

Faculteit Wetenschappen
Departement Wiskunde

Clifford Analysis on the Hyperbolic Unit Ball

David Eelbode

promotor: Prof. Dr. Franciscus Sommen

Proefschrift voorgelegd aan de Faculteit Wetenschappen van de Universiteit Gent
tot het behalen van de graad van doctor in de Wiskunde.

Academiejaar 2004–2005

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I have no particular talent.
I am only inquisitive.
(A. Einstein)

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Introduction

In this thesis we consider a projective model for the hyperbolic unit ball, realized as the manifold of rays inside the future cone in the real orthogonal space $\mathbb{R}^{1,m}$, i.e. the flat Minkowski space-time. Using Clifford algebras it is possible to define a Clifford algebra structure on this manifold, and this structure enables us to define the Dirac operator on the hyperbolic unit ball as the Dirac operator on flat Minkowski space-time acting on the sections of a homogeneous Clifford line bundle, which is an associated principal fibre bundle. This thesis then deals with a function theory for Clifford algebra valued nullsolutions for the Dirac operator on the hyperbolic unit ball, the so-called *hyperbolic monogenics*.

William Kingdon Clifford (1845-1879) introduced the algebras which were named after him in 1878 as a generalization of both Grassman's exterior algebra and Hamilton's algebra of quaternions. His objective was to incorporate inside one single structure as well the geometrical as the algebraic properties of the Euclidean space, so he called these structures *geometrical algebras*. These Clifford algebras were often rediscovered later, in particular by physicists. For instance when P.A.M Dirac in 1928 - in his famous article [24] on the electron - introduced the γ -matrices to linearize the Klein-Gordon equation, he actually constructed the generators for the Clifford algebra $\mathbb{R}_{1,3}$.

When constructing the universal Clifford algebra over \mathbb{R} , the algebra of complex numbers is obtained. The Cauchy-Riemann operator, which lies at the very heart of the theory of complex holomorphic functions, factorizes the Laplace operator in two dimensions. Therefore, holomorphic functions in the complex plane may be considered as functions in the kernel of a first order rotationally invariant differential operator factorizing the Laplacian.

It is in this sense that Clifford analysis must be understood as a natural generalization to higher dimension of the theory of holomorphic functions in the complex plane, the Dirac operator being the higher dimensional analogue

of the Cauchy-Riemann operator. The generalized holomorphic functions, known as *monogenic* functions, are thus to be interpreted as nullsolutions for the first order rotationally invariant differential operator factorizing the Laplacian in m dimensions. In the 1930's, the first attempts to develop a function theory for this operator were made by R. Fueter, G. Moisil and N. Théodorescu. A profound study of the monogenic function theory can be found in the book of F. Brackx, R. Delanghe and F. Sommen, see [8].

Whereas most of the classical literature on Clifford analysis deals with the Dirac operator on the flat Euclidean space \mathbb{R}^m , a very natural generalization consists in studying the Dirac operator on general manifolds. Much research has been done in this direction by theoretical physicists and people studying differential geometry, consider for example the so-called Atiyah-Singer Dirac operator on manifolds. However, the language used in this approach differs substantially from the approach followed in e.g. [10], [18] and [40]. In these latter references the Dirac operator on manifolds is studied within the framework of Clifford analysis, by embedding the manifold into an orthogonal space and using the properties of the Dirac operator on the embedding space.

For the specific case of the Dirac operator on a Riemannian space of constant *positive* curvature we refer to the work of J. Ryan and P. Van Lancker, see for instance [58], [59], [75] and [76]. These references are concerned with the Dirac operator on the sphere S^{m-1} in \mathbb{R}^m . The aim of this thesis is to study the Dirac operator on a Riemannian space of constant *negative* curvature, i.e. the Dirac operator on the hyperbolic unit ball. As both the sphere and the hyperbolic unit ball are real submanifolds of the complex sphere in \mathbb{C}^m , one might argue that the function theory on the sphere extends to a function theory on the hyperbolic unit ball by analytic continuation. However, this is far from trivial because in order to obtain a fundamental solution one needs the analytic continuation of distributional solutions for the Dirac equation on the sphere and this requires the use of residues for holomorphic functions in several complex variables. This can all be avoided by using the theory of distributions, and the easiest way to do so is to use the projective picture. Moreover, by fully exploiting the projective nature of our model for these Riemannian spaces of constant curvature, originated by Gel'fand (see [37]), one is easily lead to results that cannot be obtained by analytic continuation. Indeed, in the thesis we consider the limit of the hyperbolic Dirac equation as the singularities approach *infinity*, i.e. the nullcone, which is of course not possible in the Euclidean space.

Research on a monogenic function theory in Poincaré space, a conformal

model for the hyperbolic unit ball, has been started by H. Leutwiler and his students (see e.g. [34] and [49]) and is based on the study of harmonic functions on conformally flat domains. In this context we also mention the work of P. Cerejeiras and J. Cnops, see for instance [11], concerning the Hodge-Dirac operator for hyperbolic spaces. However, as was noted in reference [12] these attempts generalize the Dirac operator for $\text{Spin}(1)$ -fields whereas in this thesis we work with the true hyperbolic generalization of the Dirac operator on $\text{Spin}(\frac{1}{2})$ -fields.

In Chapter 0 we have collected all prerequisites : basic facts concerning Clifford algebras and Clifford analysis on the flat Euclidean space, a short introduction to the theory of special functions, definitions for and elementary properties of Riesz distributions and the Radon transform, etc.

Chapter 1 deals with hyperbolic geometry. First we give a historical overview of the developments in geometry that have ultimately lead to the discovery of non-Euclidean geometry, in particular the hyperbolic plane, and then we give several models for the hyperbolic unit ball (i.e. the higher dimensional analogue of the hyperbolic plane). These include the classical Klein and Poincaré model, the hemisphere model relating these two, the hyperbola H_+ in the future cone containing all space-time vectors of unit hyperbolic norm and most essential for what follows : the projective model, realizing the hyperbolic unit ball as a manifold of rays in the flat m -dimensional Minkowski space-time $\mathbb{R}^{1,m}$.

Chapter 2 deals with the so-called hyperbolic Dirac equation and its fundamental solution. We first introduce the homogeneous Clifford line bundle $\mathbb{R}_{1,m;\alpha}$ (with $\alpha \in \mathbb{C}$ arbitrary) as an associated principal fibre bundle and we then define the Dirac operator on the hyperbolic unit ball as the Dirac operator on flat Minkowski space-time acting on sections of this bundle, the projective nature of our model for the hyperbolic unit ball being essential to this construction. Using the fact that the delta distribution in a point of a general manifold can be defined as the delta distribution in the tangent plane to the manifold at that point, we derive the equation for the hyperbolic fundamental solution, labelled as the *hyperbolic Dirac equation*, and we give several explicit constructions for this fundamental solution.

A first construction uses Frobenius' method to obtain the projection of the hyperbolic fundamental solution on the Klein and Poincaré model for the hyperbolic unit ball as a modulated version of the classical Cauchy kernel.

A second construction reduces the hyperbolic Dirac equation to a problem in two dimensions, and the solution to this problem leads to a fundamental

solution consisting of a singular part and a regular part. Although this latter part is *not* unique, it arises very naturally as is shown at the end of the Chapter. It also allows to rewrite the fundamental solution in terms of the so-called hyperbolic polar co-ordinates, leading to Gegenbauer functions of the second kind.

These Gegenbauer functions of the second kind are then reobtained by means of a third construction, using Riesz distributions.

In the last construction we strongly rely upon properties of the fundamental solution for the wave-operator on the flat Minkowski space-time to obtain a recursive definition, in which the hyperbolic fundamental solution on $\mathbb{R}_{1,m+2}$ can be expressed in terms of the solution on $\mathbb{R}_{1,m}$.

In the third Chapter we generalize the idea behind the first construction for the hyperbolic fundamental solution, leading to the so-called *Modulation Theorems*, stating that homogeneous monogenic functions with respect to the Dirac operator on the flat Euclidean space can be modulated to obtain solutions for the hyperbolic Dirac equation. This is proved for both the Klein and Poincaré model for the hyperbolic unit ball. In proving the equivalence of both Theorems we then obtain a geometrical interpretation for certain properties of the hypergeometric function.

Next, we consider two generalizations of the Modulation Theorems : we construct solutions for natural powers of the hyperbolic Dirac operator and we also construct solutions for the Dirac operator on ultrahyperbolic spaces of arbitrary signature (p, q) .

At the end of the third Chapter we consider a specific bi-axial problem for the projection of the hyperbolic Dirac operator on the Klein model. Although at first sight this problem does not seem to be related to the Modulation Theorems, it leads to a new interpretation for the modulated solutions as generalized hyperbolic power functions.

In Chapter 4 we define arbitrary complex powers of the hyperbolic Dirac operator and using Riesz distributions we construct a fundamental solution for these operators. This Chapter is inspired by reference [7].

Chapter 5 contains a function theory for the hyperbolic Dirac operator, both on the hyperbola H_+ in the future cone in $\mathbb{R}^{1,m}$ and on the Klein model for the hyperbolic unit ball. For that purpose, the Cauchy-Pompeju Theorem, Stokes' Theorem and Cauchy's Theorem are essential.

In order to establish the Taylor and Laurent series for hyperbolic monogenic functions defined in an open (annular) domain of the hyperbolic unit ball, an axial decomposition for the fundamental solution is given. To do so, we

use the Modulation Theorems of Chapter 3 and this leads to an alternative interpretation for the Addition Theorem for Gegenbauer functions (see [25]). At the end of the fifth Chapter we briefly discuss the notion of eigenfunctions for the hyperbolic Dirac operator.

In Chapter 6 we introduce the photogenic Cauchy transform (PCT), defined as an integral transform whose kernel is to be seen as the fundamental solution for the hyperbolic Dirac equation with singularity at infinity. In contrast to the flat Euclidean case, it makes sense to consider singularities at infinity in the hyperbolic setting by considering a delta distribution on the nullcone. As functions $f \in L_2(S^{m-1})$ can be decomposed in a series of inner and outer spherical monogenics on the sphere S^{m-1} , we then determine the PCT of these spherical monogenics, leading to a new interpretation for the modulated solutions for the hyperbolic Dirac operator from Chapter 3.

Next we determine the photogenic boundary values of these transforms by letting the argument of the PCT of spherical monogenics approach the sphere S^{m-1} . Under certain restrictions on the complex parameter α , this leads to a mapping from the Sobolev space $W_2(S^{m-1})$ on the sphere to the set of boundary values of hyperbolic monogenic functions on the Klein model.

By considering the extension of these boundary values to the Lie sphere, it is then also proved that this latter set yields a Hilbert module with reproducing kernel, again under certain restrictions on the complex parameter α .

Chapter 7 deals with the conformal Dirac operator and illustrates how the hyperbolic Dirac operator considered in the thesis must be understood in terms of the Dirac operator on a general manifold. By defining a spin bundle on the hyperbola H_+ it is shown that the Clifford algebra valued functions considered throughout this manuscript are actually Clifford sections, and by refining this spin bundle to a spinor bundle it is shown that for one specific value for α our hyperbolic Dirac operator reduces to the Atiyah-Singer Dirac operator.

Introductory Notes : the Mathematical Toolbox

Each problem that I solved became a rule which served afterwards to solve other problems. (René Descartes)

In this Chapter all prerequisites are collected : the introduction of a Clifford algebra, the main results from Clifford analysis on the flat Euclidean space, some results from the theory of the hypergeometric function, the Legendre function and the Gegenbauer function, the definition and basic properties of the Riesz distributions, some results on the fundamental solution of the wave-operator, an introduction to the theory of bundles and to the theory of Clifford analysis on the Lie ball and Lie sphere. This Chapter also settles the notations that will be used throughout the thesis.

0.1 The Clifford Setting

In this section an introduction to Clifford algebras and Clifford analysis is given. As we will encounter orthogonal spaces of different signatures throughout this manuscript, we consider three subsections. First we consider the most general Clifford algebra $\mathbb{R}_{p,q}$ and then we consider the algebras $\mathbb{R}_{0,m}$ and $\mathbb{R}_{1,m}$, associated to the flat Euclidean space $\mathbb{R}^{0,m}$ and the flat Minkowski space-time $\mathbb{R}^{1,m}$ respectively.

0.1.1 The General Clifford Algebra $\mathbb{R}_{p,q}$

Consider the real orthogonal space $\mathbb{R}^{p,q}$, where $p + q = m$, with orthonormal basis $B_{p,q}(\epsilon_i, e_j)$, given by

$$B_{p,q}(\epsilon_i, e_j) = \{\epsilon_1, \dots, \epsilon_p, e_1, \dots, e_q\} ,$$

endowed with the quadratic form

$$Q_{p,q}(\underline{T}, \underline{X}) = \sum_{i=1}^p T_i^2 - \sum_{j=1}^q X_j^2 \quad (1)$$

where the elements of $\mathbb{R}^{p,q}$ are labeled as (p, q) -space-time vectors

$$(\underline{T}, \underline{X}) = (T_1, \dots, T_p, X_1, \dots, X_q) .$$

In terms of the inner product $\langle \cdot, \cdot \rangle_{p,q}$ on $\mathbb{R}^{p,q}$, given by

$$\langle (\underline{T}, \underline{X}), (\underline{S}, \underline{Y}) \rangle_{p,q} = \sum_{i=1}^p T_i S_i - \sum_{j=1}^q X_j Y_j ,$$

the quadratic form $Q_{p,q}(\underline{T}, \underline{X})$ can also be defined by

$$Q_{p,q}(\underline{T}, \underline{X}) = \langle (\underline{T}, \underline{X}), (\underline{T}, \underline{X}) \rangle_{p,q} .$$

The signature of the orthogonal space under consideration will be indicated where necessary to avoid confusion.

An arbitrary space-time vector $(\underline{T}, \underline{X}) \in \mathbb{R}^{p,q}$ thus consists of two parts : the vector \underline{T} belongs to $\mathbb{R}^{p,0}$ endowed with the inner product

$$\langle \underline{T}, \underline{S} \rangle_{p,0} = \sum_{i=1}^p T_i S_i ,$$

where $\langle \underline{T}, \underline{T} \rangle_{p,0} = Q_{p,0}(\underline{T})$. This vector \underline{T} forms the so-called *temporal* part of the (p, q) -space-time vector $(\underline{T}, \underline{X})$ and is a linear combination of the temporal unit vectors ϵ_i , $1 \leq i \leq p$.

The vector \underline{X} belongs to $\mathbb{R}^{0,q}$ endowed with the inner product

$$\langle \underline{X}, \underline{Y} \rangle_{0,q} = - \sum_{j=1}^q X_j Y_j ,$$

where $\langle \underline{X}, \underline{X} \rangle_{0,q} = Q_{0,q}(\underline{X})$, and forms the *spatial* part. Spatial vectors are a linear combination of the spatial unit vectors e_j , $1 \leq j \leq q$.

Throughout this thesis, temporal unit vectors will always be denoted by ϵ , and spatial unit vectors by e . A summation over temporal components will always be denoted by \sum_i whereas a summation over spatial components

will be denoted by \sum_j .

Also note that *the standard Euclidean inner product* on \mathbb{R}^m , i.e. the inner product on the real orthogonal space $\mathbb{R}^{m,0}$, will be denoted *without* explicitly mentioning the signature $(m, 0)$. By definition we thus put for two vectors \underline{x} and \underline{y} in the Euclidean vector space \mathbb{R}^m :

$$\langle \underline{x}, \underline{y} \rangle = \sum_{j=1}^m x_j y_j .$$

The Clifford algebra $\mathbb{R}_{p,q}$ associated to the orthogonal space $\mathbb{R}^{p,q}$ is the real 2^m -dimensional associative, but non-commutative algebra generated by the following multiplication rules : $\epsilon_i \epsilon_j + \epsilon_j \epsilon_i = 2\delta_{ij}$ for all $1 \leq i, j \leq p$, $e_r e_s + e_s e_r = -2\delta_{rs}$ for all $1 \leq r, s \leq q$ and $\epsilon_i e_j + e_j \epsilon_i = 0$ for all $1 \leq i \leq p$ and $1 \leq j \leq q$.

The Clifford algebra $\mathbb{R}_{p,q}$ can also be defined by means of the associative tensor algebra $T(\mathbb{R}^{p,q})$, generated by the unity element $1 \in \mathbb{R}$ together with the vector space $\mathbb{R}^{p,q}$. The ideal J of $T(\mathbb{R}^{p,q})$ consisting of sums of terms of the form $a \otimes ((\underline{T}, \underline{X}) \otimes (\underline{T}, \underline{X}) - Q_{p,q}((\underline{T}, \underline{X}), (\underline{T}, \underline{X}))) \otimes b$, with $a, b \in T(\mathbb{R}^{p,q})$ and $(\underline{T}, \underline{X}) \in \mathbb{R}^{p,q}$, is essential to this construction and allows to interpret the Clifford algebra as the vector space of exterior forms, i.e. the totally anti-symmetric tensors, with a Clifford product containing both the exterior product and the inner product on $\mathbb{R}^{p,q}$. For an explicit construction we refer e.g. to reference [5]. The idea to incorporate inside a single structure as well the inner product as the exterior product was the main motivation for William Kingdon Clifford to introduce the *geometric algebras*, as he called these structures which were later named after him in 1878.

An element of $\mathbb{R}_{p,q}$ is called a *Clifford number*; it has the form $a = \sum_{A \subset M} a_A e_A$ where $a_A \in \mathbb{R}$ and $M = \{1, \dots, p+q\}$. If $A \subset M$ is given as the set $\{i_1, \dots, i_k\}$, with $1 \leq i_1 < \dots < i_k \leq p+q$ then the element e_A is given as the product of the (temporal or spatial) basis vectors corresponding with these indices whereas $e_\emptyset = 1$. If A has k elements, e_A is called a *k-vector*. The space of *k-vectors* is denoted by $\mathbb{R}_{p,q}^{(k)}$. If $[a]_k$ is the projection of the Clifford number a on $\mathbb{R}_{p,q}^{(k)}$, then

$$a = \sum_{k=0}^m [a]_k , \quad \forall a \in \mathbb{R}_{p,q} .$$

The algebra $\mathbb{R}_{p,q}$ inherits a natural \mathbb{Z}_2 -grading by introducing the notion of *even* and *odd* Clifford numbers, which are defined as linear combinations of

basis vectors e_A with respectively an even or odd number of indices. The subsets of $\mathbb{R}_{p,q}$ containing respectively the even or odd Clifford numbers are eigenspaces for the so-called main involution, defined a bit further, with eigenvalues ± 1 . The subspace $\mathbb{R}_{p,q}^{(+)} = \sum_k \text{even} \oplus \mathbb{R}_{p,q}^{(k)}$ is a subalgebra of $\mathbb{R}_{p,q}$, called the *even subalgebra*.

In order to make a clear distinction between a (p, q) -space-time vector $(\underline{T}, \underline{X})$ in $\mathbb{R}^{p,q}$ and the 1-vector associated to this $(p+q)$ -tuple, we denote the latter by $\underline{T} + \underline{X}$. There is a canonical embedding of the orthogonal space $\mathbb{R}^{p,q}$ into its Clifford algebra $\mathbb{R}_{p,q}$, given by :

$$\mathbb{R}^{p,q} \mapsto \mathbb{R}_{p,q}^{(1)} \subset \mathbb{R}_{p,q} : (\underline{T}, \underline{X}) \mapsto \underline{T} + \underline{X} .$$

For two 1-vectors $\underline{T} + \underline{X}, \underline{S} + \underline{Y} \in \mathbb{R}_{p,q}^{(1)}$, the inner and outer product are defined as follows :

$$\begin{aligned} (\underline{T} + \underline{X}) \cdot (\underline{S} + \underline{Y}) &= \frac{1}{2} \left((\underline{T} + \underline{X})(\underline{S} + \underline{Y}) + (\underline{S} + \underline{Y})(\underline{T} + \underline{X}) \right) \\ (\underline{T} + \underline{X}) \wedge (\underline{S} + \underline{Y}) &= \frac{1}{2} \left((\underline{T} + \underline{X})(\underline{S} + \underline{Y}) - (\underline{S} + \underline{Y})(\underline{T} + \underline{X}) \right) \end{aligned}$$

whence $(\underline{T} + \underline{X})(\underline{S} + \underline{Y}) = (\underline{T} + \underline{X}) \cdot (\underline{S} + \underline{Y}) + (\underline{T} + \underline{X}) \wedge (\underline{S} + \underline{Y})$. This expresses the Clifford product on $\mathbb{R}_{p,q}$ in terms of the inner and outer product.

On $\mathbb{R}_{p,q}$ we have three important involutory (anti-)automorphisms. For all $a, b \in \mathbb{R}_{p,q}$ and $\lambda \in \mathbb{R}$ we define :

- the main involution $a \mapsto \tilde{a}$

$$\tilde{\epsilon}_i = -\epsilon_i , \quad \tilde{e}_j = -e_j , \quad (ab)^\sim = \tilde{a}\tilde{b}$$

- the reversion $a \mapsto a^*$

$$\epsilon_i^* = \epsilon_i , \quad e_j^* = e_j , \quad (ab)^* = b^*a^*$$

- the conjugation (also known as bar-map) $a \mapsto \bar{a}$

$$\bar{\epsilon}_i = -\epsilon_i , \quad \bar{e}_j = -e_j , \quad \overline{ab} = \bar{b}\bar{a}$$

These definitions extend to the complexified Clifford algebra $\mathbb{C}_{p,q} = \mathbb{C} \otimes \mathbb{R}_{p,q}$, where we also have

- the Hermitian conjugation $a \mapsto a^+$, which is the tensorproduct of the bar-map on $\mathbb{R}_{p,q}$ and the classical complex conjugation on \mathbb{C}

$$(a + ib)^+ = \bar{a} - i\bar{b} .$$

The Clifford group $\Gamma(p, q)$ is the set of all invertible elements $g \in \mathbb{R}_{p,q}$ such that for all $\underline{T} + \underline{X} \in \mathbb{R}_{p,q}^{(1)} : g(\underline{T} + \underline{X})\tilde{g}^{-1} \in \mathbb{R}_{p,q}^{(1)}$, the Pin group $\text{Pin}(p, q)$ is the quotient group $\Gamma(p, q)/\mathbb{R}^+$ and the Spin group $\text{Spin}(p, q) = \text{Pin}(p, q) \cap \mathbb{R}_{p,q}^{(+)}$.

For each $s \in \text{Pin}(p, q)$ the map $\chi(s) : \mathbb{R}^{p,q} \mapsto \mathbb{R}^{p,q} : \underline{T} + \underline{X} \mapsto s(\underline{T} + \underline{X})\bar{s}$ induces a map from $\mathbb{R}^{p,q}$ onto itself. In this way $\text{Pin}(p, q)$ defines a double covering of the orthogonal group $O(p, q)$, defined as the set of linear transformations on $\mathbb{R}^{p,q}$ leaving the quadratic form $Q_{p,q}(\underline{T}, \underline{X})$ invariant, whereas $\text{Spin}(p, q)$ defines a double covering of the group $SO(p, q)$, which is the subgroup of $O(p, q)$ containing all elements of unit determinant. For more details, the reader is referred to [23] and [56].

Definition 0.1 *The Dirac operator on the real orthogonal space $\mathbb{R}^{p,q}$ is the vector derivative*

$$D(\underline{T}, \underline{X})_{p,q} = \sum_{i=1}^p \epsilon_i \partial_{T_i} - \sum_{j=1}^q e_j \partial_{X_j} .$$

We prefer to use a notation for the Dirac operator which indicates both the signature of the space on which this operator acts and the variables used to describe elements of this space. When working with a specific Clifford algebra, in casu $\mathbb{R}_{0,m}$ or $\mathbb{R}_{1,m}$, we will introduce more appropriate notations. We refer to the next subsections for this.

Let Ω be an open subset of $\mathbb{R}^{p,q}$ and let $f \in C^1(\Omega)$ be an $\mathbb{R}_{p,q}$ -valued function on $\mathbb{R}^{p,q}$. This function is then called *left* (respectively *right*) *monogenic* with respect to the Dirac operator on $\mathbb{R}^{p,q}$ if and only if $D(\underline{T}, \underline{X})_{p,q} f(\underline{T}, \underline{X}) = 0$ in Ω (resp. $f(\underline{T}, \underline{X}) D(\underline{T}, \underline{X})_{p,q} = 0$, where this latter notation denotes the action of the Dirac operator from the right). It is the concept of monogenic functions which lies at the very heart of Clifford analysis.

In Chapter 3 the operator $D(\underline{T}, \underline{X})_{p,q}$ will be studied in greater detail and we will prove a Theorem to construct nullsolutions for this operator.

0.1.2 Clifford Analysis on Flat Euclidean Space

Most of the literature concerning Clifford analysis deals with the so-called *flat Euclidean space* \mathbb{R}^m . In this section we gather the most important results for the Dirac operator on \mathbb{R}^m . As a general reference for this subsection we mention [8], [23] and [40]. A nice overview of the most basic results can be found in [22].

In order to fit the classical situation into the framework of the previous subsection, our starting point is the real orthogonal space $\mathbb{R}^{0,m}$ generated by the orthonormal basis $B_{0,m}(e_j) = \{e_1, \dots, e_m\}$, endowed with the quadratic form

$$Q_{0,m}(\underline{x}) = -\sum_{j=1}^m x_j^2 = -r^2 = \langle \underline{x}, \underline{x} \rangle_{0,m} = -\langle \underline{x}, \underline{x} \rangle ,$$

where we have put $r = |\underline{x}| = (\sum_j x_j^2)^{\frac{1}{2}}$.

The Clifford algebra $\mathbb{R}_{0,m}$ associated to $\mathbb{R}^{0,m}$ endowed with the quadratic form $Q_{0,m}(\underline{x})$ is generated by the multiplication rules $e_i e_j + e_j e_i = -2\delta_{ij}$, for all $1 \leq i, j \leq m$. In literature this Clifford algebra is often denoted as \mathbb{R}_m , without any specification concerning the quadratic form which lies behind the algebra. However, since we will consider orthogonal spaces of different signature throughout this thesis we prefer to indicate this signature $(0, m)$.

Vectors in $\mathbb{R}^{0,m}$ are identified with 1-vectors in $\mathbb{R}_{0,m}$ under the canonical map

$$(\underline{x}) = (x_1, \dots, x_m) \mapsto \underline{x} = \sum_{j=1}^m e_j x_j .$$

For two 1-vectors \underline{x} and $\underline{y} \in \mathbb{R}_{0,m}^{(1)}$, we have

$$\underline{x}\underline{y} = \underline{x} \cdot \underline{y} + \underline{x} \wedge \underline{y} ,$$

where the Clifford inner product is given by

$$\underline{x} \cdot \underline{y} = -\sum_{j=1}^m x_j y_j = -\langle \underline{x}, \underline{y} \rangle ,$$

with $\langle \underline{x}, \underline{y} \rangle$ the standard Euclidean inner product, and the outer product by

$$\underline{x} \wedge \underline{y} = \sum_{j < k} e_{jk} (x_j y_k - x_k y_j) .$$

Apart from the general definitions given in the previous subsection, we may define the Clifford group $\Gamma(0, m)$ and its subgroups as follows :

1. the Clifford group $\Gamma(0, m)$ is generated by the non-zero vectors of \mathbb{R}^m
2. the Pin group $\text{Pin}(0, m)$ is the subgroup of $\Gamma(0, m)$ consisting of products of unit vectors in \mathbb{R}^m
3. the Spin group $\text{Spin}(0, m)$ is the subgroup of $\text{Pin}(0, m)$ consisting of products of an *even* number of unit vectors in \mathbb{R}^m .

According to Definition 0.1 the Dirac operator on $\mathbb{R}^{0,m}$ is given by the vector derivative

$$D(\underline{x})_{0,m} = - \sum_{j=1}^m e_j \partial_{x_j} .$$

However, the overall minus sign may of course be omitted and that is why we adopt the standard notation for the Dirac operator on the vector space \mathbb{R}^m endowed with the quadratic form $Q_{0,m}(\underline{x})$:

Definition 0.2 *The Dirac operator on the real orthogonal space $\mathbb{R}^{0,m}$ is the vector derivative*

$$\underline{\partial} = \sum_{j=1}^m e_j \partial_{x_j}$$

Remark : In what follows we will sometimes use two vector variables \underline{x} and \underline{y} on \mathbb{R}^m , so it becomes important to indicate on which variable the Dirac operator acts. In that case we will respectively use $\underline{\partial}_x$ and $\underline{\partial}_y$.

Let $\Omega \subset \mathbb{R}^m$ be an open subset and let $f : \Omega \mapsto \mathbb{R}^m$ be a C^1 -function on Ω which is $\mathbb{R}_{0,m}$ -valued. If f satisfies $\underline{\partial}f = 0$ (respectively $f\underline{\partial} = 0$) in Ω , f is called left (respectively right) *monogenic* in Ω . As the Dirac operator satisfies $\underline{\partial}^2 = -\Delta_m$, with $\Delta_m = \partial_{x_1}^2 + \dots + \partial_{x_m}^2$ the Laplacian on \mathbb{R}^m , monogenic functions in Ω are a refinement of harmonic functions in Ω .

In polar co-ordinates we have $\underline{x} = r\xi$ with $\xi \in S^{m-1}$, the unit sphere in \mathbb{R}^m . The Dirac operator $\underline{\partial}$ admits the following polar decomposition :

$$\underline{\partial} = \underline{\xi} \left(\partial_r + \frac{1}{r} \Gamma_{0,m} \right) ,$$

where $\Gamma_{0,m} = -\underline{x} \wedge \underline{\partial}$ is the spherical Dirac operator (or Gamma operator) on S^{m-1} . In terms of the so-called *angular momentum operators* L_{ij} , we also have :

$$\begin{aligned}\Gamma_{0,m} &= - \sum_{j < k} e_{jk} L_{jk} \\ &= - \sum_{j < k} e_{jk} (x_j \partial_k - x_k \partial_j) .\end{aligned}$$

When it is clear that we are dealing with the Gamma operator on S^{m-1} , the subscript $(0, m)$ will be suppressed to avoid overloaded notations. This will be the case in the rest of this subsection.

Remark : In case we are considering functions $f(\xi, \eta)$ depending on two co-ordinates $\underline{\xi}$ and $\underline{\eta} \in S^{m-1}$, we will label the Gamma operator respectively as $\Gamma_{\underline{\xi}}$ and $\Gamma_{\underline{\eta}}$ to indicate on which of the variables the operator acts.

Note that

$$\bar{x} \underline{\partial} = -\underline{x} \underline{\partial} = \underline{x} D(\underline{x})_{0,m} = \mathbb{E}_r + \Gamma ,$$

with $\mathbb{E}_r = r \partial_r = \sum_j x_j \partial_j$ the Euler operator on \mathbb{R}^m measuring the degree of homogeneity with respect to the co-ordinates \underline{x} on \mathbb{R}^m .

Definition 0.3 An $\mathbb{R}_{0,m}$ -valued C^∞ function $P_k(\underline{\omega})$ on S^{m-1} is called an inner spherical monogenic of order k if it is the restriction to the unit sphere of a polynomial monogenic function on \mathbb{R}^m of order k .

Definition 0.4 An $\mathbb{R}_{0,m}$ -valued C^∞ function $Q_k(\underline{\omega})$ on S^{m-1} is called an outer spherical monogenic of order k if it is the restriction to the unit sphere of a monogenic function on $\mathbb{R}^m \setminus \{0\}$, homogeneous of degree $(1 - k - m)$.

These classical *spherical monogenics* are the only *global* eigenfunctions of the Gamma operator on S^{m-1} , satisfying :

$$\begin{aligned}\Gamma P_k &= -k P_k \\ \Gamma Q_k &= (k + m - 1) Q_k .\end{aligned}$$

Local eigenfunctions for the Gamma operator have been studied extensively in [68, 75, 76], but we return to this point later.

The set of inner (respectively outer) spherical monogenics provided with the obvious laws of addition and (right) multiplication with Clifford numbers is a right Clifford-module, denoted as $M^+(k)$ (respectively $M^-(k)$). Inner and outer spherical monogenics are related in the following sense :

$$\begin{aligned} P_k(\underline{\omega}) \in M^+(k) &\Rightarrow \underline{\omega} P_k(\underline{\omega}) \in M^-(k) \\ Q_k(\underline{\omega}) \in M^-(k) &\Rightarrow \underline{\omega} Q_k(\underline{\omega}) \in M^+(k) . \end{aligned}$$

The fundamental solution for the Dirac operator $\underline{\partial}$ on \mathbb{R}^m is the so-called Cauchy kernel $E(\underline{x})$, defined by

$$E(\underline{x}) = \frac{\Gamma\left(\frac{m}{2}\right)}{2\pi^{\frac{m}{2}}} \frac{\underline{x}}{|\underline{x}|^m} = \frac{1}{A_m} \frac{\underline{x}}{|\underline{x}|^m}$$

with A_m the area of the unit sphere in \mathbb{R}^m . The Cauchy kernel is both left and right monogenic in $\mathbb{R}^m \setminus \{0\}$ with respect to $\underline{\partial}$ and it satisfies the equation $\underline{\partial}E(\underline{x}) = -\delta(\underline{x})$ in distributional sense. Since the Dirac operator $\underline{\partial}$ is invariant under translations, we have $\underline{\partial}E(\underline{x} - \underline{y}) = -\delta(\underline{x} - \underline{y})$. A series representation for the Cauchy kernel can easily be found as follows :

$$E(\underline{x} - \underline{y}) = \frac{1}{A_m} \frac{\underline{x} - \underline{y}}{|\underline{x} - \underline{y}|^m} = \frac{1}{A_m} \frac{1}{m-2} \underline{\partial}_y \frac{1}{|\underline{x} - \underline{y}|^{m-2}} .$$

With $\underline{x} = |\underline{x}| \underline{\xi}$, $\underline{y} = |\underline{y}| \underline{\eta}$ and putting $s = \frac{|\underline{x}|}{|\underline{y}|}$ and

$$t = \langle \underline{\xi}, \underline{\eta} \rangle = -\underline{\xi} \cdot \underline{\eta} = \sum_{j=1}^m \xi_j \eta_j$$

one can easily verify that for $|\underline{x}| < |\underline{y}|$

$$\frac{1}{|\underline{x} - \underline{y}|^{m-2}} = |\underline{y}|^{2-m} (1 - 2ts + s^2)^{-\left(\frac{m}{2}-1\right)} .$$

As

$$(1 - 2ts + s^2)^{-\left(\frac{m}{2}-1\right)} = \sum_{k=0}^{\infty} C_k^{\frac{m}{2}-1}(t) s^k ,$$

with $C_k^\mu(t)$ the classical Gegenbauer polynomial, we get immediately that

$$\begin{aligned} &\underline{\partial}_y \frac{1}{|\underline{x} - \underline{y}|^{m-2}} \\ &= \underline{\eta} \sum_k \left\{ (2 - m - k) C_k^{\frac{m}{2}-1}(t) + \Gamma_{\underline{\eta}}(t) \frac{d}{dt} \left(C_k^{\frac{m}{2}-1}(t) \right) \right\} \frac{|\underline{x}|^k}{|\underline{y}|^{m+k-1}} . \end{aligned}$$

Using the fact that

$$\Gamma_{\underline{\eta}}(t) = \Gamma(< \underline{\xi}, \underline{\eta} >) = \underline{\xi} \wedge \underline{\eta}$$

and the recurrence relations for the Gegenbauer polynomials (see also section 0.2.3) the following series expansion for the Cauchy kernel is obtained :

$$E(\underline{x} - \underline{y}) = -\frac{1}{A_m} \sum_{k=0}^{\infty} \frac{|\underline{x}|^k}{|\underline{y}|^{k+m-1}} \left\{ C_k^{\frac{m}{2}}(t) \underline{\eta} - C_{k-1}^{\frac{m}{2}}(t) \underline{\xi} \right\} . \quad (2)$$

Note that the function between brackets is an inner spherical monogenic with respect to the Dirac operator $\underline{\partial}_x$ and an outer spherical monogenic with respect to the Dirac operator $\underline{\partial}_y$:

$$\begin{aligned} \Gamma_{\underline{\xi}} \left\{ C_k^{\frac{m}{2}}(t) \underline{\eta} - C_{k-1}^{\frac{m}{2}}(t) \underline{\xi} \right\} &= -k \left\{ C_k^{\frac{m}{2}}(t) \underline{\eta} - C_{k-1}^{\frac{m}{2}}(t) \underline{\xi} \right\} \\ \Gamma_{\underline{\eta}} \left\{ C_k^{\frac{m}{2}}(t) \underline{\eta} - C_{k-1}^{\frac{m}{2}}(t) \underline{\xi} \right\} &= (k+m-1) \left\{ C_k^{\frac{m}{2}}(t) \underline{\eta} - C_{k-1}^{\frac{m}{2}}(t) \underline{\xi} \right\} \end{aligned}$$

Arbitrary functions $f \in L_2(S^{m-1})$ can then be decomposed as

$$f(\underline{\xi}) = \sum_{k=0}^{\infty} P(k)[f](\underline{\xi}) + Q(k)[f](\underline{\xi})$$

where the series converges in L_2 -sense on S^{m-1} . The projections $P(k)[f]$ and $Q(k)[f]$ of the function f on the spaces $M^+(k)$ and $M^-(k)$ of inner and outer spherical monogenics of order k are given by :

$$\begin{aligned} P(k)[f](\underline{\eta}) &= -\frac{1}{A_m} \underline{\eta} \int_{S^{m-1}} \left\{ C_k^{\frac{m}{2}}(t) \underline{\eta} - C_{k-1}^{\frac{m}{2}}(t) \underline{\xi} \right\} f(\underline{\xi}) dS(\underline{\xi}) \\ Q(k)[f](\underline{\eta}) &= -\frac{1}{A_m} \int_{S^{m-1}} \left\{ C_k^{\frac{m}{2}}(t) \underline{\eta} - C_{k-1}^{\frac{m}{2}}(t) \underline{\xi} \right\} \underline{\xi} f(\underline{\xi}) dS(\underline{\xi}) . \end{aligned}$$

On the module $L_2(S^{m-1})$, provided with the inner product

$$(f, g) = \int_{S^{m-1}} \bar{f}(\underline{\xi}) g(\underline{\xi}) dS(\underline{\xi}) ,$$

the spherical Dirac operator Γ on S^{m-1} is a self-adjoint operator. This is a consequence of Stokes' Theorem on the sphere, and in explicit form it says that

$$\int_{S^{m-1}} [\overline{(\Gamma f)} g - \bar{f}(\Gamma g)] dS(\underline{\xi}) = 0 \implies (\Gamma f, g) = (f, \Gamma g) . \quad (3)$$

Using the conjugation map, this can also be written as

$$\int_{S^{m-1}} [(f\Gamma)g + f(\Gamma g)] dS(\underline{\xi}) = 0, \quad (4)$$

where $(f\Gamma)$ denotes the action of the spherical Dirac operator from the right.

Next, let us consider the Moebius transformations over the compactification $\mathbb{R}^m \cup \{\infty\} = S^m$ of \mathbb{R}^m . As shown in [1], [73] and elsewhere any Moebius transformation $y = \psi(\underline{x})$ can be expressed as

$$\underline{y} = \frac{a\underline{x} + b}{c\underline{x} + d}$$

where $a, b, c, d \in \mathbb{R}_{0,m}$ are Clifford numbers satisfying

1. a, b, c, d are products of vectors from \mathbb{R}^m
2. $a\tilde{c}, c\tilde{d}, d\tilde{b}$ and $b\tilde{a} \in \mathbb{R}^m$
3. $a\tilde{d} - b\tilde{c} \in \mathbb{R}_0$

The matrix M , defined as

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with entries satisfying these conditions is then called a Vahlen matrix and gives a Moebius transformation, i.e. a finite composition of orthogonal transformations, inversions, translations and dilatations. The set of all Vahlen matrices is a group under multiplication, the so-called Vahlen group.

Suppose now that $\underline{y} = \psi(\underline{x}) = (a\underline{x} + b)(c\underline{x} + d)^{-1}$ is a Moebius transformation, and suppose that $f(\underline{y})$ is a left monogenic function in the variable \underline{y} . Then, see reference [15], the function

$$J(\psi, \underline{x}) f(\psi(\underline{x})) = \frac{(c\underline{x} + d)^\sim}{|c\underline{x} + d|^m} f(\psi(\underline{x}))$$

is left monogenic in the variable \underline{x} , with $|c\underline{x} + d|^2 = (c\underline{x} + d)(c\underline{x} + d)^\sim$. This means that the space of monogenic functions is *not* closed under the action of the Moebius group, one needs an additional *conformal weight factor* to maintain monogenic functions. It should be mentioned that H. Leutwiler and his students dealt with this problem by modifying the Clifford system

in such a way that the space of modified monogenic functions, the so-called (H)-solutions, is closed under the action of the Moebius group. We refer e.g. to [34], [49] and [50] for more information.

Another way to obtain solutions for the Dirac operator $\underline{\partial}$ on \mathbb{R}^m is given by the so-called *Cauchy-Kowalevska Theorem*, which answers the following question :

” Given an $\mathbb{R}_{0,m}$ -valued function $f(\underline{x})$ on \mathbb{R}^m which is analytic in an open subset Ω of \mathbb{R}^m , does there exist a monogenic function $f^*(x_0, \underline{x})$ in some open neighbourhood Ω^* of Ω in \mathbb{R}^{m+1} such that $f^*|_{x_0=0} = f$ in Ω ?”

First of all, note that an open neighbourhood Ω^* of $\Omega \subset \mathbb{R}^m$ is said to be x_0 -normal if for each $(x_0, \underline{x}) \in \Omega^*$ the line segment $\{(x_0, \underline{x}) + te_0 : t \in \mathbb{R}\} \cap \Omega^*$ is connected and contains just one point in Ω . We then have the following Theorem :

Theorem 0.1 (*Cauchy-Kowalevska*) Let $\Omega \subset \mathbb{R}^m$ be open and let $f(\underline{x})$ be an $\mathbb{R}_{0,m}$ -valued analytic function on Ω . Then the function $f^*(x_0, \underline{x})$ given by

$$f^*(x_0, \underline{x}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x_0^k (\bar{e}_0 \underline{\partial})^k f(\underline{x})$$

satisfies $(e_0 \partial_{x_0} + \underline{\partial}) f^*(x_0, \underline{x}) = 0$ in an open connected, normal neighbourhood Ω^* of Ω in \mathbb{R}^{m+1} . Moreover $f^*|_{x_0=0} = f$ in Ω .

Note that the extension $f^*(x_0, \underline{x})$ of $f(\underline{x})$ is formally given by

$$f^*(x_0, \underline{x}) = e^{-x_0 \bar{e}_0 \underline{\partial}} f(\underline{x}) .$$

We end this section with some considerations on the algebra \mathcal{P} of Clifford polynomials, generated by the set $\{x_1, \dots, x_m; e_1, \dots, e_m\}$, where the set $\{x_1, \dots, x_m\}$ is to be considered as a set of commuting symbols. This algebra inherits a natural \mathbb{Z} -gradation by putting

$$\mathcal{P} = \sum_{k=0}^{\infty} \mathcal{P}_k ,$$

with \mathcal{P}_k the set of k -homogeneous polynomials. Note that each space \mathcal{P}_k is an eigenspace for the Euler operator \mathbb{E}_r on \mathbb{R}^m with eigenvalue k . The

elements of \mathcal{P}_k which are monogenic with respect to the operator $\underline{\partial}$ give rise to the inner spherical monogenics defined earlier, after restriction to the unit sphere S^{m-1} .

Making use of the basic operator identities

$$\begin{aligned}\underline{\partial} \underline{x} &= -\mathbb{E}_r + \Gamma - m \\ \underline{x} \underline{\partial} &= -\mathbb{E}_r - \Gamma ,\end{aligned}$$

when acting from the left, we have immediately that the simultaneous eigenspaces of \mathbb{E}_r and Γ are the spaces $\mathcal{M}_{k,s}$ of polynomials of the form $\underline{x}^s P_k(\underline{x})$, where $P_k(\underline{x})$ stands for a spherical monogenic of degree k . We also have the following identity for these polynomials :

$$\underline{\partial}(\underline{x}^s P_k(\underline{x})) = B_{k,s} \underline{x}^{s-1} P_k(\underline{x}) \quad \text{with} \quad \begin{cases} B_{2k,s} &= -2s \\ B_{2k+1,s} &= -(2s + 2k + m) \end{cases}$$

These polynomials are important, as they are the building blocks for the so-called *Fischer decomposition*, which provides the space \mathcal{P} of Clifford polynomials with an inner product for which the spaces \mathcal{P}_k are orthogonal :

$$(R(\underline{x}), S(\underline{x})) = [\bar{R}(\underline{\partial})S(\underline{x})]_0|_{\underline{x}=\underline{0}} .$$

Any homogeneous polynomial $R_k(\underline{x}) \in \mathcal{P}_k$ then has a unique orthogonal decomposition of the form

$$R_k(\underline{x}) = P_k(\underline{x}) + \underline{x} R_{k-1}(\underline{x}) = \sum_{j=0}^k \underline{x}^j P_{k-j}(\underline{x}) ,$$

with

$$\underline{\partial} P_{k-j}(\underline{x}) = 0 \implies \underline{x}^j P_{k-j}(\underline{x}) \in \mathcal{M}_{k-j,j} .$$

This orthogonal decomposition refines the classical *Fischer decomposition* for polynomials $R_k(\underline{x})$ of degree k in terms of solid harmonic polynomials :

$$S_k(\underline{x}) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} r^{2j} S_{k-2j}(\underline{x}) ,$$

with $S_{k-2j}(\underline{x})$ a polynomial of degree $(k - 2j)$ such that $\Delta_m S_{k-2j}(\underline{x}) = 0$.

0.1.3 The Space-Time Clifford Algebra $\mathbb{R}_{1,m}$

In this thesis we will mainly be interested in $\mathbb{R}_{1,m}$ -valued functions, so it is important to fix our notations and nomenclature here. The signature $(1, m)$ will be referred to as the space-time situation, without further specification. This in contrast to the signature $(p, q) \neq (1, m)$, with p and $q \neq 0$, in this thesis always referred to as the (p, q) -space-time situation. The signature $(0, m)$, treated in the previous subsection, will always be referred to as the flat Euclidean case.

Our starting point is the real orthogonal space $\mathbb{R}^{1,m}$ of signature $(1, m)$ with orthonormal basis $B_{1,m}(\epsilon, e_j) = \{\epsilon, e_1, \dots, e_m\}$, endowed with the quadratic form

$$Q_{1,m}(T, \underline{X}) = T^2 - \sum_{j=1}^m X_j^2 = T^2 - R^2,$$

where we have put $R = |\underline{X}| = (\sum_j X_j^2)^{\frac{1}{2}}$. The orthogonal space $\mathbb{R}^{1,m}$ will be called the m -dimensional space-time, m referring to the number of spatial dimensions.

The space-time Clifford algebra $\mathbb{R}_{1,m}$ is generated by the following multiplication rules : $e_j e_j + e_j e_j = -2\delta_{ij}$ for all $1 \leq i, j \leq m$, $e_i \epsilon + \epsilon e_i = 0$ for all i and $\epsilon^2 = 1$. Vectors in $\mathbb{R}^{1,m}$, i.e. $(m+1)$ -tuples (T, \underline{X}) or space-time vectors, are identified with 1-vectors in $\mathbb{R}_{1,m}$ under the canonical map

$$(T, \underline{X}) = (T, X_1, \dots, X_m) \mapsto \epsilon T + \underline{X} \in \mathbb{R}_{1,m}^{(1)}.$$

For 1-vectors in $\mathbb{R}_{1,m}$ we also introduce the notation $X = \epsilon T + \underline{X}$. Both notations will be used throughout this thesis. The capital X will mainly be used for the hyperbolic polar representation (cfr. infra), whereas $\epsilon T + \underline{X}$ will mainly be used to express formulae explicitly in space-time co-ordinates.

Consider then a space-time vector $X = \epsilon T + \underline{X} \in \mathbb{R}_{1,m}^{(1)}$. We will temporarily restrict ourselves to space-time vectors X for which $Q_{1,m}(T, \underline{X}) > 0$, for reasons that will be made clear in what follows. Such a vector may be written as

$$X = \epsilon T + \underline{X} = Q_{1,m}(T, \underline{X})^{\frac{1}{2}} \frac{\epsilon T + \underline{X}}{Q_{1,m}(T, \underline{X})^{\frac{1}{2}}} = \rho \xi$$

with

$$\rho = Q_{1,m}(T, \underline{X})^{\frac{1}{2}} = (T^2 - |\underline{X}|^2)^{\frac{1}{2}}$$

the hyperbolic norm of the space-time vector X and with

$$\xi = \frac{\epsilon T + \underline{X}}{(T^2 - |\underline{X}|^2)^{\frac{1}{2}}}$$

the space-time *unit* vector associated to the given vector X . The notation $X = \rho \xi$ will be referred to as the *hyperbolic polar decomposition* of the space-time vector X .

For two 1-vectors $X = \epsilon T + \underline{X}$ and $Y = \epsilon S + \underline{Y}$ in $\mathbb{R}_{1,m}^{(1)}$, we have :

$$XY = X \cdot Y + X \wedge Y ,$$

where the inner product is in explicit space-time co-ordinates given by

$$(\epsilon T + \underline{X}) \cdot (\epsilon S + \underline{Y}) = ST - \sum_{j=1}^m X_j Y_j$$

and the outer product by

$$\begin{aligned} (\epsilon T + \underline{X}) \wedge (\epsilon S + \underline{Y}) &= S \underline{X} \epsilon - T \underline{Y} \epsilon + \sum_{j < k} e_{jk} (X_j Y_k - X_k Y_j) \\ &= S \underline{X} \epsilon - T \underline{Y} \epsilon + \underline{X} \wedge \underline{Y} . \end{aligned}$$

In agreement with Definition 0.1, we introduce the following :

Definition 0.5 *The Dirac operator on the m -dimensional space-time $\mathbb{R}^{1,m}$ is given by the vector derivative*

$$D(T, \underline{X})_{1,m} = \epsilon \partial_T - \sum_{j=1}^m e_j \partial_{X_j} = \partial_X$$

This canonical first order Spin(1, m)-invariant operator factorizes the wave-operator \square_m on $\mathbb{R}^{1,m}$:

$$\partial_X^2 = \square_m = \partial_T^2 - \Delta_m$$

and has a hyperbolic polar decomposition given by :

$$\partial_X = \xi \left(\partial_\rho + \frac{1}{\rho} \Gamma_{1,m} \right) = \frac{\xi}{\rho} (\mathbb{E}_\rho + \Gamma_{1,m}) .$$

Remark : It is important to note that the polar decomposition given here is only valid in the region where $Q_{1,m}(T, \underline{X}) > 0$, due to the presence of both

the hyperbolic norm ρ and the hyperbolic unit vector ξ . However, as will be made clear in the following Chapters, for our purposes it is not necessary to extend the definition *outside* the future cone.

The operator

$$\mathbb{E}_\rho = T\partial_T + \sum_{j=1}^m X_j \partial_{X_j} = \rho \partial_\rho$$

is the Euler operator on $\mathbb{R}^{1,m}$, measuring the degree of homogeneity with respect to the space-time co-ordinates (T, \underline{X}) , and the operator

$$\begin{aligned} \Gamma_{1,m} &= X \wedge \partial_X = \underline{X} \epsilon \partial_T - T \epsilon \sum_{j=1}^m e_j \partial_{X_j} - \sum_{j < k} e_{jk} (X_j \partial_{X_k} - X_k \partial_{X_j}) \\ &= \underline{X} \epsilon \partial_T - T \epsilon \underline{\partial} + \Gamma_{0,m} \end{aligned}$$

is the so-called hyperbolic Gamma operator, with $\underline{\partial}$ (resp. $\Gamma_{0,m}$) the Dirac operator (resp. the spherical Dirac operator) in terms of the co-ordinates \underline{X} on the Euclidean space \mathbb{R}^m endowed with the quadratic form $Q_{0,m}(\underline{X})$.

Recalling the fact that $\partial_X^2 = \square_m$ and making use of the polar decomposition for the wave-operator \square_m on $\mathbb{R}^{1,m}$ given by

$$\square_m = \partial_\rho^2 + \frac{m}{\rho} \partial_\rho + \frac{1}{\rho^2} \Delta_H ,$$

with Δ_H the Laplace-Beltrami operator on the hyperbolic unit ball, see e.g. reference [15], we easily get :

$$\Gamma_{1,m} + \xi \Gamma_{1,m} \xi = m \implies \Gamma_{1,m} \xi = m \xi ,$$

a relation that will be of major importance in what follows.

0.2 Special Functions

In this section we give a short introduction to the hypergeometric function, to the Legendre function and the Gegenbauer function and we list some important properties that will often be used. As a general reference for this section, we refer to [25], [35] and [46].

0.2.1 The Hypergeometric Function

The hypergeometric differential equation, given by

$$z(1-z)\frac{d^2f}{dz^2} + [c - (a+b+1)z]\frac{df}{dz} - abf = 0, \quad (5)$$

is one of the most important differential equations arising in mathematical physics. In fact, a wide class of problems in mathematical physics leads to equations of this form. The solutions to this equation are the well-known *hypergeometric functions*.

We define the Pochhammer symbol $(a)_n$, for arbitrary complex a , by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)},$$

i.e. $(a)_0 = 1$ and $(a)_n = a(a+1)\cdots(a+n-1)$ for all $n \in \mathbb{N}_0$.

- If $c \notin -\mathbb{N}$, then

$$\begin{aligned} f_1(z) &= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n! (c)_n} z^n \\ &= F(a, b; c; z) \end{aligned} \quad (6)$$

is a solution for equation (5) which is regular at $z = 0$. This function $F(a, b; c; z)$ is the hypergeometric series with parameters a , b and c . It is also known as Gauss' hypergeometric function, because he was the first to study it (1812). The series converges for $|z| < 1$, and if $\text{Re}(c - a - b) > 0$ it also converges for $|z| = 1$.

A second, independent solution to equation (5) is then given by

$$f_2(z) = z^{1-c} F(a+1-c, b+1-c; 2-c; z). \quad (7)$$

- If $c = -k$ with $k \in \mathbb{N}$, $f_1(z)$ is ill-defined and $f_2(z)$ is regular at $z = 0$. Dividing the first solution by the Gamma function $\Gamma(c)$ leads to a new solution, usually denoted as $\varphi(a, b; c; z)$, which is then well-defined and equal to $f_2(z)$ up to a constant. This is expressed in the following :

$$\lim_{c \rightarrow -k} \frac{F(a, b; c; z)}{\Gamma(c)} = \frac{(a)_{1+k} (b)_{1+k}}{(1+k)!} z^{1+k} F(a+1+k, b+1+k; 2+k; z)$$

Putting

$$\gamma_k = \frac{(a)_{1+k}(b)_{1+k}}{(1+k)!} ,$$

a second, independent solution which is singular at $z = 0$ can then be found as

$$\lim_{c \rightarrow -k} \frac{\varphi(a, b; c; z) - \gamma_k z^{1-c} F(a+1-c, b+1-c; 2-c; z)}{c+k} . \quad (8)$$

This method to construct a second independent solution for the hypergeometric differential equation by means of a limit will play an important role in this thesis. It gives rise to logarithmic functions, and will be referred to as *the limit procedure*.

The hypergeometric series has several integral representations. The most important for what follows is Euler's formula : for $\text{Re}(c) > \text{Re}(b) > 0$ we have :

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} dt . \quad (9)$$

Since the right-hand side is a one-valued analytic function of z within the domain $|\arg(1-z)| < \pi$, expression (9) also gives the analytic continuation of $F(a, b; c; z)$. Still denoting this analytic continuation by $F(a, b; c; z)$, we have thus defined the hypergeometric function as a holomorphic function in the complex plane cut along the real axis from 1 to $+\infty$.

The hypergeometric function has many properties, a few of which will be listed here for future purposes. A complete list can be found e.g. in [35].

- Derivation of a hypergeometric function yields a new hypergeometric function :

$$\frac{d^n}{dz^n} F(a, b; c; z) = \frac{(a)_n (b)_n}{(c)_n} F(a+n, b+n; c+n; z)$$

- Kummer's relations (a list of 24 solutions for the hypergeometric equation and 20 linear relations with constant coefficients connecting any three of them) :

$$F(a, b; c; z) = (1-z)^{c-a-b} F(c-a, c-b; c; z)$$

- Goursat's table of quadratic transformations :

$$F\left(a, a + \frac{1}{2}; c; z\right) = \left(1 + z^{\frac{1}{2}}\right)^{-2a} F\left(2a, c - \frac{1}{2}; 2c - 1; \frac{2z^{\frac{1}{2}}}{1 + z^{\frac{1}{2}}}\right)$$

- Some elementary functions expressed by means of a hypergeometric series :

$$\begin{aligned} F\left(a, a + \frac{1}{2}; \frac{1}{2}; z^2\right) &= \frac{(1+z)^{-2a} + (1-z)^{-2a}}{2} \\ F\left(a, a + \frac{1}{2}; \frac{3}{2}; z^2\right) &= \frac{(1+z)^{1-2a} - (1-z)^{1-2a}}{2(1-2a)z} \end{aligned}$$

- In case $\operatorname{Re}(c - a - b) > 0$ and $c \notin -\mathbb{N}$, we have

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}$$

- The contiguous relations :

$$\begin{aligned} \left(z \frac{d}{dz} + a\right) F(a, b; c; z) &= aF(1 + a, b; c; z) \\ \left(z \frac{d}{dz} + b\right) F(a, b; c; z) &= bF(a, 1 + b; c; z) \\ \left(z \frac{d}{dz} + c - 1\right) F(a, b; c; z) &= (c - 1)F(1 + a, b; c - 1; z) \end{aligned}$$

0.2.2 Legendre Functions in the Complex Plane

The Legendre functions are solutions to Legendre's differential equation

$$(1 - z^2) \frac{d^2 f}{dz^2} - 2z \frac{df}{dz} + [\nu(\nu + 1) - \mu^2(1 - z^2)^{-1}] f = 0, \quad (10)$$

with ν and μ unrestricted complex parameters.

Under the substitution $f = (z^2 - 1)^{\frac{\mu}{2}} g$, Legendre's equation becomes :

$$(1 - z^2) \frac{d^2 g}{dz^2} - 2(\mu + 1)z \frac{dg}{dz} + (\nu - \mu)(\nu + \mu + 1)g = 0, \quad (11)$$

and with $\zeta = \frac{1-z}{2}$ as the independent variable this equation reduces to a differential equation of the hypergeometric type :

$$\zeta(1 - \zeta) \frac{d^2 g}{d\zeta^2} + (1 + \mu)(1 - 2\zeta) \frac{dg}{d\zeta} + (\nu - \mu)(\nu + \mu + 1)g = 0.$$

It follows that the function $f(z) = P_\nu^\mu(z)$, for $|1 - z| < 2$ defined by

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1} \right)^{\frac{\mu}{2}} F \left(-\nu, 1+\nu; 1-\mu; \frac{1-z}{2} \right), \quad (12)$$

is a solution to Legendre's differential equation.

On the other hand, setting $\zeta = z^2$ in equation (11), we get the following differential equation of hypergeometric type :

$$4\zeta(1-\zeta)\frac{d^2g}{d\zeta^2} + [2 - (4\mu+6)\zeta]\frac{dg}{d\zeta} + (\nu-\mu)(\nu+\mu+1)g = 0.$$

Hence, the function $f(z) = Q_\nu^\mu(z)$ for $|z| > 1$ defined by

$$Q_\nu^\mu(z) = \frac{e^{i\mu\pi}\pi^{\frac{1}{2}}}{2^{1+\nu}} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})} (z^2-1)^{\frac{\mu}{2}} z^{-1-\nu-\mu} F \left(\frac{1+\nu+\mu}{2}, \frac{2+\nu+\mu}{2}; \nu+\frac{3}{2}; \frac{1}{z^2} \right), \quad (13)$$

yields a second solution to equation (10). The functions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ are known as the associated Legendre functions of the first and second kind respectively. They can be analytically extended to the whole complex plane supposed cut along the real axis from $-\infty$ to 1, and they are regular and one-valued in this region. By means of the transformation formulae for the hypergeometric function, $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ can be expressed, in several ways, as

$$\begin{aligned} P_\nu^\mu(z) &= A_1 F(a_1, b_1; c_1; \zeta) + A_2 F(a_2, b_2; c_2; \zeta) \\ Q_\nu^\mu(z) &= e^{i\mu\pi} \left(A_3 F(a_3, b_3; c_3; \zeta) + A_4 F(a_4, b_4; c_4; \zeta) \right) \end{aligned}$$

where ζ is a function of z , such that $|\zeta| < 1$. The various expansions for $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ can be found e.g. in [35]. The following expansions for the

Legendre function $Q_\nu^\mu(z)$ will often be used throughout the thesis :

$$Q_\nu^\mu(z) = \frac{\Gamma(\mu)}{2^{1-\mu}} \frac{e^{i\mu\pi} z^{\nu+\mu}}{(z^2-1)^{\frac{\mu}{2}}} F\left(-\frac{\nu+\mu}{2}, \frac{1-\nu-\mu}{2}; 1-\mu; 1-\frac{1}{z^2}\right) + \frac{\Gamma(-\mu)\Gamma(1+\mu+\nu)}{2^{\mu+1}\Gamma(1+\nu-\mu)} \frac{e^{i\mu\pi} z^{\nu-\mu}}{(z^2-1)^{-\frac{\mu}{2}}} F\left(\frac{\mu-\nu}{2}, \frac{1+\mu-\nu}{2}; 1+\mu; 1-\frac{1}{z^2}\right) \quad (14)$$

$$Q_\nu^\mu(z) = e^{i\mu\pi} \frac{\pi^{\frac{1}{2}} 2^\mu \Gamma(1+\mu+\nu)}{\Gamma(\nu+\frac{3}{2})} \frac{(z^2-1)^{\frac{\mu}{2}}}{(z+(z^2-1)^{\frac{1}{2}})^{1+\mu+\nu}} F\left(\frac{1}{2}+\mu, 1+\mu+\nu; \nu+\frac{1}{2}; \frac{z-(z^2-1)^{\frac{1}{2}}}{z+(z^2-1)^{\frac{1}{2}}}\right) . \quad (15)$$

The Legendre function of the second kind satisfies :

$$e^{-i\mu\pi} \Gamma(1+\nu-\mu) Q_\nu^\mu(z) = e^{i\mu\pi} \Gamma(1+\nu+\mu) Q_\nu^{-\mu}(z) , \quad (16)$$

and can be written in terms of the Legendre function of the first kind as

$$Q_\nu^\mu(z) = \frac{\pi e^{i\mu\pi}}{2 \sin(\mu\pi)} \left(P_\nu^\mu(z) - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} P_\nu^{-\mu}(z) \right) . \quad (17)$$

In case $\mu = 0$ and $\nu = n \in \mathbb{N}$ the Legendre function of the first kind reduces to the classical Legendre polynomial $P_n(t)$. Classically, these are defined by means of Rodrigues' formula :

$$P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n} (t^2-1)^n , \quad n \in \mathbb{N} ,$$

for arbitrary real or complex values of the variable t . They can also be defined in terms of their generating function :

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) t^n .$$

In Chapter 6 we will need the Legendre polynomials in higher dimensions. In order to define these special functions, we need some definitions from the theory on harmonic polynomials of degree k on the m -dimensional Euclidean space, i.e. k -homogeneous solutions for the Laplace operator Δ_m on \mathbb{R}^m :

$$\Delta_m P_k(\underline{x}) = 0 .$$

The restriction of a harmonic polynomial to the unit sphere is then called a *spherical harmonic* of degree k . Note that the spherical monogenics defined earlier refine these spherical harmonics. The Legendre polynomial of degree k in m dimensions is then defined as the unique harmonic polynomial $P(\underline{x})$ of degree k on \mathbb{R}^m satisfying the following properties :

1. In a certain point $\underline{\eta}$ of the unit sphere, the Legendre polynomial has value 1. Choosing the co-ordinate system in an appropriate way, we may always choose $\underline{\eta} = e_1 = (1, 0, \dots, 0)$ such that $P(e_1) = 1$.
2. The polynomial $P(\underline{x})$ is invariant under $\text{SO}(m)_{\underline{\eta}}$, i.e. the subgroup of $\text{SO}(m)$ fixing $\underline{\eta} \in S^{m-1}$. This means that for each $R \in \text{SO}(m)$ such that $R\underline{\eta} = \underline{\eta}$ we have : $P(R\underline{x}) = P(\underline{x})$.

As a consequence, the Legendre polynomial of degree k in m dimensions is a *zonal function*, depending on the Euclidean inner product $t = \langle \underline{x}, \underline{\eta} \rangle$ only. It will be denoted by $P_{k,m}(t)$ and can be defined in terms of a Rodrigues formula :

$$P_{k,m}(t) = \left(-\frac{1}{2}\right)^k \frac{\Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(k + \frac{m-1}{2}\right)} (1-t^2)^{\frac{3-m}{2}} \left(\frac{d}{dt}\right)^k (1-t^2)^{k+\frac{m-3}{2}}.$$

From this formula it is clear that the Legendre polynomials defined above are actually Legendre polynomials in 3 dimensions.

We also mention the recurrence relation for the Legendre polynomial in m dimensions :

$$(k+m-2)P_{k+1,m}(t) - (2k+m-2)tP_{k,m}(t) + mP_{k-1,m}(t) = 0 \quad (18)$$

and the classical Hecke-Funk Theorem :

Theorem 0.2 (*Hecke-Funk*) Let ξ and $\eta \in S^{m-1}$ and let $S_k(\eta)$ be a spherical harmonic of degree k on S^{m-1} . The integral of a zonal function F can be calculated by means of the following formula :

$$\int_{S^{m-1}} F(\langle \underline{\eta}, \underline{\xi} \rangle) S_k(\underline{\eta}) dS(\underline{\eta}) = A_m S_k(\underline{\xi}) \int_{-1}^1 F(t) P_{k,m}(t) (1-t^2)^{\frac{m-3}{2}} dt ,$$

A_m denoting the area of the unit sphere S^{m-1} in \mathbb{R}^m .

0.2.3 Gegenbauer Functions in the Complex Plane

Classically, Gegenbauer's *polynomial* $C_n^\mu(z)$ for integral value of n is defined as the coefficient of t^n in the expansion of $(1-2tz+t^2)^{-\mu}$ in powers of t :

$$(1-2tz+t^2)^{-\mu} = \sum_{n=0}^{\infty} C_n^\mu(z) t^n , \quad (19)$$

for $|t| < |z \pm (z^2 - 1)^{\frac{1}{2}}|$. If we compare this with the generating function for the Legendre polynomials in 3 dimensions, we immediately see that

$$P_{k,3}(t) = C_k^{\frac{1}{2}}(t).$$

The Legendre polynomials in higher dimension can also be expressed in terms of the Gegenbauer polynomials, by means of the following formula :

$$P_{k,m}(t) = \binom{k+m-3}{k} C_k^{\frac{m-2}{2}}(t).$$

The coefficient of t^n in expression (19) is found to be

$$C_n^\mu(z) = \sum_{k=0}^n \frac{(-1)^k \Gamma(\mu+k) \Gamma(2\mu+n+k)}{k! \Gamma(\mu) \Gamma(2\mu+2k) (n-k)!} \left(\frac{1-z}{2} \right)^k,$$

which by means of the definition for the hypergeometric series reduces to

$$C_n^\mu(z) = \frac{\Gamma(n+2\mu)}{\Gamma(n+1) \Gamma(2\mu)} F \left(n+2\mu, -n; \mu + \frac{1}{2}; \frac{1-z}{2} \right).$$

Hence, by means of (12) we get

$$C_n^\mu(z) = \pi^{\frac{1}{2}} 2^{-\mu+\frac{1}{2}} \frac{\Gamma(n+2\mu)}{\Gamma(\mu) \Gamma(1+n)} (z^2-1)^{\frac{1}{4}-\frac{\mu}{2}} P_{n+\mu-\frac{1}{2}}^{-\mu+\frac{1}{2}}(z),$$

and this function satisfies the differential equation

$$(1-z^2) \frac{d^2 f}{dz^2} - (2\mu+1) z \frac{df}{dz} + n(n+2\mu) f = 0.$$

We therefore define the Gegenbauer functions for arbitrary complex values μ and ν as the solutions to Gegenbauer's differential equation

$$(1-z^2) \frac{d^2 f}{dz^2} - (2\mu+1) z \frac{df}{dz} + \nu(\nu+2\mu) f = 0. \quad (20)$$

These Gegenbauer functions are defined in terms of the associated Legendre functions as follows :

$$C_\nu^\mu(z) = \pi^{\frac{1}{2}} 2^{-\mu+\frac{1}{2}} \frac{\Gamma(\nu+2\mu)}{\Gamma(\mu) \Gamma(1+\nu)} (z^2-1)^{\frac{1}{4}-\frac{\mu}{2}} P_{\nu+\mu-\frac{1}{2}}^{-\mu+\frac{1}{2}}(z) \quad (21)$$

$$D_\nu^\mu(z) = \pi^{-\frac{1}{2}} e^{2i\pi(\mu-\frac{1}{4})} 2^{-\mu+\frac{1}{2}} \frac{\Gamma(\nu+2\mu)}{\Gamma(\mu) \Gamma(1+\nu)} (z^2-1)^{\frac{1}{4}-\frac{\mu}{2}} Q_{\nu+\mu-\frac{1}{2}}^{-\mu+\frac{1}{2}}(z). \quad (22)$$

In view of the different expansions for the Legendre functions in terms of the hypergeometric function, we have several expansions for the Gegenbauer functions in terms of the hypergeometric function too. In Chapter 5, the following formulae will be essential :

$$C_\nu^\mu(z) = \frac{\Gamma(\nu + 2\mu)}{\Gamma(1 + \nu)\Gamma(2\mu)} z^\nu F\left(-\frac{\nu}{2}, \frac{1 - \nu}{2}; \mu + \frac{1}{2}; 1 - \frac{1}{z^2}\right) \quad (23)$$

$$D_\nu^\mu(z) = \frac{e^{i\mu\pi}\Gamma(\nu + 2\mu)}{2^{2\mu+\nu}\Gamma(\mu)\Gamma(\nu + \mu + 1)} \frac{(z^2 - 1)^{\frac{1}{2}-\mu}}{z^{\nu+1}} F\left(1 + \frac{\nu}{2}, \frac{\nu + 3}{2}; 1 + \nu + \mu; \frac{1}{z^2}\right) \quad (24)$$

The Gegenbauer functions $C_\nu^\mu(z)$ and $D_\nu^\mu(z)$ are holomorphic functions in the z -plane cut along the real axis from $-\infty$ to 1. Note that the Gegenbauer function of the first kind can be considered as a Gegenbauer polynomial of complexified degree.

Both Gegenbauer functions satisfy the following recurrence relations, which will often be used later :

$$\begin{aligned} \frac{d}{dz} C_\nu^\mu(z) &= 2\mu C_{\nu-1}^{\mu+1}(z) \\ \nu C_\nu^\mu(z) &= 2\mu[zC_{\nu-1}^{\mu+1}(z) - C_{\nu-2}^{\mu+1}(z)] \\ (\nu + 2\mu)C_\nu^\mu(z) &= 2\mu[C_\nu^{\mu+1}(z) - zC_{\nu-1}^{\mu+1}(z)] . \end{aligned} \quad (25)$$

These relations generalize the recurrence relations of the classical Gegenbauer polynomials $C_n^\mu(z)$ to complex values of n .

The Gegenbauer function of the first kind satisfies

$$C_{-\nu-2\mu}^\mu(z) = -\frac{\sin(\nu + 2\mu)\pi}{\sin(\nu\pi)} C_\nu^\mu(z) , \quad (26)$$

whereas the Gegenbauer function of the second kind satisfies

$$D_\nu^\mu(z) = D_{-\nu-2\mu}^\mu(z) + e^{i\pi\mu} \frac{\sin(\nu + \mu)\pi}{\sin(\nu\pi)} C_\nu^\mu(z) . \quad (27)$$

In Chapter 5, the addition formula for the Gegenbauer function of the second kind will be essential, see reference [25] :

$$\begin{aligned}
& D_\nu^\mu \left(x_1 x_2 - z(x_1^2 - 1)^{\frac{1}{2}}(x_2^2 - 1)^{\frac{1}{2}} \right) \\
&= \frac{\Gamma(2\mu - 1)}{[\Gamma(\mu)]^2} \sum_{k=0}^{\infty} (-1)^k (2k + 2\mu - 1) \frac{4^k \Gamma(1 + \nu - k) [\Gamma(\mu + k)]^2}{\Gamma(\nu + 2\mu + k)} \\
& \quad (x_1^2 - 1)^{\frac{k}{2}} (x_2^2 - 1)^{\frac{k}{2}} D_{\nu-k}^{\mu+k}(x_1) C_{\nu-k}^{\mu+k}(x_2) C_k^{\mu-\frac{1}{2}}(z) , \tag{28}
\end{aligned}$$

valid for x_1, x_2 and $z \in \mathbb{R}$, with $x_1 > x_2 > 1$.

0.3 Riesz Distributions

The aim of this section is to introduce Riesz distributions $Z_\mu \in \mathcal{D}'(\mathbb{R}^{1,m})$, defined for all $\mu \in \mathbb{C}$. For that purpose we first introduce the distributions $x_+^\lambda \in \mathcal{D}'(\mathbb{R})$ on the real line, λ being an arbitrary complex number.

Consider the function x_+^λ , defined by

$$x_+^\lambda = \begin{cases} x^\lambda & x > 0 \\ 0 & x \leq 0 \end{cases} .$$

For $\text{Re}(\lambda) > -1$ this function is locally integrable and hence defines a regular distribution $x_+^\lambda \in \mathcal{D}'(\mathbb{R})$:

$$\langle x_+^\lambda, \varphi \rangle = \int_0^\infty x^\lambda \varphi(x) dx , \quad \varphi \in \mathcal{D}(\mathbb{R}) .$$

In the strip $-n - 1 < \text{Re}(\lambda) < -n$, the distribution x_+^λ may be defined by analytical continuation :

$$\langle x_+^\lambda, \varphi \rangle = \frac{\langle \frac{d^n}{dx^n} x_+^{\lambda+n}, \varphi \rangle}{(\lambda + 1)(\lambda + 2) \cdots (\lambda + n)} ,$$

where the derivatives with respect to x must be interpreted in distributional sense. Hence, if $-n - 1 < \text{Re}(\lambda) < -n$ one defines

$$\langle x_+^\lambda, \varphi \rangle = (-1)^n \frac{\langle x_+^{\lambda+n}, \varphi^{(n)} \rangle}{(\lambda + 1)(\lambda + 2) \cdots (\lambda + n)} , \quad \varphi \in \mathcal{D}(\mathbb{R}) .$$

This means that for all test functions $\varphi \in \mathcal{D}(\mathbb{R})$, the function $\langle x_+^\lambda, \varphi \rangle$ defines a meromorphic function of λ with simple poles at $\lambda = -1 - n$, $n \in \mathbb{N}$.

The residue at $\lambda = -1 - n$ is given by

$$\frac{\varphi^{(n)}(0)}{n!} = \frac{(-1)^n}{n!} \langle \delta^{(n)}, \varphi \rangle ,$$

and we thus conclude that

$$\text{Res}(x_+^\lambda, \lambda = -1 - n) = \frac{(-1)^n}{n!} \delta^{(n)}(x) .$$

In order to remove the simple poles of x_+^λ we divide by $\Gamma(1 + \lambda)$, and so the distribution $\frac{x_+^\lambda}{\Gamma(\lambda + 1)}$ is well-defined on $\mathcal{D}(\mathbb{R})$ for all $\lambda \in \mathbb{C}$ with $\frac{\langle x_+^\lambda, \varphi \rangle}{\Gamma(\lambda + 1)}$ a holomorphic function of λ for all $\varphi \in \mathcal{D}(\mathbb{R})$.

These distributions x_+^λ on the real line can be used to define the Beta integral $I_B(\lambda, \mu)$ for arbitrary complex values :

$$I_B(\lambda, \mu) = \int_0^1 t^{\lambda-1} (1-t)^{\mu-1} dt .$$

In case $\text{Re}(\lambda) > 0$ and $\text{Re}(\mu) > 0$ this integral converges in the classical sense to the Beta function $B(\lambda, \mu)$, defined in terms of the Gamma function as

$$B(\lambda, \mu) = \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda + \mu)} .$$

For more general λ and μ this relation remains valid, and this can easily be seen as follows : the Beta integral $I_B(\lambda, \mu)$ may be interpreted as the distribution $t_+^{\lambda-1}(1-t)_+^{\mu-1}$ acting on the constant function 1 :

$$I_B(\lambda, \mu) = \langle t_+^{\lambda-1}(1-t)_+^{\mu-1}, 1 \rangle = \int_0^1 t^{\lambda-1} (1-t)^{\mu-1} dt .$$

The product of the distributions $t_+^{\lambda-1}$ and $(1-t)_+^{\mu-1}$ is well-defined for those values for which they respectively exist (i.e. for $\text{Re}(\lambda) > 0$ and $\text{Re}(\mu) > 0$), as they have a "problematic behaviour" for different values (in casu 0 and 1), and it has compact support $[0, 1] =]-\infty, 1] \cap [0, +\infty[$. This means that as a function of (λ, μ) , the distribution $t_+^{\lambda-1}(1-t)_+^{\mu-1}$ is defined in $\mathbb{C}^2 \setminus \{(\lambda, \mu) \in \mathbb{C}^2 : \lambda \in -\mathbb{N}, \mu \in -\mathbb{N}\}$ which is the complex plane minus a grid. In the complex strip $\{(\lambda, \mu) \in \mathbb{C}^2 : \text{Re}(\lambda) > 0, \text{Re}(\mu) > 0\}$ this distribution yields the Beta function $B(\lambda, \mu)$ when acting on the test function 1, and for all other possible values this equality follows by analytic continuation. We

may thus conclude that for all $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(\lambda, \mu) \in \mathbb{C}^2 : \lambda \in -\mathbb{N}, \mu \in -\mathbb{N}\}$ we have :

$$I_B(\lambda, \mu) = \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda + \mu)} . \quad (29)$$

Next we introduce the distributions $\rho^\lambda \in \mathcal{D}'(\mathbb{R}^{1,m})$, λ being an arbitrary complex number. As a general reference, we mention [21], [42] and [57].

The function ρ is defined for space-time vectors $(T, \underline{X}) \in \mathbb{R}^{1,m}$ as :

$$\rho = \begin{cases} Q_{1,m}(T, \underline{X})^{\frac{1}{2}} & T > |\underline{X}| \\ 0 & \text{otherwise} \end{cases} .$$

In the half-plane $\text{Re}(\lambda) > -2$, the function ρ^λ defines a regular distribution since ρ^λ is locally integrable for these values of λ . Hence, for all test functions $\varphi \in \mathcal{D}(\mathbb{R}^{1,m})$, the integral

$$\langle \rho^\lambda, \varphi \rangle = \int_{\mathbb{R}} \int_{\mathbb{R}^m} Q_{1,m}^{\frac{\lambda}{2}}(T, \underline{X}) \varphi(T, \underline{X}) dT d\underline{X}$$

defines an analytic function of the complex parameter λ when $\text{Re}(\lambda) > -2$. Using analytic continuation $\langle \rho^\lambda, \varphi \rangle$ may be extended to a meromorphic function in the whole complex plane.

For that purpose we use the wave-operator \square_m on $\mathbb{R}^{1,m}$. Letting this operator act on ρ^λ we get for $\text{Re}(\lambda) > -2$:

$$\square_m \rho^\lambda = \lambda(\lambda + m - 1) \rho^{\lambda-2} .$$

This suggests the following definition for the distribution ρ^λ in the strip $-4 < \text{Re}(\lambda) < -2$:

$$\langle \rho^\lambda, \varphi \rangle = \frac{\langle \square_m \rho^{\lambda+2}, \varphi \rangle}{(\lambda + 2)(\lambda + m + 1)} , \quad \varphi \in \mathcal{D}(\mathbb{R}^{1,m}) .$$

By iteration we obtain more generally for the distribution ρ^λ in the strip $-2n - 2 < \text{Re}(\lambda) < -2n$:

$$\langle \rho^\lambda, \varphi \rangle = \frac{\langle \square_m^n \rho^{\lambda+2n}, \varphi \rangle}{(\lambda + 2) \cdots (\lambda + 2n)(\lambda + m + 1) \cdots (\lambda + m + 2n - 1)} ,$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^{1,m})$. From this relation it follows that the distribution ρ^λ has poles at $\lambda = -2 - 2n$ and at $\lambda = -1 - m - 2n$, $n \in \mathbb{N}$. For m even all the

poles are simple, while for m odd the points $-2, -4, \dots, 1 - m$ are simple poles and the points $-m - 1, -m - 3, \dots$ are double poles.

The distributions ρ^λ are normalized by introducing suitable factors. Putting

$$Z_\mu = \frac{\rho^{\mu-m-1}}{\pi^{\frac{m-1}{2}} 2^{\mu-1} \Gamma\left(\frac{\mu}{2}\right) \Gamma\left(\frac{\mu+1-m}{2}\right)}, \quad (30)$$

the functional $\langle Z_\mu, \varphi \rangle$ becomes an entire function of $\mu \in \mathbb{C}$, for all test functions $\varphi \in \mathcal{D}(\mathbb{R}^{1,m})$.

These so-called *Riesz distributions* Z_μ enjoy remarkable properties, a few of which will be listed here :

1. The support of Z_μ is contained in $\overline{FC} = \{X \in \mathbb{R}^{1,m} : T \geq |\underline{X}|\}$. In particular : for $(\mu - m - 1)$ not singular (i.e. not belonging to the set of poles) the support of Z_μ is the set \overline{FC} , for $\mu = -2k, k \in \mathbb{N}$, its support is the origin and for $\mu = m + 1 - 2k, k \in \mathbb{N}_0$ and $\mu \neq 0, -2, -4, \dots$, its support is the surface $\{X \in \mathbb{R}^{1,m} : T = |\underline{X}|\}$.
2. The distributions Z_μ satisfy the convolution property $Z_\mu * Z_\nu = Z_{\mu+\nu}$.
3. For all $k \in \mathbb{N}$, we have : $Z_{-2k} = \square_m^k \delta(X)$, with $\delta(X) = \delta(T)\delta(\underline{X})$ the delta distribution in space-time co-ordinates (T, \underline{X}) . This shows that for $\mu = -2k, k \in \mathbb{N}$, the support of Z_μ is indeed the origin.
4. For all $\mu \in \mathbb{C}$ and $k \in \mathbb{N}$, $\square_m^k Z_\mu = Z_{\mu-2k}$. In particular, we get $\square_m^k Z_{2k} = \delta(X)$.

Let us then introduce $\mathcal{D}'_+(\mathbb{R}^{1,m})$ as the set of distributions $f \in \mathcal{D}'(\mathbb{R}^{1,m})$ with support contained in \overline{FC} . The convolution of two elements of $\mathcal{D}'_+(\mathbb{R}^{1,m})$ belongs to $\mathcal{D}'_+(\mathbb{R}^{1,m})$, whence $\mathcal{D}'_+(\mathbb{R}^{1,m})$ is a so-called *convolution algebra*. The distributions Z_μ belong to $\mathcal{D}'_+(\mathbb{R}^{1,m})$, and their *uniquely* determined inverses in $\mathcal{D}'_+(\mathbb{R}^{1,m})$ are the distributions $Z_{-\mu}$:

$$Z_\mu * Z_{-\mu} = \delta(X), \quad \mu \in \mathbb{C}.$$

It follows that the differential equation

$$\square_m^k f = g,$$

with f and g belonging to $\mathcal{D}'_+(\mathbb{R}^{1,m})$ has unique solution

$$f = Z_{2k} * g,$$

a property that will be essential in what follows.

0.4 The Radon Transform

The Radon transform of a rapidly decreasing function $f(\underline{x}) \in \mathcal{S}(\mathbb{R}^m)$ is a function $f^\dagger(p, \underline{\omega})$ on \mathbb{P}^m , the space of all hyperplanes in \mathbb{R}^m . A hyperplane Π in \mathbb{R}^m may be represented as the set

$$\Pi = \{ \underline{x} \in \mathbb{R}^m : \langle \underline{x}, \underline{\omega} \rangle = p, p \in \mathbb{R}, \underline{\omega} \in S^{m-1} \}.$$

As the pairs $(p, \underline{\omega})$ and $(-p, -\underline{\omega})$ give rise to the same $\Pi \in \mathbb{P}^m$, the mapping $(p, \underline{\omega}) \rightarrow \Pi$ is a double covering of $\mathbb{R} \times S^{m-1}$ onto \mathbb{P}^m .

The Radon transform $f^\dagger(p, \underline{\omega})$ of a function $f(\underline{x}) \in \mathcal{S}(\mathbb{R}^m)$ is obtained by integrating $f(\underline{x})$ over a hyperplane $\Pi \in \mathbb{P}^m$:

$$f^\dagger(p, \underline{\omega}) = \int_{\mathbb{R}^m} f(\underline{x}) \delta(\langle \underline{x}, \underline{\omega} \rangle - p) d\underline{x}. \quad (31)$$

It may thus be identified with a function f^\dagger on $\mathbb{R} \times S^{m-1}$, satisfying

$$f^\dagger(p, \underline{\omega}) = f^\dagger(-p, -\underline{\omega}).$$

Introducing the space $\mathcal{S}(\mathbb{P}^m)$ as the set of all $f \in \mathcal{S}(\mathbb{R} \times S^{m-1})$ satisfying $f(p, \underline{\omega}) = f(-p, -\underline{\omega})$, and $\mathcal{S}_H(\mathbb{P}^m)$ as the space of all $f \in \mathcal{S}(\mathbb{P}^m)$ such that $\int_{\mathbb{R}} f(p, \underline{\omega}) p^k dp$ is a k -homogeneous polynomial in $\underline{\omega}$ (for all $k \in \mathbb{N}$), we have that the Radon transform is a linear one-to-one mapping of the space $\mathcal{S}(\mathbb{R}^m)$ onto $\mathcal{S}_H(\mathbb{P}^m)$ (see e.g. [43]).

Functions $f(\underline{x}) \in \mathcal{S}(\mathbb{R}^m)$ can be recovered from their Radon transform by means of the following inversion formula:

$$f(\underline{x}) = \frac{(-1)^{\frac{m-1}{2}}}{2(2\pi)^{m-1}} \Delta_{m^2}^{\frac{m-1}{2}} \mathcal{B}[f^\dagger(p, \underline{\omega})]. \quad (32)$$

The operator \mathcal{B} , defined as

$$[\mathcal{B}g(p, \underline{\omega})](\underline{x}) = \int_{S^{m-1}} g(\langle \underline{x}, \underline{\omega} \rangle, \underline{\omega}) d\underline{\omega},$$

is the so-called *back-projection* operator, defined on functions $g(p, \underline{\omega}) \in \mathcal{S}(\mathbb{P}^m)$. This operator forms a dual pair with the Radon transform in the sense of integral geometry: while the Radon transform integrates over all points in a hyperplane, the back-projection operator integrates over all hyperplanes through a point.

The Radon transform satisfies the following elementary properties w.r.t. derivatives :

$$\begin{aligned} \left(\sum_{j=1}^m a_j \frac{\partial f}{\partial x_j} \right)^\dagger (p, \underline{\omega}) &= \langle \underline{a}, \underline{\omega} \rangle \frac{\partial f^\dagger}{\partial p}(p, \underline{\omega}) \\ \sum_{j=1}^m a_j \frac{\partial f^\dagger}{\partial \omega_j}(p, \underline{\omega}) &= -\frac{\partial}{\partial p} \left(\langle \underline{a}, \underline{x} \rangle f(\underline{x}) \right)^\dagger \end{aligned} \quad (33)$$

The Radon transform can also be defined for compactly supported distributions $S \in \mathcal{E}'(\mathbb{R}^m)$

$$\langle S^\dagger, \varphi \rangle = \langle S, \mathcal{B}\varphi \rangle \quad \forall \varphi \in \mathcal{E}(\mathbb{P}^m) .$$

For all $S \in \mathcal{E}'(\mathbb{R}^m)$, the Radon transform S^\dagger is a distribution of compact support on \mathbb{P}^m , i.e. $S^\dagger \in \mathcal{E}'(\mathbb{P}^m)$. For more details concerning the Radon transform, we refer to e.g. [20] and [43].

0.5 The Fundamental Solution for the Wave-operator

In this section, a few considerations concerning the fundamental solution $\mathcal{E}_{1,m}(T, \underline{X})$ for the wave-operator \square_m on the real orthogonal space $\mathbb{R}^{1,m}$ are gathered. As a general reference to this section, we mention [72].

We first give an explicit formula for $\mathcal{E}_{1,m}(T, \underline{X})$ and its support property (classically known as the *Huyghens principle*). We thereby have the make a distinction between even and odd space-times :

- **The case $m \in 2\mathbb{N}$**

In case of an even-dimensional space-time $\mathbb{R}^{1,m}$ we have :

$$\mathcal{E}_{1,m}(T, \underline{X}) = \frac{(-1)^{\frac{m-2}{2}}}{2\pi^{\frac{m+1}{2}}} \Gamma\left(\frac{m-1}{2}\right) \frac{H(T - |\underline{X}|)}{(T^2 - |\underline{X}|^2)^{\frac{m-1}{2}}} ,$$

where $H(x) \in \mathcal{D}'(\mathbb{R})$ stands for the classical Heaviside distribution, or step-function, on the real line.

From this it is immediately clear that the set $\{(T, \underline{X}) \in \mathbb{R}^{1,m} : T \geq |\underline{X}|\}$ is the support of $\mathcal{E}_{1,m}(T, \underline{X})$.

- **The case** $m \in 2\mathbb{N} + 1$

In case of an odd-dimensional space-time $\mathbb{R}^{1,m}$, $\mathcal{E}_{1,m}(T, \underline{X})$ is given by

$$\mathcal{E}_{1,m}(T, \underline{X}) = \frac{1}{2\pi} \left(\frac{1}{2\pi T} \frac{\partial}{\partial T} \right)^{\frac{m-3}{2}} \delta(T^2 - |\underline{X}|^2)$$

and has the set $\{(T, \underline{X}) \in \mathbb{R}^{1,m} : T = \pm |\underline{X}|\}$ as support. As

$$\delta(T^2 - |\underline{X}|^2) = \frac{\delta(T - |\underline{X}|)}{2|\underline{X}|} + \frac{\delta(T + |\underline{X}|)}{2|\underline{X}|},$$

this gives

$$\mathcal{E}_{1,m}(T, \underline{X}) = \frac{1}{2\pi} \left(\frac{1}{2\pi T} \frac{\partial}{\partial T} \right)^{\frac{m-3}{2}} \left(\frac{\delta(T - |\underline{X}|)}{2|\underline{X}|} + \frac{\delta(T + |\underline{X}|)}{2|\underline{X}|} \right).$$

In what follows we will restrict ourselves to the region in space-time defined as $\overline{FC} = \{(T, \underline{X}) \in \mathbb{R}^{1,m} : T \geq |\underline{X}|\}$, for reasons that will be made clear in Chapter 2, and that is why we will refer to

$$\mathcal{E}_{1,m}(T, \underline{X}) = \frac{1}{2\pi} \left(\frac{1}{2\pi T} \frac{\partial}{\partial T} \right)^{\frac{m-3}{2}} \frac{\delta(T - |\underline{X}|)}{2|\underline{X}|}$$

as the fundamental solution for the wave-operator in an odd-dimensional space-time.

In chapter 2 the following Lemma will be needed :

Lemma 0.1 *The fundamental solution $\mathcal{E}_m(T, \underline{X})$ for the wave-operator \square_m is for all odd integers $m \geq 3$ given by*

$$\mathcal{E}_m(T, \underline{X}) = \sum_{k=0}^a \frac{c_k^{(a)}}{(2\pi)^{\frac{m-1}{2}}} \frac{\delta^{(a-k)}(T - |\underline{X}|)}{2|\underline{X}|^{a+k+1}}, \quad c_k^{(a)} \in \mathbb{N}_0,$$

where $a = \frac{m-3}{2}$.

Proof : For $m = 3$, the statement becomes trivial, and the rest can be proved by means of induction on the dimension m . \square

Notice that we are not interested in the exact values for $c_k^{(a)}$.

Next we note that the singular support of the fundamental solution for the wave-operator, which is by definition the smallest closed set outside which it is a C^∞ function, is given by the set $\{(T, \underline{X}) \in \mathbb{R}^{1,m} : T = \pm |\underline{X}|\}$ (this is the so-called nullcone, cfr. infra).

0.6 Bundles and Sections

In this section we give a short introduction to the theory of bundles. The reason for this lies in the fact that the Dirac operator on the hyperbolic unit ball, which is the object of study in this manuscript, will be defined as an operator on a Clifford bundle. The theory of bundles plays a central role in the underlying mathematics of differential geometry and hence of general relativity. It is an important part of pure mathematics too, especially within the field of algebraic topology. Some good references on the theory of bundles are [45], [47] and [69].

