{g}{2} f^{\alpha\beta\gamma} c^\beta c^\gamma , \\ sc^i &= \frac{g}{2} f^{i\alpha\beta} c^\alpha c^\beta . \end{aligned} \quad (7.42)$$

Also, the group index of the gluon-ghost operator runs only over the off-diagonal components, namely

$$\mathcal{O}_{\text{MAG}} = \left(\frac{1}{2} A^{\alpha\mu} A_\mu^\alpha + \alpha \bar{c}^\alpha c^\alpha \right) . \quad (7.43)$$

Denoting respectively by $\tilde{Z}$ and Z the renormalization factors of the off-diagonal and diagonal components of the fields, it follows that, according to the relationship (7.27), the output of the Slavnov-Taylor and δ Ward identities gives

$$\gamma_{\mathcal{O}_{\text{MAG}}}(a) = - 2 \left(\tilde{\gamma}_{c^\alpha}(a) + \tilde{\gamma}_{gc^2}(a) \right) , \quad (7.44)$$

where $\tilde{\gamma}_{c^\alpha}(a)$ and $\tilde{\gamma}_{gc^2}(a)$ are the anomalous dimensions of the off-diagonal ghost c^α and of the composite operator $(gf^{\alpha\beta i} c^\beta c^i + \frac{g}{2} f^{\alpha\beta\gamma} c^\beta c^\gamma)$ which corresponds to the BRST variation of c^α . Moreover, as shown in [72], the use of the Landau gauge for the local residual $U(1)^{N-1}$ invariance allows for a further Ward identity. This identity, called the diagonal ghost Ward identity in [72], implies that the anomalous dimension $\tilde{\gamma}_{gc^2}(a)$ can be expressed as

$$\tilde{\gamma}_{gc^2}(a) = \frac{\beta(a)}{a} - \tilde{\gamma}_{c^\alpha}(a) - \gamma_{c^i}(a) , \quad (7.45)$$

where $\gamma_{c^i}(a)$ is the anomalous dimension of the diagonal ghost c^i . Therefore, for the expression of $\gamma_{\mathcal{O}_{\text{MAG}}}(a)$ we obtain

$$\gamma_{\mathcal{O}_{\text{MAG}}}(a) = - 2 \left(\frac{\beta(a)}{a} - \gamma_{c^i}(a) \right) , \quad (7.46)$$

a result which is in complete agreement with that already found in [179] for the case of $SU(2)$. Finally, it is worth mentioning that the anomalous dimensions of the diagonal and off-diagonal components of the fields have been computed at one-loop order in [179, 183], so that (7.27) gives explicit knowledge of the one-loop anomalous dimension of the gluon-ghost operator in the maximal Abelian gauge.

7.6 Conclusion.

We have shown that the mass dimension two gluon-ghost operator $\mathcal{O} = \frac{1}{2}A_\mu^a A^{\mu a} + \alpha \bar{c}^a c^a$ is multiplicatively renormalizable in the Landau, Curci-Ferrari and maximal Abelian gauges. Further, we were able to establish a general relation between the anomalous dimension of $\mathcal{O}$, the Faddeev-Popov ghost c^a and the dimension two ghost operator $g f^{abc} c^b c^c$, as expressed by the eq.(7.2). This relation has been derived within the framework of the algebraic renormalization [59], following from the Slavnov-Taylor identity (7.13). As such, it extends to all orders of perturbation theory and is renormalization scheme independent, for any scheme preserving the Slavnov-Taylor identity. It has been explicitly verified up to three loops in the $\overline{\text{MS}}$ scheme in the Curci-Ferrari gauge.

Furthermore, due to additional Ward identities that exist in the Landau gauge [59] and in the MAG [72], we were able to rewrite the relation (7.2) for the anomalous dimension for $\mathcal{O}$ in terms of the β -function and the anomalous dimension of the gluon and/or ghost fields. In particular, concerning the maximal Abelian gauge, it is worth underlining that the multiplicative renormalizability of the gluon-ghost operator, eq.(7.46), is a necessary ingredient towards the construction of a renormalizable effective potential for studying the possible condensation of the gluon-ghost operator and the ensuing dynamical mass generation, as done in the Landau [42] and Curci-Ferrari [178] gauges.

As a final remark, we point out that, from the three-loop expressions given in section 7.3, it is easily checked that the following relations holds in the Curci-Ferrari gauge:

$$\gamma_{\mathcal{O}}(a) = - \left(\frac{\beta(a)}{a} + \gamma_A(a) \right), \quad (7.47)$$

$$\gamma_{gc^2}(a) = \gamma_A(a) - 2\gamma_\alpha(a). \quad (7.48)$$

Up to now, we do not know if these relations are valid to all orders. They do not follow from the Slavnov-Taylor identity (7.13). Nevertheless, although eqs.(7.47), (7.48) have been obtained in a particular renormalization scheme, i.e. the $\overline{\text{MS}}$ scheme, it could be interesting to search for additional Ward identities in the Curci-Ferrari gauge which, as in the case of the Landau gauge [153], could allow for a purely algebraic proof of eqs.(7.47), (7.48). Notice in fact that, when $\alpha = 0$, eq.(7.47) yields the anomalous dimension of the composite operator A_μ^2 in the Landau gauge. Also, eq.(7.48) reduces to the relation (7.39) of the Landau gauge, since $\gamma_\alpha(a) \equiv 0$ if $\alpha = 0$.

Chapter 8

Gluon-ghost condensate of mass dimension 2 in the Curci-Ferrari gauge

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The effective potential for an on-shell BRST invariant gluon-ghost condensate of mass dimension 2 in the Curci-Ferrari gauge in $SU(N)$ Yang-Mills is analysed by combining the local composite operator technique with the algebraic renormalization. We pay attention to the gauge parameter independence of the vacuum energy obtained in the considered framework and discuss the Landau gauge as an interesting special case.

8.1 Introduction.

Nowadays an increasing evidence has been reported on the relevance of the local composite operator A^2 in the Landau gauge, both from a phenomenological point of view [33, 34] as from lattice studies [38, 37, 40]. It is no coincidence that the Landau gauge is used because then A^2 equals the non-local gauge invariant operator $(VT)^{-1} \min_U \int d^4x (A^2)^U$ with VT the space time volume. The lattice also revealed that gluons attain a dynamical mass, see e.g. [47, 48]. Some older work already discussed the pairing of gluons in connection with a mass generation, as a result of the fact that the perturbative Yang-Mills (YM) vacuum (trivially zero) is unstable [28, 29, 30]. More recently, the connection between a condensate $\langle A^2 \rangle$ and a gluon mass has been made within the OPE framework [83, 144]. A technique to effectively calculate $\langle A^2 \rangle$ and the gluon mass was presented in [42], also in the Landau gauge. An alternative method was discussed in [184].

The answer to the question how a mass is generated could be posed in a more general context than the Landau gauge. The Landau gauge is a limiting case of a class of renormalizable, generalized covariant gauges introduced in [160, 161]. We are therefore led to search for a local operator which could replace A^2 . A proposal has been made in [83], where it was shown that A^2 is a special case of a more general mass dimension 2 operator, namely $\mathcal{O} = \frac{1}{2} A_\mu^a A^{\mu a} + \alpha \bar{c}^a c^a$, also involving ghosts and which is BRST invariant on-shell, however not gauge invariant (see also [185]). The proposed condensate is not that

surprising, since it equals the operator coupled to the mass term of a massive, renormalizable $SU(N)$ model, introduced in [84, 85]. The specific form of the mass term is necessary to maintain the BRST invariance and renormalizability [84, 85, 182]. Although the Curci-Ferrari model (CF) is BRST invariant, the associated BRST operator is not nilpotent and the model is not unitary [162, 182]. Since the gauge fixing terms of the CF model and the YM theories with the gauges discussed in [83, 144, 160, 161] are the same, it seems natural to search in that direction for a suitable operator that gets a non-vanishing vacuum expectation value and invokes a dynamical mass.

The aim of this paper is to construct an effective potential for the mass dimension 2 condensate in the CF gauge. It is organized as follows. In section 8.2 we discuss the formalism to obtain a well-defined effective potential for the local composite operator $\mathcal{O} = \frac{1}{2}A_\mu^a A^{\mu a} + \alpha \bar{c}^a c^a$, a non-trivial task due to the compositeness of this operator [42, 176]. In section 8.3, we denote the Ward identities of the action, ensuring the renormalizability. A further construction of the effective action is discussed in section 8.4, where we also outline a subtlety on the minimization of the effective potential. In section 8.5, we consider the gauge parameter independence of the vacuum energy and spend some words on the BRST charge. Section 8.6 handles the explicit evaluation of the effective potential. We also discuss the interesting role of the Landau gauge as a limiting case of the CF gauge. We pay attention to the similarities between CF and the maximal Abelian gauge (MAG). A mass generating mechanism for the off-diagonal gluons in the MAG very much resembles that of the CF gauge, and could be seen as some evidence for Abelian dominance. As usual, conclusions are formulated in the last section.

8.2 The LCO formalism.

For a more detailed introduction to the local composite operator (LCO) formalism and to the algebraic renormalization technique, the reader is referred to [42, 176], respectively [59].

Let us begin by giving the expression for the $SU(N)$ Yang-Mills action in the CF gauge

$$\begin{aligned} S = S_{YM} + S_{GF+FP} = & -\frac{1}{4} \int d^4x F_{\mu\nu}^a F^{a\mu\nu} + \int d^4x \left(b^a \partial_\mu A^{\mu a} + \frac{\alpha}{2} b^a b^a + \bar{c}^a \partial^\mu D_\mu^{ab} c^b \right. \\ & \left. - \frac{\alpha}{2} g f^{abc} b^a \bar{c}^b c^c - \frac{\alpha}{8} g^2 f^{abc} f^{cde} \bar{c}^a \bar{c}^b c^d c^e \right) \end{aligned} \quad (8.1)$$

where

$$D_\mu^{ab} \equiv \partial_\mu \delta^{ab} + g f^{acb} A_\mu^c \quad (8.2)$$

is the usual covariant derivative. In order to investigate if

$$\mathcal{O} = \frac{1}{2} A_\mu^a A^{\mu a} + \alpha \bar{c}^a c^a \quad (8.3)$$

gets a non-vanishing vacuum expectation value, we introduce a suitable set of LCO sources [42, 176]. In this case this task is nontrivial. It turns out that in order to introduce the local operator $\mathcal{O}$ in the starting action in a BRST invariant way, three external sources J , η^μ and τ^μ are needed, so that

$$S_{\text{LCO}} = \int d^4x \left[J\mathcal{O} + \frac{\xi}{2} J^2 - \eta^\mu A_\mu^a c^a - \tau^\mu s(A_\mu^a c^a) \right] \quad (8.4)$$

where ξ is the LCO parameter and s denotes the BRST operator acting as

$$\begin{aligned}
sA_\mu^a &= -D_\mu^{ab} c^b \\
sc^a &= \frac{g}{2} f^{abc} c^b c^c \\
s\bar{c}^a &= b^a \\
sb^a &= -Jc^a \\
sJ &= 0 \\
s\eta^\mu &= \partial^\mu J \\
s\tau^\mu &= \eta^\mu
\end{aligned} \tag{8.5}$$

The parameter ξ has to be introduced since the introduction of the source term $J\mathcal{O}$ gives rise to novel vacuum energy divergences proportional to J^2 . These new divergences, related to those of the connected Green's function $\langle \mathcal{O}(x)\mathcal{O}(y) \rangle_c$ for $x \rightarrow y$, are canceled by a counterterm $\delta\xi \frac{J^2}{2}$.

After introduction of the sources, we still have a BRST invariant action

$$s(S_{YM} + S_{GF+FP} + S_{LCO}) = 0 \tag{8.6}$$

but it should be observed that, due to the presence of the sources (J, η^μ, τ^μ) , the BRST operator is no more nilpotent, namely

$$\begin{aligned}
s^2\Phi &= 0, \quad \Phi = (A, c, J, \eta^\mu) \\
s^2\bar{c}^a &= -Jc^a \\
s^2b &= -J\frac{g}{2} f^{abc} c^b c^c \\
s^2\tau^\mu &= \partial^\mu J
\end{aligned} \tag{8.7}$$

As a consequence, setting

$$s^2 = \delta_J \tag{8.8}$$

we have

$$\delta_J(S_{YM} + S_{GF+FP} + S_{LCO}) = 0 \tag{8.9}$$

The operator δ_J is related to the $SL(2, \mathbb{R})$ symmetry [160, 161, 171] exhibited by the Curci-Ferrari action. The generators of this $SL(2, \mathbb{R})$ symmetry are, next to the Faddeev-Popov ghost number δ_{FP} , given by

$$\begin{aligned}
\delta\bar{c}^a &= c^a \\
\delta b^a &= \frac{g}{2} f^{abc} c^b c^c \\
\delta A_\mu^a &= \delta c^a = 0
\end{aligned} \tag{8.10}$$

and

$$\begin{aligned}
\bar{\delta}c^a &= \bar{c}^a \\
\bar{\delta}b^a &= \frac{g}{2} f^{abc} \bar{c}^b \bar{c}^c \\
\bar{\delta}A_\mu^a &= \bar{\delta}\bar{c}^a = 0
\end{aligned} \tag{8.11}$$

The action of the δ symmetry can be enlarged to the sources as $\delta J = 0$, $\delta\eta_\mu = 0$ and $\delta\tau_\mu = 0$. Also, expression (8.8) shows that, in the massive case, the δ_J -invariance is a consequence of the modified BRST transformations. The lack of nilpotency of the BRST operator together with (8.8) are well known features of the CF gauge in the presence of a mass term [163].

Notice that in the present case the operator s^2 always contains the source J which will be set to zero at the end of the computation.

8.3 Ward identities.

Let us now translate the previous invariances into Ward identities. To this purpose, we introduce external sources Ω_μ^a and L^a coupled to the BRST variation of A_μ^a and c^a

$$S_{ext} = \int d^4x \left[-\Omega^{a\mu} D_\mu^{ab} c^b + L^a \frac{g}{2} f^{abc} c^b c^c \right] \quad (8.12)$$

with

$$s\Omega_\mu^a = sL^a = 0$$

The complete action

$$\Sigma = S_{YM} + S_{GF+FP} + S_{LCO} + S_{ext} \quad (8.13)$$

turns out to obey the following identities:

- The Slavnov-Taylor identity

$$\mathcal{S}(\Sigma) = 0 \quad (8.14)$$

with

$$\mathcal{S}(\Sigma) = \int d^4x \left(\frac{\delta\Sigma}{\delta A_\mu^a} \frac{\delta\Sigma}{\delta\Omega^{a\mu}} + \frac{\delta\Sigma}{\delta L^a} \frac{\delta\Sigma}{\delta c^a} + b^a \frac{\delta\Sigma}{\delta\bar{c}^a} + \partial_\mu J \frac{\delta\Sigma}{\delta\eta_\mu} + \eta^\mu \frac{\delta\Sigma}{\delta\tau^\mu} - J c^a \frac{\delta\Sigma}{\delta b^a} \right) \quad (8.15)$$

- The δ_J Ward identity

$$\mathcal{W}(\Sigma) = 0 \quad (8.16)$$

with

$$\mathcal{W}(\Sigma) = \int d^4x \left(J c^a \frac{\delta\Sigma}{\delta\bar{c}^a} + J \frac{\delta\Sigma}{\delta L^a} \frac{\delta\Sigma}{\delta b^a} - \partial_\mu J \frac{\delta\Sigma}{\delta\tau_\mu} \right) \quad (8.17)$$

Proceeding as in [153], these identities imply the renormalizability of the model and, in particular, the multiplicative renormalizability of the local operator $\mathcal{O}$.

8.4 Renormalizability of $\mathcal{O}$ and the effective action.

As established explicitly in [144, 87], the operator $\mathcal{O}$ is indeed multiplicative renormalizable in the CF gauge. Denoting the bare operator by $\mathcal{O}_B$, one has

$$\mathcal{O}_B = Z_{\mathcal{O}} \mathcal{O}_R \quad (8.18)$$

with¹ [144, 87]

$$Z_{\mathcal{O}} = 1 + \left[\frac{35}{6} - \frac{\alpha}{2} \right] \frac{g^2 N}{16\pi^2} \frac{1}{\varepsilon} + \left[\left(\frac{2765}{72} - \frac{11\alpha}{3} \right) \frac{1}{\varepsilon^2} + \left(\frac{\alpha^2}{16} + \frac{11\alpha}{16} - \frac{449}{48} \right) \frac{1}{\varepsilon} \right] \left(\frac{g^2 N}{16\pi^2} \right)^2 + \dots \quad (8.19)$$

For the anomalous dimension $\gamma_{\mathcal{O}}$ of $\mathcal{O}$, one has [144, 87]

$$\gamma_{\mathcal{O}}(g^2, \alpha) = -\mu \frac{\partial \ln Z_{\mathcal{O}}}{\partial \mu} = \left(\frac{35}{6} - \frac{\alpha}{2} \right) \frac{g^2 N}{16\pi^2} + \left(\frac{449}{24} - \frac{\alpha^2}{8} - \frac{11\alpha}{8} \right) \left(\frac{g^2 N}{16\pi^2} \right)^2 + \dots \quad (8.20)$$

Notice that $\gamma_{\mathcal{O}}$ depends on the gauge parameter α . This is due to the explicit dependence from α of the operator $\mathcal{O}$. Moreover, in the limit $\alpha \rightarrow 0$, expression (8.20) reduces to the anomalous dimension of the Landau gauge [42]. Let us also give, for further use, the β -function of the gauge parameter α in the CF gauge [144, 87].

$$\beta_{\alpha}(g^2, \alpha) = \mu \frac{\partial \alpha}{\partial \mu} = \left(\frac{13}{3} - \frac{\alpha}{2} \right) \frac{g^2 N}{16\pi^2} - \frac{\alpha^2 + 17\alpha - 118}{16} \left(\frac{g^2 N}{16\pi^2} \right)^2 + \dots \quad (8.21)$$

In order to obtain the effective potential for the operator $\mathcal{O}$, we set to zero the sources Ω_{μ}^a , L^a , η^{μ} and τ^{μ} , obtaining for the generating functional the following expression

$$\exp -i\mathcal{W}(J) = \int [D\phi] \exp iS(J) \quad (8.22)$$

with

$$S(J) = S_{YM} + S_{GF+FP} + \int d^4x \left[J\mathcal{O} + \frac{\xi}{2} J^2 \right] \quad (8.23)$$

and ϕ denoting the relevant fields.

From the bare Lagrangian associated to (8.23), one obtains that the quantity $\xi(\mu)$ obeys the following renormalization group equation (RGE)

$$\mu \frac{d\xi}{d\mu} = 2\gamma_{\mathcal{O}}(g^2, \alpha)\xi + \delta(g^2, \alpha) \quad (8.24)$$

where

$$\delta(g^2, \alpha) = \left(\varepsilon + 2\gamma_{\mathcal{O}}(g^2, \alpha) - \beta(g^2) \frac{\partial}{\partial g^2} - \alpha\beta_{\alpha}(g^2, \alpha) \frac{\partial}{\partial \alpha} \right) \xi \quad (8.25)$$

Now, following [42], it is possible to set the hitherto *free* parameter ξ such a function of g^2 and α , so that if g^2 runs according to $\beta(g^2)$ and α to $\beta_{\alpha}(g^2)$, $\xi(g^2, \alpha)$ will run according to its RGE (8.24). Specifying, $\xi(g^2, \alpha)$ is the particular solution of

$$\left(\beta(g^2) \frac{\partial}{\partial g^2} + \alpha\beta_{\alpha}(g^2, \alpha) \frac{\partial}{\partial \alpha} \right) \xi(g^2, \alpha) = 2\gamma_{\mathcal{O}}(g^2, \alpha)\xi(g^2, \alpha) + \delta(g^2, \alpha) \quad (8.26)$$

Furthermore², $\xi(g^2, \alpha)$ is multiplicatively renormalizable ($\xi + \delta\xi = Z_{\xi}\xi$). It is easy to see that $\xi(g^2, \alpha)$ will be of the form

$$\xi(g^2, \alpha) = \frac{\xi_0(\alpha)}{g^2} + \xi_1(\alpha) + \xi_2(\alpha)g^2 + \dots \quad (8.27)$$

¹We use dimensional regularization in $d = 4 - \varepsilon$ dimensions and employ the $\overline{\text{MS}}$ renormalization scheme.

²The integration constant showing up when (8.26) is solved, has been put to zero according to [42].

Performing the calculation at one-loop, we find that

$$\delta\xi = -\frac{(N^2 - 1)(3 - \alpha^2)}{16\pi^2 \varepsilon} \quad (8.28)$$

Consequently, solving (8.26) for ξ_0 as a function of the gauge parameter α , one finds

$$\xi_0(\alpha) = \frac{9}{13} \frac{N^2 - 1}{N} s_0(\alpha) \quad (8.29)$$

$$s_0(\alpha) = 1 + \frac{311}{117}\alpha + 6\alpha \left(1 - \frac{3\alpha}{26}\right) \ln \left| -\frac{26}{\alpha} + 3 \right| + c\alpha(-26 + 3\alpha) \quad (8.30)$$

with c an integration constant. Notice that $s_0(0) = 1$, so that we recover the result of [42] in the case of the Landau gauge. In the next section, we will show that the vacuum energy is formally gauge parameter independent. Henceforth, we can forget about the integration constant and set $c = 0$.

Taking now the functional derivative of $\mathcal{W}(J)$ with respect to J , we obtain

$$\left. \frac{\delta \mathcal{W}(J)}{\delta J} \right|_{J=0} = -\langle \mathcal{O} \rangle \quad (8.31)$$

The presence of the J^2 term in $\mathcal{W}(J)$ seems to spoil an energy interpretation. However, this can be dealt with by introducing a Hubbard-Stratonovich field σ so that

$$J\mathcal{O} + \frac{\xi}{2}J^2 \Rightarrow -\frac{\sigma^2}{2\xi g^2} + \frac{\sigma}{g\xi}\mathcal{O} + \frac{\sigma}{g}J - \frac{1}{2\xi}\mathcal{O}^2 \quad (8.32)$$

Therefore

$$\exp -i\mathcal{W}(J) = \int [D\phi] \exp i \left(S_\sigma + \int d^4x \frac{\sigma}{g} J \right) \quad (8.33)$$

where

$$S_\sigma = S_{YM} + S_{GF+FP} + \int d^4x \left(-\frac{\sigma^2}{2\xi g^2} + \frac{\sigma}{g\xi}\mathcal{O} - \frac{1}{2\xi}\mathcal{O}^2 \right) \quad (8.34)$$

J now appears as a linear source. Hence, we have back an energy interpretation and the $1PI$ machinery applies.

Differentiating the functional generator with respect to J , one gets the relationship

$$\langle \sigma \rangle_{S_\sigma} = g \langle \mathcal{O} \rangle \quad (8.35)$$

Recapitulating, we have constructed a multiplicatively renormalizable action S_σ incorporating the effects of a possible non-vanishing vacuum expectation value for $\mathcal{O}$. The corresponding effective action Γ obeys a linear, homogeneous RGE. Notice that to get actual knowledge of the n -loop effective action, one needs the values of $\xi_0, \dots, \xi_n$. This means, recalling (8.26), that we need the $(n+1)$ -loop values of the renormalization group functions. In [91], a slightly different Hubbard-Stratonovich transformation was used, so that

$$J\mathcal{O} + \frac{\xi}{2}J^2 \Rightarrow -\frac{\sigma^2}{2g^2} + \frac{\sigma}{g\sqrt{\xi}}\mathcal{O} + \frac{\sqrt{\xi}\sigma}{g}J - \frac{1}{2\xi}\mathcal{O}^2 \quad (8.36)$$

resulting in

$$\exp -i\mathcal{W}(J) = \int [D\phi] \exp i \left(S_\sigma + \int d^4x \frac{\sqrt{\xi}\sigma}{g} J \right) \quad (8.37)$$

where

$$S_\sigma = S_{YM} + S_{GF+FP} + \int d^4x \left(-\frac{\sigma^2}{2g^2} + \frac{\sigma}{g\sqrt{\xi}} \mathcal{O} - \frac{1}{2\xi} \mathcal{O}^2 \right) \quad (8.38)$$

With this action, it seems that it suffices to know $\xi_0, \dots, \xi_{n-1}$ to construct the n -loop effective potential. However, some attention should be paid here. It is indeed so that with (8.38), we do not need ξ_n for $\Gamma_{n\text{-loop}}$, but since the source J is now coupled to the operator $\frac{\sqrt{\xi}\sigma}{g}$, we formally have for the effective action Γ , being the Legendre transform of $\mathcal{W}(J)$

$$\Gamma \left(\frac{\sqrt{\xi}\sigma}{g} \right) = -\mathcal{W}(J) - \int d^4y J(y) \frac{\sqrt{\xi}\sigma(y)}{g} \quad (8.39)$$

Hence

$$\frac{\delta}{\delta \left(\frac{\sqrt{\xi}\sigma(y)}{g} \right)} \Gamma \left(\frac{\sqrt{\xi}\sigma(x)}{g} \right) = -J(y) \quad (8.40)$$

Since

$$\Gamma = \frac{\Gamma_0}{g^2} + \Gamma_1 + \dots \quad (8.41)$$

$$\frac{\sqrt{\xi}}{g} = \sqrt{\xi_0} \left(\frac{1}{g^2} + \frac{\xi_1}{\xi_0} + \dots \right) \quad (8.42)$$

it becomes clear that, in order to have $J = 0$ up to the considered order in a g^2 expansion (i.e. to end up in the vacuum state), one must solve (for constant configurations)

$$\frac{d}{d \left(\frac{\sqrt{\xi}\sigma}{g} \right)} V = 0 \quad (8.43)$$

which will *not*³ produce the same (correct) σ_{min} as by solving

$$\frac{dV}{d\sigma} = 0 \quad (8.44)$$

as it was done in [91]. The most efficient way to solve (8.43) is by performing the transformation

$$\sigma \rightarrow \frac{\sigma}{\sqrt{\xi}} \quad (8.45)$$

and this exactly transforms the action (8.38) into the one of (8.34). Notice that the action (8.38) is not incorrect, one should only be careful how the vacuum configuration is constructed. The conclusion is that one cannot escape the job of doing $(n+1)$ -loop calculations for n -loop results.

We draw attention to the fact that the action S_σ is BRST invariant⁴, while this BRST transformation is nilpotent for $J = 0$. This means that the action, evaluated in its minimum, i.e. the vacuum energy, should be independent of the gauge parameter α order by order. In the next section, we pay some more attention to this α independence.

³Because ξ itself is a series in g^2 .

⁴It is obvious that $s\sigma = gs\mathcal{O}$.

8.5 Gauge parameter independence of the vacuum energy.

We begin our argumentation from the generating functional (8.33). It will be useful to consider also the 'original' action $\tilde{S}(J)$ (i.e. before the Hubbard-Stratonovich transformation) defined in (8.23). To avoid confusion with (8.33)-(8.34), we added a $\sim$ to the notation. The relation between $\mathcal{W}(J)$ and $\tilde{S}(J)$ is obtained via the insertion of an unity

$$1 = \frac{1}{N} \int [D\sigma] \exp \left[i \int d^4x \left(-\frac{1}{2\xi} \left(\frac{\sigma}{g} - \mathcal{O} - \xi J \right)^2 \right) \right] \quad (8.46)$$

with N an appropriate normalization factor. Explicitly, we have

$$\exp(-i\mathcal{W}(J)) = \int [D\phi][D\sigma] \exp i \left[\tilde{S}(J) + \int d^4x \left(-\frac{1}{2\xi} \left(\frac{\sigma}{g} - \mathcal{O} - \xi J \right)^2 \right) \right] \quad (8.47)$$

Since evidently

$$\frac{d}{d\alpha} \frac{1}{N} \int [D\sigma] \exp \left[i \int d^4x \left(-\frac{1}{2\xi} \left(\frac{\sigma}{g} - \mathcal{O} - \xi J \right)^2 \right) \right] = 0 \quad (8.48)$$

we find

$$-\frac{d\mathcal{W}(J)}{d\alpha} = \left\langle s \left(\frac{\bar{c}b}{2} - \frac{g^2}{4} f^{abc} \bar{c}^a \bar{c}^b c^c \right) \right\rangle_{J=0} + \text{terms proportional to } J \quad (8.49)$$

The effective action Γ is related to $\mathcal{W}(J)$ through a Legendre transformation

$$\Gamma \left(\frac{\sigma}{g} \right) = -\mathcal{W}(J) - \int d^4y J(y) \frac{\sigma(y)}{g} \quad (8.50)$$

The effective potential $V(\sigma)$ is then defined as

$$-V(\sigma) \int d^4x = \Gamma \left(\frac{\sigma}{g} \right) \quad (8.51)$$

Let σ_{min} be the solution of

$$\left. \frac{dV(\sigma)}{d\sigma} \right|_{\sigma=\sigma_{min}} = 0 \quad (8.52)$$

Hence, we have that

$$\sigma = \sigma_{min} \Rightarrow J = 0 \quad (8.53)$$

Invoking (8.53), we derive from (8.50)-(8.51)

$$\left. \frac{d}{d\alpha} V(\sigma) \right|_{\sigma=\sigma_{min}} \int d^4x = \left. \frac{d}{d\alpha} \mathcal{W}(J) \right|_{J=0} \quad (8.54)$$

Finally, combining (8.49) and (8.54), we conclude that

$$\left. \frac{d}{d\alpha} V(\sigma) \right|_{\sigma=\sigma_{min}} = 0 \quad (8.55)$$

Some extra words concerning (8.53) and its consequences (8.54)-(8.55) are in order. Obviously, this is based on the relation

$$\frac{\delta}{\delta\left(\frac{\sigma}{g}\right)}\Gamma = -J \quad (8.56)$$

An explicit evaluation of the effective potential results in a series for $V(\sigma)$, and consequently in a gap equation via (8.52). Said otherwise, $J = 0$ means in practice that J equals zero up to a certain order in g^2 as a consequence of the solved gap equation, which is of the form

$$V_0(\sigma) + V_1(\sigma)g^2 + \dots + V_{n-1}(\sigma)(g^2)^{n-1} = 0 \quad (8.57)$$

Returning to (8.49), the terms proportional to J are themselves some series in g^2 . This means that the product of such a term with J is again a series, which has to be cut off at the considered order; thus some terms are dropped. When (8.56)-(8.57) are used, it turns out that the product of such a term with J is also zero, but up to terms of higher order. Henceforth, the gauge parameter independence is *not exact*, but holds up to terms of higher order. The same holds true for the BRST charge Q_{BRST} , which will not be exactly nilpotent, but again up to higher order terms. As it is well known, Q_{BRST} is used to define physical states as those annihilated by Q_{BRST} and which are not exact (i.e. $\neq Q_{BRST}|\text{something}\rangle$). The nilpotency of Q_{BRST} is needed to move freely in the space of gauge parameter choices. With all this in mind, the α derivative of the action is reduced to an exact BRST variation. This is the usual argument used to show that physical operators, including the vacuum energy, are independent of the choice for the gauge parameter α [59]. We underline again that here, all this is not exact, but only valid up to terms of higher order.

Concluding this section, we have shown that the effective potential, evaluated at its minimum (i.e. the vacuum energy), is gauge parameter independent at any order in a loop (g^2) expansion, at least up to terms that are of higher order.

8.6 Evaluation of the one-loop effective potential.

In order to evaluate the one-loop effective potential, it is sufficient to consider only the quadratic terms of S_σ , namely

$$S_\sigma^{quad} = \int d^4x \left(-\frac{\sigma^2}{2\xi g^2} + \bar{c}^a \Sigma^{ab} c^b + \frac{1}{2} A^{a\mu} \Omega_{\mu\nu}^{ab} A^{b\nu} \right) \quad (8.58)$$

where

$$\Sigma^{ab} = \delta^{ab} \left(\partial^2 + \frac{\sigma\alpha}{g\xi} \right) \quad (8.59)$$

and

$$\Omega_{\mu\nu}^{ab} = \delta^{ab} \left[\left(\partial^2 + \frac{\sigma}{g\xi} \right) g_{\mu\nu} - \left(1 - \frac{1}{\alpha} \right) \partial_\mu \partial_\nu \right] \quad (8.60)$$

To calculate V , we use the background formalism with the trivial background $A_\mu = 0$. This means that we restrict ourselves to the pure short-range contributions to $\langle \mathcal{O} \rangle$. If one would like to include long-range effects, one could for example use an instanton background [38].

For the one-loop effective potential we get

$$V_1(\sigma) = \frac{\sigma^2}{2\xi_0} \left(1 - \frac{\xi_1}{\xi_0} g^2 \right) + i \ln \det \Sigma^{ab} - \frac{i}{2} \ln \det \Omega_{\mu\nu}^{ab} \quad (8.61)$$

In d dimensions, it holds that

$$\begin{aligned} & \ln \det \delta^{ab} \left[g_{\mu\nu} (\partial^2 + m^2) - \left(1 - \frac{1}{\alpha} \right) \partial_\mu \partial_\nu \right] \\ &= (N^2 - 1) \left[(d-1) \text{tr} \ln (\partial^2 + m^2) + \text{tr} \ln \left(\frac{\partial^2}{\alpha} + m^2 \right) \right] \end{aligned} \quad (8.62)$$

Working up to order ε^0 and order g^2 , we find

$$\begin{aligned} i \ln \det \Sigma^{ab} &= i (N^2 - 1) \int \frac{d^d k}{(2\pi)^d} \ln \left(-k^2 + \frac{\sigma \alpha}{g \xi} \right) \\ &= -\frac{(N^2 - 1)}{32\pi^2} \left(\frac{g^2 \sigma^2 \alpha^2}{\xi_0^2} \right) \left(\ln \frac{g \sigma \alpha}{\xi_0 \bar{\mu}^2} - \frac{3}{2} - \frac{2}{\varepsilon} \right) \end{aligned} \quad (8.63)$$

$$\begin{aligned} -\frac{i}{2} \ln \det \Omega_{\mu\nu}^{ab} &= -\frac{i}{2} (N^2 - 1) \int \frac{d^d k}{(2\pi)^d} \left[(d-1) \ln \left(-k^2 + \frac{\sigma}{g \xi} \right) + \ln \left(-\frac{k^2}{\alpha} + \frac{\sigma}{g \xi} \right) \right] \\ &= \frac{3(N^2 - 1)}{64\pi^2} \left(\frac{g^2 \sigma^2}{\xi_0^2} \right) \left(\ln \frac{g \sigma}{\xi_0 \bar{\mu}^2} - \frac{5}{6} - \frac{2}{\varepsilon} \right) \\ &+ \frac{(N^2 - 1)}{64\pi^2} \left(\frac{g^2 \alpha^2 \sigma^2}{\xi_0^2} \right) \left(\ln \frac{g \alpha \sigma}{\xi_0 \bar{\mu}^2} - \frac{3}{2} - \frac{2}{\varepsilon} \right) \end{aligned} \quad (8.64)$$

Subsequently, we obtain for the one-loop effective potential in the $\overline{\text{MS}}$ scheme⁵

$$\begin{aligned} V_1(\sigma) &= \frac{\sigma^2}{2\xi_0} \left(1 - \frac{\xi_1}{\xi_0} g^2 \right) + \frac{3(N^2 - 1)}{64\pi^2} \left(\frac{g^2 \sigma^2}{\xi_0^2} \right) \left(\ln \frac{g \sigma}{\xi_0 \bar{\mu}^2} - \frac{5}{6} \right) \\ &- \frac{(N^2 - 1)}{64\pi^2} \left(\frac{g^2 \sigma^2 \alpha^2}{\xi_0^2} \right) \left(\ln \frac{g \alpha \sigma}{\xi_0 \bar{\mu}^2} - \frac{3}{2} \right) \end{aligned} \quad (8.65)$$

with ξ_0 given by (8.29). In principle, as soon one knows the value of ξ_1 , one can set $\bar{\mu}^2 = \frac{\sigma}{\sqrt{\xi_0}}$ and use the renormalization group equation for $V(\sigma)$ to sum leading logarithms and solve the gap equation. This leads to a value for the vacuum energy E , gluon mass m_{gluon} , and through the trace anomaly, one also finds an estimation for $\langle \frac{\alpha_s}{\pi} F^2 \rangle = -\frac{32}{11} E$. Since the aim of this paper is merely to describe the mass generation *mechanism* in the CF gauge, we do not perform the two-loop calculation leading to ξ_1 and corresponding numerical values. Moreover, since the vacuum energy is gauge parameter independent, we may choose a specific α . Therefore, we restrict ourselves to the case $\alpha = 0$, for which ξ_1 has already been determined [42].

The Landau gauge is by far the most interesting choice. It is a fixed point of the renormalization group for the gauge parameter at any order. Due to the transversality condition $\partial_\mu A^\mu = 0$, it is a quite physical gauge. It has some interesting non-renormalization properties [59]. Even more interesting is the already mentioned fact that $\mathcal{O}$ reduces to $\frac{A^2}{2}$, which has a gauge-invariant meaning in the Landau gauge, since it equals $(VT)^{-1} \min_U \int d^4 x (A^2)^U$, a gauge-invariant (however in general non-local) operator⁶. The relevance of the Landau gauge has also been pointed out from a more topological point of view [34]. In case of compact three-dimensional QED, A^2 was shown to be an order parameter for the monopole condensation [33, 34]. If monopole condensation has something to do with confinement,

⁵It is easily checked that using the renormalized version of the Hubbard-Stratonovich transformation (8.32), the counterterm proportional to $\delta\xi$ removes the infinities coming from (8.63) and (8.64).

⁶Although this correspondence is somewhat troubled by Gribov copies [143].

there might exist a relation between A^2 and confinement in case of QCD too. All these things are less clear in the case of the $\mathcal{O}$ operator in the CF gauge.

Having said all this, it might look like that our efforts are not that important for $\alpha \neq 0$. This is however not the case. We have given a consistent framework to *calculate* the dynamically generated gluon mass for the CF gauge. Notice that the obtained Lagrangian in the condensed vacuum is however not the one of the Curci-Ferrari model [84, 85]. The question, also posed in [42], is if the dynamically massive YM action (8.34) breaks unitarity? From a pragmatic point of view, a possible lack of unitarity in the gluon sector should not be considered very problematic. After all, since gluons are not observables due to confinement, massive gluons are a fortiori unphysical. In fact, a connection might exist between dynamically massive gluons and confinement, as it was explored in [177]. See [147] for an attempt to construct a string theory incorporating a $\langle A^2 \rangle$ condensate.

We notice that the action (8.1) can be rewritten as

$$S = S_{YM} + s\bar{s} \int d^4x \left(\frac{1}{2} A_\mu^a A^{\mu a} - \frac{\alpha}{2} c^a \bar{c}^a \right) \quad (8.66)$$

with⁷

$$\begin{aligned} \bar{s} A_\mu^a &= -D_\mu^{ab} \bar{c}^b \\ \bar{s} \bar{c}^a &= \frac{g}{2} f^{abc} \bar{c}^b c^c \\ \bar{s} c^a &= -b^a + g f^{abc} c^b \bar{c}^c \\ \bar{s} b^a &= -g f^{abc} b^b \bar{c}^c \end{aligned} \quad (8.67)$$

Another very interesting renormalizable gauge is the modified maximal Abelian gauge (MAG) [97], particularly useful in the context of the dual superconductivity mechanism for confinement. This gauge partially fixes the local $SU(N)$ freedom, i.e. up to the Abelian degrees of freedom. The MAG shares a close similarity with the CF gauge, since its gauge fixing is given by

$$S = S_{YM} + s\bar{s} \int d^4x \left(\frac{1}{2} A_\mu^{a'} A^{\mu a'} - \frac{\alpha}{2} c^{a'} \bar{c}^{a'} \right) \quad (8.68)$$

where the accent means that the color index runs strictly over the non-Abelian degrees of freedom. In particular, in [171] it has been shown that the remaining Abelian degrees of freedom can be fixed so that the resulting theory displays a global $SL(2, \mathbb{R})$ symmetry, in complete analogy with the CF gauge. Furthermore, due to the similarity (8.66)-(8.68), it is not difficult to understand that a quite analogous treatment with a source J coupled to the $U(1)^{N-1}$ invariant operator

$$\mathcal{O}' = \frac{1}{2} A_\mu^{a'} A^{\mu a'} + \alpha \bar{c}^{a'} c^{a'} \quad (8.69)$$

will provide us with a dynamical mass for the off-diagonal gluons and ghosts [83, 171, 157, 179], a hint for some kind of Abelian dominance [80]. This strategy for the MAG was already put forward in [83]. Just as the operator $\mathcal{O}$ is multiplicatively renormalizable in the CF gauge, the operator $\mathcal{O}'$ will be multiplicatively renormalizable in the MAG [179]. So far for the similarities between CF and MAG. Although it would be nice to stretch the similarity further and simply put $\alpha = 0$ from the beginning, in which case the MAG reads in differential form $D_\mu^{a'b'} A^{\mu b'} = 0$ with $D_\mu^{a'b'}$ the $U(1)^{N-1}$ Abelian covariant derivative. As such, we have some kind of $U(1)^{N-1}$ invariant version of the Landau gauge. Unfortunately, the limit $\alpha \rightarrow 0$ is now far from being trivial [77]. Moreover, $\alpha = 0$ is not a fixed point

⁷We disregard S_{LCO} here.

of the renormalization group [77, 183]. Also, although for $\alpha = 0$ the tree level action (8.68) does not contain a 4-ghost interaction, radiative corrections will reintroduce this interaction [97], unlike the Landau gauge. Making a long story short, we are forced to let the gauge parameter α free and perform a similar analysis as done in the previous sections. At the end of such a more general analysis, one could investigate if the limit $\alpha \rightarrow 0$ can be taken.

Before we formulate our conclusion, we quote the results obtained for the Landau gauge in [42]

$$\begin{aligned}
 \xi_1 &= \frac{161}{52} \frac{N^2 - 1}{16\pi^2} \\
 \left. \frac{g^2 N}{16\pi^2} \right|_{1\text{-loop}} &= \frac{36}{187} \\
 m_{\text{gluon}} &\approx 485 \text{ MeV for } N = 3 \\
 E &\approx -0.001 \text{ GeV}^4 \text{ for } N = 3 \\
 \left\langle \frac{\alpha_s}{\pi} F^2 \right\rangle &\approx 0.003 \text{ GeV}^4 \text{ for } N = 3
 \end{aligned} \tag{8.70}$$

As the relevant expansion parameter, i.e. $g^2 N / 16\pi^2$, is relatively small and results do not change much if the second loop correction to $V(\sigma)$ is included [42], qualitatively acceptable results are achieved. The value for the one-loop dynamical gluon mass m_{gluon} is also in qualitative agreement with lattice values [48, 47], reporting something like $m_{\text{gluon}} \sim 600 \text{ MeV}$.

8.7 Conclusion

In this paper, we have constructed a renormalizable effective potential for the on-shell BRST invariant local composite operator of mass dimension 2 in the Curci-Ferrari gauge, namely $\mathcal{O} = \frac{1}{2} A_\mu^a A^{\mu a} + \alpha \bar{c}^a c^a$. This gauge reduces to the Landau gauge in the limit $\alpha = 0$. It is worth underlining that, in the Landau gauge, the operator $\mathcal{O}$ equals the gauge invariant operator A^2 . Much attention has been paid recently to the condensate $\langle A^2 \rangle$. The generalization to $\alpha \neq 0$ has also its importance due to the close analogy with the maximal Abelian gauge, where the $\alpha \rightarrow 0$ limit is not as obvious as in case of the CF gauge. In particular, we have shown that the vacuum energy obtained in the presented formalism for the CF gauge is independent from the gauge parameter α . As already underlined the α -independence has to be understood in a g^2 expansion and up to terms of higher order.

We restricted ourselves in this paper to the on-shell BRST invariant condensate resulting in a mass for the particles. A gluon mass modifies the behaviour of the gluon propagator in the infrared (see e.g. [48]). A more intensive study would also include the pure ghost condensates, also of mass dimension 2, discussed in [77, 80, 81, 157, 90, 91, 171, 172]. These are not directly related to the mass generation for the gluons [157, 171, 172], but are relevant for the $SL(2, \mathbb{R})$ symmetry and can modify the ghost propagator.

Chapter 9

More on ghost condensation in Yang-Mills theory: BCS versus Overhauser effect and the breakdown of the Nakanishi-Ojima annex $SL(2, \mathbb{R})$ symmetry

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We analyze the ghost condensates $\langle f^{abc} c^b c^c \rangle$, $\langle f^{abc} \bar{c}^b \bar{c}^c \rangle$ and $\langle f^{abc} \bar{c}^b c^c \rangle$ in Yang-Mills theory in the Curci-Ferrari gauge. By combining the local composite operator formalism with the algebraic renormalization technique, we are able to give a simultaneous discussion of $\langle f^{abc} c^b c^c \rangle$, $\langle f^{abc} \bar{c}^b \bar{c}^c \rangle$ and $\langle f^{abc} \bar{c}^b c^c \rangle$, which can be seen as playing the role of the BCS, respectively Overhauser effect in ordinary superconductivity. The Curci-Ferrari gauge exhibits a global continuous symmetry generated by the Nakanishi-Ojima (NO) algebra. This algebra includes, next to the (anti-)BRST transformation, a $SL(2, R)$ subalgebra. We discuss the dynamical symmetry breaking of the NO algebra through these ghost condensates. Particular attention is paid to the Landau gauge, a special case of the Curci-Ferrari gauge.

9.1 Introduction.

Vacuum condensates play an important role in quantum field theory. They can be used to parametrize some non-perturbative effects. If one wants to attach a physical meaning to a certain condensate in case of a gauge theory, it should evidently be gauge invariant. Two well known examples in the context of QCD are the gluon condensate $\langle F_{\mu\nu}^2 \rangle$ and the quark condensate $\langle \bar{q}q \rangle$.

Recently, there was a growing interest for a mass dimension 2 condensate in (quarkless) QCD in the Landau gauge, see e.g. [42, 33, 34, 39, 37, 38]. Unfortunately, no local gauge invariant operator with mass dimension 2 exists. However, a non-local gauge invariant dimension 2 operator can be constructed by minimizing A^2 along each gauge orbit, namely $A_{\min}^2 \equiv (VT)^{-1} \min_U \int d^4x (A_\mu^U)^2$ with VT the

space time volume and U a generic $SU(N)$ transformation. This operator is related to the Gribov region as well as the so-called fundamental modular region (FMR), which is the set of absolute minima of $\int d^4x (A_\mu^U)^2$ [110, 187, 143]. In particular, in the Landau gauge $\partial_\mu A^\mu = 0$, it turned out that $A_{\min}^2$ reduces to the local operator A^2 . This gives a meaning to the condensate $\langle A^2 \rangle$. In [42], an effective action was constructed in the weak coupling for the $\langle A^2 \rangle$ condensate by means of the local composite operator technique (LCO) and it was shown that $\langle A^2 \rangle \neq 0$ is dynamically favoured since it lowers the vacuum energy. Due to this condensate, the gluons achieved a dynamical mass parameter.

In this article, we will discuss other condensates of mass dimension 2 [188], namely pure ghost condensates of the type $\langle f^{abc} c^b c^c \rangle$, $\langle f^{abc} \bar{c}^b \bar{c}^c \rangle$ and $\langle f^{abc} \bar{c}^b c^c \rangle$. Historically, these condensates came to attention in [77, 78, 79, 80] in the context of $SU(2)$ Yang-Mills theory in the maximal Abelian gauge. This is a partial non-linear gauge fixing which requires the introduction of a four ghost interaction term for consistency. A decomposition, by means of a Hubbard-Stratonovich auxiliary field, similar to the one of the 4-fermion interaction of the Gross-Neveu model [21], allowed to construct a one-loop effective potential, leading to a non-trivial minimum for the ghost condensate corresponding to $\langle f^{abc} \bar{c}^b c^c \rangle$. It was recognized in [77, 78, 79] that this condensate signals the breakdown of a global $SL(2, \mathbb{R})$ symmetry of the $SU(2)$ maximal Abelian gauge model. The ghost condensate was used to find a mass for the off-diagonal gluons, and thereby a certain evidence for the Abelian dominance was established [80]. It has been shown since then that the ghost condensate gives in fact a tachyonic mass [157].

It is worth mentioning that a simple decomposition of the 4-fermion interaction might cause troubles with the renormalizability beyond the one-loop order. For instance, in the case of the Gross-Neveu model, this procedure requires the introduction of ad hoc counterterms to maintain finiteness [24, 22]. A similar problem can be expected with the 4-ghost interaction. The LCO procedure gave an outcome to this problem [24].

Another issue that deserves clarification is the fact that with a different decomposition, different ghost condensates appear [81], corresponding to the Faddeev-Popov charged condensates $\langle f^{abc} c^b c^c \rangle$ and $\langle f^{abc} \bar{c}^b \bar{c}^c \rangle$. The existence of several channels for the ghost condensation has a nice analogy in the theory of superconductivity, known as the BCS versus Overhauser effect. The BCS channel corresponds to the charged particle-particle and hole-hole pairing [93, 94], while the Overhauser channel to the particle-hole pairing [92, 95]. In the present case, the Faddeev-Popov charged condensates $\langle f^{abc} \bar{c}^b c^c \rangle$ and $\langle f^{abc} c^b c^c \rangle$ would correspond to the BCS channel, while $\langle f^{abc} \bar{c}^b \bar{c}^c \rangle$ to the Overhauser channel. The question is whether one of these effects would be favoured. A simultaneous discussion of both effects is necessary to find out if one vacuum is more stable than the other.

It is appealing that by now the ghost condensates have been observed also in a class of non-linear generalized covariant gauges [160, 161], the so-called Curci-Ferrari gauges¹, again by the decomposition of a 4-ghost interaction [90]. The Curci-Ferrari gauge has the Landau gauge as a special case. Although the Landau gauge lacks a 4-ghost interaction, it has been shown that the ghost condensation also takes place in this gauge [91]. Evidently, this was not possible by the decomposition of a quartic interaction. However, the combination of the LCO method [42, 176] with the algebraic renormalization formalism [59, 60] allowed for a clean treatment of the ghost condensation in the Landau gauge.

It seems thus that the ghost condensation takes place in a variety of gauges: the Landau gauge, the Curci-Ferrari gauge and the maximal Abelian gauge. It is known that the Landau gauge and Curci-Ferrari gauge exhibit a global continuous symmetry, generated by the so-called Nakanishi-Ojima algebra [86, 163, 164, 158, 159, 189]. This algebra contains, next to the BRST and anti-BRST transformations, a $SL(2, \mathbb{R})$ subalgebra generated by the Faddeev-Popov ghost number and 2 other operators, δ and $\bar{\delta}$. Moreover, δ and $\bar{\delta}$ mutually transform the ghost operators $f^{abc} c^b c^c$, $f^{abc} \bar{c}^b \bar{c}^c$ and $f^{abc} \bar{c}^b c^c$ into each other. It is then apparent that the ghost condensation can appear in several channels like the BCS

¹Referring to the massive Curci-Ferrari model that has the same gauge fixing terms [84, 85].

and Overhauser channel, and that a non-vanishing vacuum expectation value for the ghost operators indicates a breakdown of this $SL(2, \mathbb{R})$ symmetry.

Recently, it has been shown that the same² NO invariance of the Landau and Curci-Ferrari gauge can be maintained in the maximal Abelian gauge for any value of N [171]. Apparently, an intimate connection exists between the NO symmetry and the appearance of the ghost condensates, since all gauges where the ghost condensates has been proven to occur, have the global NO invariance.

The aim of this article is to provide an answer to the aforementioned issues. We will discuss the Curci-Ferrari gauge. For explicit calculations, we will restrict ourselves to the Landau gauge for $SU(2)$. The presented general arguments are however neither depending on the choice of the gauge parameter, nor on the value of N . The paper is organized as follows. In section 9.2, we show that it is possible to introduce a set of external sources for the ghost operators, according to the LCO method, and this without spoiling the NO invariance. Employing the algebraic renormalization technique [59, 60], it can then be checked that the proposed action can be renormalized. In section 9.3, the effective potential for the ghost condensates is evaluated. By construction, this effective potential, incorporating the BCS as well as the Overhauser channel, is finite up to *any order* and obeys a homogeneous renormalization group equation. Next, in section 9.4, we pay attention to the dynamical symmetry breaking of the NO algebra due to the ghost condensates. Because of the $SL(2, \mathbb{R})$ invariance of the presented framework, it becomes clear that a whole class of equivalent, non-trivial vacua exist. The Overhauser and the BCS vacuum are important special cases. Notice that a nonvanishing condensate $\langle f^{abc} c^b c^c \rangle \neq 0$ could seem to pose a problem for the Faddeev-Popov ghost number symmetry and for the BRST symmetry, two basic properties of a quantized gauge theory. However, we shall be able to show that one can define a nilpotent BRST and a Faddeev-Popov symmetry in any possible ghost condensed vacuum. The existence of the NO symmetry plays a key role in this. Since the ghost condensates carry a color index, we also spend some words on the global $SU(N)$ color symmetry. Here, we can provide an argument that, thanks to the existence of the condensate $\langle A^2 \rangle$ and of its generalization $\langle \frac{1}{2} A^2 + \alpha \bar{c} c \rangle$ in the Curci-Ferrari gauge [178], the breaking of the color symmetry, induced by the ghost condensates, should be located in the unphysical part of the Hilbert space. Furthermore, we argue why no physical Goldstone particles should appear by means of the quartet mechanism [177]. Section 9.5 handles the generalization of the results to the case with quarks included. In section 9.6, we give an outline of some consequences of the gluon and ghost condensates. We end with conclusions in section 9.7. Technical details are collected in the sections 9.8 and 9.9.

9.2 The set of external sources for both BCS and Overhauser channel.

9.2.1 Introduction of the LCO sources.

For a thorough introduction to the local composite operator (LCO) formalism and to the algebraic renormalization technique, the reader is referred to [42, 176], respectively [59].

According to the LCO method, the first step in the analysis of the ghost condensation in both channels is the introduction of a suitable system of external sources. Generalizing the construction done in the pure BCS case [91], it turns out that the simultaneous presence of both channels is achieved by

²The $SL(2, \mathbb{R})$ symmetry discussed in [77, 78, 79, 164] is only acting non-trivially on the off-diagonal fields.

considering the following BRST invariant external action

$$\begin{aligned}
S_{LCO} &= s \int d^4x \left(L^a c^a + \lambda^a (b^a - g f^{abc} \bar{c}^b c^c) + \zeta \eta^a L^a - \frac{1}{2} \eta^a g f^{abc} \bar{c}^b \bar{c}^c + \frac{1}{2} \rho \lambda^a \omega^a - \omega^a \bar{c}^a \right) \\
&= \int d^4x \left(\frac{1}{2} L^a g f^{abc} c^b c^c - \frac{1}{2} \tau^a g f^{abc} \bar{c}^b \bar{c}^c + \eta^a g f^{abc} b^b \bar{c}^c + \zeta \tau^a L^a \right. \\
&\quad \left. - \omega^a g f^{abc} \bar{c}^b c^c + \lambda^a g f^{abc} b^b c^c - \frac{1}{2} \lambda^a g^2 f^{abc} \bar{c}^b f^{cmn} c^m c^n + \frac{1}{2} \rho \omega^a \omega^a \right) \quad (9.1)
\end{aligned}$$

The BRST transformation s is defined for the fields A_μ^a , c^a , $\bar{c}^a$, b^a as

$$\begin{aligned}
s A_\mu^a &= -D_\mu^{ab} c^b \\
s c^a &= \frac{g}{2} f^{abc} c^b c^c \\
s \bar{c}^a &= b^a \\
s b^a &= 0 \quad (9.2)
\end{aligned}$$

with

$$D_\mu^{ab} = \partial_\mu \delta^{ab} + g f^{acb} A_\mu^c \quad (9.3)$$

the adjoint covariant derivative.

The external sources L^a , τ^a , λ^a , ω^a , η^a transform as

$$\begin{aligned}
s \eta^a &= \tau^a, \quad s \tau^a = 0, \\
s \lambda^a &= \omega^a, \quad s \omega^a = 0, \\
s L^a &= 0 \quad (9.4)
\end{aligned}$$

From expression (9.1) one sees that the sources L^a , τ^a couple to the ghost operators $g f^{abc} c^b c^c$,

	L^a	η^a	τ^a	λ^a	ω^a
Dimension	2	1	2	1	2
Gh. Number	-2	1	2	-1	0

Table 9.1:

$g f^{abc} \bar{c}^b \bar{c}^c$ of the BCS channel, while ω^a accounts for the Overhauser channel $g f^{abc} \bar{c}^b c^c$. As far as the BRST invariance is the only invariance required for the external action (9.1), the LCO parameters ζ and ρ are independent. However, it is known that both the Landau and the Curci-Ferrari gauge display a larger set of symmetries, giving rise to the NO algebra [160, 161, 86, 163, 164, 158, 159, 189, 171]. It is worth remarking that the whole NO algebra can be extended also in the presence of the external action S_{LCO} , provided that the two parameters ζ and ρ obey the relationship

$$\rho = 2\zeta \quad (9.5)$$

In other words, the requirement of invariance of S_{LCO} under the whole NO algebra allows for a unique parameter in expression (9.1). In order to introduce the generators of the NO algebra, let us begin

with the anti-BRST transformation $\bar{s}$

$$\begin{aligned}
\bar{s}A_\mu^a &= -D_\mu^{ab}\bar{c}^b \\
\bar{s}c^a &= -b^a + g f^{abc}c^b\bar{c}^c \\
\bar{s}\bar{c}^a &= \frac{g}{2}f^{abc}\bar{c}^b\bar{c}^c \\
\bar{s}b^a &= -g f^{abc}b^b\bar{c}^c
\end{aligned} \tag{9.6}$$

Extending $\bar{s}$ to the external LCO sources as

$$\begin{aligned}
\bar{s}\eta^a &= -\omega^a, \quad \bar{s}\tau^a = 0 \\
\bar{s}\lambda^a &= L^a, \quad \bar{s}\omega^a = 0 \\
\bar{s}L^a &= 0
\end{aligned} \tag{9.7}$$

one easily verifies that

$$\{s, \bar{s}\} = s\bar{s} + \bar{s}s = 0 \tag{9.8}$$

Furthermore, the requirement of invariance of S_{LCO} under $\bar{s}$ fixes the parameter $\rho = 2\zeta$, namely

$$\bar{s}S_{LCO} = 0 \Rightarrow \rho = 2\zeta \tag{9.9}$$

This is best seen by observing that, when $\rho = 2\zeta$, the whole action S_{LCO} can be written as

$$S_{LCO} = s\bar{s} \int d^4x (\lambda^a c^a + \zeta \lambda^a \eta^a + \eta^a \bar{c}^a) \tag{9.10}$$

Concerning now the other generators δ and $\bar{\delta}$ of the NO algebra, they can be introduced as follows

$$\begin{aligned}
\delta\bar{c}^a &= c^a \\
\delta b^a &= \frac{g}{2}f^{abc}c^b c^c \\
\delta A_\mu^a &= \delta c^a = 0 \\
\delta L^a &= 2\omega^a \\
\delta\omega^a &= -\tau^a \\
\delta\lambda^a &= -\eta^a \\
\delta\tau^a &= \delta\eta^a = 0
\end{aligned} \tag{9.11}$$

and

$$\begin{aligned}
\bar{\delta}c^a &= \bar{c}^a \\
\bar{\delta}b^a &= \frac{g}{2}f^{abc}\bar{c}^b\bar{c}^c \\
\bar{\delta}A_\mu^a &= \bar{\delta}\bar{c}^a = 0 \\
\bar{\delta}\omega^a &= L^a \\
\bar{\delta}\tau^a &= -2\omega^a \\
\bar{\delta}\eta^a &= -\lambda^a \\
\bar{\delta}L^a &= \bar{\delta}\lambda^a = 0
\end{aligned} \tag{9.12}$$

It holds that

$$\delta S_{LCO} = \bar{\delta} S_{LCO} = 0 \quad (9.13)$$

The operators $s, \bar{s}, \delta, \bar{\delta}$ and the Faddeev-Popov ghost number operator δ_{FP} give rise to the NO algebra

$$\begin{aligned} s^2 &= 0, & \bar{s}^2 &= 0 \\ \{s, \bar{s}\} &= 0, & [\delta, \bar{\delta}] &= \delta_{FP}, \\ [\delta, \delta_{FP}] &= -2\delta, & [\bar{\delta}, \delta_{FP}] &= 2\bar{\delta} \\ [s, \delta_{FP}] &= -s, & [\bar{s}, \delta_{FP}] &= \bar{s}, \\ [s, \delta] &= 0, & [\bar{s}, \bar{\delta}] &= 0, \\ [s, \bar{\delta}] &= -\bar{s}, & [\bar{s}, \delta] &= -s \end{aligned} \quad (9.14)$$

In particular, $\delta_{FP}, \delta, \bar{\delta}$ generate a $SL(2, \mathbb{R})$ subalgebra. We remark that the NO algebra can be established as an exact invariance of S_{LCO} only when both channels are present. It is easy to verify indeed that setting to zero the external sources corresponding to one channel will imply the loss of the NO algebra. This implies that a complete discussion of the ghost condensates needs sources for the BCS as well as for the Overhauser channel.

Let us also give, for further use, the expressions of the gauge fixed action in the presence of the LCO external sources for the Curci-Ferrari gauge.

$$\begin{aligned} S &= S_{YM} + S_{GF+FP} + S_{LCO} \\ &= -\frac{1}{4} \int d^4x F_{\mu\nu}^a F^{a\mu\nu} + s\bar{s} \int d^4x \left(\frac{1}{2} A_\mu^a A^{a\mu} + \lambda^a c^a + \zeta \lambda^a \eta^a + \eta^a \bar{c}^a - \frac{\alpha}{2} c^a \bar{c}^a \right) \end{aligned} \quad (9.15)$$

with

$$S_{GF+FP} = \int d^4x \left(b^a \partial_\mu A^{a\mu} + \frac{\alpha}{2} b^a b^a + \bar{c}^a \partial^\mu D_\mu^{ab} c^b - \frac{\alpha}{2} g f^{abc} b^a \bar{c}^b c^c - \frac{\alpha}{8} g^2 f^{abc} f^{cde} \bar{c}^a \bar{c}^b c^d c^e \right) \quad (9.16)$$

The renormalizability of the action (9.15) is discussed in section 9.8.

The Curci-Ferrari gauge has the Landau gauge, $\alpha = 0$, as interesting special case, see for example [178]. One sees that the difference between the two actions is due to the term $\alpha c^a \bar{c}^a$, which gives rise to a quartic ghost self interaction absent in the Landau gauge. The whole set of NO invariances can be translated into functional identities which ensures the renormalizability of the model. In particular, concerning the counterterm contributions $\delta_L L^a g f^{abc} c^b c^c$, $\delta_\tau \tau^a g f^{abc} \bar{c}^b \bar{c}^c$ and $\delta_\omega \omega^a g f^{abc} \bar{c}^b c^c$, it is shown in the Appendix A (section 9.8) that

$$\delta_L = \delta_\tau = \delta_\omega \equiv \delta_2 \quad (9.17)$$

Consequently, the operators $g f^{abc} c^b c^c$, $g f^{abc} \bar{c}^b \bar{c}^c$ and $g f^{abc} \bar{c}^b c^c$ turn out to have the same anomalous dimension for any α . As expected, this result is a consequence of the presence of the NO symmetry. Moreover, in the Landau gauge, $\delta_2 \equiv 0$ due to the nonrenormalization properties of the Landau gauge [59]. In [153], one can find an explicit proof that $\delta_2 = 0$.

9.3 Effective potential for the ghost condensates.

9.3.1 General considerations.

Let us proceed with the construction of the effective potential for the ghost condensates in the Curci-Ferrari gauge. To decide which channel is favoured, we have to consider the 2 channels at once. We

shall also treat the two LCO parameters ρ and ζ for the moment as being independent and verify the relationship (9.1). Setting to zero the external sources η and λ , we start from the action

$$S = S_{YM} + S_{GF+FP} + \int d^4x \left[-\omega^a g f^{abc} \bar{c}^b c^c + \frac{1}{2} \rho \omega^a \omega^a + \frac{1}{2} L^a g f^{abc} c^b c^c - \frac{1}{2} \tau^a g f^{abc} \bar{c}^b \bar{c}^c + \zeta \tau^a L^a \right] \quad (9.18)$$

Following [42, 176], the divergences proportional to $L\tau$ are cancelled by the counterterm $\delta\zeta\tau L$, and the divergences proportional to ω^2 are cancelled by the counterterm $\frac{\delta\rho}{2}\omega^2$. Considering the bare Lagrangian associated to (9.18), we have

$$c_b = \sqrt{Z_c} c \quad \bar{c}_b = \sqrt{Z_c} \bar{c} \quad (9.19)$$

$$A_b = \sqrt{Z_A} A \quad (9.20)$$

$$g_b = \mu^{\varepsilon/2} Z_g g \quad (9.21)$$

$$L_b = \mu^{-\varepsilon/2} \frac{Z_2}{Z_g Z_c} L \quad \tau_b = \mu^{-\varepsilon/2} \frac{Z_2}{Z_g Z_c} \tau \quad \omega_b = \mu^{-\varepsilon/2} \frac{Z_2}{Z_g Z_c} \omega \quad (9.22)$$

where $Z_2 = 1 + \delta_2$ (see (9.17)).

Furthermore,

$$\zeta_b \tau_b^a L_b^a = \mu^{-\varepsilon} (\zeta + \delta\zeta) \tau^a L^a \quad (9.23)$$

$$\frac{1}{2} \rho_b \omega_b^a \omega_b^a = \frac{1}{2} \mu^{-\varepsilon} (\rho + \delta\rho) \omega^a \omega^a \quad (9.24)$$

where it is understood that we are working with dimensional regularization in $d = 4 - \varepsilon$ dimensions. The above equations allow to derive the renormalization group equation of ζ and ρ

$$\mu \frac{d\zeta}{d\mu} = 2\gamma(g^2)\zeta + \delta_\zeta(g^2) \quad (9.25)$$

$$\mu \frac{d\rho}{d\mu} = 2\gamma(g^2)\rho + \delta_\rho(g^2) \quad (9.26)$$

where $\gamma(g^2)$ denotes the anomalous dimension of the ghost operators $g f^{abc} \bar{c}^b c^c$, $g f^{abc} c^b c^c$ and $g f^{abc} \bar{c}^b \bar{c}^c$, given by

$$\gamma(g^2) = \mu \frac{d}{d\mu} \ln \frac{Z_2}{Z_g Z_c} \quad (9.27)$$

δ_ζ and δ_ρ are defined as

$$\delta_\zeta(g^2) = \left(\varepsilon - 2\hat{\gamma}(g^2) - \beta(g^2) \frac{\partial}{\partial g^2} - \alpha\gamma_\alpha(g^2) \frac{\partial}{\partial \alpha} \right) \delta\zeta \quad (9.28)$$

$$\delta_\rho(g^2) = \left(\varepsilon - 2\hat{\gamma}(g^2) - \beta(g^2) \frac{\partial}{\partial g^2} - \alpha\gamma_\alpha(g^2) \frac{\partial}{\partial \alpha} \right) \delta\rho \quad (9.29)$$

where $\beta(g^2) = \mu \frac{dg^2}{d\mu}$ is the usual running of the coupling constant, in d dimensions given by

$$\beta(g^2) = -\varepsilon g^2 - \frac{22}{3} g^2 \frac{g^2 N}{16\pi^2} - \frac{68}{3} g^2 \left(\frac{g^2 N}{16\pi^2} \right)^2 + \dots \quad (9.30)$$

while $\gamma_\alpha(g^2) = \frac{\mu}{\alpha} \frac{d\alpha}{d\mu}$ denotes the running of the gauge parameter α . We do not write the possible α dependence of the appearing renormalization group functions; for the explicit calculations in section 9.3.2, we will restrict ourselves to the Landau gauge. Therefore, we also do not write down the explicit value of $\gamma_\alpha(g^2)$ since $\alpha\gamma_\alpha(g^2) \equiv 0$ for $\alpha = 0$.

$\hat{\gamma}(g^2)$ denotes the anomalous dimension of the sources ω , τ and L . $\gamma(g^2)$ and $\hat{\gamma}(g^2)$ are related by

$$\hat{\gamma}(g^2) = \frac{\varepsilon}{2} - \gamma(g^2) \quad (9.31)$$

and therefore, the equations (9.28)-(9.29) can be rewritten as

$$\delta_\zeta(g^2) = \left(2\gamma(g^2) - \beta(g^2) \frac{\partial}{\partial g^2} - \alpha\gamma_\alpha(g^2) \frac{\partial}{\partial \alpha} \right) \delta\zeta \quad (9.32)$$

$$\delta_\rho(g^2) = \left(2\gamma(g^2) - \beta(g^2) \frac{\partial}{\partial g^2} - \alpha\gamma_\alpha(g^2) \frac{\partial}{\partial \alpha} \right) \delta\rho \quad (9.33)$$

Notice that in the equations (9.25)-(9.26), the parameter ε has immediately been set equal to zero, this is allowed because all considered quantities are finite for $\varepsilon \rightarrow 0$.

Since we have introduced 2 novel parameters³, we have a problem of uniqueness. However, this can be solved by noticing that ζ and ρ can be chosen to be a function of g^2 , such that if g^2 runs according to (9.30), $\zeta(g^2)$ and $\rho(g^2)$ will run according to (9.25), respectively (9.26). Explicitly, $\zeta(g^2)$ and $\rho(g^2)$ are the solution of the differential equations

$$\left(\beta(g^2) \frac{d}{dg^2} + \alpha\gamma_\alpha(g^2) \frac{d}{d\alpha} \right) \zeta(g^2) = 2\gamma(g^2)\zeta(g^2) + \delta_\zeta(g^2) \quad (9.34)$$

$$\left(\beta(g^2) \frac{d}{dg^2} + \alpha\gamma_\alpha(g^2) \frac{d}{d\alpha} \right) \rho(g^2) = 2\gamma(g^2)\rho(g^2) + \delta_\rho(g^2) \quad (9.35)$$

The integration constants of the solution of (9.34)-(9.35) can be put to zero; this eliminates independent parameters and assures multiplicative renormalizability

$$\zeta(g^2) + \delta\zeta(g^2, \varepsilon) = Z_\zeta(g^2, \varepsilon)\zeta(g^2) \quad (9.36)$$

$$\rho(g^2) + \delta\rho(g^2, \varepsilon) = Z_\rho(g^2, \varepsilon)\rho(g^2) \quad (9.37)$$

Notice that the n -loop knowledge of $\zeta(g^2)$ and $\rho(g^2)$ will need the $(n+1)$ -loop knowledge of $\beta(g^2)$, $\gamma(g^2)$, $\delta_\zeta(g^2)$ and $\delta_\rho(g^2)$ [178]. The generating functional $\mathcal{W}(\omega, \tau, L)$, defined as

$$e^{i\mathcal{W}(\omega, \tau, L)} = \int [D\Phi] e^{iS(\omega, \tau, L)} \quad (9.38)$$

with $S(\omega, \tau, L)$ given by (9.18) and Φ denoting the relevant fields, will now obey a homogeneous renormalization group equation [42, 176].

It is not difficult to see that $\delta_\zeta(g^2)$, $\delta_\rho(g^2)$ and $\zeta(g^2)$, $\rho(g^2)$ will be of the form

$$\delta_\zeta(g^2) = \delta_{\zeta,0}g^2 + \delta_{\zeta,1}g^4 + \dots \quad (9.39)$$

$$\delta_\rho(g^2) = \delta_{\rho,0}g^2 + \delta_{\rho,1}g^4 + \dots \quad (9.40)$$

$$\zeta(g^2) = \zeta_0 + \zeta_1g^2 + \dots \quad (9.41)$$

$$\rho(g^2) = \rho_0 + \rho_1g^2 + \dots \quad (9.42)$$

³In fact, only 1 novel parameter is introduced, since $\rho = 2\zeta$.

Taking the functional derivatives of $\mathcal{W}(\omega, \tau, L)$ with respect to the sources ω^a , τ^a and L^a , we obtain a finite vacuum expectation value for the composite operators, namely

$$\left. \frac{\delta \mathcal{W}(\omega, \tau, L)}{\delta \omega^a} \right|_{\omega=0, \tau=0, L=0} = -g \langle f^{abc} \bar{c}^b c^c \rangle \quad (9.43)$$

$$\left. \frac{\delta \mathcal{W}(\omega, \tau, L)}{\delta \tau^a} \right|_{\omega=0, \tau=0, L=0} = -\frac{g}{2} \langle f^{abc} \bar{c}^b \bar{c}^c \rangle \quad (9.44)$$

$$\left. \frac{\delta \mathcal{W}(\omega, \tau, L)}{\delta L^a} \right|_{\omega=0, \tau=0, L=0} = \frac{g}{2} \langle f^{abc} c^b c^c \rangle \quad (9.45)$$

Since the source terms appear quadratically, we seem to have lost an energy interpretation. However, this can be dealt with by introducing a pair of Hubbard-Stratonovich fields $(\sigma^a, \bar{\sigma}^a)$ for the τL term, and a Hubbard-Stratonovich field ϕ^a for the ω^2 term. For the functional generator $\mathcal{W}(\omega, \tau, L)$, we then get

$$e^{i\mathcal{W}(\omega, \tau, L)} = \int [d\Phi] e^{iS(\sigma, \bar{\sigma}, \phi) + i \int d^4x \left(\frac{\phi^a}{g} \omega^a + \frac{\sigma^a}{g} L^a + \frac{\bar{\sigma}^a}{g} \tau^a \right)} \quad (9.46)$$

where the action $S(\sigma, \bar{\sigma}, \phi)$ is given by

$$\begin{aligned} S(\sigma, \bar{\sigma}, \phi) = & S_{YM} + S_{GF+FP} + \int d^4x \left(-\frac{\sigma^a \bar{\sigma}^a}{g^2 \zeta} - \frac{\phi^a \phi^a}{2g^2 \rho} + \frac{\bar{\sigma}^a}{2g\zeta} g f^{abc} c^b c^c \right. \\ & \left. - \frac{\sigma^a}{2g\zeta} g f^{abc} \bar{c}^b \bar{c}^c - \frac{\phi^a}{g\rho} g f^{abc} \bar{c}^b c^c - \frac{1}{2\rho} g^2 (f^{abc} \bar{c}^b c^c)^2 + \frac{1}{4\zeta} g^2 f^{abc} c^b c^c f^{ade} \bar{c}^d \bar{c}^e \right) \end{aligned} \quad (9.47)$$

Notice also that in expression (9.46), the sources ω , τ , L are now linearly coupled to the fields ϕ , σ , $\bar{\sigma}$, allowing thus for the correct energy interpretation of the corresponding effective action. Taking the functional derivatives gives the relations

$$\langle \phi^a \rangle = -g^2 \langle f^{abc} \bar{c}^b c^c \rangle \quad (9.48)$$

$$\langle \sigma^a \rangle = \frac{g^2}{2} \langle f^{abc} c^b c^c \rangle \quad (9.49)$$

$$\langle \bar{\sigma}^a \rangle = -\frac{g^2}{2} \langle f^{abc} \bar{c}^b \bar{c}^c \rangle \quad (9.50)$$

where all vacuum expectation values are now calculated with the action (9.47).

Summarizing, we have constructed a new, multiplicatively renormalizable Yang-Mills action (9.47), incorporating the possible existence of ghost condensates. As such, if a non-trivial vacuum is favoured, we can perturb around a more stable vacuum than the trivial one. The action (9.47) is explicitly *NO* invariant⁴. The corresponding effective action $V(\sigma, \bar{\sigma}, \phi)$ obeys a homogeneous renormalization group equation.

To find out whether the groundstate effectively favours non-vanishing ghost condensates, we will calculate the one-loop effective potential. For the sake of simplicity, we will restrict ourselves to the case of $SU(2)$ Yang-Mills theories in the Landau gauge ($\alpha = 0$). In this context, we remark that one can prove that the vacuum energy will be gauge parameter independent, see the previous chapters. This proof is completely analogous to the one presented in [178], and is based on the fact that the derivative with respect to α of the action (9.1) is a BRST exact form plus terms proportional to the sources, which equal zero in the minima of the effective potential. As such, the usual proof of gauge parameter independence can be used [59].

⁴The *NO* variations of the σ^a , $\bar{\sigma}^a$ and ϕ^a fields can be determined immediately from (9.48)-(9.50).

9.3.2 Calculation of the one-loop effective potential for $N = 2$ in the Landau gauge.

We will determine the effective potential [170] with the background field method [190]. Let us define the 6×6 matrix

$$\mathcal{M}^{ab} = \begin{pmatrix} -\frac{\sigma^c \epsilon^{cab}}{\zeta} & \partial^2 \delta^{ab} - \frac{\epsilon^{abc} \phi^c}{\rho} \\ -\partial^2 \delta^{ab} - \frac{\epsilon^{abc} \phi^c}{\rho} & \frac{\bar{\sigma}^c \epsilon^{cab}}{\zeta} \end{pmatrix} \quad (9.51)$$

where ϵ^{abc} are the structure constants of $SU(2)$. Then the effective potential up to one-loop is easily worked out, yielding⁵

$$V_1(\sigma, \bar{\sigma}, \phi) = \frac{\sigma^a \bar{\sigma}^a}{g^2 \zeta} + \frac{\phi^a \phi^a}{2g^2 \rho} + \frac{i}{2} \ln \det \mathcal{M}^{ab} \quad (9.52)$$

or

$$V_1(\sigma, \bar{\sigma}, \phi) = \frac{\sigma^a \bar{\sigma}^a}{g^2 \zeta} + \frac{\phi^a \phi^a}{2g^2 \rho} - \int \frac{d^d k}{(2\pi)^d} \ln \left(k^6 + k^2 \left(\frac{\sigma^a \bar{\sigma}^a}{\zeta^2} + \frac{\phi^a \phi^a}{\rho^2} \right) + \frac{\epsilon^{abc} \phi^a \sigma^b \bar{\sigma}^c}{\rho \zeta^2} \right) \quad (9.53)$$

with k Euclidean.

