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Generic isotropy for algebras with involution, specialisation of involutions and related isomorphism problems

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Dust if you must.
But wouldn't it be better,
To paint a picture, or write a letter,
Bake a cake, or plant a seed?
Ponder the difference between want and need.

Dust if you must.
But there is not much time
With rivers to swim and mountains to climb!
Music to hear, and books to read,
Friends to cherish and life to lead.

Dust if you must.
But the world's out there
With the sun in your eyes,
the wind in your hair,
A flutter of snow, a shower of rain.
This day will not come round again.

Dust if you must.
But bear in mind,
Old age will come and it's not kind.
And when you go, and go you must,
You, yourself, will make more dust.

—— ***Rose Milligan***

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Preface

> Die Zeit kam so geräuschlos und entfernte sich, ohne dass man es merkte.
>
> — Robert Walser

What does one write in a preface? It's the place where people are thanked, and frankly, it's probably the only part of the thesis many people will understand. So, I wanted to try to elaborate a bit on how I experienced this Phd... not that I intend to write a survival guide, just my impressions of doing maths on the one hand, and being a part of the mathematical world on the other hand... of this world that is so alien to many people living in the real world, which is something I struggled with from time to time.

There are many people who deserve to be thanked for their support in different ways. First of all, my advisors Jan and Karim, and the FWO for financing my research. I enjoyed discussing mathematics with Jan, especially in the last year. He has such a broad knowledge, and over the years I learnt from him more than from anyone else to look at problems from different points of view (even if it takes time to get used to another view) and how enriching this can be. No matter how pretty the view of the Matterhorn is in Zermatt, you don't know what you're missing if you've never seen the view in Schönbiel... but it takes a good walk to get there (just don't try to climb the Mettelhorn on the same day). Jan always tried to create a relaxing environment, and this was important, especially during the months before submission.
Karim has given me many good questions to work on, and helped me in writing down mathematics in a rigorous way. He has a more extended background knowledge for the problems I worked on in the beginning than Jan, and he came up with good ideas to get me on the right track. He also made it possible, during my stay in Konstanz, to meet several mathematicians working in the same field, and I am very grateful to all of them for taking time to discuss my work with me. Thanks to David Grimm for interesting discussions on conics and for making me less afraid of varieties, to James O' Shea and David Leep for being a library for many quadratic form results. I also want to thank David Leep for introducing me to Bézout domains, which form the basis for the - in my

opinion - most important results of this thesis. Thanks to Jean–Pierre Tignol for many interesting and enriching discussions (both in Konstanz and Louvain-la-neuve), which were very important in the course of this thesis, and for making me less afraid of graded algebras. And also for the very pleasant surprise that one time when we were on our way to Konstanz and he cited a poem by Paul van Ostaijen from the top of his head! (And I was so embarrassed I couldn't think of a French poem to cite). Thanks to Thomas Unger for making me less afraid of Morita equivalence (although I still try to avoid it if I can). Making people less afraid of a mathematical theory is important. Just think of the proof of the Milnor Conjecture... Vladimir Voevodsky himself explicitly thanked Markus Rost and Alexander Vishik for making sure he wasn't afraid anymore of the theory of quadratic forms.

Further thanks go to Adrian Wadsworth, Anne Quéguiner–Mathieu, Raman Parimala and Skip Garibaldi. I am also very grateful to Manuel Ojanguren, who is an extraordinary mathematician and who - even though retired, but what does that even mean for a mathematician - has been a tremendous help (in several languages) when Jan and I were studying Azumaya algebras with involution over valuation rings. We learnt a lot of new maths during that period, not without struggles and extra time pressure, but Manuel would always write his emails in such a way, that they were an extra motivation to keep us going.

Thanks to the people in my jury, for their comments and suggestions to improve this dissertation!

I want to thank my parents and my sister for showing their interest in what I was doing, even though it was difficult for them to really understand what all these mathematical things are. I thank them for listening to my frustrations and complaints every so often... The algebra group in Ghent, especially the algebra girls Claudia, Elizabeth and Lien... We started studying mathematics at the same time and started a Phd at the same time. This has created a special bond between us I think... And thanks to Jeroen for helping me out with Latex and Sage a couple of times.
Last but not least, I want to thank Andrew, for many many things... for making me smile when I needed it, just by saying "How does it work in characteristic two?"... for just being there when I needed him.

When I started this Phd in 2009, I wasn't sure what to expect. I had enjoyed working on my master's thesis, but a Phd is something on a whole other level, which I soon learnt. Expecting to make progress every week, or even every month? Forget it... or relax your definition of progress (and I am well aware that mine was (is?) too narrow). And another

important point: advisors don't know everything, they also make mistakes... or to cite David Saltman... "It's not an advisor's job to be right"... something I tended to forget from time to time...
In the beginning, I was very motivated to start working on the different questions Jan and Karim gave me, and probably wanted to do too much at once. But there is so much to learn first, different theories you hope to be able to use one day. And in the beginning you think you have time, four years seems like ages... But I struggled immensely with some of those theories, and sometimes started to perceive this as a waste of time, especially since it didn't look like I'd manage to use them in my own research. And time wasn't standing still, time was passing by unnoticed, geräuschlos... A year passed, two years passed, and all of the sudden - it seemed - there was only one year left, and I thought "What have I achieved in this period? That's all?!" A pile of paper ended up in the bin (the paper bin of course, recycling is important)... And even though Jan and Karim told me that the results I had obtained were already good, I was not convinced... But then after three years, and it still leaves me a little puzzled, things suddenly started working. Suddenly, miraculously, things came together, results that were lying in the drawer could be applied... And now that I look back on that period, I can't help thinking about a quote Jan once told me: "Sometimes you are looking for things you have to find by coincidence..." And the non–mathematicians who have seen me then must have thought I was crazy (it is quite possible that I am). Here's this theorem they don't understand a thing about and is of no use to them whatsoever, and I'm practically extatic about it.

> *Tout mathématicien digne de ce nom a ressenti, même si ce n'est que quelques fois, l'état d'exaltation lucide dans lequel une pensée succède à une autre comme par miracle...* ***(André Weil)***

And this brings me to the last part of this preface... How does society look at what we - mathematicians - do? Do they just consider us to be a bunch of weirdos? Well, some people certainly do. It is very difficult to explain to non–mathematicians what kind of work we do and why. It often made me feel a bit uncomfortable... thinking where do I start, how do I start... sometimes I didn't start at all... How can we explain what we do in sound and common language, and justify what we do? Maybe we need more storytellers... I've had the pleasure of meeting some during the last nine years, and of listening to their stories... people who are good at what they do, and also good at communicating it to a general audience... Perhaps, we sometimes live too much in this academic bubble and forget about the world outside (and not just in mathematics, but in academia in general). So, we could use more storytellers... but it's a knife that cuts both ways... we also need an audience that is willing to listen, an audience that has not a priori decided that research in mathematics does not have a purpose, that abstract mathematics is just abstract nonsense without applications in the real world, that surely everything (useful) in mathematics has already been discovered... "Wir sind gewohnt, daß die Menschen verhöhnen was sie nicht verstehn," Faust says. This is something that I've experienced a few times. People ask what you do, and then ask about applications... and then you see them think... Why does one spend (waste) money on a thing like that? I haven't been able to give people a satisfactory answer as to the usefulness of abstract mathematical research... because I hadn't yet found one myself... But I went looking for answers, and I think I found some. Let's go back to Euclid...

> *A youth who had begun to read geometry with Euclid, when he had learnt the first proposition, inquired, "What do I get by learning these things?" So Euclid called a slave and said "Give him threepence, since he must make gain out of what he learns."*

So next time people ask me about applications of my work I will say... The world would be a very sad place if we would have to make (immediate, visible) gain out of everything we learn. Many things would disappear, many good things... (And if they look at me with disdain, I'll just give them threepence.) My work is not going to change the world, I know that, but I got some people who know what it is about interested, so I think I can be proud of what I have achieved... And for the people who are truly interested, who are willing to listen, I would add what I think is maybe the nicest answer I found... Manuel Ojanguren's reply to the question "Mais si on décidait de ne plus faire de recherches et d'en rester à nos connaissances actuelles?" So let me finish with this...

> *Alors on n'en resterait pas à nos connaissances actuelles. On oublierait tout. La seule façon de faire en sorte que nos connaissances restent en place, c'est de continuer à se laisser prendre au jeu... Si nous décidions tout à coup que ce que nous savons suffit, peu à peu nous ne saurions plus rien. Le savoir en mathématiques n'est pas une accumulation de faits, c'est un savoir faire. Dès qu'on cesse de faire, on cesse de savoir. Il est impossible de maintenir le même niveau sans essayer de progresser. **(M. Ojanguren)***

But, in all fairness, he also adds

> *Mais cela dit, je pense qu'on peut vivre heureux sans développer les mathématiques, ni les sciences.*

Introduction

> I don't know why I should have to learn algebra... I'm never likely to go there.
>
> Billy Connolly

In the 1970's, M. Knebusch developed the generic splitting theory of quadratic forms over fields (cf. [40, 41]). The basis of this theory is the following result. Given a non–singular quadratic form over a field that is anisotropic (i.e. does not have a nontrivial zero), one can in a generic way, namely by adding variables, construct a field extension of the ground field in which the quadratic form is isotropic (i.e. has a nontrivial zero). The latter is called a generic zero, and the constructed field is called a generic isotropy field. Every field extension of the ground field where the quadratic form is isotropic, is obtained by specialising the generic isotropy field (i.e. by specialising the variables). Using this, M. Knebusch showed that the isotropy behaviour of a non–singular quadratic form over any field extension of the ground field, is completely determined by its behaviour over a chain of generic isotropy fields.

Two aspects of the generic splitting theory of quadratic forms were the starting point for this thesis. On the one hand, the isotropy behaviour of quadratic forms with respect to places from one field to another, and on the other hand the concept of a generic isotropy field for a quadratic form. Both aspects are studied in the context of algebras with involution over fields, which are objects closely related to quadratic forms and bilinear forms, in the sense that one can associate to each non–singular symmetric or alternating bilinear form over a field its adjoint algebra with involution (cf. [45]). These are exactly the split central simple algebras with involution of the first kind, which is orthogonal if the bilinear space is symmetric, and symplectic otherwise. There also exist algebras with involution of the second kind. The distinction is determined by the way in which the involution restricts to the center of the algebra. If this restriction is the identity, the involution is of the first kind, otherwise it is of the second kind.

