

# Dirichlet type problems in Hermitian Clifford analysis

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## Abstract

Solvability conditions for some Dirichlet type boundary value problems in the framework of Hermitian Clifford analysis are established.

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## 1 Introduction

Euclidean Clifford analysis is a comprehensive generalization to higher dimension of the theory of holomorphic functions in the complex plane. More recently Hermitian Clifford analysis has emerged as a refinement of this orthogonally invariant function theory. It is centred around the concept of so-called  $h$ -monogenic functions defined in Euclidean space of even dimension, i.e. null solutions of two first-order vector-valued differential operators, called Hermitian Dirac operators, which are invariant under the action of the unitary group. In order to obtain traditional function theoretic results, such as the Cauchy integral formula, the theory of Hermitian monogenicity had to be transferred to the framework of circulant  $(2 \times 2)$  matrix functions, see e.g. [1, 2, 3]. For a thorough study of the  $h$ -monogenic function theory, we refer the reader to [5, 4, 6, 12, 13] and the references therein.

In this paper we deal with two Dirichlet type boundary value problems involving  $\mathbf{H}$ -monogenic matrix functions in  $\mathbb{R}^{2n}$ , i.e. null solutions of a matrix Dirac operator  $\mathcal{D}$  which will be defined below.

In Section 3 we develop a Hermitian Clifford operator calculus involving the Cauchy, the Teodorescu and the Hilbert transform. Section 4 studies the cited boundary value problems. To make the paper self-contained the basics of Hermitian Clifford analysis are recalled in Section 2.

## 2 H-monogenic functions theory: the basics

First we recall some definitions and basic properties of a Clifford algebra.

Let  $(e_1, \dots, e_m)$  be an orthonormal basis of the Euclidean space  $\mathbb{R}^m$ . Let  $\mathbb{C}_m$  be the complex Clifford algebra constructed over  $\mathbb{R}^m$ . The basic multiplication rules in  $\mathbb{C}_m$  are governed by

$$e_i e_j + e_j e_i = -2\delta_{ij}, \quad i, j = 1, \dots, m.$$

Any element  $a \in \mathbb{C}_m$  may thus be written as  $a = \sum_A a_A e_A$ ,  $a_A \in \mathbb{C}$ , where  $e_A = e_{j_1} \dots e_{j_k}$  with  $A = \{j_1, \dots, j_k\} \subset \{1, \dots, m\}$  is such that  $j_1 < \dots < j_k$ . Additionally, one puts  $e_\emptyset = 1$ . The conjugation in  $\mathbb{C}_m$  is given by  $a^\dagger = \sum_A a_A^c \bar{e}_A$ , where  $\cdot^c$  denotes the complex conjugation and  $\bar{\cdot}$  is the traditional Clifford conjugation, being the anti-involution for which  $\bar{e}_j = -e_j, j = 1, \dots, m$ .

In the even dimensional case  $m = 2n$  the real Clifford vector  $\underline{X}$  and its twisted counterpart  $\underline{X|}$  are given by

$$\underline{X} = \sum_{j=1}^n (e_{2j-1} x_{2j-1} + e_{2j} x_{2j}), \quad \underline{X|} = \sum_{j=1}^n (e_{2j-1} x_{2j} - e_{2j} x_{2j-1}).$$

Their respective Fourier duals are the so-called Euclidean Dirac operators,  $\partial_{\underline{X}}$  and its twisted version  $\partial_{\underline{X|}}$ , given explicitly by

$$\partial_{\underline{X}} = \sum_{j=1}^n (e_{2j-1} \partial_{x_{2j-1}} + e_{2j} \partial_{x_{2j}}), \quad \partial_{\underline{X|}} = \sum_{j=1}^n (e_{2j-1} \partial_{x_{2j}} - e_{2j} \partial_{x_{2j-1}}).$$

The corresponding fundamental solutions, which may be used as Cauchy kernels in the corresponding integral representation formulae, are given by

$$E(\underline{X}) = -\frac{1}{\sigma_{2n}} \frac{\underline{X}}{|\underline{X}|^{2n}}, \quad E|\!(\underline{X}) = -\frac{1}{\sigma_{2n}} \frac{\underline{X|}}{|\underline{X}|^{2n}},$$

with  $\sigma_{2n}$  the surface area of the unit sphere in  $\mathbb{R}^{2n}$ . The Hermitian Dirac operators  $\partial_{\underline{Z}}$  and  $\partial_{\underline{Z}^\dagger}$  then are appropriate complex linear combinations of the Euclidean Dirac operators  $\partial_{\underline{X}}$  and  $\partial_{\underline{X|}}$ :

$$\partial_{\underline{Z}} = -\frac{1}{4} (\partial_{\underline{X}} - i \partial_{\underline{X|}}), \quad \partial_{\underline{Z}^\dagger} = \frac{1}{4} (\partial_{\underline{X}} + i \partial_{\underline{X|}}).$$

**Definition 1** A  $\mathbb{C}_{2n}$ -valued function  $g$  in the open subset  $\Omega$  of  $\mathbb{R}^{2n}$  is said to be (left) *h-monogenic* if it satisfies in  $\Omega$  the system  $\{\partial_{\underline{Z}} g = 0, \partial_{\underline{Z}^\dagger} g = 0\}$  or, equivalently, the system  $\{\partial_{\underline{X}} g = 0, \partial_{\underline{X|}} g = 0\}$ .

Let  $\mathcal{D}_{(\underline{z}, \underline{z}^\dagger)}$  be the circulant  $(2 \times 2)$ -matrix Dirac operator given by

$$\mathcal{D}_{(\underline{z}, \underline{z}^\dagger)} = \begin{pmatrix} \partial_{\underline{z}} & \partial_{\underline{z}^\dagger} \\ \partial_{\underline{z}^\dagger} & \partial_{\underline{z}} \end{pmatrix}. \quad (1)$$

Following e.g. [12, 2, 3, 1] we introduce the matrix

$$\mathcal{E} = \begin{pmatrix} \mathcal{E} & \mathcal{E}^\dagger \\ \mathcal{E}^\dagger & \mathcal{E} \end{pmatrix}$$

where  $\mathcal{E} = -(E + iE|)$  and  $\mathcal{E}^\dagger = (E - iE|)$ , which may be considered as a fundamental solution of the matrix operator  $\mathcal{D}_{(\underline{z}, \underline{z}^\dagger)}$ , i.e.,  $\mathcal{D}_{(\underline{z}, \underline{z}^\dagger)} \mathcal{E} = \delta$ , with  $\delta = \text{diag}(\delta)$ ,  $\delta$  being the Dirac delta distribution.

