UNIVERSITEIT
GENT

Faculteit Wetenschappen
Vakgroep Wiskunde

Intersection problems in finite geometries

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Proefschrift voorgelegd aan
de Faculteit Wetenschappen
tot het behalen van de graad van
Doctor in de Wetenschappen
richting Wiskunde

Of course it is happening inside your head, Harry,
but why on earth should that mean that it is not real?

Albus Dumbledore in *Harry Potter and the Deathly Hallows, King's Cross.*

Wees moedig, vooral bij nederlaag.
Wees wilskrachtig, overwin uzelf.

Levenswet.

Ten geleide

L.S.,

De voorbije jaren heb ik onderzoek mogen verrichten naar combinatorische objecten in eindige meetkundes en naar lineaire codes die afkomstig zijn van eindige meetkundes. Het resultaat daarvan kan u in dit proefschrift lezen. Vaak is de definitie van deze combinatorische objecten gerelateerd aan doorsnedes. Het zijn (of ze zijn afkomstig van) verzamelingen van deelruimtes die elkaar wel of juist niet snijden. Bij het bestuderen van de lineaire codes bekijken we vaak doorsnedes van deelruimtes met een algebraïsche variëteit, of van algebraïsche variëteiten onderling, telkens in de gerelateerde meetkunde. Vandaar de titel van dit proefschrift: 'Doorsnedeproblemen in eindige meetkundes'.

Deze eindige meetkundes zijn meestal projectieve of polaire ruimtes, maar ook designs komen aan bod. Sommige combinatorische problemen en objecten die aan bod komen, zijn afkomstig uit de combinatoriek voor verzamelingen (bv. Erdős-Ko-Rado verzamelingen); andere zijn afkomstig uit de klassieke Euclidische meetkunde (bv. Kakeya verzamelingen). En nog andere zijn typisch voor eindige meetkundes (bv. spreads).

In mijn onderzoek was het vaak de bedoeling om classificatieresultaten te behalen. Van de hierboven vermelde objecten - in het geval van de codes gaat het dan om de codewoorden - zijn er vaak vele voorbeelden gekend of kunnen er op zijn minst vele geconstrueerd worden. Een klassieke vraag is dan om de grote of de kleine voorbeelden (naar gelang de context) te vinden en te classificeren. Begrippen als 'groot' en 'klein' zijn natuurlijk relatief, en sterk afhankelijk van de context. Meestal gaat het om alle voorbeelden die groter/kleiner zijn dan een vastgelegde grootte.

Opdat de resultaten die ik hier voorstel, beschikbaar zouden zijn voor onderzoekers wereldwijd, is dit proefschrift opgesteld in het Engels, de lingua franca van de wetenschappelijke wereld. Ten behoeve van de Nederlandstalige lezer is achteraan, in Appendix B, een uitgebreide Nederlandstalige samenvatting toegevoegd.

Een meer uitgebreide inleiding kan u hierna in de Preface vinden. Daarna volgen de tien inhoudelijke hoofdstukken van dit proefschrift, twee inleidende hoofdstukken en acht waarin de resultaten van mijn onderzoek beschreven worden. Op het einde volgt een appendix met de uitgestelde berekeningen.

Beste lezer, dit proefschrift is de vrucht van drie jaar onderzoek, en de bekroning van mijn wiskundige opleiding. Om welke reden u dit proefschrift ook ter hand neemt, ik hoop dat u het kan appreciëren.

Maarten De Boeck
Mere/Gent, februari 2014

Preface

After completing my master's degree with a thesis on codes arising from the incidence matrices of finite projective spaces and their substructures, I started in October 2010 as a PhD student at the Department of Mathematics of UGent, supported by an FWO grant, under the supervision of prof. dr. Leo Storme. I intended to study codes related to finite projective spaces and to look at other (related) combinatorial problems in finite geometries. However, as many researchers in mathematics know, research is like a forest, in which it is easy to forget where you came in[1]. So, it is nice to notice that in the end this thesis contains indeed some chapters on geometry-based codes, and some chapters on purely geometrical problems, however not all topics that can be found in this thesis were on my research schedule from the beginning.

The chapters on geometrical problems (combinatorial problems in finite geometries) are put together in the first part of the thesis (Chapters 2 to 7); the chapters on geometry-based codes can be found in the second part of this thesis (Chapters 8 to 10), mainly for clearness of exposition, but far away from the chronological order in which the results were obtained.

In most chapters of this thesis we discuss combinatorial objects in finite geometries, which often can be defined very easily (based on the concept of intersection, hence the title of this thesis). Usually, a plethora of examples exist, but the main questions are always how large and/or how small such an object can be, and how it looks like when it has this extremal (maximal or minimal) size. Researchers active in this area of 'extremal combinatorics' obtained many nice results in the past decades. Their answers created new questions, regarding *stability*.

Starting from an extremal example it is often possible to construct other examples of such a combinatorial object by making only small modifications. The size of these objects is then close to the extremal size. A stability result is a theorem

[1] A saying influenced by Hattori Hanzo, the fictional one.

stating that the converse is true: if the size of an example of the combinatorial object is close to the extremal size, then it arises from the extremal example by making only small modifications. Obviously, researchers aim to stretch this notion of 'close' as far as possible, and so try to find and classify the first example which is truely different. And then they go further.

Gradually, the more general question is unveiled. We try to classify all large/small examples that are maximal/minimal, in the sense that they do not arise from a larger/smaller example, i.e. that they cannot be extended to a larger example/reduced to a smaller example. A classic example in finite geometries is the classification of the large maximal arcs in $\mathrm{PG}(2,q)$. Here we will see classification results for large Erdős-Ko-Rado sets, small Kakeya sets and small weight code words.

Among the first objects that I studied during my fellowship were Kakeya sets. Professors Mazzocca and Storme started working on this topic and I could join them. A few weeks later professor Blokhuis provided us with a very useful argument. Together we constructed the first small example not arising from a hyperoval and classified it as the third smallest Kakeya set, the smallest one not arising from a hyperoval. The results on Kakeya sets can be found in Chapter 6. It was only two years later I realised that these Kakeya sets are actually Erdős-Ko-Rado sets, and thus more closely related to other chapters in this thesis than I thought in the beginning.

Around the same time professor Storme also introduced me to functional codes. These are codes arising from substructures of projective spaces, but different from the ones I already knew. I started working on the codes $C_2(\mathcal{H})$, but soon I found out Daniele Bartoli was working on the codes $C_{Herm}(\mathcal{Q})$. We combined our forces and together we were able to find improvements to the previous results on these codes, although it took us eventually quite some time to get to the end of it. These results are written down in Chapter 8.

In the spring of 2011 Frédéric Vanhove defended his PhD thesis on incidence geometry from an algebraic graph theory point of view. His chapter on Erdős-Ko-Rado problems immediately attracted my attention, and I started working on this topic. It became the main topic of my thesis. An Erdős-Ko-Rado set is a set of pairwise non-trivially intersecting subspaces of a fixed dimension in a finite geometry. First, I investigated Erdős-Ko-Rado sets of planes. These investigations are the subject of chapter 3, which is undoubtedly the longest in this thesis. It took me more than a year to sort out everything correctly, but I obtained a strong classification theorem, including more than a dozen types. Afterwards I started investigating Erdős-Ko-Rado sets on unitals and Erdős-Ko-Rado sets of generators on hyperbolic quadrics. A first version of the research on unitals was finished soon, but the investigations on hyperbolic quadrics got stuck at some point. I returned to the Erdős-Ko-Rado problem for unitals and I realised that the arguments were applicable for general 2-designs. My results on Erdős-Ko-Rado sets in 2-designs can be found in chapter 5. A few months later I finally untangled the knot for the Erdős-Ko-Rado sets of generators on hyperbolic quadrics. I classified the second largest example. The results on these Erdős-Ko-Rado sets are written down in chapter 4.

In chapter 2 an introduction to Erdős-Ko-Rado problems is given, summarising the useful background for the chapters on Erdős-Ko-Rado sets.

In the summer of 2013, after I completed the work on Erdős-Ko-Rado sets of generators on hyperbolic quadrics, prof. Storme let me know that he thought that the arguments I used in this research, could possibly also be useful for investigating small maximal partial spreads. This turned out to be true and I could prove a lower bound on the size of maximal partial spreads. My results on this topic are the subject of chapter 7.

Apart from the problems on functional codes, no coding theoretical problems were mentioned above. In the beginning of my PhD fellowship, I studied a few problems related to the code generated by the lines of $\mathrm{PG}(2, q)$, one of them together with Peter Vandendriessche, but I did never publish the results. These results are now gathered in chapter 10.

In early 2011, Peter Vandendriessche and I started our investigations on the dual code of points and generators in Hermitian varieties. We intended to generalise previous results on $\mathcal{H}(3, q^2)$ and $\mathcal{H}(5, q^2)$ to $\mathcal{H}(2n+1, q^2)$ for general n. This turned out to be more difficult than we had expected. So, we put aside this research a few times for several months, both looking to other problems, and then returned to it with new ideas. It was only after two years that we finally concluded the research on these dual Hermitian codes. The results can be found in chapter 9.

In several of the arguments for the research topics previously mentioned, some lengthy calculations were involved. These are omitted in the chapters, but are presented in Appendix A.

Contents

1
Preliminaries

You must understand, young Hobbit,
it takes a long time to say anything in Old Entish.
And we never say anything
unless it is worth taking a long time to say.

Treebeard in *The Lord of the Rings: The Two Towers.*

In this chapter we introduce the concepts which we will consider in this thesis. It is however no layman's introduction. The aim of this chapter is to recall these concepts to those familiar with them, avoiding the ambiguity that would arise by not stating some definitions or theorems, and to create a place where the introductory material can be found.

The content of this chapter is based on standard references by Buekenhout ([28]), Hirschfeld ([79, 80]) and Hirschfeld and Thas ([81]), which contain more extensive introductions to most of these topics. For some specific topics or results, we will refer to other books and articles, mostly in Sections 1.2 and 1.7.

We will assume the reader has basic knowledge of combinatorics, finite field theory, linear algebra, graph theory and group theory.

1.1 Incidence geometries

As several geometries are discussed in this thesis, and therefore introduced in this chapter, we present the definition of a general incidence geometry.

Definition 1.1.1. An *incidence geometry* is a quadruple $(\mathcal{V}, \Delta_n, t, I)$, with $\mathcal{V}$ a non-empty set, $\Delta_n = \{0, 1, \ldots, n-1\}$, t a surjective map from $\mathcal{V}$ to Δ_n, and I a symmetric relation on $\mathcal{V}$ such that for all $v, v' \in \mathcal{V}$ the statement $(v, v') \in I$ implies that $t(v) \neq t(v')$.

The elements of $\mathcal{V}$ are called the *varieties*, t is called the *type map* and I is called the *incidence relation*. If a pair of elements is contained in I, then the two elements are called *incident*. The positive integer n is the *rank* of the geometry.

The above type map correponds to the dimension map in most incidence geometries. Varieties of type 0, 1 and 2 are called *points*, *lines* and *planes*, respectively. If $(v, v') \in I$, then the varieties v and v' are called incident. Furthermore, if $t(v) < t(v')$, then v' is said to *contain* v or to *pass through* v, and v is said to *be (lying) in* v' or to be *contained in* v'. We denote this by $v \subset v'$ (or $v \in v'$ if v is a point) and $v' \supset v$ (or $v' \ni v$ if v is a point). If $\mathcal{V}$ is a finite set, then the incidence geometry is said to be a *finite* (incidence) geometry.

All varieties of incidence geometries of rank 2, are points or lines. Therefore, incidence geometries of rank 2 are called *point-line geometries*. For these, the above definition can be simplified. They can be denoted as a triple $(\mathcal{P}, \mathcal{L}, I)$ with $I \subset (\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$ the incidence relation. Hereby the elements of $\mathcal{P}$ are called points and the elements of $\mathcal{L}$ are called lines or *blocks*.

A set of points incident with a common line, is said to be *collinear*, and a set of lines incident with a common point, is said to be *concurrent*.

For an incidence geometry $\mathcal{G} = (\mathcal{V}, \Delta_n, t, I)$ of rank n, we can define its *dual*. Let $t' : \mathcal{V} \to \Delta_n$ be the map defined by $t'(v) = n - t(v) - 1$. The dual geometry of $\mathcal{G}$ is the incidence geometry $\mathcal{G}' = (\mathcal{V}, \Delta_n, t', I)$. Note that the dual of a point-line geometry is consequently obtained by interchanging the roles of points and lines: the dual of the point-line geometry $\mathcal{G} = (\mathcal{P}, \mathcal{L}, I)$ is thus the point-line geometry $\mathcal{G}' = (\mathcal{L}, \mathcal{P}, I)$.

An *isomorphism* between two incidence geometries $\mathcal{G}_1 = (\mathcal{V}_1, \Delta_n, t_1, I_1)$ and $\mathcal{G}_2 = (\mathcal{V}_2, \Delta_n, t_2, I_2)$, necessarily of the same rank, is a bijection $\alpha : \mathcal{V}_1 \to \mathcal{V}_2$

such that $(v, v') \in I_1 \Leftrightarrow (\alpha(v), \alpha(v')) \in I_2$ for all $v, v' \in \mathcal{V}_1$ and $t_1(v) = t_2(\alpha(v))$ for all $v \in \mathcal{V}_1$. If $\mathcal{G}_1 = \mathcal{G}_2$, then α is called an *automorphism*. If $\mathcal{G}_2$ is the dual of $\mathcal{G}_1$, then α is called a *duality*[1]. Note that it is always possible to construct the dual of a geometry, but dualities do not necessarily exist. If an incidence geometry admits a duality, then it is called *self-dual*.

Let $\mathcal{G}_1 = (\mathcal{V}_1, \Delta_{n_1}, t_1, I_1)$ and $\mathcal{G}_2 = (\mathcal{V}_2, \Delta_{n_2}, t_2, I_2)$ be two incidence geometries. If $\mathcal{V}_1 \subseteq \mathcal{V}_2$, $t_1(v) = t_2(v)$ and $(v, v') \in I_1 \Leftrightarrow (v, v') \in I_2$ for all $v, v' \in \mathcal{V}_1$, then $\mathcal{G}_1$ is called a *subgeometry* of $\mathcal{G}_2$, and necessarily $n_1 \leq n_2$.

1.2 Designs

The first incidence geometries we introduce are the block designs. These have been widely studied for many years. We give a short introduction. For more background on this topic we refer to the monographs [2, 29, 33, 43, 83]. The results presented in this section can be found in these references, among others. Note that the terminology 'blocks' is far more common than 'lines' for these geometries.

Definition 1.2.1. A $t-(v,k,\lambda)$ *(block) design*, $v > k > 1$, $k \geq t \geq 1$, $\lambda > 0$, is a point-line geometry $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ with incidence relation I, such that $|\mathcal{P}| = v$, such that any element of $\mathcal{B}$ is incident with k elements of $\mathcal{P}$ and such that any set of t distinct points is contained in λ different lines (blocks). We impose that no two blocks are incident with the same k points, so a block can be identified with the k-subset of $\mathcal{P}$ which it determines.

The following counting results are widely known.

Theorem 1.2.2. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be a $t-(v,k,\lambda)$ block design. Then,*

- *the number of blocks through an arbitrary set of i points equals $\lambda_i = \lambda\binom{v-i}{t-i}/\binom{k-i}{t-i}$, $i = 1, t \ldots, t$;*
- *in particular, the number of blocks through a fixed point equals $r = \lambda_1 = \lambda\binom{v-1}{t-1}/\binom{k-1}{t-1}$;*

[1] Here, a duality corresponds to reversing the order of the types of an incidence geometry. In general, a duality can be defined for any permutation of Δ_n. We will however not consider this in this thesis.

- $b = |\mathcal{B}| = \frac{vr}{k}$.

The value r, representing the number of blocks through a given point, is called the *replication number*. Note that it is necessary that all λ_i, $i = 1, t \dots, t$, are integers for a $t-(v, k, \lambda)$ block design to exist. This condition is however not sufficient.

Block designs with $\lambda = 1$, so $t - (v, k, 1)$ designs, are probably the most studied class of block designs. They are called *Steiner systems* or *t-Steiner systems*. Especially Steiner systems with $t = 2$ are well-studied. Among them we mention

- the *axiomatic projective planes* of order n, $n \geq 2$: the $2-(n^2+n+1, n+1, 1)$ designs,
- the *axiomatic affine planes* of order n, $n \geq 2$: the $2-(n^2, n, 1)$ designs,
- the *Steiner triple systems*: the $2-(v, 3, 1)$ designs and
- the *unitals* of order n, $n \geq 2$: the $2-(n^3+1, n+1, 1)$ designs.

The axiomatic projective and affine planes will be discussed in more detail in Section 1.5.

By the above results, a $2-(v, k, 1)$ design contains $b = \frac{v(v-1)}{k(k-1)}$ blocks, $r = \frac{v-1}{k-1}$ of them through a fixed point. Consequently, a $2-(v, k, 1)$ design can only exist if $v \equiv 1 \pmod{k-1}$ and $k(k-1) \mid v(v-1)$.

We end this section with a remark on the smallest 2-Steiner systems

Remark 1.2.3. Let $\mathcal{D}$ be a $2-(v, k, 1)$ design. For every point P in $\mathcal{D}$, there is a block not containing this point since $v > k$. Each of the points on this block determines a different block through P. Hence, $r \geq k$. If $r = k$, then $\mathcal{D}$ is a projective plane of order $k-1$; if $r = k+1$, then $\mathcal{D}$ is an affine plane of order k. So, the projective and affine planes are the two 'smallest' $2-(v, k, 1)$ designs. They correspond to the smallest possible values for the replication number, hence also to the smallest possible values for the number of points v.

1.3 Projective geometries over fields

In most chapters of this thesis we will be dealing with *projective geometries over fields.*

Definition 1.3.1. The n-dimensional projective geometry over the field $\mathbb{F}$, denoted by $\mathrm{PG}(n, \mathbb{F})$, is the incidence geometry $(\mathcal{V}, \Delta_n, t, I)$ arising from the vector space $\mathbb{F}^{n+1} = V(n+1, \mathbb{F})$, the $(n+1)$-dimensional vector space over $\mathbb{F}$, in the following way. The set $\mathcal{V}$ is the set of all subspaces of V, different from $\{0\}$ and V itself; t maps a subspace W to its *projective dimension* $\dim(W) = \dim_V(W) - 1$; the incidence relation I fulfills $(W, W') \in I \Leftrightarrow (W \subset W') \vee (W' \subset W)$. The varieties of $\mathrm{PG}(n, \mathbb{F})$ are called *subspaces.*

Note that it follows from this definition that the projective dimension of a subspace equals its vectorial dimension minus one. In this thesis we will always use the projective dimension for subspaces of a projective geometry. A subspace with projective dimension k is called a *k-dimensional subspace*, or briefly a *k-space.* The names points, lines and planes will be used for respectively 0-spaces, 1-spaces and 2-spaces, as indicated before. The $(n-1)$-dimensional subspaces of $\mathrm{PG}(n, \mathbb{F})$ will be called *hyperplanes.*

Note that a subspace of a vector space is a vector space, so any subspace of a projective geometry can also be seen as a projective geometry. A projective geometry is consequently sometimes called a *projective space.* The n-dimensional projective space is often also considered as a subspace of dimension n of itself.

A projective geometry can only be finite if the underlying field is finite. A finite field of order q (with q elements) exists if and only if q is a prime power. We denote the finite field of order q by $\mathbb{F}_q$. The projective geometry $\mathrm{PG}(n, \mathbb{F}_q)$ will generally be denoted by $\mathrm{PG}(n, q)$. Due to the relation with vector spaces we can easily count the number of subspaces of a certain dimension in $\mathrm{PG}(n, q)$ using the *Gaussian coefficient* $\begin{bmatrix} a \\ b \end{bmatrix}_q$, which is defined as follows:

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \prod_{i=1}^{b} \frac{q^{a-b+i} - 1}{q^i - 1} = \frac{(q^a - 1) \cdots (q^{a-b+1} - 1)}{(q^b - 1) \cdots (q - 1)} .$$

The number of k-dimensional subspaces in $\mathrm{PG}(n, q)$ equals $\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q$, the number of subspaces with vector dimension $k+1$ in the vector space $V(n+1, q)$.

The number of points in $\mathrm{PG}(n,q)$ thus equals $\begin{bmatrix} n+1 \\ 1 \end{bmatrix}_q = \frac{q^{n+1}-1}{q-1}$. We denote this value by $\theta_n(q)$.

Notation 1.3.2. For two subspaces U and V of a projective space $\mathrm{PG}(n,\mathbb{F})$, the *intersection* $U \cap V$ is the largest subspace which is contained in both U and V. We can immediately generalise this definition to $U_1 \cap \cdots \cap U_s$ for subspaces $U_1,\ldots,U_s$ of $\mathrm{PG}(n,\mathbb{F})$.

For two subspaces U and V of a projective space $\mathrm{PG}(n,\mathbb{F})$, the *span* $\langle U,V\rangle$ is the smallest subspace that contains both U and V. This is also called the *subspace generated* by U and V. This definition can as well easily be generalised to $\langle U_1,\ldots,U_s\rangle$ for subspaces $U_1,\ldots,U_s$ of $\mathrm{PG}(n,\mathbb{F})$.

The dimension theorem for vector subspaces implies the *Grassmann identity* for subspaces of a projective space:

$$\dim(U) + \dim(V) = \dim(\langle U,V\rangle) + \dim(U \cap V)$$

for all subspaces U and V of $\mathrm{PG}(n,\mathbb{F})$.

Remark 1.3.3. A point in $\mathrm{PG}(n,\mathbb{F})$ corresponds with a vector line in $V(n+1,\mathbb{F})$. If $(x_0,x_1,\ldots,x_n)$ is a non-zero vector, then the set of vectors on the vector line determined by this vector, is given by $\{\lambda(x_0,x_1,\ldots,x_n) \mid \lambda \in \mathbb{F}\}$. Therefore, the coordinates of the corresponding projective point are defined up to a scalar multiple. We call them *homogeneous coordinates*. We denote the coordinates of the point by $(x_0,x_1,\ldots,x_n)$.

A hyperplane in $\mathrm{PG}(n,\mathbb{F})$ corresponds to a vector hyperplane in $V(n+1,\mathbb{F})$. This is given by a linear equation $a_0X_0 + a_1X_1 + \cdots + a_nX_n = 0$. Note that it has to contain the zero vector. The coefficient vector $(a_0,a_1,\ldots,a_n)$ is defined up to a scalar multiple. Considering the correspondence between the vector lines and the projective points, the corresponding hyperplane is also given by the linear equation $\lambda\,(a_0X_0 + a_1X_1 + \cdots + a_nX_n) = 0$, for $\lambda \in \mathbb{F}\setminus\{0\}$. We denote its coefficient vector by $\lambda\,[a_0,a_1,\ldots,a_n]$ or briefly by $[a_0,a_1,\ldots,a_n]$.

An automorphism of a projective geometry $\mathrm{PG}(n,\mathbb{F})$ is called a *collineation*. Let V be the underlying vector space of a projective geometry $\mathrm{PG}(n,\mathbb{F})$. The mapping $V \to V : x \mapsto Ax^\sigma$, with A a non-singular $(n+1)\times(n+1)$-matrix and σ a field automorphism of $\mathbb{F}$, induces a mapping of the points of $\mathrm{PG}(n,\mathbb{F})$.

From this mapping a collineation arises. We can denote this collineation by the tuple (A, σ). The group of all such collineations of $\mathrm{PG}(n, \mathbb{F})$, is denoted by $\mathrm{P\Gamma L}(n+1, \mathbb{F})$. The *fundamental theorem of projective geometry* states that every collineation of $\mathrm{PG}(n, \mathbb{F})$, $n \geq 2$, can arise from a non-singular matrix and a field automorphism in this way. So, if $n \geq 2$, then $\mathrm{P\Gamma L}(n+1, \mathbb{F})$ is the group all collineations of $\mathrm{PG}(n, \mathbb{F})$. Note that every tuple (A, σ), A a non-singular $(n+1) \times (n+1)$-matrix and σ a field automorphism of $\mathbb{F}$, corresponds to an element of $\mathrm{P\Gamma L}(n+1, \mathbb{F})$, but an element of $\mathrm{P\Gamma L}(n+1, \mathbb{F})$ corresponds to several tuples (A, σ). The collineations $(A, \mathbb{1})$ are called *projectivities*. The group of all projectivities of $\mathrm{PG}(n, \mathbb{F})$ is denoted by $\mathrm{PGL}(n+1, \mathbb{F})$. In $\mathrm{PG}(1, \mathbb{F})$ every bijection of the points gives rise to a collineation. Hence, the group $\mathrm{P\Gamma L}(2, \mathbb{F})$ is in general not the full collineation group.

Two subsets $\mathcal{S}$ and $\mathcal{S}'$ of $\mathrm{PG}(n, \mathbb{F})$ are called PGL-*equivalent* if and only if $\alpha(\mathcal{S}) = \mathcal{S}'$ for a projectivity $\alpha \in \mathrm{PGL}(n+1, \mathbb{F})$.

Every subspace U in a vector space V has an *orthogonal complement* $U^{\perp}$ regarding the standard dot product. We know that $\dim(U) + \dim(U^{\perp}) = \dim(V)$. Hence, the dual of a projective geometry can be obtained using a type map that maps every subspace to the projective dimension of its orthogonal complement. These orthogonal complements form a vector space V', called the *dual vector space* of V. So the dual geometry of the projective geometry derived from V is also a projective geometry, the one derived from V'. Since V and V' are isomorphic, the projective geometry and its dual can be identified with each other. So the dual of a projective geometry $\mathrm{PG}(n, \mathbb{F})$ can be seen as $\mathrm{PG}(n, \mathbb{F})$ itself. A duality maps a k-dimensional subspace onto an $(n-k-1)$-dimensional subspace. The standard duality arises from the mapping of a subspace onto its orthogonal complement. Note that it maps the point $(a_0, a_1, \ldots, a_n)$ onto the hyperplane $[a_0, a_1, \ldots, a_n]$ and vice versa.

In a projective geometry we can find subgeometries that are themselves projective geometries, and that are consequently called *projective subgeometries*. If a projective geometry $\mathrm{PG}(m, \mathbb{F})$ is a subgeometry of $\mathrm{PG}(n, \mathbb{F}')$, then $m \leq n$ and $\mathbb{F}$ is a subfield of $\mathbb{F}'$. If $m = n$ and there is a prime power q such that $\mathbb{F} = \mathbb{F}_q$ and $\mathbb{F}' = \mathbb{F}_{q^2}$, then this subgeometry is called a *Baer subgeometry*.

1.4 Affine geometries

Starting from projective geometries, we can easily define affine geometries.

Definition 1.4.1. Let $\mathrm{PG}(n, \mathbb{F})$ be the n-dimensional projective geometry over the field $\mathbb{F}$, and let H_∞ be a hyperplane of $\mathrm{PG}(n, \mathbb{F})$. Let $\mathcal{V}$ be the set of subspaces of $\mathrm{PG}(n, \mathbb{F})$ different from H_∞, which are not incident with H_∞. The map $t : \mathcal{V} \to \Delta_n$ maps each subspace to its projective dimension and the relation I fulfills $(W, W') \in I$ if and only if W is incident with W' in $\mathrm{PG}(n, \mathbb{F})$. The n-dimensional affine geometry over the field $\mathbb{F}$, denoted by $\mathrm{AG}(n, \mathbb{F})$, is the incidence geometry $(\mathcal{V}, \Delta_n, t, I)$. The varieties of $\mathrm{AG}(n, \mathbb{F})$ are called *subspaces*. A subspace of $\mathrm{AG}(n, \mathbb{F})$ can be considered as its corresponding subspace U in $\mathrm{PG}(n, \mathbb{F})$ with its subspaces in $U \cap H_\infty$ removed.

So, an affine geometry is a projective geometry with a hyperplane H_∞ removed. A k-space U of the affine geometry $\mathrm{AG}(n, \mathbb{F})$ corresponds to a k-space of $\mathrm{PG}(n, \mathbb{F})$. The intersection of this k-space and H_∞ is a $(k-1)$-space in $H_\infty \subset \mathrm{PG}(n, \mathbb{F})$. It is called the *subspace 'at infinity'* of U and H_∞ is called the *hyperplane at infinity*. For example, an affine line has a point at infinity. Therefore, a projective geometry is an affine geometry with its 'structure at infinity' added.

Alternatively, the affine space $\mathrm{AG}(n, \mathbb{F})$ can be constructed directly from a vector space $V(n, \mathbb{F})$, but we will not discuss this in detail.

Remark 1.4.2. By choosing $X_0 = 0$ as the hyperplane H_∞ in $\mathrm{PG}(n, \mathbb{F})$, all projective points not on H_∞ can be written as $(1, x_1, \ldots, x_n)$. Dropping the redundant first coordinate, we can describe all affine points with the non-homogeneous coordinates $(x_1, \ldots, x_n)$.

All affine hyperplanes can be described by an equation $a_1X_1 + \cdots + a_nX_n = d$, with $(a_1, \ldots, a_n) \neq (0, \ldots, 0)$.

1.5 Axiomatic projective and affine geometries

We have introduced projective and affine geometries using vector spaces. Alternatively, we can do this axiomatically.

Definition 1.5.1. A point-line geometry is an *axiomatic projective plane* if it satisfies the following conditions:

(PP1) any two distinct points are incident with a unique common line;

(PP2) any two distinct lines are incident with a unique common point;

(PP3) there exist four points such that no three of them are collinear.

If on one line of an axiomatic projective plane $\mathcal{P}$ there are $n+1$ points, $n \in \mathbb{N}$, then $\mathcal{P}$ is a $2-(n^2+n+1, n+1, 1)$ design. It is called an axiomatic projective plane *of order n*. We know that $\mathrm{PG}(2,q)$, q a prime power, is an axiomatic projective plane of order q. In these projective planes *Desargues' theorem* is valid.

Theorem 1.5.2 (Desargues). *Let $P_1P_2P_3$ and $Q_1Q_2Q_3$ be two triangles in $\mathrm{PG}(2,q)$ such that the lines P_1Q_1, P_2Q_2 and P_3Q_3 are concurrent. Then the points $\langle P_1,P_2\rangle \cap \langle Q_1,Q_2\rangle$, $\langle P_1,P_3\rangle \cap \langle Q_1,Q_3\rangle$ and $\langle P_2,P_3\rangle \cap \langle Q_2,Q_3\rangle$ are collinear.*

Therefore these planes are called *Desarguesian*. Many non-Desarguesian axiomatic projective planes are known, but we will not deal with them in this thesis. The order of all known finite axiomatic projective planes is a prime power. Several nonisomorphic axiomatic projective planes of the same order exist. The smallest order for which non-isomorphic axiomatic projective planes exist is 9. It remains an open question whether finite axiomatic projective planes exist whose order is not a prime power.

Probably the best known projective plane is the unique projective plane of order 2. It is called the *Fano plane*.

Also projective spaces of higher dimension, can be introduced axiomatically.

Definition 1.5.3. A point-line geometry is an *axiomatic projective geometry* if it satisfies the following conditions:

(PG1) any two distinct points P and Q are incident with a unique common line $\ell_{P,Q}$;

(PG2) if P_1, P_2, P_3 and P_4 are four different points such that the lines ℓ_{P_1,P_2} and ℓ_{P_3,P_4} are incident with a common point, then the lines ℓ_{P_1,P_3} and ℓ_{P_2,P_4} are also incident with a common point;

(PG3) any line is incident with at least three points.

It can easily be seen that the point-line geometry derived from the projective geometry $\mathrm{PG}(n,\mathbb{F})$ satisfies these conditions.

For such an axiomatic projective geometry we can define subspaces in the following way: a subspace is a set of points S such that any point on a line containing at least two points of S, is also contained in S. For any subspace S, we can define its dimension k as the largest number for which we can find a strictly increasing chain of subspaces $\emptyset \subset S_0 \subset S_1 \subset \cdots \subset S_k = S$. The axiomatic projective geometry itself is also a subspace, so its dimension can be determined.

An important result was obtained by Veblen and Young in [126, 127, 128]. They proved that an axiomatic projective geometry of dimension $n \geq 3$, is necessarily a point-line geometry derived from a projective geometry over a division ring. Due to Wedderburn's Little Theorem, which states that every finite division ring is a field, we know that every finite axiomatic projective geometry of dimension $n \geq 3$ is derived from a projective geometry $\mathrm{PG}(n, q)$. By the above remarks on axiomatic projective planes, we know this is not true for $n = 2$.

Also affine geometries can be introduced axiomatically. We will not consider the general case here. We will restrict ourselves to axiomatic affine planes.

Definition 1.5.4. A point-line geometry is an *axiomatic affine plane* if it satisfies the following conditions:

(AP1) any two distinct points are incident with a unique common line;

(AP2) for every line ℓ and every point P which is not incident with ℓ, a unique line ℓ' exists such that $P \in \ell'$ and such that ℓ and ℓ' have no point in common;

(AP3) there exist three non-collinear points.

If on one line of an axiomatic affine plane $\mathcal{P}$ there are n points, $n \in \mathbb{N}$, then $\mathcal{P}$ is a $2-(n^2, n, 1)$ design. It is called an axiomatic affine plane *of order* n. We know that $\mathrm{AG}(2, q)$, q a prime power, is an axiomatic affine plane of order q. From a finite axiomatic affine plane always a finite axiomatic projective plane of the same order can be constructed, and vice versa, by respectively adding or removing a 'line at infinity'. The above remarks on the existence of finite axiomatic projective planes of a given order, are consequently also valid for finite axiomatic affine planes.

Remark 1.5.5. Two lines of an axiomatic affine plane $\mathcal{A}$ are called *parallel* if they are equal or disjoint. Axiom (AP2) states there is, given a line and a point, always a line through the point parallel to the given line. Parallel lines pass through the same point on the line at infinity. Parallelism defines an equivalence relation on the set of lines of $\mathcal{A}$. The equivalence classes are called *parallel classes*. By axiom (AP2) the set of lines of one parallel class determines a partition of the point set.

An axiomatic affine plane of order n contains n^2 points and n^2+n lines. There are $n+1$ parallel classes (corresponding to the $n+1$ points at infinity), each containing n lines. Through every affine point passes one line of each parallel class.

1.6 Polar spaces

Polar spaces are an important type of incidence geometries, with many similarities to projective spaces. They will be studied in several chapters of this thesis. Their axiomatic introduction is due to Veldkamp ([129, 130]) and Tits ([121]).

Definition 1.6.1. A *polar space of rank* n, $n \geq 3$, is an incidence geometry $(\mathcal{V}, \Delta_n, t, I)$ satisfying the following axioms.

(PS1) The incidence structure arising from an element of $v \in \mathcal{V}$ by considering all elements of $\mathcal{V}$ that are contained in v, is a projective geometry of dimension $t(v)$.

(PS2) The intersection of two elements of $\mathcal{V}$ (the set of all elements of $\mathcal{V}$ that are contained in both, containment allows equality here) is an element of $\mathcal{V}$ (together with the elements of $\mathcal{V}$ that are contained in it) or empty.

(PS3) For a point $P \in \mathcal{V}$ and an element $v \in \mathcal{V}$ with $t(v) = n-1$ and such that P and v are not incident, there is a unique element $v' \in \mathcal{V}$ with $t(v') = n-1$ such that $t(v \cap v') = n-2$, with $v \cap v'$ the intersection of v and v'. This intersection contains all points in v that are collinear with P.

(PS4) There exist two elements $v, v' \in \mathcal{V}$ such that $t(v) = t(v') = n-1$ and the intersection of v and v' is empty.

The elements of $\mathcal{V}$ are called *subspaces*. They are isomorphic to projective spaces. Note that the dimension of a subspace corresponds to its projective dimension. The subspaces of dimension $n-1$, the maximal dimension, are called *generators*. Polar spaces were introduced by Tits in [120].

Polar spaces of rank 2 are called generalised quadrangles.

Definition 1.6.2. A *generalised quadrangle* is a point-line geometry satisfying the following axioms.

(GQ1) Two distinct points are incident with at most one line.

(GQ2) Every line contains $s+1$ points, $s \geq 1$ and every point is contained in $t+1$ lines, $t \geq 1$.

(GQ3) For every point P and every line ℓ such that $P \notin \ell$, there is a unique tuple (P', ℓ'), $P' \in \ell$ a point and $\ell' \ni P$ a line, such that $P' \in \ell'$.

It follows from this definition that every line is incident with the same number of points, namely $s+1$, and that every point is incident with the same number of lines, namely $t+1$. If these numbers are finite, then the generalised quadrangle is said to have *order* (s, t). If $s = t$, then the generalised quadrangle is said to have *order* s.

We now introduce the *classical polar spaces*. They arise from some special forms on vector spaces. A *bilinear form* on a vector space V over a field $\mathbb{F}$ is a map $V \times V \to \mathbb{F}$ that is linear in both its arguments. A *sesquilinear form* on a vector space V over a field $\mathbb{F}$ is a map $V \times V \to \mathbb{F}$ that is linear in the first argument, and semilinear[2] in the second argument. A *quadratic form* on a vector space V over a field $\mathbb{F}$ is a map $Q : V \to \mathbb{F}$ that is homogeneous of the second degree, and such that $f : V \times V \to \mathbb{F} : (v, w) \mapsto Q(v+w) - Q(v) - Q(w)$ is a bilinear form.

A bilinear form f is called *symplectic* if $f(v, v) = 0$ for all v. A sesquilinear form on V is called *Hermitian* if the corresponding field automorphism θ is an involution and $f(v, w) = f(w, v)^\theta$ for all $v, w \in V$.

A vector $v \in V$ is called *singular* with respect to a bilinear or sesquilinear form f if $f(v, w) = 0$ for all $w \in V$; it is called *singular* with respect to a

[2] A map $f : V \to W$, with V, W vector spaces over a field $\mathbb{F}$, is semilinear if there is a field automorphism θ such that $f(av + bw) = a^\theta f(v) + b^\theta f(w)$ for all $v, w \in V$ and $a, b \in \mathbb{F}$.

quadratic form Q if $Q(v+w)=Q(w)$ for all $w\in V$. The quadratic, bilinear or sesquilinear form itself is called *non-degenerate* if the zero vector is the only singular vector.

A subspace W of a vector space V is called *totally isotropic* with respect to a quadratic form if its restriction to W is trivial; a subspace W of a vector space V is called *totally isotropic* with respect to a bilinear or sesquilinear form if its restriction to $W\times W$ is trivial.

Now we can describe the classical polar spaces. They consist of the totally isotropic subspaces of a vector space $V(m+1,\mathbb{F})$ over the field $\mathbb{F}$, with respect to a non-degenerate quadratic, symplectic or Hermitian form, and are equipped with the natural incidence relation. These classical polar spaces can be seen as substructures of the projective geometry $\mathrm{PG}(m,\mathbb{F})$. In this thesis, we will often consider the classical polar spaces through their embedding in the projective space. Note that these classical polar spaces are polar spaces if their rank is at least three, and that they are generalised quadrangles if their rank equals two.

We have a closer look at these classical polar spaces for $\mathbb{F}=\mathbb{F}_q$. The polar spaces arising from a quadratic form are called *quadric polar spaces* (or *quadrics*). Let Q be a non-degenerate quadratic form on the vector space $V(m+1,q)$. If m is even, by choosing an appropriate basis for $V(m+1,q)$, Q can be written as $Q(X_0,\dots,X_m)=X_0^2+X_1X_2+\cdots+X_{m-1}X_m$. This quadratic form is called *parabolic*. If m is odd, by choosing an appropriate basis for $V(m+1,q)$, Q can be written as $Q(X_0,\dots,X_m)=X_0X_1+\cdots+X_{m-1}X_m$ or as $Q(X_0,\dots,X_m)=X_0X_1+\cdots+X_{m-3}X_{m-2}+g(X_{m-1},X_m)$, with g an irreducible homogeneous polynomial of degree 2 over $\mathbb{F}_q$. In the former case, the quadratic form is called *hyperbolic*; in the latter case it is called *elliptic*.

The polar spaces arising from a symplectic form are called *symplectic polar spaces*. A non-degenerate symplectic form f on $V(m+1,q)$ only exists if m is odd. In this case, we can choose an appropriate basis $\{e_1,\dots,e_n,e'_1,\dots,e'_n\}$ for $V(2n,q)$, $m+1=2n$, such that $f(e_i,e_j)=f(e'_i,e'_j)=0$ and $f(e_i,e'_j)=\delta_{i,j}$, with $1\le i,j\le n$.

The polar spaces arising from a Hermitian form are called *Hermitian polar spaces*. The construction of a Hermitian form requires an involutory field automorphism of $\mathbb{F}_q$. This only exists if q is a square. The only involutory field automorphism of $\mathbb{F}_{q^2}$ is given by $x\mapsto x^q$. Let f be a non-degenerate Hermitian form on the vector space $V(m+1,q)$. We can choose an appropriate basis $\{e_0, \ldots, e_m\}$ for $V(m+1, q)$ such that $f(e_i, e_j) = \delta_{i,j}$, with $0 \leq i, j \leq m$. If m is even, then we can find a totally isotropic $\frac{m}{2}$-dimensional subspace of $V(m+1, q)$. If m is odd, then we can find a totally isotropic $\frac{m+1}{2}$-dimensional subspace of $V(m+1, q)$.

We now list these classical polar spaces of rank d.

- The *hyperbolic quadric* $\mathcal{Q}^+(2d-1, q)$ embedded in $\mathrm{PG}(2d-1, q)$. It arises from a hyperbolic quadratic form on $V(2d, q)$.
- The *parabolic quadric* $\mathcal{Q}(2d, q)$ embedded in $\mathrm{PG}(2d, q)$. It arises from a parabolic quadratic form on $V(2d+1, q)$.
- The *elliptic quadric* $\mathcal{Q}^-(2d+1, q)$ embedded in $\mathrm{PG}(2d+1, q)$. It arises from an elliptic quadratic form on $V(2d+2, q)$.
- The *Hermitian polar space* $\mathcal{H}(2d-1, q^2)$ embedded in $\mathrm{PG}(2d-1, q^2)$. It arises from a Hermitian form on $V(2d, q^2)$, constructed using the field automorphism $x \mapsto x^q$.
- The *Hermitian polar space* $\mathcal{H}(2d, q^2)$ embedded in $\mathrm{PG}(2d, q^2)$. It arises from a Hermitian form on $V(2d+1, q^2)$, constructed using the field automorphism $x \mapsto x^q$.
- The *symplectic polar space* $\mathcal{W}(2d-1, q)$ embedded in $\mathrm{PG}(2d-1, q)$. It arises from a symplectic form on $V(2d, q)$, which is bilinear and for which all vectors are isotropic.

In [121], Tits has proved that all finite polar spaces of rank at least 3, are classical and thus given by the above list. This result is not true for infinite polar spaces.

We now introduce the polarities associated to a polar space. Based on a quadratic form f on a vector space $V = V(m+1, \mathbb{F})$ we can define a bilinear form $f' : V \times V \to \mathbb{F}$ by $f'(v, w) \mapsto f(v+w) - f(v) - f(w)$. For a bilinear or sesquilinear form we set $f' = f$. For a subspace W of V we can define its orthogonal complement regarding f':

$$W^\perp = \{v \in V \mid \forall w \in W : f'(v, w) = 0\} .$$

Considering the subspaces of V as subspaces of $\mathrm{PG}(m, \mathbb{F})$, the mapping β that maps the subspace W onto the subspace $W^\perp$, is an involutory duality. It is called a *polarity*. The subspaces of a polar space in $\mathrm{PG}(m, \mathbb{F})$ are precisely those subspaces that are contained in their image under the polarity. Geometrically, the image of a subspace on the polar space under the corresponding polarity, is its tangent space.

To each of the above finite classical polar spaces we can attach a parameter e. For a polar space of rank d embedded in $\mathrm{PG}(m, q)$, with $m = 2d-1, 2d, 2d+1$ as above, $q^e + 1$ is the number of generators ($(d-1)$-spaces) through a $(d-2)$-space. The parameters of the polar spaces are given in the following table:

polar space	e
$\mathcal{Q}^+(2d-1, q)$	0
$\mathcal{H}(2d-1, q)$	1/2
$\mathcal{Q}(2d, q)$	1
$\mathcal{W}(2d-1, q)$	1
$\mathcal{H}(2d, q)$	3/2
$\mathcal{Q}^-(2d+1, q)$	2

Now, using the rank d and the parameter e of a polar space, the number of subspaces on this polar space can be calculated.

Lemma 1.6.3 ([23, Lemma 9.4.1]). *On a classical finite polar space of rank d with parameter e, embedded in a projective space over $\mathbb{F}_q$, the number of k-spaces is given by*

$$\begin{bmatrix} d \\ k+1 \end{bmatrix}_q \prod_{i=1}^{k+1} (q^{d+e-i} + 1).$$

Corollary 1.6.4. *On a classical finite polar space of rank d with parameter e, embedded in a projective space over $\mathbb{F}_q$, the number of k-spaces through a fixed m-space is given by*

$$\begin{bmatrix} d-m-1 \\ k-m \end{bmatrix}_q \prod_{i=1}^{k-m} (q^{d+e-m-i-1} + 1).$$

Above, we used non-degenerate forms for the construction of polar spaces. If f is a degenerate quadratic, symplectic or Hermitian form on the vector space $V(m+1, \mathbb{F})$, then we can choose a basis of $V(m+1, \mathbb{F})$ such that f can be written with at most m variables. We can repeat the above construction, but f

does not yield a polar space. We find a cone in $\mathrm{PG}(m, \mathbb{F})$ with vertex a subspace (corresponding to the singular vectors) and base a quadratic, symplectic or Hermitian polar space. This is a quadratic, symplectic or Hermitian cone.

A *quadratic variety* is described by a quadratic form, so either a quadric (quadric polar space) or a quadratic cone. In the same way we can define *Hermitian and symplectic varieties*. If a variety is a polar space, then it is called *non-singular*; if it is a cone, then it is called *singular*. Intersecting a variety in $\mathrm{PG}(m, \mathbb{F})$ with a subspace of $\mathrm{PG}(m, \mathbb{F})$, yields clearly a variety of the same type (quadratic, Hermitian or symplectic). Whether the intersection is a polar space or a cone, and in the latter case, the dimension of the vertex of the cone, depends on the actual position of the subspace related to the variety. For example, a tangent hyperplane at a point P to a classical polar space of rank d intersects the polar space in a cone with vertex P and base a classical polar space of rank $d-1$ of the same type. A non-tangent hyperplane intersects it in a classical polar space of the same type.

We end this section with a few remarks on some polar spaces.

Example 1.6.5. A quadratic variety in $\mathrm{PG}(2, q)$ is called a *conic*. It is given by an equation $\sum_{0 \leq i \leq j \leq 2} a_{i,j} X_i X_j = 0$. It can be a parabolic quadric $\mathcal{Q}(2, q)$; in this case the conic is non-singular.

Example 1.6.6. Consider the Hermitian polar space $\mathcal{H}(2, q^2)$ in the projective plane $\mathrm{PG}(2, q^2)$; this is a non-singular Hermitian variety. Up to projective transformations it is defined by $X_0^{q+1} + X_1^{q+1} + X_2^{q+1} = 0$. The set of points on $\mathcal{H}(2, q^2)$ and the lines of $\mathrm{PG}(2, q^2)$ meeting $\mathcal{H}(2, q^2)$ in $q+1$ points, determine a unital. This unital is known as the *classical unital* or *Hermitian unital*. Sometimes, the point set itself is called a unital.

Remark 1.6.7. The generators of the hyperbolic quadrics have a remarkable property. The set of generators Ω on $\mathcal{Q}^+(2n+1, q)$ can be partitioned into two equivalence *classes* Ω_1 and Ω_2, using the equivalence relation $\sim$ which is defined as follows: $\pi_1 \sim \pi_2 \Leftrightarrow \dim(\pi_1 \cap \pi_2) \equiv n \pmod 2$, for two generators π_1 and π_2 of $\mathcal{Q}^+(2n+1, q)$.

The first equivalence class Ω_1 is called the class of *Latin generators* and the second equivalence class Ω_2 is called the class of *Greek generators*. The definition of $\sim$, now implies in particular that

- on $\mathcal{Q}^+(4n+1, q)$, the generators of Ω_1 pairwise intersect, and the generators of Ω_2 pairwise intersect;

- on $\mathcal{Q}^+(4n+3,q)$, any generator of Ω_1 intersects any generator of Ω_2.

This observation will play an important role in the results in Table 2.1 and Chapter 4.

Remark 1.6.8. If q is even, then there is a strong connection between $\mathcal{W}(2d-1,q)$ and $\mathcal{Q}(2d,q)$. Let N be the *nucleus* of the parabolic quadric $\mathcal{Q}(2d,q)$. This is the point, not on the quadric, only lying on tangent lines to the quadric. Projecting $\mathcal{Q}(2d,q)$ from N onto a hyperplane α disjoint to N yields the symplectic polar space $\mathcal{W}(2d-1,q)$, q even. More information about this link between $\mathcal{W}(2d-1,q)$ and $\mathcal{Q}(2d,q)$, q even, can be found in [81, Chapter 22] and [118, Chapter 11].

In particular, by choosing α a hyperplane intersecting $\mathcal{Q}(2d,q)$, q even, in a hyperbolic quadric $\mathcal{Q}^+(2d-1,q)$ or an elliptic quadric $\mathcal{Q}^-(2d-1,q)$, we can see that the polar spaces $\mathcal{Q}^+(2d-1,q)$ and $\mathcal{Q}^-(2d-1,q)$ are embedded in the polar space $\mathcal{W}(2d-1,q)$, q even.

1.7 Arcs, blocking sets, reguli and spreads

In this section we will describe several substructures of $\mathrm{PG}(n,q)$ which we will study or use in this thesis.

Definition 1.7.1. A (k,t)*-arc* $\mathcal{A}$ in $\mathrm{PG}(2,q)$ is a set of k points such that at most t of them are collinear, $k \geq t$. A (k,t)-arc in $\mathrm{PG}(2,q)$ of *type* $(t_1,\ldots,t_m)$, $0 \leq t_1 < \cdots < t_m \leq t$, is a (k,t)-arc such that for every line ℓ in $\mathrm{PG}(2,q)$ the intersection size $|\ell \cap \mathcal{A}|$ equals t_i for some i and such that each value t_i occurs as intersection size for some line.

A $(k,2)$-arc in $\mathrm{PG}(2,q)$ will briefly be denoted as a k*-arc*. A k-arc is called *complete* if it is not contained in a $(k+1)$-arc.

A line meeting an arc (or any other point set) in precisely i-points is called an i*-secant* to the arc. A 1-secant (meeting the arc in a point P) is called a *tangent line* (to the arc at the point P).

An *oval* in $\mathrm{PG}(2,q)$ is a $(q+1)$-arc. It is necessarily of type $(0,1,2)$. Through every point of an oval passes precisely one tangent line. A *hyperoval* in $\mathrm{PG}(2,q)$ is a $(q+2)$-arc. It is necessarily of type $(0,2)$. There are thus no tangent lines to a hyperoval.

A hyperoval in $\mathrm{PG}(2,q)$ can only exist if q is even. Every hyperoval is a complete arc. In $\mathrm{PG}(2,q)$, q odd, every oval is a complete arc. For ovals in $\mathrm{PG}(2,q)$, q even, it is known that all tangent lines pass through a common point, which is called the *nucleus* (see also Remark 1.6.8). The next theorem follows immediately.

Theorem 1.7.2 ([20]). *Every $(q+1)$-arc in $\mathrm{PG}(2,q)$, q even, is contained in a hyperoval and hence not complete.*

This result was generalised to non-Desarguesian planes in [106].

A non-singular conic in $\mathrm{PG}(2,q)$ determines a set of $q+1$ points, no three of them collinear. Hence, every non-singular conic is an oval. The following result of Segre considers the inverse implication.

Theorem 1.7.3 ([109, 110]). *Every oval in $\mathrm{PG}(2,q)$, q odd, is a non-singular conic.*

This result is not true for $\mathrm{PG}(2,q)$, q even. For example, from a hyperoval which is a conic together with its nucleus, we can delete a point different from the nucleus. In general, this construction does not yield a conic.

Different from the k-arcs, also other types of arcs have received attention.

Definition 1.7.4. A (k,t)-arc in $\mathrm{PG}(2,q)$ of type $(t_1,\dots,t_m)$ is a *set of even type* if all $t_1,\dots,t_m$ are even.

If the projective plane $\mathrm{PG}(2,q)$ contains a set of even type, then necessarily q is even.

Among the sets of even type, next to the hyperovals, the $(q+t,t)$-arcs of type $(0,2,t)$ in $\mathrm{PG}(2,q)$, q even, are an intensively studied class. They were introduced by Korchmáros and Mazzocca in [91]; for $t=\frac{q}{2}$, they were considered earlier in [98]. In [91] it was noted that a $(q+t,t)$-arc of type $(0,2,t)$ only can exist if $t \mid q$, and that through every point of a $(q+t,t)$-arc of type $(0,2,t)$, there pass q different 2-secants and one t-secant. In [61], it is proved that a $(q+t,t)$-arc of type $(0,2,t)$ has a t-*nucleus*, a common point of all its t-secants.

We mentioned before that the existence of a $(q+t,t)$-arc of type $(0,2,t)$, implies that t divides q. However, it remains an open problem for which pairs (q,t) they exist. Constructions of infinite classes can be found in [61, 91, 123]. Some sporadic examples were found in [88, 95].

For example, for $t = 4$, we only know examples for $q \leq 32$ (for $q = 8, 16$: see [91]; $q = 32$: see [88]).

Now, we turn our attention to another structure: blocking sets.

Definition 1.7.5. A *blocking set* B of $\mathrm{PG}(n, q)$ with respect to the k-spaces, $0 < k < n$, is a point set in $\mathrm{PG}(n, q)$ which has a non-empty intersection with every k-space. A point $P \in B$ is called *essential* in B if $B \setminus \{P\}$ is not a blocking set. If all points of B are essential, then B is called *minimal*.

The smallest blocking sets were classified by Bose and Burton.

Theorem 1.7.6 ([21, Theorem 2]). *If B is a blocking set of $\mathrm{PG}(n, q)$ with respect to the k-spaces, then $|B| \geq \theta_{n-k}(q)$. Moreover, if $|B| = \theta_{n-k}(q)$, then B is an $(n-k)$-dimensional subspace of $\mathrm{PG}(n, q)$.*

The $(n-k)$-spaces are the smallest blocking sets of $\mathrm{PG}(n, q)$ with respect to the k-spaces. All blocking sets of $\mathrm{PG}(n, q)$ with respect to the k-spaces which contain an $(n-k)$-space, are called *trivial*; the ones which do not contain an $(n-k)$-space are called *non-trivial*. A blocking set in $\mathrm{PG}(n, q)$ with respect to the k-spaces is called *small* if it contains less than $\frac{3}{2}(q^{n-k} + 1)$ points. This is roughly $\frac{3}{2}$ times the size of the smallest blocking set in $\mathrm{PG}(n, q)$ with respect to the k-spaces.

The most studied blocking sets are blocking sets of $\mathrm{PG}(2, q)$ with respect to the lines. Much effort has been made to find the smallest non-trivial minimal blocking sets. For a prime power q, we define $r(q) = |B| - q - 1$, with B the smallest non-trivial minimal blocking set of $\mathrm{PG}(2, q)$ with respect to the lines. For $q = 2$ no such blocking set exists, so $r(2)$ is not defined. We give an overview of the results on $r(q)$.

Theorem 1.7.7. *Consider $q = p^h$, p prime, $h \in \mathbb{N} \setminus \{0\}$.*

- *([9]) If $h = 1$ and $p \neq 2$ (q an odd prime), then $r(q) = \frac{q+1}{2}$.*
- *([25, 26]) If h is even, then $r(q) = \sqrt{q}$. Moreover, if B is a non-trivial blocking set of size $q + \sqrt{q} + 1$, then B is a Baer subplane.*
- *([18]) If $h \geq 3$ is odd and $p > 3$, then $r(q) = \sqrt[3]{q^2}$.*

- *([10, 18]) If $h \geq 3$ is odd and $p \in \{2, 3\}$, then $r(q) \geq \max\left\{\sqrt{pq}, \sqrt[3]{\frac{q^2}{2}}\right\}$. If $h = 3$, then $r(q) = \sqrt[3]{q^2} = \sqrt{pq}$.*

The first result states that there are no small non-trivial blocking sets in $\mathrm{PG}(2, p)$, p prime. Now, we state a result about the smallest non-trivial blocking sets in a more general setting.

Theorem 1.7.8 ([7, 76]). *The smallest non-trivial blocking sets with respect to the k-spaces in $\mathrm{PG}(n, q)$, which are necessarily minimal, are the cones with an $(n - k - 2)$-space π as vertex and a non-trivial minimal blocking set with respect to the lines in a plane V of size $q + r(q) + 1$, with $r(q)$ as above, as base, $V \cap \pi = \emptyset$. Their size equals $\theta_{n-k}(q) + r(q)q^{n-k-1}$, with $r(q)$ as above.*

The third substructure that we introduce, is the regulus.

Definition 1.7.9. A *regulus* in $\mathrm{PG}(3, q)$ is a set $\mathcal{L}$ of $q + 1$ pairwise disjoint lines such that any line having a non-empty intersection with three lines of $\mathcal{L}$, meets all lines of $\mathcal{L}$.

We have a look at the following result, which we will use throughout Chapter 3.

Lemma 1.7.10. *Let $\mathcal{P}$ be a three-dimensional projective space $\mathrm{PG}(3, q)$ or a polar space of rank 2 embedded in $\mathrm{PG}(3, q)$. Let ℓ_1, ℓ_2 and ℓ_3 be three pairwise disjoint lines in $\mathcal{P}$. Let $\mathcal{L}$ be the set of lines in $\mathcal{P}$ meeting ℓ_1, ℓ_2 and ℓ_3 all three in a point. If $\mathcal{L}$ is non-empty, then let $\mathcal{L}'$ be the set of lines in $\mathcal{P}$ meeting all lines of $\mathcal{L}$ in a point. If $\mathcal{P}$ is a projective space $\mathrm{PG}(3, q)$, a hyperbolic quadric $\mathcal{Q}^+(3, q)$ or a symplectic polar space $\mathcal{W}(3, q)$, q even, then $\mathcal{L}$ is a regulus and $\mathcal{L}'$ is also a regulus, called the opposite regulus. If $\mathcal{P}$ is a Hermitian variety $\mathcal{H}(3, q)$, q a square, then $\mathcal{L}$ is a set of $\sqrt{q} + 1$ lines corresponding to a regulus in a hyperbolic quadric $\mathcal{Q}^+(3, \sqrt{q})$ embedded in $\mathcal{H}(3, q)$ and $\mathcal{L}'$ is the set of lines corresponding to the opposite regulus in $\mathcal{Q}^+(3, \sqrt{q}) \subset \mathcal{H}(3, q)$. If $\mathcal{P}$ is a symplectic polar space $\mathcal{W}(3, q)$, q odd, then $\mathcal{L}$ is either empty or a set of two lines, and $\mathcal{L}'$ is the set of $q + 1$ lines of $\mathcal{P}$ meeting both lines of $\mathcal{L}$ in the second case.*

Proof. All these statements are generally known. For the projective case, a clear proof is given [79, Theorem 15.3.12], based on [111, Section 190]. For the quadric case, see the remarks following the referred theorem for the projective case. For the symplectic case, q even, one can use the relationship between these polar spaces and the quadrics. For this relationship, see Remark 1.6.8. The Hermitian case was treated in [113, Section 86]; a short proof can be found in [79, Lemma 19.3.1]. For the symplectic case, q odd, see [4, Lemma 2.1]. This result in [4] is itself a corollary from several results in [102, Sections 1.3.6 and 3.3.1]. □

So, for $\mathrm{PG}(3,q)$, a regulus is the set of lines in $\mathrm{PG}(3,q)$ meeting three given, pairwise disjoint lines. Such a regulus contains $q+1$ lines. The set of lines meeting all lines of a given regulus, is a regulus itself and is called the *opposite regulus*. These two reguli determine a hyperbolic quadric $\mathcal{Q}^+(3,q)$. They correspond to the two classes of generators.

The final concept we introduce in this section is the spread.

Definition 1.7.11. A set of t-spaces in $\mathrm{PG}(n,q)$, having pairwise no point in common, is called a *partial t-spread*. If a partial t-spread cannot be extended to a larger one, then it is called *maximal*. If a partial t-spread covers all points of $\mathrm{PG}(n,q)$, then it is called a *t-spread*.

A t-spread exists in $\mathrm{PG}(n,q)$ if and only if $(t+1)|(n+1)$, a classical result ([112]). Maximal partial t-spreads have been the subject of much research. Especially, the maximal partial line spreads in $\mathrm{PG}(3,q)$ have received a lot of attention. We refer the reader to [5, 73, 75] for the study of large maximal partial line spreads in $\mathrm{PG}(3,q)$, to [64, 97] for the study of large maximal partial t-spreads in $\mathrm{PG}(n,q)$ and to [59, 71, 72, 74] for spectrum results.

1.8 Linear codes

The only codes that will be discussed in this thesis are linear codes over a field ([77, 122]).

Definition 1.8.1. A *linear code* C of *length* n over $\mathbb{F}_q$ is a subspace of the vector space $V(n,q)$. If $\dim(C)=k$, then C is called an *$[n,k]$-code*. The elements of C are *code words*. A *generator matrix* for C is a $(k \times n)$-matrix whose rows form a basis of C.

The *support* of a code word is the set of its non-zero positions. We denote the support of $c \in C$ by $\text{supp}(c)$. The *weight* $\text{wt}(c)$ of a code word $c \in C$, equals $|\text{supp}(c)|$, its number of non-zero positions. The *minimum weight* of a linear code C, is the minimum of the weights of the non-zero code words.

The *distance* $d(c, c')$ between two code words $c, c' \in C$, is $\text{wt}(c - c')$, the number of positions in which the corresponding coordinates of c and c' differ. The *minimum distance* $d(C)$ of C then equals $\min\{d(c, c') \mid c, c' \in C, c \neq c'\}$. An $[n, k]$-code with minimum distance d, is called an $[n, k, d]$*-code*.

A linear code over $\mathbb{F}_q$, is sometimes called q*-ary*, e.g. binary ($\mathbb{F}_2$), ternary ($\mathbb{F}_3$), ... For a linear code, its minimum weight and its minimum distance coincide.

Regarding the standard dot product for vectors in $V(n, q)$, the linear code C has an orthogonal complement. This is also a code; it is called the *dual code* and denoted by $C^\perp$. If C is an $[n, k]$-code, then $C^\perp$ is an $[n, n-k]$-code. For any vector $c \in C$ and any vector $c' \in C^\perp$, we know that $c \cdot c' = 0$. Moreover, if G is a generator matrix of C, then $Gc' = 0$ for any $c' \in C^\perp$. Therefore G is called a *parity check matrix* of $C^\perp$. Vice versa, a generator matrix for $C^\perp$ is a parity check matrix for C.

We mention one more concept.

Definition 1.8.2. Let C be a linear code and let $\delta > 1$ be an integer such that the weight of every code word of C is divisible by δ. Then δ is called a *divisor* of the code C.

There are various links between finite geometry and coding theory. A first important link is based on the following matrix.

Definition 1.8.3. Consider the projective geometry $\text{PG}(n, q)$, $q = p^h$, p a prime, $h \in \mathbb{N} \setminus \{0\}$, and consider $s, t \in \mathbb{N}$, with $0 \leq s < t \leq n-1$. Let $M_{s,t}(n, q)$ be the $\mathbb{F}_p$-matrix whose rows are labelled by the t-spaces and whose columns are labelled by the s-spaces of $\text{PG}(n, q)$, and such that

$$(M_{s,t}(n, q))_{i,j} = \begin{cases} 1 & \text{if } t\text{-space } i \text{ contains } s\text{-space } j, \\ 0 & \text{otherwise.} \end{cases}$$

This matrix is called the *incidence matrix* of s-spaces and t-spaces of $\text{PG}(n, q)$. The p-ary code generated by the rows of this matrix will be denoted by $C_{s,t}(n, q)$. It is called *the code generated by the s-spaces and t-spaces of* $\text{PG}(n, q)$.

The code $C_{0,t}(n,q)$ will briefly be denoted by $C_t(n,q)$. For $n = 2$, the code $C_1(2,q)$ is denoted by $C(2,q)$.

Note that the incidence matrix $M_{s,t}(n,q)$ is not a generator matrix of $C_{s,t}(n,q)$. Its rows span the code (by definition), but they are not linearly independent. Much research has been performed regarding these codes and their duals. A good survey on these codes and their duals can be found in [94]. We mention one important result.

Theorem 1.8.4 ([2, Corollary 6.4.4]). *Let C be the code $C(2,q)$. The minimum weight of $C \cap C^{\perp}$ is $2q$ and the code words of minimum weight are obtained by taking a scalar multiple of the difference of the incidence vectors of two lines.*

Incidence matrices can be defined for all incidence geometries. Here, next to the projective spaces, we only introduce the incidence matrices and corresponding codes of polar spaces, since these are discussed in this thesis.

Definition 1.8.5. Consider the finite polar space $\mathcal{P}$ of rank d embedded in $\mathrm{PG}(n,q)$, $q = p^h$, p a prime, $h \in \mathbb{N} \setminus \{0\}$, and consider $s,t \in \mathbb{N}$, with $0 \leq s < t \leq d-1$. Let $M_{s,t}(\mathcal{P})$ be the $\mathbb{F}_p$-matrix whose rows are labelled by the t-spaces of $\mathcal{P}$ and whose columns are labelled by the s-spaces of $\mathcal{P}$, and such that

$$\left(M_{s,t}(\mathcal{P})\right)_{i,j} = \begin{cases} 1 & \text{if } t\text{-space } i \text{ contains } s\text{-space } j, \\ 0 & \text{otherwise.} \end{cases}$$

This matrix is called the *incidence matrix* of s-spaces and t-spaces in $\mathcal{P}$. The p-ary code generated by the rows of this matrix will be denoted by $C_{s,t}(\mathcal{P})$. It is called *the code generated by the s-spaces and t-spaces of $\mathcal{P}$.*

The code $C_{0,t}(\mathcal{P})$ will briefly be denoted by $C_t(\mathcal{P})$.

Another link between finite projective geometries and codes is based on so-called functional codes. The definition of a functional code was first stated in [92].

Definition 1.8.6. Let $\mathcal{X}$ be an algebraic variety in $\mathrm{PG}(n,q)$ with point set $\{P_1,\ldots,P_N\}$. We normalize the coordinates of these points with respect to the leftmost non-zero coordinate, i.e. we set the leftmost non-zero coordinate equal to one by multiplying the coordinates with a scalar. Let $\mathcal{F}$ be a set of homogeneous polynomials in $n+1$ variables, closed under taking linear combinations. The *functional code* $C_{\mathcal{F}}(\mathcal{X})$ is equal to

$$C_{\mathcal{F}}(\mathcal{X}) = \{(f(P_1), \ldots, f(P_N)) \mid f \in \mathcal{F}\}.$$

Recall that the points of $\mathrm{PG}(n,q)$ are defined up to a scalar multiple, so the previous definition only makes sense because of the normalization of the coordinates. A different normalization would yield a different, but equivalent, code. Furthermore, note that these functional codes are linear codes since $\mathcal{F}$ is closed under linear combination. It follows that all polynomials in $\mathcal{F} \setminus \{0\}$ are of the same degree.

We look at two important examples of functional codes.

Example 1.8.7. Let $\mathcal{F}_h$ be the set of all polynomials of degree h (in $n+1$ variables), including the zero polynomial. Let $\mathcal{X}$ be an algebraic variety in $\mathrm{PG}(n,q)$. We denote the code $C_{\mathcal{F}_h}(\mathcal{X})$ by $C_h(\mathcal{X})$.

Example 1.8.8. Let $\mathcal{F}_{Herm}$ be the set of all Hermitian polynomials over $\mathbb{F}_{q^2}$ (in $n+1$ variables), including the zero polynomial. The Hermitian polynomials (in $n+1$ variables) are the polynomials of the form $(X_0, \ldots, X_n)A(X_0^q, \ldots, X_n^q)$ with A a Hermitian matrix, i.e. a matrix that fulfills $A^q = A^t$, whereby A^q denotes $(a_{i,j}^q)_{i=0\ldots n}^{j=0\ldots n}$. Let $\mathcal{X}$ be an algebraic variety in $\mathrm{PG}(n,q^2)$. We denote the code $C_{\mathcal{F}_{Herm}}(\mathcal{X})$ by $C_{Herm}(\mathcal{X})$.

Remark 1.8.9. An interesting problem regarding functional codes is finding their minimum distance and their small weight code words. We will show in this remark that these questions can be solved in a geometrical way. Consider the functional code $C_{\mathcal{F}}(\mathcal{X})$, with $\mathcal{X}$ an algebraic variety in $\mathrm{PG}(n,q)$ and $\mathcal{F}$ a set of homogeneous polynomials in $n+1$ variables. Let $f \neq 0$ be a polynomial in $\mathcal{F}$ and let c_f be its corresponding code word. The equation $f(X_0, \ldots, X_n) = 0$ defines an algebraic variety $\mathcal{Y}$ in $\mathrm{PG}(n,q)$. Let P be a point on $\mathcal{X}$ whose coordinates are written using the chosen normalization. If $f(P) = 0$, then on the one hand $P \in \mathcal{Y}$, but on the other hand $(c_f)_P = 0$. So the points of $\mathcal{X} \cap \mathcal{Y}$ correspond to the zero positions of c_f. Consequently, $\mathrm{wt}(c_f) + |\mathcal{X} \cap \mathcal{Y}| = |\mathcal{X}|$. In order to find (a lower bound on) the minimum distance of $C_{\mathcal{F}}(\mathcal{X})$ it is thus sufficient to find (an upper bound on) the largest size of $\mathcal{X} \cap \mathcal{Y}$ for any $\mathcal{Y}$ that is defined by a polynomial $f \in \mathcal{F} \setminus \{0\}$.

2

Erdős-Ko-Rado problems

nanos gigantium[1] *humeris insidentes*

Attributed to *Bernard de Chartres*
by *John of Salisbury* in his *Metalogicon.*

In 1961 the Hungarian Pál Erdős, the Chinese Chao Ko and the German Richard Rado published an influential paper in which they solved a problem in extremal combinatorics. They found the maximal size of a family of pairwise intersecting subsets of a finite set. Their result instigated a lot of research on this topic. Similar problems were and are studied in a variety of structures including sets, multisets, groups, ... and several geometries. In honour of the three authors of the original paper, these problems are called *Erdős-Ko-Rado problems* and the generalisations of their theorem are called *Erdős-Ko-Rado theorems.*

In this chapter we present an introduction to Erdős-Ko-Rado problems. It gives the background for the next chapters on Erdős-Ko-Rado problems in geometries, and therefore does not contain new results. It is based on [40], a survey paper on this topic, which is joint work with Leo Storme.

[1] The proper genitive plural of *gigans* is *gigantum.*

2.1 The original Erdős-Ko-Rado problem

Finding the largest sets of pairwise non-trivially intersecting elements is one of the classical problems in extremal combinatorics. In 1961, the solution to this original Erdős-Ko-Rado problem was published by Erdős, Ko and Rado.

Theorem 2.1.1 ([53, Theorem 1]). *If $\mathcal{S}$ is a family of subsets of size k in a set Ω with $|\Omega| = n$ and $n \geq 2k$, such that the elements of $\mathcal{S}$ are pairwise not disjoint, then $|\mathcal{S}| \leq \binom{n-1}{k-1}$.*

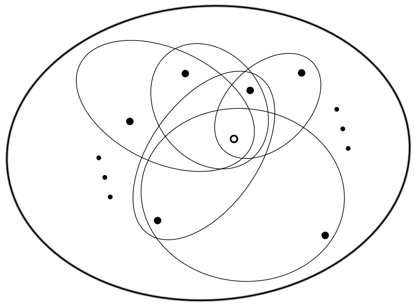

Figure 2.1: A point-pencil.

Note that the upper bound in the previous theorem is met if $\mathcal{S}$ is the set of all subsets of size k containing a fixed element of Ω. It is a consequence of the next theorem that this is the only example meeting the upper bound if $n \geq 2k + 1$. The set of all subsets of a fixed size k containing a fixed element is called a *point-pencil.* Generalizations of this structure will also be called a point-pencil.

Note that in case $n = 2k$, there are many examples attaining this upper bound: for every k-subset, there is precisely one disjoint k-subset in the set, so any set of k-subsets constructed by picking one k-subset from each such pair will do. Furthermore, in case $n < 2k$, this problem is trivial, since two subsets of size k cannot be disjoint in this case.

A first generalisation of this problem asks for the maximal size of a family of subsets of size k in a finite set, such that its elements pairwise meet in at least t elements of Ω. In [53, Theorem 2], a first result was obtained. It required the bound $n \geq t + (k - t)\binom{k}{t}^3$ (using the notation from Theorem 2.1.2). This result was improved by Wilson in 1984.

Theorem 2.1.2 ([131]). *Let $1 \leq t \leq k$ be positive integers. If $\mathcal{S}$ is a family of subsets of size k in a set Ω with $|\Omega| = n$ and $n \geq (t+1)(k-t+1)$, such that the elements of $\mathcal{S}$ pairwise intersect in at least t elements, then $|\mathcal{S}| \leq \binom{n-t}{k-t}$.*

Moreover, if $n \geq (t + 1)(k - t + 1) + 1$, then equality holds if and only if $\mathcal{S}$ is the set of all subsets of size k through a fixed t-subset of Ω.

In [131] Wilson also showed that the bound in Theorem 2.1.2 is sharp. Let $\mathcal{F}$ be the set of all subsets of size k meeting a fixed subset of Ω of size $t+2$ in at least $t+1$ elements, $k \geq t+1$. The elements of $\mathcal{F}$ pairwise intersect in at least t elements. If $n = (t+1)(k-t+1)$, then $|\mathcal{F}| = (t+2)\binom{n-t-2}{k-t-1} + \binom{n-t-2}{k-t-2}$ equals the size of the set described in Theorem 2.1.2. If $n \leq (t+1)(k-t+1)-1$, then $|\mathcal{F}|$ is larger than the size of the set described in this theorem.

Inspired by the first result, a collection of k-subsets of an arbitrary set, which are pairwise not disjoint, is called an *Erdős-Ko-Rado set*. The *Erdős-Ko-Rado problem* asks for the classification of the (largest) Erdős-Ko-Rado sets. An important result was obtained by Hilton and Milner. This result describes the largest Erdős-Ko-Rado sets which are not embedded in a point-pencil.

Theorem 2.1.3 ([78]). *Let Ω be a set of size n and let $\mathcal{S}$ be an Erdős-Ko-Rado set of k-subsets in Ω, $k \geq 3$ and $n \geq 2k+1$. If there is no element in Ω which is contained in all subsets in $\mathcal{S}$, then*

$$|\mathcal{S}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$

Moreover, equality holds if and only if

- *either $\mathcal{S}$ is the union of $\{F\}$, for some fixed k-subset F, and the set of all k-subsets G of Ω containing a fixed element $x \notin F$, such that $G \cap F \neq \emptyset$,*
- *or else $k = 3$ and $\mathcal{S}$ is the set of all subsets of size 3 having an intersection of size at least 2 with a fixed subset F of size 3.*

Consider a set of size n and let V be the set of its k-subsets. The *Johnson graph* is the graph with vertex set V and such that two vertices are adjacent if the corresponding k-subsets have $k-1$ elements in common. This graph is distance-regular; more details on this graph can be found in [23, Chapter 9]. The *Kneser graph* is the graph with vertex set V and such that two vertices are adjacent if the corresponding k-subsets are disjoint. In fact, these graphs correspond to two of the $k+1$ relations in the *Johnson scheme*. It can be seen that an Erdős-Ko-Rado set corresponds to a coclique of the Kneser graph.

Some of the results on Erdős-Ko-Rado sets are stated as results on cocliques of Kneser graphs. Some results are also obtained using a graph-theoretic approach. Therefore, the Johnson graphs and Kneser graphs are mentioned here.

2.2 Erdős-Ko-Rado sets in vector spaces and projective spaces

We mentioned before that the Erdős-Ko-Rado problem has been generalised to a variety of other structures. For example, the *q-analogues*[2] of Erdős-Ko-Rado sets were introduced. Let $V(n,q)$ be the n-dimensional vector space over the finite field $\mathbb{F}_q$ of order q. *Erdős-Ko-Rado sets* in $V(n,q)$ are sets of k-dimensional subspaces, pairwise intersecting non-trivially. The Erdős-Ko-Rado problem then asks for the size and classification of the (largest) Erdős-Ko-Rado sets.

We can analogously introduce Erdős-Ko-Rado sets of k-dimensional subspaces of a projective space $\mathrm{PG}(n,q)$. In Section 1.3 a projective geometry is defined as the geometry of the subspaces of a vector space. Therefore, results on Erdős-Ko-Rado sets in vector spaces can be interpreted as results on Erdős-Ko-Rado sets in projective spaces, and vice versa. Here, we will present them in projective spaces since the theorems and proofs in the next chapters on Erdős-Ko-Rado sets are also stated in a projective geometry setting. An Erdős-Ko-Rado set of k-dimensional subspaces in $\mathrm{PG}(n,q)$ is briefly called an *EKR(k) set* in $\mathrm{PG}(n,q)$.

In 1975, Hsieh proved in [82] a first q-analogue for Theorem 2.1.1. Among the later improvements of this theorem, we mention the results of Greene and Kleitman ([66]), Frankl and Wilson ([58]), Newman ([100]), Tanaka ([116]) and Godsil and Newman ([63]). The following theorem, which is a q-analogue for Theorem 2.1.2, combines the results of Frankl-Wilson with the results of Tanaka.

Theorem 2.2.1 ([58, Theorem 1] and [116, Theorem 3]). *Let t and k be integers, with $0 \leq t \leq k$. Let $\mathcal{S}$ be a set of k-dimensional subspaces in $\mathrm{PG}(n,q)$, pairwise intersecting in at least a t-dimensional subspace. If $n \geq 2k+1$, then $|\mathcal{S}| \leq \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q$. Equality holds if and only if $\mathcal{S}$ is the set of all k-dimensional subspaces, containing a fixed t-dimensional subspace of $\mathrm{PG}(n,q)$, or $n = 2k+1$ and $\mathcal{S}$ is the set of all k-dimensional subspaces in a*

[2]In general a q-analogue is a mathematical identity, problem, theorem, ... that depends on a variable q and that generalises a known identity, problem, theorem, ... to which it reduces in the (right) limit $q \to 1$. In a combinatorial setting it often arises by replacing a set and its subsets by a vector space and its subspaces. In the proof of Lemma 4.3.1 we will describe the q-binomial theorem, a q-analogue of the classical binomial theorem.

fixed $(2k-t)$-dimensional subspace.

If $2k-t \le n \le 2k$, then $|\mathcal{S}| \le \begin{bmatrix} 2k-t+1 \\ k-t \end{bmatrix}_q$. Equality holds if and only if $\mathcal{S}$ is the set of all k-dimensional subspaces in a fixed $(2k-t)$-dimensional subspace.

Corollary 2.2.2. *Let $\mathcal{S}$ be an EKR(k) set in $\mathrm{PG}(n,q)$. If $n \ge 2k+1$, then $|\mathcal{S}| \le \begin{bmatrix} n \\ k \end{bmatrix}_q$. Equality holds if and only if $\mathcal{S}$ is the set of all k-dimensional subspaces, containing a fixed point of $\mathrm{PG}(n,q)$, or $n = 2k+1$ and $\mathcal{S}$ is the set of all k-dimensional subspaces in a fixed hyperplane.*

Also for projective spaces, the Erdős-Ko-Rado set consisting of all subspaces through a fixed projective point is called a *point-pencil*.

Remark 2.2.3. It should be noted that the condition $n \ge 2k+1$ in the previous theorem on Erdős-Ko-Rado sets is not really a restriction. In $\mathrm{PG}(n,q)$, $n \le 2k$, no two disjoint k-dimensional subspaces can be found. So, every set of k-spaces is an Erdős-Ko-Rado set.

Analogously, the condition $n \ge 2k-t$ is not a restriction in Theorem 2.2.1, for in $\mathrm{PG}(n,q)$, $n \le 2k-t$, any pair of k-dimensional subspaces meets in at least a t-dimensional subspace.

Until now, we have only stated results about the largest Erdős-Ko-Rado sets in finite projective spaces. Now, we will also look at other Erdős-Ko-Rado sets. Obviously, new Erdős-Ko-Rado sets of any size below the size of the largest example, can be made by deleting elements from an Erdős-Ko-Rado set of largest size. Therefore we will focus on *maximal* Erdős-Ko-Rado sets, i.e. Erdős-Ko-Rado sets not extendable to a larger Erdős-Ko-Rado set. Typically, one tries to find all maximal Erdős-Ko-Rado sets of k-dimensional subspaces in a projective space $\mathrm{PG}(n,q)$, with size at least s. E.g., the above remark indicates that there is only one maximal EKR(k) set in $\mathrm{PG}(n,q)$, $n \le 2k$: the set of all k-dimensional subspaces in $\mathrm{PG}(n,q)$.

We mention the results of Blokhuis *et al.* on the second-largest maximal Erdős-Ko-Rado set of subspaces in a finite projective space. This is the q-analogue of the Hilton-Milner result (Theorem 2.1.3).

Theorem 2.2.4 ([11, Theorem 1.3 and Proposition 3.4]). *Assume $\mathcal{S}$ is a maximal EKR(k) set in $\mathrm{PG}(n,q)$, with $n \ge 2k+2$, $k \ge 2$ and $q \ge 3$ (or*

$n \geq 2k+4$, $k \geq 2$ and $q = 2$). If $\mathcal{S}$ is not a point-pencil, then

$$|\mathcal{S}| \leq \begin{bmatrix} n \\ k \end{bmatrix}_q - q^{k(k+1)} \begin{bmatrix} n-k-1 \\ k \end{bmatrix}_q + q^{k+1}.$$

Moreover, if equality holds, then

- *either $\mathcal{S}$ consists of all k-dimensional subspaces through a fixed point P, meeting a fixed $(k+1)$-dimensional subspace τ, with $P \in \tau$, in a j-dimensional subspace, $j \geq 1$, and all k-dimensional subspaces in τ,*
- *or else $k = 2$ and $\mathcal{S}$ is the set of all planes meeting a fixed plane π in at least a line.*

If $k = 1$, then an Erdős-Ko-Rado set of k-dimensional subspaces in a projective space $\mathrm{PG}(n, q)$, $n \geq 3$, is a set of pairwise intersecting lines. It was noted in [23, Section 9.3] that there are only two types of maximal Erdős-Ko-Rado sets of lines: the set of all lines through a fixed point and the set of all lines contained in a fixed plane. In fact, it was observed that there are only two types of sets of k-dimensional subspaces in a projective space $\mathrm{PG}(n, q)$, $n \geq k+2$, pairwise intersecting in a $(k-1)$-space, $k \geq 1$: the set of all k-dimensional subspaces through a fixed $(k-1)$-dimensional subspace and the set of all k-dimensional subspaces contained in a fixed $(k+1)$-dimensional subspace.

Not only large Erdős-Ko-Rado sets have been studied. At the other end of the spectrum, Mussche considered small EKR(k) sets and found the following result. It should be noted that the size of the EKR(k) set is independent of q.

Theorem 2.2.5 ([99, Theorem 2.45]). *If k is a prime power, a maximal EKR(k) set of size k^2+k+1 exists in $\mathrm{PG}(n, q)$, $n \geq k^2+k$.*

It should be noted that the restriction on n originally was not mentioned, but it is necessary, given the proof.

We end this section with a remark on graphs. Let $V(n, q)$ be a vector space and let V_k be the set of its k-spaces. The *Grassmann graph* is the graph with vertex set V_k and such that two vertices are adjacent if the corresponding k-spaces meet in a $(k-1)$-space. The distance relations in this graph give rise to an association scheme, the *Grassmann scheme* or *q-Johnson scheme*. The disjointness relation gives rise to the *generalised Kneser graph* or *q-Kneser* *graph*. Some of the above results, e.g. Theorem 2.2.1, were obtained studying the Grassmann scheme. Other bounds were found using matrix techniques. The study of the chromatic number of the generalised Kneser graphs is an important application of the results on Erdős-Ko-Rado sets. Results can be found in recent works such as [12, 31] and in the survey paper [13].

2.3 Erdős-Ko-Rado sets in polar spaces

The original Erdős-Ko-Rado sets and their q-analogues in $V(n, q)$ lead to the following definition of Erdős-Ko-Rado sets on finite classical polar spaces.

Definition 2.3.1. An *Erdős-Ko-Rado set* of k-dimensional subspaces in a finite classical polar space $\mathcal{P}$ of rank d, $k \leq d-1$, briefly an *EKR(k) set*, is a set of k-dimensional subspaces of $\mathcal{P}$, pairwise intersecting non-trivially. As before, it is called *maximal* if it is not contained in a larger Erdős-Ko-Rado set.

The Erdős-Ko-Rado problem asks for the classification of the (largest) maximal Erdős-Ko-Rado sets.

In [104], the EKR($d-1$) sets for finite classical polar spaces of rank d, i.e. the Erdős-Ko-Rado sets of generators, were investigated by Pepe, Storme and Vanhove. For most finite classical polar spaces the largest EKR($d-1$) sets were classified. In Sections 2.1 and 2.2, we found that the largest Erdős-Ko-Rado sets in finite sets and finite projective spaces are point-pencils. Surprisingly, this is true for EKR($d-1$) sets on some polar spaces but not on all of them. For some polar spaces, the point-pencils are one of the types of EKR($d-1$) sets of maximal size; for others, there are EKR($d-1$) sets of larger size. An overview of the results in [104] can be found in Table 2.1.

Remark 2.3.2. In this table, it is mentioned that there are two types of Erdős-Ko-Rado sets of generators in one class of a hyperbolic quadric $\mathcal{Q}^+(7, q)$. However, it follows directly from the proof that these are the only two maximal Erdős-Ko-Rado sets of generators in one class. So, in this case, there is a complete classification.

There is one type of finite classical polar spaces lacking in Table 2.1: the Hermitian polar spaces $\mathcal{H}(4n+1, q^2)$, $n \geq 2$, are the only ones for which the

Polar space	**Maximum size**	**Classification**
$\mathcal{Q}^-(2n+1,q)$	$(q^2+1)\cdots(q^n+1)$	point-pencil
$\mathcal{Q}(4n,q)$	$(q+1)\cdots(q^{2n-1}+1)$	point-pencil
$\mathcal{Q}(4n+2,q)$, $n\geq 2$	$(q+1)\cdots(q^{2n}+1)$	point-pencil, one class of $\mathcal{Q}^+(4n+1,q)$
$\mathcal{Q}(6,q)$	$(q+1)(q^2+1)$	point-pencil, base plane, one class of $\mathcal{Q}^+(5,q)$
$\mathcal{Q}^+(4n+1,q)$	$(q+1)\cdots(q^{2n}+1)$	one class
one class of $\mathcal{Q}^+(4n+3,q)$, $n\geq 3$	$(q+1)\cdots(q^{2n}+1)$	point-pencil
one class of $\mathcal{Q}^+(7,q)$	$(q+1)(q^2+1)$	point-pencil, meeting fixed element of other class in a plane
$\mathcal{W}(4n+1,q)$, $n\geq 2$, q odd	$(q+1)\cdots(q^{2n}+1)$	point-pencil
$\mathcal{W}(5,q)$, q odd	$(q+1)(q^2+1)$	point-pencil, base plane
$\mathcal{W}(4n+1,q)$, $n\geq 2$, q even	$(q+1)\cdots(q^{2n}+1)$	point-pencil, one class of $\mathcal{Q}^+(4n+1,q)$
$\mathcal{W}(5,q)$, q even	$(q+1)(q^2+1)$	point-pencil, base plane, one class of $\mathcal{Q}^+(5,q)$
$\mathcal{W}(4n+3,q)$	$(q+1)\cdots(q^{2n+1}+1)$	point-pencil
$\mathcal{H}(2n,q^2)$	$(q^3+1)(q^5+1)\cdots$ $\cdots(q^{2n-1}+1)$	point-pencil
$\mathcal{H}(4n+3,q^2)$	$(q+1)(q^3+1)\cdots$ $\cdots(q^{4n+1}+1)$	point-pencil
$\mathcal{H}(5,q^2)$	$q(q^4+q^2+1)+1$	base plane

Table 2.1: [104, Section 9]: results on Erdős-Ko-Rado sets of generators in the finite classical polar spaces. For some polar spaces of rank 3 (the generators are planes) the *base plane* appears in this classification. This is the set of all planes meeting a fixed plane in at least a line.

classification of the largest Erdős-Ko-Rado sets of generators is not known. The study of these Erdős-Ko-Rado sets of generators was made in detail by Ihringer and Metsch. They proved the following result.

Theorem 2.3.3 ([84, Theorem 1]). *The size of an Erdős-Ko-Rado set of generators on $\mathcal{H}(4n+1,q^2)$, $n\geq 2$, is at most $q^{4n^2+1}+O(q^{4n^2}$.*

We end this section with two remarks.

Remark 2.3.4. Above we have only stated results about Erdős-Ko-Rado sets of generators. About Erdős-Ko-Rado sets of other subspaces of finite classical polar spaces, very little is known. However, the observation about EKR(1) sets that we made in the previous section (for projective spaces), is also valid for projective spaces albeit with a small difference. A maximal Erdős-Ko-Rado set of lines is either the set of all lines through a fixed point (a point-pencil) or the set of all lines in a plane on the polar space. Note that this second example only occurs if the rank of the classical polar space is at least 3 (so if it is not a generalised quadrangle). In Chapter 3 we will focus on maximal EKR(2) sets. Up to our knowledge, nothing is known about EKR(k) sets on polar spaces of rank d, with $2 < k < d-1$.

Remark 2.3.5. Just as in the set case or the projective space case, we can attach a graph to these geometries. Let $\mathcal{P}$ be a finite classical polar space of rank d. The *dual polar graph* is the graph with the generators of $\mathcal{P}$ as vertices and such that two vertices are adjacent if the corresponding generators meet in an space of dimension $d-2$. The results in [104] were obtained studying this graph and its corresponding association scheme.

2.4 Erdős-Ko-Rado sets in incidence geometries

Inspired by the definitions in the previous sections, we can define an Erdős-Ko-Rado set of type k, briefly an EKR(k) set, for an arbitrary incidence geometry.

Definition 2.4.1. Let $\mathcal{G} = (\mathcal{V}, \Delta_n, t, I)$ be an incidence geometry. The set $\mathcal{S} \subseteq \mathcal{V}$ is an *Erdős-Ko-Rado set* of type k, briefly an *EKR(k) set*, with $0 \leq k \leq n-1$, if

- $\forall v \in \mathcal{S} : t(v) = k$;
- $\forall v, v' \in \mathcal{S},\ \exists p \in \mathcal{V} : t(p) = 0,\ p\ I\ v$ and $p\ I\ v'$.

An Erdős-Ko-Rado set is called *maximal* if it is not extendable regarding these conditions.

Note that this definition is also applicable to geometries with an infinite number of varieties such as projective spaces or classical polar spaces over infinite fields. However, in this case, often Erdős-Ko-Rado sets with an infinite number of elements can be constructed, e.g. a point-pencil in $\mathrm{PG}(n, \mathbb{F})$, $\mathbb{F}$ an infinite field. Therefore, it is often impossible to classify the largest Erdős-Ko-Rado sets. So, Erdős-Ko-Rado sets in these geometries are generally not studied.

In Theorems 2.1.2 and 2.2.1, it is indicated that there is a more general concept extending the definition of an Erdős-Ko-Rado set. The condition 'pairwise not disjoint' is replaced by 'pairwise meeting in a subset of size at least t' or 'pairwise meeting in a subspace of dimension at least t'. These are so-called *t-intersecting sets*. For polar spaces, these have not been studied in general, but some cases have been looked at. In [24] the authors considered $\{0, 1, 2\}$-cliques of polar spaces, especially symplectic and quadric polar spaces. A *$\{0, 1, 2\}$-clique* of a polar space of rank d is a set of generators such that any two of them intersect in at least a $(d-3)$-space. For the quadric and symplectic polar spaces a complete classification of the $\{0, 1, 2\}$-cliques is given. We will use some of these results in Section 3.3. For Hermitian polar spaces these were studied by Ihringer and Metsch in [85].

Definition 2.4.1 can easily be generalised to a definition of t-intersecting sets of type k in an incidence geometry, $0 \leq t \leq k$, by replacing the condition $t(p) = 0$ by $t(p) = t$. In this thesis however, we will only focus on Erdős-Ko-Rado sets and not on general t-intersecting sets.

Another generalisation that has been made by several authors, is replacing the condition 'pairwise' by 'r-wise' for an integer $r \geq 2$. This was studied by Frankl ([57]) in the set case and by Chowdhury and Patkós ([32]) in the projective space case. Also this generalisation is beyond the scope of this thesis.

A last generalisation which we mention, was studied by Güven in [67]. She investigated pairwise intersecting flags of projective and polar spaces, instead of pairwise intersecting subspaces.

3

Erdős-Ko-Rado sets of planes in projective and polar spaces

Point n'est besoin d'espérer pour entreprendre,
ni de réussir pour persévérer.

Attributed to *Willem de Zwijger, prins van Oranje.*

We learned in Chapter 2 that in general the two largest maximal Erdős-Ko-Rado sets of subspaces in a finite projective space are classified. For the finite polar spaces, the largest Erdős-Ko-Rado sets of generators are known, in most cases. Only for EKR(1) sets and the Erdős-Ko-Rado sets of generators of the hyperbolic quadric $\mathcal{Q}^+(7,q)$, we know a complete classification. When we consider the situation for maximal Erdős-Ko-Rado sets of arbitrary subspaces in polar spaces, and we look at the third largest example of maximal Erdős-Ko-Rado sets of subspaces in projective spaces and we want to find to find a seconde example of a complete classification, it is natural to look at Erdős-Ko-Rado sets of planes. They are from these several viewpoints the first or the next 'step' in the process. For $\mathrm{PG}(5,q)$, maximal EKR(2) sets were studied by Blokhuis, Brouwer and Szőnyi. They found the following result.

Theorem 3.0.1 ([12, Section 6]). *Let $\mathcal{S}$ be a maximal EKR(2) set in the* *projective space* $\mathrm{PG}(5,q)$, *with* $|\mathcal{S}| \geq 3q^4+3q^3+2q^2+q+1$. *Then one of the following cases occurs.*

- $|\mathcal{S}| = \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q$ *and* $\mathcal{S}$ *is the set of planes through a fixed point* P *or the set of planes in a* 4*-space* $\tau \subset \mathrm{PG}(5,q)$.
- $|\mathcal{S}| = 1+q(q^2+q+1)^2$ *and* $\mathcal{S}$ *is one of the following: the set of planes intersecting a fixed plane* π *in at least a line, the set of planes that either are contained in a* 3*-space* σ *or else intersect* σ *in a line through a fixed point* $P \in \sigma$, *or the set of planes that either pass through a fixed line* ℓ *or else are in a* 4*-space* $\tau \supset \ell$ *and intersect* ℓ *in a point.*
- $|\mathcal{S}| = 3q^4+3q^3+2q^2+q+1$ *and* $\mathcal{S}$ *is the set of all planes that intersect* π *in a line through* P, *all planes in* τ *that intersect* π *in a line, and all planes through* P *in* τ, *with* P *a point,* π *a plane and* τ *a* 4*-space such that* $P \in \pi \subset \tau$.