We notice that the mass dimension 6 operator $\epsilon^{abc} \phi^a \sigma^b \bar{\sigma}^c$ enters the expression for the effective potential. We shall however show that this operator plays no role in the determination of the minimum, which is a solution of

$$\begin{cases} \frac{\partial V}{\partial \sigma^a} = \frac{\sigma^a}{g^2 \zeta} - \int \frac{d^d k}{(2\pi)^d} \frac{k^2 \frac{\sigma^a}{\zeta^2} + \frac{\epsilon^{abc} \phi^b \sigma^c}{\rho \zeta^2}}{k^6 + k^2 \left(\frac{\sigma^a \bar{\sigma}^a}{\zeta^2} + \frac{\phi^a \phi^a}{\rho^2} + \frac{\epsilon^{abc} \phi^a \sigma^b \bar{\sigma}^c}{\rho \zeta^2} \right)} = 0 \\ \frac{\partial V}{\partial \bar{\sigma}^a} = \frac{\bar{\sigma}^a}{g^2 \zeta} - \int \frac{d^d k}{(2\pi)^d} \frac{k^2 \frac{\bar{\sigma}^a}{\zeta^2} + \frac{\epsilon^{abc} \phi^c \bar{\sigma}^b}{\rho \zeta^2}}{k^6 + k^2 \left(\frac{\sigma^a \bar{\sigma}^a}{\zeta^2} + \frac{\phi^a \phi^a}{\rho^2} + \frac{\epsilon^{abc} \phi^a \sigma^b \bar{\sigma}^c}{\rho \zeta^2} \right)} = 0 \\ \frac{\partial V}{\partial \phi^a} = \frac{\phi^a}{g^2 \rho} - \int \frac{d^d k}{(2\pi)^d} \frac{2k^2 \frac{\phi^a}{\rho^2} + \frac{\epsilon^{abc} \sigma^b \bar{\sigma}^c}{\rho \zeta^2}}{k^6 + k^2 \left(\frac{\sigma^a \bar{\sigma}^a}{\zeta^2} + \frac{\phi^a \phi^a}{\rho^2} + \frac{\epsilon^{abc} \phi^a \sigma^b \bar{\sigma}^c}{\rho \zeta^2} \right)} = 0 \end{cases} \quad (9.54)$$

Let us assume that $(\phi_*^a, \sigma_*^a, \bar{\sigma}_*^a)$ is a solution of (9.54). Obviously, $\phi_*^a = 0, \sigma_*^a = 0, \bar{\sigma}_*^a = 0$ is a solution, corresponding with the trivial vacuum energy $E = 0$.

Let us now assume that at least one of the field configurations is non-zero. If it occurs that $\sigma_*^a = \bar{\sigma}_*^a = (0, 0, 0)$, then necessarily $\phi_*^a \neq (0, 0, 0)$ and it can be immediately checked that the equations (9.54) are reduced to

$$\frac{1}{g^2 \zeta} - \frac{1}{\zeta^2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^4 + \left(\frac{\sigma_*^a \bar{\sigma}_*^a}{\zeta^2} + \frac{\phi_*^a \phi_*^a}{\rho^2} \right)} = 0 \quad (9.55)$$

Next, we consider the case that $\sigma_*^a \neq (0, 0, 0)$ and/or $\bar{\sigma}_*^a \neq (0, 0, 0)$. Without loss of generality, we can consider $\sigma_*^a \neq (0, 0, 0)$. Consider then the first equation of (9.54).

$$\frac{\sigma_*^a}{g^2 \zeta} - \int \frac{d^d k}{(2\pi)^d} \frac{k^2 \frac{\sigma_*^a}{\zeta^2} + \frac{\epsilon^{abc} \phi_*^b \sigma_*^c}{\rho \zeta^2}}{k^6 + k^2 \left(\frac{\sigma_*^a \bar{\sigma}_*^a}{\zeta^2} + \frac{\phi_*^a \phi_*^a}{\rho^2} \right) + \frac{\epsilon^{abc} \phi_*^a \sigma_*^b \bar{\sigma}_*^c}{\rho \zeta^2}} = 0 \quad (9.56)$$

⁵We do not write the counterterms explicitly.

By contracting the above equation with σ_*^a , we find

$$\frac{\sigma_*^a \sigma_*^a}{g^2 \zeta} - \int \frac{d^d k}{(2\pi)^d} \frac{k^2 \frac{\sigma_*^a \sigma_*^a}{\zeta^2}}{k^6 + k^2 \left(\frac{\sigma_*^a \bar{\sigma}_*^a}{\zeta^2} + \frac{\phi_*^a \phi_*^a}{\rho^2} \right) + \frac{\epsilon^{abc} \phi_*^a \sigma_*^b \bar{\sigma}_*^c}{\rho \zeta^2}} = 0 \quad (9.57)$$

or, since $\sigma_*^a \sigma_*^a \neq 0$

$$\frac{1}{g^2 \zeta} - \int \frac{d^d k}{(2\pi)^d} \frac{\frac{k^2}{\zeta^2}}{k^6 + k^2 \left(\frac{\sigma_*^a \bar{\sigma}_*^a}{\zeta^2} + \frac{\phi_*^a \phi_*^a}{\rho^2} \right) + \frac{\epsilon^{abc} \phi_*^a \sigma_*^b \bar{\sigma}_*^c}{\rho \zeta^2}} = 0 \quad (9.58)$$

Inserting (9.58) into (9.56), one learns that

$$\frac{\epsilon^{abc} \phi_*^b \sigma_*^c}{\rho \zeta^2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^6 + k^2 \left(\frac{\sigma_*^a \bar{\sigma}_*^a}{\zeta^2} + \frac{\phi_*^a \phi_*^a}{\rho^2} \right) + \frac{\epsilon^{abc} \phi_*^a \sigma_*^b \bar{\sigma}_*^c}{\rho \zeta^2}} = 0 \quad (9.59)$$

Notice that the integral in (9.59) is UV finite. If the integral of (9.59) is non-vanishing, we must have that

$$\epsilon^{abc} \phi_*^b \sigma_*^c = 0 \quad (9.60)$$

Evidently, we then also have that

$$\epsilon^{abc} \phi_*^a \sigma_*^b \bar{\sigma}_*^c = 0 \quad (9.61)$$

Expression (9.58) can then also be combined with the second and third equation of (9.54) to show that

$$\epsilon^{abc} \phi_*^b \bar{\sigma}_*^c = 0 \quad (9.62)$$

and

$$\epsilon^{abc} \sigma_*^b \bar{\sigma}_*^c = 0 \quad (9.63)$$

Henceforth, we conclude that all contributions coming from the dimension 6 operator $\epsilon^{abc} \phi^b \sigma^c \bar{\sigma}^a$ are in fact not relevant for the determination of the minimum configuration $(\phi_*^a, \sigma_*^a, \bar{\sigma}_*^a)$. It is sufficient to solve the following gap equation to search for the non-trivial minimum

$$\frac{1}{g^2 \zeta} - \frac{1}{\zeta^2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^4 + \left(\frac{\sigma^a \bar{\sigma}^a}{\zeta^2} + \frac{\phi^a \phi^a}{\rho^2} \right)} = 0 \quad (9.64)$$

In fact, this is the gap equation corresponding to the minimization of the potential (9.53) with $\epsilon^{abc} \phi^a \sigma^b \bar{\sigma}^c$ put equal to zero from the beginning, in which case the one-loop potential reduces to

$$\begin{aligned} V_1(\sigma, \bar{\sigma}, \phi)^{\epsilon^{abc} \phi^a \sigma^b \bar{\sigma}^c = 0} &= \frac{\sigma^a \bar{\sigma}^a}{g^2 \zeta_0} \left(1 - \frac{\zeta_1}{\zeta_0} g^2 \right) + \frac{\phi^a \phi^a}{2 \rho_0 g^2} \left(1 - \frac{\rho_1}{\rho_0} g^2 \right) \\ &+ \frac{1}{32 \pi^2} \left(\frac{\sigma^a \bar{\sigma}^a}{\zeta_0^2} + \frac{\phi^a \phi^a}{\rho_0^2} \right) \left(\ln \frac{\frac{\sigma^a \bar{\sigma}^a}{\zeta_0^2} + \frac{\phi^a \phi^a}{\rho_0^2}}{\bar{\mu}^4} - 3 \right) \end{aligned} \quad (9.65)$$

It remains to show that the integral of (9.59) is non-vanishing for a non-trivial vacuum configuration ($E_{\text{vac}} \neq 0$). We define

$$\begin{aligned} a &= \frac{\sigma_*^a \bar{\sigma}_*^a}{\zeta^2} + \frac{\phi_*^a \phi_*^a}{\rho^2} \\ b &= \frac{\epsilon^{abc} \phi_*^a \sigma_*^b \bar{\sigma}_*^c}{\rho \zeta^2} \end{aligned} \quad (9.66)$$

and consider the integral

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^6 + ak^2 + b} = \int \frac{d\Omega}{(2\pi)^4} \int_0^\infty \frac{k^3 dk}{k^6 + ak^2 + b} \quad (9.67)$$

For $a = 0$ and $b = 0$, (9.67) is vanishing, but then we also have that $E_{\text{vac}} = 0$.

Via the substitution $x = k^2$, one finds

$$\int \frac{k^3 dk}{k^6 + ak^2 + b} = \frac{1}{2} \int_0^\infty \frac{x dx}{x^3 + ax + b} \quad (9.68)$$

This integral is always positive for $a > 0$. For $b = 0$, this is immediately clear. For $b \neq 0$, we perform a partial integration to find

$$\frac{1}{2} \int_0^\infty \frac{x dx}{x^3 + ax + b} = \frac{x^2}{4(x^3 + ax + b)} \Big|_0^\infty + \frac{1}{4} \int_0^\infty \frac{(3x^2 + a)x^2}{(x^3 + ax + b)^2} dx = \frac{1}{4} \int_0^\infty \frac{(3x^2 + a)x^2}{(x^3 + ax + b)^2} dx \quad (9.69)$$

For $a > 0$, the integral (9.69) is also positive. Consider now the function $F(a, b)$, defined by

$$F(a, b) = \int_0^\infty \frac{x dx}{x^3 + ax + b} \quad (9.70)$$

We already know that, for $a > 0$ and fixed $b = b_*$, $F(a, b_*) > 0$. Furthermore

$$\frac{\partial F(a, b)}{\partial a} = - \int_0^\infty \frac{x^2 dx}{(x^3 + ax + b)^2} < 0 \quad (9.71)$$

meaning that the function $F(a, b_*)$ decreases for increasing a . Assuming that $F(a, b)$ has a zero at (a_0, b_0) , then we should have that $F(a, b_0)$ becomes more negative as a increases, which contradicts the fact that $F(a, b_0) > 0$ for $a > 0$. Therefore, the function $F(a, b)$ cannot become zero and the integral in (9.59) never vanishes for a non-trivial vacuum configuration.

It remains to calculate ζ_0 , ζ_1 , ρ_0 and ρ_1 . One finds (see section 9.9)

$$\delta\zeta = -\frac{g^2}{8\pi^2} \frac{1}{\varepsilon} + \frac{g^4}{(16\pi^2)^2} \left(\frac{1}{2\varepsilon} + \frac{6}{\varepsilon^2} \right) + \dots \quad (9.72)$$

$$\delta\rho = -\frac{g^2}{4\pi^2} \frac{1}{\varepsilon} + \frac{g^4}{(16\pi^2)^2} \left(\frac{1}{\varepsilon} + \frac{12}{\varepsilon^2} \right) + \dots \quad (9.73)$$

Since in the Landau gauge $Z_2 = 1$ and $Z_c = Z_g^{-1} Z_A^{-1/2}$ (see e.g. [153]), we have

$$\gamma(g^2) = \frac{1}{2} \mu \frac{d}{d\mu} \ln Z_A \equiv \gamma_A(g^2) \quad (9.74)$$

where $\gamma_A(g^2)$ is the anomalous dimension of the gluon field, given by [88, 87]

$$\gamma_A(g^2) = -\frac{13}{6} \frac{g^2 N}{16\pi^2} - \frac{59}{8} \left(\frac{g^2 N}{16\pi^2} \right)^2 + \dots \quad (9.75)$$

Henceforth, we find for (9.32)-(9.33)

$$\delta\zeta(g^2) = -\frac{g^2}{8\pi^2} + \frac{g^4}{256\pi^4} + \dots \quad (9.76)$$

$$\delta\rho(g^2) = -\frac{g^2}{4\pi^2} + \frac{g^4}{128\pi^4} + \dots \quad (9.77)$$

Another good internal check of the calculations⁶ is that the renormalization group functions (9.76)-(9.77) are indeed finite.

Finally, solving the equations (9.34)-(9.35) leads to

$$\zeta_0 = -\frac{3}{13} \quad (9.78)$$

$$\rho_0 = -\frac{6}{13} \quad (9.79)$$

$$\zeta_1 = -\frac{95}{624\pi^2} \quad (9.80)$$

$$\rho_1 = -\frac{95}{312\pi^2} \quad (9.81)$$

We indeed find that $\rho = 2\zeta$. We already knew this from the NO invariance, and we find that the $\overline{MS}$ scheme preserves this symmetry. It can also be understood from a diagrammatical point of view. Consider (9.18), first with only the source ω connected, and subsequently with only the sources τ, L connected. For each diagram giving a divergence proportional to ω^2 in the former case, there exists a similar diagram giving a divergence proportional to τL in the latter case. More precisely, when the appropriate symmetry factor is taken into account, it will hold that

$$\delta\rho = 2\delta\zeta \quad (9.82)$$

Combining this with (9.28)-(9.29) and (9.34)-(9.35), precisely gives the relation (9.5).

Notice that, due to the identity (9.5), the effective potential $V(\sigma, \bar{\sigma}, \phi)$ of (9.52) can be written in terms of 2 combinations of the fields $\sigma, \bar{\sigma}$ and ϕ , namely

$$\begin{aligned} \chi^2 &= \sigma^a \bar{\sigma}^a + \frac{\phi^a \phi^a}{4} \\ \widehat{\chi} &= \epsilon^{abc} \phi^a \sigma^b \bar{\sigma}^c \end{aligned} \quad (9.83)$$

As we have shown, $\widehat{\chi}$ does not influence the value of the minimum. So, it is sufficient to consider the potential with $\widehat{\chi} = 0$. (9.65) then becomes

$$V_1(\chi)^{\widehat{\chi}=0} = \frac{\chi^2}{g^2 \zeta_0} \left(1 - \frac{\zeta_1}{\zeta_0} g^2 \right) + \frac{1}{32\pi^2} \frac{\chi^2}{\zeta_0^2} \left(\ln \frac{\chi^2}{\zeta_0^2 \mu^4} - 3 \right) \quad (9.84)$$

Recalling (9.11) and (9.48), we find

$$\delta\phi = -2\sigma \quad (9.85)$$

$$\delta\sigma = 0 \quad (9.86)$$

$$\delta\bar{\sigma} = \phi \quad (9.87)$$

Consequently

$$\begin{aligned} \delta\chi^2 &= \phi^a \sigma^a + \frac{(2\phi^a)(-2\sigma^a)}{4} = 0 \\ \delta\widehat{\chi} &= 0 \end{aligned} \quad (9.88)$$

A similar conclusion exists for $\bar{\delta}$ and δ_{FP} . Said otherwise, χ and $\widehat{\chi}$ are $SL(2, \mathbb{R})$ invariants. Let us make a comparison with the effective potential $V(\varphi^2)$ of the $O(N)$ vector model with field $\varphi = (\varphi_1, \dots, \varphi_N)$.

⁶See also the Appendix B (section 9.9).

This potential is a function of the $O(N)$ invariant norm $\varphi^2 = \varphi_1^2 + \dots + \varphi_N^2$. Choosing a certain direction for φ breaks the $O(N)$ invariance. In the present case, choosing a certain direction for χ breaks the $SL(2, \mathbb{R})$ symmetry. However, the situation with the ghost condensates is a bit more complicated than a simple breakdown of the $SL(2, \mathbb{R})$.

Before we come to the discussion of the symmetry breaking, let us calculate the minima of (9.84). We can use the renormalization group equation to kill the logarithms and put $\bar{\mu}^4 = \frac{\chi^2}{\zeta_0^2}$. The equation of motion, $\frac{dV}{d\chi} = 0$, has, next to the perturbative one $\chi = 0$, which corresponds to a local maximum, a non-trivial solution, given by

$$\left. \frac{g^2 N}{16\pi^2} \right|_{N=2} = \frac{9}{28} \approx 0.321 \quad (9.89)$$

where it is understood that $g^2 \equiv g^2(\bar{\mu} = \sqrt{\chi}/|\zeta_0|)$. Using the one-loop expression

$$g^2(\bar{\mu}) = \frac{3}{11N} \frac{1}{\ln \frac{\bar{\mu}^2}{\Lambda_{\overline{\text{MS}}}^2}} \quad (9.90)$$

we obtain

$$\chi_{\text{vac}} = 0.539 \Lambda_{\overline{\text{MS}}}^2 \quad (9.91)$$

$$E_{\text{vac}} = -0.017 \Lambda_{\overline{\text{MS}}}^4 \quad (9.92)$$

From (9.89), it follows that the expansion parameter is relatively small. A qualitatively meaningful minimum, (9.91), is thus retrieved. The resulting vacuum energy (9.92) is negative, implying that the ground state favours the formation of the ghost condensates.

9.4 Non-trivial vacuum configurations and dynamical breaking of the NO symmetry.

In this section, we discuss the consequences for the NO symmetry of a non-trivial vacuum expectation value of the ghost operators $f^{abc} c^b c^c$, $f^{abc} \bar{c}^b \bar{c}^c$ and/or $f^{abc} \bar{c}^b c^c$. The arguments are general and applicable for all N and for all choices of the Curci-Ferrari gauge parameter α .

9.4.1 BCS, Overhauser or a combination of both?

Since the action (9.47) is NO invariant, each possible vacuum state can be transformed into another under the action of the NO symmetry. A special choice of a possible vacuum is the pure Overhauser vacuum, determined by⁷

$$\begin{cases} \phi^a = \phi_{\text{vac}} \delta^{a3} \text{ with } \phi_{\text{vac}} = 2\chi_{\text{vac}} \\ \sigma^a = \bar{\sigma}^a = 0 \end{cases} \quad (9.93)$$

Then two of the $SL(2, \mathbb{R})$ generators (δ and $\bar{\delta}$) are dynamically broken since

$$\langle \delta \bar{\sigma} \rangle = -\langle \bar{\delta} \sigma \rangle = \langle \phi \rangle \neq 0 \quad (9.94)$$

⁷Without loss of generality, we can put ϕ^a in the 3-direction.

The ghost number symmetry δ_{FP} is unbroken, just as the BRST symmetry s , since no operator $\mathcal{F}$ exists with $\langle s\mathcal{F} \rangle = \langle \phi \rangle$. In fact, setting

$$\phi^a = \phi_{\text{vac}} \delta^{a3} + \tilde{\phi}^a \text{ with } \langle \tilde{\phi}^a \rangle = 0 \quad (9.95)$$

$$s\tilde{\phi}^a = -g^2 s(f^{abc} \bar{c}^b c^c) \quad (9.96)$$

it is immediately verified that the action

$$\begin{aligned} S(\sigma, \bar{\sigma}, \tilde{\phi}) &= S_{YM} + S_{GF+FP} + \int d^4x \left(-\frac{\sigma^a \bar{\sigma}^a}{g^2 \zeta} - \frac{\phi_{\text{vac}}^2}{2g^2 \rho} - \frac{\tilde{\phi}^3 \phi_{\text{vac}}}{g^2 \rho} - \frac{\tilde{\phi}^a \tilde{\phi}^a}{2g^2 \rho} + \frac{\bar{\sigma}^a}{2g\zeta} g f^{abc} \bar{c}^b c^c \right. \\ &\quad - \frac{\sigma^a}{2g\zeta} g f^{abc} \bar{c}^b c^c - \frac{\tilde{\phi}^a}{g\rho} g f^{abc} \bar{c}^b c^c - \frac{\phi_{\text{vac}}}{g\rho} g f^{3bc} \bar{c}^b c^c \\ &\quad \left. - \frac{1}{2\rho} g^2 (f^{abc} \bar{c}^b c^c)^2 + \frac{1}{4\zeta} g^2 f^{abc} \bar{c}^b c^c f^{ade} \bar{c}^d c^e \right) \end{aligned} \quad (9.97)$$

obeys

$$sS(\sigma, \bar{\sigma}, \tilde{\phi}) = 0 \quad (9.98)$$

while evidently

$$s^2 = 0 \quad (9.99)$$

We focus on the ghost number and BRST symmetry because these are the key ingredients for the definition of a physical subspace, to have a quartet mechanism, etc.; see e.g. [177].

For vacua other than the pure Overhauser case, problems can arise concerning the BRST and/or the ghost number symmetry. Consider for example the pure BCS vacuum

$$\begin{cases} \phi^a = 0 \\ \sigma^a = b \chi_{\text{vac}} \delta^{a3} \\ \bar{\sigma}^a = \bar{b} \chi_{\text{vac}} \delta^{a3} \end{cases} \quad (9.100)$$

where b and $\bar{b}$ are a pair of Faddeev-Popov conjugated constants ($b\bar{b} = 1$). In this vacuum, $\langle f^{abc} \bar{c}^b c^c \rangle \neq 0$, while $s\bar{c}^a = \frac{g}{2} f^{abc} \bar{c}^b c^c$, so we can expect a problem with the BRST transformation. Things can even be made worse, since also vacua where σ^a and $\bar{\sigma}^a$ get a different value (up to the ghost number, which is 2, respectively -2), are allowed. In this case, the ghost number symmetry δ_{FP} is also broken.

It seems that the existence of the ghost condensates, different from the Overhauser channel, could cause serious problems. A pragmatic solution would be to simply choose the Overhauser vacuum, since one always has to choose a specific vacuum to work with. However, this is not very satisfactory. The other vacua are in principle as “good” as the Overhauser one.

Let us try to formulate a solution to the problem of the possible BRST/ghost number symmetry breakdown. Let $|\Omega\rangle$ be the Overhauser vacuum, and $|\tilde{\Omega}\rangle$ any other vacuum. As already said, a certain NO transformation $\mathcal{U}$ exists, so that

$$|\tilde{\Omega}\rangle = \mathcal{U} |\Omega\rangle \quad (9.101)$$

Let Q_{BRST} , $\bar{Q}_{BRST}$, Q_{FP} , Q_δ and $Q_{\bar{\delta}}$ be the charges corresponding to respectively s , $\bar{s}$, δ_{FP} , δ and $\bar{\delta}$. We know that

$$Q_{BRST} |\Omega\rangle = 0 \quad (9.102)$$

$$Q_{FP} |\Omega\rangle = 0 \quad (9.103)$$

With the relations (9.101)-(9.103), it is possible to define new charges⁸

$$\tilde{Q}_{BRST} = \mathcal{U}Q_{BRST}\mathcal{U}^{-1} \quad (9.104)$$

$$\tilde{\bar{Q}}_{BRST} = \mathcal{U}\bar{Q}_{BRST}\mathcal{U}^{-1} \quad (9.105)$$

$$\tilde{Q}_{FP} = \mathcal{U}Q_{FP}\mathcal{U}^{-1} \quad (9.106)$$

$$\tilde{Q}_\delta = \mathcal{U}Q_\delta\mathcal{U}^{-1} \quad (9.107)$$

$$\tilde{Q}_{\bar{\delta}} = \mathcal{U}Q_{\bar{\delta}}\mathcal{U}^{-1} \quad (9.108)$$

Since this is merely a redefinition of its generators, the new charges (9.104)-(9.108) are evidently still obeying the *NO* algebra (9.14). By construction, we have⁹

$$\tilde{Q}_{BRST}|\tilde{\Omega}\rangle = 0 \quad (9.109)$$

$$\tilde{Q}_{FP}|\tilde{\Omega}\rangle = 0 \quad (9.110)$$

As such, we have in any vacuum $\tilde{\Omega}$ the concept of a nilpotent operator $\tilde{Q}_{BRST}$. Furthermore, the physical states $|\widetilde{\text{phys}}\rangle$ are those wherefore

$$\tilde{Q}_{BRST}|\widetilde{\text{phys}}\rangle = 0 \quad (9.111)$$

$$|\widetilde{\text{phys}}\rangle \neq \tilde{Q}_{BRST}|\dots\rangle \quad (9.112)$$

$$\tilde{Q}_{FP}|\widetilde{\text{phys}}\rangle = 0 \quad (9.113)$$

and are connected to the physical states of the Overhauser case through

$$|\widetilde{\text{phys}}\rangle = \mathcal{U}|\text{phys}\rangle \quad (9.114)$$

The conclusion is that in any vacuum, the concept of a Faddeev-Popov symmetry exists, just as a nilpotent BRST transformation. The mere difference is that the functional form of these operators is no longer the usual one (9.2). But in principle, the $\sim$ generators are as good as the original ones to perform the Kugo-Ojima formalism, since this is based on algebraic properties [177]. The *NO* can thus be used to define the physical subspace $\mathcal{H}_{\text{phys}}$ of the total Hilbert space $\mathcal{H}$ of all possible states. The action of the *NO* rotates $\mathcal{H}$, whereby “ Q_{BRST} physical” states $|\text{phys}\rangle$ are rotated into “ $\tilde{Q}_{BRST}$ physical” states $|\widetilde{\text{phys}}\rangle \equiv \mathcal{U}|\text{phys}\rangle$.

Since we have to choose a certain vacuum, we assume for the rest of the article that we are in the Overhauser vacuum, the most obvious choice. Notice that this does not imply that we can simply put the sources L^a and τ^a equal to zero from the beginning. This corresponds to the ghost condensation studied in the context of the maximal Abelian gauge, originated in [77, 78, 79, 80]. Analogously, setting ω^a equal to zero from the beginning, corresponds to the BCS channel as originally studied in [81, 90, 91].

⁸As it is well known, the generators of a symmetry form an adjoint representation.

⁹ $\tilde{Q}_\delta$ for example will be a broken generator. If not, one has $Q_\delta|\Omega\rangle = 0$, a contradiction.

9.4.2 Global color symmetry.

A non-vanishing vacuum expectation value for the color charged field ϕ^a seems to spoil the global color symmetry, i.e. the global $SU(N)$ invariance. However, it can be argued that this global color symmetry breaking is located in the unphysical sector of the Hilbert space. According to [189, 177], the conserved, global $SU(N)$ current is given by

$$\mathcal{J}_\mu^a = \partial_\nu F_{\mu\nu}^a + \{Q_{BRST}, D_\mu^{ab} \bar{c}^b\} \quad (9.115)$$

while the corresponding color charge reads

$$\mathcal{Q}^a = \int d^3x \partial_i F_{0i}^a + \int d^3x \{Q_{BRST}, D_0^{ab} \bar{c}^b\} \quad (9.116)$$

The current (9.115) is the same in comparison with the one given by the usual Yang-Mills Lagrangian (i.e. without any condensate); this is immediately verified since the action (9.47) does not contain any new terms with derivatives of the fields.

The first term of (9.116) is either ill-defined due to massless particles in its spectrum, or zero as a volume integral of a total divergence [20]. Thus, if no massless particles show up (i.e. gluons are massive), (9.116) reduces to a BRST exact form

$$\mathcal{Q}^a = \int d^3x \{Q_{BRST}, D_0^{ab} \bar{c}^b\} \quad (9.117)$$

Henceforth, this color breaking should not be observed in the physical subspace of the Hilbert space, see e.g. [20] and references therein.

The required absence of massless particles is assured if the gluons are no longer massless. This is realized by another condensate of mass dimension 2, namely $\frac{1}{2} \langle A^2 \rangle$ in the case of the Landau gauge. This condensate also lowers the vacuum energy and gives rise to a dynamical gluon mass, as was shown in [42, 184]. Also lattice simulations support a dynamical gluon mass [48, 47]. The generalization to the Curci-Ferrari gauge was discussed in [178].

A rather subtle point in the foregoing is that the well-definedness of (9.117) should be assured.

9.4.3 Absence of Goldstone excitations.

The conserved current corresponding to the δ invariance is given by

$$k_\mu = c^a D_\mu^{ab} c^b + \frac{1}{2} g f^{abc} A_\mu^a c^b c^c = s(c^a A_\mu^a) \quad (9.118)$$

An analogous expression can be derived for the $\bar{\delta}$ current

$$\bar{k}_\mu = \bar{s}(\bar{c}^a A_\mu^a) \quad (9.119)$$

If these continuous δ and $\bar{\delta}$ symmetries are broken, massless Goldstone states should appear, according to the Goldstone theorem. However, since the currents are (anti-)BRST exact, those Goldstone bosons will be part of a BRST quartet, and as such decouple from the physical spectrum due to the quartet mechanism [177]. The argument is analogous to the one given in [77, 78, 79] to explain why there are no physical Goldstone particles present in the case of $SU(2)$ Yang-Mills in the maximal Abelian gauge, due to the appearance of the condensate $\langle \epsilon^{3ab} \bar{c}^a c^b \rangle$.

9.5 Inclusion of matter fields.

So far, we have considered pure Yang-Mills theories, i.e. without matter fields. The present analysis can be nevertheless straightforwardly extended to the case with quarks included. This is accomplished by adding to the pure Yang-Mills action S_{YM} the quark contribution S_m , given by

$$S_m = \int d^4x \bar{\psi}^{iI} i\gamma^\mu D_\mu^{IJ} \psi^{iJ} \quad (9.120)$$

with

$$D_\mu^{IJ} = \partial_\mu \delta^{IJ} - ig A_\mu^a T^{aIJ} \quad (9.121)$$

The T^{aIJ} are the generators of the fundamental representation of $SU(N)$, while D_μ^{IJ} is the corresponding covariant derivative. The index i labels the number of flavours ($1 \leq i \leq N_f$).

The action of the NO transformation on the fermion fields is defined as follows

$$s\psi^{iI} = -ig c^a T^{aIJ} \psi^{iJ} \quad (9.122)$$

$$s\bar{\psi}^{iI} = -ig \bar{\psi}^{iJ} T^{aJI} c^a \quad (9.123)$$

$$\bar{s}\psi^{iI} = -ig \bar{c}^a T^{aIJ} \psi^{iJ} \quad (9.124)$$

$$\bar{s}\bar{\psi}^{iI} = -ig \bar{\psi}^{iJ} T^{aJI} \bar{c}^a \quad (9.125)$$

$$\begin{aligned} \delta\psi &= \bar{\delta}\psi = \delta_{FP}\psi = 0 \\ \delta\bar{\psi} &= \bar{\delta}\bar{\psi} = \delta_{FP}\bar{\psi} = 0 \end{aligned} \quad (9.126)$$

Then it is easily checked that the algebra structure (9.14) is maintained, while the full action

$$S = S_{YM} + S_m + S_{GF+FP} + S_{LCO} \quad (9.127)$$

with S_{LCO} given by (9.1), is NO invariant.

The Ward identities in the Appendix A (section 9.8) can be generalized (see also [91]). As such, the renormalizability is assured, while the ghost operators still have the same anomalous dimension. Of course, the relation $\rho = 2\zeta$ still holds. Also the discussion in the previous section can be repeated¹⁰.

For what concerns the explicit evaluation of the effective potential in the Landau gauge, the absence of a counterterm for the ghost operators (so $Z_2 = 1$) is still valid, just as the relation $Z_c = Z_g^{-1} Z_A^{-1/2}$. Since the quarks are not contributing to $\mathcal{W}(\omega, \tau, L)$ at the one- and two-loop level, no new divergences appear at the one- and two-loop level, hence $\delta\rho_0$ and $\delta\rho_1$ are unchanged in comparison with the quarkless case. Since [88, 87]

$$\begin{aligned} \beta(g^2) &= -\varepsilon g^2 + \left(-\frac{22}{3}N + \frac{4}{3}N_f\right) g^2 \frac{g^2}{16\pi^2} \\ &\quad + \left(-\frac{68}{3}N^2 + \frac{20}{3}N_f N + 2N_f \frac{N^2-1}{N}\right) g^2 \left(\frac{g^2}{16\pi^2}\right)^2 + \dots \\ \gamma_A(g^2) &= \left(-\frac{13}{6}N + \frac{2}{3}N_f\right) \frac{g^2}{16\pi^2} + \left(-\frac{59}{8}N^2 + \frac{5}{2}N_f N + N_f \frac{N^2-1}{N}\right) \left(\frac{g^2}{16\pi^2}\right)^2 + \dots \end{aligned} \quad (9.128)$$

¹⁰A dynamical gluon mass in the presence of massless quarks has been calculated by now in [197]

we now find (again for $N = 2$)

$$\zeta_0 = \frac{3}{2N_f - 13} \quad (9.129)$$

$$\rho_0 = \frac{6}{2N_f - 13} \quad (9.130)$$

$$\zeta_1 = \frac{41N_f - 190}{96(13 - 2N_f)\pi^2} \quad (9.131)$$

$$\rho_1 = \frac{41N_f - 190}{48(13 - 2N_f)\pi^2} \quad (9.132)$$

while the one-loop effective potential reads

$$V_1(\chi) = \frac{\chi^2}{g^2\zeta_0} \left(1 - \frac{\zeta_1}{\zeta_0} g^2 \right) + \frac{1}{32\pi^2} \frac{\chi^2}{\zeta_0^2} \left(\ln \frac{\chi^2}{\zeta_0^2 \mu^4} - 3 \right) \quad (9.133)$$

with χ defined as in (9.83). The minima can be determined in the same fashion as before, this leads to

$$\left. \frac{g^2 N}{16\pi^2} \right|_{N=2} = \frac{36}{112 - 29N_f} \quad (9.134)$$

9.6 Consequences of the ghost condensates.

In this section, we would like to outline some items that deserve further investigation.

- For simplicity, we have restricted ourselves in this article to $N = 2$. Also, the effective potential has been determined at the one-loop level, by making use of the $\overline{\text{MS}}$ scheme. Then, as it is apparent from (9.134), the numbers of flavours must be so that $0 \leq N_f \leq 3$, in order to have a non-trivial solution. This can be changed if another renormalization scheme is chosen. There exist several methods to improve perturbation theory and minimize the renormalization scheme dependence, for example by introducing effective charges [191, 105] or by employing the principle of minimal sensitivity [104, 106]. Also, higher order computations are in order to improve results. Evidently, “real life” QCD will need the generalization to $N = 3$.
- Secondly, we want to comment on the observation that the ghost condensation gives rise to a tachyonic mass for the gluons in the Curci-Ferrari gauge [82]. Let us consider this in more detail in the Landau gauge for $N = 2$. The ghost propagator in the condensed vacuum (9.93) reads¹¹

$$\begin{aligned} \langle \bar{c}^a c^b \rangle_p &= -i \frac{p^2 \delta^{ab} - \frac{\phi^3}{\rho_0} \epsilon^{ab}}{p^4 + \left(\frac{\phi^3}{\rho_0} \right)^2} & a, b = 1, 2 \\ \langle \bar{c}^3 c^3 \rangle_p &= \frac{-i}{p^2} \end{aligned} \quad (9.135)$$

Following [82], one can calculate the gauge boson polarization $\Pi_{\mu\nu}^{ab}$ with this ghost propagator (see Figure 9.1), and then one finds an induced tachyonic gluon mass. Notice that this mass is a loop effect. This observation gave rise to the conclusion that gluons acquire a tachyonic mass due to the ghost condensation. It was already recognized in [157] for the maximal Abelian

¹¹ $\epsilon^{12} = -\epsilon^{21} = 1$, zero otherwise.

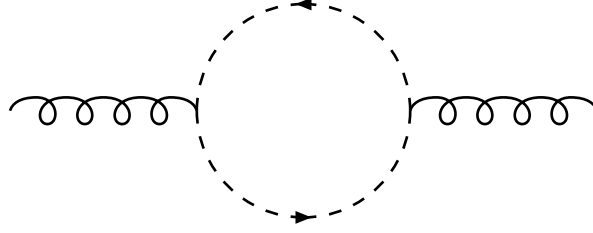


Figure 9.1: Diagram relevant for the gauge boson polarization.

gauge that the ghost condensation resulted in a tachyonic mass for the off-diagonal gluons. In our opinion, this tachyonic mass is more a consequence of an incomplete treatment than a result *in se*. The gauge boson polarization was determined with the usual perturbative gluon propagator (i.e. massless gluons). It was however shown that gluons get a mass through a non-vanishing vacuum expectation value for $\langle \frac{1}{2}A^2 \rangle$ in the Landau gauge [42] or $\langle \frac{1}{2}A^2 + \alpha \bar{c}c \rangle$ in the Curci-Ferrari gauge [178]. The LCO treatment for $\langle \frac{1}{2}A^2 \rangle$ gives a Lagrangian similar to (9.47). More precisely, a real *tree level* gluon mass m_{gluon} is present. It came out that $m_{\text{gluon}} \sim 500\text{MeV}$ [42]. Therefore, the complete procedure to analyze the nature of the induced gluon mass should be that of taking into account the simultaneous presence of both ghost and gluon condensates, i.e. $\langle f^{abc}\bar{c}^b c^c \rangle$ and $\langle \frac{1}{2}A^2 \rangle$ (or $\langle \frac{1}{2}A^2 + \alpha \bar{c}c \rangle$ in the Curci-Ferrari gauge). The induced final gluon mass receives contributions from both condensates, as the gluon propagator gets modified by the condensate $\langle \frac{1}{2}A^2 \rangle$. The diagram of Figure 9.1 is thus only part of the whole set of diagrams contributing to the gluon mass. It is worth mentioning that a similar mechanism should take place in the maximal Abelian gauge [157, 171, 178]. In fact, the mixed gluon-ghost operator $\langle \frac{1}{2}A^2 + \alpha \bar{c}c \rangle$ can be consistently introduced also in this gauge [83, 144].

Summarizing, a complete discussion of the dynamical generation of a mass parameter for gluons would require a combination the LCO formalism of this article with that of [42, 178] by introducing an extra source term $\frac{1}{2}KA_\mu^a A^{\mu a}$ for the operator $\frac{1}{2}A^2$.

- A third point of interest is the modified infrared behaviour of the propagators due to the non-vanishing condensates. If one considers the Landau gauge, the Kugo-Ojima confinement criterion [177] is fulfilled if the ghost propagator exhibits an infrared enhancement, i.e. the ghost propagator should be more singular than $\frac{1}{p^2}$ [192]. Recently, much effort has been paid to investigate this criterion (in the Landau gauge) by means of the Schwinger-Dyson equations, see e.g. [124, 193, 126, 125, 145, 194] and references therein. Defining the gluon and ghost form factors from the Euclidean propagators $D_{\mu\nu}(p^2)$ and $G(p^2)$ as

$$\begin{aligned} D_{\mu\nu}(p^2) &= \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \frac{Z_D(p^2)}{p^2} \\ G(p^2) &= \frac{Z_G(p^2)}{p^2} \end{aligned} \quad (9.136)$$

it was shown that in the infrared (in the Schwinger-Dyson framework)

$$\begin{aligned} Z_D(p^2) &\sim (p^2)^{2a} \\ Z_G(p^2) &\sim (p^2)^{-a} \end{aligned} \quad (9.137)$$

with $a \approx 0.595$ [193, 126, 125, 145]. As such, the obtained solutions of the Schwinger-Dyson equations seem to be compatible with the Kugo-Ojima confinement criterion. Furthermore, these

solutions were also in qualitative agreement with the lattice behaviour (see e.g. [193]). It would be instructive to investigate to what extent the Schwinger-Dyson solutions are modified if one would work with the Landau gauge action¹²

$$\begin{aligned}
S = & S_{YM} + S_{GF+FP} + \int d^4x \left(-\frac{\varphi^2}{2g^2\xi} - \frac{\sigma^a \bar{\sigma}^a}{g^2\zeta} - \frac{\phi^a \phi^a}{2g^2\rho} \right. \\
& + \frac{\varphi}{2g\xi} A_\mu^a A^{\mu a} + \frac{\bar{\sigma}^a}{2g\zeta} g f^{abc} c^b c^c - \frac{\sigma^a}{2g\zeta} g f^{abc} \bar{c}^b \bar{c}^c - \frac{\phi^a}{g\rho} g f^{abc} \bar{c}^b c^c \\
& \left. - \frac{1}{8\xi} (A_\mu^a A^{\mu a})^2 - \frac{1}{2\rho} g^2 (f^{abc} \bar{c}^b c^c)^2 + \frac{1}{4\zeta} g^2 f^{abc} c^b c^c f^{ade} \bar{c}^d \bar{c}^e \right) \quad (9.138)
\end{aligned}$$

that already incorporates the non-perturbative effects of the ghost condensates and the gluon condensates, thus also a gluon mass.

Very recently, some results for general covariant gauges concerning the ghost-antighost condensate $\langle c^a \bar{c}^a \rangle$ were presented in [195] within the Schwinger-Dyson approach. In the used approximation scheme, it turns out that in case of the linear gauges, no ghost-antighost condensate seems to exist. This can be understood, since in the linear gauges there is a symmetry $\bar{c} \rightarrow \bar{c} + \text{cte}$, which prevents the appearance of the operator $\bar{c}^a c^a$. It can be combined with the gluon operator A^2 to yield the mixed gluon-ghost dimension two operator $\frac{1}{2}A^2 + \alpha \bar{c}c$. To our knowledge, this operator is on-shell BRST invariant only in the Curci-Ferrari and in the maximal Abelian gauge¹³ [83, 144, 185]. In particular, concerning the nonlinear Curci-Ferrari gauge, the condensate $\langle \frac{1}{2}A^2 + \alpha \bar{c}c \rangle$ has been proven to show up in the weak coupling [178]. However, no definitive conclusion has been reached so far about this condensate within the Schwinger-Dyson framework [195]. Finally, we notice that the ghost operators $f^{abc} c^b c^c$, $f^{abc} \bar{c}^b c^c$ and $f^{abc} \bar{c}^b \bar{c}^c$ we discussed here, were not considered in [195].

- We have discussed the ghost condensation in the Curci-Ferrari gauge. Originally, the ghost condensates came to attention in the maximal Abelian gauge in [77, 78, 79, 80, 157, 81, 90]. An approach close to the one presented here should be applied to probe the ghost condensates and their consequences in the maximal Abelian gauge too. However, the maximal Abelian gauge is a bit more tricky to handle, see e.g. [178] for some more comments on this.
- So far, the gauges where the ghost condensation takes place, all have the NO symmetry. The important question rises if the ghost condensation only takes place in gauges possessing the NO symmetry? In order to do so, one should first investigate if external sources for the ghost operators can be introduced without spoiling the renormalizability.

9.7 Conclusion.

In this article, we considered Yang-Mills theory in the Curci-Ferrari gauge and as a limiting case, in the Landau gauge. These gauges possess a global continuous symmetry, generated by the NO algebra. This algebra is built out of the (anti-)BRST transformation and of the $SL(2, \mathbb{R})$ algebra. By combining the local composite operator formalism with the algebraic renormalization technique, we have proven that a ghost condensation à la $\langle f^{abc} c^b c^c \rangle$, $\langle f^{abc} \bar{c}^b \bar{c}^c \rangle$ (BCS channel) and $\langle f^{abc} \bar{c}^b c^c \rangle$ (Overhauser channel) occurs. It has been shown that different vacua are possible, with the Overhauser and BCS vacuum as two special choices. The ghost condensates (partially) break the NO symmetry. We have discussed

¹² $\langle \varphi \rangle = \frac{g}{2} A_\mu^a A^{\mu a}$. See [42, 178] for the meaning and value of ξ .

¹³In which case the color index is restricted to the off-diagonal fields.

the BRST and the ghost number symmetry in the condensed vacua. We paid some attention to the global $SU(N)$ color symmetry and to the absence of Goldstone bosons in the physical spectrum. We also briefly discussed the generalization to the case when quark fields are included.

9.8 Appendix A.

9.8.1 Ward identities for the NO algebra in the Curci-Ferrari gauge.

The renormalizability of the Curci-Ferrari gauge is well established [161, 87, 144]. In this Appendix we show that the introduction of a suitable set of external sources allows to write down Ward identities for all the generators of the NO algebra. In particular, these Ward identities will imply that all ghost polynomials $f^{abc}c^b c^c$, $f^{abc}\bar{c}^b \bar{c}^c$, $f^{abc}\bar{c}^b c^c$ have the same anomalous dimension.

In order to write down the functional identities for the NO algebra, we need to introduce three more external sources Ω_μ^a , $\bar{\Omega}_\mu^a$, ϑ_μ^a with dimensions $(2, 2, 1)$, coupled to the nonlinear BRST and anti-BRST variations of the gauge field A_μ^a .

$$S_{ext} = s\bar{s} \int d^4x \left(\vartheta^{a\mu} A_\mu^a + \frac{v}{2} \vartheta^{a\mu} \vartheta_\mu^a \right) \quad (9.139)$$

Notice that the coefficient v is allowed by power counting, since the term $\vartheta^{a\mu} \vartheta_\mu^a$ has dimension 2. The generators of the NO algebra act on Ω_μ^a , $\bar{\Omega}_\mu^a$, ϑ_μ^a as

$$s\vartheta_\mu^a = \bar{\Omega}_\mu^a \quad (9.140)$$

$$\begin{aligned} s\bar{\Omega}_\mu^a &= s\Omega_\mu^a = 0 \\ \bar{s}\vartheta_\mu^a &= -\Omega_\mu^a \end{aligned} \quad (9.141)$$

$$\begin{aligned} \bar{s}\Omega_\mu^a &= \bar{s}\bar{\Omega}_\mu^a = 0 \\ \delta\Omega_\mu^a &= -\bar{\Omega}_\mu^a \end{aligned} \quad (9.142)$$

$$\begin{aligned} \delta\vartheta_\mu^a &= \delta\Omega_\mu^a = 0 \\ \bar{\delta}\bar{\Omega}_\mu^a &= -\Omega_\mu^a \end{aligned} \quad (9.143)$$

$$\bar{\delta}\vartheta_\mu^a = \bar{\delta}\Omega_\mu^a = 0$$

Therefore, for S_{ext} one gets

$$S_{ext} = \int d^4x \left(-\Omega^{\mu a} D_\mu^{ab} c^b - \bar{\Omega}^{\mu a} D_\mu^{ab} \bar{c}^b + v \Omega^{\mu a} \bar{\Omega}_\mu^a - \vartheta^{a\mu} D_\mu^{ab} b^b + g f^{abc} \vartheta^{a\mu} (D_\mu^{bd} c^d) \bar{c}^c \right) \quad (9.144)$$

From this expression, it can be seen that the parameter v is needed to account for the behavior of the two-point Green function $\langle (D_\mu^{ab} c^b(x)) (D_\nu^{cd} \bar{c}^d(y)) \rangle$, which is related to the Kugo-Ojima criterion. In other words, the coefficient v is the LCO parameter for this Green function.

We can now translate the whole NO algebra into functional identities, which will be the starting point for the algebraic characterization of the allowed counterterm. It turns out thus that, in the Curci-Ferrari gauge, the complete action Σ

$$\begin{aligned} \Sigma &= S_{YM} + S_{GF+FP} + S_{LCO} + S_{ext} \\ &= -\frac{1}{4} \int d^4x F_{\mu\nu}^a F^{a\mu\nu} + s\bar{s} \int d^4x \left(\lambda^a c^a + \zeta \lambda^a \eta^a + \eta^a \bar{c}^a \right. \\ &\quad \left. - \frac{\alpha}{2} c^a \bar{c}^a + \vartheta^{a\mu} A_\mu^a + \frac{1}{2} A_\mu^a A^{a\mu} + \frac{v}{2} \vartheta^{a\mu} \vartheta_\mu^a \right) \end{aligned} \quad (9.145)$$

is constrained by the following identities:

- the Slavnov-Taylor identity

$$\mathcal{S}(\Sigma) = 0 \quad (9.146)$$

$$\begin{aligned} \mathcal{S}(\Sigma) = \int d^4x \left(\left(\frac{\delta \Sigma}{\delta \Omega^{a\mu}} - v \bar{\Omega}_\mu^a \right) \frac{\delta \Sigma}{\delta A_\mu^a} + \left(\frac{\delta \Sigma}{\delta L^a} - \zeta \tau^a \right) \frac{\delta \Sigma}{\delta c^a} \right. \\ \left. + b^a \frac{\delta \Sigma}{\delta \bar{c}^a} + \tau^a \frac{\delta \Sigma}{\delta \eta^a} + \omega^a \frac{\delta \Sigma}{\delta \lambda^a} + \bar{\Omega}_\mu^a \frac{\delta \Sigma}{\delta \vartheta_\mu^a} \right) \end{aligned} \quad (9.147)$$

- the anti-Slavnov-Taylor identity

$$\bar{\mathcal{S}}(\Sigma) = 0 \quad (9.148)$$

$$\begin{aligned} \bar{\mathcal{S}}(\Sigma) = \int d^4x \left(\left(\frac{\delta \Sigma}{\delta \bar{\Omega}^{a\mu}} + v \Omega_\mu^a \right) \frac{\delta \Sigma}{\delta A_\mu^a} - \left(\frac{\delta \Sigma}{\delta \tau^a} - \zeta L^a \right) \frac{\delta \Sigma}{\delta \bar{c}^a} \right. \\ \left. - \left(b^a + \frac{\delta \Sigma}{\delta \omega^a} - 2\zeta \omega^a \right) \frac{\delta \Sigma}{\delta c^a} - \frac{\delta \Sigma}{\delta \eta^a} \frac{\delta \Sigma}{\delta b^a} + L^a \frac{\delta \Sigma}{\delta \lambda^a} - \omega^a \frac{\delta \Sigma}{\delta \eta^a} - \Omega^{a\mu} \frac{\delta \Sigma}{\delta \vartheta^{a\mu}} \right) \end{aligned} \quad (9.149)$$

- the δ Ward identity

$$\mathcal{W}(\Sigma) = 0 \quad (9.150)$$

with

$$\mathcal{W}(\Sigma) = \int d^4x \left(c^a \frac{\delta \Sigma}{\delta \bar{c}^a} + \left(\frac{\delta \Sigma}{\delta L^a} - \zeta \tau^a \right) \frac{\delta \Sigma}{\delta b^a} + 2\omega^a \frac{\delta \Sigma}{\delta L^a} - \tau^a \frac{\delta \Sigma}{\delta \omega^a} - \eta^a \frac{\delta \Sigma}{\delta \lambda^a} - \bar{\Omega}_\mu^a \frac{\delta \Sigma}{\delta \Omega_\mu^a} \right) \quad (9.151)$$

- the $\bar{\delta}$ Ward identity

$$\bar{\mathcal{W}}(\Sigma) = 0 \quad (9.152)$$

with

$$\bar{\mathcal{W}}(\Sigma) = \int d^4x \left(\bar{c}^a \frac{\delta \Sigma}{\delta c^a} - \left(\frac{\delta \Sigma}{\delta \tau^a} - \zeta L^a \right) \frac{\delta \Sigma}{\delta b^a} - 2\omega^a \frac{\delta \Sigma}{\delta \tau^a} + L^a \frac{\delta \Sigma}{\delta \omega^a} - \lambda^a \frac{\delta \Sigma}{\delta \eta^a} - \Omega_\mu^a \frac{\delta \Sigma}{\delta \bar{\Omega}_\mu^a} \right) \quad (9.153)$$

9.8.2 Algebraic characterization of the invariant counterterm in the Curci-Ferrari gauge.

The most general local invariant counterterm compatible with both Slavnov-Taylor and anti-Slavnov-Taylor identities (9.146), (9.148) can be written as

$$\begin{aligned} \Sigma^c = & -\frac{\sigma}{4} \int d^4x F_{\mu\nu}^a F^{a\mu\nu} + \mathcal{B}\bar{\mathcal{B}} \int d^4x \left(a_1 \lambda^a c^a + a_2 \eta^a \lambda^a + a_3 \eta^a \bar{c}^a \right. \\ & \left. + \frac{a_4}{2} c^a \bar{c}^a + a_5 \vartheta_\mu^a A^{a\mu} + \frac{a_6}{2} A^{a\mu} A_\mu^a + \frac{a_7}{2} \vartheta_\mu^a \vartheta^{a\mu} \right) \end{aligned} \quad (9.154)$$

where σ , a_1 , a_2 , a_3 , a_4 , a_5 , a_6 , a_7 are free parameters and $\mathcal{B}$, $\bar{\mathcal{B}}$ denote the linearized nilpotent operators

$$\begin{aligned} \mathcal{B} = & \int d^4x \left(\frac{\delta \Sigma}{\delta A_\mu^a} \frac{\delta}{\delta \Omega^{a\mu}} + \left(\frac{\delta \Sigma}{\delta \Omega^{a\mu}} - \gamma \bar{\Omega}_\mu^a \right) \frac{\delta}{\delta A_\mu^a} + \left(\frac{\delta \Sigma}{\delta L^a} - \zeta \tau^a \right) \frac{\delta}{\delta c^a} \right. \\ & \left. + \frac{\delta \Sigma}{\delta c^a} \frac{\delta}{\delta L^a} + b^a \frac{\delta}{\delta \bar{c}^a} + \tau^a \frac{\delta}{\delta \eta^a} + \omega^a \frac{\delta}{\delta \lambda^a} + \bar{\Omega}_\mu^a \frac{\delta}{\delta \vartheta_\mu^a} \right) \end{aligned} \quad (9.155)$$

and

$$\begin{aligned} \bar{\mathcal{B}} = & \int d^4x \left(\frac{\delta \Sigma}{\delta A_\mu^a} \frac{\delta}{\delta \bar{\Omega}_\mu^a} + \left(\frac{\delta \Sigma}{\delta \bar{\Omega}_\mu^a} + \gamma \Omega_\mu^a \right) \frac{\delta}{\delta A_\mu^a} - \left(\frac{\delta \Sigma}{\delta \tau^a} - \zeta L^a \right) \frac{\delta}{\delta \bar{c}^a} - \frac{\delta \Sigma}{\delta \bar{c}^a} \frac{\delta}{\delta \tau^a} \right. \\ & - \left(b^a + \frac{\delta \Sigma}{\delta \omega^a} - 2\zeta \omega^a \right) \frac{\delta}{\delta c^a} - \frac{\delta \Sigma}{\delta c^a} \frac{\delta}{\delta \omega^a} - \frac{\delta \Sigma}{\delta \eta^a} \frac{\delta}{\delta b^a} - \frac{\delta \Sigma}{\delta b^a} \frac{\delta}{\delta \eta^a} \\ & \left. + L^a \frac{\delta}{\delta \lambda^a} - \omega^a \frac{\delta}{\delta \eta^a} - \Omega^{a\mu} \frac{\delta}{\delta \vartheta_\mu^a} \right) \end{aligned} \quad (9.156)$$

From the δ and $\bar{\delta}$ Ward identities (9.150), (9.152) it follows that

$$a_3 = a_1 \quad (9.157)$$

so that the final expression for (9.154) becomes

$$\begin{aligned} \Sigma^c = & -\frac{\sigma}{4} \int d^4x F_{\mu\nu}^a F^{a\mu\nu} + \mathcal{B}\bar{\mathcal{B}} \int d^4x \left(a_1 \lambda^a c^a + a_2 \eta^a \lambda^a + a_1 \eta^a \bar{c}^a \right. \\ & \left. + \frac{a_4}{2} c^a \bar{c}^a + a_5 \vartheta_\mu^a A^{a\mu} + \frac{a_6}{2} A^{a\mu} A_\mu^a + \frac{a_7}{2} \vartheta_\mu^a \vartheta^{a\mu} \right) \end{aligned} \quad (9.158)$$

The coefficients σ , a_1 , a_2 , a_4 , a_5 , a_6 , a_7 are easily seen to correspond to a multiplicative renormalization of the coupling constant g , of the gauge and LCO parameters α , ζ , γ , of the fields and external sources. In particular, the coefficients σ and a_5 are related to the renormalization of the gauge coupling constant g and of the gauge field A_μ^a , as it is apparent from

$$\mathcal{B}\bar{\mathcal{B}} \int d^4x \vartheta_\mu^a A^{a\mu} = -\mathcal{N}_A \Sigma \quad (9.159)$$

where $\mathcal{N}_A$ stands for the invariant counting operator

$$\mathcal{N}_A = \int d^4x \left(A_\mu^a \frac{\delta}{\delta A_\mu^a} - \Omega_\mu^a \frac{\delta}{\delta \Omega_\mu^a} - \bar{\Omega}_\mu^a \frac{\delta}{\delta \bar{\Omega}_\mu^a} - \vartheta_\mu^a \frac{\delta}{\delta \vartheta_\mu^a} \right) + \gamma \frac{\partial}{\partial \gamma} \quad (9.160)$$

The coefficient a_4 corresponds to the renormalization of the gauge parameter α , indeed

$$\mathcal{B}\bar{\mathcal{B}} \int d^4x \frac{1}{2} c^a \bar{c}^a = -\frac{\partial \Sigma}{\partial \alpha} \quad (9.161)$$

The coefficient a_2 is associated to the renormalization of the LCO parameter ζ , which follows from

$$\mathcal{B}\bar{\mathcal{B}} \int d^4x \eta^a \lambda^a = -\mathcal{N}_\zeta \Sigma \quad (9.162)$$

with

$$\mathcal{N}_\zeta = \zeta \frac{\partial}{\partial \zeta} + \int d^4x \left(\omega^a \frac{\delta}{\delta b^a} - \eta^a \frac{\delta}{\delta c^a} + \lambda^a \frac{\delta}{\delta \bar{c}^a} \right) \quad (9.163)$$

The coefficient a_1 is related to the anomalous dimensions of all ghost operators, namely

$$\mathcal{B}\bar{\mathcal{B}} \int d^4x (\lambda^a c^a + \eta^a \bar{c}^a) = \mathcal{N}_L \Sigma \quad (9.164)$$

where

$$\begin{aligned} \mathcal{N}_L = \int d^4x \left(L^a \frac{\delta}{\delta L^a} + \tau^a \frac{\delta}{\delta \tau^a} + \lambda^a \frac{\delta}{\delta \lambda^a} + \omega^a \frac{\delta}{\delta \omega^a} + \eta^a \frac{\delta}{\delta \eta^a} \right. \\ \left. - b^a \frac{\delta}{\delta b^a} - \bar{c}^a \frac{\delta}{\delta \bar{c}^a} - c^a \frac{\delta}{\delta c^a} \right) - 2\zeta \frac{\partial}{\partial \zeta} \end{aligned} \quad (9.165)$$

The renormalization of the LCO parameter γ is given by the coefficient a_7 , as can be seen from

$$\mathcal{B}\bar{\mathcal{B}} \int d^4x \left(\frac{1}{2} \vartheta_\mu^a \vartheta^{a\mu} \right) = \left(\frac{\partial}{\partial \gamma} - \int d^4x \vartheta_\mu^a \frac{\delta}{\delta A_\mu^a} \right) \Sigma \quad (9.166)$$

Finally, the anomalous dimension of the ghost c^a and the antighost $\bar{c}^a$ are obtained from the coefficient a_6

$$\mathcal{B}\bar{\mathcal{B}} \int d^4x \left(\frac{1}{2} A^{a\mu} A_\mu^a \right) = \mathcal{N}_c \Sigma \quad (9.167)$$

with

$$\begin{aligned} \mathcal{N}_c = \int d^4x \left(\frac{1}{2} c^a \frac{\delta}{\delta c^a} + \frac{1}{2} \bar{c}^a \frac{\delta}{\delta \bar{c}^a} + b^a \frac{\delta}{\delta b^a} - L^a \frac{\delta}{\delta L^a} - \tau^a \frac{\delta}{\delta \tau^a} - \omega^a \frac{\delta}{\delta \omega^a} \right. \\ \left. - \frac{3}{2} \lambda^a \frac{\delta}{\delta \lambda^a} - \frac{3}{2} \eta^a \frac{\delta}{\delta \eta^a} \right) - 2\alpha \frac{\partial}{\partial \alpha} + 2\zeta \frac{\partial}{\partial \zeta} \end{aligned} \quad (9.168)$$

From expressions (9.165) and (9.168) one sees that all sources L^a , τ^a , and ω^a renormalize in the same way, which means that all composite ghost polynomials $f^{abc} c^b c^c$, $f^{abc} \bar{c}^b \bar{c}^c$, $f^{abc} \bar{c}^b c^c$ have indeed the same anomalous dimension. This result is a consequence of the relationship (9.157) which, of course, stems from the existence of the NO algebra.

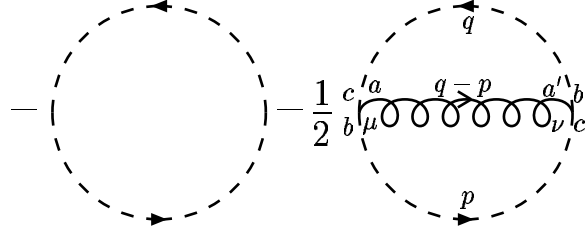


Figure 9.2: Vacuum bubbles up to two-loop order, giving divergences proportional to ω^2 .

9.9 Appendix B.

In order to construct the one-loop effective potential, we need the values of ζ_0 , ρ_0 , ζ_1 and ρ_1 . These can be calculated as soon we know the divergences proportional to ω^2 and $L\tau$ when the generating functional corresponding to the action (9.18) is calculated. In principle, it is sufficient to calculate the divergences proportional to ω^2 since the NO invariance leads to $\rho = 2\zeta$. Therefore, we can restrict ourselves to the diagrams with only the source ω connected. Let us write

$$\delta\rho = \delta\rho_0 g^2 + \delta\rho_1 g^4 + \dots \quad (9.169)$$

For $N = 2$ and $\alpha = 0$, the ghost propagator reads

$$\langle \bar{c}^a c^b \rangle_p = i \frac{-p^4 \delta^{ab} + p^2 g \epsilon^{abc} \omega^c - g^2 \omega^a \omega^b}{p^2 (p^4 + g^2 \omega^2)} \quad (9.170)$$

while the gluon propagator is given by

$$\langle A_\mu^a A_\nu^b \rangle_p = \frac{-i \delta^{ab}}{p^2} \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \quad (9.171)$$

The ghost-antighost-gluon vertex equals

$$g \epsilon^{abc} p_\mu \quad (9.172)$$

The relevant vacuum bubbles¹⁴ are shown in Figure 9.2. At one-loop, we find a contribution to $\mathcal{W}(\omega, \tau, L)$, given by

$$-i \int \frac{d^d p}{(2\pi)^d} \ln(p^4 + g^2 \omega^2) \quad (9.173)$$

Performing a Wick rotation¹⁵ and employing the $\overline{\text{MS}}$ scheme, this leads to a divergence given by

$$g^2 \omega^2 \frac{1}{32\pi^2} \frac{4}{\varepsilon} \quad (9.174)$$

Hence

$$\delta\rho_0 = -\frac{1}{4\pi^2} \frac{1}{\varepsilon} \quad (9.175)$$

¹⁴The diagrams containing a counterterm are not shown.

¹⁵If one would like to avoid a Wick rotation, one could have started immediately from the Euclidean Yang-Mills action.

At two loops, the contribution to $\mathcal{W}(\omega, \tau, L)$ is obtained by computing the second diagram of Figure 9.2, yielding

$$\mathcal{I} = \frac{1}{2} i g^2 \epsilon^{abc} \epsilon^{a'b'c'} \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \left[p_\mu q_\nu \frac{-i \delta^{aa'}}{(p-q)^2} \left(g_{\mu\nu} - \frac{(p-q)_\mu (p-q)_\nu}{(p-q)^2} \right) \right. \\ \left. \times i \frac{-p^4 \delta^{bc'} + g p^2 \epsilon^{bc'e} \omega^e - g^2 \omega^b \omega^{c'}}{p^2(p^4 + g^2 \omega^2)} \times i \frac{-q^4 \delta^{b'c} + g q^2 \epsilon^{b'ce} \omega^e - g^2 \omega^{b'} \omega^c}{q^2(q^4 + g^2 \omega^2)} \right] \quad (9.176)$$

Working out the color algebra, one finds

$$\mathcal{I} = -\frac{g^2}{2} \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \left[\frac{p_\mu q_\nu}{(p-q)^2} \left(g_{\mu\nu} - \frac{(p-q)_\mu (p-q)_\nu}{(p-q)^2} \right) \right. \\ \left. \times \frac{-6p^4 q^4 - 2g^2 \omega^2 (p^4 + q^4 - p^2 q^2)}{p^2 q^2 (p^4 + g^2 \omega^2) (q^4 + g^2 \omega^2)} \right] \quad (9.177)$$

This integral $\mathcal{I}$ has been calculated in two steps: first all tensor integrals have been reduced to a combination of scalar master integrals, applying simple algebraic rearrangements of the scalar products which appear in the numerator of the integrand; all master integrals are vacuum integrals, i.e. with vanishing external momentum; they have been replaced by their explicit expression in terms of special functions [196] and expanded in powers of ε . The calculation has been done with the Mathematica packages *DiagExpand* and *ProcessDiagram*. We find

$$\mathcal{I} = \frac{g^4 \omega^2}{(16\pi^2)^2} \left(\frac{6}{\varepsilon^2} + \frac{17}{2\varepsilon} - \frac{6}{\varepsilon} \ln \frac{g\omega}{\mu^2} + \text{finite} \right) \quad (9.178)$$

We also have to take the counterterm information into account¹⁶. Since there is no counterterm $\propto \omega^a g f^{abc} \bar{c}^b c^c$ in the Landau gauge, the only counterterm that will contribute at the order we are working, is

$$\delta Z_c \bar{c}^a \partial_\mu \partial^\mu \delta^{ab} c^b \quad (9.179)$$

where [87]

$$\delta Z_c = \frac{3}{2} \frac{g^2 N}{16\pi^2} \frac{1}{\varepsilon} + \dots \equiv z_c^1 g^2 + \dots \quad (9.180)$$

This leads to a contribution

$$(-2z_c^1 g^2) \left(-i \int \frac{d^d p}{(2\pi)^d} \ln(p^4 + g^2 \omega^2) \right) \quad (9.181)$$

Or

$$[-2z_c^1 g^2] \left[-\frac{g^2 \omega^2}{32\pi^2} \left(-\frac{4}{\varepsilon} + 2 \ln \frac{g\omega}{\mu^2} - 3 \right) \right] = \frac{g^4 \omega^2}{(16\pi^2)^2} \left(-\frac{12}{\varepsilon^2} - \frac{9}{\varepsilon} + \frac{6}{\varepsilon} \ln \frac{g\omega}{\mu^2} + \text{finite} \right) \quad (9.182)$$

Hence, the complete two-loop contribution to $\mathcal{W}(\omega, \tau, L)$ yields

$$(9.178) + (9.182) = \frac{g^4 \omega^2}{(16\pi^2)^2} \left(-\frac{6}{\varepsilon^2} - \frac{1}{2\varepsilon} + \text{finite} \right) \quad (9.183)$$

A good internal check of the calculations is that the terms proportional to $\frac{1}{\varepsilon} \ln \frac{g\omega}{\mu^2}$ are cancelled. Finally, we find that

$$\delta \rho_1 = \frac{1}{(16\pi^2)^2} \left(\frac{1}{\varepsilon} + \frac{12}{\varepsilon^2} \right) \quad (9.184)$$

¹⁶The corresponding diagram looks like the first one of Figure 9.2.

Chapter 10

Renormalizability of the local composite operator A_μ^2 in linear covariant gauges

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The local composite operator A_μ^2 is analysed within the algebraic renormalization in Yang-Mills theories in linear covariant gauges. We establish that it is multiplicatively renormalizable to all orders of perturbation theory. Its anomalous dimension is computed to two-loop order in the $\overline{MS}$ scheme.

10.1 Introduction.

The possibility that gluons might acquire in a dynamical way a mass is receiving increasing attention, both from the theoretical point of view as well as from lattice simulations. Effective gluon masses have been reported in a rather large number of gauges [198]. For instance, the relevance of the local operator $A_\mu^a A^{a\mu}$ for Yang-Mills theory in the Landau gauge has been emphasized by several authors [34, 33, 175, 37, 38]. That this operator has a special meaning in the Landau gauge follows by observing that, due to the transversality condition $\partial_\mu A^{a\mu} = 0$, the integrated mass dimension two operator $\int d^4x A_\mu^a A^{a\mu}$ is gauge invariant. Remarkably, the operator $A_\mu^a A^{a\mu}$ in the Landau gauge is multiplicatively renormalizable [42, 87], its anomalous dimension being given [87, 153] by a combination of the gauge beta function, $\beta(a)$, and of the anomalous dimension, $\gamma_A(a)$, of the gauge field, according to the relation

$$\gamma_{A^2}(a) = - \left(\frac{\beta(a)}{a} + \gamma_A(a) \right), \quad a = \frac{g^2}{16\pi^2} . \quad (10.1)$$

Moreover, lattice simulations [175, 37, 38] have provided strong indications of the existence of the condensate $\langle A_\mu^a A^{a\mu} \rangle$, which can be related to a dynamical gluon mass: a renormalizable effective potential for this condensate in pure Yang-Mills theory has been constructed and evaluated in analytic form up to two-loop order in [42], resulting in an effective gluon mass $m_{\text{gluon}} \approx 500 \text{ MeV}$. The inclusion of massless quarks has been recently worked out in [197]. Another analytic study of $\langle A_\mu^a A^{a\mu} \rangle$ can be

found in [184]. Also, lattice simulations [48] of the gluon propagator in the Landau gauge have reported a gluon mass $m_{\text{gluon}} \approx 600 \text{ MeV}$. Concerning other gauges, an effective gluon mass has been reported in lattice simulations in the Laplacian [47, 43] and maximal Abelian [49, 50] gauges. It is worth underlining that the local operator $A_\mu^a A^{a\mu}$ of the Landau gauge can be generalized [144] to the maximal Abelian gauge, which is a renormalizable gauge in the continuum [71, 72, 183]. It turns out in fact that the integrated mixed gluon-ghost operator¹ $\int d^4x \left(\frac{1}{2} A_\mu^\alpha A^{\alpha\mu} + \xi \bar{c}^\alpha c^\alpha \right)$ is BRST invariant on-shell [144], a property which ensures the multiplicative renormalizability to all orders of perturbation theory [179, 199] of the local operator $\left(\frac{1}{2} A_\mu^\alpha A^{\alpha\mu} + \xi \bar{c}^\alpha c^\alpha \right)$. The analytic evaluation of the effective potential for the condensate $\langle \frac{1}{2} A_\mu^\alpha A^{\alpha\mu} + \xi \bar{c}^\alpha c^\alpha \rangle$ has not yet been worked out. Nevertheless, we expect a nonvanishing value for this condensate, which would result in a dynamical gluon mass. This is supported by the fact that a renormalizable effective potential for the mixed gluon-ghost operator has been obtained [178] in the nonlinear Curci-Ferrari gauge, yielding a nonvanishing condensate $\langle \frac{1}{2} A_\mu^2 + \xi \bar{c}c \rangle$, which provides a dynamical mass for the gluons. The Curci-Ferrari gauge shares a close similarity with the maximal Abelian gauge. We expect thus that something similar should happen in this gauge.

A gluon condensate $\langle A_\mu^a A^{a\mu} \rangle$ has also been introduced in the Coulomb gauge [200] in order to obtain estimates for the glueball spectrum. Older works [28, 30, 201, 202] already discussed the pairing of gluons in connection with a mass generation, as a result of the instability of the perturbative Yang-Mills vacuum. Also, the dynamical mass generation for the gluons is a part of the Kugo-Ojima criterion for color confinement [177]. See [20] for a review.

In this work we analyse the ultraviolet properties of the local composite operator $A_\mu^a A^{a\mu}$ in the linear covariant gauges, whose gauge fixing term is

$$\int d^4x \left(b^a \partial_\mu A^{a\mu} + \frac{\alpha}{2} b^a b^a + \bar{c}^a \partial^\mu D_\mu^{ab} c^b \right), \quad (10.2)$$

where b^a stands for the Lagrange multiplier and α is the gauge parameter. Our aim is that of establishing some necessary requirements in order to study the possible condensation of this operator, which would imply the occurrence of dynamical mass generation in these gauges. Notice that, unlike the case of the Landau and maximal Abelian gauges, the quantity $\int d^4x A_\mu^2$ is now not BRST invariant on-shell. However, we shall be able to prove that the local operator A_μ^2 is multiplicatively renormalizable to all orders of perturbation theory. There is a simple understanding of this property. In linear covariant gauges, due to the additional shift symmetry of the antighost, *i.e.* $\bar{c} \rightarrow \bar{c} + \text{const.}$, the operator A_μ^2 does not mix with the other local dimension two composite ghost operator $\bar{c}c$, which cannot show up due to the above symmetry. We remark that the renormalizability of A_μ^2 is the first step towards the construction of a renormalizable effective potential in order to study the possible condensation of this operator and the ensuing dynamical mass generation.

The work is organized as follows. In section 10.2 we derive the Ward identities for Yang-Mills theory in linear covariant gauges in the presence of the local operator A_μ^2 . These identities turn out to ensure the multiplicative renormalizability of A_μ^2 . In section 10.3 the explicit two-loop calculation of the anomalous dimension of A_μ^2 is presented.

¹In the case of the maximal Abelian gauge the group index α labels the off-diagonal generators T^α of $SU(N)$, with $\alpha = 1, \dots, N(N-1)$. The parameter ξ is the gauge fixing parameter of the maximal Abelian gauge.

10.2 Algebraic proof of the renormalizability of the local operator $A_\mu^a A^{a\mu}$.

We begin by recalling the expression of the pure Yang-Mills action in the linear covariant gauges

$$\begin{aligned} S &= S_{YM} + S_{GF+FP} \\ &= -\frac{1}{4} \int d^4x F_{\mu\nu}^a F^{a\mu\nu} + \int d^4x \left(b^a \partial_\mu A^{a\mu} + \frac{\alpha}{2} b^a b^a + \bar{c}^a \partial^\mu D_\mu^{ab} c^b \right), \end{aligned} \quad (10.3)$$

where

$$D_\mu^{ab} \equiv \partial_\mu \delta^{ab} - g f^{abc} A_\mu^c. \quad (10.4)$$

In order to study the local composite operator $A_\mu^a A^{a\mu}$, we introduce it in the action by means of a BRST doublet [59] of external sources (J, λ) , namely

$$S_J = s \int d^4x \left(\frac{1}{2} \lambda A_\mu^a A^{a\mu} + \frac{\xi}{2} \lambda J \right) = \int d^4x \left(\frac{1}{2} J A_\mu^a A^{a\mu} + \lambda A^{a\mu} \partial_\mu c^a + \frac{\xi}{2} J^2 \right), \quad (10.5)$$

where s denotes the BRST nilpotent operator acting as

$$\begin{aligned} s A_\mu^a &= -D_\mu^{ab} c^b, \\ s c^a &= \frac{1}{2} g f^{abc} c^b c^c, \\ s \bar{c}^a &= b^a, \\ s b^a &= 0, \\ s \lambda &= J, \\ s J &= 0. \end{aligned} \quad (10.6)$$

According to the local composite operators technique [23, 24, 176], the dimensionless parameter ξ is needed to account for the divergences present in the vacuum Green function $\langle A^2(x) A^2(y) \rangle$, which turn out to be proportional to J^2 . As is apparent from expressions (10.3) and (10.5), the action $(S_{YM} + S_{GF+FP} + S_J)$ is BRST invariant

$$s(S_{YM} + S_{GF+FP} + S_J) = 0. \quad (10.7)$$

10.2.1 Ward identities.

In order to translate the BRST invariance (10.7) into the corresponding Slavnov-Taylor identity [59], we introduce two further external sources Ω_μ^a, L^a coupled to the non-linear BRST variations of A_μ^a and c^a

$$S_{\text{ext}} = \int d^4x \left(-\Omega^{a\mu} D_\mu^{ab} c^b + \frac{1}{2} g f^{abc} L^a c^b c^c \right), \quad (10.8)$$

with

$$s \Omega_\mu^a = s L^a = 0. \quad (10.9)$$

Therefore, the complete action

$$\Sigma = S_{YM} + S_{GF+FP} + S_J + S_{\text{ext}}, \quad (10.10)$$

obeys the following identities

- The Slavnov-Taylor identity

$$\mathcal{S}(\Sigma) = \int d^4x \left(\frac{\delta \Sigma}{\delta A_\mu^a} \frac{\delta \Sigma}{\delta \Omega^{a\mu}} + \frac{\delta \Sigma}{\delta c^a} \frac{\delta \Sigma}{\delta L^a} + b^a \frac{\delta \Sigma}{\delta \bar{c}^a} + J \frac{\delta \Sigma}{\delta \lambda} \right) = 0 . \quad (10.11)$$

- The linear gauge-fixing condition

$$\frac{\delta \Sigma}{\delta b^a} = \partial_\mu A^{a\mu} + \alpha b^a . \quad (10.12)$$

- The antighost equation

$$\frac{\delta \Sigma}{\delta \bar{c}^a} + \partial_\mu \frac{\delta \Sigma}{\delta \Omega^{a\mu}} = 0 . \quad (10.13)$$

Notice also that the additional shift symmetry in the antighost present in the linear covariant gauges

$$\bar{c} \rightarrow \bar{c} + \text{const.} \quad (10.14)$$

is automatically encoded in the antighost equation (10.13) . Indeed, integrating expression (10.13) on space-time yields

$$\int d^4x \frac{\delta \Sigma}{\delta \bar{c}^a} = 0 , \quad (10.15)$$

which expresses in a functional form the shift symmetry (10.14). Equations (10.13), (10.15) imply that the antighost field can enter only through the combination $(\Omega^{a\mu} + \partial^\mu \bar{c}^a)$, forbidding the appearance of the counterterm $\bar{c}^a c^a$. As a consequence, the local operator $A_\mu^a A^{a\mu}$ does not mix with $\bar{c}^a c^a$ in linear α -gauges.