Definition 0.6 *A bundle is a triple (E, π, \mathcal{M}) with E and \mathcal{M} topological spaces and with $\pi : E \mapsto \mathcal{M}$ a continuous map. The space E is called the bundle space, or total space, of the bundle and \mathcal{M} is the base space of the bundle; the map π is called the projection, and for all $x \in \mathcal{M}$ the inverse image $\pi^{-1}(\{x\})$ is the fibre over x . A C^∞ -bundle is such that both E and \mathcal{M} are C^∞ -manifolds and with π a C^∞ -map.*

This definition is very general. However, in all existing applications in physics and most uses in pure mathematics, the bundles have the special property that the fibres $\pi^{-1}(\{x\})$ are all homeomorphic (diffeomorphic in the case where E and \mathcal{M} are manifolds) to a common space F , which is then known as *the fibre* of the bundle. The bundle is then said to be a *fibre bundle*.

Examples :

1. One of the simplest examples of a fibre bundle is the so-called product bundle over \mathcal{M} with fibre F , defined as the triplet $(\mathcal{M} \times F, \text{pr}_1, \mathcal{M})$ with

$$\text{pr}_1 : \mathcal{M} \times F \mapsto \mathcal{M} : (x, f) \mapsto x .$$

2. The Moebius band is a famous example of a so-called *twisted* fibre bundle, which is a twisted strip whose base space is the circle S^1 and whose fibre can be taken to be the closed interval $[-1, 1]$. Note that the total space E is *not* the product space $S^1 \times [-1, 1]$, whence the name *twisted*.
3. Let G be a Lie group, which is by definition a group in the usual sense with the additional property that it is also a differentiable manifold, in such a way that the group operations are smooth with respect to this structure and let H be a closed subgroup of G . One can then form

the space of right cosets G/H . This is a manifold with a transitive G -action, whereby an element $g \in G$ acts by left multiplication :

$$g_0H \mapsto gg_0H .$$

Any two points on G/H are related by such a mapping. At each point in G/H there is a subgroup of G such that left multiplication by elements of this subgroup leaves the point invariant, this is the so-called *isotropy group* of the point. At the identity $eH \in G/H$ the isotropy group is the subgroup H itself, at other points it will be a translation of H . Defining the map π by

$$\pi : G \mapsto G/H : g \mapsto gH ,$$

we then have a bundle $(G, \pi, G/H)$ with fibre H . The inverse image $\pi^{-1}(gH)$ of a point in G/H , i.e. the fibre above that point, is a copy of H and one can think of the total space G of the bundle as a family of copies of H , parametrized by elements of G/H .

This example will be essential in Chapter 2, when the hyperbolic unit ball is defined as a so-called *homogeneous space*.

Definition 0.7 *A cross-section of a bundle (E, π, \mathcal{M}) is a map $s : \mathcal{M} \mapsto E$ such that the image of each point $x \in \mathcal{M}$ lies in the fibre $\pi^{-1}(\{x\})$ over x . This means that for a cross-section s we have*

$$\pi \circ s = 1_{\mathcal{M}} ,$$

where $1_{\mathcal{M}}$ denotes the identity map on \mathcal{M} .

For example, for a product fibre bundle $(\mathcal{M} \times F, \pi, \mathcal{M})$ a cross-section s defines a unique function $s' : \mathcal{M} \mapsto F$ given by

$$s(x) = (x, s'(x))$$

for all $x \in \mathcal{M}$. Conversely, each such function s' gives a unique cross-section s . Thus, in a product bundle, a cross-section is equivalent to a function from the base space to the fibre in the usual sense.

A special type of bundles is given by the so-called *principal fibre bundles*, the importance of which lies in the fact that all non-principal bundles are associated with an underlying principal bundle. Let G be an arbitrary Lie group, we then define the concept of a principal G -bundle :

Definition 0.8 A bundle (E, π, \mathcal{M}) is a principal G -bundle if E is a right G -space, where G acts freely on E , and if (E, π, \mathcal{M}) is isomorphic to the bundle $(E, \rho, E/G)$, where E/G is the orbit space of the G -action on E and where ρ is the usual projection map. Note that the fibres of the bundle are the orbits of the G -action, and that the freedom of this action implies that these orbits are all homeomorphic to G . A principal G -bundle is thus a fibre bundle with fibre G .

Let us first clarify some of the concepts introduced in this definition :

- A set E is called a left G -space if there exists a homomorphism $g \mapsto \gamma_g$ of G into the group $\text{Perm}(E)$ of bijections of E :

$$\begin{aligned} \gamma_e(p) &= p & \text{for all } p \in E \\ \gamma_g \circ \gamma_h &= \gamma_{gh} & \text{for all } g, h \in G \end{aligned} \quad .$$

For a right G -space this becomes an anti-homomorphism : $\gamma_g \circ \gamma_h = \gamma_{hg}$.

- The G -action is *free* if, for all $p \in E$, $\{g \in G : gp = p\} = \{e\}$. This means that given any pair of points p, q in E , either there is no $g \in G$ such that $q = gp$ or there is a unique g such that $q = gp$. If there indeed exists a unique g for each pair, the action is called *transitive*.
- Two bundles (E, π, \mathcal{M}) and (E', π', \mathcal{M}') are isomorphic if there exist two *bundle maps* (u, f) and (u', f') such that

$$\begin{aligned} u' \circ u &= 1_E & u \circ u' &= 1_{E'} \\ f' \circ f &= 1_{\mathcal{M}} & f \circ f' &= 1_{\mathcal{M}'} \end{aligned} \quad .$$

A bundle map (u, f) between (E, π, \mathcal{M}) and (E', π', \mathcal{M}') is defined as a pair of maps $u : E \mapsto E'$ and $f : \mathcal{M} \mapsto \mathcal{M}'$ such that $\pi' \circ u = f \circ \pi$.

Next we show how one can construct a fibre bundle with fibre F , starting from a principal bundle (E, π, \mathcal{M}) with structure group G , where G acts on the space F . For that purpose we need to introduce the ' G -product' of two spaces on which G acts.

Definition 0.9 Let X and Y be any pair of right G -spaces. The G -product $X \times_G Y$ is then defined as the space of orbits of the G -action on the Cartesian product $X \times Y$. This means that there is an equivalence relation on $X \times Y$ in which $(x, y) \sim (x', y')$ if there exists a $g \in G$ such that $x' = xg$ and $y' = yg$.

We then come to the following crucial definition of an *associated fibre bundle* :

Definition 0.10 Let $\xi = (P, \pi, \mathcal{M})$ be a principal G -bundle and let F be a left G -space. Define $P_F = P \times_G F$ where the action of G on the Cartesian product $P \times F$ is given by $(p, f)g = (pg, g^{-1}f)$ and define a projection map $\pi_F : P_F \mapsto \mathcal{M}$ by $\pi_F([p, f]) = \pi(p)$. Then $\xi[F] = (P_F, \pi_F, \mathcal{M})$ is a fibre bundle over \mathcal{M} with fibre F , and is said to be associated with the principal bundle ξ via the action of the group G on F .

The following property will be crucial for what follows, see Chapter 2 :

Theorem 0.3 If the bundle $(P_F, \pi_F, \mathcal{M})$ is an associated fibre bundle then its cross-sections are in one-to-one correspondance with maps $\phi : P \mapsto F$ that satisfy

$$\phi(pg) = g^{-1}\phi(p) ,$$

for all $p \in P$ and $g \in G$. The cross-section s_ϕ corresponding to such a map is defined by

$$s_\phi(x) = [p, \phi(p)] ,$$

where p is any point in the fibre $\pi^{-1}(\{x\})$.

0.7 Clifford analysis on the Lie sphere

In this section we first define the Lie ball and the Lie sphere and then we give a short introduction to the theory of Clifford analysis on the Lie sphere.

0.7.1 The Lie sphere

Consider $\underline{z} = (z_1, \dots, z_m) \in \mathbb{C}^m$, with $z_j = x_j + iy_j \in \mathbb{C}$. Identifying \mathbb{C}^m with \mathbb{R}^{2m} , we will sometimes write $\underline{z} = \underline{x} + i\underline{y}$. The complex conjugate of \underline{z} is given by $\bar{\underline{z}} = (\bar{z}_1, \dots, \bar{z}_m)$ and the inner product between two vectors \underline{z} and $\underline{w} \in \mathbb{C}^m$ is given by $\langle \underline{z}, \underline{w} \rangle = \sum_j z_j \bar{w}_j$. In what follows we will write \underline{z}^2 for $\langle \underline{z}, \underline{z} \rangle = \sum_j z_j^2$. The *Euclidean* norm on \mathbb{C}^m is given by $|\underline{z}|^2 = \langle \underline{z}, \bar{\underline{z}} \rangle = \sum_j |z_j|^2$. Note that

$$|\underline{z}^2| = \left| \sum_j z_j^2 \right| \leq |\underline{z}|^2 ,$$

whence $|\underline{z}|^4 - |\underline{z}^2|^2 \geq 0$. We then define the Lie norm :

Definition 0.11 For arbitrary $\underline{z} \in \mathbb{C}^m$, the Lie norm $L(\underline{z})$ is given by

$$L(\underline{z})^2 = |\underline{z}|^2 + \left(|\underline{z}|^4 - |\underline{z}^2|^2 \right)^{\frac{1}{2}}.$$

Remarks :

1. For $\underline{x} \in \mathbb{R}^m$ we have $L(\underline{x}) = |\underline{x}|$, such that the restriction of the Lie norm on \mathbb{C}^m to \mathbb{R}^m reduces to the Euclidean norm on \mathbb{R}^m .
2. The Euclidean norm on \mathbb{C}^m is equivalent with the Lie norm :

$$|\underline{z}| \leq L(\underline{z}) \leq \sqrt{2}|\underline{z}|,$$

which means that both norms determine the same topology.

3. In terms of $\underline{z} = \underline{x} + i\underline{y}$, we have

$$L(\underline{z})^2 = |\underline{x}|^2 + |\underline{y}|^2 + 2|\underline{x} \wedge \underline{y}|,$$

where $|\underline{x} \wedge \underline{y}|^2 = |\underline{x}|^2|\underline{y}|^2 - \langle \underline{x}, \underline{y} \rangle^2$ (Lagrange's identity).

In terms of this Lie norm, we then define :

Definition 0.12 The Lie ball $LB_m(1)$ in \mathbb{C}^m is given by

$$LB_m(1) = \{ \underline{z} \in \mathbb{C}^m : L(\underline{z}) < 1 \}$$

We also define the Lie sphere :

Definition 0.13 The Lie sphere LS^{m-1} in \mathbb{C}^m is given by

$$LS^{m-1} = \{ e^{it}\underline{\omega} : t \in \mathbb{R}, \underline{\omega} \in S^{m-1} \}$$

Remarks :

1. The Lie sphere LS^{m-1} is only a subset of the boundary $\partial LB_m(1)$ of the Lie ball : it is that part of $\partial LB_m(1)$ which intersects the unit sphere in \mathbb{C}^m , given by those $\underline{z} = (\underline{x}, \underline{y})$ for which \underline{x} and \underline{y} are proportional to each other.

2. The Lie sphere can be topologically identified with $S^1 \times S^{m-1} / \sim$ where the equivalence relation is given by

$$(e^{it}, \underline{\omega}) \sim (-e^{it}, -\underline{\omega}) .$$

This means that functions f on the Lie sphere LS^{m-1} can be written as $f(e^{it}, \underline{\omega}) = f(-e^{it}, -\underline{\omega}) = f(e^{it} \underline{\omega})$.

3. Both the Lie ball $LB_m(1)$ and the Lie sphere LS^{m-1} can be given a geometrical definition. For that purpose we define, for all $\underline{z} = \underline{x} + i\underline{y}$ in \mathbb{C}^m , the following sphere of dimension $m - 2$ (or, equivalently, of codimension 2) in \mathbb{R}^m :

$$S(\underline{z}) = S_{\underline{x}}(\underline{y}) = \{ \underline{t} \in \mathbb{R}^m : |\underline{t} - \underline{x}| = |\underline{y}| , \langle \underline{t} - \underline{x}, \underline{y} \rangle = 0 \} .$$

This sphere is the intersection of the sphere $|\underline{t} - \underline{x}| = |\underline{y}|$, with $\underline{t} \in \mathbb{R}^m$, centre \underline{x} and radius $|\underline{y}|$, and the hyperplane orthogonal to \underline{y} through \underline{x} .

Note that $S(\underline{z}) = S(\overline{\underline{z}})$ and that for $\underline{x} \in \mathbb{R}^m$ the sphere $S(\underline{x})$ reduces to $\{\underline{x}\}$. Defining an orientation for spheres of codimension 2 in \mathbb{R}^m , it is then possible to define a bijection between $\mathbb{C}^m \setminus \mathbb{R}^m$ and the manifold of non-trivial oriented spheres of codimension 2.

In terms of these spheres we then have the following definition for the Lie norm :

$$L(\underline{z}) = \max_{\underline{w} \in S_{\underline{x}}(\underline{y})} |\underline{w}| ,$$

for all $\underline{z} = \underline{x} + i\underline{y} \in \mathbb{C}^m$. This means that the Lie ball $LB_m(1)$ can be identified with the manifold of oriented spheres of codimension 2 inside the Euclidean unit ball $B_m(1) \subset \mathbb{R}^m$:

$$LB_m(1) = \{ \underline{z} \in \mathbb{C}^m : S(\underline{z}) \subset B_m(1) \} .$$

Recalling the fact that the Lie sphere is defined as the subset of $\partial LB_m(1)$ containing those $\underline{z} \in \mathbb{C}^m$ for which \underline{x} and \underline{y} are parallel, we also have that

$$LS^{m-1} = \{ \underline{z} \in \mathbb{C}^m : S(\underline{z}) \subset S^{m-1} \} .$$

The importance of the Lie ball lies in the following Theorem, for which we refer to [60] :

Theorem 0.4 (*Siciak*)

1. If a series $\sum_k R_k(\underline{x})$ of homogeneous polynomials converges normally in the unit ball $B_m(1) \subset \mathbb{R}^m$, its complexification $\sum_k R_k(\underline{z})$ will converge normally in $LB_m(1) \subset \mathbb{C}^m$ and hence yield a holomorphic function there.
2. The Lie ball is the maximal region in which this results holds in its full generality : there exists a harmonic function $h(\underline{x})$ such that the complexification $\sum_k S_k(\underline{z})$ of its expansion in spherical harmonics can not be holomorphically extended beyond the Lie ball.

The proof of this Theorem is beyond the scope of this section, but we mention the fact that the first part follows viz. from the fact that the *Shilov boundary* of the Lie ball is given by Lie sphere LS^{m-1} . In order to define the Shilov boundary, we recall the maximum modulus principle in complex analysis : for an open bounded region $\Omega \subset \mathbb{C}$ and a function $f(z)$ which is holomorphic in Ω and continuous up to the boundary, we have

$$\max_{z \in \overline{\Omega}} |f(z)| = \max_{z \in \partial\Omega} |f(z)| .$$

This Theorem holds for functions depending on several complex variables too : a non-constant function $f(\underline{z})$ which is holomorphic in a bounded open region $\Omega_m \subset \mathbb{C}^m$ and continuous up to the boundary, reaches its maximal modulus only on the boundary. In particular, we thus have :

$$\max_{\underline{z} \in \overline{LB_m(1)}} |f(\underline{z})| = \max_{\underline{z} \in \partial LB_m(1)} |f(\underline{z})| .$$

However, one can prove the following Theorem (which offers an alternative definition for the Lie sphere) :

Theorem 0.5 *The Lie sphere is the Shilov boundary of the Lie ball in \mathbb{C}^m :*

$$\max_{\underline{z} \in \overline{LB_m(1)}} |f(\underline{z})| = \max_{\underline{z} \in LS^{m-1}} |f(\underline{z})| .$$

0.7.2 Clifford analysis on the Lie sphere

From now on we consider functions $f(\underline{z})$ on \mathbb{C}^m which are $\mathbb{C}_{0,m}$ -valued. The relevant operators on the Lie sphere are the Gamma operator $\Gamma_{0,m}$ and the Euler operator $-i\partial_t$. On the Lie sphere, the simultaneous eigenfunctions of these operators are given by

$$(e^{it}\underline{\omega})^l P_k(e^{it}\underline{\omega}) , \quad l \in \mathbb{Z} , \quad P_k \in M^+(k) .$$

By analogy with the term 'spherical monogenics' to describe the simultaneous eigenfunctions of $\Gamma_{0,m}$ and \mathbb{E} on the sphere S^{m-1} , we call these functions on the Lie sphere *spherical monogenics of order (k, l) on LS^{m-1}* . Note that these functions are the restrictions to LS^{m-1} of complex extensions of the Clifford monomials $\underline{x}^l P_k(\underline{x})$ in $\mathcal{M}_{k,l}$. This latter notation will thus also be used to denote the set of spherical monogenics of order (k, l) on the Lie sphere :

$$\mathcal{M}_{k,l} = \{ (e^{it}\underline{\omega})^l P_k(e^{it}\underline{\omega}) , l \in \mathbb{Z} , P_k \in M^+(k) \} .$$

Defining the Hilbert module $L_2(LS^{m-1})$ of square integrable functions on the Lie sphere as

$$L_2(LS^{m-1}) = \{ f(e^{it}\underline{\omega}) : \|f\|_{L_2(LS^{m-1})} < \infty \} ,$$

where the Lie norm is given by $\|f\|_{L_2(LS^{m-1})}^2 = [(f, f)]_0$ with

$$\begin{aligned} (f, g) &= \frac{1}{2\pi A_m} \int_0^{2\pi} \int_{S^{m-1}} f(e^{it}\underline{\omega})^+ g(e^{it}\underline{\omega}) dS(\underline{\omega}) dt \\ &= \frac{1}{\pi A_m} \int_0^\pi \int_{S^{m-1}} f(e^{it}\underline{\omega})^+ g(e^{it}\underline{\omega}) dS(\underline{\omega}) dt , \end{aligned}$$

and $f(e^{it}\underline{\omega})^+$ the Hermitian conjugate on the complexified Clifford algebra \mathbb{C}_m , we have the following orthogonal decomposition of $L_2(LS^{m-1})$:

$$L_2(LS^{m-1}) = \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}} \mathcal{M}_{k,l} .$$

Putting $\theta = \langle \underline{\omega}, \underline{\xi} \rangle$ for $\underline{\omega}$ and $\underline{\xi} \in S^{m-1}$, we have for $f \in L_2(LS^{m-1})$:

$$f(e^{it}\underline{\omega}) = \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}} (e^{it}\underline{\omega})^l e^{ikt} P_{k,l} f(\underline{\omega})$$

with $P_{k,l} f(\underline{\omega})$ given by the integral

$$\frac{1}{\pi A_m} \int_0^\pi \int_{S^{m-1}} e^{-ikt} \{ C_k^{\frac{m}{2}}(\theta) + \underline{\omega} \underline{\xi} C_{k-1}^{\frac{m}{2}}(\theta) \} (e^{it}\underline{\xi})^{-l} f(e^{it}\underline{\xi}) dS(\underline{\xi}) dt ,$$

where

$$\sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}} \|P_{k,l} f\|_{L_2(LS^{m-1})}^2 < \infty .$$

This decomposition refines the decomposition of functions $f \in L_2(LS^{m-1})$ into spherical harmonics on the Lie sphere, see e.g. references [52, 64].

The inner product of two functions f and g in $L_2(LS^{m-1})$ then reduces to

$$(f, g) = \frac{1}{A_m} \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}} \int_{S^{m-1}} P_{k,l}^+ f(\underline{\omega}) P_{k,l} g(\underline{\omega}) dS(\underline{\omega}) .$$

Denoting the set of holomorphic functions on the Lie ball by $\mathcal{O}(LB_m(1))$, we then define the space $L_2^+(LS^{m-1})$ as the following Hardy-type space :

$$L_2^+(LS^{m-1}) = \left\{ f \in \mathcal{O}(LB_m(1)) : \lim_{r \rightarrow 1^-} \int_0^\pi \int_{S^{m-1}} |f(re^{it}\underline{\omega})|^2 dS(\underline{\omega}) dt < \infty \right\} .$$

As was pointed out in reference [64], this module is a Hilbert module with reproducing kernel; the so-called Cauchy-Hua kernel $H(\underline{z}, \underline{w})$. This means that a function $f \in L_2^+(LS^{m-1})$ can be represented as

$$f(\underline{z}) = \frac{1}{\pi A_m} \int_0^\pi \int_{S^{m-1}} H^+(z, e^{it}\underline{\omega}) f(e^{it}\underline{\omega}) dS(\underline{\omega}) dt ,$$

where the Cauchy-Hua kernel is defined by

$$H(\underline{z}, e^{it}\underline{\omega}) = \frac{1}{\left(-(\underline{\omega} - e^{-it}\underline{z})^2 \right)^{\frac{m}{2}}} .$$

The module $L_2^+(LS^{m-1})$ can then also be defined as

$$L_2^+(LS^{m-1}) = \left\{ f \in L_2(LS^{m-1}) : f = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (e^{it}\underline{\omega})^l P_{k,l} f(e^{it}\underline{\omega}) \right\} ,$$

which defines $L_2^+(LS^{m-1})$ as a submodule of $L_2(LS^{m-1})$ containing boundary values of holomorphic functions in the Lie ball.

We end this Chapter with a brief overview of a technique used to construct a reproducing kernel for a Hilbert module containing Clifford algebra valued nullsolutions for certain Clifford differential operators $P(\underline{x}, \underline{\partial})$ on the unit ball $B_m(1)$ with polynomial coefficients, the so-called *operators of Frobenius type* satisfying the conditions of the following Theorem (see [9] and [74]) :

Definition 0.14 *A differential operator $P(\underline{x}, \underline{\partial})$ is of the Frobenius type if its nullsolutions $f(\underline{x})$ in $B_m(1)$ can be represented as*

$$f(\underline{x}) = \sum_{k=0}^{\infty} r^k \{ \alpha_k(r^2) P_k(\underline{\omega}) + r \underline{\omega} \beta_k(r^2) \tilde{P}_k(\underline{\omega}) \} ,$$

where for all $k \in \mathbb{N}$ the functions $\alpha_k(r^2)$ and $\beta_k(r^2)$ can be written as positive power series

$$\alpha_k(r^2) = \sum_{l=0}^{\infty} a_l r^{2l} \quad , \quad \beta_k(r^2) = \sum_{l=0}^{\infty} b_l r^{2l}$$

converging on the open interval $] -1, 1[$ and with $P_k(\underline{\omega})$ and $\tilde{P}_k(\underline{\omega})$ belonging to $M^+(k)$.

Theorem 0.6 Consider the differential operator $P(\underline{x}, \underline{\partial})$ of Frobenius type with nullsolutions in $B_m(1)$ given by

$$f(\underline{x}) = \sum_{k=0}^{\infty} \{ \alpha_k(r^2) P_k(\underline{x}) + \underline{x} \beta_k(r^2) \tilde{P}_k(\underline{x}) \} \quad ,$$

where the series converges normally on $B_m(1)$. If the conditions

1.

$$\sup_{|z| \leq 1} |\alpha_k(z)| = c_1 \quad \text{and} \quad \sup_{|z| \leq 1} |\beta_k(z)| = c_2$$

2.

$$\sum_{k=0}^{\infty} \|P_k\|_{L_2(S^{m-1})}^2 < \infty \quad \text{and} \quad \sum_{k=0}^{\infty} \|\tilde{P}_k\|_{L_2(S^{m-1})}^2 < \infty \quad ,$$

are satisfied, the complexified series $f(e^{it}\underline{\omega})$ will belong to $L_2^+(LS^{m-1})$.

If the nullsolutions of the Frobenius operator $P(\underline{x}, \underline{\partial})$ satisfy the requirements of the Theorem, one may consider the submodule $H \subset L_2^+(LS^{m-1})$ containing the complexified nullsolutions, i.e. $H = \ker P(\underline{z}, \underline{\partial}_{\underline{z}}) \cap L_2^+(LS^{m-1})$. Due to the closedness of the operator $P(\underline{z}, \underline{\partial}_{\underline{z}})$ the submodule is also closed and its reproducing kernel, for the inner product on $L_2^+(LS^{m-1})$, can be obtained as the projection of the Cauchy-Hua kernel $H(\underline{z}, e^{it}\underline{\omega})$ on H . The reproducing property can then be restricted to the Euclidean unit ball $B_m(1)$ leading to a reproducing kernel for the Hilbert module of nullsolutions for the operator $P(\underline{x}, \underline{\partial})$ satisfying the requirements of the Theorem. In particular we also have that this module is a closed submodule of $L_2(S^{m-1})$. Note that the last step is not always possible, it depends on whether the inner product on the Lie sphere can be reduced to an inner product on the sphere S^{m-1} .

This construction will be applied in Chapter 6, when a reproducing kernel for a function space containing nullsolutions for the hyperbolic Dirac operator is constructed.

Chapter 1

Hyperbolic Geometry

Let no man ignorant of geometry enter
(Sign over Plato's Academy in Athens)

In this chapter a brief introduction to hyperbolic geometry is given, and several models for the hyperbolic unit ball are introduced.

1.1 Non-Euclidean Geometry

The aim of this section is to describe the concept of hyperbolic geometry from the historical point of view. Although this section is not essential for what follows, we have included it for the sake of completeness. It is partially based upon Chapter 6 in Tristan Needham's book [54].

In order to tell something on hyperbolic geometry we have to start from Euclidean geometry. One way to approach Euclidean geometry is to begin with definitions of such abstract concepts as "points" and "lines", together with a few assumptions (or *axioms*) concerning their properties. From this one deduces, using nothing but logic, further properties of these objects that are necessary consequences of the initial axioms. This was the path followed in Euclid's famous book, *The Elements*, which was published around 300 BC. Euclidean geometry did of course not come into life as a fully formed logical system of axioms and theorems, but instead it was developed gradually as an idealized description of physical measurements performed on physically constructed lines, triangles, circles, etc. In this sense Euclidean geometry is thus not simply mathematics, but a theory of space.

However, Euclidean geometry is not perfect from this point of view : experiments have revealed extremely small discrepancies between the predictions

of Euclidean geometry and the measured geometrical properties of figures constructed in the actual physical space. These deviations from Euclidean geometry are now known to be governed by the distribution of matter and energy in space, which is the essence of the theory of gravity discovered by Einstein in 1915. It is important to realize just how small these deviations typically are for figures of reasonable size : if we would measure the circumference of a circle having a radius of one meter, no deviation from the Euclidean case would be found, not even with a measuring device capable of detecting a discrepancy the size of a single atom! No wonder that for approximately two thousand years mathematicians were seduced to believe that Euclidean geometry was simply the *only logically possible* geometry.

However, non-Euclidean geometry was already discovered a century before Einstein found that it could be used to describe gravity. To locate the origin of this mathematical discovery, we return to Euclid. He began with only five axioms, the first four of which never aroused any controversy :

1. It is possible to draw one and only one straight line from any point to any point.
2. From each end of a finite straight line it is possible to produce it continuously in a straight line by an amount greater than any assigned length.
3. It is possible to describe one and only one circle with any centre and radius.
4. All right angles are equal to one another.

The fifth axiom however, the so-called *parallel axiom*, became the subject of investigations that have ultimately led to the discovery of non-Euclidean geometry :

Parallel Axiom : *Through any point p not on the line L there exists precisely one line L' that does not meet L .*

Historically, mathematicians fervently believed in the parallel axiom, so much that they even thought it had to be a logically necessary property of straight lines. But in that case they ought to be able to prove it. Many attempts were made to actually deduce the parallel axiom from the first four axioms, one of the most penetrating being that of Girolamo Saccheri in 1733. His idea was to show that if the parallel axiom was not true, then a contradiction

would necessarily arise. He thus divided the denial of the parallel axiom into two alternatives :

Spherical Axiom : *There is no line through p that does not meet L .*

or

Hyperbolic Axiom : *There are at least two lines through p that do not meet L .*

In the case of the spherical axiom, Saccheri was able to obtain a contradiction, provided "lines" are assumed to have infinite length. If we drop this requirement, we obtain a non-Euclidean geometry called *spherical geometry*, for which a model will be given in the next section.

However, in case of the hyperbolic axiom, Saccheri and later mathematicians were able to derive strange conclusions, but they were not able to find contradictions. The hyperbolic axiom yields a second viable non-Euclidean geometry, called *hyperbolic geometry*. It was devised independently by Gauss, Lobachevsky and Bolyai in the first half of the nineteenth century. Gauss never published his ideas on non-Euclidean geometry, because he was afraid that the mathematical society would not accept this revolutionary theory, and so the two men who are usually credited for their independent discovery of the hyperbolic geometry are Bolyai (1832) and Lobachevsky (1829). In the next section two well-known models for the hyperbolic plane will be given.

In order to illustrate how non-Euclidean geometries differ from Euclid's, we list some basic facts that can easily be derived from the familiar theorem in Euclidean geometry that in any triangle T the angles add up to π radians. This theorem is equivalent to the parallel axiom, from which it follows that in non-Euclidean geometry the angle sum of a triangle differs from π . This difference is the so-called *angular excess* E :

$$E(T) = (\text{Angle sum of } T) - \pi .$$

Euclidean geometry is thus characterized by the vanishing of $E(T)$, for any triangle T . Both Gauss and Lambert independently discovered that the two non-Euclidean geometries depart from Euclid's in opposite directions :

- In spherical geometry the angular sum is greater than π : $E > 0$.
- In hyperbolic geometry the angular sum is less than π : $E < 0$. Or, like Gauss formulated this himself :

The assumption that the sum of three angles is less than 180 degrees leads to some curious geometry, quite different from ours, but thoroughly consistent. (C. F. Gauss, November 8, 1824)

They also discovered the striking fact that $E(T)$ is completely determined by the *size* of the triangle T :

$$E(T) = \kappa A(T) ,$$

with $A(T)$ the area of the triangle T and κ a constant that is positive in spherical geometry and negative in hyperbolic geometry. Several interesting conclusions can be drawn in connection with this result :

- Although there are no qualitative differences between them, there are infinitely many spherical and hyperbolic geometries, depending on the value of the constant κ .
- In non-Euclidean geometry, similar triangles do not exist : the formula above tells us that two triangles of different size cannot have the same angles. In other words : similar figures are automatically congruent.
- Closely related to the previous point : in non-Euclidean geometry there exists an *absolute unit of length*. A natural way to define this is in terms of the constant κ . Since the radian measure of angle is defined as a ratio of lengths, E is a pure number. On the other hand A has units of $(\text{length})^2$. It follows that κ has units of $(\text{length})^{-2}$ and so it can be written in terms of a certain length R as :

$$\kappa = \pm \frac{1}{R^2} ,$$

where the plus (resp. minus) sign is choosen in case of a spherical (resp. hyperbolic) geometry. Later, we will give meaning to this length R .

- The smaller the triangle, the harder to distinguish it from a Euclidean triangle : only when the linear dimensions are a significant fraction of R the difference will be obvious. Eintein's theory explains why the angular excess cannot be measured by means of figures of reasonable size : the weak gravitational field in the space surrounding the earth corresponds to a microscopic value of κ , or an enormous value of R . It would of course be a different story if the experiments were to be performed in the vicinity of a black hole!

1.2 Models for the Hyperbolic Unit Ball

In this section we define the m -dimensional hyperbolic unit ball. First of all we define the hyperbolic unit ball as a *homogeneous space*, making use of group theoretical concepts, and we induce a geometry on this space and provide several explicit realizations.

Before defining the hyperbolic unit ball in m dimensions as a *homogeneous space*, let us consider its Euclidean counterpart first : the m -dimensional unit sphere in \mathbb{R}^{m+1} , which may be defined as the homogeneous space

$$\mathrm{SO}(m+1)/\mathrm{SO}(m) .$$

It may sound strange to call the Euclidean counterpart of the hyperbolic unit *ball* in $\mathbb{R}^{1,m}$ a *sphere*, but this nomenclature has grown historically. We return to this point later.

This unit sphere in \mathbb{R}^{m+1} is a typical example of an m -dimensional space of constant *positive* curvature 1, and it can be represented by the surface $S^m \subset \mathbb{R}^{m+1}$. More generally, a model for the m -dimensional positively curved space with curvature $K \in \mathbb{R}_+$ is obtained by embedding the sphere Σ , given by

$$\Sigma \leftrightarrow X_0^2 + X_1^2 + \cdots + X_m^2 = R^2 , \quad R = \frac{1}{K} ,$$

in the *flat Euclidean* space \mathbb{R}^{m+1} of one dimension higher. In view of the obvious symmetry, it is convenient to identify opposite points, but instead of pairs of points one can also consider the lines through these points. The m -dimensional elliptic space, or positively curved *Riemannian space* (which is a manifold with a symmetric and positive definite metric defined on the tangent space at each point), can then be defined as the space of all lines passing through the origin in the flat Euclidean space \mathbb{R}^{1+m} . The co-ordinates (X_0, \cdots, X_m) of the points on these lines may be considered as homogeneous co-ordinates in the Riemannian space, thus generating a *projective* model.

In case $m = 2$ we obtain a sphere in \mathbb{R}^3 as a model for the spherical plane geometry described in the first section by considering the spherical axiom as an alternative for the parallel axiom. The "lines" in this model are great circles, which makes it very easy to verify the fact that there are no "lines" through a given point parallel with a given "line", since all great circles on a sphere intersect. It was already mentioned that there is an absolute unit

of length in non-Euclidean geometry which can be expressed in terms of a certain length R . We can now explain the true meaning of this length R in case of the spherical plane : it is the radius of the sphere embedded in \mathbb{R}^3 by which the spherical geometry can be represented. In other words, if we consider a triangle T on the sphere with radius R in \mathbb{R}^3 we have the following formula for the angular excess :

$$E(T) = \kappa A(T) = \frac{A(T)}{R^2} = K^2 A(T) ,$$

with K the curvature of the sphere.

On the analogy of these Riemannian spaces of constant *positive* curvature we now turn to the spaces of constant *negative* curvature, which can also be defined as homogeneous spaces. Indeed, the m -dimensional hyperbolic unit ball is given by the homogeneous space

$$\mathrm{SO}(1, m)/\mathrm{SO}(m) ,$$

with $\mathrm{SO}(1, m)$ the set of linear transformations of unit determinant on $\mathbb{R}^{1, m}$ leaving the quadratic form $Q_{1, m}(T, \underline{X})$ invariant.

This m -dimensional space can also be represented as a surface embedded in an orthogonal space of one dimension higher :

- First of all it can be embedded in the flat *Minkowski space-time* $\mathbb{R}^{1, m}$. It is important to note that this does *not* mean that the hyperbolic unit ball is a Lorentzian manifold, by definition a pseudo-Riemannian space with symmetric nonsingular metric tensor of signature $(1, -1, \dots, -1)$. Instead, it is a classical Riemannian manifold. This will be explained below when we define a metric on the hyperbolic unit ball.
- On the other hand it is also possible to model a negatively curved space by embedding a surface into the flat *Euclidean space*. In case $m = 2$, this can be done by considering the *pseudosphere* (sometimes called the antisphere or tractrisoid) which is a surface of revolution obtained by rotating the tractrix around its asymptote. The tractrix is the curve characterized by the condition that the length of the segment of the tangent line to the curve from the tangent point to the asymptote is constant. However, the pseudosphere has a serious disadvantage : it does not model the *entire* hyperbolic plane. This follows from a result obtained by David Hilbert in 1901, who proved that it is impossible to isometrically embed the hyperbolic plane (i.e. the two-dimensional

hyperbolic unit ball) in its entirety in \mathbb{R}^3 . The pseudosphere also departs from the Euclidean plane in the following unacceptable way : it resembles a cylinder instead of a plane. Indeed, a closed loop in the Euclidean plane can always be shrunk to a point, but a loop on the pseudosphere that wraps around the axis of revolution cannot be.

Note that Riemannian manifolds *can* be embedded isometrically into Euclidean spaces, but one has to consider spaces of higher dimension. Gromov for example proved that an m -dimensional Riemannian manifold can isometrically be embedded into a Euclidean space of dimension $\frac{1}{2}(m+2)(m+3)$, see [41]. We will return to this point in Chapter 7.

In our approach we prefer the embedding in the flat Minkowski space-time. Let us therefore consider the m -dimensional space-time $\mathbb{R}^{1,m}$, where m refers to the spatial dimension, with orthonormal basis

$$B_{1,m}(\epsilon, e_j) = \{\epsilon, e_1, \dots, e_m\}$$

and endowed with the quadratic form $Q_{1,m}(T, \underline{X}) = T^2 - |\underline{X}|^2$. The following $\text{SO}(1, m)$ -invariant subsets of $\mathbb{R}^{1,m}$ are essential :

- the time-like region $TLR = \{(T, \underline{X}) \in \mathbb{R}^{1,m} : Q_{1,m}(T, \underline{X}) > 0\}$
- the space-like region $SLR = \{(T, \underline{X}) \in \mathbb{R}^{1,m} : Q_{1,m}(T, \underline{X}) < 0\}$
- the nullcone $NC = \{(T, \underline{X}) \in \mathbb{R}^{1,m} : Q_{1,m}(T, \underline{X}) = 0\}$

The time-like region itself is the union of the future cone FC and the past cone PC , given by

$$\begin{aligned} FC &= \{(T, \underline{X}) \in \mathbb{R}^{1,m} : Q_{1,m}(T, \underline{X}) > 0 \text{ and } T > 0\} \\ PC &= \{(T, \underline{X}) \in \mathbb{R}^{1,m} : Q_{1,m}(T, \underline{X}) > 0 \text{ and } T < 0\} . \end{aligned}$$

Both TLR and SLR contain a canonical $\text{SO}(1, m)$ -invariant surface which can be seen as a hyperbolic analogue of the sphere $S^m \subset \mathbb{R}^{m+1}$ in the sense that these surfaces contain, whether or not up to a minus sign, all vectors of unit hyperbolic norm. In this section however, we will focus on TLR . We therefore define the double-branched $\text{SO}(1, m)$ -invariant surface $B_T(1, m)$ by

$$B_T(1, m) = \{(T, \underline{X}) \in \mathbb{R}^{1,m} : Q_{1,m}(T, \underline{X}) = 1\} .$$

The two branches of $B_T(1, m)$ will be denoted as H_+ and H_- respectively, and they are given by

$$\begin{aligned} H_+ &= \{(T, \underline{X}) \in \mathbb{R}^{1,m} : Q_{1,m}(T, \underline{X}) = 1 \text{ and } T > 0\} \\ H_- &= \{(T, \underline{X}) \in \mathbb{R}^{1,m} : Q_{1,m}(T, \underline{X}) = 1 \text{ and } T < 0\} . \end{aligned}$$

We thus have :

$$\begin{aligned} H_+ &= B_T(1, m) \cap FC \\ H_- &= B_T(1, m) \cap PC . \end{aligned}$$

We can now obtain a space of constant negative curvature $-K$, with $K \in \mathbb{R}_+$, by embedding the m -dimensional surface Σ_H , given by

$$\Sigma_H \longleftrightarrow T^2 - X_1^2 - \dots - X_m^2 = R^2 , \quad R = \frac{1}{K} ,$$

in the *flat Minkowski space-time* $\mathbb{R}^{1,m}$. In case $m = 2$ there is again a link between this "radius" R and the angular excess $E(T)$, similar to the one derived in case of a positively curved space. Choosing $R = 1$ we see that $B_T(1, m)$ is an m -dimensional space of constant curvature -1 , and hence the true hyperbolic analogue of the sphere S^m in the flat Euclidean space. Due to the obvious symmetry it is convenient to identify diametrically opposite points. This justifies the fact that the upper sheet H_+ of $B_T(1, m)$ will be referred to as *the hyperbolic unit ball* throughout this thesis. Just like the geodesics on the sphere S^m , i.e. the great circles, can be obtained as the intersection of two-dimensional planes through the origin in \mathbb{R}^m and the sphere, geodesics on H_+ can be obtained as the intersection of two-dimensional planes through the origin in $\mathbb{R}^{1,m}$ and H_+ .

Remark : We would like to draw attention to the following remarkable fact : although both S^m and H_+ are nothing but an explicit realization of a *symmetric* homogeneous space, we cannot *visualize* H_+ in a symmetric way.

Consider for example the case $m = 2$: the sphere S^2 in \mathbb{R}^3 looks completely symmetric, whereas the surface H_+ in $\mathbb{R}^{1,2}$ (i.e. the classical hyperboloid) does *not*! Indeed, it looks as if the intersection of the hyperboloid with the T -axis has a special meaning. One should be aware of the fact that this is not really the case, all points on the hyperboloid H_+ are equivalent. Hyperbolic rotations, under which H_+ remains invariant, cannot be perceived as symmetries in a visualization because the metric of the visualization (i.e. the metric of a sheet of paper, or a computer screen) is essentially different from the Minkowski metric. It is however possible to "visualize" the hyperboloid H_+ by means of a physical experiment of thought using light rays, since the true meaning of the Lorentz invariance has to be understood in the sense of relativistic phenomena.

Instead of just identifying diametrically opposite points on $B_T(1, m)$, we can

again consider the lines through these points and this leads to a projective model for the m -dimensional hyperbolic unit ball, concentrated inside the future cone FC . This prompts the following definition :

Definition 1.1 *A projective model for the m -dimensional hyperbolic unit ball H_+ is given by the manifold of rays $\text{Ray}(FC)$, defined as*

$$\text{Ray}(FC) = \{ \{ \lambda(T, \underline{X}) : \lambda \in \mathbb{R}_+ \} : (T, \underline{X}) \in FC \} .$$

Remark : In the next Chapter it will be shown how this manifold of rays can be interpreted as a principal fibre bundle. This will be crucial when defining a Dirac operator on this manifold.

Other models for the hyperbolic unit ball are readily obtained by intersecting the manifold $\text{Ray}(FC)$ with an arbitrary surface inside the future cone FC , such that each ray intersects the surface in a unique point. This gives rise to different models for the m -dimensional hyperbolic unit ball H_+ . Before giving two such well-known models we first define a metric on H_+ by restricting to the hyperboloid H_+ the natural metric on the space $\mathbb{R}^{1,m}$, which is the Minkowski space-time metric given by

$$ds_M^2 = dT^2 - \sum_{j=1}^m dX_j^2 .$$

Let us illustrate this in case $m = 2$, where we choose co-ordinates (T, X, Y) on $\mathbb{R}^{1,2}$. Using $(X, Y) \in \mathbb{R}^2$ as parameters to describe the hyperbolic unit ball in 2 dimensions, we immediately get that

$$\begin{aligned} (dT^2 - dX^2 - dY^2)|_{H_+} &= \frac{(XdX + YdY)^2}{1 + X^2 + Y^2} - dX^2 - dY^2 \\ &= - \sum_{i,j} g_{ij}(X, Y) dX dY , \end{aligned}$$

where the coefficients of the metric tensor on H_+ are given by the matrix

$$(g_{ij}) = \frac{1}{1 + X^2 + Y^2} \begin{pmatrix} 1 + Y^2 & -XY \\ -XY & 1 + X^2 \end{pmatrix} .$$

Since the eigenvalues of the matrix (g_{ij}) are strictly positive for all (X, Y) in \mathbb{R}^2 , the restriction of the Minkowski metric to H_+ gives a metric which is, up to an overall minus sign, symmetric and positive definite. This means that H_+ is a Riemannian manifold.

Note that the unit ball $B_S(1, m)$ in the space-like region, defined by

$$B_S(1, m) = \{(T, \underline{X}) \in \mathbb{R}^{1, m} : Q_m(T, \underline{X}) = -1\} ,$$

yields an example of a Lorentzian manifold. In case $m = 2$ the restriction of the Minkowski metric to $B_S(1, 2) \leftrightarrow X^2 + Y^2 - T^2 = 1$ gives rise to the metric

$$\begin{aligned} (dT^2 - dX^2 - dY^2)|_{B_S} &= \frac{(XdX + YdY)^2}{X^2 + Y^2 - 1} - dX^2 - dY^2 \\ &= \sum_{i,j} g_{ij}(X, Y) dX dY , \end{aligned}$$

where the coefficients of the metric tensor on B_S are now given by the matrix

$$(g_{ij}) = \frac{1}{X^2 + Y^2 - 1} \begin{pmatrix} 1 - Y^2 & XY \\ XY & 1 - X^2 \end{pmatrix} .$$

This time the matrix (g_{ij}) has a positive and a negative eigenvalue, which proves that the manifold $B_S(1, 2)$ is indeed a Lorentzian manifold.

The induced metric on H_+ , denoted by ds_H^2 , gives rise to a notion of distance on H_+ . In order to compute the shortest hyperbolic distance d_H between two arbitrary points on H_+ it suffices to calculate the length of the geodesic on H_+ between ϵ and an arbitrary point. Indeed, if we are to compute this distance between two arbitrary points ξ and η on H_+ we can always perform a hyperbolic rotation (i.e. let an element of $SO(1, m)$ act on the space-time co-ordinates of these points) in order to make one of these points, say $\xi \in H_+$, coincide with ϵ . Because the geodesic on H_+ joining ϵ with η is given by a hyperbola, viz. the intersection of H_+ with the hyperplane through these two points and the origin, it suffices to determine the arc length of the curve $\gamma(x)$, parametrized by

$$\gamma(x) = ((x^2 + 1)^{\frac{1}{2}}, x, 0, \dots, 0) , \quad x \in [0, x_0]$$

with $\eta = ((x_0^2 + 1)^{\frac{1}{2}}, x_0, 0, \dots, 0) \in H_+$.

To do so, we have to integrate the arc length differential ds determined by the differential equation

$$\left(\frac{ds}{dx}\right)^2 = \left(\frac{d\gamma(x)}{dx}\right)^2 .$$

This equation arises from the requirement that the derivative of the curve $\gamma(x)$ with respect to the arc length, i.e. the vector $\frac{d\gamma}{ds}$, must be a unit vector at each point of the curve, with respect to the Minkowski metric ds_M^2 in the tangent space to the curve at that point. For the curve $\gamma(x)$ from above, this becomes

$$\begin{aligned} d_H(\epsilon, \eta) &= \int_{\gamma} ds \\ &= - \int_0^{x_0} \frac{dx}{(x^2 + 1)^{\frac{1}{2}}} . \end{aligned}$$

The minus sign appears because the geodesic γ joining ϵ and η has a space-like character. If we adopt the convention to define the distance $d_H(\epsilon, \eta)$ as the absolute value, we immediately get :

$$d_H(\epsilon, \eta) = \operatorname{argsinh}(x_0) .$$

Two other models are defined as follows :

- **The Klein model**

This model is obtained by intersecting the manifold $\operatorname{Ray}(FC)$ with the hyperplane Π , given by

$$\Pi = \{(T, \underline{X}) \in \mathbb{R}^{1,m} : T = 1\} ,$$

and projecting the point of intersection vertically down onto the unit ball $B_m(1) \subset \mathbb{R}^m$. We thus have the following mapping :

$$(T, \underline{X}) \in FC \subset \mathbb{R}^{1,m} \mapsto \underline{x} = \frac{\underline{X}}{T} \in B_m(1) \subset \mathbb{R}^m .$$

Providing $B_m(1)$ with the so-called Cayley-Klein-Hilbert metric ds_K^2 , we obtain a metric equivalence between the metric spaces (H_+, ds_H^2) and $(B_m(1), ds_K^2)$. This means that the mapping from H_+ to $B_m(1)$ described above, preserves distance for the respective metrics. Note that the metric ds_K^2 is *not* the metric on the hyperplane Π induced by the Minkowski metric, the latter being simply the standard Euclidean metric up to a minus sign. In case $m = 2$ this metric is given in co-ordinates (x, y) on \mathbb{R}^2 by

$$ds_K^2 = \frac{(1 - y^2)dx^2}{(1 - x^2 - y^2)^2} + \frac{xydx dy}{(1 - x^2 - y^2)^2} + \frac{(1 - x^2)dy^2}{(1 - x^2 - y^2)^2} .$$

Let us then explicitly verify the metric equivalence between the spaces (H_+, ds_H^2) and $(B_m(1), ds_K^2)$ by calculating the Klein distance along the geodesic between the images of the points $\eta = ((x_0^2 + 1)^{\frac{1}{2}}, x_0, 0, \dots, 0)$ and ϵ under the mapping from H_+ to $B_m(1)$ described above. These points are respectively mapped to

$$\begin{aligned}\epsilon &\mapsto \epsilon'_K = \underline{0} \\ \eta &\mapsto \eta'_K = x'_0 e_1 = \frac{x_0}{(x_0^2 + 1)^{\frac{1}{2}}} e_1 .\end{aligned}$$

The Klein geodesic joining these points is the straight line between them, where "straight" is to be understood in terms of the standard Euclidean metric; we thus find

$$\begin{aligned}d_K(\epsilon'_K, \eta'_K) &= \int_0^{x'_0} \frac{dx}{1 - x^2} \\ &= \operatorname{argtanh} \left(\frac{x_0}{(x_0^2 + 1)^{\frac{1}{2}}} \right) ,\end{aligned}$$

from which it immediately follows that

$$d_K(\epsilon'_K, \eta'_K) = d_H(\epsilon, \eta) .$$

In case $m = 2$ the classical Klein model for the hyperbolic plane is obtained, realized inside the unit disc. From the explicit form of the metric ds_K^2 it immediately follows that the Klein model for the hyperbolic plane is *not* conformal, which means that the angles measured on the model differ from the true hyperbolic angles. This is a serious disadvantage, but it is compensated for by the fact that the "lines" in the Klein model are restrictions to the unit disc of straight lines in \mathbb{R}^2 with respect to the standard Euclidean metric on \mathbb{R}^2 . Given a line in the Klein model and an arbitrary point inside the unit disc, there are infinitely many lines through the point not intersecting the given line in a point belonging to $B_2(1)$. This is in fact a manifestation of the hyperbolic alternative for the parallel axiom. In the higher-dimensional case $m > 2$ these lines are to be replaced by the intersection of $B_m(1)$ with planes of any dimension $d < m$.

The Klein model for the hyperbolic unit ball is sometimes referred to as the *velocity ball*. This nomenclature comes from the fact that the 4-velocity V in relativistic kinematics, which is an element of $\mathbb{R}^{1,3}$ whose components are the derivatives of the space-time co-ordinates (T, \underline{X})

with respect to the proper time, can be expressed in terms of the observed speed \underline{u}

$$V = \left(\frac{1}{(1 - u^2)^{\frac{1}{2}}}, \frac{\underline{u}}{(1 - u^2)^{\frac{1}{2}}} \right), \quad \text{with } \underline{u} = \frac{d\underline{X}}{dT},$$

where we have put the speed of light $c = 1$. In the same way a vector $\underline{x} \in B_m(1)$ gives rise to the space-time vector $(1, \underline{x}) \in \Pi$, i.e. an element of the Klein model for the hyperbolic unit ball which uniquely determines the 4-vector $X = (T, \underline{X}) \in H_+$ as

$$X = \left(\frac{1}{(1 - |\underline{x}|^2)^{\frac{1}{2}}}, \frac{\underline{x}}{(1 - |\underline{x}|^2)^{\frac{1}{2}}} \right),$$

Another way to see this is the following : the hyperbolic norm of the 4-velocity is always equal to one. So if we study the hyperbolic unit ball H_+ , we actually study the set of relativistic speeds.

• The Poincaré model

This model is obtained by intersecting the manifold $\text{Ray}(FC)$ with the parabolic surface \mathcal{P} , given as the set

$$\mathcal{P} = \left\{ (T, \underline{X}) \in \mathbb{R}^{1,m} : T = \frac{1 + |\underline{X}|^2}{2} \right\},$$

and projecting the point of intersection vertically down onto the unit ball $B_m(1)$ in \mathbb{R}^m . Note that each ray in $\text{Ray}(FC)$ intersects \mathcal{P} in two points, so we have to specify the point of intersection as the one having a temporal co-ordinate $T \leq 1$. We then have the following mapping :

$$(T, \underline{X}) \in FC \subset \mathbb{R}^{1,m} \mapsto \underline{x} = \frac{\underline{X}}{T + (T^2 - |\underline{X}|^2)^{\frac{1}{2}}} \in B_m(1).$$

Providing $B_m(1)$ with the Poincaré metric

$$ds_P^2 = \frac{4 \sum_j dx_j^2}{(1 - r^2)^2}$$

we obtain a metric equivalence between the metric spaces (H_+, ds_H^2) and $(B_m(1), ds_P^2)$. We will explicitly verify this, to illustrate that the

earlier calculated hyperbolic distance $d_H(\epsilon, \eta)$ remains unaffected under the mapping from H_+ to $B_m(1)$ defined above. By definition, the points ϵ and $\eta = ((x_0^2 + 1)^{\frac{1}{2}}, x_0, 0, \dots, 0)$ are mapped to

$$\begin{aligned}\epsilon &\mapsto \epsilon'_P = \underline{0} \\ \eta &\mapsto \eta'_P = x'_0 e_1 = \frac{(x_0^2 + 1)^{\frac{1}{2}} - 1}{x_0} e_1 .\end{aligned}$$

The Poincaré geodesic joining these points is the Euclidean line between them, which is a circle with infinite radius intersecting the boundary under a right angle (cfr. infra); so we find

$$\begin{aligned}d_P(\epsilon'_P, \eta'_P) &= 2 \int_0^{x'_0} \frac{dx}{1 - x^2} \\ &= 2 \operatorname{argtanh} \left(\frac{(x_0^2 + 1)^{\frac{1}{2}} - 1}{x_0} \right) .\end{aligned}$$

As

$$\sinh(d_P(\epsilon'_P, \eta'_P)) = \sinh(d_H(\epsilon, \eta)) \Rightarrow d_P(\epsilon'_P, \eta'_P) = d_H(\epsilon, \eta)$$

and this proves the metric equivalence between the spaces (H_+, ds_H^2) and $(B_m(1), ds_P^2)$.

It should again be noted that the Poincaré metric on $B_m(1)$ differs from the metric on \mathcal{P} induced by the Minkowski metric.

In case $m = 2$ the Poincaré model $(B_m(1), ds_P^2)$ for the hyperbolic unit ball reduces to the classical Poincaré model for the hyperbolic plane in which the "lines" are circles intersecting the boundary under a right angle. It is easy to verify that the axioms for the hyperbolic plane, in particular the hyperbolic alternative for Euclid's parallel axiom, are satisfied : given any "straight line" and an arbitrary point, one can draw infinitely many "straight lines" not intersecting the given line, that is : circles through the given point orthogonal to the boundary S^1 and not intersecting the given line. In the higher-dimensional case $m > 2$ these lines are to be replaced by spheres of any dimension $d < m$ intersecting the boundary S^{m-1} under a right angle.

In contrast to the Klein model for the hyperbolic unit ball, the Poincaré model offers a conformal model. This follows immediately from the fact

that

$$ds_P^2 = f(\underline{x})ds_E^2, \quad f(\underline{x}) = \frac{4}{(1-r^2)^2},$$

with ds_E^2 the standard metric on the flat Euclidean space.

Remark : These two classical models are realized inside the Euclidean unit ball and that is the reason why the hyperbolic counterpart of the *sphere* in flat Euclidean space is called the hyperbolic unit *ball* in flat Minkowski space-time.

Both the Klein and the Poincaré model for the hyperbolic unit ball are linked to each other through the *hemisphere model*. For convenience, let us consider the case where $m = 2$. In order to explain how we can go from the Poincaré model to the Klein model and vice versa, we start from the unit ball $B_2(1) \subset \mathbb{R}^2$ and we embed the plane \mathbb{R}^2 in the 3-dimensional space \mathbb{R}^3 . Consider then the sphere $S^2 \subset \mathbb{R}^3$ centred around the origin, which intersects the plane \mathbb{R}^2 in the boundary S^1 of the unit ball $B_2(1)$. If we consider the point $(0, 0, -1)$ as the south pole of a stereographic projection from the plane \mathbb{R}^2 onto the sphere S^2 , the unit ball $B_2(1)$ will be mapped onto the northern hemisphere S^2_+ whereas its boundary S^1 will be mapped onto itself, considered as the great circle on S^2 dividing the northern from the southern hemisphere. If we now consider a line in the Poincaré model, i.e. a circle intersecting S^1 under a right angle, its image under the stereographic projection will be a curve on the sphere S^2 which is to be seen as the intersection of S^2 with a hyperplane in \mathbb{R}^3 orthogonal to the plane \mathbb{R}^2 . Indeed, as the stereographic projection preserves orthogonality, the image of the Poincaré line must be orthogonal to S^1 , i.e. a vertical section. Projecting this curve on S^2 *vertically* down, back on the unit ball $B_2(1) \subset \mathbb{R}^2$, we obtain a line in the Klein model. Conversely, a line in the Klein model can be projected vertically upwards, its image being a curve on S^2 , and this will be mapped onto a line in the Poincaré model under the inverse stereographic projection. This remains true in higher dimension.

The stereographic projection also appears in the following argument, leading to a model for the m -dimensional hyperbolic unit ball on the flat Euclidean space \mathbb{R}^m , provided with the standard metric ds_E^2 . Let $\underline{x} = (x_1, \dots, x_m)$ be an element of \mathbb{R}^m and consider the map

$$\begin{aligned} \underline{x} \mapsto X &= \left(x_1, \dots, x_m, \frac{1-r^2}{2}, \frac{1+r^2}{2} \right) \\ &= (X_1, \dots, X_m, X_{m+1}, X_{m+2}) \end{aligned}$$

which is a conformal embedding of \mathbb{R}^m into a paraboloid P on the nullcone NC in the orthogonal space $\mathbb{R}^{1,m+1}$ of two dimensions higher :

$$NC \leftrightarrow X_{m+2}^2 = \sum_{j=1}^{m+1} X_j^2 .$$

When introducing on $\mathbb{R}^{1,m+1}$ the standard Minkowski metric

$$ds_H^2 = dX_{m+2}^2 - \sum_{j=1}^{m+1} dX_j^2 ,$$

it becomes clear that this mapping $\underline{x} \mapsto X$ is an isometry, up to a sign. Indeed, as X belongs to the plane V with equation

$$V \leftrightarrow X_{m+1} + X_{m+2} = 1 ,$$

we have $dX_{m+1} = -dX_{m+2}$ and so

$$ds_H^2 = dX_{m+2}^2 - \sum_{j=1}^{m+1} dX_j^2 = - \sum_{j=1}^m dx_j^2 = -ds_E^2 .$$

Conversely, any point from the intersection of the plane V and the cone NC , i.e. any point from the paraboloid $P = NC \cap V$, satisfies

$$X_{m+2} - X_{m+1} = X_{m+2}^2 - X_{m+1}^2 = \sum_{j=1}^m X_j^2 = r^2$$

so that, putting $x_j = X_j$, we obtain

$$X_{m+1} = \frac{1 - r^2}{2} \text{ and } X_{m+2} = \frac{1 + r^2}{2} .$$

Let $\underline{x} \mapsto X$, and consider then the positive half ray

$$\text{Ray}(X) = \{Y = \lambda X : \lambda \in \mathbb{R}_+\}$$

inside $NC_+ = NC \cap \{X : X_{m+2} \in \mathbb{R}_+\}$. The mapping $\underline{x} \mapsto \text{Ray}(X)$ is an embedding of \mathbb{R}^m into the manifold of rays $\text{Ray}(NC_+)$ which is injective and covers the whole positive cone NC_+ with the exception of $\text{Ray}(\underline{0}, -1, 1)$. This single ray may be considered as a unique point at infinity and leads to the one-point-compactification $\mathbb{R}^m \cup \{\infty\} = S^m$, whereby S^m may be topologically identified with a sphere on NC_+ :

$$S^m = NC_+ \cap \{X : X_{m+2} = 1\} .$$

This sphere is precisely the one used to illustrate the relation between the Klein model and the Poincaré model for the hyperbolic unit ball by means of the hemisphere model. In fact, the map from $\mathbb{R}^m \mapsto S^m \setminus \{\infty\}$ given by

$$\underline{x} \mapsto X \mapsto \text{Ray}(X) \mapsto Y \in S^m \setminus \{\infty\}$$

is precisely the stereographic projection

$$\begin{cases} Y_j &= \frac{2x_j}{1+r^2}, \quad j = 1, \dots, m \\ Y_{m+1} &= \frac{1-r^2}{1+r^2} \end{cases}$$

from the south pole $(0, -1)$ of the sphere S^m intersecting the hyperplane $X_{m+1} = 0$, i.e. the space \mathbb{R}^m , in the sphere S^{m-1} . Note that the south pole is the point at infinity, i.e. the intersection $\text{Ray}(0, -1, 1) \cap \{X : X_{m+2} = 1\}$.

In some sense the stereographic projection "factorizes" the mapping from \mathbb{R}^m to the nullcone NC in $\mathbb{R}^{1,m+1}$: instead of mapping $\underline{x} \in \mathbb{R}^m$ immediately to $X \in \mathbb{R}^{1,m+1}$ one passes by the image of \underline{x} under the conformal stereographic projection on the sphere S^m , identified with a sphere on the cone NC , i.e. $NC \cap \{X : X_{m+2} = 1\}$. The image of \underline{x} on the sphere S^m is then finally moved along the rays in NC to a point on the paraboloid $P \subset NC$.

From this point of view the conformal character of the Poincaré model for the hyperbolic unit ball can also be seen as follows: the orthogonal group $\text{SO}(1, m+1)$ preserving the cone NC , maps rays onto rays and leads to an angle preserving map on the manifold of rays $\text{Ray}(NC) = S^m$. Since the inverse stereographic projection from S_+^m back onto the unit ball $B_m(1)$ is conformal too, we easily obtain the desired result.

Chapter 2

The Dirac Operator on the Hyperbolic Unit Ball

Mathematics is the tool specially suited for dealing with abstract concepts of any kind and there is no limit to its power in this field. (P.A.M. Dirac)

In this chapter the Dirac operator on the hyperbolic unit ball is defined, and several explicit constructions for the fundamental solution for this operator are given. The projective nature of our model for the hyperbolic unit ball is essential to these constructions.

2.1 The Hyperbolic Dirac Operator

In Chapter 1 several models for the m -dimensional hyperbolic unit ball H_+ were defined. One of these was the manifold of rays, see Definition 1.1 :

$$\text{Ray}(FC) = \{ \{ \lambda(T, \underline{X}) : \lambda \in \mathbb{R}_+ \} : (T, \underline{X}) \in FC \} .$$

Note that this model is concentrated within the FC , and that is why we restricted ourselves to space-time vectors (T, \underline{X}) such that $Q_{1,m}(T, \underline{X}) > 0$ when we introduced a polar decomposition for the operator ∂_X (see section 0.1.3).

Due to the *projective* nature of this model, all concepts on the hyperbolic unit ball must be defined in such a way that they correspond to an invariant object on $\text{Ray}(FC)$. This can be done by considering, for arbitrary complex α , the *homogeneous Clifford line bundle*

$$\mathbb{R}_{1,m;\alpha} = \{ ((T, \underline{X}), a) \in \mathbb{R}_0^{1,m} \times \mathbb{R}_{1,m} \} / \sim \quad (2.1)$$

where the equivalence relation \sim is given by

$$((T, \underline{X}), a) \sim (\lambda(T, \underline{X}), \lambda^\alpha a) , \quad \lambda \in \mathbb{R}_+ .$$

In other words : the *bundle space* of the bundle $\mathbb{R}_{1,m;\alpha}$ is the set $\mathbb{R}_0^{1,m} \times \mathbb{R}_{1,m}$ and the *base space* consists of the equivalence classes under the projection π , with

$$\begin{aligned} \pi((T_1, \underline{X}_1), a_1) = \pi((T_2, \underline{X}_2), a_2) &\iff ((T_1, \underline{X}_1), a_1) \sim ((T_2, \underline{X}_2), a_2) \\ &\iff ((T_1, \underline{X}_1), a_1) = (\lambda(T_2, \underline{X}_2), \lambda^\alpha a_2) \end{aligned}$$

for a certain $\lambda \in \mathbb{R}_+$. Sections of this bundle have the following form :

$$\left((T, \underline{X}), F(T, \underline{X}) \right) \sim \left(\lambda(T, \underline{X}), \lambda^\alpha F(T, \underline{X}) \right) ,$$

which means that sections of the Clifford bundle $\mathbb{R}_{1,m;\alpha}$ are *homogeneous $\mathbb{R}_{1,m}$ -valued functions* on the flat Minkowski space-time $\mathbb{R}^{1,m}$.

Remark : Since the term *homogeneous Clifford line bundle* is no standard term in the literature, although it was already used in e.g. [12] and [13], we here present a rigorous definition for this bundle. In what follows we will then always refer to this construction when using the name 'homogeneous Clifford line bundle'.

- Let G be the multiplicative group \mathbb{R}_+ and consider its (abelian) action on punctured Minkowski space-time $\mathbb{R}_0^{1,m}$ given by

$$\lambda(T, \underline{X}) = (\lambda T, \lambda \underline{X}) \quad \text{for all } \lambda \in \mathbb{R}_+ .$$

This is a free action and the orbit space is the space of halfrays in flat Minkowski space-time $\mathbb{R}^{1,m}$. In other words, the subset of the orbit space obtained by considering the action on FC is precisely the manifold $\text{Ray}(FC)$ defined earlier. In this way, $\text{Ray}(FC)$ becomes a principal G -bundle $(FC, \pi, FC/G)$. We already mentioned this fact in the previous Chapter, when we first defined the manifold $\text{Ray}(FC)$. This means that functions defined on the hyperbolic unit ball are in fact functions defined on a bundle, i.e. *sections*. However, since monogenic functions are normally defined as *Clifford algebra-valued functions*, we would like to construct a bundle associated to $\text{Ray}(FC)$ whose fibre is precisely the Clifford algebra $\mathbb{R}_{1,m}$ such that its sections become Clifford-algebra valued functions on FC . This can be done by means of the general theory of associated principal fibre bundles!

- We thus define the following representation Λ of G on the Clifford algebra $\mathbb{R}_{1,m}$:

$$\begin{aligned}\Lambda : G &\mapsto \text{End}(\mathbb{R}_{1,m}) \\ \lambda &\mapsto \lambda^{-\alpha} ,\end{aligned}$$

where the mapping $\lambda^{-\alpha} \in \text{End}(\mathbb{R}_{1,m})$ acts by multiplication

$$\lambda^{-\alpha} : a \mapsto \lambda^{-\alpha} a .$$

- Because we now have the principle G -bundle $\text{Ray}(FC)$ and the Clifford algebra $\mathbb{R}_{1,m}$ playing the role of a (left) G -space, we can define an associated fibre bundle with fibre $\mathbb{R}_{1,m}$. To do so we first introduce the G -product

$$FC \times_G \mathbb{R}_{1,m}$$

of orbits under the action of G on the Cartesian product $FC \times \mathbb{R}_{1,m}$. Here, we have that $((T, \underline{X}), a) \sim ((S, \underline{Y}), b)$ if and only if there exists a $\lambda \in G$ such that

$$(S, \underline{Y}) = \lambda(T, \underline{X}) \quad \text{and} \quad b = (\lambda^{-\alpha})^{-1} a = \lambda^{\alpha} a .$$

By definition, the associated bundle $(FC_{\mathbb{R}_{1,m}}, \pi_{\mathbb{R}_{1,m}}, FC/G)$ is then given by : $FC_{\mathbb{R}_{1,m}} = FC \times_G \mathbb{R}_{1,m}$ and $\pi_{\mathbb{R}_{1,m}}((T, \underline{X}), a) = \pi(T, \underline{X})$ with π the projection associated to the principal G -bundle $\text{Ray}(FC)$, i.e. the bundle $(FC, \pi, FC/G)$.

- Since $(FC_{\mathbb{R}_{1,m}}, \pi_{\mathbb{R}_{1,m}}, FC/G)$ is an associated fibre bundle, its sections are in bijective correspondance with functions $\phi : FC \mapsto \mathbb{R}_{1,m}$ satisfying

$$\phi(\lambda(T, \underline{X})) = (\lambda^{-\alpha})^{-1} \phi(T, \underline{X}) = \lambda^{\alpha} \phi(T, \underline{X}) .$$

This expresses precisely the fact that sections on the Clifford line bundle are homogeneous functions on FC !

The Dirac operator on the hyperbolic unit ball can thus be defined as the Dirac operator $\partial_X = \epsilon \partial_T - \partial_{\underline{X}}$ on $\mathbb{R}^{1,m}$ acting on sections of the bundle $\mathbb{R}_{1,m;\alpha}$. This immediately leads to the following definition :

Definition 2.1 *The Dirac operator on the hyperbolic unit ball H_+ is defined as the Dirac operator on $\mathbb{R}^{1,m}$ acting on α -homogeneous functions on FC , α being an arbitrary complex number.*

As the Dirac operator ∂_X on $\mathbb{R}^{1,m}$ is homogeneous of degree (-1) , it maps α -sections onto $(\alpha - 1)$ -sections, whence the Dirac operator on the hyperbolic unit ball is well-defined. From now on, this operator will be referred to as *the hyperbolic Dirac operator*.

By analogy with the notion of monogenic functions f defined in an open subset Ω of \mathbb{R}^m with respect to the Dirac operator $\underline{\partial}$ on $\mathbb{R}^{0,m}$, we introduce the following :

Definition 2.2 *Let Ω be an open subset of the hyperbolic unit ball H_+ . An open conical region $\mathbb{R}_+\Omega$ of the future cone FC is then defined as the set*

$$\mathbb{R}_+\Omega = \{(T, \underline{X}) \in FC : (T, \underline{X}) = \lambda(\tau, \underline{\xi}) , \lambda \in \mathbb{R}_+ \text{ and } (\tau, \underline{\xi}) \in \Omega\} .$$

Definition 2.3 *Let α be an arbitrary complex number. An α -homogeneous function $F(T, \underline{X})$ defined in an open conical region $\mathbb{R}_+\Omega$ of the future cone FC in $\mathbb{R}^{1,m}$ which is monogenic with respect to the Dirac operator ∂_X , is called a monogenic function on the hyperbolic unit ball H_+ defined in $\mathbb{R}_+\Omega$.*

Recalling the polar decomposition

$$\partial_X = \xi \left(\partial_\rho + \frac{1}{\rho} \Gamma_{1,m} \right) = \frac{\xi}{\rho} (\mathbb{E}_\rho + \Gamma_{1,m})$$

of the Dirac operator on the time-like region of the m -dimensional space-time $\mathbb{R}^{1,m}$, we introduce two function spaces :

Definition 2.4 *Let $\mathbb{R}_+\Omega$ be an open conical region in FC and let α be an arbitrary complex number. We then put :*

$$\begin{aligned} \mathcal{H}^\alpha(\Omega) &= \{F \in C^1(\Omega) : \xi(\Gamma_{1,m} + \alpha)F = 0 \text{ in } \Omega\} \\ \mathcal{H}^\alpha(\mathbb{R}_+\Omega) &= \{F \in C^1(\mathbb{R}_+\Omega) : \mathbb{E}_\rho F = \alpha F \text{ and } \partial_X F = 0 \text{ in } \mathbb{R}_+\Omega\} \end{aligned}$$

Provided with the obvious laws for addition and multiplication with Clifford numbers both sets are right $\mathbb{R}_{1,m}$ -modules. The space $\mathcal{H}^\alpha(\mathbb{R}_+\Omega)$ is the set of all monogenic functions on the hyperbolic unit ball defined in $\mathbb{R}_+\Omega$. The space $\mathcal{H}^\alpha(\Omega)$ contains the restrictions to $\Omega \subset H_+$ of monogenic functions on the hyperbolic unit ball defined in $\mathbb{R}_+\Omega$, and we will label these restrictions as *hyperbolic monogenics* :

Definition 2.5 *Elements of $\mathcal{H}^\alpha(\Omega)$ are called hyperbolic monogenics in Ω .*

This means that there is a link between monogenic functions on the hyperbolic unit ball and eigenfunctions for the hyperbolic Gamma operator $\Gamma_{1,m}$. Indeed, if $F(X)$ is a monogenic function on the hyperbolic unit ball, defined on \mathbb{R}_+H_+ , we have

$$\begin{cases} \partial_X F(X) &= 0 \\ \mathbb{E}_\rho F(X) &= \alpha F(X) \end{cases}$$

whence the restriction of $F(X)$ to H_+ yields an eigenfunction for $\Gamma_{1,m}$:

$$\xi(\Gamma_{1,m} + \alpha)F(\xi) = 0 .$$

Conversely, an eigenfunction $F(\xi)$ for the operator $\Gamma_{1,m}$ with eigenvalue α gives a monogenic function $F(X) = |X|^\alpha F(\xi)$ on the hyperbolic unit ball, homogeneous of degree α .

2.2 Fundamental Solutions

In this section *the hyperbolic Dirac equation* will be derived, i.e. the equation determining the hyperbolic fundamental solution. In order to do so, we use the Klein model for the hyperbolic unit ball realized as the intersection of $\text{Ray}(FC)$ with the hyperplane $\Pi \leftrightarrow T = 1$. Four different constructions for the solution to this equation are then given.

Consider thus an arbitrary space-time vector $X = \epsilon T + \underline{X} \in FC$. We then have :

$$\epsilon T + \underline{X} = \lambda(\epsilon + \underline{x})$$

where we have put $\underline{x} = \frac{\underline{X}}{T} \in B_m(1)$ and $\lambda = T \in \mathbb{R}_+$. Interpreting (λ, \underline{x}) as a new set of co-ordinates on the FC , and rewriting the Dirac operator ∂_X in terms of these co-ordinates we find :

$$\partial_X = \epsilon \partial_T - \partial_{\underline{X}} \longrightarrow -\frac{1}{\lambda}(\underline{\partial} + \epsilon(\mathbb{E}_r - \mathbb{E}_\lambda))$$

with $\underline{\partial}$ (resp. \mathbb{E}_r) the Dirac operator (resp. the Euler operator) on $\mathbb{R}^{0,m}$ in terms of the co-ordinates \underline{x} on the Euclidean unit ball $B_m(1)$. In view of our definition of monogenic functions on the hyperbolic unit ball, we then define the hyperbolic fundamental solution $E_\alpha(T, \underline{X})$ as follows :

$$E_\alpha(T, \underline{X}) = T^\alpha E_\alpha(1, \underline{x}) = \lambda^\alpha E_\alpha(\underline{x}).$$

This function $E_\alpha(\underline{x})$ must be a solution for the operator $-(\underline{\partial} + \epsilon(\mathbb{E}_r - \alpha))$ on the punctured Euclidean unit ball $B_m(1) \setminus \{0\}$ with singularity for $\underline{x} = \underline{0}$. Therefore, it satisfies the following equation :

$$(\underline{\partial} + \epsilon(\mathbb{E}_r - \alpha))E_\alpha(\underline{x}) = -\delta(\underline{x}) .$$

In space-time co-ordinates (T, \underline{X}) this becomes

$$(\epsilon\partial_T - \partial_{\underline{X}})E_\alpha(T, \underline{X}) = T_+^{\alpha+m-1}\delta(\underline{X}) , \quad (2.2)$$

with $T_+^{\alpha+m-1}$ the distribution defined in section 0.3. Equation (2.2) is *the hyperbolic Dirac equation*; solving it is the subject of this section.

Remark 1 : As the distribution $T_+^{\alpha+m-1}$ is *not* defined for $\alpha + m \in -\mathbb{N}$, these values for α will be excluded throughout this section!

Remark 2 : The motivation for choosing $T_+^{\alpha+m-1}\delta(\underline{X})$ to be the right-hand side of the hyperbolic Dirac equation can also be illustrated in the following way : if we multiply it with the delta distribution $\delta(T^2 - |\underline{X}|^2 - 1)$, hence restricting the integration over the flat Minkowski space-time to $B_T(1, m)$, we obtain the point-evaluation in the intersection of the ray through (T, \underline{X}) and $B_T(1, m)$; i.e. a delta distribution on the hyperboloid. Indeed, for an arbitrary test function φ we have :

$$\left\langle T_+^{\alpha+m-1}\delta(\underline{X})\delta(T^2 - |\underline{X}|^2 - 1), \varphi \right\rangle = \int_{\mathbb{R}} T_+^{\alpha+m-1}\delta(T^2 - 1)\varphi(T, \underline{0})dT ,$$

which by means of the fact that

$$\delta(f(T)) = \sum_j \frac{\delta(T - T_j)}{|f'(T_j)|} \Big|_{f(T_j)=0} ,$$

reduces to

$$\begin{aligned} \left\langle T_+^{\alpha+m-1}\delta(\underline{X})\delta(T^2 - |\underline{X}|^2 - 1), \varphi \right\rangle &= \int_0^\infty T^{\alpha+m-1}\delta(T - 1)\varphi(T, \underline{0})dT \\ &= \varphi(1, \underline{0}) . \end{aligned}$$

Remark 3 : In Chapter 5 we will briefly return to this point and show how the distribution $T_+^{\alpha+m-1}\delta(\underline{X})$ arises from the point of view of the general theory of delta distributions on manifolds. We postpone this explanation until then because it would be inappropriate to discuss it here.

2.2.1 Method 1 : Projection

The idea behind this first method is to project the hyperbolic Dirac equation onto an arbitrary surface Σ , and to rewrite the hyperbolic Dirac operator in terms of co-ordinates on this surface. We will do this for two particular choices for Σ : first we consider the hyperplane $\Pi \leftrightarrow T = 1$, thus projecting the hyperbolic Dirac equation onto the Klein model for the hyperbolic unit ball, and next we consider the parabola $\mathcal{P} \leftrightarrow T = \frac{1+|\underline{X}|^2}{2}$, hence obtaining the projection onto the Poincaré model.

• The Klein Model

The projection of the hyperbolic Dirac equation onto the hyperplane Π gives rise to the following equation on the unit ball $B_m(1) \subset \mathbb{R}^m$:

$$(\underline{\partial} + \epsilon(\mathbb{E}_r - \alpha))E_\alpha(\underline{x}) = \delta(\underline{x}) , \quad (2.3)$$

with $\underline{x} = \frac{\underline{X}}{T}$ and with $E_\alpha(T, \underline{X}) = T^\alpha E_\alpha(\underline{x})$ the resulting hyperbolic fundamental solution in space-time co-ordinates (T, \underline{X}) .

The idea is to construct a hyperbolic fundamental solution $E_\alpha(\underline{x})$ which is a modulated version of the classical Cauchy kernel $E(\underline{x})$ on $\mathbb{R}^{0,m}$:

$$\begin{aligned} E_\alpha(\underline{x}) &= \sum_{j=0}^{\infty} b_j(\underline{x}\epsilon)^j E(\underline{x}) \\ &= \frac{1}{A_m} \sum_{j=0}^{\infty} (b_{2j} + b_{2j+1}\underline{x}\epsilon) |\underline{x}|^{2j} \frac{\underline{x}}{|\underline{x}|^m} , \end{aligned}$$

where the unknown coefficients are to be determined in such a way that we obtain a solution for the hyperbolic Dirac equation (2.3).

Letting the operator $\underline{\partial}$ act on $E_\alpha(\underline{x})$ we get for the scalar part of the summation

$$\underline{\partial} \sum_{j=0}^{\infty} b_{2j} |\underline{x}|^{2j} \frac{\underline{x}}{|\underline{x}|^m} = \underline{x} \sum_{j=0}^{\infty} (2j+2) b_{2j+2} |\underline{x}|^{2j} \frac{\underline{x}}{|\underline{x}|^m}$$

and for the bivector part

$$\underline{\partial} \sum_{j=0}^{\infty} b_{2j+1} \underline{x}\epsilon |\underline{x}|^{2j} \frac{\underline{x}}{|\underline{x}|^m} = -\epsilon \sum_{j=0}^{\infty} (2+2j-m) b_{2j+1} |\underline{x}|^{2j} \frac{\underline{x}}{|\underline{x}|^m} .$$

Letting the operator \mathbb{E}_r act on the scalar part we get

$$\mathbb{E}_r \sum_{j=0}^{\infty} b_{2j} |\underline{x}|^{2j} \frac{\underline{x}}{|\underline{x}|^m} = \sum_{j=0}^{\infty} (1 + 2j - m) b_{2j} |\underline{x}|^{2j} \frac{\underline{x}}{|\underline{x}|^m}$$

and for the bivector part

$$\mathbb{E}_r \sum_{j=0}^{\infty} b_{2j+1} \underline{x} \epsilon |\underline{x}|^{2j} \frac{\underline{x}}{|\underline{x}|^m} = \underline{x} \epsilon \sum_{j=0}^{\infty} (2 + 2j - m) b_{2j+1} |\underline{x}|^{2j} \frac{\underline{x}}{|\underline{x}|^m}.$$

Expressing the fact that $(\partial + \epsilon(\mathbb{E}_r - \alpha))E_\alpha(\underline{x}) = 0$ on the punctured unit ball we get the following system of equations to hold, for each $j \in \mathbb{N}$:

$$\begin{cases} (2 + 2j - m - \alpha) b_{2j+1} &= (2j + 2) b_{2j+2} \\ (1 + 2j - m - \alpha) b_{2j} &= (2 + 2j - m) b_{2j+1} \end{cases}$$

Choosing $b_0 = 1$, this will eventually give the following coefficients b_j :

$$\begin{aligned} b_{2j} &= \frac{\left(1 - \frac{\alpha+m}{2}\right)_j \left(\frac{1-\alpha-m}{2}\right)_j}{j! \left(1 - \frac{m}{2}\right)_j} \\ b_{2j+1} &= \frac{1 - \alpha - m}{2 - m} \frac{\left(1 - \frac{\alpha+m}{2}\right)_j \left(1 + \frac{1-\alpha-m}{2}\right)_j}{j! \left(2 - \frac{m}{2}\right)_j}, \end{aligned}$$

which means that we have found the following fundamental solution :

$$E_\alpha(T, \underline{X}) = \lambda^\alpha \text{Mod}(\alpha, 1 - m, \underline{x}) E(\underline{x}) \quad (2.4)$$

where we have put

$$\text{Mod}(\alpha, 1 - m, \underline{x}) = F_1(|\underline{x}|^2) + \frac{1 - m - \alpha}{2 - m} \underline{x} \epsilon F_2(|\underline{x}|^2)$$

with

$$\begin{aligned} F_1(t) &= F\left(1 - \frac{\alpha + m}{2}, \frac{1 - \alpha - m}{2}; 1 - \frac{m}{2}; t\right) \\ F_2(t) &= F\left(1 - \frac{\alpha + m}{2}, 1 + \frac{1 - \alpha - m}{2}; 2 - \frac{m}{2}; t\right). \end{aligned}$$

Remark 1 : Note that in case of an *even*-dimensional space-time, these hypergeometric functions are ill-defined. This follows from the fact that the hypergeometric function $F(a, b; c; t)$ becomes ill-defined

for $c \in -\mathbb{N}$. We will return to this point in section 2.2.2. For the moment we thus restrict ourselves to the case of an odd-dimensional space-time. There is however an exception to this, because for $\alpha = -\frac{m}{2}$ this fundamental solution is defined for both odd *and* even dimensions m . Indeed, for this value we get

$$\begin{aligned} F_1(t) &= F\left(\frac{1}{2}\left(1 - \frac{m}{2}\right), \frac{1}{2}\left(2 - \frac{m}{2}\right); 1 - \frac{m}{2}; t\right) \\ &= (1-t)^{-\frac{1}{2}} \left(\frac{1 + (1-t)^{\frac{1}{2}}}{2}\right)^{\frac{m}{2}} \end{aligned}$$

and

$$\begin{aligned} F_2(t) &= F\left(\frac{1}{2}\left(2 - \frac{m}{2}\right), \frac{1}{2}\left(3 - \frac{m}{2}\right); 2 - \frac{m}{2}; t\right) \\ &= (1-t)^{-\frac{1}{2}} \left(\frac{1 + (1-t)^{\frac{1}{2}}}{2}\right)^{\frac{m}{2}-1}, \end{aligned}$$

such that

$$\text{Mod}\left(-\frac{m}{2}, 1 - m, \underline{x}\right) = \frac{\left(1 + (1 - |\underline{x}|^2)^{\frac{1}{2}}\right)^{\frac{m}{2}}}{2^{\frac{m}{2}}(1 - |\underline{x}|^2)^{\frac{1}{2}}} \left(1 + \frac{\underline{x}\epsilon}{1 + (1 - |\underline{x}|^2)^{\frac{1}{2}}}\right).$$

We will postpone the explanation of this phenomenon until a later Chapter. Right now it suffices to remember that the value $\alpha = -\frac{m}{2}$ leads to an interesting special case of the hyperbolic Dirac equation.

Remark 2 : In Chapter 3 the idea of modulating solutions for the Dirac operator $\underline{\partial}$ on $\mathbb{R}^{0,m}$ to obtain solutions for the hyperbolic Dirac equation will be generalized. This will enable us to obtain a whole class of nullsolutions for the hyperbolic Dirac operator.

- **The Poincaré Model**

With an arbitrary point $\epsilon T + \underline{X} \in FC$ we associate the intersection of the ray through this point and the parabola \mathcal{P} . Since there are two points of intersection, we choose the one which has a temporal co-ordinate less than 1. This gives

$$\epsilon T + \underline{X} = \lambda_P \left(\epsilon \frac{1 + |\underline{x}|^2}{2} + \underline{x} \right)$$

with $\underline{x} = \frac{\underline{X}}{\lambda_P} \in B_m(1)$ and $\lambda_P = T + \sqrt{T^2 - |\underline{X}|^2}$. We then interpret $(\lambda_P, \underline{x})$ as new co-ordinates on the future cone, and the Dirac operator ∂_X on $\mathbb{R}^{1,m}$ becomes :

$$\epsilon \partial_T - \partial_{\underline{X}} \longrightarrow -\frac{1}{\lambda_P} \left(\underline{\partial} + 2 \frac{\underline{x} + \epsilon}{1 - |\underline{x}|^2} (\mathbb{E}_r - \mathbb{E}_{\lambda_P}) \right)$$

with $\underline{\partial}$ (resp. \mathbb{E}_r) the Dirac operator (resp. the Euler operator) on \mathbb{R}^m in terms of the co-ordinates $\underline{x} \in B_m(1)$. In view of Definition 2.3 we then put

$$E_\alpha(T, \underline{X}) = \lambda_P^\alpha E_\alpha(\underline{x}) ,$$

and this eventually yields the following projected hyperbolic Dirac equation :

$$\left(\underline{\partial} + 2 \frac{\underline{x} + \epsilon}{1 - |\underline{x}|^2} (\mathbb{E}_r - \alpha) \right) E_\alpha(\underline{x}) = -\delta(\underline{x}) . \quad (2.5)$$

The idea is again to look for a fundamental solution $E_\alpha(\underline{x})$ which is a modulated version of the classical Cauchy kernel :

$$\begin{aligned} E_\alpha(\underline{x}) &= \sum_{j=0}^{\infty} a_j (\underline{x}\epsilon)^j E(\underline{x}) \\ &= \frac{1}{A_m} \sum_{j=0}^{\infty} (a_{2j} + a_{2j+1} \underline{x}\epsilon) |\underline{x}|^{2j} \frac{\underline{x}}{|\underline{x}|^m} , \end{aligned}$$

where the unknown coefficients a_j are to be determined in such a way that we get a solution for the hyperbolic Dirac equation. Putting

$$(\epsilon + \underline{x})^2 = 1 - |\underline{x}|^2$$

and recalling the fact that $\underline{\partial} \underline{x} + \underline{x} \underline{\partial} = -m - 2\mathbb{E}_r$ on $\mathbb{R}^{0,m}$, we see that equation (2.5) is equivalent with the following equation :

$$(\underline{\partial}(\underline{x} + \epsilon) + m + 2\alpha)E_\alpha(\underline{x}) = \delta(\underline{x})\epsilon .$$

Letting the operator $\underline{\partial} \underline{x}$ act on $E_\alpha(\underline{x})$, we get for the scalar part of the summation :

$$\underline{\partial} \underline{x} \sum_{j=0}^{\infty} a_{2j} |\underline{x}|^{2j} \frac{\underline{x}}{|\underline{x}|^m} = - \sum_{j=0}^{\infty} (2 + 2j - m) a_{2j} |\underline{x}|^{2j} \frac{\underline{x}}{|\underline{x}|^m}$$

and for the bivector part :

$$\underline{\partial} \underline{x} \sum_{j=0}^{\infty} a_{2j+1} \underline{x} \epsilon |\underline{x}|^{2j} \frac{\underline{x}}{|\underline{x}|^m} = - \underline{x} \epsilon \sum_{j=0}^{\infty} (2j + 2) a_{2j+1} |\underline{x}|^{2j} \frac{\underline{x}}{|\underline{x}|^m} .$$

Letting the operator $\underline{\partial} \epsilon$ act on $E_\alpha(\underline{x})$, we get for the scalar part :

$$\underline{\partial} \epsilon \sum_{j=0}^{\infty} a_{2j} |\underline{x}|^{2j} \frac{\underline{x}}{|\underline{x}|^m} = \underline{x} \epsilon \sum_{j=0}^{\infty} (2j + 2) a_{2j+2} |\underline{x}|^{2j} \frac{\underline{x}}{|\underline{x}|^m}$$

and for the bivector part :

$$\underline{\partial} \epsilon \sum_{j=0}^{\infty} a_{2j+1} \underline{x} \epsilon |\underline{x}|^{2j} \frac{\underline{x}}{|\underline{x}|^m} = \sum_{j=0}^{\infty} (2 + 2j - m) a_{2j+1} |\underline{x}|^{2j} \frac{\underline{x}}{|\underline{x}|^m} .$$

Expressing the fact that $(\underline{\partial}(\underline{x} + \epsilon) + m + 2\alpha)E_\alpha(\underline{x}) = 0$ we get the following set of equations, for each $j \in \mathbb{N}$:

$$\begin{cases} (2 + 2j - m - 2\alpha) a_{2j+1} &= (2j + 2) a_{2j+2} \\ (2 + 2j - 2m - 2\alpha) a_{2j} &= (2 + 2j - m) a_{2j+1} \end{cases}$$

Choosing $a_0 = 1$, this will eventually give the following coefficients a_j :

$$\begin{aligned} a_{2j} &= \frac{(1 - \alpha - \frac{m}{2})_j (1 - \alpha - m)_j}{j! (1 - \frac{m}{2})_j} \\ a_{2j+1} &= \frac{1 - \alpha - m}{1 - \frac{m}{2}} \frac{(1 - \alpha - \frac{m}{2})_j (2 - \alpha - m)_j}{j! (2 - \frac{m}{2})_j} . \end{aligned}$$

This means that we have found the following fundamental solution :

$$E_\alpha(T, \underline{X}) = \lambda_P^\alpha \text{Mod}_P(\alpha, 1 - m, \underline{x}) E(\underline{x}) \quad (2.6)$$

where we have put

$$\text{Mod}_P(\alpha, 1 - m, \underline{x}) = F_{P,1}(|\underline{x}|^2) + \frac{1 - m - \alpha}{1 - \frac{m}{2}} \underline{x} \epsilon F_{P,2}(|\underline{x}|^2)$$

with

$$\begin{aligned} F_{P,1}(t) &= F\left(1 - \alpha - \frac{m}{2}, 1 - m - \alpha; 1 - \frac{m}{2}; t\right) \\ F_{P,2}(t) &= F\left(1 - \alpha - \frac{m}{2}, 2 - m - \alpha; 2 - \frac{m}{2}; t\right) . \end{aligned}$$

The same remarks as for the projection on the Klein model hold here as well :

Remark 1 : In case we are dealing with an *even*-dimensional space-time $\mathbb{R}^{1,m}$, the hypergeometric functions in the formulae above are again ill-defined. We return to this point in the next subsection. There is however again one exception, when $\alpha = -\frac{m}{2}$. Indeed, in that case we easily get that

$$F_{P,1}(t) = F_{P,2}(t) = F\left(1, 1 - \frac{m}{2}; 1 - \frac{m}{2}; t\right) = \sum_{k=0}^{\infty} t^k = \frac{1}{1-t} ,$$

such that the Modulation factor reduces to

$$\text{Mod}_P\left(-\frac{m}{2}, 1 - m, \underline{x}\right) = \frac{1 + \underline{x}\epsilon}{1 - |\underline{x}|^2} = (1 + \underline{x}\epsilon)^{-1} .$$

If we then recall the projection of the hyperbolic Dirac operator on the Poincaré ball in case $\alpha = -\frac{m}{2}$, given by $\underline{\partial}(\underline{x} + \epsilon)$, it is immediately clear that this is no coincidence since

$$\left((\underline{\partial}(\underline{x} + \epsilon) + (m + 2\alpha)) E_{\alpha}(\underline{x}) \right) \Big|_{\alpha = -\frac{m}{2}} = \underline{\partial}\epsilon E(\underline{x}) = \delta(\underline{x})\epsilon ,$$

as it should be! We return to this point in later Chapters.

Remark 2 : In Chapter 3 the idea of modulating the Cauchy kernel to obtain a fundamental solution for the hyperbolic Dirac equation will be generalized to arbitrary monogenic functions. In this way a second class of nullsolutions for the hyperbolic Dirac operator will be obtained. Both classes of nullsolutions, obtained by projection on the Klein and Poincaré model of the hyperbolic unit ball, will then be proved to be equivalent. This will give rise to geometrical interpretations for certain identities for the hypergeometric function.