In [8], a related problem has been studied. The authors considered Klein sets, sets of planes in $\mathrm{PG}(n,q)$ mutually intersecting in precisely one point. They classified the large Klein sets and in most cases they gave a description of the Klein sets.

In this chapter we study EKR(2) sets, both for projective and polar spaces. Due to the natural embedding of polar spaces in projective spaces, it is possible to study this simultaneously. We will find several classification theorems, classifying in general the ten, eleven or twelve largest maximal EKR(2) sets. This chapter is based on the article [36]. In Section 3.1 we give some examples of EKR(2) sets and in Section 3.2 we prove that this list contains all EKR(2) sets generating a subspace of dimension at least 6. In Section 3.3 we use this result to classify, in general, the ten largest EKR(2) sets. Hereby, we also look at the small cases (polar spaces of small rank), for which we often can give a better or even complete classification.

3.1 List of EKR(2) sets

In this section we give a list of types of maximal EKR(2) sets. In Section 3.1.1 we give a list of the maximal EKR(2) sets whose elements span at least a 6-space in the (ambient) projective space. This list contains in general the largest EKR(2) sets. Afterwards, in Section 3.1.2, we give a list of maximal EKR(2) sets contained in a 5-space of the (ambient) projective space. This list is not complete, but it contains examples of EKR(2) sets that occur in polar spaces of small rank. These examples will allow us to give a more complete classification in Section 3.3.

While presenting the examples, we should prove that the planes in such a set pairwise meet and that the set is maximal. In most cases, this proof is easy, and therefore not mentioned. Implicitly, these conditions are shown to be valid in the proofs of Section 3.2.

Throughout this section $\mathcal{P}$ denotes a projective space of dimension at least 5 or a classical polar space of rank at least 3. We introduce the following notation.

Notation 3.1.1. Let $\mathcal{S}$ be an EKR(2) set of type a in the finite projective or polar space $\mathcal{P}$. We denote the number of planes in $\mathcal{S}$ by $n(a, \mathcal{P})$.

3.1.1 Large examples

I: Consider a point P in $\mathcal{P}$. Let $\mathcal{S}$ be the set of all planes through P. This is a maximal EKR(2) set. It contains $n(\mathrm{I}, \mathcal{P}) = \begin{bmatrix} n \\ 2 \end{bmatrix}_q$ planes if $\mathcal{P} = \mathrm{PG}(n, q)$ and $\begin{bmatrix} d-1 \\ 2 \end{bmatrix}_q (q^{d+e-2}+1)(q^{d+e-3}+1)$ planes if $\mathcal{P}$ is a classical polar space of rank d with parameter e. This is the point-pencil.

II: Consider a 3-space σ in $\mathcal{P}$ and a point $P \in \sigma$. Let $\mathcal{S}$ be the set of all planes that either are contained in σ or else intersect σ in a line through P. This is a maximal EKR(2) set. The number of planes in $\mathcal{S}$ equals

$$n(\mathrm{II}, \mathcal{P}) = \begin{bmatrix} 4 \\ 3 \end{bmatrix}_q + \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q \left(\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q - \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \right)$$

$$= (q^2+q+1)\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q - (q^2+q)$$

if $\mathcal{P} = \mathrm{PG}(n,q)$ and

$$\begin{aligned} n(\mathrm{II}, \mathcal{P}) &= \begin{bmatrix} 4 \\ 3 \end{bmatrix}_q + \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q \left(\begin{bmatrix} d-2 \\ 1 \end{bmatrix}_q (q^{d+e-3}+1) - \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \right) \\ &= (q^2+q+1)\begin{bmatrix} d-2 \\ 1 \end{bmatrix}_q (q^{d+e-3}+1) - (q^2+q) \end{aligned}$$

if $\mathcal{P}$ is a classical polar space of rank $d \geq 4$ with parameter e.

This is a maximal EKR(2) set of Hilton-Milner type. We presented these already in Theorem 2.2.4 for the projective case. See also [11].

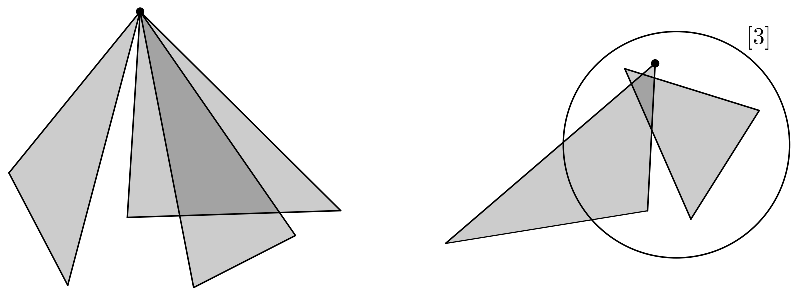

Figure 3.1: The EKR(2) sets of type I (left) and type II (right).

III: Consider a plane π in $\mathcal{P}$. Let $\mathcal{S}$ be the set containing π and all planes in $\mathcal{P}$ intersecting π in a line. This maximal EKR(2) set contains

$$n(\mathrm{III}, \mathcal{P}) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q \left(\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q - 1 \right) + 1 = (q^2+q+1)\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q - (q^2+q)$$

planes if $\mathcal{P} = \mathrm{PG}(n,q)$ and

$$\begin{aligned} n(\mathrm{III},\mathcal{P}) &= \begin{bmatrix}3\\2\end{bmatrix}_q \left(\begin{bmatrix}d-2\\1\end{bmatrix}_q (q^{d+e-3}+1) - 1 \right) + 1 \\ &= (q^2+q+1)\begin{bmatrix}d-2\\1\end{bmatrix}_q (q^{d+e-3}+1) - (q^2+q) \end{aligned}$$

planes if $\mathcal{P}$ is a classical polar space of rank d with parameter e.

This EKR(2) set was already described in Theorem 2.2.4 for the projective case, and in Table 2.1 for polar spaces of rank 3. There it was called the base plane.

IV: Consider a 4-space τ in $\mathcal{P}$, a plane $\pi \subset \tau$ and a point $P \in \pi$. Let $\mathcal{S}$ be the set containing the planes in τ intersecting π in a line, the planes in $\mathcal{P}$ intersecting π in a line through P and the planes in τ through P. This is a maximal EKR(2) set. It contains

$$\begin{aligned} n(\mathrm{IV},\mathcal{P}) &= \begin{bmatrix}2\\1\end{bmatrix}_q \left(\begin{bmatrix}n-1\\1\end{bmatrix}_q - 1 \right) + \left(\begin{bmatrix}3\\2\end{bmatrix}_q - \begin{bmatrix}2\\1\end{bmatrix}_q \right) \left(\begin{bmatrix}3\\1\end{bmatrix}_q - 1 \right) \\ &\quad + \left(\begin{bmatrix}4\\2\end{bmatrix}_q - \begin{bmatrix}2\\1\end{bmatrix}_q \left(\begin{bmatrix}3\\1\end{bmatrix}_q - 1 \right) \right) \\ &= (q+1)\begin{bmatrix}n-1\\1\end{bmatrix}_q + (2q^4+q^3-q) \end{aligned}$$

planes if $\mathcal{P} = \mathrm{PG}(n,q)$ and

$$\begin{aligned} n(\mathrm{IV},\mathcal{P}) &= \begin{bmatrix}2\\1\end{bmatrix}_q \left(\begin{bmatrix}d-2\\1\end{bmatrix}_q (q^{d+e-3}+1) - 1 \right) \\ &\quad + \left(\begin{bmatrix}3\\2\end{bmatrix}_q - \begin{bmatrix}2\\1\end{bmatrix}_q \right) \left(\begin{bmatrix}3\\1\end{bmatrix}_q - 1 \right) \\ &\quad + \left(\begin{bmatrix}4\\2\end{bmatrix}_q - \begin{bmatrix}2\\1\end{bmatrix}_q \left(\begin{bmatrix}3\\1\end{bmatrix}_q - 1 \right) \right) \\ &= (q+1)\begin{bmatrix}d-2\\1\end{bmatrix}_q (q^{d+e-3}+1) + (2q^4+q^3-q) \end{aligned}$$

planes if $\mathcal{P}$ is a classical polar space of rank $d \geq 5$ with parameter e.

The example of the third weight in Theorem 3.0.1 is an EKR(2) set of this type for the specific case $\mathcal{P} = \mathrm{PG}(5, q)$.

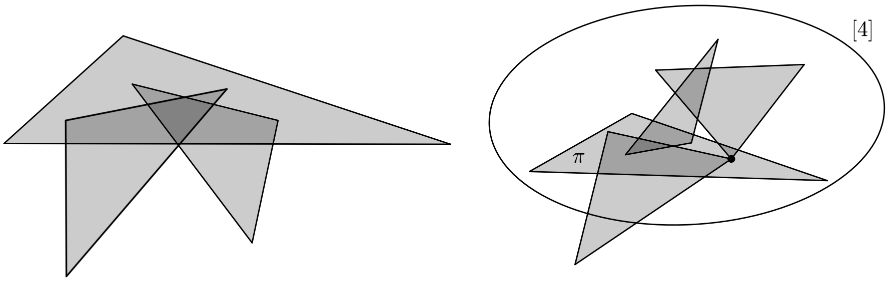

Figure 3.2: The EKR(2) sets of type III (left) and type IV (right).

V: Consider a 4-space τ in $\mathcal{P}$ and a line $\ell \subset \tau$. Let $\mathcal{S}$ be the set containing all planes through ℓ and all planes in τ containing a point of ℓ. This is a maximal EKR(2) set which contains

$$\begin{aligned} n(\mathrm{V}, \mathcal{P}) &= \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q + \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \left(\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q - \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q \right) \\ &= \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q + q^2(q+1)(q^2+q+1) \end{aligned}$$

planes if $\mathcal{P} = \mathrm{PG}(n, q)$ and

$$\begin{aligned} n(\mathrm{V}, \mathcal{P}) &= \begin{bmatrix} d-2 \\ 1 \end{bmatrix}_q (q^{d+e-3}+1) + \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \left(\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q - \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q \right) \\ &= \begin{bmatrix} d-2 \\ 1 \end{bmatrix}_q (q^{d+e-3}+1) + q^2(q+1)(q^2+q+1) \end{aligned}$$

planes if $\mathcal{P}$ is a classical polar space of rank $d \geq 5$ with parameter e.

The third example of the second weight in Theorem 3.0.1 is an EKR(2) set of this type for the specific case $\mathcal{P} = \mathrm{PG}(5, q)$.

VI: Let τ_1 and τ_2 be two 4-spaces in $\mathcal{P}$ such that $\sigma = \tau_1 \cap \tau_2$ is a 3-space. Let π and π' be two planes in σ with intersection line ℓ and let P and P' be two different points on ℓ. We define $\mathcal{S}$ to be the set containing the planes through ℓ, the planes in σ, the planes in τ_1 containing a line through P in π or a line through P' in π', and the planes in τ_2 containing a line through P in π' or a line through P' in π. This is a maximal EKR(2) set containing

$$\begin{aligned} n(\mathrm{VI}, \mathcal{P}) &= \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q + \left(\begin{bmatrix} 4 \\ 3 \end{bmatrix}_q - \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \right) + 4 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q - 1 \right) \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}_q - \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \right) \\ &= \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q + 5q^3 + q^2 \end{aligned}$$

planes if $\mathcal{P} = \mathrm{PG}(n, q)$ and

$$\begin{aligned} n(\mathrm{VI}, \mathcal{P}) &= \begin{bmatrix} d-2 \\ 1 \end{bmatrix}_q (q^{d+e-3} + 1) + \left(\begin{bmatrix} 4 \\ 3 \end{bmatrix}_q - \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \right) \\ &\quad + 4 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q - 1 \right) \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}_q - \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \right) \\ &= \begin{bmatrix} d-2 \\ 1 \end{bmatrix}_q (q^{d+e-3} + 1) + 5q^3 + q^2 \end{aligned}$$

planes if $\mathcal{P}$ is a classical polar space of rank $d \geq 5$ with parameter e.

VII: Let ρ be a 5-space contained in $\mathcal{P}$. Consider a line ℓ and a 3-space σ, disjoint to ℓ, in ρ. Choose three points P_1, P_2, P_3 on ℓ and choose four non-coplanar points Q_1, Q_2, Q_3, Q_4 in σ. Denote $\ell_1 = \langle Q_1, Q_2 \rangle$, $\overline{\ell_1} = \langle Q_3, Q_4 \rangle$, $\ell_2 = \langle Q_1, Q_3 \rangle$, $\overline{\ell_2} = \langle Q_2, Q_4 \rangle$, $\ell_3 = \langle Q_1, Q_4 \rangle$ and $\overline{\ell_3} = \langle Q_2, Q_3 \rangle$. Let $\mathcal{S}$ be the set containing all planes through ℓ and all planes through P_i in $\langle \ell, \ell_i \rangle$ or in $\langle \ell, \overline{\ell_i} \rangle$, $i = 1, 2, 3$. The number of planes in this maximal EKR(2) set equals

$$n(\mathrm{VII}, \mathcal{P}) = \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q + 6 \left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}_q - \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \right) = \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q + 6q^2$$

if $\mathcal{P} = \mathrm{PG}(n, q)$ and

$$\begin{aligned} n(\mathrm{VII}, \mathcal{P}) &= \begin{bmatrix} d-2 \\ 1 \end{bmatrix}_q (q^{d+e-3}+1) + 6\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}_q - \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q\right) \\ &= \begin{bmatrix} d-2 \\ 1 \end{bmatrix}_q (q^{d+e-3}+1) + 6q^2 \end{aligned}$$

if $\mathcal{P}$ is a classical polar space of rank $d \geq 6$ with parameter e.

VIII: Consider two 3-spaces σ and σ', intersecting in a line ℓ. Take the points P_1 and P_2 on ℓ. Let $\mathcal{S}$ be the set containing all planes through ℓ, all planes through P_1 that contain a line in σ and a line in σ', and all planes through P_2 in σ or in σ'. Note the asymmetric behaviour of P_1 and P_2. This maximal EKR(2) set contains

$$\begin{aligned} n(\mathrm{VIII}, \mathcal{P}) &= \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q + \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}_q - 1\right)^2 + 2\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}_q - \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q\right) \\ &= \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q + q^4 + 2q^3 + 3q^2 \end{aligned}$$

planes if $\mathcal{P} = \mathrm{PG}(n, q)$. If $\mathcal{P}$ is a polar space, we need to distinguish between two possibilities. Denote $\rho = \langle \sigma, \sigma' \rangle$. If $d \geq 6$ and $\rho \subset \mathcal{P}$ (case VIIIa), then $\mathcal{S}$ is an EKR(2) set with

$$\begin{aligned} n(\mathrm{VIIIa}, \mathcal{P}) &= \begin{bmatrix} d-2 \\ 1 \end{bmatrix}_q (q^{d+e-3}+1) + \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}_q - 1\right)^2 + 2\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}_q - \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q\right) \\ &= \begin{bmatrix} d-2 \\ 1 \end{bmatrix}_q (q^{d+e-3}+1) + q^4 + 2q^3 + 3q^2 \end{aligned}$$

for a classical polar space with parameter e. If $\mathcal{P}$ is either a symplectic polar space with ambient space $\mathrm{PG}(n, q)$, q odd, or a Hermitian variety, $d \geq 4$, and $\mathcal{P} \cap \rho$ is a cone with vertex ℓ (case VIIIb), then $\mathcal{S}$ is a maximal

EKR(2) set with

$$\begin{aligned} n(\text{VIIIb}, \mathcal{P}) &= \begin{bmatrix} d-2 \\ 1 \end{bmatrix}_q (q^{d+e-3}+1) + (q+1)\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}_q - \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q\right) \\ &\quad + 2\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}_q - \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q\right) \\ &= \begin{bmatrix} d-2 \\ 1 \end{bmatrix}_q (q^{d+e-3}+1) + q^3 + 3q^2 \end{aligned}$$

for a classical polar space with parameter e. Note that this construction (assuming $d \geq 4$ and $\mathcal{P} \cap \rho$ is a cone with vertex ℓ and base a polar space of rank 2) gives rise to a non-maximal EKR(2) set if $\mathcal{P}$ is a quadric or a symplectic polar space with ambient space $\text{PG}(n, q)$, q even. It can be extended to a maximal EKR(2) set of type IXa by adding planes through P_2.

For all other possibilities for $\mathcal{P} \cap \rho$, $\mathcal{S}$ is not a maximal EKR(2) set.

IX: Let ℓ be a line in $\mathcal{P}$ and σ a 3-space skew to ℓ, in the ambient projective space of $\mathcal{P}$. Denote $\langle \ell, \sigma \rangle$ by ρ. Choose two points P and P' on ℓ. Let $\mathcal{R}$ and $\mathcal{R}'$ be two sets of pairwise disjoint lines in σ, such that any line of $\mathcal{R}$ and any line of $\mathcal{R}'$ have precisely one point in common, such that $|\mathcal{R}|, |\mathcal{R}'| \geq 3$ and such that $\mathcal{R}$ and $\mathcal{R}'$ are maximal under these conditions. Let $\mathcal{S}$ be the set containing all planes through ℓ, all planes through P in a 3-space generated by ℓ and an element of $\mathcal{R}$ and all planes through P' in a 3-space generated by ℓ and an element of $\mathcal{R}'$. If $\mathcal{P} = \text{PG}(n, q)$, then $\mathcal{R}$ is a regulus and $\mathcal{R}'$ is its opposite regulus. The number of planes of this maximal EKR(2) set equals

$$n(\text{IX}, \mathcal{P}) = \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q + 2(q+1)\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}_q - \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q\right) = \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q + 2q^3 + 2q^2$$

if $\mathcal{P} = \text{PG}(n, q)$. If $\mathcal{P}$ is a polar space, we need to distinguish between several possibilities. First assume that $\rho \subset \mathcal{P}$ or that $\rho \cap \mathcal{P}$ is a cone with vertex ℓ and base a hyperbolic quadric $\mathcal{Q}^+(3, q)$ or a symplectic polar space $\mathcal{W}(3, q)$, q even (case IXa). In this case $\mathcal{R}$ is a regulus and $\mathcal{R}'$ is its opposite regulus. The number of planes of this maximal EKR(2) set equals

$$\begin{aligned} n(\mathrm{IXa}, \mathcal{P}) &= \begin{bmatrix} d-2 \\ 1 \end{bmatrix}_q (q^{d+e-3}+1) + 2(q+1)\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}_q - \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q\right) \\ &= \begin{bmatrix} d-2 \\ 1 \end{bmatrix}_q (q^{d+e-3}+1) + 2q^3 + 2q^2 \end{aligned}$$

if $\mathcal{P}$ is a classical polar space of rank d with parameter e. If $\mathcal{P}$ is a hyperbolic, parabolic or elliptic quadric, a 5-space in $\mathrm{PG}(n,q)$ whose intersection with $\mathcal{P}$ contains a cone with vertex a line can always be found if $d \geq 4$. Also, if $\mathcal{P}$ is a symplectic polar space and q is even, a 5-space whose intersection with $\mathcal{P}$ contains such a cone can always be found if $d \geq 4$. If $\mathcal{P}$ is a Hermitian variety, a 5-space in $\mathrm{PG}(n,q)$ whose intersection is such a cone does not exist. Thus, the EKR(2) set of this type only exists if the rank d of the Hermitian polar space is at least six. Also if $\mathcal{P}$ is a symplectic polar space and q is odd, an EKR(2) set of this type only exists if the rank d of the polar space is at least six.

Now we assume that $\rho \cap \mathcal{P}$ is a cone with vertex ℓ and base a Hermitian variety $\mathcal{H}(3,q)$ (case IXb). In this case $\mathcal{R}$ corresponds to a regulus of a hyperbolic quadric $\mathcal{Q}^+(3,\sqrt{q})$ embedded in $\mathcal{H}(3,q)$, and $\mathcal{R}'$ to the lines of its opposite regulus. The number of planes of this maximal EKR(2) set equals

$$\begin{aligned} n(\mathrm{IXb}, \mathcal{P}) &= \begin{bmatrix} d-2 \\ 1 \end{bmatrix}_q (q^{d+e-3}+1) + 2(\sqrt{q}+1)\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}_q - \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q\right) \\ &= \begin{bmatrix} d-2 \\ 1 \end{bmatrix}_q (q^{d+e-3}+1) + 2q^2\sqrt{q} + 2q^2 \end{aligned}$$

if $\mathcal{P}$ is a classical polar space of rank d with parameter e. This EKR(2) set only exists if $\mathcal{P}$ is a Hermitian polar space whose rank d is at least 4.

Other possibilities for the intersection $\rho \cap \mathcal{P}$ do not yield maximal EKR(2) sets.

X: Consider a 3-space σ and three 4-spaces $\tau_{12}, \tau_{13}, \tau_{14}$ through σ, all in $\mathcal{P}$. Choose four non-coplanar points P_1, P_2, P_3, P_4 in σ. Let $\mathcal{S}$ be the set containing all planes in τ_{12} through $\langle P_1, P_2 \rangle$ or $\langle P_3, P_4 \rangle$, all planes in τ_{13}

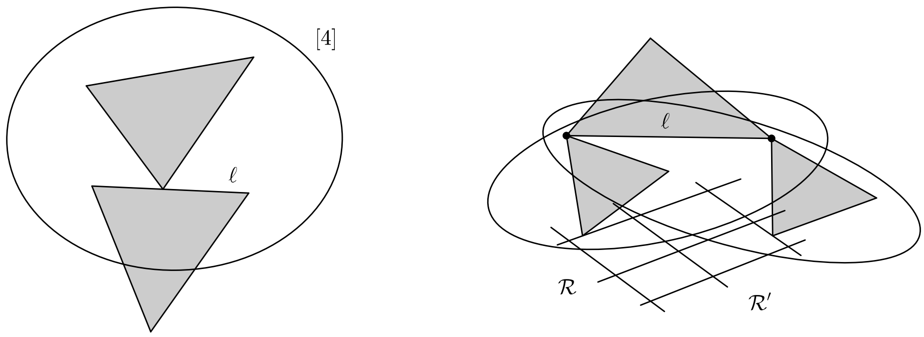

Figure 3.3: The EKR(2) sets of type V (left) and type IX (right).

through $\langle P_1, P_3\rangle$ or $\langle P_2, P_4\rangle$, all planes in τ_{14} through $\langle P_1, P_4\rangle$ or $\langle P_2, P_3\rangle$, and all planes in σ. The number of planes in this maximal EKR(2) set is

$$n(\mathrm{X}, \mathcal{P}) = \begin{bmatrix}4\\3\end{bmatrix}_q + 6\left(\begin{bmatrix}3\\1\end{bmatrix}_q - \begin{bmatrix}2\\1\end{bmatrix}_q\right) = q^3 + 7q^2 + q + 1.$$

Note that this EKR(2) set only can exist if $\mathcal{P} = \mathrm{PG}(n, q)$, if $\mathcal{P}$ is a classical polar space of rank 5 with parameter $e > 0$ or if $\mathcal{P}$ is a classical polar space of rank at least 6. Note also that $\langle \tau_{12}, \tau_{13}, \tau_{14}\rangle$ can be as well a 5-space as a 6-space.

XI: Consider a Fano plane $\mathcal{F}$ with points $P_0, \ldots, P_6$. Let $Q_0, \ldots, Q_6$ be seven linearly independent points in $\mathcal{P}$. The set $\mathcal{S}$ contains precisely those planes $\langle Q_i, Q_j, Q_k\rangle$ for those $\{i, j, k\}$ such that $\{P_i, P_j, P_k\}$ is a line in $\mathcal{F}$. This maximal EKR(2) set contains $n(\mathrm{XI}, \mathcal{P}) = 7$ planes. Note that $\mathcal{S}$ only can exist if $\mathcal{P} = \mathrm{PG}(n, q)$, $n \geq 6$, or if $\mathcal{P}$ is a classical polar space of rank at least 7.

This example was described before in [8]. In the proof of Theorem 2.2.5, Mussche also described this example, among others, using a general projective plane ([99, Theorem 2.45]). He proved therein the maximality of these examples.

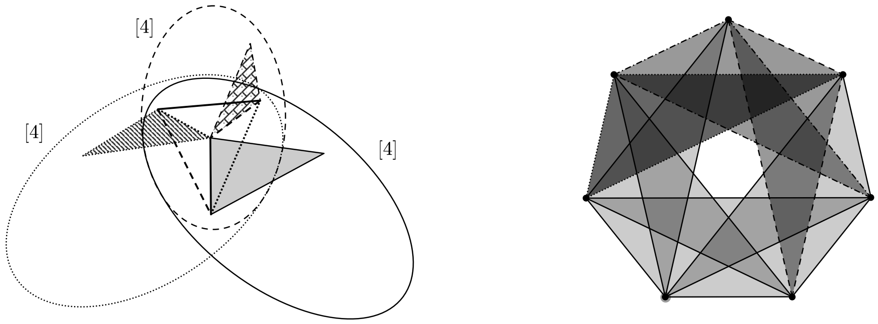

Figure 3.4: The EKR(2) sets of type X (left) and type XI (right).

3.1.2 Small examples

XII: Consider a 4-space τ in $\mathcal{P}$. The set $\mathcal{S}$ contains all planes in τ. This is a maximal EKR(2) set which contains $n(\text{XII}, \mathcal{P}) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}_q$ planes. If $\mathcal{P}$ is a polar space, its rank d must be at least 5. It should be remarked that the set of all planes on a polar space $\mathcal{P}$ in a 4-space $\tau' \not\subset \mathcal{P}$ is not an EKR(2) set, since it is not maximal.

XIII: Let $\mathcal{Q}^+(5, q)$ be a hyperbolic quadric contained in $\mathcal{P}$ and let ρ be the 5-space in the (ambient) projective space, generated by $\mathcal{Q}^+(5, q)$. We show that the set $\mathcal{S}$ of all planes of one class of planes on $\mathcal{Q}^+(5, q)$ is a maximal EKR(2) set. Note that $\mathcal{S}$ contains $n(\text{XIII}, \mathcal{P}) = q^3 + q^2 + q + 1$ planes. A plane not in ρ has a non-empty intersection with at most $q^2 + q + 1$ planes of $\mathcal{S}$. A plane π in ρ, not in $\mathcal{S}$, has a non-empty intersection with $2q^2 + q + 1$ planes (if $\pi \cap \mathcal{Q}^+(5, q)$ is the union of two lines), with $q^2 + 2q + 1$ planes (if $\pi \cap \mathcal{Q}^+(5, q)$ is a conic $\mathcal{Q}(2, q)$), with $q^2 + q + 1$ planes (if $\pi \cap \mathcal{Q}^+(5, q)$ is a line or π is a plane of the other class) or with $q + 1$ planes (if $\pi \cap \mathcal{Q}^+(5, q)$ is a point).

If $\mathcal{P}$ is a quadric with $d \geq 3$, a 5-space that contains a hyperbolic quadric $\mathcal{Q}^+(5, q)$ always can be found. If $\mathcal{P}$ is a Hermitian variety, it contains such a 5-space if $d \geq 6$. If $\mathcal{P}$ is a symplectic polar space and q is even, it contains such a 5-space if $d \geq 3$. If $\mathcal{P}$ is a symplectic polar space and q is odd, it contains such a 5-space if $d \geq 6$.

XIV: Let τ_1 and τ_2 be two 4-spaces on $\mathcal{P}$ with $\sigma = \tau_1 \cap \tau_2$ a 3-space and let ℓ and m be two disjoint lines in σ. Let $\mathcal{S}$ be the set of all planes in σ, the planes in τ_1 through ℓ, the planes in τ_1 through m, and the planes in τ_2 intersecting both ℓ and m. This is a maximal EKR(2) set; it contains

$$n(\mathrm{XIV}, \mathcal{P}) = \begin{bmatrix} 4 \\ 3 \end{bmatrix}_q + \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q^2 + 2 \right) \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}_q - \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \right) = q^4 + 3q^3 + 4q^2 + q + 1$$

planes. These EKR(2) sets exist if $\mathcal{P}$ is a projective space of dimension at least 5 or a polar space of rank $d \geq 5$. Note the asymmetric behaviour of τ_1 and τ_2.

XV: Let τ_1 and τ_2 be two 4-spaces on $\mathcal{P}$ with $\sigma = \tau_1 \cap \tau_2$ a 3-space and let $\mathcal{R}$ be a regulus in σ and let $\mathcal{R}'$ be its opposite regulus. Let $\mathcal{S}$ be the set of all planes in σ, the planes in τ_1 through a line of $\mathcal{R}$, and the planes in τ_2 through a line of $\mathcal{R}'$. This maximal EKR(2) set contains

$$n(\mathrm{XV}, \mathcal{P}) = \begin{bmatrix} 4 \\ 3 \end{bmatrix}_q + 2(q+1) \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}_q - \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \right) = 3q^3 + 3q^2 + q + 1$$

planes. These EKR(2) sets exist if $\mathcal{P}$ is a projective space of dimension at least 5 or a polar space of rank $d \geq 5$.

XVI: Let π_1 and π_2 be two disjoint planes in $\mathcal{P}$ and denote $\rho = \langle \pi_1, \pi_2 \rangle$. Let $\mathcal{S}$ be the set containing π_2 and all planes intersecting π_1 in a line and π_2 in a point. This is a maximal EKR(2) set if $\rho \subset \mathcal{P}$ (case XVIa) and also if $\rho \cap \mathcal{P}$ is a symplectic polar space of rank 3 over a finite field $\mathbb{F}_q$, q odd (case XVIb). In the former case $n(\mathrm{XVIa}, \mathcal{P}) = (q^2 + q + 1)^2 + 1$; in the latter case, which was described in [24], $n(\mathrm{XVIb}, \mathcal{P}) = q^2 + q + 2$.

XVII: Let $\mathcal{P}$ be a symplectic polar space with $\mathrm{PG}(n, q)$, q odd, its ambient projective space. Let ρ be a 5-space in $\mathrm{PG}(n, q)$ such that $\rho \cap \mathcal{P}$ is a symplectic polar space of rank 3. Let π, π_1, π_2 and π_3 be planes in ρ on $\mathcal{P}$ such that $\ell_1 = \pi \cap \pi_1$, $\ell_2 = \pi \cap \pi_2$ and $\ell_3 = \pi \cap \pi_3$ are three non-concurrent lines. Denote $P_3 = \ell_1 \cap \ell_2$, $P_2 = \ell_1 \cap \ell_3$ and $P_1 = \ell_2 \cap \ell_3$. Let $\mathcal{S}$ be the set of all planes on $\mathcal{P}$ through P_i, intersecting π_j and π_k in a line, with $\{i, j, k\} = \{1, 2, 3\}$. This maximal EKR(2) set contains $n(\mathrm{XVII}, \mathcal{P}) = 3q + 1$ elements. This EKR(2) set was first described in [24] as a $\{0, 1, 2\}$-clique.

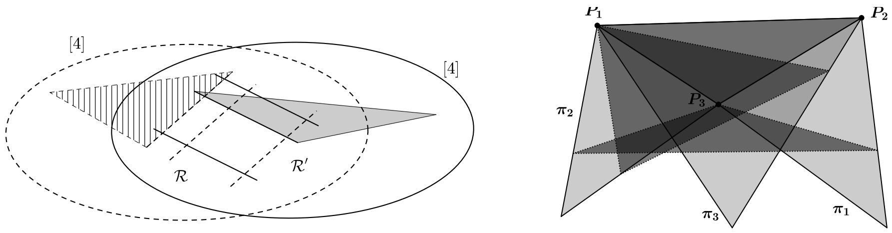

Figure 3.5: The EKR(2) sets of type XV (left) and type XVII (right).

XVIII: Let $\mathcal{P}$ be a symplectic polar space with $\mathrm{PG}(n,q)$, q odd, its ambient projective space. Let ρ be a 5-space in $\mathrm{PG}(n,q)$ such that $\rho\cap\mathcal{P}$ is a symplectic polar space of rank 3. Let $\mathcal{F}$ be a Fano plane with points $P_0,\dots,P_6$ and lines $\ell_0,\dots,\ell_6$. Consider 14 planes $\pi_0,\dots,\pi_6$ and $\pi'_0,\dots,\pi'_6$ such that a plane π_i and a plane π'_j intersect in a line iff $P_i\notin\ell_j$ and such that neither $\pi_i\cap\pi_j$ nor $\pi'_i\cap\pi'_j$ is a line. Let $\mathcal{S}$ be the set $\{\pi_0,\dots,\pi_6\}$. This is an EKR(2) set with $n(\mathrm{XVIII},\mathcal{P})=7$.

This EKR(2) set was first introduced in [24]. However, neither its maximality nor its existence were explicitly proved. We give an example of such an EKR(2) set to prove it exists. Assume the symplectic polar space of rank 3 is given by $A=(a_{i,j})_{i=0\dots5}^{j=0\dots5}$ with $a_{0,1}=a_{2,3}=a_{4,5}=1$, $a_{1,0}=a_{3,2}=a_{5,4}=-1$ and all other entries 0. Assume $\ell_i=\{P_i,P_{i+1},P_{i+3}\}$, $i=0,\dots,6$, whereby the addition in the indices is considered modulo 7, and denote $Q_j=(\delta_{0,j},\dots,\delta_{5,j})$, with $\delta_{i,j}$ the Kronecker delta. We choose $\pi_0=\langle Q_0,Q_2,Q_4\rangle$, $\pi_1=\langle Q_0,Q_3,Q_5\rangle$, $\pi_2=\langle Q_0-Q_4,Q_0-Q_3,Q_1-Q_2+Q_5\rangle$, $\pi_3=\langle Q_2-Q_4,Q_2-Q_1,Q_5-Q_0+Q_3\rangle$, $\pi_4=\langle Q_0-Q_2,Q_0-Q_5,Q_3-Q_4+Q_1\rangle$, $\pi_5=\langle Q_1,Q_3,Q_4\rangle$, $\pi_6=\langle Q_1,Q_2,Q_5\rangle$, $\pi'_0=\langle Q_1,Q_3-Q_4,Q_2-Q_5\rangle$, $\pi'_1=\langle Q_1,Q_2,Q_4\rangle$, $\pi'_2=\langle Q_0,Q_2,Q_5\rangle$, $\pi'_3=\langle Q_0,Q_3,Q_4\rangle$, $\pi'_4=\langle Q_5,Q_1-Q_2,Q_0-Q_3\rangle$, $\pi'_5=\langle Q_0-Q_2,Q_0-Q_4,Q_1-Q_2+Q_3-Q_4+Q_5\rangle$ and $\pi'_6=\langle Q_3,Q_5-Q_0,Q_4-Q_1\rangle$. Then $\mathcal{S}=\{\pi_0,\dots,\pi_6\}$ is a maximal EKR(2) set. Moreover, any set of planes fulfilling the requirements of the above paragraph is PGL-equivalent to this one.

3.2 The main theorem

We now state the main theorem of this chapter. Its proof relies on a separate treatment of different cases, subcases, subsubcases, etc. Therefore we first give only an outline of the proof. Each of the different cases can be found below.

Note that this theorem is stated for a finite projective space or a finite classical polar space. However, it can straightforwardly be generalised to infinite projective spaces and infinite classical polar spaces, respecting the difference between fields of characteristic two and the others for symplectic polar spaces. As we already remarked in Section 2.4 we will not do this; in Section 3.3, we will use this result only for finite projective and polar spaces.

Theorem 3.2.1. *Let $\mathcal{P}$ be a projective space of dimension at least* 5 *or a classical polar space of rank at least* 3 *and let* $\mathrm{PG}(n, q)$ *be the ambient space of* $\mathcal{P}$. *Let* $\mathcal{S}$ *be a maximal EKR(2) set in* $\mathcal{P}$. *Then* $\mathcal{S}$ *is of type I,..., X or XI or* $\mathcal{S}$ *is contained in a* 5*-space of* $\mathrm{PG}(n, q)$.

Outline of the proof. Since the classical polar spaces can be embedded in a projective space, we can proceed in an ambient projective space $\mathrm{PG}(n, q)$, $n \geq 5$. The planes of $\mathcal{S}$ obviously need to be on $\mathcal{P}$, but the j-spaces we will consider, are not all necessarily on $\mathcal{P}$. We will indicate their intersection with $\mathcal{P}$ when needed. The structure of the proof can be found here.

1. *Assume $\mathcal{S}$ contains two planes intersecting each other in a line ℓ.* In Remark 3.2.2 we distinguish three types of planes, denoted by A, B and C.
   - 1.1. *Assume $\mathcal{S}$ does not contain planes of type C.*
     - 1.1.1. *All planes of type B in $\mathcal{S}$ contain the same point of ℓ.* In Lemma 3.2.3 we prove that $\mathcal{S}$ must be an EKR(2) set of type I.
     - 1.1.2. *There are two planes of type B in $\mathcal{S}$ intersecting in a line and containing different points of ℓ.* In Remark 3.2.4 we study this situation. We introduce the sets $\mathcal{S}'$ and T, and the 3-space σ_1'.
       - 1.1.2.1. *Assume that $|T| = 1$.* In Lemma 3.2.5 we prove that $\mathcal{S}$ must be an EKR(2) set of type II.
       - 1.1.2.2. *Assume that $|T| \geq 2$ and that there are two planes in $\mathcal{S}'$ through a different point of ℓ which intersect in a line.* In Lemma 3.2.6 we prove that $\mathcal{S}$ must be an EKR(2) set of type III, IV or V.

1.1.2.3. *Assume that $|T| \geq 2$ and that any two planes in $\mathcal{S}'$ through a different point of ℓ intersect in a point and that two such planes can be found whose intersection point is not in σ_1'.* In Lemma 3.2.7 we prove that $\mathcal{S}$ must be an EKR(2) set of type VI.

1.1.2.4. *Assume that $|T| \geq 2$ and that any two planes in $\mathcal{S}'$ through a different point of ℓ intersect in a point of σ_1'.* In Lemma 3.2.8 we prove that this case cannot occur.

1.1.3. *Not all planes of type B in $\mathcal{S}$ pass through the same point of ℓ. Any two planes of type B through different points of ℓ intersect in a point.* In Remark 3.2.9 we introduce τ_1 and the statement $(*)$. We also study the case in which τ_1 does not fulfill $(*)$. In this case $\mathcal{S}$ is of type III and $\mathcal{P}$ is a polar space of rank 3. In Remark 3.2.10 we look at the case in which τ_1 fulfills $(*)$ and we sort the planes of type B in ten subtypes: BbA, BbB, BbC, BbD, BcA, BcB, BcC, BcD, BeA and BeB.

1.1.3.1. *Assume τ_1 fulfills $(*)$ and $\mathcal{S}$ contains no planes of subtypes BbD, BcD and BeB.* In Lemma 3.2.11 we prove this situation cannot occur.

1.1.3.2. *Assume τ_1 fulfills $(*)$ and $\mathcal{S}$ contains planes of subtype BeB.* In Lemma 3.2.12 we prove that $\mathcal{S}$ must be an EKR(2) set of type VII.

1.1.3.3. *Assume τ_1 fulfills $(*)$ and $\mathcal{S}$ contains planes of subtype BbD, but none of subtypes BcD and BeB.* In Lemma 3.2.13 we prove that $\mathcal{S}$ must be an EKR(2) set of type VIII, VIIIa or VIIIb.

1.1.3.4. *Assume τ_1 fulfills $(*)$ and $\mathcal{S}$ contains planes of subtype BcD, but none of subtypes BbD and BeB.* In Remark 3.2.14 we show this case is analogous to the previous one. Hence $\mathcal{S}$ must be an EKR(2) set of type VIII, of type VIIIa or of type VIIIb.

1.1.3.5. *Assume τ_1 fulfills $(*)$ and $\mathcal{S}$ contains planes of subtypes BbD and BcD, but none of subtype BeB.* In Lemma 3.2.15 we prove the EKR(2) set must be of type IX, IXa or IXb.

1.2. *Assume $\mathcal{S}$ contains a plane of type C.* In Remark 3.2.16 we introduce the 4-space τ_1 and the line ℓ_1.

1.2.1. *Assume that there is a plane of type C, not contained in τ_1 and that all planes of type C in $\mathcal{S}$ that are not contained in τ_1, pass* *through ℓ_1.* In Lemma 3.2.17 we show either all planes of $\mathcal{S}$ are contained in a 5-space or else we can find a line which contains a point of every plane in $\mathcal{S}$. This second possibility has been treated in Case 1.1.

1.2.2. *Assume $\mathcal{S}$ contains planes of type C not in τ_1 and not through ℓ_1.* In Remark 3.2.18 we introduce the line ℓ_2 and the point R, and we argue that $\mathcal{S}$ always contains a plane of type B.

1.2.2.1. *Assume that $\mathcal{S}$ contains a plane of type B intersecting neither ℓ_1 nor ℓ_2, and not containing R.* In Lemma 3.2.19 we prove either all planes of $\mathcal{S}$ are contained in a 5-space or else we can find a line which contains a point of every plane in $\mathcal{S}$. As noted before, this second possibility has been treated in Case 1.1.

1.2.2.2. *Assume that $\mathcal{S}$ contains a plane of type B intersecting neither ℓ_1 nor ℓ_2, but also that all such planes contain R.* In Lemma 3.2.20 we prove that either all planes of $\mathcal{S}$ are contained in a 5-space or else we can find a line which contains a point of every plane in $\mathcal{S}$. As noted before, this second possibility has been treated in Case 1.1.

1.2.2.3. *Assume that all planes of type B in $\mathcal{S}$ intersect ℓ_1 or ℓ_2.* In Lemma 3.2.21 we prove that there are three possibilities: all planes of $\mathcal{S}$ are contained in a 5-space, we can find a line which contains a point of every plane in $\mathcal{S}$, or $\mathcal{S}$ is an EKR(2) set of type X.

1.2.3. *Assume that all planes of type C in $\mathcal{S}$ are contained in τ_1.* In Lemma 3.2.22 we prove that there are three possibilities: all planes of $\mathcal{S}$ are contained in a 5-space, we can find a line which contains a point of every plane in $\mathcal{S}$, or $\mathcal{S}$ is an EKR(2) set of type X.

2. *Assume any two planes of $\mathcal{S}$ intersect each other in a point.* In Remark 3.2.23 we introduce the 5-space ρ. We show that any plane in $\mathcal{S}$ intersects ρ in at least a line.

2.1. *Assume that all planes of $\mathcal{S}$ that are not contained in ρ, pass through a common line in ρ.* In Lemma 3.2.24 we prove that this case cannot occur.

2.2. *Assume that $\mathcal{S}$ contains two planes not in ρ, through different lines* *of ρ.* In Lemma 3.2.25 we prove that $\mathcal{S}$ must be an EKR(2) set of type XI.

2.3. *Assume that all planes in $\mathcal{S}$ are contained in ρ.* In this case the EKR(2) set is obviously contained in a 5-space. □

Remark 3.2.2. In this case we assume that $\mathcal{S}$ contains two planes that intersect in a line ℓ. Let π_1 and π_2 be two planes in $\mathcal{S}$ through ℓ. The planes through ℓ are called planes of *type A*; the planes that intersect ℓ in precisely a point, are called planes of *type B*. The planes that do not contain a point of ℓ, are called planes of *type C*. All planes of $\mathcal{S}$ belong to one of these types. Any plane of type C in $\mathcal{S}$ contains a point of $\pi_1 \setminus \ell$ and a point of $\pi_2 \setminus \ell$, hence a line in $\sigma_1 = \langle \pi_1, \pi_2 \rangle$ skew to ℓ. Since all planes in $\mathcal{S}$, skew to ℓ, contain a line in the 3-space σ_1, we know that all planes through ℓ in σ_1 (planes of type A in σ_1) are contained in $\mathcal{S}$. Note that $\mathcal{S}$ only can contain planes of type C if $\sigma_1 \subset \mathcal{P}$.

Lemma 3.2.3. *Let $\mathcal{S}$ be a maximal EKR(2) set fulfilling the assumptions made in Remark 3.2.2. Using the notations introduced in that remark, we assume that $\mathcal{S}$ does not contain planes of type C. If all planes of type B in $\mathcal{S}$ pass through the same point $P \in \ell$, then $\mathcal{S}$ is of type I.*

Proof. All planes in $\mathcal{S}$ of type A contain the point $P \in \ell$ and by assumption all planes in $\mathcal{S}$ of type B also contain the point P. Since $\mathcal{S}$ contains no planes of type C, all planes in $\mathcal{S}$ contain the point P. By the maximality condition on an EKR(2) set, we know $\mathcal{S}$ must contain all planes through P. This set is indeed maximal since $n \geq 5$. Hence, $\mathcal{S}$ is an EKR(2) set of type I. □

Remark 3.2.4. We use the notations introduced in Remark 3.2.2. We assume $\mathcal{S}$ contains no planes of type C. Hence, all planes in $\mathcal{S}$ contain at least a point of ℓ and, by the maximality of an EKR(2) set, all planes through ℓ are contained in $\mathcal{S}$. Furthermore, we assume both π'_1 and π'_2 are planes of type B, with $\pi'_1 \cap \ell = \{P_1\}$ and $\pi'_2 \cap \ell = \{P_2\}$, $P_1 \neq P_2$, such that $\pi'_1 \cap \pi'_2$ is a line ℓ'. We denote $\sigma'_1 = \langle \pi'_1, \pi'_2 \rangle = \langle \ell, \ell' \rangle$, a 3-space. Note that $\sigma'_1 \subset \mathcal{P}$. The planes in $\mathcal{S}$ of type B then either are contained in σ'_1 or else intersect σ'_1 in a line which contains a point of ℓ. By the maximality of $\mathcal{S}$, all planes in the 3-space σ'_1, which are all of type A or type B, then belong to $\mathcal{S}$.

Let $\mathcal{S}'$ be the set $\mathcal{S} \setminus (\{\text{plane } \pi \mid \ell \subset \pi\} \cup \{\text{plane } \pi \mid \pi \subset \sigma'_1\})$. All planes in $\mathcal{S}'$ are of type B and hence each such plane contains precisely one point of ℓ.

Let T be the set of points on ℓ contained in at least one element of $\mathcal{S}'$. It is easy to see that T cannot be the empty set since $\mathcal{S} \setminus \mathcal{S}'$ cannot be maximal.

Lemma 3.2.5. *Let $\mathcal{S}$ be a maximal EKR(2) set fulfilling the assumptions made in Remark 3.2.4. Using the notations from that remark, we assume $|T| = 1$. Then $\mathcal{S}$ is an EKR(2) set of type II.*

Proof. In this case, all planes of $\mathcal{S}'$ pass through a common point $P' \in \ell$. By the arguments in Remark 3.2.4, all these planes intersect σ_1' in a line. However, all planes that intersect σ_1' in a line through P' intersect (obviously) each other. Thus all these planes need to be contained in $\mathcal{S}'$, and consequently in $\mathcal{S}$. Note that this set is indeed maximal since $n \geq 5$. We find an EKR(2) set of type II. $\square$

Lemma 3.2.6. *Let $\mathcal{S}$ be a maximal EKR(2) set fulfilling the assumptions made in Remark 3.2.4. Using the notations from that remark, we assume $|T| \geq 2$. If $\mathcal{S}'$ contains two planes through a different point of ℓ, intersecting each other in a line, then $\mathcal{S}$ is an EKR(2) set of type III, of type IV or of type V.*

Proof. Let $\overline{\pi}_1$ and $\overline{\pi}_2$ be two planes in $\mathcal{S}'$, with $\overline{\pi}_1 \cap \ell = \{Q_1\}$ and $\overline{\pi}_2 \cap \ell = \{Q_2\}$, $Q_1 \neq Q_2$, such that $\overline{\pi}_1 \cap \overline{\pi}_2$ is a line $\overline{\ell}$. We denote the 3-space $\langle \overline{\pi}_1, \overline{\pi}_2 \rangle$ by $\overline{\sigma}_1 \subset \mathcal{P}$. Now, regarding ℓ, the sets $\{\pi_1', \pi_2', \ell', \sigma_1'\}$ and $\{\overline{\pi}_1, \overline{\pi}_2, \overline{\ell}, \overline{\sigma}_1\}$ play the same role. Using the arguments from Remark 3.2.4, we know that all planes in $\overline{\sigma}_1$ are in $\mathcal{S}$ and all other planes of $\mathcal{S}$ intersect $\overline{\sigma}_1$ in a line. Note that $\sigma_1' \cap \overline{\sigma}_1$ is a plane. We denote this plane by V and we denote $\tau_1 = \langle \sigma_1', \overline{\sigma}_1 \rangle$.

Hence, the planes in $\mathcal{S}'$ which do not lie in $\overline{\sigma}_1$, should contain a line in σ_1' and a line in $\overline{\sigma}_1$. These planes, all of type B, therefore all belong to one of the three following types. Note that all of these planes contain at least a point of V since $\ell \subset V$.

- Ba1: the planes that intersect V in a point but not in a line. These planes are generated by a line in σ_1' not in V and a line in $\overline{\sigma}_1$ not in V, and thus contained in τ_1. These planes only exist if $\tau_1 \subset \mathcal{P}$.
- Ba2: the planes that intersect V in a line and are contained in τ_1. The projective or polar space $\mathcal{P}$ surely contains planes of this type since $\sigma_1', \overline{\sigma}_1 \subset \mathcal{P}$. In fact, the planes in $\sigma_1' \cup \overline{\sigma}_1$, different from V itself, are of this type.

Ba3: the planes that intersect V in a line and are not contained in τ_1. So these planes intersect τ_1 in a line of V.

On the one hand, note that any plane of type Ba2 intersects all planes of type Ba1, Ba2 and Ba3. By the maximality condition on an EKR(2) set, all these planes must be contained in $\mathcal{S}$, if they are in $\mathcal{P}$. On the other hand, note that a plane of type Ba1 and a plane of type Ba3 can intersect each other only in a point of ℓ. We distinguish three cases.

If $\mathcal{S}$ contains two planes of type Ba3 not through the same point of $\ell \subset V$, we know no planes of type Ba1 are contained in $\mathcal{S}$. However, all planes that intersect V in a line, intersect each other and should thus be in $\mathcal{S}$. We find an EKR(2) set of type III. This is indeed maximal since $n \geq 5$.

If all planes of type Ba3 in $\mathcal{S}$ pass through the same point P' of ℓ, then all planes of type Ba1 in $\mathcal{S}$ also pass through P' by the above remark. Using again the maximality condition on $\mathcal{S}$, we know that all planes of type Ba1 through P' are in $\mathcal{S}$. The set $\mathcal{S}$ that we have found, is maximal if $\tau_1 \subset \mathcal{P}$ since $n \geq 5$. We see that $\mathcal{S}$ is an EKR(2) set of type IV. This set is however not maximal if $\tau_1 \not\subset \mathcal{P}$ since in this case, there are no planes of type Ba1 and all planes of Ba3 then intersect the planes in $\mathcal{S}$.

If $\mathcal{S}$ contains no planes of type Ba3, then all planes of type Ba1 must be contained in $\mathcal{S}$. The set which we find in this case, is maximal if $\tau_1 \subset \mathcal{P}$. It is an EKR(2) set of type V. If $\tau_1 \not\subset \mathcal{P}$, then $\mathcal{S}$ is not maximal since all planes in $\mathcal{S}$ then necessarily intersect V in a line and each plane of type Ba3 thus intersects all planes in $\mathcal{S}$. □

Lemma 3.2.7. *Let $\mathcal{S}$ be a maximal EKR(2) set fulfilling the assumptions made in Remark 3.2.4. Using the notations from that remark, we assume that $|T| \geq 2$ and that any two planes in $\mathcal{S}'$ through a different point of ℓ intersect each other in a point. If $\mathcal{S}'$ contains two planes through a different point of ℓ, intersecting each other in a point not in σ_1', then $\mathcal{S}$ is an EKR(2) set of type VI.*

Proof. Let π_3' and π_4' be two planes in $\mathcal{S}'$, with $\pi_3' \cap \sigma_1' = \ell_3$, $\pi_4' \cap \sigma_1' = \ell_4$, $\pi_3' \cap \ell = \{P_3\}$, $\pi_4' \cap \ell = \{P_4\}$, $P_3 \neq P_4$, such that their intersection point Q_3 is not in σ_1'. Note that $\sigma_1' = \langle \ell_3, \ell_4 \rangle$. We denote $\langle \sigma_1', Q_3 \rangle = \langle \pi_3', \pi_4' \rangle$ by τ_3. Note that $\tau_3 \subset \mathcal{P}$. Also note that the planes $\langle P_3, \ell_4 \rangle$ and $\langle P_4, \ell_3 \rangle$ intersect each other in the line ℓ. Consequently, all lines that intersect ℓ, ℓ_3 and ℓ_4 must be lines through P_3 in $\langle P_3, \ell_4 \rangle$ or lines through P_4 in $\langle P_4, \ell_3 \rangle$. Recall that all planes in

$\mathcal{S}'$ intersect σ_1' in a line. Hence, all planes of type B in $\mathcal{S}'$ belong to one of the following types.

Baa: the planes, not in τ_3, containing a line through P_3 in $\langle P_3, \ell_4\rangle$, different from ℓ.

Bab: the planes, not in τ_3, containing a line through P_4 in $\langle P_4, \ell_3\rangle$, different from ℓ.

Bac: the planes in τ_3, containing precisely one point of ℓ.

We now distinguish four cases.

If $\mathcal{S}'$ contains no planes of type Baa nor of type Bab, then all planes of type Bac should be contained in $\mathcal{S}'$ since they all intersect each other. However, we can find two such planes through a different point of ℓ, which intersect in a line, a contradiction.

If $\mathcal{S}'$ contains no planes of type Bab, but does contain a plane of type Baa, necessarily only sharing a line of $\langle P_3, \ell_4\rangle$ with τ_3, then the only planes of type Bac that can be in $\mathcal{S}'$ are the ones that intersect $\langle P_3, \ell_4\rangle$ precisely in the point P_3 (type Bac1) or contain a line in $\langle P_3, \ell_4\rangle$ (type Bac2). Any plane of type Bac2 intersects all planes of type Baa and all planes of type Bac, hence all planes of type Bac2 are in $\mathcal{S}'$ because of the maximality condition on $\mathcal{S}$. However, we can find two such planes through a different point of ℓ, which intersect in a line, a contradiction.

If $\mathcal{S}'$ contains no planes of type Baa, but does contain a plane of type Bab, then we can use the same arguments.

Now, we assume $\mathcal{S}'$ contains a plane π_5' of type Baa and a plane π_6' of type Bab. We denote $\pi_5' \cap \sigma_1' = \ell_5 = \langle P_3, P_5\rangle$, with $P_5 \in \ell_4$, and $\pi_6' \cap \sigma_1' = \ell_6 = \langle P_4, P_6\rangle$, with $P_6 \in \ell_3$. Furthermore, we denote $\pi_5' \cap \pi_6' = \{Q_5\}$ and $\tau_5 = \langle \pi_5', \pi_6'\rangle$. Note that $\tau_3 \cap \tau_5 = \sigma_1'$ and that $\tau_5 \subset \mathcal{P}$. We observe that, considering ℓ, σ_1', P_3 and P_4, the sets $\{\pi_3', \pi_4', Q_3, \tau_3, \ell_3, \ell_4\}$ and $\{\pi_5', \pi_6', Q_5, \tau_5, \ell_5, \ell_6\}$ play the same role. Hence, the planes of $\mathcal{S}'$ can also be split up in three types according to this second set. They belong to one of the following types.

Baa': the planes, not in τ_5, containing a line through P_3 in $\langle P_3, \ell_6\rangle = \langle P_4, \ell_3\rangle$, different from ℓ.

Bab': the planes, not in τ_5, containing a line through P_4 in $\langle P_4, \ell_5\rangle = \langle P_3, \ell_4\rangle$, different from ℓ.

Bac': the planes in τ_5, containing precisely one point of ℓ.

A plane in $\mathcal{S}'$ belongs to one of the three types according to both systems. So, there are nine possibilities for the planes in $\mathcal{S}'$. There are no planes that are simultaneously of type Baa and of type Baa', since planes generated by two different lines through P_3 in σ_1', are contained in $\sigma_1' \subset \tau_3, \tau_5$. Analogously, there are no planes that are of type Bab and of type Bab'. Planes cannot be of type Baa and of type Bab' since they cannot contain two different points of ℓ without containing ℓ itself. Analogously, planes cannot be of type Bab and Baa'. The planes that are of type Bac and Bac' are contained in $\tau_3 \cap \tau_5 = \sigma_1'$. We already know that the planes in σ_1' are contained in $\mathcal{S}$.

The planes which are of type Baa and of type Bac' are in τ_5 and intersect σ_1' in a line in $\langle P_3, \ell_4\rangle$ through P_3. The planes which are of type Bab and of type Bac' are in τ_5 and intersect σ_1' in a line in $\langle P_4, \ell_3\rangle$ through P_4. Analogously, the planes which are of type Baa' and Bac are in τ_3 and intersect σ_1' in a line in $\langle P_4, \ell_3\rangle$ through P_3. The planes which are of type Bab' and Bac are in τ_3 and intersect σ_1' in a line in $\langle P_3, \ell_4\rangle$ through P_4. From these descriptions it can easily be seen that all these planes intersect each other, hence by the maximality condition on $\mathcal{S}$, all should be contained in $\mathcal{S}$. We find an EKR(2) set of type VI. $\square$

Lemma 3.2.8. *Let $\mathcal{S}$ be a maximal EKR(2) set fulfilling the assumptions made in Remark 3.2.4. Using the notations from that remark, we assume that $|T| \geq 2$ and that any two planes in $\mathcal{S}'$ through a different point of ℓ intersect each other in exactly a point. If any two planes in $\mathcal{S}'$ through a different point of ℓ intersect each other in a point of σ_1', then $\mathcal{S}$ cannot be a maximal EKR(2) set.*

Proof. Let π_3' and π_4' be two planes in $\mathcal{S}'$, with $\pi_3' \cap \sigma_1' = \ell_3$, $\pi_4' \cap \sigma_1' = \ell_4$, $\pi_3' \cap \ell = \{P_3\}$, $\pi_4' \cap \ell = \{P_4\}$, $P_3 \neq P_4$. By the assumptions, their intersection point Q_4 belongs to σ_1' and thus $Q_4 = \ell_3 \cap \ell_4$. Any plane in $\mathcal{S}'$ contains a line in $\langle \ell, Q_4\rangle$ since it intersects both π_3' and π_4' in a point of σ_1' and it also intersects ℓ. By this observation and the maximality condition on $\mathcal{S}$, all planes that contain a line in $\langle \ell, Q_4\rangle$ should be in $\mathcal{S}$. However, we can find two such planes through different points of ℓ, intersecting each other in a line. This contradicts the assumptions. Hence, under these imposed assumptions, $\mathcal{S}$ cannot be a maximal EKR(2) set. $\square$

Remark 3.2.9. We use the notations introduced in Remark 3.2.2. We already assumed $\mathcal{S}$ contains no planes of type C. Hence, all planes in $\mathcal{S}$ contain at least a point of ℓ and, by the maximality of $\mathcal{S}$, all planes through ℓ are contained in $\mathcal{S}$. However, these cannot be the only planes in $\mathcal{S}$; there also must be planes of type B, not all through the same point of ℓ. In this case, we assume that any two planes of type B through different points of ℓ, intersect each other in a point. Let π'_1 and π'_2 be two planes of type B, with $\pi'_1 \cap \ell = \{P_1\}$ and $\pi'_2 \cap \ell = \{P_2\}$, $P_1 \neq P_2$, such that $\pi'_1 \cap \pi'_2$ is a point Q_1. We denote the 4-space $\langle \pi'_1, \pi'_2 \rangle$ by τ_1, the 3-space $\langle P_1, \pi'_2 \rangle$ by σ'_2 and the 3-space $\langle P_2, \pi'_1 \rangle$ by σ'_1, and furthermore $\ell_1 = \langle P_1, Q_1 \rangle$, $\ell_2 = \langle P_2, Q_1 \rangle$ and $\overline{\pi} = \langle \ell, Q_1 \rangle = \langle \ell_1, P_2 \rangle = \langle \ell_2, P_1 \rangle$. Note that $\overline{\pi} = \sigma'_1 \cap \sigma'_2 \subset \mathcal{P}$.

The planes of type B in $\mathcal{S}$ then necessarily belong to one of the following types.

- Bb1: the planes through P_1 containing a line in $\overline{\pi}$, different from ℓ.
- Bb2: the planes through P_1 intersecting σ'_2 in a line, not in $\overline{\pi}$. Note that these planes exist only if $\sigma'_2 \subset \mathcal{P}$.
- Bc1: the planes through P_2 containing a line in $\overline{\pi}$, different from ℓ.
- Bc2: the planes through P_2 intersecting σ'_1 in a line, not in $\overline{\pi}$. Note that these planes exist only if $\sigma'_1 \subset \mathcal{P}$.
- Be1: the planes through $P' \in \ell \setminus \{P_1, P_2\}$ in τ_1, containing a line in $\overline{\pi}$.
- Be2: the planes through $P' \in \ell \setminus \{P_1, P_2\}$ not in τ_1, containing a line in $\overline{\pi}$.
- Be3: the planes through $P' \in \ell \setminus \{P_1, P_2\}$ in τ_1, not containing a line in $\overline{\pi}$. These planes intersect $\overline{\pi}$ in the point P'.

We now look at $\tau_1 \cap \mathcal{P}$. We assume $\tau_1 \not\subset \mathcal{P}$, thus $\mathcal{P}$ is a polar space. We know that the planes π'_1, π'_2 and $\overline{\pi}$ are in $\mathcal{P}$. Hence, the intersection cannot be a non-singular 4-dimensional polar space and thus is a cone. The vertex cannot be a line since the planes π'_1 and π'_2 intersect in a point; the vertex cannot be a 3-space since π'_1 and π'_2 generate the 4-space τ_1. We consider the remaining cases.

We assume first that the vertex is a point. Then, the base is a non-singular polar space $X(3, q)$ of rank 2 embedded in a 3-space $\widetilde{\sigma}$ (a hyperbolic quadric, a Hermitian polar space or a symplectic polar space). The vertex is contained in each plane on $\tau_1 \cap \mathcal{P}$, and hence must equal Q_1. Without loss of generality, we can assume $\ell \subset \widetilde{\sigma}$. It follows that $\sigma'_1 \cap \mathcal{P}$ is a cone with vertex the line $\langle Q_1, P_1 \rangle = \ell_1$, and that $\sigma'_2 \cap \mathcal{P}$ is a cone with vertex the line $\langle Q_1, P_2 \rangle = \ell_2$. Consequently, planes of type Bb2 and type Bc2 cannot exist. Also planes of type Be3 cannot exist since each plane of $\mathcal{P}$ in τ_1 passes through Q_1. Hence, all planes in $\mathcal{S}$ intersect $\overline{\pi}$ in at least a line. By the maximality condition on an EKR(2) set, all planes intersecting $\overline{\pi}$ in a line must be contained in $\mathcal{S}$. If the rank of the polar space is at least 4, then we can find two planes among these planes through different points of ℓ and intersecting in a line. If the rank of the polar space equals 3, then this set is maximal: $\mathcal{S}$ is an EKR(2) set of type III.

Now we assume that the vertex is a plane π^2, different from $\overline{\pi}$. Then, $\tau_1 \cap \mathcal{P}$ is the union of 3-spaces through π^2. It is not possible that both σ'_1 and σ'_2 are in $\mathcal{P}$ because $\overline{\pi} \neq \pi^2$. Without loss of generality, we can assume $\sigma'_2 \not\subset \mathcal{P}$. It follows that $\sigma'_2 \cap \mathcal{P}$ is a cone with vertex the line $\langle Q_1, P_2 \rangle = \overline{\pi} \cap \pi'_2$. Hence, there are no planes of type Bb2. Furthermore, the line $\langle Q_1, P_2 \rangle$ is contained in the vertex π^2. So, $\langle Q_1, P_2, \pi'_1 \rangle = \sigma'_1$ is contained in $\mathcal{P}$. Note that planes of type Be3 cannot exist in this case, since they are disjoint from $\langle Q_1, P_2 \rangle$. Let V be a plane through P_2, but not through ℓ, in σ'_1. This is a plane of type Bc1. It intersects all planes of type Bb1, Bc1, Bc2, Be1 or Be2, and is therefore necessarily in $\mathcal{S}$. Obviously, $\pi'_1 \cap V$ is a line since $\pi'_1, V \subset \sigma'_1$. This contradicts the assumption that any two planes of type B in $\mathcal{S}$ through different points of ℓ, do not intersect in a line.

Only one cone case remains, namely the following: the vertex is the plane $\overline{\pi}$ and $\tau_1 \cap \mathcal{P}$ is a union of 3-spaces through $\overline{\pi}$, among which σ'_1 and σ'_2. We say τ_1 fulfills $(*)$ if τ_1 is in this case or if $\tau_1 \subset \mathcal{P}$. In this remark we studied the case in which τ_1 does not fulfill $(*)$.

Remark 3.2.10. We use the notations and the assumptions we introduced in Remark 3.2.9, but now we assume that τ_1 fulfills $(*)$. Note that this implies that $\sigma'_1, \sigma'_2 \subset \mathcal{P}$. Then at least planes of types Bb1, Bb2, Bc1, Bc2, Be1 and Be2 exist. Note that all planes of types Bb1, Bc1, Bc2, Be1, Be2 or Be3, and all planes of type Bb2 in τ_1, intersect σ'_1 in at least a line. Consequently, if $\mathcal{S}$ contains no planes of type Bb2 not in τ_1, then all planes in σ'_1 should be contained in $\mathcal{S}$ by the maximality condition. This however contradicts the assumption that no two planes of type B in $\mathcal{S}$ through a different point of ℓ intersect in a line. Hence, $\mathcal{S}$ contains a plane of type Bb2 not in τ_1 and analogously also a plane of type Bc2 not in τ_1.

Let π'_3 be a plane of type Bb2 in $\mathcal{S}$ and let π'_4 be a plane of type Bc2 in $\mathcal{S}$, both not contained in τ_1, with $\pi'_2 \cap \pi'_3 = \{P_3\}$ and $\pi'_4 \cap \pi'_1 = \{P_4\}$. By the assumptions, $\pi'_3 \cap \pi'_4$ must be a point, which we denote by Q_3. We denote $\langle P_1, Q_3\rangle = \ell_3$, $\langle P_2, Q_3\rangle = \ell_4$, $\langle P_1, P_4\rangle = \overline{\ell}_1 \subset \pi'_1$, $\langle P_2, P_3\rangle = \overline{\ell}_2 \subset \pi'_2$, $\langle P_1, P_3\rangle = \overline{\ell}_3 \subset \pi'_3$, $\langle P_2, P_4\rangle = \overline{\ell}_4 \subset \pi'_4$ and $\tau_3 = \langle \pi'_3, \pi'_4\rangle$. It can be observed that $\overline{\ell}_3 \subset \sigma'_2$ and $\overline{\ell}_4 \subset \sigma'_1$ and since neither $\overline{\ell}_3$ nor $\overline{\ell}_4$ is contained in $\overline{\pi}$, these lines cannot intersect, hence $Q_3 \notin \overline{\ell}_3, \overline{\ell}_4$. Furthermore, this point cannot be contained in τ_1. We can write $\pi'_3 = \langle P_1, P_3, Q_3\rangle$ and $\pi'_4 = \langle P_2, P_4, Q_3\rangle$. Note that $\tau_1 \cap \tau_3$ is a 3-space $\overline{\sigma}$, which contains the lines ℓ, $\overline{\ell}_1$, $\overline{\ell}_2$, $\overline{\ell}_3$ and $\overline{\ell}_4$. Finally we also introduce the notations $\overline{\pi}' = \langle \ell, Q_3\rangle$, $\sigma'_3 = \langle P_2, \pi'_3\rangle$ and $\sigma'_4 = \langle P_1, \pi'_4\rangle$. Note that $\overline{\pi}', \sigma'_3, \sigma'_4 \subset \mathcal{P}$.

We observe that, regarding ℓ, P_1 and P_2, the sets $\{\pi'_1, \pi'_2, Q_1, \tau_1, \overline{\pi}, \sigma'_1, \sigma'_2\}$ and $\{\pi'_3, \pi'_4, Q_3, \tau_3, \overline{\pi}', \sigma'_3, \sigma'_4\}$ play the same role. So, the planes of type B in $\mathcal{S}$ also belong to one of the following types.

- Bb1': the planes through P_1 containing a line in $\overline{\pi}'$, different from ℓ.
- Bb2': the planes through P_1 intersecting σ'_4 in a line, not in $\overline{\pi}'$.
- Bc1': the planes through P_2 containing a line in $\overline{\pi}'$, different from ℓ.
- Bc2': the planes through P_2 intersecting σ'_3 in a line, not in $\overline{\pi}'$.
- Be1': the planes through $P' \in \ell \setminus \{P_1, P_2\}$ in τ_3, containing a line in $\overline{\pi}'$.
- Be2': the planes through $P' \in \ell \setminus \{P_1, P_2\}$ not in τ_3, containing a line in $\overline{\pi}'$.
- Be3': the planes through $P' \in \ell \setminus \{P_1, P_2\}$ in τ_3, not containing a line in $\overline{\pi}'$. These planes intersect $\overline{\pi}'$ only in the point P'.

We denote the 3-space $\langle \overline{\pi}, \overline{\pi}'\rangle = \langle \ell, Q_1, Q_3\rangle$ by σ'. Then, $\sigma' \subset \mathcal{P}$ or $\sigma' \cap \mathcal{P}$ is a union of planes through ℓ, since $\overline{\pi}, \overline{\pi}' \subset \sigma'$. Note that $\sigma' \cap \overline{\sigma} = \ell$, that $\sigma' \cap \tau_1 = \overline{\pi}$ and that $\sigma' \cap \tau_3 = \overline{\pi}'$.

Obviously the planes of type B in $\mathcal{S}$ belong to one of the seven types according to both classifications. Most of the combinations are impossible. The ten possible combinations are listed here.

- BbA: (Bb1-Bb1') the planes in σ' through P_1, not through ℓ. Note that these planes can only exist if $\sigma' \subset \mathcal{P}$.

BbB: (Bb1-Bb2') the planes through P_1 in $\langle \sigma'_4, Q_1 \rangle$, not in σ', that contain a line in $\overline{\pi}$.

BbC: (Bb2-Bb1') the planes through P_1 in $\langle \sigma'_2, Q_3 \rangle$, not in σ', that contain a line in $\overline{\pi}'$.

BbD: (Bb2-Bb2') the planes that are generated by P_1, a point of $\pi'_2 \setminus \ell_2$ and a point of $\pi'_4 \setminus \ell_4$.

BcA: (Bc1-Bc1') the planes in σ' through P_2, not through ℓ. Note that these planes can only exist if $\sigma' \subset \mathcal{P}$.

BcB: (Bc1-Bc2') the planes through P_2 in $\langle \sigma'_3, Q_1 \rangle$, not in σ', that contain a line in $\overline{\pi}$.

BcC: (Bc2-Bc1') the planes through P_2 in $\langle \sigma'_1, Q_3 \rangle$, not in σ', that contain a line in $\overline{\pi}$.

BcD: (Bc2-Bc2') the planes that are generated by P_2, a point of $\pi'_1 \setminus \ell_1$ and a point of $\pi'_3 \setminus \ell_3$.

BeA: (Be2-Be2') the planes in σ' through a point $P' \in \ell \setminus \{P_1, P_2\}$, not through ℓ. Note that these planes only can exist if $\sigma' \subset \mathcal{P}$.

BeB: (Be3-Be3') the planes in $\overline{\sigma}$ through a point $P' \in \ell \setminus \{P_1, P_2\}$, not through ℓ. Note that these planes only can exist if $\overline{\sigma} \subset \mathcal{P}$.

Let ρ be the 5-space $\langle \tau_1, \tau_3 \rangle$. Of course, $\rho \subseteq \mathcal{P}$ is possible. We investigate the different possibilities for $\rho \cap \mathcal{P}$ in case $\rho \not\subseteq \mathcal{P}$. We know that $\mathcal{P}$ contains the 3-spaces σ'_1, σ'_2, σ'_3 and σ'_4, which do not pass through a common plane. Therefore $\rho \cap \mathcal{P}$ cannot be non-singular and cannot be a cone with vertex a point or a plane. Also, it cannot be a 4-space since σ'_1 and σ'_3 span ρ. Only two possibilities for $\rho \cap \mathcal{P}$ remain: a cone with vertex a line and a base different from the elliptic quadric or a cone with vertex a 3-space and a base different from the elliptic quadric. In the former case $\rho \cap \mathcal{P}$ is the union of 3-spaces through a common line, necessarily ℓ since $\ell = \cap_{i=1}^4 \sigma'_i$. In the latter case $\rho \cap \mathcal{P}$ is the union of 4-spaces through a vertex σ^3. The 3-spaces σ'_1 and σ'_4 can only be contained in the same 4-space through σ^3 if $\sigma' \subset \mathcal{P}$, since $\sigma' \subset \langle \sigma'_1, \sigma'_4 \rangle$. In the same way, τ_1 (the 4-space through σ'_1 and σ'_2) can only be a 4-space in $\mathcal{P}$ through σ^3 if $\overline{\sigma} \subset \mathcal{P}$, since $\overline{\sigma} \subset \tau_1$. The 3-spaces σ'_1 and σ'_3 cannot be contained in the same 4-space through σ^3. Hence, if $\sigma' \not\subset \mathcal{P}$ and $\overline{\sigma} \not\subset \mathcal{P}$, then this possibility cannot occur; if $\sigma' \subset \mathcal{P}$ and $\overline{\sigma} \not\subset \mathcal{P}$, then $\sigma^3 = \sigma'$; if $\sigma' \not\subset \mathcal{P}$ and $\overline{\sigma} \subset \mathcal{P}$, then $\sigma^3 = \overline{\sigma}$. Finally, if $\sigma' \subset \mathcal{P}$ and $\overline{\sigma} \subset \mathcal{P}$, then $\rho \subseteq \mathcal{P}$, so $\rho \cap \mathcal{P}$ is not a cone.

This ends Remark 3.2.10.

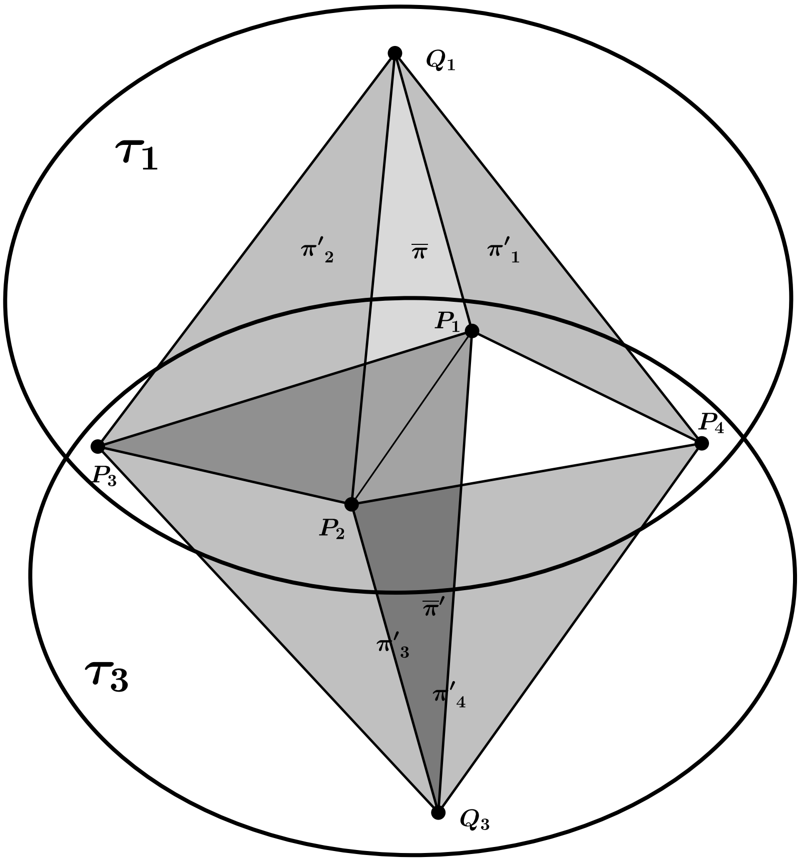

Figure 3.6: The configuration of the subspaces introduced in Remark 3.2.10. Note that only the points, planes and 4-spaces are indicated.

Lemma 3.2.11. *Let $\mathcal{S}$ be a maximal EKR(2) set fulfilling the assumptions made in Remark 3.2.10. It is impossible that $\mathcal{S}$ contains no planes of type BbD, BcD or BeB, using the notations from that remark.*

Proof. We assume that $\mathcal{S}$ contains no planes of type BbD, BcD or BeB. We already noted in Remark 3.2.10 that either $\sigma' \subset \mathcal{P}$ or else $\sigma' \cap \mathcal{P}$ is a union of planes through ℓ. We distinguish between these two cases.

First we consider the former case. All planes of type BbA, BbB, BbC, BcA, BcB, BcC or BeA, then intersect the 3-space σ' in at least a line. By the maximality condition on $\mathcal{S}$, all planes in σ' must be contained in $\mathcal{S}$. These are planes of type BbA, BcA and BeA. However, by the assumptions made in Remark 3.2.10, we know any two planes of type B which pass through different points of ℓ, intersect in a point. So, not all planes in σ' can be contained in $\mathcal{S}$, a contradiction. Clearly, $\mathcal{S}$ cannot be a maximal EKR(2) set under these assumptions.

Secondly, we consider the latter case: $\sigma' \cap \mathcal{P}$ is a union of planes through ℓ. Note that $\mathcal{P}$ is a polar space and not a projective space in this case. There cannot be planes of type BbA, BcA or BeA in this case. We look at the 4-space $\langle \sigma'_4, Q_1 \rangle = \langle \sigma'_1, Q_3 \rangle$. Since it contains $\overline{\pi}$, $\overline{\pi}'$, π'_1 and π'_4, which are all in $\mathcal{P}$, we know that $\langle \sigma'_4, Q_1 \rangle \cap \mathcal{P}$ is the union of 3-spaces through $\langle P_4, \ell \rangle$, among which σ'_1 and σ'_4. Analogously, $\langle \sigma'_3, Q_1 \rangle = \langle \sigma'_2, Q_3 \rangle$ and $\langle \sigma'_3, Q_1 \rangle \cap \mathcal{P}$ is the union of 3-spaces through $\langle P_3, \ell \rangle$, among which σ'_2 and σ'_3. Consequently, all planes of type BbB must be contained in σ'_1, all planes of type BbC must be contained in σ'_3, all planes of type BcB must be contained in σ'_2 and all planes of type BcC must be contained in σ'_4. It follows that all planes of type BbB, BbC, BcB or BcC intersect each other. Moreover, all these planes must be in $\mathcal{S}$ by the maximality condition, because there are no planes of type BbA, BcA or BeA, and $\mathcal{S}$ contains no planes of type BbD, BcD or BeB by assumption.

However, all these planes intersect $\overline{\sigma}$ in at least a line. So, if $\overline{\sigma} \subset \mathcal{P}$, all planes in $\overline{\sigma}$ (which are of type BbD, BcD or BeB, if they do not pass through ℓ), must be contained in $\mathcal{S}$ and $\mathcal{S}$ cannot be a maximal EKR(2) set under these assumptions. Consequently, $\overline{\sigma} \not\subset \mathcal{P}$ and $\overline{\sigma} \cap \mathcal{P}$ is a union of planes through ℓ.

We consider the 5-space $\rho = \langle \tau_1, \tau_3 \rangle$. Clearly $\rho \nsubseteq \mathcal{P}$, so $\rho \cap \mathcal{P}$ is a cone. By the observations at the end of Remark 3.2.10, we know that it has to be a cone with vertex the line ℓ. Now, let m be a line through P_1 in σ'_2, different from ℓ, and let ξ_m be the tangent space to $\mathcal{P}$ in $\mathrm{PG}(n, q)$ corresponding to m. Then, ξ_m intersects ρ in a 4-space and $\xi_m \cap \rho \cap \mathcal{P}$ is a cone with vertex the plane $\langle \ell, m \rangle$. Clearly, $\xi_m \cap \sigma'_4$ is a plane through ℓ. Let m' be a line through P_1 in this plane. Then, the plane $\langle m, m' \rangle$ is a plane in $\mathcal{P}$. This is a plane of type BbD and intersects all planes in $\mathcal{S}$. Hence, $\mathcal{S}$ is not maximal. So, also in this case $\mathcal{S}$ cannot be a maximal EKR(2) set under these assumptions.

Lemma 3.2.12. *Let $\mathcal{S}$ be a maximal EKR(2) set fulfilling the assumptions made in Remark 3.2.10. We use the notations from that remark. If $\mathcal{S}$ contains a plane of type BeB, then $\mathcal{S}$ is an EKR(2) set of type VII.*

Proof. Let π_6' be a plane of type BeB which is contained in $\mathcal{S}$, with $\pi_6' \cap \ell = \{P_6\}$. This plane is contained in $\overline{\sigma}$. Note that necessarily $\overline{\sigma} \subset \mathcal{P}$ and consequently $\tau_1, \tau_3 \subset \mathcal{P}$. Note that none of the planes of type BbA or type BcA intersects π_6' and that all planes of type BeB intersect π_6'. Furthermore, note that the planes of type BbB intersect π_6' if and only if they are contained in σ_1', that the planes of type BbC intersect π_6' if and only if they are contained in σ_3', that the planes of type BcB intersect π_6' if and only if they are contained in σ_2', that the planes of type BcC intersect π_6' if and only if they are contained in σ_4', that the planes of type BeA intersect π_6' if and only if they pass through P_6, that the planes of type BbD intersect π_6' if and only if their point on $\pi_2' \setminus \ell_2$ lies on $\overline{\ell}_2$ or their point on $\pi_4' \setminus \ell_4$ lies on $\overline{\ell}_4$, and that the planes of type BcD intersect π_6' if and only if their point on $\pi_1' \setminus \ell_1$ lies on $\overline{\ell}_1$ or their point on $\pi_3' \setminus \ell_3$ lies on $\overline{\ell}_3$.

It should be observed that each of these planes, except the planes of type BeA that pass through P_6, intersect $\overline{\sigma}$ in at least a line. Arguing as in the first part of the proof of Lemma 3.2.11, we find that $\mathcal{S}$ must contain a plane of type BeA which contains the point P_6. Let π_7' be such a plane in $\mathcal{S}$. Note that thus $\sigma' \subset \mathcal{P}$ and consequently $\rho \subseteq \mathcal{P}$. On the one hand, it should be noted that π_7' intersects anyhow the planes of type BbB contained in σ_1', the planes of type BcB contained in σ_3', the planes of type BcB contained in σ_2', the planes of type BcC contained in σ_4' and the planes of type BeA through P_6. On the other hand, π_7' does not intersect any of the planes of type BbD that contain a point on $\overline{\ell}_2$ or on $\overline{\ell}_4$, nor any of the planes of type BcD that contain a point on $\overline{\ell}_1$ or on $\overline{\ell}_3$. Since all planes through P_1 in σ_1' or σ_3', all planes through P_2 in σ_2' or σ_4', and all planes through P_6 in σ' or $\overline{\sigma}$ intersect each other, all these planes must belong to $\mathcal{S}$ by the maximality condition. We find an EKR(2) set of type VII. $\square$

Lemma 3.2.13. *Let $\mathcal{S}$ be a maximal EKR(2) set fulfilling the assumptions made in Remark 3.2.10. Using the notations from that remark, we assume that $\mathcal{S}$ contains no planes of type BeB. If $\mathcal{S}$ contains a plane of type BbD, but none of type BcD, then $\mathcal{S}$ is an EKR(2) set of type VIII, of type VIIIa or of type VIIIb.*

Proof. Let $\pi_6' = \langle P_1, Q_2, Q_4 \rangle$ be a plane of type BbD that is contained in $\mathcal{S}$, with $Q_2 \in \pi_2' \setminus \ell_2$ and $Q_4 \in \pi_4' \setminus \ell_4$. All planes of type BbA, BbB, BbC or BbD intersect π_6' since they contain P_1; any plane of type BcA or type BeA does not intersect π_6'. Note that π_6' intersects the planes of type BcB if and only if they are contained in σ_2' and that π_6' intersects the planes of type BcC if and only if they are contained in σ_4'. Note also that all the planes of type BbA, BbB, BbC or BbD, the planes of type BcB in σ_2' and the planes of type BcC in σ_4' intersect each other. By the maximality condition on $\mathcal{S}$, all these planes should be contained in $\mathcal{S}$. Observe that the planes of type BbA, BbB, BbC and BbD are together all planes through P_1 that contain a line in σ_2', different from ℓ, and a line in σ_4', different from ℓ.

We examine the different possibilities for the intersection $\rho \cap \mathcal{P}$. If $\rho \subset \mathcal{P}$, then it is clear that $\mathcal{S}$ is an EKR(2) set of type VIII (if $\mathcal{P}$ is a projective space) or of type VIIIa (if $\mathcal{P}$ is a polar space). If $\rho \not\subset \mathcal{P}$, then $\rho \cap \mathcal{P}$ must be a cone with vertex the line ℓ and base a non-singular polar space or a cone with vertex a 3-space by Remark 3.2.10. In the former case, we can choose $\widehat{\sigma} = \langle Q_1, Q_3, P_3, P_4 \rangle$ to be the 3-space containing the base. We define the map ϕ from the planes of type B of $\mathcal{S}$ to the lines in the base as follows: the plane V is mapped onto the line $\widehat{\sigma} \cap \langle V, \ell \rangle$. Any two lines of type B in $\mathcal{S}$, not through the same point of ℓ, are mapped onto two lines meeting in a point. Note that the planes of type BcB in σ_2' are mapped onto the line $\langle Q_1, P_3 \rangle$ and that the planes of type BcC in σ_4' are mapped onto the line $\langle Q_3, P_4 \rangle$. The planes through P_1 meeting both σ_2' and σ_4' in a line different from ℓ, are mapped onto the $q+1$ lines meeting both $\langle Q_1, P_3 \rangle$ and $\langle Q_3, P_4 \rangle$, among which $\langle Q_1, P_4 \rangle$ and $\langle Q_3, P_3 \rangle$. If we find a line m, different from $\langle Q_1, P_3 \rangle$ and $\langle Q_3, P_4 \rangle$, meeting all these $q+1$ lines, then any plane through P_2 and in $\langle \ell, m \rangle$ extends $\mathcal{S}$; this is a plane of type BcD. Therefore, $\widehat{\sigma} \cap \mathcal{P}$ cannot be a hyperbolic quadric or a symplectic polar space, q even. If it is a Hermitian variety or if q is odd and the intersection is a symplectic polar space, then such a line cannot be found and $\mathcal{S}$ is of type VIIIb. Hereby, we used Lemma 1.7.10.

Now, we look at the latter case. If $\rho \cap \mathcal{P}$ is a cone with vertex a 3-space, then this vertex equals σ' or $\overline{\sigma}$ by Remark 3.2.10. Here, the vertex cannot be σ' since $\sigma' \cap \pi_6'$ is a point, thus the vertex has to be $\overline{\sigma}$. Moreover, $Q_2 \in \overline{\ell}_2 \setminus \{P_2\}$ or $Q_4 \in \overline{\ell}_4 \setminus \{P_2\}$ for otherwise π_6' is not in a 4-space through $\overline{\sigma}$. We also find that planes of type BbA and BcA do not exist, that all planes of type BbB are contained in σ_1', that all planes of type BbC are contained in σ_3', and that all planes of type BbD pass through a point of $\overline{\ell}_2 \setminus \{P_2\}$ or through a point of $\overline{\ell}_4 \setminus \{P_2\}$. Consequently, all planes in $\mathcal{S}$ contain a line in $\overline{\sigma}$. Arguing again as in the first part of the proof of Lemma 3.2.11, we find that $\mathcal{S}$ cannot be a maximal EKR(2) set under these assumptions. $\square$

Remark 3.2.14. We consider a maximal EKR(2) set $\mathcal{S}$ as introduced in Remark 3.2.10. We observe that P_1, π'_1 and π'_3, and P_2, π'_2 and π'_4 can be interchanged. Then the types BbX and BcX take each others place and the types BeA and BeB are kept. Hence, the case that $\mathcal{S}$ contains a plane of type BcD, but no planes of type BeB nor BbD, is analogous to the case that $\mathcal{S}$ contains a plane of type BbD, but no planes of type BeB nor BcD. Also in this case, we find an EKR(2) set of type VIII, VIIIa or VIIIb.

Lemma 3.2.15. *Let $\mathcal{S}$ be a maximal EKR(2) set fulfilling the assumptions made in Remark 3.2.10. Using the notations from that remark, we assume $\mathcal{S}$ contains no planes of type BeB. If $\mathcal{S}$ contains a plane of type BbD and a plane of type BcD, then $\mathcal{S}$ is an EKR(2) set of type IX, of type IXa or of type IXb.*

Proof. Let $\pi'_5 = \langle P_1, R_2, R_4\rangle$ be a plane of type BbD and $\pi'_6 = \langle P_2, R_1, R_3\rangle$ be a plane of type BcD, both contained in $\mathcal{S}$, with $R_i \in \pi'_i \setminus \ell_i$, $i = 1, \dots, 4$. Any plane of type BbA, BcA or BeA cannot intersect both π'_5 and π'_6; a plane of type BbB intersects both π'_5 and π'_6 if and only if it is contained in σ'_1; a plane of type BbC intersects both π'_5 and π'_6 if and only if it is contained in σ'_3; a plane of type BcB intersects both π'_5 and π'_6 if and only if it is contained in σ'_2; a plane of type BcC intersects both π'_5 and π'_6 if and only if it is contained in σ'_4.

We prove that we may assume that $R_2 \notin \overline{\ell}_2$ and $R_4 \notin \overline{\ell}_4$. We distinguish between two cases: $\overline{\sigma} \subset \mathcal{P}$ and $\overline{\sigma} \not\subset \mathcal{P}$. We look first at the former case. Note that the planes of type BbB in σ'_1, the planes of type BbC in σ'_3, the planes of type BcB in σ'_2 and the planes of type BcC in σ'_4 all contain a line in $\overline{\sigma}$. If all planes of type BbD and BcD in $\mathcal{S}$ also intersect $\overline{\sigma}$ in a line, then all planes in $\overline{\sigma}$ should be contained in $\mathcal{S}$ by the maximality condition. Among the planes in $\overline{\sigma}$ however, there are planes of type BeB. This contradicts the assumption. Hence, we may assume without loss of generality that π'_5 does not contain a line in $\overline{\sigma}$, so $R_2 \notin \overline{\ell}_2$ and $R_4 \notin \overline{\ell}_4$.

Now, we look at the latter case, $\overline{\sigma} \not\subset \mathcal{P}$. Then $\overline{\sigma} \cap \mathcal{P}$ is a union of planes through the line ℓ. If all planes of type BbD and BcD in $\mathcal{S}$ intersect $\overline{\sigma}$ in a line, so does π'_5. The intersection line $\pi'_5 \cap \overline{\sigma}$ cannot be in a plane through ℓ different from $\langle P_3, \ell\rangle$ or $\langle P_4, \ell\rangle$. It follows that either $R_2 \in \overline{\ell}_2$ or else $R_4 \in \overline{\ell}_4$.

Without loss of generality, we assume $R_2 \in \overline{\ell}_2$. Note that it follows from $\overline{\sigma} \not\subset \mathcal{P}$ that $\tau_3 \cap \mathcal{P}$ is a union of 3-spaces through $\overline{\pi}'$. Since π'_5 is contained in τ_3, it must be contained in a 3-space through $\overline{\pi}'$, namely $\langle P_3, \overline{\pi}' \rangle = \sigma'_3$. However, then $R_4 \in \sigma'_3 \cap \pi'_4 = \ell_4$, a contradiction. Hence, we may assume there is a plane in $\mathcal{S}$ of type BbD or a plane in $\mathcal{S}$ of type BcD that does not intersect $\overline{\sigma}$ in a line. So, without loss of generality, we may assume that π'_5 does not contain a line in $\overline{\sigma}$, so $R_2 \notin \overline{\ell}_2$ and $R_4 \notin \overline{\ell}_4$.

We denote the 3-space $\langle Q_1, Q_3, P_3, P_4 \rangle$ by $\widehat{\sigma}$. This 3-space is disjoint from ℓ. We denote $\sigma'_1 \cap \widehat{\sigma} = \langle Q_1, P_4 \rangle = m_1$, $\sigma'_2 \cap \widehat{\sigma} = \langle Q_1, P_3 \rangle = m_2$, $\sigma'_3 \cap \widehat{\sigma} = \langle Q_3, P_3 \rangle = m_3$ and $\sigma'_4 \cap \widehat{\sigma} = \langle Q_3, P_4 \rangle = m_4$. Let V be a plane of type B in $\mathcal{S}$, then we define the map ϕ by $\phi(V) = \langle \ell, V \rangle \cap \widehat{\sigma}$. The image of these planes is a line in $\widehat{\sigma}$ since all planes of type B in $\mathcal{S}$ are contained in ρ and since all planes of type B meet ℓ. If two planes of type B in $\mathcal{S}$ pass through a different point of ℓ, then their images must be two intersecting lines. Vice versa, if the images of two planes of type B through different points of ℓ are two intersecting lines, then they intersect and hence, both can be contained in $\mathcal{S}$.

Observe that the planes of type BbB in σ'_1, among which π'_1, are all mapped to the line m_1, that the planes of type BbC in σ'_3, among which π'_3, are all mapped to the line m_3, that the planes of type BcB in σ'_2, among which π'_2, are all mapped to the line m_2, that the planes of type BcC in σ'_4, among which π'_4, are all mapped to the line m_4, that the planes of type BbD, among which π'_5, are all mapped to a line intersecting m_2 and m_4, and that the planes of type BcD, among which π'_6, are all mapped to a line intersecting m_1 and m_3. Now, we consider the line $m_5 = \phi(\pi'_5)$. This line intersects m_2, but contains neither Q_1 nor P_3 since $R_2 \in \pi'_2 \setminus (\ell_2 \cup \overline{\ell}_2)$. Also, this line intersects m_4, but contains neither Q_3 nor P_4 since $R_4 \in \pi'_4 \setminus (\ell_4 \cup \overline{\ell}_4)$. Consequently the lines m_5 and m_1 are disjoint, and the lines m_5 and m_3 are disjoint. We already knew that the lines m_1 and m_3 are disjoint. Hence, m_1, m_3 and m_5 are three pairwise disjoint lines in $\widehat{\sigma}$.