Let us also display, for further use, the quantum numbers of all fields and sources entering the action Σ

	A_μ	c	$\bar{c}$	b	λ	J	Ω	L
dim .	1	0	2	2	2	2	3	4
ghostnumber	0	1	-1	0	-1	0	-1	-2

Table 10.1:

10.2.2 Algebraic characterization of the general local invariant counterterm.

In order to characterize the most general local invariant counterterm which can be freely added to all orders of perturbation theory [59], we perturb the classical action Σ by adding an arbitrary integrated local polynomial Σ^{count} in the fields and external sources of dimension bounded by four and with zero ghost number, and we require that the perturbed action $(\Sigma + \eta \Sigma^{\text{count}})$ satisfies the same Ward identities and constraints as Σ to the first order in the perturbation parameter η , i.e.

$$\begin{aligned} \mathcal{S}(\Sigma + \eta \Sigma^{\text{count}}) &= 0 + O(\eta^2) , \\ \frac{\delta(\Sigma + \eta \Sigma^{\text{count}})}{\delta b^a} &= \partial^\mu A_\mu^a + \alpha b^a + O(\eta^2) , \\ \left(\frac{\delta}{\delta \bar{c}^a} + \partial_\mu \frac{\delta}{\delta \Omega_\mu^a} \right) (\Sigma + \eta \Sigma^{\text{count}}) &= 0 + O(\eta^2) . \end{aligned} \quad (10.16)$$

This amounts to impose the following conditions on Σ^{count}

$$\mathcal{B}_\Sigma \Sigma^{\text{count}} = 0, \quad (10.17)$$

and

$$\frac{\delta \Sigma^{\text{count}}}{\delta b^a} = 0, \quad (10.18)$$

$$\frac{\delta \Sigma^{\text{count}}}{\delta \bar{c}^a} + \partial_\mu \frac{\delta \Sigma^{\text{count}}}{\delta \Omega_\mu^a} = 0, \quad (10.19)$$

where $\mathcal{B}_\Sigma$ is the nilpotent linearized operator

$$\mathcal{B}_\Sigma = \int d^4x \left(\frac{\delta \Sigma}{\delta A_\mu^a} \frac{\delta}{\delta \Omega^{a\mu}} + \frac{\delta \Sigma}{\delta \Omega^{a\mu}} \frac{\delta}{\delta A_\mu^a} + \frac{\delta \Sigma}{\delta c^a} \frac{\delta}{\delta L^a} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta}{\delta c^a} + b^a \frac{\delta}{\delta \bar{c}^a} + J \frac{\delta}{\delta \lambda} \right), \quad (10.20)$$

$$\mathcal{B}_\Sigma \mathcal{B}_\Sigma = 0. \quad (10.21)$$

Taking into account that (J, λ) form a BRST doublet, from the general results on the cohomology of Yang-Mills theories [60] it turns out that the external sources (J, λ) can contribute only through terms which can be expressed as pure $\mathcal{B}_\Sigma$ -variations. It follows thus that the invariant local counterterm Σ^{count} can be parametrized as

$$\Sigma^{\text{count}} = -\frac{\sigma}{4} \int d^4x F_{\mu\nu}^a F^{a\mu\nu} + \mathcal{B}_\Sigma \Delta^{-1}, \quad (10.22)$$

where σ is a free parameter and Δ^{-1} is the most general local polynomial with dimension 4 and ghost number -1 , given by

$$\begin{aligned} \Delta^{-1} = \int d^4x & \left(a_1 \Omega_\mu^a A^{a\mu} + a_2 L^a c^a + a_3 \partial_\mu \bar{c}^a A^{a\mu} + \frac{a_4}{2} g f_{abc} \bar{c}^a \bar{c}^b c^c \right. \\ & \left. + a_5 b^a \bar{c}^a + a_6 \frac{\lambda}{2} A^{a\mu} A_\mu^a + a_7 \alpha \lambda \bar{c}^a c^a + a_8 \frac{\xi}{2} \lambda J \right), \end{aligned} \quad (10.23)$$

with $a_1, \dots, a_8$ arbitrary parameters. From the conditions (10.18), (10.19) it follows that

$$a_3 = a_1, \quad a_4 = a_5 = a_7 = 0, \quad (10.24)$$

so that Δ^{-1} reduces to

$$\Delta^{-1} = \int d^4x \left(a_1 (\Omega_\mu^a + \partial_\mu \bar{c}^a) A^{a\mu} + a_2 L^a c^a + a_6 \frac{\lambda}{2} A^{a\mu} A_\mu^a + a_8 \frac{\xi}{2} \lambda J \right). \quad (10.25)$$

Notice that the vanishing of the coefficient a_7 implies the absence of the counterterm $J \bar{c}^a c^a$. As already underlined, this ensures that the operator $A^{a\mu} A_\mu^a$ does not mix with the ghost operator $\bar{c}^a c^a$. Therefore, for the final form of the invariant counterterm one obtains:

$$\begin{aligned} \Sigma^{\text{count}} = \int d^4x & \left(-\frac{(\sigma + 4a_1)}{4} F_{\mu\nu}^a F^{a\mu\nu} + a_1 \partial_\mu A_\nu^a F^{a\mu\nu} + a_2 \Omega_\mu^a (D^\mu c)^a + a_2 \partial_\mu \bar{c}^a (D^\mu c)^a \right. \\ & + a_1 \Omega_\mu^a (\partial^\mu c)^a - a_1 \bar{c}^a \partial^2 c^a + \frac{1}{2} (2a_1 + a_6) J A^a{}_\nu A^{a\nu} + (a_1 + a_6 - a_2) \lambda \partial_\mu c^a A^{a\mu} \\ & \left. - \frac{a_2}{2} g f_{abc} L^a c^b c^c + \frac{a_8}{2} \xi J^2 \right). \end{aligned} \quad (10.26)$$

It remains to discuss the stability of the classical action [59], i.e. to check that Σ^{count} can be reabsorbed in the classical action Σ by means of a multiplicative renormalization of the coupling constant g , the parameters α and ξ , the fields $\{\phi = A, c, \bar{c}, b\}$ and the sources $\{\Phi = J, \lambda, L, \Omega\}$, namely

$$\Sigma(g, \xi, \alpha, \phi, \Phi) + \eta \Sigma^{\text{count}} = \Sigma(g_0, \xi_0, \alpha_0, \phi_0, \Phi_0) + O(\eta^2) , \quad (10.27)$$

with the bare fields and parameters defined as

$$\begin{aligned} A_{0\mu}^a &= Z_A^{1/2} A_\mu^a , \quad \Omega_{0\mu}^a = Z_\Omega \Omega_\mu^a , \quad g_0 = Z_g g , \\ c_0^a &= Z_c^{1/2} c^a , \quad L_0^a = Z_L L^a , \quad \alpha_0 = Z_\alpha \alpha , \\ \bar{c}_0^a &= Z_{\bar{c}}^{1/2} \bar{c}^a , \quad J_0 = Z_J J , \quad \xi_0 = Z_\xi \xi , \\ b_0^a &= Z_b^{1/2} b^a , \quad \lambda_0 = Z_\lambda \lambda . \end{aligned} \quad (10.28)$$

The parameters σ , a_1 , a_2 , a_6 , a_8 turn out to be related to the renormalization of the gauge coupling constant g , of A_μ^a , c^a , J , λ , and ξ , according to

$$\begin{aligned} Z_g &= 1 - \eta \frac{\sigma}{2} , \\ Z_A^{1/2} &= 1 + \eta \left(\frac{\sigma}{2} + a_1 \right) , \\ Z_c^{1/2} &= 1 - \eta \left(\frac{a_1 + a_2}{2} \right) , \\ Z_J &= 1 + \eta (a_6 - \sigma) , \\ Z_\lambda &= 1 + \eta \left(a_6 + \frac{a_1 - a_2 - \sigma}{2} \right) , \\ Z_\xi &= 1 + \eta (a_8 - 2a_6 + 2\sigma) . \end{aligned} \quad (10.29)$$

Concerning the other fields and the sources Ω_μ^a , L^a , it can be verified that they are renormalized as

$$\begin{aligned} Z_{\bar{c}} &= Z_c , \quad Z_b = Z_A^{-1} , \quad Z_\Omega = Z_c^{1/2} \\ Z_L &= Z_A^{1/2} , \quad Z_\alpha = Z_A . \end{aligned} \quad (10.30)$$

This completes the proof of the multiplicative renormalizability of the local composite operator A_μ^2 in linear covariant gauges. Finally, it is useful to observe that, from eqs.(10.29), one has

$$Z_\lambda = Z_J Z_c^{1/2} Z_A^{1/2} , \quad (10.31)$$

from which it follows that the anomalous dimension of A_μ^2 turns out to be related to that of the composite operator $A_\mu^a \partial^\mu c^a$

$$\gamma_{A\partial c} = \gamma_{A^2} + \gamma_c + \gamma_A , \quad (10.32)$$

where γ_c , γ_A , γ_{A^2} , and $\gamma_{A\partial c}$ are the anomalous dimensions of the Faddeev-Popov ghost c^a , of the gauge field A_μ^a , of the operator A_μ^2 , and of the composite operator $A_\mu^a \partial^\mu c^a$, which are defined as

$$\gamma_c = \mu \partial_\mu \ln Z_c^{1/2} , \quad \gamma_A = \mu \partial_\mu \ln Z_A^{1/2} , \quad \gamma_{A^2} = \mu \partial_\mu \ln Z_J , \quad \gamma_{A\partial c} = \mu \partial_\mu \ln Z_\lambda , \quad (10.33)$$

where μ is the renormalization scale. As expected, property (10.32) relies on the fact that $A_\mu^a \partial^\mu c^a$ is the BRST variation of $\frac{1}{2} A_\mu^2$, i.e.

$$s \frac{A^{a\mu} A_\mu^a}{2} = -A_\mu^a \partial^\mu c^a. \quad (10.34)$$

Although we did not consider matter fields in the previous analysis, it can be checked that the renormalizability of A_μ^2 and the relation (10.32) remain unchanged if matter fields are included.

10.3 Calculation of the two-loop anomalous dimension of A_μ^2 .

We now turn to the computation of the anomalous dimension of A_μ^2 in an arbitrary linear gauge. The method exploits the lack of mixing in the linear covariant gauges between A_μ^2 and the other dimension two Lorentz scalar zero ghost number operator $\bar{c}^a c^a$, which we have already noted. For instance, in the Curci-Ferrari gauge although both operators mix there is a combination, $\mathcal{O} = \frac{1}{2} A_\mu^2 + \alpha \bar{c}^a c^a$, which remains multiplicatively renormalizable. Prior to the proof of [153] that the anomalous dimension of $\mathcal{O}$ was related to the β -function and the gluon anomalous dimension, $\gamma_{\mathcal{O}}(a)$ was explicitly computed at three loops in $\overline{\text{MS}}$ in [87]. That method involved substituting the operator in a ghost two-point function with a non-zero momentum flowing through the operator itself and one external ghost momentum nullified. This configuration allowed for the application of the MINCER algorithm, [203], written in the symbolic manipulation language FORM, [156, 204]. A ghost two-point function was chosen to avoid the appearance of spurious infrared infinities which would arise for this momentum configuration if the external legs were gluons. To determine $\gamma_{A^2}(a)$ in the linear gauges we are forced into the same approach as [87] due to the infrared issue with gluon external legs. Hence, we have renormalized the momentum space Green's function $\langle c^a(p) \frac{1}{2} [A_\mu^2](-p) \bar{c}^b(0) \rangle$ where p is the external momentum. Clearly, this has no tree term and therefore to deduce $\gamma_{A^2}(a)$ at n -loops requires renormalizing the Green's function at $(n+1)$ -loops as the one-loop term corresponds to the tree term of $\langle A_\mu^a(p) \frac{1}{2} [A_\mu^2](-p) A_\nu^b(0) \rangle$. This is evident, for example, by drawing one- and two-loop diagrams for the various Green's functions based on the interactions of the Yang-Mills action, eq.(10.3). Since the MINCER algorithm currently only extracts the simple poles in ϵ in dimensional regularization to three loops, where $d = 4 - 2\epsilon$, this means we have only computed $\gamma_{A^2}(a)$ to two loops. Though this will be sufficient to deduce the effective potential of A_μ^2 to one-loop. The Feynman diagrams for our Green's function are generated with QGRAF, [181], and converted into FORM input notation, [87]. At one-loop there is one diagram which plays the role of the tree diagram and at two loops there are 15 diagrams. The bulk of the calculation, however, is in the evaluation of the 314 three-loop graphs. Since there is no operator mixing we can apply the rescaling technique of [88] for automatic multiloop computations to find the renormalization constant Z_{A^2} . From this we deduce

$$\begin{aligned} \gamma_{A^2}(a) = & [(35 + 3\alpha) C_A - 16 T_F N_f] \frac{a}{6} \\ & + [(449 + 33\alpha + 18\alpha^2) C_A^2 - 280 C_A T_F N_f - 192 C_F T_F N_f] \frac{a^2}{24} + O(a^3) \end{aligned} \quad (10.35)$$

in the $\overline{\text{MS}}$ scheme where N_f is the number of quarks and the colour group Casimirs are defined by $\text{Tr}(T^a T^b) = T_F \delta^{ab}$, $T^a T^a = C_F I$ and $f^{acd} f^{bcd} = C_A \delta^{ab}$. In deriving (10.35) from the corresponding renormalization constant we have verified that the double pole in ϵ is correctly reproduced for all α . Moreover, (10.35) reduces to the Landau gauge expression of [174].

10.4 Conclusion.

We have investigated the renormalizability of the dimension two operator A_μ^2 in arbitrary covariant linear gauges in Yang-Mills theories, due to the possibility that it might condense and develop a non-zero vacuum expectation value. This would generalize to these gauges previous results obtained in the Landau gauge [34, 33, 175, 37, 38, 42, 197]. One feature of our analysis is that, unlike the Curci-Ferrari gauges [87, 199], the operator A_μ^2 does not mix with the other dimension two local composite operator $\bar{c}^a c^a$. This is a general feature of the linear covariant α -gauges, present also in the Landau gauge [153], $\alpha = 0$, which is a consequence of the additional shift symmetry in the antighost (10.14). Importantly the operator A_μ^2 can thus be treated in isolation as it does not require any ghost dependent operator.

Finally, we underline that the multiplicative renormalizability of A_μ^2 and the explicit knowledge of its anomalous dimension for all α , eq.(10.35), are central ingredients towards the construction of a renormalizable effective potential for studying the possible condensation of this operator and the related dynamical mass generation, as was carried out in the Landau [42, 197] and Curci-Ferrari [178] gauges.

Chapter 11

Dynamical gluon mass generation from $\langle A_\mu^2 \rangle$ in linear covariant gauges

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We construct the multiplicatively renormalizable effective potential for the mass dimension two local composite operator $A_\mu^a A^{\mu a}$ in linear covariant gauges. We show that the formation of $\langle A_\mu^a A^{\mu a} \rangle$ is energetically favoured and that the gluons acquire a dynamical mass due to this gluon condensate. We also discuss the gauge parameter independence of the resultant vacuum energy.

11.1 Introduction.

In a previous paper [205], we took the first step towards constructing a renormalizable effective potential for the local composite operator (LCO) $A_\mu^2 \equiv A_\mu^a A^{\mu a}$ in linear covariant gauges. It was shown within the algebraic renormalization formalism [59, 60] that A_μ^2 is multiplicatively renormalizable to all orders in perturbation theory. At the same time, the anomalous dimension of A_μ^2 was explicitly computed to three loops in the $\overline{\text{MS}}$ scheme as a function of the gauge fixing parameter, α , where $\alpha = 0$ corresponds to the Landau gauge. The computation exploited the fact that in linear covariant gauges the operator does not mix, for example, with ghost operators of dimension two with the same quantum numbers.

The operator A_μ^2 has recently received widespread interest in Yang-Mills theory in the Landau gauge. Its relevance has been advocated both from a theoretical point of view as well as from lattice simulations [33, 34, 175, 37, 38]. Analytic results in favour of a non-zero value for the condensate $\langle A_\mu^2 \rangle$ in the Landau gauge have been obtained recently, [42, 184]. Further, the inclusion of quarks has been considered in [197]. Motivated by the result of [87] it has been shown in [153] that A_μ^2 is multiplicatively renormalizable to all orders in the Landau gauge, but its anomalous dimension is given by a combination of the gauge beta function, $\beta(a)$, and the anomalous dimension, $\gamma_A(a)$, of the gluon field, according to the relation [87, 153]

$$\gamma_{A^2}(a) = - \left(\frac{\beta(a)}{a} + \gamma_A(a) \right), \quad a = \frac{g^2}{16\pi^2}. \quad (11.1)$$

An important consequence of the formation of the $\langle A_\mu^2 \rangle$ condensate in the Landau gauge is the dynamical generation of a gluon mass $m_{\text{gluon}} \approx 500\text{MeV}$ [42]. Lattice simulations of the $SU(2)$ gluon propagator in the Landau gauge report $m_{\text{gluon}} \approx 600\text{MeV}$, [48]. Gluon masses have also been extracted from lattice methods in the Laplacian [47, 43] and maximal Abelian gauges [49, 50]. Earlier the pairing of gluons was discussed in connection with mass generation as a result of the instability of the perturbative Yang-Mills vacuum [28, 29, 207, 31, 30, 201]. A dynamical gluon mass might also be important, for example, in connection with the glueball spectra, [202, 200]. A dimension two gluon condensate $\langle A_i^2 \rangle$ was already introduced in [200], where the Coulomb gauge was considered. Furthermore, a dynamical gluon mass is part of a certain criterion for confinement introduced by Kugo and Ojima, [177]. For a recent review see [20].

It is no coincidence that the Landau gauge is employed in the search for a gluon condensate of mass dimension two. As is well known, there does not exist a local, gauge-invariant operator of mass dimension two in Yang-Mills theories. However, a *non-local* gauge invariant dimension two operator can be constructed by minimizing A_μ^2 along each gauge orbit [110, 187, 143], $A_{\text{min}}^2 \equiv (VT)^{-1} \min_U \int d^4x (A_\mu^U)^2$ where VT is the space time volume and U is a generic $SU(N)$ transformation. This operator A_{min}^2 is related to the so-called fundamental modular region (FMR), which is the set of absolute minima of $(VT)^{-1} \int d^4x (A_\mu^U)^2$. In the Landau gauge, $\partial_\mu A^{a\mu} = 0$, so that A_{min}^2 and A_μ^2 coincide within the FMR. As such, a gauge invariant meaning can indeed be attached to $\langle A_\mu^2 \rangle$ in the Landau gauge, as it was also expressed in [42].

Another interesting property of the Landau gauge is that the operator A_μ^2 is BRST invariant on-shell. If one considered alternative gauges to the Landau gauge, one could search for a class of gauges in which the operator A_μ^2 can be generalized to a mass dimension two operator while maintaining the on-shell BRST invariance. Doing so, one should consider a class of non-linear covariant gauges, which are the so-called Curci-Ferrari gauges [84, 85], where A_μ^2 is generalized to the mixed gluon-ghost operator $(\frac{1}{2}A_\mu^a A^{\mu a} + \alpha \bar{c}^a c^a)$ [83, 144]. The latter operator is indeed BRST invariant on-shell [83, 144], and has been proven to be multiplicatively renormalizable to all orders [199], and to give rise to a dynamical gluon mass in the Curci-Ferrari gauge [178]. Moreover, in [206], the physical meaning of $(\frac{1}{2}A_\mu^a A^{\mu a} + \alpha \bar{c}^a c^a)$ was discussed, based on on-shell BRST invariance.

In the maximal Abelian gauge, which is a renormalizable gauge in the continuum [71, 72], one should consider the operator $(\frac{1}{2}A_\mu^\beta A^{\mu\beta} + \xi \bar{c}^\beta c^\beta)$ where the group index β labels the off-diagonal generators of $SU(N)$ with $\beta = 1, \dots, N(N-1)$ and ξ is the gauge parameter of the maximal Abelian gauge. This operator also enjoys the property of being both BRST invariant on-shell [83, 144] and multiplicatively renormalizable to all orders [199, 179]. Although the effective potential for the condensate $\langle \frac{1}{2}A_\mu^\beta A^{\mu\beta} + \xi \bar{c}^\beta c^\beta \rangle$ has not yet been obtained, we expect it to have a non-vanishing vacuum expectation value, which would result in a dynamical mass for the off-diagonal gluons. This is based on the close similarity between the maximal Abelian gauge and the Curci-Ferrari gauge and hence the results of [178].

More commonly, the Landau gauge is a special case of the well known linear covariant gauges. Although the operator A_μ^2 is not even BRST invariant on-shell in these gauges, it is still renormalizable to any order in perturbation theory [205]. This is due to the fact that, thanks to the additional shift symmetry, $\bar{c} \rightarrow \bar{c} + \text{const}$, of the antighost in the linear covariant gauges, the composite operator A_μ^2 does not mix into the dimension two ghost operator $\bar{c}c$. In this article, we will construct the effective potential for the dimension two condensate $\langle A_\mu^2 \rangle$ in linear gauges and show that a non-vanishing value of $\langle A_\mu^2 \rangle$ is energetically favourable, resulting in dynamical gluon mass generation.

The paper is organized as follows. In section 11.2, we briefly review the local composite operators formalism and explicitly calculate the one-loop effective potential. In section 11.3, we discuss the gauge parameter independence of the vacuum energy which requires an extension of the LCO formalism. The behaviour of the gluon propagator is discussed briefly in section 11.4 whilst we provide concluding

comments in section 11.5.

11.2 LCO formalism and effective potential for A_μ^2 .

11.2.1 Construction of a renormalizable effective action for A_μ^2 .

We begin with the Yang-Mills action in linear covariant gauges

$$\begin{aligned} S &= S_{YM} + S_{GF+FP} \\ &= -\frac{1}{4} \int d^4x F_{\mu\nu}^a F^{a\mu\nu} + \int d^4x \left(b^a \partial_\mu A^{a\mu} + \frac{\alpha}{2} b^a b^a + \bar{c}^a \partial^\mu D_\mu^{ab} c^b \right), \end{aligned} \quad (11.2)$$

where

$$D_\mu^{ab} \equiv \partial_\mu \delta^{ab} - g f^{abc} A_\mu^c, \quad (11.3)$$

is the covariant derivate in the adjoint representation. In order to study the local composite operator A_μ^2 , we introduce it into the action by means of a BRST doublet [59] of external sources (J, λ) , namely

$$S_J = s \int d^4x \left(\frac{1}{2} \lambda A_\mu^2 + \frac{\zeta}{2} \lambda J \right) = \int d^4x \left(\frac{1}{2} J A_\mu^2 + \lambda A_\mu^a \partial^\mu c^a + \frac{\zeta}{2} J^2 \right), \quad (11.4)$$

where s denotes the BRST nilpotent operator acting as

$$\begin{aligned} s A_\mu^a &= -D_\mu^{ab} c^b, \\ s c^a &= \frac{1}{2} g f^{abc} c^b c^c, \\ s \bar{c}^a &= b^a, \\ s b^a &= 0, \\ s \lambda &= J, \\ s J &= 0. \end{aligned} \quad (11.5)$$

According to the local composite operator technique [42, 23, 24, 176], the dimensionless parameter ζ is needed to account for the divergences present in the vacuum Green function $\langle A_\mu^2(x) A_\nu^2(y) \rangle$, which turn out to be proportional to J^2 . As is apparent from the expressions (11.2) and (11.4), the action $(S_{YM} + S_{GF+FP} + S_J)$ is BRST invariant

$$s(S_{YM} + S_{GF+FP} + S_J) = 0. \quad (11.6)$$

As was shown in [205], the action $(S_{YM} + S_{GF+FP} + S_J)$ enjoys the property of being multiplicatively renormalizable to all orders of perturbation theory.

To obtain the effective potential, we set the source λ to zero and consider the renormalized generating functional

$$\exp -i\mathcal{W}(J) = \int [D\varphi] \exp iS(J), \quad (11.7)$$

with

$$S(J) = S_{YM} + S_{GF+FP} + S_{CT} + \int d^4x \left(Z_2 J \frac{A_\mu^2}{2} + (\zeta + \delta\zeta) \frac{J^2}{2} \right), \quad (11.8)$$

where φ denotes the relevant fields and S_{CT} is the usual counterterm contribution. Also, $\delta\zeta$ is the counterterm accounting for the divergences proportional to J^2 . The bare quantities are given by [205]

$$\begin{aligned} A_o^{\mu a} &= Z_A^{1/2} A^{\mu a}, & c_o^a &= Z_c^{1/2} c^a, & \bar{c}_o^a &= Z_c^{1/2} \bar{c}^a, & b_o^a &= Z_A^{-1/2} b^a, \\ g_o &= Z_g g, & \alpha_o &= Z_A \alpha, & \zeta_o &= Z_\zeta \zeta, & J_o &= Z_J J, \end{aligned} \quad (11.9)$$

where $Z_\zeta \zeta = \zeta + \delta\zeta$ and $Z_J = \frac{Z_2}{Z_A}$. The functional $\mathcal{W}(J)$ obeys the renormalization group equation (RGE)

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g^2) \frac{\partial}{\partial g^2} + \alpha \gamma_\alpha(g^2) \frac{\partial}{\partial \alpha} - \gamma_{A^2}(g^2) \int d^4 x J \frac{\delta}{\delta J} + \eta(g^2, \zeta) \frac{\partial}{\partial \zeta} \right) \mathcal{W}(J) = 0, \quad (11.10)$$

where

$$\begin{aligned} \beta(g^2) &= \mu \frac{\partial}{\partial \mu} g^2, \\ \gamma_\alpha(g^2) &= \mu \frac{\partial}{\partial \mu} \ln \alpha = \mu \frac{\partial}{\partial \mu} \ln Z_A^{-1} = -2\gamma_A(g^2), \\ \gamma_{A^2}(g^2) &= \mu \frac{\partial}{\partial \mu} \ln Z_J, \\ \eta(g^2, \zeta) &= \mu \frac{\partial}{\partial \mu} \zeta. \end{aligned} \quad (11.11)$$

From the bare Lagrangian, we infer that

$$\zeta_o J_o^2 = \mu^{-\varepsilon} (\zeta + \delta\zeta) J^2, \quad (11.12)$$

where we will use dimensional regularization throughout with the convention that $d = 4 - \varepsilon$. Hence

$$\mu \frac{\partial}{\partial \mu} \zeta = \eta(g^2, \zeta) = 2\gamma_{A^2}(g^2) \zeta + \delta(g^2, \alpha), \quad (11.13)$$

with

$$\delta(g^2, \alpha) = \left(\varepsilon + 2\gamma_{A^2}(g^2, \alpha) - \beta(g^2) \frac{\partial}{\partial g^2} - \alpha \gamma_\alpha(g^2, \alpha) \frac{\partial}{\partial \alpha} \right) \delta\zeta. \quad (11.14)$$

Now, we are faced with the problem of the hitherto arbitrary parameter ζ . As explained in [42, 23, 24, 176], setting $\zeta = 0$ would give rise to an inhomogeneous RGE for $\mathcal{W}(J)$

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g^2) \frac{\partial}{\partial g^2} + \alpha \gamma_\alpha(g^2) \frac{\partial}{\partial \alpha} - \gamma_{A^2}(g^2) \int d^4 x J \frac{\delta}{\delta J} \right) \mathcal{W}(J) = \delta(g^2, \alpha) \int d^4 x \frac{J^2}{2}, \quad (11.15)$$

and a non-linear RGE for the effective action Γ for the composite operator A_μ^2 . This problem can be overcome by making ζ a function of g^2 and α so that, if g^2 runs according to $\beta(g^2)$ and α according to $\gamma_\alpha(g^2)$, $\zeta(g^2, \alpha)$ will run according to (11.13). This is accomplished by setting ζ equal to the solution of the differential equation

$$\left(\beta(g^2) \frac{\partial}{\partial g^2} + \alpha \gamma_\alpha(g^2, \alpha) \frac{\partial}{\partial \alpha} \right) \zeta(g^2, \alpha) = 2\gamma_{A^2}(g^2) \zeta(g^2, \alpha) + \delta(g^2, \alpha). \quad (11.16)$$

Since $\zeta(g^2, \alpha)$ now automatically runs according to its RGE, $\mathcal{W}(J)$ obeys the homogeneous renormalization group equation

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g^2) \frac{\partial}{\partial g^2} + \alpha \gamma_\alpha(g^2) \frac{\partial}{\partial \alpha} - \gamma_{A^2}(g^2) \int d^4 x J \frac{\delta}{\delta J} \right) \mathcal{W}(J) = 0. \quad (11.17)$$

The final step in the formal construction of the effective potential for $\langle A_\mu^2 \rangle$ is the removal of the J^2 terms from the Lagrangian by means of a renormalized Hubbard-Stratonovich transformation. By this procedure, the energy interpretation of the effective action is made explicit again and the conventional 1PI machinery applies. We insert unity written as

$$1 = \frac{1}{N} \int [D\sigma] \exp \left[i \int d^4x \left(-\frac{1}{2Z_\zeta \zeta} \left(\frac{\sigma}{g} - Z_2 \frac{A_\mu^2}{2} - Z_\zeta \zeta J \right)^2 \right) \right], \quad (11.18)$$

with N the appropriate normalization factor, in (11.7) to arrive at the Lagrangian

$$\mathcal{L}(A_\mu, \sigma) = -\frac{1}{4} F_{\mu\nu}^2 + \mathcal{L}_{GF+FP} + \mathcal{L}_{CT} - \frac{\sigma^2}{2g^2 Z_\zeta \zeta} + \frac{1}{2} \frac{Z_2}{g^2 Z_\zeta \zeta} g \sigma A_\mu^2 - \frac{1}{8} \frac{Z_2^2}{Z_\zeta \zeta} (A_\mu^2)^2, \quad (11.19)$$

while

$$\exp -i\mathcal{W}(J) = \int [D\varphi] \exp iS_\sigma(J), \quad (11.20)$$

$$S_\sigma(J) = \int d^4x \left(\mathcal{L}(A_\mu, \sigma) + J \frac{\sigma}{g} \right). \quad (11.21)$$

Now, the source J appears as a linear source term for $\frac{\sigma}{g}$. From (11.7) and (11.20), one has the following identification

$$\left. \frac{\delta \mathcal{W}(J)}{\delta J} \right|_{J=0} = - \left\langle \frac{A_\mu^2}{2} \right\rangle = - \left\langle \frac{\sigma}{g} \right\rangle, \quad (11.22)$$

where we will not write the renormalization factors from now on. This equation states that the gauge condensate $\langle A_\mu^2 \rangle$ is related to the expectation value of the field σ , evaluated with the new action, $\int d^4x \mathcal{L}(A_\mu, \sigma)$, of (11.19).

Although we have not considered the contribution from (massless) quark fields in the previous analysis, it can be checked that the results remain unchanged if matter fields are included.

11.2.2 Explicit calculation of the one-loop effective potential.

Firstly, we will determine the renormalization group function $\delta(g^2, \alpha)$ as defined in (11.14). All the following results will be within the $\overline{\text{MS}}$ scheme. The value for $\beta(g^2)$ can be found in the literature. In d dimensions, one has

$$\begin{aligned} \beta(g^2) &= -\varepsilon g^2 - 2(\beta_0 g^4 + \beta_1 g^6 + O(g^8)), \\ \beta_0 &= \frac{1}{16\pi^2} \left(\frac{11}{3} C_A - \frac{4}{3} T_F N_f \right), \\ \beta_1 &= \frac{1}{(16\pi^2)^2} \left(\frac{34}{3} C_A^2 - 4 C_F T_F N_f - \frac{20}{3} C_A T_F N_f \right). \end{aligned} \quad (11.23)$$

where the Casimirs of the colour group are defined by $\text{Tr}(T^a T^b) = T_F \delta^{ab}$, $T^a T^a = C_F I$, $f^{acd} f^{bcd} = C_A \delta^{ab}$ and N_f is the number of quark flavours. For $\gamma_\alpha(g^2)$, we use the relation $\gamma_\alpha(g^2) = -2\gamma_A(g^2)$. The anomalous dimension $\gamma_A(g^2)$ of the gluon field in linear covariant gauges was calculated at three loops

in $\overline{\text{MS}}$ in [88]. Adapting that result to our convention, the anomalous dimension of the gauge parameter is

$$\begin{aligned}\gamma_\alpha(g^2) &= a_0 g^2 + a_1 g^4 + O(g^6), \\ a_0 &= \frac{1}{16\pi^2} \left(C_A \left(\frac{13}{3} - \alpha \right) - \frac{8}{3} T_F N_f \right), \\ a_1 &= \frac{1}{(16\pi^2)^2} \left(C_A^2 \left(\frac{59}{4} - \frac{11}{4} \alpha - \frac{1}{2} \alpha^2 \right) - 10 C_A N_f T_F - 8 C_F N_f T_F \right).\end{aligned}\quad (11.24)$$

The anomalous dimension, $\gamma_{A^2}(g^2)$, of the composite operator A_μ^2 was calculated in [205] and reads

$$\begin{aligned}\gamma_{A^2}(g^2) &= \gamma_0 g^2 + \gamma_1 g^4 + O(g^6), \\ \gamma_0 &= \frac{1}{6} \frac{1}{16\pi^2} [(35 + 3\alpha) C_A - 16 T_F N_f], \\ \gamma_1 &= \frac{1}{24} \frac{1}{(16\pi^2)^2} [(449 + 33\alpha + 18\alpha^2) C_A^2 - 280 C_A T_F N_f - 192 C_F T_F N_f].\end{aligned}\quad (11.25)$$

In order to determine $\delta(g^2, \alpha)$, we still require the counterterm $\delta\zeta$. In principle, this can be directly calculated from the divergences in $\mathcal{W}(J)$ when the propagator for a gluon with mass J is used. However, a less cumbersome way to compute $\delta\zeta$ was described in [197]. It is based on the fact that the divergences arise in the $O(J^2)$ term and therefore that part of the Green's function which contains these divergences is equivalent to the Green's function with a double insertion of the JA_μ^2 operator. More specifically, one has two external J insertions with a non-zero momentum flowing into one insertion where the only internal couplings are those of the usual QCD action. Moreover, one does not require massive propagators but instead can use massless fields which simplifies the calculation. Therefore one is reduced to computing a massless two-point function for which the MINCER algorithm, [203], was designed. We used the version written in FORM, [204, 156], where the Feynman diagrams are generated by QGRAF, [181], to determine the divergence structure to three loops. Although we only require the result to two loops the extra loop evaluation in fact acts as a non-trivial check on the two-loop result. This is because the emergence of the correct double and triple poles in ε at three loops, in a way which is consistent with the renormalization group, verifies that the single and double poles of the two-loop expression for $\delta\zeta$ are correct. We found

$$\begin{aligned}\delta\zeta &= \frac{2}{\varepsilon} \frac{N_A}{16\pi^2} \left(-\frac{3}{2} - \frac{\alpha^2}{2} \right) + \frac{N_A g^2}{(16\pi^2)^2} \left[\frac{4}{\varepsilon^2} \left(C_A \left(\frac{35}{8} + \frac{3}{8} \alpha + \frac{3}{8} \alpha^2 + \frac{3}{8} \alpha^3 \right) - 2 T_F N_f \right) \right. \\ &\quad \left. + \frac{2}{\varepsilon} \left(C_A \left(-\frac{139}{12} - \frac{5}{8} \alpha - \frac{1}{2} \alpha^2 - \frac{1}{8} \alpha^3 \right) + \frac{8}{3} T_F N_f \right) \right] + O(g^4),\end{aligned}\quad (11.26)$$

where N_A is the dimension of the adjoint representation of the colour group. Assembling our results leads to

$$\begin{aligned}\delta(g^2, \alpha) &= \delta_0 + \delta_1 g^2 + O(g^4), \\ \delta_0 &= \frac{N_A}{16\pi^2} (-3 - \alpha^2), \\ \delta_1 &= \frac{1}{6} \frac{N_A}{(16\pi^2)^2} (C_A (-278 - 15\alpha - 12\alpha^2 - 3\alpha^3) + 64 T_F N_f).\end{aligned}\quad (11.27)$$

As a check we see that $\delta(g^2, \alpha)$ contains no poles for $\varepsilon \rightarrow 0$. Further, the expressions (11.27) lead to the same results which were obtained earlier in the case of the Landau gauge ($\alpha = 0$), as can be inferred from [42] without quarks and [197] with quarks.

From the renormalization group functions (11.23), (11.24), (11.25) and (11.27), it is easy to see that the equation (11.16) can be solved for by expanding $\zeta(g^2, \alpha)$ in a Laurent series as

$$\zeta(g^2, \alpha) = \frac{\zeta_0(\alpha)}{g^2} + \zeta_1(\alpha) + O(g^2). \quad (11.28)$$

Substituting this expression in equation (11.16), we obtain

$$2\beta_0\zeta_0 + \alpha a_0 \frac{\partial \zeta_0}{\partial \alpha} = 2\gamma_0\zeta_0 + \delta_0, \quad (11.29)$$

$$2\beta_1\zeta_0 + \alpha a_0 \frac{\partial \zeta_1}{\partial \alpha} + \alpha a_1 \frac{\partial \zeta_0}{\partial \alpha} = 2\gamma_0\zeta_1 + 2\gamma_1\zeta_0 + \delta_1. \quad (11.30)$$

Thus (11.29) gives

$$\zeta_0(\alpha) = \frac{2\alpha C_0 + 3(78 - 26\alpha^2 + 3\alpha^3 + 18\alpha \ln|\alpha|) C_A N_A + 48(\alpha^2 - 3) N_A N_f T_F}{2((3\alpha - 13)C_A + 8N_f T_F)^2}, \quad (11.31)$$

with C_0 a constant of integration. As a consequence of the already rather complicated structure of ζ_0 , we will determine ζ_1 without quarks present corresponding to $N_f = 0$ since the expression for ζ_1 with $N_f \neq 0$ is several pages long. Using Mathematica, we find

$$\begin{aligned} \zeta_1(\alpha) = & -\frac{1}{1220736\pi^2(13-3\alpha)^4} \left(-1220736\pi^2\alpha^{35/13} |-13+3\alpha|^{4/13} C_1 \right. \\ & + 12716C_0\alpha^2 \left(-442 - 132\alpha + 54\alpha^2 - 1287 \left(1 - \frac{3\alpha}{13} \right)^{4/13} \alpha_2 F_1 \left[\frac{4}{13}, \frac{4}{13}, \frac{17}{13}, \frac{3\alpha}{13} \right] \right) C_A \\ & + \left(1697175909 \left(1 - \frac{3\alpha}{13} \right)^{4/13} \alpha^3 {}_3F_2 \left[\frac{4}{13}, \frac{4}{13}, \frac{4}{13}, \frac{17}{13}, \frac{17}{13}, \frac{3\alpha}{13} \right] \right. \\ & + 3335904 \left(1 - \frac{3\alpha}{13} \right)^{4/13} \alpha^4 {}_3F_2 \left[\frac{17}{13}, \frac{17}{13}, \frac{17}{13}, \frac{30}{13}, \frac{30}{13}, \frac{3\alpha}{13} \right] \\ & + 17(-396870474 + 368850105\alpha - 48761440\alpha^2 + 2066214\alpha^3 + 1928718\alpha^4 \\ & - 1004751\alpha^5 + 60588\alpha^6 - 12894024 \left(1 - \frac{3\alpha}{13} \right)^{4/13} \alpha^2 {}_2F_1 \left[-\frac{9}{13}, -\frac{9}{13}, \frac{4}{13}, \frac{3\alpha}{13} \right] \\ & - 833976 \left(1 - \frac{3\alpha}{13} \right)^{4/13} \alpha^4 {}_2F_1 \left[\frac{4}{13}, \frac{17}{13}, \frac{30}{13}, \frac{3\alpha}{13} \right] - 8926632\alpha^2 \ln|\alpha| \\ & + 2059992\alpha^3 \ln|\alpha| + 833976 \left(1 - \frac{3\alpha}{13} \right)^{4/13} \alpha^4 {}_2F_1 \left[\frac{4}{13}, \frac{17}{13}, \frac{30}{13}, \frac{3\alpha}{13} \right] \ln|\alpha| \\ & \left. - 43758\alpha^3(-1961 + 702 \ln|\alpha|) \right) N_A, \end{aligned} \quad (11.32)$$

where C_1 is a constant of integration and the (generalized) hypergeometric function is

$${}_pF_q[a_1, \dots, a_p; b_1, \dots, b_q; z] = \sum_{k=0}^{+\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{z^k}{k!}, \quad (11.33)$$

where

$$(a)_k = a(a+1)\dots(a+k-1), \quad (11.34)$$

is the Pochhammer symbol. We note that $\zeta_0(\alpha = 0) = \frac{9N_A}{13C_A}$ and $\zeta_1(\alpha = 0) = \frac{161}{832} \frac{N_A}{\pi^2}$, which recovers the Landau gauge results of [42, 197]. Further, the constants of integration C_0 and C_1 do not enter the Landau gauge results.

From expression (11.19), we deduce that the tree level gluon mass is provided by

$$m^2 = \frac{g\sigma}{\zeta_0}, \quad (11.35)$$

while the one-loop effective potential becomes

$$\begin{aligned} V_1(\sigma) &= \frac{\sigma^2}{2\zeta_0} \left(1 - \frac{\zeta_1}{\zeta_0} g^2\right) + \frac{1}{2} \ln \det \left[\delta^{ab} \left(\delta^{\mu\nu} (\partial^2 + m^2) - \left(1 - \frac{1}{\alpha}\right) \partial^\mu \partial^\nu \right) \right] \\ &= \frac{\sigma^2}{2\zeta_0} \left(1 - \frac{\zeta_1}{\zeta_0} g^2\right) + \frac{N_A}{2} [(d-1) \text{tr} \ln (\partial^2 + m^2) + \text{tr} \ln (\partial^2 + \alpha m^2)] . \end{aligned} \quad (11.36)$$

In dimensional regularization and using the $\overline{\text{MS}}$ scheme, one finds

$$V_1(\sigma) = \frac{\sigma^2}{2\zeta_0} \left(1 - \frac{\zeta_1}{\zeta_0} g^2\right) + \frac{3N_A}{64\pi^2} \frac{g^2 \sigma^2}{\zeta_0^2} \left(-\frac{5}{6} + \ln \frac{g\sigma}{\zeta_0 \bar{\mu}^2}\right) + \frac{N_A}{64\pi^2} \frac{\alpha^2 g^2 \sigma^2}{\zeta_0^2} \left(-\frac{3}{2} + \ln \frac{\alpha g\sigma}{\zeta_0 \bar{\mu}^2}\right), \quad (11.37)$$

where $\bar{\mu}$ is the renormalization scale. It can be easily checked that the infinities in the effective potential cancel when the counterterms are included.

Next, we look for a non-trivial minimum of the effective potential, which amounts to solving the gap equation $\frac{dV}{d\sigma} = 0$. To avoid possibly large logarithms, we will set $\bar{\mu}^2 = m^2 = \frac{g\sigma}{\zeta_0}$ in the gap equation,

$$\begin{aligned} \frac{dV}{d\sigma} \Big|_{\bar{\mu}^2 = \frac{g\sigma}{\zeta_0}} &= \frac{\sigma}{\zeta_0} \left(1 - \frac{\zeta_1}{\zeta_0} g^2\right) + \frac{3N_A}{32\pi^2} \frac{g^2 \sigma}{\zeta_0^2} \left(-\frac{5}{6}\right) + \frac{3N_A}{64\pi^2} \frac{g^2 \sigma}{\zeta_0^2} \\ &+ \frac{N_A}{32\pi^2} \frac{\alpha^2 g^2 \sigma}{\zeta_0^2} \left(-\frac{3}{2} + \ln \alpha\right) + \frac{N_A}{64\pi^2} \frac{\alpha^2 g^2 \sigma}{\zeta_0^2} = 0, \end{aligned} \quad (11.38)$$

and use the RGE to sum leading logarithms. Defining $y \equiv \frac{g^2 N}{16\pi^2}$, we find as a solution of (11.38)

$$\sigma = 0 \text{ or } y = \frac{C_A \zeta_0}{16\pi^2 \zeta_1 + \frac{N_A}{2} (1 + \alpha^2 - \alpha^2 \ln |\alpha|)}. \quad (11.39)$$

The first solution corresponds to the trivial vacuum, while the second one leads to

$$m = \Lambda_{\overline{\text{MS}}} e^{\frac{3}{22y}}, \quad (11.40)$$

where the one-loop formula for the coupling constant

$$g^2(\bar{\mu}) = \frac{1}{\beta_0 \ln \frac{\bar{\mu}^2}{\Lambda_{\overline{\text{MS}}}^2}}, \quad (11.41)$$

was used. The vacuum energy is given by

$$E_{\text{vac}} = -\frac{1}{2} \frac{N_A}{64\pi^2} (3 + \alpha^2) m^4. \quad (11.42)$$

We now consider the numerical evaluation of our results and restrict ourselves to the colour group $SU(3)$. For $SU(N)$ one has $T_F = \frac{1}{2}$, $C_F = \frac{N^2-1}{2N}$, $C_A = N$ and $N_A = N^2 - 1$. For completeness, we quote the results for the Landau gauge $\alpha = 0$.

$$y^{\text{Landau}} = \frac{36}{187} \approx 0.193, \quad (11.43)$$

$$m^{\text{Landau}} = e^{\frac{17}{24}} \Lambda_{\overline{\text{MS}}} \approx 2.031 \Lambda_{\overline{\text{MS}}}, \quad (11.44)$$

$$E_{\text{vac}}^{\text{Landau}} = -\frac{3}{16\pi^2} e^{\frac{17}{6}} \Lambda_{\overline{\text{MS}}}^4 \approx -0.323 \Lambda_{\overline{\text{MS}}}^4. \quad (11.45)$$

The results for general α are displayed in the Figures 11.1-11.3.

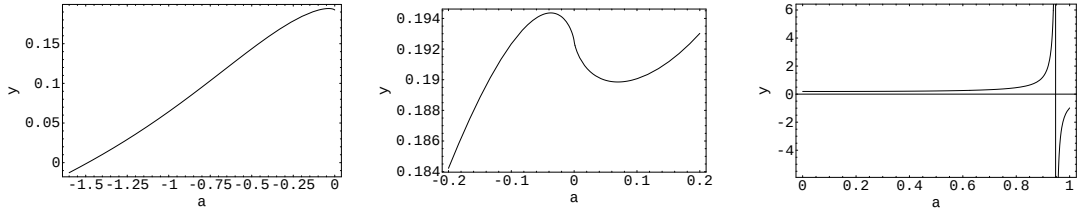


Figure 11.1: y as a function of α .

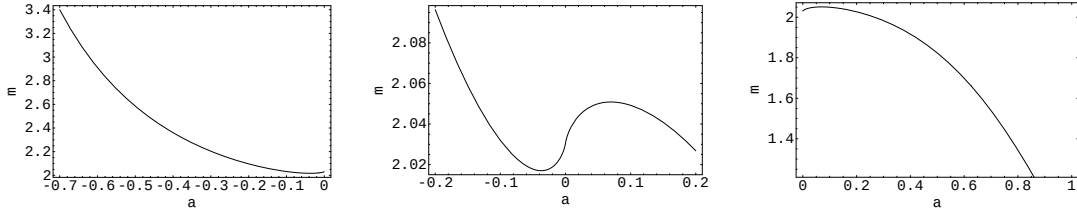


Figure 11.2: m as a function of α (in units $\Lambda_{\overline{\text{MS}}} = 1$).

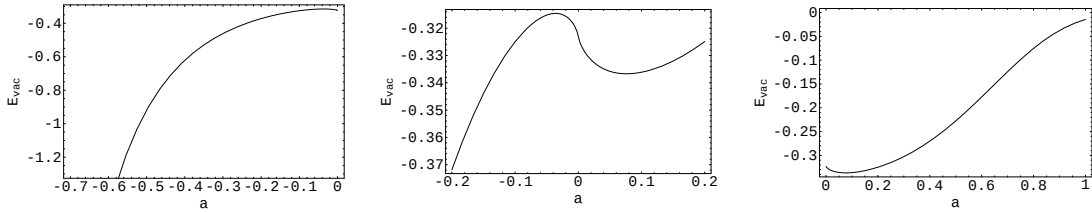


Figure 11.3: E_{vac} as a function of α (in units $\Lambda_{\overline{\text{MS}}} = 1$).

For the moment, we have set $C_0 = C_1 = 0$. Evidently, y should certainly be positive and also relatively small to have a sensible expansion. Hence, we conclude from Figure 11.1 that we should restrict the range of values for α further. We also see that m becomes rapidly larger and E_{vac} becomes rapidly more and more negative as α gets more negative. A more urgent problem is the fact that the vacuum energy E_{vac} depends on the gauge parameter α . Since E_{vac} is a physical quantity, it should be independent on the gauge parameter α . In the next section, we shall give a detailed account of this gauge parameter dependence. We shall see that it is related to the impossibility of evaluating the effective potential to arbitrary high loop orders. Further, we shall provide a simple way to circumvent this problem and obtain a vacuum energy which is independent of α .

11.3 Investigation of the gauge parameter dependence.

One possible explanation as to why E_{vac} depends on α could reside in the values of the constants of integration C_0 and C_1 we have chosen. With another choice for these constants, it could be that E_{vac} does not depend in α , or equivalently $E_{\text{vac}} = E_{\text{vac}}^{\text{Landau}}$. This can be investigated by considering the expression (11.42) for $E_{\text{vac}}(\alpha, C_0, C_1)$. In order to have the same E_{vac} for each value of α , we should solve the following equation

$$\begin{aligned} \frac{dE_{\text{vac}}}{d\alpha} = 0 &\Leftrightarrow 2\alpha m^4 + 4(\alpha^2 + 3)m^3 \frac{dm}{d\alpha} = 0 \\ &\Leftrightarrow \alpha - \frac{3}{11y^2}(3 + \alpha^2) \left(\frac{\partial y}{\partial \alpha} + \frac{\partial y}{\partial \zeta_0} \frac{\partial \zeta_0}{\partial \alpha} + \frac{\partial y}{\partial \zeta_1} \frac{\partial \zeta_1}{\partial \alpha} \right) = 0, \end{aligned} \quad (11.46)$$

in terms of C_0 and C_1 . However, the solutions of this equation depend on α , and this is not allowed since C_0 and C_1 should be α independent constants. This means that the α -dependence of E_{vac} cannot be eliminated by a suitable choice of C_0 and C_1 .

11.3.1 BRST symmetry and gauge parameter independence.

Let us now turn to a more general analysis. Consider again the generating functional (11.20). We have the following identification, ignoring the overall normalization factors

$$\begin{aligned} \exp -i\mathcal{W}(J) &= \int [D\varphi] \exp iS_\sigma(J) \\ &= \frac{1}{N} \int [D\varphi D\sigma] \exp i \left[S(J) + \int d^4x \left(-\frac{1}{2\zeta} \left(\frac{\sigma}{g} - \frac{A_\mu^2}{2} - \zeta J \right)^2 \right) \right], \end{aligned} \quad (11.47)$$

where $S(J)$ and $S_\sigma(J)$ are given respectively by (11.8), and (11.21). Since

$$\frac{d}{d\alpha} \frac{1}{N} \int [D\sigma] \exp \left[i \int d^4x \left(-\frac{1}{2\zeta} \left(\frac{\sigma}{g} - \frac{A_\mu^2}{2} - \zeta J \right)^2 \right) \right] = \frac{d}{d\alpha} 1 = 0, \quad (11.48)$$

we find

$$-\frac{d\mathcal{W}(J)}{d\alpha} = \left\langle s \int d^4x \left(\frac{\bar{c}b}{2} \right) \right\rangle_{J=0} + \text{terms proportional to } J, \quad (11.49)$$

which follows by noticing that

$$\begin{aligned} \frac{dS(J)}{d\alpha} &= \int d^4x \left(\frac{b^a b^a}{2} + \frac{\partial \zeta}{\partial \alpha} \frac{J^2}{2} \right) \\ &= s \int d^4x \left(\frac{\bar{c}b}{2} \right) + \text{terms proportional to } J. \end{aligned} \quad (11.50)$$

We see that the first term in the right hand side of (11.50) is an exact BRST variation. As such, its vacuum expectation value vanishes. This is the usual argument to prove the gauge parameter independence in the BRST framework [59]. Of course, this is based on the assumption that the BRST symmetry is not broken. Notice therefore that there does not exist an operator $\mathcal{G}$ with $A_\mu^2 = s\mathcal{G}$, so

that a non-vanishing vacuum expectation value for the condensate $\langle A_\mu^2 \rangle$ does not break the BRST invariance. Indeed, from

$$s\sigma = \frac{g}{2}sA_\mu^2 = -gA_\mu^a \partial^\mu c^a, \quad (11.51)$$

one can easily check that

$$s \int d^4x \mathcal{L}(A_\mu, \sigma) = 0, \quad (11.52)$$

so that we have a BRST invariant σ -action.

The rest of the argument is based on the fact that $J = 0$ when the vacuum is considered, so that we are left with only the BRST exact term in (11.49). More formally, the effective action $\Gamma(\sigma) \equiv \Gamma\left(\frac{\sigma}{g}\right)$ is related to $\mathcal{W}(J)$ through a Legendre transformation

$$\Gamma\left(\frac{\sigma}{g}\right) = -\mathcal{W}(J) - \int d^4y J(y) \frac{\sigma(y)}{g}. \quad (11.53)$$

The effective potential $V(\sigma)$ is then defined as

$$-V(\sigma) \int d^4x = \Gamma\left(\frac{\sigma}{g}\right). \quad (11.54)$$

Let σ_{min} be the solution of

$$\frac{dV(\sigma)}{d\sigma} = 0. \quad (11.55)$$

Since

$$\frac{\delta}{\delta\left(\frac{\sigma}{g}\right)} \Gamma = -J, \quad (11.56)$$

one finds

$$\sigma = \sigma_{min} \Rightarrow J = 0, \quad (11.57)$$

and invoking (11.57), from (11.53) and (11.54) we derive

$$\left. \frac{d}{d\alpha} V(\sigma) \right|_{\sigma=\sigma_{min}} \int d^4x = \left. \frac{d}{d\alpha} \mathcal{W}(J) \right|_{J=0}. \quad (11.58)$$

Finally, combining (11.49) and (11.58)

$$\left. \frac{d}{d\alpha} V(\sigma) \right|_{\sigma=\sigma_{min}} = 0. \quad (11.59)$$

From this, we conclude that the vacuum energy E_{vac} should be independent of the gauge parameter α .

Apparently, our explicit result (11.42) for E_{vac} is not in agreement with the above proof that E_{vac} is the same for each α . If we examine the proof in more detail we notice that a key argument is that J becomes zero at the end of the calculation. In practice, this is achieved by solving the gap equation.

Now, in a power series expansion in the coupling constant, the derivative of the effective potential with respect to σ is something of the form

$$(v_0 + v_1 g^2 + O(g^4)) \sigma, \quad (11.60)$$

where we assume that we work up to order g^2 and that we have chosen $\bar{\mu}$ so that the logarithms vanish. Then, the gap equation corresponding to (11.60) reads

$$v_0 + v_1 g^2 + O(g^4) = 0. \quad (11.61)$$

Due to (11.54) and (11.56), one also has

$$J = g (v_0 + v_1 g^2 + O(g^4)) \sigma. \quad (11.62)$$

This means that, if we solve the gap equation (11.61) up to certain order, we have

$$J = g (0 + O(g^4)) \sigma. \quad (11.63)$$

We also have

$$\frac{\partial \zeta}{\partial \alpha} = \frac{\partial \zeta_0}{\partial \alpha} \frac{1}{g^2} + \frac{\partial \zeta_1}{\partial \alpha} + O(g^4). \quad (11.64)$$

So, working to the order we are considering

$$\frac{\partial \zeta}{\partial \alpha} J^2 = \left(\frac{\partial \zeta_0}{\partial \alpha} v_0^2 + \left(\frac{\partial \zeta_0}{\partial \alpha} 2v_0 v_1 + \frac{\partial \zeta_1}{\partial \alpha} v_0^2 \right) g^2 + O(g^4) \right) \sigma^2. \quad (11.65)$$

From the square of the gap equation (11.61),

$$v_0^2 + 2v_1 v_0 g^2 + O(g^4) = 0, \quad (11.66)$$

it follows that

$$\frac{\partial \zeta}{\partial \alpha} J^2 = \left(\frac{\partial \zeta_1}{\partial \alpha} v_0^2 g^2 + O(g^4) \right) \sigma^2. \quad (11.67)$$

We see that, if one consistently works to the order we are considering, terms such as $\frac{\partial \zeta}{\partial \alpha} J^2$ do not equal zero although $J = 0$ to that order. Terms like those on the right hand side of (11.67) are cancelled by terms which are formally of higher order. This has its consequences for the terms proportional to J in (11.49). If one were able to work to infinite order, the problem would not arise. However, we do not have this ability, and we are faced with a gauge parameter dependence slipping into E_{vac} .

11.3.2 Circumventing the gauge parameter dependence.

We could resolve this issue by saying that the gauge parameter dependence of the vacuum energy should become less and less severe as we go to higher orders, and that eventually it will drop out if we go to infinite order. However, this is not very satisfactory, especially since we can surely never calculate the potential up to infinite order. Also as is clear from the quite complicated expression for $\zeta_1(\alpha)$, which will enter the differential equation for $\zeta_2(\alpha)$, a two-loop evaluation of the effective potential is already almost out of the question.

Therefore, we could try to modify the LCO formalism in order to circumvent the gauge parameter dependence of E_{vac} . Therefore, we consider the following action

$$\tilde{S}(\tilde{J}) = S_{YM} + S_{GF+FP} + \int d^4x \left[\tilde{J} \mathcal{F}(\alpha) \frac{A_\mu^2}{2} + \frac{\zeta}{2} \mathcal{F}^2(\alpha) \tilde{J}^2 \right], \quad (11.68)$$

instead of (11.8) where, for the moment, $\mathcal{F}(\alpha)$ is an arbitrary function of α of the form

$$\mathcal{F}(\alpha) = 1 + f_0(\alpha)g^2 + f_1(\alpha)g^4 + O(g^6), \quad (11.69)$$

and $\tilde{J}$ is now the source. The generating functional becomes

$$\exp -i\tilde{\mathcal{W}}(\tilde{J}) = \int [D\phi] \exp i\tilde{S}(\tilde{J}). \quad (11.70)$$

Taking the functional derivative of $\tilde{\mathcal{W}}(\tilde{J})$ with respect to $\tilde{J}$, we obtain

$$\left. \frac{\delta \tilde{\mathcal{W}}(\tilde{J})}{\delta \tilde{J}} \right|_{\tilde{J}=0} = -\mathcal{F}(\alpha) \left\langle \frac{A_\mu^2}{2} \right\rangle. \quad (11.71)$$

Again, we insert unity via

$$1 = \frac{1}{N} \int [D\tilde{\sigma}] \exp \left[i \int d^4x \left(-\frac{1}{2\zeta} \left(\frac{\tilde{\sigma}}{g\mathcal{F}(\alpha)} - \frac{A_\mu^2}{2} - \zeta \tilde{J} \mathcal{F}(\alpha) \right)^2 \right) \right], \quad (11.72)$$

to arrive at the following renormalized Lagrangian

$$\tilde{\mathcal{L}}(A_\mu, \tilde{\sigma}) = -\frac{1}{4} F_{\mu\nu}^2 + \mathcal{L}_{GF+FP} - \frac{\tilde{\sigma}^2}{2g^2 \mathcal{F}^2(\alpha) Z_\zeta \zeta} + \frac{1}{2} \frac{Z_2}{g^2 \mathcal{F}(\alpha) Z_\zeta \zeta} g \tilde{\sigma} A_\mu^2 - \frac{1}{8} \frac{Z_2^2}{Z_\zeta \zeta} (A_\mu^2)^2 + \tilde{J} \frac{\tilde{\sigma}}{g}. \quad (11.73)$$

From the generating functional

$$\exp -i\tilde{\mathcal{W}}(\tilde{J}) = \int [D\phi] \exp i \int d^4x \tilde{\mathcal{L}}(A_\mu, \tilde{\sigma}), \quad (11.74)$$

it follows that

$$\left. \frac{\delta \tilde{\mathcal{W}}(\tilde{J})}{\delta \tilde{J}} \right|_{\tilde{J}=0} = -\left\langle \frac{\tilde{\sigma}}{g} \right\rangle \Rightarrow \langle \tilde{\sigma} \rangle = g\mathcal{F}(\alpha) \left\langle \frac{A_\mu^2}{2} \right\rangle, \quad (11.75)$$

where the anomalous dimension of $\tilde{\sigma}$ equals

$$\gamma_{\tilde{\sigma}}(g^2) = \frac{\mu}{\tilde{\sigma}} \frac{\partial \tilde{\sigma}}{\partial \mu} = \frac{\beta(g^2)}{2g^2} + \gamma_{A^2}(g^2) + \mu \frac{\partial \ln \mathcal{F}(\alpha)}{\partial \mu}. \quad (11.76)$$

The lowest order gluon mass is now provided by

$$m^2 = \frac{g\tilde{\sigma}}{\zeta_0}, \quad (11.77)$$

and the vacuum configurations are now determined by solving

$$\frac{d\tilde{V}(\tilde{\sigma})}{d\tilde{\sigma}} = 0. \quad (11.78)$$

with $\tilde{V}(\tilde{\sigma})$ the effective potential. In the $\overline{\text{MS}}$ scheme, the one-loop effective potential reads

$$\begin{aligned} \tilde{V}_1(\tilde{\sigma}) &= \frac{\tilde{\sigma}^2}{2\zeta_0} \left(1 - \left(2f_0 + \frac{\zeta_1}{\zeta_0} \right) g^2 + \frac{2}{\varepsilon} \frac{N_A}{16\pi^2} \frac{g^2}{\zeta_0} \left(\frac{3}{2} + \frac{\alpha^2}{2} \right) \right) \\ &+ \frac{3(N_A)}{64\pi^2} \frac{g^2 \tilde{\sigma}^2}{\zeta_0^2} \left(-\frac{2}{\varepsilon} - \frac{5}{6} + \ln \frac{g\tilde{\sigma}}{\zeta_0 \bar{\mu}^2} \right) + \frac{N_A}{64\pi^2} \frac{\alpha^2 g^2 \tilde{\sigma}^2}{\zeta_0^2} \left(-\frac{2}{\varepsilon} - \frac{3}{2} + \ln \frac{\alpha g \tilde{\sigma}}{\zeta_0 \bar{\mu}^2} \right) \\ &= \frac{\tilde{\sigma}^2}{2\zeta_0} \left(1 - \left(2f_0 + \frac{\zeta_1}{\zeta_0} \right) g^2 \right) + \frac{3N_A}{64\pi^2} \frac{g^2 \tilde{\sigma}^2}{\zeta_0^2} \left(-\frac{5}{6} + \ln \frac{g\tilde{\sigma}}{\zeta_0 \bar{\mu}^2} \right) \\ &+ \frac{N_A}{64\pi^2} \frac{\alpha^2 g^2 \tilde{\sigma}^2}{\zeta_0^2} \left(-\frac{3}{2} + \ln \frac{\alpha g \tilde{\sigma}}{\zeta_0 \bar{\mu}^2} \right). \end{aligned} \quad (11.79)$$

We included the counterterm contribution here to illustrate explicitly that $\tilde{V}_1(\tilde{\sigma})$ is finite. With (11.76), it can also be checked that

$$\mu \frac{d}{d\mu} \tilde{V}_1(\tilde{\sigma}) = 0 + O(g^4). \quad (11.80)$$

Now, we can continue with the determination of the one-loop vacuum energy, which will not only depend on α , C_0 and C_1 , but also on $f_0(\alpha)$. We will determine an expression for $f_0(\alpha)$ so that $E_{\text{vac}}(\alpha, C_0, C_1, f_0(\alpha))$ does not depend on α . In the meantime, we could also absorb the constants of integration C_0 and C_1 in $f_0(\alpha)$ so that E_{vac} does not depend on them either. Based on this, we will immediately set $C_0 = C_1 = 0$. As usual, we put $\bar{\mu}^2 = \frac{g\tilde{\sigma}}{\zeta_0}$ in the gap equation, which now reads

$$\begin{aligned} \left. \frac{d\tilde{V}}{d\tilde{\sigma}} \right|_{\bar{\mu}^2 = \frac{g\tilde{\sigma}}{\zeta_0}} &= \frac{\tilde{\sigma}}{\zeta_0} \left(1 - \left(2f_0 + \frac{\zeta_1}{\zeta_0} \right) g^2 \right) + \frac{3N_A}{32\pi^2} \frac{g^2 \tilde{\sigma}}{\zeta_0^2} \left(-\frac{5}{6} \right) + \frac{3N_A}{64\pi^2} \frac{g^2 \tilde{\sigma}}{\zeta_0^2} \\ &+ \frac{N_A}{32\pi^2} \frac{\alpha^2 g^2 \tilde{\sigma}}{\zeta_0^2} \left(-\frac{3}{2} + \ln \alpha \right) + \frac{N_A}{64\pi^2} \frac{\alpha^2 g^2 \tilde{\sigma}}{\zeta_0^2} = 0, \end{aligned} \quad (11.81)$$

to kill the logarithms. One finds, in addition to the trivial solution $\tilde{\sigma} = 0$,

$$y = \frac{C_A \zeta_0}{16\pi^2 (2f_0 \zeta_0 + \zeta_1) + \frac{N_A}{2} (1 + \alpha^2 - \alpha^2 \ln |\alpha|)}, \quad (11.82)$$

$$m = \Lambda_{\overline{\text{MS}}} e^{\frac{3}{22y}}, \quad (11.83)$$

$$E_{\text{vac}} = -\frac{1}{2} \frac{N_A}{64\pi^2} (3 + \alpha^2) m^4. \quad (11.84)$$

In principle, the analytic solution for $f_0(\alpha)$ can be obtained by solving the following differential equation

$$\begin{aligned} \frac{dE_{\text{vac}}}{d\alpha} = 0 &\Leftrightarrow 2\alpha m^4 + 4(\alpha^2 + 3)m^3 \frac{dm}{d\alpha} = 0 \\ &\Leftrightarrow \alpha - \frac{3}{11y^2} (3 + \alpha^2) \left(\frac{\partial y}{\partial \alpha} + \frac{\partial y}{\partial \zeta_0} \frac{\partial \zeta_0}{\partial \alpha} + \frac{\partial y}{\partial \zeta_1} \frac{\partial \zeta_1}{\partial \alpha} + \frac{\partial y}{\partial f_0} \frac{\partial f_0}{\partial \alpha} \right) = 0. \end{aligned} \quad (11.85)$$

The quantity $f_0(\alpha)$ constructed in this fashion will ensure $E_{\text{vac}}(\alpha)$ is independent of the gauge parameter α . However, we still have the freedom of choosing an initial condition. We will determine $f_0(\alpha)$ so that

$E_{\text{vac}}(\alpha) = E_{\text{vac}}(0) \equiv E_{\text{vac}}^{\text{Landau}}$. This amounts to choosing $f_0(\alpha = 0) = 0$. We can justify this choice based on our remark in the introduction, which is that A_μ^2 coincides with the gauge invariant quantity A_{min}^2 in the Landau gauge in the FMR [110, 187, 143].

Unfortunately, the differential equation (11.85) is very hard to solve analytically. We could solve (11.85) and consequently y , m and E_{vac} numerically. However, there is a more elegant way to obtain the analytical solution for $f_0(\alpha)$. Considering the colour group $SU(3)$ for simplicity, then since we know that by construction that $E_{\text{vac}} = E_{\text{vac}}^{\text{Landau}}$, we are able to write down the analytical solution for m as

$$m = \left(\frac{3e^{17/6}}{3 + \alpha^2} \right)^{1/4} \Lambda_{\overline{\text{MS}}}, \quad (11.86)$$

where use was made of (11.45) and (11.84). Putting (11.86) in (11.83), we deduce that

$$y = \frac{36}{66 \ln \frac{3}{3 + \alpha^2} + 187}. \quad (11.87)$$

Combining (11.82) and (11.87) finally gives the analytic expression for $f_0(\alpha)$

$$f_0(\alpha) = \frac{\frac{\zeta_0}{12} \left(66 \ln \frac{3}{3 + \alpha^2} + 187 \right) - 4(1 + \alpha^2 - \alpha^2 \ln |\alpha|) - 16\pi^2 \zeta_1}{32\pi^2 \zeta_0}. \quad (11.88)$$

We have displayed $f_0(\alpha)$, $y(\alpha)$ and $m(\alpha)$ for the range of values $-\frac{13}{3} < \alpha < \frac{13}{3}$ in Figures 11.4-11.6. As a check, we have also plotted, in Figure 11.7, $E_{\text{vac}}(\alpha, f_0(\alpha))$ as given in (11.84) to verify that $E_{\text{vac}}(\alpha, f_0(\alpha)) = E_{\text{vac}}^{\text{Landau}}$. We observe several features. Firstly, although $f_0(\alpha)$ has some singularities in $]-\frac{13}{3}, \frac{13}{3}[$, the quantities y , m and E_{vac} are completely regular functions of α . Secondly, the expansion parameter y remains relatively small, which makes our numerical predictions at least qualitatively trustworthy. Thirdly, we also see that the value for the tree level mass does not change spectacularly in the considered region. In the Feynman gauge $\alpha = 1$, we have $m^{\text{Feynman}} = 1.89\Lambda_{\overline{\text{MS}}}$.

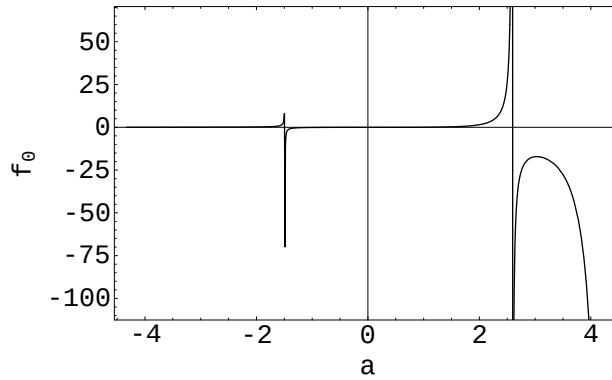


Figure 11.4: f_0 as a function of α with $-\frac{13}{3} < \alpha < \frac{13}{3}$.

Before ending this section, there are several other points. We have determined $\mathcal{F}(\alpha)$ with the renormalization scale $\bar{\mu}$ chosen in such a way that the logarithms vanish. Other choices of $\bar{\mu}$ are of course also valid. We did not explicitly write this $\bar{\mu}$ dependence of $\mathcal{F}(\alpha)$ in (11.69).

Also, the procedure we have described here applies of course at higher order. For example, at two-loops, $f_1(\alpha)$ will be required to remove the α dependence. If we were to work to infinite order in g^2 , we could

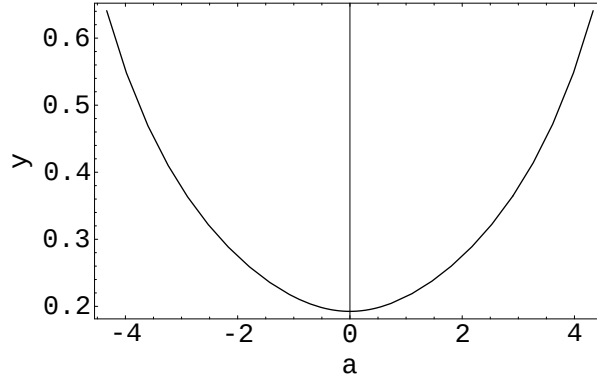


Figure 11.5: y as a function of α with $-\frac{13}{3} < \alpha < \frac{13}{3}$.

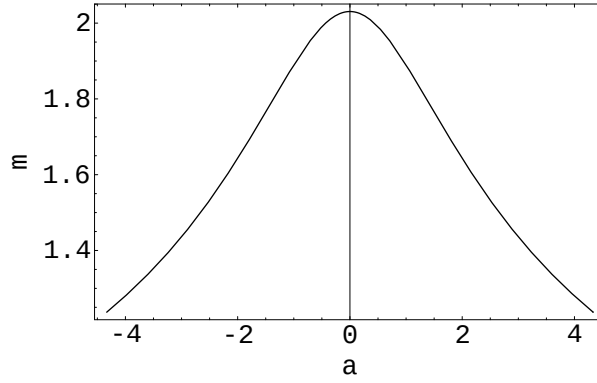


Figure 11.6: m as a function of α with $-\frac{13}{3} < \alpha < \frac{13}{3}$ (in units $\Lambda_{\overline{MS}} = 1$).

transform the action $\tilde{S}(\tilde{J})$ (11.68) *exactly* into the action $S(J)$ (11.8) by means of the transformation

$$\tilde{J} = \frac{J}{\mathcal{F}(\alpha)}. \quad (11.89)$$

The corresponding transformation for the σ and $\tilde{\sigma}$ fields reads

$$\tilde{\sigma} = \mathcal{F}(\alpha)\sigma, \quad (11.90)$$

which will transform the effective potential $\tilde{V}_\infty(\tilde{\sigma})$ *exactly* into $V_\infty(\sigma)$. As such, the constructed vacuum energy will be the same in both cases and independent of the choice of α .

11.4 Gluon propagator in linear covariant gauges.

In [48], the gluon propagator in the Landau was investigated, and a fit of the lattice results gave evidence for a gluon mass. In the Landau gauge, the lattice also gives evidence for the existence of a non-zero $\langle A_\mu^2 \rangle$ condensate, based on the discrepancy in the 10 GeV region, between the behaviour

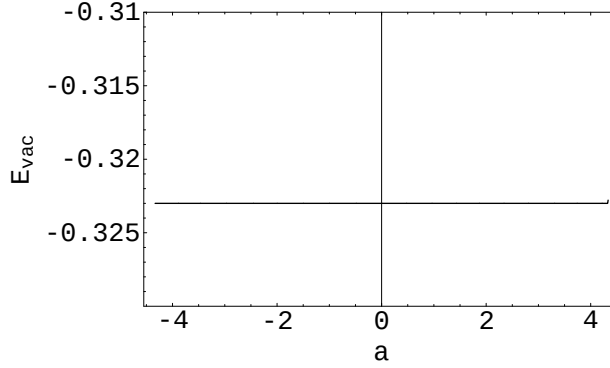


Figure 11.7: E_{vac} as a function of α with $-\frac{13}{3} < \alpha < \frac{13}{3}$ (in units $\Lambda_{\overline{\text{MS}}} = 1$).

of the observed lattice gluon propagator and strong coupling constant and the expected perturbative behaviour. The results could be matched together using an operator product expansion analysis with a non-zero $\langle A_\mu^2 \rangle$ condensate [175, 37, 38]. A combined lattice fit resulted in $\langle A_\mu^2 \rangle_{\text{OPE}} \approx (1.64 \text{ GeV})^2$. This quantity was obtained at a scale of 10 GeV in the MOM renormalization scheme. Later, it was argued that this $\langle A_\mu^2 \rangle_{\text{OPE}}$ condensate could be explained with instantons [38].

One will notice that we did not give the estimate for $\langle A^2 \rangle$ itself. From the identification (11.22) and using the relation (11.35) and the explicit result (11.44), one finds

$$\langle A_\mu^2 \rangle = -\frac{187}{52\pi^2} e^{\frac{17}{12}} \Lambda_{\overline{\text{MS}}}^2 \approx -(0.29 \text{ GeV})^2 \quad (11.91)$$

The extra minus sign arises because we have rotated from Minkowskian to Euclidean space time to make possible a comparison with the lattice. We used $\Lambda_{\overline{\text{MS}}} = 0.233 \text{ GeV}$, which was the value obtained in [37]. We should be careful not to misinterpret the relatively big difference between $\langle A_\mu^2 \rangle_{\text{OPE}}$ and (11.91). Although our result is non-perturbative in nature, it is still obtained in perturbation theory and as such it only gives information from the high energy region (or short range), while the OPE approach of Boucaud *et al* can only describe the low energy (or long range) content of $\langle A_\mu^2 \rangle$. It was already argued in [34] that $\langle A_\mu^2 \rangle$ can receive long and short range contributions. The minus sign in front of our result has to do with the regularization and renormalization of the quantity $\langle A_\mu^2 \rangle$. We refer to [42] for more details.

To our knowledge, there has been little attention on the lattice to the gluon propagator in a general linear covariant gauge. Giusti *et al* managed to put the linear covariant gauge on the lattice [208, 209, 210, 211]. The tree level gluon propagator of Euclidean Yang-Mills theory with a linear covariant gauge fixing is given by

$$D_{\mu\nu}(q) = \frac{1}{q^2} \left(\delta_{\mu\nu} - (1 - \alpha) \frac{q_\mu q_\nu}{q^2} \right). \quad (11.92)$$

This can be decomposed into the transverse and longitudinal parts as

$$D_{\mu\nu}(q) = \frac{1}{q^2} \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) D^T(q) + \frac{q_\mu q_\nu}{q^2} \frac{D^L(q)}{q^2}, \quad (11.93)$$

where $D^T(q^2)$ is q^2 times the one used in [208, 209, 210, 211]. In general, one determines $D^L(q)$ via the projector

$$P_{\mu\nu}^L(q) = q_\mu q_\nu. \quad (11.94)$$

If there is a tree level gluon mass m present, as in (11.73), the Euclidean gluon propagator in linear covariant gauges reads

$$D_{\mu\nu}(q) = \frac{1}{q^2 + m^2} \left(\delta_{\mu\nu} - (1 - \alpha) \frac{q_\mu q_\nu}{q^2 + \alpha m^2} \right), \quad (11.95)$$

with the value of m given in (11.86). The longitudinal part of this propagator is

$$D^L(q) = P_{\mu\nu}^L(q) D_{\mu\nu}(q) = \frac{1}{q^2 + m^2} \left(q^2 - (1 - \alpha) \frac{q^4}{q^2 + \alpha m^2} \right). \quad (11.96)$$

$D^L(q)$ is plotted in Figure 11.8, again using $\Lambda_{\overline{\text{MS}}} = 0.233\text{GeV}$. Of course, we should not attach any

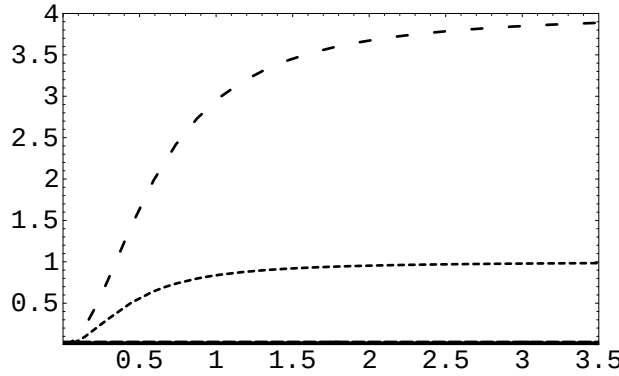


Figure 11.8: $D_L(q)$ as a function of q with $0 < q < 3.5\text{GeV}$ for $\alpha = 0$ (fat), $\alpha = 1$ (dashed) and $\alpha = 4$ (wide dashed).

firm meaning to this plot, since we are only considering the tree level propagator and do not include any renormalization effects. If we could calculate the form factors, we would also inevitably encounter the problem of a diverging perturbation theory in the infrared region. We cannot make any conclusion about the behaviour of the propagator in the IR from the above. Many other (non-perturbative) effects can influence the propagators form in the IR. Nevertheless, it might be worth noticing that the longitudinal part $D^L(q)$ is not proportional to the gauge parameter. A similar behaviour was found by Giusti *et al*, see e.g. Figure 4 of [210]. This is already different from the perturbative prediction of massless Yang-Mills theory with a linear covariant gauge fixing [20].

11.5 Conclusion.

We have considered Yang-Mills theories in linear covariant gauges and constructed a renormalizable effective potential by means of the local composite operator formalism for A_μ^2 . The formation of the gluon condensate of mass dimension two is favoured since it lowers the vacuum energy. As a result, the gluons acquire a dynamical mass m . We discussed the gauge parameter dependence of the resultant vacuum energy and observed that this is due to the fact that we do not work up to infinite order precision, but have to truncate the perturbative expansion at a finite order. We explained how this gauge parameter dependence can be avoided by a modification of our method.