In the last decades, algebras with involution have become an important point of study, especially due to their intimate connection with certain classical algebraic groups. Since

1998, they have their very own standard work: The book of involutions ([45]). This book brought together different viewpoints on involutions. It contains a study of involution analogues of many concepts for quadratic forms, such as the discriminant, the Clifford algebra, similitudes and multipliers. It further presents many results related to isotropy and hyperbolicity that are also rooted in quadratic form theory. Moreover, the authors also extensively study involutions from an algebraic group point of view.
In this thesis, I focus on isotropy and hyperbolicity/metabolicity results for involutions on the one hand, and on the other hand, on isomorphism problems that naturally came up while working on isotropy questions.

Chronologically, I first studied whether there exists an involution analogue of the concept of a generic isotropy field for quadratic forms. In order to do this, it was necessary to understand the isotropy behaviour of involutions with respect to specialisation from one field to another, by means of a place. I started by investigating this for algebras with involution in the "algebraic geometric case", namely for algebras with involution that are defined over a ground field, which is contained in both fields involved in the place (the context I needed it for). In order to go beyond this geometric case, the setting of Azumaya algebras with involution over valuation rings naturally comes into the picture.

Specialisation and good reduction

Suppose that we are given a place from one field to another field. This is a map induced by the morphism from a valuation ring of the first field to its residue field. Suppose furthermore that we are given an object over the first field, say a bilinear space or an algebra with involution. Under certain conditions, one can specialise this bilinear space (resp. algebra with involution) to a bilinear space (resp. algebra with involution) over the second field. It is then natural to ask whether the residue object inherits certain properties from the original object. The kind of specialisation questions I consider for algebras with involution over fields in this thesis, are motivated by specialisation results for symmetric bilinear spaces, which I explain below.

We fix a field F and a valuation ring $\mathcal{O}$ of F, and we let λ be the associated place from F to the residue field κ of $\mathcal{O}$. Let (V,b) be a symmetric bilinear space over F (i.e. non–singular). In order to be able to specialise (V,b) in a sensible way to a bilinear space over κ, (V,b) needs to be defined over $\mathcal{O}$, i.e. (V,b) is obtained by scalar extension from a symmetric bilinear space over $\mathcal{O}$. If this is the case, then (V,b) is said to have *good reduction with respect to λ*. Symmetric bilinear spaces over $\mathcal{O}$ have in a natural way an associated residue bilinear space over κ. Let (V,b) be a symmetric bilinear space over F with good reduction with respect to λ. It is then natural to ask whether one can associate in a sensible way a residue bilinear space over κ to (V,b). This comes down to asking whether symmetric bilinear spaces over $\mathcal{O}$ that become isometric over F, are also isometric over κ. It has been shown in [42, (1.15)] that this is case if 2 is invertible in $\mathcal{O}$, and in fact, the result is somewhat stronger. Namely, symmetric bilinear spaces over $\mathcal{O}$ that become isometric over F, are already isometric over $\mathcal{O}$, if 2 is invertible in $\mathcal{O}$ (see [66, (4.6.3)]). Let (V,b) be a symmetric bilinear space over F, with good reduction with respect to λ. Then the result in [42, (1.20)] implies that if (V,b) is isotropic over F, then its residue bilinear space is isotropic over κ.

If a symmetric bilinear space over F has good reduction with respect to λ, then its adjoint algebra with involution is obtained by scalar extension from an Azumaya algebra with involution with center $\mathcal{O}$. The isotropy behaviour of the bilinear space under λ then carries over to the adjoint algebra with involution. The aim of the first part of the thesis is to study, not only in the split case, algebras with involution over F that are obtained by scalar extension from Azumaya algebras with involution over $\mathcal{O}$, and to investigate their isotropy behaviour under λ. For algebras with involution of the first or second kind over F, we use the shorthand term *F−algebras with involution*, and similarly, for algebras with involution of the first or second kind over $\mathcal{O}$, we use the term *$\mathcal{O}$−algebras with involution*.

In Theorem 4.9, I present an involution analogue of the aformentioned result for bilinear spaces over a valuation ring. Namely, I show that an $\mathcal{O}$−algebra with involution that becomes isotropic (resp. metabolic) over F, is also isotropic (resp. metabolic) over κ. In [45], the authors introduced a set, which measurs the isotropy of an algebra with involution over a field: the index. In terms of this notion, Theorem 4.9 says the following. Let $(\mathcal{A},\sigma)$ be an $\mathcal{O}$−algebra with involution. Then the index of $(\mathcal{A},\sigma)_F$ is contained in the index of $(\mathcal{A},\sigma)_\kappa$.
It is well known that, if $\mathcal{O}$ is a Henselian valuation ring, then one can sometimes lift properties from a residue object back to the original object. In that case, the place associated to $\mathcal{O}$ forms a two–way street for those properties. For instance, one can lift isotropy (resp. hyperbolicity) of a symmetric bilinear space over $\mathcal{O}$ from κ to F, if 2 is invertible in $\mathcal{O}$ (see [66, (6.2.4)]). In Theorem 4.20, I present an involution version of this result. Namely, I show that, excluding a few cases, one can lift isotropy (resp. hyperbolicity) of an $\mathcal{O}$−algebra with involution from κ back to F. Let $(\mathcal{A},\sigma)$ be an $\mathcal{O}$−algebra with involution. In terms of the index, Theorem 4.20 states that the index of $(\mathcal{A},\sigma)_F$ is equal to the index of $(\mathcal{A},\sigma)_\kappa$.

In section 4.4, I introduce a notion of good reduction with respect to places, for algebras with involution over fields. An F−algebra with involution is said to have *good reduction with respect to λ* if it is obtained by scalar extension from an $\mathcal{O}$−algebra with involution. It is then natural to ask whether $\mathcal{O}$−algebras with involution that become isomorphic over F, are also isomorphic over κ. In Theorem 4.37, I show that this is indeed the case if 2 is invertible in $\mathcal{O}$. In order to give the proof, the index result for Henselian valuation rings (Theorem 4.20) is crucial. Inspired by the results concerning good reduction for involutions, I started looking at related isomorphism problems for algebras with involution over valuation rings, and also more general domains. This forms the second part of the thesis, and it turned out that the results in the Henselian case formed the cornerstone for many of the arguments in this second part.

Some isomorphism problems

The core of the second part of the thesis is to study the following isomorphism problem, and, continuing the flow of the first part of the thesis, mainly in a context related to valuation rings.

Question 1. Let R be a domain with fraction field F. Let $(\mathcal{A},\sigma)$ and $(\mathcal{A}',\sigma')$ be $R-$algebras with involution. Suppose that $(\mathcal{A},\sigma)_F \cong_F (\mathcal{A}',\sigma')_F$. Does this imply that $(\mathcal{A},\sigma) \cong_R (\mathcal{A}',\sigma')$?

In the literature, one uses the term *rationally isomorphic* for objects that are defined over a domain and become isomorphic over the fraction field of that domain. So, Question 1 asks for which domains R one can conclude that rationally isomorphic $R-$algebras with involution are isomorphic. One cannot expect this to hold for domains in general. In Example 5.2, inspired by discussions with M. Ojanguren, I give a simple counterexample of two involutions on a fixed algebra that become rationally isomorphic, but are not isomorphic, in the case where R is a certain Henselian local domain.
Question 1 has been studied in the literature for regular local rings. In [56], the author gives a positive answer in the case where R is a regular local ring containing a field of characteristic different from 2, using the fact that there is a positive answer for discrete valuation rings. The latter result follows from more general results on algebraic groups in [55].

The first result concerning Question 1 that I present in this thesis, is that rationally isomorphic $R-$algebras with involution are isomorphic, in the case where R is a Henselian valuation ring of F with 2 invertible in R (Theorem 4.34). This is a crucial step in the proof of the good reduction statement in section 4.4. Question 1 is pursued further in chapter 5, and the results there were obtained in collaboration with J. Van Geel. We show that rationally isomorphic $R-$algebras with involution are isomorphic, in the case where R is a valuation ring of F with 2 invertible in R. Furthermore, we noticed that the method of proof could be adapted to work also in the case where R is an intersection of finitely many valuation rings of F, i.e. R is a so–called *semilocal Bézout domain*. The main part of the proof is to give a local characterisation, for an $R-$algebra with involution $(\mathcal{A},\sigma)$, of the multipliers of $(\mathcal{A},\sigma)_F$ up to units in R. We do this by means of a norm argument based on an approximation theorem for valuations by P. Ribenboim, and by using the results for Henselian valuation rings from chapter 4.

In section 5.3, we show that the characterisation of multipliers mentioned above (and hence a positive answer to Question 1), can be obtained in a more direct way in the case where R is a discrete valuation ring. By exploiting the Noetherian property of such valuation rings, we can prove a representation result for $R-$algebras with involution (Theorem 2.39). Using this result, the proof of the multiplier result in the case of discrete valuation rings depends less on the results for Henselian valuation rings.
The case of discrete valuation rings naturally came forward when we considered the following question (suggested by K.J. Becher and A. Quéguiner–Mathieu), in which the role of F is played by the function field of a quaternion algebra defined over a smaller field k. It is well known that all $k-$valuations on such a function field are discrete.

Question 2. Let k be a field of characteristic different from 2 and let Q be a $k-$quaternion division algebra. Let B be a central simple $k-$algebra Brauer equivalent to Q and let τ and τ' be two orthogonal involutions on B. Suppose that $(B,\tau)_{k(Q)} \simeq_{k(Q)} (B,\tau')_{k(Q)}$. Does this imply that $(B,\tau) \simeq_k (B,\tau')$?

This question comes down to Question 1 in a global setting, as opposed to the local setting considered above. It is known that Question 2 has an affirmative answer in some low degree cases, namely if the degree of B is at most 4, and if the degree of B is 6 and the discriminant of τ is trivial. The results in degree 2 and 4 follow from [72, (3.6), (3.10)] and in degree 6 from [45, (15.7)]. In section 5.4, we look for conditions on (B,τ) which allow us to decide whether a nonzero element in $k(Q)$ is equal to a multiplier of $(B,\tau)_{k(Q)}$ times a unit in k. As a consequence, we obtain a positive answer to Question 2 in the case where τ becomes hyperbolic over a quadratic field extension of k splitting Q (Corollary 5.44).

Generic isotropy and hyperbolicity fields

In the third part of the thesis (chapter 6), I turn my attention to the core of the generic splitting theory of quadratic forms, namely the existence of a generic isotropy field.

Question 3. Let F be a field of characteristic different from 2 and let (B,τ) be an $F-$algebra with involution of the first kind. When does there exist a field extension N/F such that τ_N is isotropic (resp. hyperbolic), and for every field extension L/F such that τ_L is isotropic (resp. hyperbolic), there is an $F-$place from N to L? We call a field with these properties *a generic isotropy (resp. hyperbolicity) field for τ*.