Before going on, we look at $\rho \cap \mathcal{P}$. By Remark 3.2.10 we know that ρ can be a subspace of $\mathcal{P}$ ($\rho \subset \mathcal{P}$). If $\rho \not\subset \mathcal{P}$, then $\rho \cap \mathcal{P}$ is a cone, either a set of 3-spaces with vertex ℓ or else a set of 4-spaces with vertex $\overline{\sigma}$ or σ'. From the existence of the plane π'_5 it follows that $\rho \cap \mathcal{P}$ cannot be a cone with vertex a 3-space. So, either $\rho \subset \mathcal{P}$ or else $\rho \cap \mathcal{P}$ is a cone with vertex ℓ and base a non-singular classical polar space $X(3, q)$ of rank 2 (a generalised quadrangle). This base can be chosen to be contained in the 3-space $\widehat{\sigma}$, disjoint to ℓ. Then $\widehat{\sigma} \cap \mathcal{P}$ is this non-singular polar space $X(3, q)$. In this second case $\phi(\pi'_6) = m_6$ is a line meeting m_1, m_3 and m_5. Moreover, m_6 is skew to m_2 and m_4 since it cannot contain Q_1 or Q_3 and $X(3,q)$ does not contain planes.

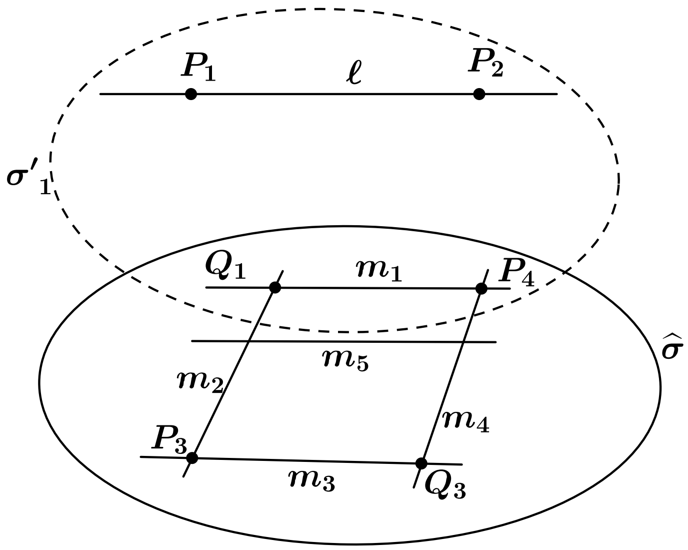

Figure 3.7: The configuration considered in Lemma 3.2.15. Only one of the 3-spaces σ'_i is drawn, but the others can be constructed in an analogous way.

We found before that all planes of type B in $\mathcal{S}$ pass through P_1 or P_2: we assumed that there are no planes of type BeB in $\mathcal{S}$ and we excluded the presence of planes of type BeA in $\mathcal{S}$ in the beginning of the proof. By the previous arguments, for all planes $V \in \mathcal{S}$ through P_1, $\phi(V)$ should be a line meeting m_2, m_4 and m_6, and for all planes $V \in \mathcal{S}$ through P_2, $\phi(V)$ should be a line meeting m_1, m_3 and m_5. Recall that a plane through P_1 and a plane through P_2 intersect each other if their images under ϕ intersect each other. Let $\mathcal{L}$ be the set of lines meeting m_1, m_3 and m_5. Then $m_2, m_4, m_6 \in \mathcal{L}$. Let $\mathcal{L}'$ be the set of lines meeting all lines of $\mathcal{L}$. By the maximality condition, all planes through P_1 in a 3-space through ℓ and a line of $\mathcal{L}$, and all planes through P_2 in a 3-space through ℓ and a line of $\mathcal{L}'$ must be contained in $\mathcal{S}$. Now we distinguish between the different possibilities using Lemma 1.7.10.

If $\mathcal{P}$ is a projective space, then $\mathcal{L}$ is a regulus and $\mathcal{L}'$ is its opposite regulus. We find an EKR(2) set of type IX. If $\mathcal{P}$ is a polar space and $\rho \subset \mathcal{P}$, then $\mathcal{L}$ is also a regulus and $\mathcal{L}'$ its opposite regulus. We conclude that $\mathcal{S}$ is an EKR(2) set of type IXa. If $\mathcal{P}$ is a quadric polar space and $\widehat{\sigma} \cap \mathcal{P}$ is a hyperbolic quadric, then $\mathcal{L}$ and $\mathcal{L}'$ are the two reguli of this hyperbolic quadric. We conclude that $\mathcal{S}$ is an EKR(2) set of type IXa. If $\mathcal{P}$ is a symplectic polar space with ambient space $\mathrm{PG}(n,q)$, q even, and $\widehat{\sigma} \cap \mathcal{P}$ is a symplectic polar space $\mathcal{W}(3,q)$, then $\mathcal{L}$ is a regulus and $\mathcal{L}'$ its opposite regulus. Also in this case $\mathcal{S}$ is an EKR(2) set of type IXa. Analogously, we find that $\mathcal{S}$ is an EKR(2) set of type IXb if $\mathcal{P}$ is a Hermitian polar space and $\widehat{\sigma} \cap \mathcal{P}$ is a Hermitian variety $\mathcal{H}(3,q)$. Finally, the situation wherein $\mathcal{P}$ is a symplectic polar space with ambient space $\mathrm{PG}(n,q)$, q odd, and $\widehat{\sigma} \cap \mathcal{P}$ is a symplectic polar space $\mathcal{W}(3,q)$, cannot occur. $\square$

Remark 3.2.16. In Remark 3.2.2, we already introduced ℓ, σ_1 and the planes of type A, B and C. Now we assume $\mathcal{S}$ contains a plane π_1^C of type C. This plane and ℓ are disjoint, so $\tau_1 = \langle \ell, \pi_1^C \rangle$ is a 4-space. Also, $\pi_1^C \cap \sigma_1$ is a line ℓ_1. All planes of type A must be contained in τ_1 since they pass through ℓ and contain a point of π_1^C. In the same way, all planes of type B must intersect τ_1 in at least a line. The planes of type C either are contained in τ_1 or else intersect τ_1 in a line of σ_1, intersecting ℓ_1 or equal to ℓ_1. Note that it follows from $\pi_1^C \subset \mathcal{P}$ that $\sigma_1 \subset \mathcal{P}$.

Lemma 3.2.17. *Let $\mathcal{S}$ be a maximal EKR(2) set fulfilling the assumptions made in Remark 3.2.16. Using the notations from that remark, we assume $\mathcal{S}$ contains a plane of type C not in τ_1. If all planes of type C in $\mathcal{S}$ that are not contained in τ_1 pass through ℓ_1, then all planes of $\mathcal{S}$ are contained in a 5-space or all planes of $\mathcal{S}$ intersect a fixed line in at least a point.*

Proof. Let π_2^C be a plane of type C in $\mathcal{S}$ that is not contained in τ_1. By the assumption, $\ell_1 \subset \pi_2^C$. We denote $\tau_2 = \langle \ell, \pi_2^C \rangle$. Arguing as in Remark 3.2.16, we find that all planes of type A in $\mathcal{S}$ must be contained in τ_2. Since they also need to be contained in τ_1, the planes of type A in $\mathcal{S}$ must be contained in σ_1. In Remark 3.2.2, we already found that all planes of type A in σ_1 are contained in $\mathcal{S}$. Now, we also know there are no other planes of type A in $\mathcal{S}$.

Arguing again as in Remark 3.2.16, we know that the planes of type B in $\mathcal{S}$ must contain a line in τ_2, as well as in τ_1, and that the planes of type C in $\mathcal{S}$ either are contained in τ_2 or else intersect τ_2 in a line of σ_1, intersecting ℓ_1. Consequently, all planes of type C in $\mathcal{S}$ contain at least a point on ℓ_1 since they cannot be contained in $\tau_1 \cap \tau_2 = \sigma_1$.

We observe that all planes of type A or C in $\mathcal{S}$ contain a point of $\ell_1 \subset \sigma_1$. If all planes of type B in $\mathcal{S}$ also contain a point of ℓ_1, then the lemma is valid.

So, we assume that $\mathcal{S}$ contains a plane of type B, not through a point of ℓ_1. Let π_1^B be such a plane. This plane must intersect π_1^C and π_2^C, hence contains a point $Q_1 \in \pi_1^C \setminus \ell_1$ and a point $Q_2 \in \pi_2^C \setminus \ell_1$. We denote the intersection point $\pi_1^B \cap \ell$ by R_1. Note that $\pi_1^B = \langle R_1, Q_1, Q_2 \rangle$, that $\pi_1^B \cap \tau_1 = \langle R_1, Q_1 \rangle$, that $\pi_1^B \cap \tau_2 = \langle R_1, Q_2 \rangle$, and that $\pi_1^B \cap \sigma_1 = \{R_1\}$. Denote $\rho = \langle \tau_1, \tau_2 \rangle$.

By Remark 3.2.2, any plane of type C in $\mathcal{S}$ contains a line in σ_1. However, these lines cannot contain the point $R_1 \in \ell$. Hence, all planes of type C in $\mathcal{S}$ are generated by a line in σ_1, disjoint to ℓ, and a point of $\pi_1^B \setminus \{R_1\}$, and thus contained in ρ.

Let π be a plane of type B in $\mathcal{S}$, intersecting ℓ in the point R. We already know that π contains a line m_1 through R and a point of π_1^C, and a line m_2 through R and a point of π_2^C. If m_1 and m_2 are different, then they span the plane π and π is contained in ρ. If $m_1 = m_2$, this is a line in σ_1 intersecting ℓ_1. The plane π also contains a point or a line on π_1^B. If π does not pass through R_1, then the intersection point $\pi \cap \pi_1^B$ and the line m_1 generate π, and π is necessarily contained in ρ. If π contains R_1, then π and π_1^B obviously intersect.

From the two preceding paragraphs, it follows that the only planes which can be in $\mathcal{S}$ and which are not necessarily contained in ρ are the planes through R_1 and a point of ℓ_1.

First, we assume π_2^B is a plane of type B in $\mathcal{S}$ that is not contained in ρ. Let m_2 be the line $\pi_2^B \cap \sigma_1$; necessarily $R_1 \in m_2$. Any plane in $\mathcal{S}$ that is contained in ρ cannot be disjoint to the line m_2 since any plane in $\mathcal{S}$ needs to intersect π_2^B. We have proved before that any plane in $\mathcal{S}$ that is not contained in ρ, passes through the point R_1. Consequently, any plane in $\mathcal{S}$ contains at least a point on m_2, so the lemma is valid in this case.

Now, we assume that all planes of type B in $\mathcal{S}$ are contained in ρ. We already know that all planes of type A or type C in $\mathcal{S}$ are contained in ρ. In this case, $\mathcal{S}$ is an EKR(2) set contained in the 5-space ρ. $\square$

Remark 3.2.18. In Remark 3.2.2 and Remark 3.2.16, we already introduced ℓ, σ_1, π_1^C, τ_1, ℓ_1 and the planes of type A, type B and type C. Now we assume that $\mathcal{S}$ contains a plane of type C, not in τ_1 and not through ℓ_1. Let π_2^C be such a plane and denote the line $\sigma_1 \cap \pi_2^C$ by ℓ_2, by assumption different from ℓ_1. By the arguments in Remark 3.2.16, we know that ℓ_1 and ℓ_2 intersect. Let $\overline{P}$ be their intersection point. We also denote $\tau_2 = \langle \ell, \pi_2^C \rangle$ and $\langle \tau_1, \tau_2 \rangle = \rho$.

We already know that all planes of type A in σ_1 are contained in $\mathcal{S}$ and, as in the beginning of the proof of Lemma 3.2.17, we can show that these are the only planes of type A in $\mathcal{S}$. The planes of type C in $\mathcal{S}$ intersect σ_1 in a line and this line intersects ℓ_1 if the plane is not contained in τ_1, respectively ℓ_2 if the plane is not contained in τ_2. The plane $\pi_{1,2} = \langle \ell_1, \ell_2 \rangle \subset \sigma_1$ does not contain ℓ, hence intersects ℓ in a point R.

Note that $\mathcal{S}$ contains in this case surely planes of type B, since each plane of type B in σ_1 intersects all planes of type A and all planes of type C; thus $\mathcal{S}$ cannot be maximal if it contains no planes of type B.

Lemma 3.2.19. *Let $\mathcal{S}$ be a maximal EKR(2) set fulfilling the assumptions made in Remark 3.2.18. We use the notations from that remark. If $\mathcal{S}$ contains a plane of type B, not through R, intersecting neither ℓ_1 nor ℓ_2, then all planes of $\mathcal{S}$ are contained in a 5-space or all planes of $\mathcal{S}$ intersect a fixed line in at least a point.*

Proof. Let π_3^B be a plane of type B in $\mathcal{S}$ that intersects neither ℓ_1 nor ℓ_2, and that does not pass through R. Then π_3^B contains a point R_1 on $\ell \setminus \{R\}$, a point $Q_1 \in \pi_1^C \setminus \ell_1$ and a point $Q_2 \in \pi_2^C \setminus \ell_2$. Moreover, π_3^B equals $\langle R_1, Q_1, Q_2 \rangle$ and $\pi_3^B \subset \rho$. Note that $\sigma_1 \cap \pi_3^B = \{R_1\}$.

Any plane of type C in $\mathcal{S}$ contains a line in σ_1. This line cannot contain a point of π_3^B since $\sigma_1 \cap \pi_3^B = \{R_1\}$. Hence, any plane of type C in $\mathcal{S}$ is contained in ρ.

Let π be a plane of type B in $\mathcal{S}$, intersecting ℓ in the point R'. We already know that π contains a line m_1 through R' and a point of π_1^C, and a line m_2 through R' and a point of π_2^C. If m_1 and m_2 are different, then they span the plane π and π is contained in ρ. If $m_1 = m_2$, this is a line in σ_1 intersecting ℓ_1 and ℓ_2 and thus the line $\langle R', \overline{P} \rangle$ if $R' \neq R$, or a line through R in $\pi_{1,2}$ if $R' = R$. The plane π also contains a point or a line on π_3^B, but π_3^B does not contain a point of $\langle R', \overline{P} \rangle$, unless $R' = R_1$. The plane π_3^B does not contain a point of $\pi_{1,2}$. So, if $R' \neq R_1$, the plane π is necessarily contained in ρ. The planes of type B in $\mathcal{S}$ that are not contained in ρ, must contain the line $\langle R_1, \overline{P} \rangle$.

If $\mathcal{S}$ contains no planes of type B not in ρ, the lemma is valid. So, we assume that $\mathcal{S}$ contains a plane of type B through the line $\langle R_1, \overline{P} \rangle$. All planes of $\mathcal{S}$ in ρ must intersect this plane in a point of $\langle R_1, \overline{P} \rangle$, hence all these planes contain a point of $\langle R_1, \overline{P} \rangle$. All planes of $\mathcal{S}$ not in ρ contain the line $\langle R_1, \overline{P} \rangle$. We conclude that all planes in $\mathcal{S}$ contain at least a point of $\langle R_1, \overline{P} \rangle$ and thus that also in this case, the lemma is valid. □

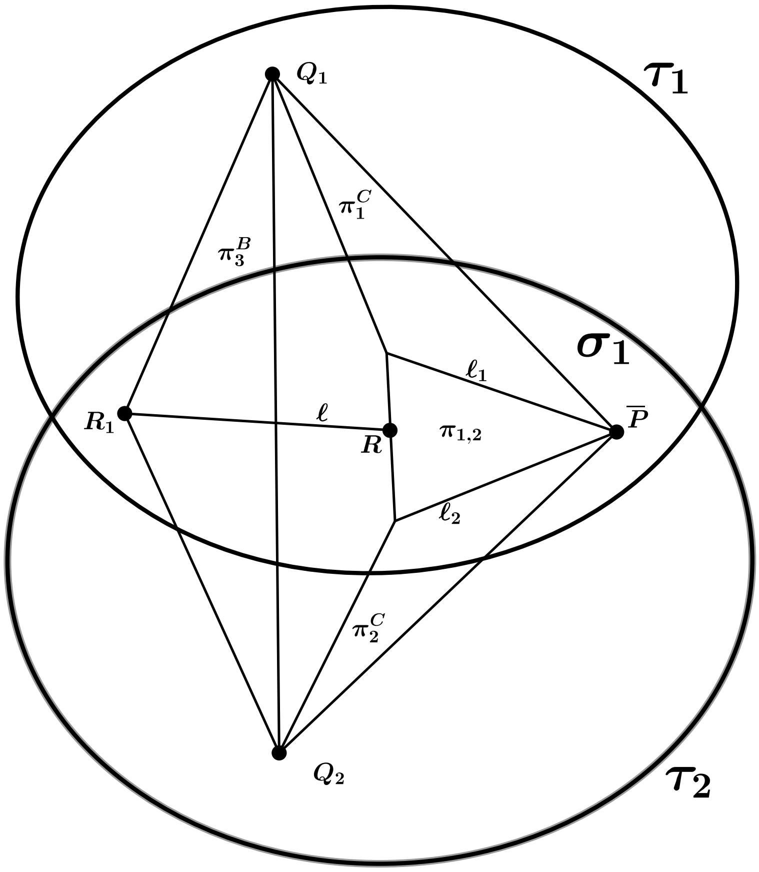

Figure 3.8: The configuration of the subspaces introduced in Lemma 3.2.19.

Lemma 3.2.20. *Let $\mathcal{S}$ be a maximal EKR(2) set fulfilling the assumptions made in Remark 3.2.18. Using the notations from that remark, we assume that all planes of type B in $\mathcal{S}$ that intersect neither ℓ_1 nor ℓ_2, pass through R. If $\mathcal{S}$ contains a plane of type B intersecting neither ℓ_1 nor ℓ_2, then all planes of $\mathcal{S}$ are contained in a 5-space or all planes of $\mathcal{S}$ intersect a fixed line in at least a point.*

Proof. Let π_3^B be a plane of type B in $\mathcal{S}$ that intersects neither ℓ_1 nor ℓ_2. Then π_3^B contains the point R on ℓ, a point $Q_1 \in \pi_1^C \setminus \ell_1$ and a point $Q_2 \in \pi_2^C \setminus \ell_2$. Moreover, π_3^B equals $\langle R, Q_1, Q_2 \rangle$ and $\pi_3^B \subset \rho$. Note that $\sigma_1 \cap \pi_3^B = \{R\}$.

Any plane of type C in $\mathcal{S}$ contains a line in σ_1. This line cannot contain a point of π_3^B since $\sigma_1 \cap \pi_3^B = \{R\}$. Hence, any plane of type C in $\mathcal{S}$ is contained in ρ.

Let π be a plane of type B in $\mathcal{S}$, intersecting ℓ in the point R'. We already know that π contains a line m_1 through R' and a point of π_1^C, and a line m_2 through R' and a point of π_2^C. If m_1 and m_2 are different, then they span the plane π and π is contained in ρ. If $m_1 = m_2$, this is a line in σ_1 intersecting ℓ_1 and ℓ_2 and thus the line $\langle R', \overline{P}\rangle$ if $R' \neq R$ or a line through R in $\pi_{1,2}$ if $R' = R$. The plane π also contains a point or a line on π_3^B, but π_3^B does not contain a point of $\langle R', \overline{P}\rangle$, unless $R' = R$. So, if $R' \neq R$, the plane π is necessarily contained in ρ. The only planes of type B in $\mathcal{S}$ that are not necessarily contained in ρ are the ones containing a line through R in $\pi_{1,2}$.

If $\mathcal{S}$ contains no planes of type B not in ρ, the lemma is valid. So, we assume that $\mathcal{S}$ contains a plane of type B through a line m in $\pi_{1,2}$ containing R. All planes of $\mathcal{S}$ in ρ must intersect this plane in a point of m, hence all these planes contain a point of m. All planes of $\mathcal{S}$ not in ρ, pass through the point $R \in m$. We conclude that all planes in $\mathcal{S}$ contain at least a point of m and thus that also in this case, the lemma is valid. $\square$

Lemma 3.2.21. *Let $\mathcal{S}$ be a maximal EKR(2) set fulfilling the assumptions made in Remark 3.2.18. We use the notations from that remark. If every plane of type B in $\mathcal{S}$ intersects ℓ_1 or ℓ_2, then all planes of $\mathcal{S}$ are contained in a 5-space, all planes of $\mathcal{S}$ intersect a fixed line in at least a point, or $\mathcal{S}$ is an EKR(2) set of type X.*

Proof. We already know that all planes of type C intersect σ_1 in a line. In Remark 3.2.18 we noted that the planes of type A are in $\mathcal{S}$ if and only if they are contained in σ_1. By the assumption of this lemma, all planes of type B in $\mathcal{S}$ also intersect σ_1 in at least a line. Hence, all planes in σ_1 must be contained in $\mathcal{S}$, by the maximality condition. We already knew this for the planes in σ_1 through ℓ; all other planes in σ_1 are necessarily of type B.

In Remark 3.2.18 we already noted that all planes of type C in $\mathcal{S}$ that are not contained in τ_1, intersect ℓ_1, and that all planes of type C in $\mathcal{S}$ that are not contained in τ_2, intersect ℓ_2. We also note that a plane of type B which intersects ℓ_1, but not ℓ_2, must contain a point of $\pi_2^C \setminus \ell_2$, and thus is contained in τ_2. Analogously, a plane of type B which intersects ℓ_2, but not ℓ_1, must contain a point of $\pi_1^C \setminus \ell_1$, and thus is contained in τ_1. Consequently, all planes that are not contained in $\tau_1 \cup \tau_2$, intersect both ℓ_1 and ℓ_2.

If all planes in $\mathcal{S}$ intersect ℓ_1 or all planes in $\mathcal{S}$ intersect ℓ_2, the lemma is obviously valid. So, we can assume $\mathcal{S}$ contains a plane π_3 not intersecting ℓ_1 and a plane π_4 not intersecting ℓ_2. By the previous observations, we know that π_3 is contained in τ_1 and that π_4 is contained in τ_2. We denote $\pi_3 \cap \sigma_1 = \ell_3$ and $\pi_4 \cap \sigma_1 = \ell_4$. The intersection points are $\ell_2 \cap \ell_3 = \{P_3\}$ and $\ell_1 \cap \ell_4 = \{P_4\}$. The lines ℓ_3 and ℓ_4 also intersect; we denote $\ell_3 \cap \ell_4 = \{\overline{Q}\}$. From $\pi_3 \subset \mathcal{P}$ and $\pi_4 \subset \mathcal{P}$, it follows that $\tau_1, \tau_2 \subset \mathcal{P}$.

Any plane in $\mathcal{S}$ that is not contained in $\tau_1 \cup \tau_2$ must intersect ℓ_1, ℓ_2, ℓ_3 and ℓ_4. Since ℓ_1 and ℓ_3 are disjoint and ℓ_2 and ℓ_4 are disjoint, these planes either pass through $\langle P_3, P_4 \rangle$ or else through $\langle \overline{P}, \overline{Q} \rangle$.

If $\mathcal{S}$ contains no planes outside of $\tau_1 \cup \tau_2$, then all planes of $\mathcal{S}$ are contained in the 5-space ρ and the lemma is valid. If all planes in $\mathcal{S}$ that are not contained in $\tau_1 \cup \tau_2$, pass through $\langle P_3, P_4 \rangle$, and $\mathcal{S}$ contains such planes, then all planes in $\mathcal{S}$ contain at least a point of $\langle P_3, P_4 \rangle$. Also in this case the lemma is valid. Analogously, the lemma is also valid if all planes in $\mathcal{S}$ that are not contained in $\tau_1 \cup \tau_2$, pass through $\langle \overline{P}, \overline{Q} \rangle$, and $\mathcal{S}$ contains such planes.

So, we can assume that $\mathcal{S}$ contains a plane π_5 through $\langle \overline{P}, \overline{Q} \rangle$ that is not contained in $\tau_1 \cup \tau_2$, and a plane π_6 through $\langle P_3, P_4 \rangle$ that is not contained in $\tau_1 \cup \tau_2$. These two planes cannot intersect each other in a point of σ_1 since $\langle P_3, P_4 \rangle$ and $\langle \overline{P}, \overline{Q} \rangle$ are disjoint. We denote the 4-space $\langle \pi_5, \pi_6 \rangle$ by τ_3; this 4-space contains σ_1 and is contained in $\mathcal{P}$. Any plane in $\mathcal{S}$ that is not contained in $\tau_1 \cup \tau_2$ and passes through $\langle P_3, P_4 \rangle$ must be contained in τ_3, in order to intersect π_6; any plane in $\mathcal{S}$ that is not contained in $\tau_1 \cup \tau_2$ and passes through $\langle \overline{P}, \overline{Q} \rangle$ must be contained in τ_3, in order to intersect π_5. Any plane in $\mathcal{S}$ that is contained in τ_1, must intersect ℓ_2, ℓ_4, $\langle P_3, P_4 \rangle$ and $\langle \overline{P}, \overline{Q} \rangle$, hence passes through ℓ_1 or ℓ_3. Analogously, any plane in $\mathcal{S}$ that is contained in τ_2, must pass through ℓ_2 or ℓ_4.

All the planes that are necessarily contained in $\mathcal{S}$ (the planes in σ_1) or that can be contained in $\mathcal{S}$ (the planes in τ_1 through ℓ_1 or ℓ_3, the planes in τ_2 through ℓ_2 or ℓ_4, the planes in τ_3 through $\langle P_3, P_4 \rangle$ or $\langle \overline{P}, \overline{Q} \rangle$), intersect each other, hence are contained in $\mathcal{S}$ by the maximality condition. We find an EKR(2) set of type X with base points P_3, P_4, $\overline{P}$ and $\overline{Q}$. $\square$

Lemma 3.2.22. *Let $\mathcal{S}$ be a maximal EKR(2) set fulfilling the assumptions made in Remark 3.2.16. We use the notations from that remark. If all planes of type C in $\mathcal{S}$ are contained in τ_1, then all planes of $\mathcal{S}$ are contained in a 5-space, all planes of $\mathcal{S}$ intersect a fixed line in at least a point, or $\mathcal{S}$ is an*

EKR(2) set of type X.

Proof. All planes of type A in τ_1 intersect all planes in $\mathcal{S}$ since all planes of type A or B in $\mathcal{S}$ contain at least a point of ℓ and all planes of type C in $\mathcal{S}$ are contained in τ_1. A plane of type A not in τ_1 cannot intersect π_1^C or any other plane of type C in $\mathcal{S}$. Hence, a plane of type A belongs to $\mathcal{S}$ if and only if it is contained in $\tau_1 \cap \mathcal{P}$. Recall that τ_1 is not necessarily contained in $\mathcal{P}$.

The planes of type B in $\mathcal{S}$ either are contained in τ_1 or else intersect τ_1 in a line through a point of ℓ and a point of π_1^C. If all planes of type B in $\mathcal{S}$ are contained in τ_1, all planes of $\mathcal{S}$ are contained in a 4-space, hence also in a 5-space. So, we can assume $\mathcal{S}$ contains planes of type B that intersect τ_1 in a line.

First, we assume that all planes of type B in $\mathcal{S}$ that are not contained in τ_1, pass through the same point of ℓ and that $\mathcal{S}$ contains such a plane of type B not in τ_1. Let $\overline{\pi}$ be such a plane and denote $\overline{\ell} = \overline{\pi} \cap \tau_1$ and $\ell \cap \overline{\ell} = \{\overline{R}\}$. Every plane of $\mathcal{S}$ in τ_1 then intersects $\overline{\ell}$ and every plane of $\mathcal{S}$ not in τ_1 passes through $\overline{R}$. Hence, all planes of $\mathcal{S}$ contain at least a point on $\overline{\ell}$. Recall that we assumed that all planes of type C in $\mathcal{S}$ are contained in τ_1.

Secondly, we assume that all planes of type B in $\mathcal{S}$ that are not contained in τ_1, pass through the same point of π_1^C and that $\mathcal{S}$ contains such a plane of type B not in τ_1. In the same way, we find a line that is intersected by any plane in $\mathcal{S}$.

Finally, we assume that $\mathcal{S}$ contains two planes π_1^B and π_2^B that intersect both ℓ and π_1^C in different points. We denote $\pi_i^B \cap \tau_1 = \langle R_i, Q_i \rangle$, $i = 1, 2$, with $R_1, R_2 \in \ell$, $Q_1, Q_2 \in \pi_1^C$ and $R_1 \neq R_2$ and $Q_1 \neq Q_2$. It can easily be argued that the lines $\langle R_1, Q_1 \rangle$ and $\langle R_2, Q_2 \rangle$ are disjoint since ℓ and π_1^C are disjoint. Hence, the intersection point $\overline{Q}$ of π_1^B and π_2^B is not contained in τ_1. Let τ_2 be the 4-space $\langle \pi_1^B, \pi_2^B \rangle$ and let σ be the 3-space generated by the disjoint lines $\langle R_1, Q_1 \rangle$ and $\langle R_2, Q_2 \rangle$. Note that $\sigma = \tau_1 \cap \tau_2$, that ℓ is contained in σ, and that π_1^C intersects σ in a line. Denote $\pi_1^C \cap \sigma = \ell'$ and note that $\ell' = \langle Q_1, Q_2 \rangle$.

Any plane in $\mathcal{S}$ that is contained in τ_1, must intersect $\langle R_1, Q_1 \rangle = \tau_1 \cap \pi_1^B$ and $\langle R_2, Q_2 \rangle = \tau_1 \cap \pi_2^B$; any plane in $\mathcal{S}$ that is contained in τ_2, must intersect ℓ' in order to contain a point of π_1^C and must intersect ℓ in order to be a plane of type B. Any plane in $\mathcal{S}$ that is not contained in $\langle \tau_1, \tau_2 \rangle$, must contain a line in τ_1 and a line in τ_2, hence a line in σ. This line intersects $\langle R_1, Q_1 \rangle$, $\langle R_2, Q_2 \rangle$, ℓ and ℓ'. Hence, all planes in $\mathcal{S}$ contain at least a line in σ. Consequently, all planes in $\sigma \cap \mathcal{P}$ are necessarily contained in $\mathcal{S}$.

If all planes of $\mathcal{S}$ are contained in $\langle \tau_1, \tau_2 \rangle$, then $\mathcal{S}$ is an EKR(2) set contained in this 5-space $\langle \tau_1, \tau_2 \rangle$. Thus, we assume $\mathcal{S}$ contains a plane not in $\langle \tau_1, \tau_2 \rangle$. Such a plane intersects $\langle R_1, Q_1 \rangle$, $\langle R_2, Q_2 \rangle$, ℓ and ℓ'. So, this plane passes either through $\langle R_1, Q_2 \rangle$ or else through $\langle R_2, Q_1 \rangle$. As before, for example as in the proof of Lemma 3.2.21, we can argue that $\mathcal{S}$ must contain a plane through $\langle R_1, Q_2 \rangle$ as well as a plane through $\langle R_2, Q_1 \rangle$, for else we can find a line that is intersected by every plane in $\mathcal{S}$, in which case the lemma is valid. Let π_3^B be a plane in $\mathcal{S}$ that passes through $\langle R_1, Q_2 \rangle$ and is not contained in $\langle \tau_1, \tau_2 \rangle$; let π_4^B be a plane in $\mathcal{S}$ that passes through $\langle R_2, Q_1 \rangle$ and is not contained in $\langle \tau_1, \tau_2 \rangle$. The intersection point $\overline{Q}'$ of the planes π_3^B and π_4^B cannot be contained in $\langle \tau_1, \tau_2 \rangle$ since the lines $\langle R_1, Q_2 \rangle$ and $\langle R_2, Q_1 \rangle$ are disjoint. Let τ_3 be the 4-space $\langle \pi_3^B, \pi_4^B \rangle$. Note that this 4-space contains σ. Since π_3^B and π_4^B are contained in $\mathcal{S}$, they are planes in $\mathcal{P}$. From this observation, it follows that $\sigma \subset \mathcal{P}$ and consequently also $\tau_2, \tau_3 \subset \mathcal{P}$.

Any plane in $\mathcal{S}$ through $\langle R_1, Q_2 \rangle$ that is not contained in $\tau_1 \cup \tau_2$, must be contained in τ_3 in order to intersect π_4^B. Analogously, every plane in $\mathcal{S}$ through $\langle R_2, Q_1 \rangle$ that is not contained in $\tau_1 \cup \tau_2$, must be contained in τ_3. Any plane in $\mathcal{S}$ that is contained in τ_1, now must intersect the lines $\langle R_1, Q_1 \rangle$, $\langle R_2, Q_2 \rangle$, $\langle R_1, Q_2 \rangle$ and $\langle R_2, Q_1 \rangle$, hence must pass through the line $\langle R_1, R_2 \rangle = \ell$ or the line $\langle Q_1, Q_2 \rangle = \ell'$. Analogously, any plane in $\mathcal{S}$ that is contained in τ_2 must pass through the line $\langle R_1, Q_1 \rangle$ or the line $\langle R_2, Q_2 \rangle$. Recall that the planes in τ_2 need to intersect ℓ in order to be a plane of type B.

Note that the planes in τ_1 through ℓ or ℓ', the planes in τ_2 through $\langle R_1, Q_1 \rangle$ or $\langle R_2, Q_2 \rangle$, and the planes in τ_3 through $\langle R_1, Q_2 \rangle$ or $\langle R_2, Q_1 \rangle$, all intersect each other. Hence, by the maximality condition on $\mathcal{S}$, all these planes are contained in $\mathcal{S}$. If $\tau_1 \subset \mathcal{P}$, then $\mathcal{S}$ contains planes through ℓ in τ_1 which are not contained in σ. Consequently, $\mathcal{S}$ is maximal. We find an EKR(2) set of type X with base points R_1, R_2, Q_1 and Q_2. If $\tau_1 \not\subset \mathcal{P}$, then $\tau_1 \cap \mathcal{P}$ is a union of 3-spaces through a common plane containing ℓ'. Moreover, $\sigma = \sigma_1$ and $\ell' = \ell_1$. Any plane through ℓ in τ_1 is thus contained in σ. It follows that any plane through ℓ' in τ_2 intersects all planes in $\mathcal{S}$, but is not contained in $\mathcal{S}$. Hence, $\mathcal{S}$ is not maximal. $\square$

Remark 3.2.23. In the final case, we assume that all planes in $\mathcal{S}$ intersect each other in a point. Let π_1 and π_2 be two planes in $\mathcal{S}$, with $\pi_1 \cap \pi_2 = \{P_3\}$. If all planes of $\mathcal{S}$ pass through P_3, then all planes through P_3 are contained in

$\mathcal{S}$ by the maximality condition. However, among all planes through P_3 there are two planes intersecting in a line. So, not all planes in $\mathcal{S}$ pass through P_3. Let π_3 be a plane in $\mathcal{S}$ not containing P_3. We denote $\pi_1 \cap \pi_3 = \{P_2\}$ and $\pi_2 \cap \pi_3 = \{P_1\}$.

The plane $\langle P_1, P_2, P_3 \rangle$, which is surely contained in $\mathcal{P}$, intersects π_1 in a line, hence cannot be contained in $\mathcal{S}$ by the assumption. By the maximality condition, $\mathcal{S}$ must contain a plane disjoint from $\langle P_1, P_2, P_3 \rangle$. Let π_4 be a plane in $\mathcal{S}$ such that π_4 and $\langle P_1, P_2, P_3 \rangle$ are disjoint. The subspace generated by π_1, π_2 and π_3 can be as well a 4-space as a 5-space. However, $\langle \pi_1, \pi_2, \pi_4 \rangle$ is necessarily a 5-space. So, we can assume $\langle \pi_1, \pi_2, \pi_3 \rangle$ is a 5-space, since we can always find a plane in $\mathcal{S}$ not through P_3 that generates a 5-space together with π_1 and π_2. We denote the 5-space $\langle \pi_1, \pi_2, \pi_3 \rangle$ by ρ. Furthermore, we denote $\pi_1 \cap \pi_4$ by Q_1, $\pi_2 \cap \pi_4$ by Q_2 and $\pi_3 \cap \pi_4$ by Q_3.

Let π be a plane in $\mathcal{S}$. The intersection $\pi \cap \rho$ cannot be a point because there is no point of ρ lying on π_1, π_2, π_3 and π_4. Hence, either π is contained in ρ or else it intersects ρ in a line. We consider the latter case and we denote the line $\rho \cap \pi$ by ℓ. Since π contains a point on π_1 and a point on π_2, ℓ passes through P_3 or ℓ lies in the 4-space $\langle \pi_1, \pi_2 \rangle$. Note that $\langle \pi_1, \pi_2 \rangle \cap \pi_3 = \langle P_1, P_2 \rangle$ and $\langle \pi_1, \pi_2 \rangle \cap \pi_4 = \langle Q_1, Q_2 \rangle$. If ℓ passes through P_3, then ℓ is contained in the 3-space $\langle P_3, \pi_3 \rangle$, but $\langle P_3, \pi_3 \rangle \cap \pi_4 = \{Q_3\}$. Hence, ℓ must be the line $\langle P_3, Q_3 \rangle$. If ℓ does not pass through P_3, then ℓ must be contained in the 3-space σ generated by $\langle P_1, P_2 \rangle$ and $\langle Q_1, Q_2 \rangle$. Moreover, ℓ must intersect $\sigma \cap \pi_3 = \langle P_1, P_2 \rangle$, $\sigma \cap \pi_4 = \langle Q_1, Q_2 \rangle$, $\sigma \cap \pi_1 = \langle Q_1, P_2 \rangle$ and $\sigma \cap \pi_4 = \langle P_1, Q_2 \rangle$. One sees easily that ℓ equals $\langle P_1, Q_1 \rangle$ or $\langle P_2, Q_2 \rangle$.

We conclude that all planes in $\mathcal{S}$ that are not contained in ρ, pass through $\langle P_1, Q_1 \rangle$, $\langle P_2, Q_2 \rangle$ or $\langle P_3, Q_3 \rangle$.

Lemma 3.2.24. *Let $\mathcal{S}$ be a maximal EKR(2) set fulfilling the assumptions made in Remark 3.2.23. We use the notations from that remark. It is impossible that all planes of $\mathcal{S}$ that are not contained in ρ pass through the same line in ρ.*

Proof. Without loss of generality we can assume that $\mathcal{S}$ contains a plane π_1' not contained in ρ, through the line $\langle P_1, Q_1 \rangle$. All planes in $\mathcal{S}$ that are contained in ρ must intersect the line $\langle P_1, Q_1 \rangle$ in order to intersect π_1'. All planes in $\mathcal{S}$ that are not contained in ρ pass through the line $\langle P_1, Q_1 \rangle$ by assumption. By the maximality condition on $\mathcal{S}$, all planes through $\langle P_1, Q_1 \rangle$ must be contained in $\mathcal{S}$. This contradicts the assumption that no two planes in $\mathcal{S}$ intersect each other in a line, which we made in Remark 3.2.23. $\square$

Lemma 3.2.25. *Let $\mathcal{S}$ be a maximal EKR(2) set fulfilling the assumptions made in Remark 3.2.23. We use the notations from that remark. If $\mathcal{S}$ contains two planes that are not contained in ρ, through different lines of ρ, then $\mathcal{S}$ is an EKR(2) set of type XI.*

Proof. Without loss of generality, we can assume that $\mathcal{S}$ contains a plane π_2' through $\langle P_2, Q_2\rangle$ and a plane π_3' through $\langle P_3, Q_3\rangle$, both not contained in ρ. The planes π_2' and π_3' intersect each other in a point R outside of ρ since $\langle P_2, Q_2\rangle$ and $\langle P_3, Q_3\rangle$ are disjoint. Note that the 3-space $\sigma_1 = \langle P_2, P_3, Q_2, Q_3\rangle$ is contained in $\mathcal{P}$. The EKR(2) set $\mathcal{S}$ cannot contain a second plane through $\langle P_2, Q_2\rangle$ or $\langle P_3, Q_3\rangle$ by the assumption we made in Remark 3.2.23. Now, we look at the planes in $\mathcal{S}$.

If a plane in $\mathcal{S}$ is not contained in ρ, then it passes through $\langle P_1, Q_1\rangle$ by the observations in Remark 3.2.23. There can be at most one such plane. Since $\langle P_1, Q_1\rangle$ and $\langle P_2, Q_2\rangle$ are disjoint, this plane must be contained in $\langle P_1, Q_1, \pi_2'\rangle$. Analogously, it also must be contained in $\langle P_1, Q_1, \pi_3'\rangle$. Consequently, this plane equals $\langle P_1, Q_1, R\rangle = \pi_1'$, since $\langle P_1, Q_1, \pi_2'\rangle \cap \langle P_1, Q_1, \pi_3'\rangle$ is the plane $\langle P_1, Q_1, R\rangle$.

If a plane π' in $\mathcal{S}$ is contained in ρ, then it must intersect $\langle P_2, Q_2\rangle$ and $\langle P_3, Q_3\rangle$. We distinguish between two cases. If π' is contained in σ_1, then it obviously intersects both lines and moreover, it also intersects the lines $\langle P_2, P_3\rangle \subset \pi_1$, $\langle P_3, Q_2\rangle \subset \pi_2$, $\langle P_2, Q_3\rangle \subset \pi_3$ and $\langle Q_2, Q_3\rangle \subset \pi_4$. If π' is not contained in σ_1, it must intersect σ_1 in a line ℓ', since it intersects both $\langle P_2, Q_2\rangle$ and $\langle P_3, Q_3\rangle$. By the assumption that any two planes in $\mathcal{S}$ cannot intersect each other in a line, $\ell' \neq \langle P_2, Q_2\rangle$ and $\ell' \neq \langle P_3, Q_3\rangle$. So, ℓ' cannot intersect each of the lines $\langle P_2, P_3\rangle \subset \pi_1$, $\langle P_3, Q_2\rangle \subset \pi_2$, $\langle P_2, Q_3\rangle \subset \pi_3$ and $\langle Q_2, Q_3\rangle \subset \pi_4$. Without loss of generality, we can assume ℓ' and $\langle P_2, P_3\rangle$ are disjoint. Then, π' must be contained in $\langle \ell', \pi_1\rangle = \langle Q_2, Q_3, \pi_1\rangle = \tau$. Note that $\tau \cap \pi_2 = \langle P_3, Q_2\rangle$ and $\tau \cap \pi_3 = \langle P_2, Q_3\rangle$. Hence, the line ℓ' must intersect $\langle P_3, Q_2\rangle$ and $\langle P_2, Q_3\rangle$, but this line must also intersect $\langle P_2, Q_2\rangle$ and $\langle P_3, Q_3\rangle$. However, then ℓ' equals either $\langle P_2, P_3\rangle \subset \pi_1$ or $\langle Q_2, Q_3\rangle \subset \pi_4$, contradicting the assumption that no two planes in $\mathcal{S}$ intersect each other in a line.

From the previous paragraphs it follows that the only planes that can be contained in $\mathcal{S} \setminus \{\pi_1, \pi_2, \pi_3, \pi_4, \pi_2', \pi_3'\}$, are the planes in σ_1 and the plane π_1'. Note that π_1' and σ_1 are disjoint. If $\mathcal{S}$ contains one of the planes in σ_1, then π_1' is not contained in $\mathcal{S}$. Moreover, by the maximality condition on

$\mathcal{S}$, all planes in σ_1 should be contained in $\mathcal{S}$, but any two of these planes intersect in a line, contradicting the assumption. So, none of the planes in σ_1 is contained in $\mathcal{S}$. Hence, the plane π'_1 is contained in $\mathcal{S}$ and consequently $\mathcal{S} = \{\pi_1, \pi_2, \pi_3, \pi_4, \pi'_1, \pi'_2, \pi'_3\}$, which is indeed maximal. This is an EKR(2) set of type XI. Note that the 6-space $\langle R, \rho \rangle$ is contained in $\mathcal{P}$. □

3.3 The classification of the largest EKR(2) sets

Using the results from the previous section we give a classification of the largest EKR(2) sets, i.e. of all maximal EKR(2) sets whose size exceeds a certain value, in finite projective spaces and finite classical polar spaces. We recall the notation $n(a, \mathcal{P})$, denoting the number of planes in an EKR(2) set in $\mathcal{P}$ of type a. All these values can be found in Section 3.1.

Remark 3.3.1. Before starting the classification, we have a look at Theorem 3.0.1, which is about the EKR(2) sets in $\mathrm{PG}(5, q)$. In that theorem, six different types of EKR(2) sets in $\mathrm{PG}(5, q)$ are mentioned. One of them is the EKR(2) set of type XII. All other EKR(2) sets mentioned in Theorem 3.0.1 contain at least one line which intersects all planes of the EKR(2) set. Hence, if we look at a 5-space ρ in $\mathrm{PG}(n, q)$, $n \geq 6$, and we consider these sets of planes in ρ, we can add all other planes of $\mathrm{PG}(n, q) \setminus \rho$ through such a line. Consequently, these sets are not maximal EKR(2) sets. Analogously, if we look at a 5-space ρ in a polar space $\mathcal{P}$ of rank $d \geq 6$, and we consider these sets of planes in ρ, we can add all other planes of $\mathcal{P}$ through such a line. Note that the planes of $\mathcal{P}$ through one line span a cone with vertex the line and base a polar space of rank $d - 2$, hence a subspace of dimension at least $2d - 3 \geq 9$ of the ambient projective space of $\mathcal{P}$.

Note that Theorem 3.0.1 classifies maximal EKR(2) sets in $\mathrm{PG}(5, q)$ with size at least $3q^4 + 3q^3 + 2q^2 + q + 1$. So, all maximal EKR(2) sets in a projective space of dimension at least 6 or on a polar space of rank at least 6, that are contained in a 5-space, and have size at least $3q^4 + 3q^3 + 2q^2 + q + 1$, must be of type XII.

The lower bound $3q^4+3q^3+2q^2+q+1$ will appear in the classification theorems for both projective spaces and polar spaces of rank at least six. Improving on Theorem 3.0.1 by finding a classification result with a smaller lower bound, would improve both classification results.

3.3.1 The projective spaces

We present the classification of the largest EKR(2) sets for the projective spaces. The 5-dimensional case was treated in Theorem 3.0.1. Note that the EKR(2) sets in $\mathrm{PG}(5,q)$ of size $\begin{bmatrix}5\\2\end{bmatrix}_q$ are of type I or XII, that the EKR(2) sets in $\mathrm{PG}(5,q)$ of size $1+q(q^2+q+1)^2$ are of type II, III or V, and that the EKR(2) sets in $\mathrm{PG}(5,q)$ of size $3q^4+3q^3+2q^2+q+1$ are of type IV.

Lemma 3.3.2. *Let $\mathcal{P}$ be a projective space $\mathrm{PG}(n,q)$, with $n \geq 6$. Then the following inequalities hold:*

$$\begin{aligned} n(I,\mathcal{P}) &> n(II,\mathcal{P}) = n(III,\mathcal{P}) > n(IV,\mathcal{P}) \\ &> n(V,\mathcal{P}) > n(VIII,\mathcal{P}) \geq n(VI,\mathcal{P}) > n(IX,\mathcal{P}) \geq n(VII,\mathcal{P}). \end{aligned}$$

Hereby, the equalities $n(VIII,\mathcal{P}) = n(VI,\mathcal{P})$ and $n(IX,\mathcal{P}) = n(VII,\mathcal{P})$ hold iff $q = 2$.

Proof. Direct computation. □

Lemma 3.3.3. *Denote $\begin{bmatrix}5\\2\end{bmatrix}_q$ by $B(q)$. Assume $n \geq 6$.*

- *For $a \in \{I, II, III\}$, the inequality $n(a,\mathrm{PG}(n,q)) > B(q)$ is valid.*
- *$n(IV,\mathrm{PG}(n,q)) > B(q)$ if and only if $n \geq 7$.*
- *For $a \in \{V, VIII\}$, the inequality $n(a,\mathrm{PG}(n,q)) > B(q)$ is valid if and only if $n \geq 8$.*
- *$n(VI,\mathrm{PG}(n,q)) < B(q)$ if $n \leq 7$ and $n(VI,\mathrm{PG}(n,q)) > B(q)$ if $n \geq 9$. Furthermore $n(VI,\mathrm{PG}(8,q)) > B(q)$ if $q = 2,3$, $n(VI,\mathrm{PG}(8,4)) = B(4)$ and $n(VI,\mathrm{PG}(8,q)) < B(q)$ if $q \geq 5$.*
- *For $a \in \{VII, IX\}$, the inequality $n(a,\mathrm{PG}(n,q)) > B(q)$ is valid if and only if $n \geq 9$.*

Proof. Direct computation. This can be done very efficiently using the results from Lemma 3.3.2. □

Lemma 3.3.4. *Denote $3q^4+3q^3+2q^2+q+1$ by $C(q)$. Assume $n \geq 6$.*

- *For* $a \in \{I, II, III, IV, V\}$*, the inequality* $n(a, \mathrm{PG}(n,q)) > C(q)$ *is valid.*
- *For* $a \in \{VI, VII, VIII, IX\}$*, the inequality* $n(a, \mathrm{PG}(n,q)) > C(q)$ *is valid if and only if* $n \geq 7$*.*
- *For* $a \in \{X, XI\}$*, the inequality* $n(a, \mathrm{PG}(n,q)) < C(q)$ *is valid.*

Proof. Direct computation. This can be done very efficiently using the results from Lemma 3.3.2. □

Now we can present the classification theorem. Thereby, EKR(2) sets of types that are separated by commas have a different size; EKR(2) sets of types that are joined by 'and' have the same size.

Theorem 3.3.5. *Let* $\mathcal{S}$ *be a maximal EKR(2) set in* $\mathrm{PG}(n,q)$*,* $n \geq 5$*, with* $|\mathcal{S}| \geq 3q^4 + 3q^3 + 2q^2 + q + 1$*.*

- *If* $n = 5$*, then* $\mathcal{S}$ *is of one of the following types, in decreasing order of size: I and XII, II and III and V, IV.*
- *If* $n = 6$*, then* $\mathcal{S}$ *is of one of the following types, in decreasing order of size: I, II and III, XII, IV, V.*
- *If* $n = 7$*, then* $\mathcal{S}$ *is of one of the following types, in decreasing order of size:*
  - *I, II and III, IV, XII, V, VI and VIII, VII and IX if* $q = 2$*,*
  - *I, II and III, IV, XII, V, VIII, VI, IX, VII if* $q > 2$*.*
- *If* $n = 8$*, then* $\mathcal{S}$ *is of one of the following types, in decreasing order of size:*
  - *I, II and III, IV, V, VIII and VI, XII, IX and VII if* $q = 2$*,*
  - *I, II and III, IV, V, VIII, VI, XII, IX, VII if* $q = 3$*,*
  - *I, II and III, IV, V, VIII, VI and XII, IX, VII if* $q = 4$*,*
  - *I, II and III, IV, V, VIII, XII, VI, IX, VII if* $q \geq 5$*.*
- *If* $n \geq 9$*, then* $\mathcal{S}$ *is of one of the following types, in decreasing order of size:*
  - *I, II and III, IV, V, VI and VIII, VII and IX, XII if* $q = 2$*,*

– *I, II and III, IV, V, VIII, VI, IX, VII, XII if $q > 2$.*

Proof. For the first case, we just restated Theorem 3.0.1. The other cases follow from Theorem 3.2.1, Remark 3.3.1 and Lemmas 3.3.2, 3.3.3 and 3.3.4.□

3.3.2 The polar spaces

In this section, $\mathcal{P}$ will always be a polar space and $\mathrm{PG}(n, q)$ will always be its ambient projective space. We look at the classification of the EKR(2) sets on the polar spaces of rank d.

First, we consider the general case $d \geq 6$. These polar spaces contain 5-spaces, so we rely on Theorem 3.0.1 and Remark 3.3.1. For these polar spaces we give a classification of the EKR(2) sets with size at least $3q^4+3q^3+2q^2+q+1$. Just as in the projective case, EKR(2) sets of types that are separated by commas have a different size, while EKR(2) sets of types that are joined by 'and' have the same size.

Theorem 3.3.6. *Let $\mathcal{P}$ be a polar space of rank d at least 6. Let $\mathcal{S}$ be a maximal EKR(2) set on $\mathcal{P}$ with $|\mathcal{S}| \geq 3q^4 + 3q^3 + 2q^2 + q + 1$.*

- *If $\mathcal{P} = \mathcal{Q}^+(11, q)$, then $\mathcal{S}$ is of one of the following types, in decreasing order of size:*
  - *I, II and III, IV, V, VI and VIIIa, VII and IXa, XII if $q = 2$,*
  - *I, II and III, IV, V, VIIIa, VI, XII, IXa, VII if $q = 3, 4$,*
  - *I, II and III, IV, V, VIIIa, VI and XII, IXa, VII if $q = 5$,*
  - *I, II and III, IV, V, VIIIa, XII, VI, IXa, VII if $q \geq 7$.*
- *If $\mathcal{P}$ is a quadric polar space different from $\mathcal{Q}^+(11, q)$, with ambient projective space $\mathrm{PG}(n, q)$, or $\mathcal{P}$ is a symplectic polar space $\mathcal{W}(n, q)$, q even, then $\mathcal{S}$ is of one of the following types, in decreasing order of size:*
  - *I, II and III, IV, V, VI and VIIIa, VII and IXa, XII if $q = 2$,*
  - *I, II and III, IV, V, VIIIa, VI, IXa, VII, XII if $q \geq 3$.*
- *If $\mathcal{P}$ is a symplectic polar space $\mathcal{W}(n, q)$, q odd, then $\mathcal{S}$ is of one of the following types, in decreasing order of size:*
  - *I, II and III, IV, V, VI and VIIIa, VII and IXa, VIIIb, XII if $q = 2$,*

  - *I, II and III, IV, V, VIIIa, VI, IXa, VII and VIIIb, XII if* $q = 3$,
  - *I, II and III, IV, V, VIIIa, VI, IXa, VIIIb, VII, XII if* $q \geq 4$.

- *If* $\mathcal{P}$ *is a Hermitian polar space* $\mathcal{H}(n, q)$, q *square, then* $\mathcal{S}$ *is of one of the following types, in decreasing order of size:*
  - *I, II and III, IV, V, VIIIa, VI, IXa, VIIIb, VII and IXb, XII if* $q = 4$,
  - *I, II and III, IV, V, VIIIa, VI, IXa, VIIIb, IXb, VII, XII if* $q \geq 9$.

Proof. It follows from Theorem 3.2.1 and Remark 3.3.1 that these are the only EKR(2) sets that can occur. Their sizes can be found in Section 3.1. □

Now, we consider the small cases. In the classification theorems below, we will use Theorem 3.2.1. Since this theorem includes the possibility that the EKR(2) set $\mathcal{S}$ is contained in a 5-space, we will study the EKR(2) sets that are contained in the intersection of $\mathcal{P}$ with a 5-space.

The following theorems are special cases ($d = 3$) of theorems in [24] and [104].

Theorem 3.3.7 ([24, Theorem 3.5]). *Let* $\mathcal{S}$ *be a maximal EKR(2) set on* $\mathcal{Q}^+(5, q)$ *or a maximal EKR(2) set on* $\mathcal{W}(5, q)$, q *even. Then* $\mathcal{S}$ *is of type I, of type III or of type XIII.*

Theorem 3.3.8 ([24, Theorem 3.7]). *Let* $\mathcal{S}$ *be a maximal EKR(2) set on* $\mathcal{W}(5, q)$, q *odd. Then* $\mathcal{S}$ *is of type I, of type III, of type XVIb, of type XVII or of type XVIII.*

Theorem 3.3.9 ([104, Theorem 45]). *Let* $\mathcal{S}$ *be a maximal EKR(2) set on* $\mathcal{H}(5, q)$, q *a square. Then* $|\mathcal{S}| \leq q^2\sqrt{q} + q\sqrt{q} + \sqrt{q} + 1$. *Moreover, if* $|\mathcal{S}| = q^2\sqrt{q} + q\sqrt{q} + \sqrt{q} + 1$, *then* $\mathcal{S}$ *is of type III.*

Corollary 3.3.10. *Let* $\mathcal{P}$ *be a polar space, with* $\mathrm{PG}(n, q)$, $n \geq 6$, *its ambient projective space and let* ρ *be a* 5*-space in* $\mathrm{PG}(n, q)$ *such that* $\mathcal{P} \cap \rho$ *is a polar space. Let* $\mathcal{S}$ *be a maximal EKR(2) set on* $\mathcal{P}$, *which is contained in* ρ. *Then,*

- $\mathcal{S}$ *is of type XIII if* $\mathcal{P}$ *is a quadric polar space or if* $\mathcal{P}$ *is a symplectic polar space and* q *is even.*
- $\mathcal{S}$ *is of type XVIb, of type XVII or of type XVIII if* $\mathcal{P}$ *is a symplectic polar space and* q *is odd.*

- $|\mathcal{S}| < q^2\sqrt{q} + q\sqrt{q} + \sqrt{q} + 1$ *if* $\mathcal{P}$ *is a Hermitian polar space.*

Proof. First, note that $\mathcal{P}' = \mathcal{P} \cap \rho$ cannot be a polar space of rank 2, since $\mathcal{P}'$ must contain planes. It has to be a polar space of rank 3. Since $\mathcal{S}$ is also a maximal EKR(2) set of the polar space $\mathcal{P}'$, we can apply Theorem 3.3.7, Theorem 3.3.8 and Theorem 3.3.9. However, if $\mathcal{S}$ is of type I or type III as EKR(2) set in $\mathcal{P}'$, then it can be extended by planes on $\mathcal{P}$ that are not in ρ. $\square$

Lemma 3.3.11. *Let* $\mathcal{P}$ *be a polar space, with* $\mathrm{PG}(n, q)$, $n \geq 6$, *its ambient projective space and let* ρ *be a* 5*-space in* $\mathrm{PG}(n, q)$ *such that* $\mathcal{P} \cap \rho$ *is a cone with vertex a point* R *and base a polar space* $\mathcal{P}'$. *Let* $\mathcal{S}$ *be a maximal EKR(2) set on* $\mathcal{P}$, *which is contained in* ρ. *Then,* $n = 6$ *and* $\mathcal{S}$ *is an EKR(2) set of type I.*

Proof. Note that the ambient space of $\mathcal{P}'$ is a 4-space. So, the rank of $\mathcal{P}'$ equals 2. Hence, all planes of $\mathcal{S}$ pass through R, since all planes on such a cone pass through its vertex R. If $\mathcal{P}$ contains a plane through R not in ρ, then $\mathcal{S}$ cannot be maximal, and thus cannot be an EKR(2) set, since such a plane extends $\mathcal{S}$.

If $n \geq 7$, there are planes through R on $\mathcal{P}$, that are not contained in ρ. If $n = 6$, then $\mathcal{P}$ is a polar space of rank 3 and all planes through R are contained in ρ. By the maximality condition, $\mathcal{S}$ must contain all planes through R and consequently $\mathcal{S}$ is an EKR(2) set of type I.

Lemma 3.3.12. *Let* $\mathcal{P}$ *be a polar space, with* $\mathrm{PG}(n, q)$, $n \geq 7$, *its ambient projective space and let* ρ *be a* 5*-space in* $\mathrm{PG}(n, q)$ *such that* $\mathcal{P} \cap \rho$ *is a cone with vertex a line* ℓ *and base a polar space* $\mathcal{P}'$. *Let* $\mathcal{S}$ *be a maximal EKR(2) set on* $\mathcal{P}$, *which is contained in* ρ. *Then* $\mathcal{P}'$ *has rank* 2 *and one of the following cases occurs:*

- $\mathcal{P} = \mathcal{Q}^+(7, q)$ *and* $\mathcal{S}$ *is an EKR(2) set of type IXa.*
- $\mathcal{P} = \mathcal{W}(7, q)$, q *even, and* $\mathcal{S}$ *is an EKR(2) set of type IXa.*
- $\mathcal{P} = \mathcal{W}(7, q)$, q *odd, and* $\mathcal{S}$ *is an EKR(2) set of type VIIIb.*
- $\mathcal{P} = \mathcal{H}(7, q)$ *and* $\mathcal{S}$ *is an EKR(2) set of type VIIIb or of type IXb.*

Proof. If $\mathcal{P}'$ has rank 1, then all planes in $\mathcal{P} \cap \rho$ pass through the vertex ℓ. Any plane on $\mathcal{P}$ that intersects ℓ in precisely a point, is not contained in ρ but extends $\mathcal{S}$. Hence, $\mathcal{S}$ cannot be a maximal EKR(2) set. So, we can assume that $\mathcal{P}'$ has rank 2.

Then, all planes of $\mathcal{S}$ are contained in one of the 3-spaces generated by ℓ and a line of the base $\mathcal{P}'$. All these planes intersect ℓ in at least a point, hence all planes through ℓ on $\mathcal{P}$ must be contained in $\mathcal{S}$. If $n \geq 8$, there are planes through ℓ on $\mathcal{P}$ that are not in ρ; so, $n = 7$. In this case, all planes through ℓ are contained in ρ. Since $\mathcal{P} \cap \rho$ contains 3-spaces, $\mathcal{P}$ has rank 4. Hence $\mathcal{P}$ is a hyperbolic quadric $\mathcal{Q}^+(7, q)$, a Hermitian polar space $\mathcal{H}(7, q)$ or a symplectic polar space $\mathcal{W}(7, q)$.

Looking at the outline of the proof of Theorem 3.2.1, we see that $\mathcal{S}$ would be treated in the case 1.1. and thus $\mathcal{S}$ will be an EKR(2) set of type I, ..., type IX. Since $\mathcal{P}$ is a polar space of rank 4, $\mathcal{S}$ cannot be an EKR(2) set of type IV, type V, type VI or type VII. If $\mathcal{S}$ is of type I or of type II, the planes of $\mathcal{S}$ span a 6-space, and if $\mathcal{S}$ is of type III, the planes of $\mathcal{S}$ span $\mathrm{PG}(7, q)$. So, in these cases, $\mathcal{S}$ is not contained in ρ. Hence $\mathcal{S}$ is of type VIIIa, type VIIIb, type IXa or type IXb. From the remarks in Section 3.1, it follows which of these types can occur in each of the different cases. $\square$

Lemma 3.3.13. *Let $\mathcal{P}$ be a polar space, with $\mathrm{PG}(n, q)$, $n \geq 8$, its ambient projective space and let ρ be a 5-space in $\mathrm{PG}(n, q)$ such that $\mathcal{P} \cap \rho$ is a cone with vertex a plane π and base a polar space $\mathcal{P}'$. There are no maximal EKR(2) sets on $\mathcal{S}$ that are contained in ρ.*

Proof. Note that the ambient space of $\mathcal{P}'$ is $\mathrm{PG}(2, q)$ and hence that the rank of $\mathcal{P}'$ is 1. Assume that $\mathcal{S}$ is an EKR(2) set on $\mathcal{P}$ that is contained in ρ. Note that all planes of $\mathcal{S}$ are contained in a 3-space generated by π and a point of $\mathcal{P}'$. So, all planes of $\mathcal{S}$ intersect π in at least a line. If $\mathcal{P}$ contains a plane not in ρ, which intersects π in a line, then $\mathcal{S}$ cannot be maximal, since such a plane extends $\mathcal{S}$. Such a plane always can be found since $n \geq 8$. $\square$

Lemma 3.3.14. *Let $\mathcal{P}$ be a polar space, with $\mathrm{PG}(n, q)$, $n \geq 9$, its ambient projective space and let ρ be a 5-space in $\mathrm{PG}(n, q)$ such that $\mathcal{P} \cap \rho$ is a cone with vertex a 3-space σ and base a polar space $\mathcal{P}'$. If $\mathcal{S}$ is a maximal EKR(2) set on $\mathcal{P}$, which is contained in ρ, then the rank of $\mathcal{P}'$ equals 1 and $\mathcal{S}$ is of type X, of type XII, of type XIV or of type XV. The first type cannot occur if $\mathcal{P}$ is a quadric.*

Proof. Note that the ambient space of $\mathcal{P}'$ is a projective line. Hence, the rank of $\mathcal{P}'$ equals 0 or 1. If it equals 0, then $\mathcal{P}'$ is empty and all planes of $\mathcal{S}$ are contained in the 3-space σ. In this case $\mathcal{S}$ can be extended by any plane not in

ρ that intersects σ in precisely a line. Hence, $\mathcal{S}$ is not maximal. So, the rank of $\mathcal{P}'$ equals 1. Denote its parameter by e'.

Note that $\mathcal{P} \cap \rho$ is the union of several 4-spaces τ_i, $i = 0, \dots, q^{e'}$, whose pairwise intersection equals σ. All planes in $\mathcal{S}$ thus intersect σ in at least a line, hence all planes in σ must be in $\mathcal{S}$ by the maximality condition. Denote $\mathcal{S} \setminus \{\pi \mid \pi \text{ plane in } \sigma\}$ by $\mathcal{S}'$. Furthermore, if all planes of $\mathcal{S}'$ are in the same 4-space of ρ, then $\mathcal{S}$ is of type XII. So, from now on, we can assume that at least two 4-spaces contain planes of $\mathcal{S}'$, say τ_0 and τ_1.

All planes in $\mathcal{S}'$ intersect σ in a line. Two planes of $\mathcal{S}'$ in the same 4-space intersect anyhow; two planes of $\mathcal{S}'$ that are in different 4-spaces intersect if and only if their corresponding lines in σ intersect. Consequently, if a plane $\pi \subset \tau_i$ is in $\mathcal{S}'$, then all other planes through $\pi \cap \sigma$ in τ_i must be in $\mathcal{S}'$ by the maximality condition on $\mathcal{S}$, $i = 0, \dots, q^{e'}$. Denote $\mathcal{L}_i = \{\sigma \cap \pi \mid \pi \in \mathcal{S}', \pi \subset \tau_i\}$ for every $i = 0, \dots, q^{e'}$. By the previous arguments, every line of $\mathcal{L}_i$ and every line of $\mathcal{L}_j$ must meet for all $0 \le i \ne j \le q^{e'}$, and the sets $\mathcal{L}_i$ must be maximal under this condition.

We know that both $\mathcal{L}_0$ and $\mathcal{L}_1$ are non-empty. Let ℓ_0 be a line in $\mathcal{L}_0$ and let ℓ_1 be a line in $\mathcal{L}_1$. If all lines in $\mathcal{L}_0$ meet ℓ_0, $\mathcal{S}$ cannot be maximal since it can be extended by a plane through ℓ_0 that is not in ρ. Such planes exist since $n \ge 9$. So, $\mathcal{L}_0$ contains a line ℓ_0' disjoint to ℓ_0. Analogously, $\mathcal{L}_1$ contains a line ℓ_1' disjoint to ℓ_1. We distinguish between two cases.

First, we assume that all sets $\mathcal{L}_i$, $i \ge 2$, are empty. If $\mathcal{L}_0 = \{\ell_0, \ell_0'\}$, then $\mathcal{L}_1$ must contain all lines intersecting ℓ_0 and ℓ_0'. We find an EKR(2) set of type XIV. So, from now we can assume $|\mathcal{L}_0|, |\mathcal{L}_1| > 2$. We look at two different subcases.

In the first subcase, we assume $\mathcal{L}_1$ contains a line ℓ_1'' disjoint to ℓ_1 and ℓ_1'. The set of lines intersecting ℓ_1, ℓ_1' and ℓ_1'' is a regulus $\mathcal{R}_0$ which necessarily contains ℓ_0 and ℓ_0'. Its opposite regulus $\mathcal{R}_1$ contains ℓ_1, ℓ_1' and ℓ_1''. By the previous arguments, we know that all lines of $\mathcal{L}_0$ must be in $\mathcal{R}_0$. Also $\mathcal{L}_0$ contains a line $\ell_0'' \in \mathcal{R}_0$ different from ℓ_0 and ℓ_0' since $|\mathcal{L}_0| > 2$. It follows that ℓ_0, ℓ_0' and ℓ_0'' are pairwise disjoint and hence all lines of $\mathcal{L}_1$ must be in $\mathcal{R}_1$. By the maximality condition, $\mathcal{L}_1 = \mathcal{R}_1$ and $\mathcal{L}_0 = \mathcal{R}_0$. Hence, $\mathcal{S}$ is an EKR(2) set of type XV.

In this second subcase, we assume that all lines in $\mathcal{L}_0$ intersect ℓ_0 or ℓ_0' and all lines in $\mathcal{L}_1$ intersect ℓ_1 or ℓ_1'. We denote $\ell_0 \cap \ell_1 = \{P_{0,1}\}$, $\ell_0' \cap \ell_1 = \{P_{0',1}\}$, $\ell_0 \cap \ell_1' = \{P_{0,1'}\}$ and $\ell_0' \cap \ell_1' = \{P_{0',1'}\}$. The lines in $\mathcal{L}_0$ must be lines through

$P_{0,1}$ in $\langle P_{0,1}, P_{0,1'}, P_{0',1'}\rangle$, lines through $P_{0,1'}$ in $\langle P_{0,1}, P_{0,1'}, P_{0',1}\rangle$, lines through $P_{0',1}$ in $\langle P_{0',1}, P_{0,1'}, P_{0',1'}\rangle$ or lines through $P_{0',1'}$ in $\langle P_{0,1}, P_{0',1}, P_{0',1'}\rangle$. Without loss of generality we can assume $\mathcal{L}_0$ contains a line $m_0 \neq \ell_0$ through $P_{0,1}$ in the plane $\langle P_{0,1}, P_{0,1'}, P_{0',1'}\rangle$. Any line in $\mathcal{L}_1$ then intersects ℓ_0, ℓ'_0 and m_0, hence is either a line through $P_{0,1}$ in $\langle P_{0,1}, P_{0',1}, P_{0',1'}\rangle$ or else a line through $P_{0',1'}$ in $\langle P_{0,1}, P_{0,1'}, P_{0',1'}\rangle$. We know that $\mathcal{L}_1$ contains such a line since $|\mathcal{L}_1| > 2$. It follows that the lines in $\mathcal{L}_0$ must be either lines through $P_{0,1}$ in $\langle P_{0,1}, P_{0,1'}, P_{0',1'}\rangle$ or else lines through $P_{0',1'}$ in $\langle P_{0,1}, P_{0',1}, P_{0',1'}\rangle$.

Now it should be noted that all lines that can be in $\mathcal{L}_0$ or $\mathcal{L}_1$, intersect the line $\langle P_{0,1}, P_{0',1'}\rangle$. Hence, a plane through this line, not in ρ, extends $\mathcal{S}$. Such a plane exists since $n \geq 9$. From this contradiction, it follows that $\mathcal{S}$ is not maximal.

Secondly, we assume that not all sets $\mathcal{L}_i$, $i \geq 2$, are empty. Note that this is not possible if $\mathcal{P}$ is a quadric polar space since there are only two 4-spaces through σ in that case. Say $\mathcal{L}_2$ is non-empty. Any line ℓ_i in $\mathcal{L}_i$, $i \geq 2$, must intersect ℓ_0, ℓ'_0, ℓ_1 and ℓ'_1, hence must be equal to either $m = \langle P_{0,1}, P_{0',1'}\rangle$ or $m' = \langle P_{0',1}, P_{0,1'}\rangle$, using the notations from above. Note that m and m' are disjoint. As before we can argue that $|\mathcal{L}_2| \geq 2$. So, $m, m' \in \mathcal{L}_2$. Moreover, since m and m' are the only lines that intersect ℓ_0, ℓ'_0, ℓ_1 and ℓ'_1, $\mathcal{L}_2 = \{m, m'\}$. Analogously, $\mathcal{L}_0 = \{\ell_0, \ell'_0\}$ and $\mathcal{L}_1 = \{\ell_1, \ell'_1\}$. Furthermore, any line in $\mathcal{L}_i$, $i \geq 3$, must intersect ℓ_0, ℓ'_0, ℓ_1, ℓ'_1, m and m', but there are no such lines. Hence, all sets $\mathcal{L}_i$ are empty for $i \geq 3$. So, we know all lines of $\mathcal{L}_i$ for all i. We find that $\mathcal{S}$ is an EKR(2) set of type X with base points $P_{0,1}$, $P_{0,1'}$, $P_{0',1}$ and $P_{0',1'}$. $\square$

Lemma 3.3.15. *Let $\mathcal{P}$ be a polar space, with $\mathrm{PG}(n,q)$, $n \geq 9$, its ambient projective space and let ρ be a 5-space in $\mathrm{PG}(n,q)$ such that $\mathcal{P} \cap \rho$ is a 4-space τ (a cone with vertex this 4-space and base a polar space of rank 0). If $\mathcal{S}$ is a maximal EKR(2) set on Q, which is contained in ρ, then it is of type XII.*

Proof. Note that all planes in $\mathcal{P} \cap \rho$ intersect each other since they are all contained in τ. So all these planes are in $\mathcal{S}$ and $\mathcal{S}$ is an EKR(2) set of type XII. $\square$

Now we can give the classification theorems of the maximal EKR(2) sets for the smallest polar spaces. We will discuss the quadric, Hermitian and symplectic polar spaces separately.

The quadrics

We look at the classification of the maximal EKR(2) sets on the quadrics $\mathcal{Q}^+(2d-1,q)$, $\mathcal{Q}(2d,q)$ and $\mathcal{Q}^-(2d+1,q)$, $d=3,4,5$. For these quadrics the classification of the maximal EKR(2) sets is complete. Recall that we classified only the largest maximal EKR(2) sets for $d \geq 6$.

First we look at the three quadric polar spaces of rank 3, secondly at the three quadric polar spaces of rank 4 and finally at the three quadric polar spaces of rank 5. Usually, we will give the list of possible types in decreasing order of size. Again, EKR(2) sets of types that are separated by commas have a different size; EKR(2) sets of types that are joined by 'and' have the same size.

The hyperbolic quadric $\mathcal{Q}^+(5,q)$ was studied in Theorem 3.3.7. We recall that all maximal EKR(2) sets are of type I, III or XIII and we mention that $n(\mathrm{XIII},\mathcal{Q}^+(5,q)) > n(\mathrm{III},\mathcal{Q}^+(5,q)) > n(\mathrm{I},\mathcal{Q}^+(5,q)) = 2q+2$.

Note that the only EKR(2) sets from the list in Section 3.1.1 that exist on quadric polar spaces of rank 3, are the ones of type I and III.

Theorem 3.3.16. *Let $\mathcal{S}$ be a maximal EKR(2) set on $\mathcal{Q}(6,q)$, then $\mathcal{S}$ is of type I, of type III or of type XIII. Moreover, $n(I,\mathcal{Q}(6,q)) = n(III,\mathcal{Q}(6,q)) = n(XIII,\mathcal{Q}(6,q))$.*

Proof. We use Theorem 3.2.1. We find that $\mathcal{S}$ is of type I or III or that it is contained in a 5-space ρ. The intersection $\mathcal{Q}(6,q) \cap \rho$ is a non-singular hyperbolic or elliptic quadric, or a cone with vertex a point. The first part of the theorem thus follows from Corollary 3.3.10 and Lemma 3.3.11.

The second part follows from the computations in Section 3.1:

$$n(\mathrm{I},\mathcal{Q}(6,q)) = n(\mathrm{III},\mathcal{Q}(6,q)) = n(\mathrm{XIII},\mathcal{Q}(6,q)) = q^3+q^2+q+1\ . \qquad \square$$

Theorem 3.3.17. *Let $\mathcal{S}$ be a maximal EKR(2) set on $\mathcal{Q}^-(7,q)$, then $\mathcal{S}$ is of one of the following types, in decreasing order of size: I, III, XIII.*

Proof. We use Theorem 3.2.1. We find that $\mathcal{S}$ is of type I or III, or $\mathcal{S}$ is contained in a 5-space. From now on, we assume that $\mathcal{S}$ is contained in a 5-space ρ. There are four possibilities for $\mathcal{Q}^-(7,q)\cap\rho$: a non-singular hyperbolic quadric, a non-singular elliptic quadric, a cone with vertex a point, a cone with vertex a line and base an elliptic quadric $\mathcal{Q}^-(3,q)$. The classification thus follows from Corollary 3.3.10, Lemma 3.3.11 and Lemma 3.3.12.

The sizes of these EKR(2) sets can be found in Section 3.1. □

Note that the only EKR(2) sets from the list in Section 3.1.1 that exist on quadric polar spaces of rank 4, are the ones of type I, II, III and IXa.

Theorem 3.3.18. *Let $\mathcal{S}$ be a maximal EKR(2) set on $\mathcal{Q}^+(7,q)$, then $\mathcal{S}$ is of one of the following types, in decreasing order of size: I, II and III, IXa, XIII.*

Proof. We use Theorem 3.2.1. We find that $\mathcal{S}$ is of type I, II, III, or IXa, or $\mathcal{S}$ is contained in a 5-space. From now on, we assume that $\mathcal{S}$ is contained in a 5-space ρ. We distinguish between the four possibilities for $\mathcal{Q}^+(7,q)\cap\rho$: a non-singular hyperbolic quadric, a non-singular elliptic quadric, a cone with vertex a point, a cone with vertex a line and base a hyperbolic quadric $\mathcal{Q}^+(3,q)$. The classification thus follows from Corollary 3.3.10, Lemma 3.3.11 and Lemma 3.3.12.

The sizes of these EKR(2) sets can be found in Section 3.1 and can easily be compared. □

Theorem 3.3.19. *Let $\mathcal{S}$ be a maximal EKR(2) set on $\mathcal{Q}(8,q)$, then $\mathcal{S}$ is of one of the following types, in decreasing order of size: I, II and III, IXa, XIII.*

Proof. We use Theorem 3.2.1. We find that $\mathcal{S}$ is of type I, II, III, or IXa, or $\mathcal{S}$ is contained in a 5-space. From now on, we assume that $\mathcal{S}$ is contained in a 5-space ρ. We distinguish between the six possibilities for $\mathcal{Q}(8,q)\cap\rho$: a non-singular hyperbolic quadric, a non-singular elliptic quadric, a cone with vertex a point, a cone with vertex a line and base a hyperbolic quadric $\mathcal{Q}^+(3,q)$, a cone with vertex a line and base an elliptic quadric $\mathcal{Q}^-(3,q)$, a cone with vertex a plane. The classification thus follows from Corollary 3.3.10 and Lemmas 3.3.11, 3.3.12 and 3.3.13.

The sizes of these EKR(2) sets can be found in Section 3.1 and can easily be compared. □

Theorem 3.3.20. *Let $\mathcal{S}$ be a maximal EKR(2) set on $\mathcal{Q}^-(9,q)$, then $\mathcal{S}$ is of one of the following types, in decreasing order of size: I, II and III, IXa, XIII.*

Proof. We use Theorem 3.2.1. We find that $\mathcal{S}$ is of type I, II, III, or IXa, or $\mathcal{S}$ is contained in a 5-space. From now on, we assume that $\mathcal{S}$ is contained in a 5-space ρ. We distinguish between the seven possibilities for $\mathcal{Q}^-(9,q) \cap \rho$: a non-singular hyperbolic quadric, a non-singular elliptic quadric, a cone with vertex a point, a cone with vertex a line and base a hyperbolic quadric $\mathcal{Q}^+(3,q)$, a cone with vertex a line and base an elliptic quadric $\mathcal{Q}^-(3,q)$, a cone with vertex a plane, a 3-space. The classification thus follows from Corollary 3.3.10 and Lemmas 3.3.11, 3.3.12, 3.3.13 and 3.3.14.

The sizes of these EKR(2) sets can be found in Section 3.1 and can easily be compared. □

Note that all types of EKR(2) sets listed in Section 3.1.1 exist on quadric polar spaces of rank 5, except the ones of type VII, VIIIa, VIIIb, IXb and XI.

Theorem 3.3.21. *Let $\mathcal{S}$ be a maximal EKR(2) set on $\mathcal{Q}^+(9,q)$, then $\mathcal{S}$ is of one of the following types, in decreasing order of size: I, II and III, XII, IV, V, VI, IXa and XIV, XV, XIII.*

Proof. We use Theorem 3.2.1. We find that $\mathcal{S}$ is of type I, ..., IXa - types VII, VIIIa, VIIIb and IXb are however not possible - or $\mathcal{S}$ is contained in a 5-space. Note that $\mathcal{S}$ cannot be of type X since $e = 0$. From now on, we assume that $\mathcal{S}$ is contained in a 5-space ρ. We distinguish between the seven possibilities for $\mathcal{Q}^+(9,q) \cap \rho$: a non-singular hyperbolic quadric, a non-singular elliptic quadric, a cone with vertex a point, a cone with vertex a line and base a hyperbolic quadric $\mathcal{Q}^+(3,q)$, a cone with vertex a line and base an elliptic quadric $\mathcal{Q}^-(3,q)$, a cone with vertex a plane, a cone with vertex a 3-space and base a hyperbolic quadric $\mathcal{Q}^+(1,q)$. The classification thus follows from Corollary 3.3.10 and Lemmas 3.3.11, 3.3.12, 3.3.13 and 3.3.14.

The sizes of these EKR(2) sets can be found in Section 3.1 and can easily be compared. □

Theorem 3.3.22. *Let $\mathcal{S}$ be a maximal EKR(2) set on $Q = \mathcal{Q}(10,q)$, then $\mathcal{S}$ is of one of the following types, in decreasing order of size:*

- *I, II and III, IV, XII, V, VI, IXa, XIV, X and XV, XIII if $q = 2$,*
- *I, II and III, IV, XII, V, VI, IXa, XIV, XV, X, XIII if $q \geq 3$.*

Proof. We use Theorem 3.2.1. We find that $\mathcal{S}$ is of type I, ..., X - types VII, VIIIa, VIIIb and IXb are however not possible - or $\mathcal{S}$ is contained in a 5-space.

From now on, we assume that $\mathcal{S}$ is contained in a 5-space ρ. We distinguish between the nine possibilities for $\mathcal{Q}(10,q)\cap\rho$: a non-singular hyperbolic quadric, a non-singular elliptic quadric, a cone with vertex a point, a cone with vertex a line and base a hyperbolic quadric $\mathcal{Q}^+(3,q)$, a cone with vertex a line and base an elliptic quadric $\mathcal{Q}^-(3,q)$, a cone with vertex a plane, a cone with vertex a 3-space and base a hyperbolic quadric $\mathcal{Q}^+(1,q)$, a 3-space, a 4-space. The classification thus follows from Corollary 3.3.10 and Lemmas 3.3.11, 3.3.12, 3.3.13, 3.3.14 and 3.3.15.

The sizes of these EKR(2) sets can be found in Section 3.1 and can easily be compared. □

Theorem 3.3.23. *Let $\mathcal{S}$ be a maximal EKR(2) set on $Q = \mathcal{Q}^-(11,q)$, then $\mathcal{S}$ is of one of the following types, in decreasing order of size:*

- *I, II and III, IV, V, VI, XII, IXa, XIV, X and XV, XIII if $q = 2$,*
- *I, II and III, IV, V, VI and XII, IXa, XIV, XV, X, XIII if $q = 3$,*
- *I, II and III, IV, V, XII, VI, IXa, XIV, XV, X, XIII if $q \geq 4$.*

Proof. We use Theorem 3.2.1. We find that $\mathcal{S}$ is of type I, ..., X - types VII, VIIIa, VIIIb and IXb are however not possible - or $\mathcal{S}$ is contained in a 5-space. From now on, we assume that $\mathcal{S}$ is contained in a 5-space ρ. We distinguish between the nine possibilities for $\mathcal{Q}^-(11,q)\cap\rho$: a non-singular hyperbolic quadric, a non-singular elliptic quadric, a cone with vertex a point, a cone with vertex a line and base a hyperbolic quadric $\mathcal{Q}^+(3,q)$, a cone with vertex a line and base an elliptic quadric $\mathcal{Q}^-(3,q)$, a cone with vertex a plane, a cone with vertex a 3-space and base a hyperbolic quadric $\mathcal{Q}^+(1,q)$, a 3-space, a 4-space. The classification thus follows from Corollary 3.3.10 and Lemmas 3.3.11, 3.3.12, 3.3.13, 3.3.14 and 3.3.15.

The sizes of these EKR(2) sets can be found in Section 3.1 and can easily be compared. □

The Hermitian polar spaces

Now we give the classification theorems of the maximal EKR(2) sets for the smallest Hermitian polar spaces. Note that the polar space $\mathcal{H}(5,q)$ was already covered in Theorem 3.3.9. First we look at $\mathcal{H}(6,q)$, the other Hermitian polar space of rank 3, secondly at the two Hermitian polar spaces of rank 4 and finally at the two Hermitian polar spaces of rank 5. Unlike the theorems about the quadric polar spaces, we do not find a complete classification of the maximal EKR(2) sets, but the upper bound on the size of the non-classified ones is much smaller than the bound in Theorem 3.3.6.

In the statement of these theorems, we will use the same convention as in the study of the quadric polar spaces.

Note that the only EKR(2) sets from the list in Section 3.1.1 that exist on Hermitian polar spaces of rank 3, are the ones of type I and type III.

Theorem 3.3.24. *Let $\mathcal{S}$ be a maximal EKR(2) set on $\mathcal{H}(6,q)$, q a square, with $|\mathcal{S}| \geq q^2\sqrt{q}+q\sqrt{q}+\sqrt{q}+1$, then $\mathcal{S}$ is of one of the following types, in decreasing order of size: I, III.*

Proof. Analogous to the proof of Theorem 3.3.16, using Corollary 3.3.10 and Lemma 3.3.11. We note that $n(\mathrm{I},\mathcal{H}(6,q)) = q^4+q^2\sqrt{q}+q\sqrt{q}+1$ and that $n(\mathrm{III},\mathcal{H}(6,q)) = q^3\sqrt{q}+q^2\sqrt{q}+q\sqrt{q}+1$. □

Note that the only EKR(2) sets from the list in Section 3.1.1 that exist on Hermitian polar spaces of rank 4, are the ones of type I, II, III, VIIIb and IXb.

Theorem 3.3.25. *Let $\mathcal{S}$ be a maximal EKR(2) set on $\mathcal{H}(7,q)$, q a square, with $|\mathcal{S}| \geq q^2\sqrt{q}+q\sqrt{q}+\sqrt{q}+1$, then $\mathcal{S}$ is of one of the following types, in decreasing order of size: I, II and III, VIIIb, IXb.*

Proof. Analogous to the proof of Theorem 3.3.18, using Corollary 3.3.10, and Lemmas 3.3.11 and 3.3.12. □

Theorem 3.3.26. *Let $\mathcal{S}$ be a maximal EKR(2) set on $\mathcal{H}(8,q)$, q a square, with $|\mathcal{S}| \geq q^2\sqrt{q}+q\sqrt{q}+\sqrt{q}+1$, then $\mathcal{S}$ is of one of the following types, in decreasing order of size: I, II and III, VIIIb, IXb.*

Proof. Analogous to the proof of Theorem 3.3.19, using Corollary 3.3.10 and Lemmas 3.3.11, 3.3.12 and 3.3.13. □

Note that all types of EKR(2) sets listed in Section 3.1.1 exist on Hermitian polar spaces of rank 5, except the ones of type VII, VIIIa, IXa and XI.

Theorem 3.3.27. *Let $\mathcal{S}$ be a maximal EKR(2) set on $\mathcal{H}(9,q)$, q a square, with $|\mathcal{S}| \geq q^2\sqrt{q}+q\sqrt{q}+\sqrt{q}+1$, then $\mathcal{S}$ is of one of the following types, in decreasing order of size: I, II and III, XII, IV, V, VI, VIIIb, IXb, XIV, XV, X.*

Proof. Analogous to the proof of Theorem 3.3.21, using Corollary 3.3.10 and Lemmas 3.3.11, 3.3.12, 3.3.13 and 3.3.14. $\square$

Theorem 3.3.28. *Let $\mathcal{S}$ be a maximal EKR(2) set on $\mathcal{H}(10, q)$, q a square, with $|\mathcal{S}| \geq q^2\sqrt{q} + q\sqrt{q} + \sqrt{q} + 1$, then $\mathcal{S}$ is of one of the following types, in decreasing order of size: I, II and III, IV, XII, V, VI, VIIIb, IXb, XIV, XV, X.*

Proof. Analogous to the proof of Theorem 3.3.22, using Corollary 3.3.10 and Lemmas 3.3.11, 3.3.12, 3.3.13, 3.3.14 and 3.3.15. $\square$

Remark 3.3.29. In this section about polar spaces we use extensively Theorem 3.2.1 and its consequence that it is sufficient to classify all maximal EKR(2) sets that are contained in a 5-space ρ, in order to find a complete classification of the maximal EKR(2) sets. This classification depends on the intersection of the projective or polar space $\mathcal{P}$ with the 5-space ρ. Above, this has been performed for most cases. Next to the case $\rho \subseteq \mathcal{P}$ (see Theorem 3.0.1 and Remark 3.3.1), the only case which has not been fully handled is $\rho \cap \mathcal{P}$ a non-singular Hermitian variety. Classifying all maximal EKR(2) sets on a Hermitian polar space $\mathcal{H}(5, q)$, q a square, would yield complete classifications in the above theorems. Note that it follows from the proof of Theorem 3.2.1 that the planes of a maximal EKR(2) set of $\mathcal{H}(5, q)$, q a square, which has not been described above, pairwise intersect in a point.

The symplectic polar spaces

In this section, we look at the classification of the maximal EKR(2) sets on the symplectic polar spaces $\mathcal{W}(2d-1, q)$ for the small cases, $3 \leq d \leq 5$. We are able to give a complete classification. Note that the polar space $\mathcal{W}(5, q)$ was already covered in Theorem 3.3.7 and Theorem 3.3.8. First we look at the symplectic polar space of rank 4 and secondly at the symplectic polar space of rank 5. In both cases we will need to distinguish between the cases q even and q odd. This is due to the observation made in Remark 1.6.8, which is only valid if q is even. Namely, the finite classical polar spaces $\mathcal{W}(2d-1, q)$ and $\mathcal{Q}(2d, q)$, q even, are isomorphic. It follows from this observation that a hyperbolic quadric $\mathcal{Q}^+(5, q)$ is embedded in $\mathcal{W}(5, q)$, if q is even. So, an EKR(2) set of type XIII can be found on $\mathcal{W}(5, q)$, q even.

In the statement of the theorems, we will use the same convention as in the study of the quadric and Hermitian polar spaces.

Note that the only EKR(2) sets from the list in Section 3.1.1 that exist on all symplectic polar spaces of rank 4, are the ones of type I, II and III. If q is even, also the ones of type IXa can occur; if q is odd, also the ones of type VIIIb can occur.

Theorem 3.3.30. *Let $\mathcal{S}$ be a maximal EKR(2) set on $\mathcal{W}(7, q)$, then $\mathcal{S}$ is of one of the following types, in decreasing order of size:*

- *I, II and III, IXa, XIII if q is even,*
- *I, II and III, VIIIb, XVIb, XVII, XVIII if q is odd.*

Proof. Analogous to the proof of Theorem 3.3.18, using Corollary 3.3.10 and Lemma 3.3.12. Note that the EKR(2) sets of type XIII only occur if q is even and that the EKR(2) sets of type XVIb, XVII or XVIII only occur if q is odd.□

Note that all types of EKR(2) sets listed in Section 3.1.1 exist on symplectic polar spaces of rank 5, except the ones of type VII, VIIIa, IXb and XI. If q is even, also the ones of type VIIIb cannot occur; if q is odd, also the ones of type IXa cannot occur.

Theorem 3.3.31. *Let $\mathcal{S}$ be a maximal EKR(2) set on $\mathcal{W}(9, q)$, then $\mathcal{S}$ is of one of the following types, in decreasing order of size:*

- *I, II and III, IV, XII, V, VI, IXa, XIV, X and XV, XIII if $q = 2$,*
- *I, II and III, IV, XII, V, VI, IXa, XIV, XV, X, XIII if $q \geq 4$ is even,*
- *I, II and III, IV, XII, V, VI, VIIIb, XIV, XV, X, XVIb, XVII, XVIII if q is odd.*

Proof. Analogous to the proof of Theorem 3.3.21, using Corollary 3.3.10, and Lemmas 3.3.12 and 3.3.14. □

In Table 3.1, we present an overview of the results on EKR(2) sets on polar spaces of small rank.

Polar space	**No. of max. EKR(2) sets**
$\mathcal{Q}^+(5,q)$	3
$\mathcal{Q}(6,q)$	3
$\mathcal{Q}^-(7,q)$	3
$\mathcal{Q}^+(7,q)$	5
$\mathcal{Q}(8,q)$	5
$\mathcal{Q}^-(9,q)$	5
$\mathcal{Q}^+(9,q)$	11
$\mathcal{Q}(10,q)$	12
$\mathcal{Q}^-(11,q)$	12

Polar space	**No. of max. EKR(2) sets**
$\mathcal{W}(5,q)$, q even	3
$\mathcal{W}(5,q)$, q odd	5
$\mathcal{W}(7,q)$, q even	5
$\mathcal{W}(7,q)$, q odd	7
$\mathcal{W}(9,q)$, q even	12
$\mathcal{W}(9,q)$, q odd	14

Polar space	**No. of max. EKR(2) sets**
$\mathcal{H}(5,q)$	1
$\mathcal{H}(6,q)$	2
$\mathcal{H}(7,q)$	5
$\mathcal{H}(8,q)$	5
$\mathcal{H}(9,q)$	12
$\mathcal{H}(10,q)$	12

Table 3.1: Overview of the results of Section 3.3.2. In the first and second table, we present for the quadric and symplectic polar spaces of small rank, the number of types of maximal EKR(2) sets. In the third table we present for the Hermitian polar spaces of small rank, the number of types of maximal EKR(2) sets with cardinality greater than or equal to $q^2\sqrt{q} + q\sqrt{q} + \sqrt{q} + 1$.

4

Erdős-Ko-Rado sets of generators on $\mathcal{Q}^+(4n+1,q)$

μή εἶναι βασιλικήν ἀτροπόν ἐπί γεωμετρίαν.

Attributed to *Εὐκλίδης* by *Πρόκλος ὁ Διάδοχος*
in his *Εἰς πρώτον Εὐκλείδου Στοιχείων.*

In Section 2.3 we discussed the Erdős-Ko-Rado sets on polar spaces. The largest Erdős-Ko-Rado sets of generators are classified, except for $\mathcal{H}(4n+1,q^2)$, $n \geq 2$, and we know a complete classification of the Erdős-Ko-Rado sets of lines. In Chapter 3 we investigated the largest Erdős-Ko-Rado sets of planes, also for polar spaces. It is clear that much more is known for projective spaces. For example nothing is known about EKR(k) sets in a polar space of rank d, $2 < k < d-1$. Also, a Hilton-Milner type result (see Theorem 2.1.3 and Theorem 2.2.4), classifying the second largest example of Erdős-Ko-Rado sets of generators, is not known.

In this chapter, we will present a Hilton-Milner type result for the Erdős-Ko-Rado sets of generators on one type of polar spaces, namely the hyperbolic quadrics $\mathcal{Q}^+(4n+1,q)$. We recall the theorems we already presented on these Erdős-Ko-Rado sets of generators. The following result was already mentioned in Table 2.1.

Theorem 4.0.1 ([104, Theorem 9 and Theorem 16]). *If $\mathcal{S}$ is an Erdős-Ko-Rado set of generators of the hyperbolic quadric $\mathcal{Q}^+(4n+1,q)$, then $|\mathcal{S}| \leq \prod_{i=1}^{2n}(q^i+1)$. Furthermore, if $|\mathcal{S}| = \prod_{i=1}^{2n}(q^i+1)$, then $\mathcal{S}$ is the set of all generators contained in one class.*

Note that for the hyperbolic quadric $\mathcal{Q}^+(5,q)$, we have given a complete classification of its Erdős-Ko-Rado sets of generators in Theorem 3.3.7.