Although there is limited lattice data available for the general linear covariant gauges compared with the Landau gauge, it would be interesting to calculate the form factor of the longitudinal and transverse

part of the gluon propagator to make a more detailed comparison possible with the lattice results of [208, 209, 210, 211]. It would also be useful to have direct evidence from the lattice community that the $\langle A_\mu^2 \rangle$ condensate exists and that the gluons become massive, in analogy with the Landau gauge. A further point worth investigating is the possible existence of ghost condensates in the linear covariant gauges, as is the case in the Landau gauge [91, 172]. These condensates can modify the gluon propagator further.

Chapter 12

An analytic study of the off-diagonal mass generation for Yang-Mills theories in the maximal Abelian gauge

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We investigate a dynamical mass generation mechanism for the off-diagonal gluons and ghosts in $SU(N)$ Yang-Mills theories, quantized in the maximal Abelian gauge. Such a mass can be seen as evidence for the Abelian dominance in that gauge. It originates from the condensation of a mixed gluon-ghost operator of mass dimension two, which lowers the vacuum energy. We construct an effective potential for this operator by a combined use of the local composite operators technique with the algebraic renormalization and we discuss the gauge parameter independence of the results. We also show that it is possible to connect the vacuum energy, due to the mass dimension two condensate discussed here, with the non-trivial vacuum energy originating from the condensate $\langle A_\mu^2 \rangle$, which has attracted much attention in the Landau gauge.

12.1 Introduction.

A widely accepted mechanism to explain color confinement in $SU(N)$ Yang-Mills theories is based on the dual superconductivity picture [65, 66, 68], according to which the low energy regime of QCD should be described by an effective Abelian theory in the presence of magnetic monopoles. These monopoles should condense, giving rise to a string formation à la Abrikosov-Nielsen-Olesen. As a result, chromoelectric charges are confined. This mechanism has received many confirmations from the lattice community in the so-called Abelian gauges, which are useful in order to isolate the effective relevant degrees of freedom at low energy.

According to the concept of Abelian dominance, the low energy region of QCD can be expressed solely in terms of Abelian degrees of freedom [73]. Lattice confirmations of the Abelian dominance can be found

in [74, 75]. A particularly interesting Abelian gauge is the maximal Abelian gauge (MAG), introduced in [68, 70, 69]. Roughly speaking, the MAG is obtained by minimizing the square of the norm of the fields corresponding to off-diagonal gluons, i.e. the gluons associated with the $N(N-1)$ off-diagonal generators of $SU(N)$. Doing so, there is a residual $U(1)^{N-1}$ Abelian gauge freedom corresponding to the Cartan subgroup of $SU(N)$. The renormalizability in the continuum of this gauge was proven in [71, 72], at the cost of introducing a quartic ghost interaction.

To our knowledge, there is no analytic proof of the Abelian dominance. Nevertheless, an argument that can be interpreted as evidence of it, is the fact that the off-diagonal gluons would attain a dynamical mass. At energies below the scale set by this mass, the off-diagonal gluons should decouple, and in this way one should end up with an Abelian theory at low energies. A lattice study of such an off-diagonal gluon mass reported a value of approximately 1.2GeV [49]. More recently, the off-diagonal gluon propagator was investigated numerically in [50], reporting a similar result.

There have been several efforts to give an analytic description of the mechanism responsible for the dynamical generation of the off-diagonal gluon mass. In [77, 80], a certain ghost condensate was used to construct an effective, off-diagonal mass. However, in [157] it was shown that the obtained mass was a tachyonic one, a fact confirmed later in [82]. Another condensation, namely that of the mixed gluon-ghost operator $(\frac{1}{2}A_\mu^a A^{\mu a} + \alpha \bar{c}^a c^a)$ ¹, that could be responsible for the off-diagonal mass, was proposed in [83]. That this operator should condense can be expected on the basis of a close analogy existing between the MAG and the renormalizable nonlinear Curci-Ferrari gauge [84, 85]. In fact, it turns out that the mixed gluon-ghost operator can be introduced also in the Curci-Ferrari gauge. A detailed analysis of its condensation and of the ensuing dynamical mass generation can be found in [199, 178].

The aim of this paper is to investigate explicitly if the mass dimension two operator $(\frac{1}{2}A_\mu^a A^{\mu a} + \alpha \bar{c}^a c^a)$ condenses, so that a dynamical off-diagonal mass is generated in the MAG. The pathway we intend to follow is based on previous research in this direction in other gauges. In [42], the local composite operator (LCO) technique was used to construct a renormalizable effective potential for the operator $A_\mu^A A^{\mu A}$ in the Landau gauge. As a consequence of $\langle A_\mu^A A^{\mu A} \rangle \neq 0$, the gauge bosons acquired a mass [42]. The fact that gluons in the Landau gauge become massive has received confirmations from lattice simulations, see for example [48]. Recently, the dynamical mass generation in the Landau gauge has been investigated within the Schwinger-Dyson formalism in [51, 52]. The condensate $\langle A_\mu^A A^{\mu A} \rangle$ has attracted attention from theoretical [33, 34] as well as from the lattice side [175, 37, 38]. It was shown by means of the algebraic renormalization technique [59] that the LCO formalism for the condensate $\langle A_\mu^A A^{\mu A} \rangle$ is renormalizable to all orders of perturbation theory [153]. The same formalism was successfully employed to study the condensation of $(\frac{1}{2}A_\mu^A A^{\mu A} + \alpha \bar{c}^A c^A)$ in the Curci-Ferrari gauge [199, 178]. We would like to note that the Landau gauge corresponds to $\alpha = 0$. Later on, the condensation of $A_\mu^A A^{\mu A}$ was confirmed in the linear covariant gauges [205, 212], which also possess the Landau gauge as a special case. It was proven formally that the vacuum energy does not depend on the gauge parameter. However, in practice, a problem occurred due to the mixing of different orders of perturbation theory, when solving the gap equation for the condensate. Nevertheless, we have been able to present a way to overcome this problem [212]. As a result, it turns out that the non-trivial vacuum energy due to the condensate $\langle A_\mu^A A^{\mu A} \rangle$ in the Landau gauge coincides with the non-trivial vacuum energy due to the appropriate mass dimension two condensate in the linear covariant gauges, $\langle A_\mu^A A^{\mu A} \rangle$, and the Curci-Ferrari gauge, $\langle \frac{1}{2}A_\mu^A A^{\mu A} + \alpha \bar{c}^A c^A \rangle$, since these two classes of gauges both have the Landau gauge, $\alpha = 0$, as a limiting case.

We would also like to underline that the concept of a gluon mass has already been widely used in a more phenomenological context since long ago, see e.g. [53]. More recently, a gluon mass of the order of a few hundred MeV has been proven to be very useful in describing the radiative decay of heavy

¹The index a runs only over the $N(N-1)$ off-diagonal generators.

quarkonia systems [54] as well as to derive estimates of the glueball spectrum [56].

To make this paper self-contained, we will explain all necessary steps in the case of the MAG, and refer to the previous papers for more details where appropriate. In section 12.2, we introduce the MAG and discuss its renormalizability when the operator $(\frac{1}{2}A_\mu^a A^{\mu a} + \alpha \bar{c}^a c^a)$ is introduced in the theory. We briefly review how the effective potential is constructed by means of the LCO technique. In section 12.3, we discuss the independence of the vacuum energy from the gauge parameter of the MAG. We face the problem of the mixing of different orders in perturbation theory, and we provide a solution of it. In section 12.4, we construct a generalized renormalizable gauge that interpolates between the MAG and the Landau gauge. Moreover, we will also show that there exists a generalized renormalizable mass dimension two operator that interpolates between the mass dimension two operators of the MAG and of the Landau gauge. This can be used to prove that the vacuum energy obtained in the MAG is the same as that of the Landau gauge. In section 12.5, we present explicit results, obtained in the case of $SU(2)$ and to the one-loop approximation. We end with conclusions in section 12.6.

12.2 $SU(N)$ Yang-Mills theories in the MAG.

Let A_μ be the Lie algebra valued connection for the gauge group $SU(N)$, whose generators T^A , satisfying $[T^A, T^B] = f^{ABC}T^C$, are chosen to be anti-Hermitian and to obey the orthonormality condition $\text{Tr}(T^A T^B) = -T_F \delta^{AB}$, with $A, B, C = 1, \dots, (N^2 - 1)$. In the case of $SU(N)$, one has $T_F = \frac{1}{2}$. We decompose the gauge field into its off-diagonal and diagonal parts, namely

$$A_\mu = A_\mu^A T^A = A_\mu^a T^a + A_\mu^i T^i, \quad (12.1)$$

where the indices $i, j, \dots$ label the $N-1$ generators of the Cartan subalgebra. The remaining $N(N-1)$ off-diagonal generators will be labelled by the indices $a, b, \dots$. For further use, we recall the Jacobi identity

$$f^{ABC} f^{CDE} + f^{ADC} f^{CEB} + f^{AEC} f^{CBD} = 0, \quad (12.2)$$

from which it can be deduced that

$$\begin{aligned} f^{abi} f^{bjc} + f^{abj} f^{bci} &= 0, \\ f^{abc} f^{bdi} + f^{abd} f^{bic} + f^{abi} f^{bcd} &= 0. \end{aligned} \quad (12.3)$$

The field strength decomposes as

$$F_{\mu\nu} = F_{\mu\nu}^A T^A = F_{\mu\nu}^a T^a + F_{\mu\nu}^i T^i, \quad (12.4)$$

with the off-diagonal and diagonal parts given respectively by

$$\begin{aligned} F_{\mu\nu}^a &= D_\mu^{ab} A_\nu^b - D_\nu^{ab} A_\mu^b + g f^{abc} A_\mu^b A_\nu^c, \\ F_{\mu\nu}^i &= \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g f^{abi} A_\mu^a A_\nu^b, \end{aligned} \quad (12.5)$$

where the covariant derivative D_μ^{ab} is defined with respect to the diagonal components A_μ^i

$$D_\mu^{ab} \equiv \partial_\mu \delta^{ab} - g f^{abi} A_\mu^i. \quad (12.6)$$

For the Yang-Mills action one obtains

$$S_{\text{YM}} = -\frac{1}{4} \int d^4x (F_{\mu\nu}^a F^{\mu\nu a} + F_{\mu\nu}^i F^{\mu\nu i}). \quad (12.7)$$

The so called MAG gauge condition amounts to fixing the value of the covariant derivative, $D_\mu^{ab} A^{\mu b}$, of the off-diagonal components by requiring that the functional

$$\mathcal{R}[A] = (VT)^{-1} \int d^4x (A_\mu^a A^{\mu a}) , \quad (12.8)$$

attains a minimum with respect to the local gauge transformations. This corresponds to imposing

$$D_\mu^{ab} A^{\mu b} = 0 . \quad (12.9)$$

However, this condition being non-linear implies a quartic ghost self-interaction term is required for renormalizability purposes. The corresponding gauge fixing term turns out to be [71, 72]

$$S_{\text{MAG}} = s \int d^4x \left(\bar{c}^a \left(D_\mu^{ab} A^{\mu b} + \frac{\alpha}{2} b^a \right) - \frac{\alpha}{2} g f^{abi} \bar{c}^a \bar{c}^b c^i - \frac{\alpha}{4} g f^{abc} c^a \bar{c}^b \bar{c}^c \right) , \quad (12.10)$$

where α is the MAG gauge parameter and s denotes the nilpotent BRST operator, acting as

$$\begin{aligned} s A_\mu^a &= - (D_\mu^{ab} c^b + g f^{abc} A_\mu^b c^c + g f^{abi} A_\mu^b c^i) , & s A_\mu^i &= - (\partial_\mu c^i + g f^{iab} A_\mu^a c^b) , \\ s c^a &= g f^{abi} c^b c^i + \frac{g}{2} f^{abc} c^b c^c , & s c^i &= \frac{g}{2} f^{iab} c^a c^b , \\ s \bar{c}^a &= b^a , & s \bar{c}^i &= b^i , \\ s b^a &= 0 , & s b^i &= 0 . \end{aligned} \quad (12.11)$$

Here c^a, c^i are the off-diagonal and the diagonal components of the Faddeev-Popov ghost field, while $\bar{c}^a, b^a$ are the off-diagonal antighost and Lagrange multiplier. We also observe that the BRST transformations (12.11) have been obtained by their standard form upon projection on the off-diagonal and diagonal components of the fields. We remark that the MAG (12.10) can be written in the form

$$S_{\text{MAG}} = s \bar{s} \int d^4x \left(\frac{1}{2} A_\mu^a A^{\mu a} - \frac{\alpha}{2} c^a \bar{c}^a \right) , \quad (12.12)$$

with $\bar{s}$ being the nilpotent anti-BRST transformation, acting as

$$\begin{aligned} \bar{s} A_\mu^a &= - (D_\mu^{ab} \bar{c}^b + g f^{abc} A_\mu^b \bar{c}^c + g f^{abi} A_\mu^b \bar{c}^i) , & \bar{s} A_\mu^i &= - (\partial_\mu \bar{c}^i + g f^{iab} A_\mu^a \bar{c}^b) , \\ \bar{s} \bar{c}^a &= g f^{abi} \bar{c}^b \bar{c}^i + \frac{g}{2} f^{abc} \bar{c}^b \bar{c}^c , & \bar{s} \bar{c}^i &= \frac{g}{2} f^{iab} \bar{c}^a \bar{c}^b , \\ \bar{s} c^a &= -b^a + g f^{abc} c^b \bar{c}^c + g f^{abi} c^b \bar{c}^i + g f^{abi} \bar{c}^b c^i , & \bar{s} c^i &= -b^i + g f^{ibc} c^b \bar{c}^c , \\ \bar{s} b^a &= -g f^{abc} b^b \bar{c}^c - g f^{abi} b^b \bar{c}^i + g f^{abi} \bar{c}^b b^i , & \bar{s} b^i &= -g f^{ibc} b^b \bar{c}^c . \end{aligned} \quad (12.13)$$

It can be checked that s and $\bar{s}$ anticommute.

Expression (12.10) is easily worked out and yields

$$\begin{aligned} S_{\text{MAG}} &= \int d^4x \left(b^a \left(D_\mu^{ab} A^{\mu b} + \frac{\alpha}{2} b^a \right) + \bar{c}^a D_\mu^{ab} D^{\mu bc} c^c + g \bar{c}^a f^{abi} (D_\mu^{bc} A^{\mu c}) c^i \right. \\ &+ g \bar{c}^a D_\mu^{ab} (f^{bcd} A^{\mu c} c^d) - \alpha g f^{abi} b^a \bar{c}^b c^i - g^2 f^{abi} f^{cdi} \bar{c}^a c^d A_\mu^b A^{\mu c} - \frac{\alpha}{2} g f^{abc} b^a \bar{c}^b c^c \\ &\left. - \frac{\alpha}{4} g^2 f^{abi} f^{cdi} \bar{c}^a \bar{c}^b c^c c^d - \frac{\alpha}{4} g^2 f^{abc} f^{adi} \bar{c}^b \bar{c}^c c^d c^i - \frac{\alpha}{8} g^2 f^{abc} f^{ade} \bar{c}^b \bar{c}^c c^d c^e \right) . \end{aligned} \quad (12.14)$$

We note that $\alpha = 0$ does in fact correspond to the “real” MAG condition, given by eq.(12.9). However, one cannot set $\alpha = 0$ from the beginning since this would lead to a nonrenormalizable gauge. Some

of the terms proportional to α would reappear due to radiative corrections, even if $\alpha = 0$. See, for example, [97]. For our purposes, this means that we have to keep α general throughout and leave to the end the analysis of the limit $\alpha \rightarrow 0$, to recover condition (12.9).

The MAG condition allows for a residual local $U(1)^{N-1}$ invariance with respect to the diagonal subgroup. In order to have a complete quantization of the theory, one has to fix this Abelian gauge freedom by means of a suitable further gauge condition on the diagonal components A_μ^i of the gauge field. A common choice for the Abelian gauge fixing, also adopted in the lattice papers [49, 50], is the Landau gauge, given by

$$S_{\text{diag}} = s \int d^4x \bar{c}^i \partial_\mu A^{\mu i} = \int d^4x \left(b^i \partial_\mu A^{\mu i} + \bar{c}^i \partial^\mu (\partial_\mu c^i + g f^{iab} A_\mu^a c^b) \right), \quad (12.15)$$

where $\bar{c}^i, b^i$ are the diagonal antighost and Lagrange multiplier.

12.2.1 Ward identities for the MAG.

In order to write down a suitable set of Ward identities, we first introduce external fields $\Omega^{\mu i}$, $\Omega^{\mu a}$, L^i , L^a coupled to the BRST nonlinear variations of the fields, namely

$$\begin{aligned} S_{\text{ext}} &= \int d^4x \left(-\Omega^{\mu a} (D_\mu^{ab} c^b + g f^{abc} A_\mu^b c^c + g f^{abi} A_\mu^b c^i) - \Omega^{\mu i} (\partial_\mu c^i + g f^{iab} A_\mu^a c^b) \right. \\ &\quad \left. + L^a \left(g f^{abi} c^b c^i + \frac{g}{2} f^{abc} c^b c^c \right) + L^i \frac{g}{2} f^{iab} c^a c^b \right), \end{aligned} \quad (12.16)$$

with

$$\begin{aligned} s\Omega^{\mu a} &= s\Omega^{\mu i} = 0, \\ sL^a &= sL^i = 0. \end{aligned} \quad (12.17)$$

Moreover, in order to discuss the renormalizability of the gluon-ghost operator

$$\mathcal{O}_{\text{MAG}} = \frac{1}{2} A_\mu^a A^{\mu a} + \alpha \bar{c}^a c^a, \quad (12.18)$$

we introduce it in the starting action by means of a BRST doublet of external sources (J, λ)

$$s\lambda = J, \quad sJ = 0, \quad (12.19)$$

so that

$$\begin{aligned} S_{\text{LCO}} &= s \int d^4x \left(\lambda \left(\frac{1}{2} A_\mu^a A^{\mu a} + \alpha \bar{c}^a c^a \right) + \zeta \frac{\lambda J}{2} \right) \\ &= \int d^4x \left(J \left(\frac{1}{2} A_\mu^a A^{\mu a} + \alpha \bar{c}^a c^a \right) + \zeta \frac{J^2}{2} - \alpha \lambda b^a c^a \right. \\ &\quad \left. + \lambda A^{\mu a} (D_\mu^{ab} c^b + g f^{abi} A_\mu^b c^i) + \alpha \lambda \bar{c}^a (g f^{abi} c^b c^i + \frac{g}{2} f^{abc} c^b c^c) \right). \end{aligned} \quad (12.20)$$

ζ is the LCO parameter accounting for the divergences present in the vacuum correlator $\langle \mathcal{O}_{\text{MAG}}(x) \mathcal{O}_{\text{MAG}}(y) \rangle$, which are proportional to J^2 . Therefore, the complete action

$$\Sigma = S_{\text{YM}} + S_{\text{MAG}} + S_{\text{diag}} + S_{\text{ext}} + S_{\text{LCO}}, \quad (12.21)$$

	$A_\mu^{a,i}$	$c^{a,i}$	$\bar{c}^{a,i}$	$b^{a,i}$	λ	J	$\Omega_\mu^{a,i}$	$L^{a,i}$
dimension	1	0	2	2	2	2	3	4
ghost number	0	1	-1	0	-1	0	-1	-2

Table 12.1:

is BRST invariant

$$s\Sigma = 0 . \quad (12.22)$$

As noticed in [83, 144], the gluon-ghost mass operator defined in eq.(12.18) is BRST invariant on-shell. In the accompanying Table 12.1 , the dimension and ghost number of all the fields and sources are listed. We are now ready to write down the Ward identities needed to discuss the renormalizability of the model. It turns out that the complete action Σ is constrained by

- the Slavnov-Taylor identity

$$\mathcal{S}(\Sigma) = 0 , \quad (12.23)$$

with

$$\mathcal{S}(\Sigma) = \int d^4x \left(\frac{\delta\Sigma}{\delta\Omega^{\mu a}} \frac{\delta\Sigma}{\delta A_\mu^a} + \frac{\delta\Sigma}{\delta\Omega^{\mu i}} \frac{\delta\Sigma}{\delta A_\mu^i} + \frac{\delta\Sigma}{\delta L^a} \frac{\delta\Sigma}{\delta c^a} + \frac{\delta\Sigma}{\delta L^i} \frac{\delta\Sigma}{\delta c^i} + b^a \frac{\delta\Sigma}{\delta \bar{c}^a} + b^i \frac{\delta\Sigma}{\delta \bar{c}^i} + J \frac{\delta\Sigma}{\delta \lambda} \right) . \quad (12.24)$$

- the diagonal ghost equation [72]

$$\mathcal{G}^i \Sigma = \Delta_{\text{cl}}^i , \quad (12.25)$$

where

$$\mathcal{G}^i = \frac{\delta}{\delta c^i} + g f^{abi} \bar{c}^a \frac{\delta\Sigma}{\delta b^b} , \quad (12.26)$$

and

$$\Delta_{\text{cl}}^i = -\partial^2 \bar{c}^i + g f^{abi} \Omega^{\mu a} A_\mu^b - \partial_\mu \Omega^{\mu i} - g f^{abi} L^a c^b . \quad (12.27)$$

Notice that expression (12.27), being linear in the quantum fields, is a classical breaking.

- the diagonal gauge fixing and anti-ghost equations

$$\begin{aligned} \frac{\delta\Sigma}{\delta b^i} &= \partial_\mu A^{\mu i} , \\ \frac{\delta\Sigma}{\delta \bar{c}^i} + \partial^\mu \frac{\delta\Sigma}{\delta \Omega^{\mu i}} &= 0 . \end{aligned} \quad (12.28)$$

- the integrated λ -equation

$$\int d^4x \left(\frac{\delta\Sigma}{\delta \lambda} + c^a \frac{\delta\Sigma}{\delta b^a} \right) = 0 , \quad (12.29)$$

expressing in a functional form the on-shell BRST invariance of the gluon-ghost operator $\mathcal{O}_{\text{MAG}}$.

- the diagonal $U(1)^{N-1}$ Ward identity

$$\mathcal{W}^i \Sigma = -\partial^2 b^i, \quad (12.30)$$

with

$$\mathcal{W}^i = \partial_\mu \frac{\delta}{\delta A_\mu^i} + g f^{abi} \left(A_\mu^a \frac{\delta}{\delta A_\mu^b} + c^a \frac{\delta}{\delta \bar{c}^b} + b^a \frac{\delta}{\delta b^b} + \bar{c}^a \frac{\delta}{\delta \bar{c}^b} + \Omega^{\mu a} \frac{\delta}{\delta \Omega^{\mu b}} + L^a \frac{\delta}{\delta L^b} \right). \quad (12.31)$$

This identity follows from the diagonal ghost equation (12.25) and the Slavnov-Taylor identity (12.23).

In order to find the foregoing Ward identities, use has been made of the Jacobi identities (12.3).

12.2.2 Algebraic characterization of the most general local counterterm.

We mention that all the classical Ward identities of the previous section can be extended to all orders of perturbation theory without encountering anomalies. In principle, this can be proven by means of the algebraic setup of [59] and of the general results on the BRST cohomology of gauge theories [60]. It can be understood in a simple way by observing that pure Yang-Mills theory in the MAG can be regularized in a gauge invariant way by employing dimensional regularization.

In order to characterize the most general invariant counterterm which can be freely added to all orders of perturbation theory, we perturb the classical action Σ by adding an arbitrary integrated local polynomial Σ^{count} in the fields and external sources of dimension bounded by four and with zero ghost number, and we require that the perturbed action $(\Sigma + \eta \Sigma^{\text{count}})$ satisfies the same Ward identities as Σ to the first order in the perturbation parameter η , i.e.,

$$\begin{aligned} \mathcal{S}(\Sigma + \eta \Sigma^{\text{count}}) &= 0 + O(\eta^2), \\ \mathcal{G}^i(\Sigma + \eta \Sigma^{\text{count}}) &= \Delta_{\text{cl}}^i + O(\eta^2), \\ \frac{\delta(\Sigma + \eta \Sigma^{\text{count}})}{\delta b^i} &= \partial_\mu A^{\mu i} + O(\eta^2), \\ \left(\frac{\delta}{\delta \bar{c}^i} + \partial^\mu \frac{\delta}{\delta \Omega^{\mu i}} \right) (\Sigma + \eta \Sigma^{\text{count}}) &= 0 + O(\eta^2), \\ \int d^4x \left(\frac{\delta}{\delta \lambda} + c^a \frac{\delta}{\delta b^a} \right) (\Sigma + \eta \Sigma^{\text{count}}) &= 0 + O(\eta^2), \\ \mathcal{W}^i(\Sigma + \eta \Sigma^{\text{count}}) &= -\partial^2 b^i + O(\eta^2). \end{aligned} \quad (12.32)$$

This amounts to imposing the following conditions on Σ^{count}

$$\mathcal{B}_\Sigma \Sigma^{\text{count}} = 0, \quad (12.33)$$

where $\mathcal{B}_\Sigma$ denotes the nilpotent linearized operator

$$\mathcal{B}_\Sigma \mathcal{B}_\Sigma = 0, \quad (12.34)$$

$$\begin{aligned} \mathcal{B}_\Sigma &= \int d^4x \left(\frac{\delta \Sigma}{\delta \Omega^{\mu a}} \frac{\delta}{\delta A_\mu^a} + \frac{\delta \Sigma}{\delta A_\mu^a} \frac{\delta}{\delta \Omega^{\mu a}} + \frac{\delta \Sigma}{\delta \Omega^{\mu i}} \frac{\delta}{\delta A_\mu^i} + \frac{\delta \Sigma}{\delta A_\mu^i} \frac{\delta}{\delta \Omega^{\mu i}} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta}{\delta c^a} \right. \\ &\quad \left. + \frac{\delta \Sigma}{\delta c^a} \frac{\delta}{\delta L^a} + \frac{\delta \Sigma}{\delta L^i} \frac{\delta}{\delta c^i} + \frac{\delta \Sigma}{\delta c^i} \frac{\delta}{\delta L^i} + b^a \frac{\delta}{\delta \bar{c}^a} + b^i \frac{\delta}{\delta \bar{c}^i} + J \frac{\delta}{\delta \lambda} \right), \end{aligned} \quad (12.35)$$

and

$$\begin{aligned}
\mathcal{G}^i \Sigma^{\text{count}} &= 0, \\
\frac{\delta \Sigma^{\text{count}}}{\delta b^i} &= 0, \\
\left(\frac{\delta}{\delta \bar{c}^i} + \partial^\mu \frac{\delta}{\delta \Omega^{\mu i}} \right) \Sigma^{\text{count}} &= 0, \\
\int d^4x \left(\frac{\delta}{\delta \lambda} + c^a \frac{\delta}{\delta b^a} \right) \Sigma + \varepsilon \Sigma^{\text{count}} &= 0, \\
\mathcal{W}^i \Sigma^{\text{count}} &= 0.
\end{aligned} \tag{12.36}$$

From the conditions (12.33) and (12.36), it turns out that the most general invariant counterterm can be written as

$$\Sigma^{\text{count}} = \frac{-a_0}{4} \int d^4x (F_{\mu\nu}^a F^{\mu\nu a} + F_{\mu\nu}^i F^{\mu\nu i}) + \mathcal{B}_\Sigma \Delta^{-1}, \tag{12.37}$$

where Δ^{-1} is an integrated local polynomial with ghost number -1 , given by

$$\begin{aligned}
\Delta^{-1} &= \int d^4x \left(a_1 L^a c^a + a_3 \Omega^{\mu a} A_\mu^a + a_5 \bar{c}^a (b^a - g f^{abi} \bar{c}^b c^i) + a_6 \bar{c}^a D_\mu^{ab} A^{\mu b} - \frac{a_5}{2} g f^{abc} \bar{c}^a \bar{c}^b c^c \right. \\
&\quad \left. + a_1 \lambda \left(\frac{1}{2} A_\mu^a A^{\mu a} + \alpha \bar{c}^a c^a \right) + \frac{a_6}{2} \lambda A_\mu^a A^{\mu a} + 2\alpha a_5 \lambda \bar{c}^a c^a + \frac{a_{13} \zeta}{2} \lambda J \right).
\end{aligned} \tag{12.38}$$

We see thus that Σ^{count} contains six free independent parameters, namely a_0, a_1, a_3, a_5, a_6 and a_{13} . These parameters can be reabsorbed by means of a multiplicative renormalization of the gauge coupling constant g , of the gauge and LCO parameters α, ζ , and of the fields $\phi = (A^{\mu a}, A^{\mu i}, c^a, \bar{c}^a, c^i, \bar{c}^i, b^a, b^i)$ and sources $\Phi = (\Omega^{\mu a}, \Omega^{\mu i}, L^a, L^i, \lambda, J)$, according to

$$\Sigma(g, \alpha, \zeta, \phi, \Phi) + \eta \Sigma^{\text{count}} = \Sigma(g_0, \alpha_0, \zeta_0, \phi_0, \Phi_0) + O(\eta^2), \tag{12.39}$$

with

$$g_0 = Z_g g, \quad \alpha_0 = Z_\alpha \alpha, \quad \zeta_0 = Z_\zeta \zeta,$$

$$A_0^{\mu a} = \tilde{Z}_A^{1/2} A^{\mu a}, \quad A_0^{\mu i} = Z_g^{-1} A^{\mu i}, \tag{12.40}$$

$$c_0^a = \tilde{Z}_c^{1/2} c^a, \quad \bar{c}_0^a = \tilde{Z}_c^{1/2} \bar{c}^a, \tag{12.41}$$

$$c_0^i = Z_c^{1/2} c^i, \quad \bar{c}_0^i = Z_c^{-1/2} \bar{c}^i, \tag{12.42}$$

$$b_0^a = Z_g Z_c^{1/2} \tilde{Z}_c^{1/2} b^a, \quad b_0^i = Z_g b^i, \tag{12.43}$$

$$\Omega_0^{\mu a} = \tilde{Z}_A^{-1/2} Z_g^{-1} Z_c^{-1/2} \Omega^{\mu a}, \quad \Omega_0^{\mu i} = Z_c^{-1/2} \Omega^{\mu i}, \tag{12.44}$$

$$L_0^a = Z_g^{-1} \tilde{Z}_c^{-1/2} Z_c^{-1/2} L^a, \quad L_0^i = Z_g^{-1} Z_c^{-1} L^i, \tag{12.45}$$

and

$$\begin{aligned} J_0 &= Z_{L^a}^{-2} \tilde{Z}_c^{-1} J = Z_g^2 Z_c J, \\ \lambda_0 &= Z_{L^a}^{-1} \tilde{Z}_c^{-1/2} \lambda = Z_g Z_c^{1/2} \lambda, \end{aligned} \quad (12.46)$$

with

$$\begin{aligned} Z_g &= 1 - \eta \frac{a_0}{2}, \\ Z_\alpha &= 1 + \eta \left(\frac{2a_5}{\alpha} + a_0 - 2a_6 \right), \\ Z_\zeta &= 1 + \eta (a_{13} + 2a_0 - 2a_1 - 2a_6), \\ \tilde{Z}_A^{1/2} &= 1 + \eta \left(\frac{a_0}{2} + a_3 \right), \\ \tilde{Z}_c^{1/2} &= 1 + \frac{\eta}{2} (a_6 - a_1), \\ Z_c^{1/2} &= 1 + \frac{\eta}{2} (a_6 + a_1). \end{aligned} \quad (12.47)$$

In particular, from eq.(12.46), one sees that the renormalization of the source J , and thus of the composite operator $\mathcal{O}_{\text{MAG}}$, can be expressed in terms of the renormalization of gauge coupling constant and of the diagonal ghost. This property follows from the diagonal ghost equation (12.25) and from the integrated λ -equation (12.29). In particular, for the anomalous dimension of the gluon-ghost operator $\mathcal{O}_{\text{MAG}}$, we obtain [199]

$$\gamma_{\mathcal{O}_{\text{MAG}}}(g^2) = \mu \frac{\partial}{\partial \mu} \log(Z_g^2 Z_c) = -2 \left(\frac{\beta(g^2)}{2g^2} - \gamma_{c^i}(g^2) \right), \quad (12.48)$$

with

$$\begin{aligned} \beta(g^2) &= \mu \frac{\partial g^2}{\partial \mu} = -g^2 \mu \frac{\partial}{\partial \mu} \ln Z_g^2, \\ \gamma_{c^i}(g^2) &= \mu \frac{\partial}{\partial \mu} \ln Z_c^{1/2}. \end{aligned} \quad (12.49)$$

12.2.3 The effective potential.

We present here the main steps in the construction of the effective potential for a local composite operator. A more detailed account of the LCO formalism can be found in [23, 176].

To obtain the effective potential for the condensate $\langle \mathcal{O}_{\text{MAG}} \rangle$, we set the sources Ω_μ^i , Ω_μ^a , L^a , L^i and λ to zero and consider the renormalized generating functional

$$\begin{aligned} \exp(-i\mathcal{W}(J)) &= \int [D\varphi] \exp iS(J), \\ S(J) &= S_{\text{YM}} + S_{\text{MAG}} + S_{\text{diag}} + S_{\text{count}} \\ &\quad + \int d^4x \left(Z_J J \left(\frac{1}{2} \tilde{Z}_A A_\mu^a A^{\mu a} + Z_\alpha \tilde{Z}_c \alpha \bar{c}^a c^a \right) + (\zeta + \delta\zeta) \frac{J^2}{2} \right), \end{aligned} \quad (12.50)$$

where φ denotes the relevant fields and S_{count} is the usual counterterm contribution, i.e. the part without the composite operator. The quantity $\delta\zeta$ is the counterterm accounting for the divergences

proportional to J^2 . Using dimensional regularization throughout with the convention that $d = 4 - \varepsilon$, one has the following identification

$$\zeta_0 J_0^2 = \mu^{-\varepsilon} (\zeta + \delta\zeta) J^2 . \quad (12.51)$$

The functional $\mathcal{W}(J)$ obeys the renormalization group equation (RGE)

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g^2) \frac{\partial}{\partial g^2} + \alpha \gamma_\alpha(g^2) \frac{\partial}{\partial \alpha} - \gamma_{\mathcal{O}_{\text{MAG}}}(g^2) \int d^4 x J \frac{\delta}{\delta J} + \eta(g^2, \zeta) \frac{\partial}{\partial \zeta} \right) \mathcal{W}(J) = 0 , \quad (12.52)$$

where

$$\begin{aligned} \gamma_\alpha(g^2) &= \mu \frac{\partial}{\partial \mu} \ln \alpha = \mu \frac{\partial}{\partial \mu} \ln Z_\alpha^{-1} , \\ \eta(g^2, \zeta) &= \mu \frac{\partial}{\partial \mu} \zeta . \end{aligned} \quad (12.53)$$

From eq.(12.51), one finds

$$\eta(g^2, \zeta) = 2\gamma_{\mathcal{O}_{\text{MAG}}}(g^2)\zeta + \delta(g^2, \alpha) , \quad (12.54)$$

with

$$\delta(g^2, \alpha) = \left(\varepsilon + 2\gamma_{\mathcal{O}_{\text{MAG}}}(g^2) - \beta(g^2) \frac{\partial}{\partial g^2} - \alpha \gamma_\alpha(g^2) \frac{\partial}{\partial \alpha} \right) \delta\zeta . \quad (12.55)$$

Up to now, the LCO parameter ζ is still an arbitrary coupling. As explained in [23, 176], simply setting $\zeta = 0$ would give rise to an inhomogeneous RGE for $\mathcal{W}(J)$

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g^2) \frac{\partial}{\partial g^2} + \alpha \gamma_\alpha(g^2) \frac{\partial}{\partial \alpha} - \gamma_{\mathcal{O}_{\text{MAG}}}(g^2) \int d^4 x J \frac{\delta}{\delta J} \right) \mathcal{W}(J) = \delta(g^2, \alpha) \int d^4 x \frac{J^2}{2} , \quad (12.56)$$

and a non-linear RGE for the associated effective action Γ for the composite operator $\mathcal{O}_{\text{MAG}}$. Furthermore, multiplicative renormalizability is lost and by varying the value of $\delta\zeta$, minima of the effective action can change into maxima or can get lost. However, ζ can be made such a function of g^2 and α so that, if g^2 runs according to $\beta(g^2)$ and α according to $\gamma_\alpha(g^2)$, $\zeta(g^2, \alpha)$ will run according to its RGE (12.54). This is accomplished by setting ζ equal to the solution of the differential equation

$$\left(\beta(g^2) \frac{\partial}{\partial g^2} + \alpha \gamma_\alpha(g^2, \alpha) \frac{\partial}{\partial \alpha} \right) \zeta(g^2, \alpha) = 2\gamma_{\mathcal{O}_{\text{MAG}}}(g^2)\zeta(g^2, \alpha) + \delta(g^2, \alpha) . \quad (12.57)$$

Doing so, $\mathcal{W}(J)$ obeys the homogeneous renormalization group equation

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g^2) \frac{\partial}{\partial g^2} + \alpha \gamma_\alpha(g^2) \frac{\partial}{\partial \alpha} - \gamma_{\mathcal{O}_{\text{MAG}}}(g^2) \int d^4 x J \frac{\delta}{\delta J} \right) \mathcal{W}(J) = 0 . \quad (12.58)$$

To lighten the notation, we will drop the renormalization factors from now on. One will notice that there are terms quadratic in the source J present in $\mathcal{W}(J)$, obscuring the usual energy interpretation. This can be cured by removing the terms proportional to J^2 in the action to get a generating functional that is linear in the source, a goal easily achieved by inserting the following unity,

$$1 = \frac{1}{N} \int [D\sigma] \exp \left[i \int d^4 x \left(-\frac{1}{2\zeta} \left(\frac{\sigma}{g} - \mathcal{O}_{\text{MAG}} - \zeta J \right)^2 \right) \right] , \quad (12.59)$$

with N the appropriate normalization factor, in eq.(12.50) to arrive at the Lagrangian

$$\mathcal{L}(A_\mu, \sigma) = -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a} - \frac{1}{4}F_{\mu\nu}^i F^{\mu\nu i} + \mathcal{L}_{\text{MAG}} + \mathcal{L}_{\text{diag}} - \frac{\sigma^2}{2g^2\zeta} + \frac{1}{g^2\zeta}g\sigma\mathcal{O}_{\text{MAG}} - \frac{1}{2\zeta}(\mathcal{O}_{\text{MAG}})^2, \quad (12.60)$$

while

$$\exp(-i\mathcal{W}(J)) = \int [D\varphi] \exp iS_\sigma(J), \quad (12.61)$$

$$S_\sigma(J) = \int d^4x \left(\mathcal{L}(A_\mu, \sigma) + J \frac{\sigma}{g} \right). \quad (12.62)$$

From eqs.(12.50) and (12.61), one has the following simple relation

$$\left. \frac{\delta \mathcal{W}(J)}{\delta J} \right|_{J=0} = -\langle \mathcal{O}_{\text{MAG}} \rangle = -\left\langle \frac{\sigma}{g} \right\rangle, \quad (12.63)$$

meaning that the condensate $\langle \mathcal{O}_{\text{MAG}} \rangle$ is directly related to the expectation value of the field σ , evaluated with the action $S_\sigma = \int d^4x \mathcal{L}(A_\mu, \sigma)$. As it is obvious from eq.(12.60), $\langle \sigma \rangle \neq 0$ is sufficient to have a tree level dynamical mass for the off-diagonal fields. At lowest order (i.e. tree level), one finds

$$\begin{aligned} m_{\text{gluon}}^{\text{off-diag.}} &= \sqrt{\frac{g\sigma}{\zeta_0}}, \\ m_{\text{ghost}}^{\text{off-diag.}} &= \sqrt{\alpha \frac{g\sigma}{\zeta_0}}. \end{aligned} \quad (12.64)$$

Meanwhile, the diagonal degrees of freedom remain massless. This could have been established already from the local $U(1)^{N-1}$ Ward identity (12.30).

12.3 Gauge parameter independence of the vacuum energy.

We begin this section with a few remarks on the determination of $\zeta(g^2, \alpha)$. From explicit calculations in perturbation theory, it will become clear ² that the RGE functions showing up in the differential equation (12.57) look like

$$\begin{aligned} \beta(g^2) &= -\varepsilon g^2 - 2(\beta_0 g^2 + \beta_1 g^4 + \dots), \\ \gamma_{\mathcal{O}_{\text{MAG}}}(g^2) &= \gamma_0(\alpha)g^2 + \gamma_1(\alpha)g^4 + \dots, \\ \gamma_\alpha(g^2) &= a_0(\alpha)g^2 + a_1(\alpha)g^4 + \dots, \\ \delta(g^2, \alpha) &= \delta_0(\alpha) + \delta_1(\alpha)g^2 + \dots. \end{aligned} \quad (12.65)$$

As such, eq.(12.57) can be solved by expanding $\zeta(g^2, \alpha)$ in a Laurent series in g^2 ,

$$\zeta(g^2, \alpha) = \frac{\zeta_0(\alpha)}{g^2} + \zeta_1(\alpha) + \zeta_2(\alpha)g^2 + \dots. \quad (12.66)$$

²See section 12.5.

More precisely, for the first coefficients ζ_0, ζ_1 of the expression (12.66), one obtains

$$\begin{aligned} 2\beta_0\zeta_0 + \alpha a_0 \frac{\partial \zeta_0}{\partial \alpha} &= 2\gamma_0\zeta_0 + \delta_0, \\ 2\beta_1\zeta_0 + \alpha a_0 \frac{\partial \zeta_1}{\partial \alpha} + \alpha a_1 \frac{\partial \zeta_0}{\partial \alpha} &= 2\gamma_0\zeta_1 + 2\gamma_1\zeta_0 + \delta_1. \end{aligned} \quad (12.67)$$

Notice that, in order to construct the n -loop effective potential, knowledge of the $(n+1)$ -loop RGE functions is needed.

The effective potential calculated with the Lagrangian (12.60) will explicitly depend on the gauge parameter α . The question arises concerning the vacuum energy E_{vac} , (i.e. the effective potential evaluated at its minimum); will it be independent of the choice of α ? Also, as it can be seen from the equations (12.67), each $\zeta_i(\alpha)$ is determined through a first order differential equation in α . Firstly, one has to solve for $\zeta_0(\alpha)$. This will introduce one arbitrary integration constant C_0 . Using the obtained value for $\zeta_0(\alpha)$, one can consequently solve the first order differential equation for $\zeta_1(\alpha)$. This will introduce a second integration constant C_1 , etc. In principle, it is possible that these arbitrary constants influence the vacuum energy, which would represent an unpleasant feature. Notice that the differential equations in α for the ζ_i are due to the running of α in eq.(12.57), encoded in the renormalization group function $\gamma_\alpha(g^2)$. Assume that we would have already shown that E_{vac} does not depend on the choice of α . If we then set $\alpha = \alpha^*$, with α^* a fixed point of the RGE for α at the considered order of perturbation theory, then equation (12.57) determining ζ simplifies to

$$\beta(g^2) \frac{\partial}{\partial g^2} \zeta(g^2, \alpha^*) = 2\gamma_{\mathcal{O}_{\text{MAG}}}(g^2) \zeta(g^2, \alpha^*) + \delta(g^2, \alpha^*), \quad (12.68)$$

since

$$\gamma_\alpha(g^2) \alpha \big|_{\alpha=\alpha^*} = 0. \quad (12.69)$$

This will lead to simple algebraic equations for the $\zeta_i(\alpha^*)$. Hence, no integration constants will enter the final result for the vacuum energy for $\alpha = \alpha^*$, and since E_{vac} does not depend on α , E_{vac} will never depend on the integration constants, even when calculated for a general α . Hence, we can put them equal to zero from the beginning for simplicity.

Summarizing, two questions remain. Firstly, we should prove that the value of α will not influence the obtained value for E_{vac} . Secondly, we should show that there exists a fixed point α^* . We postpone the discussion concerning the second question to the next section, giving a positive answer to the first one. In order to do so, let us reconsider the generating functional (12.61). We have the following identification, ignoring the overall normalization factors

$$\begin{aligned} \exp(-i\mathcal{W}(J)) &= \int [D\varphi] \exp iS_\sigma(J) \\ &= \frac{1}{N} \int [D\varphi D\sigma] \exp i \left[S(J) + \int d^4x \left(-\frac{1}{2\zeta} \left(\frac{\sigma}{g} - \mathcal{O}_{\text{MAG}} - \zeta J \right)^2 \right) \right] \end{aligned} \quad (12.70)$$

where $S(J)$ and $S_\sigma(J)$ are given respectively by eq.(12.50), and eq.(12.62). Obviously,

$$\frac{d}{d\alpha} \frac{1}{N} \int [D\sigma] \exp \left[i \int d^4x \left(-\frac{1}{2\zeta} \left(\frac{\sigma}{g} - \mathcal{O}_{\text{MAG}} - \zeta J \right)^2 \right) \right] = \frac{d}{d\alpha} 1 = 0, \quad (12.71)$$

so that

$$\frac{d\mathcal{W}(J)}{d\alpha} = - \left\langle s \int d^4x \bar{s} \left(\frac{1}{2} c^a \bar{c}^a \right) \right\rangle \Big|_{J=0} + \text{terms} \propto J, \quad (12.72)$$

which follows directly from

$$\frac{dS(J)}{d\alpha} = s \bar{s} \int d^4x \left(\frac{1}{2} c^a \bar{c}^a \right) + \text{terms} \propto J. \quad (12.73)$$

We see that the first term in the right hand side of (12.73) is an exact BRST variation. As such, its vacuum expectation value vanishes. This is the usual argument to prove the gauge parameter independence in the BRST framework [59]. Note that no local operator $\hat{\mathcal{O}}$, with $s\hat{\mathcal{O}} = \mathcal{O}_{\text{MAG}}$, exists. Furthermore, extending the action of the BRST transformation on the σ -field by

$$s\sigma = gs\mathcal{O}_{\text{MAG}} = -A^{\mu a} D_\mu^{ab} c^b + \alpha b^a c^a - \alpha g f^{abi} \bar{c}^a \bar{c}^b c^i - \frac{\alpha}{2} g f^{abc} \bar{c}^a c^b c^c \quad (12.74)$$

one can easily check that

$$s \int d^4x \mathcal{L}(A_\mu, \sigma) = 0, \quad (12.75)$$

so that we have a BRST invariant σ -action. Thus, when we consider the vacuum, corresponding to $J = 0$, only the BRST exact term in eq.(12.72) survives. The effective action Γ is related to $\mathcal{W}(J)$ through a Legendre transformation

$$\Gamma \left(\frac{\sigma}{g} \right) = -\mathcal{W}(J) - \int d^4y J(y) \frac{\sigma(y)}{g}. \quad (12.76)$$

The effective potential $V(\sigma)$ is then defined as

$$-V(\sigma) \int d^4x = \Gamma \left(\frac{\sigma}{g} \right). \quad (12.77)$$

Let $\sigma_{\min}$ be the solution of

$$\frac{dV(\sigma)}{d\sigma} = 0. \quad (12.78)$$

From

$$\frac{\delta}{\delta \left(\frac{\sigma}{g} \right)} \Gamma = -J, \quad (12.79)$$

it follows that

$$\sigma = \sigma_{\min} \Rightarrow J = 0, \quad (12.80)$$

and hence, we derive from eqs.(12.76) and (12.77) that

$$\frac{d}{d\alpha} V(\sigma) \Big|_{\sigma=\sigma_{\min}} \int d^4x = \frac{d}{d\alpha} \mathcal{W}(J) \Big|_{J=0}. \quad (12.81)$$

Thus, due to eq.(12.72),

$$\left. \frac{d}{d\alpha} V(\sigma) \right|_{\sigma=\sigma_{\min}} = 0 . \quad (12.82)$$

We conclude that the vacuum energy E_{vac} should be independent from the gauge parameter α .

A completely analogous derivation was obtained in the case of the linear gauge [212]. Nevertheless, in spite of the previous argument, explicit results in that case showed that E_{vac} did depend on α . In [212] it was argued that this apparent disagreement was due to a mixing of different orders of perturbation theory. Let us explain this with a simple example. Let us first notice that a key argument in the previous analysis is that the source $J = 0$ vanishes at the end of the calculations. In practice, $J = 0$ is achieved by solving the gap equation (12.78). Moreover, in a power series expansion in the coupling constant, the derivative of the effective potential with respect to σ will look like

$$(v_0 + v_1 g^2 + O(g^4)) \sigma , \quad (12.83)$$

where we assume that we work up to order g^2 . The corresponding gap equation reads

$$v_0 + v_1 g^2 + O(g^4) = 0 . \quad (12.84)$$

Due to eqs.(12.77) and (12.79), one also has

$$J = g (v_0 + v_1 g^2 + O(g^4)) \sigma . \quad (12.85)$$

Imposing the gap equation (12.84) leads to

$$J = g (0 + O(g^4)) \sigma . \quad (12.86)$$

However, as it can be immediately checked from expression (12.70), there are several terms proportional to J in the right-hand side of eq.(12.72). For instance, one of them is given by $\frac{\partial \zeta}{\partial \alpha} J^2$. Since

$$\frac{\partial \zeta}{\partial \alpha} = \frac{\partial \zeta_0}{\partial \alpha} \frac{1}{g^2} + \frac{\partial \zeta_1}{\partial \alpha} + O(g^2) . \quad (12.87)$$

we find

$$\frac{\partial \zeta}{\partial \alpha} J^2 = \left(\frac{\partial \zeta_0}{\partial \alpha} v_0^2 + \left(\frac{\partial \zeta_0}{\partial \alpha} 2v_0 v_1 + \frac{\partial \zeta_1}{\partial \alpha} v_0^2 \right) g^2 + O(g^4) \right) \sigma^2 . \quad (12.88)$$

Squaring the gap equation (12.84),

$$v_0^2 + 2v_1 v_0 g^2 + O(g^4) = 0 , \quad (12.89)$$

leads to

$$\frac{\partial \zeta}{\partial \alpha} J^2 = \left(\frac{\partial \zeta_1}{\partial \alpha} v_0^2 g^2 + O(g^4) \right) \sigma^2 . \quad (12.90)$$

We see that, if one consistently works to the first order, terms such as $\frac{\partial \zeta}{\partial \alpha} J^2$ do not equal zero, although $J = 0$ to that order. Terms like those on the right-hand side of eq.(12.90) are cancelled by terms which are formally of higher order, requiring thus a mixing of different orders of perturbation theory. Of course, this problem would not have occurred if we were able to compute the effective potential up to infinite order, an evidently hopeless task. Nevertheless, in [212] we succeeded in finding a suitable modification of the LCO formalism in order to circumvent this problem and obtaining a well defined

gauge independent vacuum energy E_{vac} , without the need of working at infinite order. Instead of the action (12.50), let us consider the following action

$$\tilde{S}(\tilde{J}) = S_{\text{YM}} + S_{\text{MAG}} + S_{\text{diag}} + \int d^4x \left[\tilde{J} \mathcal{F}(g^2, \alpha) \mathcal{O}_{\text{MAG}} + \frac{\zeta}{2} \mathcal{F}^2(g^2, \alpha) \tilde{J}^2 \right], \quad (12.91)$$

where, for the moment, $\mathcal{F}(g^2, \alpha)$ is an arbitrary function of α of the form

$$\mathcal{F}(g^2, \alpha) = 1 + f_0(\alpha)g^2 + f_1(\alpha)g^4 + O(g^6), \quad (12.92)$$

and $\tilde{J}$ is now the source. The generating functional becomes

$$\exp(-i\tilde{\mathcal{W}}(\tilde{J})) = \int [D\phi] \exp i\tilde{S}(\tilde{J}). \quad (12.93)$$

Taking the functional derivative of $\tilde{\mathcal{W}}(\tilde{J})$ with respect to $\tilde{J}$, we obtain

$$\left. \frac{\delta \tilde{\mathcal{W}}(\tilde{J})}{\delta \tilde{J}} \right|_{\tilde{J}=0} = -\mathcal{F}(g^2, \alpha) \langle \mathcal{O}_{\text{MAG}} \rangle. \quad (12.94)$$

Once more, we insert unity via

$$1 = \frac{1}{N} \int [D\tilde{\sigma}] \exp \left[i \int d^4x \left(-\frac{1}{2\zeta} \left(\frac{\tilde{\sigma}}{g\mathcal{F}(g^2, \alpha)} - \mathcal{O}_{\text{MAG}} - \zeta \tilde{J} \mathcal{F}(g^2, \alpha) \right)^2 \right) \right], \quad (12.95)$$

to arrive at the following Lagrangian

$$\begin{aligned} \tilde{\mathcal{L}}(A_\mu, \tilde{\sigma}) &= -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} - \frac{1}{4} F_{\mu\nu}^i F^{\mu\nu i} + \mathcal{L}_{\text{MAG}} + \mathcal{L}_{\text{diag}} \\ &\quad - \frac{\tilde{\sigma}^2}{2g^2 \mathcal{F}^2(g^2, \alpha) \zeta} + \frac{1}{g^2 \mathcal{F}(g^2, \alpha) \zeta} g \tilde{\sigma} \mathcal{O}_{\text{MAG}} - \frac{1}{2\zeta} (\mathcal{O}_{\text{MAG}})^2. \end{aligned} \quad (12.96)$$

From the generating functional

$$\exp(-i\tilde{\mathcal{W}}(\tilde{J})) = \int [D\phi] \exp iS_{\tilde{\sigma}}(\tilde{J}), \quad (12.97)$$

$$S_{\tilde{\sigma}}(\tilde{J}) = \int d^4x \left(\mathcal{L}(A_\mu, \tilde{\sigma}) + \tilde{J} \frac{\tilde{\sigma}}{g} \right). \quad (12.98)$$

it follows that

$$\left. \frac{\delta \tilde{\mathcal{W}}(\tilde{J})}{\delta \tilde{J}} \right|_{\tilde{J}=0} = -\left\langle \frac{\tilde{\sigma}}{g} \right\rangle \Rightarrow \langle \tilde{\sigma} \rangle = g \mathcal{F}(g^2, \alpha) \langle \mathcal{O}_{\text{MAG}} \rangle, \quad (12.99)$$

The renormalizability of the action (12.62) implies that the action (12.98) will be renormalizable too. Notice indeed that both actions are connected through the transformation

$$\tilde{J} = \frac{J}{\mathcal{F}(g^2, \alpha)}. \quad (12.100)$$

The tree level off-diagonal masses are now provided by

$$\begin{aligned} m_{\text{gluon}}^{\text{off-diag.}} &= \sqrt{\frac{g\tilde{\sigma}}{\zeta_0}} , \\ m_{\text{ghost}}^{\text{off-diag.}} &= \sqrt{\alpha \frac{g\tilde{\sigma}}{\zeta_0}} , \end{aligned} \quad (12.101)$$

while the vacuum configuration is determined by solving the gap equation

$$\frac{d\tilde{V}(\tilde{\sigma})}{d\tilde{\sigma}} = 0 , \quad (12.102)$$

with $\tilde{V}(\tilde{\sigma})$ the effective potential. Minimizing $\tilde{V}(\tilde{\sigma})$ will lead to a vacuum energy $E_{\text{vac}}(\alpha)$ which will depend on α and the hitherto undetermined functions $f_i(\alpha)$ ³. We will determine those functions $f_i(\alpha)$ by requiring that $E_{\text{vac}}(\alpha)$ is α -independent. More precisely, one has

$$\frac{dE_{\text{vac}}}{d\alpha} = 0 \Rightarrow \text{first order differential equations in } \alpha \text{ for } f_i(\alpha) . \quad (12.103)$$

Of course, in order to be able to determine the $f_i(\alpha)$, we need an initial value for the vacuum energy E_{vac} . This corresponds to initial conditions for the $f_i(\alpha)$. In the case of the linear gauges, to fix the initial condition we employed the Landau gauge [212], a choice which would also be possible in case of the Curci-Ferrari gauges, since the Landau gauge belongs to these classes of gauges. This choice of the Landau gauge can be motivated by observing that the integrated operator $\int d^4x A_\mu^A A^{\mu A}$ has a gauge invariant meaning in the Landau gauge, due to the transversality condition $\partial_\mu A^{\mu A} = 0$, namely

$$(VT)^{-1} \min_{U \in SU(N)} \int d^4x \left[(A_\mu^A)^U (A^{\mu A})^U \right] = \int d^4x (A_\mu^A A^{\mu A}) \quad \text{in the Landau gauge} , \quad (12.104)$$

with the operator on the left hand side of eq.(12.104) being gauge invariant⁴. Moreover, the Landau gauge is also an all-order fixed point of the RGE for the gauge parameter in case of the linear and Curci-Ferrari gauges. At first glance, it could seem that it is not possible anymore to make use of the Landau gauge as initial condition in the case of the MAG, since the Landau gauge does not belong to the class of gauges we are currently considering. Fortunately, we shall be able to prove that we can use the Landau gauge as initial condition for the MAG too. This will be the content of the next section.

Before turning our attention to this task, it is worth noticing that, if one would work up to infinite order, the expressions (12.91) and (12.98) can be transformed *exactly* into those of (12.50), respectively (12.62) by means of eq.(12.100) and its associated transformation

$$\tilde{\sigma} = \mathcal{F}(g^2, \alpha) \sigma , \quad (12.105)$$

so that the effective potentials $\tilde{V}(\tilde{\sigma})$ and $V(\sigma)$ are *exactly* the same at infinite order, and as such will give rise to the same, gauge parameter independent, vacuum energy.

12.4 Interpolating between the MAG and the Landau gauge.

In this section we shall introduce a generalized renormalizable gauge which interpolates between the MAG and the Landau gauge. This will provide a connection between these two gauges, allowing us

³At first order, E_{vac} will depend on $f_0(\alpha)$, at second order on $f_0(\alpha)$ and $f_1(\alpha)$, etc.

⁴Making abstract of the existence of Gribov copies

to use the Landau gauge as initial condition. An example of such a generalized gauge, interpolating between the Landau and the Coulomb gauge was already presented in [213]. Moreover, we must realize that in the present case, we must also interpolate between the composite operator $\frac{1}{2}A_\mu^A A^{\mu A}$ of the Landau gauge and the gluon-ghost operator $\mathcal{O}_{\text{MAG}}$ of the MAG. Although this seems to be a highly complicated assignment, there is an elegant way to treat it.

Consider again the $SU(N)$ Yang-Mills action with the MAG gauge fixing (12.12). For the residual Abelian gauge freedom, we impose

$$\begin{aligned} S'_{\text{diag}} = & \int d^4x \left(b^i \partial_\mu A^{\mu i} + \bar{c}^i \partial^2 c^i + \bar{c}^i \partial_\mu (g f^{iab} A^{\mu a} c^b) + \kappa g f^{iab} A_\mu^a (\partial_\mu c^i) \bar{c}^b \right. \\ & + \kappa g^2 f^{iab} f^{icd} \bar{c}^a c^d A_\mu^b A^{\mu c} - \kappa g f^{iab} A_\mu^i A^{\mu a} (b^b - g f^{jbc} \bar{c}^c c^j) \\ & \left. + \kappa g f^{iab} A^{\mu i} (D_\mu^{ac} c^c) \bar{c}^b + \kappa g^2 f^{abi} f^{acd} A_\mu^i A^{\mu c} c^d \bar{c}^b \right), \end{aligned} \quad (12.106)$$

where κ is an additional gauge parameter. The gauge fixing (12.106) can be rewritten as a BRST exact expression

$$S'_{\text{diag}} = \int d^4x \left[(1 - \kappa) s (\bar{c}^i \partial_\mu A^{\mu i}) + \kappa s \bar{s} \left(\frac{1}{2} A_\mu^i A^{\mu i} \right) \right]. \quad (12.107)$$

Next, we will introduce the following generalized mass dimension two operator,

$$\mathcal{O} = \frac{1}{2} A_\mu^a A^{\mu a} + \frac{\kappa}{2} A_\mu^i A^{\mu i} + \alpha \bar{c}^a c^a, \quad (12.108)$$

by means of

$$\begin{aligned} S'_{\text{LCO}} = & s \int d^4x \left(\lambda \mathcal{O} + \zeta \frac{\lambda J}{2} \right) \\ = & \int d^4x \left(J \mathcal{O} + \zeta \frac{J^2}{2} - \alpha \lambda b^a c^a + \lambda A^{\mu a} D_\mu^{ab} c^b + \alpha \lambda \bar{c}^a \left(g f^{abi} c^b c^i + \frac{g}{2} f^{abc} c^b c^c \right) \right. \\ & \left. - \kappa \lambda c^i \partial_\mu A^{\mu i} + \kappa g f^{iab} \lambda A_\mu^a A^{\mu i} c^b \right), \end{aligned} \quad (12.109)$$

with (J, λ) a BRST doublet of external sources,

$$s\lambda = J, \quad sJ = 0. \quad (12.110)$$

As in the case of the gluon-ghost operator (12.18), the generalized operator of eq.(12.108) turns out to be BRST invariant on-shell, a property which can again be expressed in a functional way, see eq.(12.121).

Let us take a closer look at the action

$$\Sigma' = S_{\text{YM}} + S_{\text{MAG}} + S'_{\text{diag}} + S'_{\text{LCO}} + S_{\text{ext}}. \quad (12.111)$$

The external source part of the action, S_{ext} , is the same as given in eq.(12.16).

Also, it can be noticed that, for $\kappa \rightarrow 0$, the generalized local composite operator $\mathcal{O}$ of eq.(12.108) reduces to the composite operator $\mathcal{O}_{\text{MAG}}$ of the MAG, while the diagonal gauge fixing (12.107) reduces to the Abelian Landau gauge (12.15). Said otherwise, for $\kappa \rightarrow 0$, the action Σ' of eq.(12.111) reduces to the one we are actually interested in and which we have discussed in the previous sections.

Another special case is $\kappa \rightarrow 1$, $\alpha \rightarrow 0$. Then the gauge fixing terms of Σ' are

$$S_{\text{MAG}} + S'_{\text{diag}} = \int d^4x s \left(-A_\mu^A \partial^\mu \bar{c}^A \right) = \int d^4x \left(\bar{c}^A \partial^\mu D_\mu^{AB} c^B + b^A \partial^\mu A_\mu^A \right), \quad (12.112)$$

which is nothing else than the Landau gauge. At the same time, we also have

$$\lim_{(\alpha, \kappa) \rightarrow (0, 1)} \mathcal{O} = \frac{1}{2} A_\mu^A A^{\mu A}, \quad (12.113)$$

which is the pure gluon mass operator of the Landau gauge [42, 153].

From [153], we already know that the Landau gauge with the inclusion of the operator $A_\mu^A A^{\mu A}$ is renormalizable to all orders of perturbation theory. On the other hand, in section 12.2, we have proven the renormalizability for $\kappa = 0$. Before we continue our argument, let us first prove the renormalizability of Σ' for general α and $\kappa \neq 0$. The complete action Σ' , as given in eq.(12.111), is BRST invariant

$$s\Sigma' = 0, \quad (12.114)$$

and obeys the following identities

- The Slavnov-Taylor identity, provided by

$$\begin{aligned} \mathcal{S}(\Sigma') = & \int d^4x \left(\frac{\delta \Sigma'}{\delta \Omega^{\mu a}} \frac{\delta \Sigma'}{\delta A_\mu^a} + \frac{\delta \Sigma'}{\delta \Omega^{\mu i}} \frac{\delta \Sigma'}{\delta A_\mu^i} + \frac{\delta \Sigma'}{\delta L^a} \frac{\delta \Sigma'}{\delta c^a} + \frac{\delta \Sigma'}{\delta L^i} \frac{\delta \Sigma'}{\delta c^i} \right. \\ & \left. + b^a \frac{\delta \Sigma'}{\delta \bar{c}^a} + b^i \frac{\delta \Sigma'}{\delta \bar{c}^i} + J \frac{\delta \Sigma'}{\delta \lambda} \right) = 0. \end{aligned} \quad (12.115)$$

- The integrated diagonal ghost equation

$$\mathcal{G}^i \Sigma' = \Delta_{\text{cl}}^i, \quad (12.116)$$

where

$$\mathcal{G}^i = \int d^4x \left[\frac{\delta}{\delta \bar{c}^i} + g f^{abi} \bar{c}^a \frac{\delta}{\delta b^b} \right], \quad (12.117)$$

and

$$\Delta_{\text{cl}}^i = \int d^4x \left[g f^{abi} \Omega^{\mu a} A_\mu^b - g f^{abi} L^a c^b + \kappa \lambda \partial_\mu A^{\mu i} \right], \quad (12.118)$$

a classical breaking.

- The diagonal anti-ghost equation

$$\frac{\delta \Sigma'}{\delta \bar{c}^i} + \partial^\mu \frac{\delta \Sigma'}{\delta \Omega^{\mu i}} = 0, \quad (12.119)$$

and

$$\frac{\delta \Sigma'}{\delta b^i} = \partial_\mu A^{\mu i}. \quad (12.120)$$

- The integrated generalized λ -equation

$$\int d^4x \left[\frac{\delta}{\delta\lambda} + c^a \frac{\delta}{\delta b^a} + \kappa c^i \frac{\delta}{\delta b^i} \right] \Sigma' = 0, \quad (12.121)$$

expressing the on-shell BRST invariance of the operator $\mathcal{O}$ of eq.(12.108).

Also in this case, these Ward identities extend to the quantum level. Accordingly, the most general local counterterm Σ'^{count} must obey the following constraints

$$\begin{aligned} \mathcal{B}_{\Sigma'} \Sigma'^{\text{count}} &= 0, \\ \mathcal{G}^i \Sigma'^{\text{count}} &= 0, \\ \frac{\delta}{\delta b^i} \Sigma'^{\text{count}} &= 0, \\ \left[\frac{\delta}{\delta \bar{c}^i} + \partial_\mu \frac{\delta}{\delta \Omega_\mu^i} \right] \Sigma'^{\text{count}} &= 0, \\ \int d^4x \left[\frac{\delta}{\delta\lambda} + c^a \frac{\delta}{\delta b^a} \right] \Sigma'^{\text{count}} &= 0. \end{aligned} \quad (12.122)$$

where $\mathcal{B}_{\Sigma'}$ denotes the nilpotent, $\mathcal{B}_{\Sigma'} \mathcal{B}_{\Sigma'} = 0$, linearized operator

$$\begin{aligned} \mathcal{B}_{\Sigma'} &= \int d^4x \left(\frac{\delta \Sigma'}{\delta \Omega_\mu^a} \frac{\delta}{\delta A_\mu^a} + \frac{\delta \Sigma'}{\delta A_\mu^a} \frac{\delta}{\delta \Omega_\mu^a} + \frac{\delta \Sigma'}{\delta \Omega_\mu^i} \frac{\delta}{\delta A_\mu^i} + \frac{\delta \Sigma'}{\delta A_\mu^i} \frac{\delta}{\delta \Omega_\mu^i} + \frac{\delta \Sigma'}{\delta L^a} \frac{\delta}{\delta c^a} \right. \\ &\quad \left. + \frac{\delta \Sigma'}{\delta c^a} \frac{\delta}{\delta L^a} + \frac{\delta \Sigma'}{\delta L^i} \frac{\delta}{\delta c^i} + \frac{\delta \Sigma'}{\delta c^i} \frac{\delta}{\delta L^i} + b^a \frac{\delta}{\delta \bar{c}^a} + b^i \frac{\delta}{\delta \bar{c}^i} + J \frac{\delta}{\delta \lambda} \right). \end{aligned} \quad (12.123)$$

From general results on BRST cohomology [60], we know that the most general, local counterterm can be written as

$$\Sigma'^{\text{count}} = -\frac{a'_0}{4} \int d^4x (F_{\mu\nu}^a F^{\mu\nu a} + F_{\mu\nu}^i F^{\mu\nu i}) + \mathcal{B}_\Sigma \Delta^{-1}, \quad (12.124)$$

where Δ^{-1} is an integrated local polynomial of ghost number -1 and dimension 4, given by

$$\begin{aligned} \Delta^{-1} &= \int d^4x \left[a'_1 \Omega_\mu^a A^{\mu a} + a'_2 \Omega_\mu^i A^{\mu i} + a'_3 L^a c^a + a'_4 L^i c^i + a'_5 (\partial_\mu \bar{c}^a) A^{\mu a} + a'_5 g f^{abi} \bar{c}^a A_\mu^i A^{\mu b} \right. \\ &\quad + a'_6 (\partial_\mu \bar{c}^i) A^{\mu i} + a'_7 g f^{aic} \bar{c}^a \bar{c}^i c^c + \frac{a'_8}{2} \alpha g f^{abi} \bar{c}^a \bar{c}^b c^i + \frac{a'_9}{2} \alpha g f^{abc} \bar{c}^a \bar{c}^b c^c + \alpha a'_{10} b^a \bar{c}^a \\ &\quad \left. + a'_{11} b^i \bar{c}^i + \frac{a'_{12}}{2} \lambda A_\mu^a A^{\mu a} + \frac{a'_{13}}{2} \kappa \lambda A_\mu^i A^{\mu i} + a'_{14} \alpha \lambda \bar{c}^a c^a + a'_{15} \alpha \lambda \bar{c}^i c^i + \frac{a'_{16}}{2} \zeta \lambda J \right] \end{aligned} \quad (12.125)$$

The constraints (12.122) lead to the relations

$$\begin{aligned} a'_7 &= a'_{11} = a'_{15} = 0, \\ a'_6 &= a'_2, \\ a'_{13} &= a'_4 - a'_2, \\ a'_8 &= -2a'_{10}, \\ a'_{14} &= 2a'_{10} + a'_3, \\ a'_{12} &= a'_3 - a'_5, \\ a''_5 &= a'_5 + \kappa (a'_{13} - a'_3), \\ a'_9 &= -a'_{10}, \\ a'_2 &= a'_4 = 0 \quad \text{and thus } a'_{13} = 0. \end{aligned} \quad (12.126)$$

Summarizing

$$\begin{aligned}
\Delta^{-1} = & \int d^4x \left[a'_1 \Omega_\mu^a A^{\mu a} + a'_3 L^a c^a - a'_5 \bar{c}^a D_\mu^{ab} A^{\mu b} - \kappa a'_3 g f^{abi} \bar{c}^a A_\mu^i A^{\mu b} - a'_{10} \alpha g f^{abi} \bar{c}^a \bar{c}^b c^i \right. \\
& - \frac{a'_{10}}{2} \alpha g f^{abc} \bar{c}^a \bar{c}^b c^c + \alpha a'_{10} b^a \bar{c}^a + a'_3 \lambda \left(\frac{1}{2} A_\mu^a A^{\mu a} + \alpha \bar{c}^a c^a \right) - \frac{a'_5}{2} \lambda A_\mu^a A^{\mu a} \\
& \left. + 2a'_{10} \alpha \lambda \bar{c}^a c^a + \frac{a'_{16}}{2} \zeta \lambda J \right] . \tag{12.127}
\end{aligned}$$

In comparison with the case of the MAG, we see that Σ'^{count} also contains six free independent parameters, namely $a'_0, a'_1, a'_3, a'_5, a'_{10}$ and a'_{13} , despite the fact that the action Σ' contains the extra gauge parameter κ . These parameters can be reabsorbed by a suitable multiplicative renormalization of the gauge coupling constant g , of the gauge and LCO parameters α, κ, ζ , and of the fields $\phi = (A^{\mu a}, A_{\mu i}, c^a, \bar{c}^a, c^i, \bar{c}^i, b^a, b^i)$ and sources $\Phi = (\Omega^{\mu a}, \Omega^{\mu i}, L^a, L^i, \lambda, J)$, according to

$$\Sigma'(g, \alpha, \kappa, \zeta, \phi, \Phi) + \eta \Sigma'^{\text{count}} = \Sigma'(g_0, \alpha_0, \kappa_0, \zeta_0, \phi_0, \Phi_0) + O(\eta^2) , \tag{12.128}$$

where

$$g_0 = Z_g g , \quad \alpha_0 = Z_\alpha \alpha , \quad \zeta_0 = Z_\zeta \zeta , \quad \kappa_0 = Z_c^{-1} \tilde{Z}_A^{-1/2} Z_g^{-1} \kappa ,$$

$$A_0^{\mu a} = \tilde{Z}_A^{1/2} A^{\mu a} , \quad A_0^{\mu i} = Z_g^{-1} A^{\mu i} , \tag{12.129}$$

$$c_0^a = \tilde{Z}_c^{1/2} c^a , \quad \bar{c}_0^a = \tilde{Z}_c^{1/2} \bar{c}^a , \tag{12.130}$$

$$c_0^i = Z_c^{1/2} c^i , \quad \bar{c}_0^i = Z_c^{-1/2} \bar{c}^i , \tag{12.131}$$

$$b_0^a = Z_g Z_c^{1/2} \tilde{Z}_c^{1/2} b^a , \quad b_0^i = Z_g b^i , \tag{12.132}$$

$$\Omega_0^{\mu a} = \tilde{Z}_A^{-1/2} Z_g^{-1} Z_c^{-1/2} \Omega^{\mu a} , \quad \Omega_0^{\mu i} = Z_c^{-1/2} \Omega^{\mu i} , \tag{12.133}$$

$$L_0^a = Z_g^{-1} \tilde{Z}_c^{-1/2} Z_c^{-1/2} L^a , \quad L_0^i = Z_g^{-1} Z_c^{-1} L^i , \tag{12.134}$$

$$J_0 = Z_g^2 Z_c J , \quad \lambda_0 = Z_g Z_c^{1/2} \lambda , \tag{12.135}$$

with

$$\begin{aligned}
Z_g &= 1 - \eta \frac{a'_0}{2} , \\
Z_\alpha &= 1 + \eta (2a'_5 + a'_0 - a'_8) , \\
Z_\zeta &= 1 + \eta (a'_{16} + 2a'_0 + 2a'_5 - 2a'_3) , \\
\tilde{Z}_A^{1/2} &= 1 + \eta \left(\frac{a'_0}{2} + a'_1 \right) , \\
\tilde{Z}_c^{1/2} &= 1 - \frac{\eta}{2} (a'_5 + a'_3) , \\
Z_c^{1/2} &= 1 + \frac{\eta}{2} (a'_3 - a'_5) . \tag{12.136}
\end{aligned}$$

Interestingly, the additional gauge parameter κ does not renormalize in an independent way. Furthermore, from eq.(12.135), we notice that the relation (12.48) is generalized to the operator $\mathcal{O}$, i.e.

$$\gamma_{\mathcal{O}}(g^2) = -2 \left(\frac{\beta(g^2)}{2g^2} - \gamma_{ci}(g^2) \right). \quad (12.137)$$

Summarizing, we have constructed a renormalizable gauge that is labelled by a couple of parameters (α, κ) . It allows us to introduce a generalized composite operator $\mathcal{O}$, given by eq.(12.108), which embodies the local operator $A_\mu^A A^{\mu A}$ of the Landau gauge as well as the operator $\mathcal{O}_{\text{MAG}}$ of the MAG. To construct the effective potential, one sets all sources equal to zero, except J , and introduces unity to remove the J^2 terms. A completely analogous argument as the one given in section 12.3 allows to conclude that the minimum value of $V(\sigma)$, thus E_{vac} , will be independent of α and κ , essentially because the derivative with respect to α as well as with respect to κ is BRST exact, up to terms in the source J . This independence of α and κ is again only assured at infinite order in perturbation theory, so we can generalize the construction, proposed in section 12.3, by making the function $\mathcal{F}$ of eq.(12.92) also dependent on κ . The foregoing analysis is sufficient to make sure that we can use the Landau gauge result for E_{vac} as the initial condition for the vacuum energy of the MAG. Moreover, we are now even in the position to answer the question about the existence of a fixed point of the RGE for the gauge parameter α , which was necessary to certify that no arbitrary constants would enter the results for E_{vac} . We already mentioned that the Landau gauge, i.e. the case $(\alpha, \kappa) = (0, 1)$, is a renormalizable model [153], i.e. the Landau gauge is stable against radiative corrections. This can be reexpressed by saying that $(\alpha, \kappa) = (0, 1)$ is a fixed point of the RGE for the gauge parameters, and this to all orders of perturbation theory.

12.5 Numerical results for $SU(2)$.

After a quite lengthy formal construction of the LCO formalism in the case of the MAG, we are now ready to present explicit results. In this paper, we will restrict ourselves to the evaluation of the one-loop effective potential in the case of $SU(2)$. As renormalization scheme, we adopt the modified minimal subtraction scheme ($\overline{\text{MS}}$). Let us give here, for further use, the values of the one-loop anomalous dimensions of the relevant fields and couplings in the case of $SU(2)$. In our conventions, one has [183, 179, 194]

$$\gamma_{ci}(g^2) = (-3 - \alpha) \frac{g^2}{16\pi^2} + O(g^4), \quad (12.138)$$

$$\gamma_\alpha(g^2) = \left(-2\alpha + \frac{8}{3} - \frac{6}{\alpha} \right) \frac{g^2}{16\pi^2} + O(g^4), \quad (12.139)$$

while

$$\beta(g^2) = -\varepsilon g^2 - 2 \left(\frac{22}{3} \frac{g^4}{16\pi^2} \right) + O(g^6), \quad (12.140)$$

and exploiting the relation (12.48)

$$\gamma_{\mathcal{O}_{\text{MAG}}}(g^2) = \left(\frac{26}{3} - 2\alpha \right) \frac{g^2}{16\pi^2} + O(g^4), \quad (12.141)$$

a result consistent with that of [179].

The reader will notice that we have given only the one-loop values of the anomalous dimensions, despite the fact that we have announced that one needs $(n+1)$ -loop knowledge of the RGE functions

to determine the n -loop potential. As we shall see soon, the introduction of the function $\mathcal{F}(g^2, \alpha)$ and the use of the Landau gauge as initial condition allow us to determine the one-loop results we are interested in, from the one-loop RGE functions only.