The set of  $(2 \times 2)$ -circulant matrices over  $\mathbb{C}_{2n}$  will be denoted by  $\mathbf{CM}^{2 \times 2}$ . It is easily checked that for  $\mathbf{A}, \mathbf{B} \in \mathbf{CM}^{2 \times 2}$ ,  $\mathbf{A} + \mathbf{B}$  and  $\mathbf{A}\mathbf{B}$  both belong to  $\mathbf{CM}^{2 \times 2}$ . Moreover, defining for

$$\mathbf{A} = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in \mathbf{CM}^{2 \times 2}$$

its Hermitian conjugate  $\mathbf{A}^\dagger$  componentwise, it clearly holds that  $(\mathbf{A}^\dagger)^\dagger = \mathbf{A}$ ;  $(\mathbf{A}a)^\dagger = a^\dagger \mathbf{A}^\dagger$  for  $a \in \mathbb{C}_{2n}$ ;  $(\mathbf{A} + \mathbf{B})^\dagger = \mathbf{A}^\dagger + \mathbf{B}^\dagger$  and  $(\mathbf{A}\mathbf{B})^\dagger = \mathbf{B}^\dagger \mathbf{A}^\dagger$ . In other words,  $\mathbf{CM}^{2 \times 2}$  is a right linear associative algebra over  $\mathbb{C}_{2n}$ , with an anti-involution.

**Definition 2** A  $\mathbf{CM}^{2 \times 2}$ -valued continuously differentiable matrix function

$$\mathbf{G}_2^1 = \begin{pmatrix} g_1 & g_2 \\ g_2 & g_1 \end{pmatrix}$$

in the open subset  $\Omega$  of  $\mathbb{R}^{2n}$  is said to be left (respectively right)  $\mathbf{H}$ -monogenic in  $\Omega$  if it satisfies in  $\Omega$

$$\mathcal{D}_{(\underline{z}, \underline{z}^\dagger)} \mathbf{G}_2^1 = \mathbf{O} \quad (\text{respectively } \mathbf{G}_2^1 \mathcal{D}_{(\underline{z}, \underline{z}^\dagger)} = \mathbf{O})$$

Here  $\mathbf{O}$  denotes the matrix in  $\mathbf{CM}^{2 \times 2}$  with zero entries.

From now on,  $\mathbf{H}_l(\Omega)$  (respectively  $\mathbf{H}_r(\Omega)$ ) stands for the set of left (respectively right)  $\mathbf{H}$ -monogenic functions in  $\Omega$ . Clearly  $\mathbf{H}_l(\Omega)$  (respectively  $\mathbf{H}_r(\Omega)$ ) is a right (respectively a left) module over  $\mathbf{CM}^{2 \times 2}$ .

Let  $\Delta$  denote the Hermitian Laplacian given by

$$\Delta = \begin{pmatrix} \Delta_{2n} & 0 \\ 0 & \Delta_{2n} \end{pmatrix},$$

then it holds that

$$4\mathcal{D}_{(\underline{z}, \underline{z}^\dagger)}(\mathcal{D}_{(\underline{z}, \underline{z}^\dagger)})^\dagger = 4(\mathcal{D}_{(\underline{z}, \underline{z}^\dagger)})^\dagger \mathcal{D}_{(\underline{z}, \underline{z}^\dagger)} = \Delta.$$

In general the  $\mathbf{H}$ -monogenicity of the matrix function  $\mathbf{G}_2^1$  does not imply the  $h$ -monogenicity of its entry functions  $g_1$  and  $g_2$ . However, choosing  $g_1 = g$  and  $g_2 = 0$ , the  $\mathbf{H}$ -monogenicity of the corresponding matrix  $\mathbf{G}_0 = \text{diag}(g)$  yet is equivalent with the  $h$ -monogenicity of the function  $g$ . Moreover one may call a matrix function  $\mathbf{G}_2^1$  harmonic if and only if it satisfies the equation  $\Delta[\mathbf{G}_2^1] = \mathbf{O}$ ; each  $\mathbf{H}$ -monogenic matrix function  $\mathbf{G}_2^1$  thus is harmonic, ensuring that its entries are harmonic functions in the usual sense.

The notions of continuity, differentiability and integrability of a function  $\mathbf{G}_2^1 \in \mathbf{CM}^{2 \times 2}$  have the usual component-wise meaning. In particular we shall need, for any suitable subset  $E$  of  $\mathbb{R}^{2n}$  and  $k \in \mathbb{N} \cup \{0\}$  the following function spaces:

- (i)  $\mathbf{L}_2(E)$ , the space of square integrable circulant matrix functions;
- (ii)  $\mathbf{C}^k(E)$ , the set of all circulant matrix functions whose entries, together with all their derivatives up to  $k$ -th order, are continuous in  $E$ ;
- (iii) the Sobolev spaces  $\mathbf{W}_2^k(E)$  and their trace spaces, the so-called Slobodetzki spaces  $\mathbf{W}_2^{k-\frac{1}{2}}(\partial E)$ ; note that  $\mathbf{W}_2^0(E) = \mathbf{L}_2(E)$ .

Here the trace on  $\partial E$  of a  $\mathbf{CM}^{2 \times 2}$ -function  $\mathbf{G}_2^1 \in \mathbf{W}_2^k(E)$  is defined as the limit in  $\mathbf{L}_2(\partial E)$  given by:

$$\text{tr}_{\partial E} \mathbf{G}_2^1 \equiv \mathbf{G}_2^1|_{\partial E} := \lim_{n \rightarrow \infty} \{\mathbf{G}_2^1\}_n|_{\partial E},$$

where  $\{\mathbf{G}_2^1\}_n$  is a sequence in  $\mathbf{C}^1(E)$ , which exists by density, converging to  $\mathbf{G}_2^1$  in the norm of  $\mathbf{W}_2^k(E)$ . We note that the trace operator is not surjective, but maps  $\mathbf{W}_2^k(E)$  onto the Slobodetzki spaces  $\mathbf{W}_2^{k-\frac{1}{2}}(\partial E)$ , i.e.

$$\text{tr}_{\Gamma} : \mathbf{W}_2^k(E) \longrightarrow \mathbf{W}_2^{k-\frac{1}{2}}(\partial E).$$

For a deeper discussion of Sobolev-Slobodetzki spaces along classical lines we refer the reader to [14].