I show that a generic isotropy field need not always exist. There are already counterexamples of degree 4 algebras with involution (Corollary 6.42). For non–singular quadratic forms over F, a generic isotropy field can be realised by considering the function field of the projective quadric associated to the quadratic form. I also take the viewpoint of varieties in the context of Question 3, by studying certain varieties naturally associated to algebras with involution over fields, and whose rational points are isotropic ideals of a certain dimension. These varieties have been studied in the literature ([52, 53]), especially due to their link with the algebraic groups related to algebras with involution. I study the isotropy behaviour of $F-$algebras with involution over the function fields of these varieties. In order to do this, I extensively use Schur index reduction formulas for these function fields, proved in [52, 53].

Let (B,τ) be an $F-$algebra with involution of the first kind of degree at least 3. In Corollary 6.47, I show that, if there exists a generic isotropy field for τ, then it can be realised as the function field of one of the varieties mentioned above. In general, in order to show that one of those function fields is a generic isotropy field for τ, it is necessary to investigate the isotropy behaviour of τ over the other function fields as well. In some cases, it is sufficient to check only one other function field. I present a result of this kind in Corollary 6.62. However, in practice, it might still be difficult to check whether the condition in Corollary 6.62 is satisfied. To this end, in the case where B has Schur index 2, I present a sufficient condition that can be easier to check in Proposition 6.63. I also obtain explicit characterisations for the existence of a generic isotropy field for algebras with involution of low degree (4,6 and partially 8).

The part of Question 3 concerning generic hyperbolicity fields is treated separately. In that case, there are at most two varieties that need to be taken into account. Using this, it is easy to show that symplectic involutions always have a generic hyperbolicity field. In the orthogonal case, this depends on the behaviour of the Clifford algebra associated to the algebra with involution.

Outline of the structure of the thesis

In chapter 1, I study Azumaya algebras with involution over valuation rings, and more generally semilocal Bézout domains. The latter type of rings occurred naturally when considering algebras with involution of the second kind over a valuation ring. In that case, the center of the algebra need not be a valuation ring, but can be an intersection of two valuation rings. In section 1.4, I zoom in on algebras with involution over fields, and recall some properties of the Clifford algebra of an orthogonal involution. Furthermore,

in the case of second kind involutions, following the approach of [45], I consider not only simple algebras, but also semisimple algebras with two simple components. The reason is that a central simple algebra with involution of the second kind becomes semisimple after scalar extension to an algebraic closure of the fixed field of the involution inside the center of the algebra.
I introduce the notion of balanced one–sided ideals, in order to treat the cases of simple and semisimple algebras with involution in the sequel as uniformly as possible. The condition balanced is trivial for ideals in central simple algebras, but restricts the set of one–sided ideals to the interesting ones in the semisimple case, when considering isotropy questions.

Sometimes it is useful to approach a problem for algebras with involution from a hermitian form point of view. To this end, in chapter 2, I study (skew–)hermitian spaces over Azumaya algebras with involution with center a semilocal Bézout domain. I devote special attention to the relation between similarity of such (skew–)hermitian spaces and isomorphism of their adjoint algebras with involution. These results will be important when considering Question 2 in later chapters. The last sections of chapter 2 concern the study of representation problems, both for (skew–)hermitian spaces and algebras with involution. Section 2.5 deals with the Noetherian case, and in section 2.6, I present some representation results for (skew–)hermitian spaces over noncommutative valuation rings. The results in the latter section are not used in the rest of the thesis.

I expect that many of the results presented in chapters 1 and 2 are not new, but since I could not find a reference for them as such, I provide explicit proofs.

Chapter 3 connects the specialisation results in this thesis with the generic isotropy results. There, I collect properties of the varieties of isotropic ideals, associated to an algebra with involution over a field. I also include the relevant Schur index reduction formulas from [52, 53]. Furthermore, in Proposition 3.9, I present a first proof of Theorem 4.9 in the version with the index of algebras with involution, in the restrictive geometric setting, that is, in the setting where the algebra with involution is already defined over some common subfield of F and κ.

Chapter 4 then deals with the full specialisation problem (section 4.2), and collects the results for Henselian valuation rings (section 4.3). The lifting results for isotropy and hyperbolicity are contained in section 4.3.1, and the isomorphism results on Question 1 are contained in section 4.3.2. The last section (section 4.4) is concerned with the good reduction statements. In some of the arguments in sections 4.2 and 4.3.1, I use the theory of value functions on vector spaces and algebras, developed in [62, 73, 74]. I give an overview of the basics of this theory in section 4.1. The flow of chapter 4 continues in chapter 5, where Question 1 is the central point of study, both in a local setting (sections 5.1 and 5.2) as in a global setting (sections 5.3 and 5.4).

The last chapter (chapter 6) mainly concerns Question 3, which is treated in sections 6.3 – 6.8. The first two sections of chapter 6 include results on involution analogues of Pfister forms, which were not yet mentioned in this introduction. Pfister forms play a central role in quadratic form theory. There are two involution analogues, which have already been studied in the literature. One can consider the *totally decomposable algebras with involution*, i.e. algebras with involution that are isomorphic to a tensor product of quaternion algebras with involution. These reflect the decomposability of Pfister forms into a tensor product of binary forms. On the other hand, one can start from the fact that Pfister forms are either anisotropic or hyperbolic over any field extension of the ground field, and consider algebras with involution with this property, which we call *Pfister algebras with involution*. It was conjectured in [8] that Pfister algebras with involution of degree a power of two are exactly the totally decomposable algebras with involution. One implication of this conjecture has been confirmed by recent work of K.J. Becher, N.A. Karpenko and J.–P. Tignol (see [8, 37]), namely that totally decomposable algebras with involution are Pfister algebras with involution. The other implication has been confirmed in low degree cases but is open in general. In section 6.1, I include one new case where the conjecture holds, namely for algebras of degree 8 with symplectic involution. The argument I present there was communicated to me by J.–P. Tignol.
In section 6.2, I take the following quadratic form result as a starting point: if a Pfister form is a factor of another Pfister form, then the complementing factor can also be chosen to be a Pfister form. I present a weak analogue of this result for algebras with involution, where “Pfister form” is replaced by “Pfister algebra with involution” (Theorem 6.22).

As indicated above, the remaining part of chapter 6 concerns the study of Question 3. The generic hyperbolicity problem is treated first (section 6.3), followed by the generic isotropy problem in section 6.4. The case of orthogonal involutions is pursued in more detail in section 6.5. In section 6.6, I then give examples of some classes of algebras with involution for which there does not exist a generic isotropy field. In section 6.7, I zoom in on the question for which algebras with involution, the isotropy behaviour is characterised by that of a quadratic form, especially in the case of totally decomposable algebras with involution (Theorem 6.75). The closing section of this thesis provides characterisations for the existence of a generic isotropy field for some low degree algebras with involution.

Nederlandse samenvatting

> Je hebt een intuïtief gevoel, ziet een platonische glimp van een oplossing en je moet het blootleggen. Zoals de dichter gedichten moet schrijven, moet de wiskundige zijn wiskunde doen. Zonder voel je je onbehaaglijk.
>
> Hendrik Lenstra

In de jaren ’70 van de vorige eeuw ontwikkelde M. Knebusch de generieke splijtingstheorie van kwadratische vormen (cf. [40, 41]). De basis van deze theorie is het bestaan van een generiek isotropieveld voor een niet–singuliere kwadratische vorm, verschillend van het hyperbolisch vlak. Dit veld wordt bekomen als een transcendente uitbreiding van het grondveld (i.e. door het toevoegen van variabelen). Elke velduitbreiding van het grondveld waar de kwadratische vorm isotroop wordt, wordt bekomen uit het generiek isotropieveld door specialisatie. M. Knebusch maakte gebruik van dit feit om aan te tonen dat het isotropiegedrag van de kwadratische vorm over een willekeurige velduitbreiding van het grondveld, bepaald wordt door zijn gedrag over een keten van generieke isotropievelden.

Twee aspecten van de generieke splijtingstheorie van kwadratische vormen waren het startpunt voor deze thesis. Enerzijds het isotropiegedrag van kwadratische vormen onder plaatsen van een veld naar een ander veld, en anderzijds het concept van een generiek isotropieveld voor een kwadratische vorm. Beide aspecten worden bestudeerd in de context van algebra’s met involutie over velden. Dit zijn objecten die in nauw verband staan met kwadratische en bilineaire vormen, in de zin dat men elke symmetrische of alternerende bilineaire ruimte over een veld een geassocieerde algebra met involutie heeft (cf. [45]). Dit zijn precies de gespleten centraal simplele algebra’s met involutie van de eerste soort. Deze involutie is orthogonaal als de bilineaire ruimte symmetrisch is, en symplectisch als de bilineaire ruimte alternerend is. Er bestaan ook algebra’s met involutie van de tweede soort. Het onderscheid wordt bepaald door de restrictie van de involutie tot het centrum van de algebra. Als deze restrictie de identiteit is, dan spreekt men van een involutie van de eerste soort. In het andere geval spreekt men van een involutie van de tweede soort.

In de laatste decennia zijn algebra's met involutie een belangrijk punt van studie geweest, in het bijzonder door hun nauwe relatie met bepaalde klassieke algebraïsche groepen. Sinds 1998 hebben ze hun eigen standaardwerk: The book of involutions ([45]). Dit boek benadert involuties vanuit verschillende invalshoeken. Het bevat een studie van involutie analogons van veel concepten uit de theorie van kwadratische vormen, zoals de discriminant, de Clifford algebra, similitudes en gelijkvormigheidsfactoren. In het boek worden ook verschillende resultaten rond isotropie en hyperboliciteit aangehaald, die geworteld zijn in de kwadratische vormen theorie. Verder wordt de connectie van algebra's met involutie met algebraïsche groepen ook grondig bestudeerd. In deze thesis focus ik op isotropie en hyperboliciteits resultaten enerzijds, en anderzijds bestudeer ik isomorfieproblemen die op een natuurlijke manier naar voren kwamen tijdens mijn werk rond de isotropieproblemen.

Ik ben mijn onderzoek begonnen met het onderzoeken of er een involutie analogon mogelijk is van het concept van een generiek isotropieveld voor kwadratische vormen. Om dit te kunnen doen was het noodzakelijk om het isotropiegedrag van involuties onder specialisatie van een veld naar een ander veld, onder de vorm van een plaats, te bestuderen. In eerste instantie deed ik dit in het "algebraïsch meetkundige geval", namelijk in het geval waarbij de algebra met involutie al gedefinieerd is over een kleiner veld, dat een deelveld is van de beide velden waartussen de plaats gedefinieerd is. De algebra's met involutie die ik bestudeerde op dat moment waren precies van deze vorm. Later onderzocht ik dit specialisatieprobleem in een ruimtere context. Daarbij kwam ik op een natuurlijke manier terecht bij de setting van Azumaya algebra's met involutie over een valuatiering.