The main theorem of this chapter is Theorem 4.2.7. Section 4.2 is devoted to its proof. In Section 4.1 the preliminary counting results are proved. Finally, in Section 4.3 some other examples of Erdős-Ko-Rado sets of generators of $\mathcal{Q}^+(4n+1,q)$ are given. Their existence indicates that a further classification will not be trivial.

The content of this chapter is based on [39].

4.1 Counting skew generators

We recall two counting results, one about subspaces and one about generators.

Theorem 4.1.1 ([111, Section 170]). *The number of j-spaces, skew to a fixed k-space, in* $\mathrm{PG}(n,q)$ *equals* $q^{(k+1)(j+1)}\begin{bmatrix} n-k \\ j+1 \end{bmatrix}_q$.

Theorem 4.1.2 ([90, Corollary 5]). *Let π_1 and π_2 be two generators of the hyperbolic quadric $\mathcal{Q}^+(2m+1,q)$ meeting in a j-dimensional space. The number of generators skew to both π_1 and π_2 equals*

$$b_j^m = \begin{cases} q^{2\binom{(m+j)/2+1}{2}-\binom{j+1}{2}} \prod_{i=1}^{(m-j)/2}(q^{2i-1}-1) & m \equiv j \pmod 2 \\ 0 & m \equiv j+1 \pmod 2 \end{cases}.$$

Now, we present a new counting result.

Lemma 4.1.3. *Let $\mathcal{Q}^+(4n+1,q)$ be a hyperbolic quadric and let π_1 and π_2 be two generators of the same class on $\mathcal{Q}^+(4n+1,q)$ meeting in a j-dimensional space, $0 \le j \le 2n$ and j even. The number of generators meeting π_1, but not π_2, equals*

$$v_j^n = \sum_{i=\frac{j}{2}}^{n-1} q^{(2n-2i)(j+1)} \begin{bmatrix} 2n-j \\ 2n-2i \end{bmatrix}_q b_j^{2i}.$$

Proof. All generators belonging to the same class as π_1 and π_2, meet both, hence cannot meet precisely one of them. Let π be a generator of the other class that meets π_1 and misses π_2. The intersection $\tau = \pi_1 \cap \pi$ is a $(2n-2i-1)$-space, for some i fulfilling $\frac{j}{2} \le i \le n-1$. Let $\overline{\tau}$ be the tangent space in τ to $\mathcal{Q}^+(4n+1,q)$; it is $(2n+2i+1)$-dimensional. The tangent space $\overline{\tau}$ contains π_1 and meets π_2 in a $(2i)$-space through $\pi_1 \cap \pi_2$. The intersection $\overline{\tau} \cap \mathcal{Q}^+(4n+1,q)$ is a cone with vertex τ and base a hyperbolic quadric $\mathcal{Q}^+(4i+1,q)$. We can choose the ambient space σ of this base $\mathcal{Q}^+(4i+1,q)$ to contain $\overline{\tau} \cap \pi_2$.

Any generator through τ now corresponds to a generator of this base quadric $\mathcal{Q}^+(4i+1,q)$. Moreover, a generator through τ meeting π_1 in τ and disjoint to π_2 corresponds to a generator of $\mathcal{Q}^+(4i+1,q)$ skew to both $\overline{\tau} \cap \pi_2 = \sigma \cap \pi_2$ and $\sigma \cap \pi_1$, which both are generators of the base quadric. Since $(\sigma \cap \pi_1) \cap (\sigma \cap \pi_2) = \pi_1 \cap \pi_2$, the number of such generators equals b_j^{2i}.

So, the total number of generators meeting π_1, but not π_2, equals

$$\sum_{i=\frac{j}{2}}^{n-1} q^{(2n-2i)(j+1)} \begin{bmatrix} 2n-j \\ 2n-2i \end{bmatrix}_q b_j^{2i} .$$

Hereby we used the result from Theorem 4.1.1 to count the number of $(2n-2i-1)$-spaces in π_1 that are skew to $\pi_1 \cap \pi_2$. $\square$

Corollary 4.1.4. *Let $\mathcal{Q}^+(4n+1,q)$ be a hyperbolic quadric and let π_1 and π_2 be two generators of the same class on $\mathcal{Q}^+(4n+1,q)$ meeting in a $2(n-t)$-dimensional space, $0 \leq t \leq n$. The number of generators not meeting both π_1 and π_2 equals*

$$W_t^n(q) = q^{2n^2+n-t^2} \left(\prod_{k=1}^{t} (q^{2k-1}-1) + 2 \sum_{i=1}^{t} \begin{bmatrix} 2t \\ 2i \end{bmatrix}_q q^{i^2+i-2it} \prod_{k=1}^{t-i} (q^{2k-1}-1) \right) .$$

Proof. Using the notations from Theorem 4.1.2 and Lemma 4.1.3, we find that the number of generators not meeting both π_1 and π_2 equals $b_{2(n-t)}^{2n} + 2v_{2(n-t)}^n$. Using the results from Theorem 4.1.2 and Lemma 4.1.3, we find that

$$\begin{aligned} W_t^n(q) &= b_{2(n-t)}^{2n} + 2v_{2(n-t)}^n \\ &= q^{2n^2+n-t^2} \prod_{k=1}^{t} (q^{2k-1}-1) + 2 \sum_{i=n-t}^{n-1} q^{(2n-2i)(2n-2t+1)} \begin{bmatrix} 2t \\ 2n-2i \end{bmatrix}_q b_{2(n-t)}^{2i} \\ &= q^{2n^2+n-t^2} \prod_{k=1}^{t} (q^{2k-1}-1) + 2 \sum_{i=1}^{t} q^{2i(2n-2t+1)} \begin{bmatrix} 2t \\ 2i \end{bmatrix}_q b_{2(n-t)}^{2(n-i)} \\ &= q^{2n^2+n-t^2} \prod_{k=1}^{t} (q^{2k-1}-1) \\ &\quad + 2 \sum_{i=1}^{t} q^{2i(2n-2t+1)} \begin{bmatrix} 2t \\ 2i \end{bmatrix}_q q^{2(n-i)^2+(n-i)-(t-i)^2} \prod_{k=1}^{t-i} (q^{2k-1}-1) \\ &= q^{2n^2+n-t^2} \left(\prod_{k=1}^{t} (q^{2k-1}-1) + 2 \sum_{i=1}^{t} \begin{bmatrix} 2t \\ 2i \end{bmatrix}_q q^{i^2+i-2it} \prod_{k=1}^{t-i} (q^{2k-1}-1) \right) . \square \end{aligned}$$

4.2 Classification of the second example

In Table 2.1 and Theorem 4.0.1 we already introduced the largest Erdős-Ko-Rado set of generators on a hyperbolic quadric $\mathcal{Q}^+(4n+1,q)$, namely the set of all generators of one class. Now we present another Erdős-Ko-Rado set.

Example 4.2.1. Let π be a generator of the hyperbolic quadric $\mathcal{Q}^+(4n+1,q)$ and let $\mathcal{S}$ be the set containing π and all generators of the other class ($\mathcal{G}$) meeting π. All elements of $\mathcal{S} \setminus \{\pi\}$ meet π and since all generators of the same class have a non-trivial intersection on this hyperbolic quadric, they also meet each other. Hence, $\mathcal{S}$ is an Erdős-Ko-Rado set. Obviously, none of the generators in $\mathcal{G} \setminus \mathcal{S}$ extends $\mathcal{S}$ to a larger Erdős-Ko-Rado set. Also, for every generator π' in the same class of π we can find a generator in $\mathcal{S}$ not meeting π'. This can be seen in different ways, e.g. as a consequence of Theorem 4.1.2. Consequently, this Erdős-Ko-Rado set is maximal.

The number of generators in $\mathcal{S}$ equals $(|\mathcal{G}|-b^{2n}_{2n})+1 = \prod_{i=1}^{2n}(q^i+1)-q^{2n^2+n}+1$.

Lemma 4.2.2. *Let $\mathcal{S}$ be a maximal Erdős-Ko-Rado set of generators of a hyperbolic quadric $\mathcal{Q}^+(4n+1,q)$. If $\mathcal{S}$ is not the set of all generators of one class or an Erdős-Ko-Rado set as described in Example 4.2.1, then it contains at most $2\prod_{k=1}^{2n}(q^k+1) - 2\min\{W^n_t(q) \mid 1 \leq t \leq n\}$ generators.*

Proof. Since $\mathcal{S}$ differs from the Erdős-Ko-Rado set of all generators of one class and from the Erdős-Ko-Rado set described in Example 4.2.1, it contains at least two generators of both classes. Let $\pi_1, \pi_2 \in \mathcal{S}$ be two generators of the same class, whose intersection is $2(n-t)$-dimensional, $t \geq 1$. Then $\mathcal{S}$ contains at most

$$\prod_{k=1}^{2n}(q^k+1) - W^n_t(q)$$

generators of the other class. The statement immediately follows. □

Notation 4.2.3. The function $f_t(q)$, $t \geq 1$, is defined in the following way:

$$f_t(q) = q^{t^2-t}\prod_{k=1}^{t}(q^{2k-1}-1) + 2\sum_{i=0}^{t-1}\begin{bmatrix}2t\\2i\end{bmatrix}_q q^{i^2-i}\prod_{k=1}^{i}(q^{2k-1}-1)\,.$$

Using Corollary 4.1.4, we see that $W^n_t(q) = q^{(n+t)(2n-2t+1)}f_t(q)$.

It should be noted that $f_t(q)$ is independent of n, however closely related to $W_t^n(q)$. We calculate $f_t(q)$ for some small values of t.

$$\begin{aligned}
f_1(q) &= q+1 \\
f_2(q) &= q^6+q^5+q^3-q^2 \\
f_3(q) &= q^{15}+q^{14}+q^{12}-q^{11}+q^{10}-q^9-q^7+q^6
\end{aligned}$$

We prove an inequality between these functions.

Lemma 4.2.4. *For every $t \geq 1$ and $q \geq 2$, the inequality $f_{t+1}(q) > q^{4t+1} f_t(q)$ is valid.*

Proof. We perform the following calculations.

$$\begin{aligned}
f_{t+1}(q) &= q^{(t+1)^2-(t+1)} \prod_{k=1}^{t+1}(q^{2k-1}-1) + 2\sum_{i=0}^{t} \begin{bmatrix} 2t+2 \\ 2i \end{bmatrix}_q q^{i^2-i} \prod_{k=1}^{i}(q^{2k-1}-1) \\
&= q^{2t}(q^{2t+1}-1)\left(q^{t^2-t}\prod_{k=1}^{t}(q^{2k-1}-1)\right) + 2 \\
&\quad + 2\sum_{i=1}^{t} \frac{(q^{2t+2}-1)(q^{2t+1}-1)}{q^{2i}-1} \begin{bmatrix} 2t \\ 2i-2 \end{bmatrix}_q q^{i^2-i} \prod_{k=1}^{i-1}(q^{2k-1}-1) \\
&= q^{2t}(q^{2t+1}-1)\left(q^{t^2-t}\prod_{k=1}^{t}(q^{2k-1}-1)\right) + 2 \\
&\quad + 2\sum_{i=0}^{t-1} \frac{(q^{2t+2}-1)(q^{2t+1}-1)q^{2i}}{q^{2i+2}-1} \begin{bmatrix} 2t \\ 2i \end{bmatrix}_q q^{i^2-i} \prod_{k=1}^{i}(q^{2k-1}-1)\ .
\end{aligned}$$

Note that

$$\frac{(q^{2t+2}-1)(q^{2t+1}-1)q^{2i}}{q^{2i+2}-1} = q^{4t+1} + \frac{q^{4t+1}-q^{2t+2i+2}-q^{2t+2i+1}+q^{2i}}{q^{2i+2}-1} > q^{4t+1}$$

since $i \leq t-1$. Substituting both the equality (for $i=t-1$) and the inequality in the previous calculation, we find

$$\begin{aligned}
f_{t+1}(q) > q^{2t}(q^{2t+1}-1)&\left(q^{t^2-t}\prod_{k=1}^{t}(q^{2k-1}-1)\right)\\
&+2q^{4t+1}\sum_{i=0}^{t-1}\begin{bmatrix}2t\\2i\end{bmatrix}_q q^{i^2-i}\prod_{k=1}^{i}(q^{2k-1}-1)\\
&+2\frac{q^{4t+1}-q^{4t}-q^{4t-1}+q^{2t-2}}{q^{2t}-1}\begin{bmatrix}2t\\2t-2\end{bmatrix}_q q^{(t-1)(t-2)}\prod_{k=1}^{t-1}(q^{2k-1}-1)\\
=q^{4t+1}f_t(q)-q^{t^2+t}&\prod_{k=1}^{t}(q^{2k-1}-1)\\
&+2\frac{q^{4t+1}-q^{4t}-q^{4t-1}+q^{2t-2}}{(q^2-1)(q-1)}q^{(t-1)(t-2)}\prod_{k=1}^{t}(q^{2k-1}-1)\\
>q^{4t+1}f_t(q)&\\
&+\left(2(q^{4t-2}-q^{4t-4}-q^{4t-5})-q^{4t-2}\right)q^{(t-1)(t-2)}\prod_{k=1}^{t}(q^{2k-1}-1)\\
\geq q^{4t+1}f_t(q)\ .&
\end{aligned}$$

Hereby we used that $q^{4t-2}-2q^{4t-4}-2q^{4t-5}\geq 0$ for all $q\geq 2$. □

We present a strong inequality that we will also use in Section 7.2.

Lemma 4.2.5. *Let $s,t,q\in\mathbb{N}$ be such that $s\leq t$ and $q\geq 3$. If $(s,q)\neq(0,3)$, then*

$$\prod_{i=s}^{t}(q^i+1)\leq (q^s+2)q^{\binom{t+1}{2}-\binom{s+1}{2}}-q^{\binom{t}{2}-\binom{s}{2}}\ .$$

Proof. We prove this result by using induction on t for a fixed values of s and q. If $t=s$, then $q^s+1\leq(q^s+2)-q^0$; so the induction base is proved. To prove the validity of the induction step, we assume the theorem to be true for t and we prove it to be true for $t+1$. In the first step we use the induction hypothesis.

$$\begin{aligned}\prod_{i=s}^{t+1}(q^i+1) &\leq (q^{t+1}+1)\left((q^s+2)q^{\binom{t+1}{2}-\binom{s+1}{2}} - q^{\binom{t}{2}-\binom{s}{2}}\right) \\ &= (q^s+2)q^{\binom{t+2}{2}-\binom{s+1}{2}} - q^{\binom{t}{2}-\binom{s}{2}}(q^{t+1}-q^t-2q^{t-s}+1) \\ &\leq (q^s+2)q^{\binom{t+2}{2}-\binom{s+1}{2}} - q^{\binom{t+1}{2}-\binom{s}{2}}\end{aligned}$$

In the final step we used that $q^{t+1}-q^t-2q^{t-s}+1 \geq q^t$ since $q \geq 3$ and $(s,q) \neq (0,3)$. □

Note that the inequality is indeed not valid if $(s,q)=(0,3)$ and $t \geq 2$.

Corollary 4.2.6. *If $q \geq 3$ and $n \geq 1$, then $q^{2n^2+n}+2q^{2n^2+n-1}+1 > \prod_{k=1}^{2n}(q^k+1)$.*

Proof. We apply Lemma 4.2.5 for $(s,t)=(1,2n)$ and we find

$$\prod_{i=1}^{2n}(q^i+1) \leq (q+2)q^{\binom{2n+1}{2}-1} - q^{\binom{2n}{2}} < (q+2)q^{2n^2+n-1}+1 \ .$$

□

Theorem 4.2.7. *The two largest types of maximal Erdős-Ko-Rado sets of generators of a hyperbolic quadric $\mathcal{Q}^+(4n+1,q)$, $n \geq 1$ and $q \geq 3$, are the Erdős-Ko-Rado set of all generators of one class and the Erdős-Ko-Rado set described in Example 4.2.1.*

Proof. Let $\mathcal{S}$ be a maximal Erdős-Ko-Rado set of generators different from the set of generators of one class and different from the Erdős-Ko-Rado set described in Example 4.2.1. By Lemma 4.2.2 we know that $|\mathcal{S}| \leq 2\prod_{k=1}^{2n}(q^k+1) - 2\min\{W_t^n(q) \mid 1 \leq t \leq n\}$. Using Lemma 4.2.4, we find that

$$W_{t+1}^n(q) = q^{(n+t+1)(2n-2t-1)}f_{t+1}(q) > q^{(n+t)(2n-2t+1)}f_t(q) = W_t^n(q) \ ,$$

for all $1 \leq t \leq n$. Hence,

$$\min\{W_t^n(q) \mid 1 \leq t \leq n\} = W_1^n(q) = (q+1)q^{2n^2+n-1} \ .$$

We mentioned already that the Erdős-Ko-Rado set containing all generators of one class, is larger than the one described in Example 4.2.1, so we compare the upper bound on $|\mathcal{S}|$ with the size of the Erdős-Ko-Rado set described in Example 4.2.1. The corresponding inequality

$$\prod_{i=1}^{2n}(q^i+1) - q^{2n^2+n} + 1 > 2\prod_{k=1}^{2n}(q^k+1) - 2(q+1)q^{2n^2+n-1}$$

is equivalent to

$$q^{2n^2+n} + 2q^{2n^2+n-1} + 1 > \prod_{k=1}^{2n}(q^k+1) \ .$$

By Lemma 4.2.6 we know that this inequality is valid if $q \geq 3$. □

Note that Theorem 3.3.7 implies this result for $n = 1$.

4.3 Other examples of large Erdős-Ko-Rado sets

Next to the two examples of maximal Erdős-Ko-Rado sets of generators, which we mentioned above and which are proved in Theorem 4.2.7 to be the largest ones, we also know the *point-pencil*. This is the set of all generators through a fixed point. For many geometries, the point-pencil is the largest Erdős-Ko-Rado set. By Theorem 4.0.1 we know that this is not true for hyperbolic quadrics $\mathcal{Q}^+(4n+1,q)$. In this case the point-pencil contains

$$\prod_{i=0}^{2n-1}(q^i+1) = 2\prod_{i=1}^{2n-1}(q^i+1) \in \Theta(q^{2n^2-n})$$

generators. Recall that the Erdős-Ko-Rado set of all generators of one class contains $\prod_{i=1}^{2n}(q^i+1) \in \Theta(q^{2n^2+n})$ generators and that the Erdős-Ko-Rado set described in Example 4.2.1 contains $\prod_{i=1}^{2n}(q^i+1) - q^{2n^2+n} + 1 \in \Theta(q^{2n^2+n-1})$ generators. So, the point-pencil is much smaller in this case.

In this section we will present some more Erdős-Ko-Rado sets of generators of $\mathcal{Q}^+(4n+1,q)$ whose size is larger than the size of a point-pencil. First we give a counting result.

Lemma 4.3.1. *Let $m \geq 0$ and $k \geq -1$ be two integers such that $k < m$. Let Ω be one of the two classes of generators of a hyperbolic quadric $\mathcal{Q}^+(2m+1,q)$.*

The number of generators in Ω skew to a fixed k-space on the quadric equals

$$\frac{1}{2}\left(\prod_{i=0}^{m-k-1}(q^i+1)\right)q^{\frac{1}{2}(k+1)(2m-k)} =: w_{m,k}\ .$$

The empty space is considered to have dimension -1.

Proof. Let π be a k-dimensional subspace of $\mathcal{Q}^+(2m+1,q)$. We prove this lemma by using induction on k. If $k=-1$, then π is the empty space. The number of generators of Ω skew to the empty space is the total number of generators of Ω, which equals $w_{m,-1}$.

Now, we assume that the lemma is proved for all subspaces of dimension at most $k-1$; we will prove it for the k-dimensional space π. The subspace π contains $\begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_q$ subspaces of dimension i, $0 \leq i \leq k$. Let σ be such an i-space and let $T_\sigma(\mathcal{Q}^+(2m+1,q))$ be its tangent space to $\mathcal{Q}^+(2m+1,q)$. We know that $\mathcal{Q}^+(2m+1,q) \cap T_\sigma(\mathcal{Q}^+(2m+1,q))$ is a cone with vertex σ and base a hyperbolic quadric $\mathcal{Q}^+(2m-2i-1,q)$. The k-space π corresponds to a $(k-i-1)$-space in this base. Arguing as in the proof of Lemma 4.1.3, the number of generators of Ω meeting π in precisely σ equals $w_{m-i-1,k-i-1}$. Hereby, we note that the generators of Ω through σ correspond to one of the two classes of generators of the base $\mathcal{Q}^+(2m-2i-1,q)$.

So, the total number of generators of Ω skew to π, is independent of the choice for π, and equals

$$\begin{aligned}
w_{m,k} &= \frac{1}{2}\prod_{j=0}^{m}(q^j+1) - \sum_{i=0}^{k}\begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_q w_{m-i-1,k-i-1} \\
&= \frac{1}{2}\prod_{j=0}^{m}(q^j+1) - \frac{1}{2}\sum_{i=0}^{k}\begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_q \left(\prod_{j=0}^{m-k-1}(q^j+1)\right)q^{\frac{1}{2}(k-i)(2m-k-i-1)} \\
&= \frac{1}{2}\left(\prod_{j=0}^{m-k-1}(q^j+1)\right)\cdot \\
&\quad \left(\prod_{j=m-k}^{m}(q^j+1) - \sum_{i=0}^{k}\begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_q q^{\frac{1}{2}(k-i)(2m-k-i-1)}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\prod_{j=0}^{m-k-1}(q^j+1)\right)\cdot \\
&\quad \left(\prod_{j=m-k}^{m}(q^j+1) - \sum_{i=1}^{k+1}\begin{bmatrix}k+1\\ i\end{bmatrix}_q q^{\frac{1}{2}(k-i+1)(2m-k-i)}\right) \\
&= \frac{1}{2}\left(\prod_{j=0}^{m-k-1}(q^j+1)\right)\cdot \\
&\quad \left(\prod_{j=m-k}^{m}(q^j+1) - q^{\frac{1}{2}(2m-k)(k+1)}\sum_{i=1}^{k+1}\begin{bmatrix}k+1\\ i\end{bmatrix}_q q^{\binom{i}{2}}q^{-mi}\right) \\
&= \frac{1}{2}\left(\prod_{j=0}^{m-k-1}(q^j+1)\right)\cdot \\
&\quad \left(\prod_{j=m-k}^{m}(q^j+1) - q^{\frac{1}{2}(2m-k)(k+1)}\left(\prod_{j=0}^{k}(q^{j-m}+1)-1\right)\right) \\
&= \frac{1}{2}\left(\prod_{j=0}^{m-k-1}(q^j+1)\right)q^{\frac{1}{2}(k+1)(2m-k)}\ .
\end{aligned}
$$

In the penultimate transition we used the q-*binomial theorem*

$$\prod_{l=0}^{n-1}(1+q^l t) = \sum_{l=0}^{n} q^{\binom{l}{2}}\begin{bmatrix}n\\ l\end{bmatrix}_q t^l\ .$$

This calculation finishes the induction step. □

Remark 4.3.2. In the previous lemma, the case $m = k$ was not covered; in that case we count the number of generators of a fixed class skew to a given generator π. We already know that this number will be dependent on the class of π and the parity of m. Using the observation made in Section 1.6 and Theorem 4.1.2, we can state the following result. If m is even, then no generators of the class of π are skew to π and $b_m^m = q^{\binom{m+1}{2}}$ generators of the other class are skew to π. If m is odd, then $b_m^m = q^{\binom{m+1}{2}}$ generators of the class of π are skew to π and no generators of the other class are skew to π.

It is an immediate consequence of the previous lemma that the total number of generators skew to a fixed k-space on a hyperbolic quadric $\mathcal{Q}^+(2m+1,q)$, $k<m$, equals

$$\left(\prod_{i=0}^{m-k-1}(q^i+1)\right)q^{\frac{1}{2}(k+1)(2m-k)}=2w_{m,k}\ .$$

We now introduce some new examples of large maximal Erdős-Ko-Rado sets of the hyperbolic quadric $\mathcal{Q}^+(4n+1,q)$.

Example 4.3.3. Consider the hyperbolic quadric $\mathcal{Q}^+(4n+1,q)$ and let τ be a fixed k-space on it, $0\leq k\leq 2n$. Denote the two classes of generators by Ω_1 and Ω_2, such that $\tau\in\Omega_1$ if $k=2n$. Let $\mathcal{S}$ be the union of the set of generators of Ω_1 meeting τ in a subspace of dimension at least j, $0\leq j\leq k$, and the set of generators of Ω_2 meeting τ in a subspace of dimension at least $k-j$. It is immediate that the elements of $\mathcal{S}$ pairwise intersect. Consequently, $\mathcal{S}$ is an Erdős-Ko-Rado set. We denote this type of Erdős-Ko-Rado sets by $I_{k,j}$. Note that not all these types are different. If $k<2n$, then $I_{k,j}$ and $I_{k,k-j}$ describe PGL-equivalent sets of generators. Also $I_{2n-1,2j-1}$, $I_{2n,2j-1}$ and $I_{2n,2j}$ describe PGL-equivalent sets of generators, $1\leq j\leq n$.

We show that an Erdős-Ko-Rado set $\mathcal{S}$ of type $I_{k,j}$ is maximal. Assume that we can find a generator π in $\Omega_1\setminus\mathcal{S}$ which meets all generators of $\mathcal{S}$. Since $\pi\notin\mathcal{S}$, we know that $\dim(\pi\cap\tau)<j$. So, we can find a $(k-j)$-space τ' in τ disjoint to $\pi\cap\tau$. We know that all generators of Ω_2 containing τ' belong to $\mathcal{S}$. Let $T_{\tau'}$ be the tangent space in τ' to $\mathcal{Q}^+(4n+1,q)$. It is $(4n-k+j)$-dimensional and meets π in a $(2n-k+j-1)$-space disjoint to τ'. The intersection $\mathcal{Q}^+(4n+1,q)\cap T_{\tau'}$ is a cone with vertex τ' and base a hyperbolic quadric $\mathcal{Q}_1\cong\mathcal{Q}^+(2(2n-k+j-1)+1,q)$. We can choose this basis such that it contains $\pi'=T_{\tau'}\cap\pi$. Moreover $T_{\tau'}\cap\pi$ is a generator of $\mathcal{Q}_1$. The set of generators of Ω_2 through τ', all in $\mathcal{S}$, correspond to the set of generators of one class of $\mathcal{Q}_1$. We denote this class by Ω_2'. If $k-j$ is even, then π' also belongs to Ω_2'. However, $2n-k+j-1$ is odd and by an observation made in Remark 4.3.2, we know that we can find a generator $\sigma\in\Omega_2'$ skew to π'. Then $\langle\pi',\tau'\rangle$ is a generator in $\mathcal{S}$ skew to π. If $k-j$ is odd, then π' belongs to Ω_1', the other class of generators of $\mathcal{Q}_1$. In this case, $2n-k+j-1$ is even and by an observation made in Remark 4.3.2, we know that we can find a generator $\sigma\in\Omega_2'$ skew to π'. Then $\langle\pi',\tau'\rangle$ is a generator in $\mathcal{S}$ skew to π. The arguments for $\pi\in\Omega_2$ are analogous.

Now, we count the number of generators in an Erdős-Ko-Rado set $\mathcal{S}$ of type $I_{k,j}$, $k < 2n$. We use the result from Lemma 4.3.1.

$$\begin{aligned}
|\mathcal{S}| &= \sum_{i=j}^{k} \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_q w_{2n-i-1,k-i-1} + \sum_{i=k-j}^{k} \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_q w_{2n-i-1,k-i-1} \\
&= \sum_{i=j}^{k} \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_q \frac{1}{2} \left(\prod_{p=0}^{2n-k-1} (q^p+1) \right) q^{\frac{1}{2}(k-i)(4n-k-i-1)} \\
&\qquad + \sum_{i=k-j}^{k} \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_q \frac{1}{2} \left(\prod_{p=0}^{2n-k-1} (q^p+1) \right) q^{\frac{1}{2}(k-i)(4n-k-i-1)} \\
&= \frac{1}{2} \left(\prod_{p=0}^{2n-k-1} (q^p+1) \right) \cdot \\
&\qquad \left[\sum_{i=j}^{k} \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_q q^{\frac{1}{2}(k-i)(4n-k-i-1)} + \sum_{i=k-j}^{k} \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_q q^{\frac{1}{2}(k-i)(4n-k-i-1)} \right] .
\end{aligned}$$

Using this result, we can see that an Erdős-Ko-Rado set of type $I_{k,j+1}$ is larger than an Erdős-Ko-Rado set of type $I_{k,j}$ if and only if $2j+1-k > 0$, $k < 2n$. Using this and the above mentioned equivalence between $I_{k,j}$ and $I_{k,k-j}$, we find that the largest among these Erdős-Ko-Rado sets are the ones of type $I_{k,k}$, which are also the ones of type $I_{k,0}$. Those contain

$$\prod_{i=1}^{2n}(q^i+1) - \frac{1}{2}(q^{\frac{1}{2}(k+1)(4n-k)} - 1) \prod_{i=0}^{2n-k-1} (q^i+1) \in \Theta(q^{2n^2-n+k})$$

generators. In this computation we used the q-binomial theorem. Since the Erdős-Ko-Rado sets of type $I_{2n,2j-1}$ and $I_{2n,2j}$ are equal to the Erdős-Ko-Rado sets $I_{2n-1,2j-1}$, $1 \leq j \leq n$, we can use the above formulas to compute their number of elements as well.

So, the only type $I_{k,j}$ of Erdős-Ko-Rado sets whose size has not been computed above is $I_{2n,0}$. However, an Erdős-Ko-Rado set of type $I_{2n,0}$ is the set of all generators of one class, Ω_1 in the above notation. Furthermore, the Erdős-Ko-Rado sets of type $I_{2n,2n}$ are the ones that are described in Example 4.1.3 and the Erdős-Ko-Rado sets of type $I_{0,0}$ are the point-pencils.

In the previous example we have introduced several types of Erdős-Ko-Rado sets, $I_{k,k}$, whose size is larger than the size of a point-pencil. These are however not the only ones. We shall give two more examples.

Example 4.3.4. Again we denote the two classes of generators on the hyperbolic quadric $\mathcal{Q}^+(4n+1,q)$ by Ω_1 and Ω_2. Let π be a generator of class Ω_1 and let τ be a fixed k-space in π, $0 \leq k \leq 2n-2$. Let $\mathcal{S}$ be the union of the set of generators of Ω_1 that are not skew to τ or that meet π in a subspace of dimension $i \geq 2$, and the set of generators of Ω_2 through τ meeting π in a subspace of dimension $2n-1$. It is immediate that the generators in $\mathcal{S}$ pairwise intersect, and hence $\mathcal{S}$ is an Erdős-Ko-Rado set. We denote this type of Erdős-Ko-Rado sets by II_k. Its maximality can be proved by arguments similar to the arguments in the proof of the maximality in Example 4.3.3.

In the above definition we imposed $k \leq 2n-2$. For $k = 2n-1$ this definition gives rise to the Erdős-Ko-Rado set described in Example 4.1.3; for $k = 2n$ this definition gives rise to the Erdős-Ko-Rado set consisting of all generators of one class.

We now count the number of generators in an Erdős-Ko-Rado set $\mathcal{S}$ of type II_k:

$$
\begin{aligned}
|\mathcal{S}| &= \sum_{i=1}^{n} \begin{bmatrix} 2n+1 \\ 2i+1 \end{bmatrix}_q b_{2n-2i-1}^{2n-2i-1} + \begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q b_{2n-1}^{2n-1} + \begin{bmatrix} 2n-k \\ 1 \end{bmatrix}_q \\
&= \sum_{i=1}^{n} \begin{bmatrix} 2n+1 \\ 2i+1 \end{bmatrix}_q q^{\binom{2(n-i)}{2}} + \begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q q^{2n^2-n} + \begin{bmatrix} 2n-k \\ 1 \end{bmatrix}_q .
\end{aligned}
$$

It can be calculated that an Erdős-Ko-Rado set of type II_k contains more elements than an Erdős-Ko-Rado set $\mathcal{S}$ of type $II_{k'}$ if and only if $k > k'$. Therefore, we calculate the size of an Erdős-Ko-Rado set $\mathcal{S}$ of type II_{2n-2}:

$$
\begin{aligned}
|\mathcal{S}| &= \sum_{i=1}^{n} \begin{bmatrix} 2n+1 \\ 2i+1 \end{bmatrix}_q q^{\binom{2(n-i)}{2}} + \begin{bmatrix} 2n-1 \\ 1 \end{bmatrix}_q q^{2n^2-n} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \\
&= \sum_{i=1}^{n-1} q^{\binom{2(n-i)}{2}} \left(q^{2(n-i)} \begin{bmatrix} 2n \\ 2n-2i \end{bmatrix}_q + \begin{bmatrix} 2n \\ 2n-2i-1 \end{bmatrix}_q \right) + 1 \\
&\quad + \frac{q^{2n-1}-1}{q-1} q^{2n^2-n} + q + 1
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=0}^{2n-2} \begin{bmatrix} 2n \\ j \end{bmatrix}_q q^{\binom{j}{2}} q^j + \frac{q^{2n-1}-1}{q-1} q^{2n^2-n} + q + 1 \\
&= \sum_{j=0}^{2n} \begin{bmatrix} 2n \\ j \end{bmatrix}_q q^{\binom{j}{2}} q^j - q^{2n^2+n} - \frac{q^{2n}-1}{q-1} q^{2n^2-n} + \frac{q^{2n-1}-1}{q-1} q^{2n^2-n} + q + 1 \\
&= \prod_{i=1}^{2n} (q^i + 1) - q^{2n^2+n} - q^{2n^2+n-1} + q + 1 \in \Theta(q^{2n^2+n-2}) \ .
\end{aligned}
$$

Analogously, the size of an Erdős-Ko-Rado set $\mathcal{S}$ of type II_0 can be calculated. We find

$$
|\mathcal{S}| = \prod_{i=1}^{2n} (q^i + 1) - \frac{(q^{2n}-1)(q^{2n^2-n+1}-1)}{q-1} \in \Theta(q^{2n^2+n-3}) \ .
$$

Example 4.3.5. Before introducing the example, we recall the *triality* map for $\mathcal{Q}^+(7,q)$, which has its origins in [120]; we follow the approach from [96]. Denote the two classes of generators of $\mathcal{Q}^+(7,q)$ by Ω_1' and Ω_2'. Let $\mathcal{P}$ be the set of points on $\mathcal{Q}^+(7,q)$ and let $\mathcal{L}$ be the set of lines on $\mathcal{Q}^+(7,q)$. Note that $|\mathcal{P}| = |\Omega_1'| = |\Omega_2'|$. A D_4-geometry[1] $\mathcal{G}$ can be constructed as follows. The elements of $\mathcal{P}$ are the 0-points, the elements of Ω_i are the i-points, $i = 1, 2$, and the elements of $\mathcal{L}$ are the lines. Incidence is defined by symmetrized containment, except for 1-points and 2-points; we define a 1-point and a 2-point to be incident if they meet in a plane of $\mathcal{Q}^+(7,q)$. Every permutation of $\{\mathcal{P}, \Omega_1', \Omega_2'\}$ defines a geometry isomorphic to $\mathcal{G}$. A triality of $\mathcal{G}$ is a map t

$$
t : \mathcal{L} \to \mathcal{L},\ \mathcal{P} \to \Omega_1',\ \Omega_1' \to \Omega_2',\ \Omega_2' \to \mathcal{P}
$$

preserving the incidence in $\mathcal{G}$ and such that t^3 is the identity relation. Such maps are known to exist and are used to construct generalised hexagons.

We use the triality to prove a result about the generators of $\mathcal{Q}^+(7,q)$. Let π_1 and π_2 be two disjoint generators of Ω_2' and let $\mathcal{S}'$ be the set of all generators of Ω_2' meeting both in a line. Let $\mathcal{S}''$ be the set of all generators of Ω_2' having a non-empty intersection with all elements of $\mathcal{S}'$. We will show that $\mathcal{S}'' = \{\pi_1, \pi_2\}$. It is clear that π_1^t and π_2^t are two points not on a line of $\mathcal{L}$. The set $\mathcal{S}'^t$ contains all points of $\mathcal{Q}^+(7,q)$ that are collinear with both π_1^t and π_2^t. Therefore, $\mathcal{S}'^t$ is the

[1] We did not introduce buildings in this thesis. Here, we point this out for readers familiar with building theory. For an extensive introduction to this theory, we refer to [28, Chapters 11 and 12].

set of points on a hyperbolic quadric $\mathcal{Q}^+(5,q)$ inside $\mathcal{Q}^+(7,q)$. The only points collinear with all points of $\mathcal{S}'^t$ are the points π_1^t and π_2^t themselves. Hence, $\mathcal{S}''^t = \{\pi_1^t, \pi_2^t\}$. The statement follows. Note that we can replace Ω_2' by Ω_1' in the statement: replacing t by t^2, this proof continues.

Now, we consider the hyperbolic quadric $\mathcal{Q}^+(4n+1,q)$, $n \geq 2$. Denote the two classes of generators by Ω_1 and Ω_2. Let π and π' be two generators of Ω_1 meeting in a $(2n-4)$-space τ. Let $\mathcal{S}$ be the set containing π, π' and all generators of Ω_2 meeting π and π'. It is clear that $\mathcal{S}$ is an Erdős-Ko-Rado set. We denote this type of Erdős-Ko-Rado sets by III.

We prove that an Erdős-Ko-Rado set $\mathcal{S}$ of type III is maximal. Here we will need that τ is a $(2n-4)$-space. It is obvious that no generators of Ω_2 that are not in $\mathcal{S}$, extend $\mathcal{S}$. Let π'' be a generator of Ω_1 extending $\mathcal{S}$. Since $\mathcal{S}$ contains all generators of Ω_2 having a non-empty intersection with τ, it can be argued that π'' has to contain τ, similar to the argument about the maximality in Example 4.3.3. Now we consider the tangent space T_τ in τ to $\mathcal{Q}^+(4n+1,q)$. The intersection $T_\tau \cap \mathcal{Q}^+(4n+1,q)$ is a cone with vertex τ and base a hyperbolic quadric $\mathcal{Q}^+(7,q)$. The generators π, π' and π'' intersect this base in $\overline{\pi}$, $\overline{\pi}'$ and $\overline{\pi}''$, respectively. These are generators of the hyperbolic quadric $\mathcal{Q}^+(7,q)$ of the same class, say Ω_2'. Let σ be a generator of $\mathcal{Q}^+(7,q)$ of class Ω_2', meeting both $\overline{\pi}$ and $\overline{\pi}'$ in a line. We know that there are b_{2n-4}^{2n-4} generators of $\mathcal{Q}^+(4n+1,q)$ through σ disjoint to τ, necessarily all of class Ω_2. Hence $\overline{\pi}''$ has to meet all generators of $\mathcal{Q}^+(7,q)$ of class Ω_2', meeting both $\overline{\pi}$ and $\overline{\pi}'$ in a line. By the above observation on $\mathcal{Q}^+(7,q)$, we know that $\overline{\pi}'' \in \{\overline{\pi}, \overline{\pi}'\}$. Hence, $\mathcal{S}$ is maximal.

We count the number of generators in an Erdős-Ko-Rado set $\mathcal{S}$ of type III.

$$\begin{aligned}
|\mathcal{S}| &= 2 + \left(\prod_{i=1}^{2n}(q^i+1) - w_{2n,2n-4}\right) + \begin{bmatrix}4\\2\end{bmatrix}_q q^{4(2n-3)} b_{2n-4}^{2n-4} \\
&= 2 + \prod_{i=1}^{2n}(q^i+1) - \left(\prod_{i=1}^{3}(q^i+1)\right) q^{(n+2)(2n-3)} \\
&\quad + (q^2+1)(q^2+q+1)q^{(n+2)(2n-3)} \\
&= \prod_{i=1}^{2n}(q^i+1) - q^{2n^2+n-6}(q^6+q^5+q^3-q^2) + 2 \in \Theta(q^{2n^2+n-2})
\end{aligned}$$

Remark 4.3.6. We already noted that the size of the largest maximal Erdős-

Ko-Rado set is of order $\Theta(q^{2n^2+n})$ and the size of the second largest maximal Erdős-Ko-Rado set is of order $\Theta(q^{2n^2+n-1})$. In the previous Examples we have described three types of maximal Erdős-Ko-Rado sets of the next order $\Theta(q^{2n^2+n-2})$, namely $I_{2n-2,2n-2}$, II_{2n-2} and III. However, it should be noted that the Erdős-Ko-Rado sets of type $I_{2n-2,2n-2}$ and Erdős-Ko-Rado sets of type II_{2n-2} are the same ones. This is an exceptional case; this pattern does not continue for other Erdős-Ko-Rado sets of type $I_{k,k}$ and $II_{k'}$. It can be easily calculated that the Erdős-Ko-Rado sets of type $I_{2n-2,2n-2}$ (II_{2n-2}) are larger than the Erdős-Ko-Rado sets of type III.

Calculations in Example 4.3.3 and Example 4.3.4 show that there are many different Erdős-Ko-Rado sets whose size is larger than the size of the point-pencil. So, a complete classification of all Erdős-Ko-Rado sets whose size is at least the size of a point-pencil is out of sight for the moment.

5

Erdős-Ko-Rado sets in Steiner 2-designs

The longer he lived, the more Tyrion realized
that nothing was simple and little was true.

Tyrion Lannister in
A Song of Ice and Fire, A Clash of Kings, Tyrion IV
by *George R.R. Martin*

In Section 1.2 we already introduced designs, but until now no Erdős-Ko-Rado sets in designs have been mentioned. Following the general approach from Section 2.4, we can define *Erdős-Ko-Rado sets* for designs as sets of blocks pairwise having at least a point in common. As usually, they are called *maximal* if they cannot be extended regarding this condition. As before, a *point-pencil* is the set of all blocks through a fixed point. A point-pencil is a maximal Erdős-Ko-Rado set in a $t-(v,k,\lambda)$ design if $r > k$.

An important result was obtained by Rands. He did not only study Erdős-Ko-Rado sets, but studied sets of blocks pairwise having at least s points in common. Recall the notations λ_i, which were defined in Section 1.2.

Theorem 5.0.1 ([107]). *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a $t-(v,k,\lambda)$ block design and let $\mathcal{S}$ be a subset of $\mathcal{B}$ such that the blocks of $\mathcal{S}$ have pairwise at least s points* *in common,* $0 < s < t \leq k$*.*

- *If* $s < t-1$ *and* $v \geq s + \binom{k}{s}(k-s+1)(k-s)$*, or*
- *if* $s = t-1$ *and* $v \geq s + \binom{k}{s}^2(k-s)$*,*

then $|\mathcal{S}| \leq \lambda_s$ *and equality is obtained if and only if* $\mathcal{S}$ *is the set of blocks through* s *fixed points.*

In this chapter, based on [38], we will improve this result for Erdős-Ko-Rado sets in Steiner 2-designs. For these the above result implies the following corollary. Recall the notation r for the number of blocks through a fixed point of the design. This number is also called the *replication number*. We know that $r = \frac{v-1}{k-1}$ for a Steiner 2-design.

Corollary 5.0.2. *Let* $\mathcal{D}$ *be a* $2-(v,k,1)$ *block design and let* $\mathcal{S}$ *be an Erdős-Ko-Rado set of* $\mathcal{D}$*,* $k \geq 2$*. If* $v \geq 1 + k^2(k-1)$*, then* $|\mathcal{S}| \leq r$ *and* $|\mathcal{S}| = r$ *if and only if* $\mathcal{S}$ *is a point-pencil.*

In the same article ([107]), it is claimed that the bound $v \geq 1+k^2(k-1)$ can be improved to $v > k^3-2k^2+2k$, but there is no proof of this statement. However, it is shown that the bound $v > k^3-2k^2+2k$ is sharp. If $v = k^3-2k^2+2k$ and $k-1$ is a prime power, the $2-(v,k,1)$ design consisting of the points and lines of $\mathrm{PG}(3,k-1)$ contains two different types of Erdős-Ko-Rado sets of size $r = k^2-k+1$: the set of all blocks through a fixed point and the set of blocks arising from the set of lines in a fixed plane.

In Sections 5.3 and 5.4, we will prove the result about the bound $v > k^3 - 2k^2 + 2k$ (see Theorem 5.4.1) and we will also investigate $2-(v,k,1)$ designs with $v < k^3-2k^2+2k$ (see Theorem 5.4.5). It turns out that $v = k^3-2k^2+2k$ is an isolated case. The results are summarized in Theorem 5.3.5 and Corollary 5.4.6. Section 5.2 provides some lemmata needed in these investigations. Section 5.1 deals with some specific Steiner systems. In Section 5.5 we use the arguments from Section 5.2 to obtain a stability result[1] for Erdős-Ko-Rado sets in unitals.

[1]Stability results were introduced in the Preface.

5.1 Some special 2-Steiner systems

We recall that projective and affine planes are the smallest 2-designs (see Remark 1.2.3). Therefore, we look at them in detail.

Remark 5.1.1. In a projective plane, every two blocks have a point in common. Hence, in a projective plane there is only one maximal Erdős-Ko-Rado set, namely the set of all blocks. Recall that we mentioned in the introduction that a point-pencil in a $2-(v,k,1)$ design is only maximal if $r > k$.

For the projective plane $\mathrm{PG}(2,q)$, we already knew this by Remark 2.2.3.

Remark 5.1.2. In an affine plane of order n, the set of blocks can be partitioned in $n+1$ parallel classes of n blocks, such that the blocks in the same class pairwise have no point in common (see Section 1.5). Two blocks of different parallel classes always meet in a point. An Erdős-Ko-Rado set contains necessarily at most one block of each parallel class. A maximal Erdős-Ko-Rado set contains precisely one block of each parallel class. Consequently, every maximal Erdős-Ko-Rado set contains $n+1$ blocks.

It should be noted that not all these maximal Erdős-Ko-Rado sets are isomorphic. Also note that the point-pencil can be described in this way.

Now we turn our attention to $2-(v,k,1)$ designs with a special property.

Definition 5.1.3. The *O'Nan configuration* in a design $\mathcal{D}$ is a set of four blocks, pairwise non-disjoint, such that no three contain a common point.

We will show that we can find a complete classification of the maximal Erdős-Ko-Rado sets in designs not containing an O'Nan configuration. Note that all affine planes of order at least 3 and all projective planes do contain O'Nan configurations.

We already know the point-pencil, a maximal Erdős-Ko-Rado set of size r. We now give an example of a maximal Erdős-Ko-Rado set on a design without an O'Nan configuration.

Example 5.1.4. Let $\mathcal{D}$ be a $2-(v,k,1)$ design without an O'Nan configuration. Let P be a point and let B be a block of $\mathcal{D}$ such that $P \notin B$. Let $\mathcal{S}$ be the union of $\{B\}$ and the set of all blocks through P meeting B. It is obvious that all blocks of $\mathcal{S}$ meet each other, hence that $\mathcal{S}$ is an Erdős-Ko-Rado set. We call it the *triangle*. It contains $k+1$ blocks. We prove that it is maximal.

Let L be a block of $\mathcal{D}$ not in $\mathcal{S}$, meeting all blocks of $\mathcal{S}$. The block L cannot pass through P, hence meets all blocks of $\mathcal{S}$ through P in a different point. Since $L \neq B$, we know $k \geq 3$. Let P' and P'' be two points on $B \setminus \{L \cap B\}$ and let B' and B'' be the blocks of $\mathcal{S}$ through P, meeting B in the points P' and P'', respectively. Then the blocks B, L, B' and B'' determine an O'Nan configuration, a contradiction.

Theorem 5.1.5. *Let $\mathcal{D}$ be a $2-(v,k,1)$ design without an O'Nan configuration and let $\mathcal{S}$ be a maximal Erdős-Ko-Rado set on $\mathcal{D}$. Then, $\mathcal{S}$ is a point-pencil or a triangle.*

Proof. Assume that $\mathcal{S}$ is not a point-pencil; then we can find three blocks in $\mathcal{S}$, say B_1, B_2 and B_3, not through a common point. Denote the point $B_2 \cap B_3$ by P_1, the point $B_3 \cap B_1$ by P_2 and the point $B_1 \cap B_2$ by P_3. Any block $B \in \mathcal{S}$ should have a non-empty intersection with as well B_1, B_2 as B_3. Since $\mathcal{D}$ does not contain an O'Nan configuration, B must pass through P_1, P_2 or P_3.

If the block $B'_i \in \mathcal{S}$ passes through P_i, $B'_i \notin \{B_1, B_2, B_3\}$, and the block $B'_j \in \mathcal{S}$ passes through P_j, $B'_j \notin \{B_1, B_2, B_3\}$, $1 \leq i \neq j \leq 3$, then the blocks B_i, B_j, B'_i and B'_j determine an O'Nan configuration, a contradiction. Hence, all blocks of $\mathcal{S} \setminus \{B_1, B_2, B_3\}$ pass through the same point P_i, $1 \leq i \leq 3$. Since $\mathcal{S}$ is maximal, it has to be a triangle based on the point P_i and the block B_i. $\square$

Note that $r > k+1$ for all $2-(v,k,1)$ designs without an O'Nan configuration, but the affine plane of order 2. Hence, the point-pencil is the largest Erdős-Ko-Rado set in these designs. Of course, the above result only makes sense if $2-(v,k,1)$ designs without an O'Nan configuration exist. We give an example. A classical unital was defined in Remark 1.6.6.

Theorem 5.1.6 ([101]). *A classical unital $\mathcal{U}$ does not contain an O'Nan configuration.*

It is conjectured that the classical unitals are the only unitals not containing an O'Nan configuration, see [22, 105]. In [22] this conjecture is proved to be true for unitals of order 3. The unique unital of order 2 is also classical.

Corollary 5.1.7. *In a classical unital there are only two types of maximal Erdős-Ko-Rado sets, the point-pencil and the triangle.*

5.2 The counting arguments

In this section we will study maximal Erdős-Ko-Rado sets in $2-(v,k,1)$ designs that are not point-pencils.

Notation 5.2.1. Let $\mathcal{D}$ be a $2-(v,k,1)$ design and let $\mathcal{S}$ be an Erdős-Ko-Rado set on $\mathcal{D}$. Denote the set of points of $\mathcal{D}$ covered by the blocks of $\mathcal{S}$ by $\mathcal{P}'$. We denote the number of points of $\mathcal{P}'$ that are contained in precisely i blocks of $\mathcal{S}$ by k_i. Furthermore we denote $k_{\mathcal{S}} = \max\{i \mid k_i > 0\}$. We use this notation throughout this section.

Lemma 5.2.2. *Let $\mathcal{D}$ be a $2-(v,k,1)$ design and let $\mathcal{S}$ be an Erdős-Ko-Rado set on $\mathcal{D}$. Then $|\mathcal{S}| \leq k_{\mathcal{S}}k - k + 1$. If $\mathcal{S}$ is maximal and different from the point-pencil, then $k_{\mathcal{S}} \leq k$.*

Proof. Fix a block $C \in \mathcal{S}$. All blocks of $\mathcal{S}$ have a non-trivial intersection with C, so

$$|\mathcal{S}| \leq 1 + k(k_{\mathcal{S}} - 1) = k_{\mathcal{S}}k - k + 1 \ .$$

Now, we prove the second part of the lemma. For every point $P \in \mathcal{P}'$, we can find a block $B \in \mathcal{S}$ not passing through P, since $\mathcal{S}$ is maximal but not a point-pencil. Any block of $\mathcal{S}$ through P should meet B and there is at most one block in $\mathcal{S}$ through P and a given point of B. Hence, there are at most k blocks in $\mathcal{S}$ passing through P. Consequently, $k_{\mathcal{S}} \leq k$. $\square$

Lemma 5.2.3. *Choose $l \in \mathbb{N} \setminus \{0,1\}$, and $a, b \in \mathbb{Z}$ with*

$$a \geq \max\left\{-l(r-l-1) + 1 - \frac{br}{l+1}, -\frac{b(b-1)}{(l+1)l} - 2(b-1)\right\} \ ,$$
$$a \leq \frac{rl - l^2 + l - 1}{l-1} - \frac{b(2l^2 + 2l - r + b - 1)}{l^2 - 1} \ .$$

Let $n_1, \ldots, n_l \in \mathbb{N}$ be such that

$$\sum_{i=1}^{l} in_i = (a-1)(l+1) + br + l(l+1)(r-l-1) \quad \textit{and}$$
$$\sum_{i=2}^{l} i(i-1)n_i = b(b-1) + l(l+1)(a+2b-2) \ .$$

Then,

$$\sum_{i=2}^{l}(i-1)n_i \leq \binom{b}{2} + (a+2b-2)\binom{l+1}{2} .$$

Proof. Note that the inequalities $-l(r-l-1)+1-\frac{br}{l+1} \leq a$ and $-\frac{b(b-1)}{(l+1)l}-2(b-1) \leq a$ are present to ensure that both $a(l+1)+br+l(l+1)(r-l-1)-l-1$ and $b(b-1)+l(l+1)(a+2b-2)$ are nonnegative.

Using the first equality, we can express n_1 as a function of l, a, b and $n_2,\dots,n_l$. Note that

$$\begin{aligned}
n_1 &= (a-1)(l+1)+br+l(l+1)(r-l-1)-\sum_{i=2}^{l} in_i \\
&\geq (a-1)(l+1)+br+l(l+1)(r-l-1)-\sum_{i=2}^{l} i(i-1)n_i \\
&= (a-1)(l+1)+br+l(l+1)(r-l-1) \\
&\qquad -b(b-1)-l(l+1)(a+2b-2) \\
&= -a(l^2-1)-b(b-1)-b(2l^2+2l-r)+(l+1)(rl-l^2+l-1) \\
&\geq 0 ,
\end{aligned}$$

by the assumption. Hence, for every choice of $n_2,\dots,n_l$, we can find a value $n_1 \in \mathbb{N}$ such that the first equality holds. Now, we focus on the second equality. Assume that $n_j > 0$ for a value $j \geq 3$. Then define $n'_j = n_j - 1$, $n'_2 = n_2 + \frac{j(j-1)}{2}$ and $n'_k = n_k$ for $k \notin \{2, j\}$. It follows that

$$\sum_{i=2}^{l} i(i-1)n'_i = \sum_{i=2}^{l} i(i-1)n_i = b(b-1)+l(l+1)(a+2b-2) .$$

However,

$$\sum_{i=2}^{l}(i-1)n'_i = \left(\sum_{i=2}^{l}(i-1)n_i\right) - (j-1) + \frac{j(j-1)}{2} > \sum_{i=2}^{l}(i-1)n_i ,$$

since $j \geq 3$. So, repeatedly applying the above construction, we find that $\sum_{i=2}^{l}(i-1)n_i$ is maximal if $n_i = 0$ for all $i \geq 3$ and $n_2 = \binom{b}{2}+(a+2b-2)\binom{l+1}{2}$. The lemma follows. □

Lemma 5.2.4. *Let $\mathcal{D}$ be a $2-(v,k,1)$ design with replication number r, $k \geq 3$, and let $\mathcal{S}$ be an Erdős-Ko-Rado set on $\mathcal{D}$ with $|\mathcal{P}'| = k(k-1)+b$. Then,*

$$|\mathcal{S}| \leq \max\left\{k^2-k+1-2\frac{(r-k)(k^2-k+1-r)}{k(k-2)}+\frac{b(b-1)}{(k-1)(k-2)}\right.$$
$$+2\frac{(b-1)(k^2-k-r)}{(k-1)(k-2)},$$
$$\left.k^2-r-\frac{r-1}{k-2}+\frac{b(b-1-r+2k(k-1))}{k(k-2)}\right\} .$$

Proof. Recall that $\mathcal{B}$ is the set of blocks of $\mathcal{D}$. We denote the subset of $\mathcal{B}$ containing precisely i points of $\mathcal{P}'$ by $\mathcal{B}_i$ and we also denote $m_i = |\mathcal{B}_i|$. Note that $\mathcal{S} \subseteq \mathcal{B}_k$. We define $a = k^2-k+1-|\mathcal{B}_k|$. Counting the tuples (P,B) with $P \in \mathcal{P}'$, $B \in \mathcal{B}$ and P on B, we find

$$\sum_{i=1}^{k} i m_i = (k(k-1)+b)r .$$

Now applying $m_k = k^2-k+1-a$, we find

$$\begin{aligned} m_1 &= (k(k-1)+b)r - \sum_{i=2}^{k-1} i m_i - k(k^2-k+1-a) \\ &= k(k-1)(r-k)+(a-1)k+br-\sum_{i=2}^{k-1} i m_i . \end{aligned}$$

Counting the tuples (P,P',B) with $P,P' \in \mathcal{P}'$, $B \in \mathcal{B}$, $P \neq P'$ and both P and P' on B, we find

$$\sum_{i=1}^{k} i(i-1)m_i = (k(k-1)+b)(k(k-1)+b-1) .$$

Hence,

$$\begin{aligned} \sum_{i=2}^{k-1} i(i-1)m_i &= (k(k-1)+b)(k(k-1)+b-1) \\ &\quad - k(k-1)(k^2-k+1-a) \\ &= b(b-1)+(a+2b-2)k(k-1) . \end{aligned}$$

Now we consider the set T of triples (P, P', B) with $P, P' \in \mathcal{P} \setminus \mathcal{P}'$, $B \in \mathcal{B}_1$, $P, P' \in B$ and $P \neq P'$. On the one hand we know

$$\begin{aligned} |T| &= m_1(k-1)(k-2) \\ &= k(k-1)^2(k-2)(r-k) + (a-1)k(k-1)(k-2) \\ &\quad + br(k-1)(k-2) - (k-1)(k-2)\sum_{i=2}^{k-1} i m_i \,. \end{aligned}$$

On the other hand, using $|\mathcal{P} \setminus \mathcal{P}'| = v - k(k-1) - b = (r-k)(k-1) - (b-1)$, we can also find that

$$\begin{aligned} |T| &\leq ((r-k)(k-1) - (b-1))\,((r-k)(k-1) - b) \\ &\quad - \sum_{i=2}^{k-1} (k-i)(k-i-1) m_i \,. \end{aligned}$$

Comparing this equality and inequality for $|T|$, we find

$$\begin{aligned} &\sum_{i=2}^{k-1} (k(k-1)(i-1) - i(i-1)) m_i \geq \\ &\quad k(k-1)^2(k-2)(r-k) + (a-1)k(k-1)(k-2) + br(k-1)(k-2) \\ &\qquad - b(b-1) - (r-k)^2(k-1)^2 + (2b-1)(r-k)(k-1) \,. \end{aligned}$$

Using the formula for $\sum_{i=2}^{k-1} i(i-1)m_i$, and dividing both sides by $k-1$, it follows that

$$k\sum_{i=2}^{k-1}(i-1)m_i \geq ak(k-1) + bkr - k^2 + (r-k)(k^3 - 2k^2 - (r-1)(k-1)) \,.$$

We distinguish between two cases. If $a > r - k + 1 + \frac{r-1}{k-2} - \frac{b(b-1-r+2k(k-1))}{k(k-2)}$, then $|\mathcal{S}| \leq |\mathcal{B}_k| \leq k^2 - r - \frac{r-1}{k-2} + \frac{b(b-1-r+2k(k-1))}{k(k-2)}$. If $a \leq r - k + 1 + \frac{r-1}{k-2} - \frac{b(b-1-r+2k(k-1))}{k(k-2)}$, we can apply Lemma 5.2.3 with $l = k-1$. Note that the conditions $-l(r-l-1) + 1 - \frac{br}{l+1} \leq a$ and $-\frac{b(b-1)}{(l+1)l} - 2(b-1) \leq a$ are fulfilled since $\sum_{i=2}^{k-1} i(i-1)m_i$ and $\sum_{i=1}^{k-1} im_i$ are nonnegative. We find

$$\begin{aligned} k\binom{b}{2} + k(a+2b-2)\binom{k}{2} &\geq ak(k-1) + bkr - k^2 \\ &\quad + (r-k)(k^3 - 2k^2 - (r-1)(k-1)) \,, \end{aligned}$$

hence

$$a \geq \frac{2(r-k)(k^2-k+1-r)}{k(k-2)} - \frac{2(b-1)(k^2-k-r)}{(k-1)(k-2)} - \frac{b(b-1)}{(k-1)(k-2)} \, .$$

We find thus that

$$\begin{aligned}|\mathcal{S}| \leq |\mathcal{B}_k| \leq k^2-k+1 &- \frac{2(r-k)(k^2-k+1-r)}{k(k-2)} \\ &+ \frac{2(b-1)(k^2-k-r)}{(k-1)(k-2)} + \frac{b(b-1)}{(k-1)(k-2)} \, ,\end{aligned}$$

which finishes the proof. $\square$

Using the substitution $R = (k-1)^2 - r$, we can rewrite this lemma.

Corollary 5.2.5. *Let $\mathcal{D}$ be a $2-(v,k,1)$ design, $k \geq 3$, and denote $(k-1)^2 - r$ by R. Let $\mathcal{S}$ be an Erdős-Ko-Rado set on $\mathcal{D}$ with $|\mathcal{P}'| = k(k-1) + b$. Then*

$$\begin{aligned}|\mathcal{S}| \leq \max \Bigg\{ & k^2-k+1-2\frac{(k^2-3k+1-R)(k+R)}{k(k-2)} + \frac{b(b-1)}{(k-1)(k-2)} \\ & + 2\frac{(b-1)(k-1+R)}{(k-1)(k-2)}, \\ & k-1+R+\frac{R}{k-2}+\frac{b(b+k^2+R-2)}{k(k-2)} \Bigg\} \, .\end{aligned}$$

Lemma 5.2.6. *Let $\mathcal{D}$ be a $2-(v,k,1)$ design and let $\mathcal{S}$ be an Erdős-Ko-Rado set on $\mathcal{D}$ with $k_{\mathcal{S}} = k$. Then $|\mathcal{P}'| = k^2-k+1$.*

Proof. Since $k_{\mathcal{S}} = k$, we can find a point $P \in \mathcal{P}'$ lying on k blocks of $\mathcal{S}$. Denote these blocks by $B_1, \ldots, B_k$ and denote the set of points covered by these blocks by $\mathcal{P}''$. Any block of $\mathcal{S}$ not through P contains a point on each of the blocks B_i, $i = 1, \ldots, k$. Since a block contains precisely k points, all points on such a block are contained in $\mathcal{P}''$. Hence, $\mathcal{P}'' = \mathcal{P}'$ and

$$|\mathcal{P}''| = |\bigcup_{i=1}^{k} B_i| = 1 + k(k-1) = k^2-k+1 \, .$$

$\square$

Lemma 5.2.7. *Let $\mathcal{D}$ be a $2-(v,k,1)$ design and let $\mathcal{S}$ be an Erdős-Ko-Rado set on $\mathcal{D}$ with $k_{\mathcal{S}} = k-1$. Write $a' = (k-1)^2 - |\mathcal{S}|$. If $a' < k-1$, then $k(k-1) \leq |\mathcal{P}'| \leq k(k-1) + \frac{a'^2-a'}{k-1-a'}$.*

Proof. First we will prove that there is a block in $\mathcal{S}$ containing at least two points that are on $k-1$ blocks of $\mathcal{S}$. Assume there is no such block and choose a block C. At most one point on C belongs to $k-1$ blocks of $\mathcal{S}$. However, all blocks of $\mathcal{S}$ have a non-trivial intersection with C, so

$$|\mathcal{S}| \leq 1 + (k-2) + (k-1)(k-3) = (k-1)(k-2) ,$$

hence $a' \geq k-1$, which contradicts the assumption $a' < k-1$.

Let B_1 be a block of $\mathcal{S}$ through the points Q_1 and Q_2, both on $k-1$ blocks of $\mathcal{S}$, and let $B_1, B_2, \ldots, B_{k-1}$ and $B_1 = C_1, C_2, \ldots, C_{k-1}$ be the blocks of $\mathcal{S}$, respectively through Q_1 and Q_2. There are $(k-2)^2$ points which lie on a block B_j and also on a block $C_{j'}$, $2 \leq j, j' \leq k-1$; there are $k-2$ points which lie on a block B_i, but not on a block $C_{i'}$, and there are also $k-2$ points which lie on a block C_i, but not on a block $B_{i'}$; the block $B_1 = C_1$ contains k points. Hence, $|\mathcal{P}'| \geq (k-2)^2 + 2(k-2) + k = k(k-1)$.

Now, recall the notation k_i. By standard counting arguments we know that

$$\sum_{i=1}^{k-1} ik_i = ((k-1)^2 - a')k \quad \text{and} \quad \sum_{i=1}^{k-1} i(i-1)k_i = ((k-1)^2 - a')(k(k-2) - a') .$$

Let $j \in \mathbb{N} \setminus \{0\}$ be the smallest value such that $k_j \neq 0$ and let R be a point of $\mathcal{P}'$ on j blocks of $\mathcal{S}$. Let $B \in \mathcal{S}$ be a block through R. All blocks of $\mathcal{S}$ meet B, hence

$$|\mathcal{S}| = (k-1)^2 - a' \leq 1 + (k-1)(k-2) + (j-1) .$$

It follows that $j \geq k-1-a'$. Therefore, the following inequality holds:

$$\sum_{i=1}^{k-1} (i - (k-a'-1))(k-1-i)k_i \geq 0 .$$

So,

$$
\begin{aligned}
0 &\le -\sum_{i=1}^{k-1} i(i-1)k_i + (2k-a'-3)\sum_{i=1}^{k-1} ik_i - (k-a'-1)(k-1)\sum_{i=1}^{k-1} k_i \\
&= -((k-1)^2 - a')(k(k-2)-a') + (2k-a'-3)((k-1)^2-a')k \\
&\quad - (k-a'-1)(k-1)\sum_{i=1}^{k-1} k_i \\
&= ((k-1)^2-a')(k-1)(k-a') - (k-a'-1)(k-1)\sum_{i=1}^{k-1} k_i \,.
\end{aligned}
$$

Consequently,

$$
|\mathcal{P}'| = \sum_{i=1}^{k-1} k_i \le \frac{((k-1)^2-a')(k-a')}{k-a'-1} = k(k-1) + \frac{a'^2-a'}{k-a'-1}
$$

and the lemma follows. □

5.3 Classification results for $k = 3$

For $k = 2$, a $2-(v,k,1)$ design is a complete graph K_v on v vertices, the edges being the blocks. It can immediately be seen that there are precisely two different types of maximal Erdős-Ko-Rado sets on K_v, namely the point-pencil, which contains $v-1$ blocks, and the triangle, a set $\{\{p_1,p_2\},\{p_1,p_3\},\{p_2,p_3\}\}$ for three points $p_1,p_2,p_3 \in \mathcal{P}$, which contains 3 blocks.

So, the first non-trivial case is $k = 3$. Recall that a $2-(v,3,1)$ design is called a Steiner triple system of size v. Steiner triple systems exist if and only if $v \equiv 1,3 \pmod 6$ and $v \ge 7$. Up to isomorphism, there is only one Steiner triple system for $v = 7$, namely the Fano plane, the projective plane of order 2, there is only one Steiner triple system for $v = 9$, namely the affine plane of order 3, and there are two Steiner triple systems for $v = 13$. For more details, we refer the interested reader to [33, Section II.1, Section II.2]. Note that the classical unital of order 2 is also a Steiner triple system. It is isomorphic to the affine plane of order 3.

Theorem 5.3.1. *Let $\mathcal{D}$ be a $2-(v,3,1)$ design and let $\mathcal{S}$ be a maximal Erdős-Ko-Rado set of $\mathcal{D}$. Then $\mathcal{S}$ belongs to one of five types. The maximal*

Erdős-Ko-Rado sets contain $\frac{v-1}{2}$, 4, 5, 6 *or* 7 *blocks. Each type corresponds to a size and vice versa.*

Proof. If all blocks of $\mathcal{S}$ pass through a common point, then $\mathcal{S}$ is a point-pencil and it contains $\frac{v-1}{2}$ blocks. So, from now on we assume that there is no point on all blocks of $\mathcal{S}$. Let $B_1, B_2, B_3 \in \mathcal{S}$ be three blocks such that $B_1 \cap B_2 = \{P_3\}$, $B_1 \cap B_3 = \{P_2\}$ and $B_2 \cap B_3 = \{P_1\}$, with P_1, P_2, P_3 three different points. Let Q_i be the third point on the block B_i, $i = 1, 2, 3$. There is precisely one block through the points P_i and Q_i. We denote it by B'_i and we denote the third point on this block by R_i, $i = 1, 2, 3$.

If the three points Q_1, Q_2 and Q_3 are contained in a common block B', then this block has to be contained in $\mathcal{S}$ by the maximality condition. The only other blocks that could be contained in $\mathcal{S}$ are B'_1, B'_2 and B'_3. If all three points R_1, R_2 and R_3 are different, then only one of these blocks belongs to $\mathcal{S}$. We find an Erdős-Ko-Rado set of size 4 or 5, depending on whether the block B' exists. If two of the points R_1, R_2 and R_3 coincide, then we find an Erdős-Ko-Rado set of size 5 or 6. If $R_1 = R_2 = R_3$, then we find an Erdős-Ko-Rado set of size 6 or 7.

Note that the two constructions of Erdős-Ko-Rado sets of size 5 give rise to isomorphic sets, so there is only one type of Erdős-Ko-Rado sets of size 5. Analogously, there is also only one type of Erdős-Ko-Rado sets of size 6. □

Remark 5.3.2. The five types of maximal Erdős-Ko-Rado sets in $2-(v,3,1)$ designs are explicitly described in the above theorem. Apart from the point-pencil, these block sets can be embedded in a Fano plane. However, they cannot be extended to a Fano plane by blocks of the design, due to the maximality condition. Note that the Erdős-Ko-Rado set of size 7 is a Fano plane that is embedded in the design.

Notation 5.3.3. Since the four types of maximal Erdős-Ko-Rado sets different from the point-pencil are determined by their size, we can denote them by EKR_i, $i = 4, \ldots, 7$, the index referring to their size. Note that each of the maximal Erdős-Ko-Rado sets different from the point-pencil, cover precisely 7 points of the design.

Remark 5.3.4. In a given $2-(v,3,1)$ design $\mathcal{D}$, not necessarily all five types occur. For example, if $\mathcal{D}$ is the Fano plane ($v = 7$), then there is only one maximal Erdős-Ko-Rado set, namely EKR_7, which is the set of all blocks in this case. If $\mathcal{D}$ is not a projective plane, at least two types occur, one of which is the point-pencil.

We list the results for Erdős-Ko-Rado sets on Steiner triple systems of size v. For small values of v, the results are more detailed.

Theorem 5.3.5. *Let $\mathcal{D}$ be a $2-(v, 3, 1)$ design.*

- *If $v = 7$, there is only one maximal Erdős-Ko-Rado set in $\mathcal{D}$.*
- *If $v = 9$, there are two types of maximal Erdős-Ko-Rado sets in $\mathcal{D}$, the point-pencil and EKR_4. Both contain 4 blocks.*
- *If $v = 13$, there are three types of maximal Erdős-Ko-Rado sets in $\mathcal{D}$, the point-pencil, EKR_4 and EKR_5. The largest Erdős-Ko-Rado sets are the point-pencils.*
- *If $v = 15$, the largest Erdős-Ko-Rado sets contain 7 blocks. There are 23 nonisomorphic $2-(15, 3, 1)$ designs containing an EKR_7, and 57 nonisomorphic $2-(15, 3, 1)$ designs not containing an EKR_7. The former have two types of maximal Erdős-Ko-Rado sets of size 7; for the latter all Erdős-Ko-Rado sets of size 7 are point-pencils.*
- *If $v \geq 19$, the largest Erdős-Ko-Rado sets are point-pencils.*

Proof. The case $v = 7$ has been treated in Remark 5.3.4. If $v = 9$, then $\mathcal{D}$ is an affine plane of order 3. One can see immediately that only two of the above types of maximal Erdős-Ko-Rado sets occur, the point-pencil and the smallest one of the others, the EKR_4. Both contain four blocks. Compare this result with Remark 5.1.2

If $v = 13$, there are two nonisomorphic $2-(v, 3, 1)$ designs. Their point sets can be denoted by $\{0, 1, \ldots, 9, a, b, c\}$. Using [33, Table II.1.27], we can write the block sets as in Table 5.1.

We know that the point-pencil contains 6 blocks. By Theorem 5.2.4, applied for $k = 3$, $b = 1$ and $r = 6$, we know that any other maximal Erdős-Ko-Rado set contains at most 5 blocks. So, on both $2-(13, 3, 1)$ designs, at most three types of maximal Erdős-Ko-Rado sets occur. Using the above notation, the two sets $\{\{0, 1, 2\}, \{0, 3, 4\}, \{1, 3, 5\}, \{2, 3, 9\}, \{2, 4, 5\}\}$ and $\{\{0, 1, 2\}, \{0, 3, 4\}, \{0, 9, a\}, \{2, 3, 9\}\}$ are maximal Erdős-Ko-Rado sets for both $2-(13, 3, 1)$ designs.

0	1	2	0	3	4
0	5	6	0	7	8
0	9	a	0	b	c
1	3	5	1	4	7
1	6	8	1	9	b
1	a	c	2	3	9
2	4	5	2	6	a
2	7	c	2	8	b
3	6	b	3	7	a
3	8	c	4	6	c
4	8	9	4	a	b
5	7	b	5	8	a
5	9	c	6	7	9

0	1	2	0	3	4
0	5	6	0	7	8
0	9	a	0	b	c
1	3	5	1	4	7
1	6	8	1	9	b
1	a	c	2	3	9
2	4	5	2	6	a
2	7	b	2	8	c
3	6	b	3	7	c
3	8	a	4	6	c
4	8	9	4	a	b
5	7	a	5	8	b
5	9	c	6	7	9

Table 5.1: Block sets of the two nonisomorphic $2-(13,3,1)$ designs.

Hence, there are precisely three types of maximal Erdős-Ko-Rado sets on $2-(13,3,1)$ designs.

There are 80 nonisomorphic $2-(15,3,1)$ designs, see [33, Table II.1.28] for an overview. The point-pencil contains 7 blocks in these designs. In [33, Table II.1.29] it is mentioned which of these 80 designs contain a Fano plane as subdesign; 23 of them do, and 57 do not. The statement follows.

If $v \geq 19$, then $r \geq 9$, hence the point-pencil contains more blocks than the Erdős-Ko-Rado sets of type EKR_i, $i = 4, \ldots, 7$. $\square$

Note that one of the 23 different $2-(15,3,1)$ designs having a Fano plane as subdesign, is the design consisting of the points and lines of $\mathrm{PG}(3,2)$. Also note that the last part of Theorem 5.3.5 is a special case of Corollary 5.0.2.

5.4 Classification results for $k \geq 4$

In this section we present the main classification theorems for Erdős-Ko-Rado sets in $2-(v,k,1)$ designs, as we mentioned in the introduction. We will use the parameter $k_{\mathcal{S}}$, introduced in Notation 5.2.1.

Theorem 5.4.1. *Let $\mathcal{D}$ be a $2-(v,k,1)$ design and let $\mathcal{S}$ be an Erdős-Ko-Rado set on $\mathcal{D}$. If $r \geq k^2-k+1$, then $|\mathcal{S}| \leq r$. If $r > k^2-k+1$ and $|\mathcal{S}| = r$,* *then $\mathcal{S}$ is a point-pencil.*

Proof. Without loss of generality, we can assume that $\mathcal{S}$ is a maximal Erdős-Ko-Rado set. If $\mathcal{S}$ is a point-pencil, then $|\mathcal{S}| = r$. So, from now on, we can assume that $\mathcal{S}$ is not a point-pencil. By Lemma 5.2.2 we know that $k_{\mathcal{S}} \leq k$. However, by the same lemma we also know that $|\mathcal{S}| \leq k^2 - k + 1$, if $k_{\mathcal{S}} \leq k$.

Both statements in the theorem immediately follow. □

As mentioned in the beginning of this chapter, there are $2-(v,k,1)$ designs with $r = k^2 - k + 1$, having a second type of Erdős-Ko-Rado sets of size r.

Now, we look at Erdős-Ko-Rado sets in $2-(v,k,1)$ designs with $r \leq k^2 - k$. A classification result will be proved in Theorem 5.4.5. First we prove some preparatory lemmata. In these lemmata we distinguish between the case $4 \leq k \leq 13$ and the case $k \geq 14$.

First, we have a look at the small cases, $4 \leq k \leq 13$. In the next lemmata, we will use the values R_k presented in Table 5.2.

k	4	5	6	7	8	9	10	11	12	13
R_k	1	2	3	4	4	5	6	7	8	9

Table 5.2: The values R_k.

Lemma 5.4.2. *Let $\mathcal{D}$ be a $2-(v,k,1)$ design, $4 \leq k \leq 13$, and denote $(k-1)^2 - r$ by R. Let $\mathcal{S}$ be an Erdős-Ko-Rado set on $\mathcal{D}$ with $k_{\mathcal{S}} = k-1$. If $0 \leq R \leq R_k$, then $|\mathcal{S}| < (k-1)^2 - R = r$.*

Proof. We denote the set of points covered by the blocks of $\mathcal{S}$ by $\mathcal{P}'$. We denote $(k-1)^2 - |\mathcal{S}|$ by a', as in Lemma 5.2.7. By Lemma 5.2.2 we know that $a' \geq 0$. If $R < a'$, then $|\mathcal{S}| < (k-1)^2 - R$. So, now we assume that $a' \leq R$. Since $R_k < k-1$, we know by Lemma 5.2.7 that $k(k-1) \leq |\mathcal{P}'| \leq k(k-1) + \frac{a'(a'-1)}{k-1-a'} \leq k(k-1) + \frac{R(R-1)}{k-1-R}$. Denoting $|\mathcal{P}'| - k(k-1)$ by b, it follows that $0 \leq b \leq \frac{R(R-1)}{k-1-R}$. By Lemma 5.2.5 we know that

$$|\mathcal{S}| \leq \max\left\{k^2 - k + 1 - 2\frac{(k^2-3k+1-R)(k+R)}{k(k-2)} + \frac{b(b-1)}{(k-1)(k-2)} \right.$$
$$\left. + 2\frac{(b-1)(k-1+R)}{(k-1)(k-2)},\right.$$

$$k-1+R+\frac{R}{k-2}+\frac{b(b+k^2+R-2)}{k(k-2)}\Bigg\} \ .$$

By hand or by using a computer algebra package, it can be checked that the above maximum is smaller than $(k-1)^2 - R = r$ for all choices of k, R, b fulfilling $4 \leq k \leq 13$, $0 \leq R \leq R_k$ and $0 \leq b \leq \frac{R(R-1)}{k-1-R}$. □

Extending the calculations in the above proof, we can see that the values R_k are optimal; enlarging one of these values leads to a contradiction.

Now, we look at the more general case $k \geq 14$.

Lemma 5.4.3. *Choose $b, c, k \in \mathbb{N}$, with $k \geq 14$, $1 \leq c \leq \frac{4}{3}k\sqrt{k} - 2k - 2\sqrt{k}$ and $0 \leq b \leq c$. Then*

$$\frac{k^3-7k^2+10k-2bk-2-\sqrt{D(b,k)}}{4(k-1)} < \frac{1-c+\sqrt{(c-1)^2+4c(k-1)}}{2} \ ,$$

with $D(b,k) = (k^3-3k^2-2bk+6k-2)^2 - 8k(k-1)(b-1)(b-2)$. Furthermore, for $k \in \mathbb{N}$ with $k \geq 14$,

$$\frac{k^3-7k^2+10k-2-\sqrt{D(0,k)}}{4(k-1)} < 0 \ .$$

Proof. First, note that $D(b,k) \geq 0$ for all $0 \leq b \leq \frac{4}{3}k\sqrt{k} - 2k - 2\sqrt{k} = C_k$, hence the above functions exist.

The second part of the lemma is immediate, so we focus on the first part. Note that

$$\begin{aligned}
&\frac{k^3-7k^2+10k-2(b+1)k-2-\sqrt{D(b+1,k)}}{4(k-1)} \\
&\qquad - \frac{k^3-7k^2+10k-2bk-2-\sqrt{D(b,k)}}{4(k-1)} \\
&= \frac{\sqrt{D(b,k)}-\sqrt{D(b+1,k)}-2k}{4(k-1)} \ .
\end{aligned}$$

Now,

$$
\begin{aligned}
& \frac{\sqrt{D(b,k)} - \sqrt{D(b+1,k)} - 2k}{4(k-1)} \geq 0 \\
\Leftrightarrow \quad & \sqrt{D(b,k)} - \sqrt{D(b+1,k)} \geq 2k \\
\Leftrightarrow \quad & D(b,k) - D(b+1,k) \geq 2k\left(\sqrt{D(b,k)} + \sqrt{D(b+1,k)}\right) \\
\Leftrightarrow \quad & 2k^3 - 6k^2 + 4bk + 2k - 8b + 4 \geq \sqrt{D(b,k)} + \sqrt{D(b+1,k)}\ .
\end{aligned}
$$

This final inequality is valid since $\sqrt{D(b,k)} + \sqrt{D(b+1,k)} \leq 2k^3 - 6k^2 - 4bk + 10k - 4$. These calculations show that

$$
\begin{aligned}
& \frac{k^3 - 7k^2 + 10k - 2(b+1)k - 2 - \sqrt{D(b+1,k)}}{4(k-1)} \\
\geq\ & \frac{k^3 - 7k^2 + 10k - 2bk - 2 - \sqrt{D(b,k)}}{4(k-1)}\ .
\end{aligned}
$$

Hence, it is sufficient to prove that

$$
\frac{k^3 - 7k^2 + 10k - 2ck - 2 - \sqrt{D(c,k)}}{4(k-1)} < \frac{1 - c + \sqrt{(c-1)^2 + 4c(k-1)}}{2}\ .
$$

Since $c \leq \frac{4}{3}k\sqrt{k} - 2k - 2\sqrt{k} < \frac{k^3-7k^2+8k}{2}$ for $k \geq 14$, this is equivalent to

$$
\begin{aligned}
& \left(2(k-1)\sqrt{(c-1)^2 + 4c(k-1)} + \sqrt{D(c,k)}\right)^2 > (k^3 - 7k^2 + 8k - 2c)^2 \\
\Leftrightarrow \quad & \sqrt{(c-1)^2 + 4c(k-1)}\sqrt{D(c,k)} > \\
& \qquad c(k^3 - 7k^2 + 14k - 6) - (2k^4 - 9k^3 + 9k^2 + 2k - 2)\ . \qquad (5.1)
\end{aligned}
$$

Considering the left-hand side of the inequality (5.1) as a function of c, for a fixed value of k, we can compute its second derivative. We find that this second derivative is negative on $[0, C_k]$, hence the function on the left-hand side is concave on $[0, C_k]$ (see Computation A.1.1 for more details). Therefore, it dominates the function

$$
c \mapsto \sqrt{D(0,k)} + c\frac{\sqrt{(C_k-1)^2 + 4C_k(k-1)}\sqrt{D(C_k,k)} - \sqrt{D(0,k)}}{C_k}\ .
$$

The slope of this line is smaller than $k^3 - 7k^2 + 14k - 6$ for $k \geq 14$. So, we only need to check the inequality for the largest considered value for c, namely C_k. It turns out that this inequality is valid if $k \geq 14$. For more details, see Computation A.1.2. □

In the final step of the argument we needed that $k \geq 14$. This is why the cases $4 \leq k \leq 13$ had to be treated separately.

Lemma 5.4.4. *Let $\mathcal{D}$ be a $2-(v,k,1)$ design, $k \geq 14$, and denote $(k-1)^2 - r$ by R. Let $\mathcal{S}$ be an Erdős-Ko-Rado set on $\mathcal{D}$ with $k_{\mathcal{S}} = k-1$. If $0 \leq R < \sqrt{k-1}$ or $\frac{1-c+\sqrt{(c-1)^2+4c(k-1)}}{2} \leq R < \frac{-c+\sqrt{c^2+4(c+1)(k-1)}}{2}$ for a value $c \in \mathbb{N}$, with $1 \leq c \leq \frac{4}{3}k\sqrt{k} - 2k - 2\sqrt{k}$, then $|\mathcal{S}| < (k-1)^2 - R$.*

Proof. We denote $(k-1)^2 - |\mathcal{S}|$ by a', as in Lemma 5.2.7. By Lemma 5.2.2 we know that $a' \geq 0$. If $R < a'$, then $|\mathcal{S}| < (k-1)^2 - R$. So, now we assume that $a' \leq R$. Denote the interval $\left[\frac{1-c+\sqrt{(c-1)^2+4c(k-1)}}{2}, \frac{-c+\sqrt{c^2+4(c+1)(k-1)}}{2}\right[$ by I_c, $c \in \mathbb{N}$ and $1 \leq c \leq \frac{4}{3}k\sqrt{k} - 2k - 2\sqrt{k} = C_k$, and the interval $\left[0, \sqrt{k-1}\right[$ by I_0. Recall the notation $\mathcal{P}'$ for the set of all points covered by a block of $\mathcal{S}$. We assume that $R \in I_c$. From Lemma 5.2.7, it follows that $|\mathcal{P}'| \leq k(k-1) + c$. We denote $k(k-1) - |\mathcal{P}'|$ by b, so $0 \leq b \leq c$. Hence, by Corollary 5.2.5,

$$\begin{aligned}|\mathcal{S}| \leq \max\Bigg\{ & k^2 - k + 1 - 2\frac{(k^2-3k+1-R)(k+R)}{k(k-2)} + \frac{b(b-1)}{(k-1)(k-2)} \\ & + 2\frac{(b-1)(k-1+R)}{(k-1)(k-2)}, \\ & k - 1 + R + \frac{R}{k-2} + \frac{b(b+k^2+R-2)}{k(k-2)} \Bigg\} .\end{aligned}$$

Since $c \leq C_k$ and $R < k-2$, the inequality

$$k - 1 + R + \frac{R}{k-2} + \frac{b(b+k^2+R-2)}{k(k-2)} < (k-1)^2 - R$$

clearly holds in all cases. Now, we consider the inequality

$$\begin{aligned}(k-1)^2 - R > {} & k^2 - k + 1 - 2\frac{(k^2-3k+1-R)(k+R)}{k(k-2)} + \frac{b(b-1)}{(k-1)(k-2)} \\ & + 2\frac{(b-1)(k-1+R)}{(k-1)(k-2)} \\ \Leftrightarrow \qquad 0 > {} & k + R - 2\frac{(k^2-3k+1-R)(k+R)}{k(k-2)} + \frac{b(b-1)}{(k-1)(k-2)} \\ & + 2\frac{(b-1)(k-1+R)}{(k-1)(k-2)} .\end{aligned}$$

This inequality is valid if and only if

$$\frac{k^3 - 7k^2 + 10k - 2bk - 2 - \sqrt{D(b,k)}}{4(k-1)} < R$$
$$< \frac{k^3 - 7k^2 + 10k - 2bk - 2 + \sqrt{D(b,k)}}{4(k-1)}, \quad (5.2)$$

with $D(b,k) = (k^3 - 3k^2 - 2bk + 6k - 2)^2 - 8k(k-1)(b-1)(b-2)$. The double inequality in (5.2) should hold for all b, with $0 \leq b \leq c$. Now,

$$R < \frac{-c + \sqrt{c^2 + 4(c+1)(k-1)}}{2} \qquad \text{and}$$
$$\frac{k^3 - 7k^2 + 10k - 2ck - 2}{4(k-1)} \leq \frac{k^3 - 7k^2 + 10k - 2bk - 2 + \sqrt{D(b,k)}}{4(k-1)},$$

but the inequality $\frac{-c+\sqrt{c^2+4(c+1)(k-1)}}{2} < \frac{k^3-7k^2+10k-2ck-2}{4(k-1)}$ is valid for all $0 \leq c \leq C_k$ since $k \geq 14$. Hence, the right inequality in (5.2) always holds. Using

$$R \geq \frac{1 - c + \sqrt{(c-1)^2 + 4c(k-1)}}{2}$$

and Lemma 5.4.3, also the left inequality in (5.2) follows. This finishes the proof. □

Theorem 5.4.5. *Let $\mathcal{D}$ be a $2-(v,k,1)$ design, $k \geq 4$, and let $\mathcal{S}$ be an Erdős-Ko-Rado set on $\mathcal{D}$. If $k^2 - k \geq r \geq k^2 - 3k + \frac{3}{4}\sqrt{k} + 2$, then $|\mathcal{S}| \leq r$. If $(r,k) \neq (8,4)$, equality is obtained if and only if $\mathcal{S}$ is a point-pencil.*

Proof. Without loss of generality, we can assume that $\mathcal{S}$ is a maximal Erdős-Ko-Rado set. Recall the notation $k_{\mathcal{S}}$. If $\mathcal{S}$ is a point-pencil, then $|\mathcal{S}| = r$. So, from now on, we can assume that $\mathcal{S}$ is not a point-pencil. Then, by Lemma 5.2.2 we know that $k_{\mathcal{S}} \leq k$. We distinguish between three cases.

If $k_{\mathcal{S}} \leq k-2$, then $|\mathcal{S}| \leq k^2 - 3k + 1$ by Lemma 5.2.2. Clearly, $k^2 - 3k + 1 < k^2 - 3k + \frac{3}{4}\sqrt{k} + 2 \leq r$.

If $k_{\mathcal{S}} = k-1$, then $|\mathcal{S}| \leq k^2 - 2k + 1$ by Lemma 5.2.2. In this case, if $k^2 - 2k + 1 < r \leq k^2 - k$, the theorem clearly holds, so we assume $r \leq k^2 - 2k + 1$. As before, we denote $R = (k-1)^2 - r$. First, assume that $k \geq 14$. In this case,

$0 \leq R \leq k - \frac{3}{4}\sqrt{k} - 1$. So, $0 \leq R < \sqrt{k-1}$ or there is a value $c \in \mathbb{N}$, with $1 \leq c \leq \frac{4}{3}k\sqrt{k} - 2k - 2\sqrt{k}$, such that $\frac{1-c+\sqrt{(c-1)^2+4c(k-1)}}{2} \leq R < \frac{-c+\sqrt{c^2+4(c+1)(k-1)}}{2}$. Applying Lemma 5.4.4 we find that $|\mathcal{S}| < (k-1)^2 - R = r$.

Now assume that $4 \leq k \leq 13$. In this case, $0 \leq R \leq R_k = \left\lfloor k - \frac{3}{4}\sqrt{k} - 1 \right\rfloor$. Applying Lemma 5.4.2, we find that $|\mathcal{S}| < (k-1)^2 - R = r$.

If $k_{\mathcal{S}} = k$, then $|\mathcal{P}'| = k^2 - k + 1$ by Lemma 5.2.6. So, we can apply Lemma 5.2.4 with $b = 1$. We find that $|\mathcal{S}|$ is at most

$$\max\left\{k^2 - k + 1 - \frac{2(r-k)(k^2-k+1-r)}{k(k-2)},\right.$$
$$\left. k^2 - r - \frac{r-1}{k-2} + \frac{2k(k-1)-r}{k(k-2)}\right\} .$$

The inequality $k^2 - k + 1 - \frac{2(r-k)(k^2-k+1-r)}{k(k-2)} < r$ holds if and only if $\frac{k^2}{2} < r < k^2 - k + 1$. If $k \geq 5$, this condition is fulfilled since $k^2 - k + 1 > k^2 - k$ and $\frac{k^2}{2} < k^2 - 3k + \frac{3}{4}\sqrt{k} + 2$. If $k = 4$ and $R = 0$, hence $r = 9$, then $k^2 - k + 1 - \frac{2(r-k)(k^2-k+1-r)}{k(k-2)} = 8 < r$; if $k = 4$ and $R = 1$, hence $r = 8$, then $k^2 - k + 1 - \frac{2(r-k)(k^2-k+1-r)}{k(k-2)} = 8 = r$.

Since $k^2 - 3k + \frac{3}{4}\sqrt{k} + 2 > \frac{k^2}{2} - \frac{k}{4} + \frac{3}{8}$ for all $k \geq 4$, the inequality $k^2 - r - \frac{r-1}{k-2} + \frac{2k(k-1)-r}{k(k-2)} < r$ is fulfilled in all cases.

Hence, for $k \geq 5$, in all three cases $|\mathcal{S}| < r$; for $k = 4$, in all three cases $|\mathcal{S}| \leq r$ and moreover $|\mathcal{S}| < r$ if $r \neq 8$. The theorem follows. □

We now summarize the results of this section.

Corollary 5.4.6. *Let $\mathcal{D}$ be a $2-(v,k,1)$ design, $k \geq 4$, with $r \geq k^2 - 3k + \frac{3}{4}\sqrt{k} + 2$, and let $\mathcal{S}$ be an Erdős-Ko-Rado set on $\mathcal{D}$. Then $|\mathcal{S}| \leq r$. If $r \neq k^2 - k + 1$ and $(r,k) \neq (8,4)$, then $|\mathcal{S}| = r$ if and only if $\mathcal{S}$ is a point-pencil.*

Proof. This follows immediately from Theorem 5.4.1 and Theorem 5.4.5. □

5.5 Maximal Erdős-Ko-Rado sets in unitals

The results from Lemma 5.2.2, Lemma 5.2.4, Lemma 5.2.6 and Lemma 5.2.7 can also be used in a different way. For a fixed class of designs, with v (or equivalently r) a function of k, an upper bound on the size of the largest maximal Erdős-Ko-Rado set different from a point-pencil can be computed. We show this for the unitals. Recall that a $2-(q^3+1, q+1, 1)$ design is a unital of order q. Note that the unique unital of order 2 has already been covered in Section 5.3. First we state Lemma 5.2.4 for a unital of order q.