In what follows,  $\Omega$  stands for an open bounded domain in  $\mathbb{R}^{2n}$  with a sufficiently smooth boundary  $\Gamma$ ; moreover we put  $\Omega^+ = \Omega$  and  $\Omega^- = \mathbb{R}^{2n} \setminus (\Omega^+ \cup \Gamma)$ , and assume that  $\Omega^+$  and  $\Omega^-$  are connected. In Section 4 of this paper we will deal with two Dirichlet type boundary value problems in the context of  $\mathbf{H}$ -monogenic matrix functions in  $\mathbb{R}^{2n}$ . The first boundary value problem is of first order and reads

$$\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \Xi_2^1 = F_2^1 \text{ in } \Omega \quad (2)$$

$$\Xi_2^1 = G_2^1 \text{ on } \Gamma \quad (3)$$

The second one is of the second order and is given by

$$\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}^2 \Xi_2^1 = F_2^1 \text{ in } \Omega \quad (4)$$

$$\Xi_2^1 = G_2^1 \text{ on } \Gamma \quad (5)$$

### 3 Hermitian Clifford operator calculus

Using the matrix function  $\mathcal{E}$  we can introduce the following integral operators:

- (i) the Hermitian Cauchy integral given, for  $\mathbf{G}_2^1 \in \mathbf{W}_2^{\frac{1}{2}}(\Gamma)$ , by

$$\mathcal{C}_\Gamma \mathbf{G}_2^1(\underline{Y}) = \int_\Gamma \mathcal{E}(\underline{Z} - \underline{V}) \mathbf{N}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{G}_2^1(\underline{X}) d\mathcal{H}^{2n-1}, \quad \underline{Y} \in \Omega^\pm,$$

where

$$d\mathcal{H}^{2n-1} = \begin{pmatrix} d\mathcal{H}^{2n-1} & 0 \\ 0 & d\mathcal{H}^{2n-1} \end{pmatrix},$$

with  $\mathcal{H}^{2n-1}$  denoting the  $(2n-1)$ -dimensional Hausdorff measure.

- (ii) the Hermitian Teodorescu transform, given, for  $\mathbf{G}_2^1 \in \mathbf{L}_2(\Omega)$ , by

$$\mathcal{T}_\Omega \mathbf{G}_2^1(\underline{Y}) = - \int_\Omega \mathcal{E}(\underline{Z} - \underline{V}) \mathbf{G}_2^1(\underline{X}) dW(\underline{Z}, \underline{Z}^\dagger), \quad \underline{Y} \in \mathbb{R}^{2n},$$

where  $dW(\underline{Z}, \underline{Z}^\dagger)$  is the associated volume element given by

$$dV(\underline{X}) = (-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n dW(\underline{Z}, \underline{Z}^\dagger).$$

- (iii) the Hermitian Hilbert transform given, for  $\mathbf{G}_2^1 \in \mathbf{L}_2(\Gamma)$ , by

$$(\mathcal{H}_\Gamma \mathbf{G}_2^1)(\underline{U}) := 2 \int_\Gamma \mathcal{E}(\underline{Z} - \underline{W}) \mathbf{N}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{G}_2^1(\underline{X}) d\mathcal{H}^{2n-1}, \quad \underline{U} \in \Gamma,$$

where the circulant matrix

$$\mathbf{N}_{(\underline{Z}, \underline{Z}^\dagger)} = \begin{pmatrix} \underline{N} & -\underline{N}^\dagger \\ -\underline{N}^\dagger & \underline{N} \end{pmatrix}$$

contains the Hermitian projections  $\underline{N}$  and  $\underline{N}^\dagger$  of the unit normal vector  $\underline{n}(\underline{X})$  at the point  $\underline{X} \in \Gamma$ .

#### Remark

The following mapping properties hold:

- (i)  $\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} : \mathbf{W}_2^k(\Omega) \longrightarrow \mathbf{W}_2^{k-1}(\Omega)$ ,  $k \in \mathbb{N}$
- (ii)  $\mathcal{T}_\Omega : \mathbf{W}_2^k(\Omega) \longrightarrow \mathbf{W}_2^{k+1}(\Omega)$ ,  $k \in \mathbb{N} \cup \{0\}$
- (iii)  $\mathcal{T}_\Omega : \mathbf{W}_2^{k,loc}(\Omega) \longrightarrow \mathbf{W}_2^{k+1,loc}(\Omega)$ ,  $k \in \mathbb{N} \cup \{0\}$
- (iv)  $\mathcal{C}_\Gamma : \mathbf{W}_2^{k-\frac{1}{2}}(\Gamma) \longrightarrow \mathbf{W}_2^k(\Omega)$ ,  $k \in \mathbb{N}$ .

They are a direct translation to our matrix setting of the corresponding properties given in [7, 11] for the corresponding entries of the operators involved.

### 3.1 Integral formulae for H-monogenic functions

From now on we reserve the notations  $\underline{Y}$  and  $\underline{Y}|$  for Clifford vectors associated to points in  $\Omega^\pm$ . Their Hermitian counterparts are denoted by

$$\underline{V} = \frac{1}{2}(\underline{Y} + i\underline{Y}|), \quad \underline{V}^\dagger = -\frac{1}{2}(\underline{Y} - i\underline{Y}|)$$

The following fundamental statements are known; for the proof we again refer to [7, 11].

**Theorem 1 (Stokes)** *Let  $\mathbf{F}_2^1, \mathbf{G}_2^1$  be arbitrary matrix functions in  $\mathbf{W}_2^1(\Omega)$ . It then holds that*

$$\int_\Gamma [\mathrm{tr}_\Gamma \mathbf{F}_2^1] \mathbf{N}_{(\underline{Z}, \underline{Z}^\dagger)} [\mathrm{tr}_\Gamma \mathbf{G}_2^1] d\mathcal{H}^{2n-1} = \int_\Omega [(\mathbf{F}_2^1 \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{G}_2^1 + \mathbf{F}_2^1 (\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{G}_2^1)] dW(\underline{Z}, \underline{Z}^\dagger).$$

By means of the matrix approach sketched above, the following relation between  $\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}, \mathcal{C}_\Gamma$  and  $\mathcal{T}_\Omega$  was established.

**Theorem 2 (Borel-Pompeiu formula)** *Let  $\mathbf{G}_2^1$  be a circulant matrix function in  $\mathbf{W}_2^1(\Omega)$ . It then holds that*

$$\mathcal{C}_\Gamma \mathbf{G}_2^1(\underline{Y}) + \mathcal{T}_\Omega \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{G}_2^1(\underline{Y}) = \begin{cases} (-1)^{\frac{n(n+1)}{2}} (2i)^n \mathbf{G}_2^1(\underline{Y}), & \underline{Y} \in \Omega^+ \\ 0, & \underline{Y} \in \Omega^- \end{cases}$$

**Corollary 1** *Let  $\mathbf{G}_2^1$  be a circulant matrix function in  $\mathbf{W}_2^1(\Omega) \cap \mathbf{H}_l(\Omega)$ . It then holds that*

$$\mathcal{C}_\Gamma \mathbf{G}_2^1(\underline{Y}) = \begin{cases} (-1)^{\frac{n(n+1)}{2}} (2i)^n \mathbf{G}_2^1(\underline{Y}), & \underline{Y} \in \Omega^+ \\ 0, & \underline{Y} \in \Omega^- \end{cases}$$

The following theorem expresses the basic property that the matricial Hermitian Teodorescu transform is the algebraic inverse to the operator  $\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}$ .

