Hadamard Three-Hyperballs Type Theorem and Overconvergence of Special Monogenic Simple Series

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Abstract

The classical Hadamard three-circles theorem (1896) gives a relation between the maximum absolute values of an analytic function on three concentric circles. More precisely, it asserts that if f is an analytic function in the annulus $\{z \in \mathbb{C} : r_1 < |z| < r_2\}$, $0 < r_1 < r < r_2 < \infty$, and if $M(r_1)$, $M(r_2)$, and $M(r)$ are the maxima of f on the three circles corresponding, respectively, to r_1, r_2, and r then

$$\{M(r)\}^{\log \frac{r_2}{r_1}} \leq \{M(r_1)\}^{\log \frac{r_2}{r}} \{M(r_2)\}^{\log \frac{r}{r_1}}.$$

In this paper we introduce a Hadamard's three-hyperballs type theorem in the framework of Clifford analysis. As a concrete application, we obtain an overconvergence property of special monogenic simple series.

Keywords: Clifford analysis, monogenic functions, Hadamard three-circles theorem.

1. Introduction

Two main problems that arise in the study of function spaces can be broadly described as follows:

1. Does the space under consideration possess a basis?
2. If this is the case, how can any other basis of this space be characterized?

These topics are closely linked together, but can be largely treated independently of each other. Let us assume for a moment that these problems are answered in a positive way. If E denotes a topological space and $\{x_n\}_{n \in \mathbb{N}}$ a basis in E, then each element $x \in E$ admits a (unique) decomposition of the form $\sum_{n=1}^{\infty} a_n(x) x_n$ whereby for each $n \in \mathbb{N}$, a_n is a linear functional on E. For the purposes of approximation theory the choice of a suitable basis is very important. This work deals essentially with these two fundamental problems in the case the underlying function spaces admit a set of polynomials as a basis. Classical examples of such function spaces are the space of holomorphic functions in an open disk and the space of analytic functions on a closed disk. Of course, as the theory of holomorphic functions in the plane allows higher dimensional generalizations [6], analogous problems may be considered in the corresponding function spaces.

In the early thirties Whittaker [34, 36, 37] and Cannon [7, 8, 9] have introduced the theory of basic sets (bases) of polynomials of one complex variable. This theory has been successfully extended to the Clifford analysis case in [1] (cf. [2]). Holomorphic functions (of one complex variable) are now replaced by Clifford algebra-valued functions that are defined in open subsets of $\mathbb{R}^{m+1}$ and that are solutions of a Dirac-type equation; for historical reasons they are called monogenic functions. In order to obtain a good analogy with the theory of one complex variable, the results in [1, 2] have been restricted to polynomials with axial symmetry (also know as special polynomials), for which a Cannon theorem on the effectiveness could be proved in closed hyperballs. It should be observed that it is expected that a similar theory on basic sets of polynomials might be possible for polynomial nullsolutions of generalized Cauchy-Riemann or Dirac operators, satisfying more general symmetry conditions. This matter is already well-exposed in [1, 2] and essential ideas therein.

The main purpose of the present work is to introduce a Hadamard's three-hyperballs type theorem in the $(m + 1)$-dimensional Euclidean space within

the Clifford analysis setting by making use of the above-mentioned theory of basis of polynomials [1, 2], and to establish an overconvergence property of special monogenic simple series. To the best of our knowledge this is done here for the first time. Theorems of this type have become significantly more involved in higher dimensions, and in particular in the quaternionic and Clifford analysis settings. In a series of papers [13, 15, 21, 22], the authors have investigated higher dimensional counterparts of the well-known Bohr theorem and Hadamard real part theorems on the majorant of a Taylor's series, as well as Bloch's theorem, in the context of quaternionic analysis. These results provide powerful additional motivation to study the asymptotic growth behavior of monogenic functions from a given space, and to explore classical problems of the theory of monogenic quasi-conformal mappings [14, 23] (see also [20, Ch. 3]).

For the general terminology used in this paper the reader is referred to Wittaker's book [37] in the complex case, and the work done by Abul-Ez et al. [1, 2] in the Clifford analysis setting.

2. Preliminaries

2.1. Basic notions of Clifford analysis

The present subsection collects some definitions and basic algebraic facts of a special Clifford algebra of signature $(0, m)$, which will be needed throughout the text.