Lemma 5.5.1. *Let $\mathcal{U}$ be a unital of order q and let $\mathcal{S}$ be an Erdős-Ko-Rado set on $\mathcal{U}$ such that $|\mathcal{P}'| = q(q+1)+b$, whereby $\mathcal{P}'$ is the set of points covered by the elements of $\mathcal{S}$. Then*

$$|\mathcal{S}| \leq \max\left\{q^2-q+1+\frac{b(b-1)}{q(q-1)}+\frac{2b}{q-1}, q+\frac{bq(q+2)}{q^2-1}+\frac{b(b-1)}{q^2-1}\right\} .$$

Lemma 5.5.2. *Let $\mathcal{U}$ be a unital of order q and let $\mathcal{S}$ be an Erdős-Ko-Rado set on $\mathcal{U}$ with $k_{\mathcal{S}} = q+1$. If $q \geq 4$, then $|\mathcal{S}| \leq q^2-q+1$. If $q=3$, then $|\mathcal{S}| \leq 8$.*

Proof. By Lemma 5.2.6 we know that $|\mathcal{P}'| = q^2+q+1$. We apply Lemma 5.5.1 and we find that $|\mathcal{S}| \leq \max\left\{q^2-q+1+\frac{2}{q-1}, q+\frac{q(q+2)}{q^2-1}\right\}$. The lemma immediately follows. □

Lemma 5.5.3. *Let $\mathcal{U}$ be a unital of order q and let $\mathcal{S}$ be an Erdős-Ko-Rado set on $\mathcal{U}$ with $k_{\mathcal{S}} = q$. If $q \geq 5$, then $|\mathcal{S}| \leq q^2-q+\sqrt[3]{q^2}-\frac{2}{3}\sqrt[3]{q}+1$. If $q=3$, then $|\mathcal{S}| \leq 7$; if $q=4$, then $|\mathcal{S}| \leq 13$.*

Proof. Denote $q^2-|\mathcal{S}|$ by a'. We can assume $a' < q$ since otherwise the lemma clearly holds. By Lemma 5.2.2, we know that $a' \geq 0$, and by Lemma 5.2.7 we know that $|\mathcal{P}'| = q^2+q+b$, with $0 \leq b \leq \frac{a'^2-a'}{q-a'}$. We apply Lemma 5.5.1 and we find that

$$|\mathcal{S}| \leq q^2-q+1+2\frac{a'(a'-1)}{(q-a')(q-1)}+\frac{a'(a'-1)(a'^2-q)}{q(q-1)(q-a')^2}$$
$$\text{or} \quad |\mathcal{S}| \leq q+\frac{qa'(q+2)(a'-1)}{(q^2-1)(q-a')}+\frac{a'(a'-1)(a'^2-q)}{(q-a')^2(q^2-1)} .$$

Using $|\mathcal{S}| = q^2 - a'$, the first inequality can be rewritten as

$$q(q-a'-1)(q-a')^2(q-1) \leq a'(a'-1)(2q^2-2qa'+a'^2-q) \; .$$

For $q=3$, this implies $a' \geq 2$ and for $q=4$, this implies $a' \geq 3$. For general $q \geq 5$, it implies $a' \geq q - \sqrt[3]{q^2} + \frac{2}{3}\sqrt[3]{q} - 1$. More details can be found in Computation A.1.3.

Now we look at the second inequality. Using $|\mathcal{S}| = q^2 - a'$, it can be rewritten as

$$(q^2-q-a')(q-a')^2(q^2-1) \leq a'(a'-1)(q^3-(a'-2)q^2-(2a'+1)q+a'^2) \; .$$

Using that $0 \leq a' < q$, it follows that $a' = q-1$.

Only one of the inequalities needs to hold, but $q - \sqrt[3]{q^2} + \frac{2}{3}\sqrt[3]{q} - 1 \leq q-1$. The lemma follows. $\square$

Theorem 5.5.4. *Let $\mathcal{U}$ be a unital of order q and let $\mathcal{S}$ be a maximal Erdős-Ko-Rado set on $\mathcal{U}$. If $q \geq 5$, then either $|\mathcal{S}| = q^2$ and $\mathcal{S}$ is a point-pencil, or else $|\mathcal{S}| \leq q^2 - q + \sqrt[3]{q^2} - \frac{2}{3}\sqrt[3]{q} + 1$. If $q = 4$, then either $|\mathcal{S}| = 16 = q^2$ and $\mathcal{S}$ is a point-pencil, or else $|\mathcal{S}| \leq 13 = q^2 - q + 1$. If $q = 3$, then either $|\mathcal{S}| = 9 = q^2$ and $\mathcal{S}$ is a point-pencil, or else $|\mathcal{S}| \leq 8$.*

Proof. If $\mathcal{S}$ is a point-pencil, then it contains q^2 elements. From now on, we assume that $\mathcal{S}$ is not a point-pencil. Recall the definition of $k_{\mathcal{S}}$. By Lemma 5.2.2, $k_{\mathcal{S}} \leq q+1$. Moreover, if $k_{\mathcal{S}} \leq q-1$, then $|\mathcal{S}| \leq q^2 - q - 1$.

First, we assume $q \geq 5$. If $k_{\mathcal{S}} = q$, then $|\mathcal{S}| \leq q^2 - q + \sqrt[3]{q^2} - \frac{2}{3}\sqrt[3]{q} + 1$ by Lemma 5.5.3. If $k_{\mathcal{S}} = q+1$, then $|\mathcal{S}| \leq q^2 - q + 1$ by Lemma 5.5.2.

The results for $q = 3, 4$ are obtained in the same way, using the results from Lemma 5.5.2 and Lemma 5.5.3. $\square$

Remark 5.5.5. Note that these results correspond with the result for classical unitals in Corollary 5.1.7 since the triangle contains only $q+2$ blocks.

Note that the unitals are not covered by Corollary 5.0.2. However, they are covered by Theorem 5.4.6. So we already knew that the point-pencils are the largest Erdős-Ko-Rado sets. The above theorem thus gives a bound on the size of the second-largest maximal Erdős-Ko-Rado set.

6

Kakeya sets in $\mathrm{AG}(2,q)$, q even

Though this be madness, yet there is method in 't.

Polonius in *Hamlet, Act II, Scene II*
by *William Shakespeare.*

In Remark 5.1.2 we already mentioned that a maximal Erdős-Ko-Rado set of lines in an affine plane, is necessarily a set containing a line of each parallel class. The affine point sets covered by such a line set are known as *Kakeya sets*. In this chapter we will study Kakeya sets in the affine plane $\mathrm{AG}(2,q)$. For the n-dimensional affine space $\mathrm{AG}(n,q)$, with $H_\infty = \mathrm{PG}(n-1,q)$ (the hyperplane 'at infinity') such that $\mathrm{AG}(n,q) \cup H_\infty = \mathrm{PG}(n,q)$, we define a Kakeya set as follows: for every point P on H_∞, let ℓ_P be a line in $\mathrm{PG}(n,q)$ through P not contained in H_∞. The point set

$$\mathcal{K} = \left(\bigcup_{P \in H_\infty} \ell_P \right) \setminus H_\infty$$

is called a Kakeya set, or a *minimal Besicovitch set*. The *finite field Kakeya problem* asks for the smallest size $k(n,q)$ of a Kakeya set in $\mathrm{AG}(n,q)$. It is the finite field version of the classical Euclidean Kakeya problem (see [117, Section

1.3] for a short survey) and was first posed by Wolff in the influential paper [132]. In that paper, it was conjectured that $k(n,q) \geq c_n q^n$, with $c_n > 0$ a constant only depending on n. The following theorem by Dvir showed that conjecture to be true.

Theorem 6.0.1 ([44]). *If $\mathcal{K}$ is a Kakeya set in* $\mathrm{AG}(n,q)$, *then*

$$|\mathcal{K}| \geq \binom{q+n-1}{n} \geq \frac{1}{n!}q^n \ .$$

The lower bound in this theorem is not sharp in general and was recently improved in [45] and [108]. The problem of finding the exact value of $k(n,q)$ seems to be very hard and gets more difficult as the dimension n increases. At this moment, only the value $k(2,q)$ is known. We will give a short survey of this in Section 6.1. In Section 6.2 we will present some results on arcs, which will be used in Section 6.3 to classify the third largest Kakeya set in $\mathrm{AG}(2,q)$, q even. Thereby we describe the first construction, up to our knowledge, of a small Kakeya set in $\mathrm{AG}(2,q)$, q even, not arising from a hyperoval. The results in this chapter were published in [16], which is joint work with Aart Blokhuis, Francesco Mazzocca and Leo Storme.

6.1 The known results for $\mathrm{AG}(2,q)$

Due to the existence of hyperovals in the affine plane $\mathrm{AG}(2,q)$, q even, and their nonexistence in the affine plane $\mathrm{AG}(2,q)$, q odd, the results on Kakeya sets in $\mathrm{AG}(2,q)$ differ between the case q even and the case q odd. Recall that hyperovals were introduced in Section 1.7. We first look at the case q odd.

Example 6.1.1. Consider a dual oval $\mathcal{O}$ (i.e. $q+1$ lines, no three concurrent) in $\mathrm{PG}(2,q)$, q odd, and let $H_\infty = \ell_\infty$ be a line in $\mathcal{O}$. Under these assumptions, every point $P \in \ell_\infty$, but one, lies on a second line $\ell_P \in \mathcal{O}$. Let A be this remaining point on ℓ_∞ and let ℓ_A be a line through it, different from ℓ_∞. Then the affine point set

$$\mathcal{K}(\mathcal{O},\ell_A) = \left(\bigcup_{P\in\ell_\infty} \ell_P\right) \setminus \ell_\infty$$

is a Kakeya set of size $\frac{q(q+1)}{2} + \frac{q-1}{2}$ in the affine plane $\mathrm{AG}(2,q) = \mathrm{PG}(2,q)\setminus\ell_\infty$. We can see this in the following way. On a line ℓ of $\mathcal{O}$, there are q affine points: one of them is only on ℓ, all others are on one other line of $\mathcal{O}$. The line ℓ_A contains one affine point which is on only one line of $\mathcal{O}$, and hence $\frac{q-1}{2}$ points on the other lines of $\mathcal{O}$.

Recall that every (dual) oval is a (dual) conic by Theorem 1.7.3.

In [17] the Kakeya sets described in the previous example were characterized as the smallest ones in $\mathrm{AG}(2,q)$, q odd, thereby solving a conjecture made in [54].

Theorem 6.1.2 ([17, Proposition 7]). *If $\mathcal{K}$ is a Kakeya set in $\mathrm{AG}(2,q)$, q odd, then*

$$|\mathcal{K}| \geq \frac{q(q+1)}{2} + \frac{q-1}{2}\ .$$

Equality holds if and only if $\mathcal{K}$ is a Kakeya set arising from a dual oval in $\mathrm{PG}(2,q) \supset \mathrm{AG}(2,q)$ as in Example 6.1.1.

Now we describe two ways to obtain a small Kakeya set in $\mathrm{AG}(2,q)$, q even.

Example 6.1.3. Consider a dual hyperoval $\mathcal{H}$ (i.e. a set of $q+2$ lines, no three concurrent) in $\mathrm{PG}(2,q)$, q even, and let ℓ_∞ be a line in $\mathcal{H}$. For every point $P \in \ell_\infty$, let ℓ_P be the line of $\mathcal{H}$ through P, different from ℓ_∞. Then the point set

$$\mathcal{K}(\mathcal{H}) = \left(\bigcup_{P \in \ell_\infty} \ell_P \right) \setminus \ell_\infty$$

is a Kakeya set in the affine plane $\mathrm{AG}(2,q) = \mathrm{PG}(2,q) \setminus \ell_\infty$. Its size equals $\frac{q(q+1)}{2}$ since every point of $\mathcal{K}(\mathcal{H})$ belongs to precisely two lines of $\mathcal{H}$.

Example 6.1.4. Let $\mathrm{AG}(2,q) = \mathrm{PG}(2,q) \setminus \ell_\infty$, q even, and let $\mathcal{H}$ be as in Example 6.1.3. Let ℓ_P be the line of $\mathcal{H}$ through P, different from ℓ_∞. Consider a point $A \in \ell_\infty$ and a line ℓ'_A through A different from ℓ_A and ℓ_∞. So, ℓ'_A is not a line of $\mathcal{H}$. Then the point set

$$\mathcal{K}(\mathcal{H}, \ell'_A) = \left(\bigcup_{P \in \ell_\infty \setminus \{A\}} (\ell_P \setminus \ell_\infty) \right) \cup (\ell'_A \setminus \ell_\infty)$$

is a Kakeya set in $\mathrm{AG}(2,q)$, whose size equals $\frac{q(q+1)}{2} + \frac{q}{2}$, since deleting the line ℓ_A from the Kakeya set $\mathcal{K}(\mathcal{H})$ does not decrease the number of covered points, and the line ℓ'_A contains $\frac{q}{2}$ affine points which lie on two lines of $\mathcal{H} \setminus \{\ell_A\}$ and $\frac{q}{2}$ affine points which lie on no lines of $\mathcal{H} \setminus \{\ell_A\}$.

It follows immediately from Theorem 6.0.1 that $k(2,q) \geq \frac{q(q+1)}{2}$. Moreover, it can easily be proved that $k(2,q) = \frac{q(q+1)}{2}$ if q is even and that equality only occurs for the Kakeya sets described in Example 6.1.3 (see e.g. the beginning of the proof of Lemma 6.3.2). Furthermore, in [15] the following result was proved (stated in its dual form). It classifies the second largest Kakeya set in $\mathrm{AG}(2,q)$, q even.

Theorem 6.1.5 ([15]). *There are no Kakeya sets $\mathcal{K}$ in $\mathrm{AG}(2,q)$, q even, with $\frac{q(q+1)}{2} < |\mathcal{K}| < \frac{q(q+1)}{2} + \frac{q}{2}$. Furthermore, all Kakeya sets of size $\frac{q(q+1)}{2} + \frac{q}{2}$ are given by Example 6.1.4.*

6.2 A few remarks on arcs

Arcs were introduced in Section 1.7. In this section we present a few lemmata about arcs, which were obtained using algebraic geometry. This discussion is base on [80, Section 10.1].

An *algebraic curve* (or *plane curve*) in $\mathrm{PG}(2,q)$ is a set of points in $\mathrm{PG}(2,q)$ defined by a homogeneous polynomial in three variables. Recall that lines in $\mathrm{PG}(2,q)$ can also be described by coefficient vectors. So, analogously, we can define an *algebraic envelope* in $\mathrm{PG}(2,q)$ as a set of lines in $\mathrm{PG}(2,q)$ determined by a homogeneous polynomial in three variables. It is clear that algebraic cureves and algebraic envelopes are dual concepts.

The *degree* of an algebraic curve is the degree of the corresponding polynomial. For an algebraic envelope the degree of the corresponding polynomial is called the *class*. The number of points on a line of an algebraic curve is bounded above by the degree of the curve. Analogously, the number of lines through a point of an algebraic envelope is at most its class. A *component* of an algebraic curve (envelope) determined by a polynomial f is an algebraic curve (envelope) defined by a polynomial of f.

Now we connect arcs to algebraic envelopes. A *tangent envelope* of a k-arc in $\mathrm{PG}(2,q)$ is an algebraic envelope containing all the tangent lines to this arc, and which is of class $q+2-k$ if q is even and of class $2(q+2-k)$ if q is odd[1]. If k is large enough, there is a unique tangent envelope, which is known as *the* tangent envelope. Dualizing the concept of a tangent envelope, we find a *tangent curve* to a dual k-arc in $\mathrm{PG}(2,q)$, containing all points which are covered precisely once by the lines of the dual arc. This is an algebraic curve of degree $q+2-k$ if q is even. The following theorem is proved in [80] in the setting of arcs and tangent envelopes, but we state it immediately in the setting of dual arcs and tangent curves.

Theorem 6.2.1 ([80, Corollary 10.3(ii)]). *Let $\mathcal{A}$ be a dual k-arc in the projective plane $\mathrm{PG}(2,q)$, q even and $k > \frac{q}{2}+1$, and let Γ_t be the tangent curve to this dual arc. The line ℓ extends $\mathcal{A}$ if and only if ℓ is a component of Γ_t.*

The following lemma applies this theorem about the tangent curve to a dual arc.

Lemma 6.2.2. *Let $\mathcal{A}$ be a dual k-arc in $\mathrm{PG}(2,q)$, q even, with $k > \frac{q}{2}+1$. A line, not extending $\mathcal{A}$, contains at least $\frac{q}{2}$ points not lying on lines of $\mathcal{A}$.*

Proof. Let Γ_t be the tangent curve of $\mathcal{A}$. Then Γ_t is an algebraic curve of degree $t = q+2-k$. By Theorem 6.2.1, a line extending $\mathcal{A}$ is a component

[1] In the statement of these theorems in [80], the values $q+k-2$ and $2(q+k-2)$ are given, but is clear from the context and the proof, that these are misprints.

of Γ_t and vice versa. Consider a line ℓ not extending $\mathcal{A}$. It intersects Γ_t in x points, with $x \leq t$. These points are the ones lying on precisely one line of $\mathcal{A}$. Consequently, $\frac{k-x}{2}$ points of ℓ are lying on two lines of $\mathcal{A}$. Hence, the number of points of ℓ not on $\mathcal{A}$ equals $(q+1)-x-\frac{k-x}{2} = q-\frac{k+x}{2}+1$. Using the bound on x, we find that $q-\frac{k+x}{2}+1 \geq q-\frac{k+t}{2}+1 = \frac{q}{2}$. The lemma follows. $\square$

6.3 Classifying the third largest example

The aim of this section is to determine the Kakeya sets $\mathcal{K}$ with $\frac{q(q+1)}{2}+\frac{q}{2} < |\mathcal{K}| \leq \frac{q(q+1)}{2}+\frac{3q}{4}$. We will prove that in $\mathrm{AG}(2,q)$, q even, there are no Kakeya sets whose size belongs to the corresponding open interval and we will characterize those of size $\frac{q(q+1)}{2}+\frac{3q}{4}$.

We describe a Kakeya set, which we will prove to be the (theoretical) third smallest example (provided that it exists). Recall the definition of a $(q+t)$-arc of type $(0,2,t)$ which was given in Section 1.7.

Example 6.3.1. Let $\mathcal{A}$ be a dual $(q+4)$-arc of type $(0,2,4)$ in $\mathrm{PG}(2,q)$, and let $\ell_0,\ell_1,\ell_2,\ell_\infty$ be four concurrent lines of $\mathcal{A}$. Consider the affine plane $\mathrm{AG}(2,q)=\mathrm{PG}(2,q)\setminus\ell_\infty$. Let $\mathcal{A}'$ be the line set $\mathcal{A}\setminus\{\ell_1,\ell_2,\ell_\infty\}$. Consider the set

$$\mathcal{K}(\mathcal{A}') = \bigcup_{\ell\in\mathcal{A}'} (\ell\setminus\ell_\infty)\,.$$

This is a Kakeya set since there is precisely one line of $\mathcal{A}'$ through every point of ℓ_∞. Note that every line of $\mathcal{A}'\setminus\{\ell_0\}$ has 1 affine point on four lines of $\mathcal{A}'$, $q-3$ affine points on two lines of $\mathcal{A}'$ and 2 affine points on one line of $\mathcal{A}'$; all affine points on the line ℓ_0 lie on a second line of $\mathcal{A}'$. So, the Kakeya set $\mathcal{K}(\mathcal{A}')$ has size $\frac{q(q+1)}{2}+\frac{3q}{4}$.

Lemma 6.3.2. *Let $\mathcal{K}=(\cup_{i=0}^{q}\ell_i)\setminus\ell_\infty$ be a Kakeya set in $\mathrm{AG}(2,q)=\mathrm{PG}(2,q)\setminus\ell_\infty$, such that its corresponding line set $\mathcal{L}=\{\ell_0,\dots,\ell_q\}$ contains a dual x-arc, but no dual $(x+1)$-arc. Then $\mathcal{K}$ contains at least $\frac{(q+4)(q+1)}{2}-2x-\left\lfloor\frac{x}{2}\right\rfloor$ points.*

Proof. Let $\mathcal{A}=\{\ell_0,\dots,\ell_{x-1}\}$ be a dual x-arc contained in $\mathcal{L}$. If we construct the Kakeya set line by line in the order $\ell_0,\dots,\ell_q$, adding the $(i+1)^{\text{th}}$ line ℓ_i increases the number of points in $\mathcal{K}$ by $q-i+m_i$, $m_i\geq 0$. Since $\{\ell_0,\dots,\ell_{x-1}\}$ is a dual x-arc, $m_i=0$ for $i=0,\dots,x-1$. For the line $\ell_i\in\{\ell_x,\dots,\ell_q\}$, we know $m_i\geq 1$ since none of these lines extends $\mathcal{A}$.

Let ℓ_k be a line of $\mathcal{K}$, with $m_k = 1$. Then ℓ_k contains precisely one intersection point $\ell_i \cap \ell_j$, $i, j < k$, of previously added lines. Assume one of these two lines, say ℓ_i, is not contained in $\mathcal{A}$, or equivalently $i \geq x$. Then the line ℓ_k extends $\mathcal{A}$ because it does not pass through an intersection point of two lines of $\mathcal{A}$. This is a contradiction since $\mathcal{A}$ does not contain a dual $(x+1)$-arc. Hence, each line $\ell_k \in \mathcal{K}$, with $m_k = 1$, contains precisely one intersection point of two lines of $\mathcal{A}$. Let $\mathcal{B}$ be the set $\{\ell_j \mid m_j = 1\}$. We call the points lying on two lines of $\mathcal{A}$ and a line of $\mathcal{B}$ *complete* points.

Let ℓ_a, ℓ_b, ℓ_c be three lines of $\mathcal{A}$ and let ℓ_k, ℓ_l be two lines of $\mathcal{B}$ such that ℓ_k passes through $\ell_a \cap \ell_b$ and ℓ_l passes through $\ell_a \cap \ell_c$, in other terms, such that $\ell_a \in \mathcal{A}$ contains two different complete points. Consider the line set $(\mathcal{A} \setminus \{\ell_a\}) \cup \{\ell_k, \ell_l\}$. This line set is a dual arc since $\mathcal{A}$ is a dual arc, and the lines ℓ_k and ℓ_l each contain precisely one intersection point of the lines of $\mathcal{A}$, both lying on ℓ_a. However, this line set contains $x + 1$ lines and is a subset of $\mathcal{L}$. We find a contradiction since we know that $\mathcal{L}$ contains no dual $(x+1)$-arc. Hence, a line of $\mathcal{A}$ contains at most one complete point. Consequently, $|\mathcal{B}| \leq \left\lfloor \frac{x}{2} \right\rfloor$.

From the previous arguments, it follows that $|\{\ell_j \mid m_j \geq 2\}| = (q+1) - x - |\mathcal{B}|$. So, we conclude

$$\begin{aligned}
|\mathcal{K}| = \sum_{i=0}^{q}(q-i) + \sum_{i=0}^{q} m_i &\geq \frac{q(q+1)}{2} + |\mathcal{B}| + 2 \cdot ((q+1) - x - |\mathcal{B}|) \\
&= \frac{(q+4)(q+1)}{2} - |\mathcal{B}| - 2x \\
&\geq \frac{(q+4)(q+1)}{2} - \left\lfloor \frac{x}{2} \right\rfloor - 2x. \qquad \square
\end{aligned}$$

Lemma 6.3.3. *Let $\mathcal{K} = (\cup_{i=0}^{q} \ell_i) \setminus \ell_\infty$ be a Kakeya set in $\mathrm{AG}(2,q) = \mathrm{PG}(2,q) \setminus \ell_\infty$, $q > 8$ even, with $|\mathcal{K}| \leq \frac{q(q+1)}{2} + \frac{3q}{4}$, and assume that the line set $\mathcal{T} = \{\ell_0, \ldots, \ell_q, \ell_\infty\}$ is not a dual hyperoval of $\mathrm{PG}(2,q)$. Then $\mathcal{T} \setminus \{\ell_\infty\}$ contains a dual q-arc or a dual $\left(\frac{q+1}{2}\right)$-arc, not extendable to a larger arc by the remaining lines of $\mathcal{T} \setminus \{\ell_\infty\}$.*

Proof. In the following, for every $j \in \{0, 1, \ldots, q\}$, we denote

$$\mathcal{L} = \mathcal{T} \setminus \{\ell_\infty\} = \{\ell_0, \ldots, \ell_q\}, \quad S_j = \left(\bigcup_{i=0}^{j} \ell_i \right) \setminus \ell_\infty, \quad |\mathcal{K}| = \frac{q(q+1)}{2} + \varepsilon\,.$$

Note that $\mathcal{K} = S_q$. By the assumptions, we know that

$$0 < \varepsilon \leq \frac{3q}{4} \,.$$

Then, $|S_j \setminus S_{j-1}| = q - j + m_j$, with $m_j \geq 0$, for $j = 1, 2, \ldots, q$. In other terms, passing from S_{j-1} to S_j by the addition of the $(j+1)^{\text{th}}$ line ℓ_j, the number of covered points increases by $q - j + m_j$. Moreover, a direct computation shows that

$$\frac{q(q+1)}{2} + \varepsilon = \sum_{i=0}^{q}(q - i + m_i) = \frac{q(q+1)}{2} + \sum_{i=0}^{q} m_i. \tag{6.1}$$

Denote by k, $k < q+1$, the maximal integer for which ℓ_∞ and k lines in $\mathcal{L}$ form a dual $(k+1)$-arc in $\mathrm{PG}(2,q)$ and, without loss of generality, assume that $\overline{\mathcal{A}} = \mathcal{A} \cup \{\ell_\infty\}$, with $\mathcal{A} = \{\ell_0, \ldots, \ell_{k-1}\}$, is such a dual $(k+1)$-arc. Imposing this assumption, we have $m_i = 0$ for $i = 0, 1, \ldots, k-1$, because each of the lines in $\mathcal{A}$ intersects the union of the remaining ones in exactly $k-1$ affine points. Moreover, because of the maximality of $\overline{\mathcal{A}}$ as a dual arc contained in $\mathcal{T}$, for $j \geq k$, no line ℓ_j extends $\overline{\mathcal{A}}$ and consequently $m_j \neq 0$ for $j \geq k$.

Now, we distinguish two cases: $k \leq \frac{q}{2}$ and $k \geq \frac{q}{2} + 1$. For $k \leq \frac{q}{2}$, we apply Lemma 6.3.2. We find that $|\mathcal{K}| \geq \frac{(q+4)(q+1)}{2} - \frac{q}{4} - q = \frac{q(q+2)}{2} + \frac{q}{4} + 2$. Hence, this possibility cannot occur. Now, we look at the case $k \geq \frac{q}{2} + 1$. Since $k + 1 > k \geq \frac{q}{2} + 1$, we can apply Lemma 6.2.2 on the dual $(k+1)$-arc $\overline{\mathcal{A}}$. Each of the lines $\ell_k, \ell_{k+1}, \ldots, \ell_q$ contains at least $\frac{q}{2}$ points not on a line of $\overline{\mathcal{A}}$. Moreover, setting $\mathcal{K}' = \left(\cup_{i=0}^{k-1}\ell_i\right) \setminus \ell_\infty$ and counting the number of points in $\mathcal{K}$, we find

$$\begin{aligned}
\frac{q(q+1)}{2} + \varepsilon = |\mathcal{K}| &= |\mathcal{K}'| + |(\cup_{j=k}^{q}\ell_j) \setminus (\mathcal{K}' \cup \ell_\infty)| \\
&\geq [q + (q-1) + \cdots + (q-k+1)] \\
&\quad + \left[\frac{q}{2} + (\frac{q}{2} - 1) + \cdots + (\frac{q}{2} - (q-k))\right] \\
&= \frac{k(3q - 2k + 2)}{2} = f(k).
\end{aligned} \tag{6.2}$$

Note that $k = q + 1$ would imply that $\overline{\mathcal{A}}$ is a dual hyperoval and that $\varepsilon = 0$. For $k \in \left[\frac{q}{2} + 2, q - 1\right]$, we find $f(k) \geq \frac{q(q+3)}{2} - 2 > \frac{q(q+1)}{2} + \frac{3q}{4} \geq \frac{q(q+1)}{2} + \varepsilon$ as $q > 8$. Consequently, $k \in \{\frac{q}{2} + 1, q\}$. □

Note that $f(\frac{q}{2}+1) = f(q) = \frac{q(q+2)}{2}$, with f as in the previous proof. This proves that $|\mathcal{K}| \notin \left]\frac{q(q+1)}{2}, \frac{q(q+1)}{2} + \frac{q}{2}\right[$, which is part of the result in Theorem 6.1.5.

Lemma 6.3.4. *Let* $\mathcal{K} = (\cup_{i=0}^{q}\ell_i)\setminus\ell_\infty$ *be a Kakeya set in* $\mathrm{AG}(2,q) = \mathrm{PG}(2,q)\setminus\ell_\infty$, $q > 8$ *even. Then* $|\mathcal{K}| \notin \left[\frac{q(q+1)}{2} + \frac{q}{2} + 1, \frac{q(q+1)}{2} + \frac{3q}{4} - 1\right]$.

Proof. We use the notation introduced in Lemma 6.3.3. Assume $\mathcal{K}$ covers precisely $\frac{q(q+2)}{2} + \varepsilon'$ points, $0 \leq \varepsilon' \leq \frac{q}{4}$. By Lemma 6.3.3, there are two cases: $k = q$ or $k = \frac{q}{2} + 1$. In the first case, $\overline{\mathcal{A}}$ is a dual $(q+1)$-arc in $\mathrm{PG}(2,q)$, containing ℓ_∞. By Theorem 1.7.2, this dual arc is contained in a unique dual hyperoval $\mathcal{H} = \overline{\mathcal{A}} \cup \{m\}$. Then $m \cap \ell_\infty = \ell_q \cap \ell_\infty$ since both $\mathcal{H}$ and $\mathcal{K}$ are Kakeya sets containing $\overline{\mathcal{A}}$. Obviously, $m \neq \ell_q$. Hence, the Kakeya set $\mathcal{K}$ is of the type given in Example 6.1.4 and $|\mathcal{K}| = \frac{(q+2)q}{2}$. Note that in this case, the inequality in (6.2) is an equality, $\varepsilon = \frac{q}{2}$ and $\varepsilon' = 0$.

Now, we look at the case $k = \frac{q}{2} + 1$. We apply Lemma 6.3.2 and we find

$$|\mathcal{K}| \geq \frac{(q+4)(q+1)}{2} - \left\lfloor \frac{q+2}{4} \right\rfloor - 2\left(\frac{q}{2}+1\right) = \frac{q(q+1)}{2} + \frac{3q}{4}\,.$$

The lemma follows from these observations. □

Lemma 6.3.5. *Let* $\mathcal{K} = (\cup_{i=0}^{q}\ell_i)\setminus\ell_\infty$ *be a Kakeya set in* $\mathrm{AG}(2,q) = \mathrm{PG}(2,q)\setminus\ell_\infty$, $q > 8$ *even, with* $|\mathcal{K}| = \frac{q(q+1)}{2} + \frac{3q}{4}$. *Then* $\mathcal{K}$ *is a Kakeya set of the type given in Example 6.3.1.*

Proof. We use the notation we introduced in Lemma 6.3.3 and Lemma 6.3.4. We recall that $\mathcal{L}$ is the line set $\{\ell_0, \ldots, \ell_q\}$. By the results of these lemmata and the arguments used in their proofs, we know that $\mathcal{L}$ contains a dual $\left(\frac{q}{2}+1\right)$-arc $\mathcal{A} = \{\ell_0, \ldots, \ell_{\frac{q}{2}}\}$. Furthermore, there is a value k', $\frac{q}{2} + 1 \leq k' \leq q$, such that $m_j = 1$ for $\frac{q}{2} + 1 \leq j \leq k'$ and $m_j \geq 2$ for $k' + 1 \leq j \leq q$. Just as in the preceding lemmata, every line ℓ_j, $\frac{q}{2} + 1 \leq j \leq k'$, contains precisely one intersection point of the lines of $\mathcal{A}$. Those intersection points were called complete points. Again arguing as in Lemma 6.3.2, we know every line of $\mathcal{A}$ contains at most one complete point, hence the set $\{\ell_j \mid m_j = 1\}$ has size at most $\frac{q}{4}$, which gives $k' \leq \frac{3q}{4}$. Using (6.1), we then obtain $k' = \frac{3q}{4}$ and $m_j = 2$ for $\frac{3q}{4} + 1 \leq j \leq q$. Thus, there are precisely $\frac{q}{4}$ complete points and all but one of the lines in $\mathcal{A}$ contain a complete point. Let ℓ_0 be the line without a complete point and let $\mathcal{A}'$ be the line set $\mathcal{A} \cup \{\ell_{\frac{q}{2}+1}, \ldots, \ell_{\frac{3q}{4}}\}$.

Let $\mathcal{B}$ be the line set $\{\ell_j \mid \frac{3q}{4}+1 \leq j \leq q\}$. Note that the intersection point of two lines of $\mathcal{B}$ cannot be on a line of $\mathcal{A}'$. This can be argued in the same way as the observation that a complete point cannot be on two lines of $\mathcal{A}' \setminus \mathcal{A}$. For a line in $\mathcal{B}$, there are two possibilities. Either, such a line contains a complete point and no other intersection point of two lines of $\mathcal{A}'$, or else it does not contain a complete point, but it contains two intersection points of two pairs of lines of $\mathcal{A}'$. Let $\mathcal{B}^* = \{\ell_j \mid \frac{3q}{4}+1 \leq j \leq \frac{3q}{4}+y\}$ be the set of the former lines and $\mathcal{B}^- = \{\ell_j \mid \frac{3q}{4}+y+1 \leq j \leq q\}$ be the set of the latter lines. Remark that we first add the lines of $\mathcal{B}^*$. The complete points lying on a line of $\mathcal{B}^*$ will be called *hypercomplete* points, and the intersection points of two lines of $\mathcal{A}'$, that are not complete points, but are lying on a line of $\mathcal{B}^-$, are called *new complete* points. It follows that there are y hypercomplete points, $\frac{q}{4}-y$ complete points that are not hypercomplete, and $2(\frac{q}{4}-y) = \frac{q}{2}-2y$ new complete points.

Since a line of $\mathcal{A}' \setminus \{\ell_0\}$ contains precisely one complete point before adding the lines of $\mathcal{B}$, it contains precisely one complete point, which is possibly hypercomplete. We will prove some properties of the hypercomplete and new complete points. Note that a point cannot be (hyper)complete and new complete at the same time.

- Firstly, we prove that a line of $\mathcal{A}'$ cannot contain a hypercomplete point and a new complete point. Let $\ell_i \in \mathcal{A} \setminus \{\ell_0\}$ be a line containing a hypercomplete point $\ell_i \cap \ell_j \cap \ell_n \cap \ell_p$ and a new complete point $\ell_i \cap \ell_s \cap \ell_r$, with $\ell_j \in \mathcal{A}$, $\ell_n \in \mathcal{A}' \setminus \mathcal{A}$, $\ell_p \in \mathcal{B}^*$, $\ell_s \in \mathcal{A}'$ and $\ell_r \in \mathcal{B}^-$. Consider now the ordering

$$\sigma = \ell_0, \ldots, \ell_{i-1}, \ell_{i+1}, \ldots, \ell_{\frac{q}{2}}, \ell_n, \ell_{\frac{q}{2}+1}, \ldots, \ell_{\frac{3q}{4}}, \ell_p, \ell_r, \ell_{\frac{3q}{4}+1}, \ldots, \ell_q, \ell_i \,.$$

  Remark that it is not indicated where ℓ_n, ℓ_p and ℓ_r are removed, but this can easily be seen. Using this alternative ordering, we can define m_a^σ for the line ℓ_a, the same way we defined m_i in the proof of Lemma 6.3.3. We find that $m_0^\sigma = \cdots = m_{i-1}^\sigma = m_{i+1}^\sigma = \cdots = m_{\frac{q}{2}}^\sigma = m_n^\sigma = 0$, that $m_{\frac{q}{2}+1}^\sigma = \cdots = m_{n-1}^\sigma = m_{n+1}^\sigma = \cdots = m_{\frac{3q}{4}}^\sigma = m_p^\sigma = m_r^\sigma = 1$ and $m_i^\sigma = 3$. This is a contradiction since also for this ordering the line set $\{\ell_j \mid m_j^\sigma = 1\}$ has size at most $\frac{q}{4}$. Now, let $\ell_i \in \mathcal{A}' \setminus \mathcal{A}$ be a line containing a hypercomplete point $\ell_i \cap \ell_j \cap \ell_n \cap \ell_p$ and a new complete point $\ell_i \cap \ell_s \cap \ell_r$, with $\ell_j, \ell_n \in \mathcal{A}$, $\ell_p \in \mathcal{B}^*$, $\ell_s \in \mathcal{A}'$ and $\ell_r \in \mathcal{B}^-$. Consider the ordering

$$\tau = \ell_0, \dots, \ell_{i-1}, \ell_{i+1}, \dots, \ell_{\frac{3q}{4}}, \ell_p, \ell_r, \ell_{\frac{3q}{4}+1}, \dots, \ell_q, \ell_i \ .$$

Analogously, we can define m_a^τ for a line ℓ_a. In the same way, we will find a contradiction.

- Similarly, we can also prove that a line of $\mathcal{L}$ cannot contain a complete point, which is possibly hypercomplete, and two new complete points. It is obvious that a line of $\mathcal{B} \cup \{\ell_0\}$ cannot contain a (hyper)complete point and two new complete points. Let $\ell_i \in \mathcal{A} \setminus \{\ell_0\}$ be a line containing a complete point $\ell_i \cap \ell_j \cap \ell_{j'}$ and two new complete points $\ell_i \cap \ell_n \cap \ell_{n'}$ and $\ell_i \cap \ell_p \cap \ell_{p'}$, with $\ell_j \in \mathcal{A}$, $\ell_{j'} \in \mathcal{A}' \setminus \mathcal{A}$, $\ell_p, \ell_n \in \mathcal{A}'$ and $\ell_{n'}, \ell_{p'} \in \mathcal{B}^-$. Consider the ordering

$$\sigma' = \ell_0, \dots, \ell_{i-1}, \ell_{i+1}, \dots, \ell_{\frac{q}{2}}, \ell_{j'}, \ell_{\frac{q}{2}+1}, \dots, \ell_{\frac{3q}{4}}, \ell_{n'}, \ell_{p'}, \ell_{\frac{3q}{4}+1}, \dots, \ell_q, \ell_i \ .$$

  As before it is not indicated where $\ell_{j'}$, $\ell_{n'}$ and $\ell_{p'}$ are removed. We define $m_a^{\sigma'}$ for ℓ_a using this ordering σ'. There are $\frac{q}{2}+1$ lines ℓ_a with $m_a^{\sigma'} = 0$ (the lines of $(\mathcal{A} \setminus \{\ell_i\}) \cup \{\ell_{j'}\}$). However, we find a contradiction as before since there are $\frac{q}{4}+1$ lines ℓ_a with $m_a^{\sigma'} = 1$ (the lines of $(\mathcal{A}' \setminus (\mathcal{A} \cup \{\ell_{j'}\})) \cup \{\ell_{n'}, \ell_{p'}\}$). If the complete point on ℓ_i is hypercomplete, then this statement follows immediately from the previous observation. Now let $\ell_i \in \mathcal{A}' \setminus \mathcal{A}$ be a line containing a complete point $\ell_i \cap \ell_j \cap \ell_{j'}$ and two new complete points $\ell_i \cap \ell_n \cap \ell_{n'}$ and $\ell_i \cap \ell_p \cap \ell_{p'}$, with $\ell_j, \ell_{j'} \in \mathcal{A}$, $\ell_p, \ell_n \in \mathcal{A}'$ and $\ell_{n'}, \ell_{p'} \in \mathcal{B}^-$. In this case, we consider the ordering

$$\tau' = \ell_0, \dots, \ell_{i-1}, \ell_{i+1}, \dots, \ell_{\frac{3q}{4}}, \ell_{p'}, \ell_{n'}, \ell_{\frac{3q}{4}+1}, \dots, \ell_q, \ell_i.$$

  Defining as before $m_a^{\tau'}$ for the line ℓ_a, we find a contradiction as before. Also in this case, the complete point is allowed to be hypercomplete.

- Finally, we prove that the line ℓ_0 cannot contain new complete points. Note first that the lines of $\mathcal{A}' \setminus \{\ell_0\}$ can be partitioned in $\frac{q}{4}$ sets of 3 lines going through a common complete point. Two of these lines belong to $\mathcal{A}$ and one belongs to $\mathcal{A}' \setminus \mathcal{A}$. Let C_a be the set of three lines containing the complete point on $\ell_a \in \mathcal{A}' \setminus \{\ell_0\}$. By swapping their positions in the ordering of the lines in $\mathcal{A}'$, each of the lines can be chosen to be the one in $\mathcal{A}' \setminus \mathcal{A}$.

  Now, assume that ℓ_0 contains a new complete point $\ell_0 \cap \ell_i \cap \ell_r$, with $\ell_i \in \mathcal{A}' \setminus \{\ell_0\}$ and $\ell_r \in \mathcal{B}^-$. Let $\ell_j \cap \ell_k \cap \ell_r$ be the second new complete point on ℓ_r, with $\ell_j, \ell_k \in \mathcal{A}' \setminus \{\ell_0\}$. Since $\ell_j \cap \ell_k$ is not a complete point, the sets C_j and C_k are different. So, at most one of them equals C_i. Without loss of generality, we can assume that C_j and C_i are different (and hence disjoint). Thus, by the above, we can choose simultaneously both ℓ_i and ℓ_j to be in $\mathcal{A}' \setminus \mathcal{A}$. However, then the set $\mathcal{A} \cup \{\ell_r\}$ is a dual $(\frac{q}{2}+2)$-arc contained in $\mathcal{L}$, a contradiction to Lemma 6.3.3.

Define the set $S' = \{(P,\ell) \mid P \text{ a hypercomplete point}, \ell \in \mathcal{L} \setminus \{\ell_0\}, P \in \ell\}$. We count the number of elements in this set in two ways. On the one hand, we find $|S'| = 4y$ since every hypercomplete point lies on precisely four lines of $\mathcal{L}$, none of which is ℓ_0. On the other hand, we find $|S'| \leq y + \left(\frac{3q}{4} - 2 \cdot (\frac{q}{2} - 2y)\right)$ since every line of $\mathcal{B}^*$ contains one hypercomplete point, none of the lines of $\mathcal{B}^-$ contains a hypercomplete point and none of the $\frac{3q}{4}$ lines in $\mathcal{A}' \setminus \{\ell_0\}$ contains a hypercomplete and a new complete point. Moreover, every line in $\mathcal{A}' \setminus \{\ell_0\}$ contains a complete point (possibly hypercomplete), hence contains at most one new complete point. Consequently, all the $2 \cdot (\frac{q}{2} - 2y)$ lines of $\mathcal{A}'$ through a new complete point are different and none of them is equal to ℓ_0 by the last of the above properties.

Thus, we find $4y \leq 5y - \frac{q}{4}$. Hence, $y \geq \frac{q}{4}$; consequently $\mathcal{B}^-$ is empty, $|\mathcal{B}^*| = \frac{q}{4}$, there are no new complete points and all $\frac{q}{4}$ complete points are hypercomplete. Since a line of $\mathcal{A}'$ contains at most one complete point regarding the lines of $\mathcal{A}'$, a line of $\mathcal{L}$ contains at most one hypercomplete point. Hence, the lines of $\mathcal{L} \setminus \{\ell_0\}$ can be partitioned in $\frac{q}{4}$ groups of four lines, each going through a common (hypercomplete) point. Furthermore, there are precisely $\frac{q}{4}$ points lying on 4 lines of $\mathcal{L}$ (the hypercomplete ones), there are $2q$ points lying on precisely one line of $\mathcal{L}$ (2 on each line through a hypercomplete point and none on ℓ_0) and there are $\frac{q(q-2)}{2}$ points on precisely two lines of $\mathcal{L}$.

Consider the binary code $C = C(2,q)$ generated by the lines of $\mathrm{PG}(2,q)$ (the points correspond to the positions), introduced in Section 1.8. Let c be the code word which is the sum of the (incidence vectors of) lines of $\mathcal{L} \cup \{\ell_\infty\}$. This corresponds to the set of points which are covered precisely once by the lines of $\mathcal{L} \cup \{\ell_\infty\}$. By the previous arguments, this is a code word of weight $2q$. Moreover, c is also a code word of $C^\perp$ since it can be written as the sum of $\frac{q}{2}+1$ differences of incidence vectors of two lines. Using Theorem 1.8.4, we find that c is the difference of the incidence vectors of two lines. Thus, the points covered only once by the lines of $\mathcal{L}$ are lying on two lines. Denote these two lines by m and m'. Then, m and m' intersect each of the lines $\ell_1, \ldots, \ell_q$ in an affine point since $\ell_1, \ldots, \ell_q$ each contain two points lying on precisely one line of $\mathcal{L}$. Consequently, $m \cap \ell_\infty = m' \cap \ell_\infty = \ell_0 \cap \ell_\infty$.

Now, we consider the line set $\{\ell_0, \ldots, \ell_q, \ell_\infty, m, m'\}$. This is a set of $q + 4$ lines in $\mathrm{PG}(2, q)$ such that every point is contained in 0, 2 or 4 lines of the set. Hence, this is a dual $(q + 4, 4)$-arc of type $(0, 2, 4)$. We conclude that the Kakeya set is of the type described in Example 6.3.1. $\square$

We summarize the known results about the smallest Kakeya sets in the next theorem.

Theorem 6.3.6. *Let $\mathcal{K}$ be a Kakeya set in $\mathrm{AG}(2, q) = \mathrm{PG}(2, q) \setminus \ell_\infty$, $q > 8$ even. Then, only the following possibilities can occur.*

- $|\mathcal{K}| = \frac{q(q+1)}{2}$ *and $\mathcal{K}$ arises from a dual hyperoval.*
- $|\mathcal{K}| = \frac{q(q+1)}{2} + \frac{q}{2}$ *and $\mathcal{K}$ is a Kakeya set of the type given in Example 6.1.4.*
- $|\mathcal{K}| = \frac{q(q+1)}{2} + \frac{3q}{4}$ *and $\mathcal{K}$ is a Kakeya set of the type given in Example 6.3.1.*
- $|\mathcal{K}| \geq \frac{q(q+1)}{2} + \frac{3q}{4} + 1$.

Remark 6.3.7. We have a look at the smallest cases for q, which are not covered by this theorem.

For $q = 2$, Theorem 6.1.5 classifies all Kakeya sets since $\frac{q(q+2)}{2} = 4 = |\mathrm{AG}(2, q)|$ in this case.

For $q = 4$, Theorem 6.1.5 classifies the Kakeya sets of size 10 and 12, and excludes size 11. Moreover, Lemma 6.3.4 is trivially valid because all other Kakeya sets have size at least $13 = \frac{q(q+2)}{2} + \frac{q}{4}$. The Kakeya sets of size 13 have not been classified by the theorems thereafter. However, by checking the possbible cases by hand, one can see that a Kakeya set of size 13 arises from a line set consisting of four concurrent lines and one line not concurrent with these four. This corresponds to a Kakeya set from Example 6.3.1. The only Kakeya set with size larger than 13 is the set of all points, with size 16, which necessarily corresponds to a line set consisting of five concurrent lines.

For $q = 8$, Theorem 6.1.5 classifies the Kakeya sets of size 36 and 40, and excludes the sizes 37, 38 and 39. In this case, $\frac{q(q+3)}{2} - 2 = 42 = \frac{q(q+2)}{2} + \frac{q}{4}$, so the proof of Lemma 6.3.3 does not continue. However, it does follow that a Kakeya set of size 41 contains a dual 8-arc or a dual 5-arc that is not extendable to a dual 6-arc with an affine line of $\mathcal{K}$. This is enough for the proof of Lemma 6.3.4 and hence we can exclude the size 41. Kakeya sets of the type given in Example 6.3.1 have size 42, but it is not proved that this is the only possibility for a Kakeya set of that type.

7

Small maximal partial t-spreads in $\mathrm{PG}(2t+1,q)$

Wer Großes will, muß sich zusammenraffen;
In der Beschränkung zeigt sich erst der Meister,
Und das Gesetz nur kann uns Freiheit geben.

Das Sonett, Goethe.

In this chapter we study small maximal partial spreads. Maximal partial spreads were defined in Section 1.7. In [64], lower bounds on the size of a maximal partial t-spread in $\mathrm{PG}(n,q)$, $n \geq 3t+1$, were derived. In Section 7.2 we will prove a lower bound on the size of maximal partial t-spreads in $\mathrm{PG}(2t+1,q)$. It will improve on the lower bound $q+\sqrt{q}-1$, which was proved in [7, Theorem 4]. Very recently, the lower bound 5 was proved in [1, Lemma 4.15], which is useful for small values of q. It was during the preparation of this paper that John Bamberg, one of the authors, raised this question about lower bounds on the size of maximal partial t-spreads in $\mathrm{PG}(2t+1,q)$.

For $t=1$, this problem was already studied by Glynn.

Theorem 7.0.1 ([62]). *A maximal partial line spread in* $\mathrm{PG}(3,q)$ *contains* *at least* $2q$ *elements.*

In [70], it is shown that this bound is not sharp for $q = 5$. In [90], small maximal partial spreads of generators in finite classical polar spaces are studied. Lower bounds on the size of these maximal partial spreads were found. An overview of these results can be found in [90, Table 1].

In Section 7.1, we prove preparatory lemmata for the results in Section 7.2. In Section 7.3, we derive additional results for small maximal partial t-spreads in $\mathrm{PG}(2t+1,q)$ by investigating the relationship with blocking sets. This chapter is based on the results in [37]. As we mentioned above, John Bamberg instigated this research.

7.1 Counting skew subspaces

In this section we count the number of t-spaces skew to one, two or three fixed pairwise disjoint t-spaces in $\mathrm{PG}(2t+1,q)$. The first result is a special case of Theorem 4.1.1.

Theorem 7.1.1. *The number of t-spaces, skew to a fixed t-space, in the projective space $\mathrm{PG}(2t+1,q)$ equals $a_t(q) = q^{(t+1)^2}$.*

The next result will allow us to find the other two values.

Lemma 7.1.2. *Let π_1 and π_2 be two disjoint t-spaces in $\mathrm{PG}(2t+1,q)$, and let σ be a k-space disjoint to both, $-1 \leq k \leq t-1$. The number of t-spaces through σ skew to both π_1 and π_2 equals*

$$d_t^k(q) = q^{\frac{(k+t+1)(t-k)}{2}} \prod_{i=1}^{t-k} (q^i - 1) \; .$$

Proof. A t-space π through σ is generated by $t-k$ linearly independent points not in σ. The first $i-1$ chosen points determine a $(k+i-1)$-space σ_{i-1} through σ, intersecting neither π_1 nor π_2, $i = 1, \ldots, t-k-1$. Then the i^{th} point must be a point not in $\langle \pi_1, \sigma_{i-1} \rangle \cup \langle \pi_2, \sigma_{i-1} \rangle$. Note that $\langle \pi_1, \sigma_{i-1} \rangle \cap \langle \pi_2, \sigma_{i-1} \rangle$ is a $(2k+2i-1)$-space. Hence, there are

$$\frac{q^{2t+2}-1}{q-1} - 2\frac{q^{t+k+i+1}-1}{q-1} + \frac{q^{2k+2i}-1}{q-1} = q^{2k+2i} \frac{(q^{t-k-i+1}-1)^2}{q-1}$$

different points that can be chosen as i^{th} point. So, there are

$$\prod_{i=1}^{t-k} q^{2k+2i} \frac{(q^{t-k-i+1}-1)^2}{q-1}$$

different tuples of $t-k$ points, together with σ, generating a t-space π through σ. The t-space π is defined by

$$(\theta_t - \theta_k)(\theta_t - \theta_{k+1}) \cdots (\theta_t - \theta_{t-1}) = \prod_{i=1}^{t-k} q^{k+i} \frac{q^{t-k-i+1}-1}{q-1}$$

different tuples of $t-k$ points. Consequently there are

$$\frac{\prod_{i=1}^{t-k} q^{2k+2i}(q^{t-k-i+1}-1)^2}{\prod_{i=1}^{t-k} q^{k+i}(q^{t-k-i+1}-1)} = q^{\frac{(k+t+1)(t-k)}{2}} \prod_{i=1}^{t-k}(q^i-1)$$

such t-spaces. □

If σ is a t-space in the above lemma, then the number of t-spaces through σ skew to both π_1 and π_2 equals one. This corresponds to the formula given in the lemma, using the convention that an empty product equals 1.

Corollary 7.1.3. *The number of t-spaces, skew to two fixed disjoint t-spaces, in $\mathrm{PG}(2t+1,q)$ equals*

$$b_t(q) = q^{\binom{t+1}{2}} \prod_{i=1}^{t+1}(q^i-1) \ .$$

Proof. This is a direct application of Lemma 7.1.2, since $b_t(q) = d_t^{-1}(q)$. □

The following result is proved in [90, Lemma 2] for generators of a finite classical polar space. The proof for t-spaces in $\mathrm{PG}(2t+1,q)$ is similar to the original proof. It is added for the sake of completeness.

Lemma 7.1.4. *Let P be a certain property of t-spaces of $\mathrm{PG}(2t+1,q)$ and let π be a fixed t-space. Denote the number of t-spaces fulfilling P and intersecting π in a subspace of dimension k by z_k, $k \leq t$. Denote the number of pairs (U,σ), with U a k-space contained in π, σ a t-space fulfilling P, and such that $U \subseteq \sigma$, by x_k. Then,*

$$z_k = \sum_{l=k}^{t}(-1)^{l-k}\begin{bmatrix} l+1 \\ k+1 \end{bmatrix}_q q^{\binom{l-k}{2}} x_l \ .$$

Proof. By counting the pairs (U,σ) with U a k-space, σ a t-space fulfilling P, and such that $U \subseteq \pi \cap \sigma$, we find that

$$x_k = \sum_{l=k}^{t} z_l \begin{bmatrix} l+1 \\ k+1 \end{bmatrix}_q = z_k + \sum_{l=k+1}^{t} z_l \begin{bmatrix} l+1 \\ k+1 \end{bmatrix}_q \ .$$

Now, we prove the desired equality using (backward) induction on k. For $k=t$, it is immediately clear that $z_k = x_k$. Now, we assume that the equality is proved for all values from $k+1$ to t, and we prove it for k. Using the first equality, we find:

$$\begin{aligned}
z_k &= x_k - \sum_{l=k+1}^{t} z_l \begin{bmatrix} l+1 \\ k+1 \end{bmatrix}_q \\
&= x_k - \sum_{l=k+1}^{t} \begin{bmatrix} l+1 \\ k+1 \end{bmatrix}_q \left(\sum_{j=l}^{t} (-1)^{j-l} \begin{bmatrix} j+1 \\ l+1 \end{bmatrix}_q q^{\binom{j-l}{2}} x_j \right) \\
&= x_k - \sum_{l=k+1}^{t} \sum_{j=l}^{t} (-1)^{j-l} \begin{bmatrix} l+1 \\ k+1 \end{bmatrix}_q \begin{bmatrix} j+1 \\ l+1 \end{bmatrix}_q q^{\binom{j-l}{2}} x_j \\
&= x_k - \sum_{j=k+1}^{t} x_j (-1)^{j-k} \begin{bmatrix} j+1 \\ k+1 \end{bmatrix}_q \sum_{l=k+1}^{j} (-1)^{k-l} \begin{bmatrix} j-k \\ l-k \end{bmatrix}_q q^{\binom{j-l}{2}} .
\end{aligned}$$

We know

$$\begin{aligned}
\sum_{l=k+1}^{j} (-1)^{k-l} \begin{bmatrix} j-k \\ l-k \end{bmatrix}_q q^{\binom{j-l}{2}} &= \sum_{l=1}^{j-k} (-1)^{-l} \begin{bmatrix} j-k \\ l \end{bmatrix}_q q^{\binom{j-k-l}{2}} \\
&= \sum_{l=0}^{j-k-1} (-1)^{k-j+l} \begin{bmatrix} j-k \\ l \end{bmatrix}_q q^{\binom{l}{2}} \\
&= (-1)^{k-j} \sum_{l=0}^{j-k} (-1)^{l} \begin{bmatrix} j-k \\ l \end{bmatrix}_q q^{\binom{l}{2}} - q^{\binom{j-k}{2}} \\
&= (-1)^{k-j} \prod_{l=0}^{j-k-1} (1-q^l) - q^{\binom{j-k}{2}} \\
&= -q^{\binom{j-k}{2}} .
\end{aligned}$$

In the penultimate step we used the q-binomial theorem, which we already stated in Lemma 4.3.1. Substituting this second result into the first one, we find the equality we wanted to prove. □

Lemma 7.1.5. *The number of t-spaces, skew to three fixed pairwise disjoint*

t-spaces, in $\mathrm{PG}(2t+1,q)$ *equals*

$$c_t(q) = q^{\binom{t+1}{2}} \left((-1)^{t+1} + \sum_{l=-1}^{t-1} (-1)^{l+1} \prod_{i=l+2}^{t+1} (q^i - 1) \right) .$$

Proof. Let π_1, π_2 and π_3 be three pairwise disjoint t-spaces in $\mathrm{PG}(2t+1,q)$. We define the property P_{12} as follows: a t-space has property P_{12} if it is skew to both π_1 and π_2. The number of t-spaces having this property and containing a fixed l-dimensional subspace of π_3 equals $d_t^l(q)$ by Lemma 7.1.2. Recall that π_3 contains $\begin{bmatrix} t+1 \\ l+1 \end{bmatrix}_q$ different l-dimensional subspaces. Using Lemma 7.1.2 and Lemma 7.1.4, we find that the number of t-spaces skew to both π_1 and π_2, and meeting π_3 in a subspace of dimension k equals

$$\begin{aligned}
z_t^k(q) &= \sum_{l=k}^{t-1} \left((-1)^{l-k} \begin{bmatrix} l+1 \\ k+1 \end{bmatrix}_q q^{\binom{l-k}{2}} \left(\begin{bmatrix} t+1 \\ l+1 \end{bmatrix}_q d_t^l(q) \right) \right) \\
&\quad + (-1)^{t-k} \begin{bmatrix} t+1 \\ k+1 \end{bmatrix}_q q^{\binom{t-k}{2}} \\
&= \sum_{l=k}^{t-1} \left((-1)^{l-k} \begin{bmatrix} l+1 \\ k+1 \end{bmatrix}_q q^{\binom{l-k}{2}} \left(\begin{bmatrix} t+1 \\ l+1 \end{bmatrix}_q q^{\frac{(l+t+1)(t-l)}{2}} \prod_{i=1}^{t-l} (q^i - 1) \right) \right) \\
&\quad + (-1)^{t-k} \begin{bmatrix} t+1 \\ k+1 \end{bmatrix}_q q^{\binom{t-k}{2}} \\
&= \sum_{l=k}^{t-1} \left((-1)^{l-k} \begin{bmatrix} l+1 \\ k+1 \end{bmatrix}_q q^{\binom{l-k}{2} + \frac{(l+t+1)(t-l)}{2}} \prod_{i=l+2}^{t+1} (q^i - 1) \right) \\
&\quad + (-1)^{t-k} \begin{bmatrix} t+1 \\ k+1 \end{bmatrix}_q q^{\binom{t-k}{2}} .
\end{aligned}$$

The number of t-spaces skew to π_1, π_2 and π_3, is $z_t^{-1}(q)$, the number of t-spaces having property P_{12} and meeting π_3 in an empty space. Consequently,

$$c_t(q) = z_t^{-1}(q) = q^{\binom{t+1}{2}} \left((-1)^{t+1} + \sum_{l=-1}^{t-1} (-1)^{l+1} \prod_{i=l+2}^{t+1} (q^i - 1) \right) .$$

Hereby we used that $\binom{l+1}{2} + \frac{(l+t+1)(t-l)}{2} = \frac{t(t+1)}{2}$. $\square$

Remark 7.1.6. An alternative proof for the result in Lemma 7.1.5 was suggested by Aart Blokhuis. We present a sketch of this proof. Let I and O be the $(t+1)\times(t+1)$ identity matrix and the $(t+1)\times(t+1)$ all-zero matrix, respectively. The three fixed pairwise disjoint t-spaces can be represented by the matrices $[O \quad I]$, $[I \quad O]$ and $[I \quad I]$. Then, there is a one-to-one correspondence between the t-spaces disjoint to these three t-spaces and the matrices $[I \quad A]$, with A a non-singular $(t+1)\times(t+1)$ matrix such that also $A-I$ is non-singular. Counting the number of such matrices A yields the number of t-spaces disjoint to the three fixed t-spaces.

7.2 A lower bound

In this section we will prove the main theorem of this chapter, a lower bound on the number of elements in a maximal partial spread of t-spaces in $\mathrm{PG}(2t+1,q)$. We follow the approach introduced by Glynn in [62]. First we present four equalities which will play a crucial role in the proof of Theorem 7.2.5. Recall that $a_t(q)$, $b_t(q)$ and $c_t(q)$ are the number of t-spaces in $\mathrm{PG}(2t+1,q)$, skew to one, two and three fixed pairwise disjoint t-spaces, respectively.

Lemma 7.2.1. *Let $\mathcal{S}$ be a partial spread of t-spaces in $\mathrm{PG}(2t+1,q)$. Denote the number of t-spaces not in $\mathcal{S}$, meeting i different elements of $\mathcal{S}$, by n_i. Then, the following equalities are valid:*

$$\sum_{i\geq 0} n_i = \begin{bmatrix} 2t+2 \\ t+1 \end{bmatrix}_q - |\mathcal{S}| \ ,$$

$$\sum_{i\geq 0} i n_i = |\mathcal{S}| \left(\begin{bmatrix} 2t+2 \\ t+1 \end{bmatrix}_q - a_t(q) - 1 \right) \ ,$$

$$\sum_{i\geq 0} i(i-1) n_i = |\mathcal{S}|(|\mathcal{S}|-1) \left(\begin{bmatrix} 2t+2 \\ t+1 \end{bmatrix}_q - 2a_t(q) + b_t(q) \right) \ ,$$

$$\sum_{i\geq 0} i(i-1)(i-2) n_i = |\mathcal{S}|(|\mathcal{S}|-1)(|\mathcal{S}|-2) \cdot \left(\begin{bmatrix} 2t+2 \\ t+1 \end{bmatrix}_q - 3a_t(q) + 3b_t(q) - c_t(q) \right) \ .$$

Proof. The first equality is obtained by counting the total number of t-spaces in $\mathrm{PG}(2t+1,q)$ not in $\mathcal{S}$. The second equality is obtained by counting the tuples (π,σ), with $\pi \in \mathcal{S}$, $\sigma \notin \mathcal{S}$ a t-space, and such that π and σ have a non-empty intersection. The third equality is obtained by counting the tuples (π_1,π_2,σ), with $\pi_1,\pi_2 \in \mathcal{S}$, $\pi_1 \neq \pi_2$, $\sigma \notin \mathcal{S}$ a t-space, and such that the intersection $\pi_i \cap \sigma$ is non-empty, $i = 1,2$. The fourth equality is obtained by counting the tuples $(\pi_1,\pi_2,\pi_3,\sigma)$, with $\pi_1,\pi_2,\pi_3 \in \mathcal{S}$, $\pi_1 \neq \pi_2 \neq \pi_3 \neq \pi_1$, $\sigma \notin \mathcal{S}$ a t-space, and such that the intersection $\pi_i \cap \sigma$ is non-empty, $i = 1,2,3$. In the two final equalities we have used the inclusion-exclusion principle. □

We present some inequalities that will be used in the proof of the main theorem.

Corollary 7.2.2. *Let $s,t,q \in \mathbb{N}$ be such that $s \leq t$ and $q \geq 3$. If $(s,q) \neq (0,3)$, then*

$$\prod_{i=s}^{t}(q^i+1) \leq (q^s+2)q^{\binom{t+1}{2}-\binom{s+1}{2}} .$$

Proof. This is a weaker version of Lemma 4.2.5. □

Note that this weaker inequality is indeed not valid if $(s,q) = (0,3)$ and $t \geq 3$.

Notation 7.2.3. We denote $c_t(q) - b_t(q)$ by $c'_t(q)$. Note that

$$c'_t(q) = q^{\binom{t+1}{2}} \left((-1)^{t+1} + \sum_{l=0}^{t-1} (-1)^{l+1} \prod_{i=l+2}^{t+1} (q^i - 1) \right) .$$

Lemma 7.2.4. *Let t and $q \geq 3$ be two natural numbers. The following inequalities are valid.*

$$\begin{aligned}
&\textit{For } t \geq 3: && b_t(q) \leq (q-1)(q^2-1)(q^3-1)(q^4-1)q^{(t+1)^2-10} .\\
&\textit{For } t \geq 2: && -c'_t(q) \leq (q^5-2q^3-q^2+3)q^{(t+1)^2-6} .\\
&\textit{For } t \geq 5: && \begin{bmatrix} 2t+2 \\ t+1 \end{bmatrix}_q \leq (q^5+2)(q^6+2)\begin{bmatrix} 10 \\ 5 \end{bmatrix}_q q^{(t+1)^2-36} .
\end{aligned}$$

Proof. The first inequality is immediate. The second inequality can be proved using induction on t. Direct computation shows that $-c'_2(q) = (q^5 - 2q^3 -$

$q^2+3)q^3$. In the proof of the induction step we use that $c'_t(q) = q^t(q^{t+1}-1)c'_{t-1}(q) + (-1)^{t+1}q^{\binom{t+1}{2}}$:

$$\begin{aligned}
-c'_t(q) &= -q^t(q^{t+1}-1)c'_{t-1}(q) + (-1)^t q^{\binom{t+1}{2}} \\
&\leq q^t(q^{t+1}-1)(q^5-2q^3-q^2+3)q^{t^2-6} + q^{\binom{t+1}{2}} \\
&\leq (q^5-2q^3-q^2+3)q^{(t+1)^2-6} - (q^5-2q^3-q^2+3)q^{t^2+t-6} + q^{\binom{t+1}{2}} \\
&\leq (q^5-2q^3-q^2+3)q^{(t+1)^2-6}\ .
\end{aligned}$$

We now consider the final inequality. We note that

$$\begin{bmatrix}2t+2\\ t+1\end{bmatrix}_q = (q^{t+1}+1)\left(q^t + \frac{q^t-1}{q^{t+1}-1}\right)\begin{bmatrix}2t\\ t\end{bmatrix}_q < (q^{t+1}+1)(q^t+1)\begin{bmatrix}2t\\ t\end{bmatrix}_q\ .$$

Using induction, we can prove immediately

$$\begin{bmatrix}2t+2\\ t+1\end{bmatrix}_q \leq \prod_{i=5}^{t}(q^i+1)\prod_{i=6}^{t+1}(q^i+1)\begin{bmatrix}10\\ 5\end{bmatrix}_q\ .$$

By applying Lemma 7.2.2 twice, we find the third inequality. □

Now, we can prove the main theorem.

Theorem 7.2.5. *A maximal partial spread of t-spaces in $\mathrm{PG}(2t+1,q)$, $t \geq 2$, contains at least $2q-1$ elements.*

Proof. Since $c_t(2) > 0$ for all t, we know that a maximal partial spread of t-spaces in $\mathrm{PG}(2t+1,2)$ contains at least four t-spaces. So, we can assume $q \geq 3$. Let $\mathcal{S}$ be a maximal partial spread of t-spaces in $\mathrm{PG}(2t+1,q)$. Denote the number of t-spaces not in $\mathcal{S}$, meeting i elements of $\mathcal{S}$, by n_i. We know $n_0 = 0$ since $\mathcal{S}$ is maximal. Hence,

$$\begin{aligned}
0 &\leq \sum_{i\geq 0}(i-1)(i-3)(i-4)n_i \\
&= \sum_{i\geq 0} i(i-1)(i-2)n_i - 5\sum_{i\geq 0} i(i-1)n_i + 12\sum_{i\geq 0} in_i - 12\sum_{i\geq 0} n_i\ .
\end{aligned}$$

Using Lemma 7.2.1, and denoting $|\mathcal{S}|$ by s, we find

$$\begin{aligned}
0 &\le s(s-1)(s-2)\left(\begin{bmatrix}2t+2\\t+1\end{bmatrix}_q - 3a_t(q) + 3b_t(q) - c_t(q)\right)\\
&\quad - 5s(s-1)\left(\begin{bmatrix}2t+2\\t+1\end{bmatrix}_q - 2a_t(q) + b_t(q)\right)\\
&\quad + 12s\left(\begin{bmatrix}2t+2\\t+1\end{bmatrix}_q - a_t(q) - 1\right) - 12\left(\begin{bmatrix}2t+2\\t+1\end{bmatrix}_q - s\right)\\
&= (s-1)(s-3)(s-4)\begin{bmatrix}2t+2\\t+1\end{bmatrix}_q - s(3s^2-19s+28)a_t(q)\\
&\quad + s(s-1)(2s-9)b_t(q) - c_t'(q)s(s-1)(s-2)\ .
\end{aligned}$$

For $t = 2,3,4$, we use the values we computed in Lemma 7.1.1, Lemma 7.1.3, Lemma 7.1.5 and Notation 7.2.3. In all three cases we find a contradiction if $s \le 2q-2$. More details can be found in Computation A.2.1.

Now we assume $t \ge 5$. We use Lemma 7.1.1 and the inequalities we derived in Lemma 7.2.4. We find:

$$\begin{aligned}
0 &\le (s-1)(s-3)(s-4)(q^5+2)(q^6+2)\begin{bmatrix}10\\5\end{bmatrix}_q q^{(t+1)^2-36}\\
&\quad - s(3s^2-19s+28)q^{(t+1)^2}\\
&\quad + (q^5-2q^3-q^2+3)q^{(t+1)^2-6}s(s-1)(s-2)\\
&\quad + s(s-1)(2s-9)(q-1)(q^2-1)(q^3-1)(q^4-1)q^{(t+1)^2-10}\\
&= q^{(t+1)^2-36}\Bigg[(s-1)(s-3)(s-4)(q^5+2)(q^6+2)\begin{bmatrix}10\\5\end{bmatrix}_q\\
&\quad - s(3s^2-19s+28)q^{36} + (q^5-2q^3-q^2+3)q^{30}s(s-1)(s-2)\\
&\quad + s(s-1)(2s-9)(q-1)(q^2-1)(q^3-1)(q^4-1)q^{26}\Bigg]\ .
\end{aligned}$$

The function

$$f_q(s) = (s-1)(s-3)(s-4)(q^5+2)(q^6+2)\begin{bmatrix}10\\5\end{bmatrix}_q$$

$$+ (q^5 - 2q^3 - q^2 + 3)q^{30}s(s-1)(s-2) - s(3s^2 - 19s + 28)q^{36}$$
$$+ s(s-1)(2s-9)(q-1)(q^2-1)(q^3-1)(q^4-1)q^{26}$$

is a function of degree three in the variable s. For $q \geq 3$, this function is monotonically increasing. It can be checked that $f_q(2q-2) < 0$ if $q \geq 3$. This contradicts $q^{(t+1)^2-36}f_q(s) \geq 0$. Consequently, $s \geq 2q-1$. More details on the calculations can be found in Computation A.2.2. □

Remark 7.2.6. The bound presented above, in Theorem 7.2.5, improves the bound $q + \sqrt{q} - 1$ from [7] for all q. It improves the bound 5 from [1] for all $q \geq 3$. For $q = 2$, the value 5 is still the best lower bound.

7.3 Other bounds for small maximal partial spreads

The first lemma links maximal partial spreads to blocking sets.

Lemma 7.3.1. *Let $\mathcal{S} = \{\pi_1, \ldots, \pi_\lambda\}$ be a maximal partial t-spread in $\mathrm{PG}(2t+1, q)$. Then $\bigcup_{i=1}^{\lambda} \pi_i$ is a blocking set of $\mathrm{PG}(2t+1, q)$ with respect to the t-spaces of $\mathrm{PG}(2t+1, q)$.*

Proof. If $\bigcup_{i=1}^{\lambda} \pi_i$ is not a blocking set with respect to the t-spaces in $\mathrm{PG}(2t+1, q)$, then we can find a t-space π' in $\mathrm{PG}(2t+1, q)$ such that $\pi' \cap \pi_i$ is empty for all i. However, then $\{\pi_1, \ldots, \pi_\lambda, \pi'\}$ is a partial t-spread, contradicting the maximality of $\mathcal{S}$. □

Since a maximal partial t-spread of $\mathrm{PG}(2t+1, q)$ defines a blocking set with respect to the t-spaces, it contains a minimal blocking set. We discuss maximal partial t-spreads containing small blocking sets. We first recall a theorem of Beutelspacher.

Theorem 7.3.2 ([7]). *If $\mathcal{U}$ is a set of subspaces partitioning the point set of $\mathrm{PG}(n, q)$, $n \geq 1$, then either $\mathcal{U} = \{\mathrm{PG}(n, q)\}$ or else $|\mathcal{U}| \geq q^{\beta+1} + 1$, with $\beta = \lceil \frac{n-1}{2} \rceil$.*

The smallest minimal blocking set of $\mathrm{PG}(2t+1, q)$ with respect to the t-spaces, is a $(t+1)$-space (see Theorem 1.7.6). We look at a maximal partial t-spread containing this blocking set.

Corollary 7.3.3. *If $\mathcal{S} = \{\pi_1, \ldots, \pi_\lambda\}$ is a maximal partial t-spread in $\mathrm{PG}(2t+1,q)$ covering a $(t+1)$-space σ, then $\lambda \geq q^{\lceil \frac{t}{2} \rceil + 1} + 1$.*

Proof. The set $\{\pi_1 \cap \sigma, \ldots, \pi_\lambda \cap \sigma\}$ is a partition of σ. Since $\sigma \not\subseteq \pi_i$, the result follows immediately from Theorem 7.3.2. □

We recall a fundamental result on small blocking sets.

Theorem 7.3.4 ([115, Theorem 2.7]). *Let B be a small minimal blocking set with respect to the t-spaces in $\mathrm{PG}(n,q)$, $q = p^h$ and $p > 2$ prime, and let τ be a subspace of $\mathrm{PG}(n,q)$. If $B \cap \tau \neq \emptyset$, then $|B \cap \tau| \equiv 1 \pmod{p}$.*

In the next theorem we use the value $r(q)$, which was defined in Section 1.7. It is the number of points in the smallest non-trivial blocking set of $\mathrm{PG}(2,q)$ minus $q+1$.

Theorem 7.3.5. *A maximal partial t-spread in $\mathrm{PG}(2t+1,q)$, $q = p^h$ and $p > 2$ prime, covering a non-trivial small minimal blocking set with respect to the t-spaces, contains at least $\sqrt{1 + (p-1)(\theta_{t+1}(q) + r(q)q^t)} + 1$ elements.*

Proof. Let $\mathcal{S} = \{\pi_1, \ldots, \pi_\lambda\}$ be a maximal partial t-spread in $\mathrm{PG}(2t+1,q)$. Note that $B = \bigcup_{i=1}^{\lambda} \pi_i$ is a blocking set with respect to the t-spaces by Lemma 7.3.1. Let B' be a small minimal blocking set with respect to the t-spaces, which is a subset of B. Denote the number of points in $B' \cap \pi_i$ by x_i. Without loss of generality we can assume $x_1 \geq x_2 \geq \cdots \geq x_\lambda$.

A line joining a point of $B' \cap \pi_1$ to a point of $B' \cap \pi_2$ contains at least $p-1$ additional points of B' by Theorem 7.3.4. Furthermore, a point in $\mathrm{PG}(2t+1,q)$ lies on precisely one line joining a point of π_1 to a point of π_2, since $\pi_1 \cap \pi_2 = \emptyset$ but $\langle \pi_1, \pi_2 \rangle = \mathrm{PG}(2t+1,q)$. Hence, $B' \setminus (\pi_1 \cup \pi_2)$ contains at least $(p-1)x_1x_2$ points. Obviously, the set $B' \setminus (\pi_1 \cup \pi_2)$ is partitioned by the sets $B' \cap \pi_i$, $3 \leq i \leq \lambda$. Each of the sets $B' \cap \pi_i$, $3 \leq i \leq \lambda$, contains at most x_3 elements. Therefore,

$$\lambda - 2 \geq \frac{(p-1)x_1x_2}{x_3} \geq (p-1)x_1 \geq (p-1)\frac{|B'|}{\lambda} .$$

So, $\lambda \geq 1 + \sqrt{1 + (p-1)|B'|}$. By Theorem 1.7.8, we know that $|B'| \geq \theta_{t+1}(q) + r(q)q^t$. Consequently, $\lambda \geq 1 + \sqrt{1 + (p-1)(\theta_{t+1}(q) + r(q)q^t)}$. The theorem follows. □

Remark 7.3.6. For general t, the bounds on the size of a maximal partial t-spread in $\mathrm{PG}(2t+1, q)$ which we found in Corollary 7.3.3 and Theorem 7.3.5, are much larger than $2q-1$, the bound we derived in Theorem 7.2.5. Hence, if a maximal partial t-spread of size $2q-1$ in $\mathrm{PG}(2t+1, q)$ exists, then it covers a blocking set with respect to the t-spaces of size at least $\frac{3}{2}(q^{t+1}+1)$. So, roughly three-quarters of the points in the blocking set formed by the union of the elements of the partial t-spread would be essential.

8

The functional codes $C_2(\mathcal{H})$ and $C_{Herm}(\mathcal{Q})$

Goh, ik heb formidabel afgezien.
Ik moet het zeggen gelijk het is hè.

Frans Verbeeck na de *Ronde van Vlaanderen, 1975.*

In Section 1.8 we introduced the functional codes and in particular the codes $C_h(\mathcal{X})$ and $C_{Herm}(\mathcal{X})$ for an algebraic variety $\mathcal{X}$. This variety $\mathcal{X}$ is in most of the studied cases chosen to be a non-singular quadric or a non-singular Hermitian variety. In general, it is easy to find the length and dimension of a functional code, but hard to find its minimum distance and to classify its small weight code words.

The first results on functional codes were obtained for the code $C_2(\mathcal{Q})$, with $\mathcal{Q}$ a non-singular quadric (hyperbolic, parabolic or elliptic) in $\mathrm{PG}(n, q)$. The length of this code equals $|\mathcal{Q}|$; the actual value depends on the type of $\mathcal{Q}$. The dimension of this code equals $\binom{n+2}{2} - 1$ since there is only one linear combination of the monomials of degree 2 in $n + 1$ variables that vanishes on $\mathcal{Q}$.

Results on the minimum distance and on the small weight code words of $C_2(\mathcal{Q})$ were obtained in [27, 46] for the small-dimensional cases $n = 3, 4$. The largedimensional cases $n \geq 5$ were investigated in [51, 68]. Some corrections to these results were presented in [35]. Also the divisors of these codes were investigated in [51]

This research was the inspiration to investigate similar functional codes defined by Hermitian varieties, namely the functional codes $C_{Herm}(\mathcal{H})$, $\mathcal{H}$ a non-singular Hermitian variety in $\mathrm{PG}(n, q^2)$. The length of this code equals $|\mathcal{H}|$, the number of points on $\mathcal{H}$. Note that the code $C_{Herm}(\mathcal{H})$ is a linear code over the field $\mathbb{F}_q$ and not over the field $\mathbb{F}_{q^2}$. Its dimension equals $n^2 + 2n$. The minimum distance, the small weight code words and the divisors of these codes were investigated in [50].

In this chapter we will look at the codes $C_2(\mathcal{H})$, with $\mathcal{H}$ a non-singular Hermitian variety in $\mathrm{PG}(n, q^2)$, and $C_{Herm}(\mathcal{Q})$, with $\mathcal{Q}$ a non-singular quadric in $\mathrm{PG}(n, q^2)$. The length of these codes is given by the number of points in their respective varieties. The former is a linear code of dimension $\binom{n+2}{2}$ over $\mathbb{F}_{q^2}$. Its minimum distance, small weight code words and divisors have been studied before in [47, 49, 52] for $n = 3$, in [48] for $n = 4$ and in [69] for $5 \leq n \leq O(q^2)$. In Sections 8.1 and 8.2 we will investigate the minimum distance and small weight code words of this code for $n \geq 4$, thereby improving some of the previous results.

The latter code $C_{Herm}(\mathcal{Q})$ is a linear code of dimension $(n + 1)^2$ over the field $\mathbb{F}_q$. Its minimum distance and small weight code words will be investigated in Sections 8.3, 8.4 and 8.5.

It should be noted that the techniques used when studying the codes $C_2(\mathcal{Q})$ and $C_{Herm}(\mathcal{H})$ cannot be applied for the two classes of functional codes that are treated in this chapter. Recall that in Remark 1.8.9 it is proved that all results on the minimum distance of the functional codes $C_2(\mathcal{Q})$ and $C_{Herm}(\mathcal{H})$ can be stated as results about the maximum size of the intersection of an arbitrary quadric with $\mathcal{H}$, respectively the maximum size of the intersection of an arbitrary Hermitian variety with $\mathcal{Q}$. In this chapter, we will also state the results in this form.

In Section 8.6 we look at the divisors of the code $C_{Herm}(\mathcal{Q})$. This chapter is based on [6], which is joint work with Daniele Bartoli, Stefania Fanali and Leo Storme, and on [35], a survey article on this topic.

8.1 The functional code $C_2(\mathcal{H})$ for $n = 4$

In this section and the next one, we will discuss the code $C_2(\mathcal{H}(4, q^2))$. Recall that $\mathcal{H}(4, q^2)$ is a non-singular Hermitian variety in $\mathrm{PG}(4, q^2)$. The best known results about this code were presented in [69, Section 3]. We will give some improvements to these results. In this section and in the next ones, quadrics and Hermitian varieties are considered as point sets.

The first lemma is an improvement of [69, Lemma 3.3].

Lemma 8.1.1. *Let $\mathcal{H}$ be a non-singular Hermitian variety in $\mathrm{PG}(4, q^2)$ and let $\mathcal{Q}$ be a non-singular (necessarily parabolic) quadric in $\mathrm{PG}(4, q^2)$. If a line ℓ on $\mathcal{Q}$ contains at most q points of $\mathcal{H}$, then $|\mathcal{Q} \cap \mathcal{H}| \leq q^5 + q^4 + 4q^3 - 3q + 1$.*

Proof. It follows immediately that ℓ contains one point of $\mathcal{H}$. Let P be a point on ℓ with $P \notin \mathcal{H}$. Take a line m of $\mathcal{Q}$ intersecting ℓ in P. Consider the plane $\pi = \langle \ell, m \rangle$. Then π lies in the tangent hyperplane $T_P(\mathcal{Q})$ and in q^2 hyperplanes containing a hyperbolic quadric $\mathcal{Q}^+(3, q^2)$ on Q; ℓ is a line on each of these hyperbolic quadrics. We denote them by $\mathcal{Q}_i$, $i = 1, \ldots, q^2$. For a given hyperbolic quadric $\mathcal{Q}_i$, we denote the regulus containing ℓ by $\mathcal{R}_i$ and the opposite one by $\mathcal{R}'_i$.

Assume that $\ell \cap \mathcal{H} = \{R\}$. In $\mathcal{R}'_i$, $i = 1, \ldots, q^2$, there is at most one line contained in $\mathcal{H}$, namely the line through R. All the lines through R which are contained in the intersection $\mathcal{Q} \cap \mathcal{H}$ are contained in both $T_R(\mathcal{Q}) \cap \mathcal{Q}$, a cone with vertex R and base $\mathcal{Q}'$, a conic $\mathcal{Q}(2, q^2)$, and in $T_R(\mathcal{H}) \cap \mathcal{H}$, a cone with vertex R and base $\mathcal{H}'$, a Hermitian curve $\mathcal{H}(2, q^2)$. We choose the bases $\mathcal{Q}'$ and $\mathcal{H}'$ such that they lie in the same 3-space disjoint from R. There are $k = |\mathcal{Q}' \cap \mathcal{H}'|$ such lines, and we know $|\mathcal{Q}' \cap \mathcal{H}'| \leq 2(q + 1)$. Hence, there are at most k hyperbolic quadrics containing one line of $\mathcal{Q} \cap \mathcal{H}$ through R.

Let $\mathcal{Q}_i$ be a hyperbolic quadric containing a line of $\mathcal{H}$ through R. Counting the points of $\mathcal{Q}_i \cap \mathcal{H}$ according to the lines of $\mathcal{R}'_i$, we find

$$|\mathcal{Q}_i \cap \mathcal{H}| \leq (q^2 + 1) + q^2(q + 1) = q^3 + 2q^2 + 1 \ .$$

Now, let $\mathcal{Q}_j$ be a hyperbolic quadric not containing a line of $\mathcal{H}$ through R. Counting the points of $\mathcal{Q}_j \cap \mathcal{H}$ according to the lines of $\mathcal{R}'_j$, we find $|\mathcal{Q}_j \cap \mathcal{H}| \leq (q^2 + 1)(q + 1)$.

We denote $a = |\pi \cap \mathcal{Q} \cap \mathcal{H}| = 1 + |m \cap \mathcal{H}|$. We may assume $a \in \{2, q + 2\}$ and we conclude

$$\begin{aligned}
|\mathcal{Q} \cap \mathcal{H}| &\leq k(q^3 + 2q^2 + 1 - a) + (q^2 - k)(q^3 + q^2 + q + 1 - a) \\
&\quad + |T_P(\mathcal{Q}) \cap \mathcal{Q} \cap \mathcal{H}| \\
&\leq q^2(q^3 + q^2 + q + 1) + k(q^2 - q) + (q + 1)(q^2 - 1) - a(q^2 - 1) \\
&\leq q^5 + q^4 + 2q^3 + 2q^2 - q - 1 + 2(q + 1)(q^2 - q) - 2(q^2 - 1) \\
&= q^5 + q^4 + 4q^3 - 3q + 1 \; .
\end{aligned}$$

Hereby we used the upper bound $(q + 1)(q^2 - 1) + a$ for $|T_P(\mathcal{Q}) \cap \mathcal{Q} \cap \mathcal{H}|$. $\square$

The second lemma is an improvement of [69, Lemma 3.6]. We first mention a preceding result.

Lemma 8.1.2 ([69, Lemma 3.5]). *Let $\mathcal{H}$ be a non-singular Hermitian variety in $\mathrm{PG}(4, q^2)$, let $\mathcal{Q}$ be a non-singular (necessarily parabolic) quadric in $\mathrm{PG}(4, q^2)$ and let P be a point of the intersection $\mathcal{Q} \cap \mathcal{H}$. If every line on $\mathcal{Q}$ contains at least $q + 1$ points of $\mathcal{H}$, then the tangent hyperplanes $T_P(\mathcal{Q})$ and $T_P(\mathcal{H})$ do not coincide.*

Lemma 8.1.3. *Let $\mathcal{H}$ be a non-singular Hermitian variety in $\mathrm{PG}(4, q^2)$ and let $\mathcal{Q}$ be a non-singular (necessarily parabolic) quadric in $\mathrm{PG}(4, q^2)$. If all lines on $\mathcal{Q}$ share $q+1$ or q^2+1 points with $\mathcal{H}$, then $|\mathcal{Q} \cap \mathcal{H}| \leq q^5 + q^4 + 2q^3 - q + 1$.*

Proof. Let P be a point of $\mathcal{Q}$, not lying on $\mathcal{H}$. Let ℓ and m be lines on $\mathcal{Q}$ through P. All lines on $\mathcal{Q}$ through P, including ℓ and m, contain precisely $q+1$ points of $\mathcal{H}$. Hence, the tangent hyperplane $T_P(\mathcal{Q})$ contains $(q+1)(q^2+1)$ points of $\mathcal{Q} \cap \mathcal{H}$ since $T_P(\mathcal{Q}) \cap \mathcal{Q}$ is a cone with P as vertex and a conic $\mathcal{Q}'$ as base.

Let R be a point of $\ell \cap \mathcal{H}$. There are $q^2 + 1$ lines of $\mathcal{Q}$ through R. We show that at most 2 of those lines can be contained in $\mathcal{H}$. Assume there are 3 lines through R contained in $\mathcal{Q} \cap \mathcal{H}$. These lines generate a hyperplane, since a plane cannot contain 3 lines of $\mathcal{Q}$. This hyperplane must be $T_R(\mathcal{Q})$ since all lines of $\mathcal{Q}$ through R are contained in $T_R(\mathcal{Q})$. In the same way this hyperplane also equals $T_R(\mathcal{H})$. However, this is a contradiction by Lemma 8.1.2. We have proved that at most 2 lines through each of the $q + 1$ points of $\ell \cap \mathcal{H}$ are contained in $\mathcal{Q} \cap \mathcal{H}$.

We consider the plane $\pi = \langle \ell, m \rangle$ and the $q^2 + 1$ hyperplanes through it. One of those hyperplanes is $T_P(\mathcal{Q})$. All the other ones intersect $\mathcal{Q}$ in a hyperbolic quadric $\mathcal{Q}^+(3, q^2)$. Each line of such a hyperbolic quadric contains by assumption $q+1$ or q^2+1 points of $\mathcal{H}$. Hence, in both reguli of such a hyperbolic quadric there is the same number of lines which contain $q+1$ respectively q^2+1 points of $\mathcal{H}$. Using Lemma 1.7.10, we find that in each of those hyperbolic quadrics, both reguli contain $q+1$, 2, 1 or 0 lines of $\mathcal{H}$ (if the intersection of $\mathcal{H}$ with the 3-space is a singular Hermitian variety, then this number is at most 1). Let a_i be the number of hyperbolic quadrics in which both reguli contain i lines of $\mathcal{H}$. We know $a_{q+1} + a_2 + a_1 + a_0 = q^2$ and $(q+1)a_{q+1} + 2a_2 + a_1 = k \leq 2(q+1)$, with k the total number of lines on $\mathcal{Q} \cap \mathcal{H}$ meeting ℓ (at most 2 through each of the points of $\ell \cap \mathcal{H}$). Counting the points of $\mathcal{H}$ according to the lines of one regulus, it can be found that the hyperbolic quadrics in which both reguli contain i lines of $\mathcal{H}$, contain precisely $i(q^2+1) + (q^2+1-i)(q+1) = (q^2+1)(q+1) + i(q^2-q)$ points of $\mathcal{H}$.

Now, we compute the total number of intersection points. Hereby, $I = \{0, 1, 2, q+1\}$. We find

$$\begin{aligned}
|\mathcal{Q} \cap \mathcal{H}| &= (q+1)(q^2+1) + \sum_{i\in I} a_i \left((q^2+1)(q+1) + i(q^2-q) - 2(q+1)\right) \\
&= (q+1)(q^2+1) + (q^2-1)(q+1)\sum_{i\in I} a_i + (q^2-q)\sum_{i\in I} a_i i \\
&= (q+1)(q^2+1) + q^2(q^2-1)(q+1) + (q^2-q)k \\
&\leq (q+1)\left(q^2(q^2-1) + q^2 + 1\right) + (q^2-q)2(q+1) \\
&= q^5 + q^4 + 2q^3 - q + 1,
\end{aligned}$$

which completes the proof. □

We recall another result from [69] and we make an observation about a case that was not treated in [69].

Lemma 8.1.4 ([69, Sections 3.2, 3.3, 3.4 and 3.5]). *Let Q be a singular quadric in $\mathrm{PG}(4, q^2)$ and let $\mathcal{H}$ be a non-singular Hermitian variety in $\mathrm{PG}(4, q^2)$. If Q is a cone with vertex a point or a line, or if Q is a plane, then $|Q \cap \mathcal{H}| \leq \max\{q^5 + q^4 + 2q^3 - q + 1, q^5 + q^4 + q^3 + 2q^2 + 1\}$.*

Lemma 8.1.5. *Let $\mathcal{H}$ be a non-singular Hermitian variety in $\mathrm{PG}(4, q^2)$ and let Q be a singular quadric in $\mathrm{PG}(4, q^2)$. If Q is a hyperplane, then $|Q \cap \mathcal{H}| \leq q^5 + q^3 + q^2 + 1$.*

Proof. The intersection $Q \cap \mathcal{H}$ is either a Hermitian variety $\mathcal{H}(3, q^2)$ or else a cone with vertex a point and base a Hermitian curve $\mathcal{H}(2, q^2)$. In the former case, $Q \cap \mathcal{H}$ contains $(q^3+1)(q^2+1)$ points. In the latter case, $Q \cap \mathcal{H}$ contains $1+q^2(q^3+1)$ points. The statement is clearly valid. □

Using these lemmata, we can now state an improved version of [69, Theorem 3.8].

Theorem 8.1.6. *Let $\mathcal{H}$ be a non-singular Hermitian variety in $\mathrm{PG}(4,q^2)$ and let Q be a quadric in $\mathrm{PG}(4,q^2)$. If $|Q \cap \mathcal{H}| > q^5+q^4+4q^3-3q+1$, then Q is the union of two hyperplanes.*

Proof. Combine the results of Lemma 8.1.1, Lemma 8.1.3, Lemma 8.1.4 and Lemma 8.1.5. Each possible type of the quadric Q is considered in one of these lemmata, except the case that Q is the union of two hyperplanes. □

We mentioned before that the code $C_2(\mathcal{H})$ for $n=4$ had also been studied in [48]. The bound from the previous theorem improves that result as well, except for the case $q=2$.

Theorem 8.1.7 ([48, Section 3]). *Let $\mathcal{H}$ be a non-singular Hermitian variety in $\mathrm{PG}(4,4)$ and let Q be a quadric in $\mathrm{PG}(4,4)$. If $|Q \cap \mathcal{H}| > 69$, then Q is the union of two hyperplanes.*

8.2 The functional code $C_2(\mathcal{H})$ for $n \geq 4$

We introduce the functions $W_n(q)$ using a recursive definition.

Definition 8.2.1. The function $W_n(q)$ is defined as follows. For $n=4$:

$$W_4(q) = \begin{cases} q^5+q^4+4q^3-3q+1 & q \geq 3 \\ 69 & q = 2 \end{cases};$$

for $n > 4$:

$$W_n(q) = \begin{cases} q^2W_{n-1}(q) + q^{n-2} + 2q^{n-3} & n \text{ odd} \\ q^2W_{n-1}(q) - q^{n-2} & n \text{ even} \end{cases}.$$

Lemma 8.2.2. *For $n \geq 4$: $W_n(q) = q^{2n-8}W_4(q) + \sum_{i=n-2}^{2n-7} q^i + 2\delta q^{n-3}$, with $\delta = 1$ if n is odd and $\delta = 0$ if n is even.*

Proof. This can easily be proved, using induction on n. $\square$

Since it helps to understand some of the inequalities in the following proofs, we give some of these functions for $q \geq 3$.

- $W_5(q) = q^7 + q^6 + 4q^5 - 2q^3 + 3q^2$,
- $W_6(q) = q^9 + q^8 + 4q^7 - 2q^5 + 2q^4$,
- $W_7(q) = q^{11} + q^{10} + 4q^9 - 2q^7 + 2q^6 + q^5 + 2q^4$,
- $W_n(q) = q^{2n-3} + q^{2n-4} + 4q^{2n-5} - 2q^{2n-7} + 2q^{2n-8} + q^{2n-9} + \ldots + q^{n-2}$ if $n \geq 8$ is even.
- $W_n(q) = q^{2n-3}+q^{2n-4}+4q^{2n-5}-2q^{2n-7}+2q^{2n-8}+q^{2n-9}+\ldots+q^{n-2}+2q^{n-3}$ if $n \geq 8$ is odd.

The next theorem is an improvement of [69, Theorem 4.1].

Theorem 8.2.3. *Let $\mathcal{H}$ be a non-singular Hermitian variety in* $\mathrm{PG}(n, q^2)$ *and let Q be a quadric in* $\mathrm{PG}(n, q^2)$*, $n \geq 4$. If $|Q \cap \mathcal{H}| > W_n(q)$, then Q is the union of two hyperplanes.*

Proof. We prove this theorem by induction on n. The theorem is true for $n = 4$ by Theorem 8.1.6 and Theorem 8.1.7. Now, we suppose the theorem to be valid for $n - 1$. We prove it for dimension n.