2.2.2 Method 2 : Radon Inversion

The idea behind this method is to use the Radon transform to reduce the hyperbolic Dirac equation, which is an m -dimensional problem for a vector-valued differential operator, to a scalar problem in two dimensions. One might argue that we are dealing with an $(m+1)$ -dimensional problem, since we need the Dirac operator on $\mathbb{R}^{1,m}$, but it should be stressed that the need for homogeneity allows us to have one degree of freedom and this reduces the dimension of the problem. At the end of this subsection we will also explain why the even-dimensional case differs substantially from the odd-dimensional case.

The starting point is of course the hyperbolic Dirac equation (2.2). The Radon transform can be used to transform this equation into an equation for $E_\alpha^\dagger(T, p, \underline{\omega})$, the Radon transform of the fundamental solution $E_\alpha(T, \underline{X})$ with respect to the variable $\underline{X} \in \mathbb{R}^m$:

$$(\epsilon \partial_T - \underline{\omega} \partial_p) E_\alpha^\dagger(T, p, \underline{\omega}) = T_+^{\alpha+m-1} \delta(p) .$$

Here we have used expression (33). Putting $E_\alpha^\dagger(T, p, \underline{\omega}) = (\epsilon \partial_T - \underline{\omega} \partial_p) \Phi_\alpha(T, p)$, it suffices to solve the scalar equation

$$(\partial_T^2 - \partial_p^2) \Phi_\alpha(T, p) = T_+^{\alpha+m-1} \delta(p)$$

for $\Phi_\alpha(T, p)$. A Radon inversion allows us then to conclude that

$$\begin{aligned} E_\alpha(T, \underline{X}) &= \frac{(-1)^{\frac{m-1}{2}}}{2(2\pi)^{m-1}} \Delta_m^{\frac{m-1}{2}} \mathcal{B}[(\epsilon \partial_T - \underline{\omega} \partial_p) \Phi_\alpha](T, \underline{X}) \\ &= \frac{(-1)^{\frac{m-1}{2}}}{2(2\pi)^{m-1}} (\epsilon \partial_T - \underline{\partial}) \Delta_m^{\frac{m-1}{2}} [\mathcal{B} \Phi_\alpha](T, \underline{X}) . \end{aligned}$$

It should be stressed that the Laplace operator Δ_m and the Dirac operator $\underline{\partial}$ on $\mathbb{R}^{0,m}$ are both given here in terms of the co-ordinates \underline{X} on \mathbb{R}^m .

With $\mathcal{E}_1(T, p)$ the fundamental solution for the wave-operator on $\mathbb{R}^{1,1}$, we find for $\Phi_\alpha(T, p)$:

$$\begin{aligned} \Phi_\alpha(T, p) &= (\mathcal{E}_1 * T_+^{\alpha+m-1} \delta(p))(T, p) \\ &= \int_0^\infty \mathcal{E}_1(T - S, p) S^{\alpha+m-1} dS . \end{aligned}$$

As we are looking for a fundamental solution $E_\alpha(T, \underline{X})$ with support in the future cone, we put

$$\mathcal{E}_1(T, p) = \frac{H(T)H(T - |p|)}{2} ,$$

whence

$$\begin{aligned}\Phi_\alpha(T, p) &= \frac{1}{2} \int_0^\infty H(T-S)H(T-S-|p|)S^{\alpha+m-1}dS \\ &= \frac{1}{2}H(T)H(T-|p|)\frac{(T-|p|)^{\alpha+m}}{\alpha+m},\end{aligned}$$

eventually leading to

$$E_\alpha(T, \underline{X}) = \frac{(-1)^{\frac{m-1}{2}}}{4(2\pi)^{m-1}}\partial_X\Delta_m^{\frac{m-1}{2}}\mathcal{B}\left[H(T)H(T-|p|)\frac{(T-|p|)^{\alpha+m}}{\alpha+m}\right].$$

This expression will now be calculated in case of an odd dimension $m \geq 3$, because in case of an even dimension m it contains a fractional power of the Laplace operator. These fractional derivatives can be defined in terms of Riesz potentials, see e.g. [44], but this makes the calculations less straightforward. The even-dimensional situation is thus again excluded until the end of this subsection.

By definition we get :

$$\mathcal{B}\left[H(T)H(T-|p|)\frac{(T-|p|)^{\alpha+m}}{\alpha+m}\right] = \int_{S^{m-1}} \frac{(T-|\langle \underline{X}, \underline{\omega} \rangle|)^{\alpha+m}}{\alpha+m} dS(\underline{\omega}).$$

Note that inside the future cone the factor $H(T)H(T-|\langle \underline{X}, \underline{\omega} \rangle|)$ reduces to the constant 1.

Choosing the vector \underline{X} in such a way that $\theta_1 = \widehat{(\underline{X}, \underline{\omega})}$, with θ_1 one of the spherical co-ordinates appearing in the Lebesgue measure on S^{m-1} ,

$$dS(\underline{\omega}) = (\sin \theta_1)^{m-2}(\sin \theta_2)^{m-3} \dots (\sin \theta_{m-2}) d\theta_1 \dots d\theta_{m-1}$$

we have $\langle \underline{X}, \underline{\omega} \rangle = |\underline{X}| \cos \theta_1$ and the above integral becomes

$$\int_{S^{m-1}} \frac{(T-|\underline{X}|\cos \theta_1)^{\alpha+m}}{\alpha+m} dS(\underline{\omega}) = \frac{4\pi^{\frac{m-1}{2}}}{\Gamma(\frac{m-1}{2})} I(T, \underline{X}),$$

with $I(T, \underline{X})$ given by

$$I(T, \underline{X}) = \int_0^1 \frac{(T-y|\underline{X}|)^{\alpha+m}}{\alpha+m} (1-y^2)^{\frac{m-3}{2}} dy.$$

As $|\underline{X}| < T$ for all $X \in FC$ and $y \in [0, 1]$, we may consider the following series expansion :

$$(T-y|\underline{X}|)^{\alpha+m} = T^{\alpha+m} \sum_{k=0}^{\infty} \binom{\alpha+m}{k} \left(-y\frac{|\underline{X}|}{T}\right)^k,$$

which leads to :

$$\begin{aligned} I(T, \underline{X}) &= \frac{T^{\alpha+m}}{\alpha+m} \sum_{k=0}^{\infty} \binom{\alpha+m}{k} \left(-\frac{|\underline{X}|}{T}\right)^k \int_0^1 y^k (1-y^2)^{\frac{m-3}{2}} dy \\ &= \frac{T^{\alpha+m}}{\alpha+m} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha+m}{k} \left(\frac{|\underline{X}|}{T}\right)^k \left[\frac{\Gamma(\frac{m-1}{2}) \Gamma(\frac{k+1}{2})}{2\Gamma(\frac{m+k}{2})} \right]. \end{aligned}$$

We thus find the following expression for $I(T, \underline{X})$:

$$\frac{T^{\alpha+m}}{\alpha+m} \sum_{k=0}^{\infty} \binom{\alpha+m}{k} \frac{(-1)^k 2^{\frac{m-3}{2}} \left(\frac{m-3}{2}\right)!}{(k+1)(k+3)\cdots(k+m-2)} \left(\frac{|\underline{X}|}{T}\right)^k,$$

such that for all space-time vectors $X = \epsilon T + \underline{X} \in FC$ we get

$$\begin{aligned} E_{\alpha}(T, \underline{X}) &= \frac{(-1)^{\frac{m-1}{2}}}{2(2\pi)^{\frac{m-1}{2}}} \partial_X \frac{T^{\alpha+m}}{\alpha+m} \times \\ &\quad \Delta_m^{\frac{m-1}{2}} \sum_{k=0}^{\infty} \binom{\alpha+m}{k} \frac{(-1)^k (k-1)!!}{(k+m-2)!!} \left(\frac{|\underline{X}|}{T}\right)^k, \end{aligned}$$

with Δ_m the Laplacian in the co-ordinates \underline{X} and adopting the classical notation

$$a!! = a(a-2)\cdots 4\cdot 2 \quad \text{if } a \in 2\mathbb{N}$$

$$a!! = a(a-2)\cdots 5\cdot 3 \quad \text{if } a \in 2\mathbb{N} + 1.$$

This expression for $E_{\alpha}(T, \underline{X})$ can now be rewritten in such a way that for odd spatial dimensions m we recover the formulae that were found in subsection 2.2.1. To that end, we need the following technical Lemma which can be proved by means of induction on the parameter $a \in \mathbb{N}_0$:

Lemma 2.1 *If $\Delta_m = \sum_{j=1}^m \partial_{x_j}^2$ denotes the Laplacian on \mathbb{R}^m , we have for all positive integers $a \in \mathbb{N}_0$:*

$$\Delta_m^a |\underline{x}|^n = \underbrace{\Delta_m \cdots \Delta_m}_{a \text{ times}} |\underline{x}|^n = \frac{n!!}{(n-2a)!!} \frac{(n+m-2)!!}{(n-2a+m-2)!!} |\underline{x}|^{n-2a}$$

Using this Lemma, we find for arbitrary $(T, \underline{X}) \in FC$:

$$E_{\alpha}(T, \underline{X}) = \frac{C_m(\epsilon \partial_T - \partial_{\underline{X}})}{\alpha+m} \frac{T^{\alpha+m}}{|\underline{X}|^{m-1}} \sum_{k=0}^{\infty} \frac{\binom{\alpha+m}{k} k!!}{(k+1-m)!!} \left(-\frac{|\underline{X}|}{T}\right)^k \quad (2.7)$$

where we have put

$$C_m = \frac{(-1)^{\frac{m-1}{2}}}{2(2\pi)^{\frac{m-1}{2}}}.$$

Due to the presence of the factor $k(k-2)\cdots(k-(m-3))$ in expression (2.7), the *even* series Σ_{even} starts from $k = m - 1$ if m is odd, which means that this series loses its singular behaviour at $|\underline{X}| = 0$. Indeed,

$$\begin{aligned} \frac{1}{|\underline{X}|^{m-1}} \Sigma_{even} &= \frac{1}{|\underline{X}|^{m-1}} \sum_{k=0}^{\infty} \frac{\binom{\alpha+m}{2k} (2k)!!}{(2k+1-m)!!} \left(\frac{|\underline{X}|}{T} \right)^{2k} \\ &= 2^{\frac{m-1}{2}} T^{1-m} \sum_{l=0}^{\infty} \frac{\binom{\alpha+m}{2l+m-1} (l+1)!}{(l+\frac{m-3}{2})!} \left(\frac{|\underline{X}|}{T} \right)^{2l}. \end{aligned} \quad (2.8)$$

The following Theorem, stating that the even series (2.8) belongs to the kernel of the wave-operator \square_m on $\mathbb{R}^{1,m}$, is proved by direct calculation :

Theorem 2.1 *For all $(T, \underline{X}) \in FC$ we have :*

$$\square_m \left\{ \frac{T^{\alpha+m}}{|\underline{X}|^{m-1}} \sum_{k=0}^{\infty} \frac{\binom{\alpha+m}{2k} (2k)!!}{(2k+1-m)!!} \left(\frac{|\underline{X}|}{T} \right)^{2k} \right\} = 0$$

Hence, in case of an odd spatial dimension m , the fundamental solution $E_\alpha(T, \underline{X})$ as obtained by means of the Radon inversion consists of a singular part $E_\alpha^{(s)}(T, \underline{X})$, given by the odd series, and a regular part $E_\alpha^{(r)}(T, \underline{X})$ given by the even series.

In what follows we will prove that the singular part can be written as a modulated version of the Euclidean Cauchy kernel $E(\underline{x})$, with $\underline{x} = \frac{\underline{X}}{T}$, and that the regular part can be written as a modulated version of the constant function 1, a trivial solution for the Euclidean Dirac operator ∂ on $\mathbb{R}^{0,m}$.

Remark : It is important to note that the regular part of the hyperbolic fundamental solution $E_\alpha(T, \underline{X})$ cannot be unique, since one can always add an arbitrary α -homogeneous nullsolution for the Dirac operator ∂_X on $\mathbb{R}^{1,m}$. However, as will be illustrated a bit further : the regular part $E_\alpha^{(r)}(T, \underline{X})$ which shows up here is in fact very natural and will allow us to rewrite the hyperbolic fundamental solution $E_\alpha(T, \underline{X})$ in terms of Gegenbauer functions of the second kind.

Using expression (2.7) for the fundamental solution and result (2.8) for the even series, we find :

$$E_{\alpha}^{(r)}(T, \underline{X}) = \frac{2^{\frac{m-1}{2}} C_m (\epsilon \partial_T - \partial_{\underline{X}})}{\alpha + m} T^{\alpha+1} \sum_{l=0}^{\infty} \frac{\binom{\alpha+m}{2l+m-1} (l+1)!}{(l + \frac{m-3}{2})!} \left(\frac{|\underline{X}|}{T} \right)^{2l}$$

Using the following identity, known as Legendre's duplication formula for the Gamma function

$$\Gamma(2z) = \frac{2^{2z-1}}{\pi^{\frac{1}{2}}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right),$$

it follows immediately that

$$\binom{\alpha+m}{2l+m-1} \frac{(l+1)!}{(l + \frac{m-3}{2})!} = \frac{\pi^{\frac{1}{2}} \Gamma(\alpha+m+1) \left(-\frac{\alpha}{2}\right)_l \left(-\frac{1+\alpha}{2}\right)_l}{2^{m-1} \Gamma\left(\frac{m}{2}\right) \Gamma(\alpha+2) l! \left(\frac{m}{2}\right)_l}.$$

Hence, $E_{\alpha}^{(r)}(T, \underline{X})$ takes the form

$$E_{\alpha}^{(r)}(T, \underline{X}) = \frac{(-1)^{\frac{m-1}{2}} (\epsilon \partial_T - \partial_{\underline{X}}) \Gamma(\alpha+m)}{2^m \pi^{\frac{m-2}{2}} \Gamma\left(\frac{m}{2}\right) \Gamma(\alpha+2)} T^{\alpha+1} F\left(-\frac{\alpha}{2}, -\frac{1+\alpha}{2}; \frac{m}{2}; |\underline{x}|^2\right)$$

with $\underline{x} = \frac{\underline{X}}{T}$. Using the contiguous relations for the hypergeometric function, straightforward calculations now yield :

$$(\epsilon \partial_T - \partial_{\underline{X}}) T^{\alpha+1} F\left(-\frac{\alpha}{2}, -\frac{1+\alpha}{2}; \frac{m}{2}; |\underline{x}|^2\right) = (1+\alpha) T^{\alpha} \text{Mod}(\alpha, 0, \underline{x}) \epsilon,$$

where $\text{Mod}(\alpha, 0, \underline{x})$ is defined by

$$\text{Mod}(\alpha, 0, \underline{x}) = F_1(|\underline{x}|^2) - \frac{\alpha}{m} \underline{x} \epsilon F_2(|\underline{x}|^2)$$

with

$$\begin{aligned} F_1(t) &= F\left(\frac{1-\alpha}{2}, -\frac{\alpha}{2}; \frac{m}{2}; t\right) \\ F_2(t) &= F\left(\frac{1-\alpha}{2}, 1 - \frac{\alpha}{2}; 1 + \frac{m}{2}; t\right). \end{aligned}$$

This eventually yields the regular part $E_{\alpha}^{(r)}(T, \underline{X})$ of the fundamental solution for the hyperbolic Dirac equation, in case of an odd spatial dimension :

$$E_{\alpha}^{(r)}(T, \underline{X}) = \frac{(-1)^{\frac{m-1}{2}} \Gamma(\alpha+m)}{2^m \pi^{\frac{m-2}{2}} \Gamma\left(\frac{m}{2}\right) \Gamma(\alpha+1)} T^{\alpha} \text{Mod}(\alpha, 0, \underline{x}) \epsilon.$$

Next, the singular part $E_\alpha^{(s)}(T, \underline{X})$ of the fundamental solution for the Dirac equation on the hyperbolic unit ball is determined. Recalling expression (2.7) we get :

$$E_\alpha^{(s)}(T, \underline{X}) = -\frac{C_m(\epsilon\partial_T - \partial_{\underline{X}})}{\alpha + m} \frac{T^{\alpha+m-1}}{|\underline{X}|^{m-2}} \sum_{k=0}^{\infty} \frac{(\alpha+m)_{2k+1} (2k+1)!!}{(2k+1-(m-3))!!} \left(\frac{|\underline{X}|}{T}\right)^{2k}.$$

In case of an odd spatial dimension m , the coefficients reduce to

$$\binom{\alpha+m}{2k+1} \frac{(2k+1)!!}{(2k+1-(m-3))!!} = \frac{2^{\frac{m-3}{2}} \pi^{\frac{1}{2}}}{\Gamma(2-\frac{m}{2})} \frac{(\frac{1-\alpha-m}{2})_k (\frac{2-\alpha-m}{2})_k}{k! (2-\frac{m}{2})_k}.$$

In view of the fact that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

the following expression is eventually obtained :

$$\begin{aligned} E_\alpha^{(s)}(T, \underline{X}) &= \frac{(\epsilon\partial_T - \partial_{\underline{X}})}{(m-2)A_m} \frac{T^{\alpha+m-1}}{|\underline{X}|^{m-2}} F\left(\frac{1-\alpha-m}{2}, \frac{2-\alpha-m}{2}; 2-\frac{m}{2}; \frac{|\underline{X}|^2}{T^2}\right) \\ &= T^\alpha \text{Mod}(\alpha, 1-m, \underline{x}) E(\underline{x}) \end{aligned}$$

This formula thus states that the singular part $E_\alpha^{(s)}(T, \underline{X})$ of the hyperbolic fundamental solution is a modulated version of the Euclidean Cauchy kernel, identical to expression (2.4).

Remark : Note that these calculations are *only* valid in case of an odd-dimensional space-time $\mathbb{R}^{1,m}$. We already encountered this phenomenon in previous subsections and the time has now come to explain this substantial difference between the even- and odd-dimensional situation. To do so we refer to the introductory theory on the hypergeometric function, where we have seen how to obtain two independent solutions for the hypergeometric differential equation.

For if we look at the solution for the hyperbolic Dirac equation obtained by means of the Radon transform, we note that both the regular and the singular part can be expressed in terms of the Dirac operator on $\mathbb{R}^{1,m}$ acting on a hypergeometric function. In case of an *odd dimension* we get for the

hyperbolic fundamental solution $E_\alpha(T, \underline{X})$:

$$E_\alpha(T, \underline{X}) = \frac{\Gamma(2 - \frac{m}{2})}{(m-2)A_m} (\epsilon \partial_T - \partial_{\underline{X}}) \frac{T^{\alpha+1}}{|\underline{x}|^{m-2}} \times \quad (2.9)$$

$$\left[\begin{aligned} & \frac{(-1)^m}{2^{m-2}\Gamma(\frac{m}{2})} \frac{\Gamma(\alpha+m)}{\Gamma(\alpha+2)} |\underline{x}|^{m-2} F\left(-\frac{\alpha}{2}, -\frac{1+\alpha}{2}; \frac{m}{2}; |\underline{x}|^2\right) \\ & + \\ & \frac{1}{\Gamma(2 - \frac{m}{2})} F\left(\frac{1-\alpha-m}{2}, \frac{2-\alpha-m}{2}; 2 - \frac{m}{2}; |\underline{x}|^2\right) \end{aligned} \right]$$

Both hypergeometric functions between the square brackets are a solution for the hypergeometric differential equation

$$t(1-t) \frac{d^2 f}{dt^2} + \left[\frac{m}{2} + \left(\alpha - \frac{1}{2} \right) t \right] \frac{df}{dt} - \frac{\alpha(1+\alpha)}{4} f = 0, \quad (2.10)$$

arising very naturally if we recall the fact that $(\epsilon \partial_T - \partial_{\underline{X}})^2 = \square_m$. Indeed, if we are to find α -homogeneous solutions $F_\alpha(T, \underline{X})$ for the Dirac operator on $\mathbb{R}^{1,m}$ it is sufficient to find $(\alpha+1)$ -homogeneous solutions $\Phi_{\alpha+1}(T, \underline{X})$ for the wave-operator first, and to put

$$F_\alpha(T, \underline{X}) = (\epsilon \partial_T - \partial_{\underline{X}}) \Phi_{\alpha+1}(T, \underline{X}).$$

So if we put

$$\begin{aligned} \Phi_{\alpha+1}(T, \underline{X}) &= T^{\alpha+1} \Phi_{\alpha+1}\left(1, \frac{\underline{X}}{T}\right) \\ &= \lambda^{\alpha+1} \Phi_{\alpha+1}(\underline{x}), \end{aligned}$$

with $\lambda = T$ and $\underline{x} = \frac{\underline{X}}{T}$, and if the function $\Phi_{\alpha+1}(\underline{x})$ is to depend on the argument $t = |\underline{x}|^2$ only, we arrive at equation (2.10) for $f(t) = \Phi_{\alpha+1}(|\underline{x}|^2)$.

If we then put $f(t) = t^{1-\frac{m}{2}} g(t)$ and $\beta = -\alpha - m$, we get the following hypergeometric differential equation :

$$t(1-t) \frac{d^2 g}{dt^2} + \left[\left(2 - \frac{m}{2}\right) - \left(\beta + \frac{5}{2}\right) t \right] \frac{dg}{dt} - \frac{(1+\beta)(2+\beta)}{4} g = 0.$$

According to the general theory, two independent solutions for this equation can be found as follows :

- If $2 - \frac{m}{2} \notin -\mathbb{N}$ or if $m > 2$ is not even, then

$$g_1(t) = F\left(1 - \frac{\alpha + m}{2}, \frac{1 - \alpha - m}{2}; 2 - \frac{m}{2}; t\right)$$

is a first solution, regular at $t = 0$, whereas a second *independent* solution is given by

$$g_2(t) = t^{\frac{m}{2}-1} F\left(-\frac{\alpha}{2}, -\frac{1 + \alpha}{2}; \frac{m}{2}; t\right).$$

This gives two independent solutions for equation (2.10) :

$$\begin{aligned} f_1(t) &= t^{1-\frac{m}{2}} F\left(1 - \frac{\alpha + m}{2}, \frac{1 - \alpha - m}{2}; 2 - \frac{m}{2}; t\right) \\ f_2(t) &= F\left(-\frac{\alpha}{2}, -\frac{1 + \alpha}{2}; \frac{m}{2}; t\right), \end{aligned}$$

and these are precisely the functions occurring in the expressions for respectively the singular and regular part of the hyperbolic fundamental solution (cfr. supra).

- If $2 - \frac{m}{2} \in -\mathbb{N}$, or if $m > 2$ is even, the first solution $g_1(t)$ becomes ill-defined and the second solution $g_2(t)$ is no longer independent in the sense that it coincides with $g_1(t)$, up to a constant :

$$\begin{aligned} \lim_{c \rightarrow 2 - \frac{m}{2}} \frac{F\left(1 - \frac{\alpha + m}{2}, \frac{1 - \alpha - m}{2}; c; t\right)}{\Gamma(c)} &= \frac{\left(1 - \frac{\alpha + m}{2}\right)_{\frac{m}{2}-1} \left(\frac{1 - \alpha - m}{2}\right)_{\frac{m}{2}-1}}{\left(\frac{m}{2} - 1\right)!} \times \\ &\quad t^{\frac{m}{2}-1} F\left(-\frac{\alpha}{2}, -\frac{1 + \alpha}{2}; \frac{m}{2}; t\right), \end{aligned}$$

which can be reduced to

$$\lim_{c \rightarrow 2 - \frac{m}{2}} \left[\begin{aligned} &\frac{(-1)^m}{2^{m-2} \Gamma\left(\frac{m}{2}\right)} \frac{\Gamma(\alpha + m)}{\Gamma(\alpha + 2)} t^{\frac{m}{2}-1} F\left(-\frac{\alpha}{2}, -\frac{1 + \alpha}{2}; \frac{m}{2}; t\right) \\ &- \frac{F\left(1 - \frac{\alpha + m}{2}, \frac{1 - \alpha - m}{2}; c; t\right)}{\Gamma(c)} \end{aligned} \right] = 0$$

Note that this factor between square brackets is, up to a relative minus sign, equal to the factor between square brackets in the expression (2.9) for the hyperbolic fundamental solution $E_\alpha(T, \underline{X})$ in case of an odd dimension. In other words, this limit expression clearly exhibits the fact that the formulae

for $E_\alpha(T, \underline{X})$ derived above are only valid for odd spatial dimensions m .

However, from the general theory on the hypergeometric function we then immediately get a fundamental solution for the hyperbolic Dirac equation in case of an *even* dimension $m > 2$ by means of the *limit procedure* :

$$E_\alpha(T, \underline{X}) = \frac{(\epsilon \partial_T - \partial_{\underline{X}})}{(m-2)A_m} \frac{T^{\alpha+1}}{|\underline{x}|^{m-2}} \lim_{c \rightarrow 2 - \frac{m}{2}} \frac{1}{c - (2 - \frac{m}{2})} \times$$

$$\left[\begin{aligned} & \frac{(-1)^m}{2^{m-2}\Gamma(\frac{m}{2})} \frac{\Gamma(\alpha+m)}{\Gamma(\alpha+2)} |\underline{x}|^{m-2} F\left(-\frac{\alpha}{2}, -\frac{1+\alpha}{2}; \frac{m}{2}; |\underline{x}|^2\right) \\ & - \frac{1}{\Gamma(c)} F\left(\frac{1-\alpha-m}{2}, \frac{2-\alpha-m}{2}; c; |\underline{x}|^2\right) \end{aligned} \right]$$

Let us now calculate this limit expression in an appropriate way. To do so, we will use the hyperbolic polar representation for space-time vectors :

$$\epsilon T + \underline{X} = X = \rho \xi = \rho \left(\epsilon \tau + (\tau^2 - 1)^{\frac{1}{2}} \underline{\xi} \right)$$

with $\underline{\xi} \in S^{m-1}$ and $\tau = \rho^{-1}T$.

If we look at the expressions for the hyperbolic fundamental solution, for both even and odd dimensions, we see that up to a relative minus sign the factor between square brackets is identical. It consists of two hypergeometric functions multiplied with the factors $T^{\alpha+1}$ and $T^{\alpha+1}|\underline{x}|^{2-m}$ respectively, which can be rewritten in terms of the Legendre function of the first kind :

$$\frac{T^{\alpha+1}}{|\underline{x}|^{m-2}} \frac{F\left(\frac{1-\alpha-m}{2}, \frac{2-\alpha-m}{2}; 2 - \frac{m}{2}; |\underline{x}|^2\right)}{\Gamma\left(2 - \frac{m}{2}\right)} = \rho^{\alpha+1} \frac{(\tau^2 - 1)^{\frac{1}{2} - \frac{m}{4}}}{2^{\frac{m}{2}-1}} P_{\alpha+\frac{m}{2}}^{\frac{m}{2}-1}(\tau)$$

and

$$\begin{aligned} & \frac{(-1)^m}{2^{m-2}\Gamma(\frac{m}{2})} \frac{\Gamma(\alpha+m)}{\Gamma(\alpha+2)} T^{\alpha+1} F\left(-\frac{\alpha}{2}, -\frac{1+\alpha}{2}; \frac{m}{2}; |\underline{x}|^2\right) \\ & = (-1)^m \rho^{\alpha+1} \frac{\Gamma(\alpha+m)}{\Gamma(\alpha+2)} \frac{(\tau^2 - 1)^{\frac{1}{2} - \frac{m}{4}}}{2^{\frac{m}{2}-1}} P_{\alpha+\frac{m}{2}}^{1-\frac{m}{2}}(\tau) . \end{aligned} \quad (2.11)$$

We will then use expression (17) to simplify the sum of these two Legendre functions of the first kind :

$$Q_{\alpha+\frac{m}{2}}^{1-\frac{m}{2}}(\tau) = \frac{\pi e^{i\pi(1-\frac{m}{2})}}{2 \sin\left(1 - \frac{m}{2}\right) \pi} \left(P_{\alpha+\frac{m}{2}}^{1-\frac{m}{2}}(\tau) - \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+m)} P_{\alpha+\frac{m}{2}}^{\frac{m}{2}-1}(\tau) \right) .$$

From this formula we can immediately observe two things : first of all there is a factor $\sin\left(\frac{m\pi}{2}\right)$ which vanishes for even dimensions m and this explains the need to take a limit in case $m \in 2\mathbb{N}$. On the other hand we note that there is a minus sign, whereas formula (2.11) only has a factor $(-1)^m$, explaining the need to introduce an additional relative minus sign in case $m \in 2\mathbb{N}$.

- In case of an odd dimension m we get :

$$E_\alpha(T, \underline{X}) = \partial_X \left[\frac{\sin\left(\frac{m\pi}{2}\right) \Gamma\left(\frac{m}{2}\right) \Gamma\left(1 - \frac{m}{2}\right) \Gamma(\alpha + m)}{e^{-i\pi\left(\frac{m}{2}-1\right)} \frac{2^{\frac{m}{2}} \pi^{\frac{m}{2}+1}}{\Gamma(\alpha + 2)}} \frac{\rho^{\alpha+1}(\tau^2 - 1)^{\frac{1}{2}-\frac{m}{4}} Q_{\alpha+\frac{m}{2}}^{1-\frac{m}{2}}(\tau)}{\Gamma(\alpha + 2)} \right],$$

which by means of definition (22) for the Gegenbauer function of the second kind in terms of the Legendre function reduces to

$$E_\alpha(T, \underline{X}) = \partial_X \left[\frac{e^{-i\pi\left(\frac{m}{2}-1\right)}}{2\pi^{\frac{m-1}{2}}} \Gamma\left(\frac{m-1}{2}\right) D_{\alpha+1}^{\frac{m-1}{2}}(\tau) \right].$$

- In case of an even dimension $m > 2$ we get :

$$E_\alpha(T, \underline{X}) = \partial_X \left[\frac{e^{i\pi\frac{m}{2}} \Gamma\left(\frac{m}{2} - 1\right) \Gamma(\alpha + m)}{2^{\frac{m}{2}} \pi^{\frac{m}{2}+1} \Gamma(\alpha + 2)} \frac{\rho^{\alpha+1}(\tau^2 - 1)^{\frac{1}{2}-\frac{m}{4}} Q_{\alpha+\frac{m}{2}}^{1-\frac{m}{2}}(\tau)}{\Gamma(\alpha + 2)} \right] \times \lim_{c \rightarrow 2 - \frac{m}{2}} \frac{\sin(c\pi)}{c - \left(2 - \frac{m}{2}\right)}$$

Note that we do not have to replace *all* factors $\left(2 - \frac{m}{2}\right)$ by the variable c , which should be done according to formula (8), because only one term will survive after taking limits due to the presence of the factor $\sin(c\pi)$. Eventually the expression for the hyperbolic fundamental solution in case of an even spatial dimension $m > 2$ will coincide up to a constant with the one we have already found in case of an odd dimension. In the next subsection it will be shown that this constant is actually equal to 1. This was of course to be expected because the expression for the hyperbolic fundamental solution for odd m in terms of the Gegenbauer function is also defined for even m , and this expression for odd m gives the correct right-hand side (which is also defined for both odd and even dimensions m).

We end this subsection by noting that the difference between determining the hyperbolic fundamental solution for odd and even dimensions is no typical

hyperbolic feature. Indeed, the fundamental solution for the Laplacian Δ_m is given by the function $c_m r^{2-m}$ where $c_m = [(2-m)A_m]^{-1}$. However, if $m = 2$ the function r^{2-m} reduces to the regular nullsolution for the Laplacian Δ_2 , i.e. the constant function 1. In that case, a fundamental solution must also be obtained by means of a limit expression :

$$\lim_{\mu \rightarrow 2} \frac{r^{2-\mu} - 1}{\mu - 2} = -\ln r .$$

2.2.3 Method 3 : Using Riesz Distributions

The idea behind this third method is to consider the scalar problem

$$\square_m \Phi_\alpha(T, \underline{X}) = T_+^{\alpha+m-1} \delta(\underline{X}) \quad (2.12)$$

related to the hyperbolic Dirac equation, and to solve this equation using Riesz distributions.

As the distribution at the right-hand side of equation (2.12) belongs to the set $\mathcal{D}'_+(\mathbb{R}^{1,m})$, we get immediately that

$$\Phi_\alpha(T, \underline{X}) = Z_2 * T_+^{\alpha+m-1} \delta(\underline{X}) ,$$

which in view of the definition for the Riesz distribution Z_2 reduces to

$$\begin{aligned} \Phi_\alpha(T, \underline{X}) &= \frac{\rho^{1-m} * T_+^{\alpha+m-1} \delta(\underline{X})}{2\pi^{\frac{m-1}{2}} \Gamma\left(\frac{3-m}{2}\right)} \\ &= \frac{H(T-R) \int_0^{T-R} ((T-S)^2 - |\underline{X}|^2)^{\frac{1-m}{2}} S^{\alpha+m-1} dS}{2\pi^{\frac{m-1}{2}} \Gamma\left(\frac{3-m}{2}\right)} , \end{aligned}$$

where we have put $R = |\underline{X}|$. The integral in the denominator can further be reduced to

$$\rho^{1-m} (T-R)^{\alpha+m} \int_0^1 [(1-t)(1-zt)]^{\frac{1-m}{2}} t^{\alpha+m-1} dt , \quad (2.13)$$

with $\rho = (T^2 - |\underline{X}|^2)^{\frac{1}{2}}$ the hyperbolic norm and with

$$z = \frac{T-R}{T+R} = \frac{\tau - (\tau^2 - 1)^{\frac{1}{2}}}{\tau + (\tau^2 - 1)^{\frac{1}{2}}} \quad \text{for} \quad \tau = \frac{T}{(T^2 - |\underline{X}|^2)^{\frac{1}{2}}} .$$

Recalling Euler's integral formula for the hypergeometric function, expression (2.13) can be written as

$$\frac{\Gamma(\alpha+m) \Gamma\left(\frac{3-m}{2}\right)}{\Gamma\left(\alpha + \frac{m+3}{2}\right)} \rho^{1-m} (T-R)^{\alpha+m} F\left(\frac{m-1}{2}, \alpha+m; \alpha + \frac{m+3}{2}; z\right)$$

if we assume that $m < 3$.

Recalling formula (15), this hypergeometric function can be written as a Legendre function of the second kind

$$F\left(\frac{m-1}{2}, \alpha+m; \alpha+\frac{m+3}{2}; z\right) = e^{-i\pi\frac{m-2}{2}} \frac{\Gamma\left(\alpha+\frac{m+3}{2}\right)}{\pi^{\frac{1}{2}} 2^{\frac{m-2}{2}} \Gamma(\alpha+m)} \frac{(\tau + (\tau^2 - 1)^{\frac{1}{2}})^{\alpha+m}}{(\tau^2 - 1)^{\frac{m}{4} - \frac{1}{2}}} Q_{\alpha+\frac{m}{2}}^{\frac{m}{2}-1}(\tau),$$

and eventually, using the definition for the Gegenbauer function of the second kind in terms of the Legendre function, we get the following expression in case $m < 3$:

$$Z_2 * T_+^{\alpha+m-1} \delta(\underline{X}) = H(T-R) \frac{e^{-i\pi\frac{m-1}{2}}}{2\pi^{\frac{m-1}{2}}} \Gamma\left(\frac{m-1}{2}\right) \rho^{\alpha+1} D_{\alpha+1}^{\frac{m-1}{2}}(\tau).$$

In view of the fact that the Gegenbauer function of the second kind needs a cut in the complex plane along the real line from $-\infty$ to $+1$, the factor $H(T-R)$ may be omitted. Indeed, as $\tau \in \mathbb{R}^+$ the condition $|\arg(\tau-1)| < \pi$ is equivalent with $\tau > 1 \Leftrightarrow T > |\underline{X}|$. The Gegenbauer function has poles at $\alpha+m \in -\mathbb{N}$. These were to be expected, because the distribution $T_+^{\alpha+m-1} \delta(\underline{X})$ has poles at the very same values. This means that both the left-hand side and the right-hand side of the previous expression determine a distribution with poles at $\alpha+m \in -\mathbb{N}$. As they are equal in the strip $m < 3$, they are equal in the whole complex plane by analytic continuation.

In other words, the fundamental solution $E_\alpha(T, \underline{X})$ for the Dirac equation on the hyperbolic unit ball is found to be the distribution

$$\begin{aligned} E_\alpha(T, \underline{X}) &= \partial_X \Phi_\alpha(T, \underline{X}) \\ &= \partial_X \left[\frac{e^{-i\pi\frac{m-1}{2}}}{2\pi^{\frac{m-1}{2}}} \Gamma\left(\frac{m-1}{2}\right) \rho^{\alpha+1} D_{\alpha+1}^{\frac{m-1}{2}}(\tau) \right]. \end{aligned}$$

This is precisely the expression for the hyperbolic fundamental solution in case of an odd dimension, found in the previous subsection, which indeed proves that this formula holds for both even and odd dimensions.

Recalling the polar decomposition

$$\partial_X = \xi \left(\partial_\rho + \frac{1}{\rho} \Gamma_{1,m} \right)$$

for the Dirac operator on $\mathbb{R}^{1,m}$ and using the fact that

$$\Gamma_{1,m}(\tau) = \Gamma_{1,m}(\xi \cdot \epsilon) = \xi \wedge \epsilon ,$$

we get

$$\partial_X \rho^{\alpha+1} D_{\alpha+1}^{\frac{m-1}{2}}(\tau) = \xi \rho^\alpha \left((m-1) D_{\alpha}^{\frac{m+1}{2}}(\tau) \xi \wedge \epsilon + (1+\alpha) D_{\alpha+1}^{\frac{m-1}{2}}(\tau) \right) ,$$

which by means of the relation $\xi(\xi \wedge \epsilon) = \epsilon - \tau \xi$ and the recurrence relations for the Gegenbauer function reduces to

$$\partial_X \rho^{\alpha+1} D_{\alpha+1}^{\frac{m-1}{2}}(\tau) = (m-1) \rho^\alpha \left(D_{\alpha-1}^{\frac{m+1}{2}}(\tau) \xi - D_{\alpha}^{\frac{m+1}{2}}(\tau) \epsilon \right) .$$

This yields the following expression for the hyperbolic fundamental solution, in case $\alpha + m \notin -\mathbb{N}$, for all $X = \epsilon T + \underline{X} \in FC$:

$$E_\alpha(T, \underline{X}) = \rho^\alpha \frac{e^{-i\pi \frac{m-1}{2}}}{\pi^{\frac{m-1}{2}}} \Gamma\left(\frac{m+1}{2}\right) \left(D_{\alpha-1}^{\frac{m+1}{2}}(\tau) \xi - D_{\alpha}^{\frac{m+1}{2}}(\tau) \epsilon \right) \quad (2.14)$$

Remark : In Chapter 4 we will use Riesz potentials to define arbitrary complex powers of the hyperbolic Dirac operator.

2.2.4 Method 4 : Using an Inductive Argument

This last method makes use of the fundamental solution for the wave-operator, and allows us to deduce a recursive definition for the hyperbolic fundamental solution. In this subsection, we again restrict ourselves to the case of an odd-dimensional space-time $\mathbb{R}^{1,m}$. However, the results for the case of an even dimension easily follow by taking appropriate limits.

First we focus on the scalar problem

$$\square_m \Phi_{\alpha,m}(T, \underline{X}) = T_+^{\alpha+m-1} \delta(\underline{X}) ,$$

such that the hyperbolic fundamental solution is given by

$$E_{\alpha,m}(T, \underline{X}) = (\epsilon \partial_T - \partial_{\underline{X}}) \Phi_{\alpha,m}(T, \underline{X}) .$$

Note that, in contrast to previous subsections, we now label the fundamental solution $E_{\alpha,m}(T, \underline{X})$ for the hyperbolic Dirac equation on $\mathbb{R}^{1,m}$ with an additional subscript m . This is necessary because we will derive a recursive formula, expressing the hyperbolic fundamental solution on $\mathbb{R}^{1,m}$ in terms of

the hyperbolic fundamental solution in a lower-dimensional space-time. We prefer to not adopt this notation throughout the whole text because it is only in this section that there is a need of space-times of different dimensions. We thus agree that, if there is no subscript indicating the spatial dimension, we are dealing with m -dimensional space-time $\mathbb{R}^{1,m}$.

As we are looking for a recursive formula for the hyperbolic fundamental solution, we first consider the trivial case $m = 3$ and next we consider the general case $m \in 2\mathbb{N} + 3$.

1. The 3-dimensional case

By definition the fundamental solution for the wave-operator \square_3 with support on the upper part of the nullcone is given by

$$\mathcal{E}_3(T, \underline{X}) = \frac{\delta(T - |\underline{X}|)}{4\pi|\underline{X}|}.$$

Hence, we immediately get

$$\begin{aligned}\Phi_{\alpha,3}(T, \underline{X}) &= T_+^{\alpha+2} \delta(\underline{X}) * \mathcal{E}_3(T, \underline{X}) \\ &= H(T)H(T - |\underline{X}|) \frac{T^{\alpha+2}(1 - |\underline{x}|)^{\alpha+2}}{4\pi|\underline{X}|}\end{aligned}$$

where $\underline{x} = \frac{\underline{X}}{T}$. Note that $\underline{x} \in B_3(1)$, for all $(T, \underline{X}) \in FC$. As we are constructing the hyperbolic fundamental solution, which has support in the future cone, the factor $H(T)H(T - |\underline{X}|)$ will again be omitted throughout the calculations.

The function $\Phi_{\alpha,3}(T, \underline{X})$ may then be decomposed into a singular part $\mathcal{F}_{\alpha,3}(T, \underline{X})$ and a regular part $\mathcal{R}_{\alpha,3}(T, \underline{X})$ as follows :

$$\Phi_{\alpha,3}(T, \underline{X}) = \mathcal{F}_{\alpha,3}(T, \underline{X}) + \mathcal{R}_{\alpha,3}(T, \underline{X}),$$

where

$$\begin{aligned}\mathcal{F}_{\alpha,3}(T, \underline{X}) &= \frac{T^{\alpha+2}}{4\pi|\underline{X}|} \frac{(1 - |\underline{x}|)^{\alpha+2} + (1 + |\underline{x}|)^{\alpha+2}}{2} \\ &= \frac{T^{\alpha+2}}{4\pi|\underline{X}|} F\left(-\frac{\alpha+2}{2}, -\frac{\alpha+1}{2}; \frac{1}{2}; |\underline{x}|^2\right),\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}_{\alpha,3}(T, \underline{X}) &= \frac{T^{\alpha+1}}{4\pi} \frac{(1 - |\underline{x}|)^{\alpha+2} - (1 + |\underline{x}|)^{\alpha+2}}{2|\underline{x}|} \\ &= -(\alpha+2) \frac{T^{\alpha+1}}{4\pi} F\left(-\frac{\alpha}{2}, -\frac{\alpha+1}{2}; \frac{3}{2}; |\underline{x}|^2\right) .\end{aligned}$$

2. The general case

We now derive an addition formula for $\Phi_{\alpha,m}(T, \underline{X})$ by means of an induction argument.

We start with the following observation : for all $m \in 2\mathbb{N}_0 + 3$ we have

$$\mathcal{E}_{m+2}(T, \underline{X}) = \frac{1}{2\pi} \frac{1}{T} \frac{\partial}{\partial T} \mathcal{E}_m(T, \underline{X}) ,$$

which is a trivial consequence of the definition for $\mathcal{E}_m(T, \underline{X})$.

Hence,

$$\begin{aligned}\Phi_{\alpha,m+2}(T, \underline{X}) &= (\mathcal{E}_{m+2} * T_+^{\alpha+m+1})(T, \underline{X}) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{(\partial_T \mathcal{E}_m)(T-S, \underline{X})}{T-S} S^{\alpha+m+1} dS .\end{aligned}$$

By means of Lemma 0.1 it is easily verified that

$$(\partial_T \mathcal{E}_m)(T-S, \underline{X}) = -(\partial_S \mathcal{E}_m)(T-S, \underline{X})$$

and that the support of

$$\partial_T \mathcal{E}_m(T-S, \underline{X}) ,$$

when considered as a distribution in the S -variable, is $\{T - |\underline{X}|\}$.

As our fundamental solution $E_{\alpha,m}(T, \underline{X})$ is supported by the future cone FC , we know that $T > |\underline{X}|$. Hence there exists $\eta \in \mathbb{R}^+$ such that $T - |\underline{X}| \in I_\eta$, where I_η is defined as the interval $[T - |\underline{X}| - \eta, T - |\underline{X}| + \eta]$.

Consequently,

$$\begin{aligned}\Phi_{\alpha,m+2}(T, \underline{X}) &= \frac{1}{2\pi} \int_{I_\eta} \mathcal{E}_m(T-S, \underline{X}) \frac{\alpha+m+1}{T-S} S^{\alpha+m} dS \\ &+ \frac{1}{2\pi} \int_{I_\eta} \mathcal{E}_m(T-S, \underline{X}) \frac{1}{(T-S)^2} S^{\alpha+m+1} dS .\end{aligned}$$

Because $S < T$ in I_η , both $(T - S)^{-1}$ and $(T - S)^{-2}$ can be expanded as a series in $\frac{S}{T}$, whence

$$\begin{aligned}\Phi_{\alpha, m+2}(T, \underline{X}) &= \frac{1}{2\pi} \int_{I_\eta} \sum_{k=0}^{\infty} \frac{\alpha + m + 1}{T^{1+k}} \mathcal{E}_m(T - S, \underline{X}) S^{\alpha+m+k+1} dS \\ &+ \frac{1}{2\pi} \int_{I_\eta} \sum_{k=0}^{\infty} \frac{1+k}{T^{2+k}} \mathcal{E}_m(T - S, \underline{X}) S^{\alpha+m+k+2} dS .\end{aligned}$$

By termwise integration, we obtain :

$$\begin{aligned}\Phi_{\alpha, m+2}(T, \underline{X}) &= \frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{\alpha + m + 1}{T^{k+1}} (\mathcal{E}_m * T_+^{\alpha+k+m})(T, \underline{X}) \\ &+ \frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{1+k}{T^{k+2}} (\mathcal{E}_m * T_+^{\alpha+k+m+1})(T, \underline{X}) .\end{aligned}\quad (2.15)$$

We then prove the following Theorem :

Theorem 2.2 *For all $m \in 2\mathbb{N} + 3$ the solution to the equation*

$$\square_m \Phi_{\alpha, m}(T, \underline{X}) = T_+^{\alpha+m-1} \delta(\underline{X})$$

is given by

$$\Phi_{\alpha, m}(T, \underline{X}) = \mathcal{F}_{\alpha, m}(T, \underline{X}) + \mathcal{R}_{\alpha, m}(T, \underline{X})$$

with

$$\mathcal{F}_{\alpha, m}(T, \underline{X}) = \frac{T^{\alpha+m-1}}{(m-2)A_m|\underline{X}|^{m-2}} F\left(\frac{1-\alpha-m}{2}, \frac{2-\alpha-m}{2}; 2 - \frac{m}{2}; |\underline{x}|^2\right)$$

and

$$\mathcal{R}_{\alpha, m}(T, \underline{X}) = \frac{T^{1+\alpha}(-1)^{\frac{m-1}{2}} \Gamma(\alpha+m)}{2^m \pi^{\frac{m}{2}-1} \Gamma(\frac{m}{2}) \Gamma(\alpha+2)} F\left(\frac{-\alpha}{2}, \frac{-1-\alpha}{2}; \frac{m}{2}; |\underline{x}|^2\right) .$$

Proof : The case $m = 3$ was already proved. By an induction argument we will prove the Theorem for an arbitrary dimension $(m+2) \in 2\mathbb{N}_0 + 3$, under the assumption that the Theorem holds for $m \in 2\mathbb{N} + 3$.

If the Theorem holds for $m \in 2\mathbb{N} + 3$, we have from (2.15) :

$$\begin{aligned}\Phi_{\alpha, m+2}(T, \underline{X}) &= \frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{\alpha + m + 1}{T^{k+1}} \{ \mathcal{F}_{\alpha+k+1, m} + \mathcal{R}_{\alpha+k+1, m} \} (T, \underline{X}) \\ &+ \frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{1+k}{T^{k+2}} \{ \mathcal{F}_{\alpha+k+2, m} + \mathcal{R}_{\alpha+k+2, m} \} (T, \underline{X}) .\end{aligned}$$

Let us first calculate Σ_1 , defined as

$$\Sigma_1 = \frac{1}{2\pi} \sum_{k=0}^{\infty} \left\{ \frac{\alpha + m + 1}{T^{k+1}} \mathcal{F}_{\alpha+k+1, m} + \frac{1+k}{T^{k+2}} \mathcal{F}_{\alpha+k+2, m} \right\} .$$

Using the induction hypothesis, we find for $\mathcal{F}_{\alpha+k+1, m}$:

$$\begin{aligned}\frac{\mathcal{F}_{\alpha+k+1, m}}{2\pi T^{k+1}} &= \frac{T^{\alpha+m+1} |\underline{x}|^2}{(m-2)mA_{m+2} |\underline{X}|^m} \times \\ &F \left(\frac{-\alpha - k - m}{2}, \frac{1 - \alpha - k - m}{2}; 2 - \frac{m}{2}; |\underline{x}|^2 \right) ,\end{aligned}$$

and for $\mathcal{F}_{\alpha+k+2, m}$:

$$\begin{aligned}\frac{\mathcal{F}_{\alpha+k+2, m}}{2\pi T^{k+2}} &= \frac{T^{\alpha+m+1} |\underline{x}|^2}{(m-2)mA_{m+2} |\underline{X}|^m} \times \\ &F \left(\frac{-1 - \alpha - k - m}{2}, \frac{-\alpha - k - m}{2}; 2 - \frac{m}{2}; |\underline{x}|^2 \right) .\end{aligned}$$

In order to carry out the summation, we would like to rewrite the hypergeometric functions appearing on the previous lines. For this purpose, we introduce a differential operator in the real variable u (representing the argument of the hypergeometric function on which this operator is supposed to act, i.e. $u = |\underline{x}|^2$) :

$$O_a = 1 - \frac{2u}{a} \frac{d}{du} , \quad \forall a \in \mathbb{N} .$$

Using the contiguous relations for the hypergeometric function, we get

$$F \left(a, a + \frac{1}{2}; 2 - \frac{m}{2}; u \right) = O_{m-2} O_{m-4} \cdots O_3 O_1 F \left(a, a + \frac{1}{2}; \frac{1}{2}; u \right) .$$

As $\underline{x} \in B_m(1)$ for all (T, \underline{X}) in the future cone FC , the following expansions are valid :

$$\frac{1}{|\underline{x}|} = -\frac{1}{1 - (1 + |\underline{x}|)} = -\sum_{k=0}^{\infty} (1 + |\underline{x}|)^k$$

and

$$\frac{1}{|\underline{x}|^2} = -\frac{1}{(1 - (1 + |\underline{x}|))^2} = \sum_{k=0}^{\infty} (1+k)(1+|\underline{x}|)^k.$$

Recalling the fact that

$$F\left(a, a + \frac{1}{2}; \frac{1}{2}; z^2\right) = \frac{(1+z)^{-2a} + (1-z)^{-2a}}{2},$$

we can now carry out the summation over k in the summation Σ_1 . Indeed, we respectively get

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\alpha + m + 1}{2\pi T^{k+1}} \mathcal{F}_{\alpha+k+1, m} &= (\alpha + m + 1) \frac{T^{\alpha+m+1} |\underline{x}|^2}{(m-2)mA_{m+2} |\underline{X}|^m} \times \\ &O_{m-2} \cdots O_1 \frac{(1 - |\underline{x}|)^{\alpha+m} - (1 + |\underline{x}|)^{\alpha+m}}{2|\underline{x}|}, \end{aligned}$$

and

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{k+1}{2\pi T^{k+2}} \mathcal{F}_{\alpha+k+2, m} &= \frac{T^{\alpha+m+1} |\underline{x}|^2}{(m-2)mA_{m+2} |\underline{X}|^m} \times \\ &O_{m-2} \cdots O_1 \frac{(1 - |\underline{x}|)^{1+\alpha+m} + (1 + |\underline{x}|)^{1+\alpha+m}}{2|\underline{x}|^2}. \end{aligned}$$

Adding both terms and using the contiguous relations for the hypergeometric function, we get :

$$\Sigma_1 = \frac{T^{\alpha+m+1} |\underline{x}|^2}{(m-2)mA_{m+2} |\underline{X}|^m} O_{m-2} \cdots O_1 \frac{F\left(\frac{-1-\alpha-m}{2}, \frac{-\alpha-m}{2}; -\frac{1}{2}; |\underline{x}|^2\right)}{|\underline{x}|^2}.$$

Now that we got rid of the summation over k , using the differential operators O_a , we are still left with the following expression :

$$O_{m-2} O_{m-4} \cdots O_3 O_1 \frac{F\left(\frac{-1-\alpha-m}{2}, \frac{-\alpha-m}{2}; -\frac{1}{2}; |\underline{x}|^2\right)}{|\underline{x}|^2}.$$

Using the fact that

$$O_n \frac{nF\left(a, b; -\frac{n}{2}; u\right)}{u} = (n+2) \frac{F\left(a, b; -\frac{n}{2} - 1; u\right)}{u},$$

a relation that can easily be verified by means of the definition of the hypergeometric series, we get

$$O_{m-2} O_{m-4} \cdots O_3 O_1 \frac{F\left(a, b; -\frac{1}{2}; |\underline{x}|^2\right)}{|\underline{x}|^2} = (m-2) \frac{F\left(a, b; 1 - \frac{m}{2}; |\underline{x}|^2\right)}{|\underline{x}|^2}.$$

Eventually we thus find :

$$\begin{aligned}\Sigma_1 &= \frac{T^{\alpha+m+1}}{mA_{m+2}|\underline{X}|^m} F\left(\frac{-1-\alpha-m}{2}, \frac{-\alpha-m}{2}; 1-\frac{m}{2}; |\underline{x}|^2\right) \\ &= \mathcal{F}_{\alpha, m+2}(T, \underline{X}) ,\end{aligned}$$

giving rise to the following inductive formula :

$$\mathcal{F}_{\alpha, m+2} = \frac{1}{2\pi} \sum_{k=0}^{\infty} \left\{ \frac{\alpha+m+1}{T^{k+1}} \mathcal{F}_{\alpha+k+1, m} + \frac{k+1}{T^{k+2}} \mathcal{F}_{\alpha+k+2, m} \right\} .$$

Next we consider the summation Σ_2 , defined by

$$\Sigma_2 = \frac{1}{2\pi} \sum_{k=0}^{\infty} \left\{ \frac{\alpha+m+1}{T^{k+1}} \mathcal{R}_{\alpha+k+1, m} + \frac{k+1}{T^{k+2}} \mathcal{R}_{\alpha+k+2, m} \right\} .$$

Before using the induction hypothesis, the regular part $\mathcal{R}_{\alpha, m}(T, \underline{X})$ is transformed into a more useful form. For this purpose we use the derivation property of the hypergeometric function, leading to

$$\begin{aligned}F\left(\frac{-\alpha}{2}, \frac{-1-\alpha}{2}; \frac{m}{2}; u\right) &= \frac{\left(\frac{3}{2}\right)_{\frac{m-3}{2}}}{\left(\frac{3-\alpha-m}{2}\right)_{\frac{m-3}{2}} \left(\frac{2-\alpha-m}{2}\right)_{\frac{m-3}{2}}} \times \\ &\quad \left(\frac{d}{du}\right)^{\frac{m-3}{2}} F\left(\frac{3-\alpha-m}{2}, \frac{2-\alpha-m}{2}; \frac{3}{2}; u\right)\end{aligned}$$

where $u = |\underline{x}|^2$. Using Legendre's duplication formula and the identity $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$, we will eventually obtain :

$$\begin{aligned}&\frac{\Gamma(\alpha+m)}{\Gamma(\alpha+2)} \left\{ \left(\frac{3-\alpha-m}{2}\right)_{\frac{m-3}{2}} \left(\frac{2-\alpha-m}{2}\right)_{\frac{m-3}{2}} \right\}^{-1} \\ &= 2^{m-3}(\alpha+m-1) ,\end{aligned}$$

hereby using the fact that m is odd. This means that $\mathcal{R}_{\alpha, m}(T, \underline{X})$ can also be written as :

$$\begin{aligned}\mathcal{R}_{\alpha, m}(T, \underline{X}) &= \frac{(-1)^{\frac{m-1}{2}} T^{1+\alpha}}{4\pi^{\frac{m-1}{2}}} (\alpha+m-1) \times \\ &\quad \left(\frac{d}{du}\right)^{\frac{m-3}{2}} F\left(\frac{3-m-\alpha}{2}, \frac{2-m-\alpha}{2}; \frac{3}{2}; u\right) ,\end{aligned}$$

with $u = |\underline{x}|^2$.

Using the induction hypothesis together with previous expressions, we thus find for $\mathcal{R}_{\alpha+k+1,m}$:

$$\frac{\mathcal{R}_{\alpha+k+1,m}}{2\pi T^{k+1}} = \frac{(-1)^{\frac{m-1}{2}} T^{\alpha+1}}{8\pi^{\frac{m+1}{2}}} (\alpha + m + k) \times \left(\frac{d}{du} \right)^{\frac{m-3}{2}} F \left(\frac{2-m-\alpha-k}{2}, \frac{1-m-\alpha-k}{2}; \frac{3}{2}; u \right),$$

and for $\mathcal{R}_{\alpha+k+2,m}$:

$$\frac{\mathcal{R}_{\alpha+k+1,m}}{2\pi T^{k+1}} = \frac{(-1)^{\frac{m-1}{2}} T^{\alpha+1}}{8\pi^{\frac{m+1}{2}}} (\alpha + m + k + 1) \times \left(\frac{d}{du} \right)^{\frac{m-3}{2}} F \left(\frac{1-m-\alpha-k}{2}, \frac{-m-\alpha-k}{2}; \frac{3}{2}; u \right).$$

Recalling the fact that

$$F \left(a, a + \frac{1}{2}; \frac{3}{2}; z^2 \right) = \frac{(1+z)^{1-2a} - (1-z)^{1-2a}}{2(1-2a)z}$$

and the expansions for $|\underline{x}|^{-1} = u^{-\frac{1}{2}}$ and $|\underline{x}|^{-2} = u^{-1}$, we then carry out the summation over k to find the following expressions (in terms of $u = |\underline{x}|^2$) :

$$\sum_{k=0}^{\infty} \frac{\alpha + m + 1}{2\pi T^{k+1}} \mathcal{R}_{\alpha+k+1,m} = (\alpha + m + 1) \frac{(-1)^{\frac{m+1}{2}} T^{\alpha+1}}{8\pi^{\frac{m+1}{2}}} \times \left(\frac{d}{du} \right)^{\frac{m-3}{2}} \frac{(1-u^{\frac{1}{2}})^{\alpha+m} + (1+u^{\frac{1}{2}})^{\alpha+m}}{2u}$$

and

$$\sum_{k=0}^{\infty} \frac{k+1}{2\pi T^{k+2}} \mathcal{R}_{\alpha+k+2,m} = \frac{(-1)^{\frac{m+1}{2}} T^{\alpha+1}}{8\pi^{\frac{m+1}{2}}} \times \left(\frac{d}{du} \right)^{\frac{m-3}{2}} \frac{(1-u^{\frac{1}{2}})^{\alpha+m+1} - (1+u^{\frac{1}{2}})^{\alpha+m+1}}{2u^{\frac{3}{2}}}.$$

Eventually we thus find for Σ_2 :

$$\Sigma_2 = \frac{(-1)^{\frac{m+1}{2}} T^{\alpha+1}}{4\pi^{\frac{m+1}{2}}} \left(\frac{d}{du} \right)^{\frac{m-3}{2}} f(u),$$

where we have put

$$f(u) = \left\{ \frac{(1 - u^{\frac{1}{2}})^{\alpha+m+1} - (1 + u^{\frac{1}{2}})^{\alpha+m+1}}{4u^{\frac{3}{2}}} \right\} \\ + (\alpha + m + 1) \left\{ \frac{(1 - u^{\frac{1}{2}})^{\alpha+m} + (1 + u^{\frac{1}{2}})^{\alpha+m}}{4u} \right\} .$$

Comparing this expression for Σ_2 with the alternative expression for $\mathcal{R}_{\alpha, m+2}(T, \underline{X})$, it becomes clear that

$$\Sigma_2 = \mathcal{R}_{\alpha, m+2}(T, \underline{X})$$

provided the following identity holds :

$$\frac{(1 - u^{\frac{1}{2}})^{\alpha+m} + (1 + u^{\frac{1}{2}})^{\alpha+m}}{4u} + \frac{(1 - u^{\frac{1}{2}})^{\alpha+m+1} - (1 + u^{\frac{1}{2}})^{\alpha+m+1}}{4(\alpha + m + 1)u^{\frac{3}{2}}} \\ = \frac{d}{du} F\left(\frac{-m - \alpha}{2}, \frac{1 - m - \alpha}{2}; \frac{3}{2}; u\right) .$$

This formula can easily be verified by means of the definition for the hypergeometric series.

Hence, the following inductive formula has been proved :

$$\mathcal{R}_{\alpha, m+2} = \frac{1}{2\pi} \sum_{k=0}^{\infty} \left\{ \frac{\alpha + m + 1}{T^{k+1}} \mathcal{R}_{\alpha+k+1, m} + \frac{k+1}{T^{k+2}} \mathcal{R}_{\alpha+k+2, m} \right\} .$$

We conclude by :

$$\Phi_{\alpha, m+2}(T, \underline{X}) = \frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{\alpha + m + 1}{T^{k+1}} \{ \mathcal{F}_{\alpha+k+1, m} + \mathcal{R}_{\alpha+k+1, m} \} (T, \underline{X}) \\ + \frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{1+k}{T^{k+2}} \{ \mathcal{F}_{\alpha+k+2, m} + \mathcal{R}_{\alpha+k+2, m} \} (T, \underline{X}) \\ = \mathcal{F}_{\alpha, m+2}(T, \underline{X}) + \mathcal{R}_{\alpha, m+2}(T, \underline{X}) ,$$

which completes the proof. \square

Recalling the fact that $E_{\alpha, m}(T, \underline{X}) = (\epsilon \partial_T - \partial_{\underline{X}}) \Phi_{\alpha, m}(T, \underline{X})$, we are lead to the following formulae :

$$E_{\alpha, m}(T, \underline{X}) = (\epsilon \partial_T - \partial_{\underline{X}}) \{ \mathcal{F}_{\alpha, m} + \mathcal{R}_{\alpha, m} \} (T, \underline{X}) ,$$

with

$$(\epsilon\partial_T - \partial_{\underline{X}})\mathcal{F}_{\alpha,m} = T^\alpha \text{Mod}(\alpha, 1-m, \underline{x})E(\underline{x})$$

and

$$(\epsilon\partial_T - \partial_{\underline{X}})\mathcal{R}_{\alpha,m} = \frac{(-1)^{\frac{m-1}{2}}}{2^m \pi^{\frac{m}{2}-1} \Gamma(\frac{m}{2})} \frac{\Gamma(\alpha+m)}{\Gamma(\alpha+1)} T^\alpha \text{Mod}(\alpha, 0, \underline{x})\epsilon .$$

These formulae are identical to the ones derived in previous subsections.

2.3 Extending the Delta Distribution

In this section an alternative interpretation for the regular part $\mathcal{R}_{\alpha,m}(T, \underline{X})$ of the hyperbolic fundamental solution $E_\alpha(T, \underline{X})$ is given. As was already mentioned in a previous section, we can not talk about *the* regular part of the hyperbolic fundamental solution because a homogeneous nullsolution for the Dirac operator on $\mathbb{R}^{1,m}$ can always be added to the fundamental solution, leading to a new hyperbolic fundamental solution. However, the Radon method and the method using the inductive argument have lead - in a very natural way - to a regular part $\mathcal{R}_{\alpha,m}(T, \underline{X})$. The aim of this section is to show how this function arises even more naturally from the distributional point of view.

Because the singular behaviour of the fundamental solution does not change if a regular function is added, the difference between two hyperbolic fundamental solutions must be a regular homogeneous function on the future cone. If we then restrict this function to the hyperplane $T = 0$ (i.e. the spatial part of $\mathbb{R}^{1,m}$), we obtain a function supported at the origin. This can only be a linear combination of derivatives of the delta distribution $\delta(\underline{X})$. In view of the degree of homogeneity of the regular part of the hyperbolic fundamental solution, this can only be the delta distribution itself.

Let us therefore consider the distribution $\delta(\underline{X})$, with $\underline{X} \in \mathbb{R}^m$, in its Fourier representation :

$$\delta(\underline{X}) = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} e^{i\langle \underline{X}, \underline{Y} \rangle} d\underline{Y} .$$

According to the classical Cauchy-Kowalevski Theorem there exists a unique distribution $\Psi(T, \underline{X})$ satisfying the following requirements :

$$(\epsilon\partial_T - \partial_{\underline{X}})\Psi(T, \underline{X}) = 0 \quad \text{and} \quad \Psi(0, \underline{X}) = \delta(\underline{X}) .$$

This distribution is the so-called CK-extension of $\delta(\underline{X})$, given by :

$$\begin{aligned}\Psi(T, \underline{X}) &= \exp(\epsilon T \partial_{\underline{X}}) \delta(\underline{X}) \\ &= \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \left[\cos(|\underline{Y}|T) + \frac{i\epsilon \underline{Y}}{|\underline{Y}|} \sin(|\underline{Y}|T) \right] e^{i\langle \underline{X}, \underline{Y} \rangle} d\underline{Y} ,\end{aligned}$$

from which it is clear that $\Psi(T, \underline{X})$ contains a scalar part $\Psi(T, \underline{X})_0$ and a bivector part $\Psi(T, \underline{X})_2$. Let us explicitly determine these distributions, in case of an odd dimension m .

- Let us first determine the scalar part $\Psi(T, \underline{X})_0$. Putting $\underline{Y} = |\underline{Y}|\underline{\eta}$, we get :

$$\begin{aligned}\Psi(T, \underline{X})_0 &= \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \cos(|\underline{Y}|T) e^{i\langle \underline{X}, \underline{Y} \rangle} d\underline{Y} \\ &= \frac{1}{(2\pi)^m} \int_{S^{m-1}} dS(\underline{\eta}) \int_0^\infty \cos(|\underline{Y}|T) e^{i|\underline{Y}|\langle \underline{X}, \underline{\eta} \rangle} |\underline{Y}|^{m-1} d|\underline{Y}| .\end{aligned}$$

In order to rewrite the left-hand side in such a way that we obtain a Fourier transform, we need an integration in $|\underline{Y}|$ from $-\infty$ to $+\infty$. For that purpose, it suffices to perform the change of variables $|\underline{Y}| \rightarrow -|\underline{Y}|$ and to note that for odd m

$$\begin{aligned}& \int_{S^{m-1}} dS(\underline{\eta}) \int_{-\infty}^0 \cos(|\underline{Y}|T) e^{i|\underline{Y}|\langle \underline{X}, \underline{\eta} \rangle} |\underline{Y}|^{m-1} d|\underline{Y}| \\ &= \int_{S^{m-1}} dS(\underline{\eta}) \int_0^\infty \cos(-|\underline{Y}|T) e^{-i|\underline{Y}|\langle \underline{X}, \underline{\eta} \rangle} (-|\underline{Y}|)^{m-1} d(-|\underline{Y}|) \\ &= \int_{S^{m-1}} dS(\underline{\eta}) \int_0^\infty \cos(|\underline{Y}|T) e^{i|\underline{Y}|\langle \underline{X}, \underline{\eta} \rangle} |\underline{Y}|^{m-1} d|\underline{Y}| ,\end{aligned}$$

where we have used the fact that

$$\int_{S^{m-1}} f(\underline{\eta}) dS(\underline{\eta}) = \int_{S^{m-1}} f(-\underline{\eta}) dS(\underline{\eta}) .$$

Using the fact that

$$\int_{-\infty}^\infty e^{ixa} dx = 2\pi \delta(a) ,$$

we easily get, for odd m :

$$\begin{aligned}& \int_0^\infty \cos(|\underline{Y}|T) e^{i|\underline{Y}|\langle \underline{X}, \underline{\eta} \rangle} |\underline{Y}|^{m-1} d|\underline{Y}| \\ &= \frac{\pi}{2} (-1)^{\frac{m-1}{2}} \partial_T^{m-1} \left(\delta(\langle \underline{X}, \underline{\eta} \rangle - T) + \delta(\langle \underline{X}, \underline{\eta} \rangle + T) \right) .\end{aligned}$$

Recalling Hecke-Funk's Theorem 0.2, this reduces to

$$\begin{aligned}\Psi(T, \underline{X})_0 &= \frac{(-1)^{\frac{m-1}{2}} \partial_T^{m-1}}{2^m \pi^{\frac{m-1}{2}} \Gamma\left(\frac{m-1}{2}\right)} \int_{-1}^1 (1-s^2)^{\frac{m-3}{2}} \delta(|\underline{X}|s - T) ds \\ &+ \frac{(-1)^{\frac{m-1}{2}} \partial_T^{m-1}}{2^m \pi^{\frac{m-1}{2}} \Gamma\left(\frac{m-1}{2}\right)} \int_{-1}^1 (1-s^2)^{\frac{m-3}{2}} \delta(|\underline{X}|s + T) ds .\end{aligned}$$

As

$$\begin{aligned}& \int_{-1}^1 (1-s^2)^{\frac{m-3}{2}} \left(\delta(|\underline{X}|s - T) + \delta(|\underline{X}|s + T) \right) ds \\ &= \frac{2}{|\underline{X}|} \left(1 - \frac{T^2}{|\underline{X}|^2} \right)^{\frac{m-3}{2}} \left(H(T)H(|\underline{X}| - T) + H(-T)H(|\underline{X}| + T) \right) ,\end{aligned}$$

we eventually find for the scalar part $\Psi(T, \underline{X})_0$:

$$\Psi(T, \underline{X})_0 = - \frac{\partial_T^{m-1} (T^2 - |\underline{X}|^2)^{\frac{m-3}{2}}}{2^{m-1} \pi^{\frac{m-1}{2}} \Gamma\left(\frac{m-1}{2}\right) |\underline{X}|^{m-2}} \mathcal{D}(T, \underline{X}) ,$$

where we have put

$$\mathcal{D}(T, \underline{X}) = H(T)H(|\underline{X}| - T) + H(-T)H(|\underline{X}| + T) .$$

- Next we determine the bivector part $\Psi(T, \underline{X})_2$, given by

$$\begin{aligned}\Psi(T, \underline{X})_2 &= \frac{i\epsilon}{(2\pi)^m} \int_{\mathbb{R}^m} \frac{\underline{Y}}{|\underline{Y}|} \sin(|\underline{Y}|T) e^{i\langle \underline{X}, \underline{Y} \rangle} d\underline{Y} \\ &= \frac{i\epsilon}{(2\pi)^m} \int_{S^{m-1}} \underline{\eta} dS(\underline{\eta}) \int_0^\infty \sin(|\underline{Y}|T) e^{i|\underline{Y}| \langle \underline{X}, \underline{\eta} \rangle} |\underline{Y}|^{m-1} d|\underline{Y}| .\end{aligned}$$

For odd m , we have

$$|\underline{Y}|^{m-1} = \partial_{\underline{X}}^{m-1} (i \langle \underline{X}, \underline{Y} \rangle) ,$$

whence

$$\Psi(T, \underline{X})_2 = \frac{\epsilon \partial_{\underline{X}}^{m-1}}{2(2\pi)^m} \int_{S^{m-1}} \underline{\eta} \left(\delta(\langle \underline{X}, \underline{\eta} \rangle + T) - \delta(\langle \underline{X}, \underline{\eta} \rangle - T) \right) dS(\underline{\eta}) .$$

Using Hecke-Funk's theorem, we eventually find :

$$\Psi(T, \underline{X})_2 = \frac{(-1)^{\frac{m-1}{2}} \epsilon \partial_{\underline{X}}^{m-1} T (T^2 - |\underline{X}|^2)^{\frac{m-3}{2}}}{2^{m-1} \pi^{\frac{m-1}{2}} \Gamma\left(\frac{m-1}{2}\right)} \frac{\underline{X}}{|\underline{X}|^m} \mathcal{D}(T, \underline{X}) .$$

If we then convolute the distributions $\Psi(T, \underline{X})$ and $T_+^{\alpha+m-1}$, we obtain an α -homogeneous solution for the hyperbolic Dirac equation. This will in fact be the function $\mathcal{R}_\alpha(T, \underline{X})$.

We will again perform these calculations in two steps :

- First, let us determine the scalar part $\Psi(T, \underline{X})_0 * T_+^{\alpha+m-1}$:

$$\mathcal{R}_\alpha(T, \underline{X})_0 = -\frac{\partial_T^{m-1}(T^2 - |\underline{X}|^2)^{\frac{m-3}{2}}}{2^{m-1}\pi^{\frac{m-1}{2}}\Gamma\left(\frac{m-1}{2}\right)|\underline{X}|^{m-2}}\mathcal{D}(T, \underline{X}) * T_+^{\alpha+m-1}$$

Using the fact that $\partial_T(f * g) = \partial_T f * g = f * \partial_T g$, we get :

$$\mathcal{R}_\alpha(T, \underline{X})_0 = -\partial_T \frac{(T^2 - |\underline{X}|^2)^{\frac{m-3}{2}}\mathcal{D}(T, \underline{X}) * \partial_T^{m-2}T_+^{\alpha+m-1}}{2^{m-1}\pi^{\frac{m-1}{2}}\Gamma\left(\frac{m-1}{2}\right)|\underline{X}|^{m-2}}.$$

As long as the exponent of $T_+^{\alpha+m-1}$ is not a negative integer, we have in distributional sense

$$\partial_T^{m-2}T_+^{\alpha+m-1} = \frac{\Gamma(\alpha+m)}{\Gamma(\alpha+2)}T_+^{1+\alpha},$$

whence for $\alpha+m \notin -\mathbb{N}$ we get

$$\mathcal{R}_\alpha(T, \underline{X})_0 = -\partial_T \frac{\Gamma(\alpha+m)}{\Gamma(\alpha+2)} \frac{(T^2 - |\underline{X}|^2)^{\frac{m-3}{2}}\mathcal{D}(T, |\underline{X}|) * T_+^{\alpha+1}}{2^{m-1}\pi^{\frac{m-1}{2}}\Gamma\left(\frac{m-1}{2}\right)|\underline{X}|^{m-2}}.$$

Using the explicit definition for the distribution $\mathcal{D}(T, |\underline{X}|)$ and putting $|\underline{X}| = R$, we thus find :

$$(T^2 - |\underline{X}|^2)^{\frac{m-3}{2}}\mathcal{D}(T, |\underline{X}|) * T_+^{\alpha+1} = \int_{T-R}^{T+R} ((T-S)^2 - R^2)^{\frac{m-3}{2}} S^{\alpha+1} dS.$$

Rewriting this integral in terms of the new variable

$$V = \frac{S - T + R}{2R}$$

and recalling Euler's integral representation for the hypergeometric function, we arrive at

$$\frac{(-1)^{\frac{m-3}{2}}\pi^{\frac{1}{2}}\Gamma\left(\frac{m-1}{2}\right)}{4\Gamma\left(\frac{m}{2}\right)}R^{m-2}(T-R)^{1+\alpha}F\left(-1-\alpha, \frac{m-1}{2}; m-1; \frac{2R}{T-R}\right),$$

which by means of the transformation formulae for the hypergeometric function due to Kummer and Goursat, can be reduced to

$$\frac{(-1)^{\frac{m-3}{2}} \pi^{\frac{1}{2}} \Gamma\left(\frac{m-1}{2}\right)}{4\Gamma\left(\frac{m}{2}\right)} R^{m-2} T^{1+\alpha} F\left(\frac{-1-\alpha}{2}, \frac{-\alpha}{2}; \frac{m}{2}; \frac{R^2}{T^2}\right).$$

Eventually, we have thus found that

$$\mathcal{R}_\alpha(T, \underline{X})_0 = \partial_T \frac{(-1)^{\frac{m-1}{2}} T^{1+\alpha}}{2^{m+1} \pi^{\frac{m}{2}-1} \Gamma\left(\frac{m}{2}\right)} \frac{\Gamma(\alpha+m)}{\Gamma(\alpha+2)} F\left(\frac{-1-\alpha}{2}, \frac{-\alpha}{2}; \frac{m}{2}; \frac{|\underline{X}|^2}{T^2}\right).$$

- Next, let us determine $\Psi(T, \underline{X})_2 * T_+^{\alpha+m-1}$:

$$\mathcal{R}_\alpha(T, \underline{X})_2 = \frac{(-1)^{\frac{m-1}{2}} \epsilon \partial_{\underline{X}}^{m-1} T (T^2 - |\underline{X}|^2)^{\frac{m-3}{2}} \mathcal{D}(T, |\underline{X}|) * T_+^{\alpha+m-1} \frac{\underline{X}}{|\underline{X}|^m}}{2^{m-1} \pi^{\frac{m-1}{2}} \Gamma\left(\frac{m-1}{2}\right)}.$$

Putting $|\underline{X}| = R$, and introducing the shorthand notation

$$I(T, \underline{X})_2 = T (T^2 - R^2)^{\frac{m-3}{2}} \mathcal{D}(T, R) * T_+^{\alpha+m-1},$$

we get

$$\begin{aligned} I(T, \underline{X})_2 &= T \int_0^\infty ((T-S)^2 - R^2)^{\frac{m-3}{2}} \mathcal{D}(T-S, R) S^{\alpha+m-1} dS \\ &\quad - \int_0^\infty ((T-S)^2 - R^2)^{\frac{m-3}{2}} \mathcal{D}(T-S, R) S^{\alpha+m} dS. \end{aligned}$$

A similar integral was already calculated above; we immediately find :

$$I(T, \underline{X})_2 = \frac{(-1)^{\frac{m-3}{2}} \sqrt{\pi} \Gamma\left(\frac{m-1}{2}\right)}{4\Gamma\left(\frac{m}{2}\right)} R^{m-2} T^{\alpha+m} \left(F_1(T, \underline{X}) - F_2(T, \underline{X})\right),$$

with

$$F_1(T, \underline{X}) = F\left(\frac{1-\alpha-m}{2}, \frac{2-\alpha-m}{2}; \frac{m}{2}; \frac{|\underline{X}|^2}{T^2}\right)$$

and

$$F_2(T, \underline{X}) = F\left(\frac{-\alpha-m}{2}, \frac{1-\alpha-m}{2}; \frac{m}{2}; \frac{|\underline{X}|^2}{T^2}\right).$$

As for odd m

$$(-1)^{\frac{m-3}{2}} (\partial_{\underline{X}})^{m-3} = \Delta_m^{\frac{m-3}{2}},$$

we get

$$\mathcal{R}_\alpha(T, \underline{X})_b = \frac{(-1)^{\frac{m-1}{2}} T^{\alpha+m} \epsilon \partial_{\underline{X}} \Delta_m^{\frac{m-3}{2}}}{2^{m+1} \pi^{\frac{m}{2}-1} \Gamma\left(\frac{m}{2}\right)} (\partial_{\underline{X}} \underline{X}) \frac{F_1(T, \underline{X}) - F_2(T, \underline{X})}{|\underline{X}|^2}.$$

As both $F_1(T, \underline{X})$ and $F_2(T, \underline{X})$ only depend on the norm $R = |\underline{X}|$, and as $\partial_{\underline{X}} \underline{X} = -m - R\partial_R$ and $\Delta_m = \partial_R^2 + \frac{m-1}{R}\partial_R$ for functions depending on R only, this reduces to

$$\begin{aligned} \mathcal{R}_\alpha(T, \underline{X})_b &= \frac{(-1)^{\frac{m-1}{2}} T^{\alpha+m-2} \partial_{\underline{X}} \epsilon}{2^{m+1} \pi^{\frac{m}{2}-1} \Gamma\left(\frac{m}{2}\right)} \\ &\quad \left(\partial_R^2 + \frac{m-1}{R} \partial_R \right)^{\frac{m-3}{2}} (m + \mathbb{E}_R) \frac{F_1(T, R) - F_2(T, R)}{\left(\frac{R}{T}\right)^2}. \end{aligned}$$

First of all, note that the function at the right-hand side is given by

$$\frac{F_1(T, R) - F_2(T, R)}{\left(\frac{R}{T}\right)^2} = -\frac{2}{\alpha + m} F' \left(\frac{-\alpha - m}{2}, \frac{1 - \alpha - m}{2}; \frac{m}{2}; t \right),$$

where the prime denotes a derivation with respect to $t = \left(\frac{R}{T}\right)^2$. As

$$R\partial_R f \left(\frac{R^2}{T^2} \right) = 2t \frac{d}{dt} f(t) = 2 \left(\frac{d}{dt} t - 1 \right) f(t),$$

we get by means of the contiguous relations for the hypergeometric function :

$$(m + \mathbb{E}_R) \frac{F_1 - F_2}{\left(\frac{R}{T}\right)^2} = -4 \frac{\frac{m}{2} - 1}{\alpha + m} F' \left(\frac{-\alpha - m}{2}, \frac{1 - \alpha - m}{2}; \frac{m}{2} - 1; t \right).$$

On the other hand, we also have that

$$\left(\partial_R^2 + \frac{m-1}{R} \partial_R \right) f \left(\frac{R^2}{T^2} \right) = \frac{4}{T^2} \frac{d}{dt} \left(\frac{m}{2} + t \frac{d}{dt} \right) f(t),$$

whence

$$\begin{aligned} \left(\partial_R^2 + \frac{m-1}{R} \partial_R \right)^{\frac{m-3}{2}} (m + \mathbb{E}_R) \frac{F_1 - F_2}{\left(\frac{R}{T}\right)^2} &= -\frac{2^{m-1} \left(\frac{m}{2} - 1\right)}{(\alpha + m) T^{m-3}} \times \\ &\quad \left(\frac{d}{dt} \left(\frac{m}{2} + t \frac{d}{dt} \right) \right)^{\frac{m-3}{2}} F' \left(\frac{-\alpha - m}{2}, \frac{1 - \alpha - m}{2}; \frac{m}{2} - 1; t \right). \end{aligned}$$

Introducing the notation $F(a, t)$ for the hypergeometric function on the right-hand side,

$$F(a; t) = F\left(\frac{-\alpha - m}{2}, \frac{1 - \alpha - m}{2}; a; t\right),$$

we get :

$$\begin{aligned} \frac{d}{dt} \left(\frac{m}{2} + t \frac{d}{dt} \right) F' \left(\frac{m}{2} - 1; t \right) &= \frac{d^2}{dt^2} \left(\frac{m}{2} - 1 + t \frac{d}{dt} \right) F \left(\frac{m}{2} - 1; t \right) \\ &= \left(\frac{m}{2} - 2 \right) F'' \left(\frac{m}{2} - 2; t \right). \end{aligned}$$

This scheme repeats itself, so that eventually we will find :

$$\left(\frac{d}{dt} \left(\frac{m}{2} + t \frac{d}{dt} \right) \right)^{\frac{m-3}{2}} F' \left(\frac{m}{2} - 1; t \right) = \frac{\Gamma(\frac{m}{2} - 1)}{\Gamma(\frac{1}{2})} \left(\frac{d}{dt} \right)^{\frac{m-1}{2}} F \left(\frac{1}{2}; t \right).$$

Recalling the derivation property of the hypergeometric function, we arrive at

$$\begin{aligned} &\left(\partial_R^2 + \frac{m-1}{R} \partial_R \right)^{\frac{m-3}{2}} (m + \mathbb{E}_R) \frac{F_1(T, R) - F_2(T, R)}{\left(\frac{R}{T} \right)^2} \\ &= -\frac{1}{T^{m-3}} \frac{\Gamma(\alpha + m)}{\Gamma(\alpha + 2)} F \left(-\frac{1 + \alpha}{2}, -\frac{\alpha}{2}; \frac{m}{2}; \frac{R^2}{T^2} \right), \end{aligned}$$

whence the final expression for $\mathcal{R}_\alpha(T, \underline{X})_2$:

$$\mathcal{R}_\alpha(T, \underline{X})_2 = -\frac{(-1)^{\frac{m-1}{2}} \Gamma(\alpha + m) T^{1+\alpha} \partial_{\underline{X}} \epsilon}{2^{m+1} \pi^{\frac{m}{2}-1} \Gamma(\frac{m}{2}) \Gamma(\alpha + 2)} F \left(-\frac{1 + \alpha}{2}, -\frac{\alpha}{2}; \frac{m}{2}; \frac{|\underline{X}|^2}{T^2} \right) \epsilon.$$

Adding up $\mathcal{R}_\alpha(T, \underline{X})_0$ and $\mathcal{R}_\alpha(T, \underline{X})_2$, we find an α -homogeneous nullsolution for the Dirac operator on $\mathbb{R}^{1,m}$ defined by

$$\mathcal{R}_\alpha(T, \underline{X}) = \frac{1}{2} \partial_X \frac{(-1)^{\frac{m-1}{2}}}{2^m \pi^{\frac{m}{2}-1} \Gamma(\frac{m}{2})} \frac{\Gamma(\alpha + m)}{\Gamma(\alpha + 2)} T^{1+\alpha} F \left(-\frac{1 + \alpha}{2}, -\frac{\alpha}{2}; \frac{m}{2}; \frac{|\underline{X}|^2}{T^2} \right) \epsilon,$$

with $\partial_X = \epsilon \partial_T - \partial_{\underline{X}}$ the Dirac operator on $\mathbb{R}^{1,m}$. This is the same expression as the one that was found in the previous section.

Chapter 3

Modulation Theorems

The art of doing mathematics consists in finding that special case which contains all the germs of generality. (D. Hilbert)

In the first section we prove the *hyperbolic Modulation Theorem* which allows to generate monogenic functions on the hyperbolic unit ball by means of classical monogenic functions. This idea is then generalized in several ways; in section 2 we consider arbitrary powers of the hyperbolic Dirac operator and in section 3 we turn our attention to the most general ultra-hyperbolic case $\mathbb{R}^{p,q}$. In section 4 we consider a specific bi-axial problem, allowing us to reinterpret the modulation hyperbolic monogenics in terms of so-called *generalized hyperbolic powerfunctions*.

3.1 The Hyperbolic Modulation Theorems

In this section we construct solutions for the Dirac operator on the hyperbolic unit ball H_+ by means of the projection method, already introduced in the previous chapter where it was used to construct the hyperbolic fundamental solution. Because the hyperbolic Dirac equation will be projected both on the hyperplane Π and the parabola \mathcal{P} , there are three subsections : first the Klein model and the Poincaré model are considered, and in a third subsection the two models are linked.

3.1.1 The Klein Model

In Chapter 2 it was already explained how to obtain the projection of the hyperbolic Dirac equation on the Klein model, by choosing appropriate coordinates on the future cone FC . Indeed, writing an arbitrary space-time

vector $\epsilon T + \underline{X}$ as $\lambda(\epsilon + \underline{x})$ with

$$\begin{cases} \lambda &= T \\ \underline{x} &= \frac{\underline{X}}{T} \end{cases},$$

and using the set (λ, \underline{x}) as co-ordinates on FC , the hyperbolic Dirac operator was found to be the differential operator

$$-\frac{1}{\lambda}(\underline{\partial} + \epsilon(\mathbb{E}_r - \mathbb{E}_\lambda)) ,$$

with $\underline{\partial}$ (respectively \mathbb{E}_r) the Dirac (respectively Euler) operator on $\mathbb{R}^{0,m}$. In view of the projective model for the hyperbolic unit ball, we now construct α -homogeneous solutions $F(T, \underline{X}) = \lambda^\alpha F(\underline{x})$ for this operator. The idea behind the Modulation Theorem is to propose the following form for $F(\underline{x})$:

$$F(\underline{x}) = \sum_{j=0}^{\infty} b_j(\underline{x}\epsilon)^j f(\underline{x}) ,$$

where $f(\underline{x})$ is an arbitrary *homogeneous* solution for the Dirac operator $\underline{\partial}$ on the unit ball $B_m(1)$. If $F(T, \underline{X})$ is to be globally defined on FC , the function $f(\underline{x})$ is to be defined on the whole unit ball. In view of its homogeneity, this means that the restriction $f(\underline{\omega})$ to S^{m-1} must be globally defined. In other words, $f(\underline{\omega})$ must then be an inner or outer spherical monogenic. One may also choose $f(\underline{x})$ to be a locally defined homogeneous solution for the Dirac operator $\underline{\partial}$ on \mathbb{R}^m , which is equivalent with saying that $f(\underline{\omega})$ must be a locally defined eigenfunction for the Gamma operator $\Gamma_{0,m}$ on an open subset Ω' of the unit sphere S^{m-1} , and this yields locally defined solutions $F(T, \underline{X}) \in \mathcal{H}^\alpha(\mathbb{R}_+\Omega)$. These local eigenfunctions for the Gamma operator on the unit sphere S^{m-1} in \mathbb{R}^m have intensively been studied by Peter Van Lancker in his PhD-thesis, see reference [75]. Note that the open subsets Ω' and Ω are linked in the following way : $\Omega' \subset S^{m-1}$ gives rise to an open sector-like subset of the unit ball $B_m(1) \subset \mathbb{R}^m$ which is then to be projected from the origin on the hyperbola H_+ . The result is the open subset Ω .

Theorem 3.1 *Consider an arbitrary μ -homogeneous solution $f(\underline{x})$ for the Dirac operator $\underline{\partial}$ on $\mathbb{R}^{0,m}$, defined in an open subset Ω' of $B_m(1)$, such that $\mu + \frac{m}{2} \notin -\mathbb{N}$. An α -homogeneous solution $F(T, \underline{X})$ for the Dirac operator $(\epsilon\partial_T - \partial_{\underline{X}})$ on $\mathbb{R}^{1,m}$ is then given by*

$$F(T, \underline{X}) = T^\alpha \text{Mod}\left(\alpha, \mu, \frac{\underline{X}}{T}\right) f\left(\frac{\underline{X}}{T}\right) ,$$

where the modulation factor is defined as

$$Mod(\alpha, \mu, \underline{x}) = F_1^{(\mu)}(|\underline{x}|^2) + \frac{\mu - \alpha}{2\mu + m} \underline{x} \epsilon F_2^{(\mu)}(|\underline{x}|^2)$$

with

$$\begin{aligned} F_1^{(\mu)}(t) &= F\left(\frac{1 + \mu - \alpha}{2}, \frac{\mu - \alpha}{2}; \mu + \frac{m}{2}; t\right) \\ F_2^{(\mu)}(t) &= F\left(\frac{1 + \mu - \alpha}{2}, 1 + \frac{\mu - \alpha}{2}; 1 + \mu + \frac{m}{2}; t\right) . \end{aligned}$$

This function $F(T, \underline{X})$ belongs to $\mathcal{H}^\alpha(\mathbb{R}_+\Omega)$, with

$$\Omega = \left\{ (T, \underline{X}) \in H_+ : \frac{\underline{X}}{T} \in \Omega' \right\} .$$

proof : The proof for this theorem is constructive : since we have put $F(T, \underline{X}) = \lambda^\alpha F(\underline{x})$ with

$$F(\underline{x}) = \sum_{j=0}^{\infty} b_j (\underline{x} \epsilon)^j f(\underline{x}) ,$$

we only need to determine the coefficients b_j in such a way that

$$(\underline{\partial} + \epsilon(\mathbb{E}_r - \alpha)) F(\underline{x}) = 0 .$$

First of all, note that the summation over j yields the sum of a scalar and a bivector-valued function :

$$\sum_{j=0}^{\infty} b_j (\underline{x} \epsilon)^j = \sum_{j=0}^{\infty} b_{2j} |\underline{x}|^{2j} + \underline{x} \epsilon \sum_{j=0}^{\infty} b_{2j+1} |\underline{x}|^{2j} .$$

Letting the Dirac operator $\underline{\partial}$ act on $F(\underline{x})$ we get :

- for the scalar part :

$$\begin{aligned} \underline{\partial} \sum_{j=0}^{\infty} b_{2j} |\underline{x}|^{2j} f(\underline{x}) &= \underline{x} \sum_{j=0}^{\infty} 2j b_{2j} |\underline{x}|^{2j-2} f(\underline{x}) \\ &= \underline{x} \sum_{j=0}^{\infty} (2j + 2) b_{2j+2} |\underline{x}|^{2j} f(\underline{x}) \end{aligned}$$

- for the bivector part :

$$\begin{aligned}\underline{\partial} \underline{x} \epsilon \sum_{j=0}^{\infty} b_{2j+1} |\underline{x}|^{2j} f(\underline{x}) &= \epsilon (\Gamma_{0,m} - \mathbb{E}_r - m) \sum_{j=0}^{\infty} b_{2j+1} |\underline{x}|^{2j} f(\underline{x}) \\ &= -\epsilon \sum_{j=0}^{\infty} (m + 2j + 2\mu) b_{2j+1} |\underline{x}|^{2j} f(\underline{x}) .\end{aligned}$$

Letting the Euler operator \mathbb{E}_r act on $F(\underline{x})$ we get :

- for the scalar part :

$$\mathbb{E}_r \sum_{j=0}^{\infty} b_{2j} |\underline{x}|^{2j} f(\underline{x}) = \sum_{j=0}^{\infty} (2j + \mu) b_{2j} |\underline{x}|^{2j} f(\underline{x})$$

- for the bivector part :

$$\mathbb{E}_r \underline{x} \epsilon \sum_{j=0}^{\infty} b_{2j+1} |\underline{x}|^{2j} f(\underline{x}) = \underline{x} \epsilon \sum_{j=0}^{\infty} (1 + 2j + \mu) b_{2j+1} |\underline{x}|^{2j} f(\underline{x}) .$$

Expressing the fact that $(\underline{\partial} + \epsilon(\mathbb{E}_r - \alpha))F(\underline{x}) = 0$ gives for each $j \in \mathbb{N}$ rise to the following set of equations :

$$\begin{cases} (2j + 2) b_{2j+2} &= (1 + 2j + \mu - \alpha) b_{2j+1} \\ (m + 2j + 2\mu) b_{2j+1} &= (2j + \mu - \alpha) b_{2j} . \end{cases}$$

This enables us to recursively define the coefficients b_j , for all $j \in \mathbb{N}$:

$$\begin{aligned}b_{2j+2} &= \frac{1 + 2j + \mu - \alpha}{2j + 2} \frac{2j + \mu - \alpha}{m + 2j + \mu} b_{2j} \\ b_{2j+1} &= \frac{2j + \mu - \alpha}{2j} \frac{2j - 1 + \mu - \alpha}{m + 2j + \mu} b_{2j-1} .\end{aligned}$$

Choosing $b_0 = 1$, this eventually yields :

$$\begin{aligned}b_{2j} &= \frac{\left(\frac{1+\mu-\alpha}{2}\right)_j \left(\frac{\mu-\alpha}{2}\right)_j}{j! \left(\mu + \frac{m}{2}\right)_j} \\ b_{2j+1} &= \frac{\mu - \alpha}{2\mu + m} \frac{\left(\frac{1+\mu-\alpha}{2}\right)_j \left(1 + \frac{\mu-\alpha}{2}\right)_j}{j! \left(1 + \mu + \frac{m}{2}\right)_j} ,\end{aligned}$$

proving that

$$\begin{aligned}\sum_{j=0}^{\infty} b_{2j} |\underline{x}|^{2j} &= F\left(\frac{1+\mu-\alpha}{2}, \frac{\mu-\alpha}{2}; \mu + \frac{m}{2}; |\underline{x}|^2\right) \\ \sum_{j=0}^{\infty} b_{2j+1} |\underline{x}|^{2j} &= \frac{\mu-\alpha}{2\mu+m} F\left(\frac{1+\mu-\alpha}{2}, 1 + \frac{\mu-\alpha}{2}; 1 + \mu + \frac{m}{2}; |\underline{x}|^2\right).\end{aligned}$$

These hypergeometric series are well-defined on the future cone FC if the coefficient $\mu + \frac{m}{2}$ does not belong to $-\mathbb{N}$. \square

Remark 1 : Note that the hyperbolic fundamental solution easily follows from the Theorem by choosing $f(\underline{x})$ to be the classical Cauchy kernel $E(\underline{x})$, a monogenic function on $\mathbb{R}^m \setminus \{0\}$ with respect to the Dirac operator $\underline{\partial}$ which is homogeneous of degree $(1-m)$. The monogenic function on the punctured hyperbolic unit ball thus obtained is the fundamental solution $E_\alpha(T, \underline{X})$ from the previous chapter (see 2.4).