Let $\{e_1, e_2, \ldots, e_m\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^m$ with a product according to the multiplication rules:

$$e_i e_j + e_j e_i = -2\delta_{i,j} \quad (i, j = 1, \ldots, m),$$

where $\delta_{i,j}$ is the Kronecker symbol. This noncommutative product generates the 2^m-dimensional Clifford algebra $Cl_{0,m}$ over $\mathbb{R}$, and the set $\{e_A : A \subseteq \{1, \ldots, m\}\}$ with

$$e_A = e_{h_1} e_{h_2} \ldots e_{h_r}, \quad 1 \leq h_1 \leq \ldots \leq h_m, \quad e_\phi = e_0 = 1,$$

forms a basis of $Cl_{0,m}$. The real vector space $\mathbb{R}^{m+1}$ will be embedded in $Cl_{0,m}$ by identifying the element $(x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1}$ with the algebra's element

$$x := x_0 + \mathbf{x} \in \mathcal{A}_m := \mathrm{span}_{\mathbb{R}}\{1, e_1, \ldots, e_m\} \subset Cl_{0,m}.$$

The elements of $\mathcal{A}$ are usually called paravectors, and $x_0 := \mathrm{Sc}(x)$ and $e_1 x_1 + \cdots + e_m x_m := \mathbf{x}$ are the so-called scalar and vector parts of x. The conjugate of x is $\overline{x} = x_0 - \mathbf{x}$, and the norm $|x|$ of x is defined by

$$|x|^2 = x\overline{x} = \overline{x}x = x_0^2 + x_1^2 + \cdots + x_m^2.$$

As $Cl_{0,m}$ is isomorphic to $\mathbb{R}^{2^m}$ we may provide it with the $\mathbb{R}^{2^m}$-norm $|a|$, and one easily sees that for any $a, b \in Cl_{0,m}$, $|a\,b| \leq 2^{\frac{m}{2}} |a|\,|b|$, where $a = \sum_{A \subseteq M} a_A e_A$ and M stands for $\{1, 2, \ldots, m\}$.

We consider $Cl_{0,m}$-valued functions defined in some open subset Ω of $\mathbb{R}^{m+1}$, i.e. functions of the form $f(x) := \sum_A f_A(x) e_A$, where $f_A(x)$ are scalar-valued functions defined in Ω. Properties (like integrability, continuity or differentiability) that are ascribed to f have to be fulfilled by all components f_A. In the sequel, we will make use of the generalized Cauchy-Riemann operator

$$D := \frac{\partial}{\partial x_0} + \sum_{i=1}^{m} e_i \frac{\partial}{\partial x_i}.$$

Suggested by the case $m = 1$, call an $Cl_{0,m}$-valued function f left- (resp. right) *monogenic* in Ω if $Df = 0$ (resp. $fD = 0$) in Ω. The interested reader is referred to [6] for more details.

Recent studies have shown that the construction of $\mathcal{A}_m$-valued monogenic functions as functions of a paravector variable is very useful, particularly if we study series expansions of $Cl_{0,m}$-valued functions in terms of special polynomial bases defined in $\mathbb{R}^{m+1}$. In this case we have

$$f : \Omega \subset \mathbb{R}^{m+1} \to \mathcal{A}_m, \quad f(x_0, \mathbf{x}) = f_0(x_0, \mathbf{x}) + \sum_{i=1}^{m} e_i f_i(x_0, \mathbf{x}),$$

and left monogenic functions are also right monogenic functions (they are often called two-sided monogenic). In particular, we shall observe that for a paravector-valued monogenic function f the equations

$$Df = fD = 0$$

are equivalent to the system

$$\text{(R)} \quad \begin{cases} \displaystyle\sum_{i=0}^{m} \frac{\partial f_i}{\partial x_i} = 0 \\[2mm] \dfrac{\partial f_i}{\partial x_j} - \dfrac{\partial f_j}{\partial x_i} = 0 \quad (i \neq j, \ 0 \leq i, j \leq m) \end{cases}$$

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or, equivalently, in a more compact form:

$$\begin{cases} \operatorname{div} \overline{f} & = & 0 \\ \operatorname{rot} \overline{f} & = & 0 \end{cases}.$$

The $(m+1)$-tuple $\overline{f}$ is said to be a system of conjugate harmonic functions in the sense of Stein-Weiß [29, 30], and the above system is called the Riesz system [27]. It is a historical precursor that generalizes the classical Cauchy-Riemann system in the plane. The solutions of the system (R) are customary called (R)-solutions.