De resultaten van de thesis bestaan uit drie delen. Deze volgen niet helemaal de chronologie van de resultaten. Het eerste deel behandelt specialisatie resultaten rond Azumaya algebra's met involutie over valuatieringen, en daarmee samenhangende goeie reductie resultaten. In het tweede deel bestudeer ik bepaalde isomorfieproblemen die het werk in het eerste deel van de thesis op een natuurlijke manier verderzetten. In het derde en laatste deel van de thesis staat de studie van het bestaan van een generiek isotropieveld voor algebra's met involutie over velden centraal. Ik beschrijf hieronder de drie delen meer in detail.

Specialisatie en goede reductie

Zij F een veld en $\mathcal{O}$ een valuatiering van F, en zij λ de geassocieerde plaats van F naar het restklassenveld van κ van $\mathcal{O}$. Onderstel dat we een bilineaire ruimte of een algebra met involutie gegeven hebben die gedefinieerd is over F. Onder bepaalde voorwaarden kan men deze bilineaire ruimte (resp. algebra met involutie) specialiseren naar een bilineaire ruimte (resp. algebra met involutie) over κ. Het is dan heel natuurlijk om de vraag te stellen of dit residu object bepaalde eigenschappen overerft van het originele object. In dit deel van de thesis bestudeer ik specialisatievragen voor algebra's met involutie over velden, die gemotiveerd zijn door specialisatieresultsaten voor symmetrische bilineaire ruimtes.

Zij (V, b) een symmetrische bilineaire ruimte over F (i.e. niet–singulier). Om (V, b) op een goede manier te kunnen specialiseren naar een bilineaire ruimte over κ, moet (V, b) gedefinieerd zijn over $\mathcal{O}$. Dit betekent dat (V, b) bekomen wordt door scalaire extensie uit een symmetrische bilineaire ruimte over $\mathcal{O}$. In dit geval zegt met dat (V, b) *goede reductie heeft voor* λ. Symmetrische bilineaire ruimtes over $\mathcal{O}$ hebben een geassocieerde residu bilineaire ruimte over κ. Zij (V, b) een symmetrische bilineaire ruimte over F met goeie reductie voor λ. Is het mogelijk om op een natuurlijke manier een symmetrische bilineaire ruimte over κ te associëren aan (V, b)? Dit komt neer op de vraag of symmetrische bilineaire ruimtes over $\mathcal{O}$ die isometrisch worden over F, al isometrisch zijn over κ. In [42, (1.15)] werd aangetoond dat dit het geval is als 2 inverteerbaar is in $\mathcal{O}$, en in feite is het resultaat iets sterker. Namelijk, symmetrische bilineaire ruimtes over $\mathcal{O}$ die isometrisch worden over F, zijn al isometrisch over $\mathcal{O}$ (zie [66, (4.6.3)]). Onderstel dat 2 inverteerbaar is in $\mathcal{O}$ en zij (V, b) een symmetrische bilineaire ruimte over F met goede reductie voor λ. Dan impliceert het resultaat in [42, (1.20)] dat, als (V, b) isotroop is over F, dat de geassocieerde residu bilineaire ruimte isotroop is over κ.

Als een symmetrische bilineaire ruimte over F goeie reductie heeft voor λ, dan komt de geassocieerde algebra met involutie van een Azumaya algebra met involutie over $\mathcal{O}$. Het isotropiegedrag van de bilineaire ruimte onder λ wordt dan overgedragen op de geassocieerde algebra met involutie. Het doel van het eerste deel van de thesis is het bestuderen van algebra's met involutie over F die komen van Azumaya algebra's met involutie over $\mathcal{O}$, en dit niet enkel in het gespleten geval, en het onderzoeken van hun isotropiegedrag onder λ.

Zij R een commutatieve ring. Zij S een commutatieve ring die ofwel gelijk is aan R, ofwel van de vorm $R[z]$, waarbij z een element is zodat $z^2 = az + b$, met $a, b \in R$ zodat $a^2 + 4b$ inverteerbaar is in R. Zij $\mathcal{A}$ een Azumaya algebra met centrum S. Zij σ een R−lineaire involutie op $\mathcal{A}$, en als $S \neq R$, onderstel dat de restrictie van σ tot S gelijk is aan het niet–triviale R−automorfisme van S. Het paar $(\mathcal{A}, \sigma)$ noemen we een *R−algebra met involutie*. Men zegt dat σ van de eerste soort is als $S = R$, en anders van de tweede soort.

In de thesis werk ik bijna uitsluitend in de situatie waarin R een domein is dat integraal gesloten is in zijn breukenveld. In dat geval is S ofwel een domein, ofwel isomorf met $R \times R$. In het laatste geval is $\mathcal{A} \cong \mathcal{B} \times \mathcal{B}^{\text{op}}$, voor een Azumaya algebra $\mathcal{B}$ over R. Onderstel dat R een veld is. Dan is S ofwel een veld, en in dat geval is $\mathcal{A}$ een centraal simpele S−algebra, ofwel is $\mathcal{A}$ semisimpel met twee simpele componenten. Het geval van semisimpele algebra's met involutie wordt ook beschouwd in [45], en is eigenlijk heel natuurlijk om de volgende reden. Een centraal simpele algebra met involutie van de tweede soort wordt semisimpel na scalaire extensie naar een algebraïsche sluiting van het fixveld van de involutie binnen het centrum van de algebra. In een poging om het gevallen van centraal simpele en semisimpele algebra's met involutie zo uniform mogelijk te behandelen, introduceer ik in sectie 1.4 de notie van een *balanced* éénzijdig ideaal. De voorwaarde balanced is triviaal voor centraal simpele algebra's, maar beperkt de idealen in het semisimpel geval tot diegene die interessant zijn bij het beschouwen van isotropievragen. Gebruikmakend van de notie balanced, kan ik ook een uniforme definitie geven van de index van een algebra met involutie over een veld. Dit concept werd geïntroduceerd in [45] als een soort van maat voor de isotropie van de involutie.

Het eerste specialisatieresultaat in deze thesis voor Azumaya algebra's met involutie wordt gegeven in Theorem 4.9. Dit is een involutie analogon van het hoger vermelde resultaat voor bilineaire ruimtes. Zij $(\mathcal{A}, \sigma)$ een $\mathcal{O}$−algebra met involutie. Dan zegt de stelling dat een isotroop balanced rechts ideaal van $(\mathcal{A}, \sigma)_F$ specialiseert tot een isotroop balanced rechts ideaal van $(\mathcal{A}, \sigma)_\kappa$, en waarbij de F−dimensie van het eerste ideaal gelijk is aan de κ−dimensie van het tweede ideaal. In het bijzonder zegt dit resultaat dat als $(\mathcal{A}, \sigma)_F$ isotroop (resp. metabolisch) is, dat dan $(\mathcal{A}, \sigma)_\kappa$ ook isotroop (resp. metabolisch) is. In termen van de index stelt het resultaat dat de index van $(\mathcal{A}, \sigma)_F$ bevat is in de index van $(\mathcal{A}, \sigma)_\kappa$.

Het is welbekend dat, in het geval dat $\mathcal{O}$ een Henselse valuatiering is, het soms mogelijk is om eigenschappen van een residu object over κ terug te liften naar het originele object over F. In dat geval vormt de plaats geassocieerd aan $\mathcal{O}$ een tweerichtingsstraat voor deze eigenschappen. Bijvoorbeeld, in het geval dat 2 inverteerbaar is in $\mathcal{O}$, dan zegt het resultaat in [66, (6.2.4)] dat isotropie (resp. hyperboliciteit) van een symmetrische bilineaire ruimte over $\mathcal{O}$, gelift kan worden van κ naar F. In Theorem 4.20, geef ik een versie voor involuties van dit resultaat. Namelijk, ik toon aan, op een paar uitzonderingsgevallen na, dat isotropie (resp. hyperboliciteit) van een $\mathcal{O}$−algebra met involutie gelift kan worden van κ naar F. In het bijzonder is dit steeds mogelijk als 2 inverteerbaar is in $\mathcal{O}$. Zij $(\mathcal{A}, \sigma)$ een $\mathcal{O}$−algebra met involutie. In termen van de index zegt Theorem 4.20 dat de index van $(\mathcal{A}, \sigma)_F$ gelijk is aan de index van $(\mathcal{A}, \sigma)_\kappa$.

In sectie 4.4 introduceer ik een notie van goede reductie voor algebra's met involutie over velden met betrekking tot plaatsen. Een F−algebra met involutie heeft *goede reductie* *voor* λ als die verkregen wordt door scalaire extensie uit een $\mathcal{O}$−algebra met involutie. De volgende vraag is dan zeer natuurlijk. Zijn $\mathcal{O}$−algebra's met involutie die isomorf worden over F, ook isomorf over κ? In Theorem 4.37 bewijs ik dat dit inderdaad het geval is als 2 inverteerbaar is in $\mathcal{O}$. In het bewijs is het index resultaat voor Henselse valuatieringen cruciaal (Theorem 4.20). De goede reductie resultaten voor involuties hebben me geïnspireerd om ook andere isomorfie problemen voor algebra's met involutie te beschouwen. De studie van die problemen vormt het tweede deel van de thesis, en de resultaten voor Henselse valuatieringen bleken ook voor de bewijzen daar een belangrijke rol te spelen.

Enkele isomorfie problemen

De kern van het tweede deel van de thesis is de studie van het volgende isomorfieprobleem, en in de geest van het eerste deel van de thesis, beschouw ik dit probleem vooral in een context gerelateerd aan valuatieringen.

Vraag 1. Zij R een domein met breukenveld F. Zij $(\mathcal{A},\sigma)$ en $(\mathcal{A}',\sigma')$ twee R−algebra's met involutie. Onderstel dat $(\mathcal{A},\sigma)_F \cong_F (\mathcal{A}',\sigma')_F$. Volgt hieruit dat $(\mathcal{A},\sigma) \cong_R (\mathcal{A}',\sigma')$?