Remark 2 : The value $\alpha = -\frac{m}{2}$ is again exceptional, since for this value the modulation factor is defined *for all* μ . Indeed, we easily find that

$$\begin{aligned}F_1^{(\mu)}(t) &= F\left(\frac{1}{2}\left(\mu + \frac{m}{2}\right), \frac{1}{2}\left(1 + \mu + \frac{m}{2}\right); \mu + \frac{m}{2}; t\right) \\ &= (1-t)^{-\frac{1}{2}} \left(\frac{1 + (1-t)^{\frac{1}{2}}}{2}\right)^{1-\mu-\frac{m}{2}} \\ F_2^{(\mu)}(t) &= F\left(\frac{1}{2}\left(1 + \mu + \frac{m}{2}\right), \frac{1}{2}\left(2 + \mu + \frac{m}{2}\right); 1 + \mu + \frac{m}{2}; t\right) \\ &= (1-t)^{-\frac{1}{2}} \left(\frac{1 + (1-t)^{\frac{1}{2}}}{2}\right)^{-\mu-\frac{m}{2}},\end{aligned}$$

such that the modulation factor reduces to

$$\text{Mod}\left(-\frac{m}{2}, \mu, \underline{x}\right) = \frac{(1 + (1 - |\underline{x}|^2)^{\frac{1}{2}})^{1-\mu-\frac{m}{2}}}{2^{1-\mu-\frac{m}{2}}(1 - |\underline{x}|^2)^{\frac{1}{2}}} \left(1 + \frac{\underline{x}\epsilon}{1 + (1 - |\underline{x}|^2)^{\frac{1}{2}}}\right).$$

3.1.2 The Poincaré Model

In Chapter 2 it was already explained how to obtain the projection of the hyperbolic Dirac equation on the Poincaré model, by choosing appropriate

co-ordinates on the future cone FC . Indeed, writing an arbitrary space-time vector $\epsilon T + \underline{X}$ as $\lambda_P \left(\epsilon \frac{1 + |\underline{x}|^2}{2} + \underline{x} \right)$ with

$$\begin{cases} \lambda_P &= T + (T^2 - |\underline{X}|^2)^{\frac{1}{2}} \\ \underline{x} &= \frac{\underline{X}}{\lambda_P} \end{cases},$$

and using the set $(\lambda_P, \underline{x})$ as co-ordinates on FC , the Dirac operator on the Poincaré ball was found to be the differential operator

$$-\frac{1}{\lambda_P} \left(\underline{\partial} + 2 \frac{\underline{x} + \epsilon}{1 - |\underline{x}|^2} \epsilon (\mathbb{E}_r - \mathbb{E}_{\lambda_P}) \right),$$

with $\underline{\partial}$ (respectively \mathbb{E}_r) the Dirac (respectively Euler) operator on $\mathbb{R}^{0,m}$. In view of the projective model for the hyperbolic unit ball, we now construct α -homogeneous solutions $G(T, \underline{X}) = \lambda_P^\alpha G(\underline{x})$ for this operator. The idea is again to propose the following form for $G(\underline{x})$:

$$G(\underline{x}) = \sum_{j=0}^{\infty} a_j (\underline{x}\epsilon)^j g(\underline{x}),$$

where $g(\underline{x})$ is an arbitrary *homogeneous* solution for the Dirac operator $\underline{\partial}$. The same remarks concerning the domain of definition as for the Klein model apply here : depending on whether the restriction $f(\underline{\omega})$ to the unit sphere S^{m-1} is locally/globally defined, a local/global monogenic on the hyperbolic unit ball is obtained.

Theorem 3.2 *Consider an arbitrary μ -homogeneous solution $g(\underline{x})$ for the Dirac operator $\underline{\partial}$ on $\mathbb{R}^{0,m}$, where $\mu + \frac{m}{2} \notin -\mathbb{N}$, defined in an open subset Ω' of $B_m(1)$. An α -homogeneous solution $G(T, \underline{X})$ for the Dirac operator $(\epsilon \partial_T - \partial_{\underline{X}})$ on $\mathbb{R}^{1,m}$ is then given by*

$$G(T, \underline{X}) = \lambda_P^\alpha \text{Mod}_P \left(\alpha, \mu, \frac{\underline{X}}{\lambda_P} \right) g \left(\frac{\underline{X}}{\lambda_P} \right)$$

with

$$\lambda_P = T + (T^2 - |\underline{X}|^2)^{\frac{1}{2}},$$

where the modulation factor is defined as

$$\text{Mod}_P(\alpha, \mu, \underline{x}) = F_{P,1}^{(\mu)}(|\underline{x}|^2) + \frac{\mu - \alpha}{\mu + \frac{m}{2}} \underline{x} \epsilon F_{P,2}^{(\mu)}(|\underline{x}|^2)$$

with

$$\begin{aligned} F_{P,1}^{(\mu)}(t) &= F\left(1 - \alpha - \frac{m}{2}, \mu - \alpha; \mu + \frac{m}{2}; t\right) \\ F_2^{(\mu)}(t) &= F\left(1 - \alpha - \frac{m}{2}, 1 + \mu - \alpha; 1 + \mu + \frac{m}{2}; t\right) . \end{aligned}$$

This function $G(T, \underline{X})$ belongs to $\mathcal{H}^\alpha(\mathbb{R}_+\Omega)$, with

$$\Omega = \left\{ (T, \underline{X}) \in H_+ : \frac{\underline{X}}{\lambda_P} \in \Omega' \right\} .$$

proof : The proof for this theorem is again constructive : since we have put $G(T, \underline{X}) = \lambda_P^\alpha G(\underline{x})$ with

$$G(\underline{x}) = \sum_{j=0}^{\infty} a_j (\underline{x}\epsilon)^j g(\underline{x}) ,$$

we only need to determine the coefficients a_j in such a way that

$$\left(\underline{\partial} + 2 \frac{\epsilon + \underline{x}}{1 - |\underline{x}|^2} (\mathbb{E}_r - \alpha) \right) G(\underline{x}) = 0 .$$

Since $(\epsilon + \underline{x})^2 = 1 - |\underline{x}|^2$, this is equivalent with

$$(\underline{\partial}(\underline{x} + \epsilon) + m + 2\alpha) G(\underline{x}) = 0 .$$

Rewriting the summation over j as the sum of a scalar and a bivector-valued function

$$\sum_{j=0}^{\infty} a_j (\underline{x}\epsilon)^j = \sum_{j=0}^{\infty} a_{2j} |\underline{x}|^{2j} + \underline{x}\epsilon \sum_{j=0}^{\infty} a_{2j+1} |\underline{x}|^{2j} ,$$

we get for the action of the operator $\underline{\partial} \underline{x}$ on $G(\underline{x})$:

- for the scalar part :

$$\underline{\partial} \underline{x} \sum_{j=0}^{\infty} a_{2j} |\underline{x}|^{2j} g(\underline{x}) = - \sum_{j=0}^{\infty} (m + 2j + 2\mu) a_{2j} |\underline{x}|^{2j} g(\underline{x})$$

- for the bivector part :

$$\begin{aligned} \underline{\partial} \underline{x} \epsilon \sum_{j=0}^{\infty} a_{2j+1} |\underline{x}|^{2j} g(\underline{x}) &= \epsilon \underline{\partial} \sum_{j=0}^{\infty} a_{2j+1} |\underline{x}|^{2j+2} g(\underline{x}) \\ &= -\underline{x}\epsilon \sum_{j=0}^{\infty} (2j + 2) a_{2j+1} |\underline{x}|^{2j} g(\underline{x}) . \end{aligned}$$

Letting the operator $\underline{\partial}\epsilon = -\epsilon\underline{\partial}$ act on $G(\underline{x})$ yields :

- for the scalar part :

$$\begin{aligned} -\epsilon\underline{\partial} \sum_{j=0}^{\infty} a_{2j} |\underline{x}|^{2j} g(\underline{x}) &= \underline{x}\epsilon \sum_{j=0}^{\infty} 2j a_{2j} |\underline{x}|^{2j-2} g(\underline{x}) \\ &= \underline{x}\epsilon \sum_{j=0}^{\infty} (2j+2) a_{2j+2} |\underline{x}|^{2j} g(\underline{x}) \end{aligned}$$

- for the bivector part :

$$\begin{aligned} -\epsilon\underline{\partial} \underline{x}\epsilon \sum_{j=0}^{\infty} a_{2j+1} |\underline{x}|^{2j} g(\underline{x}) &= (m + \mathbb{E}_r - \Gamma_{0,m}) \sum_{j=0}^{\infty} a_{2j+1} |\underline{x}|^{2j} g(\underline{x}) \\ &= \sum_{j=0}^{\infty} (m + 2j + 2\mu) a_{2j+1} |\underline{x}|^{2j} g(\underline{x}) . \end{aligned}$$

Expressing the fact that $(\underline{\partial}(\underline{x} + \epsilon) + m + 2\alpha)G(\underline{x}) = 0$ gives for each $j \in \mathbb{N}$ rise to the following set of equations :

$$\begin{cases} (2j + 2\mu - 2\alpha) a_{2j} &= (m + 2j + 2\mu) a_{2j+1} \\ (2j + 2 - m - 2\mu) a_{2j+1} &= (2j + 2) a_{2j+2} . \end{cases}$$

This enables us to recursively define the coefficients a_j , for all $j \in \mathbb{N}$:

$$\begin{aligned} a_{2j+2} &= \frac{2 + 2j - m - 2\alpha}{2j + 2} \frac{2j + 2\mu - 2\alpha}{m + 2j + 2\mu} a_{2j} \\ a_{2j+1} &= \frac{2j + 2\mu - 2\alpha}{2j} \frac{2j - m - 2\alpha}{m + 2j + 2\mu} a_{2j-1} . \end{aligned}$$

Choosing $a_0 = 1$, this eventually yields :

$$\begin{aligned} a_{2j} &= \frac{(1 - \alpha - \frac{m}{2})_j (\mu - \alpha)_j}{j! (\mu + \frac{m}{2})_j} \\ a_{2j+1} &= \frac{\mu - \alpha}{\mu + \frac{m}{2}} \frac{(1 - \alpha - \frac{m}{2})_j (1 + \mu - \alpha)_j}{j! (1 + \mu + \frac{m}{2})_j} , \end{aligned}$$

proving that

$$\begin{aligned} \sum_{j=0}^{\infty} a_{2j} |\underline{x}|^{2j} &= F\left(1 - \alpha - \frac{m}{2}, \mu - \alpha; \mu + \frac{m}{2}; |\underline{x}|^2\right) \\ \sum_{j=0}^{\infty} a_{2j+1} |\underline{x}|^{2j} &= \frac{\mu - \alpha}{\mu + \frac{m}{2}} F\left(1 - \alpha - \frac{m}{2}, 1 + \mu - \alpha; 1 + \mu + \frac{m}{2}; |\underline{x}|^2\right) . \end{aligned}$$

Note that these hypergeometric functions are well-defined on the future cone if $\mu + \frac{m}{2}$ does not belong to $-\mathbb{N}$. \square

Remark 1 : The hyperbolic fundamental solution again follows from this Theorem by choosing $g(\underline{x})$ to be the Cauchy kernel $E(\underline{x})$. The monogenic function on the punctured hyperbolic unit ball thus obtained is the fundamental solution $E_\alpha(T, \underline{X})$ from the previous chapter (see 2.6).

Remark 2 : In case $\alpha = -\frac{m}{2}$ we again obtain a very simple expression for the modulation factor :

$$\begin{aligned} F_{P,1}^{(\mu)}(t) &= F\left(1, \mu + \frac{m}{2}; \mu + \frac{m}{2}; t\right) = \frac{1}{1-t} \\ F_2^{(\mu)}(t) &= F\left(1, 1 + \mu + \frac{m}{2}; 1 + \mu + \frac{m}{2}; t\right) = \frac{1}{1-t}, \end{aligned}$$

such that

$$\text{Mod}_P\left(-\frac{m}{2}, \mu, \underline{x}\right) = \frac{1 + \underline{x}\epsilon}{1 - |\underline{x}|^2}.$$

As the modulation factor does not depend on μ , this means that in case $\alpha = -\frac{m}{2}$ there is a one-to-one correspondence between monogenic functions on an open subset $\Omega \subset B_m(1)$ and hyperbolic monogenic functions on Ω , considered as a subset of the Poincaré ball.

This can be explained as follows : we start from a monogenic function on the Poincaré ball in co-ordinates $\underline{x} \in B_m(1)$. In order to "lift" these co-ordinates from $B_m(1)$ to the hyperbolic unit ball H_+ , passing the parabola $\mathcal{P} \leftrightarrow 2T = 1 + |\underline{X}|^2$, it suffices to determine λ in such a way that

$$\lambda \left(\frac{1 + |\underline{x}|^2}{2}, \underline{x} \right) \in H_+ \implies \lambda = \frac{2}{1 - |\underline{x}|^2}.$$

In other words, the mapping from the Euclidean unit ball $B_m(1)$ to H_+ is nothing but the *stereographic projection* sending

$$\underline{x} \in B_m(1) \mapsto \left(\frac{1 + |\underline{x}|^2}{2}, \frac{2\underline{x}}{1 - |\underline{x}|^2} \right) \in H_+.$$

Because the operator $\underline{\partial}$ on $B_m(1)$ is the conformally invariant operator, where the invariance is to be understood in the sense that for any Moebius transformation $\psi(\underline{x})$ we get that

$$\underline{\partial}f(\underline{x}) = 0 \implies \underline{\partial}(J(\psi, \underline{x})f(\psi(\underline{x}))) = 0,$$

the one-to-one correspondence requires the operator on the hyperbolic unit ball to be the conformal operator too. As we will show in Chapter 7, this is the operator corresponding to the value $\alpha = -\frac{m}{2}$.

3.1.3 Equivalence of Both Models

In this subsection, the equivalence between Theorem 3.1 and Theorem 3.2 is proved, providing us with a geometrical interpretation for certain relations between hypergeometric functions.

Consider an arbitrary μ -homogeneous function $f(\underline{x})$ which is monogenic with respect to the Dirac operator $\underline{\partial}$ on $\mathbb{R}^{0,m}$ and defined in an open subset of the unit ball $B_m(1) \subset \mathbb{R}^m$.

We may associate two functions on the FC with $f(\underline{x})$:

- Theorem 3.1 provides us with a monogenic function $F(T, \underline{X})$ on the hyperbolic unit ball given by

$$F(T, \underline{X}) = \lambda^\alpha \text{Mod} \left(\alpha, \mu, \frac{\underline{X}}{\lambda} \right) f \left(\frac{\underline{X}}{\lambda} \right) ,$$

with $\lambda = T$.

- Theorem 3.2 provides us with a monogenic function $G(T, \underline{X})$ on the hyperbolic unit ball given by

$$G(T, \underline{X}) = \lambda_P^\alpha \text{Mod}_P \left(\alpha, \mu, \frac{\underline{X}}{\lambda_P} \right) f \left(\frac{\underline{X}}{\lambda_P} \right) ,$$

with $\lambda_P = T + (T^2 - |\underline{X}|^2)^{\frac{1}{2}}$.

We will now prove the following :

$$F(T, \underline{X}) = 2^{\mu-\alpha} G(T, \underline{X}) .$$

For that purpose we rewrite $G(T, \underline{X})$ in terms of the co-ordinates (λ, \underline{x}) on FC , with $\underline{x} = \frac{\underline{X}}{T}$ and $\lambda = T$:

$$G(T, \underline{X}) = \lambda^\alpha (1 + (1 - |\underline{x}|^2)^{\frac{1}{2}})^{\alpha-\mu} \text{Mod}_P \left(\alpha, \mu, \frac{\underline{x}}{1 + (1 - |\underline{x}|^2)^{\frac{1}{2}}} \right) f(\underline{x}) .$$

It then remains to prove that

$$\text{Mod}(\alpha, \mu, \underline{x}) = 2^{\mu-\alpha} (1 + (1 - |\underline{x}|^2)^{\frac{1}{2}})^{\alpha-\mu} \text{Mod}_P \left(\alpha, \mu, \frac{\underline{x}}{1 + (1 - |\underline{x}|^2)^{\frac{1}{2}}} \right),$$

which by means of the definition for the occurring modulation factors reduces to :

$$\begin{aligned} & 2^{\alpha-\mu} (1 + (1 - |\underline{x}|^2)^{\frac{1}{2}})^{\mu-\alpha} F \left(\frac{1 + \mu - \alpha}{2}, \frac{\mu - \alpha}{2}; \mu + \frac{m}{2}; |\underline{x}|^2 \right) \\ = & F \left(1 - \alpha - \frac{m}{2}, \mu - \alpha; \mu + \frac{m}{2}; \frac{|\underline{x}|^2}{(1 + (1 - |\underline{x}|^2)^{\frac{1}{2}})^2} \right) \end{aligned}$$

and

$$\begin{aligned} & 2^{\alpha-\mu} (1 + (1 - |\underline{x}|^2)^{\frac{1}{2}})^{\mu-\alpha} F \left(\frac{1 + \mu - \alpha}{2}, 1 + \frac{\mu - \alpha}{2}; 1 + \mu + \frac{m}{2}; |\underline{x}|^2 \right) \\ = & F \left(1 - \alpha - \frac{m}{2}, 1 + \mu - \alpha; 1 + \mu + \frac{m}{2}; \frac{|\underline{x}|^2}{(1 + (1 - |\underline{x}|^2)^{\frac{1}{2}})^2} \right). \end{aligned}$$

We will prove the first equality, the latter is quite similar. To prove this equality, a formula from Goursat's table of quadratic transformations and one of Kummer's relations for the hypergeometric function are necessary :

$$\begin{aligned} F(a, b; a - b + 1; z) &= \left(\frac{1+z}{1-z} \right)^{2b-1} (1+z)^{-a} \times \\ & F \left(\frac{a+1}{2} - b, 1 + \frac{a}{2} - b; a - b + 1; \frac{4z}{(1+z)^2} \right) \end{aligned}$$

and

$$F(a, b; c; z) = (1-z)^{c-a-b} F(c-a, c-b; c; z).$$

In some sense these formulae will thus be given a geometrical meaning, illustrating the subtle connection between special functions on the one hand and the geometric model on the other hand.

Putting

$$z = \frac{|\underline{x}|^2}{(1 + (1 - |\underline{x}|^2)^{\frac{1}{2}})^2},$$

and choosing $a = \mu - \alpha$ and $b = 1 - \alpha - \frac{m}{2}$, we get :

$$F(a, b; a - b + 1; z) = 2^{\alpha - \mu} (1 + (1 - |\underline{x}|^2)^{\frac{1}{2}})^{\mu - \alpha} (1 - |\underline{x}|^2)^{\alpha + \frac{m-1}{2}} \times \\ F\left(\frac{k + \alpha + m - 1}{2}, \frac{k + \alpha + m}{2}; k + \frac{m}{2}; |\underline{x}|^2\right),$$

which by means of Kummer's relation reduces to

$$F(a, b; a - b + 1; z) = 2^{\alpha - \mu} (1 + (1 - |\underline{x}|^2)^{\frac{1}{2}})^{\mu - \alpha} \times \\ F\left(\frac{1 + k - \alpha}{2}, \frac{k - \alpha}{2}; k + \frac{m}{2}; |\underline{x}|^2\right),$$

yielding

$$2^{\alpha - \mu} (1 + (1 - |\underline{x}|^2)^{\frac{1}{2}})^{\mu - \alpha} F\left(\frac{1 + \mu - \alpha}{2}, \frac{\mu - \alpha}{2}; \mu + \frac{m}{2}; |\underline{x}|^2\right) \\ = F\left(1 - \alpha - \frac{m}{2}, \mu - \alpha; \mu + \frac{m}{2}; \frac{|\underline{x}|^2}{(1 + (1 - |\underline{x}|^2)^{\frac{1}{2}})^2}\right).$$

This proves the equivalence between the Modulation Theorem in both the Klein and the Poincaré model. \square

Remark : We already encountered a manifestation of this equivalence when we studied the modulation factor for $\alpha = -\frac{m}{2}$. Indeed, recalling the fact that for this value we get in the Klein model

$$\text{Mod}\left(-\frac{m}{2}, \mu, \underline{x}\right) = \frac{(1 + (1 - |\underline{x}|^2)^{\frac{1}{2}})^{1 - \mu - \frac{m}{2}}}{2^{1 - \mu - \frac{m}{2}} (1 - |\underline{x}|^2)^{\frac{1}{2}}} \left(1 + \frac{\underline{x}\epsilon}{1 + (1 - |\underline{x}|^2)^{\frac{1}{2}}}\right)$$

and in the Poincaré model

$$\text{Mod}_P\left(-\frac{m}{2}, \mu, \underline{x}\right) = \frac{1 + \underline{x}\epsilon}{1 - |\underline{x}|^2},$$

a simple computation indeed shows that

$$\frac{(1 + (1 - |\underline{x}|^2)^{\frac{1}{2}})^{-\mu - \frac{m}{2}}}{2^{-\mu - \frac{m}{2}}} \text{Mod}_P\left(-\frac{m}{2}, \mu, \frac{\underline{x}}{1 + (1 - |\underline{x}|^2)^{\frac{1}{2}}}\right) = \text{Mod}\left(-\frac{m}{2}, \mu, \underline{x}\right).$$

3.2 Generalization to Natural Powers

The aim of this section is to generalize the Modulation Theorems from the previous section to *natural powers* of the Dirac operator on the hyperbolic unit ball. The motivation for this is to have a hyperbolic version of the classical Fischer building blocks. As we have seen in section 0.1.2, each homogeneous $\mathbb{R}_{0,m}$ -valued polynomial on \mathbb{R}^m has a unique orthogonal decomposition of the form :

$$R_k(\underline{x}) = \sum_{j=0}^k \underline{x}^j P_{k-j}(\underline{x}) , \quad \underline{x} \in \mathbb{R}^m ,$$

with $P_{k-j}(\underline{x}) \in M^+(k-j)$ an inner spherical monogenic of degree $k-j$. The functions $\underline{x}^j P_{k-j}(\underline{x})$ are the so-called *Fischer building blocks*, simultaneous eigenfunctions for \mathbb{E}_r and $\Gamma_{0,m}$. The following property of these building blocks is crucial :

$$\underline{\partial}^{s+1}(\underline{x}^s P_k(\underline{x})) = 0 .$$

Since we have proved that it is possible to construct monogenic functions on the hyperbolic unit ball by means of monogenic functions $P_k(\underline{x})$, it seems natural to look for a Theorem that enables us to do the same with the Fischer building blocks $\underline{x}^s P_k(\underline{x})$ with regards to higher powers of the Dirac operator on the hyperbolic unit ball. In view of the projective nature of our model for the hyperbolic unit ball, we are thus looking for α -homogeneous solutions for natural powers of the Dirac operator $(\epsilon \partial_T - \partial_{\underline{X}})$ on $\mathbb{R}^{1,m}$ that can be interpreted as modulated versions of the classical Fischer building blocks. In order to construct these functions we will again work with the Klein model and project the equations on the hyperplane Π . Note that it is sufficient to work with the Klein model only, since we have proved the equivalence with the Poincaré model in the previous section. Formulae for the Poincaré model may easily be derived from the formulae for the Klein model, using transformation properties of the hypergeometric function.

We already know that the projection of the hyperbolic Dirac operator on the hyperplane Π gives rise to the operator

$$-\frac{1}{\lambda} \left(\underline{\partial} + \epsilon [\mathbb{E}_r - \mathbb{E}_\lambda] \right)$$

acting on homogeneous functions of the form $\lambda^\alpha f(\underline{x})$, $\underline{x} \in B_m(1)$. Throughout this section we will adopt the following notation :

$$D_\alpha(\underline{x}) = \underline{\partial} + \epsilon (\mathbb{E}_r - \alpha) ,$$

for all $\alpha \in \mathbb{C}$. The projection of the k -iterated hyperbolic Dirac equation, i.e. the equation for the operator $(\epsilon\partial_T - \partial_{\underline{X}})^k$ acting on α -homogeneous functions, then becomes :

$$D_{\alpha-(k-1)}(\underline{x})D_{\alpha-(k-2)}(\underline{x}) \cdots D_{\alpha-1}(\underline{x})D_{\alpha}(\underline{x})f(\underline{x}) = 0 ,$$

with $\underline{x} \in B_m(1)$. The projected k -iterated hyperbolic Dirac operator will be denoted by

$$D_{\alpha}^k(\underline{x}) = D_{\alpha-(k-1)}(\underline{x})D_{\alpha-(k-2)}(\underline{x}) \cdots D_{\alpha-1}(\underline{x})D_{\alpha}(\underline{x}) ,$$

with $D_{\alpha}^1(\underline{x}) = D_{\alpha}(\underline{x})$. This operator satisfies the following properties :

$$\begin{aligned} D_{\alpha}^k(\underline{x}) &= D_{\alpha-1}^{k-1}(\underline{x})D_{\alpha}(\underline{x}) \\ D_{\alpha}^k(\underline{x}) &= D_{\alpha+1-k}(\underline{x})D_{\alpha}^{k-1}(\underline{x}) . \end{aligned}$$

The aim is to construct solutions for the equation

$$D_{\alpha}^{s+1}(\underline{x})f(\underline{x}) = 0$$

of the form

$$f(\underline{x}) = \text{Mod}(\alpha, k; s; \underline{x})\underline{x}^s P_k(\underline{x}) .$$

For $s = 0$, Theorem 3.1 offers a solution :

$$D_{\alpha}(\underline{x})[\text{Mod}(\alpha, k; 0; \underline{x})P_k(\underline{x})] = 0 \text{ for all } P_k(\underline{x}) \in M^+(k) .$$

Note that we have introduced an extra parameter s in comparison with the previous section. If this parameter s is omitted, we will implicitly assume that $s = 0$.

Theorem 3.3 *Consider an inner spherical monogenic $P_k(\underline{x}) \in M^+(k)$, a complex number $\alpha \in \mathbb{C}$ and an arbitrary integer $s \in \mathbb{N}$. We then have :*

$$D_{\alpha}^{s+1}(\underline{x})[\text{Mod}(\alpha, k; s; \underline{x})\underline{x}^s P_k(\underline{x})] = 0 ,$$

where the modulation factors are given by

$$\begin{aligned} \text{Mod}(\alpha, k; 2s; \underline{x}) &= \frac{1}{4^s s!} \frac{\Gamma(k + \frac{m}{2})}{\Gamma(k + s + \frac{m}{2})} \text{Mod}(\alpha - s, k + s; 0; \underline{x}) \\ \text{Mod}(\alpha, k; 2s + 1; \underline{x}) &= \frac{1}{4^s s!} \frac{\Gamma(k + \frac{m}{2} + 1)}{\Gamma(k + s + \frac{m}{2} + 1)} \text{Mod}(\alpha - s, k + s; 1; \underline{x}) \end{aligned}$$

with $\text{Mod}(\alpha, k; 0; \underline{x})$ given by Theorem 3.1 and

$$\text{Mod}(\alpha, k; 1; \underline{x}) = -\frac{1}{2k + m} F\left(\frac{1 + k - \alpha}{2}, 1 + \frac{k - \alpha}{2}; 1 + k + \frac{m}{2}; |\underline{x}|^2\right) .$$

In order to prove this Theorem, two technical Lemmata will be proved first :

Lemma 3.1 *Let $P_k(\underline{x}) \in M^+(k)$ be an inner spherical monogenic, let α be an arbitrary complex number and let $s \in \mathbb{N}$ be an arbitrary integer. We then have :*

$$D_\alpha(\underline{x})[\text{Mod}(\alpha, k; 2s; \underline{x})\underline{x}^{2s}P_k(\underline{x})] = \text{Mod}(\alpha - 1, k; 2s - 1; \underline{x})\underline{x}^{2s-1}P_k(\underline{x}) .$$

Lemma 3.2 *Let $P_k(\underline{x}) \in M^+(k)$ be an inner spherical monogenic, let α be an arbitrary complex number and let $s \in \mathbb{N}$ be an arbitrary integer. We then have :*

$$D_\alpha(\underline{x})[\text{Mod}(\alpha, k; 2s - 1; \underline{x})\underline{x}^{2s-1}P_k(\underline{x})] = \text{Mod}(\alpha - 1, k; 2s - 2; \underline{x})\underline{x}^{2s-2}P_k(\underline{x}) .$$

Proof of Lemma 3.1 :

Using the definitions for the modulation factors, the Lemma may be rewritten as follows :

$$\begin{aligned} & \frac{1}{4^s s!} \frac{\Gamma(k + \frac{m}{2})}{\Gamma(k + s + \frac{m}{2})} D_\alpha(\underline{x})[\text{Mod}(\alpha - s, k + s; 0; \underline{x})\underline{x}^{2s}P_k(\underline{x})] \\ &= \frac{1}{4^{s-1}(s-1)!} \frac{\Gamma(k + \frac{m}{2} + 1)}{\Gamma(k + s + \frac{m}{2})} \text{Mod}(\alpha - s, k + s - 1; 1; \underline{x})\underline{x}^{2s-1}P_k(\underline{x}) \end{aligned}$$

or equivalently :

$$\begin{aligned} & D_\alpha(\underline{x})[\text{Mod}(\alpha - s, k + s; 0; \underline{x})\underline{x}^{2s}P_k(\underline{x})] \\ &= -2sF\left(s + \frac{k - \alpha}{2}, s + \frac{1 + k - \alpha}{2}; s + k + \frac{m}{2}; |\underline{x}|^2\right) . \end{aligned}$$

The modulation factor $\text{Mod}(\alpha - s, k + s; 0; \underline{x})$ is bivector-valued. The action of the operator $D_\alpha(\underline{x})$ on this modulation factor thus consists of two parts :

- $D_\alpha(\underline{x})$ acting on the *scalar* part of the modulation factor :

$$D_\alpha(\underline{x})F_S(|\underline{x}|^2)\underline{x}^{2s}P_k(\underline{x})$$

where we have put

$$F_S(|\underline{x}|^2) = F\left(s + \frac{k - \alpha}{2}, s + \frac{1 + k - \alpha}{2}; s + k + \frac{m}{2}; |\underline{x}|^2\right) .$$

- $D_\alpha(\underline{x})$ acting on the *bivector-valued* part of the modulation factor :

$$\frac{s + \frac{k-\alpha}{2}}{s + k + \frac{m}{2}} D_\alpha(\underline{x}) \underline{x} \epsilon F_B(|\underline{x}|^2) \underline{x}^{2s} P_k(\underline{x}) ,$$

where we have put

$$F_B(|\underline{x}|^2) = F\left(1 + s + \frac{k-\alpha}{2}, s + \frac{1+k-\alpha}{2}; 1 + s + k + \frac{m}{2}; |\underline{x}|^2\right)$$

If $F(z)$ stands for the hypergeometric function $F(a, b; c; z)$, we use the shorthand notation $F^+(z)$ for the hypergeometric function $F(a+1, b+1; c+1; z)$.

We then get :

$$\begin{aligned} D_\alpha(\underline{x}) F_S(|\underline{x}|^2) \underline{x}^{2s} P_k(\underline{x}) &= (\Gamma_{0,m} - \mathbb{E}_r - m) F_S(|\underline{x}|^2) \underline{x}^{2s-1} P_k(\underline{x}) \\ &\quad + \epsilon(2s + k - \alpha) F_S(|\underline{x}|^2) \underline{x}^{2s} P_k(\underline{x}) \\ &\quad + 2\epsilon |\underline{x}|^2 \frac{(s + \frac{k-\alpha}{2})(s + \frac{1+k-\alpha}{2})}{(s + k + \frac{m}{2})} F_S^+(|\underline{x}|^2) \underline{x}^{2s} P_k(\underline{x}) \\ &= -2s F_S(|\underline{x}|^2) \underline{x}^{2s} P_k(\underline{x}) - 2\underline{x} \epsilon \left(s + \frac{k-\alpha}{2}\right) F_S(|\underline{x}|^2) \underline{x}^{2s-1} P_k(\underline{x}) \\ &\quad - 2|\underline{x}|^2 \frac{(s + \frac{k-\alpha}{2})(s + \frac{1+k-\alpha}{2})}{(s + k + \frac{m}{2})} F_S^+(|\underline{x}|^2) \underline{x}^{2s-1} P_k(\underline{x}) \\ &\quad - 2\underline{x} \epsilon |\underline{x}|^2 \frac{(s + \frac{k-\alpha}{2})(s + \frac{1+k-\alpha}{2})}{(s + k + \frac{m}{2})} F_S^+(|\underline{x}|^2) \underline{x}^{2s-1} P_k(\underline{x}) , \end{aligned}$$

and

$$\begin{aligned} D_\alpha(\underline{x}) \underline{x} \epsilon F_B(|\underline{x}|^2) \underline{x}^{2s} P_k(\underline{x}) &= \epsilon(\Gamma_{0,m} - \mathbb{E}_r - m) F_B(|\underline{x}|^2) \underline{x}^{2s} P_k(\underline{x}) - (\mathbb{E}_r - \alpha) F_B(|\underline{x}|^2) \underline{x}^{2s+1} P_k(\underline{x}) \\ &= 2\left(s + k + \frac{m}{2}\right) \underline{x} \epsilon F_B(|\underline{x}|^2) \underline{x}^{2s-1} P_k(\underline{x}) \\ &\quad + 2\left(s + \frac{1+k-\alpha}{2}\right) |\underline{x}|^2 F_B(|\underline{x}|^2) \underline{x}^{2s-1} P_k(\underline{x}) \\ &\quad + 2\underline{x} \epsilon |\underline{x}|^2 \frac{(1 + s + \frac{k-\alpha}{2})(s + \frac{1+k-\alpha}{2})}{(1 + s + k + \frac{m}{2})} F_B^+(|\underline{x}|^2) \underline{x}^{2s-1} P_k(\underline{x}) \\ &\quad + 2\frac{(1 + s + \frac{k-\alpha}{2})(s + \frac{1+k-\alpha}{2})}{(1 + s + k + \frac{m}{2})} F_B^+(|\underline{x}|^2) \underline{x}^{2s-1} P_k(\underline{x}) . \end{aligned}$$

This gives terms in $\underline{x}^{2s-1}P_k(\underline{x})$ and terms in $\underline{x}\epsilon\underline{x}^{2s-1}P_k(\underline{x})$. First we gather all terms in $\underline{x}^{2s-1}P_k(\underline{x})$:

$$\begin{aligned} & -2sF_S(|\underline{x}|^2) + 2\frac{\left(s + \frac{k-\alpha}{2}\right)\left(s + \frac{1+k-\alpha}{2}\right)\left(1 + s + \frac{k-\alpha}{2}\right)}{\left(s + k + \frac{m}{2}\right)\left(1 + s + k + \frac{m}{2}\right)}|\underline{x}|^4F_B^+(|\underline{x}|^2) \\ & + 2\frac{\left(s + \frac{k-\alpha}{2}\right)\left(s + \frac{1+k-\alpha}{2}\right)}{\left(s + k + \frac{m}{2}\right)}|\underline{x}|^2\left(F_B(|\underline{x}|^2) - F_S^+(|\underline{x}|^2)\right). \end{aligned}$$

Recalling the definitions for the functions $F_S(|\underline{x}|^2)$ and $F_B(|\underline{x}|^2)$, and writing the previous sum as a single power series, we get :

$$-2sF\left(s + \frac{k-\alpha}{2}, s + \frac{1+k-\alpha}{2}; s + k + \frac{m}{2}; |\underline{x}|^2\right)\underline{x}^{2s-1}P_k(\underline{x}).$$

Next, we gather all terms in $\underline{x}\epsilon\underline{x}^{2s-1}P_k(\underline{x})$:

$$\begin{aligned} & 2\left(s + \frac{k-\alpha}{2}\right)\left(F_B(|\underline{x}|^2) - F_S(|\underline{x}|^2)\right) \\ & - 2\frac{\left(s + \frac{k-\alpha}{2}\right)\left(s + \frac{1+k-\alpha}{2}\right)}{\left(s + k + \frac{m}{2}\right)}|\underline{x}|^2F_S^+(|\underline{x}|^2) \\ & + 2\frac{\left(s + \frac{k-\alpha}{2}\right)\left(s + \frac{1+k-\alpha}{2}\right)\left(1 + s + \frac{k-\alpha}{2}\right)}{\left(s + k + \frac{m}{2}\right)\left(1 + s + k + \frac{m}{2}\right)}|\underline{x}|^2F_B^+(|\underline{x}|^2). \end{aligned}$$

Writing this as a single power series yields zero, so that we finally have :

$$\begin{aligned} & D_\alpha(\underline{x})[\text{Mod}(\alpha - s, k + s; 0; \underline{x})\underline{x}^{2s}P_k(\underline{x})] \\ & = -2sF\left(s + \frac{k-\alpha}{2}, s + \frac{1+k-\alpha}{2}; s + k + \frac{m}{2}; |\underline{x}|^2\right), \end{aligned}$$

which proves Lemma 3.1. □

Proof of Lemma 3.2 :

Using the definitions for the modulation factors, the Lemma may be rewritten as follows :

$$\begin{aligned} & \frac{1}{4^s s!} \frac{\Gamma\left(k + \frac{m}{2} + 1\right)}{\Gamma\left(k + s + \frac{m}{2} + 1\right)} D_\alpha(\underline{x})[\text{Mod}(\alpha - s, k + s; 1; \underline{x})\underline{x}^{2s+1}P_k(\underline{x})] \\ & = \frac{1}{4^s s!} \frac{\Gamma\left(k + \frac{m}{2}\right)}{\Gamma\left(k + s + \frac{m}{2}\right)} \text{Mod}(\alpha - s - 1, k + s; 0; \underline{x})\underline{x}^{2s}P_k(\underline{x}) \end{aligned}$$

or equivalently :

$$D_\alpha(\underline{x}) [F(|\underline{x}|^2) \underline{x}^{2s+1} P_k(\underline{x})] = -2 \left(s + k + \frac{m}{2} \right) \text{Mod}(\alpha - s - 1, k + s; 0; \underline{x}) ,$$

where we have put

$$F(|\underline{x}|^2) = F \left(1 + s + \frac{k - \alpha}{2}, s + \frac{1 + k - \alpha}{2}; 1 + s + k + \frac{m}{2}; |\underline{x}|^2 \right) .$$

We get immediately :

$$\begin{aligned} D_\alpha(\underline{x}) F(|\underline{x}|^2) \underline{x}^{2s+1} P_k(\underline{x}) &= (\Gamma_{0,m} - \mathbb{E}_r - m) F(|\underline{x}|^2) \underline{x}^{2s} P_k(\underline{x}) + \epsilon(1 + 2s + k - \alpha) F(|\underline{x}|^2) \underline{x}^{2s+1} P_k(\underline{x}) \\ &\quad + 2\epsilon \frac{\left(1 + s + \frac{k-\alpha}{2}\right) \left(s + \frac{1+k-\alpha}{2}\right)}{\left(1 + s + k + \frac{m}{2}\right)} |\underline{x}|^2 F^+ (|\underline{x}|^2) \underline{x}^{2s+1} P_k(\underline{x}) \\ &= -2 \left(k + s + \frac{m}{2} \right) F(|\underline{x}|^2) \underline{x}^{2s} P_k(\underline{x}) - 2\underline{x}\epsilon \left(s + \frac{1 + k - \alpha}{2} \right) F(|\underline{x}|^2) \underline{x}^{2s} P_k(\underline{x}) \\ &\quad - 2 \frac{\left(1 + s + \frac{k-\alpha}{2}\right) \left(s + \frac{1+k-\alpha}{2}\right)}{\left(1 + s + k + \frac{m}{2}\right)} |\underline{x}|^2 F^+ (|\underline{x}|^2) \underline{x}^{2s} P_k(\underline{x}) \\ &\quad - 2\underline{x}\epsilon \frac{\left(1 + s + \frac{k-\alpha}{2}\right) \left(s + \frac{1+k-\alpha}{2}\right)}{\left(1 + s + k + \frac{m}{2}\right)} |\underline{x}|^2 F^+ (|\underline{x}|^2) \underline{x}^{2s} P_k(\underline{x}) . \end{aligned}$$

Gathering the scalar terms on the one hand and the terms in $\underline{x}\epsilon$ on the other hand, and making use of elementary properties of the hypergeometric function, we finally arrive at :

$$-2 \left(s + k + \frac{m}{2} \right) \text{Mod}(\alpha - s - 1, k + s; 0; \underline{x}) ,$$

which proves Lemma 3.2. □

We then prove Theorem 3.3, hereby making use of Lemma 3.1 and Lemma 3.2. Consider an arbitrary complex α and an arbitrary integer s . We have :

$$\begin{aligned} D_\alpha^{s+1}(\underline{x}) [\text{Mod}(\alpha, k; s; \underline{x}) \underline{x}^s P_k(\underline{x})] &= D_{\alpha-1}^s(\underline{x}) [\text{Mod}(\alpha - 1, k; s - 1; \underline{x}) \underline{x}^{s-1} P_k(\underline{x})] \\ &= \dots \\ &= D_{\alpha-s}(\underline{x}) [\text{Mod}(\alpha - s, k; 0; \underline{x}) P_k(\underline{x})] \\ &= 0 . \end{aligned}$$

This proves the Theorem. □

3.3 The Ultra-Modulation Theorem

The aim of this section is to generalize the Modulation Theorem to the ultra-hyperbolic setting, i.e. to the orthogonal space $\mathbb{R}^{p,q}$ endowed with the quadratic form

$$Q_{p,q}(\underline{T}, \underline{X}) = \sum_{i=1}^p T_i^2 - \sum_{j=1}^q X_j^2 .$$

For that purpose we first introduce the following essential subsets of $\mathbb{R}^{p,q}$:

1. The (p, q) -time-like region $TLR_{p,q} = \{(\underline{T}, \underline{X}) \in \mathbb{R}^{p,q} : Q_{p,q}(\underline{T}, \underline{X}) > 0\}$
2. The (p, q) -space-like region $SLR_{p,q} = \{(\underline{T}, \underline{X}) \in \mathbb{R}^{p,q} : Q_{p,q}(\underline{T}, \underline{X}) < 0\}$
3. The (p, q) -nullcone $NC_{p,q} = \{(\underline{T}, \underline{X}) \in \mathbb{R}^{p,q} : Q_{p,q}(\underline{T}, \underline{X}) = 0\}$

In case the signature (p, q) is equal to $(1, m)$, we are dealing with the space-time situation and these definitions then reduce to the ones already given in section 1.2.

Both the $TLR_{p,q}$ and the $SLR_{p,q}$ contain a so-called *ultrahyperbolic unit ball*, defined as follows :

$$\begin{aligned} B_T(p, q) &= \{(\underline{T}, \underline{X}) \in TLR_{p,q} : Q_{p,q}(\underline{T}, \underline{X}) = 1\} \\ B_S(p, q) &= \{(\underline{T}, \underline{X}) \in SLR_{p,q} : Q_{p,q}(\underline{T}, \underline{X}) = -1\} . \end{aligned}$$

Together with $NC_{p,q}$ these are canonical $SO(p, q)$ -invariant surfaces. Note that $B_T(p, q)$ and $B_S(p, q)$ contain respectively the elements ϵ_i and e_j of the orthonormal basis $B_{p,q}(\epsilon_i, e_j) = \{\epsilon_1, \dots, \epsilon_p, e_1, \dots, e_q\}$ for $\mathbb{R}^{p,q}$.

On the analogy of the definition for the Dirac operator on the hyperbolic unit ball H_+ in the m -dimensional space-time $\mathbb{R}^{1,m}$, a Dirac operator on the ultrahyperbolic unit balls $B_T(p, q)$ and $B_S(p, q)$ is defined and nullsolutions for this operator are constructed. To do so, we first introduce a projective model for $B_T(p, q)$ and $B_S(p, q)$:

- the manifold of rays $\text{Ray}(TLR_{p,q})$ in the (p, q) -time-like region, given by

$$\text{Ray}(TLR_{p,q}) = \{ \{ \lambda(\underline{T}, \underline{X}) : \lambda \in \mathbb{R}_0^+ \} : (\underline{T}, \underline{X}) \in TLR_{p,q} \} ,$$

yields a projective model for the ultrahyperbolic unit ball $B_T(p, q)$

- the manifold of rays $\text{Ray}(SLR_{p,q})$ in the (p, q) -space-like region, given by

$$\text{Ray}(SLR_{p,q}) = \{ \{ \lambda(\underline{T}, \underline{X}) : \lambda \in \mathbb{R}_0^+ \} : (\underline{T}, \underline{X}) \in SLR_{p,q} \} ,$$

yields a projective model for the ultrahyperbolic unit ball $B_S(p, q)$.

As for the space-time situation, the projective model for $B_T(p, q)$ (resp. $B_S(p, q)$) forces us to define the Dirac operator on these ultrahyperbolic unit balls as the Dirac operator $D(\underline{T}, \underline{X})_{p,q}$ on the orthogonal space $\mathbb{R}^{p,q}$ acting on α -homogeneous $\mathbb{R}_{p,q}$ -valued functions on the $TLR_{p,q}$ (resp. $SLR_{p,q}$).

Remark : The manifolds of rays $\text{Ray}(TLR_{p,q})$ and $\text{Ray}(SLR_{p,q})$ can also be defined as principal G -bundles, with $G = \mathbb{R}_0^+$, and the Dirac operator on these manifolds can then rigorously be defined as the Dirac operator $D(\underline{T}, \underline{X})_{p,q}$ acting on sections of an associated principal fibre bundle, see Chapter 2 for this construction.

This means that the *ultrahyperbolic Dirac operator* acts on sections of the *homogeneous Clifford line bundle*

$$\mathbb{R}_{p,q;\alpha} = \mathbb{R}_0^{p,q} \times \mathbb{R}_{p,q} / \sim$$

where the equivalence relation \sim is given by $((\underline{T}, \underline{X}), c) \sim (\lambda(\underline{T}, \underline{X}), \lambda^\alpha c)$. In other words : the bundle space of the bundle $\mathbb{R}_{p,q;\alpha}$ is the set $\mathbb{R}_0^{p,q} \times \mathbb{R}_{p,q}$ and the base space consists of the equivalence classes under the projection π , with

$$\begin{aligned} \pi((\underline{T}_1, \underline{X}_1), a_1) = \pi((\underline{T}_2, \underline{X}_2), a_2) &\iff ((\underline{T}_1, \underline{X}_1), a_1) \sim ((\underline{T}_2, \underline{X}_2), a_2) \\ &\iff ((\underline{T}_1, \underline{X}_1), a_1) = (\lambda(\underline{T}_2, \underline{X}_2), \lambda^\alpha a_2) \end{aligned}$$

for a certain $\lambda \in \mathbb{R}_+^0$. The aim of this section is to construct nullsolutions for this ultrahyperbolic Dirac operator, by means of a recursive argument.

First of all we need the ultrahyperbolic version of the polar decomposition of (p, q) -space-time vectors and of the Dirac operator $D(\underline{T}, \underline{X})_{p,q}$ on the space $\mathbb{R}^{p,q}$. Consider therefore a (p, q) -vector $(\underline{T}, \underline{X}) \in TLR_{p,q}$, and write it in the following form :

$$\begin{aligned} (\underline{T}, \underline{X}) &= Q_{p,q}(\underline{T}, \underline{X})^{\frac{1}{2}} \left(\frac{\underline{T}}{Q_{p,q}(\underline{T}, \underline{X})^{\frac{1}{2}}}, \frac{\underline{X}}{Q_{p,q}(\underline{T}, \underline{X})^{\frac{1}{2}}} \right) \\ &= Q_{p,q}(\underline{T}, \underline{X})^{\frac{1}{2}}(\underline{\tau}, \underline{\xi}) . \end{aligned} \tag{3.1}$$

Expression (3.1) represents the (p, q) -time-like vector $(\underline{T}, \underline{X}) \in TLR_{p,q}$ as a unit vector $(\underline{\tau}, \underline{\xi})$ belonging to $B_T(p, q)$ multiplied with its ultrahyperbolic norm $Q_{p,q}(\underline{T}, \underline{X})^{\frac{1}{2}}$, i.e. its distance to the origin in $\mathbb{R}^{p,q}$ with respect to the metric $ds^2 = Q_{p,q}(d\underline{T}, d\underline{X})$. One possible ultrahyperbolic decomposition for the Dirac operator on $\mathbb{R}^{p,q}$ is then given by

$$D(\underline{T}, \underline{X})_{p,q} = \frac{(\underline{\tau}, \underline{\xi})}{Q_{p,q}(\underline{T}, \underline{X})^{\frac{1}{2}}} (\mathbb{E}_{p,q} + \Gamma_{p,q}) , \quad (3.2)$$

with $\mathbb{E}_{p,q}$ the Euler operator on $\mathbb{R}^{p,q}$, measuring the degree of homogeneity with respect to the (p, q) -space-time co-ordinates $(\underline{T}, \underline{X})$ on $\mathbb{R}^{p,q}$ and with $\Gamma_{p,q}$ the ultrahyperbolic angular operator on $\mathbb{R}^{p,q}$, tangent to $B_T(p, q)$; i.e. when acting on functions depending on the ultrahyperbolic norm only, this operator is identically zero.

For space-like (p, q) -vectors $(\underline{T}, \underline{X}) \in SLR_{p,q}$, the factor $Q_{p,q}(\underline{T}, \underline{X})^{\frac{1}{2}}$ belongs to $i\mathbb{R}$ and must be replaced by $(-Q_{p,q}(\underline{T}, \underline{X}))^{\frac{1}{2}}$. The distance to the origin in $\mathbb{R}^{p,q}$ is then given by $ds^2 = -Q_{p,q}(d\underline{T}, d\underline{X})$

In both cases the (p, q) -space-time vector $(\underline{T}, \underline{X})$ and the Dirac operator $D(\underline{T}, \underline{X})_{p,q}$ are written in terms of a unit vector, the ultrahyperbolic norm and the Euler and Gamma operators.

In terms of the co-ordinates $(\underline{T}, \underline{X})$ these operators are given by :

$$\mathbb{E}_{p,q} = \sum_{i=1}^p T_i \partial_{T_i} + \sum_{j=1}^q X_j \partial_{X_j}$$

and

$$\begin{aligned} \Gamma_{p,q} &= \sum_{j < k} \epsilon_j \epsilon_k (T_j \partial_{T_k} - T_k \partial_{T_j}) - \sum_{j < k} e_j e_k (X_j \partial_{X_k} - X_k \partial_{X_j}) \\ &\quad - \sum_{j,k} \epsilon_j e_k (T_j \partial_{X_k} + X_k \partial_{T_j}) . \end{aligned}$$

This can easily be verified by calculating $(\underline{T}, \underline{X})D(\underline{T}, \underline{X})_{p,q}$ by means of the multiplications rules on the Clifford algebra $\mathbb{R}_{p,q}$ and by collecting the scalar terms (giving $\mathbb{E}_{p,q}$) and the bivector-terms (giving $\Gamma_{p,q}$). Note that there is *no* decomposition given on the nullcone. For the definition of the Dirac operator on the nullcone we refer to the work of Sommen, see [66]. We briefly return to this point in Chapter 7.

Using the explicit expressions for $\mathbb{E}_{p,q}$ and $\Gamma_{p,q}$, one can easily verify the following operator equality :

$$D(\underline{T}, \underline{X})_{p,q}(\underline{T}, \underline{X}) = (p + q - 1) + \mathbb{E}_{p,q} - \Gamma_{p,q} .$$

We now consider a (p, q) -vector $(\underline{T}, \underline{X})$ and we construct an α -homogeneous solution for the Dirac operator on $\mathbb{R}^{p,q}$ which is defined in a neighbourhood of $(\underline{T}, \underline{X})$. In view of the fact that our model for the ultrahyperbolic unit balls $B_T(p, q)$ and $B_S(p, q)$ is projective, a neighbourhood of $(\underline{T}, \underline{X})$ is to be understood as a small cone, consisting of all half rays in a neighbourhood of the half ray connecting $(\underline{T}, \underline{X})$ with the origin.

Depending on whether $(\underline{T}, \underline{X}) \in TLR_{p,q}$ or $SLR_{p,q}$ this construction yields by definition a monogenic function for the ultrahyperbolic Dirac operator on $B_T(p, q)$ or $B_S(p, q)$. If $(\underline{T}, \underline{X})$ belongs to the nullcone $NC_{p,q}$ in $\mathbb{R}^{p,q}$, a neighbourhood of $(\underline{T}, \underline{X})$ contains both rays in $TLR_{p,q}$ and $SLR_{p,q}$. There exist α -homogeneous distributional solutions for the Dirac operator $D(\underline{T}, \underline{X})_{p,q}$, defined in a neighbourhood of a nullray on $NC_{p,q}$, but here we only consider (p, q) -vectors for which there exists a neighbourhood entirely consisting of (p, q) -time-like or (p, q) -space-like half rays.

3.3.1 The (p, q) -space-like situation

If $(\underline{T}, \underline{X}) \in SLR_{p,q}$, we have by definition :

$$Q_{p,q}(\underline{T}, \underline{X}) = \sum_{i=1}^p T_i^2 - \sum_{j=1}^q X_j^2 < 0 .$$

This means that there is at least one spatial co-ordinate $X_j \neq 0$, for if $X_j = 0$ for all $1 \leq j \leq q$ we get $Q_{p,q}(\underline{T}, \underline{X}) \geq 0$ and this contradicts the fact that $(\underline{T}, \underline{X}) \in SLR_{p,q}$. Without loosing generality we may assume that $X_q \neq 0$. Let us then consider the tangent plane W_q to the ultrahyperbolic unit ball $B_S(p, q)$ in e_q . If $X_q > 0$, the half ray connecting $(\underline{T}, \underline{X})$ with the origin intersects W_q in the (p, q) -vector

$$\frac{1}{X_q} \underline{T} + \frac{1}{X_q} \sum_{j=1}^{q-1} e_j X_j + e_q \quad \text{with} \quad \left(\frac{1}{X_q} \underline{T}, \frac{1}{X_q} \sum_{j=1}^{q-1} e_j X_j \right) \in \mathbb{R}^{p,q-1} .$$

In fact, this holds for all (p, q) -vectors in a (p, q) -space-like neighbourhood of the given vector $(\underline{T}, \underline{X})$. If $X_q < 0$ we replace X_q by $|X_q|$, whence the

nullsolutions under construction are *even* with respect to the co-ordinate X_q . This expresses the rotational symmetry of the ultrahyperbolic unit ball $B_S(p, q)$. It also means that the conical neighbourhood in which the solutions are locally defined actually consist of *rays* (minus the origin) instead of *half rays*. This is not surprising, in view of the projective model for the sphere S^m considered earlier. The reason why we considered half rays in case of the hyperbolic unit ball in space-time is because the manifold H_+ is, in contrast to the manifolds considered here, not connected.

As we will use an inductive argument to construct solutions for the Dirac operator on the unit balls $B_S(p, q)$ and $B_T(p, q)$, it is important to have a clear notation for vectors belonging to orthogonal spaces of different dimensions. Therefore, we adopt the following notation : for $r > 0$ and $s > 0$, the vector $(\underline{T}_r, \underline{X}_s)$ is an $(p-r, q-s)$ -vector, belonging to $\mathbb{R}^{p-r, q-s}$ and it stands for the vector

$$\frac{(\sum_{i=1}^{p-r} \epsilon_i T_i, \sum_{j=1}^{q-s} e_j X_j)}{T_p T_{p-1} \cdots T_{p-(r-1)} X_q X_{q-1} \cdots X_{q-(s-1)}} .$$

This means that r (resp. s) indicates how many temporal (resp. spatial) co-ordinates are to be removed from the (p, q) -vector $(\underline{T}, \underline{X})$, starting with T_p (resp. X_q). The resulting vector must then also be divided by this co-ordinate. If $r = 0$ (resp. $s = 0$) the temporal part \underline{T} (resp. the spatial part \underline{X}) remains unchanged and therefore this will not be written explicitly.

The argument above thus shows that all (p, q) -vectors in a neighbourhood of the given $(\underline{T}, \underline{X}) \in SLR_{p,q}$ may be written as

$$(\underline{T}, \underline{X}) = \lambda_q (e_q + (\underline{T}, \underline{X}_1))$$

with $\lambda_q = X_q$ and $(\underline{T}, \underline{X}_1)$ the $(p, q-1)$ -vector obtained by removing the spatial component X_q and dividing the resulting vector by X_q . We then use $(\lambda_q, (\underline{T}, \underline{X}_1))$ as new co-ordinates on $SLR_{p,q}$, valid in the small conical neighbourhood of the given $(\underline{T}, \underline{X})$. In terms of these co-ordinates the vector derivative on $\mathbb{R}^{p,q}$ may be written as

$$\left(\sum_{i=1}^p \epsilon_i \partial_{T_i} - \sum_{j=1}^p e_j \partial_{X_j} \right) \rightarrow \frac{1}{\lambda_q} \left(D(\underline{T}, \underline{X}_1)_{p,q-1} + e_q \left[\mathbb{E}_{p,q-1} - \lambda_q \frac{d}{d\lambda_q} \right] \right) ,$$

$D(\underline{T}, \underline{X}_1)_{p,q-1}$ and $\mathbb{E}_{p,q-1}$ denoting respectively the Dirac and Euler operator on the orthogonal space $\mathbb{R}^{p,q-1}$ in terms of the co-ordinates $(\underline{T}, \underline{X}_1)$.

In view of the fact that nullsolutions $F(\underline{T}, \underline{X})$ for the Dirac operator on the unit ball $B_S(p, q)$ are defined as *homogeneous* solutions for the operator $D(\underline{T}, \underline{X})_{p,q}$ we put

$$F(\underline{T}, \underline{X}) = \lambda_q^{\alpha_q} F_q(\underline{T}, \underline{X}_1) ,$$

with

$$\left(D(\underline{T}, \underline{X}_1)_{p,q-1} + e_q [\mathbb{E}_{p,q-1} - \alpha_q] \right) F_q(\underline{T}, \underline{X}_1) = 0 .$$

The idea is to construct F_q as a modulated version of a homogeneous nullsolution for the operator $D(\underline{T}, \underline{X}_1)_{p,q-1}$ on $\mathbb{R}^{p,q-1}$. This is the inductive argument we alluded to, because this means that a monogenic function with respect to the Dirac operator on the unit ball $B_S(p, q)$ will be written in terms of a monogenic function with respect to the Dirac operator on a unit ball in an ultrahyperbolic space of lower dimension. We must however be careful because we do not know whether the projection of the (p, q) -vector $(\underline{T}, \underline{X})$, i.e. the $(p, q-1)$ -vector $(\underline{T}, \underline{X}_1)$, belongs to $SLR_{p,q-1}$, $TLR_{p,q-1}$ or even $NC_{p,q-1}$.

The only thing we know is that $Q_{p,q}(\underline{T}, \underline{X}) < 0$, whence

$$\sum_{i=1}^p \left(\frac{T_i}{X_q} \right)^2 - \sum_{j=1}^{q-1} \left(\frac{X_j}{X_q} \right)^2 < 1 \implies Q_{p,q-1}(\underline{T}, \underline{X}_1) < 1 .$$

This means that all three possibilities are likely and those (p, q) -vectors for which the projection $(\underline{T}, \underline{X}_1) \in NC_{p,q-1}$ will temporarily be excluded. We return to this point later. This also means that the explicit form of F_q will depend on whether $(\underline{T}, \underline{X}_1)$ belongs to $TLR_{p,q-1}$ or $SLR_{p,q-1}$.

Let us then put

$$F_q(\underline{T}, \underline{X}_1) = \text{Mod}(\alpha_q, \lambda_q; (\underline{T}, \underline{X}_1)) f_q(\underline{T}, \underline{X}_1) \quad (3.3)$$

with

$$\text{Mod}(\alpha_q, \lambda_q; (\underline{T}, \underline{X}_1)) = F_1^{(\lambda_q)}(N_{p,q-1}^2) + e_q(\underline{T}, \underline{X}_1) F_2^{(\lambda_q)}(N_{p,q-1}^2) , \quad (3.4)$$

where $N_{p,q-1}^2$ stands for $Q_{p,q-1}(\underline{T}, \underline{X}_1)$, and with

$$\begin{cases} D(\underline{T}, \underline{X}_1)_{p,q-1} f_q(\underline{T}, \underline{X}_1) = 0 \\ \mathbb{E}_{p,q-1} f_q(\underline{T}, \underline{X}_1) = \lambda_q f_q(\underline{T}, \underline{X}_1) , \end{cases}$$

i.e. with $f_q(\underline{T}, \underline{X}_1)$ a λ_q -homogeneous solution for the Dirac operator on $\mathbb{R}^{p,q-1}$. Note that $(e_q(\underline{T}, \underline{X}_1))^2 = Q_{p,q-1}(\underline{T}, \underline{X}_1) = N_{p,q-1}^2$. This means that the modulation factor is actually a power series in the bivector-variable $e_q(\underline{T}, \underline{X}_1)$. This is in agreement with the Modulation Theorem 3.1, where the modulation factor was given by a power series in the bivector $\underline{x}\epsilon$.

In order to determine the functions $F_1^{(\lambda_q)}(N_{p,q-1}^2)$ and $F_2^{(\lambda_q)}(N_{p,q-1}^2)$, we put

$$\begin{aligned} F_1^{(\lambda_q)}(N_{p,q-1}^2) &= \sum_{k=0}^{\infty} a_k N_{p,q-1}^{2k} = \sum_{k=0}^{\infty} a_k \left(\sum_{i=1}^p \left(\frac{T_i}{X_q} \right)^2 - \sum_{j=1}^{q-1} \left(\frac{X_j}{X_q} \right)^2 \right)^k \\ F_2^{(\lambda_q)}(N_{p,q-1}^2) &= \sum_{k=0}^{\infty} b_k N_{p,q-1}^{2k} = \sum_{k=0}^{\infty} b_k \left(\sum_{i=1}^p \left(\frac{T_i}{X_q} \right)^2 - \sum_{j=1}^{q-1} \left(\frac{X_j}{X_q} \right)^2 \right)^k. \end{aligned}$$

Making use of the fact that

$$\begin{aligned} D(\underline{T}, \underline{X}_1)_{p,q-1} N_{p,q-1}^{2k} &= 2k(\underline{T}, \underline{X}_1) N_{p,q-1}^{2k-2} \\ \mathbb{E}_{p,q-1} N_{p,q-1}^{2k} &= 2k N_{p,q-1}^{2k} \\ D(\underline{T}, \underline{X}_1)_{p,q-1}(\underline{T}, \underline{X})_{p,q-1} &= (p+q-1) + \mathbb{E}_{p,q-1} - \Gamma_{p,q-1}, \end{aligned}$$

we find immediately :

$$\begin{aligned} &\left(D(\underline{T}, \underline{X}_1)_{p,q-1} + e_q [\mathbb{E}_{p,q-1} - \alpha_q] \right) \sum_{k=0}^{\infty} a_k N_{p,q-1}^{2k} f_q \\ &= \sum_{k=0}^{\infty} \left\{ (\underline{T}, \underline{X}_1)(2k+2)a_{1+k} + e_q(\lambda_q + 2k - \alpha_q)a_k \right\} N_{p,q-1}^{2k} f_q \end{aligned}$$

and

$$\begin{aligned} &\left(D(\underline{T}, \underline{X}_1)_{p,q-1} + e_q [\mathbb{E}_{p,q-1} - \alpha_q] \right) e_q(\underline{T}, \underline{X})_{p,q-1} \sum_{k=0}^{\infty} b_k N_{p,q-1}^{2k} f_q \\ &= - \sum_{k=0}^{\infty} \left\{ \begin{array}{c} (\underline{T}, \underline{X}_1)(1+2k+\lambda_q-\alpha_q)b_k \\ + \\ e_q(2k+2\lambda_q+p+q-1)b_k \end{array} \right\} N_{p,q-1}^{2k} f_q \end{aligned}$$

This leads to the following recursive relations :

$$\begin{cases} (2k+2)a_{1+k} &= (1+2k+\lambda_q-\alpha_q)b_k \\ (2k+\lambda_q-\alpha_q)a_k &= (2k+2\lambda_q+p+q-1)b_k. \end{cases}$$

Choosing $a_0 = 1$, we conclude :

$$\begin{aligned} a_k &= \frac{\left(\frac{1+\lambda_q-\alpha_q}{2}\right)_k \left(\frac{\lambda_q-\alpha_q}{2}\right)_k}{k! \left(\lambda_q + \frac{p+q-1}{2}\right)_k} \\ b_k &= \frac{\frac{\lambda_q-\alpha_q}{2}}{\lambda_q + \frac{p+q-1}{2}} \frac{\left(\frac{1+\lambda_q-\alpha_q}{2}\right)_k \left(1 + \frac{\lambda_q-\alpha_q}{2}\right)_k}{k! \left(1 + \lambda_q + \frac{p+q-1}{2}\right)_k}, \end{aligned}$$

whence

$$F_1^{(\lambda_q)}(t) = F\left(\frac{1+\lambda_q-\alpha_q}{2}, \frac{\lambda_q-\alpha_q}{2}; \lambda_q + \frac{p+q-1}{2}; t\right) \quad (3.5)$$

$$F_2^{(\lambda_q)}(t) = \frac{\frac{\lambda_q-\alpha_q}{2}}{\lambda_q + \frac{p+q-1}{2}} F\left(\frac{1+\lambda_q-\alpha_q}{2}, 1 + \frac{\lambda_q-\alpha_q}{2}; 1 + \lambda_q + \frac{p+q-1}{2}; t\right) \quad (3.6)$$

Note that these hypergeometric functions are well-defined as long as $\lambda_q + \frac{m-1}{2}$ does not belong to $-\mathbb{N}$, *unless* $\alpha_q = \frac{1-m}{2}$. In that case, the operator is again related to the conformal Dirac operator on the ultrahyperbolic unit balls.

We have thus proved : for $(\underline{T}, \underline{X}) \in SLR_{p,q}$, an α_q -homogeneous solution for the Dirac operator $D(\underline{T}, \underline{X})_{p,q}$ on $\mathbb{R}^{p,q}$, defined in a neighbourhood of $(\underline{T}, \underline{X})$, is given by the function $F(\underline{T}, \underline{X}) = X_q^{\alpha_q} F_q(\underline{T}, \underline{X}_1)$, with F_q the product of a power series in the bivector-variable $e_q(\underline{T}, \underline{X}_1)$ (i.e. the modulation factor (3.4) given in terms of the hypergeometric functions (3.5) and (3.6)) and a homogeneous nullsolution for the Dirac operator on an ultrahyperbolic space of lower dimension. This latter nullsolution is homogeneous of degree λ_q , with $\lambda_q + \frac{m-1}{2} \notin -\mathbb{N}$.

3.3.2 The (p, q) -time-like situation

In this subsection we construct a locally defined monogenic function for the Dirac operator on $B_T(p, q)$. Consider a (p, q) -vector $(\underline{T}, \underline{X}) \in TLR_{p,q}$. We then have by definition :

$$Q_{p,q}(\underline{T}, \underline{X}) = \sum_{i=1}^p T_i^2 - \sum_{j=1}^p X_j^2 > 0.$$

This means that there is at least one temporal co-ordinate $T_i \neq 0$, for if $T_i = 0$ for all $1 \leq i \leq k$ we get $Q_{p,q}(\underline{T}, \underline{X}) \leq 0$ and this contradicts the fact that $(\underline{T}, \underline{X}) \in TLR_{p,q}$. Without loosing generality we may assume that

$T_p \neq 0$. Let us then consider the tangent plane V_p to the ultrahyperbolic unit ball $B_T(p, q)$ in ϵ_p . If $T_p > 0$, the half ray connecting $(\underline{T}, \underline{X})$ with the origin intersects V_p in the (p, q) -vector

$$\frac{1}{T_p} \underline{X} + \frac{1}{T_p} \sum_{i=1}^{p-1} \epsilon_i T_i + \epsilon_p \quad \text{with} \quad \left(\frac{1}{T_p} \sum_{i=1}^{p-1} \epsilon_i T_i, \frac{1}{T_p} \underline{X} \right) \in \mathbb{R}^{p-1, q} .$$

In fact, this holds for all (p, q) -vectors in a neighbourhood of the given vector $(\underline{T}, \underline{X})$. If $T_p < 0$ we replace T_p by $|T_p|$, which means that the nullsolutions that we are constructing will be *even* with respect to the co-ordinate T_p . This expresses the rotational symmetry of the ultrahyperbolic unit ball $B_T(p, q)$. Again, it means that the conical neighbourhood in which the solutions are locally defined consist of punctured rays instead of half rays.

So, all (p, q) -vectors in a neighbourhood of the given $(\underline{T}, \underline{X}) \in TLR_{p, q}$ may be written as

$$(\underline{T}, \underline{X}) = \lambda_p (\epsilon_p + (\underline{T}_1, \underline{X}))$$

with $\lambda_p = T_p$ and $(\underline{T}_1, \underline{X})$ the $(p-1, q)$ -vector obtained by removing the temporal component T_p and dividing the resulting vector by T_p . We then use $(\lambda_p, (\underline{T}_1, \underline{X}))$ as new co-ordinates on $TLR_{p, q}$, valid in a neighbourhood of the given $(\underline{T}, \underline{X})$. In terms of these new co-ordinates the Dirac operator on $\mathbb{R}^{p, q}$ may be written as

$$\left(\sum_{i=1}^p \epsilon_i \partial_{T_i} - \sum_{j=1}^p \epsilon_j \partial_{X_j} \right) \rightarrow \frac{1}{\lambda_p} \left(D(\underline{T}_1, \underline{X})_{p-1, q} + \epsilon_p \left[\lambda_p \frac{d}{d\lambda_p} - \mathbb{E}_{p-1, q} \right] \right) ,$$

$D(\underline{T}_1, \underline{X})_{p-1, q}$ and $\mathbb{E}_{p-1, q}$ denoting respectively the Dirac and Euler operator on the orthogonal space $\mathbb{R}^{p-1, q}$ in terms of the co-ordinates $(\underline{T}_1, \underline{X})$.

In view of the fact that nullsolutions $F(\underline{T}, \underline{X})$ for the Dirac operator on the unit ball $B_T(p, q)$ are defined as *homogeneous* solutions for the operator $D(\underline{T}, \underline{X})_{p, q}$ we put

$$F(\underline{T}, \underline{X}) = \lambda_p^{\alpha_p} F_p(\underline{T}_1, \underline{X}) ,$$

with

$$\left(D(p-1, q)_{\underline{T}_1, \underline{X}} + \epsilon_p [\alpha_p - \mathbb{E}_{p-1, q}] \right) F_p(\underline{T}_1, \underline{X}) = 0 . \quad (3.7)$$

The idea is to construct $F_p(\underline{T}_1, \underline{X})$ as a modulated version of a homogeneous nullsolution for a Dirac operator on an ultrahyperbolic space of lower dimension. However, the situation in the (p, q) -time-like region is slightly more complicated than the situation in the (p, q) -space-like region and the reason is the following : $(\underline{T}, \underline{X}) \in TLR_{p,q}$ does *not* imply that $Q_{p-1,q}(\underline{T}_1, \underline{X}) < 1$, making it impossible to use this variable as the expansion parameter in the modulation factor which will again, as for the (p, q) -space-like situation, be constructed as a hypergeometric series. On the other hand, we have the following :

$$-\sum_{i=1}^{p-1} \left(\frac{T_i}{T_p} \right)^2 + \sum_{j=1}^q \left(\frac{X_j}{T_p} \right)^2 < 1 \implies Q_{q,p-1}(\underline{X}, \underline{T}_1) < 1 ,$$

where $(\underline{X}, \underline{T}_1)$ stands for the $(q, p-1)$ -vector obtained by switching the role of the spatial and temporal co-ordinates in $(\underline{T}_1, \underline{X})$.

This means that for $(\underline{T}, \underline{X}) \in TLR_{p,q}$ the $(q, p-1)$ -vector $(\underline{X}, \underline{T}_1)$ belongs to $\mathbb{R}^{q,p-1}$ but we do not know whether it belongs to $SLR_{q,p-1}$, $TLR_{q,p-1}$ or even $NC_{q,p-1}$. As $Q_{q,p-1}(\underline{X}, \underline{T}_1) < 1$, the three possibilities are likely and we will again temporarily exclude those (p, q) -vectors $(\underline{T}, \underline{X}) \in TLR$ for which the $(q, p-1)$ -projected vector $(\underline{X}, \underline{T}_1)$ belongs to the nullcone $NC_{q,p-1}$. We return to this point later.

Let us now return to equation (3.7) :

$$\left(D(\underline{T}_1, \underline{X})_{p-1,q} + \epsilon_p [\alpha_p - \mathbb{E}_{p-1,q}] \right) F_p(\underline{T}_1, \underline{X}) = 0 .$$

Since we want to use $Q_{q,p-1}(\underline{X}, \underline{T}_1)$ as the expansion parameter in the modulation factor, i.e. the argument of the hypergeometric series, it seems natural to recast this equation in another form. Indeed, we will replace the Dirac operator $D(\underline{T}_1, \underline{X})_{p-1,q}$ by the Dirac operator on the space $\mathbb{R}^{q,p-1}$. Let us therefore multiply this equation by ϵ_p from the left, i.e. let us project this equation on the even subalgebra $\mathbb{R}_{p,q}^{(+)}$. We then get :

$$\epsilon_p D(\underline{T}_1, \underline{X})_{p-1,q} = T_p \left(\sum_{i=1}^{p-1} \epsilon_p \epsilon_i \partial_{T_i} - \sum_{j=1}^q \epsilon_p e_j \partial_{X_j} \right) .$$

As $(\epsilon_p \epsilon_i)^2 = -1$ and $(\epsilon_p e_j)^2 = 1$, we may formally identify the operator $\epsilon_p D(\underline{T}_1, \underline{X})_{p-1,q}$ with the Dirac operator $D(\underline{X}, \underline{T}_1)_{q,p-1}$ on $\mathbb{R}^{q,p-1}$. We thus rewrite equation (3.7) as follows :

$$\left(D(\underline{X}, \underline{T}_1)_{q,p-1} + [\alpha_p - \mathbb{E}_{q,p-1}] \right) F_p(\underline{X}, \underline{T}_1) = 0 , \quad (3.8)$$

where we have used the fact that $\mathbb{E}_{p-1,q} = \mathbb{E}_{q,p-1}$. We will now solve this equation, and at the end we replace $(\underline{X}, \underline{T}_1)$ by $\epsilon_p(\underline{T}_1, \underline{X})$ to obtain a solution for equation (3.7). In order to make a clear distinction between the solutions to equation (3.7) and (3.8), the solution to the latter equation will be labelled with a prime.

We thus put

$$F'_p(\underline{X}, \underline{T}_1) = \text{Mod}'(\alpha_p, \lambda_p; (\underline{X}, \underline{T}_1)) f'_p(\underline{X}, \underline{T}_1)$$

with

$$\text{Mod}'(\alpha_p, \lambda_p; (\underline{X}, \underline{T}_1)) = F_1^{(\lambda_p)'}((N'_{q,p-1})^2) + (\underline{X}, \underline{T}_1) F_2^{(\lambda_p)'}((N'_{q,p-1})^2) \quad (3.9)$$

where $(N'_{q,p-1})^2$ stands for $Q_{q,p-1}(\underline{X}, \underline{T}_1)$, and with

$$\begin{cases} D(\underline{X}, \underline{T}_1)_{q,p-1} f'_p(\underline{X}, \underline{T}_1) &= 0 \\ \mathbb{E}_{q,p-1} f'_p(\underline{X}, \underline{T}_1) &= \lambda_p f'_p(\underline{X}, \underline{T}_1) \end{cases},$$

i.e. with $f'_p(\underline{X}, \underline{T}_1)$ a λ_p -homogeneous solution for the Dirac operator on the orthogonal space $\mathbb{R}^{q,p-1}$.

In order to determine $F_1^{(\lambda_p)'}((N'_{q,p-1})^2)$ and $F_2^{(\lambda_p)'}((N'_{q,p-1})^2)$, we put

$$\begin{aligned} F_1^{(\lambda_p)'}((N'_{q,p-1})^2) &= \sum_{k=0}^{\infty} a_k (N'_{q,p-1})^{2k} = \sum_{k=0}^{\infty} a_k \left(\sum_{j=1}^q \left(\frac{X_j}{T_p} \right)^2 - \sum_{i=1}^{p-1} \left(\frac{T_i}{T_p} \right)^2 \right)^k \\ F_2^{(\lambda_p)'}((N'_{q,p-1})^2) &= \sum_{k=0}^{\infty} b_k (N'_{q,p-1})^{2k} = \sum_{k=0}^{\infty} b_k \left(\sum_{j=1}^q \left(\frac{X_j}{T_p} \right)^2 - \sum_{i=1}^{p-1} \left(\frac{T_i}{T_p} \right)^2 \right)^k \end{aligned}$$

Making use of the fact that

$$\begin{aligned} D(\underline{X}, \underline{T}_1)_{q,p-1} (N'_{q,p-1})^{2k} &= 2k(\underline{X}, \underline{T}_1) (N'_{q,p-1})^{2k-2} \\ \mathbb{E}_{q,p-1} (N'_{q,p-1})^{2k} &= 2k(N'_{q,p-1})^{2k} \\ D(\underline{X}, \underline{T}_1)_{q,p-1} (\underline{X}, \underline{T}_1) &= (p+q-1) + \mathbb{E}_{q,p-1} - \Gamma_{q,p-1} \end{aligned}$$

one finds immediately :

$$\begin{aligned} &\left(D(\underline{X}, \underline{T}_1)_{q,p-1} - \mathbb{E}_{q,p-1} + \alpha_p \right) \sum_{k=0}^{\infty} a_k (N'_{q,p-1})^{2k} f'_p \\ &= \sum_{k=0}^{\infty} \left\{ (\underline{X}, \underline{T}_1) (2k+2) a_{1+k} - (2k + \lambda_p - \alpha_p) a_k \right\} (N'_{q,p-1})^{2k} f'_p \end{aligned}$$

and

$$\begin{aligned} & \left(D(\underline{X}, \underline{T}_1)_{q,p-1} - \mathbb{E}_{p,q-1} + \alpha_p \right) (\underline{X}, \underline{T}_1) \sum_{k=0}^{\infty} b_k (N')^{2k} f'_p \\ &= - \sum_{k=0}^{\infty} \left\{ \frac{(\underline{X}, \underline{T}_1)(1 + 2k + \lambda_p - \alpha_p)b_k}{(2k + 2\lambda_p + p + q - 1)b_k} \right\} (N'_{q,p-1})^{2k} f'_p, \end{aligned}$$

leading to the same recursive relations as for the (p, q) -space-like situation :

$$\begin{cases} (2k + 2)a_{1+k} &= (1 + 2k + \lambda_p - \alpha_p)b_k \\ (2k + \lambda_p - \alpha_p)a_k &= (2k + 2\lambda_p + p + q - 1)b_k. \end{cases}$$

We immediately conclude :

$$F_1^{(\lambda_p)'}(t) = F\left(\frac{1 + \lambda_p - \alpha_p}{2}, \frac{\lambda_p - \alpha_p}{2}; \lambda_p + \frac{p + q - 1}{2}; t\right) \quad (3.10)$$

$$F_2^{(\lambda_p)'}(t) = \frac{\frac{\lambda_p - \alpha_p}{2}}{\lambda_p + \frac{p + q - 1}{2}} F\left(\frac{1 + \lambda_p - \alpha_p}{2}, 1 + \frac{\lambda_p - \alpha_p}{2}; 1 + \lambda_p + \frac{p + q - 1}{2}; t\right) \quad (3.11)$$

Note that these hypergeometric functions are well-defined for $\lambda_p + \frac{m-1}{2} \notin -\mathbb{N}$ (unless $\alpha_q = \frac{1-m}{2}$) as their argument $(N'_{q,p-1})^2 = Q_{q,p-1}(\underline{X}, \underline{T}_1) < 1$.

Now that we have found this solution for equation (3.8), we return to (3.7). A solution $F_p(\underline{T}_1, \underline{X})$ is given by :

$$F_p(\underline{T}_1, \underline{X}) = \text{Mod}(\alpha_p, \lambda_p; (\underline{T}_1, \underline{X})) f_p(\underline{T}_1, \underline{X})$$

with

$$\begin{aligned} \text{Mod}(\alpha_p, \lambda_p; (\underline{T}_1, \underline{X})) &= \text{Mod}'(\alpha_p, \lambda_p; \epsilon_p(\underline{T}_1, \underline{X})) \\ &= F_1^{(\lambda_p)'}((N'_{q,p-1})^2) + \epsilon_p(\underline{T}_1, \underline{X}) F_2^{(\lambda_p)'}((N'_{q,p-1})^2). \end{aligned} \quad (3.12)$$

Note that if we replace $(\underline{X}, \underline{T}_1)$ by $\epsilon_p(\underline{T}_1, \underline{X})$ in $\text{Mod}'(\alpha_p, \lambda_p; \epsilon_p(\underline{X}, \underline{T}_1))$, we do *not* alter the argument of the hypergeometric functions. Therefore we still write $N'_{q,p-1}$ with

$$(N'_{q,p-1})^2 = \sum_{j=1}^q \left(\frac{X_j}{T_p}\right)^2 - \sum_{i=1}^{p-1} \left(\frac{T_i}{T_p}\right)^2.$$

As $(\epsilon_p(\underline{T}_1, \underline{X}))^2 = Q_{q,p-1}(\underline{X}, \underline{T}_1) = (N'_{q,p-1})^2$, the modulation factor (3.12) may again be interpreted as a power series in the bivector-variable $\epsilon_p(\underline{T}_1, \underline{X})$.

The function $f_p(\underline{T}_1, \underline{X})$ can be found as follows : we start from an arbitrary solution $f'_p(\underline{X}, \underline{T}_1)$ for the Dirac operator $D(\underline{X}, \underline{T}_1)_{q,p-1}$ on $\mathbb{R}^{q,p-1}$ which is λ_p -homogeneous and we put

$$f_p(\underline{T}_1, \underline{X}) = f'_p(\epsilon_p(\underline{T}_1, \underline{X}))$$

This is of course equivalent with choosing $f_p(\underline{T}_1, \underline{X})$ as a λ_p -homogeneous solution for the Dirac operator $D(\underline{T}_1, \underline{X})_{p,q-1}$ on $\mathbb{R}^{p-1,q}$.

3.3.3 The unifying picture

In this subsection we unify both the (p, q) -time-like and (p, q) -space-like case, and this will lead to an explicit construction for nullsolutions on $B_T(p, q)$ or $B_S(p, q)$. For that purpose we rewrite the results from previous subsections. Let us first introduce the following function, defined as a power series in a bivector-variable \underline{B} :

Definition 3.1 For an arbitrary bivector $\underline{B} \in \mathbb{R}_{p,q}^{(2)}$ such that $[\underline{B}^2]_0 < 1$, we define the following bivector-valued function :

$$P_{p,q}(\alpha, \lambda, \underline{B}) = F_1^{(\lambda)}([\underline{B}^2]_0) + \frac{\frac{\lambda-\alpha}{2}}{\lambda + \frac{p+q-1}{2}} \underline{B} F_2^{(\lambda)}([\underline{B}^2]_0)$$

with $F_i^{(\lambda)}([\underline{B}^2]_0)$ ($i = 1, 2$) given by the following hypergeometric functions :

$$\begin{aligned} F_1^{(\lambda)}(t) &= F\left(\frac{1+\lambda-\alpha}{2}, \frac{\lambda-\alpha}{2}; \lambda + \frac{p+q-1}{2}; t\right) \\ F_2^{(\lambda)}(t) &= F\left(\frac{1+\lambda-\alpha}{2}, 1 + \frac{\lambda-\alpha}{2}; 1 + \lambda + \frac{p+q-1}{2}; t\right) . \end{aligned}$$

Consider then an arbitrary (p, q) -vector $(\underline{T}, \underline{X}) \in \mathbb{R}^{p,q}$:

1. if $(\underline{T}, \underline{X}) \in TLR_{p,q}$ and $T_p \neq 0$, a locally defined α -homogeneous solution for the operator $D(\underline{T}, \underline{X})_{p,q}$ is given by

$$|T_p|^\alpha P_{p,q}\left(\alpha, \lambda, \epsilon_p(\underline{T}_1, \underline{X})\right) f_\lambda(\underline{T}_1, \underline{X}) ,$$

with $f_\lambda(\underline{T}_1, \underline{X})$ a λ -homogeneous solution for $D(\underline{T}_1, \underline{X})_{p-1,q}$ on $\mathbb{R}^{p-1,q}$

2. if $(\underline{T}, \underline{X}) \in SLR_{p,q}$ and $X_q \neq 0$, a locally defined α -homogeneous solution for the operator $D(\underline{T}, \underline{X})_{p,q}$ is given by

$$|X_q|^\alpha P_{p,q} \left(\alpha, \lambda, e_q(\underline{T}, \underline{X}_1) \right) f_\lambda(\underline{T}, \underline{X}_1) ,$$

with $f_\lambda(\underline{T}, \underline{X}_1)$ a λ -homogeneous solution for $D(\underline{T}, \underline{X}_1)_{p,q-1}$ on $\mathbb{R}^{p,q-1}$

We are now able to construct α -homogeneous solutions for the operator $D(\underline{T}, \underline{X})_{p,q}$ on $\mathbb{R}^{p,q}$ in a neighbourhood of (p, q) -vectors $(\underline{T}, \underline{X})_{p,q}$ belonging to certain regions of the orthogonal space $\mathbb{R}^{p,q}$, by performing consecutive projections on tangent planes, to either space-like or time-like ultrahyperbolic unit balls.

This goes as follows : at a certain stage of the procedure we will have a vector $(\underline{T}_r, \underline{X}_s)$ belonging to either $TLR_{p-r,q-s}$ or $SLR_{p-r,q-s}$. In the first case a homogeneous solution for the operator $D(\underline{T}_r, \underline{X}_s)_{p-r,q-s}$ may be written as a modulated version of a homogeneous solution to the operator $D(\underline{T}_{r+1}, \underline{X}_s)_{p-r-1,q-s}$, whereas in the latter case a homogeneous solution for the operator $D(\underline{T}_r, \underline{X}_s)_{p-r,q-s}$ may be written as a modulated version of a homogeneous solution for the operator $D(\underline{T}_r, \underline{X}_{s+1})_{p-r,q-s-1}$. The next step, before we can use the recursion argument, is to look at the projected vector. By this we mean the following : if $(\underline{T}_r, \underline{X}_s)$ belongs to $TLR_{p-r,q-s}$ (resp. $SLR_{p-r,q-s}$) we project onto the plane tangent to the ultrahyperbolic unit ball $B_T(p-r, q-s)$ (resp. to $B_S(p-r, q-s)$) at the basis element ϵ_{p-r} (resp. e_{q-s}). If this projected vector $(\underline{T}_{r+1}, \underline{X}_s)$ (resp. $(\underline{T}_r, \underline{X}_{s+1})$), which clearly belongs to an orthogonal space of lower dimension, belongs to the nullcone of the lower-dimensional space, then the recursion argument breaks down. In that case we can only consider distributional homogeneous solutions for $D(\underline{T}_{r+1}, \underline{X}_s)_{p-r-1,q-s}$ (resp. $D(\underline{T}_r, \underline{X}_{s+1})_{p-r,q-s-1}$). If the projected vector does not belong to this nullcone, we repeat this procedure.

Eventually this means that there exist (p, q) -vectors in the orthogonal space $\mathbb{R}^{p,q}$ for which small conical neighbourhoods may consecutively be projected onto either the space-like or the time-like subset of the lower-dimensional orthogonal spaces. This means that for certain (p, q) -vectors, homogeneous solutions for the operator $D(\underline{T}, \underline{X})_{p,q}$ may be found as 'multiple-modulated' versions of homogeneous solutions for the Dirac operator on \mathbb{R}^m .

3.4 On a Bi-Axial Hyperbolic Problem

In this section we construct new hyperbolic monogenic solutions for the Dirac operator on the Klein model of the hyperbolic unit ball, by considering a bi-axial splitting of \mathbb{R}^m . This allows to reobtain the solutions constructed by means of the Modulation Theorem on the Klein ball in a totally different way. Indeed, it is proved that these solutions can be interpreted as special cases of so-called *generalized hyperbolic powerfunctions*.

3.4.1 Bi-axial hyperbolic monogenic functions

Let us consider a bi-axially symmetric domain in $\mathbb{R}^{0,m}$ by splitting $\mathbb{R}^{0,m}$ into $\mathbb{R}^{0,m_1} \oplus \mathbb{R}^{0,m_2}$. A general element $\underline{x} \in \mathbb{R}_{0,m}^{(1)}$ will be denoted by $\underline{x} = \underline{x}_1 + \underline{x}_2$ with $\underline{x}_i = r_i \underline{\xi}_i$, where $\underline{\xi}_i \in S^{m_i-1}$. The Dirac operator (resp. Euler operator) on \mathbb{R}^{0,m_i} will be denoted by $\underline{\partial}_i$ (resp. \mathbb{E}_i).

Our aim is to construct solutions $f(\underline{x}) = f(\underline{x}_1, \underline{x}_2)$ for the operator $D_\alpha(\underline{x})$, defined by

$$D_\alpha(\underline{x}) = \underline{\partial} + \epsilon(\mathbb{E} - \alpha) = \underline{\partial}_1 + \epsilon\mathbb{E}_1 + \underline{\partial}_2 + \epsilon\mathbb{E}_2 - \alpha\epsilon,$$

of the following form :

$$f(\underline{x}_1, \underline{x}_2) = \sum_{l=0}^{\infty} \left\{ F_l(r_2^2) + \underline{x}_2 \epsilon G_l(r_2^2) \right\} (\underline{x}_1 \epsilon)^l P_k(\underline{x}_1) P_q(\underline{x}_2), \quad \underline{x}_i \in \mathbb{R}_{0,m_i}^{(1)},$$

with $P_k(\underline{\xi}_1) \in M_{m_1}^+(k)$ and $P_q(\underline{\xi}_2) \in M_{m_2}^+(q)$. So, for each $l \in \mathbb{N}$ the factor between brackets represents a power series in the variable $\underline{x}_2 \epsilon$.