2.2. Overconvergence of special monogenic polynomial series

Although the term "overconvergence" is used here to describe that a given function may be defined and approximated in a certain region, the sequence of polynomials approximating the function (in the given region) may also converge uniformly in a larger region containing the given region in its interior (cf. [33]).

In the sequel, the right $Cl_{0,m}$-module defined by

$$Cl_{0,m}\left[x\right] := \operatorname{span}_{Cl_{0,m}} \left\{ z_n(x) : n \in \mathbb{N}_0 \right\}$$

is called the space of homogeneous special monogenic polynomials; x is the Clifford variable, and $z_n(x)$ is a special (two-sided) monogenic polynomial (R)-solution of degree n of the form

$$z_n(x) := \sum_{i=0}^{n} \frac{A_{(i)} B_{(n-i)}}{i!\,(n-i)!}\, \overline{x}^i x^{n-i}, \tag{1}$$

$A = \frac{m-1}{2}$ and $B = \frac{m+1}{2}$. Here, for $b \in \mathbb{R}$, $b_{(l)}$ stands for $b(b+1)\ldots(b+l-1)$. Properties of the polynomials $z_n(x)$ can be found in [1, 2] and [38].

A sequence $\{P_n(x)\}$ of special monogenic polynomials that are constructed through $z_n(x)$, forms a basis (or a basic set) in the sense of Hamel basis if any arbitrary special polynomial can be represented uniquely as a finite linear combination of these polynomials; that is,

$$z_n(x) = \sum_{k=0}^{n} P_k(x)\pi_{n,k}, \quad \pi_{n,k} \in Cl_{0,m}. \tag{2}$$

Thus if $\deg P_k = k$ for every $k \in \mathbb{N}_0$, then the set is necessarily basic (base) and is called a simple base.

Definition 2.1. *Let N_n denote the number of nonzero coefficients $\pi_{n,k}$ in the representation (2). If $N_n^{\frac{1}{n}} \to 1$ as $n \to \infty$, then the basic set is called a Cannon basic set.*

Definition 2.2. *Let $\overline{B}(r)$ denote the closed hyperball in $\mathbb{R}^{m+1}$ with radius r centered at the origin, and let f be monogenic in a neighbourhood of $\overline{B}(r)$. Then f is called special monogenic in $\overline{B}(r)$ if and only if its Taylor series expansion near zero (which is known to exist) has the form*

$$f(x) = \sum_{n=0}^{\infty} z_n(x) a_n, \quad x \in \mathcal{A}_m$$

for certain constants $a_n \in Cl_{0,m}$.

We denote the class of special (two-sided) monogenic functions in a neighbourhood of $\overline{B}(r)$ by $SM(\overline{B}(r))$.

In view of (2) there is a basic series $\sum_{n=0}^{\infty} P_n(x) c_n$ associated with $f(x)$ where

$$Cl_{0,m} \ni c_n = c_n(f) := \sum_{k=0}^{\infty} \pi_{n,k} a_k. \tag{3}$$

The above series is simple if the set is simple. The basic series represents $f(x)$ in $\overline{B}(R)$ where $R = |x|$, if it converges normally to $f(x)$ in $\overline{B}(R)$. A basic set is said to be effective in $\overline{B}(R)$ if the basic series represents in $\overline{B}(R)$ every function which is monogenic there.

Cannon [7, 8, 9] introduced a criteria for effectiveness by means of the so-called Cannon sum $\omega_n(R)$ and Cannon function $\lambda(R)$, which have been extended to the Clifford case in [1] as follows:

$$\omega_n(R) := \sum_{k=0}^{n} \sup_{|x|=R} |P_k(x)\pi_{n,k}| \leq 2^{\frac{m}{2}} \sum_{k=0}^{n} \sup_{|x|=R} |P_k(x)| \, |\pi_{n,k}|, \tag{4}$$

and, let

$$\lambda(R) := \limsup_{n \to \infty} [\omega_n(R)]^{\frac{1}{n}}. \tag{5}$$

Remark 2.1. *A necessary and sufficient condition for a Cannon base of special monogenic polynomials to be effective in $\overline{B}(R)$ is that $\lambda(R) = R$.*