In de literatuur gebruikt men de term *rationaal isomorf* voor objecten die gedefinieerd zijn over een domein en die isomorf worden over het breukenveld van dat domein. In die terminologie gaat Vraag 1 dus over het probleem voor welke domeinen R men kan besluiten dat rationaal isomorfe R−algebra's met involutie, isomorf zijn. Men kan zeker geen positief antwoord verwachten voor domeinen in het algemeen. In Example 5.2 geef ik, geïnspireerd door gesprekken met M. Ojanguren, een eenvoudig tegenvoorbeeld van twee involuties op dezelfde algebra die rationaal isomorf zijn maar niet isomorf, waarbij R een bepaald Hensels lokaal domein is.
Vraag 1 is reeds bestudeerd in de literatuur voor reguliere lokale ringen. In [56] geeft de auteur een positief antwoord in het geval waar R een reguliere lokale ring is die een veld van karakteristiek niet 2 bevat. In het bewijs maakt hij gebruik van het feit dat er een positief antwoord is voor discrete valuatieringen. Dit volgt uit meer algemene resultaten voor bepaalde algebraïsche groepen in [55].

Het eerste resultaat rond Vraag 1 in deze thesis is dat rationaal isomorfe R−algebra's met involutie isomorf zijn, in het geval dat R een Henselse valuatiering is van F en waarbij 2 inverteerbaar is in R (Theorem 4.34). Dit is een cruciale stap in het bewijs van het goede reductie resultaat in sectie 4.4. Het isomorfie probleem in Vraag 1 wordt dan verder bestudeerd in hoofdstuk 5. De resultaten daar zijn tot stand gekomen in samenwerking met J. Van Geel. We geven een positief antwoord op Vraag 1 in het geval dat R een valuatiering is van F met 2 inverteerbaar in R. Toen we aan de bewijzen aan het werken waren, hebben we gemerkt dat de argumenten, mits een aantal aanpassingen, ook konden gebruikt worden in het geval dat R een semilokaal Bézout domein is, i.e. een doorsnede van eindig veel valuatieringen van F. Het belangrijktste deel van het bewijs bestaat erin om voor een $R-$algebra met involutie $(\mathcal{A},\sigma)$, op eenheden in R na een lokale karakterisering te geven van de gelijkvormigheidsfactoren van $(\mathcal{A},\sigma)_F$ (Theorem 5.13). We maken hierbij gebruik van een normargument, dat gebaseerd is op een approximatiestelling voor valuaties van P. Ribenboim, en door gebruik te maken van de resultaten voor Henselse valuatieringen in hoofdstuk 4.

In sectie 5.3 tonen we aan dat de karakterisering van gelijkvormigheidsfactoren in Theorem 5.13 op een meer directe manier kan bekomen worden in het geval dat R een discrete valuatiering is. Door de Noetherse eigenschap van zulke valuatieringen uit te buiten, kon ik een representatieresultaat voor $R-$algebra's met involutie bewijzen (Theorem 2.39). Door dit laatste resultaat te gebruiken, is het bewijs van de karakterisering van gelijkvormigheidsfactoren minder afhankelijk van de resultaten voor Henselse valuatieringen.

Het geval van discrete valuatieringen kwam op een natuurlijke manier naar voor toen we het volgende probleem beschouwden (aangegeven door K.J. Becher en A. Quéguiner–Mathieu), waarin de rol van F gespeeld wordt door het functieveld van een quaternionenalgebra, gedefinieerd over een kleiner veld k. Het is welbekend dat alle $k-$valuaties van een dergelijk functieveld discreet zijn.

Vraag 2. Zij k een veld van karakteristiek verschillend van 2, en zij Q een $k-$quaternionen delingsalgebra. Zij B een centraal simpele $k-$algebra Brauer equivalent met Q, en zij verder τ and τ' twee orthogonale involuties op B. Onderstel dat $(B,\tau)_{k(Q)} \simeq_{k(Q)} (B,\tau')_{k(Q)}$. Volgt hieruit dat $(B,\tau) \simeq_k (B,\tau')$?

Het is bekend dat er een positief antwoord is op Vraag 2 in enkele gevallen van lage graad, als de graad van B hoogstens 4 is, en in het geval dat B graad 6 heeft en τ triviale discriminant. De resultaten voor de gevallen van graad 2 en 4 volgen uit [72, (3.6), (3.10)] en voor graad 6 uit [45, (15.7)]. In sectie 5.4 zoeken we naar voorwaarden op (B,τ) om te kunnen besluiten wanneer een niet–nul element uit $k(Q)$ gelijk is aan een gelijkvormigheidsfactor van $(B,\tau)_{k(Q)}$ maal een niet–nul element van k. De resultaten die we daar bekomen, impliceren dat Vraag 2 een positief antwoord heeft in het speciale geval waarbij τ hyperbolisch wordt over een kwadratische velduitbreiding van k die Q splijt (Corollary 5.44).

Generieke isotropie en hyperboliciteitsvelden

In het derde en laatste deel van de thesis (chapter 6) keren we terug naar de kern van de generieke splijtingstheorie voor kwadratische vormen, namelijk het bestaan van een generiek isotropieveld.

Vraag 3. Zij F een veld van karakteristiek verschillend van 2, en zij (B,τ) een $F-$algebra met involutie van de eerste soort. Wanneer bestaat er een velduitbreiding N/F zodat τ_N isotroop (resp. hyperbolisch) is, en zodat voor elke velduitbreiding L/F waarvoor τ_L isotroop (resp. hyperbolisch) is, er een $F-$plaats bestaat van N naar L? We noemen een veld met die eigenschappen een *generiek isotropie (resp. hyperboliciteits) veld voor τ.*

Ik toon aan dat een generiek isotropieveld niet altijd bestaat. Er zijn reeds tegenvoorbeelden van graad 4 (Corollary 6.42). Voor niet–singuliere kwadratische vormen over F verschillend van het hyperbolisch vlak, kan een generiek isotropieveld bekomen worden door het functieveld te beschouwen van de projectieve kwadriek, die geassocieerd is met de vorm. Ik vertrek ook vanuit de invalshoek van variëteiten bij de studie van Vraag 3.
Zij (B,τ) een $F-$algebra met involutie van de eerste soort. Ik bestudeer bepaalde variëteiten $\mathrm{IV}_i(B,\tau)$ (waarbij i een natuurlijk getal is in een zeker interval), wiens rationale punten over een velduitbreiding M/F isotrope idealen zijn van $(B,\tau)_M$ van een zekere dimensie. Deze variëteiten werden reeds bestudeerd in de literatuur ([52, 53]), in het bijzonder door hun link met de algebraïsche groepen die gerateerd zijn aan $F-$algebra's met involutie. Ik onderzoek het isotropiegedrag van $F-$algebra's met involute over de functievelden van deze variëteiten, en maak daarbij veelvuldig gebruik van Schur index reductieformules voor deze functievelden uit [52, 53].

In Proposition 6.46 toon ik aan, onder bepaalde voorwaarden, dat als er een generiek isotropieveld bestaat voor τ, dat het dan gerealiseerd kan worden als het functieveld van een zekere $\mathrm{IV}_i(B,\tau)$. In het algemeen, om te kunnen aantonen dat het functieveld van een zekere $\mathrm{IV}_i(B,\tau)$ een generiek isotropieveld is voor τ, is het noodzakelijk om het isotropiegedrag van τ over de functievelden van de andere $\mathrm{IV}_j(B,\tau)$, $j\neq i$, te onderzoeken. In sommige gevallen is het voldoende om slechts één ander functieveld te onderzoeken. Corollary 6.62 is een resultaat van deze aard. Het toont aan dat als τ orthogonaal is, en $\deg(B)\geqslant 3\,\mathrm{ind}(B)$, dat het voldoende is om het isotropiegedrag van τ over het functieveld van $\mathrm{IV}_1(B,\tau)$ te onderzoeken, om te kunnen besluiten of het functieveld van $\mathrm{IV}_{\mathrm{ind}(B)}(B,\tau)$ een generiek isotropieveld is voor τ. In de praktijk kan het nog altijd moeilijk zijn om het isotropiegedrag van τ over het functieveld van $\mathrm{IV}_1(B,\tau)$ precies te bepalen. Om hieraan tegemoet te komen, heb ik in het geval dat B Schur index 2 heeft, gezocht naar een aantal voldoende voorwaarden die mogelijk eenvoudiger na te gaan zijn (Proposition 6.63). Ik heb ook een aantal expliciete karakteriseringen bekomen van een generiek isotropieveld voor algebra's met involutie van lage graad (2, 4, 6 en ook deels 8).

Het gedeelte van Vraag 3 rond generieke hyperboliciteitsvelden behandel ik apart. In dat geval zijn er hoogstens twee variëteiten die een rol spelen. Het is dan eenvoudig om aan te tonen dat symplectische involuties steeds een generiek hyperboliciteitsveld hebben. In het orthogonaal geval hangt het bestaan af van het gedrag van de Clifford algebra geassocieerd aan de algebra met involutie.

Structuur van de thesis

In hoofdstuk 1 bestudeer ik Azumaya algebra's met involutie over valuatieringen, en algemener semilokale Bézout domeinen. De laatste soort ringen kwamen op een natuurlijke manier naar voren toen ik Azumaya algebra's met involutie van de tweede soort beschouwde. In dat geval hoeft het centrum van de algebra geen valuatiering te zijn, maar kan het een doorsnede van twee valuatieringen zijn. In sectie 1.3 bestudeer ik het gedrag van Azumaya algebra's met centrum een semilokaal Bézout domein, na scalaire extensie naar het breukenveld van dit domein. Om dit te doen beschouw ik het natuurlijke morfisme van de Brauer groep van een domein naar de Brauer groep van zijn breukenveld. Ik geef aan dat dit morfisme injectief is voor semilokale Bézout domeinen. Verder maak ik ook de link met een aantal resultaten in de literatuur rond de Brauer groep van een domein.
In sectie 1.4 ligt de focus op algebra's met involutie over velden. Ik vermeld een aantal gekende eigenschappen van de Clifford algebra van een orthogonale involutie. Verder introduceer ik in die sectie ook de notie van balanced idealen, en een definitie van de index in termen van die balanced idealen.

Soms kan het nuttig zijn om een probleem voor algebra's met involutie te benaderen vanuit het oogpunt van (scheef–)hermitische ruimtes. Om die wisselwerking te verduidelijken bestudeer ik in hoofdstuk 2 (scheef–)hermitische ruimtes over een Azumaya algebra met involutie met centrum een semilokaal Bézout domein. Als een veralgemening van het geval van bilineaire ruimtes over velden, kan men aan de (scheef–)hermitische ruimtes die beschouwd worden in chapter 2 een Azumaya algebra met involutie associëren. Er is speciale aandacht voor de relatie tussen gelijkvormigheid van (scheef–) hermitische ruimtes en isomorfie van hun geassocieerde Azumaya algebra's met involutie. Deze resultaten zijn belangrijk bij het de studie van Vraag 2 in latere hoofdstukken. De laatste twee secties van hoofdstuk 2 gaan over bepaalde representatiestellingen voor (scheef–)hermitische ruimtes enerzijds, en Azumaya algebra's met involutie anderzijds. Sectie 2.5 behandelt het Noetherse geval, en in sectie 2.6 concentreer ik me op (scheef–) hermitische ruimtes over niet–commutatieve valuatieringen. De resultaten in deze laatste sectie worden niet gebruikt in de rest van de thesis.