Let us first determine the counterterm $\delta\zeta$. For the generating functional $\mathcal{W}(J)$, we find at one-loop ⁵

$$\begin{aligned} \mathcal{W}(J) &= \int d^d x \left(-(\zeta + \delta\zeta) \frac{J^2}{2} \right) + i \ln \det [\delta^{ab} (\partial^2 + \alpha J)] \\ &\quad - \frac{i}{2} \ln \det \left[\delta^{ab} \left((\partial^2 + J) g_{\mu\nu} - \left(1 - \frac{1}{\alpha} \right) \partial_\mu \partial_\nu \right) \right], \end{aligned} \quad (12.142)$$

and employing

$$\ln \det \left[\delta^{ab} \left((\partial^2 + J) g_{\mu\nu} - \left(1 - \frac{1}{\alpha} \right) \partial_\mu \partial_\nu \right) \right] = \delta^{aa} [(d-1) \text{tr} \ln (\partial^2 + J) + \text{tr} \ln (\partial^2 + \alpha J)], \quad (12.143)$$

with

$$\delta^{aa} = N(N-1) = 2 \text{ for } N = 2, \quad (12.144)$$

one can calculate the divergent part of eq.(12.142),

$$\mathcal{W}(J) = \int d^4 x \left[-\delta\zeta \frac{J^2}{2} - \frac{3}{16\pi^2} J^2 \frac{1}{\varepsilon} - \frac{1}{16\pi^2} \alpha^2 J^2 \frac{1}{\varepsilon} + \frac{1}{8\pi^2} \alpha^2 J^2 \frac{1}{\varepsilon} \right]. \quad (12.145)$$

Consequently,

$$\delta\zeta = \frac{1}{8\pi^2} (\alpha^2 - 3) \frac{1}{\varepsilon} + O(g^2). \quad (12.146)$$

Next, we can compute the RGE function $\delta(g^2, \alpha)$ from eq.(12.55), obtaining

$$\delta(g^2, \alpha) = \frac{\alpha^2 - 3}{8\pi^2} + O(g^2). \quad (12.147)$$

Having determined this, we are ready to calculate ζ_0 . The differential equation (12.67) is solved by

$$\zeta_0(\alpha) = \alpha + (9 - 4\alpha + 3\alpha^2) C_0, \quad (12.148)$$

with C_0 an integration constant. As already explained in the previous sections, we can consistently put $C_0 = 0$. Here, we have written it explicitly to illustrate that, if α would coincide with the one-loop fixed point of the RGE for the gauge parameter, the part proportional to C_0 in eq.(12.148) would drop. Indeed, the equations $9 - 4\alpha + 3\alpha^2 = 0$ and $-2\alpha + \frac{8}{3} - \frac{6}{\alpha} = 0$, stemming from eq.(12.139), are the same. Moreover, we also notice that this equation has only complex valued solutions. Therefore, it is even more important to have made the connection between the MAG and the Landau gauge by embedding them in a bigger class of gauges, since then we have the fixed point, even at all orders. In what follows, it is understood that $\zeta_0 = \alpha$.

⁵We will do the transformation of $\mathcal{W}(J)$ to $\mathcal{W}(\tilde{J})$ only at the end.

We now have all the ingredients to construct the one-loop effective potential $\tilde{V}_1(\tilde{\sigma})$,

$$\begin{aligned} \tilde{V}_1(\tilde{\sigma}) &= \frac{\tilde{\sigma}^2}{2\zeta_0} \left(1 - \left(2f_0 + \frac{\zeta_1}{\zeta_0} \right) g^2 \right) + i \ln \det \left[\delta^{ab} \left(\partial^2 + \alpha \frac{g\tilde{\sigma}}{\zeta_0} \right) \right] \\ &\quad - \frac{i}{2} \ln \det \left[\delta^{ab} \left(\left(\partial^2 + \frac{g\tilde{\sigma}}{\zeta_0} \right) g_{\mu\nu} - \left(1 - \frac{1}{\alpha} \right) \partial_\mu \partial_\nu \right) \right], \end{aligned} \quad (12.149)$$

or, after renormalization

$$\begin{aligned} \tilde{V}_1(\tilde{\sigma}) &= \frac{\tilde{\sigma}^2}{2\zeta_0} \left(1 - \left(2f_0 + \frac{\zeta_1}{\zeta_0} \right) g^2 \right) + \frac{3}{32\pi^2} \frac{g^2 \tilde{\sigma}^2}{\zeta_0^2} \left(\ln \frac{g\tilde{\sigma}}{\zeta_0 \bar{\mu}^2} - \frac{5}{6} \right) \\ &\quad - \frac{1}{32\pi^2} \frac{g^2 \alpha^2 \tilde{\sigma}^2}{\zeta_0^2} \left(\ln \frac{g\alpha\tilde{\sigma}}{\zeta_0 \bar{\mu}^2} - \frac{3}{2} \right). \end{aligned} \quad (12.150)$$

We did not explicitly write the divergences and counterterms in eq.(12.151), since by construction we know that the formalism is renormalizable, so they would have cancelled amongst each other. This can be checked explicitly by using the unity of (12.59) with counterterms included. It can also be checked explicitly that $\tilde{V}_1(\tilde{\sigma})$ obeys the renormalization group

$$\mu \frac{d}{d\mu} \tilde{V}_1(\tilde{\sigma}) = 0 + \text{terms of higher order}, \quad (12.151)$$

by using the RGE functions (12.138)-(12.141) and the fact that the anomalous dimension of $\tilde{\sigma}$ is given by

$$\gamma_{\tilde{\sigma}}(g^2) = \frac{\beta(g^2)}{2g^2} + \gamma_{\mathcal{O}_{\text{MAG}}}(g^2) + \mu \frac{\partial \ln \mathcal{F}(g^2, \alpha)}{\partial \mu}, \quad (12.152)$$

which is immediately verifiable from eq.(12.99).

We now search for the vacuum configuration by minimizing $\tilde{V}_1(\tilde{\sigma})$ with respect to $\tilde{\sigma}$. We will put $\bar{\mu}^2 = \frac{g\tilde{\sigma}}{\zeta_0}$ to exclude possibly large logarithms, and find two solutions of the gap equation

$$\begin{aligned} \left. \frac{d\tilde{V}_1}{d\tilde{\sigma}} \right|_{\bar{\mu}^2 = \frac{g\tilde{\sigma}}{\zeta_0}} &= 0 \\ \Leftrightarrow \frac{\tilde{\sigma}}{\zeta_0} \left(1 - \left(2f_0 + \frac{\zeta_1}{\zeta_0} \right) g^2 \right) &+ \frac{3}{16\pi^2} \frac{g^2 \tilde{\sigma}}{\zeta_0^2} \left(-\frac{5}{6} \right) + \frac{3}{32\pi^2} \frac{g^2 \tilde{\sigma}}{\zeta_0^2} \\ &- \frac{1}{16\pi^2} \frac{g^2 \alpha^2 \tilde{\sigma}}{\zeta_0^2} \left(\ln \alpha - \frac{3}{2} \right) - \frac{1}{32\pi^2} \frac{g^2 \alpha^2 \tilde{\sigma}}{\zeta_0^2} = 0, \end{aligned} \quad (12.153)$$

namely

$$\tilde{\sigma} = 0, \quad (12.154)$$

$$y \equiv \left. \frac{g^2 N}{16\pi^2} \right|_{N=2} = \frac{2\zeta_0}{16\pi^2 (2f_0\zeta_0 + \zeta_1) + \alpha^2 \ln \alpha - \alpha^2 + 1}. \quad (12.155)$$

The quantity y is the relevant expansion parameter, and should be sufficiently small to have a sensible expansion. The value for $\langle \tilde{\sigma} \rangle$ corresponding to eq.(12.155) can be extracted from the one-loop coupling constant

$$g^2(\bar{\mu}) = \frac{1}{\beta_0 \ln \frac{\bar{\mu}^2}{\Lambda_{\overline{\text{MS}}}^2}} . \quad (12.156)$$

The first solution (12.154) corresponds to the usual, perturbative vacuum ($E_{\text{vac}} = 0$), while eq.(12.155) gives rise to a dynamically favoured vacuum with energy

$$E_{\text{vac}} = -\frac{1}{64\pi^2} (3 - \alpha^2) \left(m_{\text{gluon}}^{\text{off-diag}} \right)^4 , \quad (12.157)$$

$$m_{\text{gluon}}^{\text{off-diag}} = e^{\frac{3}{22y}} \Lambda_{\overline{\text{MS}}} . \quad (12.158)$$

Eq.(12.157) is obtained upon substitution of eq.(12.153) into eq.(12.150). From eq.(12.157), we notice that at the one-loop approximation, $\alpha^2 \leq 3$ must be fulfilled in order to have $E_{\text{vac}} \leq 0$. In principle, the unknown function $f_0(\alpha)$ can be determined by solving the differential equation

$$\begin{aligned} \frac{dE_{\text{vac}}}{d\alpha} = 0 &\Leftrightarrow 2\alpha \left(m_{\text{gluon}}^{\text{off-diag}} \right)^4 + 4(\alpha^2 - 3) \left(m_{\text{gluon}}^{\text{off-diag}} \right)^3 \frac{dm_{\text{gluon}}^{\text{off-diag}}}{d\alpha} = 0 \\ &\Leftrightarrow \alpha + \frac{3 - \alpha^2}{y^2} \left(\frac{\partial y}{\partial \alpha} + \frac{\partial y}{\partial \zeta_0} \frac{\partial \zeta_0}{\partial \alpha} + \frac{\partial y}{\partial \zeta_1} \frac{\partial \zeta_1}{\partial \alpha} + \frac{\partial y}{\partial f_0} \frac{\partial f_0}{\partial \alpha} \right) = 0 \end{aligned} \quad (12.159)$$

with initial condition $E_{\text{vac}}(\alpha) = E_{\text{vac}}^{\text{Landau}}$. However, to solve eq.(12.159) knowledge of ζ_1 is needed. Since we are not interested in $f_0(\alpha)$ itself, but rather in the value of the vacuum energy E_{vac} , the off-diagonal mass $m_{\text{gluon}}^{\text{off-diag}}$ and the expansion parameter y , there is a more direct way to proceed, without having to solve the eq.(12.159). Let us first give the Landau gauge value for E_{vac} in the case $N = 2$, which can be easily obtained from [42, 197],

$$E_{\text{vac}}^{\text{Landau}} = -\frac{9}{128\pi^2} e^{\frac{17}{6}} \Lambda_{\overline{\text{MS}}}^4 . \quad (12.160)$$

Since the construction is such that $E_{\text{vac}}(\alpha) = E_{\text{vac}}^{\text{Landau}}$, we can equally well solve

$$-\frac{9}{128\pi^2} e^{\frac{17}{6}} \Lambda_{\overline{\text{MS}}}^4 = -\frac{1}{64\pi^2} (3 - \alpha^2) \left(m_{\text{gluon}}^{\text{off-diag}} \right)^4 , \quad (12.161)$$

which gives the lowest order mass

$$m_{\text{gluon}}^{\text{off-diag}} = \left(\frac{9}{2} \frac{e^{\frac{17}{6}}}{3 - \alpha^2} \right)^{\frac{1}{4}} \Lambda_{\overline{\text{MS}}} , \quad (12.162)$$

and hence

$$m_{\text{ghost}}^{\text{off-diag}} = \sqrt{\alpha} \left(\frac{9}{2} \frac{e^{\frac{17}{6}}}{3 - \alpha^2} \right)^{\frac{1}{4}} \Lambda_{\overline{\text{MS}}} , \quad (12.163)$$

The result (12.162) can be used to determine y . From eq.(12.158) one easily finds

$$y = \frac{36}{187 + 66 \ln \frac{9}{2(3 - \alpha^2)}} . \quad (12.164)$$

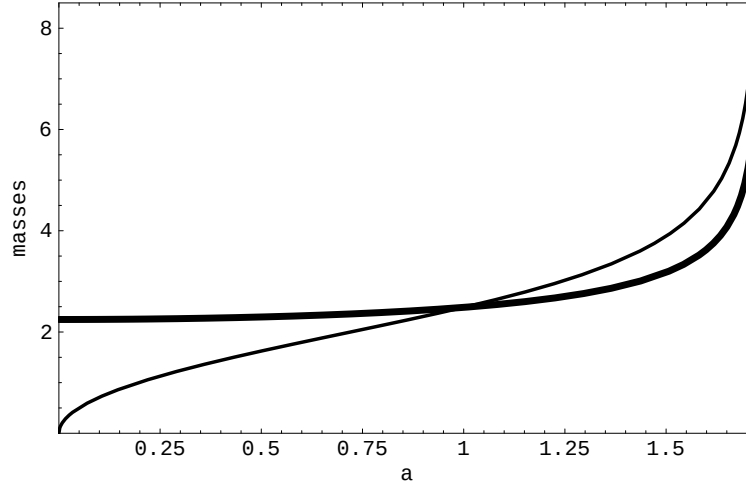


Figure 12.1: The off-diagonal gluon (fat line) and ghost mass (thin line) in function of α . Masses are in units of $\Lambda_{\overline{\text{MS}}}$.

We see thus that, for the information we are currently interested in, we do not need explicit knowledge of ζ_1 and f_0 . We want to remark that, if ζ_1 were known, the value for y obtained in eq.(12.164) can be used to determine f_0 from eq.(12.155). This is a nice feature, since the possibly difficult differential equation (12.159) never needs to be solved in this fashion. In Figure 12.1, we have plotted the off-diagonal gluon mass (12.162) and ghost mass (12.163) for $0 \leq \alpha \leq \sqrt{3}$. We notice that the masses grow to ∞ for increasing α , while the expansion parameter y drops to zero, as it is clear from Figure 12.2. The relative smallness of y means that our perturbative analysis should give qualitatively meaningful results. Before we come to the conclusions, let us consider the limit $\alpha \rightarrow 0$, corresponding to the “real” MAG $D_\mu^{ab} A^{\mu b} = 0$. One finds

$$\begin{aligned} m_{\text{gluon}}^{\text{off-diag}} &= \left(\frac{3}{2} e^{\frac{17}{6}} \right)^{\frac{1}{4}} \Lambda_{\overline{\text{MS}}} \approx 2.25 \Lambda_{\overline{\text{MS}}}, \\ y &= \frac{36}{187 + 66 \ln \frac{3}{2}} \approx 0.168. \end{aligned} \quad (12.165)$$

12.6 Discussion and conclusion.

The aim of this paper was to give analytic evidence, as expressed by eq.(12.165), of the dynamical mass generation for off-diagonal gluons in Yang-Mills theory quantized in the maximal Abelian gauge. This mass can be seen as support for the Abelian dominance [73, 74, 75] in that gauge. This result is in qualitative agreement with the lattice version of the MAG, where such a mass was also reported [49, 50]. The off-diagonal lattice gluon propagator could be fitted by $\frac{1}{p^2 + m^2}$, which is in correspondence with the tree level propagator we find. We have been able to prove the existence of the off-diagonal mass by investigating the condensation of a mass dimension two operator, namely $(\frac{1}{2} A_\mu^a A^{\mu a} + \alpha \bar{c}^a c^a)$. It was shown how a meaningful, renormalizable effective potential for this local composite operator can be constructed. By evaluating this potential explicitly at one-loop order in the case of $SU(2)$, the formation of the condensate is favoured since it lowers the vacuum energy. The latter does not depend on the choice of the gauge parameter α , at least if one would work to infinite order in perturbation

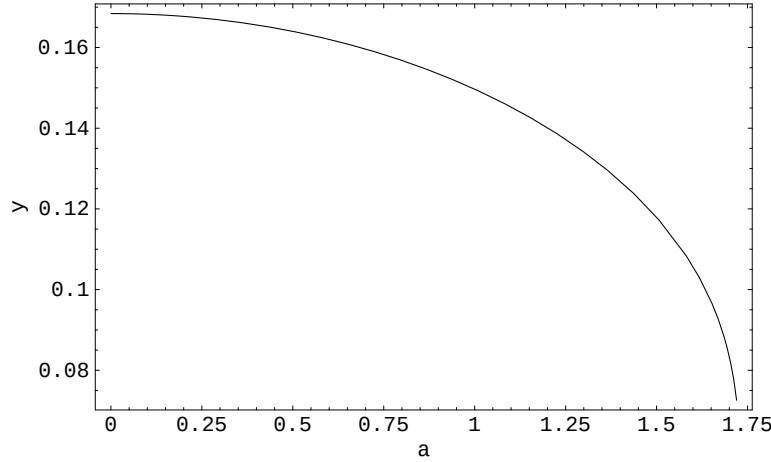


Figure 12.2: The expansion parameter γ as a function of α .

theory. We have explained in short the problem at finite order and discussed a way to overcome it. Moreover, we have been able to interpolate between the Landau gauge and the MAG by unifying them in a larger class of renormalizable gauges. This observation was used to prove that the vacuum energy of Yang-Mills theory in the MAG due to its mass dimension two condensate should be the same as the vacuum energy of Yang-Mills theory in the Landau gauge with the much explored condensate $\langle A_\mu^A A^{\mu A} \rangle$. It is worth noticing that all the gauges, where a dimension two condensate provides a dynamical gluon mass parameter, such as the Landau gauge [42], the Curci-Ferrari gauges [178], the linear gauges [212] and the MAG, can be connected to each other, either directly (e.g. Landau-MAG) or via the Landau gauge (e.g. MAG and linear gauges). This also implies that, if $\langle A_\mu^A A^{\mu A} \rangle \neq 0$ in the Landau gauge, the analogous condensates in the other gauges cannot vanish either. Then the question arises if this correspondence between different gauges could be stretched further to for instance the Coulomb gauge, where the possibility of a condensate $\langle A_i^A A^{iA} \rangle$ was already advocated some time ago in [200]. However, it is worth remarking that this might be a more complicated task, since the Coulomb gauge is not a covariant gauge fixing, and as such its analysis within the algebraic renormalization framework [59] is not straightforward.

Needless to say, the present work is far from being complete. First of all, an explicit calculation at two-loop order and for general gauge group $SU(N)$ would be interesting. We also limited our computations to the tree level order. In principle, one should evaluate the off-diagonal gluon polarization in order to get further information on the structure of the propagator. A first step in this direction was taken in the case of the Landau gauge in [214]. It is unknown what will happen at higher orders in the MAG, but it is likely that the external momentum Q^2 will enter through loop corrections and influence the possible position of a pole in the propagator. The ghost condensation, that was first investigated in [77, 80] as a possible mechanism behind the off-diagonal mass, and later on was shown to be tachyonic [157, 82], could enter this polarization too. This would require a more complete treatment of the ghost condensation in the MAG, along the lines of [172], where these condensates were considered in more detail in the case of the Curci-Ferrari and Landau gauge. Another issue which deserves attention is the behaviour of the diagonal gluon. In [50], it was found that the diagonal gluon propagator also contain a mass parameters, with $m_{\text{gluon}}^{\text{diag}} \approx \frac{1}{2} m_{\text{gluon}}^{\text{off-diag}}$, while in [49] the diagonal gluon was reported to behave like a light or massless particle. For completeness, we remind that these lattice simulations were both performed in the case of $SU(2)$. We want to remark that a condensation of the composite

diagonal operator $A_\mu^i A^{\mu i}$ cannot occur within our approach, since this is forbidden by the diagonal local $U(1)^{N-1}$ Ward identity (12.30). In principle, one could add an extra source term like $\frac{1}{2}\rho A_\mu^i A^{\mu i}$, but it does not seem possible to prove the renormalizability of this operator in the MAG. This might be consistent with the result of [50], since the diagonal gluon propagator could *not* be fitted with a Yukawa propagator $\frac{1}{p^2+m^2}$, in contrast with the off-diagonal gluon propagator which could be fitted with $\frac{1}{p^2+m^2}$. This could mean that the diagonal mass parameter is of a different nature compared to the off-diagonal one. A possible speculation is that it might have to do with Gribov copies, since a fit $\frac{p^2}{p^4+m^4}$ did work for the diagonal propagator [50].

Our analysis of the MAG condensate was also restricted to the purely perturbative level. One could imagine calculating in a certain non-trivial background. The vacuum energy calculated in one gauge should still be the same as the one calculated in the other gauge. In this context, and keeping in mind that monopole condensation is an essential ingredient of the dual superconductor picture, it might be worth noticing that the role of $\langle A_\mu^A A^{\mu A} \rangle$ as an order parameter for monopole condensation was investigated in the Landau gauge by the authors of [34], based on a similar observation in compact QED [33]. We note that an off-diagonal gluon mass can serve as a starting point to derive low energy (dual) Abelian models for Yang-Mills theories, see for example [99, 102, 101].

Let us conclude with a few considerations on the issues of the degrees of freedom and of the unitarity when the gluons attain a dynamical mass, as a consequence of a nonvanishing dimension two condensate $\langle \mathcal{O}_{\text{MAG}} \rangle = \langle \frac{1}{2} A_\mu^a A^{\mu a} + \alpha \bar{c}^a c^a \rangle$. One possible way to look at the degrees of freedom associated to a given field is through its propagator. From the pole of the propagator one gets information about the mass of the field, while from its residue one learns about polarization states. However, the propagation of the field has to occur in some vacuum. In other words, the kind of vacuum in which the field propagates has to be supplemented. In our case, this task is achieved by the LCO Lagrangian, eq.(12.60), i.e.

$$\mathcal{L}(A_\mu, \sigma) = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} - \frac{1}{4} F_{\mu\nu}^i F^{\mu\nu i} + \mathcal{L}_{\text{MAG}} + \mathcal{L}_{\text{diag}} - \frac{\sigma^2}{2g^2\zeta} + \frac{1}{g^2\zeta} g\sigma \mathcal{O}_{\text{MAG}} - \frac{1}{2\zeta} (\mathcal{O}_{\text{MAG}})^2, \quad (12.166)$$

which allows one to take into account the effects related to having a nontrivial vacuum corresponding to the nonvanishing dimension two condensate $\langle \mathcal{O}_{\text{MAG}} \rangle$, as expressed by the identity

$$\langle \sigma \rangle = g \langle \mathcal{O}_{\text{MAG}} \rangle. \quad (12.167)$$

That this is the preferred vacuum follows from the observation that the vacuum energy is lowered by the condensate $\langle \mathcal{O}_{\text{MAG}} \rangle$. Expanding thus around $\langle \sigma \rangle \neq 0$, a dynamical tree level mass $m_{\text{gluon}}^{\text{off-diag.}}$ for the off-diagonal gluons is generated in the gauge fixed Lagrangian (12.166), namely

$$m_{\text{gluon}}^{\text{off-diag.}} = \sqrt{\frac{g \langle \sigma \rangle}{\zeta_0}}. \quad (12.168)$$

Therefore, in the condensed vacuum, $\langle \sigma \rangle \neq 0$, the Lagrangian (12.166) accounts for off-diagonal massive gluons. However, we emphasize that this dynamical mass parameter occurs as the result of a particular condensate. It is not a free parameter of the gauge fixed theory, its value being determined by a gap equation. Concerning now the unitarity of the resulting theory, it should be remarked that, due to confinement, gluons and quarks are only to be called physical at a very high energy scale Q^2 , where they behave almost freely and asymptotic states can be related to them, thanks to asymptotic freedom. At very high energies, our dynamically massive action might be unitary: a renormalization group improvement could induce quantum corrections such that the mass parameter runs to zero for $Q^2 \rightarrow \infty$. Otherwise said, the corrections induced by this dynamical mass on the scattering amplitudes

are expected to become less and less important as the energy of the process increases, so that the amplitudes of the massless case are in fact recovered. Such a scenario would be analogous to the behaviour of the dynamical mass parameter discussed by Cornwall in [201]. For very high Q^2 , one does indeed expect that perturbative Yang-Mills theory with massless gluons, having two physical degrees of freedom, describes the physical spectrum and that non-perturbative corrections are absent.

To decide if our resulting theory is unitary at smaller Q^2 , one should know how to take into account the effects of confinement, which now cannot be neglected. This would amount to knowing how to construct out of our Lagrangian (12.166) the low energy spectrum of the theory, which is believed to be given by colorless bound states of gluons and quarks as, for instance, mesons, baryons and glueballs. This task is far beyond our capabilities. At intermediate Q^2 , what we can state is that this dynamical mass parametrizes the behaviour of the Greens function of the gluon. As a result of quantum effects, i.e. the condensation of the mass dimension two operator, a pole appears in the off-diagonal gluon propagator at the tree level. Including higher order effects will alter the propagators behaviour as well as the location of the pole at physical Q^2 (i.e. $Q^2 < 0$). In the case of the Landau gauge, higher order calculations showed that the condensate remains stable, and hence a nonzero mass parameter will remain, see [42, 197]. This mass parameter will describe the behaviour of the Greens function at Euclidean Q^2 . The presence of a mass parameter does however not necessarily entail the presence of a pole in the propagator at negative Q^2 , corresponding to a physical particle. Using lattice simulations of the Euclidean propagator, a mass parameter is found also by fitting at Euclidean $Q^2 > 0$, but no physical, massive particle is implied. Analogously, one should not conclude from our calculations that the gluon is a massive, physical particle and that unitarity is violated.

Chapter 13

Gribov horizon in the presence of dynamical mass generation in Euclidean Yang-Mills theories in the Landau gauge

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published in *Physics Letters B* **590** (2004) 265.

The infrared behavior of the gluon and ghost propagators is analyzed in Yang-Mills theories in the presence of the dynamical generation of a mass parameter due to $\langle A_\mu^2 \rangle$ in the Landau gauge. By restricting the domain of integration in the path-integral to the Gribov region Ω , the gauge propagator is found to be suppressed in the infrared, while the ghost propagator is enhanced.

13.1 Introduction.

The possibility that gluons might acquire a mass through a dynamical mechanism is receiving renewed interest in the last few years. Although a fully gauge invariant framework for the dynamical mass generation in Yang-Mills theories is not yet available, the number of gauges displaying this interesting phenomenon is getting considerably large.

A dynamical gluon mass has been introduced in the light-cone gauge [201] in order to obtain estimates for the spectrum of the glueballs. It has been discussed in the Coulomb gauge in [200], where the presence of a nonvanishing condensate $\langle A_i^a A_i^a \rangle$ in the operator product expansion for the two-point gauge correlation function has been pointed out. More recently, the condensate $\langle A_\mu^a A_\mu^a \rangle$ has been investigated in the Landau gauge in [33, 34, 175, 37, 38], where it has been proven to account for the discrepancy observed in the two- and three-point correlation functions between the perturbative theory and the lattice results. A renormalizable effective potential for the condensate $\langle A_\mu^a A_\mu^a \rangle$ in pure Yang-Mills theory in the Landau gauge has been constructed and evaluated in analytic form up to two-loop order in [42, 184]. This result shows that the vacuum of Yang-Mills theory favors the formation of a nonvanishing condensate $\langle A_\mu^a A_\mu^a \rangle$, which lowers the vacuum energy and provides a dynamical gluon mass, which turns out to be of the order of $\approx 500 MeV$. The inclusion of massless quarks has been

worked out in [197]. We remind here that lattice simulations of the gluon propagator in the Landau gauge have reported a gluon mass $m \approx 600 \text{ MeV}$ [48]. Concerning other gauges, the occurrence of the condensate $\langle A_\mu^a A_\mu^a \rangle$ and of the related dynamical gluon mass has been established in the linear covariant gauges in [205, 212]. These results can be generalized to a class of nonlinear covariant gauges. Here, the mixed gluon-ghost condensate $\langle \frac{1}{2} A_\mu^a A_\mu^a + \xi \bar{c}^a c^a \rangle$ has to be considered [83, 144], with ξ the gauge parameter. A renormalizable effective potential for this condensate has been obtained in the Curci-Ferrari [178] and maximal Abelian gauges [215], resulting in a dynamical mass generation. In the latter case, lattice simulations [49, 50] had already given evidences of a nonvanishing mass for the off-diagonal gluons. Moreover, a gluon mass has been reported in lattice simulations in the Laplacian gauge [43, 47]. Also, it is part of the so-called Kugo-Ojima criterion for color confinement [177] and, as discussed in [54], it proves to be useful in order to account for the data obtained on the radiative decays of heavy quarkonia systems.

In this work we pursue the study of the dynamical mass generation in Euclidean Yang-Mills theory in the Landau gauge. We attempt at incorporating the nonperturbative effects related to the Gribov horizon [107], the aim being that of investigating the infrared behavior of the gluon and ghost propagators in presence of the dynamical mass generation. These propagators have been studied to a great extent by several groups through lattice simulations [216, 217, 218, 114, 115, 116, 45, 219, 220, 117] in the Landau gauge, which have confirmed that the gluon propagator is suppressed in the infrared region while the ghost propagator is enhanced, being in fact more singular than the perturbative behavior $\approx 1/k^2$. Such behavior of the gluon and ghost propagators was already found by Gribov in [107], where it arises as a consequence of the restriction of the domain of integration in the path-integral to the region Ω whose boundary $\partial\Omega$ is the first Gribov horizon, where the first vanishing eigenvalue of the Faddeev-Popov operator, $-\partial_\mu (\partial_\mu \delta^{ab} + g f^{acb} A_\mu^c)$, appears. This restriction is necessary due to the existence of the Gribov copies, which imply that the Landau condition, $\partial_\mu A_\mu^a = 0$, does not uniquely fix the gauge. The infrared suppression of the gluon propagator and the enhancement of the ghost propagator have also been derived in [127, 128], where the restriction to the region Ω has been implemented by a Boltzmann factor through the introduction of a horizon function. Recently, the authors of [20, 121, 124, 122, 123, 125, 221, 113] have analyzed the behavior of the gluon and ghost propagators in the Landau gauge within the Schwinger-Dyson framework, also obtaining that the gluon propagator is suppressed while the ghost propagator is enhanced.

Concerning now the gluon and ghost propagators in the presence of a dynamical mass generation, we shall proceed by following Gribov's original suggestion, which amounts to implement the restriction to Ω as a no-pole condition for the two-point ghost function [107]. We shall be able to show that the gluon and ghost propagators are suppressed and enhanced, respectively, and this in the presence of a dynamical gluon mass. This behavior is in agreement with that found in [107, 127, 128, 20, 121, 124, 122, 123, 125, 221, 113].

This work is organized as follows. In section 13.2 we briefly review the properties of the Lagrangian accounting for the dynamical gluon mass generation in the Landau gauge. We remind here that the term mass should be understood as the massive parameter generated by a nonvanishing $\langle A_\mu^2 \rangle$ condensate. In section 13.3 we implement the restriction of the domain of integration in the path-integral to the region Ω . The ensuing modifications of the gauge propagator due to both the Gribov horizon and dynamical gluon mass are worked out. Section 13.4 is devoted to the analysis of the infrared behavior of the ghost propagator. Some further remarks are collected in section 13.5.

13.2 Dynamical mass generation in the Landau gauge.

The dynamical mass generation due to $\langle A_\mu^2 \rangle$ in the Landau gauge is described by the following action [42]

$$S(A, \sigma) = S_{YM} + S_{GF+FP} + S_\sigma , \quad (13.1)$$

where S_{YM} , S_{GF+FP} are the Yang-Mills and the gauge fixing terms

$$S_{YM} = \frac{1}{4} \int d^4x F_{\mu\nu}^a F_{\mu\nu}^a , \quad (13.2)$$

$$S_{GF+FP} = \int d^4x (b^a \partial_\mu A_\mu^a + \bar{c}^a \partial_\mu D_\mu^{ab} c^b) , \quad (13.3)$$

with b^a being the Lagrange multiplier enforcing the Landau gauge condition, $\partial_\mu A_\mu^a = 0$, and $\bar{c}^a$, c^a denoting the Faddeev-Popov ghosts. The color index a refers to the adjoint representation of the gauge group $SU(N)$. The term S_σ in eq.(13.1) contains the auxiliary scalar field σ and reads

$$S_\sigma = \int d^4x \left(\frac{\sigma^2}{2g^2\zeta} + \frac{1}{2} \frac{\sigma}{g\zeta} A_\mu^a A_\mu^a + \frac{1}{8\zeta} (A_\mu^a A_\mu^a)^2 \right) . \quad (13.4)$$

The introduction of the auxiliary field σ allows to study the condensation of the local operator $A_\mu^a A_\mu^a$. In fact, as shown in [42], the following relation holds

$$\langle \sigma \rangle = -\frac{g}{2} \langle A_\mu^a A_\mu^a \rangle . \quad (13.5)$$

The dimensionless parameter ζ in expression (13.4) is needed to account for the ultraviolet divergences present in the vacuum correlation function $\langle A^2(x) A^2(y) \rangle$. For the details of the renormalizability properties of the local operator $A_\mu^a A_\mu^a$ in the Landau gauge we refer to [87, 153]. Expression (13.1) is left invariant by the following BRST transformations

$$\begin{aligned} sA_\mu^a &= -D_\mu^{ab} c^b = -(\partial_\mu c^a + gf^{abc} A_\mu^b c^c) , \\ sc^a &= \frac{1}{2} gf^{abc} c^b c^c , \\ s\bar{c}^a &= b^a , \\ sb^a &= 0 , \\ s\sigma &= gA_\mu^a \partial_\mu c^a , \end{aligned} \quad (13.6)$$

and

$$sS(A, \sigma) = 0 . \quad (13.7)$$

Notice that, from the relation

$$A_\mu^a \partial_\mu c^a = -\frac{1}{2} s(A_\mu^a A_\mu^a) , \quad (13.8)$$

it follows that the BRST operator is nilpotent. The action $S(A, \sigma)$ is the starting point for constructing a renormalizable effective potential $V(\sigma)$ for the auxiliary field σ , obeying the renormalization group equations. The output of the higher loop computations done in [42, 197] shows that the minimum

of $V(\sigma)$ occurs for a nonvanishing vacuum expectation value of the auxiliary field, i.e. $\langle \sigma \rangle \neq 0$. In particular, from expression (13.1), the first order induced dynamical gluon mass is found to be

$$m^2 = \frac{g \langle \sigma \rangle}{\zeta_0}, \quad (13.9)$$

where ζ_0 is the first contribution to the parameter ζ [42], given by

$$\begin{aligned} \zeta &= \frac{\zeta_0}{g^2} + \zeta_1 + O(g^2), \\ \zeta_0 &= \frac{9}{13} \frac{(N^2 - 1)}{N}. \end{aligned} \quad (13.10)$$

We remind here that, in the Landau gauge, the Faddeev-Popov ghosts $\bar{c}^a$, c^a remain massless, due to the absence of mixing between the composite operators $A_\mu^a A_\mu^a$ and $\bar{c}^a c^a$. This stems from additional Ward identities present in the Landau [153] and in the covariant linear gauges [205], which forbid the appearance of the term $\bar{c}^a c^a$.

13.3 Infrared behavior of the gluon propagator.

13.3.1 Restriction to the region Ω .

In the previous section we have reviewed the properties of the action $S(A, \sigma)$ which accounts for the dynamical mass generation. However, it is worth underlining that the action $S(A, \sigma)$ leads to a partition function

$$\mathcal{Z} = \mathcal{N} \int DAD\sigma \delta(\partial A^a) \det(-\partial_\mu (\partial_\mu \delta^{ab} + g f^{acb} A_\mu^c)) e^{-(S_{YM} + S_\sigma)}, \quad (13.11)$$

which is still plagued by the Gribov copies, which affect the Landau gauge. It might be useful to notice here that the action $(S_{YM} + S_\sigma)$ is left invariant by the local gauge transformations

$$\begin{aligned} \delta A_\mu^a &= -D_\mu^{ab} \omega^b, \\ \delta \sigma &= g A_\mu^a \partial_\mu \omega^a, \end{aligned} \quad (13.12)$$

$$\delta(S_{YM} + S_\sigma) = 0. \quad (13.13)$$

As a consequence of the existence of Gribov copies, the domain of integration in the path-integral should be restricted further. We shall follow here Gribov's proposal to restrict the domain of integration to the region Ω [107]. Expression (13.11) is thus replaced by

$$\mathcal{Z} = \mathcal{N} \int DAD\sigma \delta(\partial A^a) \det(-\partial_\mu (\partial_\mu \delta^{ab} + g f^{acb} A_\mu^c)) e^{-(S_{YM} + S_\sigma)} \mathcal{V}(\Omega), \quad (13.14)$$

where $\mathcal{V}(\Omega)$ implements the restriction to Ω . The factor $\mathcal{V}(\Omega)$ can be accommodated for by requiring that the two-point connected ghost function $\mathcal{G}(k, A)$ has no poles for a given nonvanishing value of the momentum k [107]. This condition can be understood by recalling that the region Ω is defined as the set of all transverse gauge connections $\{A_\mu^a\}$, $\partial_\mu A_\mu^a = 0$, for which the Faddeev-Popov operator is positive definite, i.e. $-\partial_\mu (\partial_\mu \delta^{ab} + g f^{acb} A_\mu^c) > 0$. This implies that the inverse of the Faddeev-Popov operator $[-\partial_\mu (\partial_\mu \delta^{ab} + g f^{acb} A_\mu^c)]^{-1}$, and thus $\mathcal{G}(k, A)$, can become large only when approaching the horizon $\partial\Omega$, which corresponds in fact to $k = 0$ [107]. The quantity $\mathcal{G}(k, A)$ can be evaluated order by

order in perturbation theory. Repeating the same calculation of [107], we find that, up to the second order

$$\mathcal{G}(k, A) \approx \frac{1}{k^2} \frac{1}{1 - \rho(k, A)} , \quad (13.15)$$

with $\rho(k, A)$ given by

$$\rho(k, A) = \frac{g^2}{3} \frac{N}{N^2 - 1} \frac{1}{V} \frac{k_\mu k_\nu}{k^2} \sum_q \frac{1}{(k - q)^2} (A_\lambda^a(q) A_\lambda^a(-q)) \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) , \quad (13.16)$$

and V being the space-time volume. According to [107], the no-pole condition for $\mathcal{G}(k, A)$ reads

$$\begin{aligned} \rho(0, A) &< 1 , \\ \rho(0, A) &= \frac{g^2}{4} \frac{N}{N^2 - 1} \frac{1}{V} \sum_q \frac{1}{q^2} (A_\lambda^a(q) A_\lambda^a(-q)) . \end{aligned} \quad (13.17)$$

Therefore, for the factor $\mathcal{V}(\Omega)$ in eq.(13.14) we have

$$\mathcal{V}(\Omega) = \theta(1 - \rho(0, A)) , \quad (13.18)$$

where $\theta(x)$ stands for the step function¹.

13.3.2 The gauge propagator.

In order to discuss the gauge propagator, it is sufficient to retain only the quadratic terms in expression (13.14) which contribute to the two-point correlation function $\langle A_\mu^a(k) A_\nu^b(-k) \rangle$. Expanding around the nonvanishing vacuum expectation value of the auxiliary field, $\langle \sigma \rangle \neq 0$, and making use of the integral representation for the step function

$$\theta(1 - \rho(0, A)) = \int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} \frac{d\eta}{2\pi i \eta} e^{\eta(1-\rho(0, A))} , \quad (13.19)$$

we get

$$\begin{aligned} \mathcal{Z}_{\text{quadr}} &= \mathcal{N} \int DA \frac{d\eta}{2\pi i \eta} e^{\eta(1-\rho(0, A))} \delta(\partial A^a) e^{-\left(\frac{1}{4} \int d^4x ((\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 + \frac{1}{2} m^2 \int d^4x (A_\mu^a A_\mu^a))\right)} \\ &= \mathcal{N} \int DA \frac{d\eta}{2\pi i \eta} e^{\eta} e^{-\frac{1}{2} \sum_q A_\mu^a(q) \mathcal{Q}_{\mu\nu}^{ab} A_\nu^b(-q)} , \end{aligned} \quad (13.20)$$

with

$$\mathcal{Q}_{\mu\nu}^{ab} = \left((q^2 + m^2) \delta_{\mu\nu} + \left(\frac{1}{\alpha} - 1 \right) q_\mu q_\nu + \frac{\eta N g^2}{N^2 - 1} \frac{1}{2V} \frac{1}{q^2} \delta_{\mu\nu} \right) \delta^{ab} , \quad (13.21)$$

where the limit $\alpha \rightarrow 0$ has to be taken at the end in order to recover the Landau gauge. Integrating over the gauge field, one has

$$\mathcal{Z}_{\text{quadr}} = \mathcal{N} \int \frac{d\eta}{2\pi i \eta} e^{\eta} (\det \mathcal{Q}_{\mu\nu}^{ab})^{-\frac{1}{2}} = \mathcal{N} \int \frac{d\eta}{2\pi i} e^{f(\eta)} , \quad (13.22)$$

¹ $\theta(x) = 1$ if $x > 0$, $\theta(x) = 0$ if $x < 0$.

where $f(\eta)$ is given by

$$f(\eta) = \eta - \log \eta - \frac{3}{2}(N^2 - 1) \sum_q \log \left(q^2 + m^2 + \frac{\eta N g^2}{N^2 - 1} \frac{1}{2V} \frac{1}{q^2} \right). \quad (13.23)$$

Following [107], the expression (13.22) can be now evaluated at the saddle point, namely

$$\mathcal{Z}_{\text{quadr}} \approx e^{f(\eta_0)}, \quad (13.24)$$

where η_0 is determined by the minimum condition

$$1 - \frac{1}{\eta_0} - \frac{3}{4} \frac{N g^2}{V} \sum_q \frac{1}{\left(q^4 + m^2 q^2 + \frac{\eta_0 N g^2}{N^2 - 1} \frac{1}{2V} \right)} = 0. \quad (13.25)$$

Taking the thermodynamic limit, $V \rightarrow \infty$, and setting [107]

$$\gamma^4 = \frac{\eta_0 N g^2}{N^2 - 1} \frac{1}{2V}, \quad V \rightarrow \infty, \quad (13.26)$$

we get the gap equation

$$\frac{3}{4} N g^2 \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^4 + m^2 q^2 + \gamma^4} = 1, \quad (13.27)$$

where the term $1/\eta_0$ in (13.25) has been neglected in the thermodynamic limit. The gap equation (13.27) defines the parameter γ . Notice that the dynamical mass m appears explicitly in eq.(13.27). Moreover, (13.27) reduces to the original gap equation of [107, 127, 128] for $m = 0$. To obtain the gauge propagator, we can now go back to the expression for $\mathcal{Z}_{\text{quadr}}$ which, after substituting the saddle point value $\eta = \eta_0$, becomes

$$\mathcal{Z}_{\text{quadr}} = \mathcal{N} \int D A e^{-\frac{1}{2} \sum_q A_\mu^a(q) \mathcal{Q}_{\mu\nu}^{ab} A_\nu^b(-q)}, \quad (13.28)$$

with

$$\mathcal{Q}_{\mu\nu}^{ab} = \left(\left(q^2 + m^2 + \frac{\gamma^4}{q^2} \right) \delta_{\mu\nu} + \left(\frac{1}{\alpha} - 1 \right) q_\mu q_\nu \right) \delta^{ab}. \quad (13.29)$$

Computing the inverse of $\mathcal{Q}_{\mu\nu}^{ab}$ and taking the limit $\alpha \rightarrow 0$, we get the gauge propagator in the presence of the dynamical gluon mass m , i.e.

$$\langle A_\mu^a(q) A_\nu^b(-q) \rangle = \delta^{ab} \frac{q^2}{q^4 + m^2 q^2 + \gamma^4} \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right). \quad (13.30)$$

Notice that, the presence of the mass m in eq.(13.30) enforces the infrared suppression of the gluon propagator.

13.4 The infrared behavior of the ghost propagator.

Let us discuss now the infrared behavior of the ghost propagator, given by eq.(13.15) upon contraction of the gauge fields, namely

$$\mathcal{G} \approx \frac{1}{k^2} \frac{1}{1 - \rho(k)}, \quad (13.31)$$

with

$$\begin{aligned}\rho(k) &= \frac{g^2}{3} \frac{N}{N^2 - 1} \frac{k_\mu k_\nu}{k^2} \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(k - q)^2} \langle A_\lambda^a(q) A_\lambda^a(-q) \rangle \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \\ &= g^2 N \frac{k_\mu k_\nu}{k^2} \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(k - q)^2} \frac{q^2}{q^4 + m^2 q^2 + \gamma^4} \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right).\end{aligned}\quad (13.32)$$

From the gap equation (13.27), it follows

$$Ng^2 \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^4 + m^2 q^2 + \gamma^4} \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) = \delta_{\mu\nu}, \quad (13.33)$$

so that

$$1 - \rho(k) = Ng^2 \frac{k_\mu k_\nu}{k^2} \int \frac{d^4 q}{(2\pi)^4} \frac{k^2 - 2qk}{(k - q)^2} \frac{1}{q^4 + m^2 q^2 + \gamma^4} \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right). \quad (13.34)$$

Notice that the integral in the right hand side of eq.(13.34) is convergent and nonsingular at $k = 0$. Therefore, for $k \approx 0$,

$$(1 - \rho(k))_{k \approx 0} \approx \frac{3Ng^2 \mathcal{J}}{4} k^2, \quad (13.35)$$

where $\mathcal{J}$ stands for the value of the integral

$$\mathcal{J} = \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2(q^4 + m^2 q^2 + \gamma^4)}. \quad (13.36)$$

Finally, for the ghost propagator we get

$$\mathcal{G}_{k \approx 0} \approx \frac{4}{3Ng^2 \mathcal{J}} \frac{1}{k^4}, \quad (13.37)$$

exhibiting the characteristic infrared enhancement which, thanks to the gap equation (13.27), turns out to hold also in the presence of the dynamical mass generation.

13.5 Conclusion.

In this letter we have analyzed the infrared behavior of the gluon and ghost propagators in the presence of dynamical mass generation in the Landau gauge. The restriction of the domain of integration to the Gribov region Ω has been implemented by repeating Gribov's procedure [107], which amounts to impose a no-pole condition for the two-point ghost function. The output of our analysis is summarized by equations (13.27), (13.30), (13.37). Expression (13.27) is the gap equation which defines the parameter γ . Notice now that the dynamical mass m enters explicitly the gap equation for γ . Equation (13.30) yields the gauge propagator, which exhibits the infrared suppression. Finally, equation (13.37) establishes the enhancement of the ghost propagator. This behavior of the gluon and ghost propagators is in agreement with that found in [107, 127, 128, 20, 121, 124, 122, 123, 125, 221, 113]. Also, lattice simulations [216, 217, 218, 114, 115, 116, 45, 219, 220, 117] have provided confirmations of the infrared suppression of the gluon propagator and of the ghost enhancement, in the Landau gauge.

Concerning now the Gribov region Ω , it is known that it is not free from Gribov copies [109, 111, 112, 186]. The uniqueness of the gauge condition should be ensured by restricting the domain of integration

to a smaller region in field space, known as the fundamental modular region. However, this is a difficult task. Nevertheless, the restriction to the Gribov region Ω captures nontrivial nonperturbative aspects of the infrared behavior of the theory, as expressed by the suppression and the enhancement of the gluon and ghost propagators. Recently, it has been argued in [113] that the additional copies present in the Gribov region Ω might have no influence on the expectation values.

Although being outside of the aim of the present letter, we remark that the gap equation (13.27) can be also derived by using as starting point the local renormalizable action implementing the Gribov horizon, proposed in [127, 128] by Zwanziger. It turns out in fact that the local operator $A_\mu^a A_\mu^a$ can be added to the Zwanziger action without spoiling its renormalizability [222]. This will allow to study the condensation of the operator $A_\mu^a A_\mu^a$ when the restriction to the horizon is taken into account. In this case, the combination of the algebraic BRST technique with the local composite operator formalism, see e.g. [42, 205, 212], should make possible to include the renormalization effects on the gluon and ghost propagators.

Chapter 14

Remarks on the Gribov horizon and dynamical mass generation in Euclidean Yang-Mills theories

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AIP Conference Proceedings **739** (2005) 455.*

The effect of the dynamical mass generation due to $\langle A_\mu^2 \rangle$ on the gluon and ghost propagators in Euclidean Yang-Mills theory in the Landau gauge is analysed within Zwanziger's local formulation of the Gribov horizon.

14.1 The model.

In a series of papers [127, 128], Zwanziger has shown that the restriction of the domain of integration in the path integral to the Gribov region $\Omega = \{A_\mu^a \mid \partial A^a = 0, M^{ab} > 0\}$, where $M^{ab} = -\partial_\mu(\partial_\mu \delta^{ab} + g f^{acb} A_\mu^c)$ is the Faddeev-Popov operator, can be implemented by adding to the Yang-Mills action the nonlocal horizon term

$$S_h = g^2 \gamma^4 \int d^4x f^{abc} A_\mu^b (M^{-1})^{ad} f^{dec} A_\mu^e. \quad (14.1)$$

The parameter γ is known as the Gribov parameter [107], and is determined by the horizon condition [127, 128], $\frac{\delta \Gamma}{\delta \gamma} = 0$, Γ being the quantum effective action. The nonlocal term (14.1) can be localized by introducing a suitable set of additional fields [127, 128]. The resulting action displays two remarkable properties, namely: locality and multiplicative renormalizability [127, 128, 223]. Moreover, these properties are preserved when the local composite operator $A_\mu^a A_\mu^a$ is introduced in the theory. This enables us to discuss the condensate $\langle A_\mu^a A_\mu^a \rangle$ [42, 184] and the related dynamical gluon mass m in the presence of the Gribov horizon $\partial\Omega$, within a local renormalizable framework. We give here a sketchy account of this analysis by limiting ourselves to consider the Gribov approximation for the horizon action (14.1), by setting $M^{ab} \approx \delta^{ab} \partial^2$. A more complete and detailed analysis is in preparation [222]. The BRST invariant local action implementing the restriction to the region Ω , and allowing for the inclusion of the operator $A_\mu^a A_\mu^a$, is $S = (S_{YM} + S_{gf} + S_h + S_\gamma + S_{mass})$. The term $(S_{YM} + S_{gf})$ is the Yang-Mills

action together with the Landau gauge fixing, while S_h is the localized version of the horizon action (14.1) in the Gribov approximation, containing the additional fields $\{\bar{\varphi}_\mu^a, \varphi_\mu^a\}$ and $\{\bar{\omega}_\mu^a, \omega_\mu^a\}$. We have

$$S_{YM} = \frac{1}{4} \int d^4x F_{\mu\nu}^a F_{\mu\nu}^a, \quad S_{gf} = s \int d^4x \bar{c}^a \partial_\mu A_\mu^a, \quad S_h = -s \int d^4x \bar{\omega}_\mu^a \partial^2 \varphi_\mu^a,$$

Following [127, 128], the term S_γ defines the composite operator $A_\mu^a \varphi_\mu^a$ and its BRST variation, introduced here through the corresponding sources $J, \bar{\lambda}$. Finally, S_{mass} accounts for the mass operator $A_\mu^a A_\mu^a$ and its BRST variation, coupled to the sources τ, η .

$$S_\gamma = s \int d^4x \left(\bar{\lambda} A_\mu^a \varphi_\mu^a + J A_\mu^a \bar{\omega}_\mu^a + \xi \bar{\lambda} J \right), \quad S_{mass} = \frac{1}{2} s \int d^4x \left(\tau A_\mu^a A_\mu^a - \zeta \tau \eta \right). \quad (14.2)$$

The parameters ξ and ζ are needed to account for the divergences arising in the vacuum correlation functions of these composite operators. The nilpotent BRST transformations of the fields and sources are as follows:

$$\begin{aligned} s A_\mu^a &= -(\partial_\mu c^a + g f^{abc} A_\mu^b c^c), & s \bar{\omega}_\mu^a &= \bar{\varphi}_\mu^a, & s \bar{\lambda} &= \bar{J}, & s \tau &= \eta, \\ s c^a &= \frac{1}{2} g f^{abc} c^b c^c, & s \bar{\varphi}_\mu^a &= 0, & s \bar{J} &= 0, & s \eta &= 0, \\ s \bar{c}^a &= B^a, & s \varphi_\mu^a &= \omega_\mu^a, & s \lambda &= 0, \\ s B^a &= 0, & s \omega_\mu^a &= 0, & s J &= \lambda. \end{aligned} \quad (14.3)$$

By making use of the algebraic renormalization [59], the action $S = S_{YM} + S_{gf} + S_h + S_\gamma + S_{mass}$ turns out to be multiplicatively renormalizable. In particular, as discussed in [127, 128], the horizon condition is obtained by setting the sources $(J, \bar{J}, \lambda, \bar{\lambda})$ equal to $J = \bar{J} = \gamma^2$, $\lambda = \bar{\lambda} = 0$, and by requiring that $\frac{\delta \Gamma}{\delta \gamma} = 0$.

14.2 Gap equation and propagators.

Proceeding as in [224], it is not difficult to evaluate the effective action $\Gamma(\gamma)$ at one-loop level, in the presence of the dynamical gluon mass m . The horizon condition $\frac{\delta \Gamma}{\delta \gamma} = 0$ leads to the following gap equation

$$\frac{3Ng^2}{4} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^4 + m^2 k^2 + \gamma^4} = 1, \quad (14.4)$$

which generalizes that obtained in [107, 127, 128]. Notice now that the dynamical mass m appears explicitly in eq. (14.4). The gap equation (14.4) can be used to obtain the gluon and ghost propagators in the tree-level approximation. The gluon propagator is found to be [224]

$$\langle A_\mu^a(q) A_\nu^b(-q) \rangle = \delta^{ab} \frac{q^2}{q^4 + m^2 q^2 + \gamma^4} \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right). \quad (14.5)$$

For the ghost two point function we have [224]

$$G(q) = \langle c^a(q) \bar{c}^a(-q) \rangle \sim \frac{1}{q^4}. \quad (14.6)$$

Notice that, according to the [107, 127, 128] the gluon propagator is suppressed in the infrared, while the ghost propagator is enhanced.

14.3 Conclusions.

The restriction of the domain of integration to the region Ω has been implemented by considering Zwanziger's local action [107, 127, 128] in the Gribov approximation. Expression (14.4) is the gap equation defining the Gribov parameter γ in the presence of the dynamical gluon mass m . The resulting gluon propagator is suppressed in the infrared, while the ghost propagator is enhanced. This behavior of the gluon and ghost propagators is in agreement with that found in [107, 127, 128]. It has also been found in [20] within the Schwinger-Dyson framework. Evidences for a dynamical gluon mass in the Landau gauge within the Schwinger-Dyson formalism have been obtained recently in [51]. Finally, lattice simulations have provided confirmations of the infrared suppression of the gluon propagator and of the ghost enhancement, see [116] and refs. therein, reporting a gluon mass m of the order of ≈ 600 MeV [48].

Chapter 15

The Gribov parameter and the dimension two gluon condensate in Euclidean Yang-Mills theories in the Landau gauge

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The local composite operator A_μ^2 is added to the Zwanziger action, which implements the restriction to the Gribov region Ω in Euclidean Yang-Mills theories in the Landau gauge. We prove that Zwanziger's action with the inclusion of the operator A_μ^2 is renormalizable to all orders of perturbation theory, obeying the renormalization group equations. This allows to study the dimension two gluon condensate $\langle A_\mu^2 \rangle$ by the local composite operator formalism when the restriction to the Gribov region Ω is taken into account. The resulting effective action is evaluated at one-loop order in the $\overline{\text{MS}}$ scheme. We obtain explicit values for the Gribov parameter and for the mass parameter due to $\langle A_\mu^2 \rangle$, but the expansion parameter turns out to be rather large. Furthermore, an optimization of the perturbative expansion in order to reduce the dependence on the renormalization scheme is performed. The properties of the vacuum energy, with or without the inclusion of the condensate $\langle A_\mu^2 \rangle$, are investigated. In particular, it is shown that in the original Gribov-Zwanziger formulation, i.e. without the inclusion of the operator A_μ^2 , the resulting vacuum energy is always positive at one-loop order, independently from the choice of the renormalization scheme and scale. Adding the operator A_μ^2 , opens the possibility to have a negative vacuum energy, although we are unable to come to a definite conclusion at the order considered. Concerning the behaviour of the gluon and ghost propagators, we recover the well known consequences of the restriction to the Gribov region, and this in the presence of $\langle A_\mu^2 \rangle$, i.e. an infrared suppression of the gluon propagator and an enhancement of the ghost propagator. Such a behaviour is in qualitative agreement with the results obtained from the studies of the Schwinger-Dyson equations and from lattice simulations.

15.1 Introduction.

The dimension two condensate $\langle A_\mu^2 \rangle$ has received a great deal of attention in the last few years, see for example [34, 33, 42, 144, 83, 37, 175, 184, 153, 197, 225, 214, 252, 226, 227, 228]. This condensate was already introduced in [229] in order to analyze the gluon propagator within the Operator Product Expansion (OPE), while in [200] the condensate $\langle A_i^2 \rangle$ was considered in the Coulomb gauge. A renormalizable effective potential for $\langle A_\mu^2 \rangle$ has been constructed and evaluated in analytic form up to two-loop order in the Landau gauge within the local composite operator (LCO) formalism in [42, 197]. The output of these investigations is that a non-vanishing condensate is favoured as it lowers the vacuum energy. The renormalizability of the local composite operator formalism, see [176] for an introduction to the method, was proven to all orders of perturbation theory, in the case of $\langle A_\mu^2 \rangle$, in [153] using the algebraic renormalization technique [59]. Besides the Landau gauge, the method was extended to other gauges as, for instance, the Curci-Ferrari gauge [178, 199], the linear covariant gauges [205, 212] and, more recently, the maximal Abelian gauge [215].

As a consequence of the existence of a non-vanishing condensate $\langle A_\mu^2 \rangle$, a dynamical mass parameter for the gluons can be generated in the gauge fixed Lagrangian, see [42, 197, 212]. We mention that a gluon mass has been proven to be rather useful in the phenomenological context, see e.g. [53, 54, 56]. Moreover, mass parameters are commonly used in the fitting formulas for the data obtained in lattice simulations, where the gluon propagator has been studied to a great extent in the Landau gauge [114, 45, 48, 115, 116, 117].

The lattice results so far obtained have provided firm evidence of the suppression of the gluon propagator in the infrared region, in the Landau gauge. Next to the gluon propagator, also the ghost propagator has been investigated numerically on the lattice [116, 117, 118, 119, 120], exhibiting an infrared enhancement. It is worth remarking that, in agreement with lattice results, this infrared behaviour of the gluon as well as of the ghost propagator has been obtained in the analysis of the Schwinger-Dyson equations, see [121, 122, 123, 20, 124, 125, 126], as well as in a study making use of the exact renormalization group technique [230]. Recently, the possibility of a dynamical gluon mass parameter within the Schwinger-Dyson framework has been discussed in [51, 52].

The aim of the present work is to investigate further the condensation of the operator A_μ^2 in the Landau gauge using the local composite operator formalism. This will be done by taking into account the nonperturbative effects related to the existence of the Gribov ambiguities [107], which are known to affect the Landau gauge fixing condition, $\partial_\mu A_\mu^a = 0$. As a consequence of the existence of the Gribov copies, the domain of integration in the path integral has to be restricted in a suitable way. Gribov's original proposal was to restrict the domain of integration to the region Ω whose boundary $\partial\Omega$ is the first Gribov horizon, where the first vanishing eigenvalue of the Faddeev-Popov operator, $-\partial_\mu (\partial_\mu \delta^{ab} + g f^{acb} A_\mu^c)$, appears [107]. Within the region Ω the Faddeev-Popov operator is positive definite, i.e. $-\partial_\mu (\partial_\mu \delta^{ab} + g f^{acb} A_\mu^c) > 0$. One of the main results of Gribov's work [107] was that the gluon, respectively ghost propagator, got suppressed, respectively enhanced, in the infrared due to the restriction to the region Ω .

In two previous papers [224, 231], we have already worked out the consequences of the restriction to the Gribov region Ω when the dynamical generation of a gluon mass parameter due to $\langle A_\mu^2 \rangle$ takes place, also finding an infrared suppression of the gluon and an enhancement of the ghost propagator. In [224], we closely followed the setup of Gribov's paper [107]. In this work, we shall rely on the Zwanziger local formulation of the Gribov horizon. In a series of papers [127, 128], Zwanziger has been able to implement the restriction to the Gribov region Ω through the introduction of a nonlocal horizon function appearing in the Boltzmann weight defining the Euclidean Yang-Mills measure. More precisely, according to [127, 128], the starting Yang-Mills measure in the Landau gauge is given by

$$d\mu_\gamma = DA \delta(\partial_\mu A_\mu^a) \det(\mathcal{M}) e^{-(S_{YM} + \gamma^4 H)}, \quad (15.1)$$

where

$$\mathcal{M}^{ab} = -\partial_\mu (\partial_\mu \delta^{ab} + g f^{acb} A_\mu^c) , \quad (15.2)$$

$$S_{\text{YM}} = \frac{1}{4} \int d^4x F_{\mu\nu}^a F_{\mu\nu}^a , \quad (15.3)$$

and

$$H = \int d^4x h(x) = g^2 \int d^4x f^{abc} A_\mu^b (\mathcal{M}^{-1})^{ad} f^{dec} A_\mu^e , \quad (15.4)$$

is the so-called horizon function, which implements the restriction to the Gribov region. Notice that H is nonlocal. The parameter γ , known as the Gribov parameter, has the dimension of a mass and is not free, being determined by the horizon condition

$$\langle h(x) \rangle = 4(N^2 - 1) , \quad (15.5)$$

where the expectation value $\langle h(x) \rangle$ has to be evaluated with the measure $d\mu_\gamma$. To the first order, the horizon condition (15.5) reads, in d dimensions,

$$1 = \frac{N(d-1)}{4} g^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^4 + 2Ng^2\gamma^4} . \quad (15.6)$$

This equation coincides with the original gap equation derived by Gribov for the parameter γ [107].

Albeit nonlocal, the horizon function H can be localized through the introduction of a suitable set of additional fields. As shown in [127, 128, 223], the resulting local action turns out to be renormalizable to all orders of perturbation theory. Remarkably, we shall be able to prove that this feature is preserved when the local operator A_μ^2 is introduced in the Zwanziger action. Moreover, the resulting theory turns out to obey a homogeneous renormalization group equation. These important properties will allow us to study the condensation of the operator A_μ^2 within a local renormalizable framework when the restriction to the Gribov region Ω is implemented.

The paper is organized as follows. In section 15.2, we give a short account of how the nonlocal horizon functional H can be localized by means of the introduction of additional fields. In sections 15.3 and 15.4, we prove the renormalizability, to all orders of perturbation theory, of Zwanziger's action in the presence of the operator A_μ^2 , introduced through the local composite operator formalism. As the model has a rich symmetry structure, translated into several Ward identities, it turns out that only three independent renormalization factors are necessary. The resulting quantum effective action obeys a homogeneous renormalization group equation, as explicitly verified at one-loop order. From this effective action, two coupled gap equations, associated to the condensate $\langle A_\mu^2 \rangle$ and to the Gribov parameter γ , are derived. Section 15.5 is devoted to the study of these gap equations at one-loop order in the $\overline{\text{MS}}$ renormalization scheme. It is worth mentioning that, under certain conditions, we found that it is possible that the condensate $\langle A_\mu^2 \rangle$ is positive when the horizon condition is imposed. We prove that in the $\overline{\text{MS}}$ scheme, and at one-loop order, the solution of the gap equations is necessarily one with $\langle A_\mu^2 \rangle > 0$. We recall that without the restriction to the Gribov region Ω , the value found for $\langle A_\mu^2 \rangle$ using the local composite operator formalism is negative, see [42, 197, 212]. Although the expansion parameter proves to be rather large, an attempt to obtain explicit values for the Gribov and gluon mass parameter is still presented. Also, we shall prove that in the original Gribov-Zwanziger model, the vacuum energy is always positive at one-loop order, irrespective of the choice of renormalization scheme and scale. We outline the importance of the sign of the vacuum energy, as it is related to the gauge invariant gluon condensate $\langle F_{\mu\nu}^2 \rangle$, via the trace anomaly. In section 15.6 we work out an optimized expansion in order to reduce the dependence on the choice of renormalization scheme to a single

parameter b_0 , related to the coupling constant renormalization. This is achieved by exchanging the mass parameters by their renormalization scale and scheme invariant counterparts and by re-expanding the series in the one-loop coupling constant. We find that there is a region of b_0 for which the vacuum energy is negative. However, the dependence on b_0 in this region happens to be very large, implying that the results obtained at one-loop order can be taken only as a preliminary indication, as higher order contributions would be required. In section 15.7 we outline some consequences stemming from the presence of the Gribov and gluon mass parameters on the gluon and ghost propagators. We point out a particular renormalization property of the Zwanziger action in order to ensure the enhancement of the ghost propagator. Conclusions are written down in section 15.8, while some technical details are found in the Appendix.

15.2 Local action from the restriction to the Gribov region.

As explained in [127, 128], the nonlocal functional H can be localized by means of the introduction of a suitable set of additional ghost fields. More precisely, for the localized version of the measure $d\mu_\gamma$ we get,

$$d\mu_\gamma = DADbDcD\bar{c}D\varphi D\bar{\varphi}D\omega D\bar{\omega}e^{-S}, \quad (15.7)$$

where S is given by¹

$$S = S_0 - \gamma^2 g \int d^4x \left(f^{abc} A_\mu^a \varphi_\mu^{bc} + f^{abc} A_\mu^a \bar{\varphi}_\mu^{bc} \right), \quad (15.8)$$

while

$$\begin{aligned} S_0 = S_{\text{YM}} &+ \int d^4x \left(b^a \partial_\mu A_\mu^a + \bar{c}^a \partial_\mu (D_\mu c)^a \right) \\ &+ \int d^4x \left(\bar{\varphi}_\mu^{ac} \partial_\nu (\partial_\nu \varphi_\mu^{ac} + g f^{abm} A_\nu^b \varphi_\mu^{mc}) - \bar{\omega}_\mu^{ac} \partial_\nu (\partial_\nu \omega_\mu^{ac} + g f^{abm} A_\nu^b \omega_\mu^{mc}) \right. \\ &\quad \left. - g (\partial_\nu \bar{\omega}_\mu^{ac}) f^{abm} (D_\nu c)^b \varphi_\mu^{mc} \right). \end{aligned} \quad (15.9)$$

The fields $(\bar{\varphi}_\mu^{ac}, \varphi_\mu^{ac})$ are a pair of complex conjugate bosonic fields. Each field has $4(N^2 - 1)^2$ components. Similarly, the fields $(\bar{\omega}_\mu^{ac}, \omega_\mu^{ac})$ are anticommuting. The local action (15.8) is renormalizable by power counting. More precisely, it has been shown in [127, 128, 223] that the Green functions obtained with the action S_0 with the insertion of the local composite operators $f^{abc} A_\mu^a \varphi_\mu^{bc}$ and $f^{abc} A_\mu^a \bar{\varphi}_\mu^{bc}$ are renormalizable, the action S_0 being indeed renormalizable by a multiplicative renormalization of the coupling constant g and of the fields [127, 128, 223]. We remark that the action S_0 displays a global $U(f)$ symmetry, $f = 4(N^2 - 1)$, with respect to the composite index $i = (\mu, c) = 1, \dots, f$, of the additional fields $(\bar{\varphi}_\mu^{ac}, \varphi_\mu^{ac}, \bar{\omega}_\mu^{ac}, \omega_\mu^{ac})$. Setting

$$(\bar{\varphi}_\mu^{ac}, \varphi_\mu^{ac}, \bar{\omega}_\mu^{ac}, \omega_\mu^{ac}) = (\bar{\varphi}_i^a, \varphi_i^a, \bar{\omega}_i^a, \omega_i^a), \quad (15.10)$$

we get

$$\begin{aligned} S_0 &= S_{\text{YM}} + \int d^4x \left(b^a \partial_\mu A_\mu^a + \bar{c}^a \partial_\mu (D_\mu c)^a \right) \\ &+ \int d^4x \left(\bar{\varphi}_i^a \partial_\nu (D_\nu \varphi_i^a) - \bar{\omega}_i^a \partial_\nu (D_\nu \omega_i^a) - g (\partial_\nu \bar{\omega}_i^a) f^{abm} (D_\nu c)^b \varphi_i^m \right). \end{aligned} \quad (15.11)$$

¹Our conventions are different from those originally used by Zwanziger. These can be obtained from ours by setting $\varphi \rightarrow -\varphi$ and $\omega \rightarrow -\omega$.

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For the $U(f)$ invariance we have

$$U_{ij} S_0 = 0 ,$$

$$U_{ij} = \int d^4x \left(\varphi_i^a \frac{\delta}{\delta \varphi_j^a} - \bar{\varphi}_j^a \frac{\delta}{\delta \bar{\varphi}_i^a} + \omega_i^a \frac{\delta}{\delta \omega_j^a} - \bar{\omega}_j^a \frac{\delta}{\delta \bar{\omega}_i^a} \right) . \quad (15.12)$$

The presence of the global $U(f)$ invariance means that one can make use of the composite index $i = (\mu, c)$. By means of the diagonal operator $Q_f = U_{ii}$, the i -valued fields turn out to possess an additional quantum number. As shown in [127, 128, 223], the action S_0 is left invariant by the following nilpotent BRST transformations,

$$\begin{aligned} sA_\mu^a &= -(D_\mu c)^a , \\ sc^a &= \frac{1}{2} g f^{abc} c^b c^c , \\ s\bar{c}^a &= b^a , \quad sb^a = 0 , \\ s\varphi_i^a &= \omega_i^a , \quad s\omega_i^a = 0 , \\ s\bar{\varphi}_i^a &= \bar{\varphi}_i^a , \quad s\bar{\varphi}_i^a = 0 , \end{aligned} \quad (15.13)$$

with

$$sS_0 = 0 . \quad (15.14)$$

For further use, the quantum numbers of all fields entering the action S_0 are displayed in the Table 15.2 . It is worth noticing that, when $f^{abc} A_\mu^a \varphi_\mu^{bc}$ and $f^{abc} A_\mu^a \bar{\varphi}_\mu^{bc}$ are treated as composite operators, they are introduced in the starting action S_0 coupled to local external sources M_μ^{ai} , V_μ^{ai} , namely

$$- \int d^4x \left(M_\mu^{ai} (D_\mu \varphi_i)^a + V_\mu^{ai} (D_\mu \bar{\varphi}_i)^a \right) . \quad (15.15)$$

The horizon condition (15.5) is thus obtained from the quantum action by requiring that, at the end of the computation, the sources M_μ^{ai} , V_μ^{ai} attain the physical values, obtained by setting

$$M_{\mu\nu}^{ab} = V_{\mu\nu}^{ab} = \gamma^2 \delta^{ab} \delta_{\mu\nu} . \quad (15.16)$$

Indeed, expression (15.15) reduces precisely to that of eq.(15.8) when the sources M_μ^{ai} , V_μ^{ai} attain their physical value.

15.3 Renormalizability of the Zwanziger action in the presence of the composite operator $A_\mu^a A_\mu^a$.

The purpose of this section is to show that the renormalizability of the local action S_0 is preserved when, besides the operators $f^{abc} A_\mu^a \varphi_\mu^{bc}$ and $f^{abc} A_\mu^a \bar{\varphi}_\mu^{bc}$, also the local composite operator $A_\mu^a A_\mu^a$ is

	A_μ^a	c^a	$\bar{c}^a$	b^a	φ_i^a	$\bar{\varphi}_i^a$	ω_i^a	$\bar{\omega}_i^a$
dimension	1	0	2	2	1	1	1	1
ghostnumber	0	1	-1	0	0	0	1	-1
Q_f -charge	0	0	0	0	1	-1	1	-1

Table 15.1: Quantum numbers of the fields.

	U_μ^{ai}	M_μ^{ai}	N_μ^{ai}	V_μ^{ai}	η	τ	K_μ^a	L^a
dimension	2	2	2	2	2	2	3	4
ghostnumber	-1	0	1	0	-1	0	-1	-2
Q_f -charge	-1	-1	1	1	0	0	0	0

Table 15.2: Quantum numbers of the sources.

introduced. This is a remarkable feature of the Zwanziger action, allowing us to discuss the condensation of the operator $A_\mu^a A_\mu^a$ when the restriction to the Gribov region Ω is implemented. To discuss the renormalizability of the model in the presence of A_μ^2 , we start from the following complete action

$$\Sigma = S_0 + S_s + S_{\text{ext}} , \quad (15.17)$$

where S_s is the term containing all needed local composite operators with their respective local sources, and is given by

$$S_s = s \int d^4x \left(-U_\mu^{ai} (D_\mu \varphi_i)^a - V_\mu^{ai} (D_\mu \bar{\omega}_i)^a - U_\mu^{ai} V_\mu^{ai} + \frac{1}{2} \eta A_\mu^a A_\mu^a - \frac{1}{2} \zeta \tau \eta \right) , \quad (15.18)$$

where the BRST operator acts as

$$\begin{aligned} sU_\mu^{ai} &= M_\mu^{ai} , & sM_\mu^{ai} &= 0 , \\ sV_\mu^{ai} &= N_\mu^{ai} , & sN_\mu^{ai} &= 0 , \end{aligned} \quad (15.19)$$

and

$$s\eta = \tau , \quad s\tau = 0 . \quad (15.20)$$

Therefore, for S_s one gets

$$\begin{aligned} S_s &= \int d^4x \left(-M_\mu^{ai} (D_\mu \varphi_i)^a - g U_\mu^{ai} f^{abc} (D_\mu c)^b \varphi_i^c + U_\mu^{ai} (D_\mu \omega_i)^b \right. \\ &\quad - N_\mu^{ai} (D_\mu \bar{\omega}_i)^a - V_\mu^{ai} (D_\mu \bar{\varphi}_i)^a + g V_\mu^{ai} f^{abc} (D_\mu c)^b \bar{\omega}_i^c \\ &\quad \left. - M_\mu^{ai} V_\mu^{ai} + U_\mu^{ai} N_\mu^{ai} + \frac{1}{2} \tau A_\mu^a A_\mu^a + \eta A_\mu^a \partial_\mu c^a - \frac{1}{2} \zeta \tau^2 \right) . \end{aligned} \quad (15.21)$$

As already noticed, the sources M_μ^{ai} , V_μ^{ai} are needed to introduce the composite operators $(D_\mu \varphi_i)^a$ and $(D_\mu \bar{\varphi}_i)^a$. The sources U_μ^{ai} , N_μ^{ai} define the BRST variations of these operators, given by $(D_\mu \omega_i)^b$ and $(D_\mu \bar{\omega}_i)^a$. The physical value of these sources is given by

$$\begin{aligned} M_{\mu\nu}^{ab} &= V_{\mu\nu}^{ab} = \gamma^2 \delta^{ab} \delta_{\mu\nu} , \\ U_{\mu\nu}^{ab} &= N_{\mu\nu}^{ab} = 0 . \end{aligned} \quad (15.22)$$

The local composite operator $A_\mu^a A_\mu^a$ and its BRST variation, $A_\mu^a \partial_\mu c^a$, are then introduced by means of the local sources τ , η . We also notice that the complete action Σ contains terms quadratic in the external sources, namely $(M_\mu^{ai} V_\mu^{ai} - U_\mu^{ai} N_\mu^{ai})$ and $\zeta \tau^2$. These terms, allowed by power counting, are in fact needed for the multiplicative renormalizability of the model. As shown in [42], the dimensionless LCO parameter ζ of the quadratic term in the source τ is needed to account for the divergences present in the correlation function $\langle A_\mu^2(x) A_\nu^2(y) \rangle$ for $x \rightarrow y$. It should be remarked that, unlike for the term quadratic in the external source τ , we have not introduced a new free parameter for the quadratic term

$(M_\mu^{ai} V_\mu^{ai} - U_\mu^{ai} N_\mu^{ai})$ in expression (15.21). As we shall see, this term goes through the renormalization without the need of introducing a new parameter for its renormalizability. This is a remarkable feature of the Zwanziger action which plays an important role when the ghost propagator in the presence of the Gribov horizon will be discussed, see section 15.8.

Finally, the term S_{ext} is the source term needed to define the nonlinear BRST transformations of the gauge and ghost fields, i.e.

$$S_{\text{ext}} = \int d^4x \left(-K_\mu^a (D_\mu c)^a + \frac{1}{2} g L^a f^{abc} c^b c^c \right). \quad (15.23)$$

15.3.1 Ward identities.

In order to begin with the algebraic characterization of the most general counterterm needed for the renormalizability of the complete action Σ , let us first give the set of Ward identities which are fulfilled by Σ . These are

- the Slavnov-Taylor identity

$$\mathcal{S}(\Sigma) = 0, \quad (15.24)$$

with

$$\begin{aligned} \mathcal{S}(\Sigma) = \int d^4x \left(\frac{\delta \Sigma}{\delta K_\mu^a} \frac{\delta \Sigma}{\delta A_\mu^a} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta \Sigma}{\delta c^a} + b^a \frac{\delta \Sigma}{\delta \bar{c}^a} + \bar{\varphi}_i^a \frac{\delta \Sigma}{\delta \bar{\omega}_i^a} + \omega_i^a \frac{\delta \Sigma}{\delta \varphi_i^a} \right. \\ \left. + M_\mu^{ai} \frac{\delta \Sigma}{\delta U_\mu^{ai}} + N_\mu^{ai} \frac{\delta \Sigma}{\delta V_\mu^{ai}} + \tau \frac{\delta \Sigma}{\delta \eta} \right), \end{aligned} \quad (15.25)$$

- the Landau gauge condition and the antighost equation

$$\frac{\delta \Sigma}{\delta b^a} = \partial_\mu A_\mu^a, \quad (15.26)$$

$$\frac{\delta \Sigma}{\delta \bar{c}^a} + \partial_\mu \frac{\delta \Sigma}{\delta K_\mu^a} = 0, \quad (15.27)$$

- the ghost Ward identity

$$\mathcal{G}^a \Sigma = \Delta_{\text{cl}}^a, \quad (15.28)$$

with

$$\mathcal{G}^a = \int d^4x \left(\frac{\delta}{\delta c^a} + g f^{abc} \left(\bar{c}^b \frac{\delta}{\delta b^c} + \varphi_i^b \frac{\delta}{\delta \omega_i^c} + \bar{\omega}_i^b \frac{\delta}{\delta \bar{\varphi}_i^c} + V_\mu^{bi} \frac{\delta}{\delta N_\mu^{ci}} + U_\mu^{bi} \frac{\delta}{\delta M_\mu^{ci}} \right) \right), \quad (15.29)$$

and

$$\Delta_{\text{cl}}^a = g \int d^4x f^{abc} (K_\mu^b A_\mu^c - L^b c^c). \quad (15.30)$$

Notice that the term Δ_{cl}^a , being linear in the quantum fields A_μ^a , c^a , is a classical breaking.