**Theorem 3** *If  $\mathbf{G}_2^1 \in \mathbf{L}_2(\Omega)$ , then*

$$\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathcal{T}_\Omega \mathbf{G}_2^1(\underline{Y}) = \begin{cases} (-1)^{\frac{n(n+1)}{2}} (2i)^n \mathbf{G}_2^1(\underline{Y}), & \underline{Y} \in \Omega^+, \\ 0, & \underline{Y} \in \Omega^-. \end{cases}$$

**Theorem 4 (Plemelj-Sokhotski formulae)** *Let  $\mathbf{G}_2^1 \in \mathbf{W}_2^{\frac{1}{2}}(\Gamma)$ . Then the traces of the Hermitian Cauchy integral  $\mathcal{C}_\Gamma \mathbf{G}_2^1$ , for any  $\underline{U} \in \Gamma$  are given by*

$$\mathrm{tr}_\Gamma \mathcal{C}_\Gamma [\mathbf{G}_2^1] = (-1)^{\frac{n(n+1)}{2}} (2i)^n \left( \frac{1}{2} \mathcal{I}[\mathbf{G}_2^1] + \frac{1}{2} \mathcal{H}_\Gamma [\mathbf{G}_2^1] \right),$$

$$\mathrm{tr}_\Gamma^- \mathcal{C}_\Gamma [\mathbf{G}_2^1] = (-1)^{\frac{n(n+1)}{2}} (2i)^n \left( -\frac{1}{2} \mathcal{I}[\mathbf{G}_2^1] + \frac{1}{2} \mathcal{H}_\Gamma [\mathbf{G}_2^1] \right),$$

where  $\mathrm{tr}_\Gamma^-$  denotes the trace operator in the exterior domain  $\Omega^-$  and  $\mathcal{I}$  stands for the identity operator.

We now introduce the operators

$$\mathcal{P}_\Gamma^+[G_2^1] = \text{tr}_\Gamma \mathcal{C}_\Gamma[G_2^1] \quad \text{and} \quad \mathcal{P}_\Gamma^-[G_2^1] = -\text{tr}_\Gamma^- \mathcal{C}_\Gamma[G_2^1]$$

Since it holds that  $\mathcal{P}_\Gamma^+ + \mathcal{P}_\Gamma^- = \mathcal{I}$ ,  $[\mathcal{P}_\Gamma^\pm]^2 = \mathcal{P}_\Gamma^\pm$  and  $\mathcal{P}_\Gamma^+ \mathcal{P}_\Gamma^- = \mathcal{P}_\Gamma^- \mathcal{P}_\Gamma^+ = \mathbf{0}$ , the operators  $\mathcal{P}_\Gamma^\pm$  clearly are projections. More precisely  $\mathcal{P}_\Gamma^\pm$  is the projection onto the space of all  $\mathbf{CM}^{2 \times 2}$  matrix functions which are  $\mathbf{H}$ -monogenically extendable into the domain  $\Omega^\pm$  and vanish at infinity.

For a study of classical versions of the operators  $\mathcal{C}_\Gamma, \mathcal{H}_\Gamma, \mathcal{P}_\Gamma^\pm$  we refer to [1, 2, 3].

### 3.2 Orthogonal decomposition of the Hermitian Sobolev spaces

Orthogonal decomposition of the Hilbert space of square integrable functions in the context of Euclidean Clifford analysis have been studied by many authors, seen their importance in solving boundary value problems, see [7, 8, 9, 10]. Here we investigate the structure of the Hermitian Sobolev spaces  $\mathbf{W}_2^k(\Omega)$ ,  $k \in \mathbb{N} \cup \{0\}$ , with respect to the  $(2 \times 2)$  matrix Dirac operator  $\mathcal{D}_{(\underline{z}, \underline{z}^\dagger)}$ .

**Proposition 1** *One has*

$$\left( \mathbf{W}_2^k(\Omega) \cap \text{Ker } \mathcal{D}_{(\underline{z}, \underline{z}^\dagger)} \right) \cap \mathcal{D}_{(\underline{z}, \underline{z}^\dagger)} \left( \overset{\circ}{\mathbf{W}}_2^1(\Omega) \cap \mathbf{W}_2^{k+1}(\Omega) \right) = \{\mathbf{0}\},$$

where  $\overset{\circ}{\mathbf{W}}_2^1(\Omega)$  denotes the subspace of all functions in  $\mathbf{W}_2^1(\Omega)$  whose trace on  $\Gamma$  vanishes.

**Proof**

Let  $U_2^1 \in \left( \mathbf{W}_2^k(\Omega) \cap \text{Ker } \mathcal{D}_{(\underline{z}, \underline{z}^\dagger)} \right) \cap \mathcal{D}_{(\underline{z}, \underline{z}^\dagger)} \left( \overset{\circ}{\mathbf{W}}_2^1(\Omega) \cap \mathbf{W}_2^{k+1}(\Omega) \right)$  and  $U_2^1 \neq \mathbf{0}$ .

Then there exists  $V_2^1 \in \overset{\circ}{\mathbf{W}}_2^1(\Omega) \cap \mathbf{W}_2^{k+1}(\Omega)$  such that  $U_2^1 = \mathcal{D}_{(\underline{z}, \underline{z}^\dagger)} V_2^1$  and  $(\mathcal{D}_{(\underline{z}, \underline{z}^\dagger)})^\dagger \mathcal{D}_{(\underline{z}, \underline{z}^\dagger)} V_2^1 = \mathbf{0}$ . Thence  $\Delta V_2^1 = \mathbf{0}$  and the Dirichlet problem for harmonic matrix functions implies that  $V_2^1 = \mathbf{0}$  in  $\Omega$  and so also  $U_2^1 = \mathbf{0}$  in  $\Omega$ .  $\square$

The following decomposition is the central idea of the below described solution strategy for the boundary value problem (4)–(5).

**Proposition 2** *The Hermitian Hilbert Sobolev spaces  $\mathbf{W}_2^k(\Omega)$  show the orthogonal decomposition*

$$\mathbf{W}_2^k(\Omega) = \left( \mathbf{W}_2^k(\Omega) \cap \text{Ker } \mathcal{D}_{(\underline{z}, \underline{z}^\dagger)} \right) \oplus \mathcal{D}_{(\underline{z}, \underline{z}^\dagger)} \left( \overset{\circ}{\mathbf{W}}_2^1(\Omega) \cap \mathbf{W}_2^{k+1}(\Omega) \right),$$

with respect to the  $\mathbf{CM}^{2 \times 2}$ -valued inner product

$$\langle U_2^1, V_2^1 \rangle_{\mathbf{L}_2} = \int_{\Omega} [U_2^1]^\dagger V_2^1 dW(\underline{z}, \underline{z}^\dagger)$$

**Proof**

For any  $U_2^1 \in \mathbf{W}_2^k(\Omega)$  one has that  $V_2^1 = \mathcal{T}_\Omega U_2^1 \in \mathbf{W}_2^{k+1}(\Omega)$  and  $\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} V_2^1 = U_2^1$ .