Ik verwacht dat veel van de resultaten in hoofdstuk 1 en 2 niet nieuw zijn, maar vermits ik er geen referentie voor kon vinden in die vorm, geef ik op veel plaatsen expliciete bewijzen.

Hoofdstuk 3 geeft een link tussen de specialisatieresultaten in deze thesis en de resultaten rond generieke isotropie. In dat hoofdstuk geef ik een overzicht van de eigenschappen van de variëteiten $\mathrm{IV}_i(B,\tau)$, geassocieerd aan een $F-$algebra met involutie (B,τ). Ik vermeld ook de relevante Schur index reductieformules uit [52, 53] voor de functievelden van deze variëteiten. Bovendien geef ik in Proposition 3.9 een eerste bewijs van de index versie van Theorem 4.9 in de de restrictieve algebraïsch meetkundige setting, i.e. de setting waarbij de Azumaya algebra met involutie reeds gedefinieerd is over een gemeenschappelijk deelveld van F en κ.

Hoofdstuk 4 bevat dan de studie van het algemenere specialisatieprobleem voor Azumaya algebra's met involutie over valuatieringen (sectie 4.2), en bevat de resultaten voor Henselse valuatieringen (sectie 4.3). De resultaten rond het liften van isotropie en hyperboliciteit worden aangetoond in sectie 4.3.1, and de isomorfieresultaten rond Vraag 1 in sectie 4.3.2. De laatste sectie bevat de goede reductie resultaten. In een aantal van de argumenten in de secties 4.2 en 4.3.1, maak ik gebruik van de theorie van *value functions* op vectorruimtes en algebra's, ontwikkeld in [62, 73, 74]. Ik geef een overzicht van de basisconcepten en eigenschappen uit deze theorie in sectie 4.1. Hoofdstuk 5 gaat verder in de geest van hoofdstuk 4. De studie van Vraag 1 staat daar centraal, zowel in een lokale context (secties 5.1 en 5.2) als in een globale context (secties 5.3 en 5.4).

In het laatste hoofdstuk draait het in de eerste plaats om Vraag 3. Deze wordt onderzocht in secties 6.3 – 6.8. The eerste twee secties van chapter 6, bevatten resultaten rond twee analogons van Pfister formen voor algebra's met involutie, die nog niet vermeld werden in deze samenvatting. Pfister vormen spelen een centrale rol in de theorie van kwadratische vormen. Er zijn twee analogons voor involuties, die reeds bestudeerd werden in de literatuur. Men kan *totaal decomposeerbare algebra's met involutie* beschouwen. Dit zijn algebra's met involutie die isomorf zijn met een tensor product van quaternionenalgebra's met involutie. Dit analogon reflecteert de decomposeerbaarheid van Pfister vormen als een tensor product van binaire vormen. Langs de andere kant kan men vertrekken van het feit dat Pfister vormen ofwel anisotroop, ofwel hyperbolisch zijn over elke velduitbreiding van het grondveld. Algebra's met involutie met deze eigenschap noemen wij *Pfister algebra's met involutie*. In [8] bracht de auteur de conjectuur naar voren dat de Pfister algebra's met involutie van graad een macht van twee, precies de totaal decomposeerbare algebra's met involutie zijn. Eén impicatie van deze conjectuur is bevestigd door recent werk van K.J. Becher, N.A. Karpenko and J.–P. Tignol in [8, 37], namelijk dat totaal decomposeerbare algebra's met involutie Pfister algebra's met involutie zijn. De andere implicatie is gekend in een paar specifieke gevallen maar is in het algemeen nog open. Sectie 6.1 bevat één nieuw geval waar de conjectuur effectief geldt, namelijk voor algebra's van graad 8 met symplectische involutie. Het bewijs voor dit geval werd mij doorgegeven door J.–P. Tignol.

In sectie 6.2 vertrek ik van het volgende resultaat voor kwadratische vormen: als een Pfister vorm een factor is van een andere Pfister vorm, dan kan de andere factor ook als Pfister vorm gekozen worden. Ik geef een zwak analogon van dit resultaat voor algebra's met involutie, waarbij "Pfister vorm" vervangen wordt door "Pfister algebra met involutie" (Theorem 6.22).

Zoals hierboven aangegeven staat de studie van Vraag 3 centraal in de rest van hoofdstuk 6. Het probleem van het bestaan van een generiek hyperboliciteitsveld wordt eerst behandeld (sectie 6.3), gevolgd door het generiek isotropieprobleem in sectie 6.4. Het geval van orthogonale involuties wordt meer in detail bekeken in sectie 6.5. In de hiernavolgende sectie geef ik een aantal voorbeelden van klassen van algebra's met involutie waarvoor er geen generiek isotropieveld bestaat. In sectie 6.7 focus ik op de vraag voor welke algebra's met involutie het isotropiegedrag gekarakteriseerd wordt door dat van een kwadratische vorm. Ik bekijk dit probleem in het bijzonder voor totaal decomposeerbare algebra's met involutie. Ik besluit de thesis met een aantal karakteriseringen voor het bestaan van een generiek isotropieveld voor algebra's met involutie van lage graad.

1 Algebras with involution over rings

> We may always depend on it that algebra, which cannot be translated into good English and sound common sense, is bad algebra.
>
> William Kingdom Clifford

In this chapter, we study Azumaya algebras (with involution) with center a semilocal Bézout domain, or a separable quadratic extension of a semilocal Bézout domain. For such centers, we will see that many properties of central simple algebras over fields carry over to Azumaya algebras. For instance, the theorem of Skolem and Noether holds, i.e. all central automorphisms of the Azumaya algebra are inner. There is also an analogue of Wedderburn's theorem, characterising these kinds of Azumaya algebras as matrix algebras over an Azumaya algebra without zero divisors.
We further show that isomorphism of Azumaya algebras over a semilocal Bézout domain can be detected rationally, by studying the map from the Brauer group of a semilocal Bézout domain to the Brauer group of its fraction field. This is done in section 1.3. In the last section, we zoom in on algebras with involution over fields. We give an overview of known results on the Clifford algebra of a central simple algebra with orthogonal involution, and we uniformise some arguments from [45] for right ideals, to work both for simple and semisimple algebras.

We don't claim that the results presented in this chapter are new. We expect that they will look very natural to people who are familiar with Azumaya algebras over rings. Since we did not find references for the statements in the specific situation of valuation rings and semilocal Bézout domains, we give explicit arguments for many of them, exploiting the properties of valuation rings and semilocal Bézout domains.

We set some general notation and some conventions for the rest of this thesis. A ring will always mean an associative ring with unit, and a domain will always mean a commutative ring without zero divisors. Let C be a (not necessarily commutative) ring. We denote the group of units in C by $C^\times$ and the center of C by $Z(C)$. Suppose that C has *invariant basis number*, i.e. for any free right or left $C-$module, every $C-$basis of this module has the same cardinality. This cardinality is usually called the rank, but we will use the term *dimension*. Let M be a finitely generated, free right (resp. left) $C-$module. We denote its dimension over C by $\dim_C(M)$, and we call M a finite–dimensional right (resp. left) $C-$module. Commutative rings have invariant base number by [49, (III.4.2)] and the rings we will mostly be working with also have invariant basis number (e.g. left and right Bézout rings, see [11, p. 78]). Let $(e_1, \ldots, e_n)$ be a $C-$basis for M. Let C' be a ring extension of C. Since the tensor product over C commutes with direct sums by [61, (2.8)], it follows that $(e_1 \otimes 1, \ldots, e_n \otimes 1)$ is a $C'-$basis for $M_{C'} = M \otimes_C C'$ (where C' is considered as left $C-$module). We will denote $e_i \otimes 1$ again by e_i. We will use this repeatedly for different kinds of rings, without explicitly referring to [61] in the sequel.

1.1 Bézout rings

In this section, we study Bézout rings and elementary divisor rings, with an emphasis on the case of commutative semilocal rings.

Let C be a (not necessarily commutative) ring. Let $a, b \in C$. Then one says that *a is a total divisor of b* if $CbC \subset aC \cap Ca$. One calls C a *right (resp. left) Bézout ring* if every finitely generated right (resp. left) ideal of C is principal. C is called an *elementary divisor ring* if for any all $m, n \in \mathbb{N}$ and any $(m \times n)-$matrix U over C, there exist invertible matrices P, Q over C such that PUQ is a matrix where nonzero elements only appear on the main diagonal and such that these diagonal entries are subsequent total divisors of each other.

1.1 Proposition. *Every elementary divisor ring is a left and right Bézout ring.*

Proof. It suffices to show that left ideals with two generators are principal, and similarly for right ideals. This is for instance shown in [35, p. 465]. □

The *Jacobson radical* of C is the intersection of the maximal right ideals of C, and is denoted by $J(C)$. It is shown in [46, (2.4.3)] that for an element $y \in C$, we have that

$y \in J(C)$ if and only if $1 - xyz \in C^\times$, for all $x, z \in C$. It follows that $J(C)$ is a two–sided ideal of C. The ring C is called *semilocal* if $C/J(C)$ is an (Artinian) semisimple ring. If C is commutative then C is semilocal if and only if it has finitely many maximal ideals (see e.g. [46, (7.20.2)]).

Let F be a field. A *valuation ring of* F is a subring $\mathcal{O}$ of F such that for every $x \in F$, we have that $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$. Let $\mathcal{O}$ and $\mathcal{O}'$ be two valuation rings of F. They are called *comparable* if $\mathcal{O} \subseteq \mathcal{O}'$ or $\mathcal{O}' \subseteq \mathcal{O}$, and *incomparable* otherwise. We denote the smallest overring of $\mathcal{O}$ and $\mathcal{O}'$ inside F by $\mathcal{O}\,\mathcal{O}'$. Then $\mathcal{O}$ and $\mathcal{O}'$ are called *dependent* if $\mathcal{O}\,\mathcal{O}' \neq F$, and *independent* otherwise.

In the following propositions, we show that semilocal Bézout domains are closely related to valuation rings, and that local Bézout domains are exactly valuation rings.