- the linearly broken local constraints

$$\frac{\delta\Sigma}{\delta\varphi^{ai}} + \partial_\mu \frac{\delta\Sigma}{\delta M_\mu^{ai}} = gf^{abc} A_\mu^b V_\mu^{ci} , \quad (15.31)$$

$$\frac{\delta\Sigma}{\delta\omega^{ai}} + \partial_\mu \frac{\delta\Sigma}{\delta N_\mu^{ai}} - gf^{abc} \bar{\omega}^{bi} \frac{\delta\Sigma}{\delta b^c} = gf^{abc} A_\mu^b U_\mu^{ci} , \quad (15.32)$$

$$\frac{\delta\Sigma}{\delta\bar{\omega}^{ai}} + \partial_\mu \frac{\delta\Sigma}{\delta U_\mu^{ai}} - gf^{abc} V_\mu^{bi} \frac{\delta\Sigma}{\delta K_\mu^c} = -gf^{abc} A_\mu^b N_\mu^{ci} , \quad (15.33)$$

$$\frac{\delta\Sigma}{\delta\varphi^{ai}} + \partial_\mu \frac{\delta\Sigma}{\delta V_\mu^{ai}} - gf^{abc} \bar{\varphi}^{bi} \frac{\delta\Sigma}{\delta b^c} - gf^{abc} \bar{\omega}^{bi} \frac{\delta\Sigma}{\delta \bar{c}^c} - gf^{abc} U_\mu^{bi} \frac{\delta\Sigma}{\delta K_\mu^c} = gf^{abc} A_\mu^b M_\mu^{ci} , \quad (15.34)$$

- the integrated Ward identity

$$\int d^4x \left(c^a \frac{\delta\Sigma}{\delta\omega^{ai}} + \bar{\omega}^{ai} \frac{\delta\Sigma}{\delta \bar{c}^a} + U_\mu^{ai} \frac{\delta\Sigma}{\delta K_\mu^a} \right) = 0 , \quad (15.35)$$

- the exact $\mathcal{R}_{ij}$ symmetry

$$\mathcal{R}_{ij}\Sigma = 0 , \quad (15.36)$$

with

$$\mathcal{R}_{ij} = \int d^4x \left(\varphi_i^a \frac{\delta}{\delta\omega_j^a} - \bar{\omega}_j^a \frac{\delta}{\delta\varphi_i^a} + V_\mu^{ai} \frac{\delta}{\delta N_\mu^{ai}} - U_\mu^{ai} \frac{\delta}{\delta M_\mu^{ai}} \right) . \quad (15.37)$$

15.3.2 Algebraic characterization of the counterterm.

Having established all the Ward identities fulfilled by the complete action Σ , we can now turn to the characterization of the most general allowed counterterm Σ^c . Following the algebraic renormalization procedure [59], Σ^c is an integrated local polynomial in the fields and sources with dimension bounded by four, with vanishing ghost number and Q_f -charge, obeying the following constraints

$$\begin{aligned} \frac{\delta\Sigma^c}{\delta\varphi^{ai}} + \partial_\mu \frac{\delta\Sigma^c}{\delta V_\mu^{ai}} - gf^{abc} \bar{\omega}^{bi} \frac{\delta\Sigma^c}{\delta \bar{c}^c} - gf^{abc} U_\mu^{bi} \frac{\delta\Sigma^c}{\delta K_\mu^c} &= 0 , \\ \frac{\delta\Sigma^c}{\delta\omega^{ai}} + \partial_\mu \frac{\delta\Sigma^c}{\delta U_\mu^{ai}} - gf^{abc} V_\mu^{bi} \frac{\delta\Sigma^c}{\delta K_\mu^c} &= 0 , \\ \frac{\delta\Sigma^c}{\delta\omega^{ai}} + \partial_\mu \frac{\delta\Sigma^c}{\delta N_\mu^{ai}} &= 0 , \\ \frac{\delta\Sigma^c}{\delta\bar{\omega}^{ai}} + \partial_\mu \frac{\delta\Sigma^c}{\delta M_\mu^{ai}} &= 0 , \\ \frac{\delta\Sigma^c}{\delta\bar{c}^a} + \partial_\mu \frac{\delta\Sigma^c}{\delta K_\mu^a} &= 0 , \\ \frac{\delta\Sigma^c}{\delta b^a} &= 0 , \end{aligned} \quad (15.38)$$

$$\mathcal{G}^a \Sigma^c = 0 , \quad (15.39)$$

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$$\int d^4x \left(c^a \frac{\delta \Sigma^c}{\delta \omega^{ai}} + \bar{\omega}^{ai} \frac{\delta \Sigma^c}{\delta \bar{c}^a} + U_\mu^{ai} \frac{\delta \Sigma^c}{\delta K_\mu^a} \right) = 0, \quad (15.40)$$

$$\mathcal{R}_{ij} \Sigma^c = 0, \quad (15.41)$$

and

$$\mathcal{B}_\Sigma \Sigma^c = 0, \quad (15.42)$$

where $\mathcal{B}_\Sigma$ is the nilpotent linearized Slavnov-Taylor operator

$$\begin{aligned} \mathcal{B}_\Sigma = \int d^4x \left(\frac{\delta \Sigma}{\delta K_\mu^a} \frac{\delta}{\delta A_\mu^a} + \frac{\delta \Sigma}{\delta A_\mu^a} \frac{\delta}{\delta K_\mu^a} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta}{\delta c^a} + \frac{\delta \Sigma}{\delta c^a} \frac{\delta}{\delta L^a} + b^a \frac{\delta}{\delta \bar{c}^a} \right. \\ \left. + \bar{\varphi}_i^a \frac{\delta}{\delta \bar{\omega}_i^a} + \omega_i^a \frac{\delta}{\delta \varphi_i^a} + M_\mu^{ai} \frac{\delta}{\delta U_\mu^{ai}} + N_\mu^{ai} \frac{\delta}{\delta V_\mu^{ai}} + \tau \frac{\delta}{\delta \eta} \right), \end{aligned} \quad (15.43)$$

$$\mathcal{B}_\Sigma \mathcal{B}_\Sigma = 0. \quad (15.44)$$

As it was shown in [127, 128, 223], the constraints (15.38) imply that Σ^c does not depend on the Lagrange multiplier b^a , and that the antighost $\bar{c}^a$ and the i -valued fields φ_i^a , ω_i^a , $\bar{\varphi}_i^a$, $\bar{\omega}_i^a$ can enter only through the combinations

$$\begin{aligned} \tilde{K}_\mu^a &= K_\mu^a + \partial_\mu \bar{c}^a - g f^{abc} \tilde{U}_\mu^{bi} \varphi^{ci} - g f^{abc} V_\mu^{bi} \bar{\omega}^{ci}, \\ \tilde{U}_\mu^{ai} &= U_\mu^{ai} + \partial_\mu \bar{\omega}^{ai}, \\ \tilde{V}_\mu^{ai} &= V_\mu^{ai} + \partial_\mu \varphi^{ai}, \\ \tilde{N}_\mu^{ai} &= N_\mu^{ai} + \partial_\mu \omega^{ai}, \\ \tilde{M}_\mu^{ai} &= V_\mu^{ai} + \partial_\mu \bar{\varphi}^{ai}. \end{aligned} \quad (15.45)$$

Therefore, Σ^c can be parametrized as follows

$$\begin{aligned} \Sigma^c &= S^c(A) + \int d^4x \left(a_1 g f^{abc} L^a c^b c^c + a_2 \tilde{K}_\mu^a \partial_\mu c^a + a_3 g f^{abc} \tilde{K}_\mu^a A_\mu^b c^c + a_4 f^{abc} \tilde{V}_\mu^{ai} \tilde{U}_\mu^{bi} c^c \right. \\ &\quad \left. + a_5 \tilde{V}_\mu^{ai} \tilde{M}_\mu^{ai} + a_6 \tilde{U}_\mu^{ai} \tilde{N}_\mu^{ai} + \frac{a_7}{2} \tau A_\mu^a A_\mu^a + \frac{a_8}{2} \zeta \tau^2 + a_9 \eta A_\mu^a \partial_\mu c^a + a_{10} \eta c^a \partial A^a \right), \end{aligned} \quad (15.46)$$

where $S^c(A)$ depends only on the gauge field A_μ^a , and with $a_1, \dots, a_{10}$ arbitrary parameters. Notice, however, that there is no mixing in expression (15.46) between $\tilde{M}_\mu^{ai}$, $\tilde{N}_\mu^{ai}$, $\tilde{V}_\mu^{ai}$, $\tilde{U}_\mu^{ai}$ and the sources τ, η . This is due to the dimensionality and to the Q_f -charge. It is precisely the absence of this mixing that will ensure the renormalizability of the Zwanziger action in the presence of the composite operator $A_\mu^a A_\mu^a$. From the ghost equation (15.39) it follows

$$\begin{aligned} a_1 &= a_3 = a_{10} = 0, \\ a_4 &= -g(a_6 + a_5). \end{aligned} \quad (15.47)$$

From the equations (15.40) and (15.41) we obtain

$$a_6 = -a_2. \quad (15.48)$$

Finally, from eq.(15.42) it turns out that

$$\begin{aligned} a_5 &= a_2, \\ a_9 &= a_7 - a_2, \end{aligned} \quad (15.49)$$

and

$$S^c(A) = a_0 S_{YM} + a_2 \int d^4x A_\mu^a \frac{\delta S_{YM}}{\delta A_\mu^a} . \quad (15.50)$$

In summary, the most general local invariant counterterm compatible with all Ward identities contains four arbitrary parameters, a_0 , a_2 , a_7 , a_8 , and reads

$$\begin{aligned} \Sigma^c = a_0 S_{YM} &+ a_2 \int d^4x \left(A_\mu^a \frac{\delta S_{YM}}{\delta A_\mu^a} + \tilde{K}_\mu^a \partial_\mu c^a + \tilde{V}_\mu^{ai} \tilde{M}_\mu^{ai} - \tilde{U}_\mu^{ai} \tilde{N}_\mu^{ai} \right) \\ &+ \int d^4x \left(\frac{a_7}{2} \tau A_\mu^a A_\mu^a + \frac{a_8}{2} \zeta \tau^2 + (a_7 - a_2) \eta A_\mu^a \partial_\mu c^a \right) . \end{aligned} \quad (15.51)$$

15.4 Stability and renormalization constants.

Having determined the most general local invariant counterterm Σ^c compatible with all Ward identities, it remains to check that the starting action Σ is stable, i.e. that Σ^c can be reabsorbed through the renormalization of the parameters, fields and sources of Σ . According to expression (15.51), Σ^c contains four arbitrary parameters a_0 , a_2 , a_7 , a_8 , which correspond in fact to a multiplicative renormalization of the gauge coupling constant g , the parameters ζ , and of the fields $\phi = (A_\mu^a, c^a, \bar{c}^a, b^a, \varphi_i^a, \omega_i^a, \bar{\varphi}_i^a, \bar{\omega}_i^a)$ and sources $\Phi = (K^{a\mu}, L^a, M_\mu^{ai}, N_\mu^{ai}, V_\mu^{ai}, U_\mu^{ai}, \tau, \eta)$, according to

$$\Sigma(g, \zeta, \phi, \Phi) + \eta \Sigma^c = \Sigma(g_o, \zeta_o, \phi_o, \Phi_o) + O(\eta^2) , \quad (15.52)$$

with

$$g_o = Z_g g , \quad \zeta_o = Z_\zeta \zeta , \quad (15.53)$$

and

$$\begin{aligned} \phi_o &= Z_\phi^{1/2} \phi , \\ \Phi_o &= Z_\Phi \Phi . \end{aligned} \quad (15.54)$$

The coefficients a_0 , a_2 are easily seen to be related to the renormalization of the gauge coupling constant g and of the gauge field A_μ^a ,

$$\begin{aligned} Z_g &= \left(1 + \eta \frac{a_0}{2} \right) , \\ Z_A^{1/2} &= \left(1 + \eta \left(a_2 - \frac{a_0}{2} \right) \right) . \end{aligned} \quad (15.55)$$

From expression (15.51) it follows that the Faddeev-Popov ghosts $(c^a, \bar{c}^a)$ and the i -valued fields $(\varphi_i^a, \omega_i^a, \bar{\varphi}_i^a, \bar{\omega}_i^a)$ have a common renormalization constant, given by

$$Z_c = Z_{\bar{c}} = Z_\varphi = Z_{\bar{\varphi}} = Z_\omega = Z_{\bar{\omega}} = (1 - \eta a_2) = Z_g^{-1} Z_A^{-1/2} . \quad (15.56)$$

Eq.(15.56) expresses a well-known renormalization property of the Faddeev-Popov ghosts $(c^a, \bar{c}^a)$ in the Landau gauge, stemming from the transversality of the gauge propagator and from the factorization of the ghost momentum in the ghost-antighost-gluon vertex. We see therefore that, in the present case, this property holds for the i -valued fields $(\varphi_i^a, \omega_i^a, \bar{\varphi}_i^a, \bar{\omega}_i^a)$ as well. Similarly to the ghost and

the i -valued fields, the renormalization of the sources $(M_\mu^{ai}, N_\mu^{ai}, V_\mu^{ai}, U_\mu^{ai})$ is also determined by the renormalization constants Z_g and $Z_A^{1/2}$, being given by

$$Z_M = Z_N = Z_V = Z_U = Z_g^{-1/2} Z_A^{-1/4}. \quad (15.57)$$

It is worth noticing here that equation (15.57) ensures that the counterterm $a_2 (V_\mu^{ai} M_\mu^{ai} - U_\mu^{ai} N_\mu^{ai})$ can be automatically reabsorbed by the term $(-M_\mu^{ai} V_\mu^{ai} + U_\mu^{ai} N_\mu^{ai})$ in the expression (15.21) without the need of introducing new free parameters. Indeed,

$$-M_o V_o = -MV Z_M^2 = -MV Z_g^{-1} Z_A^{-1/2} = -MV + \varepsilon a_2 MV. \quad (15.58)$$

Concerning now the parameters a_7, a_8 , they are easily seen to correspond to a multiplicative renormalization of the local source τ and of the parameter ζ , according to

$$\begin{aligned} \tau_o &= Z_\tau \tau, & Z_\tau &= 1 + \eta(a_7 - 2a_2 + a_0), \\ \zeta_o &= Z_\zeta \zeta, & Z_\zeta &= 1 + \eta(-a_8 - 2a_7 + 4a_2 - 2a_0). \end{aligned} \quad (15.59)$$

Moreover, we would like to underline that there exists even an extra relation, namely

$$Z_\tau = Z_g Z_A^{-1/2}. \quad (15.60)$$

It can be proven by introducing the operator A_μ^2 through a more sophisticated set of local sources, like it was done in [153]. We will not repeat that analysis here, we only mention that a key ingredient in the proof of relation (15.60) was the presence of the ghost Ward identity, and since the Zwanziger action possesses that identity, eq.(15.28), one can proceed along the lines of [153]. Thus, there are in fact only three independent renormalization factors present.

In summary, the Zwanziger action in the presence of the local operator $A_\mu^a A_\mu^a$ is multiplicative renormalizable. In turn, this ensures that the quantum effective action obeys the homogeneous renormalization group equations (RGE). This is an important feature of the model, which will be useful when we shall try to obtain estimates for both the Gribov and mass parameter.

The effective action is defined upon setting the sources $U_{\mu\nu}^{ab}, N_{\mu\nu}^{ab}, K_\mu^a, L^a$ and η equal to zero and implementing the condition (15.16). Doing so, we get

$$\begin{aligned} S &= S_0 + S_\gamma + \int d^4x \left(\frac{\tau}{2} A_\mu^a A_\mu^a - \frac{\zeta}{2} \tau^2 \right), \\ S_\gamma &= \int d^4x \left[-\gamma^2 g f^{abc} A_\mu^a \varphi_\mu^{bc} - \gamma^2 g f^{abc} A_\mu^a \bar{\varphi}_\mu^{bc} - 4(N^2 - 1)\gamma^4 \right]. \end{aligned} \quad (15.61)$$

The term $-4(N^2 - 1)\gamma^4$ originates from the quadratic term in the external sources, namely $(-M_\mu^{ai} V_\mu^{ai} + U_\mu^{ai} N_\mu^{ai})$, in expression (15.21), evaluated at the physical values given by eq.(15.16).

Following [42, 197, 176, 212], we introduce a Hubbard-Stratonovich field σ by means of the following unity

$$1 = \int [d\sigma] e^{-\frac{1}{2\zeta} \int d^4x \left[\frac{\sigma}{g} + \frac{1}{2} A_\mu^a A_\mu^a - \zeta \tau \right]^2}, \quad (15.62)$$

to remove the term proportional to τ^2 . The source τ is henceforth linearly coupled to the field σ , as can be directly seen from the action, which now reads

$$\begin{aligned} S &= S_0 + S_\gamma + S_\sigma + \int d^4x \left(-\tau \frac{\sigma}{g} \right), \\ S_\sigma &= \frac{\sigma^2}{2g^2\zeta} + \frac{1}{2} \frac{g\sigma}{g^2\zeta} A_\mu^a A_\mu^a + \frac{1}{8\zeta} (A_\mu^a A_\mu^a)^2. \end{aligned} \quad (15.63)$$

The following identification is easily derived [42, 197, 176, 212]

$$\langle A_\mu^a A_\mu^a \rangle = -\frac{1}{g} \langle \sigma \rangle , \quad (15.64)$$

from which it follows that a nonvanishing vacuum expectation value of the field σ will result in a nonvanishing condensate $\langle A_\mu^a A_\mu^a \rangle$.

The quantum action Γ is obtained through the definition

$$e^{-\Gamma} = \int [d\Phi] e^{-S_0 - S_\gamma - S_\sigma} , \quad (15.65)$$

where Φ is a shorthand notation for all the relevant fields.

The value for $\langle \sigma \rangle$ is found through the minimization condition

$$\frac{\partial \Gamma}{\partial \sigma} = 0 . \quad (15.66)$$

The horizon is implemented by the condition [127, 128].

$$\frac{\partial \Gamma}{\partial \gamma^2} = 0 . \quad (15.67)$$

Let us show this here. The following equivalence is readily found

$$\frac{\partial \Gamma}{\partial \gamma^2} = 0 \Leftrightarrow \langle g f^{abc} A_\mu^a \varphi_\mu^{bc} \rangle + \langle g f^{abc} A_\mu^a \bar{\varphi}_\mu^{bc} \rangle = -8 (N^2 - 1) \gamma^2 , \quad (15.68)$$

From expressions (15.1) and (15.8), it follows that

$$-2\gamma^2 \langle h \rangle = \langle g f^{abc} A_\mu^a \varphi_\mu^{bc} \rangle + \langle g f^{abc} A_\mu^a \bar{\varphi}_\mu^{bc} \rangle . \quad (15.69)$$

The combination of eq.(15.68) with eq.(15.69) gives rise to the horizon condition eq.(15.5). In order to conclude this, it is tacitly assumed that $\gamma \neq 0$. We notice that the condition (15.67) does possess the solution $\gamma = 0$. This is an artefact of the reformulation of the horizon condition in terms of the equation (15.67), and must be excluded as it does not lead to the horizon condition (15.5). We shall, however, continue to keep this solution of the gap equation (15.67), as $\gamma \equiv 0$ corresponds to the case where the restriction to the Gribov region Ω would not be implemented. In this case, we must only solve the gap equation stemming from eq.(15.66) with $\gamma \equiv 0$.

The original Gribov-Zwanziger model, i.e. without the inclusion of the operator A_μ^2 , is obtained by only retaining the condition (15.67) with $\sigma \equiv 0$.

Up to now, the LCO parameter ζ is still a free parameter of the theory. We do not intend here to give a complete overview of the LCO formalism, we suffice by saying that ζ is fixed by the demand that the action Γ should obey the homogeneous renormalization group equation

$$\left(\bar{\mu} \frac{\partial}{\partial \bar{\mu}} + \beta(g^2) \frac{\partial}{\partial g^2} + \gamma_{\gamma^2}(g^2) \gamma^2 \frac{\partial}{\partial \gamma^2} + \gamma_\sigma(g^2) \sigma \frac{\partial}{\partial \sigma} \right) \Gamma = 0 , \quad (15.70)$$

with

$$\begin{aligned} \bar{\mu} \frac{\partial g^2}{\partial \bar{\mu}} &= \beta(g^2) , \\ \bar{\mu} \frac{\partial \gamma^2}{\partial \bar{\mu}} &= \gamma_{\gamma^2}(g^2) \gamma^2 , \\ \bar{\mu} \frac{\partial \sigma}{\partial \bar{\mu}} &= \gamma_\sigma(g^2) \sigma . \end{aligned} \quad (15.71)$$

This can be accommodated for by making ζ a function of the running coupling constant g^2 , in which case it is found that

$$\zeta(g^2) = \frac{\zeta_0}{g^2} + \zeta_1 + \zeta_2 g^2 + \dots \quad (15.72)$$

We refer to the available literature [42, 197, 176, 178, 212, 215] for a detailed account of the LCO formalism.

15.4.1 Renormalization group invariance of the one-loop effective action in the $\overline{\text{MS}}$ scheme without the inclusion of A_μ^2 .

Before proceeding with the detailed analysis of the horizon condition in the presence of the local operator $A_\mu^a A_\mu^a$, let us first derive the horizon condition and check the explicit renormalization group invariance of the quantum action Γ by switching off the source τ coupled to the operator $A_\mu^a A_\mu^a$. This amounts to consider the original Gribov-Zwanziger model. We consider thus the action

$$S = S_0 + S_\gamma. \quad (15.73)$$

The one-loop effective action $\Gamma^{(1)}$ is easily obtained from the quadratic part of eq.(15.73)

$$e^{-\Gamma_\gamma^{(1)}} = \int [D\Phi] e^{-S_{\text{quad}}}, \quad (15.74)$$

with S_{quad} given by

$$\begin{aligned} S_{\text{quad}} = & \int d^4x \left[\frac{1}{4} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 + \frac{1}{2\alpha} (\partial_\mu A_\mu^a)^2 + \bar{\varphi}_\mu^{ab} \partial^2 \varphi_\mu^{ab} \right. \\ & \left. - \gamma^2 g (f^{abc} A_\mu^a \varphi_\mu^{bc} + f^{abc} A_\mu^a \bar{\varphi}_\mu^{bc}) - 4(N^2 - 1)\gamma^4 \right], \end{aligned} \quad (15.75)$$

where the limit $\alpha \rightarrow 0$ is understood in order to recover the Landau gauge. After a straightforward computation, one gets

$$\Gamma^{(1)} = -4(N^2 - 1)\gamma^4 + \frac{(N^2 - 1)}{2} (d - 1) \int \frac{d^d p}{(2\pi)^d} \ln(p^4 + 2Ng^2\gamma^4). \quad (15.76)$$

Dimensional regularization, with $d = 4 - \varepsilon$, will be employed throughout this work. Taking the derivative of $\Gamma^{(1)}$, one reobtains the original gap equation for the Gribov parameter γ , namely

$$\frac{\partial \Gamma^{(1)}}{\partial \gamma} = 0 \Rightarrow 1 = \frac{N(d - 1)}{4} g^2 \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^4 + 2Ng^2\gamma^4)}. \quad (15.77)$$

More precisely, recalling that

$$\int \frac{d^d p}{(2\pi)^d} \ln(p^4 + \rho^2) = -\frac{\rho^2}{32\pi^2} \left(\ln \frac{\rho^2}{\mu^4} - 3 \right) + \frac{1}{\varepsilon} \frac{4\rho^2}{32\pi^2}, \quad (15.78)$$

the one-loop effective action $\Gamma^{(1)}$ reads

$$\Gamma^{(1)} = -4(N^2 - 1)\gamma^4 - \frac{3(N^2 - 1)}{64\pi^2} (2Ng^2\gamma^4) \left(\ln \frac{2Ng^2\gamma^4}{\mu^4} - \frac{5}{3} \right), \quad (15.79)$$

where the $\overline{\text{MS}}$ renormalization scheme has been used.

In order to check the renormalization group invariance of $\Gamma^{(1)}$, we need to know the anomalous dimension of the Gribov parameter γ . This is easily obtained from eq.(15.57), yielding

$$\gamma_{\gamma^2}(g^2) = -\frac{1}{2} \left(\frac{\beta(g^2)}{2g^2} - \gamma_A(g^2) \right), \quad (15.80)$$

where $\gamma_A(g^2)$ stands for the anomalous dimension of the gauge field A_μ^a . Thus, at one-loop order,

$$\bar{\mu} \frac{d\Gamma^{(1)}}{d\bar{\mu}} = \left(4(N^2 - 1) \left(\frac{\beta^{(1)}(g^2)}{2g^2} - \gamma_A^{(1)}(g^2) \right) + \frac{3(N^2 - 1)}{16\pi^2} 2Ng^2 \right) \gamma^4. \quad (15.81)$$

Furthermore, from (see e.g. [87])

$$\begin{aligned} \beta^{(1)}(g^2) &= -\frac{22}{3} \frac{g^4 N}{16\pi^2}, \\ \gamma_A^{(1)}(g^2) &= -\frac{13}{6} \frac{g^2 N}{16\pi^2}, \end{aligned} \quad (15.82)$$

it follows

$$\bar{\mu} \frac{d\Gamma^{(1)}}{d\bar{\mu}} = 0, \quad (15.83)$$

which establishes the RGE invariance of the effective action at the order considered.

We are now ready to face the more complex case in which the local composite operator $A_\mu^a A_\mu^a$ is present. This will be the topic of the next section.

15.5 One-loop effective action in the $\overline{\text{MS}}$ scheme with the inclusion of A_μ^2 .

15.5.1 Calculation of the one-loop effective potential.

Let us turn to the explicit one-loop evaluation of the effective action Γ in the presence of A_μ^2 . At one-loop, it turns out that²

$$\Gamma = -4(N^2 - 1)\gamma^4 + \frac{\sigma^2}{2g^2\zeta} + \frac{N^2 - 1}{2} \ln \det \left[p^2 \delta_{\mu\nu} + \frac{2Ng^2\gamma^4}{p^2} \delta_{\mu\nu} - p_\mu p_\nu \left(1 - \frac{1}{\alpha} \right) + \frac{g\sigma}{g^2\zeta} \delta_{\mu\nu} \right], \quad (15.84)$$

or

$$\Gamma = -4(N^2 - 1)\gamma^4 + \frac{\sigma^2}{2g^2\zeta} + \frac{N^2 - 1}{2}(d - 1) \int \frac{d^d p}{(2\pi)^d} \ln \left[p^4 + 2Ng^2\gamma^4 + \frac{g\sigma}{g^2\zeta} p^2 \right]. \quad (15.85)$$

Before calculating the integral, we quote the two gap equations

$$\begin{aligned} \frac{\partial \Gamma}{\partial \sigma} = 0 &\Leftrightarrow \frac{\sigma}{\zeta_0} \left(1 - \frac{\zeta_1}{\zeta_0} g^2 \right) + \frac{(N^2 - 1)}{2} \frac{g(d - 1)}{\zeta_0} \int \frac{d^d p}{(2\pi)^d} \frac{p^2}{p^4 + \frac{g\sigma}{\zeta_0} p^2 + 2Ng^2\gamma^4} = 0, \\ \frac{\partial \Gamma}{\partial \gamma} = 0 &\Leftrightarrow \gamma^3 = \gamma^3 \frac{d - 1}{4} g^2 N \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^4 + \frac{g\sigma}{\zeta_0} p^2 + 2Ng^2\gamma^4}. \end{aligned} \quad (15.86)$$

²We shall drop from now on the superscript ⁽¹⁾ indicating that we are working at one-loop order.

The second gap equation of (15.86), being the horizon condition, gives rise to the one obtained in the previous paper [224], while the first one describes the condensation of A_μ^2 when the restriction to the Gribov region Ω is implemented. We notice that the explicit value of the Gribov parameter γ is influenced by the presence of $\langle A_\mu^2 \rangle$.

It remains to calculate

$$\mathcal{I} = \int \frac{d^d p}{(2\pi)^d} \ln [p^4 + bp^2 + c] , \quad (15.87)$$

with

$$b = \frac{g\sigma}{\zeta_0} , \quad c = 2Ng^2\gamma^4 , \quad (15.88)$$

Since

$$p^4 + bp^2 + c = (p^2 + \omega_1)(p^2 + \omega_2) , \quad (15.89)$$

with

$$\omega_1 = \frac{b + \sqrt{b^2 - 4c}}{2} , \quad \omega_2 = \frac{b - \sqrt{b^2 - 4c}}{2} , \quad (15.90)$$

one has

$$\mathcal{I} = \int \frac{d^d p}{(2\pi)^d} \ln (p^2 + \omega_1) + \int \frac{d^d p}{(2\pi)^d} \ln (p^2 + \omega_2) . \quad (15.91)$$

To make sense, the expression (15.87) should be real to ensure that the one-loop effective action is real-valued. Therefore, we must demand that $c \geq 0$. If $b \geq 0$, $\mathcal{I}$ is certainly real. However, when $b^2 - 4c \leq 0$, then also $b < 0$ is allowed. We should thus have a positive Gribov parameter γ^4 , while the condensate $\langle A_\mu^2 \rangle$ can be negative or positive, depending on the case.

Using

$$\int \frac{d^d p}{(2\pi)^d} \ln (p^2 + m^2) = \frac{-m^4}{32\pi^2} \left(\frac{2}{\varepsilon} - \ln \frac{m^2}{\bar{\mu}^2} + \frac{3}{2} \right) , \quad (15.92)$$

it holds

$$\mathcal{I} = -\frac{\omega_1^2}{32\pi^2} \left(\frac{2}{\varepsilon} - \ln \frac{\omega_1}{\bar{\mu}^2} + \frac{3}{2} \right) - \frac{\omega_2^2}{32\pi^2} \left(\frac{2}{\varepsilon} - \ln \frac{\omega_2}{\bar{\mu}^2} + \frac{3}{2} \right) . \quad (15.93)$$

Finally, in the $\overline{\text{MS}}$ scheme, we obtain

$$\begin{aligned} \Gamma = & -4(N^2 - 1)\gamma^4 + \frac{\sigma^2}{2\zeta_0} \left(1 - \frac{\zeta_1}{\zeta_0} g^2 \right) + \frac{3(N^2 - 1)}{2} \times \\ & \left[\frac{\left(\frac{g\sigma}{\zeta_0} + \sqrt{\frac{g^2\sigma^2}{\zeta_0^2} - 8g^2N\gamma^4} \right)^2}{128\pi^2} \left(\ln \frac{\frac{g\sigma}{\zeta_0} + \sqrt{\frac{g^2\sigma^2}{\zeta_0^2} - 8g^2N\gamma^4}}{2\bar{\mu}^2} - \frac{5}{6} \right) \right. \\ & \left. + \frac{\left(\frac{g\sigma}{\zeta_0} - \sqrt{\frac{g^2\sigma^2}{\zeta_0^2} - 8g^2N\gamma^4} \right)^2}{128\pi^2} \left(\ln \frac{\frac{g\sigma}{\zeta_0} - \sqrt{\frac{g^2\sigma^2}{\zeta_0^2} - 8g^2N\gamma^4}}{2\bar{\mu}^2} - \frac{5}{6} \right) \right] . \end{aligned} \quad (15.94)$$

To lighten the notation a bit, let us introduce the new variables³

$$\lambda^4 = 8g^2 N \gamma^4, \quad (15.95)$$

$$m^2 = \frac{g\sigma}{\zeta_0}. \quad (15.96)$$

in which case the action (15.94) can be rewritten as

$$\begin{aligned} \Gamma = & -\frac{(N^2 - 1)\lambda^4}{2g^2 N} + \frac{\zeta_0 m^4}{2g^2} \left(1 - \frac{\zeta_1}{\zeta_0} g^2\right) \\ & + \frac{3(N^2 - 1)}{256\pi^2} \left[\left(m^2 + \sqrt{m^4 - \lambda^4}\right)^2 \left(\ln \frac{m^2 + \sqrt{m^4 - \lambda^4}}{2\bar{\mu}^2} - \frac{5}{6}\right) \right. \\ & \left. + \left(m^2 - \sqrt{m^4 - \lambda^4}\right)^2 \left(\ln \frac{m^2 - \sqrt{m^4 - \lambda^4}}{2\bar{\mu}^2} - \frac{5}{6}\right) \right]. \end{aligned} \quad (15.97)$$

We notice that the foregoing expression is also valid, i.e. real-valued, in the case in which $m^4 < \lambda^4$, as $\ell_+(m, \lambda)$ and $\ell_-(m, \lambda)$, defined by,

$$\begin{aligned} \ell_+(m, \lambda) &= \left(m^2 + \sqrt{m^4 - \lambda^4}\right)^2 \left(\ln \frac{m^2 + \sqrt{m^4 - \lambda^4}}{2\bar{\mu}^2} - \frac{5}{6}\right) \\ \ell_-(m, \lambda) &= \left(m^2 - \sqrt{m^4 - \lambda^4}\right)^2 \left(\ln \frac{m^2 - \sqrt{m^4 - \lambda^4}}{2\bar{\mu}^2} - \frac{5}{6}\right) \end{aligned} \quad (15.98)$$

are complex conjugate⁴.

The horizon condition, eq.(15.67), can be translated to

$$\frac{\partial \Gamma}{\partial \lambda} = 0, \quad (15.99)$$

and the gap equation (15.66) to

$$\frac{\partial \Gamma}{\partial m^2} = 0. \quad (15.100)$$

As a check of this one-loop calculation, the expression (15.97) with $m^2 \equiv 0$ reduces to the result obtained earlier in eq.(15.79), i.e. the original Gribov-Zwanziger model without the inclusion of A_μ^2 . If $\lambda \equiv 0$, i.e. the case where the condensation of A_μ^2 is investigated without implementing the restriction to the Gribov region Ω , eq.(15.97) coincides with the result of [42, 197, 212]. From [153], one knows that

$$\bar{\mu} \frac{\partial \langle A_\mu^2 \rangle}{\partial \bar{\mu}} = \gamma_{A_\mu^2}(g^2) \langle A_\mu^2 \rangle = - \left(\frac{\beta(g^2)}{2g^2} + \gamma_A(g^2) \right) \langle A_\mu^2 \rangle, \quad (15.101)$$

or, using the relation (15.64) and the definition (15.96),

$$\bar{\mu} \frac{\partial m^2}{\partial \bar{\mu}} = \gamma_{m^2}(g^2) m^2 = \left(\frac{\beta(g^2)}{2g^2} - \gamma_A(g^2) \right) m^2, \quad (15.102)$$

³In comparison with the previous article [224], we have the correspondence $\lambda^4 = 4\gamma^4$ with the Gribov parameter γ^4 as defined there. It is actually this γ^4 which will enter the modified propagators, see [224] and further in this paper.

⁴Using $\ln(z) = \ln|z| + i \arg(z)$ with $-\pi < \arg(z) \leq \pi$.

while from eq.(15.80), it can be inferred that

$$\bar{\mu} \frac{\partial \lambda}{\partial \bar{\mu}} = \gamma_\lambda(g^2) \lambda = \frac{1}{4} \left(\frac{\beta(g^2)}{2g^2} + \gamma_A(g^2) \right) \lambda. \quad (15.103)$$

We notice the remarkable fact that the anomalous dimensions of the Gribov parameter and of the operator A_μ^2 are proportional to each other, to all orders of perturbation theory.

It can now be checked that Γ is renormalization group invariant, namely

$$\bar{\mu} \frac{d}{d\bar{\mu}} \Gamma = 0. \quad (15.104)$$

Finally, taking the derivatives of the action given in eq.(15.97) gives rise to

$$\begin{aligned} \frac{1}{\lambda^3} \frac{\partial \Gamma}{\partial \lambda} = & -\frac{2(N^2-1)}{g^2 N} + \frac{3(N^2-1)}{256\pi^2} \left[-4 \frac{(m^2 + \sqrt{m^4 - \lambda^4})}{\sqrt{m^4 - \lambda^4}} \ln \frac{m^2 + \sqrt{m^4 - \lambda^4}}{2\bar{\mu}^2} \right. \\ & \left. + 4 \frac{(m^2 - \sqrt{m^4 - \lambda^4})}{\sqrt{m^4 - \lambda^4}} \ln \frac{m^2 - \sqrt{m^4 - \lambda^4}}{2\bar{\mu}^2} + \frac{8}{3} \right], \end{aligned} \quad (15.105)$$

and

$$\begin{aligned} \frac{\partial \Gamma}{\partial m^2} = & \frac{\zeta_0 m^2}{g^2} \left(1 - \frac{\zeta_1}{\zeta_0} g^2 \right) + \frac{3(N^2-1)}{256\pi^2} \left[2 \left(m^2 + \sqrt{m^4 - \lambda^4} \right) \left(1 + \frac{m^2}{\sqrt{m^4 - \lambda^4}} \right) \ln \frac{m^2 + \sqrt{m^4 - \lambda^4}}{2\bar{\mu}^2} \right. \\ & \left. + 2 \left(m^2 - \sqrt{m^4 - \lambda^4} \right) \left(1 - \frac{m^2}{\sqrt{m^4 - \lambda^4}} \right) \ln \frac{m^2 - \sqrt{m^4 - \lambda^4}}{2\bar{\mu}^2} - \frac{8}{3} m^2 \right]. \end{aligned} \quad (15.106)$$

15.5.2 Solving the gap equations.

We have now all the ingredients at hand to search for estimates of the mass parameter m^2 and Gribov parameter λ as solutions of the gap equations (15.105) and (15.106). To avoid misinterpretations due to the suggestive use of the notation m^2 , we remark that, due to the presence of λ , the mass parameter does not even appear as a pole in the tree level gluon propagator, see eq.(15.206).

Let us first consider the pure Gribov-Zwanziger case, i.e. we put $m^2 \equiv 0$ in the expression (15.97). The relevant gap equation (horizon condition) reads

$$\frac{\partial \Gamma}{\partial \lambda} = \lambda^3 \left(-\frac{2(N^2-1)}{g^2 N} - \frac{3(N^2-1)}{64\pi^2} \left(\ln \frac{\lambda^4}{4\bar{\mu}^4} - \frac{5}{3} \right) - \frac{3(N^2-1)}{64\pi^2} \right) = 0. \quad (15.107)$$

We remind here that the solution $\lambda = 0$ must be rejected. The natural choice for the renormalization scale is to set $\bar{\mu}^2 = \frac{\lambda^4}{4}$ to kill the logarithms, and we find

$$\left. \frac{g^2 N}{16\pi^2} \right|_{\bar{\mu}^2 = \frac{\lambda^4}{4}} = 4. \quad (15.108)$$

In principle, from

$$g^2(\bar{\mu}^2) = \frac{1}{\beta_0 \ln \frac{\bar{\mu}^2}{\Lambda_{\overline{\text{MS}}}^2}}, \quad \text{with} \quad \beta_0 = \frac{11}{3} \frac{N}{16\pi^2}, \quad (15.109)$$

eq.(15.108) could be used to determine an estimate for the Gribov parameter, however it might be clear that this is meaningless since the corresponding expansion parameter (15.108) is far too big.

It is interesting to notice that, in a general massless renormalization scheme, the one-loop action with $m^2 \equiv 0$ would read

$$\Gamma = -\frac{(N^2 - 1)}{2g^2 N} \lambda^4 - \frac{3\lambda^4 (N^2 - 1)}{264\pi^2} \left(\ln \frac{\lambda^4}{4\bar{\mu}^4} + a \right), \quad (15.110)$$

with a an arbitrary constant. The corresponding gap equation equals

$$\frac{\partial \Gamma}{\partial \lambda} = \lambda^3 \left(-\frac{2(N^2 - 1)}{g^2 N} - \frac{3(N^2 - 1)}{64\pi^2} \left(\ln \frac{\lambda^4}{4\bar{\mu}^4} + a \right) - \frac{3(N^2 - 1)}{64\pi^2} \right) = 0. \quad (15.111)$$

Denoting by λ_* a solution of eq.(15.111), for the vacuum energy corresponding to (15.110) one finds

$$E_{\text{vac}} = \Gamma(\lambda_*) = \frac{3(N^2 - 1)}{64\pi^2} \frac{\lambda_*^4}{4}. \quad (15.112)$$

This expression is valid for all $\bar{\mu}$ and for all a . The vacuum energy is thus always nonnegative at one-loop order in the original Gribov-Zwanziger model.

The gap equation (15.106) with $\lambda \equiv 0$ obviously has the solution already obtained in [42, 197, 212] where the restriction to the Gribov region Ω was not taken into account. We recall the values

$$\frac{g^2 N}{16\pi^2} = \frac{36}{187} \approx 0.193, \quad (15.113)$$

$$m^2 = e^{\frac{17}{12}} \Lambda_{\overline{\text{MS}}}^2 \approx (2.031 \Lambda_{\overline{\text{MS}}})^2, \quad (15.114)$$

$$E_{\text{vac}} = -\frac{3}{16\pi^2} e^{\frac{17}{6}} \Lambda_{\overline{\text{MS}}}^4 \approx -0.323 \Lambda_{\overline{\text{MS}}}^4, \quad (15.115)$$

which were obtained upon setting $\bar{\mu}^2 = m^2$ to kill the logarithms.

We shall now show that, in the $\overline{\text{MS}}$ scheme, the gap equations (15.105)-(15.106) have no solution with $m^2 > 0$ when the restriction to the horizon is implemented (i.e. when $\lambda \neq 0$). To this purpose, we introduce for $m^2 > 0$ the variable

$$t = \frac{\lambda^4}{m^4}. \quad (15.116)$$

Evidently, we should only consider $t > 0$.

Dividing the gap equations (15.105)-(15.106) by m^2 , they can be rewritten as⁵

$$\frac{16\pi^2}{g^2 N} = \frac{3}{8} \left(-2 \ln \frac{m^2}{2\bar{\mu}^2} + \frac{2}{3} + \frac{1}{\sqrt{1-t}} \ln \frac{t}{(1 + \sqrt{1-t})^2} - \ln t \right), \quad (15.117)$$

and

$$-\frac{24}{13} \left(\frac{16\pi^2}{g^2 N} \right) + \frac{322}{39} = 4 \ln \frac{m^2}{2\bar{\mu}^2} - \frac{4}{3} - \frac{2-t}{\sqrt{1-t}} \ln \frac{t}{(1 + \sqrt{1-t})^2} + 2 \ln t, \quad (15.118)$$

⁵We have already factored out m^2 or λ^3 since these are non-zero in the present case.

where use has been made of the explicit values of ζ_0 and ζ_1 , which can be found in [42, 197, 212]

$$\zeta_0 = \frac{9}{13} \frac{N^2 - 1}{N}, \quad \zeta_1 = \frac{161}{52} \frac{N^2 - 1}{16\pi^2}, \quad (15.119)$$

The eqns.(15.118)-(15.119) can be combined to eliminate $\ln \frac{m^2}{2\bar{\mu}^2}$, yielding the following condition

$$\frac{68}{39} \left(\frac{16\pi^2}{g^2 N} \right) + \frac{161}{39} = \frac{t}{\sqrt{1-t}} \ln \frac{t}{(1 + \sqrt{1-t})^2} \equiv F(t). \quad (15.120)$$

It can be checked that $F(t)$ is real-valued and negative for $t > 0$, thus the r.h.s. of eq.(15.120) is always negative. Since the l.h.s. of eq.(15.120) is necessarily positive for a meaningful result (i.e. $g^2 \geq 0$), there is no solution with $m^2 > 0$. As already mentioned, there are a priori also possible solutions with $m^2 < 0$.

To investigate the existence of a solution with $m^2 < 0$, it might be instructive to look again at the gap equations (15.105) and (15.106) from another perspective. We recall that, if the horizon is not implemented, i.e. $\lambda \equiv 0$, the gap equation (15.106) has two solutions, a perturbative one corresponding to $m^2 = 0$ (no condensation) and a non-perturbative one corresponding to the m^2 given in eq.(15.114). If we momentarily consider λ as a free, adjustable parameter of the theory, eq.(15.106) dictates how m^2 becomes a function of the parameter λ . From the result at $\lambda = 0$, we could expect that two branches of solutions would evolve, one starting from the perturbative and one from the non-perturbative value of m^2 at $\lambda = 0$. When $\lambda \equiv 0$, the choice for the scale $\bar{\mu}$ is quite obvious from the requirement that all the logarithms $\ln \frac{m^2}{\bar{\mu}^2}$ are vanishing. However, when $\lambda \neq 0$, we notice that there are two kinds of logarithms present, being $\ln \frac{m^2 + \sqrt{m^4 - \lambda^4}}{2\bar{\mu}^2}$ and $\ln \frac{m^2 - \sqrt{m^4 - \lambda^4}}{2\bar{\mu}^2}$. We opt to set

$$\bar{\mu}^2 = \frac{|m^2 + \sqrt{m^4 - \lambda^4}|}{2}. \quad (15.121)$$

This reduces to $\bar{\mu}^2 = m^2$ if $\lambda = 0$, while it allows for⁶ $m^2 < 0$. This is possible if $m^4 \leq \lambda^4$. This is possible if $m^4 \leq \lambda^4$, as it was mentioned below eq.(15.91). In this case, the size of both logarithms, $\ln \frac{m^2 + \sqrt{m^4 - \lambda^4}}{2\bar{\mu}^2}$ and $\ln \frac{m^2 - \sqrt{m^4 - \lambda^4}}{2\bar{\mu}^2}$, is determined by their arguments, which are complex conjugate.

Let us specify to the case $N = 3$. In Figure 15.1, we have plotted the behaviour of $m^2(\lambda^4)$. We see that next to the “non-perturbative” branch of solutions, starting from $m^2 \neq 0$, also a “perturbative” branch of solutions with $m^2 < 0$ is emerging from $m^2 = 0$, in correspondence with our expectation.

However, λ^4 is not a free parameter of the theory. We should require that λ^4 is such that the doublet $(\lambda^4, m^2(\lambda^4))$ is a solution of the gap equation (15.105), i.e. the horizon condition. In Figure 15.2, we have plotted the value of the horizon condition equation, as a function of λ^4 . It is clear that no solution with $m^2 > 0$ exists as the horizon condition never becomes zero. Of course, this is in correspondence with the foregoing general proof that there is never such a solution, independently of the choice of $\bar{\mu}$. However, we see that there is a single solution with $m^2 < 0$.

The corresponding values for the expansion parameter, for the Gribov and mass parameter, as well

⁶Evidently, $\bar{\mu}^2$ should be real and positive, hence the modulus in eq.(15.121).

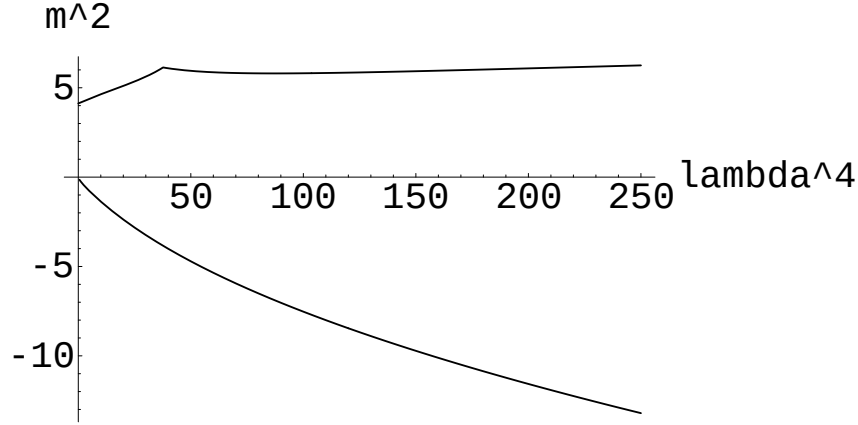


Figure 15.1: m^2 as a function of λ^4 , in units $\Lambda_{\overline{MS}} = 1$.

as for the vacuum energy are found to be

$$\frac{g^2 N}{16\pi^2} \approx 1.18, \quad (15.122)$$

$$\lambda^4 \approx 6.351 \Lambda_{\overline{MS}}^4, \quad (15.123)$$

$$m^2 \approx -0.950 \Lambda_{\overline{MS}}^2, \quad (15.124)$$

$$E_{\text{vac}} \approx 0.043 \Lambda_{\overline{MS}}^4, \quad (15.125)$$

15.5.3 Intermediate comments.

Although the $\overline{MS}$ expansion parameter (15.122) is too large to speak about reliable results, we nevertheless would like to raise some questions. Apparently, the solution of the coupled gap equations is laying on the “perturbative” branch, being the one with $m^2 \leq 0$. This gives rise to a positive value for the mass dimension two gluon condensate $\langle A_\mu^2 \rangle$. When the restriction on the domain of integration in the path integral is not implemented, as in the previous papers [42, 197, 212], $\langle A_\mu^2 \rangle$ was necessarily negative, the reason being that the action should be real-valued, as it was explained below eq.(15.91). Let us also mention here that in [37, 175, 225], a positive estimate for $\langle A_\mu^2 \rangle$ was obtained when using the OPE in combination with $\langle A_\mu^2 \rangle$. These works were based on the observation that there was existing a certain discrepancy at relatively large momentum between the expected perturbative behaviour and the obtained lattice behaviour of the effective strong coupling constant and gluon propagator. This discrepancy could be solved in both cases using $\frac{1}{q^2}$ power corrections, due to a positive $\langle A_\mu^2 \rangle_{\text{OPE}}$ gluon condensate. We do not know if there is a direct connection between the condensate $\langle A_\mu^2 \rangle$ we determine, and $\langle A_\mu^2 \rangle_{\text{OPE}}$, as the latter is expected to contain only infrared contributions, according to an OPE treatment.

An unfortunate finding is that the vacuum energy is positive, eq.(15.125). Let us explain the importance

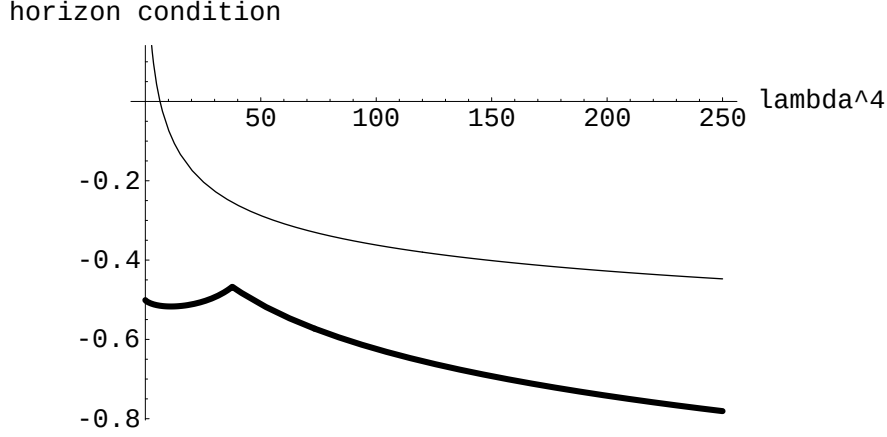


Figure 15.2: The horizon condition (15.105) as a function of λ^4 , in units $\Lambda_{\overline{MS}} = 1$. The top curve corresponds to the solutions of (15.106) with $m^2 < 0$ and the lower curve to the solutions with $m^2 > 0$.

of this. Through the trace anomaly

$$\theta_{\mu\mu} = \frac{\beta(g^2)}{2g^2} F_{\mu\nu}^2, \quad (15.126)$$

the vacuum energy can be traced back to the value of the gluon condensate $\langle F_{\mu\nu}^2 \rangle$. In particular, for $N = 3$, from this anomaly one deduces

$$\left\langle \frac{g^2}{4\pi^2} F_{\mu\nu}^2 \right\rangle = -\frac{32}{11} E_{\text{vac}}, \quad (15.127)$$

where the one-loop β -function has been used. Hence, a positive vacuum energy implies a negative value for the condensate $\langle \frac{g^2}{4\pi^2} F_{\mu\nu}^2 \rangle$. This is in contradiction with what is found. In QCD, with quarks present, one can extract phenomenological values for $\langle \frac{g^2}{4\pi^2} F_{\mu\nu}^2 \rangle$ via the sum rules [13], obtaining positive values for this condensate. It was discussed in [129] how to obtain an estimate for it by means of lattice calculations. In the case of $N = 3$ Yang-Mills theory without quarks, it was found that

$$\left\langle \frac{g^2}{4\pi^2} F_{\mu\nu}^2 \right\rangle = 0.14 \pm 0.02 \text{GeV}^4. \quad (15.128)$$

Let us mention here that the Yang-Mills β -function is negative up to the (known) four-loop order [232, 233, 234]. Hence, E_{vac} and $\langle \frac{g^2}{4\pi^2} F_{\mu\nu}^2 \rangle$ will continue to have opposite sign at higher order. From this viewpoint, it seems to us that it would be an asset that the vacuum energy obtained from any kind of calculation is at least negative.

In summary, we are left with the following questions:

- i. What is the sign and value of m^2 and thus of $\langle A_\mu^2 \rangle$?

- ii. What is the sign and value of E_{vac} and the corresponding value for $\left\langle \frac{g^2}{4\pi^2} F_{\mu\nu}^2 \right\rangle$?
- iii. Are these values better or not when the operator A_μ^2 is added to the original Gribov-Zwanziger model?

We have already discussed that the vacuum energy obtained in a one-loop approximation is always positive when the condensation of the operator A_μ^2 is left out of the discussion, using whatever renormalization scheme. To answer the above questions, one could investigate what happens at two-loop order. However, due to the already quite complicated structure of the one-loop effective action and to the fact that the calculations at higher loop order will not get any easier, this task is beyond the scope of the present article. Here, we shall mainly focus on the effects of a change of the renormalization scheme at the one-loop order. It could happen that, in a scheme different from the $\overline{\text{MS}}$ one, the vacuum energy is negative and/or that the coupling constant is small enough to speak about trustworthy results, at least qualitatively.

15.6 Changing and reducing the dependence on the renormalization scheme.

15.6.1 Preliminaries.

Before coming to the actual computations, let us first discuss some results which will turn out to be useful.

Consider again the action S of eq.(15.61). Due to the rich symmetry structure of the model, encoded in the Ward identities (15.24)-(15.37), and due to the extra relation (15.60), only three renormalization factors remain to be fixed, namely Z_g , Z_A and Z_ζ . Apparently, this means that we would need three renormalization conditions in order to fix a particular renormalization scheme. However, taking a look at the bare action associated with expression eq.(15.61), we would find the following relations

$$\begin{aligned} \zeta_o &= Z_\zeta \zeta, \\ \zeta_o \tau_o^2 &= \bar{\mu}^{-\varepsilon} Z_\zeta \zeta \tau^2, \\ \tau_o &= Z_\tau \tau, \end{aligned} \tag{15.129}$$

from which it follows that

$$Z_\zeta \zeta = \bar{\mu}^\varepsilon \zeta_o Z_\tau^2. \tag{15.130}$$

Since the bare quantity ζ_o is renormalization scheme and scale independent and since ζ always appears in the combination $Z_\zeta \zeta$ in the action, it follows that only Z_g and Z_A are relevant for the effective action, because Z_τ can be expressed in terms of these two factors. Consequently, we would only need two renormalization conditions to fix the scheme. Obviously, we can equally well choose to make use of, for example, Z_g and Z_τ as the two independent renormalization factors, corresponding to coupling constant and mass renormalization.

We will change from the $\overline{\text{MS}}$ to another massless renormalization scheme by means of the following transformations⁷

$$\begin{aligned} \bar{g}^2 &= g^2 (1 + b_0 g^2 + b_1 g^4 + \dots), \\ \bar{\lambda} &= \lambda (1 + c_0 g^2 + c_1 g^4 + \dots), \\ \bar{m}^2 &= m^2 (1 + d_0 g^2 + d_1 g^4 + \dots), \end{aligned} \tag{15.131}$$

⁷Barred quantities refer to the $\overline{\text{MS}}$ scheme.

where the parameters b_i , c_i and d_i label the new scheme. However, we should keep in mind that the renormalization of the Gribov parameter λ is not independent of that of g^2 and m^2 . Eliminating $\gamma_A(g^2)$ between eqns.(15.102) and (15.103), yields

$$\gamma_\lambda(g^2) = \frac{1}{4} \left(\frac{\beta(g^2)}{g^2} - \gamma_{m^2}(g^2) \right). \quad (15.132)$$

This relation, valid to all orders of perturbation theory, implies the existence of relationships between the coefficients b_i , c_i and d_i . For further use, we shall explicitly construct the relation between b_0 , c_0 and d_0 . Let us adopt as parametrization of $\beta(g^2)$, $\gamma_{m^2}(g^2)$ and $\gamma_\lambda(g^2)$

$$\begin{aligned} \beta(g^2) &= -2(\beta_0 g^4 + \beta_1 g^6 + \dots), \\ \gamma_{m^2}(g^2) &= \gamma_0 g^2 + \gamma_1 g^4 + \dots, \\ \gamma_\lambda(g^2) &= \lambda_0 g^2 + \lambda_1 g^4 + \dots, \end{aligned} \quad (15.133)$$

and an analogous one in the case of the $\overline{\text{MS}}$ scheme. Then, one computes

$$\begin{aligned} \bar{\mu} \frac{\partial \bar{\lambda}}{\partial \bar{\mu}} &= \bar{\mu} \frac{\partial}{\partial \bar{\mu}} [\lambda (1 + c_0 g^2 + \dots)] \\ &= \dots \\ &= \lambda (\lambda_0 g^2 + (\lambda_1 + c_0 \lambda_0 - 2\beta_0 c_0) g^4 + \dots), \end{aligned} \quad (15.134)$$

which can be expressed in terms of γ_i and β_i by exploiting the relation (15.132). We find

$$\bar{\mu} \frac{\partial \bar{\lambda}}{\partial \bar{\mu}} = \lambda \left[\frac{-2\beta_0 - \gamma_0}{4} g^2 + \left(\frac{-2\beta_1 - \gamma_1}{4} + c_0 \frac{-2\beta_0 - \gamma_0}{4} - 2\beta_0 c_0 \right) g^4 + \dots \right]. \quad (15.135)$$

We can also calculate $\bar{\mu} \frac{d\bar{\lambda}}{d\bar{\mu}}$ by first exploiting the relation (15.132), obtaining

$$\begin{aligned} \bar{\mu} \frac{\partial \bar{\lambda}}{\partial \bar{\mu}} &= \frac{1}{4} [(-2\beta_0 - \gamma_0) \bar{g}^2 + (-2\beta_1 - \bar{\gamma}_1) \bar{g}^4 + \dots] [\lambda (1 + c_0 g^2 + \dots)] \\ &= \dots \\ &= \frac{1}{4} [(-2\beta_0 - \gamma_0) g^2 + (c_0(-2\beta_0 - \gamma_0) - 2\beta_1 - \gamma_1 - 2\beta_0(-d_0 + b_0)) g^4 + \dots] \end{aligned} \quad (15.136)$$

In the previous expression, we had to express $\bar{\gamma}_1$ in terms of γ_1 ; a task accomplished by using the relation

$$\bar{\gamma}_1 = \gamma_1 - 2\beta_0 d_0 - \gamma_0 b_0, \quad (15.137)$$

which can be obtained along the same lines of the previous calculations. It should also be noted that γ_0 , β_0 and β_1 are renormalization scheme independent quantities. Thus, the identification of eqns.(15.135) and (15.136) gives the desired relation, given by

$$c_0 = \frac{1}{4}(b_0 - d_0). \quad (15.138)$$

We now perform the transformations (15.131) on the action (15.97), which was calculated in the $\overline{\text{MS}}$

scheme, to obtain it in a general scheme.

$$\begin{aligned} \Gamma = & -\frac{(N^2-1)\lambda^4}{2g^2N} (1+4c_0g^2-b_0g^2) + \frac{\zeta_0m^4}{2g^2} \left(1 - \frac{\zeta_1}{\zeta_0}g^2 + 2d_0g^2 - b_0g^2\right) \\ & + \frac{3(N^2-1)}{256\pi^2} \left[\left(m^2 + \sqrt{m^4 - \lambda^4}\right)^2 \left(\ln \frac{m^2 + \sqrt{m^4 - \lambda^4}}{2\bar{\mu}^2} - \frac{5}{6}\right) \right. \\ & \left. + \left(m^2 - \sqrt{m^4 - \lambda^4}\right)^2 \left(\ln \frac{m^2 - \sqrt{m^4 - \lambda^4}}{2\bar{\mu}^2} - \frac{5}{6}\right) \right], \end{aligned} \quad (15.139)$$

while the gap equations now read

$$\begin{aligned} \frac{\partial \Gamma}{\partial \lambda} = & -\frac{2(N^2-1)}{g^2N} \lambda^3 (1+4c_0g^2-b_0g^2) + \frac{3(N^2-1)\lambda^3}{256\pi^2} \left[\frac{8}{3} \right. \\ & \left. - 4 \frac{(m^2 + \sqrt{m^4 - \lambda^4})}{\sqrt{m^4 - \lambda^4}} \ln \frac{m^2 + \sqrt{m^4 - \lambda^4}}{2\bar{\mu}^2} + 4 \frac{(m^2 - \sqrt{m^4 - \lambda^4})}{\sqrt{m^4 - \lambda^4}} \ln \frac{m^2 - \sqrt{m^4 - \lambda^4}}{2\bar{\mu}^2} \right], \\ \frac{\partial \Gamma}{\partial m^2} = & \frac{\zeta_0m^2}{g^2} \left(1 - \frac{\zeta_1}{\zeta_0}g^2 + 2d_0g^2 - b_0g^2\right) \\ & + \frac{3(N^2-1)}{256\pi^2} \left[2 \left(m^2 + \sqrt{m^4 - \lambda^4}\right) \left(1 + \frac{m^2}{\sqrt{m^4 - \lambda^4}}\right) \ln \frac{m^2 + \sqrt{m^4 - \lambda^4}}{2\bar{\mu}^2} \right. \\ & \left. + 2 \left(m^2 - \sqrt{m^4 - \lambda^4}\right) \left(1 - \frac{m^2}{\sqrt{m^4 - \lambda^4}}\right) \ln \frac{m^2 - \sqrt{m^4 - \lambda^4}}{2\bar{\mu}^2} - \frac{8}{3}m^2 \right]. \end{aligned} \quad (15.140)$$

We mention that, in the case in which $m^2 \geq 0$, similar algebraic manipulations as those leading to the condition (15.120), give a more general equation

$$\frac{68}{39} \left(\frac{16\pi^2}{g^2N} \right) + \frac{161}{39} + \frac{16\pi^2}{N} \left(\frac{32}{3}c_0 - \frac{68}{39}b_0 - \frac{24}{13}d_0 \right) = \frac{t}{\sqrt{1-t}} \ln \frac{t}{(1+\sqrt{1-t})^2}, \quad (15.141)$$

or, using the relation (15.138),

$$\frac{68}{39} \left(\frac{16\pi^2}{g^2N} \right) + \frac{161}{39} + \frac{16\pi^2}{N} \left(\frac{12}{13}b_0 - \frac{176}{39}d_0 \right) = \frac{t}{\sqrt{1-t}} \ln \frac{t}{(1+\sqrt{1-t})^2}. \quad (15.142)$$

From this expression, it is apparent that a sensible solution with $m^2 > 0$ might exist, depending on the values of the renormalization parameters d_0 ($\sim$ mass renormalization) and b_0 ($\sim$ coupling constant renormalization).

Frequently used are the so-called physical renormalization schemes whereby, loosely speaking, one demands that the quantum corrected quantities reduce to the tree level values at a certain scale $\bar{\mu}$. However, it turns out that such an approach is not particularly useful to implement in the current case due to the presence of the several scales. Therefore, the question arises how one can make a somewhat motivated choice for the arbitrary parameters, labeling a certain renormalization scheme. In the next section we shall discuss a way to reduce the freedom in the choice of the renormalization parameters. The method relies on the possibility of performing an optimization of the renormalization scheme dependence, as illustrated in [106, 148].

15.6.2 Optimization of the renormalization scheme.

Consider a quantity ϱ that runs according to

$$\bar{\mu} \frac{d\varrho}{d\bar{\mu}} = \gamma_\varrho(g^2)\varrho, \quad (15.143)$$

where

$$\gamma_\varrho(g^2) = \gamma_{\varrho,0}g^2 + \gamma_{\varrho,1}g^4 + \dots \quad (15.144)$$

To ϱ , we can associate a quantity $\hat{\varrho}$ that does not depend on the choice of the renormalization scheme and which is scale independent. It is defined as

$$\hat{\varrho} = \mathcal{F}_\varrho(g^2)\varrho, \quad (15.145)$$

whereby

$$\bar{\mu} \frac{d}{d\bar{\mu}} \mathcal{F}_\varrho(g^2) = -\gamma_\varrho(g^2)\mathcal{F}_\varrho(g^2). \quad (15.146)$$

It is apparent that $\hat{\varrho}$ will not depend on the scale $\bar{\mu}$. It can also be checked [106, 148] that $\hat{\varrho}$ is left unmodified by a change of the renormalization scheme, implemented through transformations analogous to those of eqns.(15.131). The equation (15.146) can be solved in a series expansion in g^2 by noticing that

$$\bar{\mu} \frac{d}{d\bar{\mu}} \mathcal{F}_\varrho(g^2) \equiv \beta(g^2) \frac{d}{dg^2} \mathcal{F}_\varrho(g^2). \quad (15.147)$$

Then, the above differential equation can be solved in a series expansion in g^2 , more precisely by

$$\mathcal{F}_\varrho(g^2) = (g^2)^{\frac{\gamma_{\varrho,0}}{2\beta_0}} \left(1 + \frac{1}{2} \left(\frac{\gamma_{\varrho,1}}{\beta_0} - \frac{\beta_1\gamma_{\varrho,0}}{\beta_0^2} \right) g^2 + \dots \right). \quad (15.148)$$

Consider once more the $\overline{\text{MS}}$ action Γ given in eq.(15.97). We shall now replace the $\overline{\text{MS}}$ variables $\overline{m}^2$ and $\overline{\lambda}$ by their renormalization scheme and scale independent counterparts $\hat{m}^2$ and $\hat{\lambda}$, which are obtained as before. By inverting eq.(15.148), one has

$$\overline{m}^2 = (\overline{g}^2)^{-\frac{\gamma_0}{2\beta_0}} \left(1 - \frac{1}{2} \left(\frac{\overline{\gamma}_1}{\beta_0} - \frac{\beta_1\gamma_0}{\beta_0^2} \right) \overline{g}^2 + \dots \right) \hat{m}^2, \quad (15.149)$$

$$\overline{\lambda} = (\overline{g}^2)^{-\frac{\lambda_0}{2\beta_0}} \left(1 - \frac{1}{2} \left(\frac{\overline{\lambda}_1}{\beta_0} - \frac{\beta_1\lambda_0}{\beta_0^2} \right) \overline{g}^2 + \dots \right) \hat{\lambda}. \quad (15.150)$$

Moreover, introducing the notations

$$a = -\frac{\gamma_0}{2\beta_0}, \quad b = -\frac{\lambda_0}{\beta_0}, \quad (15.151)$$

$$A = -\left(\frac{\overline{\gamma}_1}{\beta_0} - \frac{\beta_1\gamma_0}{\beta_0^2} \right), \quad B = -2\left(\frac{\overline{\lambda}_1}{\beta_0} - \frac{\beta_1\lambda_0}{\beta_0^2} \right), \quad (15.152)$$

the one-loop action is rewritten as

$$\begin{aligned} \Gamma = & -\frac{(N^2-1)}{2N}(\bar{g}^2)^{2b}\hat{\lambda}^4\left(\frac{1}{\bar{g}^2}+B\right)+\frac{\zeta_0}{2}\hat{m}^4(\bar{g})^{2a}\left(\frac{1}{\bar{g}^2}+A-\frac{\zeta_1}{\zeta_0}\right)+\frac{3(N^2-1)}{256\pi^2}\times \\ & \left[\left(\hat{m}^2(\bar{g}^2)^a+\sqrt{\hat{m}^4(\bar{g}^2)^{2a}-\hat{\lambda}^4(\bar{g}^2)^{2b}}\right)^2\left(\ln\frac{\hat{m}^2(\bar{g}^2)^a+\sqrt{\hat{m}^4(\bar{g}^2)^{2a}-\hat{\lambda}^4(\bar{g}^2)^{2b}}}{2\bar{\mu}^2}-\frac{5}{6}\right)\right. \\ & \left.+\left(\hat{m}^2(\bar{g}^2)^a-\sqrt{\hat{m}^4(\bar{g}^2)^{2a}-\hat{\lambda}^4(\bar{g}^2)^{2b}}\right)^2\left(\ln\frac{\hat{m}^2(\bar{g}^2)^a-\sqrt{\hat{m}^4(\bar{g}^2)^{2a}-\hat{\lambda}^4(\bar{g}^2)^{2b}}}{2\bar{\mu}^2}-\frac{5}{6}\right)\right]. \end{aligned} \quad (15.153)$$

The action (15.153) is still written in terms of the $\overline{\text{MS}}$ coupling $\bar{g}^2$. Performing the first transformation of (15.131), Γ can be reexpressed as

$$\begin{aligned} \Gamma = & -\frac{(N^2-1)}{2N}(g^2)^{2b}\hat{\lambda}^4\left(\frac{1}{g^2}+B-b_0+2bb_0\right) \\ & +\frac{\zeta_0}{2}\hat{m}^4(g^2)^{2a}\left(\frac{1}{g^2}+A-b_0+2ab_0-\frac{\zeta_1}{\zeta_0}\right)+\frac{3(N^2-1)}{256\pi^2}\times \\ & \left[\left(\hat{m}^2(g^2)^a+\sqrt{\hat{m}^4(g^2)^{2a}-\hat{\lambda}^4(g^2)^{2b}}\right)^2\left(\ln\frac{\hat{m}^2(g^2)^a+\sqrt{\hat{m}^4(g^2)^{2a}-\hat{\lambda}^4(g^2)^{2b}}}{2\bar{\mu}^2}-\frac{5}{6}\right)\right. \\ & \left.+\left(\hat{m}^2(g^2)^a-\sqrt{\hat{m}^4(g^2)^{2a}-\hat{\lambda}^4(g^2)^{2b}}\right)^2\left(\ln\frac{\hat{m}^2(g^2)^a-\sqrt{\hat{m}^4(g^2)^{2a}-\hat{\lambda}^4(g^2)^{2b}}}{2\bar{\mu}^2}-\frac{5}{6}\right)\right]. \end{aligned} \quad (15.154)$$

So far, we have constructed an action which is written in terms of renormalization scale and scheme independent variables $\hat{\lambda}$ and $\hat{m}^2$ and the coupling constant $g^2(\bar{\mu})$. This is a certain improvement, since we are not faced anymore with a choice of the parameters d_i , related to the renormalization of the Gribov and mass parameter. The remaining freedom in the choice of the renormalization scheme resides in the coupling constant, labeled by the parameters $b_0, b_1, \dots$, and in the scale $\bar{\mu}$. Of course, the higher order coefficients b_i , $i = 1, \dots$ do not show up here, since we have restricted ourselves to the one-loop level. Nevertheless, we will perform one more step, since the dependence on the coupling constant renormalization can be reduced to solely b_0 , by expanding the perturbative series in inverse powers of

$$x \equiv \beta_0 \ln \frac{\bar{\mu}^2}{\Lambda^2}, \quad (15.155)$$

rather than in terms of g^2 . For another illustration of this, see e.g. [106, 148]. The coupling constant g^2 can be replaced by x since g^2 is explicitly determined by

$$g^2 = \frac{1}{x} \left(1 - \frac{\beta_1}{\beta_0} \frac{\ln \frac{x}{\beta_0}}{x} + \dots \right). \quad (15.156)$$

In [141], the relation between the scale parameter Λ , corresponding to a certain coupling constant renormalization, and that of the $\overline{\text{MS}}$ scheme, $\Lambda_{\overline{\text{MS}}}$, was found to be

$$\Lambda = e^{-\frac{b_0}{2\beta_0}} \Lambda_{\overline{\text{MS}}}. \quad (15.157)$$

One finally gets

$$\begin{aligned}
\Gamma = & -\frac{(N^2-1)}{2N}x^{-2b}\widehat{\lambda}^4\left(x+B+(1-2b)\left(\frac{\beta_1}{\beta_0}\ln\frac{x}{\beta_0}-b_0\right)\right) \\
& + \frac{\zeta_0}{2}\widehat{m}^4x^{-2a}\left(x+A-\frac{\zeta_1}{\zeta_0}+(1-2a)\left(\frac{\beta_1}{\beta_0}\ln\frac{x}{\beta_0}-b_0\right)\right) + \frac{3(N^2-1)}{256\pi^2}\times \\
& \left[\left(\widehat{m}^2x^{-a}+\sqrt{\widehat{m}^4x^{-2a}-\widehat{\lambda}^4x^{-2b}}\right)^2\left(\ln\frac{\widehat{m}^2x^{-a}+\sqrt{\widehat{m}^4x^{-2a}-\widehat{\lambda}^4x^{-2b}}}{2\bar{\mu}^2}-\frac{5}{6}\right)\right. \\
& \left.+ \left(\widehat{m}^2x^{-a}-\sqrt{\widehat{m}^4x^{-2a}-\widehat{\lambda}^4x^{-2b}}\right)^2\left(\ln\frac{\widehat{m}^2x^{-a}-\sqrt{\widehat{m}^4x^{-2a}-\widehat{\lambda}^4x^{-2b}}}{2\bar{\mu}^2}-\frac{5}{6}\right)\right].
\end{aligned} \tag{15.158}$$

Notice that this alternative expansion is correct up to order $(\frac{1}{x})^0$.

The gap equations we intend to solve are now obtained from

$$\frac{1}{\widehat{\lambda}^3}\frac{\partial\Gamma}{\partial\widehat{\lambda}} = 0, \tag{15.159}$$

$$\frac{1}{\widehat{m}^2}\frac{\partial\Gamma}{\partial\widehat{m}^2} = 0. \tag{15.160}$$

In principle, we can solve these two equations for two quantities $\widehat{m}_*$ and $\widehat{\lambda}_*$, which will be functions of the two remaining parameters $\bar{\mu}$ and b_0 . However, by construction, we know that $\widehat{m}$ as well as $\widehat{\lambda}$ should be independent of the renormalization scale and scheme order by order. This gives us an interesting way to fix these parameters by demanding that the solutions $\widehat{m}_*(\bar{\mu}, b_0)$ and $\widehat{\lambda}_*(\bar{\mu}, b_0)$ depend minimally on b_0 and $\bar{\mu}$. Since this would give a quite complicated set of equations to solve, we can make life somewhat easier by reasonably choosing the scale⁸ $\bar{\mu}$ in the gap equations (15.159)-(15.160). In analogy to the choice for $\bar{\mu}^2$ done in the previous equation (15.121), we shall now set

$$\bar{\mu}^2 = \left| \frac{\widehat{m}^2x^{-a} + \sqrt{\widehat{m}^4x^{-2a} - \widehat{\lambda}^4x^{-2b}}}{2} \right|, \tag{15.161}$$

In order to proceed, we still have two quantities at our disposal to fix the remaining parameter b_0 . In fact, we can also take the vacuum energy E_{vac} in consideration since, being a physical quantity, it should depend minimally on the renormalization scheme and scale. Therefore, we could determine the value for b_0 by demanding that

$$\Upsilon(b_0) \equiv \left| \frac{\partial\widehat{\lambda}_*^4}{\partial b_0} \right| + \left| \frac{\partial\widehat{m}_*^4}{\partial b_0} \right| + \left| \frac{\partial E_{\text{vac}}}{\partial b_0} \right|, \tag{15.162}$$

is minimal w.r.t. the parameter b_0 . This seems to be a reasonable candidate. When its dependence on b_0 is small, then the dependence of $\widehat{m}$, $\widehat{\lambda}$ and E_{vac} on b_0 is necessarily small too. The ideal situation would be that Υ is zero for a certain b_0 . If no such an ideal b_0 would exist, we weaken the condition by requiring that Υ is as small as possible. The condition (15.162) to fix b_0 can be considered as some kind of *principle of minimal sensitivity* à la Stevenson [104]. An alternative that is sometimes used is a *fastest apparent convergence criterion*, where it is demanded that the quantum corrections are as small

⁸This can be motivated thanks to the scale independence of the $\widehat{}$ -quantities.

as possible compared to the tree level value. For example, if we denote by $\Gamma^{[0]}$ the action to order $(\frac{1}{x})^{-1}$ and by $\Gamma^{[1]}$ to order $(\frac{1}{x})^0$, we could demand that

$$\left| \frac{\Gamma^{[1]} - \Gamma^{[0]}}{\Gamma^{[0]}} \right| \quad (15.163)$$

is as small as possible when the parameters fulfill the gap equation describing the vacuum of the theory. Before continuing with explicit calculations, let us just remark here that the other logarithm, namely $\ln \frac{\hat{m}^2 x^{-a} - \sqrt{\hat{m}^4 x^{-2a} - \hat{\lambda}^4 x^{-2b}}}{2\bar{\mu}^2}$, could become large for a small argument, thus when $\hat{\lambda}^4 x^{-2b}$ would be small compared to $\hat{m}^4 x^{-2a}$. However, it is harmless since it appears in the form of $u \ln u$, while we know that $u \ln u|_{u \approx 0} \approx 0$.

15.6.3 Numerical results.

Let us first give some numerical factors we need. From e.g. [87], we infer that

$$\beta_1 = \frac{34}{3} \left(\frac{N}{16\pi^2} \right)^2, \quad \gamma_0 = -\frac{3}{2} \frac{N}{16\pi^2}, \quad \gamma_1 = -\frac{95}{24} \left(\frac{N}{16\pi^2} \right)^2, \quad (15.164)$$

and hence, from the relation (15.132),

$$\lambda_0 = -\frac{35}{24} \frac{N}{16\pi^2}, \quad \lambda_1 = -\frac{449}{96} \left(\frac{N}{16\pi^2} \right)^2. \quad (15.165)$$

This means that, for any N , the quantities a and b in eq.(15.151) are found to be

$$a = \frac{9}{44}, \quad b = \frac{35}{88}. \quad (15.166)$$

It is instructive to consider once more the original Gribov-Zwanziger model by setting $\hat{m} \equiv 0$ and by solving the gap equation (15.159). If $\hat{\lambda}_*$ is a solution of this equation, then it is not difficult to show that the corresponding vacuum energy is given by

$$E_{\text{vac}} = \frac{3(N^2 - 1)}{64\pi^2} \frac{\hat{\lambda}_*^4}{4}, \quad (15.167)$$

for any choice of $\bar{\mu}^2$. Thus, also with the improved perturbative expansion, the vacuum energy of the original Gribov-Zwanziger is always nonnegative at the lowest order.

Let us return to the model we were investigating. We solved the gap equations stemming from (15.159)-(15.160) numerically.