Assuming that  $U_2^1 \in \left( \mathbf{W}_2^k(\Omega) \cap \text{Ker } \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \right)^\perp$ , then

$$\langle U_2^1, G_2^1 \rangle_{L_2} = \int_\Omega [\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} V_2^1]^\dagger G_2^1 dW(\underline{Z}, \underline{Z}^\dagger) = \mathbf{0}$$

for all  $G_2^1 \in \mathbf{W}_2^k(\Omega) \cap \text{Ker } \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}$ . In particular,

$$\int_\Omega [\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} V_2^1]^\dagger [\mathcal{E}(\underline{Z} - \underline{V})] dW(\underline{Z}, \underline{Z}^\dagger) = \mathbf{0},$$

for  $\underline{Y} \in \Omega^-$ , since obviously  $\mathcal{E}(\underline{Z} - \underline{V}) \in \mathbf{W}_2^k(\Omega) \cap \text{Ker } \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}$ . The Borel-Pompeiu formula (Theorem 2) then yields

$$\begin{aligned} \mathbf{0} &= \left[ \int_\Omega [\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} V_2^1]^\dagger [\mathcal{E}(\underline{Z} - \underline{V})] dW(\underline{Z}, \underline{Z}^\dagger) \right]^\dagger \\ &= \int_\Omega [\mathcal{E}(\underline{Z} - \underline{V})]^\dagger [\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} V_2^1] dW(\underline{Z}, \underline{Z}^\dagger) \\ &= \int_\Gamma [\mathcal{E}(\underline{Z} - \underline{V})]^\dagger \mathbf{N}_{(\underline{Z}, \underline{Z}^\dagger)} \text{tr}_\Gamma V_2^1(\underline{X}) d\mathcal{H}^{2n-1} \end{aligned}$$

whence also  $\mathcal{C}_\Gamma[\text{tr}_\Gamma V_2^1](\underline{Y}) = \mathbf{0}$  for  $\underline{Y} \in \Omega^-$ . This last equality yields  $\text{tr}_\Gamma^- \mathcal{C}_\Gamma[V_2^1] = \mathbf{0}$  and hence  $\text{tr}_\Gamma V_2^1 \in \text{im } \mathcal{P}_\Gamma^+ \cap \mathbf{W}_2^{k+\frac{1}{2}}(\Gamma)$ . Consequently, there exists a matrix  $\mathbf{H}_2^1 \in \mathbf{W}_2^{k+1}(\Omega) \cap \text{Ker } \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}$  with the property that  $\text{tr}_\Gamma \mathbf{H}_2^1 = \text{tr}_\Gamma V_2^1$ . For

$$\mathbf{Z}_2^1 = V_2^1 - \mathbf{H}_2^1 \in \overset{\circ}{\mathbf{W}}_2^1(\Omega) \cap \mathbf{W}_2^{k+1}(\Omega)$$

one then has  $\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{Z}_2^1 \in \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}(\overset{\circ}{\mathbf{W}}_2^1(\Omega) \cap \mathbf{W}_2^{k+1}(\Omega))$ . Clearly

$$U_2^1 = \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} V_2^1 = \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{Z}_2^1$$

whence  $U_2^1 \in \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}(\overset{\circ}{\mathbf{W}}_2^1(\Omega) \cap \mathbf{W}_2^{k+1}(\Omega))$ . This last result means that

$$\left( \mathbf{W}_2^k(\Omega) \cap \text{Ker } \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \right)^\perp \subseteq \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \left( \overset{\circ}{\mathbf{W}}_2^1(\Omega) \cap \mathbf{W}_2^{k+1}(\Omega) \right).$$

Conversely, taking  $J_2^1 \in \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \left( \overset{\circ}{\mathbf{W}}_2^1(\Omega) \cap \mathbf{W}_2^{k+1}(\Omega) \right)$ , there exists a matrix function  $S_2^1 \in \overset{\circ}{\mathbf{W}}_2^1(\Omega) \cap \mathbf{W}_2^{k+1}(\Omega)$  for which  $J_2^1 = \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} S_2^1$ , and thus

$$\int_\Omega [U_2^1]^\dagger J_2^1 dW(\underline{Z}, \underline{Z}^\dagger) = \int_\Omega [U_2^1]^\dagger [\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} S_2^1] dW(\underline{Z}, \underline{Z}^\dagger).$$

By Theorem 1 one then has that

$$\begin{aligned} \int_{\Omega} [\mathbf{U}_2^1]^\dagger [\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{S}_2^1] dW(\underline{Z}, \underline{Z}^\dagger) &= - \int_{\Omega} [[\mathbf{U}_2^1]^\dagger \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}] \mathbf{S}_2^1 dW(\underline{Z}, \underline{Z}^\dagger) \\ &= - \int_{\Omega} [(\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)})^\dagger [\mathbf{U}_2^1]]^\dagger \mathbf{S}_2^1 dW(\underline{Z}, \underline{Z}^\dagger) = \mathbf{0} \end{aligned}$$

for all  $\mathbf{U}_2^1 \in \text{Ker } \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}$ , whence

$$\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \left( \overset{\circ}{\mathbf{W}}_2^1(\Omega) \cap \mathbf{W}_2^{k+1}(\Omega) \right) \subseteq \left( \mathbf{W}_2^k(\Omega) \cap \text{Ker } \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \right)^\perp$$

which completes the proof.  $\square$

The above orthogonal decomposition generates two orthoprojections on the corresponding subspaces, more precisely we have

$$\begin{aligned} \mathbb{P} : \mathbf{W}_2^k(\Omega) &\longrightarrow \mathbf{W}_2^k(\Omega) \cap \text{Ker } \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \\ \mathbb{Q} : \mathbf{W}_2^k(\Omega) &\longrightarrow \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \left( \overset{\circ}{\mathbf{W}}_2^1(\Omega) \cap \mathbf{W}_2^{k+1}(\Omega) \right). \end{aligned}$$

Moreover, the operator  $\mathbb{P}$  can be seen as a Hermitian matrix generalization of the classical Bergman projection.

We end this section with an interesting property needed in the sequel.