1.2 Proposition. *Let T be a semilocal domain. Then T is a Bézout domain if and only if the localisation of T at each of its maximal ideals is a valuation ring. In that case, T is equal to the intersection of these valuation rings, which are, moreover, pairwise incomparable.*

Proof. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_\ell$ be the different maximal ideals of T. It is clear that $T \subset \cap_{i=1}^{\ell} T_{\mathfrak{m}_i}$. For the converse, let $x \in \cap_{i=1}^{\ell} T_{\mathfrak{m}_i}$ be arbitrary. Then there exist $a_1, \dots, a_\ell \in T$ and $b_i \in T \setminus \mathfrak{m}_i$ for $i = 1, \dots, \ell$ such that $x = a_1/b_1 = \dots = a_\ell/b_\ell$. Using that $b_1T + \dots + b_\ell T = T$, one can easily show that $x \in T$.
Suppose that T is a semilocal Bézout domain. Let $\mathfrak{m}$ be an arbitrary maximal ideal of T and let $a, b \in T$. If one of $a, b \notin \mathfrak{m}$, then it follows immediately that one of $a/b, b/a \in T_\mathfrak{m}$. So, suppose that $a, b \in \mathfrak{m}$ and nonzero. Then $aT + bT = dT$ for some $d \in T$, since T is a Bézout domain. We write $a = da', b = db'$ with $a', b' \in T$. Then $a/b = a'/b'$ and $b/a = b'/a'$. Since $a'T + b'T = T$, it follows that one of a', b' does not belong to $\mathfrak{m}$. Hence, $T_\mathfrak{m}$ is a valuation ring. Furthermore, the valuation rings $T_{\mathfrak{m}_1}, \dots T_{\mathfrak{m}_\ell}$ are pairwise incomparable, since $\mathfrak{m}_1, \dots, \mathfrak{m}_\ell$ are different maximal ideals of T.

Suppose conversely that the localisation of T at each of its maximal ideals is a valuation ring. Then T is an intersection of pairwise incomparable valuation rings. The fact that T is a Bézout domain then follows from [24, (III.5.1)], where it is shown more generally that semilocal Prüfer domains are Bézout domains. □

We can make the statement of Proposition 1.2 stronger.

1.3 Proposition. *Let T be a semilocal domain. Then the following are equivalent:*

(i) T is a Bézout domain.

(ii) T is an elementary divisor domain.

(iii) T is the intersection of finitely many valuation rings of its fraction field.

Proof. We denote the fraction field of T by F. That (ii) implies (i) is the statement of Proposition 1.1. For semilocal domains, the converse holds by [24, (III.6.6)]. We have that (i) implies (iii) by Proposition 1.2, and the converse follows from [24, (III.5.1)] since an intersection of finitely many valuation rings of F is also an intersection of finitely many incomparable valuation rings of F. □

It is well known that the overrings of a valuation ring within its fraction field are again valuation rings, and that the set of these overrings is linearly ordered. We show below that the overrings of a semilocal Bézout domain within its fraction field are again semilocal Bézout domains, but here, the set of overrings need not be linearly ordered anymore. However, we can describe what the overrings look like. The maximal ideals of a semilocal Bézout domain have been described in [21]. Since we use this result frequently, we include it here for convenience.

1.4 Proposition. *Let F be a field and let $\mathcal{O}_1, \ldots, \mathcal{O}_\ell$ be pairwise incomparable valuation rings of F. Let $T = \mathcal{O}_1 \cap \ldots \cap \mathcal{O}_\ell$. For $i = 1, \ldots, \ell$, let $\mathcal{M}_i$ be the maximal ideal of $\mathcal{O}_i$. Then $\mathcal{O}_i = T_{\mathcal{M}_i \cap T}$ for $i = 1, \ldots, \ell$, and $\mathcal{M}_1 \cap T, \ldots, \mathcal{M}_\ell \cap T$ are the different maximal ideals of T.*

Proof. See [21, (3.2.6), (3.2.7)]. □

1.5 Lemma. *Let T be a semilocal Bézout domain and denote its fraction field by F. Let $T \subset T' \subset F$ be an overring. Then T' is a semilocal Bézout domain, and T' does not have more maximal ideals than T.*

Proof. A proof of the fact that T' is a Bézout domain can be found in [7, Proposition 2, Theorem], where it is in fact shown that T' is a localisation of T at a multiplicatively closed subset $\mathcal{S}$.

Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_\ell$ be the different maximal ideals of T. Suppose for the sake of contradiction that T' has at least $\ell + 1$ different maximal ideals, say $\mathfrak{M}_1, \ldots, \mathfrak{M}_{\ell+1}$. Let $\mathfrak{p}_i = \mathfrak{M}_i \cap T$ for $i = 1, \ldots, \ell + 1$. Since T' is a localisation of T, there is a one-to-one correspondence between prime ideals of T not intersecting $\mathcal{S}$ and prime ideals of T' (see e.g. [36, Theorem 34]). It follows that $\mathfrak{p}_1, \ldots, \mathfrak{p}_{\ell+1}$ are different prime ideals of T. Since T only has ℓ maximal ideals, at least two of these prime ideals must be contained in the same maximal ideal. Without loss of generality, we may assume that $\mathfrak{p}_1, \mathfrak{p}_2 \subset \mathfrak{m}_1$. Then $T_{\mathfrak{m}_1} \subset T_{\mathfrak{p}_1} \cap T_{\mathfrak{p}_2}$. Since T is a Bézout domain, it follows from Proposition 1.2 that $T_{\mathfrak{m}_1}$ is a valuation ring, and hence its set of overrings within F is linearly ordered. Without loss of generality, we may assume that $T_{\mathfrak{p}_1} \subset T_{\mathfrak{p}_2}$, and hence, $\mathfrak{p}_2 \subset \mathfrak{p}_1$. This implies that $\mathfrak{M}_2 = \mathfrak{p}_2 T' \subset \mathfrak{p}_1 T' = \mathfrak{M}_1$, which yields $\mathfrak{M}_1 = \mathfrak{M}_2$, a contradiction. Hence, the statement follows. □

1.6 Lemma. *Let F be a field and let $\mathcal{O}_1, \ldots, \mathcal{O}_\ell$ be pairwise incomparable valuation rings of F. Let $T = \mathcal{O}_1 \cap \ldots \cap \mathcal{O}_\ell$.*

(a) Let $V_1, \ldots, V_\ell$ be valuation rings of F such that $\mathcal{O}_i \subset V_i$ for $i \in \{1, \ldots, \ell\}$, with a strict inclusion for at least one i. Then $\mathcal{O}_1 \cap \ldots \cap \mathcal{O}_\ell \subsetneq V_1 \cap \ldots \cap V_\ell$.

(b) Let $T \subsetneq T' \subset F$. Then there exist valuation rings $V_1, \ldots, V_\ell$ of F such that $T' = V_1 \cap \ldots \cap V_\ell$ and for each $i \in \{1, \ldots, \ell\}$, there exists $j \in \{1, \ldots, \ell\}$ such that $\mathcal{O}_j \subset V_i$.

Proof. We first show (a). Clearly, we have that $T \subset V_1 \cap \ldots \cap V_\ell$. Suppose for the sake of contradiction that $T = V_1 \cap \ldots \cap V_\ell$. Since $\mathcal{O}_1, \ldots, \mathcal{O}_\ell$ are pairwise incomparable, it follows from Proposition 1.4 that $\mathcal{O}_1 \cap \ldots \cap \mathcal{O}_\ell = V_1 \cap \ldots \cap V_\ell$ has ℓ different maximal ideals. If $V_1, \ldots, V_\ell$ are not pairwise incomparable, then again invoking Proposition 1.4, it would follow that $V_1 \cap \ldots \cap V_\ell$ has at most $\ell - 1$ maximal ideals, a contradiction. Hence, $V_1, \ldots, V_\ell$ are pairwise incomparable. Then Proposition 1.4 yields that $\{V_1, \ldots, V_\ell\} = \{\mathcal{O}_1, \ldots, \mathcal{O}_\ell\}$. However, since $\mathcal{O}_i \subsetneq V_i$ for at least one i, this gives a contradiction.

Let $T \subsetneq T' \subset F$. By Lemma 1.5, T' is a semilocal Bézout domain with at most ℓ maximal ideals. Let $\mathfrak{M}_1, \ldots, \mathfrak{M}_r$ be the different maximal ideals of T', and let $V_i = T'_{\mathfrak{M}_i}$ for $i = 1, \ldots, r$, and $V_i = F$ for $i = r+1, \ldots, \ell$. Then $T' = V_1 \cap \ldots \cap V_\ell$ by Lemma 1.3. Let $\mathcal{M}_1$ be the maximal ideal of V_1. Then $\mathcal{M}_1 \cap T$ is a prime ideal of T, which is contained in a maximal ideal of T, say $\mathfrak{m}$. Then $T_\mathfrak{m} \subset T_{\mathcal{M}_1 \cap T} \subset T'_{\mathcal{M}_1} = V_1$. This proves (b). □

We now give some properties of left and right Bézout rings without zero divisors, which will be used to obtain a decomposition statement for certain hermitian and skew–hermitian spaces in chapter 2.

1.7 Proposition. *Let Δ be a left and right Bézout ring without zero divisors. Let furthermore $(a_1, \ldots, a_m)$ be a unimodular row over Δ, i.e. there exist $b_1, \ldots, b_m \in \Delta$ such that $\sum_{i=1}^m a_i b_i = 1$. Then there exists an invertible $(m \times m)-$matrix U over Δ having $(a_1, \ldots, a_m)$ as its first row, such that the inverse of U has $(b_1, \ldots, b_m)$ as its first column.*

Proof. The statement holds for Hermite rings by [11, (0.4.1)]. Left and right Bézout rings without zero divisors are Hermite rings by [11, (2.3.4), (2.3.17)]. □

1.8 Remark. In the sequel, we only apply Proposition 1.7 in the case where Δ is an Azumaya algebra with center a semilocal Bézout domain. In that case, the proof can be made more explicit.

1.9 Proposition. *Let Δ be a left and right Bézout ring without zero divisors. Every finitely generated, torsion–free left or right $\Delta-$module is free.*

Proof. This can be shown using Proposition 1.7. See [11, (2.3.19)], where the proof is given for right modules. The proof for left modules is similar. □

Let $m, n \in \mathbb{N}$. We denote the set of $(m \times n)$−matrices over C by $\mathrm{M}_{m,n}(C)$, and if $m = n$, then we we write $\mathrm{M}_n(C)$ for $\mathrm{M}_{n,n}(C)$. Suppose that $n \geqslant m$ and let $d_1, \dots, d_m \in C$. We denote by $\mathrm{diag}(d_1, \dots, d_m)$ the $(n \times m)$−matrix with $d_1, \dots, d_m$ as consecutive entries on the main diagonal, and 0 elsewhere.