Let us first search for a possible solution of the gap equation in the region of space determined by $\hat{m}^4 x^{-2a} \geq \hat{\lambda}^4 x^{-2b}$. Taking a look at the action (15.158), it might be clear that the gap equations derived from it will be coupled and hence quite complicated to solve numerically. From the calculational point of view, it is useful to introduce new variables, defined by

$$\omega_1 = \frac{\hat{m}^2 x^{-a} + \sqrt{\hat{m}^4 x^{-2a} - \hat{\lambda}^4 x^{-2b}}}{2}, \quad (15.168)$$

$$\omega_2 = \frac{\hat{m}^2 x^{-a} - \sqrt{\hat{m}^4 x^{-2a} - \hat{\lambda}^4 x^{-2b}}}{2}, \quad (15.169)$$

with the inverse transformation

$$\begin{aligned}\hat{m}^2 x^{-a} &= \omega_1 + \omega_2, \\ \hat{\lambda}^4 x^{-2b} &= 4\omega_1\omega_2.\end{aligned}\tag{15.170}$$

This defines a mapping from the space $\hat{m}^4 x^{-2a} \geq \hat{\lambda}^4 x^{-2b} > 0$ to $\omega_1 \geq \omega_2 > 0$. One checks that the gap equations (15.159)-(15.160) are equivalent to

$$\left(\frac{\omega_1}{\omega_1 - \omega_2} \frac{\partial}{\partial \omega_1} - \frac{\omega_2}{\omega_1 - \omega_2} \frac{\partial}{\partial \omega_2} \right) \Gamma(\omega_1, \omega_2) = 0, \tag{15.171}$$

$$\left(\frac{1}{\omega_1 - \omega_2} \frac{\partial}{\partial \omega_1} - \frac{1}{\omega_1 - \omega_2} \frac{\partial}{\partial \omega_2} \right) \Gamma(\omega_1, \omega_2) = 0. \tag{15.172}$$

We notice that the case in which ω_1 and ω_2 would become equal, i.e. $\hat{m}^4 x^{-2a} = \hat{\lambda}^4 x^{-2b}$, should be treated with some extra care. Let us therefore first assume that $\omega_1 > \omega_2$. Then the two equations (15.171)-(15.172) can be recombined to

$$\frac{\partial}{\partial \omega_1} \Gamma = 0, \tag{15.173}$$

$$\frac{\partial}{\partial \omega_2} \Gamma = 0. \tag{15.174}$$

The action $\Gamma(\omega_1, \omega_2)$ is explicitly given by

$$\begin{aligned}\Gamma &= -2 \frac{(N^2 - 1)}{N} \mathcal{U}_1 \omega_1 \omega_2 + \frac{\zeta_0}{2} \mathcal{U}_2 (\omega_1 + \omega_2)^2 \\ &+ \frac{3(N^2 - 1)}{64\pi^2} \left[\omega_1^2 \left(\ln \frac{\omega_1}{\bar{\mu}^2} - \frac{5}{6} \right) + \omega_2^2 \left(\ln \frac{\omega_2}{\bar{\mu}^2} - \frac{5}{6} \right) \right].\end{aligned}\tag{15.175}$$

where

$$\mathcal{U}_1 = x + B + (1 - 2b) \left(\frac{\beta_1}{\beta_0} \ln \frac{x}{\beta_0} - b_0 \right), \tag{15.176}$$

$$\mathcal{U}_2 = x + A - \frac{\zeta_1}{\zeta_0} + (1 - 2a) \left(\frac{\beta_1}{\beta_0} \ln \frac{x}{\beta_0} - b_0 \right). \tag{15.177}$$

It is not difficult to work out the gap equations (15.173)-15.174, being given by

$$-2 \frac{N^2 - 1}{N} \mathcal{U}_1 \omega_2 + \zeta_0 \mathcal{U}_2 (\omega_1 + \omega_2) + \frac{3(N^2 - 1)\omega_1}{32\pi^2} \left(-\frac{1}{3} + \ln \frac{\omega_1}{\bar{\mu}^2} \right) = 0, \tag{15.178}$$

$$-2 \frac{N^2 - 1}{N} \mathcal{U}_1 \omega_1 + \zeta_0 \mathcal{U}_2 (\omega_1 + \omega_2) + \frac{3(N^2 - 1)\omega_2}{32\pi^2} \left(-\frac{1}{3} + \ln \frac{\omega_2}{\bar{\mu}^2} \right) = 0. \tag{15.179}$$

From the explicit expression of the gap equations and of the action itself in terms of ω_1 and ω_2 , the advantages of using these variables should be obvious, since we can decouple the two gap equations. Explicitly, since $\bar{\mu}^2 = \omega_1$, one finds from eq.(15.178),

$$\omega_2 = \frac{\frac{N^2 - 1}{32\pi^2} - \zeta_0 \mathcal{U}_2}{-2 \frac{N^2 - 1}{N} \mathcal{U}_1 + \zeta_0 \mathcal{U}_2} \omega_1, \tag{15.180}$$

which can be substituted in the second gap equation (15.179), yielding an equation for ω_1 which does not contain ω_2 anymore. The nominator of eq.(15.180) is different from zero, since filling in the numbers gives

$$-2 \frac{N^2 - 1}{N} \bar{U}_1 + \zeta_0 \bar{U}_2 = \frac{N^2 - 1}{4576} \left(-\frac{975}{\pi^2} - \frac{5984}{N} x \right) \neq 0. \quad (15.181)$$

where we kept in mind that for a meaningful result, $x \sim \frac{1}{g^2}$, should be positive.

A numerical investigation of the gap equation (15.179) using eq.(15.180) revealed that there are no zeros. We conclude that there are no solutions with $\hat{m}^4 x^{-2a} > \hat{\lambda}^4 x^{-2b}$.

Next, let us find out if a possible solution with $\hat{m}^4 x^{-2a} = \hat{\lambda}^4 x^{-2b}$ or $\omega_1 = \omega_2$ might exist. We explicitly evaluate the gap equations (15.171)-(15.172), where now $\bar{\mu}^2 = \omega_1$,

$$\zeta_0 \bar{U}_2(\omega_1 + \omega_2) - \frac{N^2 - 1}{32\pi^2}(\omega_1 + \omega_2) - \frac{3(N^2 - 1)}{32\pi^2} \frac{\omega_2^2}{\omega_1 - \omega_2} \ln \frac{\omega_2}{\omega_1} = 0, \quad (15.182)$$

$$2 \frac{N^2 - 1}{N} \bar{U}_1 - \frac{N^2 - 1}{32\pi^2} - \frac{3(N^2 - 1)}{32\pi^2} \frac{\omega_2}{\omega_1 - \omega_2} \ln \frac{\omega_2}{\omega_1} = 0. \quad (15.183)$$

From the foregoing expressions, we infer that the limit $\omega_1 \rightarrow \omega_2$ exists, giving rise to

$$2\zeta_0 \bar{U}_2 - \frac{2(N^2 - 1)}{32\pi^2} + \frac{3(N^2 - 1)}{32\pi^2} = 0, \quad (15.184)$$

$$2 \frac{N^2 - 1}{N} \bar{U}_1 - \frac{N^2 - 1}{32\pi^2} + \frac{3(N^2 - 1)}{32\pi^2} = 0. \quad (15.185)$$

This means that we have two equations to solve for the single quantity ω_1 , which is present in $\bar{U}_1$ and $\bar{U}_2$ through the quantity x . It would be an extreme coincidence if these two different equations, which can be rewritten as

$$\frac{18}{13} \bar{U}_2 = -\frac{N}{32\pi^2}, \quad (15.186)$$

$$\bar{U}_1 = -\frac{N}{32\pi^2}. \quad (15.187)$$

possess a common solution. That this is not the case can be inferred from the numerical solutions of both equations (15.186) and (15.187), shown in Figure 15.3.

As a final step, we should investigate if there is a solution in the region $\hat{m}^4 x^{-2a} < \hat{\lambda}^4 x^{-2b}$. We can still define the coordinates ω_1 and ω_2 by

$$\omega_1 = \frac{\hat{m}^2 x^{-a} + i\sqrt{-\hat{m}^4 x^{-2a} + \hat{\lambda}^4 x^{-2b}}}{2}, \quad (15.188)$$

$$\omega_2 = \frac{\hat{m}^2 x^{-a} - i\sqrt{-\hat{m}^4 x^{-2a} + \hat{\lambda}^4 x^{-2b}}}{2}. \quad (15.189)$$

In this case, ω_1 and ω_2 are complex conjugate. Henceforth, it would be more appropriate to use the modulus R and the argument ϕ , $\phi \in]-\pi, \pi]$, defined by

$$Re^{i\phi} = \omega_1, \quad (15.190)$$

$$Re^{-i\phi} = \omega_2, \quad (15.191)$$

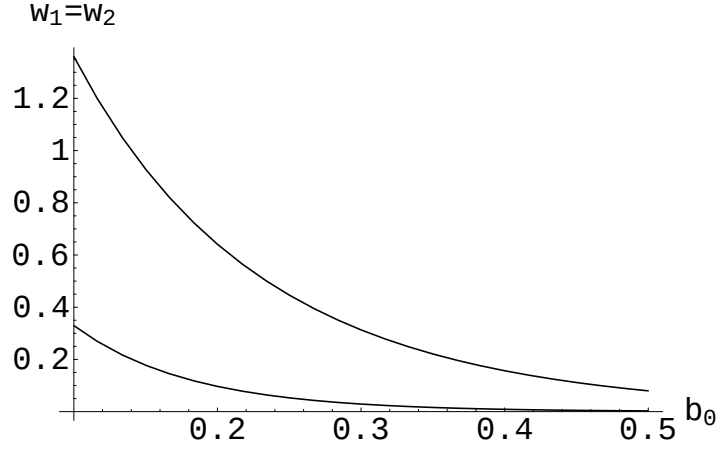


Figure 15.3: The solution $\omega_1 = \omega_2$ as a function of b_0 of eq.(15.186), top curve, and eq.(15.187), bottom curve, in units $\Lambda_{\overline{MS}} = 1$. Clearly, these two curves do not coincide.

If the argument ϕ is so that $|\phi| > \frac{\pi}{2}$, then $\hat{m}^2 x^{-a} < 0$. As a consequence, the estimate for $\langle A_\mu^2 \rangle$ will be positive.

Most of the foregoing analysis can be repeated. The action (15.175) is rewritten in terms of R and ϕ by

$$\begin{aligned} \Gamma = & -2 \frac{(N^2 - 1)}{N} \mathcal{U}_1 R^2 + 2\zeta_0 \mathcal{U}_2 R^2 \cos^2 \phi \\ & + \frac{3R^2 (N^2 - 1)}{32\pi^2} \left[\cos(2\phi) \left(\ln \frac{R}{\bar{\mu}^2} - \frac{5}{6} \right) - \phi \sin(2\phi) \right]. \end{aligned} \quad (15.192)$$

The gap equations (15.178)-(15.179) reduce to

$$-2 \frac{N^2 - 1}{N} \mathcal{U}_1 R e^{-i\phi} + \zeta_0 \mathcal{U}_2 R (e^{i\phi} + e^{-i\phi}) + \frac{3(N^2 - 1) R e^{i\phi}}{32\pi^2} \left(-\frac{1}{3} + i\phi \right) = 0, \quad (15.193)$$

and its complex conjugate. With the parametrization (15.190), we have $\bar{\mu}^2 = R$.

We must solve the following two real equations⁹ for ϕ and R .

$$-2 \frac{N^2 - 1}{N} \mathcal{U}_1 \cos \phi + 2\zeta_0 \mathcal{U}_2 \cos \phi + \frac{3(N^2 - 1)}{32\pi^2} \left(-\frac{\cos \phi}{3} - \phi \sin \phi \right) = 0, \quad (15.194)$$

$$2 \frac{N^2 - 1}{N} \mathcal{U}_1 \sin \phi + \frac{3(N^2 - 1)}{32\pi^2} \left(-\frac{\sin \phi}{3} + \phi \cos \phi \right) = 0. \quad (15.195)$$

We can divide these equations¹⁰ by $\cos \phi$ to obtain

$$-2 \frac{N^2 - 1}{N} \mathcal{U}_1 + 2\zeta_0 \mathcal{U}_2 + \frac{3(N^2 - 1)}{32\pi^2} \left(-\frac{1}{3} - \phi \tan \phi \right) = 0, \quad (15.196)$$

$$2 \frac{N^2 - 1}{N} \mathcal{U}_1 \tan \phi + \frac{3(N^2 - 1)}{32\pi^2} \left(-\frac{\tan \phi}{3} + \phi \right) = 0. \quad (15.197)$$

⁹The R -dependence is hidden in $\mathcal{U}_1$ and $\mathcal{U}_2$

¹⁰We may assume $\cos \phi \neq 0$, otherwise eqns.(15.194)-(15.195) would give $\phi = 0$, which is inconsistent with $\cos \phi = 0$.

These equations can also be decoupled. The most efficient way to proceed is to eliminate R between these two equations to obtain an equation for ϕ , as the range in which we must search for a solution is limited for this angle. The equation for ϕ finally becomes

$$\frac{-90985N - 107712\pi^2 b_0 + 12N \left(484\phi \cot \phi + 1734 \ln \left(\frac{-117(50+11\phi \csc \phi \sec \phi)}{8228} \right) - 1573\phi \tan \phi \right)}{107712\pi^2} = 0 \quad (15.198)$$

while the value of R is obtained from

$$x \equiv \beta_0 \ln \frac{R}{\Lambda_{\overline{MS}}^2} + b_0 = -\frac{1950 + 429\phi \csc \phi \sec \phi}{11968\pi^2} N. \quad (15.199)$$

We shall concentrate on the case $N = 3$. Depending on the value of the parameter b_0 , there is more than one solution possible. This is explained in the Appendix. There is a single solution ϕ if $b_0 > -0.33564...$ or $b_0 < -0.41595...$. If $-0.41595... < b_0 < -0.33564...$, there are three solutions, while for $b_0 = -0.41595...$ and $b_0 = -0.33564...$ there are two solutions. To determine the solution ϕ which characterizes the vacuum, we should take that one which gives us the absolute minimum of the energy functional Γ . In Figure 15.4, we have displayed the solution for ϕ and R , and in Figure 15.5 the vacuum energy E_{vac} and the corresponding expansion parameter, which is now given by $y \equiv \frac{N}{16\pi^2 x}$. For completeness, we have also shown the values which do correspond to higher values of Γ , these are indicated with the thinner lines.

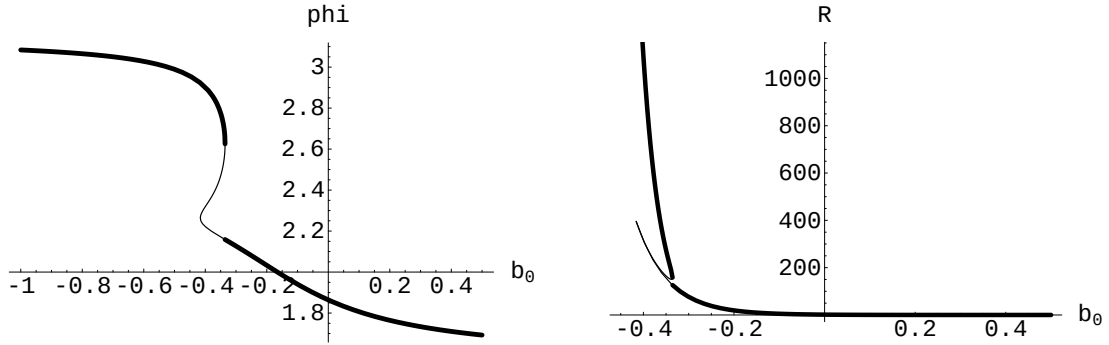


Figure 15.4: The angle ϕ and scale R as a function of b_0 , in units $\Lambda_{\overline{MS}} = 1$.

In Figure 15.6, we collected the solutions of the scale invariant quantities $\hat{m}^2$ and $\hat{\lambda}^4$ as a function of b_0 .

15.6.4 Interpretation of the results.

After the rather technical issue of solving the gap equations, we now come to a discussion of the results. Let us first have a look at the plot of vacuum energy, on the l.h.s. of Figure 15.5. We notice that for $b_0 < -0.33564...$, the vacuum energy becomes negative. However, we cannot attach any definitive meaning to this result. In fact, as it can be seen from the Figures 15.6 and 15.7, the values of the vacuum energy and the supposedly minimally b_0 -dependent quantities $\hat{m}^2$ and $\hat{\lambda}^4$ are extremely b_0 -dependent. Very small variations in b_0 induce large fluctuations on e.g. the energy. This is indicative of the fact that the equations we have solved are not yet stable against b_0 -variations in the range of the values obtained for b_0 . The behaviour is better for, let us say $b_0 > -0.2$. However, in this case, we

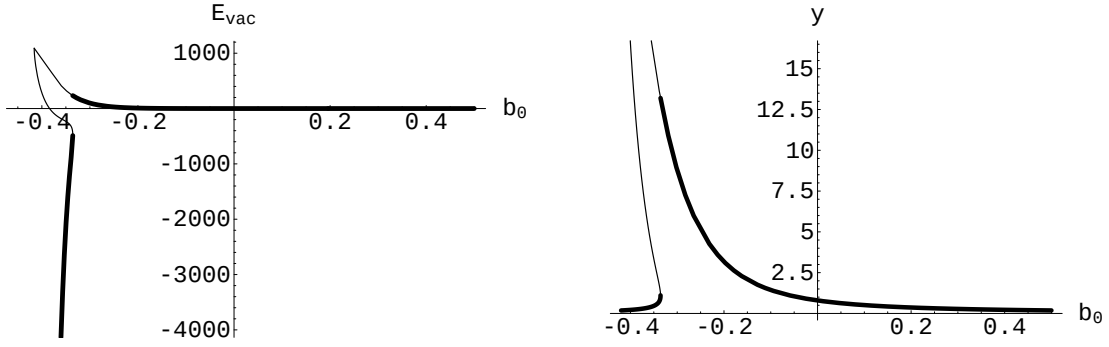


Figure 15.5: The vacuum energy E_{vac} and the expansion parameter y as a function of b_0 , in units $\Lambda_{\overline{\text{MS}}} = 1$.

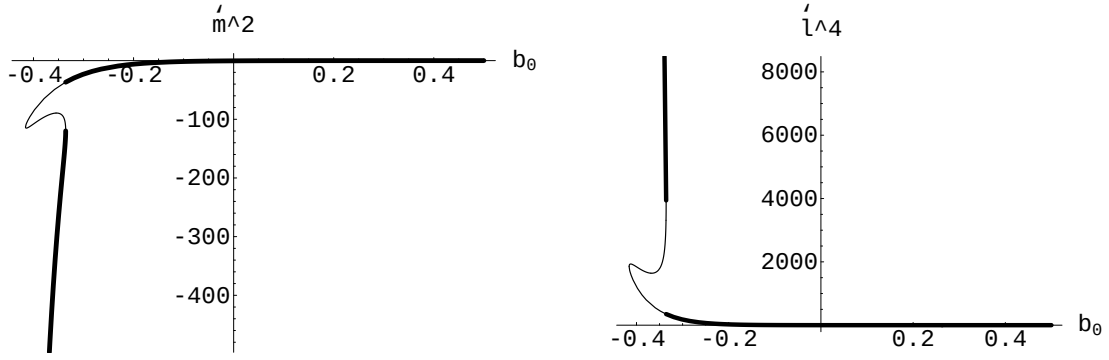


Figure 15.6: The quantities $\hat{m}^2$ and $\hat{\lambda}^4$ as a function of b_0 , in units $\Lambda_{\overline{\text{MS}}} = 1$.

find again that the vacuum energy is positive. We have shown the quantity $\Upsilon(b_0)$ in Figure 15.7, but there is no value for b_0 where this quantity becomes minimal. In fact, as for the vacuum energy E_{vac} , also $\hat{m}^2$ and $\hat{\lambda}^4$ fall off to zero for growing b_0 . A similar conclusion can be drawn when one would like to fix b_0 by a fastest apparent convergence, there does not seem to exist a favoured b_0 .

As an example, we set $b_0 = 0$, i.e. we choose to use the $\overline{\text{MS}}$ coupling constant. Then we find, with the optimized expansion,

$$y \approx 0.796, \quad (15.200)$$

$$\hat{\lambda}^4 x^{-2b} \approx 7.939 \Lambda_{\overline{\text{MS}}}^4, \quad (15.201)$$

$$\hat{m}^2 x^{-a} \approx -0.814 \Lambda_{\overline{\text{MS}}}^2, \quad (15.202)$$

$$E_{\text{vac}} \approx 0.063 \Lambda_{\overline{\text{MS}}}^4, \quad (15.203)$$

which are in fair agreement with the naive $\overline{\text{MS}}$ results (15.122)-(15.125). We notice that the expansion parameter y is already smaller than 1, but still relatively large, while the vacuum energy is indeed positive.

We see therefore that, in order to be able to give a reasonable answer to the questions concerning the sign of m^2 and E_{vac} and to get more trustworthy numerical values, the two-loop evaluation of the effective action Γ , at least in the $\overline{\text{MS}}$ scheme, would be very useful. Although being beyond of the aim of the present work, it might be worth noticing that the same decomposition as in eq.(15.90) can be

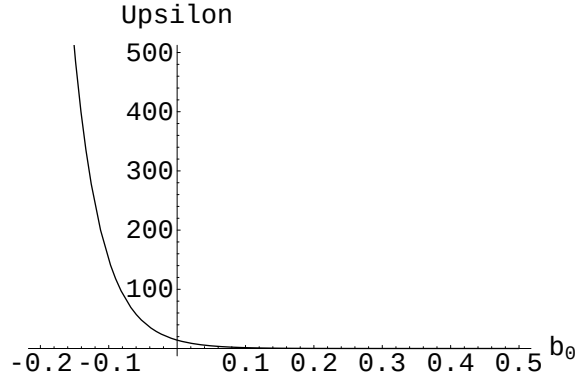


Figure 15.7: The quantity $\Upsilon(b_0)$ as a function of b_0 , in units $\Lambda_{\overline{MS}} = 1$.

used to write the gluon propagator, ripped of its tensorial structure, as

$$\frac{p^2}{p^4 + p^2 m^2 + \frac{\lambda^4}{4}} = \frac{\omega_1}{\omega_1 - \omega_2} \frac{1}{p^2 + \omega_1} - \frac{\omega_2}{\omega_1 - \omega_2} \frac{1}{p^2 + \omega_2} \quad (15.204)$$

Using this decomposition, the calculation of the vacuum diagrams could be performed with standard massive propagators. The effective action Γ will remain symmetric under the exchange of ω_1 and ω_2 and equations like (15.171)–(15.174) shall remain valid. Also, one does not need to evaluate any new anomalous dimension, since these are already known, either from previous calculations [42, 197, 212], or from exploiting relations like eq.(15.132).

Before turning to the final conclusions, we shall give in the following section a brief account of the consequences stemming from the presence of the Gribov parameter, to emphasize the important role of this parameter.

15.7 Consequences of a non-vanishing Gribov parameter.

15.7.1 The gluon propagator.

If there is no generation of a mass parameter due to $\langle A_\mu^2 \rangle$, we can consider just the action (15.8). Then the tree level gluon propagator turns out to be

$$\langle A_\mu^a A_\nu^b \rangle_p = \delta^{ab} \frac{p^2}{p^4 + \frac{\lambda^4}{4}} \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right). \quad (15.205)$$

This result, first pointed out in [107], was obtained by retaining only the first term of the nonlocal horizon function (15.4), corresponding to the approximation $-\partial D \approx -\partial^2$. The gluon propagator, eq.(15.205), is suppressed in the infrared region due to the presence of the Gribov parameter λ . In particular, the presence of this parameters implies that $\langle A_\mu^a A_\nu^b \rangle_p$ vanishes at zero momentum, $p = 0$. When the possibility of the existence of a dynamical mass parameter in the gluon propagator is included, by investigating the condensation of A_μ^2 , the tree level gluon propagator reads

$$\langle A_\mu^a A_\nu^b \rangle_p \equiv \delta^{ab} \frac{D(p^2)}{p^2} \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) = \delta^{ab} \frac{p^2}{p^4 + m^2 p^2 + \frac{\lambda^4}{4}} \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right). \quad (15.206)$$

This type of propagator is sometimes called the Stingl propagator, from the author who used it as an ansatz for solving the Schwinger-Dyson equations, see [235] for more details.

However, it should be realized that eq.(15.206) describes only the tree level gluon propagator. In particular, to produce a plot of the form factor $\mathcal{D}(p^2)$ as a function of the momentum p , which would allow to make a comparison with the results obtained in lattice simulations, see e.g. [45] for $N = 3$ and [48, 116] for $N = 2$, one should go beyond the zeroth order approximation, for example by including higher order polarization effects and/or trying to perform a renormalization group improvement. In general, these corrections will also be dependent on the external momentum p .

15.7.2 The ghost propagator.

Even more prominent is the influence of the Gribov parameter on the infrared behaviour of the ghost propagator, which can be calculated at one-loop order using the modified gluon propagator (15.205) or (15.206) with their respective gap equations (15.6) and (15.86). In both cases, the infrared behaviour of the ghost propagator [107, 224, 231, 127, 128] is shown to be

$$\frac{\delta^{ab}}{N^2 - 1} \langle c^a \bar{c}^b \rangle_{p \approx 0} \equiv \frac{1}{p^2} \mathcal{G}(p^2) \Big|_{p \approx 0} \approx \frac{4}{3Ng^2 \mathcal{J} p^4}, \quad (15.207)$$

where $\mathcal{J}$ stands for the real, finite integral given by

$$\mathcal{J} = \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 \left(k^4 + m^2 k^2 + \frac{\lambda^4}{4} \right)}. \quad (15.208)$$

The original Gribov-Zwanziger model corresponds to $m^2 \equiv 0$. Thus, the ghost propagator is strongly enhanced in the infrared region compared to the perturbative behaviour, if the restriction to the first Gribov region is taken into account. It is important to notice that this behaviour of the ghost propagator is preserved in the present treatment, due to the peculiar form of the gap equation (15.86) implementing the horizon condition. In particular, from the expression for the effective action in eq.(15.85), one sees that, while the term quadratic in the field σ , i.e. $\frac{\sigma^2}{2g^2\zeta}$, contains the LCO parameter ζ , the first term which depends on the Gribov parameter, i.e. $-4(N^2 - 1)\gamma^4$, does not contain any such new LCO parameter. This important feature follows from the fact that no new parameter has to be introduced in order to renormalize the term $(M_\mu^{ai} V_\mu^{ai} - U_\mu^{ai} N_\mu^{ai})$, as remarked in eq.(15.58). While the parameter ζ is required to take into account the ultraviolet divergences of the vacuum correlator $\langle A_\mu^2(x) A_\nu^2(y) \rangle$, which are proportional to τ^2 , no such a parameter is needed for $(M_\mu^{ai} V_\mu^{ai} - U_\mu^{ai} N_\mu^{ai})$ which, upon setting the external sources to their physical values, gives rise to term $-4(N^2 - 1)\gamma^4$ in the expression (15.85). Said otherwise, this term is not affected by the presence of a new parameter which would be required if eq.(15.58) would not hold. As a consequence, the factor “1” appearing in the left hand side of the gap equation (15.86) is, so to speak, left unchanged by the quantum corrections. It is precisely that property which ensures, through a delicate cancelation mechanism, see [107, 224, 127, 128], the infrared enhancement of the ghost propagator.

Analogously to the case of the gluon propagator, a more detailed study of higher order corrections would be needed in order to obtain a plot of the ghost form factor $\mathcal{G}(p^2)$.

15.7.3 The strong coupling constant.

Usually, a nonperturbative definition of the renormalized strong coupling constant α_R can be written down from the knowledge of the gluon and ghost propagators as, see e.g. [121, 116]

$$\alpha_R(p^2) = \alpha_R(\mu) \mathcal{D}(p^2, \mu) \mathcal{G}^2(p^2, \mu), \quad (15.209)$$

where $\mathcal{D}$ and $\mathcal{G}$ stand for the gluon and ghost form factors as defined before. This definition represents a kind of nonperturbative extension of the perturbative results (15.56). According to Schwinger-Dyson studies [122, 123, 20, 124, 125, 126], those form factors satisfy a power law behaviour in the infrared

$$\begin{aligned}\lim_{p \rightarrow 0} \mathcal{D}(p^2) &\propto (p^2)^\theta, \\ \lim_{p \rightarrow 0} \mathcal{G}(p^2) &\propto (p^2)^\omega,\end{aligned}\tag{15.210}$$

where the infrared exponents θ and ω obey the sum rule

$$\theta + 2\omega = 0.\tag{15.211}$$

Such a sum rule suggests the development of an infrared fixed point for the renormalized coupling constant, (15.209), as also pointed out by lattice simulations for $SU(2)$ as well as for $SU(3)$ [116, 117, 118],

$$\lim_{p \rightarrow 0} \alpha(p^2) = \alpha_c.\tag{15.212}$$

The existence of a fixed point in this reasoning is dependent on the sum rule rather than on the precise value of the exponents. We refer to the already quoted literature for more details on the value of these exponents. We end by noticing that the form factors of the gluon and ghost propagator in our zeroth order approximation give rise to the sum rule (15.211), since we have $\theta = 2$ and $\omega = -1$. Moreover, without Gribov parameter, the sum rule (15.211) is lost, and thus there is no indication for an infrared fixed point.

15.7.4 Positivity violation.

The behaviour of the gluon propagator is sometimes used as an *indication* of confinement of gluons by means of the so called positivity violation, see e.g. [236, 237] and references therein.

Briefly, when the Euclidean gluon propagator $D(p) \equiv \frac{\mathcal{D}(p^2)}{p^2}$ is written through a spectral representation as

$$D(p) = \int_0^{+\infty} dM^2 \frac{\rho(M^2)}{p^2 + M^2},\tag{15.213}$$

the spectral density $\rho(M^2)$ should be positive in order to have a Källen-Lehmann representation, making possible the interpretation of the fields in term of stable particles. We refer to [236, 237] for more details. One can define the temporal correlator [237]

$$\mathcal{C}(t) = \int_0^{+\infty} dM \rho(M^2) e^{-Mt},\tag{15.214}$$

which is certainly positive for positive $\rho(M^2)$. The inverse is not necessarily true. $\mathcal{C}(t)$ can be also positive for a $\rho(M^2)$ attaining negative values. However, if $\mathcal{C}(t)$ becomes negative for certain t , then a fortiori $\rho(M^2)$ cannot be always positive. Using a contour integration argument, it is not difficult to show that $\mathcal{C}(t)$ can be rewritten as

$$\mathcal{C}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ipt} D(p) dp.\tag{15.215}$$

Let us consider the function $\mathcal{C}(t)$ using the tree level propagator (15.206), thus using

$$D(p) = \frac{p^2}{p^4 + p^2 m^2 + \frac{\lambda^4}{4}}. \quad (15.216)$$

We can consider several cases¹¹:

- if $\lambda = 0$ (thus $m^2 > 0$), one shall find that

$$\mathcal{C}(t) = \frac{e^{-mt}}{2m}. \quad (15.217)$$

This function is always positive.

- if $m^2 = 0$,

$$\mathcal{C}(t) = \frac{e^{-\frac{Lt}{2}}}{2L} \left(\cos \frac{Lt}{2} - \sin \frac{Lt}{2} \right), \quad (15.218)$$

and clearly, this function will attain negative values for certain t .

- in any other case, the correlator $\mathcal{C}(t)$ is found to be

$$\mathcal{C}(t) = \frac{1}{2} \left[\frac{\sqrt{\omega_1}}{\omega_1 - \omega_2} e^{-\sqrt{\omega_1}t} + \frac{\sqrt{\omega_2}}{\omega_2 - \omega_1} e^{-\sqrt{\omega_2}t} \right] \quad (15.219)$$

where the decomposition (15.204) has been employed once more. It is understood that $\sqrt{\omega_1}$ ($\sqrt{\omega_2}$) is the root having a positive real part.

If we assume that $\hat{m}^4 > \lambda^4$, then $\omega_1 > \omega_2$ and $\mathcal{C}(t)$ becomes negative for $t > \frac{1}{2} \frac{\ln \frac{\omega_1}{\omega_2}}{\omega_1 - \omega_2}$. In the case that $\hat{m}^4 = \lambda^4$, or $\omega_1 = \omega_2$, one finds that $\mathcal{C}(t) = \frac{e^{-\sqrt{\omega_1}t}}{4\sqrt{\omega_1}} (1 - \sqrt{\omega_1}t)$, which can also become negative. If $\hat{m}^4 < \lambda^4$, we can reintroduce the complex polar coordinates R and ϕ for the complex conjugate quantities ω_1 and ω_2 . If $\cos \frac{\phi}{2} \geq 0$, eq.(15.219) can be rewritten as

$$\mathcal{C}(t) = \frac{1}{2\sqrt{R} \sin \phi} e^{-\sqrt{R} \cos(\frac{\phi}{2})t} \sin \left(\frac{\phi}{2} - \sqrt{R} \sin \left(\frac{\phi}{2} \right) t \right) \quad (15.220)$$

By choosing an appropriate value of $t > 0$, also this expression can be made negative. An analogous expression and conclusion can be derived in case that $\cos \frac{\phi}{2} < 0$.

We conclude that, when the restriction to the Gribov region Ω is implemented, the function $\mathcal{C}(t)$ exhibits a violation of positivity when the tree level propagator is used, with or without the inclusion of $\langle A_\mu^2 \rangle$. The goal of this section was merely to provide some interesting consequences when the restriction to the first Gribov region Ω is implemented. Higher loop effects, which shall be momentum dependent, would also influence the behaviour of the gluon and ghost propagator. Hence, to give a sensible interpretation of the behaviour of e.g. the form factors and of the strong coupling constant α_R , a more detailed analysis than a tree level one is necessary.

¹¹Each of the following expressions for $\mathcal{C}(t)$ is obtainable via contour integration.

15.8 Conclusion.

In this work we have considered $SU(N)$ Euclidean Yang-Mills theories in the Landau gauge, $\partial_\mu A_\mu = 0$. We have studied the condensation of the dimension two composite operator A_μ^2 when the restriction to the Gribov region Ω is taken into account. Such a restriction is needed due to the presence of the Gribov copies [107], which are known to affect the Landau gauge.

In a previous work [224], the consequences of the restriction to the region Ω in the presence of a dynamical mass parameter due to the gluon condensate $\langle A_\mu^a A_\mu^a \rangle$ were studied by following Gribov's seminal work [107]. Here, we have relied on Zwanziger's action [127, 128], which allows to implement the restriction to the Gribov region Ω within a local and renormalizable framework. We have been able to show that Zwanziger's action remains renormalizable to all orders of perturbation theory in the presence of the operator A_μ^2 , introduced through the local composite operator technique [42, 197, 176, 212]. The effective action, constructed via the local composite operator formalism [42] obeys a homogeneous renormalization group. The explicit form of the one-loop effective action has been worked out. We have seen that, considering the original Gribov-Zwanziger model, i.e. without including the operator A_μ^2 , the vacuum energy is always positive at one-loop order, independently from the choice of the renormalization scheme. A positive vacuum energy would give rise to a negative value for the gauge invariant gluon condensate $\langle F_{\mu\nu}^2 \rangle$, through the trace anomaly. Furthermore, by adding the operator A_μ^2 , we have proven that there is no solution of the two coupled gap equations at the one-loop order in the $\overline{\text{MS}}$ scheme with $\langle A_\mu^2 \rangle < 0$. Nevertheless, when $\langle A_\mu^2 \rangle > 0$, a solution of the gap equations was found, although the corresponding expansion parameter was too large and the vacuum energy still positive.

In order to find out what happens in other schemes, we performed a detailed study, at lowest order, of the influence of the renormalization scheme. We have been able to reduce the freedom of the choice of the renormalization scheme to two parameters, namely the renormalization scale $\bar{\mu}$ and a parameter b_0 , associated to the coupling constant renormalization. We reexpressed the effective action in terms of the mass parameter $\hat{m}$ and Gribov parameter $\hat{\lambda}$, which are renormalization scheme and scale independent order by order. The resulting gap equations for these parameters have been solved numerically. Although a solution with negative vacuum energy was found, we have been unable to attach any definitive meaning to it. This is due to the fact that the results obtained turn out to be strongly dependent from the parameter b_0 . This brought us to the conclusion that we should extend our calculations to a higher order to obtain more sensible numerical estimates.

The mass parameters $\hat{m}$ and $\hat{\lambda}$ are of a nonperturbative nature and appear in the gluon and ghost propagator. Even if we lack reliable estimates for these parameters, some interesting features can already be observed. For a nonzero mass and Gribov parameter, there is a *qualitative* agreement with the behaviour found in lattice simulations and Schwinger-Dyson studies: a suppressed gluon and enhanced ghost propagator in the infrared, while further consequences of the Gribov parameter are e.g. the possible existence of an infrared fixed point for the strong coupling constant and the violation of positivity related to the gluon propagator.

Let us conclude by remarking that the Gribov region is not free from Gribov copies [110, 109, 111, 112], i.e. Gribov copies still exist inside Ω . To avoid the presence of these additional copies, a further restriction to a smaller region Λ , known as the fundamental modular region, should be implemented. This is, however, a very difficult task. Nevertheless, recently, it has been argued in [113] that the additional copies existing inside Ω have no influence on the expectation values, so that averages calculated over Λ or Ω are expected to give the same value.

Appendix.

In this Appendix, we shall outline some details on solving the gap equation (15.198). In Figure 15.8, we have plotted the expression (15.198) for several values of the parameter b_0 , namely $b_0 = 0.25, 0, -0.25, -0.3, -0.33564\dots, -0.41594\dots, -0.5$. As we have already noticed, the number of so-

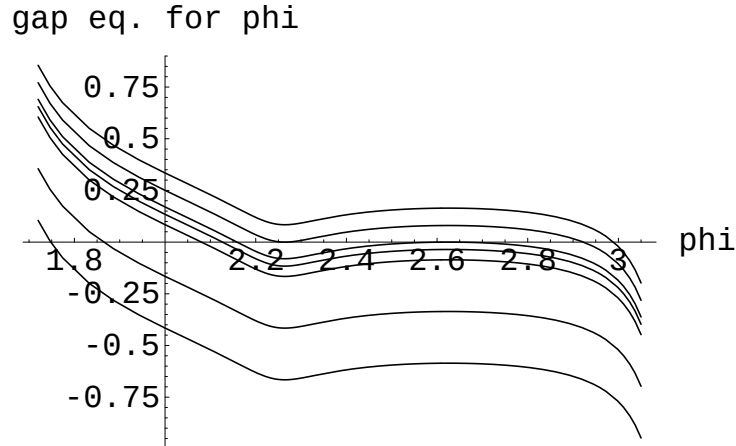


Figure 15.8: The gap equation (15.198) with $N = 3$ plotted in function of ϕ for the values $b_0 = 0.25, 0, -0.25, -0.3, -0.33564\dots, -0.41594\dots, -0.5$ (from bottom to top).

lutions depends on the value of b_0 . It is possible to obtain those values of b_0 where the number of solutions change. If we consider the plots in Figure 15.8, it is apparent that for each b_0 , the corresponding curve possesses two extremal values. The number of solution exactly changes at those values of b_0 where the curve becomes tangent to the ϕ -axis. An explicit evaluation learns that this occurs at $b_0 = -0.41595\dots$, where $\phi = 2.26407\dots$ and at $b_0 = -0.33564\dots$ where $\phi = 2.62545$. It is important to know these numbers to a high enough accuracy, to instruct Mathematica in which ϕ -interval it can search for a solution. If the initial values are not chosen in an appropriate way, the iterations will jump between the different branches of solutions and there will be no convergence to any of them.

Chapter 16

Renormalization properties of the mass operator $A_\mu^a A_\mu^a$ in three-dimensional Yang-Mills theories in the Landau gauge

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Massive renormalizable Yang-Mills theories in three dimensions are analysed within the algebraic renormalization in the Landau gauge. In analogy with the four-dimensional case, the renormalization of the mass operator $A_\mu^a A_\mu^a$ turns out to be expressed in terms of the fields and coupling constant renormalization factors. We verify the relation we obtain for the operator anomalous dimension by explicit calculations in the large N_f expansion. The generalization to other gauges such as the nonlinear Curci-Ferrari gauge is briefly outlined.

16.1 Introduction.

Recently, much work has been devoted to the study of the operator $A_\mu^a A_\mu^a$ in four-dimensional Yang-Mills theories in the Landau gauge, where a renormalizable effective potential for this operator can be consistently constructed [42, 197]. This has produced analytic evidence of a nonvanishing condensate $\langle A_\mu^a A_\mu^a \rangle$, resulting in a dynamical mass generation for the gluons [42, 197]. A gluon mass in the Landau gauge has been reported in lattice simulations [48] as well as in a recent investigation of the Schwinger-Dyson equations [51]. Besides being multiplicatively renormalizable to all orders of perturbation theory in the Landau gauge, the operator $A_\mu^a A_\mu^a$ displays remarkable properties. In fact, it has been proven [153] by using BRST Ward identities that the anomalous dimension $\gamma_{A^2}(a)$ of the operator $A_\mu^a A_\mu^a$ in the Landau gauge is not an independent parameter, being expressed as a combination of the gauge beta function, $\beta(a)$, and of the anomalous dimension, $\gamma_A(a)$, of the gauge field, according to the relation

$$\gamma_{A^2}(a) = - \left(\frac{\beta(a)}{a} + \gamma_A(a) \right), \quad a = \frac{g^2}{16\pi^2}, \quad (16.1)$$

which can be explicitly verified by means of the three-loop computations available in [87]. The operator $A_\mu^a A_\mu^a$ turns out to be multiplicatively renormalizable also in the linear covariant gauges [205]. Its condensation and the ensuing dynamical gluon mass generation in this gauge have been discussed in [212].

Moreover, the operator $A_\mu^a A_\mu^a$ in the Landau gauge can be generalized to other gauges such as the Curci-Ferrari and maximal Abelian gauges. Indeed, as was shown in [83, 144], the mixed gluon-ghost operator¹ $(\frac{1}{2}A_\mu^a A_\mu^a + \alpha \bar{c}^a c^a)$ turns out to be BRST invariant on-shell, where α is the gauge parameter. In both gauges, the operator $(\frac{1}{2}A_\mu^a A_\mu^a + \alpha \bar{c}^a c^a)$ turns out to be multiplicatively renormalizable to all orders of perturbation theory and, as in the case of the Landau gauge, its anomalous dimension is not an independent parameter of the theory [199]. A detailed study of the analytic evaluation of the effective potential for the condensate $\langle \frac{1}{2}A_\mu^a A_\mu^a + \alpha \bar{c}^a c^a \rangle$ in these gauges can be found in [178, 215]. In particular, it is worth emphasizing that in the case of the maximal Abelian gauge, the off-diagonal gluons become massive due to the gauge condensate $\langle \frac{1}{2}A_\mu^a A_\mu^a + \alpha \bar{c}^a c^a \rangle$, a fact that can be interpreted as evidence for the Abelian dominance hypothesis underlying the dual superconductivity mechanism for color confinement.

The aim of this work is to analyse the renormalization properties of the operator $A_\mu^a A_\mu^a$ in three-dimensional Yang-Mills theories in the Landau gauge. This investigation might be useful in order to study by analytical methods the formation of the condensate $\langle A_\mu^a A_\mu^a \rangle$ in three dimensions, whose relevance for the Yang-Mills theories at high temperatures has been pointed out long ago [130]. Furthermore, the possibility of a dynamical gluon mass generation related to the operator $A_\mu^a A_\mu^a$ could provide a suitable infrared cutoff which would prevent three-dimensional Yang-Mills theory from the well known infrared instabilities [132], due to its superrenormalizability.

The organization of the paper is as follows. In section 16.2 we discuss the renormalizability of the three-dimensional Yang-Mills theory in the Landau gauge, when the operator $A_\mu^a A_\mu^a$ is added to the starting action in the form of a mass term, $m^2 \int d^3x A_\mu^a A_\mu^a$. We shall be able to prove that the renormalization factor Z_{m^2} of the mass parameter m^2 can be expressed in terms of the renormalization factors Z_A and Z_g of the gluon field and of the gauge coupling constant, according to

$$Z_{m^2} = Z_g Z_A^{-1/2}. \quad (16.2)$$

This relation represents the analogue in three dimensions of the eq.(16.1). In section 16.3 we give an explicit verification of the relation (16.2) by using the large N_f expansion method. In section 16.4 we present the generalization to the nonlinear Curci-Ferrari gauge.

16.2 Renormalizability of massive three-dimensional Yang-Mills theory in the Landau gauge.

16.2.1 Ward identities.

In order to analyze the renormalizability of three-dimensional Yang-Mills theory, in the presence of the mass term $\frac{1}{2}m^2 \int d^3x A_\mu^a A_\mu^a$, we start from the following gauge fixed action

$$S = \int d^3x \left(-\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{1}{2} m^2 A_\mu^a A_\mu^a + b^a \partial_\mu A_\mu^a + \bar{c}^a \partial_\mu (D_\mu c)^a \right), \quad (16.3)$$

with

$$(D_\mu c)^a = \partial_\mu c^a + g f^{abc} A_\mu^b c^c, \quad (16.4)$$

¹In the case of the maximal Abelian gauge, the color index a runs only over the $N(N-1)$ off-diagonal components.

where b^a is the Lagrange multiplier enforcing the Landau gauge condition, $\partial_\mu A_\mu^a = 0$, and $\bar{c}^a$, c^a are the Faddeev-Popov ghosts. Concerning the mass term in expression (16.3), two remarks are in order. The first one is that, although in three dimensions the gauge field might become massive due to the introduction of the Chern-Simons topological action [238], one should note that the mass term considered here is of a different nature. In fact, unlike the Chern-Simons term, the mass term $m^2 A_\mu^a A_\mu^a$ does not break parity. As a consequence, the starting action 16.3 is parity preserving. Therefore, the parity breaking Chern-Simons term cannot show up due to radiative corrections. The second remark is related to the superrenormalizability of three-dimensional Yang-Mills theories, as expressed by the dimensionality of the gauge coupling g . As shown in [132], a standard perturbation theory would be affected by infrared singularities in the massless case. However, the presence of the mass term prevents the theory from this infrared instability, allowing one to define an infrared safe perturbative expansion. Following [163], the action (16.3) is left invariant by a set of modified BRST transformations, given by

$$\begin{aligned} sA_\mu^a &= -(D_\mu c)^a, & sc^a &= \frac{g}{2} f^{abc} c^b c^c, \\ s\bar{c}^a &= b^a, & sb^a &= -m^2 c^a, \end{aligned} \quad (16.5)$$

and

$$sS = 0. \quad (16.6)$$

Notice that, due to the introduction of the mass term m , the operator s is not strictly nilpotent, *i.e.*

$$\begin{aligned} s^2 \Phi &= 0, \quad (\Phi = A^{a\mu}, c^a), \\ s^2 \bar{c}^a &= -m^2 c^a, \\ s^2 b^a &= -m^2 \frac{g}{2} f^{abc} c^b c^c. \end{aligned} \quad (16.7)$$

Therefore, setting

$$s^2 \equiv -m^2 \delta, \quad (16.8)$$

we have

$$\delta S = 0. \quad (16.9)$$

The operator δ is related to a global $SL(2, \mathbb{R})$ symmetry [163], which is known to be present in the Landau, Curci-Ferrari and maximal Abelian gauges [171]. Finally, in order to express the BRST and δ invariances in a functional way, we introduce the external action [59]

$$\begin{aligned} S_{\text{ext}} &= \int d^3x \left(\Omega_\mu^a sA_\mu^a + L^a sc^a \right) \\ &= \int d^3x \left(-\Omega_\mu^a (D_\mu c)^a + L^a \frac{g}{2} f^{abc} c^b c^c \right), \end{aligned} \quad (16.10)$$

where Ω_μ^a and L^a are external sources invariant under both BRST and δ transformations, coupled to the nonlinear variations of the fields A_μ^a and c^a . It is easy to check that the complete classical action,

$$\Sigma = S + S_{\text{ext}}, \quad (16.11)$$

is invariant under BRST and δ transformations

$$s\Sigma = 0, \quad \delta\Sigma = 0. \quad (16.12)$$

When translated into functional form, the BRST and the δ invariances give rise to the following Ward identities for the complete action Σ , namely

- the Slavnov-Taylor identity

$$\mathcal{S}(\Sigma) = 0, \quad (16.13)$$

with

$$\mathcal{S}(\Sigma) = \int d^3x \left(\frac{\delta \Sigma}{\delta \Omega_\mu^a} \frac{\delta \Sigma}{\delta A_\mu^a} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta \Sigma}{\delta c^a} + b^a \frac{\delta \Sigma}{\delta \bar{c}^a} - m^2 c^a \frac{\delta \Sigma}{\delta b^a} \right), \quad (16.14)$$

- the δ Ward identity

$$\mathcal{W}(\Sigma) = 0, \quad (16.15)$$

with

$$\mathcal{W}(\Sigma) = \int d^3x \left(c^a \frac{\delta \Sigma}{\delta \bar{c}^a} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta \Sigma}{\delta b^a} \right). \quad (16.16)$$

In addition, the following Ward identities holds in the Landau gauge [59], i.e.

- the gauge fixing condition and the antighost equation

$$\frac{\delta \Sigma}{\delta b^a} = \partial_\mu A_\mu^a, \quad \frac{\delta \Sigma}{\delta \bar{c}^a} + \partial_\mu \frac{\delta \Sigma}{\delta \Omega_\mu^a} = 0, \quad (16.17)$$

- the integrated ghost equation [89, 59]

$$\mathcal{G}^a \Sigma = \Delta_{\text{cl}}^a, \quad (16.18)$$

with

$$\mathcal{G}^a = \int d^3x \left(\frac{\delta}{\delta c^a} + g f^{abc} \bar{c}^b \frac{\delta}{\delta b^c} \right), \quad (16.19)$$

and

$$\Delta_{\text{cl}}^a = g \int d^3x f^{abc} (A_\mu^b \Omega_\mu^c - L^b c^c). \quad (16.20)$$

Notice that the breaking term Δ_{cl}^a in the right-hand side of eq.(16.18), being linear in the quantum fields, is a classical breaking, not affected by quantum corrections [89, 59].

16.2.2 Algebraic characterization of the invariant counterterm.

Having established all Ward identities obeyed by the classical action Σ , we can now proceed with the characterization of the most general local counterterm compatible with the identities (16.13), (16.15), (16.17) and (16.18). Let us begin by displaying the quantum numbers of all fields, sources and parameters

	A_μ^a	c^a	$\bar{c}^a$	b^a	L^a	Ω_μ^a	g	s	m
Ghost number	0	1	-1	0	-2	-1	0	1	0
Dimension	1/2	0	1	3/2	5/2	2	1/2	1/2	1

(16.21)

In order to characterize the most general invariant counterterm which can be freely added to all orders of perturbation theory, we perturb the classical action Σ by adding an arbitrary integrated, parity preserving, local polynomial Σ^{count} in the fields and external sources of dimension bounded by three and with zero ghost number, and we require that the perturbed action $(\Sigma + \eta \Sigma^{\text{count}})$ satisfies the same Ward identities and constraints as Σ to first order in the perturbation parameter η , which are

$$\begin{aligned} \mathcal{S}(\Sigma + \eta \Sigma^{\text{count}}) &= 0 + O(\eta^2), \\ \mathcal{W}(\Sigma + \eta \Sigma^{\text{count}}) &= 0 + O(\eta^2), \\ \frac{\delta(\Sigma + \eta \Sigma^{\text{count}})}{\delta b^a} &= \partial_\mu A_\mu^a + O(\eta^2), \\ \left(\frac{\delta}{\delta \bar{c}^a} + \partial_\mu \frac{\delta}{\delta \Omega_\mu^a} \right) (\Sigma + \eta \Sigma^{\text{count}}) &= 0 + O(\eta^2), \\ \mathcal{G}^a(\Sigma + \eta \Sigma^{\text{count}}) &= \Delta_{\text{cl}}^a + O(\eta^2). \end{aligned} \quad (16.22)$$

This amounts to imposing the following conditions on Σ^{count}

$$\mathcal{B}_\Sigma \Sigma^{\text{count}} = 0, \quad (16.23)$$

with

$$\begin{aligned} \mathcal{B}_\Sigma &= \int d^3x \left(\frac{\delta \Sigma}{\delta A_\mu^a} \frac{\delta}{\delta \Omega^{a\mu}} + \frac{\delta \Sigma}{\delta \Omega^{a\mu}} \frac{\delta}{\delta A_\mu^a} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta}{\delta c^a} + \frac{\delta \Sigma}{\delta c^a} \frac{\delta}{\delta L^a} \right. \\ &\quad \left. + b^a \frac{\delta}{\delta \bar{c}^a} - m^2 c^a \frac{\delta}{\delta b^a} \right), \end{aligned} \quad (16.24)$$

$$\mathcal{W}_\Sigma \Sigma^{\text{count}} = \int d^3x \left(c^a \frac{\delta \Sigma^{\text{count}}}{\delta \bar{c}^a} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta \Sigma^{\text{count}}}{\delta b^a} + \frac{\delta \Sigma}{\delta b^a} \frac{\delta \Sigma^{\text{count}}}{\delta L^a} \right) = 0, \quad (16.25)$$

$$\frac{\delta \Sigma^{\text{count}}}{\delta b^a} = 0, \quad \frac{\delta \Sigma^{\text{count}}}{\delta \bar{c}^a} + \partial_\mu \frac{\delta \Sigma^{\text{count}}}{\delta \Omega_\mu^a} = 0, \quad (16.26)$$

and

$$\mathcal{G}^a \Sigma^{\text{count}} = 0. \quad (16.27)$$

Following the algebraic renormalization procedure [59], it turns out that the most general local, parity preserving, invariant counterterm Σ^{count} compatible with all constraints (16.23), (16.25), (16.26) and (16.27), contains only two independent free parameters σ and a_1 , and is given by

$$\Sigma^{\text{count}} = \int d^3x \left(-\frac{(\sigma + 4a_1)}{4} F_{\mu\nu}^a F_{\mu\nu}^a + a_1 F_{\mu\nu}^a \partial_\mu A_\nu^a + \frac{a_1}{2} m^2 A_\mu^a A_\mu^a + a_1 (\Omega_\mu^a + \partial_\mu \bar{c}^a) \partial_\mu c^a \right). \quad (16.28)$$

16.2.3 Stability and renormalization of the mass parameter.

It remains now to discuss the stability of the classical action [59], i.e. to check that Σ^{count} can be reabsorbed in the classical action Σ by means of a multiplicative renormalization of the coupling constant g , the mass parameter m^2 , the fields $\{\phi = A, c, \bar{c}, b\}$ and the sources L, Ω , namely

$$\Sigma(g, m^2, \phi, L, \Omega) + \eta \Sigma^{\text{count}} = \Sigma(g_0, m_0^2, \phi_0, L_0, \Omega_0) + O(\eta^2), \quad (16.29)$$

with the bare fields and parameters defined as

$$\begin{aligned} A_{0\mu}^a &= Z_A^{1/2} A_\mu^a, & \Omega_{0\mu}^a &= Z_\Omega \Omega_\mu^a, \\ c_0^a &= Z_c^{1/2} c^a, & L_0^a &= Z_L L^a, \\ g_0 &= Z_g g, & m_0^2 &= Z_{m^2} m^2, \\ \bar{c}_0^a &= Z_{\bar{c}}^{1/2} \bar{c}^a, & b_0^a &= Z_b^{1/2} b^a. \end{aligned} \quad (16.30)$$

The parameters σ and a_1 , are easily seen to be related to the renormalization of the gauge coupling constant g and of the gauge field A_μ^a , according to

$$\begin{aligned} Z_g &= 1 - \eta \frac{\sigma}{2}, \\ Z_A^{1/2} &= 1 + \eta \left(\frac{\sigma}{2} + a_1 \right). \end{aligned} \quad (16.31)$$

Concerning the other fields and the sources Ω_μ^a , L^a , it can be verified that they are renormalized as

$$Z_{\bar{c}} = Z_c = Z_g^{-1} Z_A^{-1/2}, \quad (16.32)$$

$$Z_b = Z_A^{-1}, \quad Z_\Omega = Z_c^{1/2}, \quad Z_L = Z_A^{1/2}. \quad (16.33)$$

Finally, for the mass parameter m^2 ,

$$Z_{m^2} = Z_g Z_A^{-1/2}, \quad (16.34)$$

which, due to eq.(16.32), can be rewritten as

$$Z_{m^2} = Z_c^{-1} Z_A^{-1}. \quad (16.35)$$

Equation (16.32) expresses the well known nonrenormalization property of the ghost-antighost-gluon vertex in the Landau gauge. As shown in [89], this is a direct consequence of the ghost Ward identity (16.18). Also, as anticipated, equation (16.34) shows that the renormalization of the mass parameter m^2 can be expressed in terms of the gauge field and coupling constant renormalization factors. It is worth mentioning here that eqs.(16.32), (16.34) are in complete agreement with the results obtained in the case of the four-dimensional Yang-Mills theory in the Landau gauge [153].

Although we did not consider matter fields in the previous analysis, it can be easily checked that the renormalizability of the mass operator $A_\mu^a A_\mu^a$ and the relations (16.34), (16.35) remain unchanged if massless spinor fields are included, namely

$$S_{\text{matter}} = \int d^3x \left(i\bar{\psi}^i \not{\partial} \psi^i + g A_\mu^a \bar{\psi}^i \gamma^\mu T^a \psi^i \right), \quad (16.36)$$

with $i = 1, \dots, N_f$. In fact, as was pointed out in [132], the addition of massless fermions does not break the parity invariance of the starting action (16.3). Of course, the inclusion of the matter action (16.36) requires the introduction of a suitable renormalization factor Z_ψ for the spinor fields.

16.2.4 Absence of one-loop ultraviolet divergences.

In the previous section we have proven that the massive three-dimensional Yang-Mills action (16.3) is multiplicatively renormalizable to all orders of perturbation theory, displaying interesting renormalization features, as expressed by equations (16.32) and (16.34). Only two renormalization constants, Z_g and

Z_A , are needed at the quantum level. These factors should be computed order by order by means of a suitable regularization, which in the present case could be provided by dimensional regularization. Due to the absence of parity breaking terms, this would give an invariant regularization scheme. Furthermore, we recall that Yang-Mills theory in three dimensions is a superrenormalizable theory, a property which reduces the number of divergent integrals. It is thus worth looking at the Feynman diagrams of the theory. Let us begin with the one-loop ghost-antighost self-energy. It can be checked that, due to the transversality of the gluon propagator in the Landau gauge, the Feynman integral for the ghost self-energy

$$g^2 \int \frac{d^3k}{(2\pi)^3} \frac{p_\mu(p-k)_\nu}{(p-k)^2} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \frac{1}{k^2 + m^2}, \quad (16.37)$$

where p_μ stands for the external momentum, is free from ultraviolet divergences. As a consequence we have that, at one-loop order in $\overline{\text{MS}}$,

$$Z_c = Z_{\bar{c}} = 1, \quad \text{at one-loop order.} \quad (16.38)$$

Analogously, by simple inspection, it turns out that the one-loop correction to the ghost-antighost-gluon vertex is also finite. The same feature holds for the one-loop Feynman diagrams contributing to the four gluon vertex, from which it follows that in $\overline{\text{MS}}$

$$Z_g^2 Z_A^2 = 1, \quad \text{at one-loop order.} \quad (16.39)$$

Moreover, from equation (16.32), we have

$$Z_A = 1, \quad \text{at one-loop order,} \quad (16.40)$$

so that

$$Z_g = 1, \quad \text{at one-loop order} \quad (16.41)$$

in $\overline{\text{MS}}$. We see therefore that, at one-loop order, the theory is completely free from ultraviolet divergences, a feature which also holds in the presence of massless fermions. At higher orders, ultraviolet divergences could show up.

To provide a non-trivial check of the validity of the relation (16.1) from another point of view, we shall make use of the large N_f expansion, given the existence of a fixed point in the β -function. Within this large N_f expansion technique, it is commonly known that this fixed point can be obtained by analytic continuation of the one existing in $d = 4 - 2\varepsilon$ dimensions. This will be considered in the following section.

16.3 Large N_f verification.

Having established the renormalizability of the mass operator in the Landau gauge, we verify the result in QCD using the large N_f critical point method developed in [240, 241] for the non-linear σ model and extended to QED and QCD in [242, 243, 244, 245]. Briefly, this method allows one to compute the critical exponents associated with the renormalization of the fields, coupling constants or composite operators at the d -dimensional fixed point of the QCD β -function. The critical exponents encode all orders information on the respective anomalous dimensions, β -function and operator anomalous dimensions and are more fundamental than their associated renormalization group functions in that they are renormalization group invariant. Knowing the explicit location of the d -dimensional fixed point

allows one to convert the information encoded in the exponents to the explicit coefficients in the four-dimensional perturbative expansion of the renormalization group functions. Since we are interested in the renormalization of $\frac{1}{2}A_\mu^a A_\mu^a$ in the Landau gauge and its connection with the gluon and ghost wave function renormalization we will show that, in agreement with eqs.(16.34), (16.35), the critical exponent associated with the Landau gauge renormalization of $A_\mu^a A_\mu^a$ at leading order in large N_f is simply the sum of the gluon and ghost wave function critical exponents. The latter have already been determined in [244]. Moreover, since the computation is in d -dimensions, $2 < d < 4$, the three-dimensional result of the previous sections will emerge *naturally*.

To fix notation for this section, we recall that the d -dimensional $\overline{\text{MS}}$ QCD β -function, [246], is

$$\begin{aligned} \beta(a) = & (d-4)a + \left[\frac{2}{3}T_F N_f - \frac{11}{6}C_A \right] a^2 + \left[\frac{1}{2}C_F T_F N_f + \frac{5}{6}C_A T_F N_f - \frac{17}{12}C_A^2 \right] a^3 \\ & - \left[\frac{11}{72}C_F T_F^2 N_f^2 + \frac{79}{432}C_A T_F^2 N_f^2 + \frac{1}{16}C_F^2 T_F N_f + \frac{2857}{1728}C_A^3 \right. \\ & \left. - \frac{205}{288}C_F C_A T_F N_f - \frac{1415}{864}C_A^2 T_F N_f \right] a^4 + O(a^5), \end{aligned} \quad (16.42)$$

where the group Casimirs are defined by $T^a T^a = C_F I$, $f^{acd} f^{bcd} = C_A \delta^{ab}$ and $\text{Tr}(T^a T^b) = T_F \delta^{ab}$. The leading $O(a)$ term corresponds to the dimension of the coupling in d -dimensions and is necessary to deduce the location of the non-trivial d -dimensional fixed point a_c . Expanding in powers of $1/N_f$ it is given by

$$\begin{aligned} a_c = & \frac{3\varepsilon}{T_F N_f} + \frac{1}{4T_F^2 N_f^2} \left[33C_A \varepsilon - (27C_F + 45C_A) \varepsilon^2 \right. \\ & \left. + \left(\frac{99}{4}C_F + \frac{237}{8}C_A \right) \varepsilon^3 + O(\varepsilon^4) \right] + O\left(\frac{1}{N_f^3}\right), \end{aligned} \quad (16.43)$$

where $d = 4 - 2\varepsilon$. QCD is in the same universality class as the non-Abelian Thirring model (NATM), [247], which has the Lagrangian

$$\mathcal{L}^{\text{NATM}} = i\bar{\psi}^i \not{\partial} \psi^i + \frac{\lambda^2}{2} (\bar{\psi}^i \gamma^\mu T^a \psi^i)^2, \quad (16.44)$$

or rewriting it in terms of an auxiliary vector field, $\tilde{A}_\mu^a$,

$$\mathcal{L}^{\text{NATM}} = i\bar{\psi}^i \not{\partial} \psi^i + \tilde{A}_\mu^a \bar{\psi}^i \gamma^\mu T^a \psi^i - \frac{(\tilde{A}_\mu^a)^2}{2\lambda^2}, \quad (16.45)$$

where the coupling constant λ is dimensionless in two dimensions. By analogy the NATM plays the same role as the $O(N)$ nonlinear σ model in the d -dimensional critical point equivalence with the four-dimensional $O(N)$ ϕ^4 theory at the d -dimensional Wilson-Fisher fixed point. One feature of the universality criterion at criticality is that the interactions of the fields play the major role. Hence, comparing the QCD and NATM Lagrangians where for this section we take

$$\mathcal{L}^{\text{QCD}} = i\bar{\psi}^i \not{\partial} \psi^i + \tilde{A}_\mu^a \bar{\psi}^i \gamma^\mu T^a \psi^i - \frac{(F_{\mu\nu}^a)^2}{4g^2}, \quad (16.46)$$

the quark-gluon 3-point interaction of both models is dominant in the large N_f critical point method. In QCD the field strength of the Lagrangian is infrared irrelevant and drops out of the large N_f analysis. However, in practice the triple and quartic gluon interactions emerge in diagrams with closed quark

loops with respectively three and four external $\tilde{A}_\mu^a$ fields, [247, 245]. It is worth noting that in this section alone we have redefined the gluon field and incorporated a power of the QCD coupling constant into its definition, $\tilde{A}_\mu^a = g A_\mu^a$ which is the origin of the power of g^2 factor with the field strength term. This rescaling is necessary for the application of the critical point large N_f programme which requires a unit coupling constant for the quark gluon interaction and therefore defines the canonical scaling dimensions in such a way as to make the calculational tool of uniqueness applicable which was used extensively in the original large N_f critical point method of [240, 241]. As we are interested in the

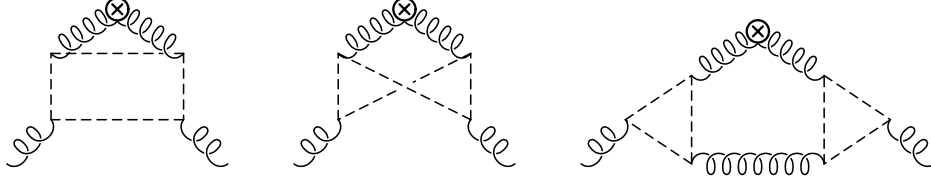


Figure 16.1: $O(1/N_f)$ diagrams contributing to η_{A^2} .

critical exponents and therefore the anomalous dimensions of the composite operator $\frac{1}{2}A_\mu^a A_\mu^a$ in the large N_f expansion, we follow the method of [245]. There the critical exponent ω associated with the QCD β -function was computed at $O(1/N_f)$ in d -dimensions by inserting the composite operator $(F_{\mu\nu}^a)^2$ into a gluon 2-point function and applying the method of [248] to determine the critical dimension of its associated coupling. For the anomalous dimension of $\frac{1}{2}A_\mu^a A_\mu^a$ we follow the same approach and note that the appropriate $O(1/N_f)$ large N_f diagrams are given in Figure 16.1 where a gluon line counts one power of $1/N_f$ which is why there are two and three-loop Feynman diagrams at this order. The latter three-loop graph in fact contains the relevant contribution from the triple gluon vertex which is absent in the NATM Lagrangian. Unlike perturbation theory the propagators of Figure 16.1 are not the usual ones. Their asymptotic scaling forms are deduced from dimensional analysis and consistency with Lorentz symmetry. In the Landau gauge we have, [243, 244],

$$\psi(k) \sim \frac{A k}{(k^2)^{\mu-\alpha}}, \quad A_{\mu\nu}(k) \sim \frac{B}{(k^2)^{\mu-\beta}} \left[\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right], \quad c(k) \sim \frac{C}{(k^2)^{\mu-\gamma}}, \quad (16.47)$$

in momentum space at leading order as $k^2 \rightarrow \infty$ as one approaches the d -dimensional fixed point. We have given the ghost propagator asymptotic scaling form for completeness and to define its scaling dimension even though it is not needed at $O(1/N_f)$ for the explicit computation of the critical exponent of $\frac{1}{2}A_\mu^a A_\mu^a$. The powers of the propagators are defined as

$$\alpha = \mu + \frac{1}{2}\eta, \quad \beta = 1 - \eta - \chi, \quad \gamma = \mu - 1 + \frac{1}{2}\eta_c, \quad (16.48)$$

where A , B and C are the momentum independent amplitudes though only the combinations $z = A^2 B$ and $y = C^2 B$ appear in calculations, [244]. We use $\mu = d/2$ for shorthand, η is the critical exponent of the quark field, χ is the critical exponent of the quark-gluon vertex anomalous dimension and η_c is the ghost critical exponent. We note that the explicit $O(1/N_f)$ values of the critical exponents in d -dimensions in the Landau gauge are, [244],

$$\begin{aligned} \eta_1 &= \frac{(2\mu-1)(\mu-2)\Gamma(2\mu)C_F}{4\Gamma^2(\mu)\Gamma(\mu+1)\Gamma(2-\mu)T_F} \equiv \eta_1^0 \frac{C_F}{T_F}, \\ \chi_1 &= - \left[C_F + \frac{C_A}{2(\mu-2)} \right] \frac{\eta_1^0}{T_F}, \quad \eta_{c1} = - \frac{C_A \eta_1^0}{2(\mu-2)T_F}, \end{aligned} \quad (16.49)$$

where we will use the notation $\eta = \sum_{i=1}^{\infty} \eta_i / N_f^i$. The expression for the ghost anomalous dimension follows from the usual Slavnov-Taylor identity as expressed in exponent language,

$$\eta_c = \eta + \chi - \chi_c, \quad (16.50)$$

where χ_c is the anomalous dimension of the ghost-gluon vertex and was shown in [244] to vanish in the Landau gauge at $O(1/N_f)$.

The explicit computation of the exponent associated with the renormalization of $\frac{1}{2}A_\mu^a A_\mu^a$, which we will call η_{A^2} , is deduced by inserting (16.47) into the diagrams of Figure 16.1 and applying the procedure of [248] to determine the scaling dimension of the operator insertion, η_O . The value of η_{A^2} is deduced from the relation

$$\eta_{A^2} = \eta + \chi + \eta_O, \quad (16.51)$$

where the first two terms correspond to the anomalous part of the gluon critical dimension or wave function renormalization. For completeness we note that the corresponding critical exponent in the Thirring model, ω^{NATM} , is deduced by dimensionally analysing the final term of (16.45) giving

$$\omega^{\text{NATM}} = \mu - 1 + \eta + \chi + \eta_O. \quad (16.52)$$

In practice a regularization has to be introduced for the Feynman integrals which is obtained by shifting the exponent of the vertex renormalization, χ , to the new value of $\chi + \Delta$. Here Δ plays a role akin to ε in dimensional regularization. Though it should be stressed that we are working in fixed dimensions, d , and not dimensionally regularizing here. The actual contribution to η_O is determined from the residue of the simple pole in Δ from the sum of all the diagrams of Figure 16.1. In [245] the two and three-loop diagrams were computed using various techniques such as integration by parts and uniqueness, [241], after the regularized Feynman integrals were broken up into a set of basic integrals which were straightforward to determine and a set which required a substantial amount of effort particularly in the case of the three-loop diagram. We have used the same integrals here but supplemented with an extra set since the operator insertion of $\frac{1}{2}A_\mu^a A_\mu^a$ alters the power of the internal gluon line containing the operator insertion. An example of one of the tedious graphs in this respect is that illustrated in Figure

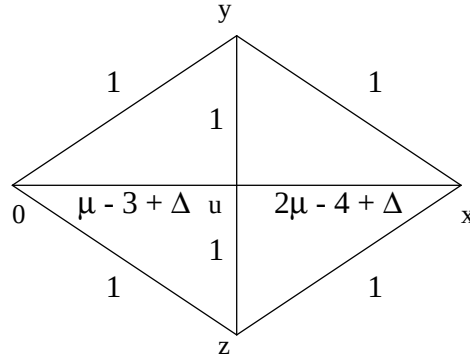


Figure 16.2: Basic three-loop Feynman diagram.

16.2 where we have indicated the power of the propagator beside the line. We have used coordinate space representation where one integrates over the location of the internal vertices, u , y and z , but with x corresponding to the external coordinate or momentum of the diagram. To determine the residue with respect to the Δ -pole we convert the integral to momentum representation, [241], which produces the first diagram of Figure 16.3. There we have nullified the regularization since the associated factor from the transformation is

$$\frac{a^6(1)a(\mu-3)a(2\mu-4)}{a(-1+2\Delta)}, \quad (16.53)$$

which, due to the denominator factor, is clearly divergent as $\Delta \rightarrow 0$ since $a(\alpha) = \Gamma(\mu - \alpha)/\Gamma(\alpha)$. To proceed we use the language of [241] and apply a conformal transformation to the first diagram of Figure 16.3 based on the left external point. Then integrating the unique triangle and subsequent unique vertex before undoing the original conformal transformation finally produces the second diagram of Figure 16.3. The factor associated with these manipulations from the first diagram of Figure 16.3 is $a^4(\mu - 1)/a(2\mu - 4)$. To deduce the value of the final diagram which is Δ -finite we integrate by parts on the top right internal vertex based on the line with exponent 1. This produces four two-loop diagrams. However, these intermediate diagrams are in fact divergent though their sum is finite. To ensure the correct finite part emerges, one introduces a temporary intermediate regularization prior to integrating by parts by shifting the exponent of the line labeled $3 - \mu$ to an exponent of $3 - \mu + \delta$. In fact two of the resulting diagrams then cancel exactly, leaving two integrals which are related to the function $\text{ChT}(\alpha, \beta)$ defined in [241] and evaluated exactly in [180, 241]. Explicitly one has the difference of $\text{ChT}(-1 - \delta, 3 - \mu)$ and $\text{ChT}(\mu - 3 - \delta, 3 - \mu)$ and expanding in powers of δ a finite expression emerges. Accumulating all the contributions the final contribution of the integral of Figure 16.2 to the critical exponent computation is

$$\frac{(2\mu - 3)(\mu - 1)^2(2\mu^2 - 7\mu + 4)}{4\Gamma(\mu + 1)\Delta}. \quad (16.54)$$

Having completed the computation of all the intermediate basic integrals we note that the transverse

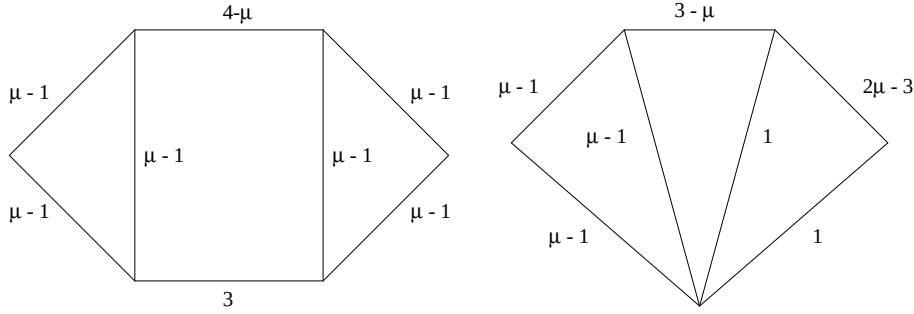


Figure 16.3: Intermediate three-loop Feynman diagrams.

contribution of each of the four diagrams of Figure 16.1 to η_O are respectively,

$$-\frac{(2\mu - 1)(2\mu - 3)C_F\eta_1^O}{2(\mu - 2)T_F}, \quad \frac{(2\mu - 1)(2\mu - 3)[C_F - \frac{1}{2}C_A]\eta_1^O}{2(\mu - 2)T_F}, \quad \frac{(\mu - 1)^2C_A\eta_1^O}{(\mu - 2)T_F}. \quad (16.55)$$

Hence,

$$\eta_{A^2} = -\frac{C_A\eta_1^O}{4(\mu - 2)T_F N_f} + O\left(\frac{1}{N_f^2}\right), \quad (16.56)$$

in d -dimensions. Clearly this is equivalent to the sum of anomalous dimension parts of the Landau gauge gluon and ghost critical exponents at $O(1/N_f)$. More explicitly, from (16.48) and (16.49),

$$\eta_{A^2 1} = \eta_1 + \chi_1 - \frac{1}{2}\eta_{c 1}, \quad (16.57)$$

which due to our choice of conventions and notation was the way this identity was originally uncovered in [87] prior to the all orders proof of [153] and its subsequent expression in the form of (16.1). Therefore,

(16.57) is an explicit d -dimensional verification of the all orders result of the previous section. Moreover, it nicely recovers the d -dimensional case of [87, 153, 234].

As three-dimensional QCD is of interest in other problems, we note that the explicit three-dimensional value of ω^{NATM} is

$$\omega^{\text{NATM}} \Big|_{d=3} = \frac{1}{2} - \frac{4C_A}{3\pi^2 T_F N_f} + O\left(\frac{1}{N_f^2}\right). \quad (16.58)$$

In two dimensions, interestingly the critical exponent does not run to its mean field value and one has

$$\omega^{\text{NATM}} \Big|_{d=2} = -\frac{C_A}{16T_F N_f} + O\left(\frac{1}{N_f^2}\right). \quad (16.59)$$

16.4 Generalization to other gauges: the example of the Curci-Ferrari gauge.

The mass operator $A_\mu^a A_\mu^a$ in the Landau gauge can be generalized to other gauges, such as the Curci-Ferrari and the maximal Abelian gauge. In this case the mixed gluon-ghost mass operator $(\frac{1}{2}A_\mu^a A_\mu^a + \alpha \bar{c}^a c^a)$ has to be considered, where α stands for the gauge parameter. Let us consider here the case of the Curci-Ferrari nonlinear gauge. For the gauge fixed action we have

$$\begin{aligned} S_{\text{CF}} = & \int d^3x \left(-\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + b^a \partial_\mu A_\mu^a + \frac{\alpha}{2} b^a b^a + \bar{c}^a \partial_\mu (D_\mu c)^a - \frac{\alpha}{2} g f^{abc} b^a \bar{c}^b c^c \right. \\ & \left. - \frac{\alpha}{8} g^2 f^{abc} f^{cde} \bar{c}^a \bar{c}^b c^d c^e + m^2 \left(\frac{1}{2} A_\mu^a A_\mu^a + \alpha \bar{c}^a c^a \right) \right). \end{aligned} \quad (16.60)$$

Notice that in this case also the Faddeev-Popov ghosts $\bar{c}^a, c^a$ are massive. Moreover, the Curci-Ferrari gauge reduces to the Landau gauge in the limit $\alpha \rightarrow 0$. The action (16.60) is invariant under the BRST and δ transformations of eqs.(16.5), (16.8). Introducing the external action

$$S_{\text{ext}} = \int d^3x \left(-\Omega_\mu^a (D_\mu c)^a + L^a \frac{g}{2} f^{abc} c^b c^c \right),$$

it follows that the complete classical action

$$\Sigma_{\text{CF}} = S_{\text{CF}} + S_{\text{ext}}, \quad (16.61)$$

turns out to be constrained by the Slavnov-Taylor identity

$$\mathcal{S}(\Sigma) = \int d^3x \left(\frac{\delta \Sigma}{\delta \Omega_\mu^a} \frac{\delta \Sigma}{\delta A_\mu^a} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta \Sigma}{\delta c^a} + b^a \frac{\delta \Sigma}{\delta \bar{c}^a} - m^2 c^a \frac{\delta \Sigma}{\delta b^a} \right) = 0, \quad (16.62)$$

and by the δ Ward identity

$$\mathcal{W}(\Sigma) = \int d^3x \left(c^a \frac{\delta \Sigma}{\delta \bar{c}^a} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta \Sigma}{\delta b^a} \right) = 0. \quad (16.63)$$

Due to the presence of the quartic ghost-antighost term $g^2 f^{abc} f^{cde} \bar{c}^a \bar{c}^b c^d c^e$ and of $g f^{abc} b^a \bar{c}^b c^c$ the additional Ward identities (16.17) and (16.18) of the Landau gauge do not hold in the present case. Nevertheless, identities (16.62) and (16.63) ensure the multiplicative renormalizability of the model.