**Proposition 3** *Let  $\mathbf{F}_2^1 \in \mathbf{L}_2(\Omega)$ . Then  $\text{tr} \mathcal{T}_\Omega \mathbf{F}_2^1 = \mathbf{0}$  if and only if  $\mathbf{F}_2^1$  belongs to  $\text{im} \mathbb{Q}$ .*

**Proof**

First assume that  $\mathbf{F}_2^1 \in \text{im} \mathbb{Q}$ , then there exists  $\mathbf{U}_2^1 \in \overset{\circ}{\mathbf{W}}_2^1(\Omega)$  such that  $\mathbf{F}_2^1 = \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{U}_2^1$ . The Borel–Pompeiu formula implies that

$$\mathbf{U}_2^1 = \mathcal{C}_\Gamma \mathbf{U}_2^1 + \mathcal{T}_\Omega [\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{U}_2^1] = \mathcal{T}_\Omega [\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{U}_2^1] = \mathcal{T}_\Omega \mathbf{F}_2^1.$$

Because  $\mathbf{U}_2^1 \in \overset{\circ}{\mathbf{W}}_2^1(\Omega)$  we have  $\text{tr} \mathcal{T}_\Omega \mathbf{F}_2^1 = \mathbf{0}$ .

Conversely, let  $\text{tr} \mathcal{T}_\Omega \mathbf{F}_2^1 = \mathbf{0}$ . Now decompose  $\mathbf{F}_2^1$  according to Proposition 2:  $\mathbf{F}_2^1 = \mathbb{P} \mathbf{F}_2^1 + \mathbb{Q} \mathbf{F}_2^1$ . As always  $\text{tr} \mathcal{T}_\Omega \mathbb{Q} \mathbf{F}_2^1 = \mathbf{0}$ . Then  $\text{tr} \mathcal{T}_\Omega \mathbb{P} \mathbf{F}_2^1 = \mathbf{0}$ . From the given factorization of the Hermitian Laplacian we have

$$\Delta \mathcal{T}_\Omega [\mathbb{P} \mathbf{F}_2^1] = 4(\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)})^\dagger [\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathcal{T}_\Omega [\mathbb{P} \mathbf{F}_2^1]] = 4(\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)})^\dagger [\mathbb{P} \mathbf{F}_2^1] = \mathbf{0},$$

which asserts that  $\mathcal{T}_\Omega [\mathbb{P} \mathbf{F}_2^1] = \mathbf{0}$ , due to the uniqueness of the solution of the Dirichlet problem for the Hermitian Laplace equation. Therefore, applying the Hermitian Dirac operator  $\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}$ , we have

$$\mathbf{0} = \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathcal{T}_\Omega [\mathbb{P} \mathbf{F}_2^1] = \mathbb{P} \mathbf{F}_2^1,$$

i.e.,  $\mathbf{F}_2^1 \in \text{im} \mathbb{Q}$ .  $\square$

## 4 Main results

In this section we deal with the boundary value problems (2)-(3) and (4)-(5) stated above.

### 4.1 Necessary and sufficient condition for solvability of the problem (2)–(3)

**Theorem 5** *Let  $\mathbf{F}_2^1 \in \mathbf{W}_2^k(\Omega)$  and  $\mathbf{G}_2^1 \in \mathbf{W}_2^{k-\frac{1}{2}}(\Gamma)$ . Then the boundary value problem (2)–(3) has a solution if and only if*

$$\mathcal{P}_\Gamma^-[G_2^1] = \text{tr}_\Gamma \mathcal{T}_\Omega[F_2^1]. \quad (6)$$

The unique solution  $\Xi_2^1 \in \mathbf{W}_2^{k+1}(\Omega)$  admits the representation

$$\Xi_2^1 = \mathcal{C}_\Gamma G_2^1 + \mathcal{T}_\Omega F_2^1.$$

#### Proof

From the Borel-Pompeiu formula it follows that

$$\mathcal{T}_\Omega F_2^1 = \mathcal{T}_\Omega[\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \Xi_2^1] = \Xi_2^1 - \mathcal{C}_\Gamma \Xi_2^1 = \Xi_2^1 - \mathcal{C}_\Gamma G_2^1$$

The Plemelj-Sokhotski formulae now yield

$$G_2^1 = \mathcal{P}_\Gamma^+[G_2^1] + \text{tr}_\Gamma \mathcal{T}_\Omega[F_2^1]$$

from which the necessary condition (6) follows. Conversely, assume condition (6) to be fulfilled. Then  $\mathcal{C}_\Gamma G_2^1 \in \mathbf{H}_l(\Omega)$ , and using the invertibility of  $\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}$  in  $\Omega$  it follows that  $\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \Xi_2^1 = F_2^1$ . The boundary condition (3) then follows from  $\mathcal{P}_\Gamma^+ + \mathcal{P}_\Gamma^- = \mathcal{I}$ .  $\square$

### 4.2 Treatment of the problem (4)–(5)

The aim of this subsection consists in the study of the second-order boundary value problem (4)–(5) by means of the orthogonal decomposition of the Hermitian Hilbert Sobolev spaces and the mapping properties of the operators  $\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}$ ,  $\mathcal{T}_\Omega$  and  $\mathcal{C}_\Gamma$ .

Assume that  $\mathbf{F}_2^1 \in \mathbf{W}_2^k(\Omega)$  and  $\mathbf{G}_2^1 \in \mathbf{W}_2^{k+\frac{3}{2}}(\Gamma)$ ,  $k \geq 0$ . First let us investigate the boundary value problem:

$$\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}^2 \Xi_2^1 = F_2^1, \text{ in } \Omega, \quad (7)$$

$$\Xi_2^1 = \mathbf{0}, \text{ on } \Gamma. \quad (8)$$

Using the mapping properties of  $\mathcal{T}_\Omega$  we have  $\mathcal{T}_\Omega[\mathbf{F}_2^1] \in \mathbf{W}_2^{k+1}(\Omega)$ . We thus get that  $\mathbb{Q}[\mathcal{T}_\Omega \mathbf{F}_2^1] \in \mathbf{W}_2^{k+1}(\Omega)$  and, again by the mapping properties of  $\mathcal{T}_\Omega$  we now obtain that

$$\mathcal{T}_\Omega \mathbb{Q} \mathcal{T}_\Omega [\mathbf{F}_2^1] \in \mathbf{W}_2^{k+2}(\Omega).$$

Now using  $\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathcal{T}_\Omega = \mathcal{I}$ , the Borel-Pompeiu formula and the existence of a matrix function  $\Xi_2^1 \in \mathring{\mathbf{W}}_2^1(\Omega) \cap \mathbf{W}_2^{k+1}(\Omega)$  for which  $\mathbb{Q}[\mathcal{T}_\Omega \mathbf{F}_2^1] = \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \Xi_2^1$ , we see that

$$\mathcal{T}_\Omega [\mathbb{Q} \mathcal{T}_\Omega \mathbf{F}_2^1] = \mathcal{T}_\Omega \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \Xi_2^1 = \Xi_2^1,$$

and therefore  $\Xi_2^1 \in \mathbf{W}_2^{k+2}(\Omega)$ . Finally

$$\begin{aligned} \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}^2 \Xi_2^1 &= \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbb{Q} \mathcal{T}_\Omega [\mathbf{F}_2^1] = \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} [\mathcal{I} - \mathbb{P}] \mathcal{T}_\Omega [\mathbf{F}_2^1] \\ &= \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathcal{T}_\Omega [\mathbf{F}_2^1] - \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbb{P} \mathcal{T}_\Omega [\mathbf{F}_2^1] = \mathbf{F}_2^1, \end{aligned}$$

since  $\text{im } \mathbb{P} \subset \text{Ker } \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}$ . So  $\mathcal{T}_\Omega \mathbb{Q} \mathcal{T}_\Omega [\mathbf{F}_2^1]$  is a solution of (7)–(8).