Note that we choose these inner spherical monogenics to belong to the spaces $\mathbb{R}_{0,m_1}^{(+)}$ and $\mathbb{R}_{0,m_2}^{(+)}$ respectively, such that

$$\begin{aligned} [\underline{\partial}_1, P_q(\underline{x}_2)] &= [\underline{x}_1, P_q(\underline{x}_2)] = 0 \\ [\underline{\partial}_2, P_k(\underline{x}_1)] &= [\underline{x}_2, P_k(\underline{x}_1)] = 0 \end{aligned}.$$

Letting the operator $D_\alpha(\underline{x})$ act upon the function $f(\underline{x})$, hereby using the fact that

$$\begin{aligned} \underline{\partial}_i(\underline{x}_i^{2l} P_k(\underline{x}_i)) &= -2l \underline{x}_i^{2l-1} P_k(\underline{x}_i) \\ \underline{\partial}_i(\underline{x}_i^{2l+1} P_k(\underline{x}_i)) &= -(2k + 2l + m_i) \underline{x}_i^{2l} P_k(\underline{x}_i) \end{aligned},$$

putting $t = r_2^2$ and $\mathbb{E}_t = t \frac{d}{dt}$, we arrive at the following system S of equations for the unknown functions F_l and G_l ($l \in \mathbb{N}$) :

$$S \leftrightarrow \begin{cases} (k + l + \frac{m_1}{2})F_{2l+1} &= (\mathbb{E}_t + l + \frac{k+q-\alpha}{2})F_{2l} - (\mathbb{E}_t + q + \frac{m_2}{2})G_{2l} \\ (1 + l)F_{2l+2} &= (\mathbb{E}_t + l + \frac{1+k+q-\alpha}{2})F_{2l+1} - (\mathbb{E}_t + q + \frac{m_2}{2})G_{2l+1} \\ (k + l + \frac{m_1}{2})G_{2l+1} &= \frac{d}{dt}F_{2l} - (\mathbb{E}_t + l + \frac{1+k+q-\alpha}{2})G_{2l} \\ (1 + l)G_{2l+2} &= \frac{d}{dt}F_{2l+1} - (\mathbb{E}_t + 1 + l + \frac{k+q-\alpha}{2})G_{2l+1} \end{cases}$$

This system must be interpreted in the following sense : given two analytic functions $F_0(t)$ and $G_0(t)$, the functions $F_l(t)$ and $G_l(t)$ can be determined for all $l \in \mathbb{N}$ by means of these equations. Introducing the short-hand notation

$$\begin{aligned} Q_0 &= \frac{k + q - \alpha}{2} \\ Q_1 &= \frac{1 + k + q - \alpha}{2} , \end{aligned}$$

the system S can also be rewritten as a recursive set of equations :

$$\left\{ \begin{aligned} F_{2l+1} &= \frac{(\mathbb{E}_t + l + Q_0)(\mathbb{E}_t + l - 1 + Q_1) - (\mathbb{E}_t + q + \frac{m_2}{2})\frac{d}{dt}}{l(k + l + \frac{m_1}{2})} F_{2l-1} \\ F_{2l+2} &= \frac{(\mathbb{E}_t + l + Q_0)(\mathbb{E}_t + l + Q_1) - (\mathbb{E}_t + q + \frac{m_2}{2})\frac{d}{dt}}{(1 + l)(k + l + \frac{m_1}{2})} F_{2l} \\ G_{2l+1} &= \frac{(\mathbb{E}_t + l + Q_0)(\mathbb{E}_t + l + Q_1) - (\mathbb{E}_t + 1 + q + \frac{m_2}{2})\frac{d}{dt}}{l(k + l + \frac{m_1}{2})} G_{2l-1} \\ G_{2l+2} &= \frac{(\mathbb{E}_t + l + 1 + Q_0)(\mathbb{E}_t + l + Q_1) - (\mathbb{E}_t + 1 + q + \frac{m_2}{2})\frac{d}{dt}}{(1 + l)(k + l + \frac{m_1}{2})} G_{2l} \end{aligned} \right.$$

In order to solve this system one needs to specify the so-called *Cauchy data*, i.e. the explicit form of the solution $f(\underline{x}_1, \underline{x}_2)$ for $\underline{x}_1 = \underline{0}$. This function $f(\underline{0}, \underline{x}_2)$ can then be extended to a solution for the hyperbolic Dirac equation on the Klein model, by determining the functions $F_l(t)$ and $G_l(t)$ by means of the recursive set of equations above.

3.4.2 Generalized Hyperbolic Power Functions

The aim of this section is to construct solutions for the operator $D_\alpha(\underline{x})$ which are generalizations of the monomials z^λ in complex analysis. For that purpose we define \underline{x}_2^λ for arbitrary complex λ , and we then use this function multiplied by an inner spherical monogenic $P_q(\underline{x}_2)$ as initial Cauchy data in the system S from the previous subsection, in order to obtain a solution $f(\underline{x}_1, \underline{x}_2)$ for the Dirac equation on the hyperbolic Klein ball.

The function \underline{x}_2^λ was already defined in e.g. [23], where the authors have put

$$\begin{aligned}\underline{x}_2^\lambda &= r_2^\lambda \left(\cos \frac{\pi\lambda}{2} + \underline{\xi}_2 \sin \frac{\pi\lambda}{2} \right) \\ &= r_2^\lambda \cos \frac{\pi\lambda}{2} + r_2^{\lambda-1} \underline{x}_2 \sin \frac{\pi\lambda}{2} .\end{aligned}$$

We will now use this function as Cauchy data, in two distinct steps : first we consider *inner power functions* and then *outer power functions*.

- **Inner Power Functions**

Recalling the fact that $t = r_2^2$, let us first consider the function

$$\begin{aligned}F_0(t) &= t^\lambda \\ G_0(t) &= 0 .\end{aligned}$$

In view of the fact that hyperbolic monogenic functions are defined as homogeneous solutions for the Dirac operator ∂_X on $\mathbb{R}^{1,m}$, these initial data are canonical : they come from the projection on the hyperplane $\Pi \leftrightarrow T = 1$ of the α -homogeneous function

$$F(T, \underline{0}, \underline{X}_2) = T^{\alpha-2\lambda-q} |\underline{X}_2|^{2\lambda} P_q(\underline{X}_2)$$

defined for $\underline{X}_1 = \underline{0}$. We are thus starting from an α -homogeneous function in space-time variables $(T, \underline{0}, \underline{X}_2)$, and this function will be extended by means of the system S to an α -homogeneous solution for the Dirac operator ∂_X on $\mathbb{R}^{1,m}$, in space-time variables $(T, \underline{X}_1, \underline{X}_2)$.

With the aid of S , one finds :

$$\begin{aligned}F_1(t) &= -\frac{\lambda + Q_0}{k + \frac{m_1}{2}} t^\lambda \\ G_1(t) &= \frac{\lambda}{k + \frac{m_1}{2}} t^{\lambda-1} .\end{aligned}$$

We can then derive expressions for $F_{2l}(t)$, $F_{2l+1}(t)$ and $G_{2l+1}(t)$ by means of the recursive set of equations.

Consider for example the formula giving $F_{2l+2}(t)$ in terms of $F_{2l}(t)$. Putting

$$\mathcal{O}_{2l+2}(F) = \frac{(\mathbb{E}_t + l + Q_0)(\mathbb{E}_t + l + Q_1) - (\mathbb{E}_t + q + \frac{m_2}{2}) \frac{d}{dt}}{(1+l)(k+l+\frac{m_1}{2})} ,$$

we have $F_{2l+2}(t) = \mathcal{O}_{2l+2}(F)F_{2l}(t)$. Starting with $F_0(t)$, the operator $\mathcal{O}_2(F)$ contains a part which multiplies $F_0(t)$ with a (complex) constant, and a part which lowers the degree of $F_0(t)$ and then multiplies with a (complex) constant. This scheme repeats itself when the operator $\mathcal{O}_{2l+2}(F)$ acts on $F_{2l}(t)$, whence $F_{2l+2}(t)$ will have the following form :

$$F_{2l+2}(t) = c_0 t^\lambda + c_1 t^{\lambda-1} + \dots + c_{l+1} t^{\lambda-l-1} \quad \text{with } c_i \in \mathbb{C}.$$

However, if $\lambda \in \mathbb{N}$ the scheme breaks down when the exponent of t becomes zero and the remaining coefficients all vanish.

We then prove that

$$F_{2l}(t) = \sum_{i=0}^l c_i(\lambda, l) t^{\lambda-i} \quad (3.13)$$

with $c_i(\lambda, l)$ given by

$$(-1)^i \binom{\lambda}{i} \frac{(\lambda - i + q + \frac{m_2}{2})_i}{(k + \frac{m_1}{2})_i} \frac{(\lambda + \frac{k+q-\alpha}{2})_{l-i}}{(l-i)!} \frac{(\lambda + \frac{1+k+q-\alpha}{2})_{l-i}}{(i + k + \frac{m_1}{2})_{l-i}}.$$

Here it is understood that for $\lambda \in \mathbb{N}$ the series terminates when the exponent of t becomes zero, such that the remaining coefficients then all vanish.

proof : We will prove this formula by induction. Since

$$\begin{aligned} F_2(t) &= \mathcal{O}_2(F) t^\lambda \\ &= \frac{(\lambda + Q_0)(\lambda + Q_1)}{k + \frac{m_1}{2}} t^\lambda - \lambda \frac{\lambda - 1 + q + \frac{m_2}{2}}{k + \frac{m_1}{2}} t^{\lambda-1}, \end{aligned}$$

the formula holds for $l = 1$. Suppose the formula is correct for $F_{2l}(t)$, we then prove that it also holds for $F_{2l+2}(t) = \mathcal{O}_{2l+2}(F)F_{2l}(t)$.

First of all, the term in t^λ is given by

$$\begin{aligned} \frac{(\mathbb{E}_t + l + Q_0)(\mathbb{E}_t + l + Q_1)}{(1+l)(k + l + \frac{m_1}{2})} c_0(\lambda, l) t^\lambda &= \frac{(\lambda + Q_0)_{l+1}(\lambda + Q_1)_{l+1}}{(l+1)! (k + \frac{m_1}{2})_{l+1}} t^\lambda \\ &= c_0(\lambda, l+1) t^\lambda, \end{aligned}$$

while the term in $t^{\lambda-l-1}$ is given by

$$- \frac{(\mathbb{E}_t + q + \frac{m_2}{2})}{(1+l)(k + l + \frac{m_1}{2})} \frac{d}{dt} c_l(\lambda, l) t^{\lambda-l},$$

which by means of the fact that

$$\frac{\lambda - l}{1 + l} \binom{\lambda}{l} = \binom{\lambda}{l+1},$$

reduces to

$$(-1)^{l+1} \binom{\lambda}{l+1} \frac{(\lambda - l - 1 + q + \frac{m_2}{2})_{l+1}}{(k + \frac{m_1}{2})_{l+1}} t^{\lambda-l-1} = c_{l+1}(\lambda, l+1) t^{\lambda-l-1}.$$

For the term in $t^{\lambda-i}$, with $0 < i < 1 + l$, we obtain

$$\begin{aligned} & c_i(\lambda, l) \frac{(\mathbb{E}_t + l + Q_0)(\mathbb{E}_t + l + Q_1)}{(1+l)(k + l + \frac{m_1}{2})} t^{\lambda-i} \\ & - c_{i-1} \frac{(\mathbb{E}_t + q + \frac{m_2}{2})}{(1+l)(k + l + \frac{m_1}{2})} \frac{d}{dt}(\lambda, l) t^{\lambda+1-i}, \end{aligned}$$

which by means of the fact that

$$(1 + l - i) \binom{\lambda}{i} + (1 + \lambda - i) \binom{\lambda}{i-1} = (1 + l) \binom{\lambda}{i}$$

may be written as

$$(-1)^i \binom{\lambda}{i} \frac{(\lambda - i + q + \frac{m_2}{2})_i}{(k + \frac{m_1}{2})_i} \frac{(\lambda + Q_0)_{1+l-i} (\lambda + Q_1)_{1+l-i}}{(1 + l - i)! (i + k + \frac{m_1}{2})_{1+l-i}} t^{\lambda-i},$$

which is equal to $c_i(\lambda, 1 + l) t^{\lambda-i}$. Adding these terms we finally obtain

$$F_{2l+2}(t) = \sum_{i=0}^{1+l} c_i(\lambda, 1 + l) t^{\lambda-i},$$

as was to be proved. \square

A similar approach can be followed to prove that

$$F_{2l+1}(t) = \frac{\lambda + \frac{k+q-\alpha}{2}}{k + \frac{m_1}{2}} \sum_{i=0}^l d_i(\lambda, l) t^{\lambda-i} \quad (3.14)$$

with $d_i(\lambda, l)$ given by

$$(-1)^i \binom{\lambda}{i} \frac{(\lambda - i + q + \frac{m_2}{2})_i}{(1 + k + \frac{m_1}{2})_i} \frac{(\lambda + 1 + \frac{k+q-\alpha}{2})_{l-i} (\lambda + \frac{1+k+q-\alpha}{2})_{l-i}}{(l-i)! (1 + i + k + \frac{m_1}{2})_{l-i}},$$

and

$$G_{2l+1}(t) = \frac{\lambda}{k + \frac{m_1}{2}} \sum_{i=0}^l f_i(\lambda - 1, l) t^{\lambda-1-i} \quad (3.15)$$

with $f_i(\lambda - 1, l)$ given by

$$(-1)^i \binom{\lambda - 1}{i} \frac{(\lambda - i + q + \frac{m_2}{2})_i}{(1 + k + \frac{m_1}{2})_i} \frac{(\lambda + \frac{k+q-\alpha}{2})_{l-i}}{(l-i)!} \frac{(\lambda + \frac{1+k+q-\alpha}{2})_{l-i}}{(1 + i + k + \frac{m_1}{2})_{l-i}}.$$

In both cases it is understood that the series terminates if the exponent of t becomes zero, whence all remaining coefficients vanish.

Returning to the explicit form for $f(\underline{x}_1, \underline{x}_2)$ proposed at the beginning of this section, we have thus found a solution for the operator $D_\alpha(\underline{x})$, with $\underline{x} = \underline{x}_1 + \underline{x}_2$, which generalizes the function $|\underline{x}_2|^{2\lambda} P_q(\underline{x}_2)$ on \mathbb{R}^{m_2} :

$$\begin{aligned} I_{\alpha,k,q}^\lambda(\underline{x}) &= \left(\sum_{l=0}^{\infty} \left[\sum_{i=0}^l c_i(\lambda, l) r_2^{2\lambda-2i} \right] r_1^{2l} \right) P_k(\underline{x}_1) P_q(\underline{x}_2) \\ &+ \frac{\lambda + \frac{k+q-\alpha}{2}}{k + \frac{m_1}{2}} \underline{x}_1 \epsilon \left(\sum_{l=0}^{\infty} \left[\sum_{i=0}^l d_i(\lambda, l) r_2^{2\lambda-2i} \right] r_1^{2l} \right) P_k(\underline{x}_1) P_q(\underline{x}_2) \\ &+ \frac{\lambda \underline{x}_1 \underline{x}_2}{k + \frac{m_1}{2}} \left(\sum_{l=0}^{\infty} \left[\sum_{i=0}^l f_i(\lambda - 1, l) r_2^{2\lambda-2i-2} \right] r_1^{2l} \right) P_k(\underline{x}_1) P_q(\underline{x}_2). \end{aligned}$$

Remark : Let us consider a special case of this result by putting $\lambda = 0$ such that $F_0(t) = 1$ and $G_0(t) = 0$. The function $I_{\alpha,k,q}^0(\underline{x})$ then reduces to

$$I_{\alpha,k,q}^0(\underline{x}) = \left(\sum_{l=0}^{\infty} \left[c_0(0, l) + d_0(0, l) \frac{\frac{k+q-\alpha}{2}}{k + \frac{m_1}{2}} \underline{x}_1 \epsilon \right] r_1^{2l} \right) P_k(\underline{x}_1) P_q(\underline{x}_2),$$

which by means of the definition for $\text{Mod}(\alpha, k, \underline{x})$ can be written as

$$I_{\alpha,k,q}^0(\underline{x}) = \text{Mod}(\alpha - q, k, \underline{x}_1) P_k(\underline{x}_1) P_q(\underline{x}_2).$$

One can immediately see that this is a solution for the hyperbolic Dirac equation on the Klein ball by decomposing the operator $D_\alpha(\underline{x})$ as

$$D_\alpha(\underline{x}) = \underline{\partial}_1 + \epsilon(\mathbb{E}_1 - (\alpha - q)) + \underline{\partial}_2 + \epsilon(\mathbb{E}_2 - q).$$

Note that in case $q = 0$, one finds the solution given by Theorem 3.1, showing that the modulated hyperbolic monogenics $\text{Mod}(\alpha, k, \underline{x}) P_k(\underline{x})$ can be interpreted as trivial inner hyperbolic monogenic powers $I_{\alpha,k,0}^0(\underline{x})$.

- **Outer Power Functions**

Next, we consider for arbitrary $\lambda \in \mathbb{C}$ the *outer* power function as initial Cauchy data :

$$\begin{aligned} F_0(t) &= 0 \\ G_0(t) &= t^\lambda . \end{aligned}$$

These initial data form the projection on the hyperplane $\Pi \leftrightarrow T = 1$ of the α -homogeneous function

$$F(T, \underline{0}, \underline{X}_2) = T^{\alpha-2\lambda-1-k} \underline{X}_2 |\underline{X}_2|^{2\lambda} P_k(\underline{X}_2) ,$$

defined for $\underline{X}_1 = \underline{0}$. We are thus again starting from an α -homogeneous function in space-time variables $(T, \underline{0}, \underline{X}_2)$, and this function will be extended by means of the system S to an α -homogeneous solution for the Dirac operator ∂_X on $\mathbb{R}^{1,m}$, in space-time variables $(T, \underline{X}_1, \underline{X}_2)$.

With the aid of S one finds :

$$\begin{aligned} F_1(t) &= -\frac{\lambda + q + \frac{m_2}{2}}{k + \frac{m_1}{2}} t^\lambda \\ G_1(t) &= -\frac{\lambda + \frac{1+k+q-\alpha}{2}}{k + \frac{m_1}{2}} t^\lambda . \end{aligned}$$

Since we have already found that

$$\begin{aligned} F_1(t) = \frac{\lambda + \frac{k+q-\alpha}{2}}{k + \frac{m_1}{2}} t^\lambda &\xrightarrow{S} F_{2l+1}(t) = \frac{\lambda + \frac{k+q-\alpha}{2}}{k + \frac{m_1}{2}} \sum_{i=0}^l d_i(\lambda, l) t^{\lambda-i} \\ G_1(t) = \frac{\lambda}{k + \frac{m_1}{2}} t^{\lambda-1} &\xrightarrow{S} G_{2l+1}(t) = \frac{\lambda}{k + \frac{m_1}{2}} \sum_{i=0}^l f_i(\lambda-1, l) t^{\lambda-i-1} , \end{aligned}$$

we get immediately :

$$F_{2l+1}(t) = -\frac{\lambda + q + \frac{m_2}{2}}{k + \frac{m_1}{2}} \sum_{i=0}^l d_i(\lambda, l) t^{\lambda-i} \quad (3.16)$$

with $d_i(\lambda, l)$ given by

$$(-1)^i \binom{\lambda}{i} \frac{(\lambda - i + q + \frac{m_2}{2})_i (\lambda + 1 + \frac{k+q-\alpha}{2})_{l-i} (\lambda + \frac{1+k+q-\alpha}{2})_{l-i}}{(1 + k + \frac{m_1}{2})_i (l-i)! (1 + i + k + \frac{m_1}{2})_{l-i}} ,$$

and

$$G_{2l+1}(t) = -\frac{\lambda + \frac{1+k+q-\alpha}{2}}{k + \frac{m_1}{2}} \sum_{i=0}^l f_i(\lambda, l) t^{\lambda-i} \quad (3.17)$$

with $f_i(\lambda, l)$ given by

$$(-1)^i \binom{\lambda}{i} \frac{(1 + \lambda - i + q + \frac{m_2}{2})_i}{(1 + k + \frac{m_1}{2})_i} \frac{(1 + \lambda + \frac{k+q-\alpha}{2})_{l-i}}{(l-i)!} \frac{(1 + \lambda + \frac{1+k+q-\alpha}{2})_{l-i}}{(1 + i + k + \frac{m_1}{2})_{l-i}}.$$

In order to find $G_{2l}(t)$, we compare the formula giving $G_{2l+2}(t)$ in terms of $G_{2l}(t)$ with the formula giving $F_{2l+2}(t)$ in terms of $F_{2l}(t)$. Up to a substitution

$$q \longrightarrow q + 1$$

both formulae are identical, so that we immediately get :

$$G_{2l}(t) = \sum_{i=0}^l g_i(\lambda, l) t^{\lambda-i} \quad (3.18)$$

with $g_i(\lambda, l)$ given by

$$(-1)^i \binom{\lambda}{i} \frac{(\lambda - i + q + 1 + \frac{m_2}{2})_i}{(k + \frac{m_1}{2})_i} \frac{(\lambda + 1 + \frac{k+q-\alpha}{2})_{l-i}}{(l-i)!} \frac{(\lambda + \frac{1+k+q-\alpha}{2})_{l-i}}{(i + k + \frac{m_1}{2})_{l-i}}.$$

Note that in the formulae for $F_{2l+1}(t)$, $G_{2l}(t)$ and $G_{2l+1}(t)$ it is understood that if the exponent of t becomes zero, all remaining coefficients vanish.

Returning to the explicit form for $f(\underline{x}_1, \underline{x}_2)$ proposed at the beginning of this section, we have thus found the following solution for $D_\alpha(\underline{x})$, with $\underline{x} = \underline{x}_1 + \underline{x}_2$, generalizing the function $\underline{x}_2 |\underline{x}_2|^{2\lambda} P_q(\underline{x}_2)$:

$$\begin{aligned} O_{\alpha, k, q}^\lambda(\underline{x}) = & \frac{\lambda + q + \frac{m_2}{2}}{k + \frac{m_1}{2}} \underline{x}_1 \epsilon \left(\sum_{l=0}^{\infty} \left[\sum_{i=0}^l d_i(\lambda, l) r_2^{2\lambda-2i} \right] r_1^{2l} \right) P_k(\underline{x}_1) P_q(\underline{x}_2) \\ & - \underline{x}_2 \epsilon \left(\sum_{l=0}^{\infty} \left[\sum_{i=0}^l g_i(\lambda, l) r_2^{2\lambda-2i} \right] r_1^{2l} \right) P_k(\underline{x}_1) P_q(\underline{x}_2) \\ & + \frac{\lambda + \frac{1+k+q-\alpha}{2}}{k + \frac{m_1}{2}} \underline{x}_1 \underline{x}_2 \left(\sum_{l=0}^{\infty} \left[\sum_{i=0}^l f_i(\lambda, l) r_2^{2\lambda-2i} \right] r_1^{2l} \right) P_k(\underline{x}_1) P_q(\underline{x}_2) \end{aligned}$$

This allows us to define the *generalized hyperbolic powerfunction* $\mathcal{P}_{\alpha,k,q}^\lambda(\underline{x})$, with $(\alpha, \lambda) \in \mathbb{C}^2$ and $(k, q) \in \mathbb{N}^2$. When multiplying this function by T^α and putting $\underline{x} = \frac{X}{T}$ we obtain an α -homogeneous solution for the Dirac operator ∂_X on $\mathbb{R}^{1,m}$, whence the index α . The index λ refers to the power function \underline{x}_2^λ in the Cauchy data $\underline{x}_2^\lambda P_q(\underline{x}_2)$, whereas the indices k and q refer to the inner monogenics $P_k(\underline{x}_1)$ and $P_q(\underline{x}_2)$ respectively. This function is in explicit form given by :

$$\mathcal{P}_{\alpha,k,q}^\lambda(\underline{x}) = I_{\alpha,k,q}^{\frac{\lambda}{2}}(\underline{x}) \cos \frac{\pi\lambda}{2} + O_{\alpha,k,q}^{\frac{\lambda-1}{2}}(\underline{x}) \sin \frac{\pi\lambda}{2} .$$

Chapter 4

Arbitrary Powers of the Hyperbolic Dirac Operator

The shortest path between two truths in the real domain passes through the complex domain. (J. Hadamard)

In this Chapter arbitrary complex powers of the hyperbolic Dirac operator are defined and a fundamental solution for these operators is constructed. First the so-called natural powers of the Dirac operator on the hyperbolic unit ball will be considered, and afterwards arbitrary complex powers.

4.1 Natural Powers of the Hyperbolic Dirac Operator

In this section natural powers of the Dirac operator on the hyperbolic unit ball are considered, i.e. integer powers of the Dirac operator ∂_X on $\mathbb{R}^{1,m}$ acting on sections of the homogeneous Clifford line bundle $\mathbb{R}_{1,m;\alpha}$ introduced in Chapter 2 as an associated principal fibre bundle or, equivalently, acting on α -homogeneous functions on the future cone in space-time co-ordinates (T, \underline{X}) . In the previous Chapter we already encountered these natural powers when we generalized the Modulation Theorem to the case of the k -iterated hyperbolic Dirac equation (see Theorem 3.3).

When projecting the k -iterated Dirac operator ∂_X^k acting on α -homogeneous functions $F(T, \underline{X})$ on the FC onto the Klein model for the hyperbolic unit ball, we obtain the differential operator

$$D_\alpha^k(\underline{x}) = D_{\alpha-(k-1)}(\underline{x})D_{\alpha-(k-2)}(\underline{x}) \cdots D_{\alpha-1}(\underline{x})D_\alpha(\underline{x})$$

on the unit ball $B_m(1) \subset \mathbb{R}^m$ acting on functions $f(\underline{x})$, with

$$F(T, \underline{X}) = \lambda^\alpha F\left(1, \frac{\underline{X}}{T}\right) = \lambda^\alpha f(\underline{x}) ,$$

and $\lambda = T$ and where the operator $D_\alpha(\underline{x}) = D_\alpha^1(\underline{x})$ stands for

$$D_\alpha(\underline{x}) = \underline{\partial} + \epsilon(\mathbb{E}_r - \alpha) ,$$

with $\underline{\partial}$ (resp. \mathbb{E}_r) the Dirac (resp. Euler) operator on $\mathbb{R}^{0,m}$ in co-ordinates \underline{x} .

On the analogy of our first method to construct a fundamental solution for the hyperbolic Dirac equation, for which we refer to Chapter 2, we can try to apply Theorem 3.3 to the outer spherical monogenic $\underline{\xi} \in M^-(0)$, i.e. the restriction of the Cauchy kernel to S^{m-1} , to obtain a fundamental solution for the k -iterated hyperbolic Dirac equation.

We make a distinction between even and odd natural powers :

- **Even Natural Powers of the Hyperbolic Dirac Operator**

In case the power ∂_X^s of the hyperbolic Dirac operator is even, we are dealing with a scalar-valued differential operator. Indeed, the product of two consecutive operators reduces to

$$\begin{aligned} D_{\alpha-1}(\underline{x})D_\alpha(\underline{x}) &= \left(\underline{\partial} + \epsilon(\mathbb{E}_r - (\alpha - 1))\right)\left(\underline{\partial} + \epsilon(\mathbb{E}_r - \alpha)\right) \\ &= \left(\Delta_m - (\mathbb{E}_r + 1 - \alpha)(\mathbb{E}_r - \alpha)\right) . \end{aligned}$$

This explains why Theorem 3.3 provided us with a scalar solution in case of an even power of the hyperbolic Dirac operator.

To obtain a fundamental solution for the s -iterated Dirac equation on the hyperbolic unit ball, with $s = 2k$, we try to modulate the Cauchy kernel $E(\underline{x})$ on \mathbb{R}^m . For those values for which the modulation factor in Theorem 3.3 is well-defined, we get the following solution for the operator $D_\alpha^{2k}(\underline{x})$, defined on the punctured unit ball $B_m(1) \setminus \{0\}$:

$$E_\alpha^{2k}(\underline{x}) = \text{Mod}(1 + \alpha - k, -m + k; 1; \underline{x}) \underline{x}^{2k-1} E(\underline{x}) .$$

For *odd* dimensions m this function is defined for all even powers $s = 2k$, whereas in case of an *even* dimension m we have to restrict ourselves

to those powers for which $k > \frac{m}{2} - 1$ in order to make sure that the hypergeometric function in the modulation factor is well-defined.

However, despite the fact that this solution is well-defined for these latter values, the function $\underline{x}^{2k-1}E(\underline{x})$ will no longer have a singularity at $\underline{x} = \underline{0}$. Thus in case of an even dimension m the function $E_\alpha^{2k}(\underline{x})$ does not lead to a fundamental solution for the iterated hyperbolic Dirac operator. This phenomenon can be illustrated by means of its Euclidean analogue : consider the Laplacian Δ_m on \mathbb{R}^m , with m even and $m > 2$, its fundamental solution up to a normalizing constant given by the function r^{2-m} . The fundamental solution for the operator Δ_m^2 is then given by the function r^{4-m} , again up to a normalizing constant. According to this scheme the fundamental solution for the operator $\Delta_m^{\frac{m}{2}}$ should be given by the function r^{m-m} but this is the regular constant solution $f(r) = 1$ for the operator $\Delta_m^{\frac{m}{2}}$. The fundamental solution can then be found as

$$\lim_{\mu \rightarrow \frac{m}{2}} \frac{r^{2\mu-m} - 1}{2\mu - m} ,$$

leading to the logarithmic function $\ln r$. The same thing happens in case of an even power $s = 2k$ of the hyperbolic Dirac operator acting on a space of even dimension m . This case will again be temporarily excluded, and the special treatment involving logarithmic functions will be tackled by the introduction of Gegenbauer functions of the second kind, just like we did in Chapter 2.

As the singular behaviour of the fundamental solution does not change when a particular regular solution for the s -iterated hyperbolic Dirac equation is added, we will add a particular nullsolution to $E_\alpha^{2k}(\underline{x})$. For those values for which this function yields a fundamental solution, i.e. in case of an odd dimension m , it is a modulated version of the fundamental solution for the operator $\underline{\partial}^{2k}$ on $\mathbb{R}^{0,m}$. It seems obvious to add a regular nullsolution which is given by the modulation of a constant, hereby inspired by the construction of the fundamental solution for the hyperbolic Dirac operator. However, applying Theorem 3.3 to the constant function $P_0(\underline{x}) = 1$ would yield a function

$$\text{Mod} \left(1 + \alpha - k, k - 1; 1; \frac{X}{T} \right) \left(\frac{X}{T} \right)^{2k-1} ,$$

and this is not what we are looking for because this function does not

modulate a constant. Instead, we will add the regular solution

$$\text{Mod} \left(1 + \alpha - k, -k; 1; \frac{X}{T} \right) ,$$

but we first need a direct proof for the fact that this is indeed a solution for the iterated hyperbolic Dirac equation. This is expressed in the following :

Lemma 4.1 *Let $k \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$. For all $\underline{x} \in B_m(1)$ we have :*

$$D_\alpha^{2k}(\underline{x}) \text{Mod}(1 + \alpha - k, -k; 1; \underline{x}) = 0 .$$

Proof : First of all note that the operator $D_\alpha^{2k}(\underline{x})$ can be written as a k -fold product of hypergeometric differential operators in the variable $t = r^2$. Writing

$$\mathcal{H}_t(a, b; c) = t(1-t) \frac{d^2}{dt^2} + [c - (1+a+b)t] \frac{d}{dt} - ab ,$$

we get for the operator $D_\alpha^{2k}(\underline{x})$:

$$\begin{aligned} & D_{\alpha-(2k-1)}(\underline{x}) D_{\alpha-(2k-2)}(\underline{x}) \cdots D_{\alpha-1}(\underline{x}) D_\alpha(\underline{x}) \\ &= (-4)^k \mathcal{H}_t \left(k-1 - \frac{\alpha}{2}, k-1 + \frac{1-\alpha}{2}; \frac{m}{2} \right) \cdots \mathcal{H}_t \left(-\frac{\alpha}{2}, \frac{1-\alpha}{2}; \frac{m}{2} \right) . \end{aligned}$$

In view of the fact that the modulation factor $\text{Mod}(1 + \alpha - k, -k; 1; \underline{x})$ in terms of the variable t is, up to a constant, given by the function

$$F \left(-\frac{\alpha}{2}, \frac{1-\alpha}{2}; 1 - k + \frac{m}{2}; t \right) ,$$

it suffices in case $k > 1$ to prove that if all of the differential operators \mathcal{H}_t above, except for the last one, act on this modulation factor we get up to a constant the hypergeometric function

$$F \left(k-1 - \frac{\alpha}{2}, k-1 + \frac{1-\alpha}{2}; \frac{m}{2}; t \right) ,$$

which is the regular solution for the last hypergeometric differential operator in the expansion for $D_\alpha^{2k}(\underline{x})$. To that end it suffices to prove that

$$\mathcal{H}_t(a, b; c) F(a, b; c - k; t) = \frac{abk}{c-k} F(1+a, 1+b; 1+c-k; t) ,$$

which can easily be verified by means of the definition of the hypergeometric series.

In case $k = 1$ we get immediately that

$$D_\alpha^2(\underline{x})\text{Mod}(\alpha, -1; 1; \underline{x}) = 0 .$$

This proves the Lemma. \square

Returning to the $(2k)$ -iterated hyperbolic Dirac equation, we propose the following expression for the fundamental solution $E_\alpha^{2k}(T, \underline{X})$ in space-time co-ordinates (T, \underline{X}) :

$$\begin{aligned} E_\alpha^{2k}(T, \underline{X}) &= T^\alpha \text{Mod}\left(1 + \alpha - k, -m + k; 1; \frac{X}{T}\right) \left(\frac{X}{T}\right)^{2k-1} E\left(\frac{X}{T}\right) \\ &+ c_{\alpha, 2k} T^\alpha \text{Mod}\left(1 + \alpha - k, -k; 1; \frac{X}{T}\right) , \end{aligned}$$

with $c_{\alpha, 2k}$ a constant that will be determined in an appropriate way. This goes as follows : using the definition of the modulation factors, we get immediately

$$\begin{aligned} E_\alpha^{2k}(T, \underline{X}) &= \frac{T^\alpha \underline{x}^{2k}}{A_m |\underline{x}|^m} F\left(k - \frac{\alpha + m}{2}, k + \frac{1 - \alpha - m}{2}; 1 + k - \frac{m}{2}; |\underline{x}|^2\right) \\ &+ c'_{a, 2k} T^\alpha F\left(-\frac{\alpha}{2}, \frac{1 - \alpha}{2}; 1 - k + \frac{m}{2}; |\underline{x}|^2\right) . \end{aligned}$$

Rewriting everything in terms of the hyperbolic polar co-ordinates

$$\epsilon T + \underline{X} = \rho \xi = (T^2 - |\underline{X}|^2)^{\frac{1}{2}} (\epsilon \tau + (\tau^2 - 1)^{\frac{1}{2}} \underline{\xi}) ,$$

we get

$$\begin{aligned} &\frac{T^\alpha \underline{x}^{2k}}{A_m |\underline{x}|^m} F\left(k - \frac{\alpha + m}{2}, k + \frac{1 - \alpha - m}{2}; 1 + k - \frac{m}{2}; |\underline{x}|^2\right) \\ &= (-1)^k \frac{\Gamma(1 + k - \frac{m}{2}) \Gamma(\frac{m}{2})}{2^{1 + \frac{m}{2} - k} \pi^{\frac{m}{2}}} \rho^\alpha (\tau^2 - 1)^{\frac{k}{2} - \frac{m}{4}} P_{\alpha + \frac{m}{2} - k}^{\frac{m}{2} - k}(\tau) \end{aligned}$$

and

$$\begin{aligned} &c'_{a, 2k} T^\alpha F\left(-\frac{\alpha}{2}, \frac{1 - \alpha}{2}; 1 - k + \frac{m}{2}; |\underline{x}|^2\right) \\ &= c'_{a, 2k} \frac{\Gamma(1 + \frac{m}{2} - k)}{2^{k - \frac{m}{2}}} \rho^\alpha (\tau^2 - 1)^{\frac{k}{2} - \frac{m}{4}} P_{\alpha + \frac{m}{2} - k}^{k - \frac{m}{2}}(\tau) . \end{aligned}$$

Adding these functions, we thus get :

$$E_{\alpha}^{2k}(T, \underline{X}) = (-1)^k \frac{\Gamma(1+k-\frac{m}{2}) \Gamma(\frac{m}{2})}{2^{1+\frac{m}{2}-k} \pi^{\frac{m}{2}}} \rho^{\alpha} (\tau^2 - 1)^{\frac{k}{2}-\frac{m}{4}} \\ \left[P_{\alpha+\frac{m}{2}-k}^{\frac{m}{2}-k}(\tau) + (-1)^k \frac{2^{1+m-2k} \pi^{\frac{m}{2}} \Gamma(1+\frac{m}{2}-k)}{\Gamma(1+k-\frac{m}{2}) \Gamma(\frac{m}{2})} c'_{\alpha,2k} P_{\alpha+\frac{m}{2}-k}^{k-\frac{m}{2}}(\tau) \right] .$$

Choosing $c'_{\alpha,2k}$ in such a way that the factor between square brackets reduces to

$$\left[P_{\alpha+\frac{m}{2}-k}^{\frac{m}{2}-k}(\tau) - \frac{\Gamma(1+\alpha+m-2k)}{\Gamma(1+\alpha)} P_{\alpha+\frac{m}{2}-k}^{k-\frac{m}{2}}(\tau) \right] ,$$

we find with the aid of formula (17) that the fundamental solution for the $(2k)$ -iterated hyperbolic Dirac equation is, up to a constant, given by

$$E_{\alpha}^{2k}(T, \underline{X}) \sim \frac{\sin(k-\frac{m}{2}) \pi}{e^{i\pi(k-\frac{m}{2})}} \rho^{\alpha} (\tau^2 - 1)^{\frac{k}{2}-\frac{m}{4}} Q_{\alpha+\frac{m}{2}-k}^{k-\frac{m}{2}}(\tau) .$$

Note the appearance of the sine function, clearly indicating the fact that this formula does not yield a fundamental solution in case of an even dimension m . Recalling formula (22) we then find the function $E_{\alpha}^{2k}(T, \underline{X})$ for odd dimensions m in terms of a Gegenbauer function of the second kind :

$$E_{\alpha}^{2k}(T, \underline{X}) \sim \rho^{\alpha} D_{\alpha}^{\frac{m+1}{2}-k}(\tau) .$$

This formula is also valid in case of an even dimensions m , and that will be verified explicitly in the next subsection, where the fundamental solution for the iterated hyperbolic Dirac operator will be obtained by means of a different method. One can also see this by means of the same argument that was used in Chapter 2 when we determined the hyperbolic fundamental solution. There it was explicitly shown that the fundamental solution obtained by means of the limit procedure, i.e. the method that must be used in case of an even dimension and that leads to logarithmic functions, also yields the expression in terms of the Gegenbauer function of the second kind. We will however not repeat these calculations here.

- **Odd Natural Powers of the Hyperbolic Dirac Operator**

In case the power ∂_X^s of the hyperbolic Dirac operator is odd, we are dealing with a vector-valued differential operator. We can again apply the same trick to obtain a fundamental solution for this operator, by using Theorem 3.3 to modulate the Cauchy kernel $E(\underline{x})$, which yields

$$E_\alpha^{2k+1}(\underline{x}) = \text{Mod}(\alpha - k, 1 - m + k; 0; \underline{x}) \underline{x}^{2k} E(\underline{x}) ,$$

and by adding a particular regular nullsolution which allows us to find an expression for $E_\alpha^{2k+1}(T, \underline{X})$ in terms of the Gegenbauer function of the second kind.

Since we will obtain these results in the next subsection by means of a different method, we will not repeat this argument. Instead, it suffices to note that the α -homogeneous fundamental solution for the operator ∂_X^{2k+1} may easily be derived from the $(\alpha+1)$ -homogeneous fundamental solution for the operator ∂_X^{2k+2} . Hence,

$$\begin{aligned} E_\alpha^{2k+1}(T, \underline{X}) &\sim \partial_X E_{\alpha+1}^{2k+2}(T, \underline{X}) \\ &= \rho^\alpha \xi(\Gamma_{1,m} + 1 + \alpha) D_{\alpha+1}^{\frac{m+1}{2}-k}(\tau) . \end{aligned}$$

Using the recurrence relations (25) for the Gegenbauer function, this gives rise to the following fundamental solution for the $(2k+1)$ -iterated hyperbolic Dirac equation :

$$\begin{aligned} E_\alpha^{2k+1}(T, \underline{X}) &\sim \partial_X E_{\alpha+1}^{2k+2}(T, \underline{X}) \\ &= \rho^\alpha \left(D_{\alpha-1}^{\frac{m+1}{2}-k}(\tau) \xi - D_\alpha^{\frac{m+1}{2}-k}(\tau) \epsilon \right) . \end{aligned}$$

4.2 Complex Powers of the Hyperbolic Dirac Operator

In this section the definition for powers of the hyperbolic Dirac operator will be extended from natural powers, treated in the previous section, to arbitrary complex powers. Instead of using the technique from the previous section, by projecting the s -iterated hyperbolic Dirac equation on the Klein model for the hyperbolic unit ball, we will now use a different technique, based on Riesz distributions. This technique was already used in Chapter 2, when we determined the hyperbolic fundamental solution, but here it will be generalized to arbitrary complex powers.

In order to define arbitrary complex powers of the *hyperbolic* Dirac operator, we first need to define arbitrary complex powers ∂_X^μ of the Dirac operator on the real orthogonal space $\mathbb{R}^{1,m}$. Complex powers of the *hyperbolic* Dirac operator will then easily be found by letting the operator ∂_X^μ act on homogeneous functions, i.e. sections of the homogeneous Clifford line-bundle $\mathbb{R}_{1,m;\alpha}$.

In order to define the operator ∂_X^μ for $\mu \in \mathbb{C}$ we must first focus on the natural powers. Since the technique used here differs from the one of the previous section, it is worthwhile to reconsider these powers :

1. Natural Powers of the Dirac Operator ∂_X on $\mathbb{R}^{1,m}$

First of all, the operator ∂_X^k is defined for all integer powers $k \in \mathbb{N}$.

For $k = 2l$ we get $\partial_X^{2l} = \square_m^l$. Since $Z_{-2l} = \square_m^l \delta(X)$ we get for all $f \in \mathcal{D}'_+(\mathbb{R}^{1,m})$

$$\partial_X^{2l} f = \square_m^l f = \square_m^l (\delta(X) * f) = Z_{-2l} * f ,$$

whence the operator ∂_X^{2l} may be defined as a convolution operator on $\mathcal{D}'_+(\mathbb{R}^{1,m})$:

$$\partial_X^{2l} f = Z_{-2l} * f \text{ for all } f \in \mathcal{D}'_+(\mathbb{R}^{1,m}) .$$

For odd powers $k = 2l + 1$ we have $\partial_X^{2l+1} f = \square_m^l (\partial_X f) = (\partial_X Z_{-2l}) * f$. This prompts the following definition :

$$\partial_X^{2l+1} f = \partial_X Z_{-2l} * f \text{ for all } f \in \mathcal{D}'_+(\mathbb{R}^{1,m}) .$$

In the following Lemma the distribution $\partial_X Z_{-2k}$ is rewritten as a new distribution :

Lemma 4.2 *For all $\mu \in \mathbb{C}$ and for all $\varphi \in \mathcal{D}(\mathbb{R}^{1,m})$ we have*

$$\langle \partial_X Z_\mu, \varphi \rangle = \langle \frac{X Z_{\mu-2}}{\mu-2}, \varphi \rangle$$

proof : Consider an arbitrary $\varphi \in \mathcal{D}(\mathbb{R}^{1,m})$. By definition we have :

$$\langle \partial_X Z_\mu, \varphi \rangle = -\epsilon \langle Z_\mu, \partial_T \varphi \rangle + \sum_{i=1}^m e_i \langle Z_\mu, \partial_{X_i} \varphi \rangle$$

Let us first consider μ such that $\operatorname{Re}(\mu) > 1 + m$.

Putting

$$c(\mu, m) = \pi^{\frac{m-1}{2}} 2^{\mu-1} \Gamma\left(\frac{\mu}{2}\right) \Gamma\left(\frac{\mu+1-m}{2}\right)$$

and using partial integration the first term yields :

$$\begin{aligned} \langle Z_\mu, \partial_T \varphi \rangle &= \frac{1}{c(\mu, m)} \int_{\mathbb{R}^m} d\underline{X} \int_{|\underline{X}|}^\infty (T^2 - |\underline{X}|^2)^{\frac{\mu-m-1}{2}} \partial_T \varphi(T, \underline{X}) \\ &= \frac{1+m-\mu}{c(\mu, m)} \int_{\mathbb{R}^m} d\underline{X} \int_{|\underline{X}|}^\infty T(T^2 - |\underline{X}|^2)^{\frac{\mu-m-3}{2}} \varphi(T, |\underline{X}|) , \end{aligned}$$

where we have used the fact that φ has a compact support and that $\operatorname{Re}(\mu) > 1 + m$. Using the definition of the Riesz distribution $Z_{\mu-2}$, this can also be written as

$$\langle Z_\mu, \partial_T \varphi \rangle = -\frac{1}{\mu-2} \langle T Z_{\mu-2}, \varphi \rangle$$

The same argument can be used to obtain

$$\begin{aligned} \langle Z_\mu, \partial_{X_i} \varphi \rangle &= \frac{1}{c(\mu, m)} \int_0^\infty dT \int_{B_m(T)} d\underline{X} (T^2 - |\underline{X}|^2)^{\frac{\mu-m-1}{2}} \partial_{X_i} \varphi(T, \underline{X}) \\ &= \frac{1}{\mu-2} \langle X_i Z_{\mu-2}, \varphi \rangle \end{aligned}$$

where we have introduced $B_m(T)$ as the ball with radius T in \mathbb{R}^m .

This means that for all $\mu \in \mathbb{C}$ such that $\operatorname{Re}(\mu) > 1 + m$ and for all $\varphi \in \mathcal{D}(\mathbb{R}^{1,m})$ we have :

$$\langle \partial_X Z_\mu, \varphi \rangle = \frac{1}{\mu-2} \langle X Z_{\mu-2}, \varphi \rangle . \quad (4.1)$$

Note that the distribution at the right-hand side does *not* have a pole at $\mu = 2$ since

$$\lim_{\mu \rightarrow 2} X Z_{\mu-2} = X \delta(X) = 0 ,$$

from which it follows that $\frac{X Z_{\mu-2}}{(\mu-2)}$ is well-defined for $\mu = 2$ if we remove the apparent singularity, by putting

$$\lim_{\mu \rightarrow 2} \langle \frac{X Z_{\mu-2}}{\mu-2}, \varphi \rangle = \lim_{\mu \rightarrow 2} \langle \partial_X Z_\mu, \varphi \rangle .$$

As $\square_m Z_2 = \delta(X) = \partial_X^2 Z_2$, we thus have :

$$\lim_{\mu \rightarrow 2} \langle \frac{X Z_{\mu-2}}{\mu-2}, \varphi \rangle = \langle E(X), \varphi \rangle ,$$

with $E(X) = \partial_X Z_2$ the fundamental solution for the Dirac operator ∂_X on $\mathbb{R}^{1,m}$.

This means that both sides of equation (4.1) define a holomorphic function of μ for all $\varphi \in \mathcal{D}(\mathbb{R}^{1,m})$. Since those functions coincide in the region where $\text{Re}(\mu) > 1 + m$, they are equal. As φ was chosen arbitrarily, this proves the lemma. \square

Eventually this yields the natural powers of the Dirac operator on $\mathbb{R}^{1,m}$ as convolution operators : for all $k \in \mathbb{N}$ and for all $f \in \mathcal{D}'_+(\mathbb{R}^{1,m})$ we have

$$\begin{aligned} \partial_X^{2k} f &= Z_{-2k} * f \\ \partial_X^{2k+1} f &= \partial_X Z_{-2k} * f = -\frac{X Z_{-2k-2} * f}{2k+2} \end{aligned}$$

2. Complex Powers of the Dirac Operator ∂_X on $\mathbb{R}^{1,m}$

On the analogy of what was done in [7] and [23] we then define $\partial_X^\mu f$ as the following convolution operator, for all $f \in \mathcal{D}'_+(\mathbb{R}^{1,m})$:

$$\begin{aligned} \partial_X^\mu f &= \left(\frac{1 + e^{i\pi\mu}}{2} Z_{-\mu} - \frac{1 - e^{i\pi\mu}}{2} \frac{X Z_{-\mu-1}}{1 + \mu} \right) * f \\ &= \left(\frac{1 + e^{i\pi\mu}}{2} Z_{-\mu} + \frac{1 - e^{i\pi\mu}}{2} \frac{\Gamma(-\frac{\mu}{2}) \Gamma(\frac{1-m-\mu}{2})}{\Gamma(\frac{1-\mu}{2}) \Gamma(\frac{-m-\mu}{2})} \xi Z_{-\mu} \right) * f \end{aligned}$$

Introducing $c_\pm = \frac{1 \pm e^{i\pi\mu}}{2}$, $\partial_X^\mu f$ can also be written as

$$\partial_X^\mu f = (c_+ Z_{-\mu} + c_- \partial_X Z_{1-\mu}) * f .$$

Note that, as a linear combination of Riesz distributions, the operator ∂_X^μ belongs to $\mathcal{D}'_+(\mathbb{R}^{1,m})$.

Now that we have defined complex powers ∂_X^μ of the Dirac operator on the real orthogonal space $\mathbb{R}^{1,m}$, their fundamental solutions may be constructed. For that purpose we consider the distributional equation

$$\partial_X^\mu E^\mu(X) = (c_+ Z_{-\mu} + c_- \partial_X Z_{1-\mu}) * E^\mu(X) = \delta(X) .$$

The superscript μ indicates the fact that $E^\mu(X)$ is the fundamental solution for the operator ∂_X^μ . For $\mu = 1$, this superscript is omitted.

Since $Z_\mu * Z_{-\mu} = Z_0 = \delta(X)$ and $\partial_X Z_{1-\mu} * \partial_X Z_{1+\mu} = \square Z_2 = \delta(X)$, it seems natural to look for a fundamental solution which has the following form :

$$\begin{aligned} E^\mu(X) &= aZ_\mu + b\partial_X Z_{1+\mu} \\ &= aZ_\mu + b\frac{XZ_{\mu-1}}{\mu-1}, \end{aligned}$$

with a and b two complex constants that still need to be determined. Letting the operator ∂_X^μ act on $E^\mu(X)$, one finds four terms :

$$\begin{aligned} ac_+ Z_{-\mu} * Z_\mu &= ac_+ \delta(X) \\ bc_+ Z_{-\mu} * \partial_X Z_{\mu+1} &= bc_+ \partial_X Z_1 \\ ac_- \partial_X Z_{1-\mu} * Z_\mu &= ac_- \partial_X Z_1 \\ bc_- \partial_X Z_{1-\mu} * \partial_X Z_{\mu+1} &= bc_- \delta(X) \end{aligned}$$

In order to obtain a fundamental solution, we choose $a = c_+$ and $b = -c_-$ such that

$$\begin{aligned} \partial_X^\mu E^\mu(X) &= (c_+^2 - c_-^2)\delta(X) \\ &= e^{i\pi\mu}\delta(X) \end{aligned}$$

Let us therefore define the fundamental solution for the operator ∂_X^μ , for all $\mu \in \mathbb{C}$, as

$$\begin{aligned} E^\mu(X) &= \frac{1 + e^{-i\pi\mu}}{2} Z_\mu + \frac{1 - e^{-i\pi\mu}}{2} \partial_X Z_{1+\mu} \\ &= \frac{1 + e^{-i\pi\mu}}{2} Z_\mu + \frac{1 - e^{-i\pi\mu}}{2} \frac{XZ_{\mu-1}}{\mu-1}. \end{aligned}$$

General complex powers of the *hyperbolic* Dirac operator may be defined by letting the convolution operator ∂_X^μ act on sections of the homogeneous Clifford bundle $\mathbb{R}_{1,m;\alpha}$. This means that complex powers of the hyperbolic Dirac operator are defined as convolution operators acting on homogeneous distributions $f \in \mathcal{D}'_+(\mathbb{R}^{1,m})$.

In what follows an α -homogeneous fundamental solution $E_\alpha^\mu(X)$ for these operators is constructed, thus yielding a basic example of a homogeneous distribution in $\mathcal{D}'_+(\mathbb{R}^{1,m})$. For that purpose we consider the equation :

$$\partial_X^\mu E_\alpha^\mu(X) = T_+^{\alpha+m-\mu} \delta(\underline{X}). \quad (4.2)$$

The right-hand side is a generalization to arbitrary complex powers of (2.1), and expresses the fact that we are looking for a fundamental solution $E_\alpha^\mu(X)$ which is homogeneous of degree α . For fixed μ we exclude those α for which $\alpha + m \in 1 + \mu - \mathbb{N}$.

Since $E_\mu(X)$ is the fundamental solution for the operator ∂_X^μ , we immediately get :

$$\begin{aligned} E_\alpha^\mu(X) &= E^\mu(X) * T_+^{\alpha+m-\mu} \delta(\underline{X}) \\ &= \left(\frac{1 + e^{-i\pi\mu}}{2} Z_\mu + \frac{1 - e^{-i\pi\mu}}{2} \partial_X Z_{1+\mu} \right) * T_+^{\alpha+m-\mu} \delta(\underline{X}) \end{aligned}$$

Let us therefore calculate $Z_\sigma * T_+^{\alpha+m-\mu} \delta(\underline{X})$, σ being an arbitrary complex number and $\alpha + m \notin 1 + \mu - \mathbb{N}$. Denoting $R = |\underline{X}|$, we get :

$$\begin{aligned} Z_\sigma * T_+^{\alpha+m-\mu} \delta(\underline{X}) &= H(T-R) \frac{\int_0^{T-R} ((T-S)^2 - R^2)^{\frac{\sigma-m-1}{2}} S^{\alpha+m-\mu} dS}{\pi^{\frac{m-1}{2}} 2^{\sigma-1} \Gamma(\frac{\sigma}{2}) \Gamma(\frac{\sigma+1-m}{2})} \\ &= H(T-R) \frac{(T^2 - R^2)^{\frac{\sigma-m-1}{2}} (T-R)^{1+\alpha+m-\mu}}{\pi^{\frac{m-1}{2}} 2^{\sigma-1} \Gamma(\frac{\sigma}{2}) \Gamma(\frac{\sigma+1-m}{2})} \times \\ &\quad \int_0^1 [(1-t)(1-zt)]^{\frac{\sigma-m-1}{2}} t^{\alpha+m-\mu} dt \end{aligned}$$

where we have put $z = \frac{T-R}{T+R}$. Using Euler's representation formula for the hypergeometric function, the integral can be written as

$$\begin{aligned} &\frac{\Gamma(1+\alpha+m-\mu) \Gamma(\frac{\sigma+1-m}{2})}{\Gamma(\alpha-\mu + \frac{\sigma+3+m}{2})} \times \\ &F\left(\frac{1+m-\sigma}{2}, 1+\alpha+m-\mu; \alpha-\mu + \frac{\sigma+3+m}{2}; z\right), \end{aligned}$$

if we assume that $\text{Re}(\sigma) > m-1$. Since

$$z = \frac{T-R}{T+R} = \frac{\tau - (\tau^2 - 1)^{\frac{1}{2}}}{\tau + (\tau^2 - 1)^{\frac{1}{2}}} \quad \text{for } \tau = \frac{T}{(T^2 - R^2)^{\frac{1}{2}}},$$

we find with the aid of (15) that the hypergeometric function is equal to an associated Legendre function of the second kind :

$$e^{-i\frac{m-\sigma}{2}\pi} \frac{\Gamma(\alpha-\mu + \frac{\sigma+3+m}{2})}{\sqrt{\pi} 2^{\frac{m-\sigma}{2}} \Gamma(1+\alpha+m-\mu)} \frac{(\tau + (\tau^2 - 1)^{\frac{1}{2}})^{1+\alpha+m-\mu}}{(\tau^2 - 1)^{\frac{m-\sigma}{4}}} Q_{\alpha-\mu + \frac{\sigma+m}{2}}^{\frac{m-\sigma}{2}}(\tau).$$

With the aid of (16), we then find :

$$Z_\sigma * T_+^{\alpha+m-\mu} \delta(\underline{X}) = H(T-R) e^{i\pi \frac{\sigma-m-1}{2}} \rho^{\alpha+\sigma-\mu} \frac{\Gamma\left(\frac{1+m-\sigma}{2}\right) D_{\alpha+\sigma-\mu}^{\frac{1+m-\sigma}{2}}(\tau)}{2^{\sigma-1} \pi^{\frac{m-1}{2}} \Gamma\left(\frac{\sigma}{2}\right)},$$

with $\rho = (T^2 - R^2)^{\frac{1}{2}} = Q_{1,m}(T, \underline{X})^{\frac{1}{2}}$ the hyperbolic norm on the future cone.

Because the Gegenbauer functions are defined in the complex plane cut along $]-\infty, 1]$, the factor $H(T-R)$ may be omitted. Indeed, as $\tau \in \mathbb{R}^+$ the condition $|\arg(\tau-1)| < \pi$ is equivalent with $\tau > 1 \Leftrightarrow T > R$. The Gegenbauer function has zeroes for $\frac{1+m-\sigma}{2} \in -\mathbb{N}$, cancelling the poles of the Gamma function $\Gamma\left(\frac{1+m-\sigma}{2}\right)$, and poles at $\alpha - \mu = -k - m$ with $k \in \mathbb{N}_0$. Note that these poles were to be expected since the distribution $T_+^{\alpha+m-\mu}$ also has poles at these values. This leaves us with two distributions, the left-hand side and the right-hand side of previous expression, having poles at $\alpha - \mu = -k - m$ and being equal in the strip where $\text{Re}(\sigma) > m-1$. By analytic continuation, these distributions are equal in the whole complex plane.

We thus have :

$$Z_\mu * T_+^{\alpha+m-\mu} \delta(\underline{X}) = \rho^\alpha \frac{e^{i\pi \frac{\mu-m-1}{2}} \Gamma\left(\frac{1+m-\mu}{2}\right)}{2^{\mu-1} \pi^{\frac{m-1}{2}} \Gamma\left(\frac{\mu}{2}\right)} D_\alpha^{\frac{1+m-\mu}{2}}(\tau)$$

and

$$\partial_X Z_{1+\mu} * T_+^{\alpha+m-\mu} \delta(\underline{X}) = \partial_X \left[\rho^{1+\alpha} \frac{e^{i\pi \frac{\mu-m}{2}} \Gamma\left(\frac{m-\mu}{2}\right)}{2^\mu \pi^{\frac{m-1}{2}} \Gamma\left(\frac{1+\mu}{2}\right)} D_{1+\alpha}^{\frac{m-\mu}{2}}(\tau) \right]$$

Recalling the polar decompositions $X = \rho\xi$ and $\partial_X = \xi(\partial_\rho + \frac{1}{\rho}\Gamma_{1,m})$ and using the fact that $\Gamma_{1,m}(\tau) = \Gamma(\xi \cdot \epsilon) = \xi \wedge \epsilon$, we get :

$$\partial_X \rho^{1+\alpha} D_{1+\alpha}^{\frac{m-\mu}{2}}(\tau) = \xi \rho^\alpha \left((m-\mu) D_\alpha^{\frac{m-\mu}{2}+1}(\tau) \xi \wedge \epsilon + (1+\alpha) D_{1+\alpha}^{\frac{m-\mu}{2}}(\tau) \right).$$

Since

$$\xi(\xi \wedge \epsilon) = \epsilon - \tau \xi$$

we eventually get, hereby using (25) :

$$\partial_X \rho^{1+\alpha} D_{1+\alpha}^{\frac{m-\mu}{2}}(\tau) = (\mu-m) \rho^\alpha \left(D_{\alpha-1}^{\frac{m-\mu}{2}+1}(\tau) \xi - D_\alpha^{\frac{m-\mu}{2}+1}(\tau) \epsilon \right)$$

This yields the fundamental solution $E_\alpha^\mu(X)$ for an arbitrary complex power of the hyperbolic Dirac operator, for all $\mu \in \mathbb{C}$ and $\alpha \neq \mu - m - k$, where $k \in \mathbb{N}_0$:

$$\begin{aligned} E_\alpha^\mu(X) &= \frac{1 + e^{-i\pi\mu}}{2} \rho^\alpha \frac{e^{-i\pi \frac{m+1-\mu}{2}}}{2^{\mu-1} \pi^{\frac{m-1}{2}}} \frac{\Gamma\left(\frac{1+m-\mu}{2}\right)}{\Gamma\left(\frac{\mu}{2}\right)} D_\alpha^{\frac{1+m-\mu}{2}}(\tau) \\ &- \frac{1 - e^{-i\pi\mu}}{2} \rho^\alpha \frac{e^{-i\pi \frac{m-\mu}{2}}}{2^{\mu-1} \pi^{\frac{m-1}{2}}} \frac{\Gamma\left(1 + \frac{m-\mu}{2}\right)}{\Gamma\left(\frac{1+\mu}{2}\right)} \left(D_{\alpha^{-1}}^{\frac{m-\mu}{2}+1}(\tau) \xi - D_\alpha^{\frac{m-\mu}{2}+1}(\tau) \epsilon \right) \end{aligned}$$

Remark : Note that for the case of the natural powers these expressions coincide with the ones that were found in the previous section, by means of modulation Theorem 3.3. The advantage of working with the Riesz distributions is that we obtained the solution for both the even and the odd powers of the hyperbolic Dirac operator by means of the same argument.

Chapter 5

Function Theory on the Hyperbolic Unit Ball

The construction itself is an art, its application to the world an evil parasite. (L. Brouwer)

In this chapter a function theory on the hyperbolic unit ball is developed. By means of Stokes Theorem and properties of the Gegenbauer functions, basic integral formulae will be established and these will be used to obtain a Taylor (and Laurent) expansion for hyperbolic monogenics on H_+ . The main results are then restated for the Klein model of the hyperbolic unit ball.

Throughout this Chapter, the hyperbolic angular operator will be denoted by Γ instead of $\Gamma_{1,m}$.

5.1 The Boosted Fundamental Solution

The aim of this section is to obtain a fundamental solution for the hyperbolic Dirac equation having its singularities on an *arbitrary* ray inside the future cone FC . The hyperbolic fundamental solution $E_\alpha(X) = E_\alpha(T, \underline{X})$ obtained in Chapter 2 becomes singular for space-time vectors belonging to the time-like ray through ϵ . But if we want to establish integral formulae for arbitrary points on the hyperbolic unit ball H_+ , we need to remove these singularities from the time-like ray through ϵ to an arbitrary ray inside the future cone.

In the flat Euclidean case, one first solves the Dirac equation

$$\underline{\partial} E(\underline{x}) = -\delta(\underline{x}) , \quad \underline{x} \in \mathbb{R}^m ,$$

for the Cauchy kernel $E(\underline{x})$, and one then *translates* the singularity from the origin to an arbitrary point $\underline{y} \in \mathbb{R}^m$:

$$\partial E(\underline{x} - \underline{y}) = -\delta(\underline{x} - \underline{y}) , \quad \underline{x} \in \mathbb{R}^m .$$

For the hyperbolic unit ball, the situation is more complicated because H_+ is not translationally invariant. However, as H_+ is $\text{SO}(1, m)$ -invariant we can use hyperbolic rotations (or *Lorentz boost*) to remove the singularity. We will illustrate this in two different ways : first of all we will consider the space-time picture and explicitly construct the boosted hyperbolic fundamental solution in co-ordinates (T, \underline{X}) , hereby making use of the group $\text{SO}(1, m)$. Afterwards we will show how this boosted hyperbolic fundamental solution can be obtained by means of the group $\text{Spin}(1, m)$, the double covering group for $\text{SO}(1, m)$.

By definition, a Lorentz boost or hyperbolic rotation is an element of the orthogonal group $\text{SO}(1, m)$. This means that a Lorentz boost $B \in \text{SO}(1, m)$ is a linear transformation on $\mathbb{R}^{1, m}$ leaving the quadratic form $Q_{1, m}(T, \underline{X})$ unchanged :

$$Q_{1, m}(B(T, \underline{X})) = Q_{1, m}(T, \underline{X}) .$$

Denoting by $I_{1, m}$ the matrix of the quadratic form $Q_{1, m}(T, \underline{X})$,

$$I_{1, m} = \text{Diag}(1, -1, \dots, -1) ,$$

we thus have for the matrix B of a Lorentz transformation :

$$I_{1, m} = B^t I_{1, m} B .$$

A *pure boost* is defined as a transformation $B(\underline{\omega})$ mixing up the temporal co-ordinate T and the spatial co-ordinates \underline{X} in a certain direction $\underline{\omega}$, and belongs to the Lie algebra generated by bivectors of the form $v \epsilon \underline{\omega}$, with $\underline{\omega} \in S^{m-1}$ and $v \in \mathbb{R}$.

Because a pure boost $B(\underline{\omega})$ can always be obtained as the composition $R(e_1, \underline{\omega}) \circ B(e_1) \circ R(\underline{\omega}, e_1)$, where $R(\underline{\omega}, \underline{\xi})$ stands for the rotation mapping $\underline{\omega} \in S^{m-1} \mapsto \underline{\xi} \in S^{m-1}$, it is sufficient to consider pure boosts $B(e_1)$ in the direction e_1 only, whence the label (e_1) will be omitted. These boosts can all be represented by a matrix of the following form :

$$B_\theta = \begin{pmatrix} \cosh \theta & \sinh \theta & 0 \\ \sinh \theta & \cosh \theta & 0 \\ 0 & 0 & I_{m-1} \end{pmatrix} , \quad \theta \in \mathbb{R} ,$$

where I_{m-1} stands for the unity matrix in $(m-1)$ dimensions. Under this transformation, a space-time vector (T, \underline{X}) transforms in the following way :

$$\begin{pmatrix} T' \\ X'_1 \\ \vdots \\ X'_m \end{pmatrix} = B_\theta \begin{pmatrix} T \\ X_1 \\ \vdots \\ X_m \end{pmatrix} = \begin{pmatrix} T \cosh \theta + X_1 \sinh \theta \\ X_1 \cosh \theta + T \sinh \theta \\ \vdots \\ X_m \end{pmatrix}.$$

When considering Lorentz boosts $B_\theta \in \text{SO}(1, m)$ one often uses the so-called *rapidity* $v = \tanh \theta \in]-1, 1[$:

$$\begin{cases} T' = \frac{T + vX_1}{(1 - v^2)^{\frac{1}{2}}} \\ X'_1 = \frac{X_1 + vT}{(1 - v^2)^{\frac{1}{2}}} \end{cases}.$$

Letting this boost B_θ act on the hyperbolic Dirac equation 2.2, we get :

$$(\epsilon \partial_T - \partial_{\underline{X}}) B_\theta(E_\alpha(T, \underline{X})) = B_\theta(T_+^{\alpha+m-1} \delta(\underline{X})),$$

with

- $B_\theta(E_\alpha(T, \underline{X})) = E_\alpha^\theta(T, \underline{X})$ the function that needs to be determined, i.e. the boosted fundamental solution
- $B_\theta(T_+^{\alpha+m-1} \delta(\underline{X}))$ the distribution at the right-hand side in the new variables (T', \underline{X}') , representing a delta function along the ray through (T', \underline{X}') :

$$\begin{aligned} B_\theta(T_+^{\alpha+m-1} \delta(\underline{X})) &= (T')_+^{\alpha+m-1} \delta(\underline{X}') \\ &= \left(\frac{T + vX_1}{(1 - v^2)^{\frac{1}{2}}} \right)_+^{\alpha+m-1} \delta \left(\frac{X_1 + vT}{(1 - v^2)^{\frac{1}{2}}} \right) \delta(\underline{X} \wedge e_1), \end{aligned}$$

with $\delta(\underline{X} \wedge e_1) = \Pi_{j=2}^m \delta(X_j)$, which can be reduced to

$$\begin{aligned} B_\theta(T_+^{\alpha+m-1} \delta(\underline{X})) &= \left(\frac{T - v^2 T}{(1 - v^2)^{\frac{1}{2}}} \right)_+^{\alpha+m-1} \delta \left(\frac{X_1 + vT}{(1 - v^2)^{\frac{1}{2}}} \right) \delta(\underline{X} \wedge e_1) \\ &= (1 - v^2)^{\frac{\alpha+m}{2}} T_+^{\alpha+m-1} \delta(X_1 + vT) \delta(\underline{X} \wedge e_1). \end{aligned}$$

We thus arrive at the *boosted hyperbolic Dirac equation* :

$$(\epsilon\partial_T - \partial_{\underline{X}})E_\alpha^\theta(T, \underline{X}) = (1 - v^2)^{\frac{\alpha+m}{2}} T_+^{\alpha+m-1} \delta(X_1 + vT) \delta(\underline{X} \wedge e_1) \quad (5.1)$$

To solve this equation, we use the method involving Riesz distributions. Let us therefore first consider the following related scalar equation :

$$\square_m \Phi_\alpha^\theta(T, \underline{X}) = (1 - v^2)^{\frac{\alpha+m}{2}} T_+^{\alpha+m-1} \delta(X_1 + vT) \delta(\underline{X} \wedge e_1) , \quad (5.2)$$

such that $E_\alpha^\theta(T, \underline{X}) = (\epsilon\partial_T - \partial_{\underline{X}})\Phi_\alpha^\theta(T, \underline{X})$. As the distribution at the right-hand side belongs to $\mathcal{D}'_+(\mathbb{R}^{1,m})$, we immediately get :

$$\begin{aligned} \Phi_\alpha^\theta(T, \underline{X}) &= (1 - v^2)^{\frac{\alpha+m}{2}} Z_2 * T_+^{\alpha+m-1} \delta(X_1 + vT) \delta(\underline{X} \wedge e_1) \\ &= \frac{(1 - v^2)^{\frac{\alpha+m}{2}}}{2\pi^{\frac{m-1}{2}} \Gamma\left(\frac{3-m}{2}\right)} \int_0^\infty \frac{H(T - S - |\underline{X} + vSe_1|)}{((T - S)^2 - |\underline{X} + vSe_1|^2)^{\frac{m-1}{2}}} S^{\alpha+m-1} dS \end{aligned}$$

Let us for a moment put $f(S) = T - S - |\underline{X} + vSe_1|$. We then have :

$$\begin{aligned} f(S) = 0 &\implies (T - S)^2 = |\underline{X} + vSe_1|^2 \\ &\iff S = S_\pm \end{aligned}$$

with

$$S_\pm = \frac{T + vX_1 \pm ((T + vX_1)^2 - (1 - v^2)(T^2 - |\underline{X}|^2))^{\frac{1}{2}}}{1 - v^2} .$$

Note that

$$(T + vX_1)^2 - (1 - v^2)(T^2 - |\underline{X}|^2) = (1 - v^2)|\underline{X}'|^2 \geq 0 ,$$

such that both S_+ and S_- are real-valued. One can easily verify that only $S = S_-$ gives a root of $f(S)$, whence

$$\Phi_\alpha^\theta(T, \underline{X}) = \frac{(1 - v^2)^{\frac{\alpha+m}{2}}}{2\pi^{\frac{m-1}{2}} \Gamma\left(\frac{3-m}{2}\right)} \int_0^{S_-} \frac{S^{\alpha+m-1}}{((T - S)^2 - |\underline{X} + vSe_1|^2)^{\frac{m-1}{2}}} dS ,$$

where we have omitted a factor $H(S_-)$. We will return to this point later. Denoting the integral on previous line as I_S , we get immediately :

$$I_S = (1 - v^2)^{\frac{1-m}{2}} \frac{S_-^{\alpha+\frac{m+1}{2}}}{S_+^{\frac{m-1}{2}}} \int_0^1 (1 - t)^{\frac{1-m}{2}} \left(1 - \frac{S_-}{S_+} t\right)^{\frac{1-m}{2}} t^{\alpha+m-1} dt .$$

Using Euler's integral formula for the hypergeometric function, we get for those m for which $\text{Re}(\frac{3-m}{2}) < 0$:

$$I_S = \frac{\Gamma(\alpha+m)\Gamma(\frac{3-m}{2})}{(1-v^2)^{\frac{m-1}{2}}\Gamma(\alpha+\frac{m+3}{2})} \frac{S_-^{\alpha+\frac{m+1}{2}}}{S_+^{\frac{m+1}{2}}} F\left(\alpha+m, \frac{m-1}{2}; \alpha+\frac{m+3}{2}; \frac{S_-}{S_+}\right).$$

As

$$\frac{S_-}{S_+} = \frac{\tau - (\tau^2 - 1)^{\frac{1}{2}}}{\tau + (\tau^2 - 1)^{\frac{1}{2}}} \text{ for } \tau = \frac{T + vX_1}{(1-v^2)^{\frac{1}{2}}(T^2 - |\underline{X}|^2)^{\frac{1}{2}}},$$

we get by means of formula (15) :

$$F\left(\alpha+m, \frac{m-1}{2}; \alpha+\frac{m+3}{2}; \frac{S_-}{S_+}\right) = \frac{e^{i\pi\frac{m-1}{2}} \Gamma(\alpha+\frac{m+3}{2})}{2^{\frac{m}{2}-1}\pi^{\frac{1}{2}} \Gamma(\alpha+m)} \times \frac{(\tau + (\tau^2 - 1)^{\frac{1}{2}})^{\alpha+m}}{(\tau^2 - 1)^{\frac{m}{4}-\frac{1}{2}}} Q_{\alpha+\frac{m}{2}}^{1-\frac{m}{2}}(\tau).$$

On the other hand we also have

$$\frac{S_-^{\alpha+\frac{m+1}{2}}}{S_+^{\frac{m+1}{2}}} = \left(\frac{T^2 - |\underline{X}|^2}{1-v^2}\right)^{\frac{1+\alpha}{2}} (\tau + (\tau^2 - 1)^{\frac{1}{2}})^{-\alpha-m},$$

such that by means of expression (22) we will eventually find :

$$\Phi_\alpha^\theta(T, \underline{X}) = \frac{e^{-i\pi\frac{m-1}{2}}}{2\pi^{\frac{m-1}{2}}} \Gamma\left(\frac{m-1}{2}\right) (T^2 - |\underline{X}|^2)^{\frac{\alpha+1}{2}} D_{\alpha+1}^{\frac{m-1}{2}}(\tau).$$

Note that we have omitted a factor $H(S_-)$ when we calculated the integral I_S . This puts no limitations on our result however, because the information encoded in the Heaviside function $H(S_-)$ is also encoded in the cut of the Gegenbauer function $D_{\alpha+1}^{\frac{m-1}{2}}(\tau)$:

$$\tau > 1 \implies S_- > 0.$$

Now that we have found an expression for $\Phi_\alpha^\theta(T, \underline{X})$ we can easily derive an expression for $E_\alpha^\theta(T, \underline{X})$ by letting the Dirac operator $(\epsilon\partial_T - \partial_{\underline{X}})$ act on $\Phi_\alpha^\theta(T, \underline{X})$. To do so, we will use the polar decomposition for the Dirac operator on $\mathbb{R}^{1,m}$. First of all, note that with $X = \epsilon T + \underline{X} = \rho\xi$ we get :

$$\tau = \frac{T + vX_1}{(1-v^2)^{\frac{1}{2}}(T^2 - |\underline{X}|^2)^{\frac{1}{2}}} = \xi \cdot \frac{\epsilon - ve_1}{(1-v^2)^{\frac{1}{2}}},$$

the dot indicating the Clifford inner product on \mathbb{R}^m .

On the other hand, note that the space-time vector at the right-hand side is nothing but the image of ϵ under the inverse Lorentz boost $B_{-\theta}$:

$$B_{-\theta} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} (1-v^2)^{-\frac{1}{2}} \\ -v(1-v^2)^{-\frac{1}{2}} \\ \vdots \\ 0 \end{pmatrix} .$$

This means that in polar representation we have

$$\Phi_{\alpha}^{\theta}(\rho\xi) = \frac{e^{-i\pi\frac{m-1}{2}}}{2\pi^{\frac{m-1}{2}}} \Gamma\left(\frac{m-1}{2}\right) \rho^{\alpha+1} D_{\alpha+1}^{\frac{m-1}{2}}(\xi \cdot B_{-\theta}(\epsilon)) ,$$

such that with $\partial_X = \xi \left(\partial_{\rho} + \frac{1}{\rho} \Gamma \right)$ we get :

$$E_{\alpha}^{\theta}(\rho\xi) = \frac{e^{-i\pi\frac{m+1}{2}}}{\pi^{\frac{m-1}{2}}} \Gamma\left(\frac{m+1}{2}\right) \rho^{\alpha} \left(D_{\alpha-1}^{\frac{m+1}{2}}(\tau)\xi - D_{\alpha}^{\frac{m+1}{2}}(\tau)B_{-\theta}(\epsilon) \right) ,$$

where we have used (25) and where we have put $\tau = \xi \cdot B_{-\theta}(\epsilon)$.

Now that we have obtained a fundamental solution for the Dirac equation on the hyperbolic unit ball becoming singular on an arbitrary ray inside the future cone, we introduce the following notation :

Definition 5.1 Consider two space-time vectors $X = |X|\xi$ and $Y = |Y|\eta$, with ξ and η belonging to the hyperbolic unit ball H_+ , and an arbitrary $\alpha \in \mathbb{C}$ such that $\alpha + m \notin -\mathbb{N}$. The restriction to H_+ of the fundamental solution for the Dirac operator ∂_X on $\mathbb{R}^{1,m}$ which is homogeneous of degree α in X and has its singularities on the ray through the space-time vector η will be denoted as $E_{\alpha}(\xi, \eta)$, and is given by

$$E_{\alpha}(\xi, \eta) = \frac{e^{-i\pi\frac{m+1}{2}}}{\pi^{\frac{m-1}{2}}} \Gamma\left(\frac{m+1}{2}\right) \left(D_{\alpha-1}^{\frac{m+1}{2}}(\tau)\xi - D_{\alpha}^{\frac{m+1}{2}}(\tau)\eta \right) ,$$

where τ denotes the Clifford inner product $\xi \cdot \eta$.

Note that $E_{\alpha}(\xi, \eta) \in \mathcal{H}^{\alpha}(H_+ \setminus \{\eta\})$.

Remark : In terms of this new definition the fundamental solution (2.14) obtained in section 2.2.3 can be represented as

$$E_\alpha(T, \underline{X}) = E_\alpha(X) = \rho^\alpha E_\alpha(\xi, \epsilon) .$$

To conclude this section, we illustrate how to obtain the boosted fundamental solution by means of the action of the Spingroup $\text{Spin}(1, m)$. For that purpose we start from the hyperbolic Dirac equation (2.2) in polar representation :

$$\partial_X E_\alpha(X) = [T_+^{\alpha+m-1} \delta(\underline{X})]_{\text{polar}} .$$

Note that we have not yet written the right-hand side in its polar representation, because this is far from trivial.

Using definition 5.1, the previous expression can also be written as follows :

$$\xi(\Gamma + \alpha) E_\alpha(\xi, \epsilon) = \frac{1}{\rho^{\alpha-1}} [T_+^{\alpha+m-1} \delta(\underline{X})]_{\text{polar}} .$$

It seems intuitively obvious to define the distribution at the right-hand side as $\delta(\xi - \epsilon)$. The mathematical principle underlying this definition is the fact that a delta distribution on a manifold, in casu the delta distribution $\delta(\xi - \epsilon)$ on the hyperbolic unit ball H_+ , can be defined as a delta distribution in the tangent plane at the point where the delta distribution is to be defined. In the tangent plane, one then considers a local co-ordinate system. In the present situation this local system is obtained by the *radial projection* on the tangent plane, defined as the map sending an arbitrary element $\xi \in H_+$ to the intersection of the tangent plane and the ray through ξ . This is precisely the projection on the hyperplane Π leading to the Klein model of the hyperbolic unit ball, and that is how we have derived the hyperbolic Dirac equation in the first place (see section 2.2).

The idea is now to consider an arbitrary element $s \in \text{Spin}(1, m)$ and to let this element act on the hyperbolic Dirac equation in its polar representation. An arbitrary element $s \in \text{Spin}(1, m)$ has the following form :

$$s = \cosh \frac{\theta}{2} + \epsilon \underline{\omega} \sinh \frac{\theta}{2} , \quad \theta \in \mathbb{R} , \quad \underline{\omega} \in S^{m-1} ,$$

and acts on space-time vectors and on functions according to the transformation rules

$$\begin{aligned} X &\longrightarrow s X \bar{s} \\ F(X) &\longrightarrow H(s) F(X) = s F(\bar{s} X s) \bar{s} . \end{aligned}$$

Since the Dirac operator ∂_X is $\text{Spin}(1, m)$ -invariant, we get :

$$H(s)[\partial_X E_\alpha(\xi, \epsilon)] = \partial_X[H(s)E_\alpha(\xi, \epsilon)] = H(s)\delta(\xi - \epsilon) .$$

By definition, we have

$$H(s)E_\alpha(\xi, \epsilon) = sE_\alpha(\bar{s}\xi s, \epsilon)\bar{s} .$$

Recalling the explicit expression for $E_\alpha(\xi, \epsilon)$, given by

$$E_\alpha(\xi, \epsilon) = \frac{e^{-i\pi\frac{m+1}{2}}}{\pi^{\frac{m-1}{2}}} \Gamma\left(\frac{m+1}{2}\right) \left(D_{\alpha-1}^{\frac{m+1}{2}}(\tau)\xi - D_{\alpha}^{\frac{m+1}{2}}(\tau)\epsilon \right)$$

with $\tau = \xi \cdot \epsilon$, making use of the fact that $\bar{s}\xi s \cdot \epsilon = \xi \cdot s\epsilon\bar{s}$ and putting $s\epsilon\bar{s} = \eta$ we eventually get

$$sE_\alpha(\bar{s}\xi s, \epsilon)\bar{s} = \frac{e^{-i\pi\frac{m+1}{2}}}{\pi^{\frac{m-1}{2}}} \Gamma\left(\frac{m+1}{2}\right) \left(D_{\alpha-1}^{\frac{m+1}{2}}(\xi \cdot \eta)\xi - D_{\alpha}^{\frac{m+1}{2}}(\xi \cdot \eta)\eta \right) .$$

In other words, the only effect of the action of an element $s \in \text{Spin}(1, m)$ on the hyperbolic fundamental solution $E_\alpha(\xi, \epsilon)$ is that ϵ must be replaced by its image under the inverse Spin-transformation \bar{s} .

Note that the distribution $H(s)\delta(\xi - \epsilon) = \delta(\xi - \eta)$. This can be understood as follows : since a delta distribution is a scalar object, it transforms as

$$H(s)\delta(\xi - \epsilon) = s\delta(\bar{s}\xi s - \epsilon)\bar{s} = \delta(\bar{s}\xi s - \epsilon).$$

Hence, its action on test functions $\varphi(\xi)$ reduces to

$$\begin{aligned} \langle H(s)\delta(\xi - \epsilon), \varphi(\xi) \rangle &= \langle \delta(\xi - \epsilon), \varphi(s\xi\bar{s}) \rangle \\ &= \varphi(s\epsilon\bar{s}) \\ &= \langle \delta(\xi - \eta), \varphi(\xi) \rangle . \end{aligned}$$

The latter delta distribution can be defined by considering a local co-ordinate system in the tangent plane to H_+ at η , obtained by a radial projection from a uniquely determined point in the embedding space $\mathbb{R}^{1,m}$: the vertex of the cone touching the hyperbolic surface H_+ at infinity. This point lies on the normal on the tangent plane at η .

If we choose $s \in \text{Spin}(1, m)$ to be the element

$$s = \cosh \frac{\theta}{2} + \epsilon e_1 \sinh \frac{\theta}{2} , \quad \theta \in \mathbb{R} , \quad \underline{\omega} \in S^{m-1} ,$$

the space-time vector $X = \epsilon T + \underline{X}$ transforms as follows :

$$\begin{aligned} X &\longrightarrow \left(\cosh \frac{\theta}{2} + \epsilon e_1 \sinh \frac{\theta}{2} \right) X \left(\cosh \frac{\theta}{2} - \epsilon e_1 \sinh \frac{\theta}{2} \right) \\ &\longrightarrow (T \cosh \theta - X_1 \sinh \theta) \epsilon + (X_1 \cosh \theta - T \sinh \theta) e_1 + \sum_{j=2}^m X_j e_j , \end{aligned}$$

which is of course the transformation of X under the Lorentz boost $B_{-\theta}$, because $\text{Spin}(1, m)$ defines a double covering of the group $\text{SO}(1, m)$.

5.2 Integral Formulae for $\xi(\Gamma + \alpha)$

In this section the basic integral formulae related to the classes of functions $\mathcal{H}^\alpha(\Omega)$ and $\mathcal{H}^\alpha(\mathbb{R}_+\Omega)$ are established. These include the Cauchy-Pompeju Theorem, Stokes' Theorem and Cauchy's Theorem. Also the Modulation Theorems from Chapter 3 will be reformulated in terms of the polar co-ordinates (ρ, ξ) on $\mathbb{R}^{1,m}$, which will be used throughout this whole section, where $\rho = (T^2 - |\underline{X}|^2)^{\frac{1}{2}}$ denotes the hyperbolic norm and

$$\xi = \frac{\epsilon T + \underline{X}}{(T^2 - |\underline{X}|^2)^{\frac{1}{2}}} \in H_+$$

the hyperbolic unit vector associated to $X = \epsilon T + \underline{X}$. In what follows we will use, depending on the problem, two different expansions for arbitrary hyperbolic unit vectors ξ and $\eta \in H_+$:

$$\begin{aligned} \xi &= \tau \epsilon + (\tau^2 - 1)^{\frac{1}{2}} \underline{\xi} = \epsilon \cosh \theta + \underline{\xi} \sinh \theta \\ \eta &= \sigma \epsilon + (\sigma^2 - 1)^{\frac{1}{2}} \underline{\eta} = \epsilon \cosh \varphi + \underline{\eta} \sinh \varphi \end{aligned}$$

with $\underline{\xi}$ and $\underline{\eta} \in S^{m-1}$, with $\tau, \sigma \in [1, +\infty[$ and $\theta, \varphi \in \mathbb{R}$.

First of all, we will apply the hyperbolic Modulation Theorem 3.1 to both inner and outer spherical monogenics on \mathbb{R}^m and write the resulting formulae in terms of the polar co-ordinates on H_+ :

1. Consider an inner spherical monogenic $P_k(\underline{\xi}) \in M^+(k)$. The following function then belongs to $\mathcal{H}^\alpha(\mathbb{R}_+ H_+)$:

$$T^\alpha \text{Mod} \left(\alpha, k, \frac{\underline{X}}{T} \right) P_k \left(\frac{\underline{X}}{T} \right) .$$

The idea is to rewrite this function in such a way that we obtain an element of the function space $\mathcal{H}^\alpha(H_+)$ expressed in terms of the coordinates on H_+ that were introduced earlier. For that purpose formula (23), expressing the Gegenbauer function in terms of a hypergeometric function with argument $(1 - z^{-2})$, will be essential. As the modulation factor consists of two hypergeometric functions whose argument is given by $|\underline{x}|^2$, we get :

$$|\underline{x}|^2 = \frac{|\underline{X}|^2}{T^2} = 1 - \frac{1}{z^2} \implies z = \tau .$$

This allows us to rewrite the modulation factor in the variable $\tau = \xi \cdot \epsilon$, the same variable occurring in the the definition of the fundamental solution $E_\alpha(\xi, \epsilon)$.

First of all, we have :

$$\begin{aligned} & F\left(\frac{1+k-\alpha}{2}, \frac{k-\alpha}{2}; k + \frac{m}{2}; \frac{|\underline{X}|^2}{T^2}\right) \\ &= \tau^{k-\alpha} \frac{\Gamma(1+\alpha-k)\Gamma(2k+m-1)}{\Gamma(\alpha+k+m-1)} C_{\alpha-k}^{k+\frac{m-1}{2}}(\tau) , \end{aligned}$$

whence

$$\begin{aligned} & T^\alpha F\left(\frac{1+k-\alpha}{2}, \frac{k-\alpha}{2}; k + \frac{m}{2}; \frac{|\underline{X}|^2}{T^2}\right) P_k\left(\frac{\underline{X}}{T}\right) \\ &= \rho^\alpha (\tau^2 - 1)^{\frac{k}{2}} \frac{\Gamma(1+\alpha-k)\Gamma(2k+m-1)}{\Gamma(\alpha+k+m-1)} C_{\alpha-k}^{k+\frac{m-1}{2}}(\tau) P_k(\underline{\xi}) . \end{aligned}$$

Next, we also have

$$\begin{aligned} & F\left(\frac{1+k-\alpha}{2}, 1 + \frac{k-\alpha}{2}; 1 + k + \frac{m}{2}; \frac{|\underline{X}|^2}{T^2}\right) \\ &= \tau^{1+k-\alpha} \frac{\Gamma(\alpha-k)\Gamma(2k+m+1)}{\Gamma(\alpha+k+m)} C_{\alpha-k-1}^{k+\frac{m+1}{2}}(\tau) , \end{aligned}$$

whence

$$\begin{aligned} & T^\alpha \frac{k-\alpha}{2k+m} \frac{\underline{X}}{T} \epsilon F\left(\frac{1+k-\alpha}{2}, \frac{k-\alpha}{2}; k + \frac{m}{2}; \frac{|\underline{X}|^2}{T^2}\right) P_k\left(\frac{\underline{X}}{T}\right) \\ &= -\rho^\alpha (\tau^2 - 1)^{\frac{k+1}{2}} \underline{\xi} \epsilon \frac{\Gamma(1+\alpha-k)\Gamma(2k+m)}{\Gamma(\alpha+k+m)} C_{\alpha-k-1}^{k+\frac{m+1}{2}}(\tau) P_k(\underline{\xi}) . \end{aligned}$$

As $(\tau^2 - 1)^{\frac{1}{2}} \underline{\xi} \epsilon = \xi \wedge \epsilon = (\xi \epsilon - \tau)$, and as

$$\begin{aligned} & (\alpha + k + m - 1) C_{\alpha-k}^{k+\frac{m-1}{2}}(\tau) + (2k + m - 1) \tau C_{\alpha-k-1}^{k+\frac{m+1}{2}}(\tau) \\ &= (2k + m - 1) C_{\alpha-k}^{k+\frac{m+1}{2}}(\tau) , \end{aligned}$$

we eventually get :

$$T^\alpha \text{Mod} \left(\alpha, k, \frac{X}{T} \right) P_k \left(\frac{X}{T} \right) = \rho^\alpha P_{\alpha,k}(\xi) ,$$

with $P_{\alpha,k}(\xi)$ given by :

$$\frac{\Gamma(1 + \alpha - k) \Gamma(2k + m)}{\Gamma(\alpha + k + m)} (\tau^2 - 1)^{\frac{k}{2}} \left\{ C_{\alpha-k}^{k+\frac{m+1}{2}}(\tau) - C_{\alpha-k-1}^{k+\frac{m+1}{2}}(\tau) \xi \epsilon \right\} P_k(\underline{\xi}) .$$

Note that the poles of $\Gamma(\alpha + k + m)$ are cancelled by the poles of the Gegenbauer functions and that the poles of $\Gamma(1 + \alpha - k)$ are cancelled by the zeroes of the Gegenbauer functions, whence no restrictions on the parameter α are to be made.

Theorem 3.1 may thus be reformulated as follows :

Theorem 5.1 *Let $P_k(\underline{\xi}) \in M^+(k)$ be an inner spherical monogenic on \mathbb{R}^m and let $\alpha \in \mathbb{C}$ be an arbitrary complex number. The function $P_{\alpha,k}(\xi)$ defined for all $\xi = \tau \epsilon + (\tau^2 - 1)^{\frac{1}{2}} \underline{\xi} \in H_+$ by*

$$\begin{aligned} P_{\alpha,k}(\xi) &= \frac{\Gamma(1 + \alpha - k) \Gamma(2k + m)}{\Gamma(\alpha + k + m)} \times \\ & (\tau^2 - 1)^{\frac{k}{2}} \left\{ C_{\alpha-k}^{k+\frac{m+1}{2}}(\tau) - C_{\alpha-k-1}^{k+\frac{m+1}{2}}(\tau) \xi \epsilon \right\} P_k(\underline{\xi}) \end{aligned}$$

belongs to $\mathcal{H}^\alpha(H_+)$.