Proceeding as in the previous section, it turns out that the most general invariant counterterm contains five free independent parameters, σ , a_1 , a_2 , a_3 , a_5 and is given by

$$\begin{aligned} \Sigma_{\text{CF}}^{\text{count}} = & \int d^3x \left(-\frac{(\sigma + 4a_1)}{4} F_{\mu\nu}^a F_{\mu\nu}^a + a_1 F_{\mu\nu}^a \partial_\mu A_\nu^a + (a_1 - a_2) b^a \partial_\mu A_\mu^a \right. \\ & + a_1 (\partial_\mu \bar{c}^a + \Omega_\mu^a) \partial_\mu c^a + (a_1 - a_2) \bar{c}^a \partial_\mu (D_\mu c)^a + a_5 (\partial_\mu \bar{c}^a + \Omega_\mu^a) (D_\mu c)^a \\ & - a_3 \frac{\alpha}{2} b^a b^a + \frac{(a_3 + a_5)}{2} \alpha g f^{abc} b^a \bar{c}^b c^c + \frac{(a_3 + 2a_5)}{8} \alpha g^2 f^{abc} f^{cde} \bar{c}^a \bar{c}^b c^d c^e \\ & \left. - \frac{a_5}{2} g f^{abc} L^a c^b c^c + m^2 \left((a_1 - \frac{a_2}{2} + \frac{a_5}{2}) A_\mu^a A_\mu^a - \alpha a_3 \bar{c}^a c^a \right) \right). \end{aligned} \quad (16.64)$$

The parameters σ , a_1 , a_2 , a_3 , a_5 are easily seen to correspond to a multiplicative renormalization of the fields, sources and parameters, according to

$$\begin{aligned} Z_g &= 1 - \eta \frac{\sigma}{2}, \\ Z_A^{1/2} &= 1 + \eta \left(\frac{\sigma}{2} + a_1 \right), \\ Z_c^{1/2} &= Z_{\bar{c}}^{1/2} = 1 - \eta \left(\frac{(a_2 + a_5)}{2} \right), \\ Z_L &= 1 + \eta \left(\frac{\sigma}{2} + a_2 \right), \\ Z_\alpha &= 1 + \eta (-a_3 + 2a_2 + \sigma), \end{aligned} \quad (16.65)$$

and

$$\begin{aligned} Z_\Omega &= Z_A^{-1/2} Z_c^{1/2} Z_L, \\ Z_b^{1/2} &= Z_L^{-1}, \\ Z_{m^2} &= Z_L^{-2} Z_c^{-1}. \end{aligned} \quad (16.66)$$

In particular, from eqs.(16.66) it follows that the renormalization factor Z_{m^2} is not independent, being expressed in terms of the ghost renormalization factor Z_c and of the renormalization factor Z_L of the source L^a coupled to the composite ghost operator $\frac{1}{2} g f^{abc} c^b c^c$. Again, these results are in complete agreement with those obtained in the four-dimensional case [199].

16.5 Conclusion.

In this paper we have analysed the renormalization properties of the mass operator $A_\mu^a A_\mu^a$ in three-dimensional Yang-Mills theories in the Landau gauge. In analogy with the four-dimensional case, the renormalization factor Z_{m^2} is not an independent parameter of the theory, as expressed by the relations (16.34) and (16.35), which have been explicitly verified in the large N_f expansion method. These results will be used in order to investigate by analytical methods the possible formation of the gauge condensate $\langle A_\mu^a A_\mu^a \rangle$. This would provide a dynamical generation of a parity preserving mass for the gluons in three dimensions, a topic which has been extensively investigated in recent years. For instance, see [133, 134, 135, 136].

Finally, we underline that the Curci-Ferrari gauge allows one to study the generalized mixed gluon-ghost condensate $\langle \frac{1}{2} A_\mu^a A_\mu^a + \alpha \bar{c}^a c^a \rangle$. In particular, as discussed in the four-dimensional case, the presence of the gauge parameter α could be useful to investigate the gauge independence of the vacuum energy, due to the formation of the aforementioned condensates.

Part III

Conclusion

Chapter 17

Conclusion

In this final part, we present some general conclusive remarks.

Most of the attention was focused on the dynamical formation of a gluon condensate of mass dimension two. This was achieved using the local composite operator formalism, which allows to construct a sensible effective potential for local composite operators using perturbative calculations, and as such one may obtain some nonperturbative information in the form of a condensate, whose formation is favoured since it lowers the vacuum energy.

We have generalized this research, which was initiated originally in the Landau gauge [42], to a wide variety of covariant gauges: the linear covariant, the Curci-Ferrari and the maximal Abelian gauges. The latter gauge fixing is relevant for the dual superconductivity picture of the QCD vacuum at lower energies, important as a possible scenario of confinement.

Our work can be seen as giving evidence that condensates of mass dimension two are relevant for gauge theories, since a similar phenomenon occurs in every gauge we have investigated. However, our understanding of these condensates is far from complete.

- Although the operator A_μ^2 can be given a gauge invariant meaning in the Landau gauge, such an interpretation for the generalized composite operators is lost in the other gauges. We cannot say anything about a gauge invariant interpretation of our results.

There has even appeared a proof that $\langle A_\mu^2 \rangle$ should be gauge invariant, but it remains unclear to us if this proof is correct [249].

If Zwanziger is correct about the equivalence of expectation values evaluated in the Fundamental Modular Region Λ or Gribov region Ω [113], $\langle \dots \rangle_\Lambda = \langle \dots \rangle_\Omega$, then in principle the quantity $\langle A_\mu^2 \rangle_\Omega$, discussed in Chapter 15, is a true gauge invariant quantity, being equal to $\langle A_\mu^2 \rangle_{\min}$. However, this statement is a little academic, as the question remains how the quantity $\langle A_\mu^2 \rangle_{\min}$ can be calculated in gauges other than the Landau gauge.

- The fact that in several classes of gauges, there can even be found a priori an operator that is renormalizable to all orders of perturbation theory, sometimes even with extra properties like for the anomalous dimension in the Landau gauge or MAG, is quite remarkable.

There are a few other gauges available, although these might be less accessible due to specific problems like noncovariantness, the appearance of nonlocal counterterms, only numerically implementable, ...¹.

¹We mention that the condensate A_i^2 , $i = 1, \dots, 3$, has been discussed in connection with glueballs in the Coulomb gauge in [200].

- Some papers have challenged the relevance of dimension two operators, as they are not gauge invariant and do not correspond to physical observables [250, 251]. Also the renormalizability of the operators was challenged, as this was proven each time in a specific gauge for a specific operator. For example, A_μ^2 is not renormalizable for general gauge parameter in the MAG.

Our viewpoint is always that our investigations start from the classical Yang-Mills action. At the quantum level, a suitable gauge condition has to be imposed in order to make a proper quantization of the theory possible. It seems natural to us that the gauge fixing influences the behaviour of the theory at the quantum level. Evidently, gauge invariance should be recovered at the end, especially when one is considering *physical* quantities like the particle spectrum of the theory. Considering gauge variant quantities like the gluon propagator, it is perfectly allowed that gauge variant condensates would influence a propagator [229]. Depending on the gauge, a specific (renormalizable) operator might condense or not.

One of the main consequences of $\langle A_\mu^2 \rangle$ is the presence of a mass parameter in the gluon propagator. The appearance of mass parameters in the gluon propagator has received confirmations from various other studies: lattice simulations [43, 44, 45, 46, 47, 48, 49, 50], solutions of the Schwinger-Dyson equations [51, 52], more phenomenologically oriented studies [53, 54, 55, 56, 57], Also other consequences from our research are in qualitative agreement with other studies, we mention the presence of an off-diagonal and the absence of a diagonal gluon mass in the maximal Abelian gauge [49, 50] or the consequences of the restriction to the Gribov region such as a more singular ghost propagator [116, 117, 118, 119, 120].

As already mentioned in the introduction, it is possible that these mass dimension two condensates contain a gauge invariant piece [34], however this is mainly speculation, and we do not want to add anything to this. Nevertheless, we have motivated that at least the nontrivial vacuum energy, due to the respective condensates in each gauge, should be formally the same in our framework.

In our opinion, gauge invariance may not be simply used as some kind of dogma to “forbid” research in certain directions as it involves gauge variant quantities. We already mentioned the example of the gauge variant propagator which can receive, in principle, gauge variant contributions. We do agree that gauge invariance is a key feature, but if one sticks too firmly to gauge invariance, perhaps not much can be done beyond perturbation theory at the analytical level. It would be already a certain achievement if one would be able to gain some more knowledge about for example the gluon propagator and/or ghost propagator beyond the pure perturbative results, let us only think about the Kugo-Ojima confinement criterium.

A perfect example to illustrate that demanding strict gauge invariance can work counterproductive, is the Gribov problem. It would be a formidable task to find a solution to the problem of gauge copies in a gauge invariant fashion. Due to the special properties of the Landau gauge, something can be said or even done about the problem, so why not profit from it and get some information on nonperturbative dynamics in gauge theories, albeit only in the Landau gauge, which gives one the possibility to say something more than in other gauges.

- Our work was restricted to the perturbative analysis of nonperturbative effects. Probably, other major sources of nonperturbative effects, like topological ones in QCD, shall have their influence on e.g. the condensate $\langle A_\mu^2 \rangle$.
- Most of the times, we restricted the analysis to a one-loop approximation. We remind here that a one-loop knowledge of the effective potential via the LCO formalism makes necessary two-loop calculations. Also the presence of a gauge parameter complicates the analysis. Although it would be desirable to have for example a complete two-loop knowledge of the effective potential for a general choice of the gauge parameter in the case of the linear covariant gauges, the calculations can become extremely complicated. We refer for example to the expression (11.32) for $\zeta_1(\alpha)$,

which shall enter the differential equation of $\zeta_2(\alpha)$. The case with quarks included is even more complicated.

- Concerning the generation of a gluon mass parameter, it must be noticed that this certainly does not mean that we would have solved the problem of the *mass gap in QCD*. Therefore, we would have to prove that the physical excitations², constructable from the QCD-action, are massive. Said otherwise, this would ask for the proof of confinement, a task clearly beyond our capabilities.

Related to this is the question of *unitarity*. Massive Yang-Mills models are not unitary [84, 162, 182]. First of all, we are not considering (bare) massive, but *dynamically* massive Yang-Mills theory. But perhaps more important is the remark that unitarity should be proven at the level of physical excitations, and we nor anybody else do know how to construct these out of the action at the present time, even if one starts from the original massless action. We are investigating Yang-Mills theory written in terms of the elementary, yet unphysical quark and gluon field excitations. This is complementary with the fact that the presence of a pole in the gluon propagator, using the LCO formalism [214, 252], does not necessarily entail the presence of a *physical* massive particle (gluon) [253].

At very high energies, asymptotic states can be attached to the quark and gluon field, as the interaction is very weak due to the asymptotic freedom. At these energies, perturbation theory should do the job, and nonperturbative corrections coming from condensates or whatsoever are absent. Then, the spectrum of Yang-Mills theories contains the two transversal polarizations of the gauge bosons and unitarity is present, see e.g. [177]. This can be proven starting from the BRST symmetry. In the nonperturbative (low energy) region of QCD, it is not clear if there even exists a BRST symmetry.

- As we have focused most attention to the condensation of A_μ^2 in QCD in the Landau gauge, one might wonder what happens in quantum electrodynamics (QED), the very successful quantum theory of the electromagnetic interaction. One could imagine that $\langle A_\mu^2 \rangle$ might arise in QED, giving the photon a mass. As it is well known, the photon is an observable particle, but massless.

To start with, QED is not asymptotically free, so in order for perturbation theory to be useful for the LCO construction of the effective potential, the quantity $\langle A_\mu^2 \rangle$, if arising, should be relatively small³.

Without providing any details, it can be shown within the algebraic renormalization formalism and using the Landau gauge, that the operator A_μ^2 is renormalizable to all orders of perturbation theory. For explicit results, we can make use of the numbers in [197]. Although that particular work was written for QCD, all the Casimir operators of the symmetry group were written explicitly, and the QED-results can be obtained from it upon setting these Casimir operators to the appropriate value, see also the end of [87]. In practice, the LCO one-loop effective potential for QED reads (in the $\overline{\text{MS}}$ scheme)

$$V(\sigma') = -\frac{9}{2} \frac{1}{8N_f} \sigma'^2 + \frac{e^2}{\pi^2} \sigma'^2 \left(\frac{3}{64} \ln \frac{e\sigma'}{\bar{\mu}^2} - \frac{13}{128} + \frac{207}{32} \frac{1}{16N_f} - \frac{117}{32} \frac{1}{8N_f} \right), \quad (17.1)$$

where $\langle \sigma' \rangle \propto \langle A_\mu^2 \rangle$. As it was noticed in [197], the tree level potential is unbounded from below. In [197], it was therefore remarked that the potential does not make sense for QED. Nevertheless, we can use the quantum corrected potential, whose behaviour is changed due to the presence of

²The physical spectrum is built out of baryons, mesons, glueballs,... and not out of (massive/massless) gluons or quarks.

³We remember that the size of the condensate reflects on the size of the coupling constant.

the logarithm. However, if we derive the gap equation $\frac{dV}{d\sigma} = 0$, and put $\bar{\mu}^2 = e\sigma'$ to kill the possibly large logarithm, we find as non-trivial solution

$$\frac{e_*^2}{16\pi^2} = \frac{-18}{27 + 40N_f} < 0, \quad (17.2)$$

where it is understood that $e_* \equiv e(e\sigma')$. Thus, there is no non-trivial solution at one-loop in the $\overline{\text{MS}}$ scheme. Unfortunately, it is difficult to state that we will also find only the trivial solution $\sigma' = 0$ when higher orders are taken into account. But, as a certain check, we can also perform the analysis at two-loop order in the $\overline{\text{MS}}$ scheme, since the necessary numbers can be also found in [197]. In this case, we establish that $\frac{e_*^2}{16\pi^2} > 0$ is possible, although only for $N_f \geq 3$. This implies that the photon remains massless, if one would consider QED with only the electron present. One can argue that there are three species of fermions, namely the leptons electron, muon and tau, but as the mass of the τ -particle is about 2 GeV, the approximation to treat it as a massless fermion is not really permitted.

Next to a condensation of a mass dimension two operator involving the gauge bosons, we also investigated further the condensation of mass dimension two ghost operators, related to the breakdown of a continuous $SL(2, \mathbb{R})$ symmetry, present in some gauges. These condensates, if existing, shall also influence the propagators.

Is this a closed work? Clearly it is not. Several items can be explored further. We list only a few.

It would be nice to receive confirmations from other sources (lattice) if a mass dimension two condensate exists in other gauges. In the Landau gauge, $\frac{1}{Q^2}$ power corrections related to $\langle A_\mu^2 \rangle$ were found when the coupling constant g^2 was considered, obtained from various interaction vertices, see e.g. [37, 38, 39, 257]. One might think of searching for $\frac{1}{Q^2}$ corrections to g^2 in other gauges that can be put on a lattice, e.g. the MAG.

Likewise it would be nice to receive some alternative evidence that something like a ghost condensation does occur in the Landau gauge. What is the influence of the ghost condensation on the gluon condensation and vice versa? What is the combined effect of the ghost and gluon condensates on the propagators? At a preliminary stage, we have found that the ghost condensate in the Overhauser vacuum induces a mass splitting between the diagonal and off-diagonal gluon mass, whereby the diagonal mass is smaller than the off-diagonal mass [254]. Perhaps this might serve as an indication for some kind of Abelian dominance in the Landau gauge [255, 256].

What happens when the implementation to the Gribov region is implemented at two-loop order? Is it possible to find a reliable result with negative vacuum energy? What is the possible role of the gluon condensate $\langle A_\mu^2 \rangle$ in this? If it would turn out that the vacuum energy remains positive at higher loop order, then this would be an indication that an order by order implementation of the horizon condition is far from being "sufficient" for a decent infrared description of QCD.

Can we extend the LCO formalism to three dimensions? As the massive gauge theory turned out to be finite at one-loop order, it would be also interesting to find out what happens at higher loop order in this superrenormalizable theory.

Does an extension of A_μ^2 exists in supersymmetric Yang-Mills theories using a generalized version of the Landau gauge [258]? An advantage of considering supersymmetric Yang-Mills theories is that exact results are at disposal due to holomorphy [259], and these can serve to test nonperturbative mechanisms.

...

THE END

Appendix A

Nederlandse samenvatting

In this Appendix, the reader shall encounter a Dutch summary of the thesis.

A.1 Situering van het werk.

Klassiek wordt de grondtoestand van een model bepaald door na te gaan of er een globaal minimum van de potentiaal bestaat. Het concept van een klassieke potentiaal kan uitgebreid worden tot de zogenaamde kwantum effectieve potentiaal indien het model gekwantiseerd wordt. Deze effectieve potentiaal omvat de kwantumcorrecties op de klassieke potentiaal. Deze correcties worden doorgaans via een perturbatieve ontwikkeling in een koppelingsconstante g^2 bekomen, i.e. een Taylorreeks in g^2 . Uiteraard zijn perturbatieve berekeningen enkel zinvol voor een voldoende kleine g^2 . Elke term in de reeks wordt berekend door middel van de evaluatie van Feynmandiagrammen. Een gekend probleem bij perturbatieve berekeningen in kwantumveldentheorie is het optreden van ultraviolet divergenties. Deze zijn het gevolg van het feit dat de analytische uitdrukkingen, corresponderend met Feynmandiagrammen, integralen zijn over alle mogelijke momenta. De bijdragen van oneindig grote momenta zorgen voor divergenties door het slechte gedrag op oneindig van de integranda. Om zinvol om te gaan met deze divergenties moet men renormalizeren, i.e. men voert tegentermen in die de optredende oneindigheden orde per orde elimineren. Men noemt een model renormalizeerbaar indien de tegentermen door een (oneindige) herdefinitie van de al aanwezige velden, bronnen, massa's, koppelingen in het oorspronkelijk model kunnen geabsorbeerd worden.

Ons onderzoek heeft zich de afgelopen drie jaar vooral gericht op methoden om de vacuüm verwachtingswaarden van lokaal samengestelde operatoren (LCO) te berekenen. Een vacuüm verwachtingswaarde $\langle \mathcal{O} \rangle$ van een bepaalde operator $\mathcal{O}$ wordt ook wel een condensaat genoemd. Dergelijke condensaten karakteriseren in zekere zin het vacuüm van de theorie die onderzocht wordt.

De beschouwde operatoren zijn lokaal, omdat werken met niet-lokale operatoren verre van triviaal is, en meestal moeten samengestelde operatoren gebruikt worden omdat de elementaire basisvelden zelf niet kunnen condenseren, b.v. omdat dit de Lorentzsymmetrie zou breken.

Een methode om een condensaat te berekenen bestaat uit het construeren van de effectieve potentiaal voor de operator die onderstelt wordt te condenseren. Indien een lagere waarde van de potentiaal, hetgeen niets anders dan de vacuümenergie voorstelt, kan bekomen worden bij een niet-verdwijnde vacuüm verwachtingswaarde, dan correspondeert dit met een meer stabiele toestand van de theorie en treedt bijgevolg condensatie op.

Onze aandacht ging voornamelijk uit naar kwantum chromodynamica (QCD), de ijktheorie die de sterke (kleur) interactie beschrijft. Eén van de meest in het oog springende eigenschappen van QCD is de

asymptotische vrijheid. Sammeer wil dit zeggen dat de sterkte van de interactie daalt voor stijgende energie. De sterkte van de interactie zit vervat in de koppelingsconstante g^2 van de theorie. Omgekeerd wordt g^2 groter bij lager wordende energie. Met andere woorden, perturbatietheorie kan enkel bij zeer hoge energieën een accurate beschrijving van QCD geven. Bij lagere energie zullen niet-perturbatieve effecten een rol beginnen spelen.

Het meest prominente voorbeeld van een niet-perturbatief QCD fenomeen is confinement: de elementaire deeltjes horende bij de basisvelden uit de QCD-actie, namelijk de “gekleurde” gluonen en quarks, worden niet afzonderlijk waargenomen, maar komen steeds gebonden (confined) voor in de vorm van kleurloze toestanden als baryonen, mesonen, glueballs, Confinement in QCD vormt één van de grote onopgeloste problemen van de theoretische fysica.

Klassiek komt er geen massaschaal voor in de QCD-actie. Evenwel is voor het expliciet uitvoeren van de renormalizatie van een veldentheorie de introductie van een massaschaal $\bar{\mu}$ nodig. Deze kruipt onder andere in de koppelingsconstante g^2 , die also een functie $g^2(\bar{\mu})$ wordt. Meerbepaald is $g^2 \sim \ln^{-1}(\bar{\mu}^2/\Lambda_{\overline{\text{MS}}}^2)$ met $\Lambda_{\overline{\text{MS}}}$ een vaste massaschaal, onafhankelijk van de keuze van de arbitraire renormalizatieschaal $\bar{\mu}$. Na renormalizatie treedt er dus wel een massaschaal op, echter alleen indirect via g^2 . Probeert men dus rechtstreeks een condensaat in perturbatietheorie te berekenen, dan zal door de afwezigheid van een expliciete massaschaal dit noodzakelijk nul opleveren. Men zou zich echter kunnen voorstellen dat, door één of ander mechanisme, dergelijk condensaat proportioneel wordt met de schaal $\Lambda_{\overline{\text{MS}}}$. Dit moet dan via een niet-perturbatief mechanisme gebeuren, vermits $\Lambda_{\overline{\text{MS}}} \propto e^{-\frac{1}{g^2}}$ en deze laatste uitdrukking heeft zelf geen Taylorontwikkeling.

Het is instructief om eens intuïtief te illustreren hoe het mogelijk is dat men iets $\propto \Lambda_{\overline{\text{MS}}}$ kan vinden gebruik makende van de effectieve potentiaal $V(\sigma)$, met σ het veld dat ondersteld wordt te condenseren. De minima worden bepaald door de vergelijking $\frac{dV}{d\sigma} = 0$ op te lossen, en dat zal aanleiding geven tot een voorwaarde op g^2 , die in functie zal staan van σ . Dit laatste is gebaseerd op de renormalizatiegroep. Door de relatie $g^2 \sim \ln^{-1}(\sigma/\Lambda_{\overline{\text{MS}}}^2)$ te inverteren komt men dan tot $\sigma \propto \Lambda_{\overline{\text{MS}}}^2$. Dit is een voorbeeld van “dimensionele transmutatie” vermits een dimensieloze koppeling (g^2) wordt ingewisseld voor een massavolle schaal ($\Lambda_{\overline{\text{MS}}}$).

Om tenminste van kwalitatief aanvaardbare resultaten te kunnen spreken, moet de ontwikkelingsparameter $g(\sigma)$ voldoende klein zijn, en dit vertaalt zich via de asymptotische vrijheid in een voldoende groot condensaat.

Laten we nog even terugkomen op het feit dat we samengestelde operatoren zullen beschouwen. Over het algemeen brengen deze operatoren problemen met zich mee op het kwantumniveau. Er moet bewezen worden dat de operator zelf renormalizeerbaar is, waarna er nog moet aangetoond worden dat een renormalizeerbare effectieve potentiaal kan berekend worden, die dan nog moet voldoen aan de nodige vereisten. Dit zijn hoogst niet-triviale eigenschappen. Een methode om een zinvolle effectieve potentiaal te berekenen, werd ontwikkeld door Verschelde in [23] en met succes getest op het twee-dimensionele massaloze Gross-Neveu model [21]. Dit is een interessant model vermits het ook asymptotisch vrij is en het geweten is dat er door niet-perturbatieve kwantumeffecten een massa gegenereerd wordt, die dan ook nog exact gekend is [25], dus men kan nagaan hoe adequaat de ontwikkelde technieken zijn.

Vermits het zogenaamde LCO formalisme een belangrijke rol speelt binnen dit werk voor het construeren van een zinvolle potentiaal, geven we er hier een beknopte uitleg over. De details staan elders in de thesis uitvoerig vermeld. Beschouw een vier-dimensionele veldentheorie en onderstel dat we geïnteresseerd zijn om de eventuele condensatie van een LCO $\mathcal{O}$ van massadimensie twee na te gaan. Een operator kan aan de actie gekoppeld worden via een bron J , i.e. men voegt een term $J\mathcal{O}$ toe aan de actie. Dit geeft aanleiding tot een functionaal $W(J)$, waarbij de Legendre getransformeerde van $W(J)$ niets anders is dan de effectieve potentiaal $V(\sigma)$, waarbij het hulpveld σ de samengestelde operator $\mathcal{O}$ voorstelt. Er treden echter nieuwe oneindigheden $\propto J^2$ op door divergenties in de correlator $\langle \mathcal{O}(x)\mathcal{O}(y) \rangle$ voor $x \approx y$. Deze oneindigheden stemmen overeen met deze in de vacuümenergie. Dit maakt een tegenterm $\propto J^2$

nodig in de Lagrangiaan om de bewuste divergentie te elimineren, maar dit maakt op zijn beurt dan ook een term ζJ^2 nodig om de tegenterm in te kunnen opslorpen. De waarde van deze LCO parameter ζ kan vastgelegd worden door op een intelligente wijze de renormalisatiegroep te gebruiken [23].

Het LCO formalisme laat toe om via perturbatieve berekening ook enige informatie te bekomen over niet-perturbatieve zaken. Uiteraard is het ijdele hoop te denken dat het hiermee mogelijk is alle niet-perturbatieve effecten van QCD te kunnen beschrijven. Er kunnen andere, misschien voorname, oorzaken van niet-perturbatieve effecten in QCD gevonden worden, zoals b.v. instantonen¹, meronen, magnetische monopolen, dyonen, vortices,

Hoe interessant het Gross-Neveu model ook is als testmodel, het is altijd de bedoeling om ook iets meer te weten te komen over fysisch meer relevante theorieën zoals QCD. Als we de quarks buiten beschouwing laten, dan heeft het condensaat met laagste dimensie dat ijk-invariant is, dimensie vier, met name de veldsterkte $\langle F_{\mu\nu}^2 \rangle$. Door het werk van Zakharov et al werd echter de aandacht gevestigd op het mogelijk bestaan van een condensaat van massadimensie twee in ijktheorieën voorzien van de Landau ijk, $\partial_\mu A_\mu = 0$, namelijk het kwadraat van de vectorpotentiaal, A_μ^2 [33, 34]. Terzelfdertijd kwam er ook evidentie vanuit de hoek van de roostersimulaties voor het mogelijk bestaan van dit condensaat in de Landau ijk [37, 39, 257]. Eveneens via numerieke simulaties werd aangevoerd dat instantonen een bijdrage leveren aan dit condensaat [38].

De inspanningen omtrent het bestaan van een condensaat van massadimensie twee waren gericht op de Landau ijk, vermits er een ijk-invariante mening aan de lokale operator A_μ^2 kan gegeven worden in deze ijk.

Een eerste expliciete *berekening* van een waarde voor $\langle A_\mu^2 \rangle$ werd voorgesteld in [42] via de LCO methode, waarbij inderdaad een niet-nul waarde werd gevonden. Een gevolg van dit condensaat is het optreden van een nulde orde² dynamische gluonmassa van enkele honderden MeV. Het optreden van massavolle parameters in de gluonpropagator werd ook al duidelijk door roostersimulaties [43, 44, 45, 46, 47, 48, 49, 50] en door andere niet-perturbatieve methoden zoals de Schwinger-Dyson vergelijkingen [51, 52]. Ook fenomenologisch bleek uit bepaalde werken al dat een dynamische massaparameter betere resultaten oplevert voor de beschrijving van bepaalde processen [53, 54].

A.2 Overzicht van het gedane onderzoek.

A.2.1 De $2PPI$ ontwikkeling.

Een alternatieve methode die toelaat om enige informatie te bekomen over de vacuüm verwachtingswaarde van lokale, samengestelde operatoren werd ontwikkeld in [61, 62, 63, 64] voor het $\lambda\varphi^4$ model. De expansie omvat alle Feynmandiagrammen die samenhangend blijven wanneer er twee lijnen, die samenkomen op hetzelfde punt, doorgesneden worden. Diagrammen die uiteenvallen bij dergelijke snijoperatie worden “2-point-particle-reducible” ($2PPR$) genoemd. Alleen de “2-point-particle-irreducible” ($2PPI$) diagrammen blijven over in de som die de vacuümenergie E opbouwt, vandaar de naam $2PPI$ expansie. In Hoofdstuk 2 hebben we een andere afleiding van de $2PPI$ expansie gegeven gebruik makende van het Gross-Neveu model. De essentie van de $2PPI$ expansie bestaat uit het verwijderen van alle $2PPR$ diagrammen door ze te vervangen door een effectieve massaparameter m . De zelf-consistentie van de expansie wordt gewaarborgd door een gapvergelijking waaraan m moet voldoen, namelijk $\frac{dE}{dm} = 0$. We hebben ook aandacht besteed aan de renormalizatie van de $2PPI$ expansie. De gapvergelijking werd expliciet bepaald tot op twee lussen, evenals de diagrammen die nodig zijn voor het bepalen van de pool van de propagator, waaruit dan een waarde voor de niet-perturbatieve massa van

¹Dit zijn oplossingen van de klassieke bewegingsvergelijkingen met niet-triviaal windingsgetal, m.a.w. deze beschrijven topologisch niet-triviale eigenschappen van QCD.

²Hiermee bedoelen we op het niveau van de Lagrangiaan zelf.

het Gross-Neveu fermion volgt. De uiteindelijke numerieke resultaten voor de massa en vacuümenergie waren in vrij goede overeenkomst met de exact gekende waarden.

Nadat de $2PPI$ expansie getest werd op het Gross-Neveu model, pasten we ze vervolgens aan voor Yang-Mills ijktheorieën in de Landau ijk. Dit is beschreven in Hoofdstuk 3. De $2PPI$ massa parameter is in dit geval evenredig met $\langle A_\mu^2 \rangle$. Alzo waren we in staat om een bijkomende aanwijzing voor het bestaan van dit massadimensie twee condensaat te geven in de Landau ijk. De resultaten waren compatibel met wat eerder was gevonden met de LCO methode in [42]: een lagere vacuümenergie en een dynamische gluon massaparameter van een paar honderd MeV.

A.2.2 De duale supergeleider en de maximaal Abelse ijk.

Zoals reeds aangehaald werd, is er nog geen oplossing voor het probleem van confinement. Een fysisch model dat voor confinement zou kunnen zorgen, is de duale supergeleider [65, 66, 67, 68]. Hierbij zou het lage energie regime van QCD beschreven worden door middel van een effectieve Abelse theorie in de aanwezigheid van (kleur)magnetische monopolen. Deze monopolen worden ondersteld te condenseren om op deze manier een duaal Meissnereffect te verkrijgen: een fluxbuis (string) wordt gevormd tussen de kleurelektrische ladingen. Deze string zorgt voor een lineair stijgende potentiaal tussen de kleurladingen, die aldus “confined” zijn omdat het dan intuïtief duidelijk is dat het enorm veel energie kost om ze zo ver van elkaar te verwijderen dat ze als afzonderlijke deeltjes kunnen beschouwd worden.

Abelse ijken [68] zijn een bijzonder hulpvol concept om magnetische monopolen te introduceren in QCD. Kort gezegd wordt via een Abelse ijk het niet-Abelse stuk van de ijkgroep al vastgelegd door een geschikte ijkeuze, waarbij de Abelse subgroep van de totale groep overblijft als residuele ijkgroep. In sommige punten van de ruimtetijd kan een Abelse ijk slecht gedefinieerd zijn. Dergelijke singulariteiten kunnen geïdentificeerd worden met Abelse monopolen.

Een gekend voorbeeld van een Abelse ijk is de maximaal Abelse ijk (MAG) [68, 69, 70]. Deze ijkconditie legt op dat de norm van de niet-diagonale gluonvelden³ zo klein mogelijk is, vandaar de benaming maximaal Abels. De renormaliseerbaarheid van de MAG werd bewezen in [71, 72], waarvoor er echter wel een kwartische spookinteractie in de actie moest ingevoerd worden. De residuele Abelse ijk vrijheid wordt meestal vastgelegd door een Landau ijk [49, 50].

Volgens het principe van Abelse dominantie, is het mogelijk het lage energie regime van QCD te beschrijven louter in termen van Abelse vrijheidsgraden [73]. Hiervoor bestaan er aanwijzingen bekomen via numerieke roostersimulaties [74, 75, 76]. Voor zover wij weten is er geen rechtstreeks bewijs voor de Abelse dominantie. Een aanwijzing zou kunnen zijn dat de niet-diagonale gluonen dynamisch massief zouden worden. Bij energieën lager dan deze massaschaal zouden de niet-diagonale gluonen ontkoppelen van de theorie⁴, en aldus zou men uitkomen bij een theorie met alleen lichte (massaloze), Abelse vrijheidsgraden.

Roostersimulaties van de MAG geven inderdaad aan dat een niet-diagonale gluonmassa van ongeveer 1.2 GeV optreedt, terwijl de Abelse gluonen zeer licht dan wel massaloos gedrag vertonen [49, 50].

A.2.3 Spookcondensaten en $SL(2, \mathbb{R})$ symmetrie.

Rest nog een mechanisme te vinden dat eventueel verantwoordelijk zou kunnen zijn voor de niet-diagonale massageneratie. In [77, 78, 79, 80] werd een spookcondensaat, $\langle f^{abc} \bar{c}^a c^b \rangle$, beschreven dat aan de basis zou liggen van de gezochte massa. In Hoofdstuk 3 hebben we echter aangetoond dat de alzo bekomen massa tachyonisch is (negatief massakwadraat). We stelden ook vragen bij de

³Hiermee wordt bedoeld de gluonvelden horende bij de niet-diagonale generatoren van de ijkgroep $SU(N)$.

⁴Men kan zich wel indenken dat zware vrijheidsgraden niet veel “dynamica” vertonen bij lage energie.

renormaliseerbaarheid van de potentiaal die geconstrueerd werd om het spookcondensaat te vinden. We stelden de LCO methode voor als een geschikt alternatief om toch deze spookcondensatie te onderzoeken.

In [81] werd nog een andere klasse van spookcondensaten beschreven, namelijk $\langle f^{abc} c^a c^b \rangle$ en $\langle f^{abc} \bar{c}^a \bar{c}^b \rangle$. Dit kon verwacht worden, vermits de spookcondensaten een ordeparameter vormen voor een continue $SL(2, \mathbb{R})$ symmetrie van de MAG actie. De $SL(2, \mathbb{R})$ transformaties roteren de verschillende massadi-mensie twee spookoperatoren in elkaar. Het verdient dan ook onderzocht te worden of een bepaald spookcondensaat een stabielere grondtoestand oplevert dan de andere configuraties.

Een operator, die wel aanleiding kon geven tot een niet-diagonale massa werd voorgesteld door Kondo in [83], namelijk $\mathcal{O} = \frac{1}{2} A_\mu^a A^{\mu a} + \alpha \bar{c}^a c^a$. De keuze voor deze operator kon verwacht worden vermits er een analogie bestaat tussen de MAG en een andere klasse van renormalizeerbare ijken, namelijk de niet-lineaire Curci-Ferrari ijken [84, 85], die overeenstemmen met de massaloze limiet van een renormalizeerbaar massief QCD model. De massaterm van het Curci-Ferrari model is juist $m^2 (\frac{1}{2} A_\mu^a A^{\mu a} + \alpha \bar{c}^a c^a)$. Vermelden we nog dat de Landau ijk een speciaal geval is van de Curci-Ferrari ijken.

In Hoofdstuk 5 hebben we de $SL(2, \mathbb{R})$ symmetrie verder bestudeerd. We bekwamen het al gekende resultaat dat de Landau ijk en de Curci-Ferrari ijk deze invariantie vertonen, maar daarnaast toonden we ook aan dat het mogelijk is de volledige symmetrie op een renormalizeerbare wijze op te leggen indien de MAG wordt opgelegd, aangevuld met een geschikte diagonale ijk voor de residuele Abelse ijk-vrijheid⁵.

De algebra opgebouwd door de (anti-)BRST en $SL(2, \mathbb{R})$ transformaties is gekend als de Nakanishi-Ojima (NO) algebra [86]. De spookcondensaten signaleren een gedeeltelijke breking van deze NO symmetrie.

A.2.4 De anomale dimensie van de samengestelde operator A_μ^2 in de Landau ijk.

Hoewel de lokaal samengestelde operator A_μ^2 al bestudeerd werd in de Landau ijk, gebruik makende van de LCO methode, was het enkel door expliciete, gecompliceerde meerdere-lus Feynmandiagram berekeningen dat de renormaliseerbaarheid werd aangetoond, en dit is dan maar tot op de beschouwde drie-lus orde.

In Hoofdstuk 6 hebben we gebruik makende van het algebraïsche renormalisatie formalisme kunnen aantonen dat de LCO methode in het geval van de Landau ijk renormalizeerbaar is tot op elke orde, en dit zonder al te gecompliceerde berekeningen.

De algebraïsche renormalisatie, zie [59] voor een introductie, is een krachtig formalisme dat, simplistisch voorgesteld, condities oplegt aan de meest algemene kwantumcorrecties die kunnen optreden. Deze condities volgen uit de invarianties van de actie waarmee men start en worden ugedrukt door zogenaamde Wardidentiteiten, waaraan de volledige kwantumactie moet voldoen. Over het algemeen kan de vorm van de meest algemene kwantumcorrectie bepaald worden. In het geval van Yang-Mills ijktheorieën gebeurt dit met behulp van de cohomologie geassocieerd aan de Slavnov-Taylor identiteit, die kan gezien worden als een functionele vertaling van de ijk-invariantie eens een ijk opgelegd is.

Verder gaven we ook een expliciet bewijs van het vermoeden dat de anomale dimensie van de operator A_μ^2 een lineaire combinatie is van meer elementaire renormalisatiegroepsfuncties. Gracey formuleerde in [87], op basis van de berekende waarden, de hypothese dat de renormalisatiegroepvergelijking van A_μ^2 niet onafhankelijk is van deze van het gluonveld A_μ en de β -functie, wier waarden al sinds langer gekend zijn tot op drie lussen [88]. Ons bewijs steunde op een restrictieve Ward identiteit, de *ghost Ward identity*, aanwezig in de Landau ijk [89].

⁵Indien de diagonale Landau ijk wordt opgelegd, gaat het “diagonale” gedeelte van de $SL(2, \mathbb{R})$ symmetrie al verloren bij het begin.

A.2.5 De anomale dimensie van de gluon-spook massa operator in Yang-Mills theorie en het gluon-spookcondensaat van massadimensie twee in de Curci-Ferrari ijk.

In Hoofdstuk 7 hebben we een eerste stap gezet om de LCO methode uit te breiden van de Landau ijk naar andere ijken. Een logische vraag is of er een renormalizeerbare massadimensie twee operator bestaat in andere ijken. We haalden al het voorbeeld aan van de Curci-Ferrari ijk en de bijhorende operator. We hebben aangetoond dat dit inderdaad een renormalizeerbaar model oplevert tot op alle ordes. We gaven ook een korte discussie omtrent de veralgemening van deze operator in geval van de MAG, aangevuld met de Abelse Landau ijk.

We vonden ook relaties tussen de verschillende anomale dimensies voor de Curci-Ferrari ijk en de MAG. In geval van de Curci-Ferrari ijk is de relatie echter niet zo bruikbaar aangezien ze de anomale dimensie van een andere samengestelde operator, $gf^{abc}c^ac^b$, bevat. We hebben de gevonden resultaten via expliciete berekeningen gecontroleerd tot op drie lussen. Voor de MAG is de bekomen relatie van meer praktisch nut vermits ze de anomale dimensie van de operator uitdrukt in termen van deze van het diagonaal spook en de β -functie. De details van de MAG staan verderop in Hoofdstuk 12, waar deze ijk uitvoerig werd bestudeerd.

Het is interessant op te merken dat de operator $\mathcal{O}$ niet langer kan geïdentificeerd worden met een ijk-invariante grootheid zoals het geval was voor de Landau ijk. Hoewel we dus in staat zijn om op een renormalizeerbare wijze de operator A_μ^2 uit te breiden, verliezen we de link met een ijk-invariante operator. Desalniettemin is de renormalizeerbaarheid op zich van dergelijke niet-triviale operator al vrij verrassend.

Verder vermelden we nog dat de operator $\mathcal{O}$ in de Curci-Ferrari ijk en de MAG toch nog een bepaalde eigenschap deelt met de operator A_μ^2 in de Landau ijk, namelijk de on-shell BRST invariantie⁶. Deze on-shell invariantie kan vertaald worden in een Ward identiteit, zie Hoofdstuk 12. We herinneren er hier aan dat een voldoende aantal Ward identiteiten belangrijk kan zijn om de renormalizeerbaarheid te kunnen bewijzen tot op alle ordes.

Hoofdstuk 8 is gewijd aan de uitbreiding van het LCO formalisme zelf naar de Curci-Ferrari ijk om daadwerkelijk de mogelijke condensatie van de operator $\mathcal{O}$ en de bijhorende dynamische massageneratie te onderzoeken. We dreven de berekeningen niet tot het expliciet kennen van de effectieve potentiaal, vermits de Curci-Ferrari ijk eerder diende als voorbereiding op de meer interessante maar ook meer gecompliceerde MAG. De condensatie van de relevante operator in de MAG zou een massa geven aan de niet-diagonale gluonen en kan dus dienen als een indicatie voor Abelse dominantie.

In elk geval leerde de Curci-Ferrari ijk ons al enkele zaken die het vermelden waard zijn. In de Landau ijk is er duidelijkerwijze geen ijkparameter. Daar de beschouwde operator in geval van de Curci-Ferrari ijk niet ijk-invariant is, zou men zich de vraag kunnen stellen wat het effect is op fysische, dus ijk-invariante grootheden. Voor wat betreft ons onderzoek, is de fysische grootheid waarmee we onmiddellijk geconfronteerd worden de vacuümenergie E als het minimum van de effectieve potentiaal. We hebben een argument gegeven dat E formeel niet varieert als de ijkparameter α wordt gevarieerd over de Curci-Ferrari ijken.

We herhalen hier dat de LCO methode de introductie van een nieuwe parameter ζ vereist, waarvan de waarde wordt vastgelegd door een differentiaalvergelijking in g^2 die volgt uit de renormalizatiegroepvergelijking. In geval van de Landau ijk kan die differentiaalvergelijking opgelost worden met een reeks in g^2 , gebruik makende van de Frobeniusmethode, en men bekomt een unieke ζ die multiplicatieve renormalizatie toelaat. Echter, indien er een ijkparameter α aanwezig is, worden de coëfficiënten van de g^2 -machten in de reeksontwikkeling voor ζ differentiaalvergelijkingen in α . Hierdoor duiken er arbitraire integratieconstanten op in de uiteindelijke uitdrukking voor ζ en dus ook in de effectieve

⁶On-shell wil zeggen dat de bewegingsvergelijkingen van het hulpveld b mogen gebruikt worden.

potentiaal. In Hoofdstuk 8 hebben we deze integratieconstanten gewoon nul gesteld op grond van de ijkparameteronafhankelijkheid van de vacuümenergie. Een beter uitgewerkt argument waarom deze constanten nul mogen gesteld worden, werd gevonden in Hoofdstuk 12.

A.2.6 Meer over de spookcondensatie.

In Hoofdstuk 8 hebben we verder aandacht besteed aan de spookcondensatie. De aanwezigheid van een vierpuntsinteractie tussen de spoken in de MAG diende als een hint dat een spookcondensatie kan optreden⁷, met als gevolg dat een continue $SL(2, \mathbb{R})$ symmetrie wordt gebroken. Vermits dergelijke vierpuntsinteractie en de $SL(2, \mathbb{R})$ symmetrie ook aanwezig zijn in de Curci-Ferrari ijk, kon een spookcondensatie ook in dit geval verwacht worden [90].

Door de afwezigheid van dergelijke interactie in de Landau ijk, was er absoluut geen reden om ook in een spookcondensatie in deze ijk te geloven, ondanks de $SL(2, \mathbb{R})$ symmetrie. Verrassend genoeg was het via de LCO methode toch mogelijk een spookcondensatie in de Landau ijk te bestuderen [91].

Dit werd verder onderzocht in Hoofdstuk 8. Onze studie werd uitgevoerd met de LCO methode, die zelfs toelaat om simultaan alle kanalen te beschouwen in dewelke de condensatie kan plaatsgrijpen. Deze kanalen zijn uiteraard verbonden door de $SL(2, \mathbb{R})$ rotaties. Het condensaat $\langle f^{abc} \bar{c}^a c^b \rangle$ correspondeert met het Overhauser kanaal, terwijl $\langle f^{abc} c^a c^b \rangle$ en $\langle f^{abc} \bar{c}^a \bar{c}^b \rangle$ met het BCS kanaal overeenkomen. Wij hebben deze namen gekozen naar analogie met het Overhauser en BCS effect uit de theorie van de gewone supergeleiding [92, 93, 94, 95].

We kwamen tot de conclusie dat de gelijktijdige invoering van de spookoperatoren twee LCO parameters vereiste, die evenwel evenredig moeten zijn door de symmetrie-eigenschappen van het model. We gaven daarnaast ook een diagrammatische uitleg waarom deze twee parameters evenredig zijn.

We schreven de één-lus effectieve potentiaal neer, die uitgedrukt kon worden in functie van twee $SL(2, \mathbb{R})$ invarianten. Daar de expliciete berekening van de potentiaal niet eenvoudig zou zijn, slaagden we erin de inspanning enigszins te verlichten door aan te tonen dat één van die twee invarianten noodzakelijkerwijze nul is voor de vacuümconfiguratie, tenminste op het één-lus niveau. Uiteindelijk vonden we een lagere vacuümenergie door het spookcondensaat.

Een vacuüm kiezen volgens een bepaalde $SL(2, \mathbb{R})$ -richting impliceert de breking van deze symmetrie. Elke richting is echter equivalent daar ze dezelfde energie opleveren, alzo wordt er geen onderscheid gemaakt tussen b.v. het Overhauser of BCS kanaal.

Vermits de BRST- en spookladingssymmetrie een belangrijke rol spelen in perturbatieve veldentheorie voor QCD [177], hebben we nog enige aandacht aan vacua andere dan het Overhauser besteed. Om het probleem te situeren merken we op dat in het BCS vacuüm $\langle Q_{\text{BRST}}(\dots) \rangle \propto \langle f^{abc} c^a c^b \rangle \neq 0$, dus de BRST-symmetrie is gebroken, net als de spookladingssymmetrie vermits de operator $f^{abc} c^a c^b$ ook een spooklading draagt. We motiveerden dat er in dergelijke vacua toch een concept van spook- en nilpotente BRST-lading bestaat, zijnde de “geroteerde” versie van de originele spook- en BRST-lading.

Vervolgens beperkten we ons tot het Overhauser vacuum. Daar de gebroken $SL(2, \mathbb{R})$ symmetrie continu was, voorspelt de stelling van Goldstone de aanwezigheid van massaloze bosonen. We argumenteerden echter dat deze tot een onfysische sector van de theorie behoren. We maakten hiervoor gebruik van een argument van Kugo en Ojima [177].

Een gevolg van de spookcondensatie is dat het gedrag van de spoken in functie van de kleur veranderd is, vermits de globale $SU(N)$ kleursymmetrie inderdaad gebroken is door een kleurgeladen condensaat als $\langle f^{abc} \bar{c}^a c^b \rangle$. Wanneer we de uitdrukking voor de globale kleurlading Q_c voor de ongebroken symmetrie echter in beschouwing nemen, dan reduceert deze zich tot een BRST exacte uitdrukking onder bepaalde

⁷De ontbinding van een vierpuntsinteractie via een hulpveld is een gekende truc om een potentiaal voor dit hulpveld, dat proportioneel is met de “wortel” van de interactie, te vinden.

voorwaarden⁸, i.e. tot iets van de vorm $Q_{\text{BRST}}(\dots)$. Hierdoor levert de actie van Q_c op fysische toestanden nul op⁹, ook in de vacua met spookcondensatie, zelfs al genereert Q_c geen globale symmetrie van de actie meer.

A.2.7 Massadimensie twee gluoncondensaat in de lineaire, covariante ijken.

We hebben tot nu al de condensatie van A_μ^2 bestudeerd in de Landau ijk en de veralgemening ervan in de Curci-Ferrari ijken, waarnaast we ook al de renormaliseerbaarheid in de MAG hebben nagegaan. Telkenmale bleek de relevante operator on-shell BRST invariant te zijn. Men zou dus kunnen denken dat deze invariantie een conditio sine qua non is.

Er bestaat echter nog een klasse van covariante ijken, die misschien wel het best gekend is, namelijk de lineaire, covariante ijken met als typevoorbeelden de Landau en Feynman ijk. Wij stelden de massa dimensie twee operator A_μ^2 voor als een kandidaat uitbreiding binnen deze klasse van ijken. We vermelden dat A_μ^2 in dit geval zelfs niet on-shell BRST invariant is. Desniettemin waren we toch in staat de renormaliseerbaarheid te bewijzen tot op alle ordes, we berekenden de anomale dimensie tot op twee lussen en de één-lus effectieve potentiaal, en we vonden ook hier een niet-verdwijnd condensaat $\langle A_\mu^2 \rangle$.

Hoewel we erin slaagden ons bewijs van de onafhankelijkheid van de ijkparameter voor wat betreft de vacuümenergie E te herhalen voor deze ijken, waren de expliciete resultaten voor E voor verschillende keuzen van de ijkparameter niet in overeenstemming. Voor zover wij weten, is deze inconsistentie tussen theorie en praktijk een gevolg van een opmenging van verschillende ordes van perturbatietheorie wanneer er slechts tot op een eindige orde nauwkeurig wordt gewerkt, en zou dit probleem zich niet voordoen op oneindige orde. We stelden een middel voor om het probleem te reduceren op eindige orde, en hebben dit toegepast op de berekening van de vacuümenergie en nulde orde massa.

De details staan uitgewerkt in de Hoofdstukken 10 en 11.

A.2.8 Niet-diagonale massageneratie in de maximaal Abelse ijk.

Met de kennis die we tot hier al opgedaan hebben door het onderzoek van eenvoudiger ijken, zijn we voldoende gewapend om uiteindelijk ook de MAG, aangevuld met de Abelse Landau ijk, te analyseren. De resultaten hiervan staan vermeld in Hoofdstuk 12.

We verifieerden de renormaliseerbaarheid, we bepaalden de één-lus effectieve potentiaal en vonden een niet-triviale nulde orde massa voor de niet-diagonale gluonen, terwijl de diagonale gluonen massaloos bleven. Deze bevindingen zijn in kwalitatieve overeenkomst met de roostersimulaties van de MAG [49, 50].

Een bijkomend probleem waarmee we geconfronteerd werden is het volgende: elke eerder onderzochte ijk bezat de Landau ijk als limietgeval. Hierdoor waren we steeds in staat om een verband te leggen tussen de lagere vacuümenergie door de respectievelijke massadimensie twee condensaten in elk van deze ijken. De MAG heeft echter duidelijk de Landau ijk niet als speciaal geval, en aldus was het onduidelijk of er ook een verband bestond tussen het massadimensie twee condensaat in deze ijk en $\langle A_\mu^2 \rangle$ in de Landau ijk. We losten dit vraagstuk op door een ijk te construeren die niet alleen interpoleerde tussen de Landau ijk en de MAG, maar ook toeliet om een interpolerende massadimensie twee operator in te voeren, en dit alles op een renormaliseerbare wijze. Er werd een extra ijkparameter ingevoerd, maar we toonden aan dat deze niet onafhankelijk renormalizeert. We legden dus een verband tussen de MAG en de Landau ijk, en bijgevolg ook met de lineaire, covariante en de Curci-Ferrari ijken.

⁸Voorwaarden die trouwens dezelfde zijn ingeval er geen sprake is van spookcondensaten.

⁹Fysische toestanden $|fys\rangle$ worden gegeven door de BRST-gesloten toestanden, i.e. $Q_{\text{BRST}}|fys\rangle = 0$, die niet BRST-exact zijn, i.e. $|fys\rangle \neq Q_{\text{BRST}}|\dots\rangle$.

A.2.9 Keuze van het renormalizatieschema.

We herhalen dat renormalizatie erin bestaat om via tegentermen oneindigheden, die optreden bij het uitrekenen van Feynmandiagrammen, te elimineren. Logischerwijs is dit gedefinieerd op eindige tegentermen na: men kan een oneindigheid " ∞_1 " steeds wegwerken door een tegenterm " $-\infty_1 + c$ " met c een willekeurige constante. De waarde van deze constanten wordt vastgelegd door een keuze van een renormalizatieschema.

Meestal wordt gebruik gemaakt van het $\overline{\text{MS}}$ schema¹⁰, vermits dit een zeer efficiënt schema is voor wat betreft de berekeningen.

Over het algemeen hebben we gevonden dat de bekomen expansieparameter voor de perturbatiereeks van de effectieve potentiaal voldoende klein is om te kunnen spreken van kwalitatief aanvaardbare resultaten. Hiermee bedoelen we dat we geen compleet verschillende waarden verwachten te vinden op hogere orde of in een andere renormalizatieschema voor b.v. de nulde orde massaparameter.

Wanneer het Gross-Neveu model bestudeerd wordt, dan kan een aanpak via de LCO of $2PPI$ methode alle niet-perturbatieve effecten, die bijdragen tot de fermionmassa, omvatten, daar de niet-perturbatieve sector van dit model veel minder rijk is dan deze van QCD. In het Gross-Neveu geval kan het dan interessant zijn om de afhankelijkheid van het renormalizatieschema zo klein mogelijk te maken, om zo tot betere numerieke resultaten te komen. Essentieel zijn de technieken die daarvoor ter beschikking zijn gebaseerd op een vervanging van de relevante grootheden door hun renormalizatieschema- en schaalonafhankelijke tegenhangers, zie b.v. [104, 105, 24, 106]

In het geval van ijktheorieën is dergelijke optimalizatie soms eerder tijdrovend en minder nuttig, vermits er vele andere bronnen van niet-perturbatieve effecten zijn. Ons werk is eerder bedoeld om een idee te krijgen over de grootte-orde van b.v. de nulde orde massa ten gevolge van een massadimensie twee condensaat gebruik makende van perturbatieve berekeningen. We vonden zo, in alle beschouwde iken, een waarde voor de massa van enkele honderden MeV in het $\overline{\text{MS}}$ schema, met een expansieparameter die niet al te groot was. Als de $\overline{\text{MS}}$ koppeling te groot zou uitvallen, dan kan het wel de moeite zijn om een analyse te maken van wat er gebeurt in andere schema's.

A.2.10 Het Gribov probleem: ijkkopieën.

Wanneer een ijktheorie gekwantiseerd wordt door middel van het padintegraal formalisme, moet een ijk gekozen worden om te verzekeren dat slechts één representant van elke ijkequivalentieklasse zou bijdragen tot de padintegraal. Voor de iken die we tot nu beschouwd hebben, werd er altijd van uit gegaan dat de gekozen ijkvoorwaarde een unieke representant selecteert.

Gribov toonde aan dat de Landau ijk de ijkvrijheid niet volledig vastlegde [107]: er bestaan meerdere ijkequivalente configuraties die allen voldoen aan $\partial_\mu A_\mu = 0$. Het Gribov probleem is niet beperkt tot de Landau ijk, maar een generiek probleem van niet-Abelse ijktheorieën [108]. Het bestaan van ijkkopieën impliceert dat het integratiedomein van de padintegraal verder ingeperkt moet worden. De vraag is uiteraard of dit kan vertaald worden in een bruikbaar formalisme. Het bleek dat dit mogelijk is in de Landau ijk, en in zekere mate in de niet-covariante Coulomb ijk [107], maar buiten deze iken is er niet veel bekend over hoe het Gribov probleem kan vermeden worden.

Gribov toonde aan dat het bestaan van ijkkopieën op het infinitesimale niveau equivalent is met het bestaan van nulmodes van de Faddeev-Popov operator, $-\partial_\mu (\partial_\mu \delta^{ab} + g f^{acb} A_\mu^c)$. Hierdoor geleid stelde hij voor om de padintegratie te beperken tot het zogenaamde Gribovgebied, waar de eigenwaarden van de Faddeev-Popov operator¹¹ positief zijn. Op de rand van dit gebied, ook wel de (eerste) Gribov horizon genoemd, duikt de eerste verdwijnende eigenwaarde op.

¹⁰ $\overline{\text{MS}}$ =modified minimal subtraction.

¹¹Deze operator is Hermitisch in de Landau ijk, en bezit aldus reële eigenwaarden.

Uiteraard moet er bewezen worden dat elke ijkconfiguratie een representant bezit binnen het Gribovgebied. Dit werd door Gribov bewezen voor configuraties die “dicht” tegen de buitenkant van het Gribovgebied liggen [107], en later werd een bewijs voor willekeurige configuratie gegeven in [109]. Verder is het ook niet zeker dat het Gribovgebied zelf geen ijkkopieën bevat. Er werd inderdaad aangetoond dat er kopieën bestaan binnen dat gebied. Het kleinere gebied dat wel vrij is van kopieën wordt de “fundamental modular region” (FMR) genoemd. Er werd geargumenteed dat verwachtingswaarden berekend binnen de FMR samenvallen met deze berekend binnen het Gribovgebied [113].

Terugkerend op het probleem van de restrictie van de padintegraal, vermelden we dat Gribov een laagste orde approximatie uitvoerde in [107], maar alle essentiële kenmerken van de restrictie kwamen al naar voor: er moet een dimensievolle parameter in de theorie ingevoerd worden, waarvan de waarde door een gapvergelijking bepaald wordt. Het voornaamste gevolg is een infrarood onderdrukking, respectievelijk versterking, van de gluon-, respectievelijk spookpropagator. Dergelijk gedrag is in overeenstemming met de resultaten bekomen door roostersimulaties [114, 45, 48, 115, 116, 117, 118, 119, 120] of met oplossingen van de Schwinger-Dyson vergelijkingen [121, 122, 123, 20, 124, 125, 126]. We kunnen ook nog aanhalen dat Kugo en Ojima een criterium voor confinement opstelden in [177]. Een voldoende voorwaarde in de Landau ijk voor dit criterium is een spookpropagator die sterker dan kwadratisch singulier is. Dit ondersteunt het geloof dat het Gribovprobleem, en meerbepaald een oplossing ervoor, belangrijk kan zijn voor wat betreft de infrarood dynamica van ijktheorieën.

De mogelijkheid om een betere padintegraal te construeren in de Landau ijk door het werk van Gribov, gaf de motivatie voor een ander deel van ons onderzoek: de vraag rees wat de invloed kon zijn op de condensatie van A_μ^2 .

In Hoofdstuk 13 hebben we van dichtbij Gribov's work gevolgd indien het mogelijk bestaan van een condensaat $\langle A_\mu^2 \rangle$ in rekening wordt gebracht. We kwamen tot een gelijkaardige conclusie als voorheen: een infrarood onderdrukking/versterking van de gluon-/spookpropagator.

Zwanziger stelde in [127, 128] een lokale Lagrangiaan op die toelaat om de restrictie tot het Gribovgebied orde per orde te implementeren en dit op een renormalizeerbare wijze. Zo was men ook zeker dat de restrictie tot het Gribovgebied de renormalizeerbaarheid niet in het gedrang brengt. De restrictie wordt expliciet geïmplementeerd door de horizon voorwaarde. Deze voorwaarde is niets anders dan de gapvergelijking voor de massavolle Gribov parameter, en kan afgeleid worden uit de effectieve actie berekend met de Zwanziger actie.

Hoofdstuk 13 behandelt kort de algebraïsche setup om de renormalizeerbaarheid tot op alle ordes te bewijzen van de gelokaliseerde versie van Gribov's originele approximatie van de restrictie.

In Hoofdstuk 14 hebben we de renormalizeerbaarheid tot op alle ordes bewezen wanneer de Zwanziger Lagrangiaan aangevuld wordt met de operator A_μ^2 volgens de LCO methode. Een interessante eigenschap van deze actie is dat er geen nieuwe renormalisatiefactoren nodig zijn. Dit is een gevolg van de rijke symmetrie-inhoud van deze actie, met of zonder A_μ^2 erbij. In het bijzonder bleek wat men de LCO parameter voor de Gribov parameter zou kunnen noemen, exact “1” te zijn. Dit is een belangrijke eigenschap om de infrarood versterking van de spookpropagator te kunnen aantonen. We gaven ook enkele mogelijke implicaties van de aanwezigheid van de Gribov parameter.

Uitgaande van de effectieve actie, bekwamen we dan twee gapvergelijkingen, één voor de Gribovparameter en één voor de massaparameter geassocieerd aan $\langle A_\mu^2 \rangle$. Dit liet dan toe om de effecten na te gaan van een condensatie van A_μ^2 op de Gribovparameter en omgekeerd. We bekwamen expliciete waarden in het $\overline{\text{MS}}$ schema voor beide massavolle grootheden, maar de expansieparameter bleek te groot te zijn. Daardoor voerden we een optimalisatie uit van de perturbatiereeks om de afhankelijkheid van het renormalisatieschema te minimaliseren. We onderzochten tevens de eigenschappen van de vacuümenergie, met of zonder aansluiting van A_μ^2 . We toonden ondermeer aan dat in het originele Gribov-Zwanziger model, dus zonder aansluiting van A_μ^2 , de vacuümenergie noodzakelijk positief is in de één-lusbenadering, voor eender welke keuze van renormalisatieschema of -schaal. Verder toonden

we aan dat voor de oplossing in het geval A_μ^2 wel aangesloten wordt, in het $\overline{\text{MS}}$ schema en in de één-lus benadering, zeker geldt dat $\langle A_\mu^2 \rangle > 0$. Wanneer de horizon voorwaarde niet wordt opgelegd, is de schatting voor $\langle A_\mu^2 \rangle$ bekomen via de LCO methode negatief, zie [42] of Hoofdstuk 11.

De bevinding dat de vacuümenergie positief zou zijn is eerder ongewenst. Door de dilatatie-anomalie¹² staat de vacuümenergie rechtstreeks in verband met de waarde van het ijk-invariante gluoncondensaat $\langle F_{\mu\nu}^2 \rangle$, dat een voorname rol speelt in de QCD-fenomenologie [13, 14]. In het bijzonder impliceert een positieve vacuümenergie een negatieve waarde voor $\langle \frac{g^2}{4\pi^2} F_{\mu\nu}^2 \rangle$. Dit is in tegenspraak met wat gevonden wordt. De fenomenologische waarden voor dit condensaat voor “real life” QCD (met quarks erbij) zijn positief [13, 14], terwijl ook voor QCD zonder quarks de schattingen, ditmaal bekomen via roostersimulaties [129], positief blijken te zijn. Het lijkt dus aangeraden om toch tenminste een negatieve vacuümenergie te vinden. De aansluiting van de operator A_μ^2 maakt het mogelijk om een negatieve vacuümenergie te vinden, evenwel waren we niet in staat om hierover tot een sluitende conclusie te komen in de beschouwde één-lus benadering, vermits de afhankelijkheid van het renormalizatieschema pathologisch sterk was.

A.2.11 Drie-dimensionele ijktheorieën.

In Hoofdstuk 16 namen we drie-dimensionele ijktheorieën onder de loep. Deze zijn fysisch relevant als de hoge temperatuurs-limiet van van hun vier-dimensionele tegenhanger [130]. Tevens zijn ze interessant voor roostersimulaties door de kleinere rekentijd [131, 115]. In eerste instantie ging onze interesse uit naar de vraag of er ook in drie-dimensionele Yang-Mills theorieën een condensatie van A_μ^2 mogelijk zou zijn, met de bijhorende dynamische generatie van een massaparameter.

De situatie is echter iets ingewikkelder dan een eenvoudige aanpassing van het reeds bestaande vier-dimensionele onderzoek. In drie dimensies zijn Yang-Mills theorieën superrenormalizeerbaar daar de koppelingsconstante g^2 zelf dimensievol is. Superrenormalizeerbare modellen zijn méér dan renormalizeerbaar in de zin dat er maar enkele “basis” ultraviolet divergente diagrammen zijn. Daarentegen treden er ernstige infrarood problemen op in het massaloze geval [132]. Dit kan eenvoudig begrepen worden aan de hand van volgende denkwijze: onderstel dat een werkzame doorsnede wordt berekend met een zekere externe schaal Q . Vermits g^2 zelf massadimensie 1 draagt, moet de perturbatieve ontwikkeling a fortiori gebeuren in $\frac{g^2}{Q}$. Het moge duidelijk zijn dat dit problemen oplevert wanneer de infrarood limit $Q \approx 0$ wordt beschouwd. Als er echter een massa zou gegenereerd worden, dan zou de infrarood sector gevrijwaard kunnen worden van problemen, vandaar de interesse in het vinden van een dynamische massa, zie b.v. [133, 134, 135, 136].

We hebben al een eerste stap gezet in de richting van het uitbreiden van de LCO methode naar drie dimensies door aan te tonen dat, wanneer de Landau ijk wordt gekozen, de insertie van de operator A_μ^2 via een massaterm $\frac{1}{2}m^2 A_\mu^2$ een renormalizeerbaar model oplevert, terwijl we, enigszins verrassend, ook vonden dat de relatie voor de anomale dimensie van de operator A_μ^2 in vier dimensies bewaard blijft in drie dimensies. Deze relatie werd ook numeriek nagegaan gebruik makende van de “large N_f ” methode, waarbij N_f het aantal quarks voorstelt. We gaven ook een korte analyse van de Curci-Ferrari gauge. Laat ons eindigen met de opmerking dat de drie-dimensionele theorie eindig bleek te zijn in eerste orde. We hebben nog geen informatie over wat er gebeurt in hogere orde, of over een mogelijke veralgemening van de LCO methode zelf om de condensatie van A_μ^2 te onderzoeken.

¹²De dilatatie-anomalie, ook wel gekend als de “trace anomaly”, vindt zijn ontstaan in het feit dat er op het klassieke niveau geen, maar na renormalizatie wel een massaschaal in de QCD-actie voorkomt. De dilatatie-invariantie is dus spontaan gebroken. We merken op dat het Gribov-Zwanziger formalisme sowieso een massavolle Gribovparameter γ introduceert in de QCD-actie, maar de trace anomalie blijft geldig, zoals rigoreus werd bewezen in [128].

A.3 Besluit.

Tot slot geven we nog enkele algemene beschouwingen tot besluit van het voorgestelde werk.

De aandacht was vooral gericht op de dynamische generatie van een massa dimensie twee gluon condensaat. Dit werd mogelijk gemaakt door het LCO formalisme, dat toelaat om een zinvolle effectieve potentiaal voor lokaal samengestelde operatoren te berekenen via perturbatie-theorie. Men kan also enige niet-perturbatieve informatie bekomen in de vorm van een condensaat, wier formatie optreedt omdat het de vacuümenergie verlaagt.

We hebben dit onderzoek, dat begon in de Landau ijk [42], uitgebreid naar verschillende andere klassen van ijken: de lineair covariante, de Curci-Ferrari en de maximaal Abelse ijken. Deze laatste ijk is belangrijk voor het duale supergeleider model van het QCD vacuum bij lagere energie, als een mogelijke verklaring van confinement.

Dit werk kan gezien worden als een indicatie dat massa dimensie twee condensaten relevantie hebben voor ijktheorieën, vermits een gelijkaardig fenomeen optreedt in een groot aantal ijken. Desalniettemin is ons begrip van deze condensaten verre van compleet.

- Hoewel de operator A_μ^2 een ijk-invariante mening kan gegeven worden in de Landau ijk, is zo een interpretatie niet langer mogelijk in de andere ijken. We kunnen niets zeggen over een ijk-invariante betekenis van onze resultaten.

Er is wel een bewijs verschenen dat $\langle A_\mu^2 \rangle$ ijk-invariant zou zijn, het ligt echter buiten onze mogelijkheden om de juistheid van dit bewijs in te schatten [249].

Indien Zwanziger juist is omtrent de gelijkheid van verwachtingswaarden berekend binnen de FMR of het Gribov gebied [113], dan is de grootheid $\langle A_\mu^2 \rangle$ zoals berekend in Hoofdstuk 15 een ijk-invariante grootheid. Dit is echter een vrij academisch gegeven, vermits het onduidelijk blijft hoe deze grootheid te berekenen in een andere ijk dan de Landau ijk.

- In onze opinie is het al vrij opmerkelijk dat er een operator van massa dimensie twee kan gevonden worden die renormaliseerbaar is tot op elke orde van perturbatietheorie in de verschillende onderzochte ijken, soms zelfs met extra eigenschappen zoals voor de anomale dimensie in de Landau ijk of MAG.

Er bestaan nog andere ijken, maar deze vertonen een gans gamma van andere problemen zoals het niet covariant zijn, problemen met niet-lokale tegentermen, alleen numeriek te implementeren,... Deze zijn dan waarschijnlijk een stuk moeilijker, zoniet onmogelijk om te onderzoeken voor wat betreft een massa dimensie twee condensaat op de manier zoals wij het gedaan hebben. We vermelden wel nog dat het condensaat $\langle A_i^2 \rangle$, $i = 1, \dots, 3$, werd ingevoerd voor een constructie van glueballs in de Coulomb ijk [200].

- Enkele werken hebben het bestaan van massa dimensie twee condensaten in vraag gesteld, op basis van het niet ijk-invariant zijn van de operator. Dergelijke operatoren kunnen niet corresponderen met fysisch waarneembare grootheden en zijn aldus irrelevant [250, 251]. Evenzo werd de renormaliseerbaarheid in vraag gesteld, vermits deze steeds voor een specifieke operator in een specifieke ijk werd bewezen. De operator A_μ^2 is bijvoorbeeld niet renormaliseerbaar in de MAG voor willekeurige ijkparameter.

Ons standpunt is steeds dat het onderzoek start bij de klassieke Yang-Mills actie. Bij kwantisatie moet een ijk opgelegd worden. Het lijkt ons logisch dat de keuze van de ijk het gedrag zal beïnvloeden van de theorie op het kwantumniveau. Het is denkbaar dat in een bepaalde ijk een bepaalde operator renormaliseerbaar is en kan condenseren. Uiteraard zou op het einde de ijk-invariantie van de theorie terug naar voren moeten komen, in het bijzonder wanneer men fysische

grootheden gaat beschouwen zoals het deeltjesspectrum. Maar wanneer men geïnteresseerd is in het gedrag van een ijkvariante grootheid zoals de gluonpropagator, dan sluit niets de aanwezigheid van ijkvariante condensaten uit [229].

Eén van de voornaamste gevolgen van het gluoncondensaat is inderdaad het optreden van een expliciete massaparameter in de gluonpropagator. Het optreden van (een) massaparameter(s) in de gluonpropagator werd ook gevonden via roostersimulaties [43, 44, 45, 46, 47, 48, 49, 50], oplossingen van de Schwinger-Dyson vergelijkingen [51, 52], eerder fenomenologische studies [53, 54, 55, 56, 57], ... Ook andere gevolgen van ons onderzoek zijn in kwalitatieve overeenkomst met het resultaat van andere studies, we vermelden b.v. de generatie van een niet-diagonale en afwezigheid van een diagonale gluonmassa in de MAG [49, 50] of de consequenties van de restrictie tot het Gribovgebied [116, 117, 118, 119, 120].

- Het is mogelijk dat de massa dimensie twee condensaten een ijk invariant stuk bevatten, zie de speculatie hieromtrent voor $\langle A_\mu^2 \rangle$ in de Landau ijk [34]. Dit blijft evenwel speculatie, en we zullen er dan verder ook niet veel woorden meer aan vuil maken. We vermelden enkel nog dat we gemotiveerd hebben dat tenminste de lagere vacuümenergie door de respectievelijke condensaten in verschillende ijken formeel dezelfde zou moeten zijn in ons formalisme.

In onze opinie betekent ijk invariantie niet dat deze zomaar ingeroepen kan worden om onderzoek in een bepaalde richting uit te sluiten omdat het ijkvariante grootheden beschouwt. We haalden al aan dat het perfect toegelaten is dat ijkvariante grootheden optreden in bijvoorbeeld de gluonpropagator. Uiteraard zijn we akkoord dat ijk invariantie een sleutelbegrip is, maar als men ijk invariantie in een te strikte zin wil opleggen, dan kan er misschien niet veel buiten perturbatietheorie gedaan worden op een min of meer analytische wijze. Het is al leuk indien men iets meer kan zeggen over bijvoorbeeld de gluon- of spookpropagator voorbij het louter perturbatieve, denken we bijvoorbeeld maar aan het Kugo-Ojima confinement criterium.

Een goed voorbeeld om de illustreren dat strikte ijk invariantie opleggen misschien eerder contraproductief werkt, is het Gribovprobleem. Het zou een ongelooflijke uitdaging zijn een oplossing voor dit probleem te vinden dat ijk invariant is. Door de speciale eigenschappen van de Landau ijk kan er iets gedaan worden, dus waarom er dan geen gebruik van maken? Het is een uitgelezen kans om dan tenminste toch in deze ijk iets meer te kunnen zeggen over de infrarood dynamica van QCD.

- Ons werk is beperkt gebleven tot een perturbatieve analyse van niet-perturbatieve effecten. Hoogstwaarschijnlijk zijn er andere (meer) belangrijke oorzaken van niet-perturbatieve effecten, bijvoorbeeld komende van de topologische inhoud van QCD.
- Meestal hebben we ons beperkt tot een één-lus benadering. We merken hier nog op dat een nadeel van LCO formalisme is dat men de $(n + 1)$ -lus waarden van o.a. de anomale dimensies moet kennen voor de n -lus effectieve potentiaal. Daarnaast compliceert ook de aanwezigheid van een ijkparameter de berekeningen. Hoewel het eigenlijk aangeraden is om expliciet de twee-lus benadering van de effectieve potentiaal te kennen voor een algemene keuze van lineaire covariante ijk, worden de berekeningen zeer ingewikkeld. We verwijzen bijvoorbeeld naar de uitdrukking (11.32) voor $\zeta_1(\alpha)$, zelf optredend in de differentiaalvergelijking voor $\zeta_2(\alpha)$. Wanneer er dan ook nog quarks beschouwd worden, worden de zaken verder gecompliceerd.
- Betreffende de dynamische generatie van een gluon massaparameter zouden we willen opmerken dat dit geenszins impliceert dat we het probleem van de “mass gap” in QCD hebben opgelost. Daarvoor zouden we moeten aantonen dat het fysisch spectrum, te construeren uit de QCD-actie, enkel massieve deeltjes bevat. Dit zou het dan ook aantonen van confinement nodig maken, duidelijkerwijze gaat dit ons petje te boven.

Bovengaan de kwestie staat in verband met de unitariteit. Massieve Yang-Mills modellen zijn niet unitair [84, 162, 182]. We merken echter op dat wij niet een (naakte) massieve, maar een dynamisch massieve Yang-Mills theorie opstellen. Maar uiteindelijk moet unitariteit bewezen worden op het niveau van de fysische excitaties, waarvan vooralsnog niemand weet hoe deze te construeren, zelfs al gaat het uit van de gebruikelijke massaloze actie. Wij beschouwen de QCD-actie steeds geschreven in termen van de elementaire, maar onfysische quarks en gluonen. Dit is complementair met het feit dat een pool in de gluonpropagator, hetgeen inderdaad optreedt binnen het LCO formalisme [214, 252], niet noodzakelijk een fysisch massief deeltje (gluon) impliceert [253].

Bij zeer hoge energieën kunnen quarks en gluonen als vrije deeltjes beschouwd worden door de asymptotische vrijheid. Bij deze energieën is perturbatietheorie echter geldig, en zijn niet-perturbatieve effecten afwezig. In dit geval bestaat het Yang-Mills spectrum uit de twee transversale gluonpolarizaties, en er is unitariteit, zie b.v. [177].

Naast de condensatie van massa dimensie twee gluon operatoren, hebben we ook verdere aandacht besteed aan de condensatie van massa dimensie twee spook operatoren, die een ordeparameter zijn van een continue $SL(2, \mathbb{R})$ symmetrie die aanwezig is in sommige van de ijken die we onderzocht hebben. Als deze condensaten optreden, zullen ze ook een verdere invloed uitoefenen op de propagatoren.

Is dit een werk een afgerond geheel? Nodeloos te zeggen dat het dit niet is. Verscheidene zaken kunnen verder onderzocht worden. We noemen er een paar.

Het zou fijn zijn om aanwijzingen te krijgen vanuit andere hoek (roostersimulaties) dat een massa dimensie twee condensaat bestaat in andere ijken. In de Landau ijk werden de $\frac{1}{Q^2}$ correcties gerelateerd aan $\langle A_\mu^2 \rangle$ gevonden wanneer de koppelingsconstante g^2 werd beschouwd, berekend via verschillende interactievertices [37, 38, 39, 257]. Men zou er kunnen aan denken om $\frac{1}{Q^2}$ correcties te zoeken voor wat betreft de koppelingsconstante in andere ijken die op een rooster kunnen gesimuleerd worden, b.v. de MAG.

Het zou verder ook fijn zijn om andere indicaties te verkrijgen dat er zoiets als een spookcondensaat bestaat in de Landau ijk. Wat zou de invloed van een spook- op het gluoncondensaat en omgekeerd zijn? Wat is het effect op de propagatoren? Lopend onderzoek lijkt uit te wijzen dat het Overhauser spookcondensaat zorgt voor een verschil tussen de diagonale en niet-diagonale massa, waarbij de diagonale kleiner uitvalt dan de niet-diagonale [254]. Misschien kan dit gezien worden als indicatie voor een soort van Abelse dominantie in de Landau ijk [255, 256].

Wat gebeurt er indien de Gribov restrictie geïmplementeerd wordt op hogere orde? Is het mogelijk een zinvolle, negatieve waarde te vinden voor de vacuümenergie? Wat is de mogelijke rol van het condensaat $\langle A_\mu^2 \rangle$? Als men zou vinden dat de vacuümenergie positief blijft op hogere orde, dan zou dat een indicatie kunnen zijn dat een orde-per-orde implementatie van de horizon conditie verre van "voldoende" is om een behoorlijke infrarood beschrijving van QCD te geven.

Kunnen we het LCO formalisme uitbreiden naar drie dimensies? Daar de massieve ijktheorie eindig bleek te zijn in de één-lus benadering, zou het interessant zijn na te gaan wat er gebeurt in hogere orde in deze superrenormalizeerbare theorie.

Bestaat er een uitbreiding van A_μ^2 voor supersymmetrische Yang-Mills ijktheorieën, gebruik makende van een veralgemeende Landau ijk [258]? Een voordeel van het beschouwen van supersymmetrische ijktheorieën is dat er exacte resultaten kunnen afgeleid worden, gebruik makende van holomorfie [259], en zo kunnen niet-perturbatieve mechanismen getoetst worden.

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ENDE

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