Moreover, we can use this result to solve the boundary value problem

$$\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}^2 \Xi_2^1 = \mathbf{0}, \quad \text{in } \Omega, \quad (9)$$

$$\Xi_2^1 = \mathbf{G}_2^1, \quad \text{on } \Gamma. \quad (10)$$

As  $\mathbf{G}_2^1 \in \mathbf{W}_2^{k+\frac{3}{2}}(\Gamma)$ , there exists a  $\mathbf{W}_2^{k+2}(\Omega)$ –extension, not necessarily unique, with  $\text{tr}_\Gamma \mathbf{H}_2^1 = \mathbf{G}_2^1$ . If we put  $\Xi_2^1 = \mathbf{J}_2^1 + \mathbf{H}_2^1$ , we have the last boundary value problem transformed into

$$\begin{aligned} \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}^2 \mathbf{J}_2^1 &= -\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}^2 \mathbf{H}_2^1, \quad \text{in } \Omega, \\ \mathbf{J}_2^1 &= \mathbf{0}, \quad \text{on } \Gamma. \end{aligned}$$

Applying our solution of (7)–(8) one has the representation

$$\mathbf{J}_2^1 = -\mathcal{T}_\Omega \mathbb{Q} \mathcal{T}_\Omega \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}^2 \mathbf{H}_2^1.$$

The Borel-Pompeiu formula implies that

$$\begin{aligned} \mathbf{J}_2^1 &= -\mathcal{T}_\Omega \mathbb{Q} [\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{H}_2^1] + \mathcal{T}_\Omega \mathbb{Q} [\mathcal{C}_\Gamma [\text{tr}_\Gamma \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{H}_2^1]] = -\mathcal{T}_\Omega \mathbb{Q} [\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{H}_2^1] \\ &= -\mathcal{T}_\Omega \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{H}_2^1 + \mathcal{T}_\Omega \mathbb{P} \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{H}_2^1 = -\mathbf{H}_2^1 + \mathcal{C}_\Gamma [\mathbf{G}_2^1] + \mathcal{T}_\Omega \mathbb{P} \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{H}_2^1, \end{aligned}$$

and so

$$\Xi_2^1 = \mathbf{J}_2^1 + \mathbf{H}_2^1 = \mathcal{C}_\Gamma [\mathbf{G}_2^1] + \mathcal{T}_\Omega \mathbb{P} \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{H}_2^1.$$

Note that adding the solutions of both boundary value problems described above, gives a solution of (4)–(5), which leads to the following theorem.

**Theorem 6** Let  $\mathbf{F}_2^1 \in \mathbf{W}_2^k(\Omega)$  and  $\mathbf{G}_2^1 \in \mathbf{W}_2^{k+\frac{3}{2}}(\Gamma)$ ,  $k \geq 0$ . Then the boundary value problem (4)–(5) has the unique solution  $\Xi_2^1 \in \mathbf{W}_2^{k+2}(\Omega)$  of the form

$$\Xi_2^1 = \mathcal{C}_\Gamma[\mathbf{G}_2^1] + \mathcal{T}_\Omega \mathbb{P}[\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{H}_2^1] + \mathcal{T}_\Omega \mathbb{Q} \mathcal{T}_\Omega[\mathbf{F}_2^1],$$

where  $\mathbf{H}_2^1$  denotes a  $\mathbf{W}_2^{k+2}(\Omega)$ -extension of  $\mathbf{G}_2^1$ .

**Proof**

The only point remaining to be proven is the uniqueness of the solution. To that end we remark that the boundary value problem

$$\begin{aligned} \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}^2 \Xi_2^1 &= \mathbf{0} \\ \text{tr}_\Gamma \Xi_2^1 &= \mathbf{0} \end{aligned}$$

admits the unique solution  $\Xi_2^1 = \mathbf{0}$ , when invoking the fact that  $\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}^2 \Xi_2^1 = \mathbf{0}$  also implies that  $\Delta \Xi_2^1 = \mathbf{0}$ .  $\square$

At this point, we are able to prove representation formulae for the orthoprojections  $\mathbb{P}$  and  $\mathbb{Q}$  in terms of  $\text{tr}_\Gamma, \mathcal{C}_\Gamma, \mathcal{T}_\Omega$ , similar to those developed in [7]. To this end we first prove a useful result on the existence of an isomorphism between Slobodetzki spaces.

**Proposition 4** Let  $k \in \mathbb{N} \cup \{0\}$ . Then the operator

$$\text{tr}_\Gamma \mathcal{T}_\Omega \mathcal{C}_\Gamma : \mathbf{W}_2^{k+\frac{1}{2}}(\Gamma) \cap \text{im} \mathcal{P}_\Gamma^+ \longrightarrow \mathbf{W}_2^{k+\frac{3}{2}}(\Gamma) \cap \text{im} \mathcal{P}_\Gamma^-$$

is an isomorphism.

**Proof**

Using the mapping properties of  $\mathcal{T}_\Omega$  and  $\mathcal{C}_\Gamma$  we obtain

$$(\text{tr}_\Gamma \mathcal{T}_\Omega \mathcal{C}_\Gamma)(\mathbf{W}_2^{k+\frac{1}{2}}(\Gamma)) \subset \mathbf{W}_2^{k+\frac{3}{2}}(\Gamma).$$

The task now is to prove that  $\text{Ker}(\text{tr}_\Gamma \mathcal{T}_\Omega \mathcal{C}_\Gamma) = \{\mathbf{0}\}$ . Let  $\Xi_2^1 \in \mathbf{W}_2^{k+\frac{1}{2}}(\Gamma) \cap \text{im} \mathcal{P}_\Gamma^+$  with  $\text{tr}_\Gamma \mathcal{T}_\Omega \mathcal{C}_\Gamma[\Xi_2^1] = \mathbf{0}$ . From the study of the second-order boundary value problem above and since  $\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}^2 \mathcal{T}_\Omega \mathcal{C}_\Gamma[\Xi_2^1] = \mathbf{0}$  we have that  $\mathcal{T}_\Omega \mathcal{C}_\Gamma[\Xi_2^1] = \mathbf{0}$ . Hence  $\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}[\mathcal{T}_\Omega \mathcal{C}_\Gamma[\Xi_2^1]] = \mathbf{0}$  and it follows that  $\mathcal{C}_\Gamma[\Xi_2^1] = \mathbf{0}$ . We then obtain that  $\Xi_2^1 = \mathbf{0}$ , since  $\Xi_2^1 \in \text{im} \mathcal{P}_\Gamma^+$  yields  $\mathcal{C}_\Gamma[\Xi_2^1] = (-1)^{\frac{n(n+1)}{2}} (2i)^n \Xi_2^1$ , in view of Corollary 1.