Remark 2.2. *A necessary and sufficient condition for a Cannon base of special monogenic polynomials to be effective in $D_+(R)$ (R greater than or equal to zero) by which we mean any open hyperball enclosing the closed hyperball $\overline{B}(R)$ is that $\lambda(R+) = R$, where $\lambda(R+) = \lim_{r \to R} \lambda(r)$ for $r > R$.*

Concerning the subscript Cannon function $\lambda(R)$ we next state an interesting property for Cannon bases, which is the generalization of Whittaker results in the complex case [17, Thm. 7, Thm. 26].

Theorem 2.1. *Let $0 < a < b$. All Cannon bases of special monogenic polynomials satisfy*

$$\lambda(R^{1+a}) \leq \{\lambda(R)\}^{1-\frac{a}{b}} \left\{\lambda(R^{1+b})\right\}^{\frac{a}{b}}, \tag{6}$$

where $\lambda(R)$ is defined as (5).

To prove the above result we first introduce a straightforward generalization of the famous Hadamard three-circles theorem from complex one-dimensional analysis to the special case of monogenic functions defined on hyperballs in the $(m+1)$-dimensional Euclidean space (see Theorem 3.2 below).

3. Hadamard three-circles theorem and generalizations

3.1. The Hadamard three-circles theorem for analytic functions

The famous Hadamard three-circles theorem gives the following relation between the maximum absolute values of an analytic function on three concentric circles.

Theorem 3.1 (Hadamard, 1896)**.** *Let $0 < r_1 < r_2 < r_3 < \infty$ and let f be an analytic function in the annulus $\{z \in \mathbb{C} : r_1 < |z| < r_2\}$. Denote the maximum of $|f(z)|$ on the circle $|z| = r$ by $M(r)$. Then*

$$\{M(r_2)\}^{\log \frac{r_3}{r_1}} \leq \{M(r_1)\}^{\log \frac{r_3}{r_2}} \{M(r_3)\}^{\log \frac{r_2}{r_1}}.$$

Originally, this theorem was given by Hadamard without proof in 1896 [16], and apparently it was first published in 1912 [17]. It reappeared in 1973 in the work of Výborný [32], but from the point of view of partial differential equations. For references as well as for some interesting applications we refer

the reader to [28] and [18, pp. 323-325]. The significance of the theorem is that it sharps the classical maximum modulus principle. Recently, multi-dimensional analogues and other generalizations of the classical Hadamard three-circles theorem for subharmonic functions in $\mathbb{R}^m$ $(m \geq 2)$ are treated by several authors. Without claiming completeness we mention here the contributions by Protter and Weinberger [26, pp. 128-131], MiklYukov, Rasila and Vuorinen [19]. For references as well as for some interesting applications we refer to [25]. The majority of proofs of Hadamard three-circles theorem makes use of the commutativity in the algebra of holomorphic functions. It is therefore of interest to see whether a Hadamard three-hyperballs type theorem can be proved if the underlying structure is not commutative as in the case of monogenic functions.

3.2. Hadamard three-hyperballs type theorem

Suppose f is special (two-sided) monogenic in an open hyperspherical shell

$$S_{|x|} := \left\{ x \in \mathcal{A}_m : 0 < r_1 < |x| < r_2 < \infty \right\} \subset \mathbb{R}^{m+1},$$

and componentwise continuous in the closed hyperspherical shell $\overline{S}_{|x|}$. Let $M(r)$ denote $\sup\{|f(x)| : |x| = r, x \in \mathcal{A}_m\}$. By the maximum modulus principle [12, 26] it follows that

$$M(r) \leq \max\left\{ M(r_1), M(r_2) \right\}.$$

This property may be written in either of the following equivalent forms:

$$M(r) \leq \{M(r_1)\}^\alpha \{M(r_2)\}^{1-\alpha}, \quad \text{where } \alpha := \frac{\log\left(\frac{r_2}{r}\right)}{\log\left(\frac{r_2}{r_1}\right)}, \tag{7}$$

or

$$\log M(r) \leq \frac{\log r_2 - \log r}{\log r_2 - \log r_1} \log M(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} \log M(r_2). \tag{8}$$