2. Consider an outer spherical monogenic $Q_k(\underline{\xi}) \in M^-(k)$. If we denote $H_+ \setminus \{\epsilon\}$ by $H_+^{(\epsilon)}$, the following function belongs to $\mathcal{H}^\alpha(\mathbb{R}_+ H_+^{(\epsilon)})$:

$$T^\alpha \text{Mod} \left(\alpha, 1 - k - m, \frac{X}{T} \right) Q_k \left(\frac{X}{T} \right) .$$

The idea is to rewrite this function as an element of $\mathcal{H}^\alpha(H_+^{(\epsilon)})$, expressed in terms of the polar co-ordinates on H_+ . Note that all outer spherical monogenics $Q_k(\underline{\xi})$ give rise to an α -homogeneous solution for the Dirac

operator on $\mathbb{R}^{1,m}$ with singularities on the ray through ϵ . This singular behaviour does not change if we add a particular α -homogeneous null-solution for the Dirac operator, i.e. an element of $\mathcal{H}^\alpha(H_+)$. First of all note that if $Q_k(\underline{\xi}) \in M^-(k)$, then

$$P'_k(\underline{\xi}) = \epsilon \underline{\xi} Q_k(\underline{\xi}) \in M^+(k) .$$

The factor ϵ is introduced for convenience, and has no effect on the monogeneity of $P'_k(\underline{\xi})$ with respect to the Dirac operator $\underline{\partial}$ on $\mathbb{R}^{0,m}$. Theorem 3.1 can then be used to modulate this function, and the result will be an element of the function space $\mathcal{H}^\alpha(H_+)$. Using definitions for the Gegenbauer functions we will then rewrite the function

$$\begin{aligned} & T^\alpha \text{Mod} \left(\alpha, 1 - k - m, \frac{X}{T} \right) Q_k \left(\frac{X}{T} \right) \\ & - 2^{1-2k-m} \frac{\Gamma(1 - k - \frac{m}{2})}{\Gamma(k + \frac{m}{2})} \frac{\Gamma(\alpha + k + m)}{\Gamma(\alpha - k + 1)} T^\alpha \text{Mod} \left(\alpha, k, \frac{X}{T} \right) P'_k \left(\frac{X}{T} \right) \end{aligned}$$

belonging to $\mathcal{H}^\alpha(H_+^{(\epsilon)})$, in terms of the co-ordinates on H_+ hence obtaining an element of the function space $\mathcal{H}^\alpha(H_+^{(\epsilon)})$.

Before doing so, let us consider an explicit example by choosing $Q_k(\underline{\xi})$ to be $\underline{\xi}$ (i.e. $k = 0$). The previous expression then reduces to

$$\begin{aligned} & \frac{2\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})} T^\alpha \text{Mod} \left(\alpha, 1 - m, \frac{X}{T} \right) E \left(\frac{X}{T} \right) \\ & + 2^{1-m} \frac{\Gamma(1 - \frac{m}{2})}{\Gamma(\frac{m}{2})} \frac{\Gamma(\alpha + m)}{\Gamma(\alpha + 1)} T^\alpha \text{Mod} \left(\alpha, 0, \frac{X}{T} \right) \epsilon , \end{aligned}$$

which gives for odd m up to the factor A_m :

$$\begin{aligned} & T^\alpha \text{Mod} \left(\alpha, 1 - m, \frac{X}{T} \right) E \left(\frac{X}{T} \right) \\ & + \frac{(-1)^{\frac{m-1}{2}}}{2^m \pi^{\frac{m-2}{2}} \Gamma(\frac{m}{2})} \frac{\Gamma(\alpha + m)}{\Gamma(\alpha + 1)} T^\alpha \text{Mod} \left(\alpha, 0, \frac{X}{T} \right) \epsilon , \end{aligned}$$

precisely the expression for the hyperbolic fundamental solution as it was obtained in Chapter 2.

Consider the modulation factor $\text{Mod}(\alpha, 1 - k - m, \frac{X}{T})$. First of all we

have

$$\begin{aligned} & F\left(\frac{1-k-m-\alpha}{2}, 1-\frac{k+m+\alpha}{2}; 1-k-\frac{m}{2}; \frac{|X|^2}{T^2}\right) \\ &= \frac{\Gamma\left(1-k-\frac{m}{2}\right)}{2^{k+\frac{m}{2}}} \frac{(\tau^2-1)^{\frac{k}{2}+\frac{m}{4}}}{\tau^{\alpha+k+m-1}} P_{\alpha+\frac{m}{2}-1}^{k+\frac{m}{2}}(\tau), \end{aligned}$$

whence

$$\begin{aligned} & T^\alpha F\left(\frac{1-k-m-\alpha}{2}, 1-\frac{k+m+\alpha}{2}; 1-k-\frac{m}{2}; \frac{|X|^2}{T^2}\right) Q_k\left(\frac{X}{T}\right) \\ &= \rho^\alpha \frac{\Gamma\left(1-k-\frac{m}{2}\right)}{2^{k+\frac{m}{2}}} (\tau^2-1)^{\frac{1}{2}-\frac{m}{4}} P_{\alpha+\frac{m}{2}-1}^{k+\frac{m}{2}}(\tau) Q_k(\underline{\xi}). \end{aligned}$$

Next, we also have

$$\begin{aligned} & F\left(1+\frac{1-k-m-\alpha}{2}, 1-\frac{k+m+\alpha}{2}; 2-k-\frac{m}{2}; \frac{|X|^2}{T^2}\right) \\ &= \frac{\Gamma\left(2-k-\frac{m}{2}\right)}{2^{k+\frac{m}{2}-1}} \frac{(\tau^2-1)^{\frac{k}{2}+\frac{m}{4}-\frac{1}{2}}}{\tau^{\alpha+k+m-2}} P_{\alpha+\frac{m}{2}-1}^{k+\frac{m}{2}-1}(\tau), \end{aligned}$$

whence

$$\begin{aligned} & T^\alpha \frac{1-k-m-\alpha}{2-2k-m} \frac{X}{T} \epsilon F\left(\frac{1+k-\alpha}{2}, \frac{k-\alpha}{2}; k+\frac{m}{2}; \frac{|X|^2}{T^2}\right) Q_k\left(\frac{X}{T}\right) \\ &= \rho^\alpha (1-k-m-\alpha) \frac{\Gamma\left(1-k-\frac{m}{2}\right)}{2^{k+\frac{m}{2}}} (\tau^2-1)^{\frac{1}{2}-\frac{m}{4}} \underline{\xi} \epsilon P_{\alpha+\frac{m}{2}-1}^{k+\frac{m}{2}-1}(\tau) Q_k(\underline{\xi}). \end{aligned}$$

On the other hand, the function $T^\alpha \text{Mod}(\alpha, k, \frac{X}{T}) P'_k\left(\frac{X}{T}\right)$ in terms of the Legendre functions can easily be found by means of preceeding calculations :

$$\begin{aligned} & T^\alpha F\left(\frac{1+k-\alpha}{2}, \frac{k-\alpha}{2}; k+\frac{m}{2}; \frac{|X|^2}{T^2}\right) P'_k\left(\frac{X}{T}\right) \\ &= \rho^\alpha (\tau^2-1)^{\frac{1}{2}-\frac{m}{4}} \frac{\Gamma\left(k+\frac{m}{2}\right)}{2^{1-k-\frac{m}{2}}} P_{\alpha+\frac{m}{2}-1}^{1-k-\frac{m}{2}}(\tau) P'_k(\underline{\xi}), \end{aligned}$$

and

$$\begin{aligned} & T^\alpha \frac{k-\alpha}{2k+m} \frac{X}{T} \epsilon F\left(\frac{1+k-\alpha}{2}, \frac{k-\alpha}{2}; k+\frac{m}{2}; \frac{|X|^2}{T^2}\right) P'_k\left(\frac{X}{T}\right) \\ &= (k-\alpha) \rho^\alpha (\tau^2-1)^{\frac{1}{2}-\frac{m}{4}} \underline{\xi} \epsilon \frac{\Gamma\left(k+\frac{m}{2}\right)}{2^{1-k-\frac{m}{2}}} P_{\alpha+\frac{m}{2}-1}^{-k-\frac{m}{2}}(\tau) P'_k(\underline{\xi}), \end{aligned}$$

where $P'_k(\underline{\xi}) = \epsilon \underline{\xi} Q_k(\underline{\xi})$.

Let us then return to the function

$$T^\alpha \text{Mod} \left(\alpha, 1 - k - m, \frac{X}{T} \right) Q_k \left(\frac{X}{T} \right) \\ - 2^{1-2k-m} \frac{\Gamma(1 - k - \frac{m}{2})}{\Gamma(k + \frac{m}{2})} \frac{\Gamma(\alpha + k + m)}{\Gamma(\alpha - k + 1)} T^\alpha \text{Mod} \left(\alpha, k, \frac{X}{T} \right) P'_k \left(\frac{X}{T} \right) .$$

It has a component in $Q_k(\underline{\xi})$ given by

$$\rho^\alpha \frac{\Gamma(1 - k - \frac{m}{2})}{2^{k+\frac{m}{2}}} (\tau^2 - 1)^{\frac{1}{2} - \frac{m}{4}} \left[P_{\alpha+\frac{m}{2}-1}^{k+\frac{m}{2}}(\tau) - \frac{\Gamma(\alpha + k + m)}{\Gamma(\alpha - k)} P_{\alpha+\frac{m}{2}-1}^{-k-\frac{m}{2}}(\tau) \right]$$

and a component in $\xi \epsilon Q_k(\underline{\xi})$ given by

$$(1 - k - m - \alpha) \rho^\alpha \frac{\Gamma(1 - k - \frac{m}{2})}{2^{k+\frac{m}{2}}} (\tau^2 - 1)^{\frac{1}{2} - \frac{m}{4}} \times \\ \left[P_{\alpha+\frac{m}{2}-1}^{k+\frac{m}{2}-1}(\tau) - \frac{\Gamma(\alpha + k + m - 1)}{\Gamma(\alpha - k + 1)} P_{\alpha+\frac{m}{2}-1}^{1-k-\frac{m}{2}}(\tau) \right] .$$

With the aid of formulae (16) and (17) we get for the term in $Q_k(\underline{\xi})$

$$\frac{e^{i\pi(k+\frac{m}{2})} \Gamma(\alpha + k + m)}{2^{k+\frac{m}{2}-1} \Gamma(\alpha - k) \Gamma(k + \frac{m}{2})} \rho^\alpha (\tau^2 - 1)^{\frac{1}{2} - \frac{m}{4}} Q_{\alpha+\frac{m}{2}-1}^{-k-\frac{m}{2}}(\tau)$$

and for the term in $\xi \epsilon Q_k(\underline{\xi})$

$$\frac{e^{i\pi(k+\frac{m}{2}-1)} \Gamma(\alpha + k + m)}{2^{k+\frac{m}{2}-1} \Gamma(\alpha - k + 1) \Gamma(k + \frac{m}{2})} \rho^\alpha (\tau^2 - 1)^{\frac{1}{2} - \frac{m}{4}} Q_{\alpha+\frac{m}{2}-1}^{1-k-\frac{m}{2}}(\tau) .$$

Recalling the definition for the Gegenbauer function of the second kind in terms of the Legendre function of the second kind, the term in $Q_k(\underline{\xi})$ gives

$$2\pi^{\frac{1}{2}} e^{-i\pi(k+\frac{m+1}{2})} \frac{\Gamma(k + \frac{m+1}{2})}{\Gamma(k + \frac{m}{2})} \rho^\alpha (\tau^2 - 1)^{\frac{k+1}{2}} D_{\alpha-1-k}^{k+\frac{m+1}{2}}(\tau)$$

and the term in $\xi \epsilon Q_k(\underline{\xi})$ gives

$$\pi^{\frac{1}{2}} e^{-i\pi(k+\frac{m-1}{2})} (\alpha + m + k - 1) \frac{\Gamma(k + \frac{m+1}{2})}{\Gamma(k + \frac{m}{2})} \rho^\alpha (\tau^2 - 1)^{\frac{k}{2}} D_{\alpha-k}^{k+\frac{m-1}{2}}(\tau) .$$

Adding both pieces and using the recurrence relations for the Gegenbauer functions eventually yields :

$$\rho^\alpha(\tau^2 - 1)^{\frac{k}{2}} 2\pi^{\frac{1}{2}} e^{-i\pi(k + \frac{m+1}{2})} \frac{\Gamma(k + \frac{m+1}{2})}{\Gamma(k + \frac{m}{2})} \times \\ \left\{ D_{\alpha-k-1}^{k + \frac{m+1}{2}}(\tau) \xi \epsilon - D_{\alpha-k}^{k + \frac{m+1}{2}}(\tau) \right\} \xi \epsilon Q_k(\underline{\xi}) .$$

Note that this expression has poles for $\alpha = -m - k - n$ with $n \in \mathbb{N}$.

We thus arrive at the following Theorem :

Theorem 5.2 *Let $Q_k(\underline{\xi}) \in M^-(k)$ be an outer spherical monogenic on \mathbb{R}^m and let $\alpha \in \mathbb{C}$ such that $\alpha \notin -m - k - \mathbb{N}$. The function $Q_{\alpha,k}(\xi)$ defined for all $\xi = \tau\epsilon + (\tau^2 - 1)^{\frac{1}{2}}\underline{\xi} \in H_+$ by*

$$Q_{\alpha,k}(\xi) = (\tau^2 - 1)^{\frac{k}{2}} 2\pi^{\frac{1}{2}} e^{-i\pi(k + \frac{m+1}{2})} \frac{\Gamma(k + \frac{m+1}{2})}{\Gamma(k + \frac{m}{2})} \times \\ \left\{ D_{\alpha-k-1}^{k + \frac{m+1}{2}}(\tau) \xi \epsilon - D_{\alpha-k}^{k + \frac{m+1}{2}}(\tau) \right\} \xi \epsilon Q_k(\underline{\xi})$$

belongs to $\mathcal{H}^\alpha(H_+^{(\epsilon)})$.

Let us then consider the fundamental solution $E_\alpha(\xi, \eta)$. We already know that it satisfies the equation

$$\xi(\Gamma + \alpha)E_\alpha(\xi, \eta) = \delta(\xi - \eta) ,$$

but in view of Cauchy's Theorem we need an expression for this operator acting from the right, as will be made clear a bit further. In order to indicate on which of the variables the Gamma operator is acting, the variable will be used as an additional label. We then have the following :

Lemma 5.1 *Let $E_\alpha(\xi, \eta)$ be the hyperbolic fundamental solution given by Definition 5.1. Putting $\beta = -\alpha - m$, the following relation holds :*

$$E_\alpha(\xi, \eta)(\Gamma_\eta - \beta)\eta = \delta(\xi - \eta) .$$

Proof : Using conjugation, the previous expression is found to be equivalent with

$$\bar{\eta}(\bar{\Gamma}_\eta - \bar{\beta})\bar{E}_\alpha(\xi, \eta) = \delta(\xi - \eta) .$$

Since $\bar{\eta} = -\eta$ and $\bar{\Gamma}_\eta = -\Gamma$, we get

$$\eta(\Gamma_\eta + \bar{\beta})\bar{E}_\alpha(\xi, \eta) = \delta(\xi - \eta) .$$

In order to determine $\bar{E}_\alpha(\xi, \eta)$ we consider its explicit expression in terms of the Gegenbauer function of the second kind. Using expression (24) one can easily verify that

$$\overline{e^{-i\pi\frac{m+1}{2}} D_{\alpha}^{\frac{m+1}{2}}(z)} = e^{-i\pi\frac{m+1}{2}} D_{\bar{\alpha}}^{\frac{m+1}{2}}(z) ,$$

such that $\bar{E}_\alpha(\xi, \eta) = -E_{\bar{\alpha}}(\xi, \eta)$. We thus have to prove that

$$-\eta(\Gamma_\eta + \bar{\beta})E_{\bar{\alpha}}(\xi, \eta) = \delta(\xi - \eta) ,$$

or equivalently

$$-\eta(\Gamma_\eta + \beta)E_\alpha(\xi, \eta) = \delta(\xi - \eta) ,$$

where $\beta = -\alpha - m$. First of all we know that

$$\eta(\Gamma_\eta + \beta)E_\beta(\eta, \xi) = \delta(\xi - \eta) ,$$

or in explicit form, with $\tau = \eta \cdot \xi$:

$$\frac{e^{-i\pi\frac{m+1}{2}} \Gamma\left(\frac{m+1}{2}\right)}{\pi^{\frac{m-1}{2}}} \eta(\Gamma_\eta + \beta) \left\{ D_{\beta-1}^{\frac{m+1}{2}}(\tau) \eta - D_{\beta}^{\frac{m+1}{2}}(\tau) \xi \right\} = \delta(\xi - \eta) .$$

On the other hand, putting $k = 0$ in Theorem 5.1, we also have

$$\frac{e^{-i\pi\frac{m+1}{2}} \Gamma\left(\frac{m+1}{2}\right)}{\pi^{\frac{m-1}{2}}} \eta(\Gamma_\eta + \beta) \left\{ C_{\beta-1}^{\frac{m+1}{2}}(\tau) \eta - C_{\beta}^{\frac{m+1}{2}}(\tau) \xi \right\} = 0 .$$

Eventually making use of expression (27) to put

$$D_{\beta-1}^{\frac{m+1}{2}}(\tau) - e^{i\pi\frac{m+1}{2}} \frac{\sin\left(\beta + \frac{m-1}{2}\right) \pi}{\sin(\beta-1)\pi} C_{\beta-1}^{\frac{m+1}{2}}(\tau) = D_{\alpha}^{\frac{m+1}{2}}(\tau)$$

and

$$D_{\beta}^{\frac{m+1}{2}}(\tau) - e^{i\pi\frac{m+1}{2}} \frac{\sin\left(\beta + \frac{m+1}{2}\right) \pi}{\sin(\beta\pi)} C_{\beta}^{\frac{m+1}{2}}(\tau) = D_{\alpha-1}^{\frac{m+1}{2}}(\tau) ,$$

we arrive at

$$\frac{e^{-i\pi\frac{m+1}{2}} \Gamma\left(\frac{m+1}{2}\right)}{\pi^{\frac{m-1}{2}}} \eta(\Gamma_\eta + \beta) \left\{ D_{\alpha}^{\frac{m+1}{2}}(\tau) \eta - D_{\alpha-1}^{\frac{m+1}{2}}(\tau) \xi \right\} = \delta(\xi - \eta) ,$$

or

$$-\eta(\Gamma_\eta + \beta)E_\alpha(\xi, \eta) = \delta(\xi - \eta) .$$

This proves the Lemma. \square

Remark : This Lemma also proves that the function $E_\alpha(X, Y)$, defined for space-time vectors X and $Y \in FC$, is monogenic with respect to the Dirac operator ∂_X on $\mathbb{R}^{1,m}$ from the left *and* the operator ∂_Y from the right if we impose the following homogeneity condition :

$$E_\alpha(X, Y) = |X|^\alpha E_\alpha(\xi, \eta) |Y|^\beta .$$

It becomes singular if X and Y are on the same ray through the origin.

In the next Theorem we will give a homogeneous version of the following basic identity, valid on \mathbb{R}^m :

$$d(f\sigma(\underline{x}, d\underline{x})g) = \left((f\partial)g + f(\partial g) \right) d\underline{x} , \quad (5.3)$$

with $d\underline{x} = dx_1 \cdots dx_m$ the volume element on \mathbb{R}^m , ∂ the Dirac operator on $\mathbb{R}^{0,m}$ and $\sigma(\underline{x}, d\underline{x})$ the oriented surface element. It is defined by means of the contraction operator :

$$\sigma(\underline{x}, d\underline{x}) = \partial \rfloor d\underline{x} = \sum_{j=1}^m (-1)^{j+1} e_j dx_1 \cdots dx_{j-1} dx_{j+1} \cdots dx_m ,$$

the contraction operator itself being determined by the relations

$$\begin{aligned} \partial_{x_j} \rfloor (x_k F) &= x_k \partial_{x_j} \rfloor F \\ \partial_{x_j} \rfloor (dx_k F) &= \delta_{jk} F - dx_k \partial_{x_j} \rfloor F . \end{aligned}$$

Remark : The contraction operator is sometimes referred to as a *fermionic derivation operator*. This is motivated by the fact that this operator acts as a "derivative" on the "variables" dx_k in such a way that it anti-commutes with dx_j .

In order to derive a homogeneous version of identity (5.3) which is valid on $\mathbb{R}^{1,m}$, we first define the Leray form and a homogeneous version of the oriented surface element as contractions of respectively the volume element $dT d\underline{X}$ and the surface element $\sigma(T, \underline{X}; dT, d\underline{X})$ on $\mathbb{R}^{1,m}$ with the Euler operator \mathbb{E}_ρ on $\mathbb{R}^{1,m}$ (in the following formulae, the variable X_0 is to be replaced by the time-variable T) :

Definition 5.2 *The Leray form $L(X, dX)$ on $\mathbb{R}^{1,m}$ is defined as*

$$\begin{aligned}
L(X, dX) &= \mathbb{E}_\rho \rfloor dX_0 dX_1 \cdots dX_m \\
&= X_0 dX_1 \cdots dX_m - dX_0 L(\underline{X}, d\underline{X}) \\
&= X_0 d\underline{X} + (-1)^m L(\underline{X}, d\underline{X}) dX_0 \\
&= \sum_{j=0}^m (-1)^j X_j dX_{\hat{j}} .
\end{aligned}$$

Definition 5.3 *The homogeneous Σ -form on $\mathbb{R}^{1,m}$ is defined as*

$$\begin{aligned}
\Sigma(X, dX) &= \mathbb{E}_\rho \rfloor \sigma(X, dX) \\
&= \epsilon L(\underline{X}, d\underline{X}) + T \sigma(\underline{X}, d\underline{X}) + (-1)^m \partial \rfloor L(\underline{X}, d\underline{X}) dT .
\end{aligned}$$

The notation \hat{j} hereby indicates that this index is omitted in the summation.

Both objects transform in a homogeneous way under the transformations

$$\begin{aligned}
X &\longrightarrow \lambda X \\
dX &\longrightarrow \lambda dX + X d\lambda ,
\end{aligned}$$

whence they are well-defined on the hyperbolic unit ball. As their restrictions to the hyperbolic surface H_+ coincide respectively with the volume element and the oriented surface element on H_+ , it seems natural to replace the volume element and the oriented surface element in expression (5.3) by the Leray form $L(X, dX)$ and $\Sigma(X, dX)$ respectively. A homogeneous version of this identity is then given by

Theorem 5.3 (*Cauchy-Pompeju*) *Let F and $G \in C^1(\mathbb{R}^{1,m})$ with $F(X)$ and $G(X)$ respectively homogeneous of degree α and β , where $\alpha + \beta + m = 0$. We then have*

$$d\left(F\Sigma(X, dX)G\right) = -\left[(F\partial_X)G + F(\partial_X G)\right]L(X, dX) .$$

For a proof we refer to e.g. [12]. □

Since both sides of this equation are homogeneous of degree zero, if the condition $\alpha + \beta + m = 0$ holds, the result is essentially valid on the manifold $\text{Ray}(FC)$ and can thus be realized on arbitrary surfaces inside the future cone, in particular on the hyperbolic surface H_+ . Let us therefore consider an open subset Ω of H_+ and let $C \subset \Omega$ be compact with smooth boundary ∂C . We are then lead to the following Theorems :

Theorem 5.4 (Stokes) Let F and $G \in C^1(\Omega)$ with $F(X)$ and $G(X)$ respectively homogeneous of degree β and α , where $\alpha + \beta + m = 0$. Then :

$$\begin{aligned} \int_{\partial C} F \Sigma(\xi, d\xi) G &= \int_C [(F\Gamma)\xi G + F\Gamma(\xi G)] L(\xi, d\xi) \\ &= \int_C [(F(\Gamma - \beta))\xi G - F\xi(\Gamma + \alpha)G] L(\xi, d\xi) \end{aligned}$$

Proof : We start from the Cauchy-Pompeju identity on $\mathbb{R}^{1,m}$:

$$F \Sigma(X, dX) G = - \left[(F \partial_X) G + F(\partial_X G) \right] L(X, dX) .$$

Since

$$F \partial_X = -\overline{\partial_X F} = -\overline{\xi(\Gamma + \bar{\beta}) \overline{F}(\xi)} = F(\beta - \Gamma)\xi$$

on H_+ , integration over a compact subset $C \subset \Omega$ with smooth boundary yields immediately :

$$\int_{\partial C} F \Sigma(X, dX) G = \int_C \left[(F(\Gamma - \beta)\xi) G - F\xi(\Gamma + \alpha)G \right] L(X, dX) .$$

As $\alpha + \beta = -m$ and $\Gamma + \xi\Gamma\xi = m$ this reduces to

$$\begin{aligned} \int_{\partial C} F \Sigma(X, dX) G &= \int_C \left[(F\Gamma)\xi G - F(\xi\Gamma\xi - m)(\xi G) \right] L(X, dX) \\ &= \int_C \left[(F\Gamma)\xi G + F\Gamma(\xi G) \right] L(X, dX) . \end{aligned}$$

This proves the Theorem. □

Theorem 5.5 (Cauchy) Let $F \in \mathcal{H}^\alpha(\Omega)$. The following formula holds :

$$\int_{\partial C} E_\alpha(\eta, \xi) \Sigma(\xi, d\xi) F(\xi) = \begin{cases} F(\eta) & \text{if } \eta \in \overset{\circ}{C} \\ 0 & \text{if } \eta \in \Omega \setminus C \end{cases}$$

Proof : Follows immediately from Stokes' Theorem and Lemma 5.1. □

5.3 Taylor Series on the Hyperbolic Unit Ball

In this section the Taylor (and Laurent) series for hyperbolic monogenics on $\text{SO}(m)$ -invariant subdomains Ω_η of H_+ is established. An $\text{SO}(m)$ -invariant subdomain Ω_η of H_+ is defined as an open subset $\Omega_\eta \subset H_+$ such that the subgroup $\text{SO}(1, m)_\eta$ of $\text{SO}(1, m)$ fixing η , leaves the subset Ω_η invariant. A decomposition for the fundamental solution $E_\alpha(\eta, \xi)$ will be given, using the classical Cauchy kernel $E(\underline{x})$ on \mathbb{R}^m and Theorems 5.1 and 5.2. Cauchy's Theorem will then be used to establish the Taylor (resp. Laurent) series for hyperbolic monogenics on open domains (resp. open *annular* domains) in H_+ .

From the previous section (in particular Lemma 5.1) it is clear that the fundamental solution $E_\alpha(\eta, \xi)$ for the operator $\eta(\Gamma + \alpha)$ with singularity for $\eta = \xi$ may be considered as the restriction to the hyperbola H_+ of a function which is α -homogeneous in Y , β -homogeneous in X , monogenic with respect to the Dirac operator ∂_Y on $\mathbb{R}^{1, m}$ acting from the left and the operator ∂_X acting from the right and having singularities for Y and X belonging to the same ray through the origin. It seems therefore natural to consider the Cauchy kernel $E(\underline{y} - \underline{x})$ on \mathbb{R}^m , and to modulate this function by means of Theorems 5.1 and 5.2 in order to obtain $E_\alpha(\eta, \xi)$.

So let us consider the series expansion for the Euclidean Cauchy kernel $E(\underline{y} - \underline{x})$, see formula (2), with $t = \langle \underline{x}, \underline{y} \rangle$ the standard Euclidean inner product :

$$E(\underline{y} - \underline{x}) = \frac{1}{A_m} \sum_{k=0}^{\infty} \frac{|\underline{x}|^k}{|\underline{y}|^{k+m-1}} \left\{ C_k^{\frac{m}{2}}(t) \underline{\eta} - C_{k-1}^{\frac{m}{2}}(t) \underline{\xi} \right\} ,$$

valid for $|\underline{x}| < |\underline{y}|$. Since the function between brackets is an outer spherical monogenic on \mathbb{R}^m with respect to the variable \underline{y} , having a singularity for $\underline{y} = \underline{0}$, Theorem 5.2 can be used to construct a hyperbolic monogenic on $H_+^{(\epsilon)} = H_+ \setminus \{\epsilon\}$. With $\eta = \sigma\epsilon + (\sigma^2 - 1)^{\frac{1}{2}} \underline{\eta} \in H_+$, this function is for all $\eta \neq \epsilon$ and $\alpha + m + k \notin -\mathbb{N}$ given by

$$\frac{2\pi^{\frac{1}{2}}}{e^{i\pi(k+\frac{m+1}{2})}} \frac{\Gamma(k+\frac{m+1}{2})}{\Gamma(k+\frac{m}{2})} (\sigma^2 - 1)^{\frac{k}{2}} \times \\ \left\{ D_{\alpha-k-1}^{k+\frac{m+1}{2}}(\sigma) \eta \epsilon - D_{\alpha-k}^{k+\frac{m+1}{2}}(\sigma) \right\} \underline{\eta} \epsilon \left\{ C_k^{\frac{m}{2}}(t) \underline{\eta} - C_{k-1}^{\frac{m}{2}}(t) \underline{\xi} \right\} .$$

Using the various recurrence relations for the Gegenbauer functions, this may also be written as

$$\eta \frac{\pi^{\frac{1}{2}}}{e^{i\pi(k+\frac{m-1}{2})}} \frac{\Gamma(k+\frac{m-1}{2})}{\Gamma(k+\frac{m}{2})} (\sigma^2 - 1)^{\frac{k}{2}} \times \\ \left\{ (1 + \alpha - k) D_{1+\alpha-k}^{k+\frac{m-1}{2}}(\sigma) + (2k + m - 1) (\sigma^2 - 1)^{\frac{1}{2}} D_{\alpha-k}^{k+\frac{m+1}{2}}(\sigma) \underline{\eta} \epsilon \right\} \times \\ \underline{\eta} \left\{ C_k^{\frac{m}{2}}(t) \underline{\eta} - C_{k-1}^{\frac{m}{2}}(t) \underline{\xi} \right\}$$

On the other hand, the function $\left\{ C_k^{\frac{m}{2}}(t) \underline{\eta} - C_{k-1}^{\frac{m}{2}}(t) \underline{\xi} \right\}$ is an inner spherical monogenic with respect to the variable \underline{x} . A slightly modified version of Theorem 5.1 may then be used to obtain a hyperbolic monogenic with respect to the operator $(\Gamma - \beta)\xi$ acting from the right :

$$\frac{\Gamma(1 + \beta - k) \Gamma(2k + m)}{\Gamma(\beta + k + m)} (\tau^2 - 1)^{\frac{k}{2}} \left\{ C_k^{\frac{m}{2}}(t) \underline{\eta} - C_{k-1}^{\frac{m}{2}}(t) \underline{\xi} \right\} \times \\ \left\{ C_{\beta-k}^{k+\frac{m+1}{2}}(\tau) - C_{\beta-k-1}^{k+\frac{m+1}{2}}(\tau) \underline{\xi} \epsilon \right\}$$

In order to replace the indices $(\beta - k)$ and $(\beta - k - 1)$ of the Gegenbauer functions by indices $(\alpha - k)$ and $(1 + \alpha - k)$, we use Legendre's duplication formula, relation (26) for the Gegenbauer functions of the first kind

$$C_{-\nu-2\mu}^{\mu}(z) = -\frac{\sin(\nu+2\mu)\pi}{\sin(\nu\pi)} C_{\nu}^{\mu}(z) ,$$

and the Gegenbauer recursion formula

$$\left(C_{\alpha-k-1}^{k+\frac{m+1}{2}}(\tau) - \tau C_{\alpha-k}^{k+\frac{m+1}{2}}(\tau) \right) = -\frac{1 + \alpha - k}{2k + m - 1} C_{1+\alpha-k}^{k+\frac{m-1}{2}}(\tau) .$$

This allows us to obtain the following hyperbolic monogenic with respect to the operator $(\Gamma - \beta)\xi$ acting from the right :

$$\frac{\Gamma(1 + \alpha - k)}{\Gamma(\alpha + k + m)} \Gamma(2k + m - 1) (\tau^2 - 1)^{\frac{k}{2}} \left\{ C_k^{\frac{m}{2}}(t) \underline{\eta} - C_{k-1}^{\frac{m}{2}}(t) \underline{\xi} \right\} \times \\ \left\{ (2k + m - 1) (\tau^2 - 1)^{\frac{1}{2}} \underline{\xi} \epsilon C_{\alpha-k}^{k+\frac{m+1}{2}}(\tau) + (1 + \alpha - k) C_{1+\alpha-k}^{k+\frac{m-1}{2}}(\tau) \right\}$$

When modulating the Euclidean Cauchy kernel $E(\underline{y} - \underline{x})$ to the hyperbolic bi-monogenic function $E_{\alpha}(Y, X)$, with $\alpha + m \notin -\mathbb{N}$ and

$$Y = |Y| \eta = |Y| \left(\sigma \epsilon + (\sigma^2 - 1)^{\frac{1}{2}} \underline{\eta} \right) = \epsilon S + \underline{Y}$$

$$X = |X| \xi = |X| \left(\tau \epsilon + (\tau^2 - 1)^{\frac{1}{2}} \underline{\xi} \right) = \epsilon T + \underline{X}$$

we identify \underline{y} with $\frac{Y}{S}$ and \underline{x} with $\frac{X}{T}$ (because the Modulation Theorems essentially follow from the projection on the Klein model for the hyperbolic unit ball). Since the series expansion for the Cauchy kernel converges for $|\underline{x}| < |\underline{y}|$, with

$$|\underline{x}| < |\underline{y}| \iff \tau < \sigma ,$$

we propose the following decomposition for the function $E_\alpha(\eta, \xi)$, valid for all $\alpha + m \notin -\mathbb{N}$ and for $\tau < \sigma$:

$$\begin{aligned} & \frac{1}{A_m} \sum_{k=0}^{\infty} \eta (-1)^k 2^{2k+m-2} e^{-i\pi \frac{m-1}{2}} \Gamma \left(k + \frac{m-1}{2} \right)^2 \frac{\Gamma(1+\alpha-k)}{\Gamma(\alpha+k+m)} \times \\ & (\sigma^2 - 1)^{\frac{k}{2}} \left\{ (1+\alpha-k) D_{1+\alpha-k}^{k+\frac{m-1}{2}}(\sigma) + (2k+m-1)(\sigma^2-1)^{\frac{1}{2}} D_{\alpha-k}^{k+\frac{m+1}{2}}(\sigma) \underline{\eta} \epsilon \right\} \\ & \left\{ C_k^{\frac{m}{2}}(<\underline{\xi}, \underline{\eta}>) + C_{k-1}^{\frac{m}{2}}(<\underline{\xi}, \underline{\eta}>) \underline{\eta} \underline{\xi} \right\} \\ & (\tau^2 - 1)^{\frac{k}{2}} \left\{ (1+\alpha-k) C_{1+\alpha-k}^{k+\frac{m-1}{2}}(\tau) + (2k+m-1)(\tau^2-1)^{\frac{1}{2}} \underline{\xi} \epsilon C_{\alpha-k}^{k+\frac{m+1}{2}}(\tau) \right\} \end{aligned}$$

with a region of convergence that will be determined later.

In what follows, we will give a technical proof for the expansion we have proposed here. The key ingredient is the addition formula (28) for the Gegenbauer function of the second kind, for which we refer to section 0.2.3. We could have proved the expansion for $E_\alpha(\eta, \xi)$ starting from this formula, without mentioning the modulation of the Euclidean Cauchy kernel, but this doesn't help to *understand* the expansion. In some sense the modulation argument, which is very natural, even gives a geometrical interpretation to the addition formula for the Gegenbauer function.

Let us now put $<\underline{\eta}, \underline{\xi}> = \cos \psi \in [-1, 1]$ such that

$$\eta \cdot \xi = \sigma\tau - ((\sigma^2 - 1)(\tau^2 - 1))^{\frac{1}{2}} <\underline{\eta}, \underline{\xi}> = \cosh \varphi \cosh \theta - \sinh \varphi \sinh \theta \cos \psi .$$

The addition formula for the Gegenbauer function of the second kind, applied to $D_{\alpha+1}^{\frac{m-1}{2}}(\eta \cdot \xi)$, yields :

$$D_{\alpha+1}^{\frac{m-1}{2}}(\eta \cdot \xi) = \frac{\Gamma(m-1)}{\Gamma\left(\frac{m-1}{2}\right)^2} \sum_{k=0}^{\infty} a_k(\alpha, m) c_k(\theta, \varphi, \psi)$$

with

$$\begin{aligned}
a_k(\alpha, m) &= (-1)^k 4^k \Gamma\left(\frac{m-1}{2} + k\right)^2 \frac{\Gamma(2 + \alpha - k)}{\Gamma(\alpha + k + m)} \\
c_k(\theta, \varphi, \psi) &= \frac{m-2+2k}{m-2} (\sinh \varphi \sinh \theta)^k \times \\
&\quad D_{1+\alpha-k}^{\frac{m-1}{2}+k}(\cosh \varphi) C_k^{\frac{m}{2}-1}(\cos \psi) C_{1+\alpha-k}^{\frac{m-1}{2}+k}(\cosh \theta) .
\end{aligned}$$

This series converges in the region where

$$|\cos \psi + (\cos^2 \psi - 1)^{\frac{1}{2}}| < \left| \frac{(\cosh \varphi \mp 1)(\cosh \theta + 1)}{(\cosh \varphi \pm 1)(\cosh \theta - 1)} \right|^{\frac{1}{2}} .$$

The Gamma operator $\Gamma = Y \wedge \partial_Y$ on the hyperbolic unit ball H_+ has the following representation in space-time co-ordinates (S, \underline{Y}) :

$$\Gamma = -\epsilon \underline{Y} \partial_S - S \epsilon \partial_{\underline{Y}} + \Gamma_{0,m} ,$$

with $\Gamma_{0,m}$ the spherical Dirac operator on S^{m-1} in the co-ordinates $\underline{\eta}$.

With respect to ϵ as a privileged direction, a co-ordinate system on the hyperbolic unit ball is obtained by choosing $\varphi \in \mathbb{R}$ and $\underline{\eta} \in S^{m-1}$ as the co-ordinates on H_+ . With respect to this co-ordinate system the hyperbolic Dirac operator Γ is given by

$$\Gamma = \left(1 + \underline{\eta} \epsilon \frac{\cosh \varphi}{\sinh \varphi}\right) \Gamma_{0,m} + \underline{\eta} \epsilon \frac{\partial}{\partial \varphi} .$$

In view of the fact that the function $E_\alpha(\eta, \xi)$, given by definition 5.1, can be written as

$$E_\alpha(\eta, \xi) = \frac{e^{-i\pi \frac{m-1}{2}}}{2\pi^{\frac{m-1}{2}}} \Gamma\left(\frac{m-1}{2}\right) \eta (\Gamma + 1 + \alpha) D_{\alpha+1}^{\frac{m-1}{2}}(\tau) , \quad (5.4)$$

where the hyperbolic Gamma operator acts on the Y -variable, we have the following series expansion for all $\alpha + m \notin -\mathbb{N}$:

$$\begin{aligned}
E_\alpha(\eta, \xi) &= \frac{e^{-i\pi \frac{m-1}{2}}}{2\pi^{\frac{m-1}{2}}} \frac{\Gamma(m-1)}{\Gamma\left(\frac{m-1}{2}\right)} \eta \\
&\quad \left[\left(1 + \underline{\eta} \epsilon \frac{\cosh \varphi}{\sinh \varphi}\right) \Gamma_{0,m} + \underline{\eta} \epsilon \frac{\partial}{\partial \varphi} + 1 + \alpha \right] \sum_{k=0}^{\infty} a_k(\alpha, m) c_k(\theta, \varphi, \psi)
\end{aligned}$$

Before actually calculating this expression, we define :

Definition 5.4 Let $\underline{\xi}$ and $\underline{\eta} \in S^{m-1}$, and let $k \in \mathbb{N}$. We then define

$$\begin{aligned} Z_k(\underline{\eta}, \underline{\xi}) &= C_k^{\frac{m}{2}}(< \underline{\eta}, \underline{\xi} >) + \underline{\eta} \underline{\xi} C_{k-1}^{\frac{m}{2}}(< \underline{\eta}, \underline{\xi} >) \\ B_k(\underline{\eta}, \underline{\xi}) &= \begin{cases} -\underline{\eta} Z_{k-1}(\underline{\eta}, \underline{\xi}) \underline{\xi} & k \geq 1 \\ 0 & k = 0 \end{cases} \end{aligned}$$

The motivation for this definition lies in the following equality :

$$\frac{m-2+2k}{m-2} C_k^{\frac{m}{2}-1}(< \underline{\eta}, \underline{\xi} >) = Z_k(\underline{\eta}, \underline{\xi}) - B_k(\underline{\eta}, \underline{\xi}) .$$

As $Z_k(\underline{\eta}, \underline{\xi})$ is an inner spherical monogenic of degree k and as $B_k(\underline{\eta}, \underline{\xi})$ is an outer spherical monogenic of degree $(k-1)$, both with respect to the variable \underline{y} on \mathbb{R}^m , we get :

$$\begin{aligned} \Gamma_{0,m} Z_k(\underline{\eta}, \underline{\xi}) &= -k Z_k(\underline{\eta}, \underline{\xi}) \\ \Gamma_{0,m} B_k(\underline{\eta}, \underline{\xi}) &= (m+k-2) B_k(\underline{\eta}, \underline{\xi}) . \end{aligned}$$

Hence, considering the hyperbolic Gamma operator in terms of the co-ordinates $(\varphi, \underline{\eta})$ we find

$$\begin{aligned} & \left(\Gamma + 1 + \alpha \right) \left[(\sinh \varphi)^k D_{1+\alpha-k}^{\frac{m-1}{2}+k} (\cosh \varphi) \frac{m-2+2k}{m-2} C_k^{\frac{m}{2}-1}(\cos \psi) \right] \\ &= (\sinh \varphi)^{k-1} \left\{ \begin{aligned} & (1+\alpha-k) \sinh \varphi D_{1+\alpha-k}^{\frac{m-1}{2}+k} (\cosh \varphi) \\ & + (2k+m-1) \sinh^2 \varphi D_{\alpha-k}^{\frac{m+1}{2}+k} (\cosh \varphi) \underline{\eta} \epsilon \end{aligned} \right\} Z_k \\ &+ (\sinh \varphi)^{k-1} \left\{ \begin{aligned} & (m+\alpha+k-1) \sinh \varphi D_{1+\alpha-k}^{\frac{m-1}{2}+k} (\cosh \varphi) \\ & + \frac{(m+\alpha+k-1)(\alpha+2-k)}{(m+2k-3)} D_{2+\alpha-k}^{\frac{m-3}{2}+k} (\cosh \varphi) \underline{\eta} \epsilon \end{aligned} \right\} B_k . \end{aligned}$$

In view of the addition formula, both the left-hand side and the right-hand side of the previous equation must still be multiplied with

$$a_k(\alpha, m) \frac{\Gamma(m-1)}{\Gamma\left(\frac{m-1}{2}\right)^2} (\sinh \theta)^k C_{1+\alpha-k}^{\frac{m-1}{2}+k} (\cosh \theta)$$

and summed over the parameter k , in order to obtain an expression for the operator $(\Gamma + 1 + \alpha)$ acting on the Gegenbauer function $D_{1+\alpha}^{\frac{m-1}{2}}(\underline{\eta} \cdot \underline{\xi})$. This expression can then be cast into the form

$$\sum_{k=0}^{\infty} c_k \left(S_1 + S_2 \underline{\eta} \epsilon \right) \left(S_3 + S_4 \underline{\eta} \underline{\xi} \right) \left(S_5 + S_6 \underline{\xi} \epsilon \right)$$

with c_k a constant, depending on k , and S_i a scalar function ($i = 1, 2, \dots, 6$).

This goes as follows : first of all we write B_k as $-\eta Z_{k-1} \xi$. Since $Z_{-1} \equiv 0$, the second series starts from $k = 1$. Rewriting this series, by putting $k' = k - 1$, one finds :

$$(\Gamma_\eta + 1 + \alpha) D_{1+\alpha}^{\frac{m-1}{2}}(\eta \cdot \xi) = \Sigma_1 + \Sigma_2 ,$$

where we have put

$$\begin{aligned} \Sigma_1 = & \frac{\Gamma(m-1)}{\Gamma\left(\frac{m-1}{2}\right)^2} \sum_{k=0}^{\infty} a_k(\alpha, m) (\sinh \varphi)^k C_{1+\alpha-k}^{\frac{m-1}{2}+k}(\cosh \theta) \times \\ & (\sinh \theta)^k \left\{ \begin{array}{c} (1 + \alpha - k) D_{1+\alpha-k}^{\frac{m-1}{2}+k}(\cosh \varphi) \\ + \\ (2k + m - 1) \sinh \varphi D_{\alpha-k}^{\frac{m+1}{2}+k}(\cosh \varphi) \underline{\eta} \underline{\epsilon} \end{array} \right\} Z_k(\underline{\eta}, \underline{\xi}) \end{aligned}$$

and

$$\begin{aligned} \Sigma_2 = & \frac{\Gamma(m-1)}{\Gamma\left(\frac{m-1}{2}\right)^2} \sum_{k=0}^{\infty} \frac{(2k + m - 1)^2}{(1 + \alpha - k)} a_k(\alpha, m) (\sinh \varphi)^k C_{\alpha-k}^{\frac{m+1}{2}+k}(\cosh \theta) \times \\ & (\sinh \theta)^{k+1} \left\{ \begin{array}{c} \sinh \varphi D_{\alpha-k}^{\frac{m+1}{2}+k}(\cosh \varphi) \underline{\eta} \\ + \\ \frac{1 + \alpha - k}{2k + m - 1} D_{1+\alpha-k}^{\frac{m-1}{2}+k}(\cosh \varphi) \underline{\epsilon} \end{array} \right\} Z_k(\underline{\eta}, \underline{\xi}) \underline{\xi} . \end{aligned}$$

Eventually gathering the terms in $D_{1+\alpha-k}^{\frac{m-1}{2}+k}(\cosh \varphi)$ (resp. $D_{\alpha-k}^{\frac{m+1}{2}+k}(\cosh \varphi)$) and making use of definition 5.4 to rewrite $Z_k(\underline{\xi}, \underline{\eta})$ in its explicit form, we get :

$$\begin{aligned} & \left(\Gamma + 1 + \alpha \right) D_{1+\alpha}^{\frac{m-1}{2}}(\eta \cdot \xi) \\ = & \frac{\Gamma(m-1)}{\Gamma\left(\frac{m-1}{2}\right)^2} \sum_{k=0}^{\infty} (-1)^k 4^k \frac{\Gamma(\alpha + 1 - k)}{\Gamma(\alpha + m + k)} \Gamma\left(\frac{m-1}{2} + k\right)^2 \times \\ & \left\{ (1 + \alpha - k) D_{1+\alpha-k}^{\frac{m-1}{2}+k}(\cosh \varphi) + (2k + m - 1) \sinh \varphi D_{\alpha-k}^{\frac{m+1}{2}+k}(\cosh \varphi) \underline{\eta} \underline{\epsilon} \right\} \\ & (\sinh \varphi \sinh \theta)^k \left\{ C_k^{\frac{m}{2}}(< \underline{\eta}, \underline{\xi} >) + \underline{\eta} \underline{\xi} C_{k-1}^{\frac{m}{2}}(< \underline{\eta}, \underline{\xi} >) \right\} \\ & \left\{ (1 + \alpha - k) C_{1+\alpha-k}^{\frac{m-1}{2}+k}(\cosh \theta) + (2k + m - 1) \sinh \theta C_{\alpha-k}^{\frac{m+1}{2}+k}(\cosh \theta) \underline{\xi} \underline{\epsilon} \right\} \end{aligned}$$

In view of expression (5.4) this has to be multiplied with

$$\frac{e^{-i\pi\frac{m-1}{2}}}{2\pi^{\frac{m-1}{2}}}\Gamma\left(\frac{m-1}{2}\right)\eta,$$

and the result is the function $E_\alpha(\eta, \xi)$, defined for all $\alpha + m \notin -\mathbb{N}$:

$$\begin{aligned} E_\alpha(\eta, \xi) = & \frac{e^{-i\pi\frac{m-1}{2}}}{2\pi^{\frac{m-1}{2}}}\frac{\Gamma(m-1)}{\Gamma(\frac{m-1}{2})}\eta\sum_{k=0}^{\infty}(-1)^k4^k\frac{\Gamma(\alpha+1-k)}{\Gamma(\alpha+m+k)}\Gamma\left(\frac{m-1}{2}+k\right)^2 \\ & \left\{ (1+\alpha-k)D_{1+\alpha-k}^{\frac{m-1}{2}+k}(\cosh\varphi) + (2k+m-1)\sinh\varphi D_{\alpha-k}^{\frac{m+1}{2}+k}(\cosh\varphi)\underline{\eta}\epsilon \right\} \\ & (\sinh\varphi\sinh\theta)^k\left\{ C_k^{\frac{m}{2}}(<\underline{\eta},\underline{\xi}>) + \underline{\eta}\underline{\xi}C_{k-1}^{\frac{m}{2}}(<\underline{\eta},\underline{\xi}>) \right\} \\ & \left\{ (1+\alpha-k)C_{1+\alpha-k}^{\frac{m-1}{2}+k}(\cosh\theta) + (2k+m-1)\sinh\theta C_{\alpha-k}^{\frac{m+1}{2}+k}(\cosh\theta)\overline{\underline{\xi}}\epsilon \right\} \end{aligned}$$

Invoking the relation

$$\Gamma(2z) = \frac{2^{2z-1}}{\pi^{\frac{1}{2}}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right),$$

using the explicit definition for A_m and recalling the fact that $\sinh\varphi$ (resp. $\sinh\theta$) stands for $(\sigma^2-1)^{\frac{1}{2}}$ (resp. $(\tau^2-1)^{\frac{1}{2}}$), this expression for $E_\alpha(\eta, \xi)$ reduces to the expression we have proposed earlier.

Remark : Although the calculations involved are rather lengthy, this proof for the expansion for $E_\alpha(\eta, \xi)$ is an important result : it gives an alternative interpretation for the Addition Theorem for the Gegenbauer function of the second kind as established in reference [25]. In [75] the Addition Theorem for the Gegenbauer functions is used to obtain an axial decomposition for the fundamental solution of the Dirac equation on the sphere S^{m-1} in \mathbb{R}^m . There too, the author gives an alternative proof, which is close to the spirit of harmonic analysis.

The present approach enables us to obtain expansions for the hyperbolic fundamental solution by means of a natural modulation argument. This will be used in the next section, when we describe the function theory on the Klein model for the hyperbolic unit ball.

In order to characterize the region of convergence of the series expansion for $E_\alpha(\eta, \xi)$, we first introduce the following definition :

Definition 5.5 For all $\zeta \in H_+$ and for arbitrary $R > 1$ the set $HC(R, \zeta)$ is defined as

$$HC(R, \zeta) = \{\xi \in H_+ : 1 \leq \xi \cdot \zeta < R\} .$$

The notation HC is inspired by the fact that this subset $HC(R, \zeta) \subset H_+$ is a *hyperbolic cap*, by analogy with the term spherical cap. Note that the set $HC(R, \zeta)$ is invariant under the subgroup $SO(1, m)_\zeta$ of $SO(1, m)$ fixing $\zeta \in H_+$, or $HC(R, \zeta)$ is an $SO(m)$ -invariant subdomain of H_+ .

Recalling our notations

$$\begin{aligned} \eta &= \sigma\epsilon + (\sigma^2 - 1)^{\frac{1}{2}}\underline{\eta} = \epsilon \cosh \varphi + \underline{\eta} \sinh \varphi \\ \xi &= \tau\epsilon + (\tau^2 - 1)^{\frac{1}{2}}\underline{\xi} = \epsilon \cosh \theta + \underline{\xi} \sinh \theta \end{aligned} ,$$

the series expansion for $E_\alpha(\eta, \xi)$ converges normally on each hyperbolic cap $HC(R, \epsilon)$, for all $\tau \leq R < \sigma$, where σ is being held fixed.

Definition 5.6 For two arbitrary space-time vectors η and $\xi \in H_+$, the function $E_\alpha^k(\eta, \xi)$ is for all $k \in \mathbb{N}$ and $\alpha + m \notin -\mathbb{N}$ defined by

$$\begin{aligned} E_\alpha^k(\eta, \xi) &= \eta(-1)^k 2^{2k+m-2} e^{-i\pi \frac{m-1}{2}} \Gamma\left(k + \frac{m-1}{2}\right)^2 \frac{\Gamma(1 + \alpha - k)}{\Gamma(\alpha + k + m)} \\ &(\sigma^2 - 1)^{\frac{k}{2}} \left\{ (1 + \alpha - k) D_{1+\alpha-k}^{k+\frac{m-1}{2}}(\sigma) + (2k + m - 1)(\sigma^2 - 1)^{\frac{1}{2}} D_{\alpha-k}^{k+\frac{m+1}{2}}(\sigma) \underline{\eta} \epsilon \right\} \\ &\quad \left\{ C_k^{\frac{m}{2}}(< \underline{\xi}, \underline{\eta} >) + C_{k-1}^{\frac{m}{2}}(< \underline{\xi}, \underline{\eta} >) \underline{\eta} \underline{\xi} \right\} \\ &(\tau^2 - 1)^{\frac{k}{2}} \left\{ (1 + \alpha - k) C_{1+\alpha-k}^{k+\frac{m-1}{2}}(\tau) + (2k + m - 1)(\tau^2 - 1)^{\frac{1}{2}} C_{\alpha-k}^{k+\frac{m+1}{2}}(\tau) \underline{\xi} \epsilon \right\} \end{aligned}$$

These functions $E_\alpha^k(\eta, \xi)$ are the building blocks of the axial decomposition for $E_\alpha(\eta, \xi)$. The term *axial decomposition* hereby refers to the fact that we have singled out a privileged direction, or *axis*, in casu $\epsilon \in H_+$.

However, the following Theorem remains true if we replace ϵ by an arbitrary element ζ of the hyperbolic unit ball. This is an immediate consequence of the fact that the hyperbolic unit ball is a *homogeneous* space, in which all points are equivalent by definition.

Theorem 5.6 The hyperbolic fundamental solution $E_\alpha(\eta, \xi)$ has an axial decomposition given by :

$$E_\alpha(\eta, \xi) = \frac{1}{A_m} \sum_{k=0}^{\infty} E_\alpha^k(\eta, \xi) .$$

This expansion is valid for all $\alpha + m \notin -\mathbb{N}$ and converges normally on each closed hyperbolic cap $\overline{HC}(R, \epsilon)$ with $\tau \leq R < \sigma$. By construction we also have

$$\begin{aligned} E_\alpha^k(\eta, \xi) &\in \mathcal{H}_\eta^\alpha(H_+^{(\epsilon)}) \\ \overline{E}_\alpha^k(\eta, \xi) &\in \mathcal{H}_\xi^\beta(H_+) , \end{aligned}$$

where we have indicated the variable on which the Dirac operator is supposed to act.

This Theorem can then be used to establish a Taylor series for hyperbolic monogenics defined in $SO(m)$ -invariant subdomains Ω_ϵ . However, we must be careful because the decomposition for $E_\alpha(\eta, \xi)$ established above holds for $\sigma > \tau$. In order to find the Taylor series of a hyperbolic monogenic around ϵ we thus need to integrate the fundamental solution with respect to η , in contrast to the Cauchy integral obtained in Theorem 5.5. Using expression (27) one can easily verify that $E_\alpha(\eta, \xi) = -E_\beta(\xi, \eta)$, up to a nullsolution for the hyperbolic angular operator, such that for functions $F \in \mathcal{H}^\alpha(\Omega)$, with $\Omega \subset H_+$ open and $C \subset \Omega$ compact with smooth boundary ∂C , Stokes' Theorem yields immediately :

$$\begin{aligned} \eta \in \overset{\circ}{C} \implies F(\eta) &= \int_{\partial C} E_\alpha(\eta, \xi) \Sigma(\xi, d\xi) F(\xi) \\ &= - \int_{\partial C} E_\beta(\xi, \eta) \Sigma(\xi, d\xi) F(\xi) , \end{aligned}$$

or equivalently, for $\xi \in \overset{\circ}{C}$:

$$F(\xi) = - \int_{\partial C} E_\beta(\eta, \xi) \Sigma(\eta, d\eta) F(\eta) ,$$

which by means of the fact that $\overline{E}_{\bar{\beta}}(\eta, \xi) = -E_\beta(\eta, \xi)$ eventually leads to :

$$F(\xi) = \int_{\partial C} \overline{E}_{\bar{\beta}}(\eta, \xi) \Sigma(\eta, d\eta) F(\eta) ,$$

with $\overline{E}_{\bar{\beta}}(\eta, \xi) = \frac{1}{A_m} \sum_k \overline{E}_{\bar{\beta}}^k(\eta, \xi)$ and

$$\begin{aligned} \overline{E}_{\bar{\beta}}^k(\eta, \xi) &= (-1)^k 2^{2k+m-2} e^{-i\pi \frac{m-1}{2}} \Gamma\left(k + \frac{m-1}{2}\right)^2 \frac{\Gamma(1+\beta-k)}{\Gamma(\beta+k+m)} \\ &(\tau^2 - 1)^{\frac{k}{2}} \left\{ (1+\beta-k) C_{1+\beta-k}^{k+\frac{m-1}{2}}(\tau) + (2k+m-1)(\tau^2-1)^{\frac{1}{2}} C_{\beta-k}^{k+\frac{m+1}{2}}(\tau) \underline{\xi} \underline{\epsilon} \right\} \\ &\quad \left\{ C_k^{\frac{m}{2}}(<\underline{\xi}, \underline{\eta}>) + C_{k-1}^{\frac{m}{2}}(<\underline{\xi}, \underline{\eta}>) \underline{\xi} \underline{\eta} \right\} \\ &(\sigma^2 - 1)^{\frac{k}{2}} \left\{ (1+\beta-k) D_{1+\beta-k}^{k+\frac{m-1}{2}}(\sigma) - (2k+m-1)(\sigma^2-1)^{\frac{1}{2}} D_{\beta-k}^{k+\frac{m+1}{2}}(\sigma) \underline{\eta} \underline{\epsilon} \right\} \overline{\eta} \end{aligned}$$

This eventually leads to a Taylor series for functions $F(\xi) \in \mathcal{H}^\alpha(HC(R, \epsilon))$, with $R > 1$ arbitrary :

Theorem 5.7 (Taylor) *Let $F(\xi) \in \mathcal{H}^\alpha(HC(R, \epsilon))$, with $R > 1$ arbitrary and $\alpha + m \notin -\mathbb{N}$. There exists a sequence of functions $(F_\epsilon^{(k)}(\xi))_{k \in \mathbb{N}}$ such that the function $\xi \mapsto F_\epsilon^{(k)}(\xi)$ belongs to $\mathcal{H}^\alpha(H_+)$ for each $k \in \mathbb{N}$ and such that the following expansion holds in $HC(R, \epsilon)$:*

$$F(\xi) = \sum_{k=0}^{\infty} F_\epsilon^{(k)}(\xi) ,$$

where $F_\epsilon^{(k)}(\xi)$ has the following integral representation, for each $k \in \mathbb{N}$:

$$F_\epsilon^{(k)}(\xi) = \frac{1}{A_m} \int_{\sigma=r} \overline{E}_{\bar{\beta}}(\eta, \xi) \Sigma(\eta, d\eta) F(\eta)$$

where the integration goes over $\partial HC(r, \epsilon)$, $r \in]1, R[$ arbitrarily and where $\Sigma(\eta, d\eta) = n ds$, n denoting the outer unit normal with respect to $\partial HC(r, \epsilon)$ and ds the Lebesgue measure on $\partial HC(r, \epsilon)$. Denoting the projection of a function f on S^{m-1} onto the space of inner spherical monogenics of order k by $P(k)[f]$, this integral is in explicit form given by

$$F_\epsilon^{(k)}(\xi) = \frac{(-1)^k 2^{2k+m-2}}{e^{i\pi \frac{m-1}{2}}} \Gamma\left(k + \frac{m-1}{2}\right)^2 \frac{\Gamma(1+\beta-k)}{\Gamma(\beta+k+m)} (\tau^2 - 1)^{\frac{k}{2}} (r^2 - 1)^{\frac{k+m-1}{2}} \\ \left\{ (\beta + m + k - 1) C_{\beta-k}^{k+\frac{m-1}{2}}(\tau) + (2k + m - 1) (\tau^2 - 1)^{\frac{1}{2}} C_{\beta-k-1}^{k+\frac{m+1}{2}}(\tau) \underline{\xi} \epsilon \right\} \\ P(k) \left[\begin{pmatrix} (1 + \beta - k) D_{1+\beta-k}^{k+\frac{m-1}{2}}(r) \underline{\eta} \epsilon \\ + \\ (2k + m - 1) (r^2 - 1)^{\frac{1}{2}} D_{\beta-k}^{k+\frac{m+1}{2}}(r) \end{pmatrix} F(r\epsilon + (r^2 - 1)^{\frac{1}{2}} \underline{\eta}) \right]$$

This series expansion for $F(\eta)$ converges on closed hyperbolic caps $\overline{HC}(\rho, \epsilon)$, with $\tau \leq \rho < r$.

Proof : Consider a function $F(\xi) \in \mathcal{H}^\alpha(HC(R, \epsilon))$, such that

$$\xi(\Gamma + \alpha)F(\xi) = 0$$

for all $\xi \in HC(R, \epsilon)$, i.e. for all $\xi \in H_+$ such that $\xi \cdot \epsilon < R$. Consider then an arbitrary $\xi = \tau\epsilon + (\tau^2 - 1)^{\frac{1}{2}} \underline{\xi} \in HC(R, \epsilon)$. As $\tau = \xi \cdot \epsilon < R$, there exists

a real number r such that $\tau < r < R$. The closed hyperbolic cap $\overline{HC}(r, \epsilon)$, with boundary

$$\partial\overline{HC}(r, \epsilon) = \{\eta \in H_+ : \sigma = \eta \cdot \epsilon = r\} ,$$

may then be interpreted as a compact subset of the open set $HC(R, \epsilon)$. Since $\tau < r$, we have $\xi \in \overline{HC}(r, \epsilon)$ such that Cauchy's Theorem gives :

$$F(\xi) = \int_{\sigma=r} \overline{E}_{\bar{\beta}}(\eta, \xi) \Sigma(\eta, d\eta) F(\eta) .$$

In view of Theorem 5.6 for $\overline{E}_{\bar{\beta}}(\eta, \xi)$, this becomes

$$F(\xi) = \frac{1}{A_m} \sum_{k=0}^{\infty} \int_{\sigma=r} \overline{E}_{\bar{\beta}}^k(\eta, \xi) \Sigma(\eta, d\eta) F(\eta) = \sum_{k=0}^{\infty} F_{\epsilon}^{(k)}(\xi) ,$$

with

$$F_{\epsilon}^{(k)}(\xi) = \frac{1}{A_m} \int_{\sigma=r} \overline{E}_{\bar{\beta}}^k(\eta, \xi) \Sigma(\eta, d\eta) F(\eta) .$$

Because the boundary $\partial\overline{HC}(r, \epsilon)$ is a sphere with radius $(r^2 - 1)^{\frac{1}{2}}$, we get :

$$\int_{\sigma=r} \Sigma(\eta, d\eta) = (r^2 - 1)^{\frac{m-1}{2}} \int_{S^{m-1}} n dS(\underline{\eta}) ,$$

with $dS(\underline{\eta})$ the Lebesgue measure on the m -dimensional sphere S^{m-1} and n the outer unit normal to $\partial\overline{HC}(r, \epsilon)$ with respect to $\overline{HC}(r, \epsilon)$. In an arbitrary point of the boundary, this is nothing but the tangent vector to the hyperbola obtained by intersecting H_+ with the 2-dimensional plane through this point, its diametrically opposite point on the sphere and the origin. Considering the point $p = r\epsilon + (r^2 - 1)^{\frac{1}{2}}\underline{\eta} \in \overline{HC}(r, \epsilon)$, the tangent vector n_p in p is given as the unit space-time vector

$$n_p = (r^2 - 1)^{\frac{1}{2}}\epsilon + r\underline{\eta} .$$

This can easily be verified geometrically : for $r = 1$ the point $p = \epsilon$ and the tangent vector n_p is then given by a pure spatial vector, orthogonal to ϵ with respect to the hyperbolic metric. When considering another point $p' \in H_+$, obtained by applying a Lorentz boost to p , the tangent vector n_p is boosted to a new tangent vector $n_{p'}$ such that p' and $n_{p'}$ are still orthogonal with respect to the hyperbolic metric. In addition, $n_{p'}$ remains a space-like unit vector and this fixes $n_{p'}$. Note that in the limit for $r \rightarrow +\infty$ the tangent

vector becomes a nullvector.

The function $F_\epsilon^{(k)}(\xi)$ is thus given by

$$F_\epsilon^{(k)}(\xi) = \frac{1}{A_m} (r^2 - 1)^{\frac{m-1}{2}} \int_{S^{m-1}} \overline{E}_\beta^k(\eta, \xi) ((r^2 - 1)^{\frac{1}{2}} \epsilon + r \underline{\eta}) dS(\underline{\eta}) F(\eta) ,$$

where η is to be replaced by $r\epsilon + (r^2 - 1)^{\frac{1}{2}} \underline{\eta}$.

Referring to the definition of $\overline{E}_\beta^k(\eta, \xi)$, the following factor may be brought from under the integration :

$$\begin{aligned} & (-1)^k 2^{2k+m-2} e^{-i\pi \frac{m-1}{2}} \Gamma \left(k + \frac{m-1}{2} \right)^2 \frac{\Gamma(1+\beta-k)}{\Gamma(\beta+k+m)} (\tau^2 - 1)^{\frac{k}{2}} (r^2 - 1)^{\frac{k}{2}} \\ & \left\{ (1+\beta-k) C_{1+\beta-k}^{k+\frac{m-1}{2}}(\tau) + (2k+m-1)(\tau^2 - 1)^{\frac{1}{2}} C_{\beta-k}^{k+\frac{m+1}{2}}(\tau) \underline{\xi} \epsilon \right\} . \end{aligned}$$

Under the integration sign, we are left with the following :

$$\begin{aligned} & \left\{ C_k^{\frac{m}{2}}(< \underline{\xi}, \underline{\eta} >) + C_{k-1}^{\frac{m}{2}}(< \underline{\xi}, \underline{\eta} >) \underline{\xi} \underline{\eta} \right\} \\ & \left\{ (1+\beta-k) D_{1+\beta-k}^{k+\frac{m-1}{2}}(r) - (2k+m-1)(r^2 - 1)^{\frac{1}{2}} D_{\beta-k}^{k+\frac{m+1}{2}}(r) \overline{\eta} \epsilon \right\} \\ & \overline{\eta} ((r^2 - 1)^{\frac{1}{2}} \epsilon + r \underline{\xi}) F(\eta) , \end{aligned}$$

which is equal to

$$\begin{aligned} & -\underline{\xi} \left\{ C_k^{\frac{m}{2}}(< \underline{\xi}, \underline{\eta} >) \underline{\xi} - C_{k-1}^{\frac{m}{2}}(< \underline{\xi}, \underline{\eta} >) \underline{\eta} \right\} \\ & \left\{ (1+\beta-k) D_{1+\beta-k}^{k+\frac{m-1}{2}}(r) \underline{\eta} \epsilon + (2k+m-1)(r^2 - 1)^{\frac{1}{2}} D_{\beta-k}^{k+\frac{m+1}{2}}(r) \right\} \\ & F(r\epsilon + (r^2 - 1)^{\frac{1}{2}} \underline{\eta}) . \end{aligned}$$

Note that the first factor is the projection kernel on the space of inner or outer spherical monogenics on \mathbb{R}^m (see also section 0.1.2). Recalling the notation

$$P(k)[f](\underline{\xi}) = -\frac{1}{A_m} \underline{\xi} \int_{S^{m-1}} \left\{ C_k^{\frac{m}{2}}(< \underline{\xi}, \underline{\eta} >) \underline{\xi} - C_{k-1}^{\frac{m}{2}}(< \underline{\xi}, \underline{\eta} >) \underline{\eta} \right\} f(\underline{\eta}) dS(\underline{\eta})$$

for the projection of a function f on S^{m-1} onto the space of inner spherical monogenics, the integral representation for $F_\epsilon^{(k)}(\xi)$ can thus be reduced to

$$\begin{aligned} & \frac{(-1)^k 2^{2k+m-2}}{e^{i\pi \frac{m-1}{2}}} \Gamma\left(k + \frac{m-1}{2}\right)^2 \frac{\Gamma(1+\beta-k)}{\Gamma(\beta+k+m)} (\tau^2-1)^{\frac{k}{2}} (r^2-1)^{\frac{k+m-1}{2}} \\ & \left\{ (\beta+m+k-1) C_{\beta-k}^{k+\frac{m-1}{2}}(\tau) + (2k+m-1)(\tau^2-1)^{\frac{1}{2}} C_{\beta-k-1}^{k+\frac{m+1}{2}}(\tau) \underline{\xi} \epsilon \right\} \\ & P(k) \left[\begin{pmatrix} (1+\beta-k) D_{1+\beta-k}^{k+\frac{m-1}{2}}(r) \underline{\eta} \epsilon \\ + \\ (2k+m-1)(r^2-1)^{\frac{1}{2}} D_{\beta-k}^{k+\frac{m+1}{2}}(r) \end{pmatrix} F(r\epsilon + (r^2-1)^{\frac{1}{2}} \underline{\eta}) \right] \end{aligned}$$

The series expansion for the function $F(\xi)$ converges normally on compact subsets $\overline{HC}(\rho, \epsilon)$, with $\tau = \xi \cdot \epsilon \leq \rho < r$. \square

Next, we consider a hyperbolic monogenic $F(\xi) \in \mathcal{H}^\alpha(\Omega_\epsilon)$, where Ω_ϵ stands for the open annular $\text{SO}(m)$ -invariant subdomain $HC(R_1, \epsilon) \setminus \overline{HC}(R_2, \epsilon)$ in H_+ , with $R_1 > R_2 > 1$. For these functions, the following Theorem holds :

Theorem 5.8 (*Laurent*) *Let $F \in \mathcal{H}^\alpha(HC(R_1, \epsilon) \setminus \overline{HC}(R_2, \epsilon))$, with $\alpha \in \mathbb{C}$ such that $\alpha + m \notin -\mathbb{N}$ and $R_1 > R_2 > 1$. There exists a sequence of functions $(F_\epsilon^{(k)}(\xi))_{k \in \mathbb{N}}$ and a sequence of functions $(G_\epsilon^{(k)}(\xi))_{k \in \mathbb{N}}$ such that for all $k \in \mathbb{N}$*

$$\begin{aligned} \xi & \mapsto F_\epsilon^{(k)}(\xi) \in \mathcal{H}^\alpha(H_+) \\ \xi & \mapsto G_\epsilon^{(k)}(\xi) \in \mathcal{H}^\alpha(H_+^{(\epsilon)}) , \end{aligned}$$

and such that for all $\xi \in HC(R_1, \epsilon) \setminus \overline{HC}(R_2, \epsilon)$ we have

$$F(\xi) = \sum_{k=0}^{\infty} F_\epsilon^{(k)}(\xi) + G_\epsilon^{(k)}(\xi) .$$

This series expansion converges normally on compact annular subdomains $\overline{HC}(r_1, \epsilon) \setminus HC(r_2, \epsilon)$ of H_+ , with $R_1 > r_1 \geq \tau \geq r_2 > R_2$, and the functions $F_\epsilon^{(k)}(\xi)$ and $G_\epsilon^{(k)}(\xi)$ have the following integral representation :

$$\begin{aligned} F_\epsilon^{(k)}(\xi) &= \frac{1}{A_m} \int_{\sigma=r_1} \overline{E}_{\bar{\beta}}(\eta, \xi) \Sigma(\eta, d\eta) F(\eta) \\ G_\epsilon^{(k)}(\xi) &= \frac{1}{A_m} \int_{\sigma=r_2} E_\alpha(\xi, \eta) \Sigma(\eta, d\eta) F(\eta) \end{aligned}$$

where the integration goes resp. over the spheres $\partial HC(r_1, \epsilon)$ and $\partial HC(r_2, \epsilon)$; and where $\Sigma(\eta, d\eta) = nds$, with n the outer unit normal field with respect to

$\partial HC(r_1, \epsilon)$ and $\partial HC(r_2, \epsilon)$ and ds the Lebesgue measure on these spheres. Denoting the projection of a function f on S^{m-1} onto the space of inner (resp. outer) spherical monogenics of order k by $P(k)[f]$ (resp. $Q(k)[f]$), these integrals are in explicit form given by :

$$F_\epsilon^{(k)}(\xi) = \frac{(-1)^k 2^{2k+m-2}}{e^{i\pi \frac{m-1}{2}}} \Gamma\left(k + \frac{m-1}{2}\right)^2 \frac{\Gamma(1+\beta-k)}{\Gamma(\beta+k+m)} (\tau^2-1)^{\frac{k}{2}} (r_1^2-1)^{\frac{k+m-1}{2}} \\ \left\{ (\beta+m+k-1) C_{\beta-k}^{k+\frac{m-1}{2}}(\tau) + (2k+m-1)(\tau^2-1)^{\frac{1}{2}} C_{\beta-k-1}^{k+\frac{m+1}{2}}(\tau) \underline{\xi} \epsilon \right\} \\ P(k) \left[\left(\begin{array}{c} (1+\beta-k) D_{1+\beta-k}^{k+\frac{m-1}{2}}(r_1) \underline{\eta} \epsilon \\ + \\ (2k+m-1)(r_1^2-1)^{\frac{1}{2}} D_{\beta-k}^{k+\frac{m+1}{2}}(r_1) \end{array} \right) F(r_1 \epsilon + (r_1^2-1)^{\frac{1}{2}} \underline{\eta}) \right]$$

and

$$G_\epsilon^{(k)}(\xi) = \frac{(-1)^k 2^{2k+m-2}}{e^{i\pi \frac{m-1}{2}}} \Gamma\left(k + \frac{m-1}{2}\right)^2 \frac{\Gamma(1+\alpha-k)}{\Gamma(\alpha+k+m)} (\tau^2-1)^{\frac{k}{2}} (r_1^2-1)^{\frac{k+m-1}{2}} \\ \left\{ (\alpha+m+k-1) D_{\alpha-k}^{k+\frac{m-1}{2}}(\tau) \underline{\xi} \epsilon - (2k+m-1)(\tau^2-1)^{\frac{1}{2}} D_{\alpha-k-1}^{k+\frac{m+1}{2}}(\tau) \right\} \\ Q(k) \left[\left(\begin{array}{c} (\alpha+k+m-1) C_{\alpha-k}^{k+\frac{m-1}{2}}(r_2) \\ + \\ (2k+m-1)(r_2^2-1)^{\frac{1}{2}} C_{\alpha-k-1}^{k+\frac{m+1}{2}}(r_2) \end{array} \right) F(r_2 \epsilon + (r_2^2-1)^{\frac{1}{2}} \underline{\eta}) \right]$$

Proof : Let us put $\Omega_\epsilon = HC(R_1, \epsilon) \setminus \overline{HC}(R_2, \epsilon)$, where $R_1 > R_2 > 1$, such that

$$\xi(\Gamma + \alpha)F(\xi) = 0 ,$$

for all $\xi \in \Omega_\epsilon$. Consider then an arbitrary $\xi = \tau\epsilon + (\tau^2-1)^{\frac{1}{2}}\underline{\xi} \in \Omega_\epsilon$, $\underline{\xi} \in S^{m-1}$. As $R_1 > \tau > R_2$, there exist r_1 and r_2 such that $r_1 > \tau > r_2$. Hence, Cauchy's Theorem may be applied to the compact subset $\overline{HC}(r_1, \epsilon) \setminus HC(r_2, \epsilon)$ of H_+ , bounded by the spheres

$$\partial \overline{HC}(r_1, \epsilon) = \{ \eta \in H_+ : \sigma = \eta \cdot \epsilon = r_1 \}$$

and

$$\partial \overline{HC}(r_2, \epsilon) = \{ \eta \in H_+ : \sigma = \eta \cdot \epsilon = r_2 \} .$$

We thus have that

$$F(\xi) = \int_{\sigma=r_1} \overline{E}_{\bar{\beta}}(\eta, \xi) \Sigma(\eta, d\eta) F(\eta) - \int_{\sigma=r_2} E_{\alpha}(\xi, \eta) \Sigma(\eta, d\eta) F(\eta) .$$

For the first integral, we refer to the proof of the Taylor Theorem 5.7. In order to calculate the second integral, we use the expansion of $E_{\alpha}(\xi, \eta)$ as given by Theorem 5.6. This yields :

$$\int_{\sigma=r_2} E_{\alpha}(\xi, \eta) \Sigma(\eta, d\eta) F(\eta) = (r_2^2 - 1)^{\frac{m-1}{2}} \sum_{k=0}^{\infty} \int_{S^{m-1}} E_{\alpha}^k(\xi, \eta) n F(\eta) dS(\underline{\eta}) ,$$

where $\eta = r_2 \epsilon + (r_2^2 - 1)^{\frac{1}{2}} \underline{\eta}$ and n stands for the outer unit normal to the sphere $\partial \overline{HC}(r_2, \epsilon)$. This normal vector was already determined in the proof of Theorem 5.7, and is in the point η given by $n = (r_2^2 - 1)^{\frac{1}{2}} \epsilon + r_2 \underline{\eta}$. Denoting for each $k \in \mathbb{N}$

$$I_k(\xi) = \int_{S^{m-1}} E_{\alpha}^k(\xi, \eta) n F(\eta) dS(\underline{\eta}) ,$$

we can bring the following factor outside the integration in $I_k(\xi)$:

$$\begin{aligned} & \frac{(-1)^k 2^{2k+m-2}}{A_m e^{i\pi \frac{m-1}{2}}} \Gamma \left(k + \frac{m-1}{2} \right)^2 \frac{\Gamma(1+\alpha-k)}{\Gamma(\alpha+k+m)} (\tau^2 - 1)^{\frac{k}{2}} (r_2^2 - 1)^{\frac{k}{2}} \\ & \xi \left\{ (1+\alpha-k) D_{1+\alpha-k}^{k+\frac{m-1}{2}}(\tau) + (2k+m-1)(\tau^2 - 1)^{\frac{1}{2}} D_{\alpha-k}^{k+\frac{m+1}{2}}(\tau) \underline{\xi} \epsilon \right\} \end{aligned}$$

which, in view of what follows, may be written as

$$\begin{aligned} & \frac{(-1)^k 2^{2k+m-2}}{A_m e^{i\pi \frac{m-1}{2}}} \Gamma \left(k + \frac{m-1}{2} \right)^2 \frac{\Gamma(1+\alpha-k)}{\Gamma(\alpha+k+m)} (\tau^2 - 1)^{\frac{k}{2}} (r_2^2 - 1)^{\frac{k}{2}} \\ & \left\{ (\alpha+m+k-1) D_{\alpha-k}^{k+\frac{m-1}{2}}(\tau) \underline{\xi} \epsilon - (2k+m-1)(\tau^2 - 1)^{\frac{1}{2}} D_{\alpha-k-1}^{k+\frac{m+1}{2}}(\tau) \right\} \underline{\xi} . \end{aligned}$$

Note that we now have a factor $\underline{\xi}$ at the end. Under the integration sign in $I_k(\xi)$ we are left with the following :

$$\begin{aligned} & \left\{ C_k^{\frac{m}{2}}(< \underline{\xi}, \underline{\eta} >) + C_{k-1}^{\frac{m}{2}}(< \underline{\xi}, \underline{\eta} >) \underline{\xi} \underline{\eta} \right\} \\ & \left\{ (1+\alpha-k) C_{1+\alpha-k}^{k+\frac{m-1}{2}}(r_2) + (2k+m-1)(r_2^2 - 1)^{\frac{1}{2}} C_{\alpha-k}^{k+\frac{m+1}{2}}(r_2) \overline{\underline{\eta}} \epsilon \right\} \\ & \left\{ (r_2^2 - 1)^{\frac{1}{2}} \epsilon + r_2 \underline{\eta} \right\} . \end{aligned}$$

The product of the last two factors may be simplified, hereby using the recurrence relations for the Gegenbauer functions, and the factor $\underline{\xi}$ mentioned above may be brought under the integral sign. This eventually gives the following integrand :

$$\left\{ C_k^{\frac{m}{2}}(< \underline{\xi}, \underline{\eta} >) \underline{\xi} - C_{k-1}^{\frac{m}{2}}(< \underline{\xi}, \underline{\eta} >) \underline{\eta} \right\} \\ \left\{ (\alpha + k + m - 1) C_{\alpha-k}^{k+\frac{m-1}{2}}(r_2) \underline{\eta} - (2k + m - 1) (r_2^2 - 1)^{\frac{1}{2}} C_{\alpha-k-1}^{k+\frac{m+1}{2}}(r_2) \epsilon \right\} \quad .$$

Recalling the formula for projection onto the space $M_-(k)$ of outer spherical monogenics on S^{m-1} ,

$$Q(k)[f](\underline{\xi}) = -\frac{1}{A_m} \int_{S^{m-1}} \left\{ C_k^{\frac{m}{2}}(< \underline{\xi}, \underline{\eta} >) \underline{\xi} - C_{k-1}^{\frac{m}{2}}(< \underline{\xi}, \underline{\eta} >) \underline{\eta} \right\} \underline{\eta} f(\underline{\eta}) dS(\underline{\eta}) ,$$

this will eventually yield :

$$-\int_{\sigma=r_2} E_\alpha(\xi, \eta) \Sigma(\eta, d\eta) F(\eta) = \\ \frac{(-1)^k 2^{2k+m-2}}{e^{i\pi \frac{m-1}{2}}} \Gamma\left(k + \frac{m-1}{2}\right)^2 \frac{\Gamma(1 + \alpha - k)}{\Gamma(\alpha + k + m)} (\tau^2 - 1)^{\frac{k}{2}} (r_1^2 - 1)^{\frac{k+m-1}{2}} \\ \left\{ (\alpha + m + k - 1) D_{\alpha-k}^{k+\frac{m-1}{2}}(\tau) \underline{\xi} \epsilon - (2k + m - 1) (\tau^2 - 1)^{\frac{1}{2}} D_{\alpha-k-1}^{k+\frac{m+1}{2}}(\tau) \right\} \\ Q(k) \left[\begin{pmatrix} (\alpha + k + m - 1) C_{\alpha-k}^{k+\frac{m-1}{2}}(r_2) \\ + \\ (2k + m - 1) (r_2^2 - 1)^{\frac{1}{2}} C_{\alpha-k-1}^{k+\frac{m+1}{2}}(r_2) \end{pmatrix} F(r_2 \epsilon + (r_2^2 - 1)^{\frac{1}{2}} \underline{\eta}) \right]$$

which also gives the explicit form for $G_\epsilon^{(k)}(\xi)$. This ends the proof. \square

5.4 Function Theory on the Klein Model

The aim of this section is to restate the results obtained in the previous sections of this Chapter in terms of the Klein model $(B_m(1), ds_K^2)$ for the hyperbolic unit ball. The main purpose is to find a Taylor (resp. Laurent) expansion for functions $f(\underline{x})$ defined in an open subset Ω_K (resp. in an open annular subdomain) of the unit ball $B_m(1)$, satisfying the equation

$$(\underline{\partial} + \epsilon(\mathbb{E} - \alpha)) f(\underline{x}) = 0 , \quad \text{for all } \underline{x} \in \Omega_K .$$

Such functions belong to the following function space, defined on the analogy of the spaces $\mathcal{H}^\alpha(\mathbb{R}_+\Omega)$ and $\mathcal{H}^\alpha(\Omega)$ from Chapter 2 :

Definition 5.7 *Let $\Omega_K \subset B_m(1)$ be open and let α be an arbitrary complex number. We then put*

$$\mathcal{H}_K^\alpha(\Omega_K) = \{f \in C^1(\Omega_K) : (\partial + \epsilon(\mathbb{E} - \alpha))f(\underline{x}) = 0 \text{ in } \Omega_K\} .$$

Note that for an arbitrary open subset $\Omega \subset H_+$, we have immediately :

$$F(T, \underline{X}) \in \mathcal{H}^\alpha(\mathbb{R}_+\Omega) \implies f(\underline{x}) = F\left(1, \frac{\underline{X}}{T}\right) \in \mathcal{H}_K^\alpha(\Omega_K)$$

with

$$\Omega_K = \left\{ \underline{x} \in B_m(1) : \left(\frac{1}{(1 - |\underline{x}|^2)^{\frac{1}{2}}}, \frac{\underline{x}}{(1 - |\underline{x}|^2)^{\frac{1}{2}}} \right) \in \Omega \right\} .$$

In order to establish a Taylor and Laurent series on $(B_m(1), ds_K^2)$, we need 3 key ingredients :

- Stokes' Theorem
- Cauchy's Theorem
- a decomposition for the fundamental solution

First of all, let us establish Stokes' Theorem. Since the Cauchy-Pompeju Theorem 5.3 is essentially valid on the manifold of rays it can be realized on any surface inside the FC , in particular on the hyperplane $\Pi \leftrightarrow T = 1$. Let us therefore determine the restrictions of both $L(X, dX)$ and $\Sigma(X, dX)$ to this hyperplane (note that these restrictions are labelled by a superscript K , referring to the Klein model) :

$$\begin{aligned} L(X, dX)_K &= \sum_{j=0}^m (-1)^j X_j dX_{\hat{j}} = d\underline{x} \\ \Sigma(X, dX)_K &= \epsilon L(\underline{x}, d\underline{x}) + \sigma(\underline{x}, d\underline{x}) . \end{aligned}$$

Consider then an open subset Ω_K of the unit ball $B_m(1)$ and let $C_K \subset \Omega_K$ be a compact subset with smooth boundary ∂C_K . We already know that the projection on the Klein ball of the Dirac operator on $\mathbb{R}^{1,m}$ acting on α -homogeneous functions $F(T, \underline{X}) = T^\alpha F\left(1, \frac{\underline{X}}{T}\right)$ is given by

$$(\epsilon \partial_T - \partial_{\underline{X}})_K = -D_\alpha(\underline{x}) = -\left(\partial + \epsilon(\mathbb{E} - \alpha) \right) ,$$

where $\underline{\partial}$ (resp. \mathbb{E}) denotes the Dirac (resp. Euler) operator on $\mathbb{R}^{0,m}$, so we have the following version of Stokes' Theorem on the Klein model for the hyperbolic unit ball :

Theorem 5.9 (*Stokes*) Let f and g in $C^1(\Omega_K)$ be arbitrary functions on an open subset Ω_K of the Klein ball and choose $\alpha, \beta \in \mathbb{C}$ such that $\alpha + \beta + m = 0$. We then have :

$$\int_{\partial C_K} f(\sigma(\underline{x}, d\underline{x}) + \epsilon L(\underline{x}, d\underline{x}))g = \int_{C_K} \left[(f D_\alpha(\underline{x}))g + f(D_\beta(\underline{x})g) \right] d\underline{x}$$

Cauchy's Theorem on the Klein model for the hyperbolic unit ball then easily follows from Stokes' Theorem if we choose $f(\underline{x})$ to be the restriction $E_K(\underline{y}, \underline{x})$ of the hyperbolic fundamental solution $E_\alpha(\eta, \xi)$ to Π . This function has a singularity for $\underline{x} = \underline{y}$, with \underline{x} (resp. \underline{y}) the intersection of Π and the ray through ξ (resp. η), and satisfies

$$D_\alpha(\underline{y})E_{\alpha,K}(\underline{y}, \underline{x}) = -\delta(\underline{y} - \underline{x}) = E_{\alpha,K}(\underline{y}, \underline{x})D_\beta(\underline{x}) .$$

This leads immediately to Cauchy's Theorem :

Theorem 5.10 (*Cauchy*) Let $f \in \mathcal{H}_K^\alpha(\Omega_K)$. We then have :

$$-\int_{\partial C_K} E_{\alpha,K}(\underline{y}, \underline{x})(\sigma(\underline{x}, d\underline{x}) + \epsilon L(\underline{x}, d\underline{x}))f(\underline{x}) = \begin{cases} f(\underline{y}) & \text{if } \underline{y} \in \mathring{C}_K \\ 0 & \text{if } \underline{y} \in \Omega_K \setminus C_K \end{cases}$$

Let us then first consider the Taylor expansion of functions $f \in \mathcal{H}_K^\alpha(B_m(r))$ with $r < 1$. This follows from Cauchy's Theorem, once we have established a decomposition for the hyperbolic fundamental solution $E_{\alpha,K}(\underline{y}, \underline{x})$. In view of the construction of the axial decomposition for $E_\alpha(\eta, \xi)$ in the previous section, see Theorem 5.6, a decomposition for $E_{\alpha,K}(\underline{y}, \underline{x})$ is easily obtained by modulating the decomposition of the Cauchy kernel on \mathbb{R}^m from both sides. This is expressed in the following :

Theorem 5.11 The fundamental solution $E_{\alpha,K}(\underline{y}, \underline{x})$ can be decomposed as follows :

$$\begin{aligned} E_{\alpha,K}(\underline{y}, \underline{x}) &= -\frac{1}{A_m} \sum_{k=0}^{\infty} E_{\alpha,K}^{(k)}(\underline{y}, \underline{x}) \\ &= -\frac{1}{A_m} \sum_{k=0}^{\infty} \text{Mod}(\alpha, k; \underline{y}) \frac{|\underline{y}|^k C_k(\underline{\xi}, \underline{\eta})}{|\underline{x}|^{k+m-1}} \overline{\text{Mod}}(\beta, 1-k-m; \underline{x}) , \end{aligned}$$

where we have put $\underline{y} = |\underline{y}|\underline{\eta}$ and $\underline{x} = |\underline{x}|\underline{\xi}$ and with

$$C_k(\underline{\xi}, \underline{\eta}) = C_k^{\frac{m}{2}}(<\underline{\eta}, \underline{\xi}>)\underline{\xi} - C_{k-1}^{\frac{m}{2}}(<\underline{\eta}, \underline{\xi}>)\underline{\eta}.$$

This series expansion is valid for all $\alpha + m \notin -\mathbb{N}$ and converges normally on each $\overline{B}_m(r)$ with $|\underline{y}| \leq r < |\underline{x}|$. We also have that

$$\begin{aligned} E_{\alpha, K}^{(k)}(\underline{y}, \underline{x}) &\in \mathcal{H}_K^\alpha(B_m(1))_y \\ \overline{E}_{\alpha, K}^{(k)}(\underline{y}, \underline{x}) &\in \mathcal{H}_K^\beta(B_m(1) \setminus \{0\})_x, \end{aligned}$$

where we have indicated the variable on which the (projected) Dirac operator acts.

We can now establish the Taylor expansion :

Theorem 5.12 (Taylor) *Let $f(\underline{y}) \in \mathcal{H}_K^\alpha(B_m(R))$, with $R < 1$ and $\alpha \in \mathbb{C}$ with $\alpha + m \notin -\mathbb{N}$. Then there exists a sequence $(f^{(k)}(\underline{y}))_{k \in \mathbb{N}}$ such that the function $\underline{y} \mapsto f^{(k)}(\underline{y})$ belongs to $\mathcal{H}_K^\alpha(B_m(1))$ for each k and such that the following expansion is valid :*

$$f(\underline{y}) = \sum_{k=0}^{\infty} f^{(k)}(\underline{y}),$$

where $f^{(k)}(\underline{y})$ has the following integral representation :

$$f^{(k)}(\underline{y}) = \frac{1}{A_m} \int_{|\underline{x}|=r} E_{\alpha, K}^{(k)}(\underline{y}, \underline{x}) (\sigma(\underline{x}, d\underline{x}) + \epsilon L(\underline{x}, d\underline{x})) f(\underline{x}).$$

The integration goes over the sphere $\partial B_m(r)$, with $r < R$, $L(\underline{x}, d\underline{x}) = ds$ and $\sigma(\underline{x}, d\underline{x}) = nds$, where n denotes the outer unit normal with respect to $\partial B_m(r)$ and ds denotes the Lebesgue measure on $\partial B_m(r)$. The Taylor expansion for $f(\underline{y}) \in \mathcal{H}_K^\alpha(B_m(R))$ converges normally on compact sets $\overline{B}_m(\rho)$, with $|\underline{y}| \leq \rho < r$. An explicit expression for $f^{(k)}(\underline{y})$ is given by :

$$f^{(k)}(\underline{y}) = \text{Mod}(\alpha, k; \underline{y}) \frac{|\underline{y}|^k}{r^k} P(k) \left[\text{Mod}(\beta, 1 - k - m; r\underline{\xi})(r\underline{\xi}\epsilon - 1)f(r\underline{\xi}) \right](\underline{\eta}),$$

where $P(k)[f]$ denotes the projection of a function $f \in L_2(S^{m-1})$ onto the space of inner spherical monogenics.