By the assumption, $|Q \cap \mathcal{H}| > W_n(q)$. Assume now that every non-tangent hyperplane to $\mathcal{H}$ contains at most $W_{n-1}(q)$ points of $Q \cap \mathcal{H}$. We count the number N of tuples (P, π), with $P \in Q \cap \mathcal{H}$, π a hyperplane not tangent to $\mathcal{H}$, and $P \in \pi$. On the one hand,

$$
\begin{aligned}
N &> W_n(q)\left(\frac{q^{2n}-1}{q^2-1} - q^2|\mathcal{H}(n-2, q^2)| - 1\right)\\
&= W_n(q)\frac{q^{2n} - q^2 - q^2(q^{n-1} + (-1)^{n-2})(q^{n-2} + (-1)^{n-1})}{q^2-1}.
\end{aligned}
$$

On the other hand, counting this number of incidences in a different way,

$$
\begin{aligned}
N &\leq W_{n-1}(q)\left(\frac{q^{2n+2}-1}{q^2-1} - |\mathcal{H}|\right)\\
&= W_{n-1}(q)\left(\frac{q^{2n+2} - 1 - (q^{n+1} + (-1)^n)(q^n + (-1)^{n+1})}{q^2-1}\right).
\end{aligned}
$$

Thus,

$$W_n(q) < W_{n-1}(q)\left[\frac{q^{2n+2}-1-(q^{n+1}+(-1)^n)(q^n+(-1)^{n+1})}{q^{2n}-q^2-q^2(q^{n-1}+(-1)^{n-2})(q^{n-2}+(-1)^{n-1})}\right].$$

We look first at the case n even. We find

$$\begin{aligned}
W_n(q) &< W_{n-1}(q)\left[\frac{q^{2n+2}-1-(q^{n+1}+1)(q^n-1)}{q^{2n}-q^2-q^2(q^{n-1}+1)(q^{n-2}-1)}\right]\\
&= W_{n-1}(q)q^2 - W_{n-1}(q)\frac{q^{n+3}-q^{n+2}-q^{n+1}+q^n}{q^{2n}-q^{2n-1}+q^{n+1}-q^n}\\
&< W_{n-1}(q)q^2 - (q^{2n-5}+q^{2n-6})\frac{q^{n+3}-q^{n+2}-q^{n+1}+q^n}{q^{2n}-q^{2n-1}+q^{n+1}-q^n}\\
&= W_{n-1}(q)q^2 - \frac{q^{3n-2}-2q^{3n-4}+q^{3n-6}}{q^{2n}-q^{2n-1}+q^{n+1}-q^n}\\
&= W_{n-1}(q)q^2 - q^{n-2} - \frac{q^{3n-3}-2q^{3n-4}+q^{3n-6}-q^{2n-1}+q^{2n-2}}{q^{2n}-q^{2n-1}+q^{n+1}-q^n}\\
&< W_{n-1}(q)q^2 - q^{n-2}\,.
\end{aligned}$$

Now we look at the case n odd. If $q \geq 4$, we find

$$\begin{aligned}
W_n(q) &< W_{n-1}(q)\left[\frac{q^{2n+2}-1-(q^{n+1}-1)(q^n+1)}{q^{2n}-q^2-q^2(q^{n-1}-1)(q^{n-2}+1)}\right]\\
&= q^2W_{n-1}(q) + W_{n-1}(q)\frac{q^{n+3}-q^{n+2}-q^{n+1}+q^n}{q^{2n}-q^{2n-1}-q^{n+1}+q^n}\\
&< q^2W_{n-1}(q) + (q^{2n-5}+2q^{2n-6})\frac{q^{n+3}-q^{n+2}-q^{n+1}+q^n}{q^{2n}-q^{2n-1}-q^{n+1}+q^n}\\
&= q^2W_{n-1}(q) + \frac{q^{3n-2}+q^{3n-3}-3q^{3n-4}-q^{3n-5}+2q^{3n-6}}{q^{2n}-q^{2n-1}-q^{n+1}+q^n}\\
&= q^2W_{n-1}(q) + q^{n-2}+2q^{n-3}\\
&\qquad - \frac{q^{3n-4}+q^{3n-5}-2q^{3n-6}-q^{2n-1}-q^{2n-2}+2q^{2n-3}}{q^{2n}-q^{2n-1}-q^{n+1}+q^n}\\
&< q^2W_{n-1}(q) + q^{n-2}+2q^{n-3}\,.
\end{aligned}$$

If $q = 2, 3$, the inequality $W_{n-1}(q) < q^{2n-5}+2q^{2n-6}$, used in this derivation is not valid. However, the inequality

$$W_{n-1}(q)\left[\frac{q^{2n+2}-1-(q^{n+1}-1)(q^n+1)}{q^{2n}-q^2-q^2(q^{n-1}-1)(q^{n-2}+1)}\right] < q^2W_{n-1}(q)+q^{n-2}+2q^{n-3}$$

is still valid in these cases. In order to prove this, it is sufficient to show that

$$W_{n-1}(q)\frac{q^{n+3}-q^{n+2}-q^{n+1}+q^n}{q^{2n}-q^{2n-1}-q^{n+1}+q^n} < q^{n-2}+2q^{n-3}.$$

We first consider $q=2$:

$$\begin{aligned}
& W_{n-1}(2)\frac{2^{n+3}-2^{n+2}-2^{n+1}+2^n}{2^{2n}-2^{2n-1}-2^{n+1}+2^n} < 2^{n-2}+2\cdot 2^{n-3} \\
\Leftrightarrow \quad & 3\cdot 2^n\, W_{n-1}(2) < 2^{n-1}(2^{2n-1}-2^n) \\
\Leftrightarrow \quad & 6\, W_{n-1}(2) < 2^{2n-1}-2^n \\
\Leftrightarrow \quad & 2^{2n-1}-2^{2n-4}-2^{2n-7}-2^{2n-9}-2^{n-1}-2^{n-2} < 2^{2n-1}-2^n\,,
\end{aligned}$$

which clearly holds if $n \geq 5$. Hereby we used that $W_{n-1}(2) = 69.\cdot 2^{2n-10} + 2^{2n-8} - 2^{n-3}$ if n is odd. Now, we consider $q=3$:

$$\begin{aligned}
& W_{n-1}(3)\frac{3^{n+3}-3^{n+2}-3^{n+1}+3^n}{3^{2n}-3^{2n-1}-3^{n+1}+3^n} < 3^{n-2}+2.3^{n-3} \\
\Leftrightarrow \quad & W_{n-1}(3)(2\cdot 3^{n+2}-2\cdot 3^n) < (2\cdot 3^{2n-1}-2\cdot 3^n)(3^{n-2}+2\cdot 3^{n-3}) \\
\Leftrightarrow \quad & W_{n-1}(3)(3^2-1) < (3^{n-1}-1)(3^{n-2}+2\cdot 3^{n-3}) \\
\Leftrightarrow \quad & 3^{2n-3}+2\cdot 3^{2n-4}-2\cdot 3^{2n-6} \\
& -2\cdot 3^{2n-7}-3^{2n-10}-3^{n-2}-3^{n-3} < 3^{2n-3}+2\cdot 3^{2n-4}-3^{n-2}-2\cdot 3^{n-3}\,,
\end{aligned}$$

which clearly holds if $n \geq 5$. We used that $W_{n-1}(3) = 3^{2n-5} + 2\cdot 3^{2n-6} + 3^{2n-7} - 3^{2n-9} - \frac{3^{2n-10}+3^{n-3}}{2}$ if n is odd and $q=3$, which follows from Lemma 8.2.2.

We conclude that in all cases, we find a contradiction. Hence, there is a non-tangent hyperplane π containing more than $W_{n-1}(q)$ points of $Q \cap \mathcal{H}$. We can continue as in part 2 of the proof of [69, Theorem 4.1]. We add this for sake of completeness.

The intersection $\mathcal{H}' = \pi \cap \mathcal{H}$ is a non-singular $(n-1)$-dimensional Hermitian variety. We conclude from the previous paragraph that $|\mathcal{H}' \cap Q| > W_{n-1}(q)$. By the induction hypothesis, $Q \cap \pi$ is the union of two $(n-2)$-spaces. The only quadrics in $\mathrm{PG}(n, q^2)$ containing $(n-2)$-spaces, are the cones $\pi_{n-4}\mathcal{Q}^+(3, q^2)$, $\pi_{n-3}\mathcal{Q}(2, q^2)$, $\pi_{n-2}\mathcal{Q}^-(1, q^2)$, which is just an $(n-2)$-space, and $\pi_{n-2}\mathcal{Q}^+(1, q^2)$, which is the union of two hyperplanes. We want to eliminate the first three possibilities. Each of those three quadrics can be described as the union of 1 or q^2+1 distinct $(n-2)$-dimensional spaces. The largest intersection of an $(n-2)$-space and $\mathcal{H}$ is achieved if n is odd and the intersection is a cone with a line as vertex and a non-singular Hermitian variety $\mathcal{H}(n-4,q^2)$ as base. The cardinality of such an intersection is

$$\frac{(q^{n-3}-1)(q^{n-4}+1)}{q^2-1}q^4+q^2+1=q^n\frac{q^{n-3}-1}{q^2-1}+\frac{q^{n+1}-1}{q^2-1}\ .$$

Hence, the cardinality of the intersection of each of those quadrics with $\mathcal{H}$ is at most

$$\begin{aligned}
&(q^2+1)(q^{2n-5}+q^{2n-7}+\cdots+q^{n+2}+q^n+q^{n-1}+q^{n-3}+\cdots+q^2+1)\\
&=q^{2n-3}+2q^{2n-5}+2q^{2n-7}+\cdots+2q^{n+2}+q^{n+1}\\
&\qquad+q^n+2q^{n-1}+2q^{n-3}+\cdots+2q^2+1\\
&<W_n(q)\ .
\end{aligned}$$

Since the cardinality of the intersection is smaller than $W_n(q)$, those three possibilities can be eliminated. The only possibility remaining for Q is the union of two hyperplanes. □

The structure of the small weight code words of $C_2(\mathcal{H})$, with $\mathcal{H}$ a non-singular Hermitian variety in $\mathrm{PG}(n,q^2)$, can be derived from this theorem. We refer to [69, Section 5] for a detailed analysis, which was performed for $4\leq n<O(q^2)$, but which is valid in general dimension $n\geq 4$ by the previous theorem. The next results are therefore generalisations of [68, Theorem 5.4.2.1] and [68, Theorem 5.4.2.2].

Theorem 8.2.4. *Let $\mathcal{H}$ be a non-singular Hermitian variety in* $\mathrm{PG}(n,q^2)$, $n\geq 4$ *even. The minimum weight of $C_2(\mathcal{H})$ equals $w_1^e=q^{2n-1}-q^{2n-3}-q^{n-1}-q^{n-2}$. The second and third weight are given by $w_2^e=w_1^e+q^{n-2}$ and $w_3^e=w_1^e+q^{n-1}$; if $(n,q)\neq(4,2)$, the fourth and fifth weight are given by $w_4^e=w_1^e+q^{n-1}+q^{n-2}$ and $w_5^e=w_1^e+2q^{n-1}$. The code words of weight w_1^e arise from a quadric consisting of two non-tangent hyperplanes (w.r.t. $\mathcal{H}$) through an $(n-2)$-space intersecting $\mathcal{H}$ in a non-singular Hermitian variety.*

Theorem 8.2.5. *Let $\mathcal{H}$ be a non-singular Hermitian variety in* $\mathrm{PG}(n,q^2)$, $n\geq 5$ *odd. The minimum weight of $C_2(\mathcal{H})$ equals $w_1^o=q^{2n-1}-q^{2n-3}-q^{n-1}+q^{n-2}$. The second, third and fourth weight are given by $w_2^o=w_1^o+q^{n-1}-q^{n-2}$,*

$w_3^o = w_1^o + q^{n-1}$ and $w_4^o = w_1^o + 2q^{n-1} - q^{n-2}$; if $(n,q) \neq (5,2)$, the fifth weight is given by $w_5^o = w_1^o + 2q^{n-1}$. The code words of weight w_1^o arise from a quadric consisting of two tangent hyperplanes (w.r.t. $\mathcal{H}$) through an $(n-2)$-space intersecting $\mathcal{H}$ in a non-singular Hermitian variety.

Using [69, Table 1], [69, Table 3(a)] and [69, Table 3(b)], the classification of code words corresponding to the five smallest weights of the code $C_2(\mathcal{H})$ can be found. All of these correspond to intersections of $\mathcal{H}$ with quadrics consisting of two hyperplanes. Also the number of these code words has been computed.

8.3 The functional code $C_{Herm}(\mathcal{Q})$ for small n

In the preceding section we investigated the code $C_2(\mathcal{H})$, with $\mathcal{H}$ a non-singular Hermitian variety in $\mathrm{PG}(n,q^2)$, by intersecting the fixed Hermitian variety $\mathcal{H}$ with all possible types of quadrics. In this section and the next section we will interchange the roles of the Hermitian variety and the quadrics. Namely, we will investigate the code $C_{Herm}(\mathcal{Q})$, with $\mathcal{Q}$ a non-singular quadric in $\mathrm{PG}(n,q^2)$. In this section we look at the cases $n = 3, 4$. Note that these codes do not only depend on the dimension n and the order q^2 of the field, but in the odd-dimensional case also on the type of $\mathcal{Q}$. In order to study the minimum weight of these codes, we look at the possible sizes of the intersections $\mathcal{Q} \cap H$ with H an arbitrary Hermitian variety.

First, we look at the three-dimensional case.

In [47], an upper bound for the intersection size of a non-singular Hermitian variety and an arbitrary quadric, is given. In particular, the following theorem is valid.

Theorem 8.3.1 ([47, Section 5]). *Let $\mathcal{H}$ be a non-singular Hermitian variety in $\mathrm{PG}(3,q^2)$ and let $\mathcal{Q}$ be a non-singular (elliptic or hyperbolic) quadric in $\mathrm{PG}(3,q^2)$. Then $|\mathcal{Q} \cap \mathcal{H}| \leq 2q^3 + q^2 + 1$.*

The next two lemmata consider the singular Hermitian varieties that are cones whose vertex is a point. Such a Hermitian variety is the union of $q^3 + 1$ lines through the vertex and the points of a non-singular Hermitian variety $\mathcal{H}(2,q^2)$ in a plane disjoint to the vertex.

Lemma 8.3.2. *Let $\mathcal{Q}$ be a non-singular (elliptic or hyperbolic) quadric in* $\mathrm{PG}(3, q^2)$ *and let H be a singular Hermitian variety in* $\mathrm{PG}(3, q^2)$. *If H is a cone with vertex a point $P \in \mathcal{Q}$, then $|H \cap \mathcal{Q}| \leq q^3 + 2q^2 - q + 1$.*

Proof. We know the base of the cone H is a non-singular Hermitian variety $\mathcal{H}(2, q^2)$, which we will denote by $\mathcal{H}'$. Consider the tangent plane $T_P(\mathcal{Q})$. We know that $T_P(\mathcal{Q}) \cap \mathcal{Q}$ is a cone with vertex P and base $\mathcal{Q}'$, with $\mathcal{Q}'$ a non-singular quadric in a line in $T_P(\mathcal{Q})$ disjoint from P, which is of the same type as the quadric $\mathcal{Q}$. All the lines of H not in this tangent plane contain one extra point of the quadric $\mathcal{Q}$. The number of lines contained in the intersection $T_P(\mathcal{Q}) \cap H$ is at most two, leading to at most $2q^2+1$ intersection points, since this tangent plane contains two lines of the quadric $\mathcal{Q}$ if it is a hyperbolic quadric and zero lines of the quadric $\mathcal{Q}$ if it is an elliptic quadric. The number of lines of H, not contained in $T_P(\mathcal{Q})$, is $q^3 + 1 - (q + 1) = q^3 - q$ or $q^3 + 1 - 1 = q^3$ depending on whether this plane contains $q + 1$ lines or one line of H. Consequently,

$$|H \cap \mathcal{Q}| \leq 2q^2 + 1 + (q^3 - q) = q^3 + 2q^2 - q + 1 ,$$

in case the tangent plane $T_P(\mathcal{Q})$ contains two lines of $\mathcal{Q} \cap H$. All the other cases lead to smaller upper bounds on the intersection size. $\square$

Lemma 8.3.3. *Let $\mathcal{Q}$ be a non-singular (elliptic or hyperbolic) quadric in* $\mathrm{PG}(3, q^2)$ *and let H be a singular Hermitian variety in* $\mathrm{PG}(3, q^2)$. *If H is a cone with vertex a point $P \notin \mathcal{Q}$, then $|H \cap \mathcal{Q}| \leq 2q^3 + 2$.*

Proof. Any line through P in the cone H contains at most two points of the quadric $\mathcal{Q}$. Hence, $|H \cap \mathcal{Q}| \leq 2(q^3 + 1) = 2q^3 + 2$. $\square$

In the next lemmata we consider the singular Hermitian varieties that are cones whose vertex is a line ℓ. Such a Hermitian variety is the union of $q + 1$ planes through the line ℓ.

Lemma 8.3.4. *Let $\mathcal{Q}$ be a non-singular (elliptic or hyperbolic) quadric in* $\mathrm{PG}(3, q^2)$ *and let H be a singular Hermitian variety in* $\mathrm{PG}(3, q^2)$. *If H is a cone with vertex a line ℓ such that $\ell \cap \mathcal{Q} = \emptyset$, then $|H \cap \mathcal{Q}| \leq q^3 + q^2 + q + 1$.*

Proof. No plane of the cone H contains a line or two lines of $\mathcal{Q}$, for in this case there would be at least one intersection point on ℓ. Hence, such a plane contains at most $q^2 + 1$ points of $\mathcal{Q}$. Consequently, $|H \cap \mathcal{Q}| \leq (q^2 + 1)(q + 1) = q^3 + q^2 + q + 1$. $\square$

Lemma 8.3.5. *Let $\mathcal{Q}$ be a non-singular (elliptic or hyperbolic) quadric in* $\mathrm{PG}(3, q^2)$ *and let H be a singular Hermitian variety in* $\mathrm{PG}(3, q^2)$. *If H is a cone with vertex a line ℓ such that $\ell \cap \mathcal{Q} = \{R\}$, with R a point, then $|H \cap \mathcal{Q}| \leq q^3 + 2q^2 + 1$.*

Proof. Since ℓ only shares the point R with $\mathcal{Q}$, it is a tangent line to $\mathcal{Q}$. So it could be that one of the planes of the cone H is the tangent plane $T_R(\mathcal{Q})$ to $\mathcal{Q}$ in R. If $T_R(\mathcal{Q})$ is a plane of H, then, as in the previous lemma, it contains at most $2q^2$ extra points (next to R) of the intersection $H \cap \mathcal{Q}$. Any other plane of H contains at most q^2 extra points of the intersection. This implies that $|H \cap \mathcal{Q}| \leq 1 + 2q^2 + q \cdot q^2 = q^3 + 2q^2 + 1$. □

Lemma 8.3.6. *Let $\mathcal{Q}$ be a non-singular (elliptic or hyperbolic) quadric in* $\mathrm{PG}(3, q^2)$ *and let H be a singular Hermitian variety in* $\mathrm{PG}(3, q^2)$. *If H is a cone with vertex a line ℓ such that $\ell \cap \mathcal{Q} = \{P, R\}$, with P and R distinct points, then $|H \cap \mathcal{Q}| \leq q^3 + 3q^2 - q + 1$.*

Proof. The two tangent planes $T_P(\mathcal{Q})$ and $T_R(\mathcal{Q})$ meet in a line disjoint to ℓ which contains at most 2 points of $\mathcal{Q}$. Hence, at most two planes of the cone H could be tangent planes. This maximum is attained if $\mathcal{Q}$ is a hyperbolic quadric.

If a plane of H is a tangent plane to $\mathcal{Q}$, then it contains at most $2q^2 - 1$ points of the intersection $H \cap \mathcal{Q}$, next to P and R. Any other plane of H contains at most $q^2 - 1$ extra points of this intersection. Thus, if H contains two tangent hyperplanes to $\mathcal{Q}$, then $|H \cap \mathcal{Q}| \leq 2 + 2(2q^2 - 1) + (q-1)(q^2-1) = q^3 + 3q^2 - q + 1$. It is clear that we find a smaller upper bound if H contains zero tangent planes to $\mathcal{Q}$ or one tangent plane to $\mathcal{Q}$. □

Lemma 8.3.7. *Let $\mathcal{Q}$ be a non-singular (elliptic or hyperbolic) quadric in* $\mathrm{PG}(3, q^2)$ *and let H be a singular Hermitian variety in* $\mathrm{PG}(3, q^2)$. *If H is a cone with vertex a line $\ell \subset \mathcal{Q}$, then $|H \cap \mathcal{Q}| \leq q^3 + 2q^2 + 1$.*

Proof. In every plane of H, there are at most q^2 additional intersection points next to the intersection points on ℓ. Consequently, $|H \cap \mathcal{Q}| \leq (q^2 + 1) + q^2(q + 1) = q^3 + 2q^2 + 1$. □

We can summarize the previous lemmata in the following theorem. This result gives a lower bound on the minimum weight of the code $C_{Herm}(\mathcal{Q})$, with $\mathcal{Q}$ a non-singular quadric in $\mathrm{PG}(3, q^2)$. We introduce thereby the function $\overline{W}_3(q)$.

Theorem 8.3.8. *Let $\mathcal{Q}$ be a non-singular (elliptic or hyperbolic) quadric in $\mathrm{PG}(3, q^2)$ and let H be an arbitrary Hermitian variety in $\mathrm{PG}(3, q^2)$. Then $|H \cap \mathcal{Q}| \le 2q^3 + q^2 + 1 = \overline{W}_3(q)$.*

Now we look at the four-dimensional case. We define a function $\overline{W}_4(q)$.

Definition 8.3.9. The function $\overline{W}_4(q)$ is defined as follows:

$$\overline{W}_4(q) = \begin{cases} q^5 + 2q^4 - \frac{1}{3}q^3 + 2q^2 + q + 1 & q \neq 3 \\ 424 & q = 3 \end{cases}.$$

Note that $\overline{W}_4(q) \ge W_4(q)$ for all prime power values of q. Moreover, equality only occurs for $q = 3$. The next lemma is a special case of Theorem 8.1.6 and Theorem 8.1.7, and follows immediately from the previous observation.

Lemma 8.3.10. *Let $\mathcal{H}$ be a non-singular Hermitian variety in $\mathrm{PG}(4, q^2)$ and let $\mathcal{Q}$ be a non-singular (necessarily parabolic) quadric in $\mathrm{PG}(4, q^2)$. Then $|\mathcal{Q} \cap \mathcal{H}| \le \overline{W}_4(q)$.*

We proceed now in the same way as in the three-dimensional case, by investigating the different possibilities for a singular Hermitian variety in $\mathrm{PG}(4, q^2)$. Note that $\overline{W}_4(q) \ge q^5 + 2q^4 - \frac{1}{3}q^3 + 2q^2 + q + 1$ for all prime powers q.

The first lemmata consider the singular Hermitian varieties which are cones with a point as vertex. In this case, the Hermitian variety is the union of $(q^3 + 1)(q^2 + 1)$ lines through the vertex. Note that there are $(q + 1)(q^3 + 1)$ planes on such a cone, all through P, corresponding to the lines of the base. Each point different from the vertex is on $q + 1$ of those planes.

Lemma 8.3.11. *Let $\mathcal{Q}$ be a non-singular (parabolic) quadric in $\mathrm{PG}(4, q^2)$ and let H be a singular Hermitian variety in $\mathrm{PG}(4, q^2)$. If H is a cone with vertex a point $P \in \mathcal{Q}$, then $|H \cap \mathcal{Q}| \le q^5 + 2q^3 + 3q^2 + 1 \le \overline{W}_4(q)$.*

Proof. In this case, the tangent hyperplane $T_P(\mathcal{Q})$ intersects the quadric $\mathcal{Q}$ in a cone with vertex P and base a non-singular quadric $\mathcal{Q}(2, q^2)$, which we will denote by $\mathcal{Q}'$. This conic $\mathcal{Q}'$ lies in a plane $\pi \subset T_P(\mathcal{Q})$. Consider the intersection $H' = \pi \cap H$. By Bézout's theorem, $|\mathcal{Q}' \cap H'| \le 2(q+1)$ regardless the stucture of H', a non-singular Hermitian curve or a singular Hermitian curve consisting of $q + 1$ concurrent lines. Therefore, $|H \cap \mathcal{Q} \cap T_P(\mathcal{Q})| \le 2(q + 1)q^2 + 1$.

Every line in the cone H, that is not contained in $T_P(\mathcal{Q})$, contains one additional point (next to P) of the quadric $\mathcal{Q}(4,q^2)$. The number of such lines is at most $(1+q^2)(q^3+1)-(q^3+1) = q^5+q^2$, with equality in case $\pi \cap \mathcal{H}'$ is a Hermitian curve.

Consequently, $|H \cap \mathcal{Q}| \leq 2(q+1)q^2 + 1 + q^5 + q^2 = q^5 + 2q^3 + 3q^2 + 1$. Note that $q^5 + 2q^3 + 3q^2 + 1 < W_4(q) \leq \overline{W}_4(q)$. □

Note that the bound $q^5 + 2q^3 + 3q^2 + 1$ in the previous lemma is sharp, since there exist conics and Hermitian curves in $\mathrm{PG}(2,q^2)$ sharing $2(q+1)$ points.

Lemma 8.3.12. *Let $\mathcal{Q}$ be a non-singular (parabolic) quadric in $\mathrm{PG}(4,q^2)$ and let H be a singular Hermitian variety in $\mathrm{PG}(4,q^2)$. If H is a cone with vertex a point $P \notin \mathcal{Q}$ and $|H \cap \mathcal{Q}| > \overline{W}_4(q)$, then H contains more than $2q^3 + \frac{2}{3}q^2 - \frac{1}{3}q + 2$ planes that meet $\mathcal{Q}$ in two lines.*

Proof. The Hermitian variety H is a cone with vertex a point. Its base $\mathcal{H}'$ is a non-singular Hermitian variety $\mathcal{H}(3,q^2)$. Every line m in $\mathcal{H}'$ determines a plane $\langle P, m\rangle$ on H. Such a plane intersects $\mathcal{Q}$ in one point, q^2+1 points (line or conic) or $2q^2+1$ points (two lines). Let α_m be the number of points in this intersection $\mathcal{Q} \cap \langle P, m\rangle$.

Now we consider the q^3+q lines of $\mathcal{H}'$ intersecting m. They define q^3+q planes through P. Note that all points of $\langle P, m\rangle \setminus \{P\}$ lie on q of these planes, and each point of $H \setminus \langle P, m\rangle$ belongs to precisely one of these planes. Let x be the number of these planes sharing $2q^2+1$ points with $\mathcal{Q}$ and let y be the number of these planes sharing one point with $\mathcal{Q}$. We find that

$$\begin{aligned}\overline{W}_4(q) &< x(2q^2+1) + (q^3+q-x-y)(q^2+1) + y - \alpha_m(q-1)\\ &\leq q^2x + (q^3+q)(q^2+1) - \alpha_m(q-1)\ .\end{aligned}$$

Hence, at least

$$\frac{\overline{W}_4(q) - (q^3+q)(q^2+1) + (q-1)\alpha_m}{q^2}$$

of the q^3+q considered planes share two lines with $\mathcal{Q}$.

We can repeat this argument for all q^4+q^3+q+1 lines of $\mathcal{H}'$. Note that every plane is considered q^3+q times. So, taking the sum over all lines m of $\mathcal{H}'$, we find at least

$$\begin{aligned}
&\frac{1}{q^2(q^3+q)} \sum_{m \in \mathcal{H}'} \left(\overline{W}_4(q) - (q^3+q)(q^2+1) + (q-1)\alpha_m\right) \\
&= \frac{1}{q^2(q^3+q)} \Bigg[(q^4+q^3+q+1)(\overline{W}_4(q) - (q^3+q)(q^2+1)) \\
&\qquad + (q-1) \sum_{m \in \mathcal{H}'} \alpha_m \Bigg] \\
&> \frac{1}{q^2(q^3+q)} \left[(q^4+q^3+q+1)(\overline{W}_4(q) - (q^3+q)(q^2+1)) \right. \\
&\qquad \left. + (q-1)(q+1)\overline{W}_4(q) \right] \\
&= \frac{q^4+q^3+q^2+q}{q^2(q^3+q)} \overline{W}_4(q) - \frac{(q+1)(q^2+1)(q^3+1)}{q^2} \\
&\geq \frac{q+1}{q^2}(q^5+2q^4-\frac{1}{3}q^3+2q^2+q+1) - \frac{(q+1)(q^2+1)(q^3+1)}{q^2} \\
&> 2q^3 + \frac{2}{3}q^2 - \frac{1}{3}q + 2
\end{aligned}$$

planes of H containing two lines of $\mathcal{Q}$. This proves the lemma. □

Lemma 8.3.13. *Let $\mathcal{Q}$ be a non-singular (parabolic) quadric in* $\mathrm{PG}(4, q^2)$ *and let H be a singular Hermitian variety in* $\mathrm{PG}(4, q^2)$. *Assume that H is a cone with vertex a point $P \notin \mathcal{Q}$. If $\ell \ni P$ is a line on H that is contained in at least three planes intersecting $\mathcal{Q}$ in two lines, then ℓ is tangent to $\mathcal{Q}$. So, all lines on $\mathcal{Q}$ in the planes of H through ℓ pass through a common point.*

Proof. As in the previous lemma, we denote the base of the cone H by $\mathcal{H}'$. We denote the point $\ell \cap \mathcal{H}'$ by R. Note that all planes on H through ℓ are contained in the 3-space $\langle P, T_R(\mathcal{H}')\rangle = \sigma$, with $T_R(\mathcal{H}')$ the tangent plane in R to $\mathcal{H}'$ in the ambient 3-space of $\mathcal{H}'$. The quadric $\mathcal{Q}$ intersects a 3-space in a non-singular hyperbolic quadric $\mathcal{Q}^+(3, q^2)$, a non-singular elliptic quadric $\mathcal{Q}^-(3, q^2)$, or a cone with vertex a point and base a conic. The intersection $\sigma \cap \mathcal{Q}$ cannot be an elliptic quadric since it contains lines. Also, since ℓ contains at most two points of $\mathcal{Q}$, but there are at least three planes through ℓ in σ sharing two lines with $\mathcal{Q}$, there is a point of $\ell \cap \mathcal{Q}$ which lies on at least three lines of $\mathcal{Q}$. Consequently, the intersection $\sigma \cap \mathcal{Q}$ cannot be a hyperbolic quadric. So, $\sigma \cap \mathcal{Q}$ is a cone Q' with vertex a point R' and base a conic $\mathcal{Q}'$.

All lines of the cone Q' pass through the vertex R'. So all lines of $\mathcal{Q}$ on a plane in σ contain R'. Hence, R' has to be on ℓ, and $\ell \cap \mathcal{Q} = \{R'\}$. We conclude that ℓ is a tangent line. $\square$

Lemma 8.3.14. *Let $\mathcal{Q}$ be a non-singular (parabolic) quadric in* $\mathrm{PG}(4, q^2)$ *and let H be a singular Hermitian variety in* $\mathrm{PG}(4, q^2)$. *Assume that H is a cone with vertex a point $P \notin \mathcal{Q}$, and let π be a plane on H intersecting $\mathcal{Q}$ in two lines. Then, π contains at most one line through P in H that is contained in at least three planes sharing two lines with $\mathcal{Q}$.*

Proof. Since $\pi \cap \mathcal{Q}$ is the union of two lines, and $P \notin \mathcal{Q}$, there is only one tangent line through P to $\mathcal{Q}$ in π. It follows immediately from the previous lemma, that π contains at most one line through P that is contained in at least three planes sharing two lines with $\mathcal{Q}$. This tangent line to $\mathcal{Q}$ through P in π is the only possibility. $\square$

Lemma 8.3.15. *Let $\mathcal{Q}$ be a non-singular (parabolic) quadric in* $\mathrm{PG}(4, q^2)$ *and let H be a singular Hermitian variety in* $\mathrm{PG}(4, q^2)$. *If H is a cone with vertex a point $P \notin \mathcal{Q}$ and $|H \cap \mathcal{Q}| > \overline{W}_4(q)$, then H contains at most $2q^3 + \frac{2}{3}q^2 - \frac{1}{3}q + 2$ planes that meet $\mathcal{Q}$ in two lines.*

Proof. Let $\mathcal{V}$ be the set of planes of H intersecting $\mathcal{Q}$ in two lines and let $\mathcal{L}$ be the set of lines on H through P lying on at least three planes of $\mathcal{V}$. Denote $|\mathcal{L}|$ by α and the number of lines of H through P lying on at most two planes of $\mathcal{V}$ by β. On the one hand, by counting the tuples (ℓ, π) with $\ell \in \mathcal{L}$, $\pi \in \mathcal{V}$ and $\ell \subset \pi$, we find

$$(q+1)\alpha + 2\beta \geq (q^2+1)|\mathcal{V}| \ .$$

Since $\alpha + \beta = |\mathcal{H}(3, q^2)| = (q^3+1)(q^2+1)$, we can rewrite this as

$$\begin{aligned} & \alpha(q-1) + 2(q^5+q^3+q^2+1) \geq (q^2+1)|\mathcal{V}| \ , \\ \Leftrightarrow \qquad & \frac{|\mathcal{V}|(q^2+1) - 2(q^5+q^3+q^2+1)}{q-1} \leq \alpha \ . \end{aligned}$$

On the other hand, we know that every line of $\mathcal{L}$ is contained in at least three planes of $\mathcal{V}$, but every plane of $\mathcal{V}$ contains at most one line of $\mathcal{L}$ by the previous lemma. Hence, $3\alpha \leq |\mathcal{V}|$.

Combining both inequalities, we find

$$|\mathcal{V}| \geq 3\frac{|\mathcal{V}|(q^2+1) - 2(q^5+q^3+q^2+1)}{q-1}$$
$$\Rightarrow |\mathcal{V}| \leq 2q^3 + \frac{2}{3}q^2 - \frac{1}{3}q + 2\,. \qquad \square$$

Lemma 8.3.16. *Let $\mathcal{Q}$ be a non-singular (parabolic) quadric in* $\mathrm{PG}(4,q^2)$ *and let H be a singular Hermitian variety in* $\mathrm{PG}(4,q^2)$*. If H is a cone with vertex a point $P \notin \mathcal{Q}$, then $|H \cap \mathcal{Q}| \leq \overline{W}_4(q)$.*

Proof. If $|H \cap \mathcal{Q}| > \overline{W}_4(q)$, then we know by Lemma 8.3.12 that H contains more than $2q^3 + \frac{2}{3}q^2 - \frac{1}{3}q + 2$ planes that meet $\mathcal{Q}$ in two lines, but by Lemma 8.3.15 that H contains at most $2q^3 + \frac{2}{3}q^2 - \frac{1}{3}q + 2$ planes that meet $\mathcal{Q}$ in two lines, a contradiction. Consequently, $|H \cap \mathcal{Q}| \leq \overline{W}_4(q)$. $\square$

The next lemmata deal with Hermitian varieties which are cones with vertex a line. In this case, the Hermitian variety is the union of q^3+1 planes through the vertex, corresponding to the points of the base, which is a non-singular Hermitian variety $\mathcal{H}(2,q^2)$ in a plane skew to the vertex.

Lemma 8.3.17. *Let $\mathcal{Q}$ be a non-singular (parabolic) quadric in* $\mathrm{PG}(4,q^2)$ *and let H be a singular Hermitian variety in* $\mathrm{PG}(4,q^2)$*. If H is a cone with vertex a line ℓ such that $\ell \cap \mathcal{Q} = \emptyset$, then $|H \cap \mathcal{Q}| \leq q^5+q^3+q^2+1 < \overline{W}_4(q)$.*

Proof. A plane of H, necessarily through ℓ, cannot intersect $\mathcal{Q}$ in a line. So, in every plane of H there are at most q^2+1 points of $\mathcal{Q}$. Consequently, $|H \cap \mathcal{Q}| \leq (q^3+1)(q^2+1) = q^5+q^3+q^2+1 < \overline{W}_4(q)$. $\square$

Lemma 8.3.18. *Let $\mathcal{Q}$ be a non-singular (parabolic) quadric in* $\mathrm{PG}(4,q^2)$ *and let H be a singular Hermitian variety in* $\mathrm{PG}(4,q^2)$*. If H is a cone with vertex a line ℓ such that $\ell \cap \mathcal{Q} = \{R\}$, with R a point, then $|H \cap \mathcal{Q}| \leq q^5+2q^2+1 < \overline{W}_4(q)$.*

Proof. We denote the base of the cone H by $\mathcal{H}'$. We know that $\mathcal{H}'$ is a Hermitian curve $\mathcal{H}(2,q^2)$ in a plane π disjoint to ℓ. The tangent hyperplane $T_R(\mathcal{Q})$ contains the line ℓ and intersects π in a line r. We also know that $T_R(\mathcal{Q}) \cap \mathcal{Q}$ is a cone with vertex R and base a conic $\mathcal{Q}(2,q^2)$, that we denote by $\mathcal{Q}'$. We distinguish between two cases.

If r is a tangent line to $\mathcal{H}'$, then $|r \cap \mathcal{Q}' \cap \mathcal{H}'| \leq 1$. Hence, there is at most one plane of the Hermitian variety H containing lines of the tangent cone $T_R(\mathcal{Q}) \cap \mathcal{Q}$, and such plane contains at most 2 lines of this cone. All $q^3 \cdot q^2$ lines of H through R that are not contained in $T_R(\mathcal{Q})$, contain precisely one additional point of the quadric $\mathcal{Q}$, next to R. Consequently, $|H \cap \mathcal{Q}| \leq 1 + 2q^2 + q^3 q^2 = q^5 + 2q^2 + 1$.

If r is a secant line to $\mathcal{H}'$, then $|r \cap \mathcal{Q}' \cap \mathcal{H}'| \leq 2$. Now, there are at most two planes of the Hermitian variety H containing lines of this tangent cone. All $(q^3 - q)q^2$ lines of H through R, that are not contained in $T_R(\mathcal{Q})$, contain precisely one additional point of the quadric $\mathcal{Q}$. Consequently, $|H \cap \mathcal{Q}| \leq 1 + 2 \cdot 2q^2 + (q^3 - q)q^2 = q^5 - q^3 + 4q^2 + 1$.

The observation $q^5 - q^3 + 4q^2 + 1 \leq q^5 + 2q^2 + 1 < \overline{W}_4(q)$ finishes the proof. $\square$

Lemma 8.3.19. *Let $\mathcal{Q}$ be a non-singular (parabolic) quadric in* $\mathrm{PG}(4, q^2)$ *and let H be a singular Hermitian variety in* $\mathrm{PG}(4, q^2)$*. If H is a cone with vertex a line ℓ such that $\ell \cap \mathcal{Q} = \{P, R\}$, with P and R distinct points, then $|H \cap \mathcal{Q}| \leq q^5 + q^3 + 3q^2 + 1 < \overline{W}_4(q)$.*

Proof. Lines contained in the intersection $H \cap \mathcal{Q}$ necessarily pass through P or R, so lie in $T_P(\mathcal{Q})$ or in $T_R(\mathcal{Q})$. Since $T_P(\mathcal{Q})$ and $T_R(\mathcal{Q})$ meet in a plane π disjoint to ℓ, we can choose the non-singular conic $\mathcal{Q}' = \mathcal{Q} \cap \pi$ as base for both cones $T_P(\mathcal{Q}) \cap \mathcal{Q}$ and $T_R(\mathcal{Q}) \cap \mathcal{Q}$. Since π is disjoint to ℓ, we can consider the non-singular Hermitian variety $\mathcal{H}' = H \cap \pi$ as base for H. Every point in $\mathcal{H}' \cap \mathcal{Q}'$ then corresponds to a plane of H sharing two lines with $\mathcal{Q}$. Every point in $\mathcal{H}' \setminus \mathcal{Q}'$ then corresponds to a plane of H sharing a conic with $\mathcal{Q}$. We know by Bézout's theorem, that $|\mathcal{H}' \cap \mathcal{Q}'| \leq 2(q+1)$. Hence,

$$\begin{aligned} |H \cap \mathcal{Q}| &= 2 + |\mathcal{H}' \cap \mathcal{Q}'|(2q^2 - 1) + \left(q^3 + 1 - |\mathcal{H}' \cap \mathcal{Q}'|\right)(q^2 - 1) \\ &= q^2 |\mathcal{H}' \cap \mathcal{Q}'| + (q^3 + 1)(q^2 - 1) + 2 \\ &\leq q^5 + q^3 + 3q^2 + 1 < \overline{W}_4(q) \ . \end{aligned}$$

$\square$

Lemma 8.3.20. *Let $\mathcal{Q}$ be a non-singular (parabolic) quadric in* $\mathrm{PG}(4, q^2)$ *and let H be a singular Hermitian variety in* $\mathrm{PG}(4, q^2)$*. If H is a cone with vertex a line $\ell \subset \mathcal{Q}$, then $|H \cap \mathcal{Q}| \leq q^5 + 2q^2 + 1 < \overline{W}_4(q)$.*

Proof. Every plane through ℓ contains at most one extra line of the quadric $\mathcal{Q}$. So, $|H \cap \mathcal{Q}| \leq q^2 + 1 + (q^3 + 1)q^2 = q^5 + 2q^2 + 1 < \overline{W}_4(q)$. $\square$

The final lemmata consider the Hermitian varieties whose vertex is a plane or a 3-space.

Lemma 8.3.21. *Let $\mathcal{Q}$ be a non-singular (parabolic) quadric in $\mathrm{PG}(4,q^2)$ and let H be a singular Hermitian variety in $\mathrm{PG}(4,q^2)$. If H is a cone with vertex a plane π, then $|H \cap \mathcal{Q}| \leq q^5 + q^4 + q^3 + 2q^2 + 1 < \overline{W}_4(q)$.*

Proof. The Hermitian variety H is the union of $q+1$ different hyperplanes through π, corresponding to a Hermitian variety $\mathcal{H}(1,q^2)$ on a line disjoint to π. A hyperplane intersects $\mathcal{Q}$ in a non-singular hyperbolic quadric $\mathcal{Q}^+(3,q^2)$, a non-singular elliptic quadric $\mathcal{Q}^-(3,q^2)$, or a cone with vertex a point and base a conic. The plane π contains one point, q^2+1 points or $2q^2+1$ points of $\mathcal{Q}$. We distinguish between these three cases.

If $|\pi \cap \mathcal{Q}| = 1$, then every hyperplane of H contains at most $q^4 + q^2$ additional points of the intersection $H \cap \mathcal{Q}$ since these hyperplanes cannot intersect $\mathcal{Q}$ in a non-singular hyperbolic quadric $\mathcal{Q}^+(3,q^2)$. So, $|H \cap \mathcal{Q}| \leq 1 + (q+1)(q^4+q^2) = q^5 + q^4 + q^3 + q^2 + 1$.

If $|\pi \cap \mathcal{Q}| = q^2 + 1$, then every hyperplane of H contains at most $q^4 + q^2$ additional points of the intersection. So, $|H \cap \mathcal{Q}| \leq q^2 + 1 + (q+1)(q^4+q^2) = q^5 + q^4 + q^3 + 2q^2 + 1$.

If $|\pi \cap \mathcal{Q}| = 2q^2+1$, then every hyperplane of H contains at most q^4 additional points of the intersection. So, $|H \cap \mathcal{Q}| \leq 2q^2 + 1 + (q+1)q^4 = q^5 + q^4 + 2q^2 + 1$.

Hence, in general, the intersection size can be at most $q^5 + q^4 + q^3 + 2q^2 + 1 < \overline{W}_4(q)$. □

Intersections of a non-singular quadric and a Hermitian variety which is the union of $q+1$ hyperplanes are studied in more detail in Section 8.5.

Lemma 8.3.22. *Let $\mathcal{Q}$ be a non-singular (parabolic) quadric in $\mathrm{PG}(4,q^2)$ and let H be a singular Hermitian variety in $\mathrm{PG}(4,q^2)$. If H is a cone with vertex a 3-space, then $|H \cap \mathcal{Q}| \leq q^4 + 2q^2 + 1 < \overline{W}_4(q)$.*

Proof. The Hermitian variety H is a hyperplane because the base is empty. We know that the maximal intersection size of a hyperplane with the quadric $\mathcal{Q}$ is $q^4 + 2q^2 + 1$, which is attained if the intersection is a non-singular hyperbolic quadric $\mathcal{Q}^+(3,q^2)$. □

Resuming the previous results, we can state the following theorem.

Theorem 8.3.23. *Let $\mathcal{Q}$ be a non-singular quadric in $\mathrm{PG}(4, q^2)$, which is necessarily parabolic, and let H be a Hermitian variety in $\mathrm{PG}(4, q^2)$. Then $|\mathcal{Q} \cap H| \leq \overline{W}_4(q)$. Hence, the minimum distance of the code $C_{Herm}(\mathcal{Q})$ is at least $|\mathcal{Q}| - \overline{W}_4(q)$.*

Proof. This follows immediately from Lemmas 8.3.10, 8.3.11, 8.3.16, 8.3.17, 8.3.18, 8.3.19, 8.3.20, 8.3.21 and 8.3.22. □

8.4 The functional code $C_{Herm}(\mathcal{Q})$ for $n \geq 5$

We now determine upper bounds on the intersection size of a non-singular quadric $\mathcal{Q}$ in $\mathrm{PG}(n, q^2)$ with an arbitrary Hermitian variety H in $\mathrm{PG}(n, q^2)$. If the intersection size is larger than $W_n(q)$, then the Hermitian variety H must be singular by Theorem 8.2.3. As in the three-dimensional and four-dimensional case, we present a discussion of several cases based on the dimension of the space of singular points of H.

Definition 8.4.1. We define the value $\overline{W}_n(q)$, $n \geq 5$, in the following way:

$$\overline{W}_n(q) = \begin{cases} q^7 + 2q^6 + 2q^5 - \frac{1}{2}q^4 - \frac{21}{4}q^3 + \frac{15}{8}q^2 + \frac{195}{16}q + 8 & n = 5,\ q \geq 3\,, \\ \frac{25}{84}2^{2n} + \frac{2425}{1764}2^n - \frac{2655125}{100107} & n \text{ even},\ q = 2\,, \\ \frac{25}{84}2^{2n} + \frac{725}{1764}2^n - \frac{27050}{3087} & n \text{ odd},\ q = 2\,, \\ W_n(q) & \text{else}\,. \end{cases}$$

Lemma 8.4.2. *Let $\mathcal{H}$ be a non-singular Hermitian variety in $\mathrm{PG}(n, q^2)$ and let $\mathcal{Q}$ be a non-singular quadric in $\mathrm{PG}(n, q^2)$, $n \geq 5$. Then $|\mathcal{Q} \cap \mathcal{H}| \leq \overline{W}_n(q)$.*

Proof. We know that $|\mathcal{Q} \cap \mathcal{H}| \leq W_n(q)$ by Theorem 8.2.3. Note that $\overline{W}_n(q) \geq W_n(q)$ for all $n \geq 5$. The lemma follows immediately. □

The next lemmata consider the Hermitian varieties whose vertex is a point. These Hermitian varieties are the union of lines through P and a point of the base, which is a non singular Hermitian variety $\mathcal{H}(n-1, q^2)$.

Lemma 8.4.3. *Let $\mathcal{Q}$ be a non-singular quadric in $\mathrm{PG}(n, q^2)$ and let H be a singular Hermitian variety in $\mathrm{PG}(n, q^2)$, $n \geq 5$. If H is a cone with vertex a point $P \in \mathcal{Q}$, then $|H \cap \mathcal{Q}| \leq \overline{W}_n(q)$.*

Proof. We consider the tangent hyperplane $T_P(\mathcal{Q})$ in P to the quadric $\mathcal{Q}$. We know that $T_P(\mathcal{Q}) \cap \mathcal{Q}$ is a cone with vertex P and base a non-singular quadric $\mathcal{Q}'$ of the same type as $\mathcal{Q}$, in an $(n-2)$-space π disjoint to P. The intersection $H \cap \mathcal{Q}$ contains all points of the lines through P and a point of $(H \cap \pi) \cap \mathcal{Q}'$. Note that $H \cap \pi$ is either a non-singular Hermitian variety or else a cone with vertex a point. On the other lines on H through P in the tangent hyperplane $T_P(\mathcal{Q})$ there are no extra points of the intersection $H \cap \mathcal{Q}$. The lines on H through P not in the tangent hyperplane $T_P(\mathcal{Q})$ contain one additional point of $\mathcal{Q}$, next to P. The number of these lines equals either

$$\begin{aligned}
&|\mathcal{H}(n-1,q^2)| - |\mathcal{H}(n-2,q^2)| \\
&= \frac{(q^n-(-1)^n)(q^{n-1}-(-1)^{n-1})}{q^2-1} - \frac{(q^{n-1}-(-1)^{n-1})(q^{n-2}-(-1)^{n-2})}{q^2-1} \\
&\leq q^{2n-3}+q^{n-2}\,,
\end{aligned}$$

or else

$$\begin{aligned}
&|\mathcal{H}(n-1,q^2)| - \left(1+q^2|\mathcal{H}(n-3,q^2)|\right) \\
&= \frac{(q^n-(-1)^n)(q^{n-1}-(-1)^{n-1})}{q^2-1} - 1 \\
&\qquad\qquad - q^2\frac{(q^{n-2}-(-1)^{n-2})(q^{n-3}-(-1)^{n-3})}{q^2-1} \\
&= q^{2n-3}\,,
\end{aligned}$$

depending on the intersection $H \cap \pi$.

We now proceed using induction on the dimension n. Hereby, we use the results from Theorem 8.3.8 and Theorem 8.3.23 as induction base. Then, by the induction hypothesis and Lemma 8.4.2, we know that $|(H\cap\pi)\cap\mathcal{Q}'| \leq \overline{W}_{n-2}(q)$. In general, we know by the arguments above that $|H \cap \mathcal{Q}| \leq \overline{W}_{n-2}(q)q^2 + (q^{2n-3}+q^{n-2})+1$. For $n=5$, we find

$$\begin{aligned}
|H \cap \mathcal{Q}| &\leq \overline{W}_3(q)q^2 + (q^7+q^3)+1 \\
&= (2q^3+q^2+1)q^2 + (q^7+q^3)+1 \\
&= q^7+2q^5+q^4+q^3+q^2+1 < \overline{W}_5(q)\,.
\end{aligned}$$

For $n \geq 6$, we need to distinguish between two cases. For $q \geq 3$, we know that $\overline{W}_{n-2}(q) < q^{2n-7}+2q^{2n-8}+4q^{2n-9}$ and $q^{2n-3}+q^{2n-5}+2q^{2n-6}+5q^{2n-7} < \overline{W}_n(q)$, and thus we find

$$\begin{aligned}|H \cap \mathcal{Q}| &\leq \overline{W}_{n-2}(q)q^2 + (q^{2n-3} + q^{n-2}) + 1 \\ &< q^2(q^{2n-7} + 2q^{2n-8} + 4q^{2n-9}) + q^{2n-3} + q^{n-2} + 1 \\ &< q^{2n-3} + q^{2n-5} + 2q^{2n-6} + 5q^{2n-7} < \overline{W}_n(q) .\end{aligned}$$

For $q = 2$, we can check the inequality $\overline{W}_{n-2}(q)q^2 + (q^{2n-3} + q^{n-2}) + 1 < \overline{W}_n(q)$ straightforward. □

Notation 8.4.4. In the following lemmata we turn to the case in which the vertex (a point) of the Hermitian variety does not belong to the non-singular quadric. We recall that every line of the base corresponds to a plane of the Hermitian variety through the vertex. Such a plane intersects the quadric in 1, $q^2 + 1$ or $2q^2 + 1$ points. This last case occurs when the intersection equals two lines. We say that a line in the base of the cone is *of type (2)* if the corresponding plane through P contains $2q^2 + 1$ points of the quadric.

Lemma 8.4.5. *Let $\mathcal{Q}$ be a non-singular quadric in* $\mathrm{PG}(n, q^2)$ *and let H be a singular Hermitian variety in* $\mathrm{PG}(n, q^2)$*, $n \geq 5$. If H is a cone with vertex a point $P \notin \mathcal{Q}$ and $|H \cap \mathcal{Q}| > \overline{W}_n(q)$, then there are more than $\frac{a_n(q)}{b_n(q)}\overline{W}_n(q) - c_n(q)$ lines of type (2) in the base of H, with*

$$\begin{aligned}a_n(q) &= \big(|\mathcal{H}(n-1, q^2)| + (q^2+1)(|\mathcal{H}(n-3, q^2)| - 2)\big) \cdot |\mathcal{H}(n-3, q^2)| , \\ b_n(q) &= q^2(q^2+1)^2(|\mathcal{H}(n-3, q^2)| - 1) , \\ c_n(q) &= \frac{|\mathcal{H}(n-1, q^2)| \cdot |\mathcal{H}(n-3, q^2)|}{q^2} .\end{aligned}$$

Proof. The base of the cone H is a non-singular Hermitian variety $\mathcal{H}(n-1, q^2)$, which we will denote by $\mathcal{H}'$. Let m be a line of this base $\mathcal{H}'$. The number of lines in $\mathcal{H}'$ which intersect m in a point, equals $(q^2+1)(|\mathcal{H}(n-3, q^2)| - 1)$. Let α_m be the number of points in the intersection of the plane $\langle P, m\rangle \subset H$ and $\mathcal{Q}$. An easy counting argument, analogous to the argument in the proof of Lemma 8.3.12, shows that for at least

$$\frac{\overline{W}_n - (q^2+1)^2(|\mathcal{H}(n-3, q^2)| - 1) + (|\mathcal{H}(n-3, q^2)| - 2)\alpha_m}{q^2}$$

lines of $\mathcal{H}'$ intersecting m, the corresponding plane through P contains two lines of the quadric $\mathcal{Q}$. These are lines of type (2).

We can repeat the same argument for all lines of $\mathcal{H}'$, of which there are

$$\frac{|\mathcal{H}(n-1,q^2)| \cdot |\mathcal{H}(n-3,q^2)|}{q^2+1} .$$

In order to count the total number of lines of type (2), we sum over all lines in $\mathcal{H}'$. In this way, every such line is counted $(q^2+1)(|\mathcal{H}(n-3,q^2)|-1)$ times. Hence, the total number of lines of type (2) is at least

$$\begin{aligned}
&\sum_{m\subset\mathcal{H}'} \frac{\overline{W}_n(q) - (q^2+1)^2(|\mathcal{H}(n-3,q^2)|-1) + (|\mathcal{H}(n-3,q^2)|-2)\alpha_m}{q^2(q^2+1)(|\mathcal{H}(n-3,q^2)|-1)} \\
&= \frac{|\mathcal{H}(n-1,q^2)| \cdot |\mathcal{H}(n-3,q^2)| \cdot (\overline{W}_n(q) - (q^2+1)^2(|\mathcal{H}(n-3,q^2)|-1))}{q^2(q^2+1)^2(|\mathcal{H}(n-3,q^2)|-1)} \\
&\quad + \frac{|\mathcal{H}(n-3,q^2)|-2}{q^2(q^2+1)(|\mathcal{H}(n-3,q^2)|-1)} \sum_{m\subset\mathcal{H}'} \alpha_m \\
&> \frac{|\mathcal{H}(n-1,q^2)| \cdot |\mathcal{H}(n-3,q^2)|}{q^2(q^2+1)^2(|\mathcal{H}(n-3,q^2)|-1)} \overline{W}_n(q) - \frac{|\mathcal{H}(n-1,q^2)| \cdot |\mathcal{H}(n-3,q^2)|}{q^2} \\
&\quad + \frac{(|\mathcal{H}(n-3,q^2)|-2) \cdot |\mathcal{H}(n-3,q^2)|}{q^2(q^2+1)(|\mathcal{H}(n-3,q^2)|-1)} \overline{W}_n(q) \\
&= \frac{(|\mathcal{H}(n-1,q^2)| + (q^2+1)(|\mathcal{H}(n-3,q^2)|-2)) \cdot |\mathcal{H}(n-3,q^2)|}{q^2(q^2+1)^2(|\mathcal{H}(n-3,q^2)|-1)} \overline{W}_n(q) \\
&\quad - \frac{|\mathcal{H}(n-1,q^2)| \cdot |\mathcal{H}(n-3,q^2)|}{q^2} .
\end{aligned}$$

In the penultimate step, we made use of the fact that every intersection point lies on precisely $|\mathcal{H}(n-3,q^2)|$ planes through P of H, hence is counted $|\mathcal{H}(n-3,q^2)|$ times. □

Definition 8.4.6. We define for $n \geq 5$:

$$\delta_n(q) = \begin{cases} 1 + q^2|\mathcal{Q}^+(n-4,q^2)| & n \text{ odd}, \\ |\mathcal{Q}^+(n-3,q^2)| & n \text{ even}. \end{cases}$$

Lemma 8.4.7. *Let $\mathcal{Q}$ be a non-singular quadric in $\mathrm{PG}(n,q^2)$ and let H be a singular Hermitian variety in $\mathrm{PG}(n,q^2)$, $n \geq 5$. Assume that H is a cone with vertex a point $P \notin \mathcal{Q}$ and base $\mathcal{H}'$. If R is a point of $\mathcal{H}'$, which is contained in at least $\delta_n(q)+1$ lines of type (2) in $\mathcal{H}'$, then $\langle P, T_R(\mathcal{H}')\rangle \cap \mathcal{Q}$ is a cone with vertex a point which belongs to the line $\langle P, R\rangle$.*

Proof. Note that $\delta_n(q) \geq |\mathcal{Q}_{n-3}|$ and $\delta_n(q) \geq 1+q^2|\mathcal{Q}_{n-4}|$ for any non-singular quadric $\mathcal{Q}_{n-3}$ in $\mathrm{PG}(n-3, q^2)$ and any non-singular quadric $\mathcal{Q}_{n-4}$ in $\mathrm{PG}(n-4, q^2)$, as well in case n is even, as in case n is odd.

Let Q be the $(n-1)$-dimensional quadric $\langle P, T_R(\mathcal{H}')\rangle \cap \mathcal{Q}$. This quadric Q either is a non-singular quadric, or else a singular quadric with vertex a point. We first show that only the last possibility can occur.

Assume that the quadric Q is non-singular. Then every point of this quadric lies on $|\mathcal{Q}(n-3, q^2)|$ lines of Q, whereby $\mathcal{Q}(n-3, q^2)$ has the same type as Q. We know that $|\mathcal{Q}(n-3, q^2)| \leq \delta_n(q)$. We also know that the line $\langle P, R\rangle$ intersects $\mathcal{Q}$ in either one or two points since this line is contained in at least one plane intersecting $\mathcal{Q}$ in two lines, but $P \notin \mathcal{Q}$. These intersection points therefore belong to at least $\delta_n(q)+1$ lines of Q, a contradiction. So the quadric Q is singular with a point S as vertex and a base $\mathcal{Q}'$, a non-singular $(n-2)$-dimensional quadric $\mathcal{Q}(n-2, q^2)$.

Now, we show that this point S belongs to the line $\langle P, R\rangle$ and that this point S is the only intersection point of this line with the quadric $\mathcal{Q}$. Again, the line $\langle P, R\rangle$ shares one or two points with the quadric $\mathcal{Q}$. Assume that these, one or two, intersection points are non-singular (not equal to the vertex), then they belong to at most $1 + q^2|\mathcal{Q}(n-4, q^2)|$ lines of the quadric Q, whereby $\mathcal{Q}(n-4, q^2)$ has the same type as $\mathcal{Q}'$. But, in any case, this number of lines is smaller than $\delta_n(q)+1$. So the line $\langle P, R\rangle$ contains the vertex S of the quadric Q contained in $\mathcal{Q}$. □

Lemma 8.4.8. *Let $\mathcal{Q}$ be a non-singular quadric in $\mathrm{PG}(n, q^2)$ and let H be a singular Hermitian variety in $\mathrm{PG}(n, q^2)$, $n \geq 5$. Assume that H is a cone with vertex a point $P \notin \mathcal{Q}$ and base $\mathcal{H}'$, and let m be a line of type (2) in $\mathcal{H}'$. Then m contains at most one point which is contained in at least $\delta_n(q)+1$ lines of type (2) in $\mathcal{H}'$.*

Proof. The line m defines a plane $\langle P, m\rangle$ intersecting $\mathcal{Q}$ in two lines m' and m''. Denote the intersection point $m' \cap m''$ by T. By Lemma 8.4.7, the point $R = \langle P, T\rangle \cap m$ is the only point that can be lying on at least $\delta_n(q)+1$ lines of type (2) in $\mathcal{H}'$. □

Lemma 8.4.9. *Let $\mathcal{Q}$ be a non-singular quadric in $\mathrm{PG}(n, q^2)$ and let H be a singular Hermitian variety in $\mathrm{PG}(n, q^2)$, $n \geq 5$. Assume that H is a cone with* *vertex a point $P \notin \mathcal{Q}$ and base $\mathcal{H}'$. The number of lines of type (2) is at most*

$$d_n(q) = \frac{\delta_n(q)(\delta_n(q)+1)\,|\mathcal{H}(n-1,q^2)|}{(q^2+2)\delta_n(q)+q^2+1-|\mathcal{H}(n-3,q^2)|} \ .$$

Proof. Let A be the number of lines of type (2) in $\mathcal{H}'$. Define $\mathcal{P}$ as the set of points on $\mathcal{H}'$ lying on at least $\delta_n(q)+1$ lines of type (2), and denote $\alpha = |\mathcal{P}|$. Let β be the number of points of $\mathcal{H}'$ lying on at most $\delta_n(q)$ lines of type (2). Then, using a double counting argument, we find

$$\alpha|\mathcal{H}(n-3,q^2)| + \beta\delta_n(q) \geq A(q^2+1) \ .$$

Since $\alpha + \beta = |\mathcal{H}(n-1,q^2)|$, we can rewrite this as

$$\alpha \geq \frac{A(q^2+1) - \delta_n(q)|\mathcal{H}(n-1,q^2)|}{|\mathcal{H}(n-3,q^2)| - \delta_n(q)} \ .$$

Now, using Lemma 8.4.8 and a double counting argument, we also find

$$A \geq (\delta_n(q)+1)\alpha \ .$$

Combining both inequalities, we find

$$A \geq (\delta_n(q)+1)\cdot\frac{A(q^2+1) - \delta_n(q)|\mathcal{H}(n-1,q^2)|}{|\mathcal{H}(n-3,q^2)| - \delta_n(q)} \ .$$

It follows immediately that

$$A \leq \frac{\delta_n(q)(\delta_n(q)+1)\,|\mathcal{H}(n-1,q^2)|}{(q^2+2)\delta_n(q)+q^2+1-|\mathcal{H}(n-3,q^2)|} \ .$$

It should be noted that this deduction does not hold in the case $n=5$ and $q=2$ since $|\mathcal{H}(2,2^2)| = 9 = \delta_5(2)$. From the first inequality and the result $\alpha+\beta = |\mathcal{H}'|$ however, it follows immediately that $A \leq 297$, which is precisely the inequality we wanted to prove. $\square$

Lemma 8.4.10. *Let $\mathcal{Q}$ be a non-singular quadric in* $\mathrm{PG}(n,q^2)$ *and let H be a singular Hermitian variety in* $\mathrm{PG}(n,q^2)$*, $n \geq 5$. If H is a cone with vertex a point $P \notin \mathcal{Q}$, then $|H \cap \mathcal{Q}| \leq \overline{W}_n(q)$.*

Proof. We denote the base of the cone H by $\mathcal{H}'$, a non-singular Hermitian variety $\mathcal{H}(n-1, q^2)$. If $|H \cap \mathcal{Q}| > \overline{W}_n(q)$, then there are more than $\frac{a_n(q)}{b_n(q)}\overline{W}_n(q) - c_n(q)$ lines of type (2) in $\mathcal{H}'$ by Lemma 8.4.5, with

$$\begin{aligned} a_n(q) &= (|\mathcal{H}(n-1,q^2)| + (q^2+1)(|\mathcal{H}(n-3,q^2)| - 2)) \cdot |\mathcal{H}(n-3,q^2)| \ , \\ b_n(q) &= q^2(q^2+1)^2(|\mathcal{H}(n-3,q^2)| - 1) \ , \\ c_n(q) &= \frac{|\mathcal{H}(n-1,q^2)| \cdot |\mathcal{H}(n-3,q^2)|}{q^2} \ . \end{aligned}$$

However, by Lemma 8.4.9, we know that there are at most $d_n(q)$ lines of type (2) in $\mathcal{H}'$, with

$$d_n(q) = \frac{\delta_n(q)(\delta_n(q)+1)\,|\mathcal{H}(n-1,q^2)|}{(q^2+2)\delta_n(q) + q^2 + 1 - |\mathcal{H}(n-3,q^2)|} \ .$$

Hence, if $d_n(q) < \frac{a_n(q)}{b_n(q)}\overline{W}_n(q) - c_n(q)$, we find a contradiction. This condition is equivalent to $\overline{W}_n(q) > \frac{b_n(q)(c_n(q)+d_n(q))}{a_n(q)}$.

Using a computer algebra package, it can be checked that this inequality is valid. More details can be found in Computation A.3.1. Hence, the desired contradiction is found. □

In the next lemmata we consider the singular Hermitian varieties whose vertex is a line. These Hermitian varieties are the union of $|\mathcal{H}(n-2,q^2)|$ planes through the vertex.

Lemma 8.4.11. *Let $\mathcal{Q}$ be a non-singular quadric in* $\mathrm{PG}(n, q^2)$ *and let H be a singular Hermitian variety in* $\mathrm{PG}(n, q^2)$, $n \geq 5$. *If H is a cone with vertex a line ℓ such that $\ell \cap \mathcal{Q} = \emptyset$, then $|H \cap \mathcal{Q}| \leq q^{2n-3} + 3q^{2n-5} + 2q^{2n-7} < \overline{W}_n(q)$.*

Proof. A plane of H through ℓ cannot share a line with $\mathcal{Q}$. So, in each of these planes there are at most q^2+1 points of $\mathcal{Q}$. Consequently,

$$\begin{aligned} |H \cap \mathcal{Q}| &\leq |\mathcal{H}(n-2,q^2)|(q^2+1) \\ &\leq (q^{2n-5} + 2q^{2n-7})(q^2+1) \\ &= q^{2n-3} + 3q^{2n-5} + 2q^{2n-7} < \overline{W}_n(q) \ . \end{aligned}$$

The final inequality is immediate if $\overline{W}_n(q) = W_n(q)$. The other cases can easily be checked. □

Lemma 8.4.12. *Let $\mathcal{Q}$ be a non-singular quadric in* $\mathrm{PG}(n, q^2)$ *and let H be a singular Hermitian variety in* $\mathrm{PG}(n, q^2)$, *$n \geq 5$. If H is a cone with vertex a line ℓ such that $\ell \cap \mathcal{Q} = \{R\}$, with R a point, then $|H \cap \mathcal{Q}| \leq q^{2n-3} + 2q^{2n-5} + 1 < \overline{W}_n(q)$.*

Proof. We denote the base of the cone H by $\mathcal{H}'$, a non-singular Hermitian variety $\mathcal{H}(n-2, q^2)$. The tangent hyperplane $T_R(\mathcal{Q})$ contains the line ℓ, so intersects the $(n-2)$-space containing $\mathcal{H}'$ in an $(n-3)$-space. The intersection $H' = T_R(\mathcal{Q}) \cap \mathcal{H}'$ is either a non-singular Hermitian variety or a singular Hermitian variety with a point as vertex. Clearly, $1 + q^2|H'|$ is an upper bound on $|(H \cap \mathcal{Q}) \cap T_R(\mathcal{Q})|$.

The number of points in $\mathcal{H}' \setminus H'$ equals $|\mathcal{H}(n-2, q^2)| - |H'|$. Each of the points of $\mathcal{H}' \setminus H'$ corresponds to a plane of H through ℓ. Such a plane contains q^2 lines through R, all of them not contained in $T_R(\mathcal{Q})$. They all contain one additional point of $\mathcal{Q}$, next to R. So, $(H \cap \mathcal{Q}) \setminus T_R(\mathcal{Q})$ contains $q^2(|\mathcal{H}(n-2, q^2)| - |H'|)$ points.

We find that

$$\begin{aligned}
|H \cap \mathcal{Q}| &\leq 1 + q^2|H'| + q^2(|\mathcal{H}(n-2, q^2)| - |H'|) \\
&\leq (q^{2n-5} + 2q^{2n-7})q^2 + 1 \\
&= q^{2n-3} + 2q^{2n-5} + 1 < \overline{W}_n(q) .
\end{aligned}$$

The final inequality can be derived from the inequality $q^{2n-3} + 3q^{2n-5} + 2q^{2n-7} < \overline{W}_n(q)$, see Lemma 8.4.11. □

Lemma 8.4.13. *Let $\mathcal{Q}$ be a non-singular quadric in* $\mathrm{PG}(n, q^2)$ *and let H be a singular Hermitian variety in* $\mathrm{PG}(n, q^2)$, *$n \geq 5$. If H is a cone with vertex a line ℓ such that $\ell \cap \mathcal{Q} = \{P, R\}$, with P and R distinct points, then $|H \cap \mathcal{Q}| < \overline{W}_n(q)$.*

Proof. Since $T_P(\mathcal{Q})$ and $T_R(\mathcal{Q})$ meet in an $(n-2)$-space π disjoint to ℓ, we can choose the non-singular quadric $\mathcal{Q}' = \mathcal{Q} \cap \pi$ as base for both cones $T_P(\mathcal{Q}) \cap \mathcal{Q}$ and $T_R(\mathcal{Q}) \cap \mathcal{Q}$. Since π is disjoint to ℓ, we can consider the non-singular Hermitian variety $\mathcal{H}' = H \cap \pi$ as base for H. Every point in $\mathcal{H}' \cap \mathcal{Q}'$ corresponds to a plane of H sharing two lines with $\mathcal{Q}$ and every point in $\mathcal{H}' \setminus \mathcal{Q}'$ corresponds to a plane of H sharing a conic with $\mathcal{Q}$. We already know by Theorem 8.2.3

that $|\mathcal{H}' \cap \mathcal{Q}'| \leq W_{n-2}(q)$. Hence,

$$\begin{aligned} |H \cap \mathcal{Q}| &= 2 + |\mathcal{H}' \cap \mathcal{Q}'|(2q^2 - 1) + (|\mathcal{H}'| - |\mathcal{H}' \cap \mathcal{Q}'|)\,(q^2 - 1) \\ &= 2 + q^2|\mathcal{H}' \cap \mathcal{Q}'| + (q^2 - 1)|\mathcal{H}(n-2, q^2)| \\ &\leq 2 + q^2 W_{n-2}(q) + (q^2 - 1)|\mathcal{H}(n-2, q^2)| \\ &< W_n(q) \leq \overline{W}_n(q)\ . \end{aligned}$$

For $n = 5$, we hereby use the value $2q^3 + q^2 + 1$ for $W_3(q)$, which we found in Theorem 8.3.1, as upper bound for $|\mathcal{H}' \cap \mathcal{Q}'|$. □

Lemma 8.4.14. *Let $\mathcal{Q}$ be a non-singular quadric in* $\mathrm{PG}(n, q^2)$ *and let H be a singular Hermitian variety in* $\mathrm{PG}(n, q^2)$*, $n \geq 5$. If H is a cone with vertex a line $\ell \subset \mathcal{Q}$, then $|H \cap \mathcal{Q}| < \overline{W}_n(q)$.*

Proof. We denote the base of the cone H by $\mathcal{H}'$, a non-singular Hermitian variety $\mathcal{H}(n-2, q^2)$. It is contained in an $(n-2)$-space π disjoint to ℓ. We consider the $(n-2)$-space $T_\ell(\mathcal{Q})$, the tangent space to $\mathcal{Q}$ at the line ℓ. We know that $T_\ell(\mathcal{Q}) \cap \mathcal{Q}$ is a cone with vertex ℓ and base a non-singular quadric $\mathcal{Q}'$ of the same type as $\mathcal{Q}$. We can choose the base $\mathcal{Q}'$ in the $(n-4)$-space $\pi' = \pi \cap T_\ell(\mathcal{Q})$. We denote the Hermitian variety $H \cap \pi'$ by H'.

A plane through ℓ and a point of π' is contained in the intersection $H \cap \mathcal{Q}$ if and only if the point in π' is contained in $H' \cap \mathcal{Q}'$. Hence, $|(H \cap \mathcal{Q}) \cap T_\ell(\mathcal{Q})| = q^2 + 1 + q^4|H' \cap \mathcal{Q}'|$. A plane of H through ℓ which is not contained in $T_\ell(\mathcal{Q})$, shares at most one other line with $\mathcal{Q}$. Hence, $|(H \cap \mathcal{Q}) \setminus T_\ell(\mathcal{Q})| \leq q^2|\mathcal{H}' \setminus \mathcal{Q}'| = q^2\,(|\mathcal{H}'| - |H' \cap \mathcal{Q}'|)$. Consequently,

$$\begin{aligned} |H \cap \mathcal{Q}| &\leq q^2 + 1 + (q^4 - q^2)|H' \cap \mathcal{Q}'| + q^2|\mathcal{H}'| \\ &\leq q^2 + 1 + (q^4 - q^2)|\mathcal{Q}'| + q^2|\mathcal{H}'| \\ &\leq q^2 + 1 + (q^4 - q^2)\left(\frac{q^{2n-8} - 1}{q^2 - 1} + q^{n-5}\right) + q^2(q^{2n-5} + 2q^{2n-7}) \\ &= q^{2n-3} + 2q^{2n-5} + q^{2n-6} + q^{n-1} - q^{n-3} + 1 \\ &\leq W_n(q) \leq \overline{W}_n(q) \end{aligned}$$ □

The last lemma of this section deals with singular Hermitian varieties in $\mathrm{PG}(n, q^2)$ whose vertex is an s-space, with $2 \leq s \leq n - 1$.

Lemma 8.4.15. *Let $\mathcal{Q}$ be a non-singular quadric in* $\mathrm{PG}(n, q^2)$ *and let H be a singular Hermitian variety in* $\mathrm{PG}(n, q^2)$*, $n \geq 5$. If H is a cone with vertex* *an s-space π_s, $2 \leq s \leq n-1$, then $|H \cap \mathcal{Q}| < \overline{W}_n(q)$ or $s = n-2$ and H is the union of $q+1$ hyperplanes.*

Proof. If $s = n-1$, then H is a hyperplane. In this case, the intersection $H \cap \mathcal{Q}$ is a quadric in an $(n-1)$-space. Its size is clearly at most $\overline{W}_n(q)$.

From now on, we assume that $s \leq n-2$. Since the vertex π_s of H contains a plane, and a plane in $\mathrm{PG}(2, q^2)$ cannot be disjoint to a non-singular quadric, we can find a point P in $\pi_s \cap \mathcal{Q}$. Consider the tangent hyperplane $T_P(\mathcal{Q})$ to $\mathcal{Q}$ in P. The intersection $T_P(\mathcal{Q}) \cap \mathcal{Q}$ is a cone with vertex P and base a non-singular quadric $\mathcal{Q}'$ of the same type as $\mathcal{Q}$. Furthermore, the tangent hyperplane $T_P(\mathcal{Q})$ intersects H in a singular Hermitian variety with vertex an $(s-i)$-space, $i = -1, 0, 1$.

The number of points in $H \setminus T_P(\mathcal{Q})$ equals q^{2n-1} if $i = -1$, equals $q^{2n-1} + (-1)^{n-s}q^{n+s}$ if $i = 0$ and equals $q^{2n-1} + (-1)^{n-s}q^{n+s-1}(q-1)$ if $i = 1$. The points in $H \setminus T_P(\mathcal{Q})$ lie on lines of H through P. Since these lines do not lie in $T_P(\mathcal{Q})$, they all contain, besides the point P, one additional point of $\mathcal{Q}$. So, $|(H \cap \mathcal{Q}) \setminus T_P(\mathcal{Q})| \leq q^{2n-3} + q^{n+s-2}$. Consequently,

$$\begin{aligned}
|H \cap \mathcal{Q}| &= |(H \cap \mathcal{Q}) \setminus T_P(\mathcal{Q})| + |(H \cap \mathcal{Q}) \cap T_P(\mathcal{Q})| \\
&\leq q^{2n-3} + q^{n+s-2} + |\mathcal{Q} \cap T_P(\mathcal{Q})| \\
&\leq q^{2n-3} + q^{n+s-2} + \frac{q^{2n-2}-1}{q^2-1} + q^{n-1}
\end{aligned}$$

If $s \leq n-3$, then it follows that $|H \cap \mathcal{Q}| < \overline{W}_n(q)$. This finishes the proof. □

In the next section we will see that the two upper bounds we used in the previous lemma are not met simultaneously. In fact, also in case the Hermitian variety H is the union of $q+1$ hyperplanes, the inequality $|H \cap \mathcal{Q}| < \overline{W}_n(q)$ is valid.

Resuming the results of this section, we can state the following theorem.

Theorem 8.4.16. *Let $\mathcal{Q}$ be a non-singular quadric in $\mathrm{PG}(n, q^2)$ and let H be an arbitrary Hermitian variety in $\mathrm{PG}(n, q^2)$. Then $|H \cap \mathcal{Q}| \leq \overline{W}_n(q)$ or H is a singular Hermitian variety which is the union of $q+1$ hyperplanes.*

Proof. This is an immediate consequence of Lemmas 8.4.2, 8.4.3, 8.4.10, 8.4.11, 8.4.12, 8.4.13, 8.4.14 and 8.4.15. □

8.5 Some small weight code words

In this section we will give some examples of small weight code words of the code $C_{Herm}(\mathcal{Q})$, $\mathcal{Q}$ a non-singular quadric. It follows from Theorem 8.4.16 that the code words arising from the Hermitian varieties in $\mathrm{PG}(n, q^2)$ with an $(n-2)$-space as vertex and a non-singular Hermitian variety $\mathcal{H}(1, q^2)$ as base, are of particular interest. These Hermitian varieties can be seen as the union of $q+1$ hyperplanes through this vertex.

It will turn out that these give rise to code words with weights a little above $|\mathcal{Q}| - \overline{W}_n(q)$. The difference between the bound $\overline{W}_n(q)$ on the intersection size of $\mathcal{Q}$ with a Hermitian variety H and the intersection size of the examples that we will present is $O(q^{2n-5})$. So, in fact, the obtained upper bound $\overline{W}_n(q)$ on the maximum size of the intersection of the non-singular quadric $\mathcal{Q}$ with an arbitrary Hermitian variety, arises from the case of non-singular Hermitian varieties. Note that the upper bound in Theorem 8.4.16 is obtained by taking the maximum of the upper bounds from the different cases, and this case gave the largest upper bound.

It should be observed that not any union of $q+1$ hyperplanes through a fixed $(n-2)$-space, is a Hermitian variety. Such a set of hyperplanes defines a Hermitian variety if and only if its intersection with a line disjoint to the fixed $(n-2)$-space is a Baer subline of this line.

Remark 8.5.1. Each non-singular quadric has an *index* w, related to its type. Its index equals 2 if the quadric is hyperbolic, 1 if the quadric is parabolic, and 0 if it is elliptic. Using this index the number of points on a non-singular quadric in $\mathrm{PG}(n, q^2)$ is given by $\frac{q^{2n}-1}{q^2-1} + (w-1)q^{n-1}$.

Example 8.5.2. Let π_{n-2} be an $(n-2)$-space intersecting the non-singular quadric $\mathcal{Q}$ in $\mathrm{PG}(n, q^2)$, in a cone with vertex a line ℓ and base a non-singular quadric $\mathcal{Q}'$ in an $(n-4)$-space. Each of the q^2+1 hyperplanes through π_{n-2} intersects $\mathcal{Q}$ in a cone with vertex a point. Hence, any Hermitian variety with π_{n-2} as vertex consists of $q+1$ hyperplanes intersecting $\mathcal{Q}$ in a singular quadric whose vertex is a point.

We now calculate the weight of the code word c that such a Hermitian variety H gives rise to. The weight of c equals the number of points of $\mathcal{Q}$, not on H. Each of these points lies on a hyperplane through π_{n-2} that does not belong to the $q+1$ hyperplanes of H. There are q^2-q hyperplanes through π_{n-2}

that are not in H and each of those hyperplanes contains q^{2n-6} planes, not in π_{n-2}, through ℓ. Such a plane contains precisely two lines of $\mathcal{Q}$, one of them ℓ. Consequently, the weight of c is $(q^2-q)q^2q^{2n-6} = q^{2n-2} - q^{2n-3}$.

Note that the size of the intersection $H \cap \mathcal{Q}$ equals $\frac{q^{2n-2}-1}{q^2-1} + q^{2n-3} + (w-1)q^{n-1}$, with w the index of $\mathcal{Q}$.

Example 8.5.3. Let π_{n-2} be an $(n-2)$-space intersecting the non-singular quadric $\mathcal{Q}$ in $\mathrm{PG}(n, q^2)$, in a cone with vertex a point P and base a non-singular quadric $\mathcal{Q}''$ in an $(n-3)$-space. All but one of the hyperplanes through π_{n-2} intersect $\mathcal{Q}$ in a non-singular quadric. One of those q^2+1 hyperplanes through π_{n-2}, the tangent hyperplane to $\mathcal{Q}$ in P, intersects $\mathcal{Q}$ in a singular quadric with vertex a point, namely P, and base a non-singular quadric $\mathcal{Q}'$ in an $(n-2)$-space. Hence, the Hermitian varieties with π_{n-2} as vertex can be split up in two groups: the ones that contain the tangent hyperplane $T_P(\mathcal{Q})$ and the ones that do not.

We now calculate the weight of the code words that both of these Hermitian varieties H give rise to. If a hyperplane π is not $T_P(\mathcal{Q})$, then the number of points of $\mathcal{Q}$ in this hyperplane π, not in π_{n-2}, equals q^{2n-4} since every line through P in π, but not in π_{n-2}, is a bisecant to $\mathcal{Q}$. The number of points of $\mathcal{Q}$ in $T_P(\mathcal{Q})$, not in π_{n-2}, equals

$$\begin{aligned}
|T_P(\mathcal{Q}) \cap \mathcal{Q}| - |\pi_{n-2} \cap \mathcal{Q}| &= (1 + q^2|\mathcal{Q}'|) - (1 + q^2|\mathcal{Q}''|) \\
&= q^2\left(|\mathcal{Q}'| - |\mathcal{Q}''|\right) \\
&= q^{2n-4} + (w'-1)q^{n-1} - (w''-1)q^{n-2}\ .
\end{aligned}$$

Hereby w' is the index of $\mathcal{Q}'$, which equals the index of $\mathcal{Q}$, and w'' is the index of $\mathcal{Q}''$.

In case the Hermitian variety contains the hyperplane $T_P(\mathcal{Q})$, then the weight of the corresponding code word is $q^{2n-2} - q^{2n-3}$. In case the Hermitian variety does not contain the hyperplane $T_P(\mathcal{Q})$, then the weight of the corresponding code word equals $q^{2n-2} - q^{2n-3} - q^{n-1}$ or $q^{2n-2} - q^{2n-3} + q^{n-1}$ if n is odd, and it equals $q^{2n-2} - q^{2n-3} - q^{n-2}$ or $q^{2n-2} - q^{2n-3} + q^{n-2}$ if n is even.

Note that the size of the intersection $H \cap \mathcal{Q}$ equals $\frac{q^{2n-2}-1}{q^2-1} + q^{2n-3} + (w'-1)q^{n-1}$ in the former case and $\frac{q^{2n-2}-1}{q^2-1} + q^{2n-3} + (w''-1)q^{n-2}$ in the latter case, with w' and w'' as before.

In case the $(n-2)$-space intersects $\mathcal{Q}$ in a non-singular quadric we need to distinguish several cases.

Example 8.5.4. We assume n to be odd. Let π_{n-2} be an $(n-2)$-space intersecting the non-singular quadric $\mathcal{Q}$ in $\mathrm{PG}(n, q^2)$, in a non-singular quadric $\mathcal{Q}'$. If $\mathcal{Q}$ and $\mathcal{Q}'$ are of the same type, then two hyperplanes through π_{n-2} are tangent hyperplanes to $\mathcal{Q}$. The q^2-1 remaining hyperplanes through π_{n-2} intersect $\mathcal{Q}$ in a non-singular parabolic quadric. If $\mathcal{Q}$ and $\mathcal{Q}'$ are of a different type, then all q^2+1 hyperplanes through π_{n-2} intersect $\mathcal{Q}$ in a non-singular parabolic quadric.

Let w be the index of $\mathcal{Q}$ and let w' be the index of $\mathcal{Q}'$. Let π be a hyperplane through π_{n-2}. If π intersects $\mathcal{Q}$ in a non-singular parabolic quadric, then $\pi \setminus \pi_{n-2}$ contains $q^{2n-4}-(w'-1)q^{n-3}$ points of $\mathcal{Q}$. If π is a tangent hyperplane to $\mathcal{Q}$, then $\pi \setminus \pi_{n-2}$ contains $q^{2n-4}+(w'-1)(q^{n-1}-q^{n-3})$ points of $\mathcal{Q}$. Note that in this case necessarily $w = w'$.

Hence, if $w \neq w'$, this code word has weight $q^{2n-2}-q^{2n-3}-(w'-1)(q^{n-1}-q^{n-2})$. If $w = w'$, the weight of this code word equals $q^{2n-2}-q^{2n-3}-(w'-1)(q^{n-1}-q^{n-2})$, $q^{2n-2}-q^{2n-3}+(w'-1)q^{n-2}$ or $q^{2n-2}-q^{2n-3}+(w'-1)(q^{n-1}+q^{n-2})$ depending on the number of tangent hyperplanes contained in the Hermitian variety, two, one or zero. Among these, the smallest code word has weight $q^{2n-2}-q^{2n-3}-q^{n-1}-q^{n-2}$. This corresponds to a code word arising from an elliptic quadric $\mathcal{Q}$ and a Hermitian variety which is the union of $q+1$ hyperplanes, none of them tangent hyperplanes, through an $(n-2)$-space intersecting the quadric $\mathcal{Q}$ in a non-singular elliptic quadric.

Note that in this case the intersection size equals $\frac{q^{2n-2}-1}{q^2-1}+q^{2n-3}+q^{n-2}$.

Example 8.5.5. We assume n to be even and q to be odd. Let π_{n-2} be an $(n-2)$-space intersecting the non-singular parabolic quadric $\mathcal{Q}$ in a non-singular parabolic quadric $\mathcal{Q}'$. Let ℓ be the polar line of π_{n-2}, necessarily disjoint from π_{n-2}. There are two possibilities. If ℓ is a secant line to $\mathcal{Q}$, with $\ell \cap \mathcal{Q} = \{Q, R\}$, Q and R distinct points, then two of the hyperplanes through π_{n-2} are tangent hyperplanes, namely $T_Q(\mathcal{Q})$ and $T_R(\mathcal{Q})$, precisely $\frac{q^2-1}{2}$ of the remaining hyperplanes intersect $\mathcal{Q}$ in an $(n-1)$-dimensional non-singular hyperbolic quadric (*hyperbolic hyperplanes*) and precisely $\frac{q^2-1}{2}$ of them intersect $\mathcal{Q}$ in an $(n-1)$-dimensional non-singular elliptic quadric (*elliptic hyperplanes*). If ℓ is a line disjoint from $\mathcal{Q}$, then $\frac{q^2+1}{2}$ of the hyperplanes through π_{n-2} intersect $\mathcal{Q}$ in a non-singular hyperbolic quadric and $\frac{q^2+1}{2}$ of them intersect $\mathcal{Q}$ in a non-singular elliptic quadric.

Let π be a hyperplane through π_{n-2}. If π is a tangent hyperplane, then $\pi \setminus \pi_{n-2}$ contains q^{2n-4} points of $\mathcal{Q}$. If π is a hyperbolic hyperplane, then $\pi \setminus \pi_{n-2}$ contains $q^{2n-4} + q^{n-2}$ points of $\mathcal{Q}$. If π is an elliptic hyperplane, then $\pi \setminus \pi_{n-2}$ contains $q^{2n-4} - q^{n-2}$ points of $\mathcal{Q}$.

We look at an example in the case ℓ is a secant line. We consider the standard equation $X_0^2 + X_1X_2 + \cdots + X_{n-1}X_n = 0$ of $\mathcal{Q}$ and let π_{n-2} be the $(n-2)$-space given by the equations $X_{n-1} = X_n = 0$. The two tangent hyperplanes through π_{n-2} are given by $X_{n-1} = 0$, which is $T_Q(\mathcal{Q})$ for $Q = (0, \ldots, 0, 0, 1)$, and by $X_n = 0$, which is $T_R(\mathcal{Q})$ for $R = (0, \ldots, 0, 1, 0)$. The other hyperplanes through π_{n-2} are given by $X_{n-1} + \alpha X_n = 0$, $\alpha \in \mathbb{F}_{q^2}^*$, which we denote by shortened coordinates $\overline{[1, \alpha]}$. The tangent hyperplanes correspond to $\overline{[1, 0]}$ and $\overline{[0, 1]}$. The hyperplane $\overline{[1, \alpha]}$ intersects $\mathcal{Q}$ in a hyperbolic quadric if and only if α is a non-zero square; the hyperplane $\overline{[1, \alpha]}$ intersects $\mathcal{Q}$ in an elliptic quadric if and only if α is a non-square. We now investigate how a dual Baer subline can intersect these sets.

First assume that the Hermitian variety contains both tangent hyperplanes (the dual Baer subline contains $\overline{[1, 0]}$ and $\overline{[0, 1]}$). The dual Baer subline is then defined by choosing a third hyperplane $\overline{[1, y]}$: all $q - 1$ hyperplanes of this dual Baer subline, different from $\overline{[1, 0]}$ and $\overline{[0, 1]}$, then can be written as $\overline{[1, \beta y]}$, with $\beta \in \mathbb{F}_q^* \subset \mathbb{F}_{q^2}$. Since $\beta \in \mathbb{F}_q^* \subset \mathbb{F}_{q^2}^*$, βy is a non-zero square in $\mathbb{F}_{q^2}^*$ if and only if y is a non-zero square in $\mathbb{F}_{q^2}^*$. Hence, either all hyperplanes in the dual Baer subline, different from $\overline{[1, 0]}$ and $\overline{[0, 1]}$, intersect $\mathcal{Q}$ in a hyperbolic quadric or all hyperplanes in the dual Baer subline, different from $\overline{[1, 0]}$ and $\overline{[0, 1]}$, intersect $\mathcal{Q}$ in an elliptic quadric. The $q + 1$ dual Baer sublines through $\overline{[1, 0]}$ and $\overline{[0, 1]}$, which we denote by $l_0, \ldots, l_q$, partition the $q^2 - 1$ remaining points of the dual line. Next to the two tangent hyperplanes, $\frac{q+1}{2}$ of these dual Baer sublines, say $l_0, \ldots, l_{\frac{q-1}{2}}$, only contain hyperbolic hyperplanes and $\frac{q+1}{2}$ of these dual Baer sublines, say $l_{\frac{q+1}{2}}, \ldots, l_q$, only contain elliptic hyperplanes. We find code words of weight

$$
\begin{aligned}
&\frac{q^2-1}{2}(q^{2n-4} - q^{n-2}) + \left(\frac{q^2-1}{2} - (q-1)\right)(q^{2n-4} + q^{n-2}) \\
&= q^{2n-2} - q^{2n-3} - q^{n-1} + q^{n-2},
\end{aligned}
$$

corresponding to the dual Baer sublines l_i, $i \leq \frac{q-1}{2}$, and of weight

$$\begin{aligned}&\frac{q^2-1}{2}(q^{2n-4}+q^{n-2})+\left(\frac{q^2-1}{2}-(q-1)\right)(q^{2n-4}-q^{n-2})\\&= q^{2n-2}-q^{2n-3}+q^{n-1}-q^{n-2}\,,\end{aligned}$$

corresponding to the dual Baer sublines l_i, $i \geq \frac{q+1}{2}$.

Secondly, we assume that the dual Baer subline only contains one of the two tangent hyperplanes, say $[1, 0]$. Since two distinct dual Baer sublines have at most two hyperplanes in common, such a dual Baer subline contains at most one hyperbolic or elliptic hyperplane of each l_i, $0 \leq i \leq q$. A dual Baer subline contains precisely $q+1$ hyperplanes, so all but one of the dual Baer sublines l_i contribute one hyperplane. Let l_j be the one dual Baer subline that does not contribute an additional hyperplane. If $j \leq \frac{q-1}{2}$, then the corresponding code word has weight

$$\begin{aligned}&q^{2n-4}+\left(\frac{q^2-1}{2}-\frac{q+1}{2}\right)(q^{2n-4}-q^{n-2})\\&\qquad+\left(\frac{q^2-1}{2}-\frac{q-1}{2}\right)(q^{2n-4}+q^{n-2})\\&= q^{2n-2}-q^{2n-3}+q^{n-2}\,;\end{aligned}$$

if $j \geq \frac{q+1}{2}$, then the corresponding code word has weight

$$\begin{aligned}&q^{2n-4}+\left(\frac{q^2-1}{2}-\frac{q-1}{2}\right)(q^{2n-4}-q^{n-2})\\&\qquad+\left(\frac{q^2-1}{2}-\frac{q+1}{2}\right)(q^{2n-4}+q^{n-2})\\&= q^{2n-2}-q^{2n-3}-q^{n-2}\,.\end{aligned}$$

By looking at some examples of Baer sublines one can see that both possibilities occur.

Finally, we assume that the dual Baer subline contains no tangent hyperplanes. Then, it contains k hyperbolic hyperplanes and $q+1-k$ elliptic hyperplanes,

$0 \le k \le q+1$. The corresponding code word has weight

$$\begin{aligned} & 2q^{2n-4} + \left(\frac{q^2-1}{2} - k\right)(q^{2n-4}+q^{n-2}) \\ & \quad + \left(\frac{q^2-1}{2} - (q+1-k)\right)(q^{2n-4}-q^{n-2}) \\ = \; & q^{2n-2} - q^{2n-3} + (q+1-2k)q^{n-2}\,. \end{aligned}$$

The weight of all these code words is thus between $q^{2n-2}-q^{2n-3}-q^{n-1}-q^{n-2}$ and $q^{2n-2}-q^{2n-3}+q^{n-1}+q^{n-2}$.

Example 8.5.6. We assume n to be even and q to be even. Let π_{n-2} be an $(n-2)$-space intersecting the non-singular parabolic quadric $\mathcal{Q}$ in a non-singular parabolic quadric $\mathcal{Q}'$. Recall that $\mathcal{Q}$ has a nucleus N in this case (see Remark 1.6.8). We distinguish between two cases.

If $N \in \pi_{n-2}$, then all hyperplanes through π_{n-2} are tangent hyperplanes to $\mathcal{Q}$. We know that $(\pi \backslash \pi_{n-2}) \cap \mathcal{Q}$ contains q^{2n-4} points, for such a tangent hyperplane π. Hence, all code words of this type have weight $(q^2-q)q^{2n-4} = q^{2n-2}-q^{2n-3}$. The intersection size of such a Hermitian variety and $\mathcal{Q}$ equals $q^{2n-3} + \frac{q^{2n-2}-1}{q^2-1}$.