In other words, the inequality says that $\log M(r)$ is a convex function of $\log r$. There are many ways to prove Hadamard three-circles theorem, including works by Littlewood [17], Bohr and Landau [5], Titchmarsh [31, p. 172], Robinson [28], Edwards [11, p. 187], and Derbyshire [10, p. 376]. The

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reference list does not claim to be complete. Here we follow closely the proof given by Titchmarsh. With little fundamental alteration his proof is considerably simplified compared to the previous ones, so that we adapt his idea by examining the function

$$F : \overline{S}_r \subset \mathbb{R}^{m+1} \to \mathcal{A}_m, \quad F(x) := f(x) \sup_{|x|=r} |z_\alpha(x)|$$

for any $f \in SM(\overline{S}_r)$, where $z_\alpha(x)$ is given by (1) and the positive integer number α is chosen such that

$$r_1^\alpha M(r_1) \equiv r_2^\alpha M(r_2) \quad (r_1 < r < r_2).$$

A Clifford version of Hadamard three-circles theorem is contained in the following:

Theorem 3.2 (Hadamard three-hyperballs type theorem). *Suppose f is a special monogenic function in a closed hyperspherical shell $\overline{S}_r = \{r_1 \leq r \leq r_2\}$. Let $r_1 < r_2 < r_3$ and $M(r_i)$ be the maximum value of $|f(x)|$ on $\overline{B}(r_i)$ $(i = 1, 2, 3)$. Then $\log M(r)$ is a convex function of $\log r$. In other words,*

$$\{M(r_2)\}^{\log \frac{r_3}{r_1}} \leq \{M(r_1)\}^{\log \frac{r_3}{r_2}} \{M(r_3)\}^{\log \frac{r_2}{r_1}} \tag{9}$$

Proof. Let $f(x) = \sum_{n=0}^\infty z_n(x) a_n$, $a_n \in Cl_{0,m}$ be a special monogenic function in $\overline{S}_r$. For the sake of simplicity, we begin by setting up the following auxiliary function:

$$F(x) = f(x) \sup_{|x|=r} |z_\alpha(x)|,$$

where the positive integer number α is chosen so that

$$r_1^\alpha M(r_1) = r_3^\alpha M(r_3). \tag{10}$$

We shall proceed in such a manner that we state an upper bound estimate on the supremum of $|z_\alpha(x)|$ with $x \in \mathcal{A}_m$ (see [1]):

$$\sup_{|x|=r} |z_\alpha(x)| = \frac{(m)_\alpha}{\alpha!} r^\alpha. \tag{11}$$

A direct observation shows that the (two-sided) monogenicity of f in the hyperspherical shell $\overline{S}_r$ implies the (two-sided) monogenicity of F in the same domain. Evidently, if $r_1 \leq |x| \leq r_3$ it follows that

$$
\begin{aligned}
|F(x)| &= |f(x)| \sup_{|x|=r} |z_\alpha(x)| \\
&\leq \max \left\{ \frac{(m)_\alpha}{\alpha!} r_1^\alpha M(r_1), \frac{(m)_\alpha}{\alpha!} r_3^\alpha M(r_3) \right\}.
\end{aligned}
\tag{12}
$$

We shall now note that if $r_1 < r_2 < r_3$ then the previous relation gives

$$
r_2^\alpha M(r_2) \leq \max \left\{ r_1^\alpha M(r_1), r_3^\alpha M(r_3) \right\}
$$

and, in particular

$$
M(r_2) \leq \left(\frac{r_2}{r_1} \right)^{-\alpha} M(r_1).
\tag{13}
$$

By (10) it follows that

$$
\alpha \log r_1 + \log M(r_1) = \alpha \log r_3 + \log M(r_3)
$$

that is,

$$
-\alpha \log \left(\frac{r_3}{r_1} \right) = \log \left[\frac{M(r_3)}{M(r_1)} \right].
\tag{14}
$$