Proof : Let $\underline{y} \in B_m(R) \Rightarrow |\underline{y}| < R$. Hence, there exists $r < R$ such that $|\underline{y}| \in B_m(r)$ and with $\overline{B}_m(r)$ a compact subset of $B_m(R)$. From Cauchy's Theorem, we then immediately get :

$$f(\underline{y}) = - \int_{|\underline{x}|=r} E_{\alpha,K}(\underline{y}, \underline{x}) (\sigma(\underline{x}, d\underline{x}) + \epsilon L(\underline{x}, d\underline{x})) f(\underline{x}) .$$

As the integration goes over the sphere $\partial B_m(r)$, we have

$$\int_{|\underline{x}|=r} (\sigma(\underline{x}, d\underline{x}) + \epsilon L(\underline{x}, d\underline{x})) = r^{m-1} \int_{S^{m-1}} (\underline{\xi} + r\epsilon) dS(\underline{\xi}) .$$

Recalling the decomposition 5.11, this yields

$$f(\underline{y}) = \sum_{k=0}^{\infty} f^{(k)}(\underline{y})$$

with $f^{(k)}(\underline{y})$ given by

$$\frac{\text{Mod}(\alpha, k; \underline{y})}{A_m} \frac{|\underline{y}|^k}{r^k} \int C_k(\underline{\xi}, \underline{\eta}) \overline{\text{Mod}}(\beta, 1 - k - m; r\underline{\xi}) (\underline{\xi} + r\epsilon) f(r\underline{\xi}) dS(\underline{\xi}) .$$

Using the fact that

$$C_k(\underline{\xi}, \underline{\eta}) = \underline{\eta}^2 C_k(\underline{\xi}, \underline{\eta}) \underline{\xi}^2 = -\underline{\eta} C_k(\underline{\eta}, \underline{\xi}) \underline{\xi} ,$$

the integral over S^{m-1} reduces to

$$-\underline{\eta} \int C_k(\underline{\eta}, \underline{\xi}) \underline{\xi} \overline{\text{Mod}}(\beta, 1 - k - m; r\underline{\xi}) (\underline{\xi} + r\epsilon) f(r\underline{\xi}) dS(\underline{\xi}) ,$$

so that by means of

$$\underline{\xi} \overline{\text{Mod}}(\beta, 1 - k - m; r\underline{\xi}) (\underline{\xi} + r\epsilon) = \text{Mod}(\beta, 1 - k - m; r\underline{\xi}) (r\underline{\xi}\epsilon - 1)$$

we finally arrive at :

$$f^{(k)}(\underline{y}) = \text{Mod}(\alpha, k; \underline{y}) \frac{|\underline{y}|^k}{r^k} P(k) \left[\text{Mod}(\beta, 1 - k - m; r\underline{\xi}) (r\underline{\xi}\epsilon - 1) f(r\underline{\xi}) \right] .$$

The series expansion for $f(\underline{y})$ converges normally on compact balls $\overline{B}_m(\rho)$ with $|\underline{y}| \leq \rho < r$. \square

This means that hyperbolic monogenic functions $f(\underline{y})$ on $B_m(1)$, monogenic

with respect to the operator $(\underline{\partial} + \epsilon(\mathbb{E} - \alpha))_{\underline{y}}$, can be decomposed in a series of modulated inner spherical monogenics on S^{m-1} with respect to the Dirac operator $\underline{\partial}$ on $\mathbb{R}^{0,m}$:

$$f(\underline{y}) = \sum_{k=0}^{\infty} \text{Mod}(\alpha, k; \underline{y}) P_k(\underline{y}) ,$$

with $P_k(\underline{y})$ given by Taylor's Theorem.

Next, we consider the Laurent expansion for functions $f(\underline{y}) \in \mathcal{H}_K^\alpha(\Omega_K)$ with Ω_K the annular domain in $B_m(1)$ defined by

$$\Omega_K = B_m(R_1) \setminus \bar{B}_m(R_2) , \quad R_1 > R_2 .$$

For that purpose we need both the decomposition for $E_{\alpha,K}(\underline{y}, \underline{x})$ of Theorem 5.11 valid for $|\underline{y}| < |\underline{x}|$ and the following decomposition, valid for $|\underline{y}| > |\underline{x}|$:

Theorem 5.13 *The fundamental solution $E_{\alpha,K}(\underline{y}, \underline{x})$ can be decomposed as follows :*

$$\begin{aligned} E_{\alpha,K}(\underline{y}, \underline{x}) &= \frac{1}{A_m} \sum_{k=0}^{\infty} E_{\alpha,K}^{(k)'}(\underline{y}, \underline{x}) \\ &= \frac{1}{A_m} \sum_{k=0}^{\infty} \text{Mod}(\alpha, 1 - k - m; \underline{y}) \frac{|\underline{x}|^k C_k(\underline{\eta}, \underline{\xi})}{|\underline{y}|^{k+m-1}} \overline{\text{Mod}(\beta, k; \underline{x})} , \end{aligned}$$

where we have put $\underline{y} = |\underline{y}|\underline{\eta}$ and $\underline{x} = |\underline{x}|\underline{\xi}$ and with

$$C_k(\underline{\eta}, \underline{\xi}) = C_k^{\frac{m}{2}}(<\underline{\eta}, \underline{\xi}>) \underline{\eta} - C_{k-1}^{\frac{m}{2}}(<\underline{\eta}, \underline{\xi}>) \underline{\xi} .$$

This series expansion is valid for all $\alpha + m \notin -\mathbb{N}$ and converges normally on each $\bar{B}_m(R) \setminus B_m(r)$ with $1 > R \geq |\underline{y}| \geq r > |\underline{x}|$. We also have that

$$\begin{aligned} E_{\alpha,K}^{(k)'}(\underline{y}, \underline{x}) &\in \mathcal{H}_K^\alpha(B_m(1) \setminus \{0\})_{\underline{y}} \\ \overline{E_{\alpha,K}^{(k)'}}(\underline{y}, \underline{x}) &\in \mathcal{H}_K^\beta(B_m(1))_{\underline{x}} , \end{aligned}$$

where we have indicated the variable on which the (projected) Dirac operator acts.

Remark : Note that we have labelled the building blocks $E_{\alpha,K}^{(k)'}(\underline{y}, \underline{x})$ of the decomposition for $|\underline{y}| > |\underline{x}|$ with a prime, to distinguish it from the building blocks of the decomposition for $|\underline{y}| < |\underline{x}|$.

We then have the following Laurent expansion :

Theorem 5.14 (Laurent) Let $f(\underline{y}) \in \mathcal{H}^\alpha(\Omega)$, with $\Omega \subset B_m(1)$ the open annular domain $B_m(R_1) \setminus \bar{B}_m(R_2)$, where $0 < R_2 < R_1 < 1$. Then there exists a sequence $(f^{(k)}(\underline{y}))_{k \in \mathbb{N}}$ with $\underline{y} \mapsto f^{(k)}(\underline{y})$ belonging to $\mathcal{H}_K^\alpha(B_m(1))$ and a sequence $(g^{(k)}(\underline{y}))_{k \in \mathbb{N}}$ with $\underline{y} \mapsto g^{(k)}(\underline{y})$ belonging to $\mathcal{H}_K^\alpha(B_m(1) \setminus \{0\})$, such that the following expansion is valid :

$$f(\underline{y}) = \sum_{k=0}^{\infty} f^{(k)}(\underline{y}) + g^{(k)}(\underline{y}) ,$$

where $f^{(k)}(\underline{y})$ is given by the integral

$$f^{(k)}(\underline{y}) = \text{Mod}(\alpha, k; \underline{y}) \frac{|\underline{y}|^k}{r_1^k} P(k) \left[\text{Mod}(\beta, 1 - k - m; r_1 \underline{\xi}) (r_1 \underline{\xi} \epsilon - 1) f(r_1 \underline{\xi}) \right] (\underline{\eta}) ,$$

with $|\underline{y}| < r_1 < R_1$ and $g^{(k)}(\underline{y})$ by the integral

$$g^{(k)}(\underline{y}) = \text{Mod}(\alpha, 1 - k - m; \underline{y}) \frac{r_2^{k+m-1}}{|\underline{y}|^{k+m-1}} Q(k) \left[\text{Mod}(\beta, k; r_2 \underline{\xi}) (1 - r_2 \underline{\xi} \epsilon) f(r_2 \underline{\xi}) \right] (\underline{\eta}) ,$$

with $|\underline{y}| > r_2 > R_2$.

Proof : Suppose $f(\underline{y}) \in \mathcal{H}^\alpha(B_m(R_1) \setminus \bar{B}_m(R_2))$ with $0 < R_2 < R_1 < 1$ and take \underline{y} belonging to this open annular domain, so that we have $R_2 < |\underline{y}| < R_1$. Hence there exist r_1, r_2 such that $R_2 < r_2 < |\underline{y}| < r_1 < R_1$, whence \underline{y} belongs to the compact subset $C = \bar{B}_m(r_1) \setminus B_m(r_2)$ in $B_m(1)$. From Cauchy's Theorem we then immediately get :

$$\begin{aligned} f(\underline{y}) &= - \int_{\partial C} E_{\alpha, K}(\underline{y}, \underline{x}) (\sigma(\underline{x}, d\underline{x}) + \epsilon L(\underline{x}, d\underline{x})) f(\underline{x}) \\ &= - \int_{|\underline{x}|=r_1} E_{\alpha, K}(\underline{y}, \underline{x}) (\sigma(\underline{x}, d\underline{x}) + \epsilon L(\underline{x}, d\underline{x})) f(\underline{x}) \\ &\quad + \int_{|\underline{x}|=r_2} E_{\alpha, K}(\underline{y}, \underline{x}) (\sigma(\underline{x}, d\underline{x}) + \epsilon L(\underline{x}, d\underline{x})) f(\underline{x}) . \end{aligned}$$

For the calculation of the first integral we refer to the proof of the Taylor series. We have :

$$- \int_{|\underline{x}|=r_1} E_{\alpha, K}(\underline{y}, \underline{x}) (\sigma(\underline{x}, d\underline{x}) + \epsilon L(\underline{x}, d\underline{x})) f(\underline{x}) = \sum_{k=0}^{\infty} f^{(k)}(\underline{y}) ,$$

with

$$f^{(k)}(\underline{y}) = \text{Mod}(\alpha, k; \underline{y}) \frac{|\underline{y}|^k}{r_1^k} P(k) \left[\text{Mod}(\beta, 1 - k - m; r_1 \underline{\xi}) (r_1 \underline{\xi} \epsilon - 1) f(r_1 \underline{\xi}) \right] .$$

On the other hand, we have the following :

$$\begin{aligned} & \int_{|\underline{x}|=r_2} E_{\alpha,K}(\underline{y}, \underline{x}) (\sigma(\underline{x}, d\underline{x}) + \epsilon L(\underline{x}, d\underline{x})) f(\underline{x}) \\ &= r_2^{m-1} \sum_{k=0}^{\infty} \int_{S^{m-1}} E_{\alpha,K}^{(k)'}(\underline{y}, \underline{x}) (\underline{\xi} + r_2 \epsilon) f(r_2 \underline{\xi}) dS(\underline{\xi}) , \end{aligned}$$

with $E_{\alpha,K}^{(k)'}(\underline{y}, \underline{x})$ given by Theorem 5.13. Putting

$$\int_{|\underline{x}|=r_2} E_{\alpha,K}(\underline{y}, \underline{x}) (\sigma(\underline{x}, d\underline{x}) + \epsilon L(\underline{x}, d\underline{x})) f(\underline{x}) = \sum_{k=0}^{\infty} g^{(k)}(\underline{y}) ,$$

we then get :

$$\begin{aligned} g^{(k)}(\underline{y}) &= \frac{1}{A_m} \text{Mod}(\alpha, 1 - k - m; \underline{y}) \frac{r_2^{k+m-1}}{|\underline{y}|^{k+m-1}} \\ &\quad \int_{S^{m-1}} C_k(\underline{\eta}, \underline{\xi}) \overline{\text{Mod}}(\beta, k; r_2 \underline{\xi}) (\underline{\xi} + r_2 \epsilon) f(r_2 \underline{\xi}) dS(\underline{\xi}) . \end{aligned}$$

As

$$\overline{\text{Mod}}(\beta, k; r_2 \underline{\xi}) (\underline{\xi} + r_2 \epsilon) = \underline{\xi} \text{Mod}(\beta, k; r_2 \underline{\xi}) (1 - r_2 \underline{\xi} \epsilon) ,$$

and recalling the formula for projection onto the space of outer spherical monogenics on \mathbb{R}^m , we finally arrive at

$$g^{(k)}(\underline{y}) = \text{Mod}(\alpha, 1 - k - m; \underline{y}) \frac{r_2^{k+m-1}}{|\underline{y}|^{k+m-1}} Q(k) \left[\text{Mod}(\beta, k; r_2 \underline{\xi}) (1 - r_2 \underline{\xi} \epsilon) f(r_2 \underline{\xi}) \right] .$$

This yields the Laurent series

$$f(\underline{y}) = \sum_{k=0}^{\infty} f^{(k)}(\underline{y}) + g^{(k)}(\underline{y})$$

for $f \in \mathcal{H}_K^\alpha(B_m(R_1) \setminus \bar{B}_m(R_2))$, converging normally on compact annular subdomains $\bar{B}_m(r_1) \setminus B_m(r_2)$ with $r_1 \geq |\underline{y}| \geq r_2$. \square

Remark : The Laurent Theorem on the Klein model for the hyperbolic unit ball again illustrates that the function theory for hyperbolic monogenics on the Klein ball can be described as a "modulation of the classical function theory for the operator $\underline{\partial}$ on $\mathbb{R}^{0,m}$ ".

5.5 Eigenfunctions for the Operator $\xi(\Gamma + \alpha)$

In this section eigenfunctions for the Dirac operator on the hyperbolic unit ball are constructed. It is important to make a clear distinction between the eigenvalue problem related to the hyperbolic Dirac operator, i.e. the Dirac operator ∂_X on $\mathbb{R}^{1,m}$ acting on homogeneous functions, and the eigenvalue problem related to the hyperbolic Gamma operator Γ . Eigenfunctions for the latter problem give rise to nullsolutions for the Dirac operator on the hyperbolic unit ball :

$$\Gamma F(\xi) = \alpha F(\xi) \implies \xi(\Gamma - \alpha)F(\xi) = 0 .$$

Eigenfunctions related to the first problem can by no means be related to a *single* eigenvalue problem for the Gamma operator. However, in this section it will be shown how they are related to *systems* of eigenvalue problems for the Gamma operator.

The eigenvalue problem for the Dirac operator on the hyperbolic unit ball arises from the following question : is it possible to generalize the idea of an exponential function to the hyperbolic unit ball? In classical Clifford analysis, i.e. Clifford analysis on the flat Euclidean space \mathbb{R}^m , a generalized exponential function was constructed as an eigenfunction for the operator $\underline{\partial}$ on $\mathbb{R}^{0,m}$. This inspires us to look for a hyperbolic version of this exponential function as an eigenfunction of the Dirac operator on $\mathbb{R}^{1,m}$. However, in view of the projective nature of our model for the hyperbolic ball, this hyperbolic version of the exponential function must be defined on the manifold of rays $\text{Ray}(FC)$. In other words, the hyperbolic version of the exponential function must be *homogeneous*. In this sense we cannot find eigenfunctions for the Dirac operator on the hyperbolic unit ball in the strict sense of the word, for the operator itself is homogeneous of degree (-1) . However, for an arbitrary function $\psi(T, \underline{X})$ which is homogeneous of degree $(+1)$ one may consider the following inhomogeneous equation :

$$(\epsilon \partial_T - \partial_{\underline{X}})F_\lambda(T, \underline{X}) = \lambda \frac{F_\lambda(T, \underline{X})}{\psi(T, \underline{X})} .$$

It seems obvious to choose $\psi(T, \underline{X})$ in such a way that it corresponds to a $\text{SO}(1, m)$ -invariant object, because the Dirac operator itself is invariant under $\text{Spin}(1, m)$ -transformations. Choosing

$$\psi(T, \underline{X}) = Q_{1,m}(T, \underline{X})^{\frac{1}{2}} = (T^2 - |\underline{X}|^2)^{\frac{1}{2}}$$

to be the hyperbolic norm, we thus arrive at the following equation :

$$\xi(\Gamma + \alpha)F_\lambda(\xi) = \lambda F_\lambda(\xi) , \quad \xi \in H_+ , \quad (5.5)$$

which is the eigenvalue problem related to the Dirac operator on the hyperbolic unit ball.

In order to solve equation (5.5) we will first rewrite this equation as a set of equations, by means of the zero divisors $(1 \pm \xi)$. For that purpose we put

$$F_\lambda = \frac{1+\xi}{2}F_\lambda^+ + \frac{1-\xi}{2}F_\lambda^- .$$

Letting the hyperbolic Dirac operator act on F_λ and using the fact that

$$\xi\Gamma\xi + \Gamma = m ,$$

we get respectively :

$$\frac{1}{2}\xi(\Gamma + \alpha)(1 + \xi)F_\lambda^+ = -\frac{1-\xi}{2}\Gamma F_\lambda^+ + \frac{m}{2}F_\lambda^+ + (\alpha - \lambda)\frac{1+\xi}{2}F_\lambda^+$$

and

$$\frac{1}{2}\xi(\Gamma + \alpha)(1 - \xi)F_\lambda^- = \frac{1+\xi}{2}\Gamma F_\lambda^- - \frac{m}{2}F_\lambda^- - (\alpha + \lambda)\frac{1-\xi}{2}F_\lambda^- .$$

The eigenvalue problem for the Dirac operator on the hyperbolic unit ball is thus equivalent with the following set of equations :

$$\begin{cases} \Gamma F_\lambda^- + \left(\frac{m}{2} + \alpha - \lambda\right) F_\lambda^+ - \frac{m}{2}F_\lambda^- = 0 \\ \Gamma F_\lambda^+ + \left(\frac{m}{2} + \alpha + \lambda\right) F_\lambda^- - \frac{m}{2}F_\lambda^+ = 0 \end{cases}$$

This can also be arranged in a matrix formalism :

$$\Gamma \begin{pmatrix} F_\lambda^+ \\ F_\lambda^- \end{pmatrix} = \begin{pmatrix} \frac{m}{2} & \beta + \frac{m}{2} - \lambda \\ \beta + \frac{m}{2} + \lambda & \frac{m}{2} \end{pmatrix} \begin{pmatrix} F_\lambda^+ \\ F_\lambda^- \end{pmatrix}$$

where we have put $\beta = -\alpha - m$. In other words, the eigenvalue problem for the hyperbolic Dirac operator is equivalent with a two-dimensional system of equations for the hyperbolic Gamma operator Γ . Let us denote the (2×2) matrix in previous expression as M . In order to solve the system of equations, M has to be reduced to its Jordan canonical form N :

$$Q^{-1}MQ = N .$$

- If M has two eigenvalues $\mu_1 \neq \mu_2$, N is the diagonal matrix

$$N = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}.$$

Putting

$$Q^{-1} \begin{pmatrix} F_\lambda^+ \\ F_\lambda^- \end{pmatrix} = \begin{pmatrix} G_\lambda^+ \\ G_\lambda^- \end{pmatrix},$$

the system for $(G_\lambda^+, G_\lambda^-)$ decouples and reduces to

$$\begin{cases} \Gamma G_\lambda^+ &= \mu_1 G_\lambda^+ \\ \Gamma G_\lambda^- &= \mu_2 G_\lambda^- \end{cases}$$

This means that G_λ^+ (resp. G_λ^-) is the restriction to H_+ of a solution for the Dirac operator ∂_X on $\mathbb{R}^{1,m}$ which is homogeneous of degree $(-\mu_1)$ (resp. homogeneous of degree $(-\mu_2)$). Once we have choosen $(G_\lambda^+, G_\lambda^-)$, the solutions $(F_\lambda^+, F_\lambda^-)$ are found to be :

$$\begin{pmatrix} F_\lambda^+ \\ F_\lambda^- \end{pmatrix} = Q \begin{pmatrix} G_\lambda^+ \\ G_\lambda^- \end{pmatrix}$$

- If M has only one eigenvalue μ , the Jordan normal form N reduces to

$$N = \begin{pmatrix} \mu & 1 \\ 0 & \mu \end{pmatrix}.$$

The equations for $(G_\lambda^+, G_\lambda^-)$ are then given by

$$\begin{cases} \Gamma G_\lambda^+ &= \mu G_\lambda^+ + G_\lambda^- \\ \Gamma G_\lambda^- &= \mu G_\lambda^- \end{cases}$$

which means that the function G_λ^- is easily found as the restriction to the hyperbolic unit ball H_+ of an arbitrary $(-\mu)$ -homogeneous solution for the Dirac operator on $\mathbb{R}^{1,m}$. However, in order to find G_λ^+ one must solve an inhomogeneous equation for the hyperbolic Gamma operator.

To do so, one may follow an approach which is very similar to the one followed by P. Van Lancker for the case of the sphere S^{m-1} in \mathbb{R}^m (see reference [75]). This approach is influenced by the work of S. Bell

and D. Calderbank, see [4] and [10], in which the Dirac operator on C^∞ -*submanifolds-with-boundary* of a given manifold is studied.

Another way to solve the latter system of equations makes use of Riesz distributions. Suppose $F \in \mathcal{D}'_+(\mathbb{R}^{1,m})$, such that we immediately have that both G_+^λ and G_-^λ belong to $\mathcal{D}'_+(\mathbb{R}^{1,m})$. As

$$(\Gamma - \lambda)G_+^\lambda = G_-^\lambda \implies \partial_X(\rho^{-\lambda}G_+^\lambda) = \rho^{-\lambda-1}\xi G_-^\lambda ,$$

it suffices to solve the scalar equation

$$\square_m \Phi_+^\lambda = \rho^{-\lambda-1}\xi G_-^\lambda ,$$

hereby using Riesz distributions, and to put

$$G_+^\lambda = \partial_X \Phi_+^\lambda = \partial_X(Z_2 * \rho^{-\lambda-1}\xi G_-^\lambda) .$$

Chapter 6

Boundary Value Theory

*If in the infinite you want to stride,
just walk in the finite to every side.*
(J.W. von Goethe)

In this Chapter the fundamental solution for the hyperbolic Dirac equation is considered in the limit as its singularities approach the nullcone. This gives rise to the so-called *photogenic Cauchy kernel*, referring to both its monogeneity with respect to the Dirac operator ∂_X on $\mathbb{R}^{1,m}$ and its relation to the nullcone, which is the surface on which the worldlines of photons are lying. This kernel will be used to define an integral transform on the unit ball $B_m(1)$, the *photogenic Cauchy transform*, and the boundary values of this transform will be calculated as its argument approaches the unit sphere. By considering the extension of these boundary values to the Lie sphere, it is found that under certain restrictions on the parameter α the function space containing boundary values of hyperbolic monogenic functions on the Klein ball has a reproducing kernel.

6.1 The Photogenic Cauchy Kernel

In the previous Chapter a function theory on the hyperbolic unit ball was developped. The starting point was the hyperbolic Dirac equation (2.2) for the fundamental solution $E_\alpha(T, \underline{X})$, being an α -homogeneous solution for the Dirac operator ∂_X with singularities on the temporal ray through ϵ . These singularities were boosted to an arbitrary ray inside the future cone, leading to equation (5.1). It is a well-known fact that this ray can never be boosted *on* the nullcone. Relativistically speaking, this would require an infinite amount of energy. Nevertheless, we can still try to solve the following equation :

$$(\epsilon\partial_T - \partial_{\underline{X}})\mathcal{F}_{\alpha,\underline{\omega}}(T, \underline{X}) = T_+^{\alpha+m-1}\delta(T\underline{\omega} - \underline{X}) , \quad \underline{\omega} \in S^{m-1} . \quad (6.1)$$

The right-hand side of this equation contains a delta distribution becoming singular on the upper part of the nullcone, and a factor $T_+^{\alpha+m-1}$ reassuring our photogenic Cauchy kernel, i.e. the solution to the equation above, to be α -homogeneous. The nullray containing the singularities will be denoted by

$$R_{\underline{\omega}} = \{(T, \underline{X}) \in \mathbb{R}^{1,m} : \underline{X} = T\underline{\omega}\}, \quad \underline{\omega} \in S^{m-1}.$$

In order to solve equation (6.1), from now on referred to as the *photogenic Dirac equation*, we first consider the following related scalar problem :

$$\square_m \Phi_{\alpha, \underline{\omega}}(T, \underline{X}) = T_+^{\alpha+m-1} \delta(T\underline{\omega} - \underline{X}).$$

Since the distribution at the right-hand side does not exist for $\alpha \in -m - \mathbb{N}$, we expect the distribution $\Phi_{\alpha, \underline{\omega}}(T, \underline{X})$ to have poles at the same values. For all other values $\alpha \in \mathbb{C}$, $T_+^{\alpha+m-1} \delta(T\underline{\omega} - \underline{X})$ belongs to the set $\mathcal{D}'_+(\mathbb{R}^{1,m})$ whence $\Phi_{\alpha, \underline{\omega}}(T, \underline{X})$ is given by

$$\Phi_{\alpha, \underline{\omega}}(T, \underline{X}) = Z_2 * T_+^{\alpha+m-1} \delta(T\underline{\omega} - \underline{X}).$$

By definition, we thus get :

$$\begin{aligned} \Phi_{\alpha, \underline{\omega}}(T, \underline{X}) &= \frac{1}{2\pi^{\frac{m-1}{2}} \Gamma\left(\frac{3-m}{2}\right)} \int_0^\infty S^{\alpha+m-1} \rho(T-S, \underline{X}-S\underline{\omega})^{1-m} dS \\ &= \frac{1}{2\pi^{\frac{m-1}{2}} \Gamma\left(\frac{3-m}{2}\right)} \int_0^{S_0} \frac{S^{\alpha+m-1}}{(T^2 - |\underline{X}|^2 - 2S(T - \langle \underline{X}, \underline{\omega} \rangle))^{\frac{m-1}{2}}} dS \end{aligned}$$

where we have put

$$S_0 = \frac{1}{2} \frac{T^2 - |\underline{X}|^2}{T - \langle \underline{X}, \underline{\omega} \rangle}.$$

The integral with respect to S can be reduced to a Beta integral :

$$\begin{aligned} &\int_0^{S_0} \frac{S^{\alpha+m-1}}{(T^2 - |\underline{X}|^2 - 2S(T - \langle \underline{X}, \underline{\omega} \rangle))^{\frac{m-1}{2}}} dS \\ &= (T^2 - |\underline{X}|^2)^{\frac{1-m}{2}} \left(\frac{1}{2} \frac{T^2 - |\underline{X}|^2}{T - \langle \underline{X}, \underline{\omega} \rangle} \right)^{\alpha+m} \int_0^1 t^{\alpha+m-1} (1-t)^{\frac{1-m}{2}} dt. \end{aligned}$$

In case of an even spatial dimension m , for $\alpha \in \mathbb{C}$ such that $\alpha + m \notin -\mathbb{N}$, the integral at the right-hand side is the integral given by expression (29) :

$$\int_0^1 t^{\alpha+m-1} (1-t)^{\frac{1-m}{2}} dt = \frac{\Gamma(\alpha+m) \Gamma\left(\frac{3-m}{2}\right)}{\Gamma\left(\alpha + \frac{3+m}{2}\right)},$$

such that the distribution $\Phi_{\alpha,\underline{\omega}}(T, \underline{X})$ is found to be

$$\Phi_{\alpha,\underline{\omega}}(T, \underline{X}) = \frac{\Gamma(\alpha + m)}{2^{1+\alpha+m} \pi^{\frac{m-1}{2}} \Gamma\left(\alpha + \frac{m+3}{2}\right)} \frac{(T^2 - |\underline{X}|^2)^{\alpha + \frac{m+1}{2}}}{(T - \langle \underline{X}, \underline{\omega} \rangle)^{\alpha+m}} . \quad (6.2)$$

In case of an odd spatial dimension m , the Beta integral does not converge in the classical sense. However, the distribution $\Phi_{\alpha,\underline{\omega}}(T, \underline{X})$ as given above is defined for both even *and* odd m as long as $\alpha \notin -m - \mathbb{N}$, these values being excluded because the Gamma function has poles there. Note that the Gamma function in the denominator does *not* remove these poles in case of an odd dimension! We demonstrate this by considering the residue for $\alpha = -m$. In that case, we have :

$$\text{Res}\{T_+^{\alpha+m-1} \delta(T\underline{\omega} - \underline{X}) , \alpha = -m\} = \delta(T) \delta(T\underline{\omega} - \underline{X}) = \delta(T) \delta(\underline{X}) .$$

This latter equation can easily be verified by letting both distributions act on a test function $\varphi(T, \underline{X}) \in \mathcal{D}(\mathbb{R}^{1,m})$. This means that

$$\text{Res}\{Z_2 * T_+^{\alpha+m-1} \delta(T\underline{\omega} - \underline{X}) , \alpha = -m\} = Z_2 .$$

On the other hand, we also have by definition :

$$\begin{aligned} \text{Res}\{\Phi_{\alpha,\underline{\omega}}(T, \underline{X}) , \alpha = -m\} &= \lim_{\alpha \rightarrow -m} (\alpha + m) \Phi_{\alpha,\underline{\omega}}(T, \underline{X}) \\ &= Z_2 . \end{aligned}$$

Thus, although the simplification

$$\frac{\Gamma(\alpha + m)}{\Gamma\left(\alpha + \frac{m+3}{2}\right)} = (\alpha + m - 1) \cdots \left(\alpha + \frac{m+3}{2}\right)$$

seems to remove the pole at $\alpha = -m$, it only makes it less obvious to see that there actually *is* a pole at this value. Because we now have two distributions which are equal in a strip of the complex plane and which have poles at the same values for α , they are equal in the whole complex plane minus the poles by analytic continuation.

Using the fact that $\square_m = (\epsilon \partial_T - \partial_{\underline{X}})^2$, we thus find the following explicit formula for the photogenic Cauchy kernel :

$$\mathcal{F}_{\alpha,\underline{\omega}}(T, \underline{X}) = (\epsilon \partial_T - \partial_{\underline{X}}) \left[\frac{\Gamma(\alpha + m)}{2^{1+\alpha+m} \pi^{\frac{m-1}{2}} \Gamma\left(\alpha + \frac{m+3}{2}\right)} \frac{(T^2 - |\underline{X}|^2)^{\alpha + \frac{m+1}{2}}}{(T - \langle \underline{X}, \underline{\omega} \rangle)^{\alpha+m}} \right] .$$

The constant in previous expression will be denoted as $c(\alpha, m)$ throughout this Chapter :

$$c(\alpha, m) = \frac{\Gamma(\alpha + m)}{2^{1+\alpha+m} \pi^{\frac{m-1}{2}} \Gamma\left(\alpha + \frac{m+3}{2}\right)} .$$

We are then lead to the following definition :

Definition 6.1 *Let $\underline{\omega} \in S^{m-1}$ be an arbitrary unit vector in \mathbb{R}^m and let α be an arbitrary complex number such that $\alpha + m \notin -\mathbb{N}$. The photogenic Cauchy kernel is defined as the distribution*

$$\begin{aligned} \mathcal{F}_{\alpha, \underline{\omega}}(T, \underline{X}) &= (2\alpha + m + 1)c(\alpha, m)(\epsilon T + \underline{X}) \frac{(T^2 - |\underline{X}|^2)^{\alpha + \frac{m-1}{2}}}{(T - \langle \underline{X}, \underline{\omega} \rangle)^{\alpha+m}} \\ &- (\alpha + m)c(\alpha, m)(\epsilon + \underline{\omega}) \frac{(T^2 - |\underline{X}|^2)^{\alpha + \frac{m+1}{2}}}{(T - \langle \underline{X}, \underline{\omega} \rangle)^{\alpha+m+1}} , \end{aligned}$$

and satisfies the photogenic Dirac equation

$$(\epsilon \partial_T - \partial_{\underline{X}}) \mathcal{F}_{\alpha, \underline{\omega}}(T, \underline{X}) = T_+^{\alpha+m-1} \delta(T \underline{\omega} - \underline{X}) .$$

Remark : In Chapter 3 we already mentioned the fact that there exist α -homogeneous distributional solutions for the Dirac operator $D(\underline{T}, \underline{X})_{p,q}$ on $\mathbb{R}^{p,q}$ which are defined in a neighbourhood of a nullray. These distributions were explicitly excluded when we proved the Ultra-Modulation Theorem. The photogenic Cauchy kernel $\mathcal{F}_{\alpha, \underline{\omega}}(T, \underline{X})$ in fact offers a nice example of such a distribution, in case of the space-time Dirac operator on $\mathbb{R}^{1,m}$!

6.2 The Photogenic Cauchy Transform (PCT)

Now that we have found the photogenic Cauchy kernel $\mathcal{F}_{\alpha, \underline{\omega}}(T, \underline{X})$, we can define a photogenic Cauchy transform. To do so, we project the photogenic Dirac equation onto the Klein model of the hyperbolic unit ball. Putting

$$\mathcal{F}_{\alpha, \underline{\omega}}(T, \underline{X}) = \lambda^\alpha \mathcal{F}_\alpha(\underline{x}, \underline{\omega}) ,$$

where $\lambda = T$ and $\underline{x} = \frac{\underline{X}}{T} = r \underline{\xi} \in B_m(1)$, we get immediately :

$$(\partial + \epsilon(\mathbb{E}_r - \alpha)) \mathcal{F}_\alpha(\underline{x}, \underline{\omega}) = -\delta(\underline{x} - \underline{\omega}) ,$$

with $\underline{\partial}$ and \mathbb{E}_r respectively the Dirac operator and the Euler operator on $\mathbb{R}^{0,m}$. The projected photogenic fundamental solution $\mathcal{F}_\alpha(\underline{x}, \underline{\omega})$ is hereby defined as the distribution

$$\begin{aligned} \mathcal{F}_\alpha(\underline{x}, \underline{\omega}) &= (2\alpha + m + 1)c(\alpha, m)(\epsilon + \underline{x}) \frac{(1 - r^2)^{\alpha + \frac{m-1}{2}}}{(1 - \langle \underline{x}, \underline{\omega} \rangle)^{\alpha+m}} \\ &- (\alpha + m)c(\alpha, m)(\epsilon + \underline{\omega}) \frac{(1 - r^2)^{\alpha + \frac{m+1}{2}}}{(1 - \langle \underline{x}, \underline{\omega} \rangle)^{\alpha+m+1}} . \end{aligned}$$

We then consider the sphere S^{m-1} as a subset of \mathbb{R}^m , and we define the photogenic Cauchy transform of a function $f(\underline{\omega})$ on S^{m-1} by means of the photogenic Cauchy kernel, by analogy with the classical Cauchy transform. This is expressed in the following :

Definition 6.2 *Let $f(\underline{\omega})$ be an arbitrary function defined on the sphere S^{m-1} . The photogenic Cauchy transform $\mathcal{C}_P^\alpha[f](\underline{x})$ of f is for all $\underline{x} \in B_m(1)$ defined by the following integral :*

$$\mathcal{C}_P^\alpha[f](\underline{x}) = \frac{1}{A_m} \int_{S^{m-1}} \mathcal{F}_\alpha(\underline{x}, \underline{\omega}) \underline{\omega} f(\underline{\omega}) d\omega$$

Remark : The additional factor $\underline{\omega}$ in the definition for $\mathcal{C}_P^\alpha[f](\underline{x})$ plays the role of unit normal vector on S^{m-1} , on the analogy of the definition for the Cauchy transform on graphs of Lipschitz functions (see reference [19]).

Note that the photogenic Cauchy transform of a function $f(\underline{\omega})$ on S^{m-1} yields a new function $\mathcal{C}_P^\alpha[f](\underline{x})$, defined for all $\underline{x} \in B_m(1)$, which is a solution for the projected hyperbolic Dirac operator $(\underline{\partial} + \epsilon(\mathbb{E}_r - \alpha))$. This means that the function $\mathcal{C}_P^\alpha[f](T, \underline{X})$, defined by

$$\mathcal{C}_P^\alpha[f](T, \underline{X}) = T^\alpha \mathcal{C}_P^\alpha[f] \left(\frac{\underline{X}}{T} \right)$$

belongs to the function space $\mathcal{H}^\alpha(\mathbb{R}_+ H_+)$.

In view of the fact that square integrable functions $f \in L_2(S^{m-1})$ can be decomposed in terms of inner and outer spherical monogenics, by means of the decomposition

$$f(\underline{\xi}) = \sum_{k=0}^{\infty} P(k)[f](\underline{\xi}) + Q(k)[f](\underline{\xi}) ,$$

we will now explicitly determine the photogenic Cauchy transform of inner and outer spherical monogenics on \mathbb{R}^m .

We will use two Lemmata : the first one is a refinement of the classical Hecke-Funk Theorem (see Theorem 0.2), the latter yields an explicit formula for an integral that will often occur in what follows :

$$\mathcal{P}_{k,m}(\lambda; r) = \int_{-1}^1 \frac{P_{k,m}(t)(1-t^2)^{\frac{m-3}{2}}}{(1-rt)^\lambda} dt .$$

Lemma 6.1 *Let $P_k(\underline{\omega}) \in M^+(k)$ be an inner spherical monogenic of degree k and let $P_{k,m}(t)$ be the Legendre polynomial of degree k in m dimensions. If we put $\underline{x} = r\underline{\xi}$, we have the following identities :*

$$\int_{S^{m-1}} f(\langle \underline{x}, \underline{\omega} \rangle) P_k(\underline{\omega}) d\underline{\omega} = A_m \left(\int_{-1}^1 f(rt) P_{k,m}(t) (1-t^2)^{\frac{m-3}{2}} dt \right) P_k(\underline{\xi})$$

$$\int_{S^{m-1}} f(\langle \underline{x}, \underline{\omega} \rangle) \underline{\omega} P_k(\underline{\omega}) d\underline{\omega} = A_m \left(\int_{-1}^1 f(rt) P_{1+k,m}(t) (1-t^2)^{\frac{m-3}{2}} dt \right) \underline{\xi} P_k(\underline{\xi})$$

For a proof of this Lemma we refer to [63].

Lemma 6.2 *Let $P_{k,m}(t)$ be the Legendre polynomial of degree k in m dimensions, let $F(a, b; c; t)$ be the hypergeometric series and let $r < 1$. We then have the following identity*

$$\mathcal{P}_{k,m}(\lambda; r) = \frac{\pi^{\frac{1}{2}}(\lambda)_k \Gamma\left(\frac{m-1}{2}\right)}{2^k \left(\frac{m}{2}\right)_k \Gamma\left(\frac{m}{2}\right)} r^k F\left(\frac{k+\lambda}{2}, \frac{1+k+\lambda}{2}; k + \frac{m}{2}; r^2\right) .$$

Proof : This Lemma will be proven by induction on the parameter k . For $k = 0$ and $r < 1$ we get :

$$\begin{aligned} \int_{-1}^1 \frac{(1-t^2)^{\frac{m-3}{2}}}{(1-rt)^\lambda} dt &= \sum_{l=0}^{\infty} \binom{-\lambda}{l} (-r)^l \int_{-1}^1 t^l (1-t^2)^{\frac{m-3}{2}} dt \\ &= \sum_{l=0}^{\infty} \binom{-\lambda}{2l} (r)^{2l} \int_0^1 (t^2)^{l-\frac{1}{2}} (1-t^2)^{\frac{m-3}{2}} dt^2 , \end{aligned}$$

which by means of the definition for the Beta integral reduces to

$$\int_{-1}^1 \frac{(1-t^2)^{\frac{m-3}{2}}}{(1-rt)^\lambda} dt = \sum_{l=0}^{\infty} \binom{-\lambda}{2l} B\left(l + \frac{1}{2}, \frac{m-1}{2}\right) r^{2l} .$$

Writing the Beta function in terms of the Gamma function, we arrive at

$$\int_{-1}^1 \frac{(1-t^2)^{\frac{m-3}{2}}}{(1-rt)^\lambda} dt = \sqrt{\pi} \frac{\Gamma(\frac{m-1}{2})}{\Gamma(\frac{m}{2})} F\left(\frac{\lambda}{2}, \frac{1+\lambda}{2}; \frac{m}{2}; r^2\right).$$

For $k = 1$ we get, hereby using that $P_{1,m}(t) = t$:

$$\int_{-1}^1 \frac{t(1-t^2)^{\frac{m-3}{2}}}{(1-rt)^\lambda} dt = \frac{\partial_r}{\lambda-1} \int_{-1}^1 \frac{(1-t^2)^{\frac{m-3}{2}}}{(1-rt)^{\lambda-1}} dt,$$

which by means of the derivation property of the hypergeometric function reduces to

$$\int_{-1}^1 \frac{t(1-t^2)^{\frac{m-3}{2}}}{(1-rt)^\lambda} dt = \sqrt{\pi} \frac{\lambda \Gamma(\frac{m-1}{2})}{m \Gamma(\frac{m}{2})} r F\left(\frac{1+\lambda}{2}, 1+\frac{\lambda}{2}; 1+\frac{m}{2}; r^2\right).$$

The rest of the proof uses the recurrence relation (18) for the Legendre polynomials in higher dimensions. We then get for arbitrary $k > 1$:

$$\begin{aligned} \int_{-1}^1 \frac{P_{1+k,m}(t)(1-t^2)^{\frac{m-3}{2}}}{(1-rt)^\lambda} dt &= \frac{2k+m-2}{k+m-2} \frac{\partial_r}{\lambda-1} \int_{-1}^1 \frac{P_{k,m}(t)(1-t^2)^{\frac{m-3}{2}}}{(1-rt)^{\lambda-1}} dt \\ &\quad - \frac{k}{k+m-2} \int_{-1}^1 \frac{P_{k-1,m}(t)(1-t^2)^{\frac{m-3}{2}}}{(1-rt)^{\lambda-1}} dt \end{aligned}$$

Using the induction hypothesis and the elementary properties of the hypergeometric series, this can be simplified to

$$\frac{\pi^{\frac{1}{2}}(\lambda)_{1+k} \Gamma(\frac{m-1}{2})}{2^{1+k} (\frac{m}{2})_{1+k} \Gamma(\frac{m}{2})} r^{1+k} F\left(\frac{1+k+\lambda}{2}, 1+\frac{k+\lambda}{2}; 1+k+\frac{m}{2}; r^2\right).$$

This proves the Lemma. □

6.2.1 The PCT of Inner Spherical Monogenics

Let us consider an arbitrary inner spherical monogenic $P_k(\underline{\omega}) \in M^+(k)$. By definition, we have :

$$\begin{aligned} \mathcal{C}_P^\alpha[P_k](\underline{x}) &= \frac{1}{A_m} \int_{S^{m-1}} \mathcal{F}_\alpha(\underline{x}, \underline{\omega}) \underline{\omega} P_k(\underline{\omega}) dS(\underline{\omega}) \\ &= (2\alpha + m + 1) \frac{c(\alpha, m)}{A_m} (1-r^2)^{\alpha+\frac{m-1}{2}} \int \frac{(\epsilon + \underline{x}) \underline{\omega} P_k(\underline{\omega}) dS(\underline{\omega})}{(1 - \langle \underline{x}, \underline{\omega} \rangle)^{\alpha+m}} \\ &\quad - (\alpha + m) \frac{c(\alpha, m)}{A_m} (1-r^2)^{\alpha+\frac{m+1}{2}} \int \frac{(\epsilon + \underline{\omega}) \underline{\omega} P_k(\underline{\omega}) dS(\underline{\omega})}{(1 - \langle \underline{x}, \underline{\omega} \rangle)^{\alpha+m+1}}. \end{aligned}$$

Using Lemma 6.1 it is readily verified that up to the factor $P_k(\underline{\xi})$ the PCT $\mathcal{C}_P^\alpha[P_k](\underline{x})$ has a scalar component and a bivector-valued component. These are respectively given by :

$$\begin{aligned}\mathcal{C}_P^\alpha[P_k](\underline{x})|_0 &= c(\alpha, m)(\alpha + m)(1 - r^2)^{\alpha + \frac{m+1}{2}} \mathcal{P}_{k,m}(\alpha + m + 1; r) P_k(\underline{\xi}) \\ &\quad - c(\alpha, m)(2\alpha + m + 1)(1 - r^2)^{\alpha + \frac{m-1}{2}} r \mathcal{P}_{1+k,m}(\alpha + m; r) P_k(\underline{\xi}) \\ \mathcal{C}_P^\alpha[P_k](\underline{x})|_2 &= -c(\alpha, m)(\alpha + m)(1 - r^2)^{\alpha + \frac{m+1}{2}} \mathcal{P}_{1+k,m}(\alpha + m + 1; r) \epsilon \underline{\xi} P_k(\underline{\xi}) \\ &\quad + c(\alpha, m)(2\alpha + m + 1)(1 - r^2)^{\alpha + \frac{m-1}{2}} \mathcal{P}_{1+k,m}(\alpha + m; r) \epsilon \underline{\xi} P_k(\underline{\xi})\end{aligned}$$

By means of Lemma 6.2, this reduces to

$$\begin{aligned}\mathcal{C}_P^\alpha[P_k](\underline{x})|_0 &= -c(\alpha, m) \frac{\pi^{\frac{1}{2}} \Gamma(\alpha + m + k + 1) \Gamma\left(\frac{m-1}{2}\right)}{2^k \Gamma\left(k + \frac{m}{2}\right) \Gamma(\alpha + m)} (1 - r^2)^{\alpha + \frac{m-1}{2}} P_k(\underline{x}) \\ &\quad \left[\begin{array}{c} \frac{2\alpha + m + 1}{2k + m} r^2 F\left(\frac{1 + \alpha + m + k}{2}, 1 + \frac{\alpha + m + k}{2}; 1 + k + \frac{m}{2}; r^2\right) \\ -(1 - r^2) F\left(\frac{1 + \alpha + m + k}{2}, 1 + \frac{\alpha + m + k}{2}; k + \frac{m}{2}; r^2\right) \end{array} \right]\end{aligned}$$

and

$$\begin{aligned}\mathcal{C}_P^\alpha[P_k](\underline{x})|_2 &= c(\alpha, m) \frac{\pi^{\frac{1}{2}} \Gamma(\alpha + m + k + 2) \Gamma\left(\frac{m-1}{2}\right)}{2^{k+1} \Gamma\left(1 + k + \frac{m}{2}\right) \Gamma(\alpha + m)} (1 - r^2)^{\alpha + \frac{m-1}{2}} \epsilon \underline{x} P_k(\underline{x}) \\ &\quad \left[\begin{array}{c} \frac{2\alpha + m + 1}{\alpha + m + k + 1} F\left(\frac{1 + \alpha + m + k}{2}, 1 + \frac{\alpha + m + k}{2}; 1 + k + \frac{m}{2}; r^2\right) \\ -(1 - r^2) F\left(1 + \frac{\alpha + m + k}{2}, 1 + \frac{1 + \alpha + m + k}{2}; 1 + k + \frac{m}{2}; r^2\right) \end{array} \right]\end{aligned}$$

Using the definition of the hypergeometric series, these terms between square brackets can respectively be rewritten as

$$-F\left(\frac{\alpha + m + k - 1}{2}, \frac{\alpha + m + k}{2}; k + \frac{m}{2}; r^2\right)$$

for the scalar term and

$$-(k - \alpha) F\left(\frac{\alpha + m + k}{2}, \frac{1 + \alpha + m + k}{2}; 1 + k + \frac{m}{2}; r^2\right)$$

for the bivector-valued term. Eventually using one of Kummer's relations, in order to get rid of the factor $(1 - r^2)^{\alpha + \frac{m-1}{2}}$, we find :

$$\begin{aligned}\mathcal{C}_P^\alpha[P_k](\underline{x}) &= \mathcal{C}_P^\alpha[P_k](\underline{x})|_0 + \mathcal{C}_P^\alpha[P_k](\underline{x})|_2 \\ &= c(\alpha, m) \frac{\pi^{\frac{1}{2}} \Gamma(\alpha + m + k + 1) \Gamma\left(\frac{m-1}{2}\right)}{2^k \Gamma\left(k + \frac{m}{2}\right) \Gamma(\alpha + m)} \text{Mod}(\alpha, k, \underline{x}) P_k(\underline{x}) ,\end{aligned}$$

with $\text{Mod}(\alpha, k, \underline{x})$ the modulation factor defined in Theorem 3.1. This means that the hyperbolic monogenic functions on the Klein ball constructed by means of this Modulation Theorem are reobtained as photogenic Cauchy transforms of inner spherical monogenics.

6.2.2 The PCT of Outer Spherical Monogenics

Let us then consider an outer spherical monogenic $Q_k(\underline{\omega}) \in M^-(k)$. We will hereby restrict ourselves to $Q_k(\underline{\omega}) = \underline{\omega}P_k(\underline{\omega})$ such that the inner spherical monogenic $P_k(\underline{\omega}) \in M^+(k)$ takes its values in the even subalgebra $\mathbb{R}_{0,m}^{(+)}$, whence the commutator $[P_k(\underline{\omega}), \epsilon] = 0$. By definition, we have :

$$\begin{aligned}\mathcal{C}_P^\alpha[Q_k](\underline{x}) &= -\frac{1}{A_m} \int_{S^{m-1}} \mathcal{F}_\alpha(\underline{x}, \underline{\omega}) P_k(\underline{\omega}) dS(\underline{\omega}) \\ &= -(2\alpha + m + 1) \frac{c(\alpha, m)}{A_m} (1 - r^2)^{\alpha + \frac{m-1}{2}} \int \frac{(\epsilon + \underline{x}) P_k(\underline{\omega}) dS(\underline{\omega})}{(1 - \langle \underline{x}, \underline{\omega} \rangle)^{\alpha+m}} \\ &\quad + (\alpha + m) \frac{c(\alpha, m)}{A_m} (1 - r^2)^{\alpha + \frac{m+1}{2}} \int \frac{(\epsilon + \underline{\omega}) P_k(\underline{\omega}) dS(\underline{\omega})}{(1 - \langle \underline{x}, \underline{\omega} \rangle)^{\alpha+m+1}}.\end{aligned}$$

Using Lemma 6.1 we immediately see that up to the factor $P_k(\underline{\xi})$ the PCT $\mathcal{C}_P^\alpha[Q_k](\underline{x})$ has a component in ϵ and a component in $\underline{\xi}$, respectively given by

$$\begin{aligned}\mathcal{C}_P^\alpha[Q_k](\underline{x})|_\epsilon &= c(\alpha, m)(\alpha + m)(1 - r^2)^{\alpha + \frac{m+1}{2}} \mathcal{P}_{k,m}(\alpha + m + 1; r) \epsilon P_k(\underline{\xi}) \\ &\quad - c(\alpha, m)(2\alpha + m + 1)(1 - r^2)^{\alpha + \frac{m-1}{2}} \mathcal{P}_{k,m}(\alpha + m; r) \epsilon P_k(\underline{\xi})\end{aligned}$$

$$\begin{aligned}\mathcal{C}_P^\alpha[Q_k](\underline{x})|_{\underline{\xi}} &= c(\alpha, m)(\alpha + m)(1 - r^2)^{\alpha + \frac{m+1}{2}} \mathcal{P}_{1+k,m}(\alpha + m + 1; r) \underline{\xi} P_k(\underline{\xi}) \\ &\quad - c(\alpha, m)(2\alpha + m + 1)(1 - r^2)^{\alpha + \frac{m-1}{2}} r \mathcal{P}_{k,m}(\alpha + m; r) \underline{\xi} P_k(\underline{\xi})\end{aligned}$$

With the aid of Lemma 6.2 this reduces to

$$\begin{aligned}\mathcal{C}_P^\alpha[Q_k](\underline{x})|_\epsilon &= c(\alpha, m) \frac{\pi^{\frac{1}{2}} \Gamma(\alpha + m + k + 1) \Gamma\left(\frac{m-1}{2}\right)}{2^k \Gamma\left(k + \frac{m}{2}\right) \Gamma(\alpha + m)} (1 - r^2)^{\alpha + \frac{m-1}{2}} \epsilon P_k(\underline{x}) \\ &\quad \left[\begin{aligned} &-\frac{2\alpha + m + 1}{\alpha + m + k} F\left(\frac{\alpha + m + k}{2}, \frac{1 + \alpha + m + k}{2}; k + \frac{m}{2}; r^2\right) \\ &+ (1 - r^2) F\left(\frac{1 + \alpha + m + k}{2}, 1 + \frac{\alpha + m + k}{2}; k + \frac{m}{2}; r^2\right) \end{aligned} \right]\end{aligned}$$

and

$$\begin{aligned} \mathcal{C}_P^\alpha[P_k](\underline{x})|_{\underline{\xi}} &= -c(\alpha, m) \frac{\pi^{\frac{1}{2}} \Gamma(\alpha + m + k + 2) \Gamma\left(\frac{m-1}{2}\right)}{2^{k+1} \Gamma\left(1 + k + \frac{m}{2}\right) \Gamma(\alpha + m)} (1 - r^2)^{\alpha + \frac{m-1}{2}} \underline{x} P_k(\underline{x}) \\ &\quad \left[\begin{aligned} &\frac{(2\alpha + m + 1)(2k + m)}{(\alpha + m + k)(\alpha + m + k + 1)} F\left(\frac{\alpha + m + k}{2}, \frac{1 + \alpha + m + k}{2}; k + \frac{m}{2}; r^2\right) \\ &-(1 - r^2) F\left(1 + \frac{\alpha + m + k}{2}, 1 + \frac{1 + \alpha + m + k}{2}; 1 + k + \frac{m}{2}; r^2\right) \end{aligned} \right] \end{aligned}$$

Eventually making use of the definition for the hypergeometric series to rewrite the terms between square brackets respectively as

$$-(1 + \alpha - k) F\left(\frac{\alpha + m + k - 1}{2}, \frac{\alpha + m + k}{2}; k + \frac{m}{2}; r^2\right)$$

for the term in ϵ and

$$-\frac{(\alpha - k)(1 + \alpha - k)}{(2k + m)(\alpha + k + m)} F\left(\frac{\alpha + m + k}{2}, \frac{1 + \alpha + m + k}{2}; 1 + k + \frac{m}{2}; r^2\right)$$

for the term in $\underline{\xi}$ and recalling Kummer's relation, in order to get rid of the factor $(1 - r^2)^{\alpha + \frac{m-1}{2}}$, we arrive at :

$$\begin{aligned} \mathcal{C}_P^\alpha[Q_k](\underline{x}) &= \mathcal{C}_P^\alpha[Q_k](\underline{x})|_{\epsilon} + \mathcal{C}_P^\alpha[Q_k](\underline{x})|_{\underline{\xi}} \\ &= (k - \alpha - 1)c(\alpha, m) \frac{\pi^{\frac{1}{2}} \Gamma(\alpha + m + k) \Gamma\left(\frac{m-1}{2}\right)}{2^k \Gamma\left(k + \frac{m}{2}\right) \Gamma(\alpha + m)} \text{Mod}(\alpha, k, \underline{x}) P_k(\underline{x}) \epsilon . \end{aligned}$$

6.3 Photogenic Boundary Values

Now that we have found the photogenic Cauchy transform of inner and outer spherical monogenics on \mathbb{R}^m , we will determine their boundary values. Since both $\mathcal{C}_P^\alpha[P_k](\underline{x})$ and $\mathcal{C}_P^\alpha[Q_k](\underline{x})$ are solutions for the operator $(\partial + \epsilon(\mathbb{E}_r - \alpha))$ defined on the unit ball $B_m(1) \subset \mathbb{R}^m$, i.e.

$$\begin{aligned} \mathcal{C}_P^\alpha[P_k](\underline{x}) &\in \mathcal{H}_K^\alpha(B_m(1)) \\ \mathcal{C}_P^\alpha[Q_k](\underline{x}) &\in \mathcal{H}_K^\alpha(B_m(1)) , \end{aligned}$$

it seems natural to investigate whether the following limits exist :

$$\begin{aligned} \mathcal{C}_P^\alpha[P_k] \uparrow (\underline{\xi}) &= \lim_{r \rightarrow 1^-} [H(1 - r) \mathcal{C}_P^\alpha[P_k](r \underline{\xi})] \\ \mathcal{C}_P^\alpha[Q_k] \uparrow (\underline{\xi}) &= \lim_{r \rightarrow 1^-} [H(1 - r) \mathcal{C}_P^\alpha[Q_k](r \underline{\xi})] , \end{aligned}$$

where we have put $\underline{x} = r\underline{\xi}$ and where the arrow \uparrow is used to indicate that these limits are calculated with r approaching 1 from beneath.

In order to determine these limits, the following property of the hypergeometric function is essential : for $\operatorname{Re}(c - a - b) > 0$ and $c \notin -\mathbb{N}$ we have

$$\lim_{t \rightarrow 1} F(a, b; c; t) = \frac{\Gamma(c - a - b)\Gamma(c)}{\Gamma(c - a)\Gamma(c - b)}.$$

Recalling the definitions for the constant $c(\alpha, m)$ and the modulation factor $\operatorname{Mod}(\alpha, k, \underline{x})$, for which we respectively refer to Definition 6.1 and Theorem 3.1, we get for those α for which $\operatorname{Re}(\alpha) + \frac{m-1}{2} > 0$:

$$\mathcal{C}_P^\alpha[P_k] \uparrow (\underline{\xi}) = \frac{\Gamma\left(\frac{m-1}{2}\right) (\alpha + m + k) \{(\alpha + m + k - 1) + (k - \alpha)\underline{\xi}\epsilon\} P_k(\underline{\xi})}{8\pi^{\frac{m-1}{2}} (\alpha + \frac{m+1}{2}) (\alpha + \frac{m-1}{2})} \quad (6.3)$$

$$\mathcal{C}_P^\alpha[Q_k] \uparrow (\underline{\xi}) = \frac{\Gamma\left(\frac{m-1}{2}\right) (1 + \alpha - k) \{(\alpha - k) - (\alpha + m + k - 1)\underline{\xi}\epsilon\} Q_k(\underline{\xi})}{8\pi^{\frac{m-1}{2}} (\alpha + \frac{m+1}{2}) (\alpha + \frac{m-1}{2})} \quad (6.4)$$

Remark : It was to be expected that we would have to make restrictions on the parameter α ! Consider for example the value $\alpha = -\frac{m}{2}$, for which the projection of the hyperbolic Dirac operator on the Poincaré model reduces to the operator $\underline{\partial}(\underline{x} + \epsilon)$ on the unit ball $B_m(1)$, see Chapter 3. If a hyperbolic monogenic function $f(\underline{x})$ on the Poincaré model - which is a function $f(\underline{x})$ on $B_m(1)$ such that $\underline{\partial}(\underline{x} + \epsilon)f(\underline{x}) = 0$ - would have a boundary value $f(\underline{\xi})$ then also the function $g(\underline{x}) = (\underline{x} + \epsilon)f(\underline{x})$ would have a boundary value, viz. $g(\underline{\xi}) = (\underline{\xi} + \epsilon)f(\underline{\xi})$. From this it follows immediately that $\underline{\xi}g(\underline{\xi}) = -\epsilon g(\underline{\xi})$. On the other hand the function $g(\underline{x})$ is monogenic on $B_m(1)$ with respect to the operator $\underline{\partial}$ on $\mathbb{R}^{0,m}$, such that $g(\underline{\xi})$ stands for the boundary value of a classical monogenic function. In other words, for all inner spherical monogenics $P_k(\underline{\xi}) \in M^+(k)$ we get

$$\int_{S^{m-1}} \overline{P}_k(\underline{\xi}) \epsilon \underline{\xi} g(\underline{\xi}) dS(\underline{\xi}) = 0 \implies \int_{S^{m-1}} \overline{P}_k(\underline{\xi}) g(\underline{\xi}) dS(\underline{\xi}) = 0.$$

As this expression holds for *all* $P_k(\underline{\xi})$ we get $g(\underline{\xi}) = 0$ and thus $f(\underline{\xi}) = 0$, which illustrates that for $\alpha = -\frac{m}{2}$ there are no non-trivial hyperbolic monogenic functions with boundary values on the sphere.

Formulae (6.3) and (6.4) can also be expressed in terms of the spherical Gamma operator $\Gamma_{0,m}$ on the sphere S^{m-1} , hereby using the fact that inner (resp. outer) spherical monogenics of degree k are eigenfunctions for the operator $\Gamma_{0,m}$ with eigenvalue $(-k)$ (resp. $(k+m-1)$). For $\text{Re}(\alpha) + \frac{m-1}{2} > 0$ we get

$$\begin{aligned} \mathcal{C}_P^\alpha[P_k] \uparrow (\underline{\xi}) \\ = \frac{\Gamma\left(\frac{m-1}{2}\right)}{8\pi^{\frac{m-1}{2}}} \frac{\{(\alpha+m-1-\Gamma_{0,m}) - \underline{\xi}\epsilon(\Gamma_{0,m}+\alpha)\}(\alpha+m-\Gamma_{0,m})P_k(\underline{\xi})}{\left(\alpha+\frac{m+1}{2}\right)\left(\alpha+\frac{m-1}{2}\right)}, \end{aligned}$$

$$\begin{aligned} \mathcal{C}_P^\alpha[Q_k] \uparrow (\underline{\xi}) \\ = \frac{\Gamma\left(\frac{m-1}{2}\right)}{8\pi^{\frac{m-1}{2}}} \frac{\{(\alpha+m-1-\Gamma_{0,m}) - \underline{\xi}\epsilon(\Gamma_{0,m}+\alpha)\}(\alpha+m-\Gamma_{0,m})Q_k(\underline{\xi})}{\left(\alpha+\frac{m+1}{2}\right)\left(\alpha+\frac{m-1}{2}\right)}. \end{aligned}$$

This shows that in case $\text{Re}(\alpha) + \frac{m-1}{2} > 0$ the boundary values $\mathcal{C}_P^\alpha[P_k] \uparrow (\underline{\xi})$ and $\mathcal{C}_P^\alpha[Q_k] \uparrow (\underline{\xi})$ can be found by letting the polynomial operator

$$\mathcal{P}_\alpha(\Gamma_{0,m}) = \frac{\Gamma\left(\frac{m-1}{2}\right)}{8\pi^{\frac{m-1}{2}}} \frac{\{(\alpha+m-1-\Gamma_{0,m}) - \underline{\xi}\epsilon(\Gamma_{0,m}+\alpha)\}(\alpha+m-\Gamma_{0,m})}{\left(\alpha+\frac{m+1}{2}\right)\left(\alpha+\frac{m-1}{2}\right)}$$

act on respectively the inner and outer spherical monogenics $P_k(\underline{\xi})$ and $Q_k(\underline{\xi})$.

We will now reinterpret these formulae in such a way that the boundary values $\mathcal{C}_P^\alpha[P_k] \uparrow (\underline{\xi})$ and $\mathcal{C}_P^\alpha[Q_k] \uparrow (\underline{\xi})$ can be recovered as the action of a distribution on the spherical monogenics $P_k(\underline{\xi})$ and $Q_k(\underline{\xi})$.

Two different approaches can be followed here :

- The first approach uses the polynomial operator $\mathcal{P}_\alpha(\Gamma_{0,m})$ introduced above. Since we will need two different variables $\underline{\omega}$ and $\underline{\xi} \in S^{m-1}$, the Gamma operator on S^{m-1} will be labelled with the variable on which it is supposed to act from now on. So instead of $\Gamma_{0,m}$ we will write $\Gamma_{\underline{\omega}}$:

$$\begin{aligned} \mathcal{C}_P^\alpha[P_k] \uparrow (\underline{\xi}) &= \int_{S^{m-1}} \delta(\underline{\xi} - \underline{\omega}) \mathcal{P}_\alpha(\Gamma_{\underline{\omega}}) P_k(\underline{\omega}) dS(\underline{\omega}) \\ \mathcal{C}_P^\alpha[Q_k] \uparrow (\underline{\xi}) &= \int_{S^{m-1}} \delta(\underline{\xi} - \underline{\omega}) \mathcal{P}_\alpha(\Gamma_{\underline{\omega}}) Q_k(\underline{\omega}) dS(\underline{\omega}) \end{aligned}$$

Recalling formulae (3) and (4), this can also be written as

$$\begin{aligned}\mathcal{C}_P^\alpha[P_k] \uparrow (\underline{\xi}) &= - \int_{S^{m-1}} [\delta(\underline{\xi} - \underline{\omega}) \mathcal{P}_\alpha(\Gamma_{\underline{\omega}})] P_k(\underline{\omega}) dS(\underline{\omega}) \\ &= \int_{S^{m-1}} \overline{\mathcal{P}_\alpha(\Gamma_{\underline{\omega}}) \delta(\underline{\xi} - \underline{\omega})} P_k(\underline{\omega}) dS(\underline{\omega})\end{aligned}$$

and

$$\begin{aligned}\mathcal{C}_P^\alpha[Q_k] \uparrow (\underline{\xi}) &= - \int_{S^{m-1}} [\delta(\underline{\xi} - \underline{\omega}) \mathcal{P}_\alpha(\Gamma_{\underline{\omega}})] Q_k(\underline{\omega}) dS(\underline{\omega}) \\ &= \int_{S^{m-1}} \overline{\mathcal{P}_\alpha(\Gamma_{\underline{\omega}}) \delta(\underline{\xi} - \underline{\omega})} Q_k(\underline{\omega}) dS(\underline{\omega}) .\end{aligned}$$

Defining the action of a distribution $\mathcal{D}(\underline{\omega})$ on a test function $\varphi(\underline{\omega})$ by

$$\langle \mathcal{D}(\underline{\omega}), \varphi(\underline{\omega}) \rangle = \int_{S^{m-1}} \mathcal{D}(\underline{\omega}) \underline{\omega} \varphi(\underline{\omega}) dS(\underline{\omega}) ,$$

we get for all $\operatorname{Re}(\alpha) + \frac{m-1}{2}$:

$$\begin{aligned}\mathcal{C}_P^\alpha[P_k] \uparrow (\underline{\xi}) &= - \langle \overline{\mathcal{P}_\alpha(\Gamma_{\underline{\omega}}) \delta(\underline{\xi} - \underline{\omega})}, \underline{\omega} P_k(\underline{\omega}) \rangle \\ &= \langle \underline{\omega} \overline{\mathcal{P}_\alpha(\Gamma_{\underline{\omega}}) \delta(\underline{\xi} - \underline{\omega})}, P_k(\underline{\omega}) \rangle\end{aligned}$$

$$\begin{aligned}\mathcal{C}_P^\alpha[Q_k] \uparrow (\underline{\xi}) &= - \langle \overline{\mathcal{P}_\alpha(\Gamma_{\underline{\omega}}) \delta(\underline{\xi} - \underline{\omega})}, \underline{\omega} Q_k(\underline{\omega}) \rangle \\ &= \langle \underline{\omega} \overline{\mathcal{P}_\alpha(\Gamma_{\underline{\omega}}) \delta(\underline{\xi} - \underline{\omega})}, Q_k(\underline{\omega}) \rangle\end{aligned}$$

- The second approach again uses the photogenic Cauchy kernel but in contrast to previous considerations, the distribution $\mathcal{F}_\alpha(\underline{x}, \underline{\omega})$, $\underline{x} = r\underline{\xi}$, will now be interpreted as a distribution in the variable $\underline{\xi} \in S^{m-1}$. When acting on a test function $\varphi(\underline{\xi})$, we get by definition (cfr. supra) :

$$\langle \mathcal{F}_\alpha(r\underline{\xi}, \underline{\omega}), \varphi(\underline{\xi}) \rangle = \int_{S^{m-1}} \mathcal{F}_\alpha(r\underline{\xi}, \underline{\omega}) \underline{\xi} \varphi(\underline{\xi}) dS(\underline{\xi}) .$$

We will let this distribution act on an inner spherical monogenic $P_k(\underline{\xi})$ (resp. an outer spherical monogenic $Q_k(\underline{\xi})$), and perform similar calculations as in the previous subsection to simplify the resulting expressions.

For an arbitrary $P_k(\underline{\xi}) \in M^+(k)$, we get :

$$\begin{aligned} \langle \mathcal{F}_\alpha(r\underline{\xi}, \underline{\omega}), P_k(\underline{\xi}) \rangle &= c(\alpha, m)(1-r^2)^{\alpha+\frac{m-1}{2}} P_k(\underline{\omega}) \\ &\quad \left[\begin{array}{l} -r(2\alpha+m+1)\mathcal{P}_{k,m}(\alpha+m; r) \\ +(\alpha+m)(1-r^2)\mathcal{P}_{1+k,m}(\alpha+m+1; r) \end{array} \right] \\ &- c(\alpha, m)(1-r^2)^{\alpha+\frac{m-1}{2}} \underline{\omega} \epsilon P_k(\underline{\omega}) \\ &\quad \left[\begin{array}{l} (2\alpha+m+1)\mathcal{P}_{1+k,m}(\alpha+m; r) \\ -(\alpha+m)(1-r^2)\mathcal{P}_{1+k,m}(\alpha+m+1; r) \end{array} \right] \end{aligned}$$

Lemma 6.2 can be used to calculate the expressions between square brackets as a sum of hypergeometric functions.

Using elementary properties of the hypergeometric series, this can then be simplified to

$$\begin{aligned} \langle \mathcal{F}_\alpha(r\underline{\xi}, \underline{\omega}), P_k(\underline{\xi}) \rangle &= c(\alpha, m)(\alpha-k)(\alpha-k+1) \frac{\pi^{\frac{1}{2}} \Gamma(\alpha+m+k) \Gamma(\frac{m-1}{2})}{2^{k+1} \Gamma(\alpha+m) \Gamma(1+k+\frac{m}{2})} \\ &\quad r^k F\left(\frac{1+k-\alpha}{2}, 1+\frac{k-\alpha}{2}; 1+k+\frac{m}{2}; r^2\right) P_k(\underline{\omega}) \\ &+ c(\alpha, m) \frac{k-\alpha}{2k+m} \frac{\pi^{\frac{1}{2}} \Gamma(\alpha+m+k+1) \Gamma(\frac{m-1}{2})}{2^k \Gamma(\alpha+m) \Gamma(k+\frac{m}{2})} \\ &\quad r^{1+k} F\left(\frac{1+k-\alpha}{2}, 1+\frac{k-\alpha}{2}; 1+k+\frac{m}{2}; r^2\right) \underline{\omega} \epsilon P_k(\underline{\omega}) . \end{aligned}$$

For $\text{Re}(\alpha) + \frac{m-1}{2} > 0$ we thus get in the limit $\lim_{r \rightarrow 1}$:

$$\begin{aligned} \lim_{r \rightarrow 1} \langle \mathcal{F}_\alpha(r\underline{\xi}, \underline{\omega}), P_k(\underline{\xi}) \rangle &= \frac{\Gamma(\frac{m-1}{2})}{8\pi^{\frac{m-1}{2}}} \frac{(\alpha-k)\{(\alpha-k-1) - \underline{\omega} \epsilon(\alpha+m+k)\} P_k(\underline{\omega})}{(\alpha+\frac{m+1}{2})(\alpha+\frac{m-1}{2})} . \end{aligned}$$

Next we consider an arbitrary $Q_k(\underline{\xi}) \in M^-(k)$. We get :

$$\begin{aligned} \langle \mathcal{F}_\alpha(r\underline{\xi}, \underline{\omega}), Q_k(\underline{\xi}) \rangle &= c(\alpha, m)(1-r^2)^{\alpha+\frac{m-1}{2}} \underline{\omega} P_k(\underline{\omega}) \\ &\quad \left[\begin{array}{l} -r(2\alpha+m+1)\mathcal{P}_{1+k,m}(\alpha+m; r) \\ +(\alpha+m)(1-r^2)\mathcal{P}_{k,m}(\alpha+m+1; r) \end{array} \right] \\ &- c(\alpha, m)(1-r^2)^{\alpha+\frac{m-1}{2}} \epsilon P_k(\underline{\omega}) \\ &\quad \left[\begin{array}{l} (2\alpha+m+1)\mathcal{P}_{k,m}(\alpha+m; r) \\ -(\alpha+m)(1-r^2)\mathcal{P}_{k,m}(\alpha+m+1; r) \end{array} \right] \end{aligned}$$

Lemma 6.2 can be used to simplify the expressions between square brackets as a sum of hypergeometric functions, and these can then be reduced by means of the definition for the hypergeometric series to

$$\begin{aligned}
\langle \mathcal{F}_\alpha(r\underline{\xi}, \underline{\omega}), Q_k(\underline{\xi}) \rangle &= c(\alpha, m) \frac{\pi^{\frac{1}{2}} \Gamma(\alpha + m + k) \Gamma\left(\frac{m-1}{2}\right)}{2^{k+1} \Gamma(\alpha + m) \Gamma\left(1 + k + \frac{m}{2}\right)} \\
&\quad r^k F\left(\frac{1+k-\alpha}{2}, \frac{k-\alpha}{2}; k + \frac{m}{2}; r^2\right) \underline{\omega} P_k(\underline{\omega}) \\
&- c(\alpha, m)(1 + \alpha - k) \frac{\pi^{\frac{1}{2}} \Gamma(\alpha + m + k) \Gamma\left(\frac{m-1}{2}\right)}{2^k \Gamma(\alpha + m) \Gamma\left(k + \frac{m}{2}\right)} \\
&\quad r^k F\left(\frac{1+k-\alpha}{2}, \frac{k-\alpha}{2}; k + \frac{m}{2}; r^2\right) \epsilon P_k(\underline{\omega}) .
\end{aligned}$$

For $\operatorname{Re}(\alpha) + \frac{m-1}{2} > 0$ we thus get in the limit $\lim_{r \rightarrow 1}$:

$$\begin{aligned}
\lim_{r \rightarrow 1} \langle \mathcal{F}_\alpha(r\underline{\xi}, \underline{\omega}), Q_k(\underline{\xi}) \rangle &= \frac{\Gamma\left(\frac{m-1}{2}\right) (\alpha + m + k - 1) \{(\alpha + m + k) - \underline{\omega} \epsilon (1 + \alpha - k)\} Q_k(\underline{\omega})}{8\pi^{\frac{m-1}{2}} (\alpha + \frac{m+1}{2}) (\alpha + \frac{m-1}{2})} .
\end{aligned}$$

If we now compare these expressions for

$$\begin{aligned}
\lim_{r \rightarrow 1} \langle \mathcal{F}_\alpha(r\underline{\xi}, \underline{\omega}), P_k(\underline{\xi}) \rangle & \\
\lim_{r \rightarrow 1} \langle \mathcal{F}_\alpha(r\underline{\xi}, \underline{\omega}), Q_k(\underline{\xi}) \rangle &
\end{aligned}$$

with the boundary values $\mathcal{C}_P^\alpha[P_k] \uparrow (\underline{\xi})$ and $\mathcal{C}_P^\alpha[Q_k] \uparrow (\underline{\xi})$ for the PCT of inner and outer spherical monogenics on \mathbb{R}^m , we conclude that

$$\begin{aligned}
\lim_{r \rightarrow 1} \langle \mathcal{F}_\alpha(r\underline{\xi}, \underline{\omega}), P_k(\underline{\xi}) \rangle &= \mathcal{C}_P^\beta[P_k] \uparrow (\underline{\xi}) \\
\lim_{r \rightarrow 1} \langle \mathcal{F}_\alpha(r\underline{\xi}, \underline{\omega}), Q_k(\underline{\xi}) \rangle &= \mathcal{C}_P^\beta[Q_k] \uparrow (\underline{\xi}) ,
\end{aligned}$$

where we have put $\beta = -\alpha - m$.

This can easily be understood as follows : if we consider for example an inner spherical monogenic $P_k(\underline{\omega}) \in M^+(k)$, we have by definition

$$\begin{aligned}
\mathcal{C}_P^\beta[P_k] \uparrow (\underline{\xi}) &= \lim_{r \rightarrow 1} \int_{S^{m-1}} \mathcal{F}_\beta(r\underline{\xi}, \underline{\omega}) \underline{\omega} P_k(\underline{\omega}) dS(\underline{\omega}) \\
\lim_{r \rightarrow 1} \langle \mathcal{F}_\alpha(r\underline{\omega}, \underline{\xi}), P_k(\underline{\omega}) \rangle &= \lim_{r \rightarrow 1} \int_{S^{m-1}} \mathcal{F}_\alpha(r\underline{\omega}, \underline{\xi}) \underline{\omega} P_k(\underline{\omega}) dS(\underline{\omega})
\end{aligned}$$

showing that both formulae are identical, except for the fact that the arguments of the photogenic kernel are switched and that $\alpha \leftrightarrow \beta$. We have already encountered this phenomenon in Chapter 5, where the following observation was made : the fundamental solution for the hyperbolic Dirac equation on H_+ is the restriction to H_+ of a function $E_\alpha(X, Y)$, defined for X and $Y \in FC$, which is α -homogeneous in X and monogenic with respect to the operator ∂_X acting from the left and β -homogeneous in Y and monogenic with respect to the operator ∂_Y acting from the right.

6.4 Hyperbolic Hilbert Modules

In this section, we introduce a function space containing square integrable functions on $S^{m-1} \subset \mathbb{R}^m$ which are boundary values of hyperbolic monogenic functions on the Klein model, i.e. functions belonging to $\mathcal{H}_K^\alpha(B_m(1))$. Considering the extension of these functions to the Lie sphere LS^{m-1} it can be proved that this function space is a Hilbert module with reproducing kernel, under certain restrictions on the parameter α . Throughout this section, the spherical Dirac operator $\Gamma_{0,m}$ on S^{m-1} will always be denoted by means of the variable on which it acts; i.e. as $\Gamma_{\underline{\omega}}$ or $\Gamma_{\underline{\xi}}$ (with $\underline{\omega}$ and $\underline{\xi} \in S^{m-1}$).

First of all, let us consider a function $f \in L_2(S^{m-1})$. This function can be decomposed as $f = \sum_{k=0}^{\infty} P(k)[f] + Q(k)[f]$, where the series converges in L_2 -sense. By definition, we thus have :

$$\|f\|_{L_2(S^{m-1})}^2 = \sum_{k=0}^{\infty} \|P(k)f\|_{L_2(S^{m-1})}^2 + \|Q(k)f\|_{L_2(S^{m-1})}^2 < \infty ,$$

which means that $(\|P(k)f\|_{L_2(S^{m-1})})_{k \in \mathbb{N}}$ and $(\|Q(k)f\|_{L_2(S^{m-1})})_{k \in \mathbb{N}}$ belong to the space of square summable series l_2 . From the previous section we already know that if we let the photogenic Cauchy transform act on these monogenic building blocks $P(k)[f]$ and $Q(k)[f]$, we get for $\alpha \in \mathbb{C}$:

$$\begin{aligned} \mathcal{C}_P^\alpha[P(k)f](\underline{x}) &\in \mathcal{H}_K^\alpha(B_m(1)) \\ \mathcal{C}_P^\alpha[Q(k)f](\underline{x}) &\in \mathcal{H}_K^\alpha(B_m(1)) , \end{aligned}$$

with, up to a constant depending on α , m and k :

$$\begin{aligned} \mathcal{C}_P^\alpha[P(k)f](\underline{x}) &\sim \text{Mod}(\alpha, k; \underline{x}) P(k)f(\underline{x}) \\ \mathcal{C}_P^\alpha[Q(k)f](\underline{x}) &\sim \text{Mod}(\alpha, k; \underline{x}) P(k)f(\underline{x}) \epsilon . \end{aligned}$$

For each $k \in \mathbb{N}$ fixed and $\alpha \in \mathbb{C}$ such that $\operatorname{Re}(\alpha) > \frac{1-m}{2}$, we also have that for $f \in L_2(S^{m-1})$:

$$\begin{aligned} \lim_{|\underline{x}| \rightarrow 1} \mathcal{C}_P^\alpha[P(k)f](\underline{x}) &= \mathcal{C}_P^\alpha[P(k)f] \uparrow (\underline{\xi}) \in L_2(S^{m-1}) \\ \lim_{|\underline{x}| \rightarrow 1} \mathcal{C}_P^\alpha[Q(k)f](\underline{x}) &= \mathcal{C}_P^\alpha[Q(k)f] \uparrow (\underline{\xi}) \in L_2(S^{m-1}) . \end{aligned}$$

Indeed, consider for example

$$\mathcal{C}_P^\alpha[P(k)f] \uparrow (\underline{\xi}) = c_p P(k)[f](\underline{\xi}) + c_q Q(k)[f](\underline{\xi}) ,$$

with

$$\begin{aligned} c_p &= \frac{\Gamma\left(\frac{m-1}{2}\right) (\alpha + m + k)(\alpha + m + k - 1)}{8\pi^{\frac{m-1}{2}} \left(\alpha + \frac{m+1}{2}\right) \left(\alpha + \frac{m-1}{2}\right)} \\ c_q &= \frac{\Gamma\left(\frac{m-1}{2}\right) (\alpha + m + k)(k - \alpha)}{8\pi^{\frac{m-1}{2}} \left(\alpha + \frac{m+1}{2}\right) \left(\alpha + \frac{m-1}{2}\right)} . \end{aligned}$$

Since

$$\|\mathcal{C}_P^\alpha[P(k)f] \uparrow\|_{L_2(S^{m-1})}^2 = \int_{S^{m-1}} \overline{\mathcal{C}_P^\alpha[P(k)f] \uparrow (\underline{\xi})} \mathcal{C}_P^\alpha[P(k)f] \uparrow (\underline{\xi}) dS(\underline{\xi}) ,$$

we get, in view of the orthogonality of $P(k)[f]$ and $Q(k)[f]$:

$$\begin{aligned} \|\mathcal{C}_P^\alpha[P(k)f] \uparrow\|_{L_2(S^{m-1})}^2 &= |c_p|^2 \|P(k)f\|_{L_2(S^{m-1})}^2 - |c_q|^2 \|Q(k)f\|_{L_2(S^{m-1})}^2 \\ &< \infty . \end{aligned}$$

A natural question to ask is the following : if

$$f = \sum_{k=0}^{\infty} P(k)[f] + Q(k)[f] \text{ in } L_2\text{-sense} ,$$

do we also have that

$$\mathcal{C}_P^\alpha[f] \uparrow = \sum_{k=0}^{\infty} \mathcal{C}_P^\alpha[P(k)f] \uparrow + \mathcal{C}_P^\alpha[Q(k)f] \uparrow \text{ in } L_2\text{-sense} ?$$

In view of expressions (6.3) and (6.4) the answer is clearly *no*, as both $\mathcal{C}_P^\alpha[P(k)f] \uparrow$ and $\mathcal{C}_P^\alpha[Q(k)f] \uparrow$ contain factors $k^2 P(k)f$ and $k^2 Q(k)f$ from the second-order derivation with respect to the Gamma operator.

However, if we restrict ourselves to functions f belonging to the Sobolev space $W_2(S^{m-1})$, see e.g. [74], defined by

Definition 6.3

$$W_2(S^{m-1}) = \{f : L_{\underline{\sigma}\underline{\tau}}f \in L_2(S^{m-1}) , \ |\underline{\sigma}| = |\underline{\tau}| \leq 2\} ,$$

where $L_{\underline{\sigma}\underline{\tau}} = L_{\sigma_1\tau_1} \cdots L_{\sigma_n\tau_n}$ for multi-indices $\underline{\sigma}$ and $\underline{\tau} \in \mathbb{N}^n$,

it is immediately clear that $f \in W_2(S^{m-1}) \implies \mathcal{C}_P^\alpha[f] \uparrow \in L_2(S^{m-1})$.

Lemma 6.3 *The operator $\mathcal{C}_P^\alpha[\cdot] \uparrow$ on $W_2(S^{m-1})$ is continuous, for all $\alpha \in \mathbb{C}$ such that $\operatorname{Re}(\alpha) > \frac{1-m}{2}$.*

Proof : It suffices to verify that if $(f_j)_{j \in \mathbb{N}} \rightarrow f$ in $W_2(S^{m-1})$, we also get that

$$\|\mathcal{C}_P^\alpha[f_j] \uparrow - \mathcal{C}_P^\alpha[f] \uparrow\|_{L_2(S^{m-1})} \longrightarrow 0 .$$

Since $f_j \in L_2(S^{m-1})$ we have $f_j = \sum_k P_k^{(j)} + Q_k^{(j)}$, and since $(f_j)_{j \in \mathbb{N}} \rightarrow f$ in the L_2 -sense we also have $f = \sum_k P_k + Q_k$. Hence, we get that

$$\begin{aligned} \mathcal{C}_P^\alpha[f_j] \uparrow &= \sum_{k=0}^{\infty} \operatorname{Mod}(\alpha, k; \underline{\xi}) (c_{\alpha, k, m} + c'_{\alpha, k, m} \epsilon) P_k^{(j)}(\underline{\xi}) \\ \mathcal{C}_P^\alpha[f] \uparrow &= \sum_{k=0}^{\infty} \operatorname{Mod}(\alpha, k; \underline{\xi}) (c_{\alpha, k, m} + c'_{\alpha, k, m} \epsilon) P_k(\underline{\xi}) , \end{aligned}$$

where the condition on α is needed to ensure the existence of the radial limits. This means that

$$\mathcal{C}_P^\alpha[f_j] \uparrow - \mathcal{C}_P^\alpha[f] \uparrow = \sum_{k=0}^{\infty} \operatorname{Mod}(\alpha, k; \underline{\xi}) (c_{\alpha, k, m} + c'_{\alpha, k, m} \epsilon) (P_k^{(j)}(\underline{\xi}) - P_k(\underline{\xi})) ,$$

and in view of the fact that both $\|P_k^{(j)} - P_k\|_{L_2} \rightarrow 0$ and $\|Q_k^{(j)} - Q_k\|_{L_2} \rightarrow 0$ this yields the desired result. \square

Since the photogenic Cauchy transform $\mathcal{C}_P^\alpha[f](\underline{x})$ of a function f in the Sobolev space $W_2(S^{m-1})$ belongs to $\mathcal{H}_K^\alpha(B_m(1))$ it seems natural to define a function space $\mathcal{ML}_2^\alpha(S^{m-1})$ containing radial limits, in the L_2 -sense, of hyperbolic monogenics on the Klein ball :

Definition 6.4 *Let α be an arbitrary complex number. We then put*

$$\mathcal{ML}_2^\alpha(S^{m-1}) = \{f \in L_2(S^{m-1}) : f(\underline{\omega}) = \lim_{r \rightarrow 1} f^*(\underline{x}) , \ f^* \in \mathcal{H}_K^\alpha(B_m(1))\} .$$

For $\alpha \in \mathbb{C}$ such that $\operatorname{Re}(\alpha) > \frac{1-m}{2}$, we thus have the following mapping :

$$\mathcal{C}_P^\alpha[\cdot] \uparrow : W_2(S^{m-1}) \mapsto \mathcal{ML}_2^\alpha(S^{m-1}) : f \mapsto \mathcal{C}_P^\alpha[f] \uparrow .$$

Note however that this mapping is not one-to-one. Indeed, from the explicit expressions for $\mathcal{C}_P^\alpha[P_k](\underline{x})$ and $\mathcal{C}_P^\alpha[Q_k\epsilon](\underline{x})$ we get for all $k \in \mathbb{N}$ that

$$\mathcal{C}_P^\alpha[(1 + \alpha - k)P_k + (\alpha + m + k)Q_k\epsilon] \uparrow (\underline{\xi}) = 0 .$$

In view of the fact that the spherical monogenics are eigenfunctions for the operator $\Gamma_{\underline{\omega}}$ on the unit sphere S^{m-1} , this can be rewritten as

$$\mathcal{C}_P^\alpha[(\Gamma_{\underline{\omega}} + 1 + \alpha)(1 + \underline{\omega}\epsilon)P_k] \uparrow (\underline{\xi}) = 0 .$$

Recalling the explicit expression for the photogenic Cauchy transform, and using the fact that $(1 + \underline{\omega}\epsilon)\underline{\omega}\epsilon = (1 + \underline{\omega}\epsilon)$, this gives rise to the following equalities :

$$\int_{S^{m-1}} \mathcal{F}_\alpha(\underline{x}, \underline{\omega}) \underline{\omega} (\Gamma + 1 + \alpha)(1 + \underline{\omega}\epsilon) P_k(\underline{\omega}) dS(\underline{\omega}) = 0$$

and also

$$\int_{S^{m-1}} \mathcal{F}_\alpha(\underline{x}, \underline{\omega}) \underline{\omega} (\Gamma + 1 + \alpha)(1 + \underline{\omega}\epsilon) Q_k(\underline{\omega}) dS(\underline{\omega}) = 0 .$$

We will now rewrite these expressions in terms of the inner product on the space $L_2(S^{m-1})$:

$$(\mathcal{D}_\alpha(\underline{\omega})\mathcal{F}_\alpha(\underline{x}, \underline{\omega}), P_k) = (\mathcal{D}_\alpha(\underline{\omega})\mathcal{F}_\alpha(\underline{x}, \underline{\omega}), Q_k) = 0 ,$$

with $\mathcal{D}_\alpha(\underline{\omega})$ a differential operator on the unit sphere. Since

$$(f, P_k) = (f, Q_k) = 0 \implies f = 0 ,$$

this will give rise to a differential equation for the photogenic Cauchy kernel *when considered as a function of $\underline{\omega} \in S^{m-1}$* .

Using the fact that $\underline{\omega}\Gamma_{\underline{\omega}}\underline{\omega} = \Gamma_{\underline{\omega}} - (m-1)$, we immediately get

$$\begin{aligned} & \int_{S^{m-1}} \mathcal{F}_\alpha(\underline{x}, \underline{\omega}) \underline{\omega} (\Gamma_{\underline{\omega}} + 1 + \alpha)(1 + \underline{\omega}\epsilon) P_k(\underline{\omega}) dS(\underline{\omega}) \\ &= \int_{S^{m-1}} \mathcal{F}_\alpha(\underline{x}, \underline{\omega}) \left((1 + \alpha)\underline{\omega}(1 + \underline{\omega}\epsilon) - (m-1)\epsilon + (\underline{\omega} + \epsilon)\Gamma_{\underline{\omega}} \right) P_k(\underline{\omega}) dS(\underline{\omega}) , \end{aligned}$$

which by means of the fact that $(f, \Gamma g) = (\Gamma f, g)$ can be written as

$$\left(\left\{ \Gamma_{\underline{\omega}}(\underline{\omega} + \epsilon) - (m-1)\epsilon + (1+\alpha)(\underline{\omega} - \epsilon) \right\} \mathcal{F}_{\alpha}(\underline{x}, \underline{\omega}), P_k \right) .$$

In other words, the operator $\mathcal{D}_{\alpha}(\underline{\omega})$ mentioned earlier is given by

$$\mathcal{D}_{\alpha}(\underline{\omega}) = \Gamma_{\underline{\omega}}(\underline{\omega} + \epsilon) - (m-1)\epsilon + (1+\alpha)(\underline{\omega} - \epsilon)$$

and its action on the photogenic Cauchy kernel is given by the following :

Lemma 6.4 *The photogenic Cauchy kernel $\mathcal{F}_{\alpha}(\underline{x}, \underline{\omega})$ belongs to the kernel of the operator $\mathcal{D}_{\alpha}(\underline{\omega})$, for all $\underline{x} \in B_m(1)$:*

$$\mathcal{D}_{\alpha}(\underline{\omega})\mathcal{F}_{\alpha}(\underline{x}, \underline{\omega}) = 0 .$$

Proof : Follows from direct calculations. □

Let us then consider a function f belonging to $\mathcal{ML}_2^{\alpha}(S^{m-1})$. By definition, there exists a function $f^*(\underline{x}) \in \mathcal{H}_K^{\alpha}(B_m(1))$ such that $f(\underline{\xi}) = \lim_{r \rightarrow 1} f^*(r\underline{\xi})$. In view of the Taylor Theorem on the hyperbolic Klein ball, we thus have :

$$f^*(\underline{x}) = \sum_{k=0}^{\infty} \text{Mod}(\alpha, k; \underline{x}) P_k(\underline{x}) \implies f(\underline{\xi}) = \sum_{k=0}^{\infty} \text{Mod}(\alpha, k; \underline{\xi}) P_k(\underline{\xi}) ,$$

with $P_k(\underline{x}) \in M^+(k)$, its explicit form being given by Theorem 5.12. Since $f \in L_2(S^{m-1})$ we also have that

$$\left(|F_1^{(k)}(1)| \|P_k\|_{L_2} \right)_k \quad \text{and} \quad \left(|F_2^{(k)}(1)| \|P_k\|_{L_2} \right)_k \in l_2 .$$

Remark : Note that the spaces $\mathcal{ML}_2^{\alpha}(S^{m-1})$ are trivial for $\text{Re}(\alpha) \leq \frac{1-m}{2}$, since the modulation factor $\text{Mod}(\alpha, k; \underline{x})$ does not coverge for these values as $|\underline{x}| \rightarrow 1$.