Now, let  $\mathbf{V}_2^1 \in \text{im} \mathcal{P}_\Gamma^-$ , so we have that  $\mathcal{C}_\Gamma[\mathbf{V}_2^1] = \mathbf{0}$ . Again, the study of the solution of the second-order boundary value problem enables us to conclude that the problem

$$\begin{aligned} \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}^2 \Xi_2^1 &= \mathbf{0} \\ \Xi_2^1 &= \mathbf{V}_2^1 \end{aligned}$$

has the solution  $\Xi_2^1 = \mathcal{C}_\Gamma[\mathbf{V}_2^1] + \mathcal{T}_\Omega \mathbb{P} \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{H}_2^1$ , where  $\mathbf{H}_2^1$  denotes a  $\mathbf{W}_2^{k+2}(\Omega)$ -extension of  $\mathbf{V}_2^1$ , whence  $\Xi_2^1 = \mathcal{T}_\Omega \mathbb{P} \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{H}_2^1$ . Corollary 1 now implies that

$$\Xi_2^1 = \frac{1}{(-1)^{\frac{n(n+1)}{2}} (2i)^n} \mathcal{T}_\Omega \mathcal{C}_\Gamma \mathbb{P} \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{H}_2^1$$

From this we conclude that  $\Xi_2^1 \in \text{im}[\text{tr}_\Gamma \mathcal{T}_\Omega \mathcal{C}_\Gamma]$ , which completes the proof.  $\square$

**Theorem 7** *Let  $\Xi_2^1 \in \mathbf{W}_2^k(\Omega)$ ,  $k \geq 1$ . Then the orthoprojections  $\mathbb{P}$  and  $\mathbb{Q}$  have the respective algebraic representations*

$$\begin{aligned} \mathbb{P}[\Xi_2^1] &= \mathcal{C}_\Gamma(\text{tr}_\Gamma \mathcal{T}_\Omega \mathcal{C}_\Gamma)^{-1} \text{tr}_\Gamma \mathcal{T}_\Omega [\Xi_2^1] \in \mathbf{W}_2^k(\Omega), \\ \mathbb{Q}[\Xi_2^1] &= (\mathcal{I} - \mathcal{C}_\Gamma(\text{tr}_\Gamma \mathcal{T}_\Omega \mathcal{C}_\Gamma)^{-1} \text{tr}_\Gamma \mathcal{T}_\Omega)[\Xi_2^1] \in \mathbf{W}_2^k(\Omega). \end{aligned}$$

**Proof**

Assume  $\Xi_2^1 \in \mathbf{W}_2^k(\Omega)$  and put  $\mathbb{P}_* = \mathcal{C}_\Gamma(\text{tr}_\Gamma \mathcal{T}_\Omega \mathcal{C}_\Gamma)^{-1} \text{tr}_\Gamma \mathcal{T}_\Omega$ . Then we have that  $\mathcal{T}_\Omega[\Xi_2^1] \in \mathbf{W}_2^{k+1}(\Omega)$ , which yields  $\text{tr}_\Gamma \mathcal{T}_\Omega \Xi_2^1 \in \mathbf{W}_2^{k+\frac{1}{2}}(\Gamma)$ . From  $\text{tr}_\Gamma \mathcal{T}_\Omega \Xi_2^1 \in \text{im} \mathcal{P}_\Gamma^-$  we have, by Proposition 4, that  $(\text{tr}_\Gamma \mathcal{T}_\Omega \mathcal{C}_\Gamma)^{-1} \text{tr}_\Gamma \mathcal{T}_\Omega \Xi_2^1 \in \mathbf{W}_2^{k-\frac{1}{2}}(\Gamma)$ . Consequently, applying the mapping properties of  $\mathcal{C}_\Gamma$  we obtain

$$\mathbb{P}_*[\Xi_2^1] \in \mathbf{W}_2^k(\Omega) \cap \text{Ker} \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}.$$

It is easily verified that  $\mathbb{P}_*^2 = \mathbb{P}_*$ . Also  $\mathcal{I} - \mathbb{P}_*$  is a projection onto the subspace  $\mathbf{W}_2^k(\Omega) \cap \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}(\mathbf{W}_2^1(\Omega) \cap \mathbf{W}_2^{k+1}(\Omega))$ . Obviously  $(\mathcal{I} - \mathbb{P}_*)^2 = \mathcal{I} - \mathbb{P}_*$  holds. Furthermore, we have

$$\begin{aligned} (\mathcal{I} - \mathbb{P}_*)\Xi_2^1 &= \Xi_2^1 - \mathbb{P}_*\Xi_2^1 = \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathcal{T}_\Omega(\Xi_2^1 - \mathbb{P}_*\Xi_2^1) \\ &= \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}[\mathcal{T}_\Omega \Xi_2^1 - \mathcal{T}_\Omega \mathbb{P}_*\Xi_2^1]. \end{aligned}$$

It is clear that

$$\begin{aligned} \text{tr}_\Gamma[\mathcal{T}_\Omega \Xi_2^1 - \mathcal{T}_\Omega \mathbb{P}_*\Xi_2^1] &= \text{tr}_\Gamma \mathcal{T}_\Omega \Xi_2^1 - \text{tr}_\Gamma \mathcal{T}_\Omega \mathbb{P}_*\Xi_2^1 \\ &= \text{tr}_\Gamma \mathcal{T}_\Omega \Xi_2^1 - \text{tr}_\Gamma \mathcal{T}_\Omega \mathcal{C}_\Gamma(\text{tr}_\Gamma \mathcal{T}_\Omega \mathcal{C}_\Gamma)^{-1} \text{tr}_\Gamma \mathcal{T}_\Omega \Xi_2^1 \\ &= \text{tr}_\Gamma \mathcal{T}_\Omega \Xi_2^1 - \text{tr}_\Gamma \mathcal{T}_\Omega \Xi_2^1 = \mathbf{0}. \end{aligned}$$

Invoking the uniqueness of the orthoprojections  $\mathbb{P}$  and  $\mathbb{Q}$  concludes the proof.  $\square$

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