Now, using relations (13) and (14) a straightforward computation shows that

$$
\begin{aligned}
\{M(r_2)\}^{\log \frac{r_3}{r_1}} &\leq \left(\frac{r_2}{r_1} \right)^{-\alpha \log \frac{r_3}{r_1}} \{M(r_1)\}^{\log \frac{r_3}{r_1}} \\
&= \left(\frac{r_2}{r_1} \right)^{\log \frac{M(r_3)}{M(r_1)}} \{M(r_1)\}^{\log \frac{r_3}{r_1}} \\
&= \left[\frac{M(r_3)}{M(r_1)} \right]^{\log \frac{r_2}{r_1}} \{M(r_1)\}^{\log \frac{r_3}{r_1}}.
\end{aligned}
$$

With these calculation at hand, it follows that

$$
\{M(r_2)\}^{\log \frac{r_3}{r_1}} \leq \{M(r_3)\}^{\log \frac{r_2}{r_1}} \{M(r_1)\}^{\log \frac{r_3}{r_2}},
$$

and the theorem is proved. $\qquad\square$

Now we are ready to prove Theorem 2.1.

Proof. Let $0 < a < b$. To begin with, we note that inequality (6) composes with the Hadamard three-hyperballs type theorem on $M(R)$, namely

$$M(R^{1+a}) \leq \{M(R)\}^{1-\frac{a}{b}} \left\{M(R^{1+b})\right\}^{\frac{a}{b}} \tag{15}$$

where $M(R)$ is the maximum modulus of the integral function $f(x)$ on $\overline{B}(R)$. We set $R_1 := R^{1+a}$ and $R_2 := R^{1+b}$. Hence

$$\omega_n(R_1) \leq N_n \, 2^{\frac{m}{2}} \max_k \left(\sup_{|x|=R_1} |P_k(x)||\pi_{n,k}| \right)$$

$$\leq N_n \, 2^{\frac{m}{2}} \sup_{|x|=R_1} |P_k(x)||\pi_{n,k}|, \tag{16}$$

and applying (14) to the function $P_k(x)\pi_{n,k}$, a direct computation shows that

$$\omega_n(R_1) \leq N_n \, 2^{\frac{m}{2}} \left\{ \sup_{|x|=R} |P_k(x)||\pi_{n,k}| \right\}^{1-\frac{a}{b}} \left\{ \sup_{|x|=R_2} |P_k(x)||\pi_{n,k}| \right\}^{\frac{a}{b}}$$

$$\leq N_n \, 2^{\frac{m}{2}} \{\omega_n(R)\}^{1-\frac{a}{b}} \{\omega_n(R_2)\}^{\frac{a}{b}}.$$

With these calculations at hand, we obtain

$$\{\omega_n(R_1)\}^{\frac{1}{n}} \leq \left(N_n \, 2^{\frac{m}{2}}\right)^{\frac{1}{n}} \left[\{\omega_n(R)\}^{\frac{1}{n}}\right]^{1-\frac{a}{b}} \left[\{\omega_n(R_2)\}^{\frac{1}{n}}\right]^{\frac{a}{b}}$$

and, taking the limit as n approaches infinity we get

$$\lambda(R_1) \leq \{\lambda(R)\}^{1-\frac{a}{b}} \{\lambda(R_2)\}^{\frac{a}{b}}.$$

This proves the theorem. $\qquad\square$

Remark 3.1. *It is known that $\lambda(R)$ is a non-decreasing function. If we define $\lambda(R-) := \lim_{r\uparrow R} \lambda(r)$ and $\lambda(R+) := \lim_{r\downarrow R} \lambda(r)$, and assume that both limits exist, respectively, for $R > 0$ and $R \geq 0$; $r \uparrow R$ means $R-$ and $r \downarrow R$ means $R+$. It then follows that $R \leq \lambda(R-) \leq \lambda(R) \leq \lambda(R+)$.*

The following example illustrates how the functions $\lambda(R)$, $\lambda(R-)$ and $\lambda(R+)$ can be different for a given R.

Example 3.1. *We set*

$$P_n(x) = \begin{cases} z_n(x), & n \text{ even,} \\ z_n(x) + 2^n z_{2n^2}(x), & n \text{ odd.} \end{cases}$$

We then have $z_n(x) = P_n(x) - 2^n P_{2n^2}(x)$. Having in mind (11), the Cannon sum (4) is given by

$$\omega_n(R) = R^n + 2^{n+1} \frac{(m)_{2n^2}}{(2n^2)!} R^{2n^2} \frac{n!}{(m)_n}, \quad n \text{ odd.}$$

Hence, $\lambda(R) = R$ for $R < 1$ (i.e. $\lambda(1-) = 1$); $\lambda(1) = 2$ for $R = 1$, and $\lambda(R) = \infty$ for $R > 1$ (i.e. $\lambda(1+) = \infty$).