If $N \notin \pi_{n-2}$, then one of the hyperplanes through π_{n-2} is a tangent hyperplane. Precisely $\frac{q^2}{2}$ of the remaining hyperplanes through π_{n-2}, intersect $\mathcal{Q}$ in a non-singular hyperbolic quadric, and precisely $\frac{q^2}{2}$ of the remaining hyperplanes through π_{n-2}, intersect $\mathcal{Q}$ in a non-singular elliptic quadric. As in the previous example, these are called hyperbolic and elliptic hyperplanes respectively. If π is a hyperbolic hyperplane, then $(\pi \setminus \pi_{n-2}) \cap \mathcal{Q}$ contains $q^{2n-4}+q^{n-2}$ points. If π is an elliptic hyperplane, then $(\pi \setminus \pi_{n-2}) \cap \mathcal{Q}$ contains $q^{2n-4}-q^{n-2}$ points.

If the Hermitian variety H contains the unique tangent hyperplane through π_{n-2}, then it contains k hyperbolic hyperplanes and $q-k$ elliptic hyperplanes. The corresponding code word has weight

$$\begin{aligned} & \left(\frac{q^2}{2} - k\right)(q^{2n-4}+q^{n-2}) + \left(\frac{q^2}{2} - (q-k)\right)(q^{2n-4}-q^{n-2}) \\ = \; & q^{2n-2} - q^{2n-3} + (q-2k)q^{n-2}\,, \end{aligned}$$

and $|H \cap \mathcal{Q}|$ equals $q^{2n-3} + \frac{q^{2n-2}-1}{q^2-1} - (q-2k)q^{n-2}$.

If the Hermitian variety H does not contain the unique tangent hyperplane through π_{n-2}, then it contains k hyperbolic hyperplanes and $q+1-k$ elliptic hyperplanes. The corresponding code word has weight

$$\begin{aligned} q^{2n-4} + \left(\frac{q^2}{2} - k\right)(q^{2n-4} + q^{n-2}) \\ + \left(\frac{q^2}{2} - (q+1-k)\right)(q^{2n-4} - q^{n-2}) \\ = q^{2n-2} - q^{2n-3} + (q+1-2k)q^{n-2}, \end{aligned}$$

and $|H \cap \mathcal{Q}|$ equals $q^{2n-3} + \frac{q^{2n-2}-1}{q^2-1} - (q+1-2k)q^{n-2}$.

8.6 A divisibility condition on the weights

All functional codes studied so far, have been shown to have a divisor of the form q^e, see [50, 51, 52] for the functional codes $C_2(\mathcal{Q})$, $C_{Herm}(\mathcal{H})$ and $C_2(\mathcal{H})$, $\mathcal{Q}$ a non-singular quadric and $\mathcal{H}$ a non-singular Hermitian variety. The proofs of these results all rely on the following theorem by Ax and Katz.

Theorem 8.6.1 ([86]). *Let S be a finite set of variables and let $T = \{f_i | i \in I\}$ be a collection of polynomials in $\mathbb{F}_q[S]$, with $d_i = \deg(f_i)$. Denote the number of common zeros of the polynomials of T by N. Then $N \equiv 0 \pmod{q^\mu}$, with*

$$\mu = \left\lceil \frac{|S| - \sum_{i \in I} d_i}{\sup_{i \in I} d_i} \right\rceil .$$

In this section we prove that also the code $C_{Herm}(\mathcal{Q})$ has a divisor of the form q^e.

Lemma 8.6.2. *Let H be a Hermitian variety in $\mathrm{PG}(n, q^2)$ and let Q be a quadric in $\mathrm{PG}(n, q^2)$, which both can be singular. Then, on the one hand, $|Q \cap H| \equiv \frac{q^{n-2}-1}{q^2-1} \pmod{q^{n-2}}$ if n is even, and, on the other hand, $|Q \cap H| \equiv \frac{q^{n-1}-1}{q^2-1} \pmod{q^{n-2}}$ if n is odd.*

Proof. We can choose a coordinate system such that the equation of H can be written as $X_0^{q+1} + \cdots + X_i^{q+1} = g(X_0, \ldots, X_n) = 0$, with $0 \leq i \leq n$. The quadric Q is given by an equation $f(X_0, \ldots, X_n) = 0$, with f a quadratic polynomial. Every point of $Q \cap H$ corresponds vectorially to $q^2 - 1$ solutions of the system of equations

$$\begin{cases} f(X_0, \ldots, X_n) = 0 \\ g(X_0, \ldots, X_n) = 0 \end{cases} , \tag{8.1}$$

and every non-zero solution of this system corresponds to a point of $Q \cap H$. Let α be an element of $\mathbb{F}_{q^2} \setminus \mathbb{F}_q$. Then every element $x \in \mathbb{F}_{q^2}$ can be written as $x = y + \alpha z$, with $y, z \in \mathbb{F}_q$. Now we write $X_i = Y_i + \alpha Z_i$, with Y_i and Z_i variables over $\mathbb{F}_q$. In these new variables, the equation of H is given by

$$\begin{aligned} 0 &= \sum_{j=0}^{i} (Y_j + \alpha Z_j)^{q+1} \\ &= \sum_{j=0}^{i} (Y_j^{q+1} + \alpha Y_j^q Z_j + \alpha^q Y_j Z_j^q + \alpha^{q+1} Z_j^{q+1}) \\ &= \sum_{j=0}^{i} (Y_j^2 + (\alpha + \alpha^q) Y_j Z_j + \alpha^{q+1} Z_j^2) , \end{aligned}$$

a quadratic equation over $\mathbb{F}_q$, since $\alpha + \alpha^q, \alpha^{q+1} \in \mathbb{F}_q$. Hereby we used the identity $x^q = x$ which is valid for any $x \in \mathbb{F}_q$. We denote $\sum_{j=0}^{i}(Y_j^2 + (\alpha + \alpha^q)Y_j Z_j + \alpha^{q+1} Z_j^2)$ by $\overline{g}(Y_0, Z_0, \ldots, Y_n, Z_n)$. The equation of Q in the variables Y_j and Z_j is of the form

$$f_0(Y_0, \ldots, Y_n) + \alpha f_1(Y_0, Z_0, \ldots, Y_n, Z_n) + \alpha^2 f_2(Z_0, \ldots, Z_n) = 0 .$$

Writing $\alpha^2 = a_2 + b_2\alpha$, with $a_2, b_2 \in \mathbb{F}_q$, we can rewrite this equation as

$$\begin{aligned} 0 &= [f_0(Y_0, \ldots, Y_n) + a_2 f_2(Z_0, \ldots, Z_n)] \\ &\quad + \alpha\, [f_1(Y_0, Z_0, \ldots, Y_n, Z_n) + b_2 f_2(Z_0, \ldots, Z_n)] \\ &= \overline{f}_0(Y_0, Z_0, \ldots, Y_n, Z_n) + \alpha \overline{f}_1(Y_0, Z_0, \ldots, Y_n, Z_n) . \end{aligned}$$

The quadric Q is thus defined by a system of two equations over the variables Y_j, Z_j:

$$\begin{cases} \overline{f}_0(Y_0, Z_0, \ldots, Y_n, Z_n) = 0 \\ \overline{f}_1(Y_0, Z_0, \ldots, Y_n, Z_n) = 0 \end{cases} .$$

Now we look at the system of equations

$$\begin{cases} \overline{g}(Y_0, Z_0, \ldots, Y_n, Z_n) = 0 \\ \overline{f}_0(Y_0, Z_0, \ldots, Y_n, Z_n) = 0 \\ \overline{f}_1(Y_0, Z_0, \ldots, Y_n, Z_n) = 0 \end{cases} . \tag{8.2}$$

Every solution $(x_0, \ldots, x_n)$, $x_i \in \mathbb{F}_{q^2}$, with $x_i = y_i + \alpha z_i$, $y_i, z_i \in \mathbb{F}_q$, of (8.1) corresponds to a unique solution $(y_0, z_0, \ldots, y_n, z_n)$ of (8.2) and vice versa. Let M be the number of solutions of (8.2) in $V(2n+2, q)$. By Theorem 8.6.1, we know that

$$M \equiv 0 \pmod{q^\mu}, \quad \text{with} \quad \mu = \left\lceil \frac{2(n+1) - 3 \cdot 2}{2} \right\rceil = n - 2.$$

Thus we can write $M = mq^{n-2}$ for an integer m. Note that the all-zero vector is a solution of (8.2). Since Q and H are defined by homogeneous polynomials over $\mathbb{F}_{q^2}$, we know that $|Q \cap H| = \frac{M-1}{q^2-1}$. Hence, $M \equiv 1 \pmod{(q^2-1)}$.

On the one hand, if n is even, we find that

$$1 \equiv M \equiv mq^{n-2} \equiv m(q^2)^{(n-2)/2} \equiv m \pmod{(q^2-1)} .$$

So, $m = m'(q^2 - 1) + 1$ for an integer m'. Consequently,

$$|Q \cap H| = \frac{mq^{n-2} - 1}{q^2 - 1} = m'q^{n-2} + \frac{q^{n-2} - 1}{q^2 - 1} .$$

On the other hand, if n is odd, we find that

$$1 \equiv M \equiv mq^{n-2} \equiv m(q^2)^{(n-3)/2} q \equiv mq \pmod{(q^2-1)} .$$

So, $m = m'(q^2 - 1) + q$ for an integer m'. Consequently,

$$|Q \cap H| = \frac{mq^{n-2} - 1}{q^2 - 1} = m'q^{n-2} + \frac{q^{n-1} - 1}{q^2 - 1} .$$

In both cases the statement follows. □

Theorem 8.6.3. *The value q^{n-2} is a divisor of the code $C_{Herm}(\mathcal{Q})$, $\mathcal{Q}$ a non-singular quadric in* $\mathrm{PG}(n, q^2)$, $n \geq 3$.

Proof. Let c be a code word of the code $C_{Herm}(\mathcal{Q})$. This code word is generated by a polynomial f which gives rise to a Hermitian variety H. We need to distinguish two cases.

First we assume n even. By Lemma 8.6.2, we know that $|\mathcal{Q} \cap H| \equiv \frac{q^{n-2}-1}{q^2-1} \pmod{q^{n-2}}$. Furthermore,

$$|\mathcal{Q}| = \frac{q^{2n}-1}{q^2-1} = q^{n-2}\left(\frac{q^{n+2}-1}{q^2-1}\right) + \frac{q^{n-2}-1}{q^2-1} \equiv \frac{q^{n-2}-1}{q^2-1} \pmod{q^{n-2}} .$$

We conclude:

$$\mathrm{wt}(c) = |\mathcal{Q}| - |\mathcal{Q} \cap H| \equiv 0 \pmod{q^{n-2}} .$$

Now we assume n odd. Denote the index of $\mathcal{Q}$ by w. By Lemma 8.6.2, we know that $|\mathcal{Q} \cap H| \equiv \frac{q^{n-1}-1}{q^2-1} \pmod{q^{n-2}}$. Furthermore,

$$\begin{aligned} |\mathcal{Q}| &= \frac{q^{2n}-1}{q^2-1} + (w-1)q^{n-1} \\ &= q^{n-1}\left(\frac{q^{n+1}-1}{q^2-1} + w - 1\right) + \frac{q^{n-1}-1}{q^2-1} \\ &\equiv \frac{q^{n-1}-1}{q^2-1} \pmod{q^{n-2}} . \end{aligned}$$

We conclude:

$$\mathrm{wt}(c) = |\mathcal{Q}| - |\mathcal{Q} \cap H| \equiv 0 \pmod{q^{n-2}} .$$

Since c is an arbitrary code word, the theorem is proved. □

Comparing the proof of Lemma 8.6.2 to the proof of [52, Theorem 3.4], we note that in their proof the Hermitian variety needs to be non-singular, whereas it is allowed to be singular in our proof. In their proof however, the Hermitian variety is intersected by a hypersurface of degree h, whereas we only considered $h = 2$. The techniques of the above proof could be used to prove a generalisation of this lemma, involving hypersurfaces of degree $h < n$. We did not do this here since we do not need it for the proof of Theorem 8.6.3. Note also that there is a small mistake in the proof of [52, Theorem 3.4]: in fact they count the number of intersection points in $\mathrm{PG}(2n+1, q)$, not in $\mathrm{PG}(n, q^2)$.

9

The dual code $C_n(\mathcal{H}(2n+1,q^2))^\perp$

Orde brengen is de taak van sterk onderlegde idealisten. [...]
Wetenschap en kunst, leidende factoren des levens zullen voortaan als dusdanig dienen opgevat te worden. Er zal van ons nog veel gevergd worden. Dat men ons immer bereid vinde tot den goeden kamp.

Prosper De Troyer in een *brief aan Felix De Boeck.*

For a finite polar space $\mathcal{P}$ of rank d, we introduced in Section 1.8 the linear code $C_k(\mathcal{P})$, $k < d$. This is the linear code generated by the incidence vectors of the k-spaces on $\mathcal{P}$, with the points of $\mathcal{P}$ as positions. The linear code $C_{d-1}(\mathcal{P})$ is the code generated by the incidence vectors of the generators. In this chapter we will discuss the dual code of this code for the Hermitian polar space in $\mathrm{PG}(2n+1,q^2)$, i.e. the code $C_n(\mathcal{H}(2n+1,q^2))^\perp$. It is called the dual code of points and generators (n-spaces) of the Hermitian polar space $\mathcal{H}(2n+1,q^2)$.

The codes $C_k(\mathcal{P})$ for polar spaces or generalised quadrangles, their duals and related codes have been studied before in [89, 103, 124, 125] among others. In [103], the code $C_n(\mathcal{H}(2n+1,q^2))^\perp$ was investigated. The following theorems about $C_n(\mathcal{H}(2n+1,q^2))^\perp$ are known. Note that we can identify the set of positions $\mathrm{supp}(c)$ of a code word c with a set of points of $\mathcal{H}(2n+1,q^2)$ using the correspondence between the positions and the points of $\mathcal{H}(2n+1,q^2)$. We will use this identification throughout this chapter.

Theorem 9.0.1 ([89, Proposition 3.7]). *Let c be a code word of the code $C_1(\mathcal{H}(3,q^2))^\perp$. If $0 < \mathrm{wt}(c) < 3q$, then $\mathrm{wt}(c) = 2(q+1)$. Moreover, the supports of all code words with weight equal to $2(q+1)$, are* PGL*-equivalent. If $\mathrm{wt}(c) \leq \frac{\sqrt{q}(q+1)}{2}$, then c is a linear combination of these code words.*

Theorem 9.0.2 ([103, Theorem 43]). *Let c be a code word of the code $C_2(\mathcal{H}(5,q^2))^\perp$, $q > 893$. If $0 < \mathrm{wt}(c) \leq 2(q^3+q^2)$, then $\mathrm{wt}(c) = 2(q^3+1)$ or $\mathrm{wt}(c) = 2(q^3+q^2)$. Moreover, the supports of all code words with weight equal to $2(q^3+1)$ are* PGL*-equivalent and the supports of all code words with weight equal to $2(q^3+q^2)$ are* PGL*-equivalent.*

The investigations on the code $C_n(\mathcal{H}(2n+1,q^2))^\perp$ in this chapter, improve on the work in [103, Section 5]. We determine the minimum weight of $C_n(\mathcal{H}(2n+1,q^2))^\perp$ for general n, and we show that if q is sufficiently large, a similar statement to the second part of Theorem 9.0.1 holds for general n. The main result of this chapter is as follows.

Theorem 9.0.3. *Let n be any positive integer and let $\delta > 0$ be a constant. If c is a code word of $C_n(\mathcal{H}(2n+1,q^2))^\perp$ with $0 < \mathrm{wt}(c) \leq 4q^{2n-2}(q-1)$ and q is sufficiently large, then there are only n possible* PGL*-equivalence classes for $\mathrm{supp}(c)$; call S this set of* PGL*-equivalence classes. If c is a code word with $\mathrm{wt}(c) < \delta q^{2n-1}$, then c is a linear combination of code words which are an incidence vector of a set in S. The minimum distance of $C_n(\mathcal{H}(2n+1,q^2))^\perp$ is $2q^{2n-4}(q^3+1)$ for $n \geq 2$.*

The code words mentioned in this theorem are described in Section 9.1. The theorem itself is proved in Section 9.3 using the results from Section 9.2. We refer to Section 1.6 for basic knowledge about Hermitian polar spaces. For sake of simplicity, we denote the number of points on the Hermitian polar space $\mathcal{H}(m,q^2)$ by $\mu_m(q^2) = \frac{(q^{m+1}-(-1)^{m+1})(q^m-(-1)^m)}{q^2-1}$.

This chapter is based on [41], which is joint work with Peter Vandendriessche.

9.1 The code words

In this section we introduce a set of code words of the code $C_n(\mathcal{H}(2n+1,q^2))^\perp$. Recall that the point set of a Hermitian polar space $\mathcal{H}(2n+1,q^2)$ can be considered as a non-singular Hermitian variety in $\mathrm{PG}(2n+1,q^2)$. We start with a remark on the description of these code words.

Remark 9.1.1. A code word c of $C_n(\mathcal{H}(2n+1,q^2))^\perp$, $q = p^h$, p prime, is an element of the null space of the corresponding incidence matrix over $\mathbb{F}_p$, which is equivalent to a mapping from the point set of $\mathcal{H}(2n+1,q^2)$ to $\mathbb{F}_p$, mapping a point P to c_P, the value of c on the position corresponding to P, with the additional property that $\sum_{P\in\pi} c_P = 0$, for all generators π of $\mathcal{H}(2n+1,q^2)$. Hence, code words can be studied as multisets of points such that each generator on $\mathcal{H}(2n+1,q^2)$ contains $0 \pmod p$ of the points in the multiset.

Note that these arguments are valid for any code $C_k(\mathcal{P})^\perp$.

Lemma 9.1.2. *Consider a non-singular Hermitian variety $\mathcal{H}(2n+1,q^2)$ in $\mathrm{PG}(2n+1,q^2)$ and let σ be the corresponding polarity. Let π be a k-dimensional subspace in $\mathrm{PG}(2n+1,q^2)$ such that $\pi \cap \mathcal{H}(2n+1,q^2)$ is a cone with vertex π_i and base H_{k-i-1} with $H_{k-i-1} \cong \mathcal{H}(k-i-1,q^2)$ and π_i an i-space, $-1 \le i \le \min\{k,n\}$. Then $\pi \cap \pi^\sigma = \pi_i$. Conversely, if $\pi \cap \pi^\sigma$ is an i-space π_i, then $\pi \cap \mathcal{H}(2n+1,q^2)$ is a cone with vertex π_i and base H'_{k-i-1} with $H'_{k-i-1} \cong \mathcal{H}(k-i-1,q^2)$.*

Proof. The first statement is [81, Lemma 23.2.8]; the second statement is a corollary of the first. □

We will use this theorem mostly in the case $k = n$. In the construction of the code words we need the following lemma.

Lemma 9.1.3. *Consider $\mathcal{H}(2n+1,q^2)$ in $\mathrm{PG}(2n+1,q^2)$. Let π be an n-space in $\mathrm{PG}(2n+1,q^2)$ and let μ be a generator of $\mathcal{H}(2n+1,q^2)$. Then $\pi \cap \mu$ and $\pi^\sigma \cap \mu$ are subspaces of the same dimension.*

Proof. We denote $\mu \cap \pi = \pi_j$, a j-space, possibly empty ($j = -1$). It follows that $2n - j = \dim((\mu\cap\pi)^\sigma) = \dim(\langle \mu^\sigma, \pi^\sigma\rangle)$. Using the Grassmann identity and $\mu = \mu^\sigma$ (μ is a generator), we find $\dim(\mu \cap \pi^\sigma) = \dim(\mu) + \dim(\pi^\sigma) - \dim(\langle \mu^\sigma, \pi^\sigma\rangle) = j$. □

Now, we can give the construction of small weight code words of the code $C_n(\mathcal{H}(2n+1,q^2))^\perp$. This construction is based on [103, Theorem 58].

Theorem 9.1.4. *Consider $\mathcal{H}(2n+1,q^2)$ and its corresponding polarity σ in $\mathrm{PG}(2n+1,q^2)$. Let π be an n-space in $\mathrm{PG}(2n+1,q^2)$. Denote the incidence vector of $\pi \cap \mathcal{H}(2n+1,q^2)$ by v_π and the incidence vector of $\pi^\sigma \cap \mathcal{H}(2n+1,q^2)$ by v_{π^σ}. Then $\alpha(v_\pi - v_{\pi^\sigma})$, $\alpha \in \mathbb{F}_p$, is a code word of $C_n(\mathcal{H}(2n+1,q^2))^\perp$.*

Proof. Let μ be a generator of $\mathcal{H}(2n+1,q^2)$ and denote its incidence vector by v_μ. Using Lemma 9.1.3, we find μ intersects both π and π^σ, or neither. In the first case $|\pi \cap \mu| \equiv |\pi^\sigma \cap \mu| \equiv 1 \pmod q$ and in the second case $|\pi \cap \mu| = |\pi^\sigma \cap \mu| = 0$. In both cases $v_\pi \cdot v_\mu = v_{\pi^\sigma} \cdot v_\mu$. The theorem follows.□

Example 9.1.5. We list the different possibilities for $\pi \cap \pi^\sigma$. Hereby, we use Lemma 9.1.2 for $k = n$ and Theorem 9.1.4. We write $\mathcal{H} = \mathcal{H}(2n+1,q^2)$.

- $\pi \cap \pi^\sigma = \emptyset$. We write $\pi \cap \mathcal{H} = H$ and $\pi^\sigma \cap \mathcal{H} = H'$. We know, $H, H' \cong \mathcal{H}(n,q^2)$. The corresponding code words have weight $2\mu_n(q^2)$.
- $\pi \cap \pi^\sigma = \pi_i$, an i-space, $0 \le i \le n-2$. We write $\pi \cap \mathcal{H} = \pi_i H$ and $\pi^\sigma \cap \mathcal{H} = \pi_i H'$, which are both cones with vertex π_i and bases H and H', with $H, H' \cong \mathcal{H}(n-i-1,q^2)$. The corresponding code words have weight $2q^{2i+2}\mu_{n-i-1}(q^2)$.
- $\pi \cap \pi^\sigma = \pi_{n-1}$, an $(n-1)$-space. Then $\pi \cap \mathcal{H} = \pi^\sigma \cap \mathcal{H} = \pi_{n-1}$ since $\mathcal{H}(0,q^2)$ is empty. The construction gives rise to the zero code word.
- $\pi \cap \pi^\sigma = \pi_n$, an n-space. Then $\pi = \pi^\sigma = \pi_n \subset \mathcal{H}$. Also in this case, the construction gives rise to the zero code word.

It can easily be checked that among these four cases, the code words with smallest weight are the ones corresponding to $i = n-3$.

Note that the code words of weight $2(q+1)$ mentioned in Theorem 9.0.1 and the code words of weight $2(q^3+1)$ or $2(q^3+q^2)$ mentioned in Theorem 9.0.2 are code words of the type described above.

Remark 9.1.6. We consider the construction from Theorem 9.1.4, using the notation from Example 9.1.5. We look at $\pi \cap \mathcal{H} = \pi_i H_{n-i-1}$ and $\pi^\sigma \cap \mathcal{H} = \pi_i H'_{n-i-1}$; both are cones with vertex π_i. Let P be a point of $\pi_i H_{n-i-1}$ and let P' be a point of $\pi_i H'_{n-i-1}$. We know that $P' \in \pi^\sigma \subseteq P^\sigma$ and $P' \in \mathcal{H}$. Hence, the line $\langle P, P' \rangle$ is a line of $\mathcal{H}$.

9.2 Some counting results

Lemma 9.2.1. *Consider the non-singular Hermitian variety $\mathcal{H}(2n+1,q^2)$ in $\mathrm{PG}(2n+1,q^2)$ and let σ be the corresponding polarity. Let τ be a j-space such that $\tau \cap \mathcal{H}(2n+1,q^2) = H_j \cong \mathcal{H}(j,q^2)$, $-1 \leq j \leq n$. The number of generators on $\mathcal{H}(2n+1,q^2)$ skew to τ equals*

$$c_{n,j} = q^{\binom{j+1}{2}} \prod_{k=0}^{n-j-1} (q^{2k+1}+1) \prod_{l=2(n-j)+1}^{2n-j+1} (q^l - (-1)^l).$$

Proof. By [81, Theorem 23.4.2 (i)] we know that the number of generators skew to τ only depends on the parameters n and j and not on the choice of τ itself.

We will prove this theorem using induction on j. If $j=-1$, τ is the empty space and hence $c_{n,-1}$ equals the total number of generators. Using Lemma 1.6.3 we find $c_{n,-1} = \prod_{k=0}^{n}(q^{2k+1}+1)$. Now, we prove a recursive relation between $c_{n,j}$ and $c_{n-1,j-1}$.

By Lemma 9.1.2 we know $\tau \cap \tau^\sigma = \emptyset$. Hence, every point $P \in \mathrm{PG}(2n+1,q^2) \setminus (\tau \cup \tau^\sigma)$ belongs to only one line $\langle P_\tau, P_{\tau^\sigma}\rangle$, with $P_\tau \in \tau$ and $P_{\tau^\sigma} \in \tau^\sigma$. For every point $P \in \mathrm{PG}(2n+1,q^2) \setminus (\tau \cup \tau^\sigma)$, we define $\phi_\tau(P) = P_\tau$. This is the projection of P from τ^σ on τ. We define a correlation $\overline{\sigma} : \tau \to \tau$ that maps the subspace $U \subset \tau$ to $U^\sigma \cap \tau$. It is straightforward to check that $\overline{\sigma}$ defines a polarity on τ. Moreover, it can be seen easily that the points of H_j are the absolute points of $\overline{\sigma}$. Hence, $\overline{\sigma}$ is the polarity of τ corresponding to H_j.

Now, we consider the set $S = \{(P,\mu) \mid P \in \mu \setminus \tau^\sigma, \phi_\tau(P) \notin H_j, \mu \cap \tau = \emptyset, \mu \text{ a generator}\}$. We count the number of elements of S in two ways. On the one hand, there are $c_{n,j}$ generators skew to τ. Let μ be such a generator. The intersection $\mu \cap \tau^\sigma$ is an $(n-j-1)$-space since $\dim(\mu\cap\tau^\sigma)+\dim(\langle \mu,\tau\rangle) = 2n$. We also know that $\phi_\tau(P) = R$ for every point $P \in \langle R, \mu\cap\tau^\sigma\rangle \setminus (\mu\cap\tau^\sigma)$, $R \in \tau$. Hence, for each generator there are $\theta_n(q^2) - \theta_{n-j-1}(q^2) - \mu_j(q^2)(\theta_{n-j}(q^2) - \theta_{n-j-1}(q^2)) = q^{2(n-j)}(\theta_j(q^2) - \mu_j(q^2))$ points fulfilling the requirements. On the other hand, we count the points $P \in \mathcal{H}(2n+1,q^2) \setminus (\tau \cup \tau^\sigma)$ fulfilling the requirements. There are $\mu_{2n+1}(q^2) - \mu_j(q^2) - \mu_{2n-j}(q^2)$ points in this set. We must assure that $\phi_\tau(P) \notin H_j$. Let R be a point of H_j. Since $\tau^\sigma \subseteq R^\sigma$, a line $\langle R, Q\rangle$, $Q \in \tau^\sigma$, is a tangent line in R to $\mathcal{H}(2n+1,q^2)$ or a line which is completely contained in $\mathcal{H}(2n+1,q^2)$. Hence, $\phi_\tau(P)$ is a point of H_j iff P lies on a line through $\phi_\tau(P)$ and a point of $\tau^\sigma \cap \mathcal{H}(2n+1,q^2)$. Consequently there are

$$\begin{aligned}&\mu_{2n+1}(q^2) - \mu_j(q^2) - \mu_{2n-j}(q^2) - \mu_j(q^2)\mu_{2n-j}(q^2)(q^2-1)\\ =\ & q^{2n-j}(\theta_j(q^2) - \mu_j(q^2))(q^{2n-j+1} - (-1)^{2n-j+1})\end{aligned}$$

points $P \in \mathcal{H}(2n+1,q^2) \setminus (\tau \cup \tau^\sigma)$ fulfilling the requirement $\phi_\tau(P) \notin H_j$. Now, we fix such a point P and we count the number of generators skew to τ, through it. All these generators are contained in P^σ. We know $P^\sigma \cap \mathcal{H}(2n+1,q^2)$ is a cone with vertex P and base H_{2n-1}, with $H_{2n-1} \cong \mathcal{H}(2n-1,q^2)$. There is a 1-1 correspondence between the generators of $\mathcal{H}(2n+1,q^2)$ through P and the generators of H_{2n-1}. We also find

$$\tau \cap P^\sigma = \tau \cap (\phi_\tau(P))^\sigma = (\phi_\tau(P))^{\overline{\sigma}}\ ,$$

since P^σ is a hyperplane through the intersection $(\phi_\tau(P))^\sigma \cap (\phi_{\tau^\sigma}(P))^\sigma$ and through $\tau \subset (\phi_{\tau^\sigma}(P))^\sigma$. Hence, the $(j-1)$-space $\tau \cap P^\sigma$ intersects $\mathcal{H}(2n+1,q^2)$ in $H_{j-1} \cong \mathcal{H}(j-1,q^2)$, since $\phi_\tau(P) \notin H_j$. We can choose the base H_{2n-1} of the cone $P^\sigma \cap \mathcal{H}(2n+1,q^2)$ such that it contains $\tau \cap P^\sigma$. The generators through P and skew to τ correspond to the generators of H_{2n-1}, skew to $\tau \cap P^\sigma$. There are $c_{n-1,j-1}$ such generators. We conclude

$$\begin{aligned}& c_{n,j}q^{2(n-j)}\left[\theta_j(q^2) - \mu_j(q^2)\right]\\ &\quad = c_{n-1,j-1}q^{2n-j}\left[\theta_j(q^2) - \mu_j(q^2)\right](q^{2n-j+1} - (-1)^{2n-j+1})\\ \Rightarrow\qquad & c_{n,j} = c_{n-1,j-1}q^j(q^{2n-j+1} - (-1)^{2n-j+1}).\end{aligned}$$

An induction calculation now finishes the proof. □

Notation 9.2.2. From now on in this section, we use the following notation: $H \cong \mathcal{H}(2n+1,q^2)$ is a non-singular Hermitian variety in $\mathrm{PG}(2n+1,q^2)$ and σ is the polarity corresponding to it; π is an n-space in $\mathrm{PG}(2n+1,q^2)$, such that $\pi \cap H$ is a cone $\pi_i H_{n-i-1}$ with vertex π_i and base H_{n-i-1} with $H_{n-i-1} \cong \mathcal{H}(n-i-1,q^2)$ and π_i an i-space, $-1 \leq i \leq n$. By Lemma 9.1.2, for $k=n$, we know $\pi \cap \pi^\sigma = \pi_i$ and consequently $\pi^\sigma \cap H$ is a cone $\pi_i H'_{n-i-1}$ with vertex π_i and base H'_{n-i-1} with $H'_{n-i-1} \cong \mathcal{H}(n-i-1,q^2)$.

Definition 9.2.3. We use the conventions from Notation 9.2.2. The number of generators on H intersecting π in a fixed point $P \in \pi_i H_{n-i-1} \setminus \pi_i$ and no other point of $\pi_i H_{n-i-1}$, and intersecting π^σ in a fixed point $P' \in \pi_i H'_{n-i-1} \setminus \pi_i$ and no other point of $\pi_i H'_{n-i-1}$ is denoted by $N(\pi, P, P', H)$. The number of generators on H skew to π is denoted by $N'(\pi, H)$.

By Lemma 9.1.3 we know that the generators skew to π are also skew to π^σ and that the generators intersecting π in precisely one point also intersect π^σ in precisely one point.

Lemma 9.2.4. *The number $N'(\pi, H)$ only depends on the intersection parameters (n, i) of π. Consequently, we can denote $N'(\pi, H)$ by $N'_{n,i}(q)$.*

Proof. This follows immediately from [81, Theorem 23.4.2 (i)]. □

Lemma 9.2.5. *We use the conventions from Notation 9.2.2. For $n \geq 2$, $-1 \leq i \leq n-2$, $N(\pi, P, P', H) = N'_{n-2,i}(q)$. Consequently, $N(\pi, P, P', H)$ only depends on the intersection parameters (n, i) of π.*

Proof. Consider the points $P \in (\pi_i H_{n-i-1} \setminus \pi_i) \subseteq \pi$ and $P' \in (\pi_i H'_{n-i-1} \setminus \pi_i) \subseteq \pi^\sigma$. Denote $\ell = \langle P, P' \rangle$. Then ℓ^σ is a $(2n-1)$-space intersecting H in a cone with ℓ as vertex and a non-singular $(2n-3)$-dimensional Hermitian variety H_{2n-3} as base. Since $\dim(\ell \cap \pi) = \dim(\ell \cap \pi^\sigma) = 0$, $\ell^\sigma \cap \pi = V$ is an $(n-1)$-space and $\ell^\sigma \cap \pi^\sigma = V'$ is an $(n-1)$-space. Also, $\ell \subset \langle \pi, \pi^\sigma \rangle = \pi_i^\sigma$, hence $\pi_i \subset l^\sigma$. Let W and W' be $(n-2)$-spaces in V and V' respectively, containing π_i and such that $P \notin W$ and $P' \notin W'$. Denote the $(2n-i-4)$-space $\langle W, W' \rangle$ by τ'. It can be seen that on the one hand $\tau' \subset \ell^\sigma$ and on the other hand $\ell \cap \tau' = \emptyset$, so the $(2n-3)$-space τ containing the base H_{2n-3} can be chosen such that $\tau' \subseteq \tau$. Let σ' be the polarity of τ corresponding to H_{2n-3}. Analogously to the proof of Lemma 9.2.1 we can define this polarity as follows: $U^{\sigma'} = \tau \cap U^\sigma$. It now immediately follows that $W^{\sigma'} = W'$ because both are $(n-2)$-spaces contained in W^σ and in τ.

Arguing again as in the proof of Lemma 9.2.1, we see there is a 1-1 correspondence between the generators of H_{2n-3} and the generators of H through ℓ (the generators containing P and P'). If a generator of H through ℓ contains no points of $\pi \cup \pi^\sigma$ but P and P', then its corresponding generator of H_{2n-3} is skew to W and W'. Vice versa, every generator μ of H_{2n-3} skew to W and W', is contained in precisely one generator of H intersecting $\pi \cup \pi^\sigma$ in only the points P and P', namely $\langle \mu, P, P' \rangle$. Since $W^{\sigma'} = W'$, the generators of H_{2n-3} skew to W and W' are the ones skew to W, by Lemma 9.1.3. Hence, $N(\pi, P, P', H) = N'_{n-2,i}(q)$.

The second statement in the lemma follows immediately from the first one. □

Notation 9.2.6. Since $N(\pi, P, P', H)$ only depends on the intersection parameters (n, i) of π, we can denote it by $N_{n,i}(q)$.

The previous theorem now states $N_{n,i}(q) = N'_{n-2,i}(q)$ for $n \geq 2$, $-1 \leq i \leq n-2$.

Lemma 9.2.7. *For $n \geq 1$ and $-1 \leq i \leq n-2$, the following equality is valid:*

$$N_{n,i}(q) = q^{(n-1)^2 - \binom{n-i-1}{2}} \prod_{j=1}^{n-i-2} (q^j - (-1)^j).$$

Proof. We use the conventions introduced in Notation 9.2.2. We prove this theorem using induction. Using Lemma 9.2.5, we know that $N_{n,-1}(q)$ equals $N'_{n-2,-1}(q)$, the number of generators of a Hermitian variety $H' \cong \mathcal{H}(2n-3,q^2)$ skew to an $(n-2)$-space intersecting H' in a Hermitian variety $\mathcal{H}(n-2,q^2)$, if $n \geq 2$. This number equals $c_{n-2,n-2}$. Hence, by Lemma 9.2.1,

$$N_{n,-1}(q) = q^{\binom{n-1}{2}} \prod_{l=1}^{n-1} (q^l - (-1)^l) = q^{(n-1)^2 - \binom{n-(-1)-1}{2}} \prod_{j=1}^{n-(-1)-2} (q^j - (-1)^j),$$

which proves the induction base for $n \geq 2$. If $n = 1$, it is easy to prove that $N_{1,-1}(q) = 1$. Hence, the formula holds also in this case.

Now, we will prove that $N_{n,i}(q) = q^{2n-3} N_{n-1,i-1}(q)$. By Lemma 9.2.5, this is equivalent to proving that $N'_{n,i}(q) = q^{2n+1} N'_{n-1,i-1}(q)$. Consider the set $S = \{(R,\mu) \mid R \in \mu, \mu \text{ a generator skew to } \pi, R \notin \langle \pi, \pi^\sigma \rangle = \pi_i^\sigma\}$. The subspace π_i^σ intersects H in a cone $\pi_i H_{2(n-i)-1}$, with vertex π_i and base $H_{2(n-i)-1} \cong \mathcal{H}(2(n-i)-1, q^2)$. We will count $|S|$ in two ways.

On the one hand, there are $N'_{n,i}(q)$ generators skew to π. Fix such a generator μ. Then $\dim(\mu \cap \pi_i^\sigma) = n-i-1$ since $\dim(\mu \cap \pi_i^\sigma) + \dim(\langle \mu, \pi_i \rangle) = 2n$. So, μ contains precisely $\theta_n(q^2) - \theta_{n-i-1}(q^2) = q^{2(n-i)}\theta_i(q^2)$ points not in π_i^σ. Consequently, $|S| = q^{2(n-i)}\theta_i(q^2) N'_{n,i}(q)$.

On the other hand, there are

$$\mu_{2n+1}(q^2) - \theta_i(q^2) - \mu_{2(n-i)-1}(q^2) - (q^2-1)\theta_i(q^2)\mu_{2(n-i)-1}(q^2) = q^{4n-2i+1}\theta_i(q^2)$$

points in $H \setminus \pi_i^\sigma$. Fix such a point P. The hyperplane P^σ intersects π in an $(n-1)$-space V and intersects π_i in an $(i-1)$-space $\pi_{i-1} \subset V$. Hence, the intersection $V \cap H$ has intersection parameters $(n-1, i-1)$. The intersection $P^\sigma \cap H$ is a cone with vertex P and base $H_{2n-1} \cong \mathcal{H}(2n-1, q^2)$. Let τ be the $(2n-1)$-space containing H_{2n-1}. We can choose τ such that it contains V. Then, there is a 1-1 correspondence between the generators of $\mathcal{H}(2n+1, q^2)$

through P, skew to π and the generators of H_{2n-1} skew to V. Consequently, there are $N'_{n-1,i-1}(q)$ such generators. Thus, $|S| = q^{4n-2i+1}\theta_i(q^2)N'_{n-1,i-1}(q)$.

Comparing both expressions for $|S|$, we find the desired relation between $N'_{n,i}(q)$ and $N'_{n-1,i-1}(q)$. An easy calculation now finishes the proof. □

Lemma 9.2.8. *Using the conventions introduced in Notation 9.2.2, we assume $n \geq 2$ and $-1 \leq i \leq n-2$. Let P be a point of $H \setminus (\pi \cup \pi^\sigma)$. Let $n_P(n,i)$ be the number of generators through P intersecting both $\pi \setminus \pi_i$ and $\pi^\sigma \setminus \pi_i$ in precisely one point. Then,*

- $n_P(n,i) = N_{n-1,i-1}(q)q^{4i}\left(\mu_{n-i-1}(q^2)\right)^2$ *if* $P \notin \pi_i^\sigma = \langle \pi, \pi^\sigma \rangle$;
- $n_P(n,i) = N_{n-1,i}(q)q^{4i+4}\left(\mu_{n-i-2}(q^2)\right)^2$ *if* $P \in \pi_i^\sigma = \langle \pi, \pi^\sigma \rangle$ *but P does not belong to a line of H through a point of $\pi \setminus \pi_i$ and a point of $\pi^\sigma \setminus \pi_i$;*
- $n_P(n,i) = N_{n-1,i+1}(q)q^{4i+4}\mu_{n-i-3}(q^2)\left[q^4\mu_{n-i-3}(q^2) + q^2 - 1\right]$ *if* $P \in \pi_i^\sigma = \langle \pi, \pi^\sigma \rangle$ *and P belongs to a line of H through a point of $\pi \setminus \pi_i$ and a point of $\pi^\sigma \setminus \pi_i$, and $i \leq n-4$.*
- $n_P(n,i) = N_{n,i}(q)q^{2i+2}$ *if* $P \in \pi_i^\sigma = \langle \pi, \pi^\sigma \rangle$ *and P belongs to a line of H through a point of $\pi \setminus \pi_i$ and a point of $\pi^\sigma \setminus \pi_i$, and $i \in \{n-3, n-2\}$.*

The first case can only occur if $i \geq 0$. The second case can only occur if $i \leq n-3$.

Proof. Since $P \notin \pi \cup \pi^\sigma$, $P^\sigma \cap \pi = V$ is an $(n-1)$-space and $P^\sigma \cap \pi^\sigma = V'$ is an $(n-1)$-space. Furthermore $P^\sigma \cap H$ is a cone with vertex P and base $H_{2n-1} \cong \mathcal{H}(2n-1, q^2)$. Let τ be the $(2n-1)$-space containing H_{2n-1}.

First we consider the case $P \notin \langle \pi, \pi^\sigma \rangle = \pi_i^\sigma$. In this case P^σ intersects π_i in an $(i-1)$-space $\pi_{i-1} = V \cap V'$. Also, τ can be chosen so that it contains V and V'. Hence, the number of generators through P fulfilling the requirements equals the number of generators of H_{2n-1} intersecting V and V' in a point. Let σ' be the polarity of τ corresponding to H_{2n-1}. Analogously to the argument in the proof of Lemma 9.2.5, it can be seen that $V' = V^{\sigma'}$. Consequently there are $N_{n-1,i-1}(q)$ generators of this type through a fixed point of $V \setminus \pi_{i-1}$ and a fixed point of $V' \setminus \pi_{i-1}$. There are $q^{2i}\mu_{n-i-1}(q^2)$ possible choices for each of these points. The first part of the lemma follows. Note that $\langle \pi, \pi^\sigma \rangle = \mathrm{PG}(2n+1, q^2)$ if $i = -1$. Hence, this case cannot occur when $i = -1$.

We fix some notation for the remaining cases. Let $W \subseteq \pi$ and $W' \subseteq \pi^\sigma$ be the $(n-i-1)$-spaces containing H_{n-i-1} and H'_{n-i-1}, respectively. Furthermore, let $\overline{\sigma}$ and $\overline{\sigma}'$ be the polarities of W and W' corresponding to H_{n-i-1} and H'_{n-i-1}, respectively. In all three remaining cases, $\pi_i \subset P^\sigma$, hence $P^\sigma \cap W = W_1$ and $P^\sigma \cap W' = W'_1$ are $(n-i-2)$-spaces. Now, the point P is contained in a unique plane $\langle P_{\pi_i}, P_W, P_{W'}\rangle$, with $P_W \in W$, $P_{W'} \in W'$ and $P_{\pi_i} \in \pi_i$. The points P_W, $P_{W'}$ and P_{π_i} are the projections of P from π^σ on W, from π on W' and from $\langle W, W'\rangle$ on π_i, respectively. Arguing as in the proof of Lemma 9.2.1 we can see that $W_1 = P^\sigma \cap W = P_W^{\overline{\sigma}}$ and that $W'_1 = P^\sigma \cap W' = P_{W'}^{\overline{\sigma}'}$. Moreover, since P and P_{π_i} are contained in P^σ, neither or both of P_W and $P_{W'}$ are contained in P^σ. Hence, we need to distinguish two cases.

- $P_W \in W_1$ and $P_{W'} \in W'_1$ are both contained in P^σ; consequently, $P_W \in P_W^{\overline{\sigma}}$, thus $P_W \in H_{n-i-1} \subset H$ and $P_W^{\overline{\sigma}} \cap H_{n-i-1}$ is a cone $P_W H_{n-i-3}$, with vertex P_W and base $H_{n-i-3} \cong \mathcal{H}(n-i-3, q^2)$. Let $W_2 \subset W_1$ be the $(n-i-3)$-space containing H_{n-i-3}. Then, the intersection of $V = \langle \pi_i, W_1\rangle$ and H is the cone with vertex $\langle \pi_i, P_W\rangle$ and base H_{n-i-3}. Analogously we introduce $H'_{n-i-3} \subset W'_2 \subset W'_1$. Then $V' \cap H$ is the cone with vertex $\langle \pi_i, P_{W'}\rangle$ and base H'_{n-i-3}. Furthermore, since $P_W \in V$, $P_{W'} \in V'$, and $P_{\pi_i} \in V \cap V'$, P is contained in $\langle V, V'\rangle$. Also, the line $\langle P, P_W\rangle$ is contained in P^σ and is not a 1-secant since $P, P_W \in H$, hence it is a line of H. This line intersects π^σ in a point of $\langle P_{W'}, \pi_i\rangle \setminus \pi_i$.
- $P_W \notin W_1$ and $P_{W'} \notin W'_1$ are both not contained in P^σ; consequently, $P_W \notin P_W^{\overline{\sigma}}$, thus $P_W \notin H_{n-i-1}$, $P_W \notin H$ and $P_W^{\overline{\sigma}} \cap H_{n-i-1}$ is a non-singular Hermitian variety $H_{n-i-2} \cong \mathcal{H}(n-i-2, q^2)$ in W_1. Then, the intersection of $V = \langle \pi_i, W_1\rangle$ and H is the cone $\pi_i H_{n-2-i}$ with vertex π_i and base H_{n-2-i}. Analogously we introduce $H'_{n-i-2} \subset W'_1$. The intersection $V' \cap H$ is the cone $\pi_i H'_{n-i-2}$ with vertex π_i and base H'_{n-2-i}. Furthermore, $P \notin \langle V, V'\rangle$ since $P_W \notin W_1$ and $P_{W'} \notin W'_1$. Also, all lines in π_i^σ through P intersecting $\pi \setminus \pi_i$ and $\pi^\sigma \setminus \pi_i$, are contained in $\langle P_W, P_{W'}, \pi_i\rangle$, but not in $\langle P, \pi_i\rangle$. Since $P_W, P_{W'} \notin P^\sigma$, none of the lines through P can be contained in H.

These two cases clearly correspond to the three remaining cases of the lemma. We will treat them separately.

First of all, we look at the latter, which is the second case in the statement of the lemma. Since $P \notin \langle V, V'\rangle$, we can choose τ such that it contains $\langle V, V'\rangle$.

Hence, every generator through P, intersecting both $\pi \setminus \pi_i$ and $\pi^\sigma \setminus \pi_i$ in a point, corresponds to a generator of H_{2n-1} intersecting both $V \setminus \pi_i$ and $V' \setminus \pi_i$ in a point, and vice versa. For a fixed point in $V \setminus \pi_i$ and a fixed point in $V' \setminus \pi_i$, there are $N_{n-1,i}(q)$ such generators. We also know that $|V \setminus \pi_i| = |V' \setminus \pi_i| = q^{2i+2}\mu_{n-i-2}(q^2)$. The second part of the lemma follows. Note that $V \setminus \pi_i$ and $V' \setminus \pi_i$ are empty if $i = n-2$. Hence, this case only occurs if $i \leq n-3$.

Finally, we look at the former case, which corresponds to the third and the fourth case in the statement of the lemma. Let ℓ be a line on H through P, a point of $\pi \setminus \pi_i$ and a point of $\pi^\sigma \setminus \pi_i$. By changing, if necessary, the choices for W and W', we can assume $\ell = \langle P_W, P_{W'} \rangle$. We distinguish between two types of generators: the ones that contain ℓ and the ones that do not contain ℓ. First we look at the ones that contain ℓ. We know $\ell^\sigma \cap H$ is a cone with vertex ℓ and base $H_{2n-3} \cong \mathcal{H}(2n-3, q^2)$. Let τ' be the $(2n-3)$-space containing H_{2n-3}. We can choose τ' so that it contains π_i, W_2 and W_2'. As before, one can see that $\langle \pi_i, W_2 \rangle^{\widehat{\sigma}'} = \langle \pi_i, W_2' \rangle$, with $\widehat{\sigma}'$ the polarity of τ' corresponding to H_{2n-3}. The number of generators of the requested type through ℓ then equals the number of generators of H_{2n-3} skew to $\langle \pi_i, W_2 \rangle$. This number equals $N'_{n-2,i}(q) = N_{n,i}(q)$. Furthermore, since ℓ is a line on H through P intersecting $\pi \setminus \pi_i$ and $\pi^\sigma \setminus \pi_i$, every line through P and a point of $\langle P_W, \pi_i \rangle \setminus \pi_i$ belongs to H and intersects $\langle P_{W'}, \pi_i \rangle \setminus \pi_i \subset \pi^\sigma \setminus \pi_i$. Thus, there are $\theta_{i+1}(q^2) - \theta_i(q^2) = q^{2i+2}$ such lines. Hence, there are $q^{2i+2}N_{n,i}(q)$ generators of the first type. Now, we assume no line through P, intersecting π and π^σ, is contained in the generator. Let Q_W and $Q_{W'}$ be the points of the generator in W and W', respectively. By the previous remarks on this case, we know there are $\mu_{n-i-3}(q^2)q^{2i+4}$ possible choices for Q_W and for $Q_{W'}$. Now, we consider the plane $\langle P, Q_W, Q_{W'} \rangle$. Using arguments, similar to the ones in the previous case, we find $N'_{n-3,i+1}(q) = N_{n-1,i+1}(q)$ generators fulfilling the requirements for every choice of Q_W and $Q_{W'}$. Hence, the total number of generators in this third case equals

$$\begin{aligned} n_P &= q^{2i+2}N_{n,i}(q) + \left(\mu_{n-i-3}(q^2)q^{2i+4}\right)^2 N_{n-1,i+1}(q) \\ &= \left[q^{2i+2}q^{2i+2}(q^2-1)\mu_{n-i-3}(q^2) + \left(\mu_{n-i-3}(q^2)q^{2i+4}\right)^2\right] N_{n-1,i+1}(q) \\ &= q^{4i+4}\mu_{n-i-3}(q^2)\left[q^2 - 1 + q^4\mu_{n-i-3}(q^2)\right] N_{n-1,i+1}(q) \ . \end{aligned}$$

Hereby we used the relation between $N_{n,i}(q)$ and $N_{n-1,i+1}(q)$ which can immediately be derived from Lemma 9.2.7.

Note that $V \setminus \langle \pi_i, P_W \rangle$ and $V' \setminus \langle \pi_i, P_{W'} \rangle$ are empty if $n-3 \leq i \leq n-2$. In this case, we cannot consider the points Q_W and $Q_{W'}$. So, there are no generators of the second type. Consequently, all generators are of the first type and there are precisely $q^{2i+2}N_{n,i}(q)$ such generators. □

9.3 Classifying the small weight code words

Recall Theorem 9.0.1 and Theorem 9.0.2 which we stated in the introduction of this chapter. In this section it is our aim to generalise these results. We start our arguments with two lemmata about n-spaces: the second lemma shows the existence of an n-space containing many points of the support of a code word, while the first lemma shows that a generator cannot contain many points of the support of a code word. In the proof of the second lemma we use the following result.

Theorem 9.3.1. *Let $c \in C_n(\mathcal{H}(2n+1,q^2))^\perp$ be a code word and denote* $\mathrm{supp}(c) = S$. *Let P be a point in S. Then $|P^\sigma \cap S| \geq 2 + q^{2n-1}$.*

Proof. This is a special case of [103, Proposition 9(d)]. □

Throughout the three following lemmata the functions $\Sigma_{n,i}(q)$ are used.

Definition 9.3.2. For $n, i \in \mathbb{N}$ and a prime power q, with $-1 \leq i \leq n-2$, we define

$$\Sigma_{n,i}(q) = \begin{cases} 2q^{2i+2}\mu_{n-i-1}(q^2) + 4\frac{\mu_{n-i-2}(q^2)(q^{n-i-1}-1)}{q^{n-3i-5}(q^2-1)} & n-i \text{ odd },\\ 2q^{2i+2}\left[\mu_{n-i-1}(q^2) + 2\frac{q^4\mu_{n-i-3}(q^2)+q^2-1}{q^2-1}\right] & n-i \text{ even }. \end{cases}$$

Note that in both cases $\Sigma_{n,i}(q) = 2q^{2n-1} + \sigma_{n,i}(q)$, with $\sigma_{n,i}(q) \in \mathcal{O}(q^{2n-2})$ and $\sigma_{n,i}(q) > 0$ if $q > 0$.

Lemma 9.3.3. *Let $c \in C_n(\mathcal{H}(2n+1,q^2))^\perp$ be a code word with* $\mathrm{wt}(c) \leq w = \delta q^{2n-1}$, *and denote* $\mathrm{supp}(c) = S$. *Let μ be a generator of $\mathcal{H}(2n+1,q^2)$. Then $|\mu \cap S| \leq \delta\theta_{n-1}(q^2)$.*

Proof. The proof is a generalisation of the proof of [103, Lemma 41].

Denote $x = |\mu \cap S|$ and let P be a point in $\mu \cap S$. Then $P^\sigma \cap \mathcal{H}(2n+1,q^2)$ is a cone with vertex P. Let $H' \cong \mathcal{H}(2n-1,q^2)$ be a base of this cone and consider the projection from P onto H'. Denote the projection of $(S \cap P^\sigma) \setminus \{P\}$ by S'. The projection of μ is a generator μ' of H'. Note that S' is a blocking set of the generators on H', i.e. every generator of H' contains at least one point of S'.

By [90, Lemma 10], we know there are q^{n^2} generators in H' that are skew to μ', of which $q^{(n-1)^2}$ pass through a fixed point of $H' \setminus \mu'$. Hence, the blocking set S' contains at least q^{2n-1} points not in μ'. Counting the tuples (P, Q), $P \in \mu \cap S$, $Q \in S \setminus \mu$, with $\langle P, Q \rangle \subset \mathcal{H}(2n+1, q^2)$, in two ways we find

$$xq^{2n-1} \leq \delta q^{2n-1} \theta_{n-1}(q^2) ,$$

where the upper bound follows from the fact that every point $Q \in S \setminus \mu$ is collinear with the points of an $(n-1)$-space in μ and not with the other points in μ. The theorem follows immediately. $\square$

Note that the size of a blocking set on a Hermitian variety $\mathcal{H}(2n+1, q^2)$ is at least $q^{2n+1} + 1$.[1]

Recall that the symmetric difference $A\Delta B$ of two sets A and B is the set $(A \cup B) \setminus (A \cap B)$.

Lemma 9.3.4. *Let p be a fixed prime and denote $q = p^h$, $h \in \mathbb{N}$. Let $c \in C_n(\mathcal{H}(2n+1, q^2))^\perp$ be a code word with $\mathrm{wt}(c) \leq w = \delta q^{2n-1}$, $\delta > 0$ a constant, and denote $\mathrm{supp}(c) = S$. Denote $\mathcal{H}(2n+1, q^2)$ by H and let σ be the polarity related to H. Then a constant $C_n > 0$, a value $Q > 0$ and an n-space π can be found such that $|(\pi\Delta\pi^\sigma) \cap S| > C_n q^{2n-1}$ and such that $\frac{p-1}{p}|(\pi\Delta\pi^\sigma) \cap H| < \Sigma_{n,i}(q) - C_n q^{2n-1}$, if $q \geq Q$. Hereby, i is such that $\pi \cap H$ is a cone with an i-dimensional vertex and $i \leq n-2$.*

Proof. We introduce the notion of a *semi-arc*. A semi-arc $\mathcal{A}$ is a set of $k \geq n$ points in $\mathrm{PG}(2n+1, q^2)$ such that no $n+1$ points of $\mathcal{A}$ are contained in an $(n-1)$-space. We make two remarks about these semi-arcs. First, if $|S| > \binom{k}{n}\theta_{n-1}(q^2)$, then S contains a semi-arc with $k+1$ points, since it is possible to construct the semi-arc point by point: we start with a set of n linearly independent points in S and we extend the semi-arc point by point

[1] Blocking sets were in this thesis only introduced for projective spaces. However, they can also be defined for polar spaces. We refer to [34] for a survey on blocking sets and related substructures on polar spaces.

until we have $k+1$ points, which is possible by the condition on $|S|$. Secondly, if we choose K points $\{P_1,\ldots,P_K\}$ in a semi-arc $\mathcal{A} \subseteq S$, then

$$\begin{aligned}|(P_1^\sigma \cup P_2^\sigma \cup \cdots \cup P_K^\sigma) \cap S| \le & \sum_{\{i\}\in S_{K,1}} |P_i^\sigma \cap S| \\ & - \sum_{\{i,j\}\in S_{K,2}} |P_i^\sigma \cap P_j^\sigma \cap S| + \ldots \\ & + \sum_{\{i_1,\ldots,i_{2l+1}\}\in S_{K,2l+1}} |P_{i_1}^\sigma \cap P_{i_2}^\sigma \cap \cdots \cap P_{i_{2l+1}}^\sigma \cap S|\end{aligned} \tag{9.1}$$

since every point of $(P_1^\sigma \cup P_2^\sigma \cup \cdots \cup P_K^\sigma) \cap S$ is counted at least once on the right-hand side. Also

$$\begin{aligned}|(P_1^\sigma \cup P_2^\sigma \cup \cdots \cup P_K^\sigma) \cap S| \ge & \sum_{\{i\}\in S_{K,1}} |P_i^\sigma \cap S| \\ & - \sum_{\{i,j\}\in S_{K,2}} |P_i^\sigma \cap P_j^\sigma \cap S| + \ldots \\ & - \sum_{\{i_1,\ldots,i_{2l}\}\in S_{K,2l}} |P_{i_1}^\sigma \cap P_{i_2}^\sigma \cap \cdots \cap P_{i_{2l}}^\sigma \cap S|\end{aligned} \tag{9.2}$$

since every point of $(P_1^\sigma \cup P_2^\sigma \cup \cdots \cup P_K^\sigma) \cap S$ is counted at most once on the right-hand side. In both expressions we denoted the set of all subsets of $\{1,\ldots,K\}$ of size j by $S_{K,j}$.

Now, we prove using induction on t, for every $0 \le t \le n$, that for any $(t+1)$-tuple $(c_0,\ldots,c_t)$ and for any constant $c_j > 0$ (independent of q), we can find a constant $K_t \in \mathbb{N}$ such that

$$\begin{aligned}&\forall K \ge K_t, \forall \{P_1,\ldots,P_K\} \subseteq \mathcal{A} \subseteq S : \\ &\qquad \sum_{\{i_0,\ldots,i_t\}\in S_{K,t+1}} |P_{i_0}^\sigma \cap P_{i_1}^\sigma \cap \cdots \cap P_{i_t}^\sigma \cap S| \ge c_t q^{2n-1} .\end{aligned}$$

We consider the case $t = 0$, the induction base. Let $\{P_1,\ldots,P_K\}$ be a set of points in $\mathcal{A} \subseteq S$ (without restriction on K). By Theorem 9.3.1, we know

$$\sum_{i=1}^{K} |P_i^\sigma \cap S| \ge K q^{2n-1} .$$

Hence, it is sufficient to choose $K_0 = \lceil c_0 \rceil$.

Next, we prove the induction step. We distinguish between two cases: t even and t odd. We look at the former, so we assume the inequality to be proved for $t \leq 2l-1$ and we prove it for $t = 2l$. Let K_m be the constant arising from the $(m+1)$-tuple $(c_0, \ldots, c_m)$, $m < 2l$, and let $\{P_1, \ldots, P_K\}$ be a set of points in $\mathcal{A} \subseteq S$ with $K \geq K_{2l-1}$. By (9.1), we know that

$$
\begin{aligned}
|(P_1^\sigma \cup P_2^\sigma \cup \cdots \cup P_K^\sigma) \cap S| \leq \sum_{\{i\} \in S_{K,1}} |P_i^\sigma \cap S| - \sum_{\{i,j\} \in S_{K,2}} |P_i^\sigma \cap P_j^\sigma \cap S| \\
+ \cdots + \sum_{\{i_0, \ldots, i_{2l}\} \in S_{K,2l+1}} |P_{i_0}^\sigma \cap P_{i_1}^\sigma \cap \cdots \cap P_{i_{2l}}^\sigma \cap S| \, .
\end{aligned}
$$

Using the induction hypothesis and Theorem 9.3.1, we find

$$
\begin{aligned}
&\sum_{\{i_0, \ldots, i_{2l}\} \in S_{K,2l+1}} |P_{i_0}^\sigma \cap P_{i_1}^\sigma \cap \cdots \cap P_{i_{2l}}^\sigma \cap S| \\
&\geq \frac{\binom{K}{K_{2l-1}}}{\binom{K-2l}{K_{2l-1}-2l}} c_{2l-1} q^{2n-1} + \frac{\binom{K}{K_{2l-3}}}{\binom{K-2l+2}{K_{2l-3}-2l+2}} c_{2l-3} q^{2n-1} + \cdots + \frac{\binom{K}{K_1}}{\binom{K-2}{K_1-2}} c_1 q^{2n-1} \\
&\quad - \left[\binom{K}{2l-1} + \binom{K}{2l-3} + \cdots + K \right] \delta q^{2n-1} + q^{2n-1}
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\sum_{\{i_0, \ldots, i_{2l}\} \in S_{K,2l+1}} |P_{i_0}^\sigma \cap P_{i_1}^\sigma \cap \cdots \cap P_{i_{2l}}^\sigma \cap S| \\
&\geq \frac{\binom{K}{2l}}{\binom{K_{2l-1}}{2l}} c_{2l-1} q^{2n-1} + \frac{\binom{K}{2l-2}}{\binom{K_{2l-3}}{2l-2}} c_{2l-3} q^{2n-1} + \cdots + \frac{\binom{K}{2}}{\binom{K_1}{2}} c_1 q^{2n-1} \\
&\quad - \left[\binom{K}{2l-1} + \binom{K}{2l-3} + \cdots + K \right] \delta q^{2n-1} + q^{2n-1} \\
&= q^{2n-1} f(K, \delta, l, K_1, K_3, \ldots, K_{2l-1}, c_1, c_3, \ldots, c_{2l-1}) \, .
\end{aligned}
$$

Note that $\frac{\binom{K}{K_{2i-1}}}{\binom{K-2i}{K_{2i-1}-2i}} = \frac{\binom{K}{2i}}{\binom{K_{2i-1}}{2i}}$. We now study the function f, which is clearly independent of q. Considering f as a function of K and comparing the exponents, we see that the term $\frac{\binom{K}{2l}}{\binom{K_{2l-1}}{2l}} c_{2l-1}$ dominates the others. Hence, we can find a value $K_{2l} \geq K_{2l-1}$ such that the right-hand side is at least $c_{2l}q^{2n-1}$ for all $K \geq K_{2l}$, with c_{2l} as chosen above. Then the statement follows. Note that K_{2l} depends on the parameters $l, c_1, \ldots, c_{2l}$ chosen before (the values K_i, $0 \leq i < 2l$, depend themselves on $i, c_1, \ldots, c_i$).

For the latter case, t odd, the argument is similar, in this case starting from (9.2).

We will now apply the previous result for $t = n$. In order to do this, we need a semi-arc containing at least K_n points. We argued in the beginning of the proof that $\delta q^{2n-1} = |S| > \binom{K_n-1}{n}\theta_{n-1}(q^2)$ is a sufficient condition for a semi-arc of size K_n to exist. Since K_n is a constant, independent of q, and $\theta_{n-1}(q^2) = q^{2n-2} + q^{2n-4} + \cdots + q^2 + 1$, we can find $Q_1' > 0$ such that this inequality is true for all $q \geq Q_1'$. Then we know

$$\sum_{\{i_0,\ldots,i_n\} \in S_{K_n,n+1}} |P_{i_0}^\sigma \cap P_{i_1}^\sigma \cap \cdots \cap P_{i_n}^\sigma \cap S| \geq c_n q^{2n-1}$$

for the points $\{P_1, P_2, \ldots, P_{K_n}\}$ defining a semi-arc in S. Hence, we can find $n+1$ points, without loss of generality the points $\{P_1, \ldots, P_{n+1}\}$, such that

$$|P_1^\sigma \cap P_2^\sigma \cap \cdots \cap P_{n+1}^\sigma \cap S| \geq \frac{c_n}{\binom{K_n}{n+1}} q^{2n-1}.$$

We can find a constant $\overline{K} > 0$ and a value $Q' \geq Q_1'$ such that $\frac{c_n}{\binom{K_n}{n+1}} q^{2n-1} \geq \overline{K}q^{2n-1} + \theta_{n-2}(q^2)$ for $q \geq Q'$. We write $C_n = \overline{K} - \epsilon$, $\max\{0, \overline{K} - \frac{2}{p}\} < \epsilon < \overline{K}$, and we denote the n-space $P_1^\sigma \cap P_2^\sigma \cap \cdots \cap P_{n+1}^\sigma$ by π. Note that π is an n-space since the points $P_1, P_2, \ldots, P_{n+1}$ belong to a semi-arc. Then $|\pi \cap S| > C_n q^{2n-1} + \theta_{n-2}(q^2)$.

We know that the intersection $\pi \cap H$ is a cone $\pi_i H_{n-i-1}$, with an i-space π_i as vertex and $H_{n-i-1} \cong \mathcal{H}(n-i-1, q^2)$ as base, $-1 \leq i \leq n$. Let $Q'' \geq Q'$ be such that $C_n q^{2n-1} + \theta_{n-2}(q^2) > \delta\theta_{n-1}(q^2)$ for all $q \geq Q''$. Such a value exists since the first term on the left-hand side dominates the right-hand side. If $i \geq n-1$, then $\pi \cap H$ is contained in a generator of H. Thus, using Lemma 9.3.3 and the assumption $q \geq Q''$ we find a contradiction. Hence, $i \leq n-2$. We find:

$$|(\pi\Delta\pi^\sigma) \cap S| \geq |(\pi \setminus \pi_i) \cap S| \geq C_n q^{2n-1} + \theta_{n-2}(q^2) - \theta_i(q^2) \geq C_n q^{2n-1}\ .$$

We still need to check the second claim in the statement of the lemma:

$$\frac{p-1}{p}|(\pi\Delta\pi^\sigma)\cap H| < \Sigma_{n,i}(q) - C_n q^{2n-1} .$$

Looking at the terms of highest degree in $\Sigma_{n,i}(q) - C_n q^{2n-1} - \frac{p-1}{p}|(\pi\Delta\pi^\sigma)\cap H|$, we find $2 - C_n - 2\frac{p-1}{p} = \epsilon - \frac{c_n}{\binom{K_n}{n+1}} + \frac{2}{p} > 0$. Hence, we can find $Q \geq Q''$ such that the inequality $\frac{p-1}{p}|(\pi\Delta\pi^\sigma)\cap H| < \Sigma_{n,i}(q) - C_n q^{2n-1}$ holds for all $q \geq Q$.□

In this proof $\frac{c_n}{\binom{K_n}{n+1}}$ depends also on the choice of $c_0, \ldots, c_{n-1}$. So, investigating the possible values for $c_0, \ldots, c_n$, we can find many different values for C_n. With each of these values, a value Q corresponds. We pick one of the possible values for C_n. By investigating different possibilities for C_n, we can see there is a trade-off between the choice of C_n and the corresponding value Q.

From now on, we consider C_n and the corresponding value Q to be fixed.

Lemma 9.3.5. *Let $c \in C_n(\mathcal{H}(2n+1,q^2))^\perp$ be a code word with $\mathrm{wt}(c) \leq w = \delta q^{2n-1}$, $\delta > 0$ a constant, and denote $\mathrm{supp}(c) = S$. Consider $H \cong \mathcal{H}(2n+1,q^2)$. Let π be an n-space such that $\pi \cap H$ is a cone $\pi_i H_{n-i-1}$ with vertex an i-space π_i and base $H_{n-i-1} \cong \mathcal{H}(n-i-1,q^2)$. Assume that $|S \cap (\pi \setminus \pi_i)| = x$ and $|S \cap (\pi^\sigma \setminus \pi_i)| = t$. Then there exists a value $Q_{n,i} \geq 0$ such that $x+t \leq C_n q^{2n-1}$ or $x+t \geq \Sigma_{n,i}(q) - C_n q^{2n-1}$ if $q \geq Q_{n,i}$.*

Proof. If $i = n-1$ or $i = n$, the sets $\pi \setminus \pi_i$ and $\pi^\sigma \setminus \pi_i$ are empty and hence also their intersections with S. The first inequality is clearly valid in this case. So from now on, we assume $i \leq n-2$.

Let P be a point of $S \cap (\pi \setminus \pi_i)$ and let P' be a point of $((\pi^\sigma \cap H) \setminus \pi_i) \setminus S$ and denote $\ell = \langle P, P' \rangle$. By Lemma 9.2.7 we know the number $N_{n,i}(q)$ of generators through ℓ intersecting π and π^σ in precisely one point, namely P and P'. Each of these generators contains an additional point of S. Let R be a point of $H \setminus (\pi \cup \pi^\sigma)$. By Lemma 9.2.8 we know the number $n_R(n,i)$ of generators through R intersecting both π and π^σ in a point. Hence, $S \setminus (\pi \cup \pi^\sigma)$ contains at least

$$x(|(\pi^\sigma \cap H) \setminus \pi_i| - t)\frac{N_{n,i}(q)}{n_{\max}(n,i)} = x(q^{2i+2}\mu_{n-i-1}(q^2) - t)\frac{N_{n,i}(q)}{n_{\max}(n,i)}$$

points, whereby $n_{\max}(n,i) = \max_{R \in S\setminus(\pi\cup\pi^\sigma)} n_R(n,i)$. Switching the roles of π and π^σ, and adding these two inequalities, we find after dividing by two a lower bound on $|S \setminus (\pi \cup \pi^\sigma)|$. Adding the points in $S \cap (\pi \Delta \pi^\sigma)$, we find

$$\begin{aligned} w \geq |S| \geq x(q^{2i+2}\mu_{n-i-1}(q^2) - t)\frac{N_{n,i}(q)}{2n_{\max}(n,i)} \\ + t(q^{2i+2}\mu_{n-i-1}(q^2) - x)\frac{N_{n,i}(q)}{2n_{\max}(n,i)} + x + t\ . \end{aligned}$$

Rewriting this inequality yields

$$(x+t)\left(q^{2i+2}\mu_{n-i-1}(q^2)N_{n,i}(q) + 2n_{\max}(n,i)\right) - 2xtN_{n,i}(q) \leq 2w\, n_{\max}(n,i)\ .$$

Using the inequality $2xt \leq \frac{1}{2}(x+t)^2$ and writing $y = x+t$, we find

$$\frac{1}{2}y^2 N_{n,i}(q) - \left[q^{2i+2}\mu_{n-i-1}(q^2)N_{n,i}(q) + 2n_{\max}(n,i)\right] y + 2w\, n_{\max}(n,i) \geq 0\ .$$

We now distinguish between two cases: $n-i$ odd and $n-i$ even. First we look at the former. By detailed analysis (carried out in Computation A.4.1) one can see that in this case

$$\begin{aligned} &N_{n-1,i}(q)q^{4i+4}\left(\mu_{n-i-2}(q^2)\right)^2 \\ &\geq N_{n-1,i-1}(q)q^{4i}\left(\mu_{n-i-1}(q^2)\right)^2 \\ &\geq N_{n-1,i+1}(q)q^{4i+4}\mu_{n-i-3}(q^2)\left[q^4\mu_{n-i-3}(q^2) + q^2 - 1\right] \end{aligned}$$

if $n-i > 3$ and

$$\begin{aligned} N_{n-1,n-3}(q)q^{4n-8}(q+1)^2 &\geq N_{n-1,n-4}(q)q^{4n-12}\left(q^3+1\right)^2 \\ &\geq N_{n,n-3}(q)q^{2n-4}\ . \end{aligned}$$

These inequalities correspond to $i = n-3$. Hence,

$$n_{\max}(n,i) = N_{n-1,i}(q)q^{4i+4}\left(\mu_{n-i-2}(q^2)\right)^2\ .$$

Using the formula for $N_{n,i}(q)$ from Lemma 9.2.7, and simplifying, we can rewrite this inequality as

$$\begin{aligned} \frac{1}{2}q^{n-3i-5}y^2 - \left[q^{n-i-3}\mu_{n-i-1}(q^2) + 2\mu_{n-i-2}(q^2)\frac{q^{n-i-1}-1}{q^2-1}\right] y \\ + 2\delta q^{2n-1}\mu_{n-i-2}(q^2)\frac{q^{n-i-1}-1}{q^2-1} \geq 0\ . \end{aligned}$$

Let $\alpha_{n,i}(q^2)$ and $\alpha'_{n,i}(q^2)$ be the two solutions of the corresponding equation, with $\alpha_{n,i}(q^2) \leq \alpha'_{n,i}(q^2)$. Then $x+t \leq \alpha_{n,i}(q^2)$ or $x+t \geq \alpha'_{n,i}(q^2)$. Moreover,

$$\begin{aligned}\alpha_{n,i}(q^2) + \alpha'_{n,i}(q^2) &= 2q^{2i+2}\mu_{n-i-1}(q^2) + 4\frac{\mu_{n-i-2}(q^2)(q^{n-i-1}-1)}{q^{n-3i-5}(q^2-1)} \\ &= \Sigma_{n,i}(q)\ .\end{aligned}$$

For the given δ we calculate

$$\overline{\alpha_{n,i}} = \lim_{q\to\infty} \alpha_{n,i}(q^2) = \lim_{q\to\infty} \frac{B' - \sqrt{B'^2 - 4\delta q^{3n-3i-6}C'}}{q^{n-3i-5}}\ ,$$

with

$$\begin{aligned}B' &= q^{n-i-3}\mu_{n-i-1}(q^2) + 2\mu_{n-i-2}(q^2)\frac{q^{n-i-1}-1}{q^2-1}\ , \\ C' &= \mu_{n-i-2}(q^2)\frac{q^{n-i-1}-1}{q^2-1}\ .\end{aligned}$$

Since $\overline{\alpha_{n,i}} \in O(q^{2n-2})$, we can find $Q_{n,i} > 0$ such that $\alpha_{n,i}(q^2) \leq C_n q^{2n-1}$ for $q \geq Q_{n,i}$.

In the latter case, $n-i$ even, similar arguments can be used. However, in this case we need to distinguish between $n-i > 2$ and $i = n-2$. First, we discuss $n-i > 2$. We can deduce (see Computation A.4.1) that

$$\begin{aligned}&N_{n-1,i+1}(q)q^{4i+4}\mu_{n-i-3}(q^2)\left[q^4\mu_{n-i-3}(q^2) + q^2 - 1\right] \\ &\geq N_{n-1,i-1}(q)q^{4i}\left(\mu_{n-i-1}(q^2)\right)^2 \\ &\geq N_{n-1,i}(q)q^{4i+4}\left(\mu_{n-i-2}(q^2)\right)^2\ ,\end{aligned}$$

hence $n_{\max}(n,i) = N_{n-1,i+1}(q)q^{4i+4}\mu_{n-i-3}(q^2)\left[q^4\mu_{n-i-3}(q^2) + q^2 - 1\right]$. We find the inequality

$$\begin{aligned}\frac{q^2-1}{2}y^2 - q^{2i+2}\left[\mu_{n-i-1}(q^2)(q^2-1) + 2(q^4\mu_{n-i-3}(q^2) + q^2 - 1)\right]y& \\ + 2\delta q^{2n-1}q^{2i+2}(q^4\mu_{n-i-3}(q^2) + q^2 - 1) \geq 0\ .&\end{aligned}$$

Just as in the previous case $\Sigma_{n,i}(q)$ equals the sum of the solutions of the corresponding equation. Also for these values of n and i, we define $\overline{\alpha_{n,i}}$:

$$\overline{\alpha_{n,i}} = \lim_{q\to\infty} \frac{B'' - \sqrt{B''^2 - 4\delta q^{2n-1}(q^2-1)C''}}{q^2-1}\ ,$$

with

$$
\begin{aligned}
B'' &= q^{2i+2}\left[\mu_{n-i-1}(q^2)(q^2-1)+2(q^4\mu_{n-i-3}(q^2)+q^2-1)\right] \ , \\
C'' &= q^{2i+2}(q^4\mu_{n-i-3}(q^2)+q^2-1) \ .
\end{aligned}
$$

Since $\overline{\alpha_{n,i}} \in O(q^{2n-2})$ also holds in this case, we again can find a value $Q_{n,i} > 0$ such that $\alpha_{n,i}(q^2) \leq C_n q^{2n-1}$ for $q \geq Q_{n,i}$.

Finally, we consider the case $i = n-2$. The second possibility in Lemma 9.2.8 can thus not occur. We note that

$$N_{n-1,n-3}(q)q^{4(n-2)}(q+1)^2 \leq q^{2n-2}N_{n,n-2}(q)$$

if $q \geq 3$ and

$$N_{n-1,n-3}(q)q^{4(n-2)}(q+1)^2 \geq q^{2n-2}N_{n,n-2}(q)$$

if $q = 2$. The arguments in these cases are analogous.

Hence, in all cases we can find a value $Q_{n,i} > 0$ such that $x+t \leq C_n q^{2n-1}$ or $x+t \geq \Sigma_{n,i}(q) - C_n q^{2n-1}$ for $q \geq Q_{n,i}$. $\square$

Using the three previous lemmata, we can now prove a classification theorem for the small weight code words in $C_n(\mathcal{H}(2n+1,q^2))^\perp$.

Theorem 9.3.6. *Let p be a fixed prime, $\delta > 0$ be a fixed constant and n be a fixed positive integer. Then there is a constant $\overline{Q}$ such that, for any $q = p^h \geq \overline{Q}$, $h \in \mathbb{N}$, and any $c \in C_n(\mathcal{H}(2n+1,q^2))^\perp$ with $\mathrm{wt}(c) \leq w = \delta q^{2n-1}$, c is a linear combination of code words described in Theorem 9.1.4.*

Proof. For the given values p and δ we have found a set of possible C_n-values, of which we have chosen one, in Lemma 9.3.4, with Q, a power of p, corresponding to it. By the proof of this lemma, we know that $C_n q^{2n-1} > \delta\theta_{n-1}(q^2)$ for all $q \geq Q$. Define $\overline{Q} = \max(\{Q\} \cup \{Q_{n,i} \mid -1 \leq i \leq n-2\})$, with $Q_{n,i}$ as in Lemma 9.3.5, corresponding to the chosen value C_n. We assume $q \geq \overline{Q}$.

Denote $\mathrm{supp}(c) = S$. By Lemma 9.3.4, we find an n-space π such that $N = |(\pi\Delta\pi^\sigma) \cap S| > C_n q^{2n-1}$. The intersection $\pi \cap H$ can be written as a cone $\pi_i H_{n-i-1}$, with an i-space π_i as vertex and $H_{n-i-1} \cong \mathcal{H}(n-i-1,q^2)$ as base, $-1 \leq i \leq n-2$.

Since $N > C_n q^{2n-1}$ and $q \geq Q_{n,i}$, we know by Lemma 9.3.5 that $N \geq \Sigma_{n,i}(q) - C_n q^{2n-1}$. For each element $\alpha \in \mathbb{F}_p^*$, we denote by N_α the sum of the number of points $P \in \pi$ such that $c_P = \alpha$ and the number of points $Q \in \pi^\sigma$ such that $c_Q = -\alpha$. We can find $\beta \in \mathbb{F}_p^*$ such that $N_\beta \geq \frac{N}{p-1}$. We now consider the code word $c' = c - \beta(v_\pi - v_{\pi^\sigma})$, with v_π and v_{π^σ} as in Theorem 9.1.4. We know

$$\begin{aligned}\mathrm{wt}(c') &= (N - N_\beta) + (|(\pi\Delta\pi^\sigma) \cap H| - N)\\ &= |(\pi\Delta\pi^\sigma) \cap H| - N_\beta\\ &\leq |(\pi\Delta\pi^\sigma) \cap H| - \frac{N}{p-1} \ .\end{aligned}$$

We also know that $N \geq \Sigma_{n,i}(q) - C_n q^{2n-1} > \frac{p-1}{p} |(\pi\Delta\pi^\sigma) \cap H|$ by Lemma 9.3.4 since $q \geq Q$. It follows that

$$\mathrm{wt}(c') < \frac{p}{p-1}N - \frac{N}{p-1} = N \leq \mathrm{wt}(c) \ .$$

Hence, the theorem follows using induction on $w = \mathrm{wt}(c)$. $\square$

We now focus on the code words that we described in Section 9.1.

Remark 9.3.7. Let c be a small weight code word and q sufficiently large. Following the arguments in the proof of Theorem 9.3.6, we know that $c = c_1 + \cdots + c_m$, with c_i, $1 \leq i \leq m-1$, a code word that we described in Theorem 9.1.4 and Example 9.1.5, such that $\mathrm{wt}(c_1 + \cdots + c_{m'}) < \mathrm{wt}(c_1 + \cdots + c_{m'+1})$ for all $1 \leq m' \leq m$. From this observation, it immediately follows that the code words that we described in Theorem 9.1.4 and Example 9.1.5 are the code words of smallest weights.

Now we consider small weight code words different from the ones described in Theorem 9.1.4. Let c be a code word c of weight at most $4q^{2n-2}(q-1)$, q sufficiently large. Since c is not of the type we described in Theorem 9.1.4, c can be written as a linear combination of at least two of these code words. By the above arguments, we can find a code word c' which is a linear combination of precisely two of these code words, such that $\mathrm{wt}(c') \leq \mathrm{wt}(c)$. In particular, we can find $\alpha, \alpha' \in \mathbb{F}_p^*$ and n-spaces π, π', $\pi' \notin \{\pi, \pi^\sigma\}$, such that $c' = \alpha(v_\pi - v_{\pi^\sigma}) + \alpha'(v_{\pi'} - v_{\pi'^\sigma})$ and $\mathrm{wt}(c') \leq 4q^{2n-2}(q-1)$. Let S be the support of c'. We know that $S = ((\pi\Delta\pi^\sigma) \cup (\pi'\Delta\pi'^\sigma)) \cap \mathcal{H}(2n+1, q^2)$ if $\alpha + \alpha' \neq 0$ and $S = ((\pi\Delta\pi^\sigma) \,\Delta\, (\pi'\Delta\pi'^\sigma)) \cap \mathcal{H}(2n+1, q^2)$ if $\alpha + \alpha' = 0$. In both cases, $S \supseteq ((\pi\Delta\pi^\sigma) \,\Delta\, (\pi'\Delta\pi'^\sigma)) \cap \mathcal{H}(2n+1, q^2)$. However, it can be seen that $|(\pi\Delta\pi^\sigma) \cap (\pi'\Delta\pi'^\sigma)| \leq 4q^{2n-2}$. Hence,

$$\begin{aligned}|S| &\geq \mathrm{wt}\left(\alpha\left(v_\pi - v_{\pi^\sigma}\right)\right) + \mathrm{wt}\left(\alpha'\left(v_{\pi'} - v_{\pi'^\sigma}\right)\right) - |(\pi\Delta\pi^\sigma) \cap (\pi'\Delta\pi'^\sigma)|\\ &> 4q^{2n-2}(q-1) \ ,\end{aligned}$$

which is a contradiction. It follows that the only code words of weight at most $4q^{2n-2}(q-1)$ are of the type described in Theorem 9.1.4.

Note that Theorem 9.3.6 only proves the second half of Theorem 9.0.3. From Remark 9.3.7 now the first half also follows.

10

Some remarks on the code $C(2, q)$

Gij komt tot ons, gans onverwacht, in alle mooie dingen.

Adeleyd.

The linear code $C(2, q)$, q a prime power, arising from the incidence matrix of points and lines in the projective plane $\mathrm{PG}(2, q)$, has been introduced in Section 1.8. In this chapter two unpublished results concerning the codes $C(2, q)$ are gathered. In Section 10.1 we mention some results and a conjecture on the small weight code words in the code $C(2, q)$. In Section 10.2 we solve an open problem related to Theorem 10.1.2. In Section 10.3 we present a small weight code word which indicates that Conjecture 10.1.3 cannot be generalised to the prime case. The result of this section is joint work with Peter Vandendriessche.

10.1 The small weight code words

The results in this chapter deal with small weight code words of $C(2,q)$. The minimum weight and the classification of these small weight code words has been found independently in [42] and [114]. In [3, Proposition 1] a short proof can be found.

Theorem 10.1.1. *The minimum weight of $C(2,q)$, q a prime power, is $q+1$ and the minimum weight code words are the scalar multiples of incidence vectors of the lines in* $\mathrm{PG}(2,q)$.

In [3, Theorem 1] the analogous result for $C_{s,t}(n,q)$ has been proved. We look at another result related to the code $C(2,q)$. Recall that a unital which is embedded in $\mathrm{PG}(2,q^2)$, is a set of q^3+1 points meeting every line in 1 or $q+1$ points.

Theorem 10.1.2 ([14]). *If U is a unital embedded in* $\mathrm{PG}(2,q)$, *q a square, then U is a Hermitian unital if and only if the incidence vector of U is a code word of $C(2,q)$.*

It is a consequence of this theorem that the incidence vector of a non-singular Hermitian variety in $\mathrm{PG}(2,q)$, q a square, is a code word of $C(2,q)$; its weight equals $q\sqrt{q}+1$. It is conjectured that this example is a borderline case.

Conjecture 10.1.3. Let c be a code word in $C(2,q)$, $q=p^h$, p a prime and $h>1$, with $\mathrm{wt}(c)<q\sqrt{q}+1$. Then c is a linear combination of incidence vectors of $\left\lceil \frac{\mathrm{wt}(c)}{q+1} \right\rceil$ lines.

The next theorem is an important result towards the proof of this conjecture. It is only a special case of the original theorem, which is about the code $C_t(n,q)$

Theorem 10.1.4 ([93, Corollary 20]). *The code $C(2,q)$ has no code words with weight in the interval $]q+1,2q[$.*

For the prime case ($q=p$), which is not covered by the Conjecture 10.1.3, classifications of the small weight code words, improving the result of Theorem 10.1.1, are known.

Theorem 10.1.5 ([30]). *If c is a code word in $C(2,p)$ with $0 < wt(c) \leq 2p$, p prime, then c is*

(i) a code word αv_ℓ, with v_ℓ the incidence vector of a line ℓ in $\mathrm{PG}(2,p)$, and $\alpha \in \mathbb{F}_p^*$, with weight $p+1$,

(ii) or a code word $\alpha(v_\ell - v_{\ell'})$, with v_ℓ and $v_{\ell'}$ the incidence vectors of the lines ℓ and ℓ' in $\mathrm{PG}(2,p)$ respectively, $\ell \neq \ell'$ and $\alpha \in \mathbb{F}_p^*$, with weight $2p$.

Theorem 10.1.6 ([55, Theorem 4]). *If c is a code word in $C(2,p)$ with $0 < wt(c) \leq 2p + \frac{p-1}{2}$, $p \geq 11$ prime, then c is*

(i) a code word αv_ℓ, with v_ℓ the incidence vector of a line ℓ in $\mathrm{PG}(2,p)$, and $\alpha \in \mathbb{F}_p^*$, with weight $p+1$,

(ii) a code word $\alpha(v_\ell - v_{\ell'})$, with v_ℓ and $v_{\ell'}$ the incidence vectors of the lines ℓ and ℓ' in $\mathrm{PG}(2,p)$ respectively, $\ell \neq \ell'$ and $\alpha \in \mathbb{F}_p^*$, with weight $2p$.

(iii) or a code word $\alpha v_\ell + \alpha' v_{\ell'}$, with v_ℓ and $v_{\ell'}$ the incidence vectors of the lines ℓ and ℓ' in $\mathrm{PG}(2,p)$ respectively, $\ell \neq \ell'$, $\alpha, \alpha' \in \mathbb{F}_p^*$ and $\alpha + \alpha' \neq 0$, with weight $2p+1$.

It is communicated that Conjecture 10.1.3 has been solved for $h > 2$ ([60]), but no proof has been published yet.

10.2 The Hermitian unital as sum of lines

By Theorem 10.1.2 we know that the incidence vector of a Hermitian unital in $\mathrm{PG}(2,q^2)$ is a code word of $C(2,q^2)$. The proof of this theorem is however non-constructive. In [94, Open Problem 3.19] it is asked to describe a linear combination of incidence vectors of lines which equals the incidence vector of a Hermitian unital. We solve this problem for q even.

First we mention a result on complete arcs. Recall from Section 1.7 that hyperovals are the largest k-arcs in $\mathrm{PG}(2,q^2)$, q even, and therefore complete. The second-largest complete arcs were studied in [19, 56, 87, 119]. The first part of this theorem was proved independently in several of these articles. The second part was proved in [119].

Theorem 10.2.1. *In* $\mathrm{PG}(2,q^2)$*, q even, the largest complete arcs different from hyperovals, contain q^2-q+1 points. The tangent lines to such an arc form a dual Hermitian unital.*

We use this theorem to find the desired linear combination.

Theorem 10.2.2. *Let $\mathcal{L} = \{\ell_0, \ldots, \ell_{q^2-q}\}$ be a set of lines in* $\mathrm{PG}(2,q^2)$*, q even, such that its dual is a complete arc of size q^2-q+1 and let v_{ℓ_i} be the incidence vector of the line ℓ_i. Then $\sum_{i=0}^{q^2-q} v_{\ell_i}$ is the incidence vector of a Hermitian unital.*

Proof. Let $\mathcal{L}^D$ be the dual of $\mathcal{L}$, a complete (q^2-q+1)-arc in $\mathrm{PG}(2,q^2)^D$, the dual plane of $\mathrm{PG}(2,q^2)$. Through every point of $\mathcal{L}^D$ there are $q+1$ tangent lines. Each of these $(q+1)(q^2-q+1) = q^3+1$ tangent lines contains one point of $\mathcal{L}^D$. All other lines in $\mathrm{PG}(2,q^2)^D$ contain zero or two points of $\mathcal{L}^D$.

We denote the set of duals of the tangent lines by $\mathcal{H}$. This is a set of q^3+1 points in $\mathrm{PG}(2,q^2)$. Note that $\mathcal{H}$ is a Hermitian unital since its dual is a dual Hermitian unital by Theorem 10.2.1. We know that each point of $\mathcal{H}$ belongs to one line of $\mathcal{L}$. All other points of $\mathrm{PG}(2,q^2)$ belong to zero or two lines of $\mathcal{L}$. So, the sum $\sum_{i=0}^{q^2-q} v_{\ell_i}$ of the incidence vectors v_{ℓ_i}, is a vector with 1 on every position corresponding with a point of $\mathcal{H}$. On all other positions of this vector, there is a 0, since q is even. So this sum is the incidence vector of $\mathcal{H}$, which is a Hermitian unital. $\square$

10.3 A small weight code word in $C(2,p)$

In this section we present the construction of a code word in $C(2,p)$, p prime, of weight $3(p-1)$, which cannot be realised as a linear combination of incidence vectors of at most 3 lines. This is in general larger than the upper bounds from Theorems 10.1.5 and 10.1.6. However, in general it is much smaller than $p\sqrt{p}+1$, clearly indicating that Conjecture 10.1.3 cannot be generalised to the prime case ($h=1$). Up to our knowledge, this is the first construction of a code word contradicting the conjecture for the prime case.

We first mention two results which we will need in the proof of Lemma 10.3.3. This lemma will allow us to prove that the vector we constructed, is a code word.

Theorem 10.3.1 ([65]). *The p-rank of the incidence matrix $M_{0,1}(2,q)$, $q = p^h$ and p prime, equals $\binom{p+1}{2}^h + 1$. Consequently, the code $C(2,q)$ has dimension $\binom{p+1}{2}^h + 1$.*

Theorem 10.3.2 ([2, Theorem 6.3.1]). *Let C be the code $C(2,q)$, $q = p^h$ and p prime, and let $\mathbf{1}$ be the all-one vector of length q^2+q+1. Then, $(C \cap C^\perp) \oplus \mathbf{1} = C$.*

To be precise, the previous theorem is not the theorem which is presented in [2], but a result which is found during its proof.

Lemma 10.3.3. *If p is a prime, then $C(2,p)^\perp \subset C(2,p)$.*

Proof. We denote the code $C(2,p)$ by C. By Theorem 10.3.1 we know that the dimension of C equals $\frac{p(p+1)}{2} + 1$. The dimension of $C^\perp$ thus equals $\frac{p(p+1)}{2} = (p^2+p+1) - \left(\frac{p(p+1)}{2} + 1\right)$. So, $\dim(C^\perp) + 1 = \dim(C)$. However, by Theorem 10.3.2 we know that also $\dim(C \cap C^\perp) + 1 = \dim(C)$. Hence, $C \cap C^\perp = C^\perp$ since $C \cap C^\perp \subseteq C^\perp$ and $\dim(C \cap C^\perp) = \dim(C^\perp)$. Consequently, $C^\perp = C \cap C^\perp \subset C$. $\square$

Now, we introduce the small weight code word.

Example 10.3.4. Let c be a vector of the vector space $\mathbb{F}_p^{p^2+p+1}$, $p \neq 2$ a prime, whose positions correspond to the points of $\mathrm{PG}(2,p)$, such that

$$c_P = \begin{cases} a & P = (0,1,a) \\ b & P = (1,0,b) \\ -c & P = (1,1,c) \\ 0 & \text{otherwise} \end{cases},$$

whereby c_P is the value of c at the position corresponding with the point P. Note that the points corresponding to positions with non-zero coordinates belong to the line $m : X_0 = 0$, the line $m' : X_1 = 0$ or the line $m'' : X_0 = X_1$. These three lines are concurrent in the point $(0,0,1)$.

Now we check that $c \in C(2,p)^\perp$. For any line ℓ in $\mathrm{PG}(2,p)$, the dot product $c \cdot v_\ell$ should equal 0, with v_ℓ the incidence vector of ℓ, or equivalently $\sum_{P \in \ell} c_P = 0$. We distinguish between three cases. If $\ell \in \{m, m', m''\}$, then $\sum_{P\in\ell} c_P =$

$\sum_{x\in\mathbb{F}_p} = \frac{p(p-1)}{2} = 0$ since $p \neq 2$. If $(0,0,1) \in \ell$ and $\ell \notin \{m, m', m''\}$, then $c_P = 0$ for all $P \in \ell$, so $\sum_{P\in\ell} c_P = 0$. Finally, if $(0,0,1) \notin \ell$, then ℓ can be described by an equation $uX_0 + vX_1 + X_2 = 0$. For a point $P \in \ell$, the value c_P differs from 0 if and only if $P \in \{\ell \cap m, \ell \cap m', \ell \cap m''\}$. We know that $\ell \cap m = (0,1,-v)$, $\ell \cap m' = (1,0,-u)$ and $\ell \cap m'' = (1,1,-(u+v))$. Hence, $\sum_{P\in\ell} c_P = (-v) + (-u) + (u+v) = 0$. So, in all three cases $\sum_{P\in\ell} c_P = 0$ and thus $c \in C(2,p)^\perp$.