In view of the fact that

$$\mathcal{C}_P^{\alpha} \left[\frac{2^k \Gamma(k + \frac{m}{2})}{\Gamma(\alpha + m + k + 1)} P_k(\underline{\omega}) \right] (\underline{x}) = \frac{\Gamma(\frac{m-1}{2}) \text{Mod}(\alpha, k; \underline{x}) P_k(\underline{x})}{2^{\alpha+m+1} \pi^{\frac{m}{2}-1} \Gamma(\alpha + \frac{m+3}{2})} ,$$

it follows that

$$\mathcal{C}_P^{\alpha}[\cdot] \uparrow : \sum_{k=0}^{\infty} \frac{2^k \Gamma(k + \frac{m}{2})}{\Gamma(\alpha + m + k + 1)} P_k(\underline{\omega}) \mapsto \frac{\Gamma(\frac{m-1}{2})}{2^{\alpha+m+1} \pi^{\frac{m}{2}-1} \Gamma(\alpha + \frac{m+3}{2})} f(\underline{\xi}) .$$

where the convergence is understood in the L_2 -sense. In order to verify whether this series belongs to the space $W_2(S^{m-1})$ it suffices to prove that

$$\sum_{k=0}^{\infty} k^2 \frac{2^k \Gamma(k + \frac{m}{2})}{\Gamma(\alpha + m + k + 1)} P_k(\underline{\omega}) \in L_2(S^{m-1}).$$

For that purpose we use Legendre's duplication formula :

$$\begin{aligned} \left| \frac{k^2 2^k \Gamma(k + \frac{m}{2})}{\Gamma(\alpha + m + k + 1)} \right| &\leq \left| \frac{k^2}{(\alpha + m + k)(\alpha + m + k - 1)} \frac{2^{2-\alpha-m} \pi^{\frac{1}{2}} \Gamma(k + \frac{m}{2})}{\Gamma(\frac{\alpha+m+k}{2}) \Gamma(\frac{\alpha+m+k-1}{2})} \right| \\ &\leq C_{\alpha,m} |F_1^{(k)}(1)| \left| \frac{k^2}{(\alpha + m + k)(\alpha + m + k - 1)} \right|, \end{aligned}$$

with $C_{\alpha,m}$ a constant depending on α and m only. Since the last term is bounded by a constant for all k , we immediately get that the desired result, hereby using the fact that $f \in L_2(S^{m-1})$:

$$\sum_{k=0}^{\infty} \frac{2^k \Gamma(k + \frac{m}{2})}{\Gamma(\alpha + m + k + 1)} P_k(\underline{\omega}) \in W_2(S^{m-1}).$$

Since the kernel of the mapping $\mathcal{C}_P^\alpha[\cdot] \uparrow$ is closed, which follows immediately from the continuity, its domain can be characterized as the orthogonal complement of its kernel, where

$$\ker \mathcal{C}_P^\alpha[\cdot] \uparrow = \left\{ f \in W_2(S^{m-1}) : f(\underline{\xi}) = (\Gamma + 1 + \alpha) \sum_{k=0}^{\infty} c_k (1 + \underline{\xi}) P_k(\underline{\xi}) \right\}.$$

Returning to $f^*(\underline{x}) \in \mathcal{H}_K^\alpha(B_m(1))$, we expand the hypergeometric functions in the modulation factor as a series :

$$\begin{aligned} f^*(\underline{x}) &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{\left(\frac{k-\alpha}{2}\right)_l \left(\frac{1+k-\alpha}{2}\right)_l}{l! \left(k + \frac{m}{2}\right)_l} |\underline{x}|^{2l} P_k(\underline{x}) \\ &+ \frac{k-\alpha}{2k+m} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{\left(\frac{1+k-\alpha}{2}\right)_l \left(1 + \frac{k-\alpha}{2}\right)_l}{l! \left(1 + k + \frac{m}{2}\right)_l} |\underline{x}|^{2l} P_k(\underline{x}) \epsilon, \end{aligned}$$

which is a double series converging normally on the unit ball $B_m(1)$. Hence, Siciak's Theorem tells us that the complexified double series

$$\begin{aligned} f^*(\underline{z}) &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{\left(\frac{k-\alpha}{2}\right)_l \left(\frac{1+k-\alpha}{2}\right)_l}{l! \left(k + \frac{m}{2}\right)_l} |\underline{z}|^{2l} P_k(\underline{z}) \\ &+ \frac{k-\alpha}{2k+m} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{\left(\frac{1+k-\alpha}{2}\right)_l \left(1 + \frac{k-\alpha}{2}\right)_l}{l! \left(1 + k + \frac{m}{2}\right)_l} |\underline{z}|^{2l} P_k(\underline{z}) \epsilon \end{aligned}$$

will converge normally on the Lie ball $LB_m(1)$, from which it immediately follows that $f^*(\underline{z})$ is holomorphic in the Lie ball.

To examine the L_2 -boundary behaviour of this holomorphic function, we consider the extension of f to the Lie sphere LS^{m-1} , given by

$$f(e^{it}\underline{\xi}) = \sum_{k=0}^{\infty} \text{Mod}(\alpha, k; e^{it}\underline{\xi}) P_k(e^{it}\underline{\xi}) .$$

By definition, we have :

$$f(e^{it}\underline{\xi}) \in L_2(LS^{m-1}) \iff \|f(e^{it}\underline{\xi})\|_{L_2(LS^{m-1})} < \infty ,$$

where the norm on the Lie sphere is given by

$$\|f(e^{it}\underline{\xi})\|_{L_2(LS^{m-1})}^2 = \frac{1}{A_m \pi} \int_0^\pi \int_{S^{m-1}} f(e^{it}\underline{\xi})^+ f(e^{it}\underline{\xi}) dt dS(\underline{\xi}) ,$$

with $f(e^{it}\underline{\xi})^+$ the Hermitian conjugate on the complexified Clifford algebra.

In view of the definition of the modulation factor $\text{Mod}(\alpha, k; e^{it}\underline{\xi})$, the Lie norm $\|f\|_{L_2(LS^{m-1})}^2$ reduces to

$$\frac{1}{\pi} \sum_{k=0}^{\infty} \left\{ \int_0^\pi |F_1^{(k)}(e^{2it})|^2 dt - \left| \frac{k-\alpha}{2k+m} \right| \int_0^\pi |F_2^{(k)}(e^{2it})|^2 dt \right\} \|P_k\|_{L_2(S^{m-1})}^2 ,$$

with as before :

$$\begin{aligned} F_1^{(k)}(e^{2it}) &= F\left(\frac{k-\alpha}{2}, \frac{1+k-\alpha}{2}; k + \frac{m}{2}; e^{2it}\right) \\ F_2^{(k)}(e^{2it}) &= F_1^{(k+1)}(e^{2it}) . \end{aligned}$$

Note that

$$f(e^{it}\underline{\xi}) \in L_2(LS^{m-1}) \iff f(e^{it}\underline{\xi})\epsilon \in L_2(LS^{m-1}) ,$$

where $\|f\epsilon\|_{L_2(LS^{m-1})}^2$ is given by

$$\frac{1}{\pi} \sum_{k=0}^{\infty} \left\{ \int_0^\pi |F_1^{(k)}(e^{2it})|^2 dt + \left| \frac{k-\alpha}{2k+m} \right| \int_0^\pi |F_2^{(k)}(e^{2it})|^2 dt \right\} \|P_k\|_{L_2(S^{m-1})}^2 ,$$

whence $f(e^{it}\underline{\xi}) \in L_2(LS^{m-1})$ if both

$$\frac{1}{\pi} \left(\int_0^\pi |F_1^{(k)}(e^{2it})|^2 dt \|P_k\|_{L_2(S^{m-1})}^2 \right)_k \in l_1$$

and

$$\frac{1}{\pi} \left(\int_0^\pi |F_2^{(k)}(e^{2it})|^2 dt \|P_k\|_{L_2(S^{m-1})}^2 \right)_k \in l_1 .$$

The aim is now to find conditions on α such that

$$\begin{aligned} \frac{1}{\pi} \left(\int_0^\pi |F_1^{(k)}(e^{2it})|^2 dt \|P_k\|_{L_2(S^{m-1})}^2 \right)_k &\leq \left(|F_1^{(k)}(1)|^2 \|P_k\|_{L_2(S^{m-1})}^2 \right)_k \\ \frac{1}{\pi} \left(\int_0^\pi |F_2^{(k)}(e^{2it})|^2 dt \|P_k\|_{L_2(S^{m-1})}^2 \right)_k &\leq \left(|F_2^{(k)}(1)|^2 \|P_k\|_{L_2(S^{m-1})}^2 \right)_k , \end{aligned}$$

where the inequalities mean that starting from a certain $k_0 \in \mathbb{N}$, each term on the left-hand side has the corresponding term on the right-hand side as upper bound. For α such that $\operatorname{Re}(\alpha) > \frac{1-m}{2}$ both $F_1^{(k)}(1)$ and $F_2^{(k)}(1)$ converge for all $k \in \mathbb{N}$, respectively to the values

$$\begin{aligned} F_1^{(k)}(1) &= \frac{\Gamma\left(\alpha + \frac{m-1}{2}\right) \Gamma\left(k + \frac{m}{2}\right)}{\Gamma\left(\frac{\alpha+m+k-1}{2}\right) \Gamma\left(\frac{\alpha+m+k}{2}\right)} \\ F_2^{(k)}(1) &= \frac{\Gamma\left(\alpha + \frac{m-1}{2}\right) \Gamma\left(1 + k + \frac{m}{2}\right)}{\Gamma\left(\frac{\alpha+m+k}{2}\right) \Gamma\left(\frac{\alpha+m+k+1}{2}\right)} . \end{aligned}$$

This means that for $\alpha \in \mathbb{C}$ such that $\operatorname{Re}(\alpha) > \frac{1-m}{2}$ it suffices to prove that for k large enough, the integrals at the left-hand side are bounded from above by the modulus of these constants.

To so, we use the following : let a , b and c be *real* such that $c > b > 0$, $a > 0$ and $c - a - b > 0$. We then have Euler's integral representation formula for the hypergeometric function $F(a, b; c; e^{ix})$:

$$F(a, b; c; e^{ix}) = \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-te^{ix})^{-a} dt .$$

From this, it easily follows that

$$|F(a, b; c; e^{ix})| \leq \left| \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \right| \int_0^1 |t^{b-1} (1-t)^{c-b-1}| |(1-te^{ix})^{-a}| dt ,$$

which, in view of the fact that the parameters are real, reduces to

$$|F(a, b; c; e^{ix})| \leq \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} |(1-te^{ix})|^{-a} dt .$$

As $a > 0$ and

$$|1 - te^{ix}| = ((1 + t^2) - 2t \cos x)^{\frac{1}{2}} \geq 1 - t ,$$

we eventually find that

$$\begin{aligned} |F(a, b; c; e^{ix})| &\leq \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-t)^{-a} dt \\ &= F(a, b; c; 1) . \end{aligned}$$

In the present situation this means that for *real* α such that $\alpha > \frac{1-m}{2}$, we have that $f \in \mathcal{ML}_2^\alpha(S^{m-1}) \implies f \in L_2(LS^{m-1})$. Indeed, it suffices to choose $k_0 \in \mathbb{N}$ in such a way that $k_0 > \alpha$. For all $k \geq k_0$, the hypergeometric series $F_1^{(k)}(e^{2it})$ and $F_2^{(k)}(e^{2it})$ then satisfy the requirements to conclude that

$$\begin{aligned} \frac{1}{\pi} \int_0^\pi |F_1^{(k)}(e^{2it})|^2 dt \|P_k\|_{L_2(S^{m-1})}^2 &\leq |F_1^{(k)}(1)|^2 \|P_k\|_{L_2(S^{m-1})}^2 \\ \frac{1}{\pi} \int_0^\pi |F_2^{(k)}(e^{2it})|^2 dt \|P_k\|_{L_2(S^{m-1})}^2 &\leq |F_2^{(k)}(1)|^2 \|P_k\|_{L_2(S^{m-1})}^2 . \end{aligned}$$

In other words, for $\alpha \in \mathbb{R}$ such that $\alpha > \frac{1-m}{2}$ we have the inclusion :

$$\mathcal{ML}_2^\alpha(S^{m-1}) \subset L_2^+(LS^{m-1}) .$$

Note that this also means that the projection $D_\alpha(\underline{x})$ of the hyperbolic Dirac operator on the Klein ball is a differential operator of the Frobenius type, and hence we can construct a reproducing kernel for hyperbolic monogenic functions on the Klein ball belonging to the function space $\mathcal{ML}_2^\alpha(S^{m-1})$ for $\alpha > \frac{1-m}{2}$ if the inner product on $L_2^+(LS^{m-1})$ can be rewritten in terms of the inner product on the sphere S^{m-1} .

Remark : From these estimates it is also clear that for real $\alpha > \frac{1-m}{2}$ we have the following equivalence :

$$f(\underline{\omega}) \in \mathcal{ML}_2^\alpha(S^{m-1}) \iff f(\underline{\omega}) = \sum_{k=0}^{\infty} \text{Mod}(\alpha, k; \underline{\omega}) P_k(\underline{\omega}) .$$

Indeed, the " \implies " is a trivial consequence of Taylor's Theorem and for the " \impliedby " it suffices to put $\tilde{f}(\underline{x}) = \sum_{k=0}^{\infty} \text{Mod}(\alpha, k; \underline{x}) P_k(\underline{x})$, where the convergence is understood in the sense of the supremum norm on compact sets $K \subset B_m(1)$. The estimates above then guarantee that this series indeed converges and by construction we also have that $f(\underline{\omega}) = \lim_{r \rightarrow 1} \tilde{f}(r\underline{\omega})$.

Let us then determine this reproducing kernel, which will be denoted by $K_\alpha(\underline{x}, \underline{\omega})$ (where from now on $\alpha \in \mathbb{R}$ satisfies the condition $\alpha > \frac{1-m}{2}$), with

$$f(\underline{x}) = \int_{S^{m-1}} \overline{K_\alpha(\underline{x}, \underline{\omega})} f(\underline{\omega}) dS(\underline{\omega})$$

for $f \in \mathcal{ML}_2^\alpha(S^{m-1})$.

As $f \in \mathcal{ML}_2^\alpha(S^{m-1})$ belongs to $L_2^+(LS^{m-1})$, we have by definition for the complex holomorphic extension $f(\underline{z})$ on the Lie ball :

$$f(\underline{z}) = \frac{1}{A_m \pi} \int_0^\pi \int_{S^{m-1}} H^+(z, e^{it}\underline{\omega}) f(e^{it}\underline{\omega}) dS(\underline{\omega}) dt ,$$

which, by means of the definition for the Cauchy-Hua kernel, reduces to

$$f(r\underline{\xi}) = \frac{1}{A_m \pi} \sum_{k,l=0}^\infty \int_0^\pi \int_{S^{m-1}} (r\underline{\xi})^l r^k e^{-ikt} Z_k(\underline{\xi}, \underline{\omega}) (e^{it}\underline{\omega})^{-l} f(e^{it}\underline{\omega}) dS(\underline{\omega}) dt ,$$

where $\theta = \langle \underline{\xi}, \underline{\omega} \rangle$ and (see Definition 5.4)

$$Z_k(\underline{\xi}, \underline{\omega}) = C_k^{\frac{m}{2}}(\theta) + \underline{\xi} \underline{\omega} C_{k-1}^{\frac{m}{2}}(\theta)$$

and with $\underline{x} = r\underline{\xi} \in B_m(1)$. As $f \in \mathcal{ML}_2^\alpha(S^{m-1})$, we have by means of Theorem 5.12

$$f(\underline{x}) = \sum_{q=0}^\infty \text{Mod}(\alpha, q; \underline{x}) P_q(\underline{x}) ,$$

such that the function $f(r\underline{\xi})$ is for all $\underline{x} \in B_m(1)$ given by

$$\frac{1}{A_m \pi} \sum_{q,k,l=0}^\infty r^{k+l} \underline{\xi}^l \int_0^\pi \int_{S^{m-1}} e^{i(q-k-l)t} Z_k(\underline{\xi}, \underline{\omega}) \underline{\omega}^{-l} \text{Mod}(\alpha, q; e^{it}\underline{\omega}) P_q(\underline{\omega}) dS(\underline{\omega}) dt .$$

In view of the definition for the modulation factor, this reduces to the sum of two terms :

$$\Sigma_1 = \sum_{q,k,l=0}^\infty \frac{r^{k+l} \underline{\xi}^l}{A_m \pi} \int_0^\pi e^{i(q-k-l)t} F_1^{(q)}(e^{2it}) dt \int_{S^{m-1}} Z_k(\underline{\xi}, \underline{\omega}) \underline{\omega}^{-l} P_q(\underline{\omega}) dS(\underline{\omega})$$

and

$$\begin{aligned} \Sigma_2 = \sum_{q,k,l=0}^\infty \frac{q-\alpha}{(2q+m)} \frac{r^{k+l} \underline{\xi}^l}{A_m \pi} \int_0^\pi e^{i(1+q-k-l)t} F_2^{(q)}(e^{2it}) dt \quad \times \\ \int_{S^{m-1}} Z_k(\underline{\xi}, \underline{\omega}) \underline{\omega}^{-l} \underline{\omega}^\epsilon P_q(\underline{\omega}) dS(\underline{\omega}) . \end{aligned}$$

Consider for example the first term Σ_1 . Due to the orthogonality of spherical monogenics on the sphere and the fact that $Z_k(\underline{\xi}, \underline{\omega}) = \overline{Z_k(\underline{\omega}, \underline{\xi})}$, with $Z_k(\underline{\omega}, \underline{\xi})$ an inner spherical monogenic of order k , the integral in $\underline{\omega}$ only differs from zero if both $l \in 2\mathbb{N}$ and $q = k$. Hence,

$$\Sigma_1 = \sum_{k,l=0}^{\infty} \frac{r^{k+2l}}{A_m \pi} \int_0^\pi e^{-2ilt} F_1^{(k)}(e^{2it}) dt \int_{S^{m-1}} Z_k(\underline{\xi}, \underline{\omega}) P_k(\underline{\omega}) dS(\underline{\omega}) .$$

As

$$\int_0^\pi e^{-2ilt} F_1^{(k)}(e^{2it}) dt = \sum_{j=0}^{\infty} \frac{\left(\frac{k-\alpha}{2}\right)_j \left(\frac{1+k-\alpha}{2}\right)_j}{j! \left(k + \frac{m}{2}\right)_j} \int_0^\pi e^{-2i(j-l)t} dt ,$$

the summation in l disappears, and only the term for which $j = l$ remains. Note that this is a crucial step : it enables us to get rid of the variable t , and therefore we can rewrite the inner product on the Lie sphere in terms of the inner product on the sphere S^{m-1} . We thus find :

$$\Sigma_1 = \sum_{k=0}^{\infty} \frac{r^k}{A_m} F_1^{(k)}(r^2) \int_{S^{m-1}} Z_k(\underline{\xi}, \underline{\omega}) P_k(\underline{\omega}) dS(\underline{\omega}) .$$

In a completely similar way, we arrive at

$$\Sigma_2 = \sum_{k=0}^{\infty} \frac{k-\alpha}{2k+m} \frac{r^{1+k}}{A_m} F_2^{(k)}(r^2) \underline{\xi} \epsilon \int_{S^{m-1}} Z_k(\underline{\xi}, \underline{\omega}) P_k(\underline{\omega}) dS(\underline{\omega}) ,$$

from which it then immediately follows that

$$f(r\underline{\xi}) = \sum_{k=0}^{\infty} \text{Mod}(\alpha, k; r\underline{\xi}) P_k(r\underline{\xi}) ,$$

as was to be expected!

We then propose the following form for the reproducing kernel $K_\alpha(\underline{x}, \underline{\omega})$:

$$K_\alpha(\underline{\omega}, \underline{x}) = \sum_{k=0}^{\infty} r^k \frac{\text{Mod}(\alpha, k; r\underline{\xi}) Z_k(\underline{\xi}, \underline{\omega}) \overline{\text{Mod}(\alpha, k; \underline{\omega})}}{|F_1^{(k)}(1)|^2 - |F_2^{(k)}(1)|^2} .$$

This kernel satisfies the necessary conditions :

- In view of its very own definition, we have

$$K_\alpha(\underline{x}, \underline{y}) = \overline{K_\alpha(\underline{y}, \underline{x})} ,$$

which is the property of anti-symmetry.

- The kernel $K_\alpha(\underline{\xi}, \underline{\omega})$ belongs to the module $\mathcal{ML}_2^\alpha(S^{m-1})$.
- It has the reproducing property :

$$\begin{aligned} & \int_{S^{m-1}} \overline{K_\alpha(r\underline{\xi}, \underline{\omega})} f(\underline{\omega}) dS(\underline{\omega}) \\ &= \sum_{l=0}^{\infty} \int_{S^{m-1}} \overline{K_\alpha(r\underline{\xi}, \underline{\omega})} \text{Mod}(\alpha, l; \underline{\omega}) P_l(\underline{\omega}) dS(\underline{\omega}) , \end{aligned}$$

which by means of the orthogonality of spherical monogenics on the sphere reduces to

$$\int_{S^{m-1}} \overline{K_\alpha(r\underline{\xi}, \underline{\omega})} f(\underline{\omega}) dS(\underline{\omega}) = f(r\underline{\xi}) .$$

Theorem 6.1 *For real α such that $\alpha > \frac{1-m}{2}$, the space $\mathcal{ML}_2^\alpha(S^{m-1})$ is a Hilbert module with reproducing kernel, given by*

$$K_\alpha(\underline{\omega}, r\underline{\xi}) = \sum_{k=0}^{\infty} r^k \frac{\text{Mod}(\alpha, k; r\underline{\xi}) Z_k(\underline{\xi}, \underline{\omega}) \overline{\text{Mod}(\alpha, k; \underline{\omega})}}{|F_1^{(k)}(1)|^2 - |F_2^{(k)}(1)|^2} .$$

Remark : Note that for $\alpha = k$ the reproducing kernel $K_k(\underline{\omega}, \underline{\xi})$ reduces to the classical Szego kernel

$$K_k(\underline{\omega}, \underline{\xi}) = \sum_{k=0}^{\infty} Z_k(\underline{\xi}, \underline{\omega}) = \frac{1 + \underline{\xi}\underline{\omega}}{|1 + \underline{\xi}\underline{\omega}|^m} .$$

In other words, the reproducing kernel for the space of hyperbolic monogenics in the Klein ball can again be found as a modulation of the "classical" kernel for the Dirac operator on the flat Euclidean space \mathbb{R}^m , albeit only under certain restrictions on the parameter α .

Chapter 7

The Conformal Case

Mathematics is the art of giving the same name to different things. (J. H. Poincaré)

This Chapter is dedicated to the so-called *conformal Dirac operator* on the hyperbolic unit ball, which is well-known and has already extensively been studied throughout literature. This conformal operator is only a special case of the $\text{Spin}(1, m)$ -invariant operator studied in the thesis, as will be shown, which makes it possible to state some results for the conformal Dirac operator (e.g. a Modulation Theorem).

7.1 Dirac Operators on Manifolds

In this section we consider the Dirac operator on the hyperbolic unit ball from the classical point of view, i.e. we explain how the operator studied in the thesis must be understood in terms of the general theory of Dirac operators on manifolds.

A first possibility to define the Dirac operator on a general manifold is to study the *abstract metric manifold*, and in this sense follow the original idea of B. Riemann that a manifold exists in itself. This allows the study of manifolds without any notion of "exterior", which is the main idea behind differential geometry. In this theory vector bundles play an important central role : to define a Dirac operator on a general manifold it is equipped with a suitable structure, the so-called spinor bundle. Much research has been done on spinor Dirac operators (or Atiyah-Singer operators), see for instance [3], [6] and [71].

It is however well-known that Riemannian manifolds, with a certain degree of

smoothness, are isometrically embeddable in a Euclidean space of sufficiently high dimension, see reference [53]. This was later generalized to the pseudo-Riemannian case, see reference [14], and in some sense these two Theorems remove the principal objection to use *embedded manifolds*; that these are less general than abstract metric manifolds. A second approach to define a Dirac operator on a manifold fully exploits the idea of embedding the manifold isometrically in a (pseudo-)Euclidean space, making it possible to use the properties of the Dirac operator in the embedding space. This is actually the approach followed in the thesis, as will be shown.

The main difference between both approaches lies in the mathematical framework and language that is used. When working without an embedding, one needs to define a "comparison rule" to compare objects (such as tangent vectors) in different points of the manifold, which can be done by means of a *connection* on fibre bundles. When working with an embedding of the manifold one uses properties of the embedding space. For example, consider the sphere S^{m-1} as the $(m-1)$ -dimensional manifold embedded in \mathbb{R}^m . The curvature of this manifold can be determined implicitly, using nothing but the metric on this manifold, or it can be determined in terms of properties of the unit normal.

In this section we will construct the Atiyah-Singer spinor Dirac operator on H_+ and we will show that the $\mathbb{R}_{1,m}$ -valued hyperbolic solutions $F(T, \underline{X})$ considered in this thesis are so-called *Clifford sections*. We will explain how these sections can be related to the spinor bundle on H_+ . For that purpose H_+ will be considered as an m -dimensional manifold embedded in the flat Minkowski space-time $\mathbb{R}^{1,m}$. As a general reference to the rest of this section we refer to [16], where the general method was outlined, and [18].

In order to define spinor bundles, we first need the concept of a *spinor*. For that purpose we use the *regular representation* L of the Clifford algebra $\mathbb{R}_{1,m}$ which maps the algebra into its endomorphism algebra $\text{End}(\mathbb{R}_{1,m})$; that is, into the algebra of linear transformations on the vector space structure of the Clifford algebra :

$$L : \mathbb{R}_{1,m} \mapsto \text{End}(\mathbb{R}_{1,m}) : a \mapsto L(a) ,$$

with

$$L(a)b = ab \text{ for all } a, b \in \mathbb{R}^{1,m} .$$

This representation will of course not be irreducible, in the sense that certain vector subspaces will be preserved under multiplication from the left, namely

the left ideals. In other words, the minimal left ideals transform irreducible under the regular representation L . The mapping into the endomorphism algebra of any minimal left ideal induced by the regular representation is called the spinor representation and the minimal left ideal is called the *space of spinors*. A spinor space can be realized as a minimal left ideal $\mathbb{R}_{1,m}J$, with $J^2 = J$ a primitive idempotent of the algebra. The name spinor space is inspired by the fact that a spinor space can also be interpreted as a representation of the Spin group by left multiplication. To certain extent, a spinor can be seen as a column vector. Any associative algebra can be embedded into a total matrix algebra, and the columns of these matrices that carry an irreducible representation of the algebra may be considered as spinors.

Let us then follow the approach of [16] and define a unit normal field on H_+ as the mapping

$$N : H_+ \mapsto \mathbb{R}^{1,m} : \xi \mapsto \xi .$$

This unit normal field will be essential to the construction of the spin fibres, cfr. infra. The tangent space $T_\xi H_+$ in an arbitrary $\xi \in H_+$ is then given by

$$T_\xi H_+ = \{X \in \mathbb{R}^{1,m} : \langle X, \xi \rangle_{1,m} = 0\} .$$

As the restriction of the inner product $\langle \cdot, \cdot \rangle_{1,m}$ on $\mathbb{R}^{1,m}$ to $T_\xi H_+$ is non-degenerate, the Clifford algebra of this tangent space can be constructed. This subalgebra of $\mathbb{R}_{1,m}$ is denoted by $Cl(T_\xi H_+)$. In the point $\epsilon \in H_+$ we put $Cl(T_\epsilon H_+) = \mathbb{R}_{m,0}$, the Clifford algebra associated to the flat Euclidean space \mathbb{R}^m endowed with the quadratic form $Q_{m,0}(\underline{X}) = |\underline{X}|^2$. Note that this definition for the tangent space makes use of the embedding into $\mathbb{R}^{1,m}$, since we use a unit normal field, but in the literature tangent spaces are usually defined intrinsically as equivalence classes of curves on a manifold.

Let us then define the Spin bundle, by means of its fibres :

Definition 7.1 *The spin fibre σ_ξ in $\xi \in H_+$ is defined as*

$$\sigma_\xi = \{(\xi, \sigma) \in H_+ \times Spin(1, m) : \sigma \xi \bar{\sigma} = \epsilon\} .$$

The Spin bundle Σ is the union $\bigcup_\xi \sigma_\xi$ of all spin fibres over H_+ with topology inherited from $H_+ \times Spin(1, m)$.

In the language of bundles, we have the base space H_+ and the total space $H_+ \times Spin(1, m)$ with a projection π such that

$$\pi : H_+ \times Spin(1, m) \mapsto H_+ : \pi(\xi, \sigma) = \xi \text{ if } \sigma \xi \bar{\sigma} = \epsilon .$$

Recalling the mapping χ from the group $\text{Spin}(1, m)$ into the group $\text{SO}(1, m)$, we get for all $(\xi, \sigma) \in \sigma_\xi$ a mapping $\chi(\sigma)$ between two tangent spaces :

$$\chi(\sigma) : T_\xi H_+ \mapsto T_\epsilon H_+ .$$

Indeed, consider an arbitrary $X_0 \in T_\xi H_+$, such that $\langle X_0, \xi \rangle_{1,m} = 0$. In order to prove that $\chi(\sigma)X_0 \in T_\epsilon H_+$, whenever (ξ, σ) belongs to the Spin fibre σ_ξ , it suffices to prove that $\langle \chi(\sigma)X_0, \epsilon \rangle_{1,m} = 0$ or that $\langle X_0, \bar{\sigma}\epsilon\sigma \rangle_{1,m} = 0$. This is immediately clear from the definition of the Spin fibre : $(\xi, \sigma) \in \sigma_\xi$ implies that $\bar{\sigma}\epsilon\sigma = \xi$.

This means that for all $(\xi, \sigma) \in \sigma_\xi$ we have a mapping

$$\chi(\sigma) : Cl(T_\xi H_+) \mapsto Cl(T_\epsilon H_+) = \mathbb{R}_{m,0}$$

relating the Clifford algebras of tangent spaces at different points of H_+ .

Consequently, we also have a mapping τ between the Spin bundle Σ and the subgroup of $\text{Spin}(1, m)$ containing all elements mapping ϵ to an element of H_+ (since there are elements mapping ϵ to an element of H_- too), defined by

$$\tau : \Sigma \mapsto \text{Spin}(1, m) : (\xi, \sigma) \mapsto \sigma .$$

The inverse mapping is easily found to be

$$\tau^{-1} : \text{Spin}(1, m) \mapsto \Sigma : \sigma \mapsto (\bar{\sigma}\epsilon\sigma, \sigma) .$$

The Spin bundle can then be used to define the Clifford bundle :

Definition 7.2 *The Clifford fibre Cl_ξ in $\xi \in H_+$ is defined as*

$$Cl_\xi = \{ (\xi, a) \in H_+ \times \mathbb{R}_{1,m} : \sigma a \bar{\sigma} \in Cl(T_\epsilon H_+) = \mathbb{R}_{m,0} \text{ for } (\xi, \sigma) \in \sigma_\xi \} .$$

The Clifford bundle $Cl(\Sigma)$ is then defined as the union $\bigcup_\xi Cl_\xi$ of all Clifford fibres.

Again in the language of bundles we have a base space H_+ , a total space $H_+ \times \mathbb{R}_{1,m}$ and a projection π such that

$$\pi : H_+ \times \mathbb{R}_{1,m} \mapsto H_+ : (\xi, a) \mapsto \xi$$

if, for all σ in the spin fibre σ_ξ , we have $\sigma a \bar{\sigma} \in Cl(T_\epsilon) = \mathbb{R}_{m,0}$.

A *Clifford section* is a mapping $s : H_+ \mapsto H_+ \times \mathbb{R}_{1,m}$ such that $\pi \circ s = 1_{H_+}$. In other words, a section of the Clifford bundle is a function F on H_+ such that

$$(\xi, F(\xi)) \in Cl_\xi \text{ for all } \xi \in H_+ .$$

It is important to note that, without explicitly mentioning, we have been using the concept of Clifford sections on $Cl(\Sigma)$ throughout this whole thesis. Indeed, consider for example the construction of the hyperbolic fundamental solution $E_\alpha(\xi, \eta)$. This happened in two steps :

- First we have projected the hyperbolic Dirac equation onto the hyperplane $\Pi \leftrightarrow T = 1$. This is of course nothing but the tangent space $T_\epsilon H_+$ whence the functions constructed by means of the projection on the Klein model for the hyperbolic unit ball automatically take their values in the Clifford algebra $Cl(T_\epsilon H_+)$ if they are evaluated at ϵ .

This argument also explains why we have identified $Cl(T_\epsilon H_+)$ with $\mathbb{R}_{m,0}$: in projecting the hyperbolic Dirac equation on Π one obtains an equation whose solutions can be written as modulated versions of homogeneous monogenics with respect to the operator $\underline{\partial}$ on $\mathbb{R}^{0,m}$. These modulation factors, for which we refer to Chapter 3, are power series in the bivector-valued variable $\underline{x}\epsilon$. This variable can be identified with a vector variable $\underline{u} \in \mathbb{R}^m$ such that $|\underline{u}|^2 = \underline{u}^2$. In other words : these hyperbolic functions take their values in $\mathbb{R}_{m,0} \cong \mathbb{R}_{1,m}^{(+)}$, the latter algebra being generated by ϵe_j ($1 \leq j \leq m$).

- In a second step we mentioned the fact that ϵ does not play a privileged role in this construction : the solution in another point $\eta \in H_+$ can be obtained by replacing $\epsilon \leftrightarrow \eta$, i.e. by applying a $\text{Spin}(1, m)$ transformation sending $\epsilon \mapsto \eta$. This means that the functions at each point of H_+ take their values in the Clifford algebra of the tangent space at this point, turning these functions by definition into Clifford sections. The Spin bundle actually formalizes this idea : ϵ is chosen as a reference point and the spin fibre σ_η at each $\eta \in H_+$ is used to "translate" the results at ϵ to results at η , by means of a $\text{Spin}(1, m)$ -transformation.

In a completely similar way as for the Spin bundle, we can define the *spinor bundle*. The only difference with the Spin bundle Σ is the fact that we now use the regular representation for the Spin group, by left multiplication. As the author notes in [16], the name 'spinor bundle' is somewhat abused in the sequel : classically in the literature the fibre of the spinor bundle in a

point gives an irreducible representation of the Clifford fibre in that point, while here it is defined in such a way that it is isomorphic to the Clifford fibre. However, referring to the definition of a spinor, it is clear that one can obtain an irreducible spinor bundle by means of a suitable idempotent J at the right.

Definition 7.3 *The spinor fibre S_ξ in $\xi \in H_+$ is defined as*

$$S_\xi = \{(\xi, \psi) \in H_+ \times \mathbb{R}_{1,m} : \sigma\psi \in Cl(T_\epsilon H_+) \text{ for } (\xi, \sigma) \in \sigma_\xi\} .$$

The spinor bundle \mathbf{S} is then defined as the union $\bigcup_\xi S_\xi$ of spinor fibres.

Note that in some sense the spinor fibre S_ξ consists of "square roots" of the Clifford fibre Cl_ξ , and hence of the Spin fibre σ_ξ . Indeed, for all $(\xi, \sigma) \in \sigma_\xi$ we have

$$\begin{aligned} (\xi, \psi) \in S_\xi &\implies \sigma\psi \in Cl(T_\epsilon H_+) \\ &\implies \sigma\psi\bar{\sigma}\psi \in Cl(T_\epsilon H_+) \\ &\implies \sigma\psi\bar{\psi}\bar{\sigma} \in Cl(T_\epsilon H_+) \end{aligned}$$

which means that $(\xi, \psi\bar{\psi}) \in Cl_\xi$.

Sections of the spinor bundle, or *spinor fields*, are defined in the natural way as mappings from the base space H_+ to the total space such that the image of $\xi \in H_+$ under the section belongs to the spinor fibre S_ξ . Let us denote the space of C^∞ spinor fields as $\Gamma^\infty(\mathbf{S})$. This space is a right Clifford module, for if $f \in \Gamma^\infty(\mathbf{S})$ we have for all $(\xi, \sigma) \in \sigma_\xi$ that $\sigma f(\xi) \in Cl(T_\epsilon H_+) = \mathbb{R}_{m,0}$. So, if $a \in \mathbb{R}_{m,0}$ we also have that $\sigma f(\xi)a \in \mathbb{R}_{m,0}$ which means that $fa \in \Gamma^\infty(\mathbf{S})$.

The following Theorem characterizes the spinor sections :

Theorem 7.1 *A function f on H_+ is a spinor section if and only if*

$$\xi f(\xi) = \tilde{f}(\xi)\epsilon \text{ for all } \xi \in H_+ .$$

Proof : By definition f is a spinor section if $\sigma f(\xi) \in \mathbb{R}_{m,0}$ for all ξ , with σ in the spin fibre σ_ξ . As the Spin structure is defined in such a way that $\sigma\xi\bar{\sigma} = \epsilon$ we get that $\xi = \bar{\sigma}\epsilon\sigma$. On the other hand, a belongs to $Cl(T_\epsilon H_+)$ if and only if $\epsilon a = \tilde{a}\epsilon$. If we apply this to $a = \sigma f(\xi)$, we get

$$\sigma\tilde{f}(\xi)\epsilon = \epsilon\sigma f(\xi) = \sigma\xi f(\xi) ,$$

if and only if f is a spinor section. Left multiplication with σ^{-1} completes the proof. \square

As an example of this Theorem we note that

$$f(\xi) = \xi(\xi + \epsilon) = (\xi + \epsilon)\epsilon = 1 + \xi\epsilon$$

is a spinor section, because it is easily checked that $\xi f(\xi) = \widetilde{f}(\xi)\epsilon$.

Next, we define a connection on the spinor bundle **S**. We mentioned that a connection is necessary to compare objects in different points of the manifold. This can best be understood in terms of the Clifford bundle : the comparison between two elements of the Clifford bundle at different points ξ and η of H_+ seems obvious, given the fact that the Clifford algebras of the tangent spaces at these points belong to the enveloping algebra of the embedding space. The logical infinitesimal comparison operator is thus given by the ordinary derivation operator $\partial_{\xi, \underline{x}} f$ in the direction \underline{x} , where (ξ, \underline{x}) belongs to the tangent bundle $TH_+ = \bigcup_{\xi} T_{\xi} H_+$, defined as

$$\partial_{\xi, \underline{x}} f = \partial_t F(\xi + t\underline{x})|_{t=0}$$

with F an arbitrary smooth extension of f in a neighbourhood of ξ in $\mathbb{R}^{1,m}$. However, the result of this derivation is not necessarily an element of the tangent bundle itself, and this leads to an operator

$$\nabla_{\xi, \underline{x}} f = P_{\xi} \partial_{\xi, \underline{x}} f ,$$

with P_{ξ} the orthogonal projection on the Clifford algebra $Cl(T_{\xi})$ of the tangent space at ξ .

In a similar way, a connection on the spinor bundle can be defined. Let f be a smooth spinor section and let \underline{x} be a tangent vector at $\xi \in H_+$, i.e. let $(\xi, \underline{x}) \in TH_+$. Denoting the orthogonal projection on the spinor fibre by Π_{ξ} we put

$$\nabla_{\xi, \underline{x}} f = \Pi_{\xi} \partial_{\xi, \underline{x}} f$$

Note that the projection on the spinor fibre of an arbitrary Clifford algebra-valued function on the hyperbolic unit ball is given by

$$\Pi f = \frac{1}{2}(f + \xi \widetilde{f} \epsilon) .$$

Indeed, we have $\xi \Pi f(\xi) = \widetilde{\Pi f}(\xi) \epsilon$ and $\Pi^2 f = \Pi f$.

Taking into account that for spinor sections we have $f\epsilon = \xi\tilde{f}$ we find

$$\begin{aligned}\nabla_{\xi,\underline{x}}f &= \frac{1}{2} \left(\partial_{\xi,\underline{x}}f + \xi(\partial_{\xi,\underline{x}}\tilde{f})\epsilon \right) \\ &= \frac{1}{2} \left(\partial_{\xi,\underline{x}}f + \partial_{\xi,\underline{x}}(\xi\tilde{f})\epsilon - (\partial_{\xi,\underline{x}}\xi)\tilde{f}\epsilon \right) ,\end{aligned}$$

where we have used the chain rule for derivation. This can then further be reduced to

$$\begin{aligned}\nabla_{\xi,\underline{x}}f &= \frac{1}{2}\partial_{\xi,\underline{x}}f + \frac{1}{2}\partial_{\xi,\underline{x}}(f\epsilon) - \frac{1}{2}(\partial_{\xi,\underline{x}}\xi)\xi(\tilde{f}\epsilon) \\ &= \partial_{\xi,\underline{x}}f - \frac{1}{2}\underline{x}\xi f ,\end{aligned}$$

where we have used that $\partial_{\xi,\underline{x}}\xi = \partial_t(\xi + t\underline{x})|_{t=0} = \underline{x}$.

The spinor Dirac operator in ξ can then be defined as follows : let e_1, \dots, e_m be an orthonormal basis for the tangent space $T_\xi H_+$. For $f \in \Gamma^\infty(\mathbf{S})$ we then put

$$\begin{aligned}\nabla f(\xi) &= \sum_{j=1}^m e_j \nabla_{\xi,e_j} f \\ &= \sum_{j=1}^m \left(e_j \partial_{\xi,e_j} f - e_j \frac{e_j \xi}{2} f \right) \\ &= \sum_{j=1}^m e_j \partial_{\xi,e_j} f - \frac{m}{2} \xi f .\end{aligned}$$

If we then introduce the Dirac operator ∂_X on $\mathbb{R}^{1,m}$, its hyperbolic polar decomposition given by

$$\partial_X = \xi \left(\partial_\rho + \frac{1}{\rho} \Gamma_{1,m} \right) ,$$

and if we note that the set (ξ, e_1, \dots, e_m) forms an orthogonal basis for $\mathbb{R}^{1,m}$ we have that

$$\frac{\xi}{\rho} \Gamma_{1,m} = \sum_{j=1}^m e_j \partial_{\xi,e_j} .$$

This means that the Atiyah-Singer spinor Dirac operator in $\xi \in H_+$ is given by

$$\nabla f(\xi) = \xi \left(\Gamma_{1,m} - \frac{m}{2} \right) f(\xi) .$$

In other words : the Dirac operator on the hyperbolic unit ball constructed within the framework of differential geometry concides with the one we have studied in the thesis, i.e. the Dirac operator on $\mathbb{R}^{1,m}$ acting on α -homogeneous functions $F(T, \underline{X})$, if we choose $\alpha = -\frac{m}{2}$.

This also means that hyperbolic monogenics $f(\xi) \in \mathcal{H}^\alpha(\Omega)$, with $\Omega \subset H_+$ and $\alpha \neq -\frac{m}{2}$, give rise to (local) eigensections for the spinor Dirac operator on H_+ (see [17]). Indeed, suppose $f \in \mathcal{H}^\alpha(\Omega)$ such that $\xi(\Gamma_{1,m} + \alpha)f = 0$ for all $\xi \in \Omega$. We then have the following eigensections for the spinor Dirac operator ∇ :

$$\nabla(1 \pm i\xi)f = \pm i \left(\alpha + \frac{m}{2} \right) (1 \pm i\xi)f .$$

On the other hand, if $f(\xi)$ is an eigensection for the spinor Dirac operator

$$\nabla f(\xi) = \lambda f(\xi) ,$$

we have the following hyperbolic monogenics :

$$\xi \left(\Gamma_{1,m} - \left(\frac{m}{2} \pm i\lambda \right) \right) (1 \mp i\xi)f = 0 \implies (1 \mp i\xi)f \in \mathcal{H}^{\mp i\lambda - \frac{m}{2}}(H_+) .$$

Hence eigenfunctions for the hyperbolic angular operator $\Gamma_{1,m}$, which is the Dirac operator on H_+ induced by the Dirac operator ∂_X on the embedding space $\mathbb{R}^{1,m}$, correspond to eigenfunctions of the spinor Dirac operator on H_+ with shifted eigenvalues.

As mentioned in [17], this also means that sections of the spinor bundle can be considered as Clifford algebra valued functions on the hyperbolic unit ball. However, it is important to distinguish clearly between *eigenfunctions* of ∇ considered as an operator on $C^\infty(H_+)$ and *eigensections* of ∇ acting in $\Gamma^\infty(\mathbf{S})$. When looking for eigensections it suffices to describe all eigenfunctions and then pick out those which are sections, hereby making use of Theorem 7.1.

7.2 The Conformal Fundamental Solution

In this section the hyperbolic fundamental solution $E_\alpha(\xi, \eta)$ in case $\alpha = -\frac{m}{2}$ is determined, to obtain the fundamental solution for the "classical" Dirac operator on H_+ and it is explained why this operator is often referred to as the *conformal Dirac operator*. This will be done by means of a Moebius transformation.

Recalling the Definition 5.1 for the hyperbolic fundamental solution, we get immediately :

$$E_{-\frac{m}{2}}(\xi, \eta) = \frac{e^{-i\pi\frac{m+1}{2}}}{\pi^{\frac{m-1}{2}}} \Gamma\left(\frac{m+1}{2}\right) \left\{ D_{-\frac{m}{2}-1}^{\frac{m+1}{2}}(\tau)\xi - D_{-\frac{m}{2}}^{\frac{m+1}{2}}(\tau)\eta \right\}.$$

Using formula (22) to rewrite these Gegenbauer functions in terms of the Legendre function of the second kind, we get :

$$D_{-\frac{m}{2}-1}^{\frac{m+1}{2}}(\tau) = \frac{e^{i\pi(m+\frac{1}{2})}\Gamma\left(\frac{m}{2}\right)}{2^{\frac{m}{2}}\pi^{\frac{1}{2}}\Gamma\left(\frac{m+1}{2}\right)}(\tau^2 - 1)^{-\frac{m}{4}} \frac{Q_{-1}^{-\frac{m}{2}}(\tau)}{\Gamma\left(-\frac{m}{2}\right)}$$

and

$$D_{-\frac{m}{2}}^{\frac{m+1}{2}}(\tau) = \frac{e^{i\pi(m+\frac{1}{2})}\Gamma\left(1+\frac{m}{2}\right)}{2^{\frac{m}{2}}\pi^{\frac{1}{2}}\Gamma\left(\frac{m+1}{2}\right)}(\tau^2 - 1)^{-\frac{m}{4}} \frac{Q_0^{-\frac{m}{2}}(\tau)}{\Gamma\left(1-\frac{m}{2}\right)}.$$

Next, we apply formula (14) to write the Legendre functions of the second kind in terms of a hypergeometric function with argument $1 - \tau^{-2}$. As we are looking for a *fundamental* solution we will only use that part which has a singularity for $\xi = \eta$, i.e. for $\tau = 1$. This is allowed because the part that will be omitted, which is regular for $\tau = 1$, is a solution for the hyperbolic Dirac equation too : it is the function $T^\alpha \text{Mod}(\alpha, 0, \underline{x})$. We thus have :

$$\begin{aligned} \frac{Q_{-1}^{-\frac{m}{2}}(\tau)}{\Gamma\left(-\frac{m}{2}\right)} &= \frac{2^{\frac{m}{2}-1}}{e^{i\pi\frac{m}{2}}} \frac{t^{\frac{m}{2}-1}}{(t^2 - 1)^{\frac{m}{4}}} F\left(\frac{1}{2} - \frac{m}{4}, 1 - \frac{m}{4}; 1 - \frac{m}{2}; 1 - \frac{1}{\tau^2}\right) \\ &+ \text{Regular Function} \end{aligned}$$

and

$$\begin{aligned} \frac{Q_0^{-\frac{m}{2}}(\tau)}{\Gamma\left(1-\frac{m}{2}\right)} &= \frac{2^{\frac{m}{2}-1}}{e^{i\pi\frac{m}{2}}} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(1+\frac{m}{2}\right)} \frac{t^{\frac{m}{2}}}{(t^2 - 1)^{\frac{m}{4}}} F\left(-\frac{m}{4}, \frac{1}{2} - \frac{m}{4}; 1 - \frac{m}{2}; 1 - \frac{1}{\tau^2}\right) \\ &+ \text{Regular Function} . \end{aligned}$$

Using the following formulae, for which we refer to [35] ,

$$\left(\frac{1 + (1-z)^{\frac{1}{2}}}{2}\right)^{1-2a} = F\left(a - \frac{1}{2}, a; 2a; z\right) = (1-z)^{\frac{1}{2}} F\left(a, a + \frac{1}{2}; 2a; z\right)$$

we then eventually find :

$$E_{-\frac{m}{2}}(\xi, \eta) = \frac{1}{2^{\frac{m}{2}} A_m} \frac{\xi - \eta}{(\tau - 1)^{\frac{m}{2}}}.$$

This result was also obtained by Ryan and his collaborators by means of a Cayley transformation, for which we refer to [58] and [59]. We briefly sketch this approach here, by introducing the Cayley transformation

$$C : \mathbb{R}^m \mapsto \mathbb{R}^{1,m} : \underline{x} \mapsto \frac{\epsilon - \underline{x}}{1 + \epsilon \underline{x}}$$

mapping the unit ball $B_m(1)$ in \mathbb{R}^m conformally onto the hyperbolic unit ball H_+ . Indeed, one can easily verify that the inner product $C(\underline{x}) \cdot C(\underline{x}) = 1$ on $\mathbb{R}_{1,m}$ whence $C(\underline{x}) \in H_+$. To see that this mapping is conformal, it suffices to rewrite the image of $\underline{x} \in B_m(1)$ under the Cayley transformation as

$$C(\underline{x}) = \frac{\epsilon - \underline{x}}{1 - |\underline{x}|^2}(1 - \epsilon \underline{x}) = \frac{-2\underline{x}}{1 - |\underline{x}|^2} + \frac{1 + |\underline{x}|^2}{1 - |\underline{x}|^2} \epsilon .$$

Recalling the mapping from (H_+, ds_H^2) onto the Poincaré model $(B_m(1), ds_P^2)$ of the hyperbolic unit ball, for which we refer to Chapter 1, it is immediately clear that the Cayley transformation is actually the inverse mapping : a point $\underline{x} \in B_m(1)$ is projected vertically upwards onto the parabola \mathcal{P} and the intersection of the ray through this point with H_+ gives $C(\underline{x})$. This means that there is a metric equivalence between $(B_m(1), ds_P^2)$ and (H_+, ds_H^2) under the Cayley transformation. Since the Poincaré metric is conformally equivalent with the standard Euclidean metric on $B_m(1)$, the Cayley transformation maps the unit ball $B_m(1) \subset \mathbb{R}^m$ provided with the Euclidean metric ds_E^2 conformally onto the hyperbolic unit ball $H_+ \subset \mathbb{R}^{1,m}$ with its natural metric inherited from the Minkowski metric.

In [58] the author then considers two functions $f(\underline{x})$ and $g(\underline{x})$ defined on an open domain $\Omega \subset \mathbb{R}^m$. If Σ is a Lipschitz surface in Ω bounding a subregion of Ω , where the Lipschitz condition by definition means that there exists an outward pointing normal for almost all $\underline{x} \in \Sigma$, and if $\underline{\partial} f = 0 = g \underline{\partial}$ the following holds :

$$\int_{\Sigma} g(\underline{x}) \sigma(\underline{x}, d\underline{x}) f(\underline{x}) = 0 .$$

Making use of the Cayley transformation to rewrite this integral in terms of the variable $\xi = C(\underline{x}) \in C(\Sigma) \subset H_+$, this yields :

$$\int_{C(\Sigma)} g(C^{-1}(\xi)) \tilde{J}(C^{-1}, \xi) \Sigma(\xi, d\xi) J(C^{-1}, \xi) f(C^{-1}(\xi)) = 0 ,$$

with $\tilde{J}(C^{-1}, \xi) \Sigma(\xi, d\xi) J(C^{-1}, \xi)$ the vector-valued measure on $C(\Sigma)$. The factor $J(C^{-1}, \xi)$, for which we refer to the introductory Chapter, is given by

$$J(C^{-1}, \xi) = \frac{1 + \epsilon \tilde{\xi}}{[(1 + \epsilon \tilde{\xi})(1 + \epsilon \xi)]^{\frac{m}{2}}} .$$

Denoting the region bounded by the surface Σ as V and applying Stokes' Theorem to the integral above finally gives

$$\begin{aligned} & \int_{C(\Sigma)} g(C^{-1}(\xi)) \tilde{J}(C^{-1}, \xi) \Sigma(\xi, d\xi) J(C^{-1}, \xi) f(C^{-1}(\xi)) \\ &= \int_{C(V)} [g(C^{-1}(\xi)) \tilde{J}(C^{-1}, \xi) D_{H_+}] J(C^{-1}, \xi) f(C^{-1}(\xi)) L(\xi, d\xi) \\ &+ \int_{C(V)} g(C^{-1}(\xi)) \tilde{J}(C^{-1}, \xi) [D_{H_+} J(C^{-1}, \xi) f(C^{-1}(\xi))] L(\xi, d\xi) , \end{aligned}$$

where D_{H_+} stands for the Dirac operator on H_+ arising from the application of Stokes' Theorem and where $L(\xi, d\xi)$ denotes the Lebesgue measure on H_+ . As Σ is arbitrary, it follows that

$$D_{H_+} J(C^{-1}, \xi) f(C^{-1}(\xi)) = 0 = g(C^{-1}(\xi)) \tilde{J}(C^{-1}, \xi) D_{H_+} .$$

This allows to conclude that a function $f(\xi)$ is left (resp. right) monogenic with respect to the operator D_{H_+} acting from the left (resp. the right) if and only if the function

$$J(C, \underline{y}) f(C(\underline{y}))$$

is left (resp. right) monogenic with respect to the Dirac operator $\underline{\partial}$ on \mathbb{R}^m , where $C(\underline{x}) = \xi$. We also encountered this in Chapter 3, when we explained why for $\alpha = -\frac{m}{2}$ there is a one-to-one correspondence between monogenic functions with respect to the operator $\underline{\partial}$ on \mathbb{R}^m and hyperbolic monogenics.

The author then applies this to the Cauchy kernel $E(\underline{x})$, which is both left and right monogenic on \mathbb{R}_0^m , hereby making use of the following essential relation :

$$E(C(\underline{x}) - C(\underline{y})) = \tilde{J}(C, \underline{x})^{-1} E(\underline{x} - \underline{y}) J(C, \underline{y})^{-1} ,$$

for which he refers to [55]. Putting $C(\underline{x}) = \xi \in H_+$ and $C(\underline{y}) = \eta \in H_+$, Ryan thus arrives at the fundamental solution

$$E(\xi - \eta) = \frac{1}{A_m} \frac{\xi - \eta}{|\xi - \eta|^m} ,$$

which by means of the fact that

$$|\xi - \eta|^2 = (\xi - \eta)(\tilde{\xi} - \tilde{\eta}) = 2\tau - 2 ,$$

reduces to

$$E(\xi - \eta) = \frac{1}{2^{\frac{m}{2}} A_m} \frac{\xi - \eta}{(\tau - 1)^{\frac{m}{2}}} .$$

This explains why the operator

$$D_{H_+} = \xi \left(\Gamma_{1,m} - \frac{m}{2} \right)$$

is referred to as the *conformal Dirac operator on the hyperbolic unit ball* : it arises very naturally by considering a Moebius transformation. Note that this conformal operator is only a special case of the operator we have studied in the thesis. The main reason for this is the following : the hyperbolic Dirac operator is invariant under the group $\text{Spin}(1, m)$ which is only a subgroup of $\text{Spin}(1, m + 1)$, this latter group being isomorphic to the Moebius group. So in considering a subgroup of this conformal group we obtain a richer class of invariant operators, and hence a richer class of functions.

7.3 Conformal Invariance and the Nullcone

In this section we show how to obtain the conformal character of the operator

$$\xi \left(\Gamma - \frac{m}{2} \right)$$

by means of the isomorphy of the Moebius group with the $\text{Spin}(1, m + 1)$. For that purpose we need some basic notions from the theory of Dirac operators on nullcones, see [66]. We preferred not to put this into Chapter 0 because it can be seen as a non-trivial continuation of the theory we have developped throughout the thesis.

Consider the nullcone $NC_{1,m+1}$ in $\mathbb{R}^{1,m+1}$ provided with the co-ordinates $(T, \underline{X}, X_{m+1})$. This nullcone can be described by means of a projective model, by putting

$$\text{Ray}(NC_{1,m+1}) = \{ \{ \lambda(\epsilon, \omega) : \lambda \in \mathbb{R}_0^+ \} : \omega \in S^m \} ,$$

or equivalently as a principal bundle by considering the free action of \mathbb{R}_0^+ on $NC_{1,m+1}$ and identifying the orbit space with the manifold of nullrays.

Hence, functions on the nullcone will have to be homogeneous with respect to the co-ordinates $(T, \underline{X}, X_{m+1})$. There is however a serious constraint on the degree of homogeneity α , as was pointed out by Sommen (in joint work with Souček). Let Ω be an open *conical* subset of $\mathbb{R}_0^{1,m+1}$, i.e. $\Omega = \lambda\Omega$ for positive λ , and let $N\Omega$ be the intersection $\Omega \cap NC_{1,m+1}$. If $f(X, X_{m+1})$ is a given smooth α -homogeneous function in $N\Omega$, then f admits smooth α -homogeneous extensions to conical subsets $\Omega \subset \mathbb{R}_0^{1,m+1}$ and any two of them will satisfy

$$f_1(X, X_{m+1}) - f_2(X, X_{m+1}) = (X^2 - X_{m+1}^2)g(X, X_{m+1})$$

for some smooth function g . Hence, if $\mathcal{E}_\alpha(\mathcal{U})$ denotes the set of smooth α -homogeneous functions in \mathcal{U} we have the isomorphism

$$\mathcal{E}_\alpha(N\Omega) \cong \mathcal{E}_\alpha(\Omega)/(X^2 - X_{m+1}^2)\mathcal{E}_{\alpha-2}(\Omega)$$

where the projection operator from $\mathcal{E}_\alpha(\Omega) \mapsto \mathcal{E}_\alpha(N\Omega)$ is the restriction $f \mapsto f|_{NC}$ and where $X^2 = T^2 - |\underline{X}|^2$.

In [66] the author considered a very natural refinement of this restriction operator when working within the context of Clifford analysis :

$$f_1 \sim f_2 \Leftrightarrow f_1(X, X_{m+1}) - f_2(X, X_{m+1}) = (X + X_{m+1}e_{m+1})g(X, X_{m+1}) ,$$

which also implies that $f_1|_{NC} = f_2|_{NC} + (X + X_{m+1}e_{m+1})g|_{NC}$. This leads to the following definition :

Definition 7.4 *Let α be an arbitrary complex number and let Ω be an open conical subset of $\mathbb{R}_0^{1,m+1}$. We then define*

$$\begin{aligned} \mathcal{G}_\alpha(N\Omega) &\cong \mathcal{E}_\alpha(\Omega)/(X + X_{m+1}e_{m+1})\mathcal{E}_{\alpha-1}(\Omega) \\ &\cong \mathcal{E}_\alpha(N\Omega)/(X + X_{m+1}e_{m+1})\mathcal{E}_{\alpha-1}(N\Omega) . \end{aligned}$$

We now want to define the action of the Dirac operator ∂ on $\mathbb{R}^{1,m+1}$, given by

$$\partial = \partial_X - e_{m+1}\partial_{X_{m+1}} = \epsilon\partial_T - \sum_{j=1}^{m+1} e_j\partial_{X_j} ,$$

on this set $\mathcal{G}_\alpha(N\Omega)$. First of all, note that it doesn't make sense to define the Dirac operator as an operator from $\mathcal{E}_\alpha(N\Omega)$ to $\mathcal{E}_{\alpha-1}(N\Omega)$ because

$$\partial((X^2 - X_{m+1}^2)g) = 2(X + X_{m+1}e_{m+1})g + (X^2 - X_{m+1}^2)\partial g .$$

It *does* however make sense to define ∂ from $\mathcal{E}_\alpha(N\Omega) \mapsto \mathcal{G}_{\alpha-1}(N\Omega)$, i.e. to call a function $f(X, X_{m+1}) \in \mathcal{E}_\alpha(N\Omega)$ (left) monogenic if

$$\partial f = (X + X_{m+1}e_{m+1})g$$

for some $g(X, X_{m+1}) \in \mathcal{E}_{\alpha-2}(N\Omega)$. But in view of the fact that

$$\partial((X + X_{m+1}e_{m+1})g) = 2\left(\mathbb{E}_{1,m+1} + 1 + \frac{m}{2}\right)g - (X + X_{m+1}e_{m+1})\partial g,$$

where $\mathbb{E}_{1,m+1}$ denotes the Euler operator on $\mathbb{R}^{1,m+1}$, it *only* makes sense to consider the operator

$$\partial : \mathcal{G}_\alpha(N\Omega) \mapsto \mathcal{G}_{\alpha-1}(N\Omega)$$

for $\alpha = -\frac{m}{2}$. Indeed, only for $g \in \mathcal{E}_{-\frac{m}{2}-1}(N\Omega)$ we get that

$$\partial(f + (X + X_{m+1}e_{m+1})g) = \partial f - (X + X_{m+1}e_{m+1})\partial g.$$

We are thus lead to the following :

Definition 7.5 *The Dirac operator on the nullcone in $\mathbb{R}^{1,m+1}$ is defined as the operator ∂ acting on the restriction of $(-\frac{m}{2})$ -homogeneous functions $f(X, X_{m+1})$ to the nullcone.*

In [66] the author then develops a function theory for this operator, by deriving a Cauchy formula on the nullcone, but this is beyond the scope of the thesis. For our purposes it suffices to remember the definition of the Dirac operator on the nullcone.

Consider then a hyperbolic monogenic, i.e. $F(\xi) \in \mathcal{H}_\alpha(\Omega)$ with $\Omega \subset H_+$, for $\alpha = -\frac{m}{2}$. This means that

$$\begin{cases} \partial_X F(X) &= 0 \\ \mathbb{E}_{1,m} F(X) &= |X|^{-\frac{m}{2}} F(\underline{x}) \end{cases}$$

By considering the *constant extension* of this function in the e_{m+1} -direction, we obtain a $(-\frac{m}{2})$ -homogeneous solution for the operator ∂ on $\mathbb{R}^{1,m+1}$ which after restriction to $NC_{1,m+1}$ gives rise to a solution for the Dirac operator on the nullcone. On this function $F(X, X_{m+1})|_{NC}$ the group $\text{Spin}(1, m+1)$ may act by means of the $H(s)$ -representation.

We will now show that only for $\alpha = -\frac{m}{2}$ this identification between monogenic functions on the nullcone and hyperbolic monogenics also works in

the opposite direction. For that purpose we start from a solution for the Dirac operator on $NC_{1,m+1}$, i.e. the restriction to the nullcone of a $(-\frac{m}{2})$ -homogeneous solution $F(X, X_{m+1})$ for the operator ∂ on $\mathbb{R}^{1,m+1}$. Our goal now is to construct a uniquely determined $(-\frac{m}{2})$ -homogeneous solution $\tilde{F}(X)$ for the Dirac operator ∂_X on $\mathbb{R}^{1,m}$, determined by the function $F(X, X_{m+1})$.

As monogenic functions on $NC_{1,m+1}$ are defined up to a function

$$(X + X_{m+1}e_{m+1})h(X, X_{m+1})$$

it suffices to find $h(X, X_{m+1})$ such that

$$\partial_X \left(F(X, |X|) + |X|(\xi + e_{m+1})h(X, |X|) \right) = 0 .$$

Note that the function between brackets represents the *constant* extension of the function $F(X, X_{m+1})|_{NC}$ in the e_{m+1} -direction. In other words, we have to construct a $(-\frac{m}{2})$ -homogeneous function $\tilde{h}(X, |X|)$ such that

$$\partial_X \left(F(X, |X|) + (\xi + e_{m+1})\tilde{h}(X, |X|) \right) = 0 .$$

This goes as follows : we know that both $F(X, X_{m+1})$ and $F(X, |X|)$ are smooth extensions of the function $F|_{NC}$, whence

$$F(X, X_{m+1}) - F(X, |X|) = (X + X_{m+1}e_{m+1})^2 h(X, X_{m+1}) .$$

From this we get that

$$\partial F(X, X_{m+1}) = \partial \left(F(X, |X|) + (X + X_{m+1}e_{m+1})^2 h(X, X_{m+1}) \right) = 0 ,$$

such that

$$\partial_X F(X, |X|) = -(X + X_{m+1}e_{m+1})g(X, X_{m+1})$$

with

$$g(X, X_{m+1}) = 2h(X, X_{m+1}) + (X + X_{m+1}e_{m+1})\partial h(X, X_{m+1})$$

a $(-\frac{m}{2})$ -homogeneous function. Since the left-hand side does not depend on X_{m+1} neither does the right-hand side, whence

$$\partial_{X_{m+1}} \left((X + X_{m+1}e_{m+1})g(X, X_{m+1}) \right) = 0 .$$

Choosing $X_{m+1} = |X|$ we are lead to

$$\partial_X F(X, |X|) = -(\xi + e_{m+1})\tilde{g}(X, |X|) ,$$

$\tilde{g}(X, |X|)$ being a $(-\frac{m}{2} - 1)$ -homogeneous function. If we can find a function $h(X)$ such that

$$\partial_X(\xi + e_{m+1})h(X) = (\xi + e_{m+1})\tilde{g}(X, |X|) ,$$

it follows immediately that

$$\partial_X \left(F(X, |X|) + (\xi + e_{m+1})h(X) \right) = 0 ,$$

as required. The crucial step in proving the existence of the function $h(X)$ is the following observation : for $(-\frac{m}{2})$ -homogeneous functions, the factor $(\xi + e_{m+1})$ anticommutes with the operator ∂_X . Indeed :

$$\partial_X e_{m+1} = -e_{m+1} \partial_X$$

and

$$\partial_X \xi = \xi \left(\partial_\rho + \frac{1}{\rho} \Gamma_{1,m} \right) \xi = \partial_\rho + \frac{1}{\rho} \xi \Gamma_{1,m} \xi ,$$

which by means of the fact that $\xi \Gamma_{1,m} \xi = m - \Gamma_{1,m}$ reduces to

$$\begin{aligned} \partial_X \xi &= \partial_\rho + \frac{m}{\rho} - \frac{1}{\rho} \Gamma_{1,m} \\ &= -\xi \left(\xi \left(\partial_\rho + \frac{1}{\rho} \Gamma_{1,m} \right) \right) + \frac{1}{\rho} (m + 2\mathbb{E}_{1,m}) \\ &= -\xi \partial_X + \frac{1}{\rho} (m + 2\mathbb{E}_{1,m}) . \end{aligned}$$

Corollary : For $(-\frac{m}{2})$ -homogeneous functions $h(X)$ we get that

$$\partial_X(\xi + e_{m+1})h(X) = -(\xi + e_{m+1})\partial_X h(X) .$$

Eventually this means that it suffices to solve the equation

$$\partial_X h(X) = -\tilde{g}(X, |X|) ,$$

which can easily be done using Riesz distributions. The existence and uniqueness of $h(X)$ then follow from elementary properties of these distributions.

Corollary : We have now thus showed how to obtain the action of the conformal group on the hyperbolic unit ball H_+ by means of the spin group on the nullcone $NC_{1,m+1}$. Indeed, one first lifts a hyperbolic monogenic $F(\xi)$ on H_+ to a homogeneous function $F(X, X_{m+1})$ on the future cone $FC_{1,m+1}$ by constant extension in an additional spatial direction e_{m+1} and then restricts the result to the nullcone. After that, the conformal transformation group is given by $\text{Spin}(1, m+1)$ and acts by a simple spinor rotation. The problem is however that the transformed function will no longer be constant in the direction of e_{m+1} , such that after the spinor transformation we do not necessarily end up with a solution for the operator ∂_X . This can be realized by adding a uniquely determined function, but it is only possible in case $\alpha = -\frac{m}{2}$.

Samenvatting

In deze doctoraatsscriptie beschouwen we een projectief model voor de hyperbolische eenheidsbal, gerealiseerd als variëteit van halfrechten in de toekomst-gerichte lichtkegel in de reële orthogonale ruimte $\mathbb{R}^{1,m}$, beter gekend als de vlakke Minkowski-ruimte-tijd. Met behulp van Clifford-algebra's is het mogelijk om op de desbetreffende variëteit een Clifford-algebra-structuur te definiëren, en dit stelt ons dan in staat om de Dirac-operator op de hyperbolische eenheidsbal in te voeren als de Dirac-operator op de vlakke Minkowski-ruimte inwerkend op secties van een homogene lijnbundel. De doctoraatsscriptie beoogt het opstellen van een functietheorie voor Clifford-algebra-waardige nuloplossingen voor deze hyperbolische Dirac-operator, de zogenaamde hyperbolische monogenen.

Clifford-algebra's zijn genoemd naar William Kingdon Clifford (1845-1879), die deze algebra's invoerde ter veralgemening van zowel Grassmans uitwendige algebra als Hamiltons algebra der quaternionen. Zijn bedoeling was om de meetkundige en de algebraïsche eigenschappen van de Euclidische ruimte samen te brengen in een allesomvattende overkoepelende structuur. Vandaar ook zijn oorspronkelijke benaming : "geometrische algebra's". Deze Clifford-algebra's werden later vaak herontdekt, niet in het minst door fysici. Toen bijvoorbeeld P.A.M. Dirac in 1928, in zijn beroemd artikel [24] over het elektron, de γ -matrices invoerde om de Klein-Gordon-vergelijking te linearizeren, construeerde hij eigenlijk de generatoren voor de Clifford-algebra $\mathbb{R}_{1,3}$.

Wanneer men de Clifford algebra construeert over het veld der reële getallen bekomt men de algebra der complexe getallen. De Cauchy-Riemann-operator, die de basis vormt voor de theorie der complexe holomorfe functies, factorizeert de Laplaciaan in twee dimensies. Met andere woorden, complexe holomorfe functies kunnen worden beschouwd als functies die behoren tot de kern van een rotatie-invariante eerste-orde differentiaaloperator die de Laplaciaan factorizeert.

Bovenstaande observatie stelt ons in staat om Clifford-analyse te beschouwen als een canonische veralgemening tot hogere dimensies van de theorie van holomorfe functies in het complexe vlak, waarbij de Dirac-operator fungeert als analogon voor de klassieke Cauchy-Riemann-operator. De veralgemeende holomorfe functies, die men monogene functies noemt, kunnen dus worden beschouwd als nuloplossingen van de rotatie-invariante eerste-orde differentiaaloperator die de Laplaciaan in m dimensies factorizeert. De eerste pogingen om een functietheorie voor deze operator op te stellen werden in de jaren 1930 ondernomen door R. Fueter, G. Moisil en N. Théodorescu (zie [36] en [51]). Een diepgaande studie van de monogene functietheorie kan men terugvinden in het boek [8] van F. Brackx, R. Delanghe en F. Sommen.

Hoewel de klassieke literatuur omtrent Clifford-analyse zich voornamelijk toespitste op de Dirac-operator in de vlakke Euclidische ruimte \mathbb{R}^m , ging men zich al vlug toeleggen op Dirac-operatoren op algemene variëteiten, wat een voor de hand liggende veralgemening was. Er werd reeds veel onderzoek verricht in die richting door theoretische fysici en differentiaalmeetkundigen, we verwijzen bijvoorbeeld naar de Atiyah-Singer-Dirac-operator op variëteiten. Er is echter een essentieel verschil tussen het wiskundig formalisme dat in deze aanpak werd gebruikt en de aanpak die werd gevolgd in bijvoorbeeld [10], [18] en [40]. In deze laatste werden Dirac-operatoren op variëteiten werkelijk binnen het kader van de Clifford-analyse bestudeerd, doordat men de variëteiten inbedt in een orthogonale ruimte en gebruik maakt van de eigenschappen van de Dirac-operator in deze ruimte.

Het meer specifieke geval van de Dirac-operator op een Riemann-ruimte met constante *positieve* kromming werd o.a. bestudeerd door J. Ryan en P. Van Lancker in [58], [59], [75] en [76]. Daarin bestudeert men de Dirac operator op de sfeer S^{m-1} in \mathbb{R}^m . De bedoeling van onderhavige scriptie is de studie van de Dirac-operator op een Riemann-ruimte met constante *negatieve* kromming, d.w.z. de Dirac-operator op de hyperbolische eenheidsbal. Daar zowel de sfeer als de hyperbolische eenheidsbal kunnen worden beschouwd als reële deelvariëteiten van de complexe eenheidsbal, zou men kunnen argumenteren dat de functietheorie op de hyperbolische eenheidsbal uit de functietheorie op de sfeer volgt via analytische voortzetting. Dit is echter verre van triviaal omdat men de analytische voortzetting van distributionele oplossingen voor de Dirac-vergelijking op de sfeer nodig heeft, wil men een fundamentele oplossing bekomen, en dit vereist de berekening van residu's van holomorfe functies in meerdere complexe veranderlijken. Dit kan worden vermeden door te werken met distributies, en de eenvoudigste methode om die in te voeren maakt gebruik van het projectieve model. Bovendien kan men, steu-

nend op de projectieve aard van dit model, resultaten bekomen die men via analytische voorzetting niet kán bekomen. Zo beschouwen we in deze verhandeling een limietsituatie voor de hyperbolische Diracvergelijking waarbij de singulariteiten naderen tot *oneindig*, d.w.z. de lichtkegel, wat uiteraard niet mogelijk is in het Euclidische geval.

Onderzoek naar een monogene functietheorie in de Poincaré-ruimte, welke een metrisch model vormt voor de hyperbolische eenheidsbal, werd gestart door H. Leutwiler en zijn studenten (zie bijvoorbeeld [34] en [49]) en steunt op de studie van harmonische functies op domeinen die conform equivalent zijn met de vlakke ruimte. In deze context vermelden we ook het werk van P. Cerejeiras en J. Cnops, zie bijvoorbeeld [11], omtrent de Hodge-Dirac-operator op hyperbolische ruimtes. Echter, zoals reeds werd opgemerkt in referentie [12], veralgemeent men op die manier de Dirac-operator voor $\text{Spin}(1)$ -velden, terwijl we ons hier toeleggen op de hyperbolische veralgemening van de Dirac-operator voor $\text{Spin}(\frac{1}{2})$ -velden.

In Hoofdstuk 0 wordt al het voorbereidende materiaal verzameld : basisbegrippen betreffende Clifford-algebra's en Clifford-analyse op de vlakke Euclidische ruimte, een korte inleiding tot de theorie der speciale functies, definities voor en eigenschappen van Riesz-distributies en de Radon-transformatie, enz.

Hoofdstuk 1 is gewijd aan het begrip hyperbolische meetkunde. Eerst geven we een historisch overzicht van de ontwikkelingen in de meetkunde die hebben geleid tot de ontdekking van de niet-Euclidische meetkunde, in het bijzonder het hyperbolische vlak, en daarna geven we verscheidene modellen voor de hyperbolische eenheidsbal (het hoger-dimensionale analogon voor het hyperbolische vlak). We beschouwen achtereenvolgens het klassieke Klein- en Poincaré-model, het hemisfeer-model dat beide modellen met elkaar in verband brengt, de hyperboloïde H_+ in de toekomstgerichte lichtkegel die alle ruimte-tijd vectoren met hyperbolische eenheidsnorm bevat en voor wat volgt het meest essentiële model : het projectieve model, dat de hyperbolische eenheidsbal realiseert als variëteit van halfrechten in de toekomstgerichte lichtkegel in de vlakke Minkowski-ruimte-tijd $\mathbb{R}^{1,m}$.

In Hoofdstuk 2 bestuderen we de zogenaamde hyperbolische Dirac-vergelijking en haar fundamentele oplossing. We introduceren eerst de homogene Clifford-lijnbundel $\mathbb{R}_{1,m;\alpha}$ (met $\alpha \in \mathbb{C}$ willekeurig) als een geassocieerde vezelbundel en nadien definiëren we de Dirac-operator op de hyperbolische eenheidsbal als de Dirac-operator op de vlakke Minkowski-ruimte-tijd inwerkend

op secties van deze bundel. De projectieve aard van ons model voor de hyperbolische eenheidsbal is hierbij van essentieel belang. Steunend op het feit dat de delta-distributie in een punt van een algemene variëteit kan gedefinieerd worden als de delta-distributie in het raakvlak in dat punt aan de variëteit stellen we dan de vergelijking op waaraan de hyperbolische fundamentele oplossing moet voldoen, de zogenaamde hyperbolische Dirac-vergelijking, en geven we enkele expliciete constructies voor deze oplossing.

Een eerste constructie maakt gebruik van de Frobenius-methode voor differentiaalvergelijkingen en levert de projectie van de hyperbolische fundamentele oplossing op het Klein- en Poincaré-model voor de hyperbolische eenheidsbal op, als een gemoduleerde versie van de klassieke Cauchy-kern.

Een tweede constructie herleidt de hyperbolische Dirac-vergelijking tot een probleem in twee dimensies en de oplossing van dit probleem geeft ons een fundamentele oplossing die bestaat uit een sigulier en een regulier gedeelte. Hoewel het reguliere gedeelte *niet* uniek is, komt het op canonische wijze tevoorschijn zoals wordt aangetoond aan het einde van het tweede hoofdstuk. Dit regulier stuk stelt ons ook in staat om de hyperbolische fundamentele oplossing te herschrijven in termen van zogenaamde hyperbolische poolcoördinaten, en dat leidt tot Gegenbauer-functies van de tweede soort. Deze Gegenbauer-functies duiken dan opnieuw op in een derde methode, waar we de hyperbolische fundamentele oplossing bepalen aan de hand van Riesz-distributies.

De laatste methode ter constructie van de hyperbolische fundamentele oplossing steunt heel sterk op eigenschappen voor de fundamentele oplossing voor de golfoperator op de vlakke Minkowski-ruimte-tijd, en leidt tot een fundamentele oplossing voor de hyperbolische Dirac-vergelijking in $\mathbb{R}_{1,m+2}$ die kan worden uitgedrukt in termen van de oplossing in $\mathbb{R}_{1,m}$.

In het derde hoofdstuk veralgemenen we de idee achter de eerste constructie voor de hyperbolische fundamentele oplossing, en dit leidt tot de zogenaamde Modulatiestellingen die uitdrukken dat homogene monogene oplossingen voor de Dirac-operator op de vlakke Euclidische ruimte kunnen worden gemoduleerd tot oplossingen voor de hyperbolische Dirac-vergelijking. Dit wordt aangetoond voor zowel het Klein-model als voor het Poincaré-model voor de hyperbolische eenheidsbal. Beide stellingen zijn equivalent, en dat leidt dan tot geometrische interpretaties voor eigenschappen van de hypergeometrische functie.

Vervolgens beschouwen we twee veralgemeningen van de Modulatiestellingen. Eerst construeren we oplossingen voor de zogenaamde natuurlijke machten van de hyperbolische Dirac-operator en daarna construeren we ook oplossingen voor de Dirac-operator op ultrahyperbolische ruimten van willekeurige

(p, q) -signatuur.

Aan het eind van het derde hoofdstuk beschouwen we een specifiek bi-axiaal probleem voor de projectie van de hyperbolische Dirac-vergelijking op het Klein-model. Hoewel op het eerste zicht dit probleem geen uitstaans heeft met de Modulatiestellingen, stelt het ons in staat de gemoduleerde oplossingen te interpreteren als zogenaamde veralgemeende hyperbolische machtsfuncties.

In Hoofdstuk 4 definiëren we willekeurige complexe machten van de hyperbolische Dirac-operator en construeren we een fundamentele oplossing voor deze operatoren aan de hand van Riesz-distributies. Dit Hoofdstuk is geïnspireerd door [7].

Hoofdstuk 5 behandelt de functietheorie voor de hyperbolische Dirac-operator, zowel op de hyperboloïde H_+ in de toekomstgerichte nulkegel als in het Klein-model voor de hyperbolische eenheidsbal. Hiervoor maken we gebruik van de Cauchy-Pompeju Stelling, de Stelling van Stokes en de Cauchy integraalstelling.

Om vervolgens de Taylor- en Laurent-reeks op te stellen voor hyperbolische monogenen, gedefinieerd in een open (ringvormig) domein van de hyperbolische eenheidsbal, stellen we een axiale decompositie op voor de fundamentele oplossing. Daartoe gebruiken we de Modulatiestellingen van Hoofdstuk 3 en vinden we een alternatieve interpretatie voor het Additietheorema voor Gegenbauer-functies (zie [25]).

Aan het einde van het vijfde hoofdstuk gaan we kort in op de vraag naar het bestaan van eigenfuncties voor de hyperbolische Dirac operator.

In Hoofdstuk 6 introduceren we de fotogene Cauchy-transformatie (FCT), gedefinieerd als een integraaltransformatie met als kern de fundamentele oplossing voor de hyperbolische Dirac-vergelijking met singulariteiten op oneindig. In tegenstelling tot de vlakke Euclidische ruimte kan men in het hyperbolische geval spreken van singulariteiten op oneindig door een delta-distributie te beschouwen op de lichtkegel. Daar L_2 -functies op de sfeer S^{m-1} kunnen worden ontwikkeld in een reeks van inwendig en uitwendig sferische monogenen op S^{m-1} , bepalen we vervolgens de FCT van deze sferische monogenen, en dat leidt opnieuw tot een andere interpretatie voor de gemoduleerde oplossingen uit Hoofdstuk 3.

Vervolgens beschouwen we de fotogene randwaarden van deze transformaties, door het argument van de FCT van de sferische monogenen tot de sfeer te laten naderen. Onder bepaalde voorwaarden op de complexe parameter α , verkrijgen we dan een afbeelding van de Sobolevruimte op de sfeer $W_2(S^{m-1})$

naar de verzameling van randwaarden van hyperbolische monogenen in het Klein-model.

Door de extensie van deze randwaarden tot de Lie-sfeer te beschouwen, kunnen we dan ook aantonen dat de eerder genoemde verzameling van randwaarden van hyperbolische monogenen een Hilbert-module met reproducerende kern oplevert.

Hoofdstuk 7 tenslotte behandelt de conforme Dirac-operator en illustreert hoe de hyperbolische Dirac-operator uit dit proefschrift moet worden gezien in termen van de Dirac-operator op een algemene variëteit. Door een spinbundel in te voeren op de hyperboloïde H_+ wordt er aangetoond dat de hier beschouwde Clifford-algebra-waardige functies eigenlijk Clifford-secties zijn, en door deze bundel te verfijnen tot de zogenaamde spinorbundel wordt er aangetoond dat onze hyperbolische Dirac-operator zich voor een specifieke waarde voor α herleidt tot de Atiyah-Singer-Dirac-operator.

Bibliography

- [1] Ahlfors, L. V., *Moebius transformations expressed through 2×2 matrices of Clifford numbers*, Complex Variables **5** (1986), pp. 215-224.
- [2] Ahlfors, L. V., *Moebius Transformations in Several Dimensions*, Ordway Professorship Lectures in Mathematics (1981), University of Minnesota.
- [3] Atiyah, M. F., Singer, I. M., *The index of elliptic operators on compact manifolds*, Bull. Amer. Math. Soc. **69** (1963), pp. 422-433.
- [4] Bell, S. R., *The Cauchy Transform, Potential Theory and Conformal Mapping*, CRC Press, Boca Raton, 1992.
- [5] Benn, I. M., Tucker, R. W., *An Introduction to Spinors and Geometry with Applications in Physics*, Adam Hilger, Bristol, 1987.
- [6] Booss-Bavnek, B., Wojciechowski, K., *Elliptic boundary problems for Dirac operators*, Birkhäuser, Boston, 1993.
- [7] Brackx, F., De Schepper, H., *On Riesz Potentials and the Hilbert-Dirac Operator*, *submitted for publication*.
- [8] Brackx, F., Delanghe, R., Sommen, F., *Clifford Analysis*, Research Notes in Mathematics No. 76, Pitman, London, 1982.
- [9] Brackx, F., Delanghe, R., Sommen, F., Van Acker. N., *Reproducing kernels on the unit sphere*, Generalized functions and their applications (Varanasi, 1991), pp. 1-10, Plenum, New York, 1993.
- [10] Calderbank, D., *Dirac operators and Clifford analysis on manifolds with boundary*, Max Planck Institute for Mathematics, Bonn, Preprint number 96-131, 1996.
- [11] Cerejeiras, P., Cnops, J., *Hodge-Dirac operators for hyperbolic spaces*, Complex Variables **41** (2000), pp. 267-278.

- [12] Cerejeiras, P., Kähler, U., Sommen, F., *Clifford Analysis on Projective Hyperbolic Space*, J. Nat. Geom. **22** (2002), pp. 19-34.
- [13] Cerejeiras, P., Kähler, U., Sommen, F., *Clifford Analysis on Projective Hyperbolic Space II*, Math. Meth. Appl. Sci. **25** (2002), pp. 1465-1477.
- [14] Clarke, C., *On the global isometric embedding of pseudo-Riemannian manifolds*, Proc. Roy. Soc. London, ser. A **314** (1979), pp. 417-428.
- [15] Cnops, J., *Hurwitz Pairs and applications of Moebius transformations*, Habilitation Thesis, Ghent, Belgium, 1994.
- [16] Cnops, J., *Manifolds with and without embeddings*, Clifford algebras and their application in mathematical physics (Aachen, 1996), pp. 57-65.
- [17] Cnops, J., *Dirac operators on embedded sphere and hyperboloid*, preprint.
- [18] Cnops, J., *An Introduction to Dirac Operators on Manifolds*, Progress Notes in Mathematical Physics Vol. 24, Birkhäuser, Boston, 2002.
- [19] Coifman, R.R., McIntosh, A., Meyer, Y., *L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes Lipschitziennes*, Ann. Math. **116** (1982), pp. 361-387.
- [20] Deans, S. R., *The Radon Transform and some of its applications*, Wiley-Interscience, New York, 1983.
- [21] de Jager, E.M., *Applications of Distributions in Mathematical Physics*, Math. Centre Tact **10**, Amsterdam, 1964.
- [22] Delanghe, R., *Clifford Analysis : History and Perspective*, Comp. Meth. and Func. Theory **1** (2001), pp. 107-153.
- [23] Delanghe, R., Sommen, F., Souček, V., *Clifford Algebra and Spinor-Valued Functions*, Kluwer Academic Publishers, Dordrecht, 1992.
- [24] Dirac, P.A.M., *The Quantum Theory of the Electron*, Proc. Roy. Soc. A **117** (1928), pp. 610-624.
- [25] Durand, L., Fishbane, P. M., Simmons, L. M. Jr., *Expansion formulas and addition theorems for Gegenbauer functions*, J. Math. Phys. **17** (1976), pp. 1933-1948.
- [26] Eelbode, D., *Solutions for the Hyperbolic Dirac Equation on $\mathbb{R}_{1,m}$* , Complex Variables **48** No. 5 (2003), pp. 377-395.

- [27] Eelbode, D., *Arbitrary complex powers of the Dirac operator on the hyperbolic unit ball*, to appear in Ann. Acad. Scient. Fenn.
- [28] Eelbode, D., *The fundamental solution of the Dirac equation on $\mathbb{R}_{1,m}$: a new approach*, to appear in Bull. Belg. Math. Soc.-Simon Stevin.
- [29] Eelbode, E., Sommen, F., *The Inverse Radon Transform and the Fundamental Solution of the Hyperbolic Dirac Equation*, to appear in Math. Zeit.
- [30] Eelbode, D., Sommen, F., *Taylor Series on the Hyperbolic Unit Ball*, to appear in Bull. Belg. Math. Soc.-Simon Stevin.
- [31] Eelbode, D., Sommen, F., *On a bi-axial system related to the hyperbolic Dirac equation*, to appear in Integr. Transf. Spec. F.
- [32] Eelbode, D., Sommen, F., *Hyperbolic L_2 -modules with Reproducing Kernel*, to appear in Acta Mathematica Sinica - English Series.
- [33] Eelbode, D., Sommen, F., *The Photogenic Cauchy Transform*, to appear in Journ. Geom. Phys.
- [34] Eriksson-Bique, S.L., Leutwiler, H., *Hypermonogenic functions, Clifford algebras and their applications in mathematical physics*, Vol. 2 (Ixtapa, 1999), pp. 287-302, Progr. Phys. **19**, Birkhäuser Boston, Boston, 2000.
- [35] Erdelyi, A., Magnus, W., Oberhettinger, F., Tricomi, F.G., *Higher Transcendental Functions*, McGraw-Hill, New York, 1953.
- [36] Fueter, R., *Zur Theorie der Regulären Funktionen einer Quaternionenvariablen*, Monat. für Math. und Phys. **43** (1935), pp. 69-74.
- [37] Gel'Fand, I.M., Gindikin, S. G., Graev, M.I., *Integral geometry in affine and projective spaces*, Transl. Otagi. Nauk. Tekh., Seria Sovrem. Probl. Mat. **16** (1980), pp. 53-226.
- [38] Gel'Fand, I.M., Graev, M.I., Vilenkin, N.Y., *Generalized Functions : Integral Geometry and Representation Theory (volume 5)*, Academic Press, London and New York, 1966.
- [39] Gel'Fand, I.M., Shilov, G.E., *Generalized Functions : Properties and Operations (volume 1)*, Academic Press, New York, 1964.
- [40] Gilbert, J., Murray, M. A. M., *Clifford Algebras and Dirac Operators in Harmonic Analysis*, Cambridge University Press, Cambridge, 1991.

- [41] Gromov, M., *Partial Differential Relations*, Springer-Verlag, New York, 1986.
- [42] Hadamard, J., *Lectures on Cauchy's problem in linear partial differential equations*, Dover Publications, New York, 1952.
- [43] Helgason, S., *Groups and Geometric Analysis*, Academic Press, Orlando, 1984.
- [44] Helgason, S., *The Radon Transform*, Birkhäuser, Boston, 1980.
- [45] Husemoller, D., *Fibre Bundles*, Mc-Graw-Hill, New York, 1966.
- [46] Hochstadt, H., *The Functions of Mathematical Physics*, Pure and Applied Math. 23, Wiley Interscience, New York, 1971.
- [47] Isham, C. J., *Modern Differential Geometry for Physicists*, World Scientific Lecture Notes in Physics, Vol. 61, Singapore, 2001.
- [48] Kanwal, R. P., *Generalized Functions : theory and technique*, Academic Press, New York, 1983.
- [49] Leutwiler, H., *Modified quaternionic analysis in \mathbb{R}^3* , Complex Variables **20** (1991), pp. 19-51.
- [50] Leutwiler, H., *Modified Clifford analysis*, Complex Variables **17** (1992), pp. 153-171.
- [51] Moisil, G., Théodoresco, N., *Fonctions holomorphes dans l'espace*, Mathematica **V** (1931), pp. 142-159.
- [52] Morimoto M., *Analytic functionals on the Lie sphere*, Tokyo J. Math. **3**, No. 1 (1980), pp. 1-35.
- [53] Nash, J., *The embedding problem for Riemannian manifolds*, Ann. of Math. (2) **63** (1956), pp. 20-63.
- [54] Needham, T., *Visual Complex Analysis*, Clarendon Press, Oxford, 1997.
- [55] Peetre, J., Tao, Q., *Moebius Covariance of Iterated Dirac Operators*, J. Austral. Math. Soc. (Series A) **56** (1994), pp. 403-414.
- [56] Porteous, I., *Topological Geometry* (2nd Edition), Cambridge University Press, New York, 1981.

- [57] Riesz, M., *L'intégrale de Riemann-Liouville et le problème de Cauchy*, Acta Math. **81** (1949), pp. 1-223.
- [58] Ryan, J., *Clifford analysis on spheres and hyperbolae*, Math. Meth. Appl. Sci. **20** (1997), pp. 1617-1624.
- [59] Liu, H., Ryan, J., *Clifford Analysis Techniques for Spherical PDE*, J. Fourier Anal. Appl. **8** (2002), pp. 535-563.
- [60] Siciak, J., *Holomorphic continuation of harmonic functions*, Ann. Polon. Math. **29** (1974), pp. 67-73.
- [61] Eriksson-Bique, S-L., Leutwiler, H., *Hypermonogenic Functions, Clifford Algebras and Their Applications in Mathematical Physics*, pp. 287-302, Ryan John *et al.* (eds) Vol. 2., Birkhäuser, Basel, 2002.
- [62] Sommen, F., *Some Connections Between Clifford Analysis and Complex Analysis*, Complex Variables : Theory and Application 1 (1982), pp. 97-118.
- [63] Sommen, F., *Plane Wave Decompositions of Monogenic Functions*, Ann. Polon. Math. XLIX (1988), pp. 101-114.
- [64] Sommen, F., *Spherical monogenics on the Lie sphere*, J. Funct. Anal. **92** (1990), pp. 372-402.
- [65] Sommen, F., *Clifford Analysis and Integral Geometry, Clifford algebras and their applications in mathematical physics* (Montpellier, 1989), pp. 293-311, Fund. Theories Phys. **47**, Kluwer Academic Publishers, Dordrecht, 1992.
- [66] Sommen, F., *Clifford analysis on the nullcone*, submitted.
- [67] Sommen, F., Van Acker, N., *$SO(m)$ -Invariant Differential Operators on Clifford Algebra-Valued Functions*, Found. Phys. Vol. 23 **11** (1993), pp. 1491-1519.
- [68] Sommen, F., Van Lancker, P., *Homogeneous Functions in Euclidean Space*, Integr. Transf. Spec. F. **7** No. 3-4 (1998), pp. 285-298.
- [69] Steenrod, N., *The Topology of Fibre Bundles*, Princeton University Press, Princeton, 1951.
- [70] Stillwell, J., *Geometry of Surfaces*, Springer-Verlag, New York, 1992.

- [71] Trautman, A., *Spinors and the Dirac operator on hypersurfaces. I. General theory*, J. Math. Phys. **33/12** (1992), pp.4011-4019.
- [72] Treves, F., *Basic Linear Partial Differential Equations*, Academic Press, New York, 1975.
- [73] Vahlen, K., *Über Bewegungen und komplexe Zahlen*, Math. Ann. **55** (1902), pp. 585-593.
- [74] Van Acker, N., *Clifford-differentiaaloperatoren en randwaardentheorie van de nuloplossingen ervan op de sfeer en de Liesfeer*, PhD-thesis, Ghent, Belgium, 1992.
- [75] Van Lancker, P., *Clifford Analysis on the Unit Sphere*, PhD-thesis, Ghent, Belgium, 1997.
- [76] Van Lancker, P., *Taylor and Laurent Series on the Sphere*, Complex Variables **38** (1999), pp. 321-365.