From Theorem 3.2, the following property can be easily deduced, which generalizes the one by Newns in [24, Thm. 11.3].

Corollary 3.1. *The Cannon function $\lambda(R)$ has at most one discontinuity in $0 < R < \infty$. In fact, $\lambda(R-) < \lambda(R+)$ implies $\lambda(R+) = \infty$.*

Proof. The proof can be carried out similarly to the one given by Newns in [24]. □

4. Overconvergence of special monogenic simple series

The Taylor basic series $\sum_{n=0}^{\infty} P_n(x)c_n$ associated with a special monogenic function f in an open ball $B(r)$ bears the feature of overconvergence when it has Hadamard's gaps. The next theorem studies the overconvergence of certain partial sums of the basic series associated with f special monogenic in an open hyperball, when this series possesses gaps of Hadamard's type.

Theorem 4.1. *Let $f(x) = \sum_{n=0}^{\infty} z_n(x)a_n$ be a special monogenic function in the open hyperball $B(R)$ and let $\{P_n(x)\}$ be a simple base of special monogenic polynomials effective in $\overline{B}(R')$, where R' is some number less than R. Suppose that in the basic series $\sum_{n=0}^{\infty} P_n(x)c_n$ associated with f holds $c_n \equiv 0$ for $\mu_k < n < \nu_k$ where $\nu_k \geq (1+\theta)\mu_k$ $(k = 1, 2, \ldots)$ and $\theta > 0$. Then the sequence $\{S_{\mu_k}(x)\}$ of partial sums, given by*

$$S_{\mu_k}(x) := \sum_{n=0}^{\mu_k} P_n(x)c_n \tag{17}$$

is convergent to $f(x)$ in a region including $B(R)$ and the neighbourhood of every point lying on the closed hyperball $\overline{B}(R)$ at which f is monogenic.

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Proof. Let $R' < R$ be given. To begin with, we note that since the base $\{P_n(x)\}$ is both simple and effective in $\overline{B}(R')$ it follows from [4, Thm. 9] that

$$\lambda(r) = r, \quad r \geq R'. \tag{18}$$

Also, since $f(x)$ is monogenic in $B(R)$ from [1, 2] it follows that

$$\limsup_{n \to \infty} |a_n|^{\frac{1}{n}} = \frac{1}{R}. \tag{19}$$

For simplicity's sake we shall suppose that $f(x)$ is monogenic at $x = R$. Therefore there exists a positive number $\delta < \frac{1}{2}$ for which $(1 - \delta)R > R'$, and such that $f(x)$ is monogenic in and on the hyperball B_1 with center $\frac{1}{2}R$ and radius $r_1 := \left(\frac{1}{2} + \delta\right)R$. In view of (19) a positive number $\eta < \delta$ can be chosen such that

$$|a_n| < \frac{K}{(1-\eta)^n R^n}, \quad n \geq 0. \tag{20}$$

Here K denotes a constant that does not retain the same value throughout.

In addition, according to (5) and (18) there exist two positive numbers δ' and δ'' so that $\eta < \delta < \delta' < \delta''$ such that

$$\omega_n\{(1 + \delta)R\} \leq K(1 + \delta')^n R^n \tag{21}$$

and

$$\omega_n\{(1 - \delta)R\} \leq K(1 + \delta'')^n R^n. \tag{22}$$

Now, for the simple base $\{P_n(x)\}$, the coefficients (3) can be written as follows:

$$c_n = \sum_{k=0}^{\infty} \pi_{n,n+k} a_{n+k}.$$

Hence applying (20) and (22), and using (4) it follows that

$$\sup_{|x|=(1-\delta)R} |P_n(x)c_n| \leq 2^{\frac{m}{2}} \sum_{k=0}^{\infty} \omega_{n+k}\{(1-\delta)R\} |a_{n+k}| \tag{23}$$

$$\leq K\left(\frac{1-\delta''}{1-\eta}\right)^n. \tag{24}$$

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Moreover, if $P_n(x) = \sum_{k=0}^{n} z_k(x) P_{n,k}$ then by (4), Cauchy's inequality [2] and in view of [24, p. 565] and [3] we have