By Lemma 10.3.3 it follows immediately that $c \in C(2,p)$. It is clear that $\mathrm{wt}(c) = 3(p-1)$. If $p = 3$, then the code word which we described here is a code word of weight $2p$. In this case it equals $v_{\ell_1} - v_{\ell_2}$ with ℓ_1 the line $X_0+X_1-X_2 = 0$ and ℓ_2 the line $X_0+X_1+X_2 = 0$. If $p \geq 5$, then $3(p-1) > 2p$, the upper bound from Theorem 10.1.5. If $p \geq 11$, then $3(p-1) > 2p + \frac{p-1}{2}$, the upper bound from Theorem 10.1.5.

We now argue that this code word cannot be realised as a linear combination of at most $\left\lceil \frac{3(p-1)}{p+1} \right\rceil = 3$ lines if $p > 3$. This shows that Conjecture 10.1.3 cannot be generalised to the prime case. It is clear that c cannot be a linear combination of incidence vectors of lines through $(0,0,1)$. Let ℓ' be a line not containing $(0,0,1)$, whose incidence vector appears non-trivially in a linear combination yielding c. The line ℓ' contains $p-2$ points whose corresponding position in c contains a zero. Consequently, there are at least $p-2$ other lines whose incidence vector appears non-trivially in the given linear combination. So, any linear combination yielding c contains at least $p-1 > 3$ terms.

A
The omitted calculations

I show you how deep the rabbit hole goes.

Morpheus in *The Matrix.*

In this appendix we deal with some calculations of the previous chapters. They were not included in the text for sake of clearness and overview. Here we have a closer look at them. In Section A.1, we consider calculations from Chapter 5. In Section A.2, we consider calculations from Chapter 7. In Section A.3, we consider a calculation from Chapter 8 and in Section A.4, we consider a calculation from Chapter 9.

We assume the reader to be familiar with some basic calculus. For some of the calculations the assistance of a computer algebra package has been used[1]. This is clearly indicated.

[1] We used the package Maple (by Maplesoft), however no code lines are presented. These calculations can be repeated using most computer algebra packages.

A.1 Erdős-Ko-Rado sets in designs

At the end of Lemma 5.4.3 we want to prove the inequality (5.1). Firstly, we want to show that the function on the left-hand side is concave.

Computation A.1.1. We introduced the notations $C_k = \frac{4}{3}k\sqrt{k} - 2k - 2\sqrt{k}$ and $D(b,k) = (k^3 - 3k^2 - 2bk + 6k - 2)^2 - 8k(k-1)(b-1)(b-2)$, $k \geq 14$ an integer. We define the functions f_k and g_k for $k \geq 14$ an integer, as follows:

$$\begin{aligned}
f_k(c) &= \sqrt{(c-1)^2 + 4c(k-1)}\sqrt{D(c,k)}\ , \\
g_k(c) &= \left((c-1)^2 + 4c(k-1)\right) D(c,k)\ .
\end{aligned}$$

Note that $f_k(c)$ exists on the interval $[0, C_k]$. Obviously, $f_k(c) \geq 0$. It is clear that $f_k^2(c) = g_k(c)$. We want to show that $f_k(c)$ is concave on the interval $[0, C_k]$. Therefore we want to study its second derivative (with respect to c). Note that

$$f_k'' = (\sqrt{g_k})'' = \frac{g_k''}{2\sqrt{g_k}} - \frac{(g_k')^2}{4\sqrt{g_k^3}} = \frac{g_k''}{2f_k} - \frac{(g_k')^2}{4f_k^3} = \frac{2g_k''g_k - (g_k')^2}{4f_k^3}\ .$$

Consequently, in order to prove that $f_k''(c) < 0$ it is sufficient to prove that $2g_k''g_k - (g_k')^2 < 0$. The function $2g_k''(c)g_k(c) - (g_k'(c))^2$ is however a polynomial in the variable c. We define $h_k(c) = \frac{1}{16}\left(2g_k''(c)g_k(c) - (g_k'(c))^2\right)$. Then, we can compute that

$$\begin{aligned}
h_k(c) = {} & c^6\left(8k^4 - 32k^3 + 32k^2\right) \\
& + c^5\left(12k^6 - 12k^5 - 192k^4 + 528k^3 - 384k^2\right) \\
& + c^4\left(78k^7 - 492k^6 + 906k^5 + 120k^4 - 1848k^3 + 1332k^2 + 24k\right) \\
& + c^3\left(-k^{10} + 5k^9 + 95k^8 - 1025k^7 + 3922k^6 - 6956k^5 + 5124k^4 \right. \\
& \quad \left. -68k^3 - 1000k^2 - 256k\right) \\
& + c^2\left(-6k^{11} + 39k^{10} - 51k^9 - 477k^8 + 3153k^7 - 9402k^6 + 15696k^5 \right. \\
& \quad \left. -14532k^4 + 6852k^3 - 1776k^2 + 624k\right) \\
& + c\left(-3k^{10} + 15k^9 - 3k^8 - 243k^7 + 1194k^6 - 2760k^5 + 3276k^4 \right. \\
& \quad \left. -1884k^3 + 552k^2 - 192k\right) \\
& + \left(-k^{14} + 15k^{13} - 116k^{12} + 588k^{11} - 2116k^{10} + 5594k^9 - 10945k^8 \right. \\
& \quad + 15709k^7 - 16378k^6 + 12494k^5 - 7196k^4 + 3172k^3 - 1012k^2 \\
& \quad \left. +232k - 32\right)\ .
\end{aligned}$$

We now define the function

$$\overline{h}_k(c) = -\left(\varphi_5(k)c^5 + \varphi_4(k)c^4 + \varphi_3(k)c^3 + \varphi_2(k)c^2\right)(C_k - c) \ ,$$

with

$$\begin{aligned}
\varphi_5(k) &= 8k^4 - 32k^3 + 32k^2 \ , \\
\varphi_4(k) &= 12k^6 + \frac{32}{3}k^5\sqrt{k} - 28k^5 - \frac{176}{3}k^4\sqrt{k} - 128k^4 + \frac{320}{3}k^3\sqrt{k} + 464k^3 \\
&\quad - 64k^2\sqrt{k} - 384k^2 \ , \\
\varphi_3(k) &= 16k^7\sqrt{k} + \frac{614}{9}k^7 - \frac{248}{3}k^6\sqrt{k} - \frac{4820}{9}k^6 + \frac{8}{3}k^5\sqrt{k} + \frac{12794}{9}k^5 \\
&\quad + \frac{1984}{3}k^4\sqrt{k} - \frac{3320}{3}k^4 - 1312k^3\sqrt{k} - 952k^3 + 768k^2\sqrt{k} \\
&\quad + 1332k^2 + 24k \ , \\
\varphi_2(k) &= \frac{37}{3}k^9 \ .
\end{aligned}$$

It can easily be checked that $\varphi_i(k) \geq 0$ for $k \geq 14$, $i \in \{2, 3, 4, 5\}$. Hence, $\overline{h}_k(c) \leq 0$ on $[0, C_k]$, $k \geq 14$. Calculating the difference between the functions $\overline{h}_k(c)$ and $h_k(c)$, we find that $\overline{h}_k(c) - h_k(c) = \chi_3(k)c^3 + \chi_2(k)c^2 + \chi_1(k)c + \chi_0(k)$, with

$$\begin{aligned}
\chi_3(k) &= k^{10} - 14k^9 - \frac{1592}{27}k^8\sqrt{k} + \frac{551}{3}k^8 + \frac{18500}{27}k^7\sqrt{k} - 215k^7 \\
&\quad - \frac{79952}{27}k^6\sqrt{k} - \frac{5866}{3}k^6 + \frac{16924}{3}k^5\sqrt{k} + \frac{23444}{3}k^5 - 3568k^4\sqrt{k} \\
&\quad - 10676k^4 - 2144k^3\sqrt{k} + 4268k^3 + 2632k^2\sqrt{k} + 1048k^2 \\
&\quad + 48k\sqrt{k} + 256k \ , \\
\chi_2(k) &= 6k^{11} - \frac{148}{9}k^{10}\sqrt{k} - \frac{43}{3}k^{10} + \frac{74}{3}k^9\sqrt{k} + 51k^9 + 477k^8 - 3153k^7 \\
&\quad + 9402k^6 - 15696k^5 + 14532k^4 - 6852k^3 + 1776k^2 - 624k \ , \\
\chi_1(k) &= 3k^{10} - 15k^9 + 3k^8 + 243k^7 - 1194k^6 + 2760k^5 - 3276k^4 + 1884k^3 \\
&\quad - 552k^2 + 192k \ , \\
\chi_0(k) &= k^{14} - 15k^{13} + 116k^{12} - 588k^{11} + 2116k^{10} - 5594k^9 + 10945k^8 \\
&\quad - 15709k^7 + 16378k^6 - 12494k^5 + 7196k^4 - 3172k^3 + 1012k^2 \\
&\quad - 232k + 32 \ .
\end{aligned}$$

It can also be checked that $\chi_i(k) > 0$ for $k \geq 14$, $i \in \{0, 1, 2, 3\}$. So, $\overline{h}_k(c) - h_k(c) > 0$ if $c \geq 0$, $k \geq 14$. Consequently, on the interval $[0, C_k]$ the following inequalities are valid

$$h_k(c) < \overline{h}_k(c) \leq 0 \ ,$$

which proves that $f_k(c)$ is concave on $[0, C_k]$.

Secondly, we use the fact that the left-hand side of (5.1) is concave to find a linear lower bound. We compare this linear function with the linear function on the right-hand side.

Computation A.1.2. We use the notation C_k and the function $D(c, k)$ which we defined in Computation A.1.1. We want to prove the inequality

$$\frac{\sqrt{(C_k-1)^2 + 4C_k(k-1)}\sqrt{D(C_k,k)} - \sqrt{D(0,k)}}{C_k} < k^3 - 7k^2 + 14k - 6 \ .$$

Calculating some limits, we can see that

$$\frac{\sqrt{(C_k-1)^2 + 4C_k(k-1)}\sqrt{D(C_k,k)} - \sqrt{D(0,k)}}{C_k} = k^3 - \frac{7}{6}k^2\sqrt{k} + O(k^2) \ .$$

Then, using a computer algebra package, it is easy to check that

$$\begin{aligned}\frac{\sqrt{(C_k-1)^2 + 4C_k(k-1)}\sqrt{D(C_k,k)} - \sqrt{D(0,k)}}{C_k} &< k^3 - \frac{7}{6}k^2\sqrt{k} - 3k^2 \\ &< k^3 - 7k^2 + 14k - 6 \ .\end{aligned}$$

Consequently, it is sufficient to check the inequality (5.1) for $c = C_k$. Since the left-hand side is clearly nonnegative, the inequality is equivalent to the inequality of the squares of both sides. Note that

$$\begin{aligned}&\left(\sqrt{(C_k-1)^2 + 4C_k(k-1)}\sqrt{D(C_k,k)}\right)^2 \\ &= \frac{16}{9}k^9 - \frac{256}{27}k^8\sqrt{k} - \frac{1492}{81}k^8 + \frac{3304}{27}k^7\sqrt{k} + \frac{11000}{81}k^7 - \frac{17036}{27}k^6\sqrt{k} \\ &\quad - \frac{21073}{27}k^6 + \frac{41120}{27}k^5\sqrt{k} + 2386k^5 - \frac{3428}{3}k^4\sqrt{k} - \frac{10457}{3}k^4 \\ &\quad - \frac{3848}{3}k^3\sqrt{k} + \frac{16264}{9}k^3 + \frac{5312}{3}k^2\sqrt{k} + \frac{944}{3}k^2 - 96k\sqrt{k} + 56k \\ &\quad + 48\sqrt{k} + 4\end{aligned}$$

and

$$\begin{aligned}&\left(C_k(k^3-7k^2+14k-6)-(2k^4-9k^3+9k^2+2k-2)\right)^2\\=&\frac{16}{9}k^9-\frac{32}{3}k^8\sqrt{k}-\frac{128}{9}k^8+152k^7\sqrt{k}+\frac{284}{9}k^7-\frac{2644}{3}k^6\sqrt{k}-\frac{103}{9}k^6\\&+2656k^5\sqrt{k}+\frac{1198}{9}k^5-\frac{13172}{3}k^4\sqrt{k}-811k^4+3824k^3\sqrt{k}+1432k^3\\&-\frac{4432}{3}k^2\sqrt{k}-912k^2+96k\sqrt{k}+184k+48\sqrt{k}+4\;.\end{aligned}$$

Comparing these two functions, we see that the first one dominates the second one if k is large enough. Using a computer algebra package it is easy to check that the desired inequality is fulfilled if $k \geq 14$.

In Lemma 5.5.3 we derive an upper bound for an Erdős-Ko-Rado set which is not a point-pencil. In its proof an inequality is considered.

Computation A.1.3. We know that the inequality

$$q(q-a'-1)(q-a')^2(q-1) \leq a'(a'-1)(2q^2-2qa'+a'^2-q)$$

is fulfilled for integer parameters q and a' which are related to a unital and an Erdős-Ko-Rado set on this unital. We want to derive a lower bound on a' (function of q) from this inequality. We define the functions

$$f_q(a') = a'(a'-1)(2q^2-2qa'+a'^2-q) - q(q-a'-1)(q-a')^2(q-1)$$

for all prime powers q. We can compute that

$$f_q(q-\sqrt[3]{q^2}+\frac{2}{3}\sqrt[3]{q}-1-t) = \varphi_4(q)t^4+\varphi_3(q)t^3+\varphi_2(q)t^2+\varphi_1(q)t+\varphi_0(q)\;,$$

with

$$\begin{aligned}\varphi_4(q) &= 1\;,\\ \varphi_3(q) &= -q^2-q+4\sqrt[3]{q^2}-\frac{8}{3}\sqrt[3]{q}+5\;,\\ \varphi_2(q) &= -3q^2\sqrt[3]{q^2}+2q^2\sqrt[3]{q}-3q\sqrt[3]{q^2}+8q\sqrt[3]{q}-14q+\frac{53}{3}\sqrt[3]{q^2}-10\sqrt[3]{q}+9\;,\\ \varphi_1(q) &= -3q^3\sqrt[3]{q}+2q^3-\frac{4}{3}q^2\sqrt[3]{q^2}-3q^2\sqrt[3]{q}+14q^2-\frac{64}{3}q\sqrt[3]{q^2}+\frac{85}{3}q\sqrt[3]{q}\end{aligned}$$

$$\begin{aligned}&-\frac{842}{27}q+\frac{74}{3}\sqrt[3]{q^2}-12\sqrt[3]{q}+7\ ,\\ \varphi_0(q)&=-\frac{127}{27}q^3+9q^2\sqrt[3]{q^2}-14q^2\sqrt[3]{q}+\frac{593}{27}q^2-\frac{644}{27}q\sqrt[3]{q^2}+\frac{1825}{81}q\sqrt[3]{q}\\ &-\frac{499}{27}q+11\sqrt[3]{q^2}-\frac{14}{3}\sqrt[3]{q}+2\ .\end{aligned}$$

Now, it can be checked that $\varphi_2(q), \varphi_1(q), \varphi_0(q) < 0$ for all $q \geq 5$. Moreover,

$$\varphi_4(q)t^4+\varphi_3(q)t^3=t^3\left(t+\varphi_3(q)\right)\leq t^3\left(q-\sqrt[3]{q^2}+\frac{2}{3}\sqrt[3]{q}-1+\varphi_3(q)\right)<0\ ,$$

for $t \leq q-\sqrt[3]{q^2}+\frac{2}{3}\sqrt[3]{q}-1$. So, $f_q(q-\sqrt[3]{q^2}+\frac{2}{3}\sqrt[3]{q}-1-t)<0$ if $0 \leq t \leq q-\sqrt[3]{q^2}+\frac{2}{3}\sqrt[3]{q}-1$. We conclude that $f_q(a')$ can only be positive if $a' > q-\sqrt[3]{q^2}+\frac{2}{3}\sqrt[3]{q}-1$.

A.2 Small maximal partial spreads

In this Section we deal with several calculations related to Theorem 7.2.5.

Computation A.2.1. In the middle of the proof of Theorem 7.2.5, we claim that $h_{t,q}(s) < 0$ if $s \leq 2q-2$, $t = 2, 3, 4$ and $q \geq 3$ a prime power, with

$$\begin{aligned}h_{t,q}(s)&=(s-1)(s-3)(s-4)\begin{bmatrix}2t+2\\ t+1\end{bmatrix}_q-s(3s^2-19s+28)a_t(q)\\ &+s(s-1)(2s-9)b_t(q)-c'_t(q)s(s-1)(s-2)\ .\end{aligned}$$

Hereby $a_t(q)$, $b_t(q)$ and $c'_t(q)$ are defined as in Chapter 7. For $t = 2$ we find $h_{2,q}(s) = \psi_3(q)s^3 + \psi_2(q)s^2 + \psi_1(q)s + \psi_0(q)$, with

$$\begin{aligned}\psi_3(q)&=q^6+4q^5+5q^4+4q^3+2q^2+q+1\ ,\\ \psi_2(q)&=-5q^7-18q^6-32q^5-35q^4-22q^3-16q^2-8q-8\ ,\\ \psi_1(q)&=12q^8+29q^7+53q^6+64q^5+66q^4+54q^3+38q^2+19q+19\ ,\\ \psi_0(q)&=-12q^9-12q^8-24q^7-36q^6-36q^5-36q^4-36q^3-24q^2\\ &-12q-12\ .\end{aligned}$$

Then, $h'_{2,q}(s) = 3\psi_3(q)s^2 + 2\psi_2(q)s + \psi_1(q)$. Now, $h_2 = 4\psi_2^2 - 12\psi_3\psi_1$ is a polynomial of degree 14 such that $h_2(q) = -44q^{14} + O(q^{13})$. It can be checked that $h_2(q) < 0$ if $q \geq 3$. Hence, $h'_{2,q} > 0$ for $q \geq 3$, and thus is $h_{2,q}$ monotonically increasing, $q \geq 3$.

Consequently, it is sufficient to prove that $h_{2,q}(2q-2) < 0$. We find that

$$h_{2,q}(2q-2) = -2q^8 - 12q^7 + 30q^6 + 40q^5 - 72q^4 - 60q^3 - 110q^2 + 36q - 90 .$$

It now easily can be observed that $h_{2,q}(2q-2) < 0$ if $q \geq 3$.

For $t = 3, 4$ the claim can be proved analogously.

Computation A.2.2. We consider the polynomials

$$\begin{aligned}
f_q(s) = {} & (s-1)(s-3)(s-4)(q^5+2)(q^6+2)\begin{bmatrix}10\\5\end{bmatrix}_q \\
& + (q^5 - 2q^3 - q^2 + 3)q^{30}s(s-1)(s-2) - s(3s^2 - 19s + 28)q^{36} \\
& + s(s-1)(2s-9)(q-1)(q^2-1)(q^3-1)(q^4-1)q^{26} .
\end{aligned}$$

It is claimed in the final paragraph of Theorem 7.2.5 that $f_q(s) < 0$ for all $s \leq 2q-2$. We will first prove that the polynomials f_q are monotonically increasing if $q \geq 3$. We find that

$$f'_q(s) = g_2(q)s^2 - g_1(q)s + g_0(q) ,$$

with

$$\begin{aligned}
g_2(q) = {} & 3q^{33} + 12q^{32} + 39q^{31} + 48q^{30} + 51q^{29} + 66q^{28} + 90q^{27} + 132q^{26} \\
& + 165q^{25} + 192q^{24} + 234q^{23} + 273q^{22} + 318q^{21} + 354q^{20} + 384q^{19} \\
& + 405q^{18} + 429q^{17} + 435q^{16} + 435q^{15} + 417q^{14} + 396q^{13} + 363q^{12} \\
& + 327q^{11} + 288q^{10} + 240q^9 + 198q^8 + 150q^7 + 120q^6 + 90q^5 \\
& + 60q^4 + 36q^3 + 24q^2 + 12q + 12 , \\
g_1(q) = {} & 10q^{34} + 36q^{33} + 74q^{32} + 188q^{31} + 226q^{30} + 272q^{29} + 362q^{28} \\
& + 490q^{27} + 694q^{26} + 880q^{25} + 1024q^{24} + 1248q^{23} + 1456q^{22} \\
& + 1696q^{21} + 1888q^{20} + 2048q^{19} + 2160q^{18} + 2288q^{17} + 2320q^{16} \\
& + 2320q^{15} + 2224q^{14} + 2112q^{13} + 1936q^{12} + 1744q^{11} + 1536q^{10} \\
& + 1280q^9 + 1056q^8 + 800q^7 + 640q^6 + 480q^5 + 320q^4 + 192q^3
\end{aligned}$$

$$
\begin{aligned}
&+ 128q^2 + 64q + 64\ , \\
g_0(q) = {} & 12q^{35} + 29q^{34} + 53q^{33} + 93q^{32} + 189q^{31} + 253q^{30} + 323q^{29} + 447q^{28} \\
&+ 599q^{27} + 807q^{26} + 1045q^{25} + 1216q^{24} + 1482q^{23} + 1729q^{22} \\
&+ 2014q^{21} + 2242q^{20} + 2432q^{19} + 2565q^{18} + 2717q^{17} + 2755q^{16} \\
&+ 2755q^{15} + 2641q^{14} + 2508q^{13} + 2299q^{12} + 2071q^{11} + 1824q^{10} \\
&+ 1520q^9 + 1254q^8 + 950q^7 + 760q^6 + 570q^5 + 380q^4 + 228q^3 \\
&+ 152q^2 + 76q + 76\ .
\end{aligned}
$$

The function $g = g_1^2 - 4g_0g_2$ is a polynomial of degree 68. One can see that $g(q) = -44q^{68} + O(q^{67})$. Using a computer algebra package, it can easily be checked that $g(q) < 0$ for $q \geq 3$. Hence, if $q \geq 3$, then $f_q' > 0$. Consequently, the functions f_q are monotonically increasing, $q \geq 3$.

Now, we want to show that $f_q(2q-2) < 0$. Calculating this value yields

$$
\begin{aligned}
f_q(2q-2) = {} & -2q^{35} + 32q^{34} - 204q^{33} + 316q^{32} + 52q^{31} - 112q^{30} - 8q^{29} \\
&- 208q^{28} + 76q^{27} + 112q^{26} - 526q^{25} - 180q^{24} - 546q^{23} \\
&- 418q^{22} - 760q^{21} - 908q^{20} - 1358q^{19} - 1092q^{18} - 1608q^{17} \\
&- 1508q^{16} - 1960q^{15} - 1782q^{14} - 1970q^{13} - 1892q^{12} \\
&- 1666q^{11} - 1856q^{10} - 1364q^9 - 1640q^8 - 980q^7 - 844q^6 \\
&- 788q^5 - 704q^4 - 264q^3 - 440q^2 + 144q - 360\ .
\end{aligned}
$$

We observe that the function $f_q(2q-2)$ is negative if q is large enough. Using a computer algebra package, it can be seen that $f_q(2q-2) < 0$ if $q \geq 3$.

A.3 Functional codes

Computation A.3.1. At the end of Lemma 8.4.10 we claim that the inequality

$$\overline{W}_n(q) \geq \frac{b_n(q)(c_n(q) + d_n(q))}{a_n(q)}$$

is valid for $q \geq 2$ and $n \geq 5$, with $a_n(q)$, $b_n(q)$, $c_n(q)$ and $d_n(q)$ as defined in that proof. The functions $\overline{W}_n(q)$ were defined in Definition 8.4.1. We distinguish in this discussion between $q = 2$ and $q \geq 3$ since the definition of $W_n(q)$ differs between those cases. We also need to split it based on the parity of n. First we look at the case $q \geq 3$ and $n = 2m$, $m \in \mathbb{N}$. Since $\overline{W}_n(q) = W_n(q)$ in this case it is sufficient to prove the inequality

$$W_n(q)a_n(q) \geq b_n(q)(c_n(q) + d_n(q)) . \tag{A.1}$$

We find that (A.1) is equivalent to the inequality

$$\begin{aligned} 0 \leq \varphi_{16}(q)q^{16m-20} &+ \varphi_{14}(q)q^{14m-18} + \varphi_{12}(q)q^{12m-16} + \varphi_{10}(q)q^{10m-14} \\ &+ \varphi_8(q)q^{8m-12} + \varphi_6(q)q^{6m-10} + \varphi_4(q)q^{4m-8} + \varphi_2(q)q^{2m-6} + \varphi_0(q) , \end{aligned}$$

with

$$\begin{aligned}
\varphi_{16}(q) &= q^{13} - 2q^{12} + q^{11} + 4q^{10} - 8q^9 + 4q^8 - 4q^7 + 4q^6 - 6q^5 - 4q^4 + 5q^3 \\
&\quad - 4q^2 - q - 2 , \\
\varphi_{14}(q) &= -2q^{16} + 4q^{15} - 5q^{14} + 5q^{12} - 4q^{11} - 11q^{10} + 27q^9 - 45q^8 + 39q^7 \\
&\quad - 21q^6 + 4q^5 + 18q^4 - 25q^3 + 11q^2 - q + 6 , \\
\varphi_{12}(q) &= -q^{18} + q^{17} + 7q^{16} - 17q^{15} + 21q^{14} - 15q^{13} - 3q^{12} - 7q^{11} + 41q^{10} \\
&\quad - 62q^9 + 109q^8 - 72q^7 + 26q^6 + 15q^5 - 29q^4 + 42q^3 - 9q^2 \\
&\quad + 7q - 6 , \\
\varphi_{10}(q) &= -q^{20} + 2q^{19} + 5q^{18} - 9q^{16} + 16q^{15} - 37q^{14} + 44q^{13} - 17q^{12} \\
&\quad + 58q^{11} - 64q^{10} + 74q^9 - 100q^8 + 52q^7 - 7q^6 - 9q^5 + 23q^4 \\
&\quad - 26q^3 + q^2 - 7q + 2 , \\
\varphi_8(q) &= q^{22} + q^{21} + 2q^{20} - 4q^{19} - 9q^{18} + 11q^{16} - 4q^{15} + 37q^{14} - 32q^{13} \\
&\quad + 18q^{12} - 74q^{11} + 27q^{10} - 49q^9 + 45q^8 - 26q^7 + q^6 - 12q^5 \\
&\quad - 10q^4 + 2q^3 + q^2 + 2q , \\
\varphi_6(q) &= 2q^{23} - q^{22} - q^{21} - 3q^{20} - 7q^{19} + 7q^{18} + 11q^{17} - 6q^{16} - 8q^{15} \\
&\quad - 12q^{14} - 33q^{13} + 6q^{12} + 26q^{11} - 3q^{10} + 25q^9 - 21q^8 + 11q^7 \\
&\quad - 5q^6 + 8q^5 + 2q^4 + 2q^3 , \\
\varphi_4(q) &= -2q^{23} + 2q^{22} - 2q^{21} + 3q^{20} + 7q^{19} - 9q^{18} - 9q^{17} - 10q^{16} + 17q^{15} \\
&\quad - 7q^{14} + 46q^{13} - 7q^{12} + 8q^{11} + 6q^{10} - 5q^9 + 8q^8 + 2q^6 , \\
\varphi_2(q) &= -2q^{22} + 2q^{21} - 3q^{20} + 2q^{19} + 6q^{18} - 3q^{17} + 15q^{16} - 8q^{15} + 10q^{14} \\
&\quad - 11q^{13} + 2q^{12} - 8q^{11} - 2q^9 , \\
\varphi_0(q) &= 2q^{16} + q^{14} - 6q^{12} - 7q^{10} - 2q^8 .
\end{aligned}$$

For $m \geq 5$ and $q \geq 3$, we prove that $\sum_{i=k}^{8} \varphi_{2i}(q)q^{2(m-1)(i-k)} \geq 0$, with $1 \leq k \leq 8$. We denote $\overline{\varphi}_k(q) = \sum_{i=k}^{8} \varphi_{2i}(q)q^{8(i-k)}$. These are functions independent of m. It can be checked that $\overline{\varphi}_k(q) \geq 0$ for all $k = 1, \ldots, 8$. Now we prove $\sum_{i=k}^{8} \varphi_{2i}(q)q^{2(m-1)(i-k)} \geq \overline{\varphi}_k(q) \geq 0$. We use backward induction on k. For $k = 8$, we see that $\varphi_{16}(q) = \overline{\varphi}_8(q) \geq 0$. The induction step is proved by

$$\begin{aligned}
\sum_{i=k-1}^{8} \varphi_{2i}(q)q^{2(m-1)(i-k+1)} &= \varphi_{2k-2}(q) + q^{2m-2} \sum_{i=k}^{8} \varphi_{2i}(q)q^{2(m-1)(i-k)} \\
&\geq \varphi_{2k-2}(q) + q^{2m-2}\overline{\varphi}_k(q) \\
&\geq \varphi_{2k-2}(q) + q^{8}\overline{\varphi}_k(q) \\
&= \overline{\varphi}_{k-1}(q) \; .
\end{aligned}$$

The induction hypothesis was used in the penultimate step. The desired inequality now follows since both $\sum_{i=k}^{8} \varphi_{2i}(q)q^{2(m-1)(i-k)} \geq 0$ and $\varphi_0(q) \geq 0$. Indeed,

$$\begin{aligned}
0 &\leq q^{-4} \sum_{i=k}^{8} \varphi_{2i}(q)q^{2(m-1)(i-k)} + \varphi_0(q) \\
&= \varphi_{16}(q)q^{16m-20} + \varphi_{14}(q)q^{14m-18} + \varphi_{12}(q)q^{12m-16} + \varphi_{10}(q)q^{10m-14} \\
&\quad + \varphi_8(q)q^{8m-12} + \varphi_6(q)q^{6m-10} + \varphi_4(q)q^{4m-8} + \varphi_2(q)q^{2m-6} + \varphi_0(q) \; .
\end{aligned}$$

Above, we assumed $m \geq 5$. The cases $n = 2m = 6$ and $n = 2m = 8$ can be checked individually.

The arguments for $n \geq 7$ odd are completely equivalent. We shall not present the detailed calculations here. The case $n = 5$ however needs to be treated separately. Recall that the definition of $\overline{W}_5(q)$ differs from the definition of the other functions $\overline{W}_n(q)$.

Now, we turn to the case $q = 2$. Recall from Definition 8.4.1 that $\overline{W}_n(2)$ is defined in an other way than $\overline{W}_n(q)$ for $q \geq 3$. This is necessary since the inequality (A.1) is not valid in this case. First we look at the case n odd. It can be computed that

$$\frac{b_{2m+1}(2)(c_{2m+1}(2) + d_{2m+1}(2))}{a_{2m+1}(2)} = \frac{25}{21}2^{4m} + \frac{725}{882}2^{2m} - \frac{27050}{3087} - \frac{50}{3087}\frac{\psi_1(m)}{\psi_2(m)} \, ,$$

with

$$\psi_1(m) = 104704 \cdot 2^{10m} + 649583 \cdot 2^{8m} + 952350 \cdot 2^{6m} - 2203672 \cdot 2^{4m} - 7423808 \cdot 2^{2m} - 13458432 \,,$$

$$\psi_2(m) = (7 \cdot 2^{4m} - 6 \cdot 2^{2m} - 96)(2^{4m} + 19 \cdot 2^{2m} + 40)(2^{4m} - 2 \cdot 2^{2m} - 8) \,.$$

Now, it can be observed that $\psi_1(m), \psi_2(m) \geq 0$ if $m \geq 3$. Hence,

$$\begin{aligned}\frac{b_{2m+1}(2)(c_{2m+1}(2) + d_{2m+1}(2))}{a_{2m+1}(2)} &\leq \frac{25}{21}2^{4m} + \frac{725}{882}2^{2m} - \frac{27050}{3087} \\ &= \frac{25}{84}2^{2n} + \frac{725}{1764}2^{n} - \frac{27050}{3087} \\ &= \overline{W}_n(2) \,.\end{aligned}$$

Now we look at the case n even. It can be computed that

$$\frac{b_{2m}(2)(c_{2m}(2) + d_{2m}(2))}{a_{2m}(2)} = \frac{25}{84}2^{4m} + \frac{2425}{1764}2^{2m} - \frac{172225}{6174} + \frac{50}{3087}\frac{\psi_3(m)}{\psi_4(m)} \,,$$

with

$$\psi_3(m) = 49829 \cdot 2^{10m} + 322888 \cdot 2^{8m} - 8747088 \cdot 2^{6m} - 112697216 \cdot 2^{4m} - 238621696 \cdot 2^{2m} + 3183771648 \,,$$

$$\psi_4(m) = (2^{4m} + 16 \cdot 2^{2m} + 160)(7 \cdot 2^{4m} + 12 \cdot 2^{2m} - 384)(2^{4m} + 4 \cdot 2^{2m} - 32) \,.$$

Since $\psi_3(m), \psi_4(m) \geq 0$ if $m \geq 3$, we know that

$$\frac{b_{2m}(2)(c_{2m}(2) + d_{2m}(2))}{a_{2m}(2)} > \frac{25}{84}2^{4m} + \frac{2425}{1764}2^{2m} - \frac{172225}{6174}$$

for $m \geq 3$. However, $\frac{50}{3087}\frac{\psi_3(m)}{\psi_4(m)}$, the difference between the left-hand side and the right-hand side of this equation approaches 0 as m approaches infinity. Moreover, this difference is a decreasing function on $[3\,,\infty[$. Consequently,

$$\begin{aligned}\frac{b_{2m}(2)(c_{2m}(2) + d_{2m}(2))}{a_{2m}(2)} &\leq \frac{25}{84}2^{4m} + \frac{2425}{1764}2^{2m} - \frac{172225}{6174} + \frac{50}{3087}\frac{\psi_3(3)}{\psi_4(3)} \\ &= \frac{25}{84}2^{4m} + \frac{2425}{1764}2^{2m} - \frac{2655125}{100107} \\ &= \frac{25}{84}2^{2n} + \frac{2425}{1764}2^{n} - \frac{2655125}{100107} \\ &= \overline{W}_n(2) \,.\end{aligned}$$

So, we have checked that the inequality (A.3.1) is valid in all cases.

A.4 Generators on $\mathcal{H}(2n+1, q^2)$

Computation A.4.1. In Lemma 9.3.5 we compare the values

$$\begin{aligned}
A_{n,i}(q) &= N_{n-1,i-1}(q)q^{4i}\left(\mu_{n-i-1}(q^2)\right)^2 \ , \\
B_{n,i}(q) &= N_{n-1,i}(q)q^{4i+4}\left(\mu_{n-i-2}(q^2)\right)^2 \ , \\
C_{n,i}(q) &= N_{n-1,i+1}(q)q^{4i+4}\mu_{n-i-3}(q^2)\left[q^4\mu_{n-i-3}(q^2)+q^2-1\right] \text{ and} \\
D_{n,i}(q) &= N_{n,i}(q)q^{2i+2} \ ,
\end{aligned}$$

given that $i \leq n-2$. Recall that $\mu_j(q^2)$ is the number of points on $\mathcal{H}_j(q^2)$.

For $i = n-2$, we only need to compare $A_{n,i}(q)$ and $D_{n,i}(q)$. We find that

$$\begin{aligned}
A_{n,n-2}(q) &= N_{n-1,n-3}(q)q^{4(n-2)}(q+1)^2 \\
&= N_{n-1,n-3}(q)q^{4n-8}(q+1)^2 \ , \\
D_{n,n-2}(q) &= N_{n,n-2}(q)q^{2(n-2)+2} \\
&= q^{2n-3}N_{n-1,n-3}(q)q^{2n-2} \\
&= N_{n-1,n-3}(q)q^{4n-8}q^3 \ .
\end{aligned}$$

Here we used the relation $N_{n,i}(q) = q^{2n-3}N_{n-1,i-1}(q)$ which was derived in the proof of Lemma 9.2.7. We obtain that $A_{n,n-2}(q) \leq D_{n,n-2}(q)$ if and only if $q \geq 3$.

For $i = n-3$, we need to compare $A_{n,i}(q)$, $B_{n,i}(q)$ and $D_{n,i}(q)$. We find that

$$\begin{aligned}
A_{n,n-3}(q) &= N_{n-1,n-4}(q)q^{4(n-3)}(q^3+1)^2 \\
&= q^{(n-2)^2-1}(q+1)q^{4n-12}\left(q^3+1\right)^2 \\
&= \left(q^{n^2-9}(q+1)\right)(q^3+1)^2 \ , \\
B_{n,n-3}(q) &= N_{n-1,n-3}(q)q^{4(n-2)}(q+1)^2 \\
&= q^{(n-2)^2}q^{4n-8}(q+1)^2 \\
&= \left(q^{n^2-9}(q+1)\right)q^5(q+1) \ , \\
D_{n,n-3}(q) &= N_{n,n-3}(q)q^{2(n-3)+2} \\
&= q^{(n-1)^2-1}(q+1)q^{2n-4} \\
&= \left(q^{n^2-9}(q+1)\right)q^5 \ .
\end{aligned}$$

We obtain that $B_{n,n-3}(q) \geq A_{n,n-3}(q) \geq D_{n,n-3}(q)$.

Now we deal with the case $i \leq n-4$. First note that

$$A_{n,i}(q) = \left(\prod_{j=1}^{n-i-4} (q^j - (-1)^j) \frac{q^{(n-2)^2 - \binom{n-i-1}{2} + 4i}}{(q^2-1)^2} \right) a_{n-i}(q) \ ,$$
$$B_{n,i}(q) = \left(\prod_{j=1}^{n-i-4} (q^j - (-1)^j) \frac{q^{(n-2)^2 - \binom{n-i-1}{2} + 4i}}{(q^2-1)^2} \right) b_{n-i}(q) \ ,$$
$$C_{n,i}(q) = \left(\prod_{j=1}^{n-i-4} (q^j - (-1)^j) \frac{q^{(n-2)^2 - \binom{n-i-1}{2} + 4i}}{(q^2-1)^2} \right) c_{n-i}(q) \ ,$$

with

$$\begin{aligned} a_m(q) &= (q^m - (-1)^m)^2 \left(q^{m-1} - (-1)^{m-1}\right)^2 \cdot \\ &\qquad \left(q^{m-2} - (-1)^{m-2}\right) \left(q^{m-3} - (-1)^{m-3}\right) \ , \\ b_m(q) &= q^{m+2} \left(q^{m-1} - (-1)^{m-1}\right)^2 \left(q^{m-2} - (-1)^{m-2}\right)^2 \left(q^{m-3} - (-1)^{m-3}\right) \ , \\ c_m(q) &= q^{2m-1} \left(q^{m-2} - (-1)^{m-2}\right) \left(q^{m-3} - (-1)^{m-3}\right) \cdot \\ &\qquad \left(q^4 \left(q^{m-2} - (-1)^{m-2}\right) \left(q^{m-3} - (-1)^{m-3}\right) + \left(q^2-1\right)^2\right) \ . \end{aligned}$$

We look at the difference $b_m(q) - a_m(q)$. We can assume $m \geq 4$ since $n-i \geq 4$. If m is even, we find that

$$\begin{aligned} b_m(q) - a_m(q) = &-q^{5m-8}\varphi_5(q) + q^{4m-7}\varphi_4(q) - q^{3m-6}\varphi_3(q) + q^{2m-5}\varphi_2(q) \\ &- q^{m-4}\varphi_1(q) + 1 \ , \end{aligned}$$

with

$$\begin{aligned} \varphi_5(q) &= q^3 - 2q \ , \\ \varphi_4(q) &= -q^5 + q^4 - 2q^2 + 4q - 1 \ , \\ \varphi_3(q) &= -q^6 + 2q^5 - 3q^3 + 5q^2 - 3q + 2 \ , \\ \varphi_2(q) &= 2q^6 - q^5 - 3q^4 + 3q^3 - 4q^2 + 2q - 1 \ , \\ \varphi_1(q) &= -q^6 + 2q^4 - 2q^3 + q^2 - q \ . \end{aligned}$$

Now, it can be checked that the five functions

$$\chi_i(q) = \sum_{j=i}^{5} (-1)^j \varphi_j(q) q^{3(j-i)} \,, \quad i = 2, \ldots, 5 \,, \quad \text{and}$$

$$\chi_1(q) = \sum_{j=i}^{5} (-1)^j \varphi_j(q) q^{3(j-i)} + 1$$

are negative for $q \geq 2$. It follows that

$$\begin{aligned}
&q^{(i+1)(m-1)-3}\chi_{i+1}(q) + \sum_{j=1}^{i} (-1)^j \varphi_j(q) q^{j(m-1)-3} \\
&= q^{(i+1)(m-1)-3}\chi_{i+1}(q) + (-1)^i \varphi_i(q) q^{i(m-1)-3} + \sum_{j=1}^{i-1} (-1)^j \varphi_j(q) q^{j(m-1)-3} \\
&= q^{i(m-1)-3}(q^{m-1}\chi_{i+1}(q) + (-1)^i \varphi_i(q)) + \sum_{j=1}^{i-1} (-1)^j \varphi_j(q) q^{j(m-1)-3} \\
&\leq q^{i(m-1)-3}(q^{3}\chi_{i+1}(q) + (-1)^i \varphi_i(q)) + \sum_{j=1}^{i-1} (-1)^j \varphi_j(q) q^{j(m-1)-3} \\
&= q^{i(m-1)-3}\chi_{i}(q) + \sum_{j=1}^{i-1} (-1)^j \varphi_j(q) q^{j(m-1)-3} \,,
\end{aligned}$$

for $i = 2, \ldots, 5$, with $\chi_6(q) = 0$. We used that $q^{m-1} \geq q^3$ and $\chi_i(q) \leq 0$. Hence, applying this rule four times, we find

$$\begin{aligned}
b_m(q) - a_m(q) &= \sum_{j=1}^{5} (-1)^j q^{j(m-1)-3} \varphi_j(q) + 1 \\
&= q^{6(m-1)-3}\chi_6(q) + \sum_{j=1}^{5} (-1)^j q^{j(m-1)-3} \varphi_j(q) + 1 \\
&\leq q^{5(m-1)-3}\chi_5(q) + \sum_{j=1}^{4} (-1)^j \varphi_j(q) q^{j(m-1)-3} + 1 \\
&\leq q^{4(m-1)-3}\chi_4(q) + \sum_{j=1}^{3} (-1)^j \varphi_j(q) q^{j(m-1)-3} + 1
\end{aligned}$$

$$
\begin{aligned}
&\le q^{3(m-1)-3}\chi_3(q) + \sum_{j=1}^{2}(-1)^j \varphi_j(q) q^{j(m-1)-3} + 1 \\
&\le q^{2(m-1)-3}\chi_2(q) - \varphi_1(q) q^{(m-1)-3} + 1 \\
&\le q^{(m-1)-3}\left(q^3\chi_2(q) - \varphi_1(q)\right) + 1 \\
&\le \chi_1(q) \\
&< 0 \,.
\end{aligned}
$$

Consequently, $b_m(q) < a_m(q)$ if $m \ge 4$ is even. If $m \ge 4$ is odd, then

$$
\begin{aligned}
b_m(q) - a_m(q) = {}& q^{5m-8}\varphi_5(q) + q^{4m-7}\varphi_4(q) + q^{3m-6}\varphi_3(q) + q^{2m-5}\varphi_2(q) \\
&+ q^{m-4}\varphi_1(q) + 1 \,,
\end{aligned}
$$

Arguing in the same way, we find that $b_m(q) > a_m(q)$. This time, it should be observed that

$$
\sum_{j=i}^{5} \varphi_j(q) q^{3(j-i)} > 0 \,, \qquad i = 1, \ldots, 5 \,,
$$

for all $q \ge 2$.

Now, we look at the difference $c_m(q) - a_m(q)$. We can still assume that $m \ge 4$. If m is even, then

$$
\begin{aligned}
c_m(q) - a_m(q) = {}& q^{5m-8}\psi_5(q) + q^{4m-7}\psi_4(q) + q^{3m-6}\psi_3(q) + q^{2m-5}\psi_2(q) \\
&+ q^{m-4}\psi_1(q) + 1 \,,
\end{aligned}
$$

with

$$
\begin{aligned}
\psi_5(q) &= q^4 - q^3 - 2q^2 + 2q \,, \\
\psi_4(q) &= q^6 - 2q^5 - q^4 + 2q^3 - 3q^2 + 5q - 1 \,, \\
\psi_3(q) &= -q^7 + q^6 - q^4 + 6q^3 - 6q^2 + 3q - 2 \,, \\
\psi_2(q) &= 2q^6 + q^5 - 5q^4 + 3q^3 - 4q^2 + 2q - 1 \,, \\
\psi_1(q) &= -2q^3 + 2q^2 - q + 1 \,.
\end{aligned}
$$

If m is odd, then

$$
\begin{aligned}
c_m(q) - a_m(q) = {}& -q^{5m-8}\psi_5(q) + q^{4m-7}\psi_4(q) - q^{3m-6}\psi_3(q) + q^{2m-5}\psi_2(q) \\
&- q^{m-4}\psi_1(q) + 1 \,.
\end{aligned}
$$

Proceeding in the same way as for the difference $b_m(q) - a_m(q)$, we can see that $c_m(q) - a_m(q) > 0$ for $q \geq 2$, if $m \geq 4$ is even, and that $c_m(q) - a_m(q) < 0$ for $q \geq 2$, if $m \geq 4$ is odd.

We conclude that $c_m(q) > a_m(q) > b_m(q)$ if $m \geq 4$ is even, and $b_m(q) > a_m(q) > c_m(q)$ if $m \geq 4$ is odd, both for $q \geq 2$. This leads to the inequalities on $A_{n,i}$, $B_{n,i}$ and $C_{n,i}$ that are mentioned in Lemma 9.3.5.

B
Nederlandstalige samenvatting

Waarom ik met Van Maerlant juich,
zijn leuze voor de mijne erken.

Lambrecht Lambrechts, *Omdat ik Vlaming ben.*

In deze appendix geven we een Nederlandstalige samenvatting van dit proefschrift. We vermelden enkel de belangrijkste begrippen en resultaten. Voor de details verwijzen we naar de Engelstalige tekst, waarvan we de structuur zullen volgen. De verschillende secties komen overeen met de hoofdstukken in dit proefschrift. We gaan ervan uit dat de lezer vertrouwd is met een aantal wiskundige basisbegrippen.

B.1 Inleiding

In de inleiding worden verschillende *incidentiemeetkundes* en substructuren van incidentiemeetkundes beschreven. Deze meetkundes hebben deelruimtes van verschillende dimensies, zoals *punten*, *rechten*, *vlakken*, ... en *hypervlakken*.

Een $t-(v,k,\lambda)$ *design*, $v > k > 1$, $k \geq t \geq 1$, $\lambda > 0$, is een meetkunde met v punten, zodat iedere rechte (blok) k punten bevat, en zodat elke verzameling van t verschillende punten in precies λ rechten is bevat. Het aantal rechten door een vast punt wordt genoteerd als r. Designs met $\lambda = 1$ worden *Steiner designs* genoemd. We vermelden enkele belangrijke Steiner 2-designs. De $2-(n^2+n+1, n+1, 1)$ designs zijn de *axiomatisch projectieve vlakken* van orde n; de $2-(n^2, n, 1)$ designs zijn de *axiomatisch affiene vlakken* van orde n. De $2-(n^3+1, n+1, 1)$ designs worden *unitalen* genoemd en de Steiner 2-designs met $k = 3$, worden *Steiner 3-systemen* genoemd.

Een *projectieve ruimte* $\mathrm{PG}(n, \mathbb{F})$ van dimensie n over een veld $\mathbb{F}$ is de meetkunde van de deelruimtes van een $(n+1)$-dimensionale vectorruimte over $\mathbb{F}$. In dit proefschrift behandelen we voornamelijk eindige projectieve ruimtes, i.e. projectieve ruimtes over een eindig veld $\mathbb{F}_q$. De n-dimensionale projectieve ruimte over $\mathbb{F}_q$ wordt genoteerd als $\mathrm{PG}(n, q)$. De deelruimtes van de onderliggende vectorruimte induceren deelruimtes in de projectieve ruimte. De *projectieve dimensie* van een deelruimte is één minder dan de vectoriële dimensie van deze deelruimte. De k-dimensionale deelruimtes van een projectieve ruimte noemen we kortweg *k-ruimtes*. In projectieve ruimtes bestaat het principe van *dualiteit*. Er bestaan afbeeldingen die k-ruimtes afbeelden op $(n-k-1)$-ruimtes zodat incidentie behouden blijft. Dit laat ons toe om objecten en stellingen te dualiseren. Het aantal punten in $\mathrm{PG}(n, q)$ is gelijk aan $\frac{q^{n+1}-1}{q-1} = \theta_n(q)$. Het aantal k-ruimtes in $\mathrm{PG}(n, q)$ wordt gegeven door de *Gaussische coëfficiënt* $\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q$.

De *affiene ruimte* $\mathrm{AG}(n, \mathbb{F})$ is de meetkunde die ontstaat door in de projectieve ruimte $\mathrm{PG}(n, \mathbb{F})$ een hypervlak H_∞ met alle deelruimtes die erin liggen te verwijderen. Men noemt dit hypervlak ook wel *'het hypervlak op oneindig'*. De k-ruimtes van $\mathrm{PG}(n, \mathbb{F})$ die niet in H_∞ liggen, induceren k-ruimtes in $\mathrm{AG}(n, \mathbb{F})$. Iedere affiene k-ruimte bepaalt een $(k-1)$-ruimte in dit hypervlak op oneindig. Twee k-ruimtes door eenzelfde $(k-1)$-ruimte op oneindig zijn *parallel*.

Een *klassieke polaire ruimte*[1] is de meetkunde van de totaal isotrope deelruimtes van een vectorruimte, ten opzichte van een niet-singuliere kwadratische, Hermitische of symplectische vorm. Totaal isotrope deelruimtes zijn deelruimtes waarop de vorm reduceert tot de nulvorm. De deelruimtes van maximale dimensie van een klassieke polaire ruimte noemen we *generatoren* en hun vectoriële dimensie de *rang* van de klassieke polaire ruimte. Als gevolg van de rechtstreekse identificatie van vectorruimtes met projectieve ruimtes, kunnen we de klassieke polaire ruimtes beschouwen als deelstructuren van projectieve ruimtes.

De eindige klassieke polaire ruimtes zijn de klassieke polaire ruimtes opgebouwd vanuit een vectorruimte over een eindig veld $\mathbb{F}_q$. We onderscheiden zes types van eindige klassieke polaire ruimtes van rang r: de *hyperbolische kwadrieken* $\mathcal{Q}^+(2r-1,q)$ ingebed in $\mathrm{PG}(2r-1,q)$, de *parabolische kwadrieken* $\mathcal{Q}(2r,q)$ ingebed in $\mathrm{PG}(2r,q)$, de *elliptische kwadrieken* $\mathcal{Q}^-(2r+1,q)$ ingebed in $\mathrm{PG}(2r+1,q)$, de *Hermitische polaire ruimtes* $\mathcal{H}(2r-1,q^2)$ ingebed in $\mathrm{PG}(2r-1,q^2)$, de *Hermitische polaire ruimtes* $\mathcal{H}(2r,q^2)$ ingebed in $\mathrm{PG}(2r,q^2)$ en de *symplectische polaire ruimtes* $\mathcal{W}(2r-1,q)$ ingebed in $\mathrm{PG}(2r-1,q)$.

Als we op dezelfde wijze als bij klassieke polaire ruimtes, een meetkunde opbouwen vanuit een kwadratische, Hermitische of symplectische vorm die niet noodzakelijk niet-singulier is, vinden we een *kwadratische, Hermitische of symplectische variëteit*. Als de vorm niet-singulier is, vinden we *niet-singuliere variëteiten*, de klassieke polaire ruimtes. Als de vorm singulier is, vinden we *kegels* met een deelruimte van de projectieve ruimte als top en een klassieke polaire ruimte (een niet-singuliere variëteit) als basis. De kwadratische en Hermitische variëteiten kunnen we identificeren met hun puntenverzamelingen.

Een (k,t)-*boog* in $\mathrm{PG}(2,q)$ is een verzameling van k punten in $\mathrm{PG}(2,q)$ zodat elke rechte hoogstens t punten van de boog bevat. Een $(k,2)$-boog wordt kortweg een k-*boog* genoemd. We weten dat k-bogen in $\mathrm{PG}(2,q)$ enkel bestaan voor $k \leq q+2$ als q even is, en voor $k \leq q+1$ als q oneven is. Een $(q+2)$-boog in $\mathrm{PG}(2,q)$, q even, wordt een *hyperovaal* genoemd. Als iedere rechte een $(q+t,t)$-boog in $\mathrm{PG}(2,q)$ in precies 0, 2 of t punten snijdt, dan noemen we deze boog een $(q+t,t)$-*boog van type* $(0,2,t)$. Deze kunnen enkel bestaan als q even is en $t \mid q$.

Een *blokkerende verzameling* ten opzichte van de k-ruimtes in $\mathrm{PG}(n,q)$ is een

[1] De klassieke polaire ruimtes vormen een speciale, maar zeer belangrijke klasse van de *polaire ruimtes*. Deze zullen we hier niet algemeen introduceren.

puntenverzameling in $\mathrm{PG}(n, q)$ zodat elke k-ruimte minstens één punt van de verzameling bevat. Een blokkerende verzameling wordt *minimaal* genoemd als ze geen enkele strikte deelverzameling bevat, die zelf een blokkerende verzameling is. De kleinste blokkerende verzameling ten opzichte van de k-ruimtes in $\mathrm{PG}(n, q)$, is de verzameling van alle punten in één $(n-k)$-ruimte. Deze blokkerende verzameling wordt de *triviale* blokkerende verzameling genoemd. Een blokkerende verzameling in $\mathrm{PG}(n, q)$ ten opzichte van de k-ruimtes, die hoogstens $\frac{3}{2}(q^{n-k}+1)$ punten bevat, wordt *klein* genoemd.

Een *partiële t-spread* in $\mathrm{PG}(n, q)$ is een verzameling van t-ruimtes die paarsgewijs disjunct zijn. Als deze verzameling geen strikte deelverzameling is van een andere partiële spread, dan noemen we de partiële spread maximaal.

Naast al de voorgaande meetkundige begrippen voeren we ook enkele begrippen uit de codeertheorie in. Een *lineaire code* van lengte n over $\mathbb{F}_q$ is een deelvectorruimte van de vectorruimte $V(n, q)$. Als de lineaire code een k-dimensionale deelruimte is, noemen we hem een *$[n, k]$-code*. De vectoren in deze deelruimte noemen we *codewoorden*. Het *gewicht* van een codewoord is zijn aantal coördinaten verschillend van 0. Het *minimum gewicht* van een code is dan het minimum van de gewichten van de codewoorden van de code, de nulvector buiten beschouwing gelaten. Als er een getal bestaat dat het gewicht van ieder codewoord in de code deelt, dan noemen we dit getal een *deler* van de code.

Aangezien een lineaire code een deelvectorruimte is, kunnen we het orthogonaal complement van de lineaire code bekijken, gedefinieerd op basis van het standaard inwendig product. Dit orthogonaal complement is ook een lineaire code en deze wordt de *duale code* genoemd.

We beschrijven twee methodes die codes creëren op basis van meetkundige structuren. Een *incidentievector* van een k-ruimte van een projectieve of polaire ruimte is een vector waarvan de posities worden geïndexeerd door de puntenverzameling van de meetkunde, en waarvan een element gelijk is aan 1, respectievelijk aan 0, als het punt corresponderend met de positie waarop het element staat, bevat is in de k-ruimte. Een *incidentiematrix* van punten en k-ruimtes van een projectieve of polaire ruimte is een matrix waarvan de kolommen worden geïndexeerd door de puntenverzameling van de meetkunde, en waarvan de rijen de incidentievectoren van de k-ruimtes zijn. De lineaire code $C_t(n, q)$ is dan de deelvectorruimte voortgebracht door de rijen van een incidentiematrix van de t-ruimtes in $\mathrm{PG}(n, q)$.

De andere methode vertrekt van een algebraïsche variëteit in $\mathrm{PG}(n, q)$, een puntenverzameling beschreven door een algebraïsche vergelijking, en een verzameling homogene veeltermen over $\mathbb{F}_q$ in $n+1$ variabelen, gesloten onder lineaire combinatie. We geven de punten in de puntenverzameling een vaste volgorde en normaliseren hun coördinaten ten opzichte van de meest linkse coördinaat die verschilt van 0. Elke veelterm in de gekozen verzameling correspondeert nu met één codewoord, door de veelterm te evalueren in elk van de punten (gezien als coördinaten van een vector), de gekozen volgorde respecterend. De codes die zo ontstaan worden *functionele codes* genoemd. Een belangrijke observatie leert ons dat het aantal nulcoördinaten in een codewoord gelijk is aan het aantal punten in de doorsnede van de gekozen algebraïsche variëteit en de algebraïsche variëteit gedefinieerd door de veelterm die overeenkomt met het codewoord. Dit is een belangrijk hulpmiddel bij het bepalen van het minimum gewicht van deze codes.

B.2 Erdős-Ko-Rado problemen

In een invloedrijk artikel bestudeerden Erdős, Ko en Rado families van deelverzamelingen van grootte k in een verzameling met n elementen, zodat twee deelverzamelingen in deze familie altijd minstens één element gemeen hebben. De centrale vraag was hoe groot zo een familie kan zijn, en hoe ze kan beschreven worden, gegeven dat de maximale grootte wordt bereikt. Naderhand werden dergelijke families van deelverzamelingen *Erdős-Ko-Rado verzamelingen* genoemd. Dit concept werd later uitgebreid tot andere structuren (verschillend van eindige verzamelingen). Ook werd de vraag verbreed, tot het algemene *Erdős-Ko-Rado probleem*: de classificatie van de (maximale) Erdős-Ko-Rado verzamelingen. We overlopen hier de bekende resultaten voor eindige verzamelingen, eindige projectieve ruimtes en eindige klassieke polaire ruimtes.

Een Erdős-Ko-Rado verzameling van deelverzamelingen van grootte k in een eindige verzameling van grootte n werd hierboven al gedefinieerd. De volgende stelling vat resultaten van Erdős, Ko en Rado, en van Wilson samen.

Stelling B.2.1. *Beschouw een Erdős-Ko-Rado verzameling $\mathcal{S}$ van deelverzamelingen van grootte k in een verzameling van grootte n. Als $n \geq 2k$, dan is $|\mathcal{S}| \leq \binom{n-1}{k-1}$. Als $n \geq 2k+1$ en $|\mathcal{S}| = \binom{n-1}{k-1}$, dan is $\mathcal{S}$ de verzameling van alle deelverzamelingen van grootte k die een vast element bevatten.*

Hilton en Milner bewezen een bovengrens op de grootte van een Erdős-Ko-

Rado verzameling van deelverzamelingen van grootte $k \geq 3$ in een verzameling met $n \geq 2k+1$ elementen, waarvoor geen enkel van deze n elementen van de verzameling tot alle deelverzamelingen van de Erdős-Ko-Rado verzameling behoort, en classificeerden de Erdős-Ko-Rado verzamelingen die deze bovengrens bereiken.

Een *Erdős-Ko-Rado verzameling van k-ruimtes*, kortweg een *EKR(k) verzameling*, in $\mathrm{PG}(n,q)$, is een verzameling k-ruimtes in $\mathrm{PG}(n,q)$ die paarsgewijs minstens één punt gemeen hebben. Een EKR(k) verzameling wordt *maximaal* genoemd als ze geen strikte deelverzameling is van een andere EKR(k) verzameling. Het Erdős-Ko-Rado probleem vraagt dan om de maximale Erdős-Ko-Rado verzamelingen te classificeren. Typisch probeert men alle Erdős-Ko-Rado verzamelingen die minstens s elementen bevatten te classificeren. De volgende stelling combineert resultaten van Frankl en Wilson, en van Tanaka.

Stelling B.2.2. *Een EKR(k) verzameling in* $\mathrm{PG}(n,q)$, $n \geq 2k+1$, *bevat hoogstens* $\begin{bmatrix} n \\ k \end{bmatrix}_q$ *elementen. Als* $\mathcal{S}$ *een EKR(k) verzameling in* $\mathrm{PG}(n,q)$ *is,* $n \geq 2k+1$, *die deze grens bereikt, dan is* $\mathcal{S}$ *de verzameling van alle k-ruimtes door een vast punt of is* $n = 2k+1$ *en is* $\mathcal{S}$ *de verzameling van alle k-ruimtes in een hypervlak.*

We merken hierbij op dat het Erdős-Ko-Rado probleem voor k-ruimtes in $\mathrm{PG}(n,q)$ triviaal is als $n \leq 2k$. Blokhuis *et al.* bewezen een projectief equivalent van het resultaat van Hilton en Milner. Ze classificeerden in de meeste gevallen de tweede grootste maximale Erdős-Ko-Rado verzamelingen.

We merken ook op dat het Erdős-Ko-Rado probleem in $\mathrm{PG}(n,q)$, $n \geq 3$, triviaal is voor $k = 1$. In dit geval zijn er slechts twee maximale Erdős-Ko-Rado verzamelingen, namelijk de verzameling van alle rechten door een punt en de verzameling van alle rechten in een vlak.

Een *Erdős-Ko-Rado verzameling van k-ruimtes*, kortweg een *EKR(k) verzameling*, in een klassieke polaire ruimte, is een verzameling k-ruimtes in die polaire ruimte die paarsgewijs minstens één punt gemeen hebben. Net zoals hierboven wordt ze *maximaal* genoemd als ze geen strikte deelverzameling van een andere EKR(k) verzameling is. De grootste Erdős-Ko-Rado verzamelingen van generatoren werd geclassificeerd door Pepe, Storme en Vanhove, voor alle eindige klassieke polaire ruimtes, behalve voor de Hermitische polaire ruimtes $\mathcal{H}(4n+1,q^2)$, $n \geq 2$. Voor deze klassieke polaire ruimtes werd de beste bovengrens op de grootte van de Erdős-Ko-Rado verzamelingen van generatoren bewezen door Ihringer en Metsch.

We merken op dat de classificatie van de EKR(1) verzamelingen triviaal is, net zoals bij projectieve ruimtes. Over EKR(k) verzamelingen in een klassieke polaire ruimte van rang r, met $2 < k < r - 1$, is niets geweten. In de volgende sectie komt het geval $k = 2$ aan bod.

B.3 Erdős-Ko-Rado verzamelingen van vlakken

Gezien de gekende resultaten in verband met het Erdős-Ko-Rado probleem, voorgesteld in de vorige sectie, is het niet onlogisch om de EKR(2) verzamelingen te bestuderen. Blokhuis, Brouwer en Szőnyi classificeerden de zes grootste voorbeelden van maximale EKR(2) verzamelingen in $\mathrm{PG}(5, q)$.

Eerst introduceren we *elf types van maximale EKR(2) verzamelingen*, zowel voor $\mathrm{PG}(n, q)$, $n \geq 5$, als voor eindige klassieke polaire ruimtes van rang $d \geq 3$, met volgnummers I tot XI. Deze types komen voor in een projectieve en een polaire variant (soms 2 polaire varianten). We geven deze lijst hier niet, maar verwijzen daarvoor naar Sectie 3.1.1. Sommige types komen niet voor op bepaalde eindige klassieke polaire ruimtes van lage rang. De maximale EKR(2) verzamelingen van deze elf types hebben gemeen dat ze niet noodzakelijk bevat zijn in een 5-ruimte van de (omhullende) n-dimensionale projectieve ruimte. Meestal spannen ze de volledige ruimte, een $(n-1)$-ruimte, een $(n-2)$-ruimte of een 6-ruimte op. In Sectie 3.1.2 geven we dan een lijst van *zeven andere types van maximale EKR(2) verzamelingen*, die wel steeds in een 5-ruimte van de (omhullende) projectieve ruimte bevat zijn, ongeacht de projectieve of klassieke polaire ruimte waarin ze zijn ingebed. Deze worden genummerd van XII tot XVIII. De hoofdstelling van dit gedeelte is het volgende resultaat.

Stelling B.3.1. *Als $\mathcal{S}$ een maximale EKR(2) verzameling is, die bevat is in een projectieve ruimte $\mathrm{PG}(n, q)$, $n \geq 5$, of in een eindige klassieke polaire ruimte van rang $d \geq 3$, dan is $\mathcal{S}$ van type I, II, ..., XI of is $\mathcal{S}$ bevat in een 5-ruimte van de (omhullende) projectieve ruimte.*

Deze stelling laat ons toe om een classificatie te maken van de grootste maximale EKR(2) verzamelingen voor eindige projectieve ruimtes en eindige klassieke polaire ruimtes van rang minstens 6.

Stelling B.3.2.

- *Een maximale EKR(2) verzameling in* $\mathrm{PG}(n,q)$*,* $n \geq 5$*, die minstens* $3q^4+3q^3+2q^2+q+1$ *elementen bevat, behoort tot één van de beschreven types. Als* $n = 5, 6$*, dan zijn er zes mogelijke types; als* $n \geq 7$*, dan zijn er tien mogelijke types.*
- *Een maximale EKR(2) verzameling in een klassieke polaire ruimte* $\mathcal{P}$ *van rang* $d \geq 6$*, ingebed in een projectieve ruimte* $\mathrm{PG}(n,q)$*, die minstens* $3q^4+3q^3+2q^2+q+1$ *elementen bevat, behoort tot één van de beschreven types. Als* $\mathcal{P}$ *een hyperbolische kwadriek is, of als* $\mathcal{P}$ *een symplectische polaire ruimte met* q *even is, dan zijn er tien mogelijke types; als* $\mathcal{P}$ *een symplectische polaire ruimte met* q *oneven is, dan zijn er elf mogelijke types; als* $\mathcal{P}$ *een Hermitische polaire ruimte is, dan zijn er twaalf mogelijke types.*

In Sectie 3.3 staat precies beschreven welke de verschillende mogelijke types zijn in elk van deze gevallen en welke de grootste, tweede grootste, enz. onder hen is.

Voor eindige klassieke polaire ruimtes van lage rang laat de hoofdstelling ons zelfs toe om een betere classificatie te bekomen, voor kwadrieken en symplectische polaire ruimtes zelfs een complete classificatie.

Stelling B.3.3. *Een maximale EKR(2) verzameling in een klassieke polaire ruimte* $\mathcal{P}$ *van rang* $3 \leq d \leq 5$*, ingebed in een projectieve ruimte* $\mathrm{PG}(n,q)$*, behoort tot één van de beschreven types.*

- *Veronderstel dat* $\mathcal{P}$ *een kwadriek is. Als* $\mathcal{P}$ *van rang* 3 *is, dan zijn er drie mogelijke types; als* $\mathcal{P}$ *van rang* 4 *is, dan zijn er vijf mogelijke types; als* $\mathcal{P}$ *hyperbolisch en van rang* 5 *is, dan zijn er elf mogelijke types; als* $\mathcal{P}$ *elliptisch of parabolisch, en van rang* 5 *is, dan zijn er twaalf mogelijke types.*
- *Veronderstel dat* $\mathcal{P}$ *een symplectische polaire ruimte is, met* q *even. Als* $\mathcal{P}$ *van rang* 3 *is, dan zijn er drie mogelijke types; als* $\mathcal{P}$ *van rang* 4 *is, dan zijn er vijf mogelijke types; als* $\mathcal{P}$ *van rang* 5 *is, dan zijn er twaalf mogelijke types.*
- *Veronderstel dat* $\mathcal{P}$ *een symplectische polaire ruimte is, met* q *oneven. Als* $\mathcal{P}$ *van rang* 3 *is, dan zijn er vijf mogelijke types; als* $\mathcal{P}$ *van rang* 4 *is, dan zijn er zeven mogelijke types; als* $\mathcal{P}$ *van rang* 5 *is, dan zijn er veertien mogelijke types.*

- *Veronderstel dat $\mathcal{P}$ een Hermitische polaire ruimte is, en dat de maximale EKR(2) verzameling minstens $q^2\sqrt{q}+q\sqrt{q}+\sqrt{q}+1$ elementen bevat. Als $\mathcal{P}=\mathcal{H}(5,q)$, dan is er één mogelijk type; als $\mathcal{P}=\mathcal{H}(6,q)$, dan zijn er twee mogelijke types; als $\mathcal{P}$ van rang 4 is, dan zijn er vijf mogelijke types; als $\mathcal{P}$ van rang 5 is, dan zijn er twaalf mogelijke types.*

B.4 Erdős-Ko-Rado verzamelingen van generatoren op $\mathcal{Q}^+(4n+1,q)$

We vermeldden hierboven al dat voor Erdős-Ko-Rado verzamelingen in eindige projectieve ruimtes in het algemeen het grootste en het tweede grootste voorbeeld geclassificeerd werden. Voor Erdős-Ko-Rado verzamelingen in eindige klassieke polaire ruimtes is hoogstens het grootste voorbeeld geclassificeerd. In dit gedeelte beschrijven we een classificatie van de tweede grootste Erdős-Ko-Rado verzameling van generatoren op een hyperbolische kwadriek $\mathcal{Q}^+(4n+1,q)$.

De generatoren van een hyperbolische kwadriek $\mathcal{Q}^+(2m+1,q)$ kunnen in *twee klassen* worden opgedeeld, de zogenaamde *Latijnse* en *Griekse* generatoren. Twee generatoren uit eenzelfde klasse snijden elkaar dan in een deelruimte van dimensie $m-2i$, $i\leq 0\leq \frac{m+1}{2}$; twee generatoren uit een verschillende klasse snijden elkaar dan in een deelruimte van dimensie $m-2i-1$, $i\leq 0\leq \frac{m}{2}$.

Voor een hyperbolische kwadriek $\mathcal{Q}^+(4n+1,q)$ betekent dit dat de verzameling van alle generatoren uit eenzelfde klasse een Erdős-Ko-Rado verzameling is. Deze bevat precies de helft van de generatoren van $\mathcal{Q}^+(4n+1,q)$. Dit is de grootste (en dus ook een maximale) Erdős-Ko-Rado verzameling van generatoren op $\mathcal{Q}^+(4n+1,q)$.

We beschrijven nu een andere Erdős-Ko-Rado verzameling van generatoren op $\mathcal{Q}^+(4n+1,q)$. Kies een vaste generator π. De verzameling die bestaat uit π en uit alle generatoren van de klasse waartoe π niet behoort, maar die π wel snijden, is een maximale Erdős-Ko-Rado verzameling van generatoren. De hoofdstelling van dit gedeelte stelt dat dit de tweede grootste Erdős-Ko-Rado verzameling van generatoren op $\mathcal{Q}^+(4n+1,q)$ is, als $n\geq 1$ en $q\geq 3$.

Merk op dat het grootste voorbeeld $q^{2n^2+n}+q^{2n^2+n-1}+q^{2n^2+n-2}+O(q^{2n^2+n-3})$ elementen bevat, en dat het tweede grootste voorbeeld $q^{2n^2+n-1}+q^{2n^2+n-2}+O(q^{2n^2+n-3})$ elementen bevat. Alle andere maximale Erdős-Ko-Rado verzamelingen bevatten $O(q^{2n^2+n-2})$ elementen. In Sectie 4.3 construeren we verschillende voorbeelden van Erdős-Ko-Rado verzamelingen van generatoren die $q^{2n^2+n-2} + O(q^{2n^2+n-3})$ elementen bevatten.

B.5 Erdős-Ko-Rado verzamelingen in Steiner 2-designs

Voor een design is een *Erdős-Ko-Rado verzameling* een verzameling blokken die paarsgewijs minstens één punt gemeen hebben. In dit gedeelte bespreken we Erdős-Ko-Rado verzamelingen in $2-(v,k,1)$ designs. Het aantal blokken door een vast punt in een $2-(v,k,1)$ design design is gelijk aan $\frac{v-1}{k-1}$ en noteren we als r. Rands bewees dat een Erdős-Ko-Rado verzameling in een $2-(v,k,1)$ design, met $r \geq k^2$, hoogstens r blokken kan bevatten en dat de enige Erdős-Ko-Rado verzameling die r blokken bevat, de verzameling van alle blokken door een vast punt is.

De hoofdstelling van dit gedeelte verbetert deze grens.

Stelling B.5.1. *Beschouw een $2-(v,k,1)$ design $\mathcal{D}$, $k \geq 4$, met $r \geq k^2 - 3k + \frac{3}{4}\sqrt{k} + 2$, en een Erdős-Ko-Rado verzameling $\mathcal{S}$ in $\mathcal{D}$. Dan bevat $\mathcal{S}$ hoogstens r elementen. Als $r \neq k^2 - k + 1$ en $(r,k) \neq (8,4)$, dan is $|\mathcal{S}| = r$ als en slechts als $\mathcal{S}$ de verzameling van alle blokken door een vast punt is.*

In deze stelling is de uitzondering $r \neq k^2 - k + 1$ noodzakelijk. Als $k-1$ een priemmacht is, bestaat er immers een $2-(k^3-2k^2+2k, k, 1)$ design waarin een tweede type van Erdős-Ko-Rado verzamelingen van grootte r bestaat, namelijk het design van punten en rechten van $\mathrm{PG}(3, k-1)$.

Daarnaast bekijken we ook nog enkele speciale types van Steiner 2-designs. In axiomatisch projectieve vlakken bestaat er slechts één maximale Erdős-Ko-Rado verzameling, de verzameling van alle rechten. In een axiomatisch affien vlak van de orde n bestaan alle maximale Erdős-Ko-Rado verzamelingen uit $n+1$ rechten, één uit elke parallelklasse (één door elk punt van de rechte op oneindig). In designs zonder *O'Nan configuratie* (4 blokken die paarsgewijs snijden in zes verschillende punten) zijn er twee types van maximale Erdős-Ko-Rado verzamelingen. In Steiner 3-systemen kunnen vijf verschillende types van maximale Erdős-Ko-Rado verzamelingen voorkomen. Als een Steiner 3-systeem meer dan 19 punten bevat, dan is de verzameling van alle blokken door een vast punt sowieso de grootste Erdős-Ko-Rado verzameling.

Voor unitalen vinden we de volgende stelling.

Stelling B.5.2. *Beschouw een unitaal $\mathcal{U}$ van de orde q, en een maximale Erdős-Ko-Rado verzameling op $\mathcal{U}$. Als $q \geq 5$, dan is $|\mathcal{S}| = q^2$ en is $\mathcal{S}$ de verzameling van alle blokken door een vast punt, of is $|\mathcal{S}| \leq q^2 - q + \sqrt[3]{q^2} - \frac{2}{3}\sqrt[3]{q} + 1$. Als $q = 4$, dan is $|\mathcal{S}| = q^2 = 16$ en is $\mathcal{S}$ de verzameling van alle blokken door een vast punt, of is $|\mathcal{S}| \leq 13$. Als $q = 3$, dan is $|\mathcal{S}| = q^2 = 9$ en is $\mathcal{S}$ de verzameling van alle blokken door een vast punt, of is $|\mathcal{S}| \leq 8$.*

B.6 Kakeya verzamelingen in $\mathrm{AG}(2, q)$

Een *Kakeya verzameling* in een affiene ruimte $\mathrm{AG}(n, q)$ is de puntenverzameling bedekt door een verzameling rechten die één rechte door elk punt van het hypervlak op oneindig heeft (één rechte in elke richting). Voor $\mathrm{AG}(2, q)$ bevat zo een bijhorende rechtenverzameling precies $q + 1$ rechten. We bespreken in deze sectie de kleine Kakeya verzamelingen in $\mathrm{AG}(2, q)$, q even. Zo een Kakeya verzameling bevat minstens $\frac{q(q+1)}{2}$ punten.

We geven twee voorbeelden van kleine Kakeya verzamelingen. We herinneren eraan dat $\mathrm{AG}(2, q)$ kan geïdentificeerd worden met $\mathrm{PG}(2, q)$ waaruit een rechte 'op oneindig' ℓ_∞ is verwijderd. Kies een duale hyperovaal in het projectief vlak $\mathrm{PG}(2, q)$ zodat één van de rechten samenvalt met ℓ_∞. De $q + 1$ andere rechten gaan elk door een ander punt van deze rechte op oneindig en bepalen dus een Kakeya verzameling. Deze bevat $\frac{q(q+1)}{2}$ punten. Ook voor het tweede voorbeeld kiezen we een duale hyperovaal in $\mathrm{PG}(2, q)$ zodat één van de rechten samenvalt met ℓ_∞. Vervang één van de rechten op de duale hyperovaal, verschillend van ℓ_∞, door een andere rechte door hetzelfde punt van ℓ_∞. Deze verzameling bevat nog steeds $q + 1$ rechten, elk door een ander punt van de rechte op oneindig. De bijhorende Kakeya verzameling bevat $\frac{q(q+2)}{2}$ punten. Blokhuis en Mazzocca bewezen dat alle Kakeya verzamelingen in $\mathrm{AG}(2, q)$, q even, die hoogstens $\frac{q(q+2)}{2}$ punten bevatten, tot één van deze twee types moeten behoren.

We construeren nu een nieuw type Kakeya verzamelingen. Kies een duale $(q + 4, 4)$-boog $\mathcal{A}$ van type $(0, 2, 4)$, zodat één van de rechten samenvalt met ℓ_∞. Dan bevat ℓ_∞ precies één punt waardoor drie rechten gaan. Door twee van deze rechten (en ℓ_∞) te verwijderen uit de rechtenverzameling van $\mathcal{A}$, blijven er $q + 1$ rechten over, één door elk punt van ℓ_∞. De Kakeya verzameling die bepaald wordt door deze rechten bevat $\frac{q(q+2)}{2} + \frac{q}{4}$ punten.

Het belangrijkste resultaat uit dit gedeelte geeft een classificatie van alle Kakeya verzamelingen $\mathrm{AG}(2,q)$, $q>8$ even, die hoogstens $\frac{q(q+2)}{2}+\frac{q}{4}$ punten bevatten. Deze behoren tot één van de twee hierboven beschreven types afkomstig van een hyperovaal of tot het type afkomstig van een duale $(q+4,4)$-boog van type $(0,2,4)$. Merk op dat dit betekent dat er geen Kakeya verzamelingen bestaan waarvan de grootte in $]\frac{q(q+1)}{2},\frac{q(q+2)}{2}[\,\cup\,]\frac{q(q+2)}{2},\frac{q(q+2)}{2}+\frac{q}{4}[$ bevat is. Deze observatie geldt ook voor $q=4,8$. Voor $q=4$ kunnen we de volledige classificatie van Kakeya verzamelingen met de hand opstellen. In het geval $q=8$ kennen we echter de classificatie van de Kakeya verzamelingen van grootte $\frac{q(q+2)}{2}+\frac{q}{4}$ niet.

B.7 Kleine maximale partiële t-spreads in $\mathrm{PG}(2t+1,q)$

Een *maximale partiële t-spread* in $\mathrm{PG}(2t+1,q)$ is een verzameling paarsgewijs disjuncte t-ruimtes in $\mathrm{PG}(2t+1,q)$ zodat geen enkele t-ruimte in $\mathrm{PG}(2t+1,q)$ disjunct is aan elk van de t-ruimtes in deze verzameling. In dit gedeelte zoeken we een ondergrens op de grootte van een maximale partiële t-spread in $\mathrm{PG}(2t+1,q)$. Glynn bewees eerder de ondergrens $2q$ in het geval $t=1$. In Sectie 7.2 vinden we het volgende resultaat.

Stelling B.7.1. *Een maximale partiële t-spread in $\mathrm{PG}(2t+1,q)$ bevat minstens $2q-1$ elementen.*

Er is ook een belangrijk verband tussen maximale partiële t-spreads in $\mathrm{PG}(2t+1,q)$, en blokkerende verzamelingen ten opzichte van de t-ruimtes in $\mathrm{PG}(2t+1,q)$. Immers, de puntenverzameling bedekt door de unie van t-ruimtes in een maximale partiële t-spread is een blokkerende verzameling ten opzichte van de t-ruimtes. Aangezien we maximale partiële t-spreads met weinig elementen willen bestuderen, bekijken we maximale partiële t-spreads die een kleine minimale blokkerende verzameling ten opzichte van de t-ruimtes bevatten. De volgende resultaten vinden we in Sectie 7.3.

Stelling B.7.2.

- *Een maximale partiële t-spread in $\mathrm{PG}(2t+1,q)$ die een $(t+1)$-ruimte (een triviale blokkerende verzameling) bedekt, bevat minstens $q^{\lceil \frac{t}{2}\rceil+1}+1$ elementen.*

- *Een maximale partiële t-spread in* $\mathrm{PG}(2t+1,q)$, $q=p^h$ *en* $p>2$ *priem, die een niet-triviale kleine blokkerende verzameling ten opzichte van de t-ruimtes bedekt, bevat minstens* $\sqrt{1+(p-1)(\theta_{t+1}(q)+r(q)q^t)}+1$ *elementen. Hierbij is* $r(q)$ *gelijk aan* $|B|-q-1$, *met* B *de kleinste niet-triviale blokkerende verzameling ten opzichte van de rechten in* $\mathrm{PG}(2,q)$.

Merk op dat $r(q)\geq\sqrt{q}$ voor alle q.

B.8 De functionele codes $C_2(\mathcal{H})$ en $C_{Herm}(\mathcal{Q})$

We beschreven hierboven al hoe functionele codes $C_{\mathcal{F}}(\mathcal{X})$ worden opgebouwd met behulp van een algebraïsche variëteit $\mathcal{X}$ en een verzameling veeltermen $\mathcal{F}$. Vaak wordt een niet-singuliere kwadriek $\mathcal{Q}$ of een niet-singuliere Hermitische variëteit als algebraïsche variëteit gekozen. Als veeltermenverzameling kiest men vaak de verzameling van homogene veeltermen van graad 2, inclusief de nulveelterm, of de verzameling Hermitische veeltermen, inclusief de nulveelterm. De codes $C_2(\mathcal{Q})$ en $C_{Herm}(\mathcal{H})$ werden al uitgebreid bestudeerd. In dit gedeelte gaat het over de codes $C_2(\mathcal{H})$ en $C_{Herm}(\mathcal{Q})$. De lengte en dimensie van deze codes is eenvoudig te vinden. We zoeken naar (een ondergrens op) het minimum gewicht en naar codewoorden met een klein gewicht. Dit doen we door doorsnedes van variëteiten te onderzoeken.

Voor het bestuderen van de code $C_2(\mathcal{H})$, $\mathcal{H}$ een niet-singuliere Hermitische variëteit in $\mathrm{PG}(n,q^2)$, voeren we de volgende functies in:

$$W_4(q)=\begin{cases} q^5+q^4+4q^3-3q+1 & q\geq 3\\ 69 & q=2\end{cases},$$

$$W_n(q)=\begin{cases} q^2W_{n-1}(q)+q^{n-2}+2q^{n-3} & n>4 \text{ oneven}\\ q^2W_{n-1}(q)-q^{n-2} & n>4 \text{ even}\end{cases}.$$

De volgende stelling is het belangrijkste resultaat van Secties 8.1 en 8.2.

Stelling B.8.1. *Als de doorsnede* $\mathcal{H}\cap Q$ *van een niet-singuliere Hermitische variëteit* $\mathcal{H}$ *met een willekeurige kwadriek* Q *in* $\mathrm{PG}(n,q^2)$, $n\geq 4$, *meer dan* $W_n(q)$ *punten bevat, dan is* Q *de unie van twee hypervlakken.*

Alle codewoorden in $C_2(\mathcal{H})$, $\mathcal{H}$ een niet-singuliere Hermitische variëteit in $\mathrm{PG}(n,q^2)$, waarvan het gewicht kleiner is dan $|\mathcal{H}|-W_n(q)$, zijn afkomstig van kwadrieken die de unie zijn van twee hypervlakken. Dit laat ons toe om het minimum gewicht van deze code te vinden en de codewoorden van de vijf kleinste gewichten te classificeren.

Voor het bestuderen van de code $C_{Herm}(\mathcal{Q})$, $\mathcal{Q}$ een niet-singuliere kwadriek in $\mathrm{PG}(n, q^2)$, voeren we de functies $\overline{W}_n(q)$ op de volgende wijze in:

$$\overline{W}_4(q) = \begin{cases} q^5 + 2q^4 - \frac{1}{3}q^3 + 2q^2 + q + 1 & q \neq 3 \\ 424 & q = 3 \end{cases};$$

$$\overline{W}_n(q) = \begin{cases} q^7 + 2q^6 + 2q^5 - \frac{1}{2}q^4 - \frac{21}{4}q^3 + \frac{15}{8}q^2 + \frac{195}{16}q + 8 & n = 5,\ q \geq 3 \\ W_n(q) & n > 5,\ q \geq 3 \end{cases};$$

$$\overline{W}_n(2) = \begin{cases} \frac{25}{84}2^{2n} + \frac{2425}{1764}2^n - \frac{2655125}{100107} & n > 5 \text{ even} \\ \frac{25}{84}2^{2n} + \frac{725}{1764}2^n - \frac{27050}{3087} & n > 5 \text{ oneven} \end{cases}.$$

De volgende stelling is het belangrijkste resultaat van Secties 8.3, 8.4 en 8.5.

Stelling B.8.2. *De doorsnede $\mathcal{Q} \cap H$ van een niet-singuliere kwadriek $\mathcal{Q}$ met een willekeurige Hermitische variëteit H in* $\mathrm{PG}(n, q^2)$*, $n \geq 4$, bevat hoogstens $\overline{W}_n(q)$ punten.*

Bijgevolg is $|\mathcal{Q}| - \overline{W}_n(q)$ een ondergrens voor het minimum gewicht van de code $C_{Herm}(\mathcal{Q})$, $\mathcal{Q}$ een niet-singuliere kwadriek in $\mathrm{PG}(n, q^2)$. In Sectie 8.5 staan een aantal codewoorden beschreven met een gewicht net boven deze ondergrens.

In Sectie 8.6 tonen we aan dat q^{n-2} een deler is van de code $C_{Herm}(\mathcal{Q})$, $\mathcal{Q}$ een niet-singuliere kwadriek in $\mathrm{PG}(n, q^2)$. Voor de andere hierboven beschreven functionele codes was al eerder een deler gekend.

B.9 De duale code $C_n(\mathcal{H}(2n+1, q^2))^\perp$

De code $C_n(\mathcal{H}(2n+1, q^2))^\perp$ bestaat uit alle vectoren die orthogonaal staan op alle incidentievectoren van generatoren van de Hermitische polaire ruimte $\mathcal{H}(2n+1, q^2)$. In deze sectie bekijken we het minimum gewicht van deze code en de codewoorden van klein gewicht in deze code. Voor $n = 1, 2$ werden deze codes al eerder bestudeerd.

In Sectie 9.1 beschrijven we n *klassen codewoorden* van de code $C_n(\mathcal{H}(2n+1, q^2))^\perp$. Elk van deze codewoorden heeft gewicht $2q^{2n-1} + O(q^{2n-2})$. De hoofdstelling van deze sectie, bewezen in Sectie 9.3, stelt dat dit de codewoorden met het kleinste gewicht zijn en dat alle codewoorden van klein gewicht kunnen geschreven worden als lineaire combinatie van deze codewoorden.

Stelling B.9.1. *Beschouw een natuurlijk getal n en een constante $\delta > 0$. Als q voldoende groot is, dan behoort een codewoord $c \neq 0$ van $C_n(\mathcal{H}(2n+1,q^2))^\perp$, waarvan het gewicht hoogstens $4q^{2n-2}(q-1)$ is, tot één van de n beschreven klassen. Als q voldoende groot is, dan is een codewoord van $C_n(\mathcal{H}(2n+1,q^2))^\perp$, waarvan het gewicht hoogstens δq^{2n-1} is, een lineaire combinatie van codewoorden uit deze n klassen. Het minimum gewicht van $C_n(\mathcal{H}(2n+1,q^2))^\perp$ is $2q^{2n-4}(q^3+1)$, $n \geq 2$.*

B.10 Bemerkingen bij de code $C(2,q)$

De code $C(2,q) = C_1(2,q)$ is de code voortgebracht door de incidentievectoren van de rechten in $\mathrm{PG}(2,q)$. In dit laatste gedeelte staan de oplossingen van twee kleine problemen in verband met deze code.

Blokhuis, Brouwer en Wilbrink bewezen dat de incidentievector van een unitaal van orde q ingebed in $\mathrm{PG}(2,q^2)$, een codewoord van $C(2,q^2)$ is als en slechts als de unitaal Hermitisch is. De Hermitische kromme is dus een codewoord, maar werd tot nu toe nog niet beschreven als lineaire combinatie van incidentievectoren van rechten. Een eerste resultaat in dit gedeelte beschrijft zo een lineaire combinatie, in het geval q even is, vertrekkend van een maximale (q^2-q+1)-boog in $\mathrm{PG}(2,q^2)$.

De classificatie van de codewoorden van klein gewicht in $C(2,q)$ is volledig voor codewoorden van gewicht hoogstens $2q-1$. Er wordt vermoed dat alle codewoorden in $C(2,q)$, $q = p^h$ en p een priemgetal, met een gewicht $w < q\sqrt{q}+1$, kunnen geschreven worden als lineaire combinatie van $\left\lceil \frac{w}{q+1} \right\rceil$ incidentievectoren van rechten, op voorwaarde dat $h > 1$. Een tweede resultaat in dit gedeelte beschrijft een codewoord in $C(2,q)$, q een priemgetal, van gewicht $3(q-1)$, dat niet geschreven kan worden als lineaire combinatie van 3 incidentievectoren van rechten. Dit is het eerste codewoord, voor zover we weten, waarvan het bestaan aantoont dat bovenstaand vermoeden niet kan uitgebreid worden tot het geval $h = 1$.

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Dankwoord

Dit proefschrift zou er niet gekomen zijn zonder de steun en hulp van heel wat mensen. Het vermelden van namen houdt het risico in dat ik mensen vergeet, maar daarom niemand bij naam vermelden, is het kind met het badwater weggooien.

Eerst en vooral zou ik professor Storme, mijn promotor, willen bedanken. Hij was het die mij vier jaar geleden de kans gaf om een doctoraat aan te vatten. De voorbije jaren voorzag hij mij steeds van interessante en uitdagende wiskundige problemen die toch net binnen mijn mogelijkheden lagen. Op belangrijke momenten stuurde hij mij de juiste richting in, of legde hij contacten met andere wiskundigen waarmee ik kon samenwerken. Meteen wil ik ook het FWO-Vlaanderen bedanken, dat mij voorzag van de beurs om dit onderzoek te verrichten.

Ik wil ook de leden van de jury bedanken, in de eerste plaats voor het lezen en vakkundig beoordelen van dit proefschrift. Verder wil ik professor Blokhuis ook bedanken voor de nieuwe kijk op Kakeya verzamelingen die hij mij gaf. I would like to thank professor Metsch and professor Szőnyi especially for their hospitality during my research stays in Gießen and Budapest, and professor Sziklai and professor Weiermann in particular for investing their time. In het bijzonder zou ik ook Geertrui Van de Voorde willen bedanken. Zij volgde enkele jaren geleden een zeer gelijkaardig traject, en ik kon dan ook vaak bij haar terecht, als mijn 'grote zus' in de wereld van het onderzoek.

I also want to thank my other coauthors, professor Mazzocca, Daniele Bartoli and Peter Vandendriessche for the fruitful collaborations. Ook professor Lavrauw, die mij ontving in Vicenza, zou ik willen bedanken.

In de afgelopen jaren waren er verschillende meetkundigen op wiens raad ik kon rekenen, zowel in verband met onderzoeksproblemen als in verband met (over)leven in de academische wereld, met wie ik één en ander eens rustig kon bespreken, of wiens boeken ik kon gebruiken. Ik denk in het bijzonder aan Jan De Beule, professor De Clerck, professor Van Maldeghem, professor De Bruyn, professor Bamberg, professor Cara, Anamari Nakić, Ferdinand Ihringer, Sara Rottey en Bert Seghers. Een zeer bijzonder woord van dank zou ik ook willen richten tot Frédéric Vanhove, die helaas veel te vroeg van ons heenging. Hij was een voorbeeld op gebied van precisie en doorzettingsvermogen. Wat het onderzoek naar Erdős-Ko-Rado verzamelingen betreft ben ik de *nanus* en hij één van de *gigantes* uit de uitspraak bovenaan hoofdstuk 2.

De collega's die 'de Sterre' in het algemeen, en 'de Galglaan' in het bijzonder tot een aangename werkplek maakten, verdienen ook zeker een woord van dank. Naast de hierboven vermelde meetkundigen denk ik aan Michaël Vyverman, de 'algebrameisjes' (Sofie Beke, Lien Boelaert, Elizabeth Callens en Claudia Degroote), professor De Medts, Jeroen Demeyer, Karsten Naert, Erik Rijcken, Goedele Waeyaert, professor Mestdag, Jeroen Van der Meeren, Korneel Debaene, ...[1] en aan Samuel Perez, Sonia Surmont en Geert Vernaeve bij wie ik terecht kon met praktische problemen.

In de voorbije jaren mocht ik ook aan heel wat wiskundestudenten lesgeven en enkelen onder hen begeleiden. Ze hebben mij (waarschijnlijk onbewust) af en toe gedwongen mijn gedachten goed te ordenen. Lien Lambert, Linda Van Puyvelde en Andries Vansweevelt wil ik hier zeker vermelden.

Wat zou ik de voorbije jaren geweest zijn zonder mijn vrienden? Frederik en Jellen kennen mij al sinds het begin van de middelbare school, en hebben al te vaak mijn wiskundepraatjes moeten aanhoren. Mijn kameraden in het VNJ wil ik ook zeker bedanken: Geoffrey, Emile, Jeroen, Karel-Hendrik, Gert-Jan, Brecht, Willem, Elke, Emma, Asgerd, Rutger, Tijl, Maarten, Alwin, Kenneth, Dries, Wouter, Grietje, Femke, Reinout, Laurens, Soetkin, Wilhard, Wouter, Nele, ...[2]

De belangrijkste woorden van dank gaan naar mijn familie, en in het bijzonder naar mijn ouders. De steun die ik, door alles heen, van hen mocht ontvangen, was cruciaal. Ik herinner mij levendig dat mijn 'wiskundige opleiding' als zesjarige van start ging met rekenoefeningen tijdens de maaltijd, dat mijn vader in de eerste jaren van het middelbaar oefeningen opstelde die mij dwongen mijn grenzen te verleggen, dat ik leerde meerdere stappen vooruit te denken, dat er voor elk examen een kaarsje gebrand werd, dat ... Zonder hen zou u dit boek niet in uw handen hebben.

Tot slot wil ik u bedanken, beste lezer. Wiskundige ideeën moeten verspreid worden, willen ze nut hebben. Door dit proefschrift te lezen, draagt u daaraan bij.

[1] en alle anderen waarmee ik de afgelopen jaren heb mogen samenwerken.

[2] en alle anderen waarmee ik de afgelopen jaren heb mogen samenwerken.

"It is a lesson," Armen said, "the last lesson we must learn before we don our maester's chains. The glass candle is meant to represent truth and learning, rare and beautiful and fragile things. It is made in the shape of a candle to remind us that a maester must cast light wherever he serves, and it is sharp to remind us that knowledge can be dangerous. Wise men may grow arrogant in their wisdom, but a maester must always remain humble. The glass candle reminds us of that as well. Even after he has said his vow and donned his chain and gone forth to serve, a maester will think back on the darkness of his vigil and remember how nothing that he did could make the candle burn... for even with knowledge, some things are not possible."

A Song of Ice and Fire, A Feast for Crows, Prologue by *G.R.R. Martin.*