$$|P_{n,n}| R^n \leq \sup_{|x|=R} |P_n(x)| \leq \frac{\omega_n(R)}{|\pi_{n,n}|} \tag{25}$$

where

$$P_{n,n} \pi_{n,n} = 1. \tag{26}$$

Thus, combining (21), (23), (25) and (26) it follows that

$$
\begin{aligned}
\sup_{|x|=(1+\delta)R} |P_n(x) c_n| &\leq \frac{\sup_{|x|=(1-\delta)R} |P_n(x) c_n| \,\omega_n \{(1+\delta)R\}}{|\pi_{n,n} P_{n,n}| (1-\delta'')^n R^n} \\
&\leq K \left(\frac{1-\delta''}{1-\eta} \right)^n \left(\frac{1-\delta'}{1-\delta} \right)^n.
\end{aligned}
\tag{27}
$$

Now, let B_2 and B_3 be the closed hyperballs with center $\frac{1}{2}R$ and radii, respectively, $r_2 := \left(\frac{1}{2} + \epsilon \right) R$ and $r_3 := \left(\frac{1}{2} - \delta \right) R$, where ϵ is any positive number less than δ'', and suppose that $M(R)$ is the maximum value of $|f(x)|$ on $B(R)$. Now, consider the function

$$\Phi(x) := f(x) - \delta S_{\mu_k}(x),$$

and let $M(r_1)$, $M(r_2)$ and $M(r_3)$ be, respectively, the maximum values of $|\Phi(x)|$ on $B(r_1)$, $B(r_2)$ and $B(r_3)$. Then, according to (17), (23) and (27) a straightforward computation shows that

$$
\begin{aligned}
M(r_1) &\leq M(R) + \sum_{n=0}^{\mu_k} \sup_{(1+\delta)R} |P_n(x) c_n| \\
&\leq K \left(\frac{1-\delta''}{1-\eta} \right)^{\mu_k} \left(\frac{1+\delta'}{1-\delta} \right)^{\mu_k}
\end{aligned}
\tag{28}
$$

and, consequently

$$M(r_3) \leq \sum_{n=i_k}^{\infty} \sup_{(1-\delta)R} |P_n(x) c_n| \leq K \left(\frac{1-\delta''}{1-\eta} \right)^{(1+\theta)\mu_k}. \tag{29}$$

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Applying Theorem 3.2, respectively, to $M(r_1)$, $M(r_2)$ and $M(r_3)$ we obtain

$$\{M(r_2)\}^{\log \frac{1-2\delta}{1-2\delta}} \leq \{M(r_1)\}^{\log \frac{1+2\epsilon}{1-2\delta}} \leq \{M(r_3)\}^{\log \frac{1+2\delta}{1+2\epsilon}}.$$

Hence, combining both (28) and (29) it follows that

$$\{M(r_2)\}^{\log \frac{1+2\delta}{1-2\delta}} \leq K \left[\left\{ \left(\frac{1-\delta''}{1-\eta} \right) \left(\frac{1+\delta'}{1-\delta} \right) \right\}^{\log \frac{1+2\epsilon}{1-2\delta}} \left(\frac{1-\delta''}{1-\eta} \right)^{(1+\theta)\log \frac{1+2\delta}{1+2\epsilon}} \right]^{\mu_k} . \quad (30)$$

Let T denote the value of the expression inside the brackets on the right-hand side of (30). When η and ϵ tend to zero we observe that

$$\lim_{\eta,\epsilon \to 0} T = \exp \left\{ 2\delta \left(-\theta\delta'' - 2\delta'' + \delta + \delta' \right) + O\left[(\delta')^2 \right] \right\}, \quad (31)$$

when δ' is small. Since δ and δ' can be taken as near as we please to δ, and since θ is positive we conclude that the exponent on the right-hand side of (31) is negative either when δ' and δ'' are sufficiently near to each other and when δ' is small enough. That is to say

$$\lim_{\eta,\epsilon \to 0} T < 1.$$

Due to the continuity of T in η and ϵ we deduce that there exist positive values η and ϵ such that T is less than 1. In view of (30) it follows that $M(r_2)$ approaches zero as k approaches infinity. The theorem is therefore established. $\qquad\square$

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