We can use the elementary divisor property to prove that certain modules over an elementary divisor domain are free of finite dimension. This result will be used when we study the isotropy behaviour of algebras with involution with respect to places from one field to another in chapter 4. There, we will also give a different proof of the statement below, using so–called value functions.

1.10 Proposition. *Let T be an elementary divisor domain and denote its fraction field by F. Let $\mathcal{V}$ be a finite–dimensional T−module and let $V = \mathcal{V} \otimes_T F$. Let W be a nonzero F−subspace of V. Then $W \cap \mathcal{V}$ is free as a T−module and*

$$\dim_F(W) = \dim_T(W \cap \mathcal{V}).$$

Proof. Suppose that $\dim_T \mathcal{V} = n$ and let $(e_1, \dots, e_n)$ be a T−basis for $\mathcal{V}$. Then $(e_1, \dots, e_n)$ is an F−basis for V. Suppose that $\dim_F W = m$ and let $(f_1, \dots, f_m)$ be an F−basis for W. Clearly, $m \leqslant n$. Let U be the $(n \times m)$−matrix $(u_{ji})_{ji}$ such that $f_i = \sum_{j=1}^{n} e_j u_{ji}$, for $i = 1, \dots, m$. Let e be the row matrix with e_i in the i-th column, and let f be the row matrix with f_j in the j-th column. We have that

$$eU = f.$$

Since F is the fraction field of T, we can write each entry of U as a fraction of elements of T. Let δ be the product of all denominators of the entries. Let D be the $(n \times n)$−matrix $\delta \cdot I_n$. Then $DU \in \mathrm{M}_{n \times m}(T)$. Since T is an elementary divisor domain, there exist invertible matrices P, P' over T such that $P(DU)P'$ is a diagonal $(n \times m)$−matrix over T. It follows that PUP' is a diagonal matrix with entries in F, say

$$PUP' = \mathrm{diag}(d_1, \dots, d_m),$$

with $d_1, \dots, d_m \in F$.

We define row matrices $e' = (e'_1, \dots, e'_n)$ and $f' = (f'_1, \dots, f'_m)$ by

$$e' = eP^{-1} \quad \text{and} \quad f' = fP'.$$

Since the matrices P and P' are invertible over T it follows that $(e'_1, \ldots, e'_n)$ is a T–basis for $\mathcal{V}$ (and hence an F–basis for V), and $(f'_1, \ldots, f'_m)$ is an F–basis for W. Moreover, we have

$$e' \operatorname{diag}(d_1, \ldots, d_m) = e'PUP' = eUP' = fP' = f'.$$

Hence, $f'_1 = e'_1 d_1, \ldots, f'_m = e'_m d_m$. Note that, since $(f'_1, \ldots, f'_m)$ is an F–basis for W, $d_1, \ldots, d_m$ are nonzero. It follows that $(e'_1, \ldots, e'_m)$ is an F–basis for W. We claim that

$$W \cap \mathcal{V} = e'_1 T \oplus \cdots \oplus e'_m T.$$

It is clear that $e'_1 T \oplus \cdots \oplus e'_m T \subset W \cap \mathcal{V}$ since $e'_1, \ldots, e'_m \in W \cap \mathcal{V}$. In order to see the other inclusion, let $w \in W \cap \mathcal{V}$ be arbitrary. Then $w = \sum_{i=1}^m e'_i b_i$, with $b_1, \ldots, b_m \in F$, since $(e'_1, \ldots, e'_m)$ is an F–basis for W. Since $(e'_1, \ldots, e'_n)$ is a T–basis for $\mathcal{V}$ and $w \in \mathcal{V}$, it follows that $b_1, \ldots, b_m \in T$. □

We do not know whether Proposition 1.10 holds more generally for a Bézout domain.

1.2 Azumaya algebras

Let R be a commutative ring. An associative R–algebra $\mathcal{A}$ is called *separable over R* if it is projective as a module over $\mathcal{A} \otimes_R \mathcal{A}^{\mathrm{op}}$, where the module action is given by $(a \otimes b)x = axb$ for all $a, b, x \in \mathcal{A}$. If $\mathcal{A}$ is central, separable and finitely generated as a module over R, then it is called an *Azumaya algebra over R*. An Azumaya algebra is projective as a module over its center, by [43, (III.5.1.1)]. If R is a field, then a finite–dimensional commutative R–algebra is separable over R if it is isomorphic to a finite product of separable field extensions of R (see [43, p. 42]).

1.11 Proposition. *Let $\mathcal{A}$ be an associative R–algebra, finitely generated as an R–module. The following hold:*

(a) Let T be a commutative R–algebra. If the R–algebra $\mathcal{A}$ is separable (resp. an Azumaya algebra), then so is the T–algebra $\mathcal{A} \otimes_R T$.

(b) Suppose that R is a field and that $\mathcal{A}$ is an Azumaya algebra over R. Then $\mathcal{A}$ is central simple.

(c) $\mathcal{A}$ is an Azumaya algebra over R if and only if, for every maximal ideal $\mathfrak{m}$ of R, the $R/\mathfrak{m}$–algebra $\mathcal{A}/\mathfrak{m}\mathcal{A}$ is central simple.

Proof. See [43, (III.5.1.9), (III.5.1.3), (III.5.1.10)] for (a)–(c). □

1.12 Proposition. *Let $\mathcal{A}$ be an Azumaya algebra over R. Every two–sided ideal of $\mathcal{A}$ has the form $\mathfrak{b}\mathcal{A}$, for an ideal $\mathfrak{b}$ of R, and furthermore, $\mathfrak{b}\mathcal{A} \cap R = \mathfrak{b}$.*

Proof. See [4, (3.2)]. □

For the rest of section 1.2, we assume that R is a domain, and we denote its fraction field by F.

1.13 Corollary. *Let $\mathcal{A}$ be an Azumaya algebra over R. Then $J(\mathcal{A}) = J(R)\,\mathcal{A}$.*

Proof. Recall that $J(\mathcal{A})$ is a two–sided ideal by [46, (2.4.3)]. By Proposition 1.12, there exists an ideal $\mathfrak{b}$ of R such that $J(\mathcal{A}) = \mathfrak{b}\mathcal{A}$. Furthermore, using the criterion in [46, (2.4.3)] mentioned below Proposition 1.1, and the fact that $\mathcal{A}^\times \cap R = R^\times$, it follows that $\mathfrak{b} \subset J(R)$, and hence, $J(\mathcal{A}) \subset J(R)\,\mathcal{A}$. We show the other inclusion. Let $\mathfrak{m}$ be a maximal ideal of R. Then $\mathfrak{m}R$ is a maximal two–sided ideal of $\mathcal{A}$ by Proposition 1.12. Hence, it follows from the definition of $J(R)$ that $J(R)\,\mathcal{A}$ is contained in every maximal two–sided ideal of $\mathcal{A}$. Since, by [61, (22.15)], every one–sided maximal ideal of $\mathcal{A}$ contains a two–sided maximal ideal of $\mathcal{A}$, we get that $J(R)\,\mathcal{A} \subset J(\mathcal{A})$, whence the statement. □

By *a separable quadratic R–algebra*, we mean a commutative ring of the form $R[z]$ where z is an element such that $z^2 = az + b$, with $a, b \in R$ such that $a^2 + 4b \in R^\times$. This notion of separable quadratic R–algebra is more restrictive than the one defined in [43]. However, we will apply the results on Azumaya algebras in this section only in the case where R is such that the notion introduced here coincides with the one in [43].

For the rest of section 1.2, we fix a commutative ring S, which is either equal to R, or a separable quadratic R–algebra. In the latter case, we let a, b, z be as above and we write $f(x) = x^2 - ax - b \in R[x]$. If S is a domain then we denote its fraction field by K.

1.14 Proposition. *The following hold:*

(a) *Suppose that $S \neq R$. Then S is a domain if and only if $f(x)$ is irreducible in $R[x]$.*

(b) *Suppose that R is integrally closed in F. Then S is the integral closure of R in $S \otimes_R F$. Furthermore, S is a domain if and only if $S \otimes_R F$ is a field, and the latter is then the fraction field of S. If S is not a domain, then $S \cong R \times R$.*

Proof. If $f(x)$ is irreducible then one can check that it generates a prime ideal in $R[x]$. This immediately yields (a).
Suppose that R is integrally closed in F. (b) trivially holds if $S = R$, so suppose that $S \neq R$. If S is not a domain, then $S \cong R \times R$ by [43, (III.4.4.3)], and $R \times R$ is the integral closure of R in $S \otimes_R F \cong F \times F$. Since R is integrally closed in F, $f(x)$ is irreducible in $R[x]$ if and only if it is irreducible in $F[x]$. Hence, S is a domain if and only if $S \otimes_R F$ is a field. If S is a domain then S is the integral closure of R in the field $S \otimes_R F$ by [22, (6.1.2)], since the discriminant of $f(x)$ is a unit in R. □

1.15 Proposition. *There is a unique R–linear automorphism ι of S such that $R = \{x \in S \mid \iota(x) = x\}$. If $S \neq R$ then ι is given by $\iota(c + dz) = c + d(a - z)$, for all $c, d \in R$. If $S \cong R \times R$ then ι is given by the switch map.*

Proof. The statement is trivial if $S = R$. So, suppose that $S \neq R$. It is clear that $\iota : S \to S : c + dz \mapsto c + d(a - z)$ is an R–linear automorphism Since the discriminant of $f(x)$ is an element of $R^\times$, it follows that $z \neq a - z$, and hence $R = \{x \in S \mid \iota(x) = x\}$. Since any R–linear automorphism of S must map z to a root of $f(x)$ in S, the uniqueness of ι is clear. One easily checks that if $S \cong R \times R$ then ι is given by the switch map. □

For the rest of section 1.2, we fix ι as in Proposition 1.15, and we also use the notation ι for the induced nontrivial F–automorphism of $S \otimes_R F$.

An *involution* on a ring is an anti–automorphism of order at most 2. Let $\mathcal{A}$ be an Azumaya algebra over R or a separable quadratic R–algebra. Let furthermore σ be an R–linear involution on $\mathcal{A}$ such that, if $Z(\mathcal{A}) \neq R$, then σ restricts to the nontrivial R–automorphism of $Z(\mathcal{A})$, given by Proposition 1.15. If $Z(\mathcal{A}) = R$ then σ is called *an involution of the first kind*, otherwise σ is called *an involution of the second kind*. We call the pair $(\mathcal{A}, \sigma)$ an *R–algebra with involution*. If S is not a domain then we call $(\mathcal{A}, \sigma)$ *degenerate*.

Let $\mathcal{A}$ and $\mathcal{A}'$ be Azumaya algebras over S. Then the tensor product $\mathcal{A} \otimes_S \mathcal{A}'$ is an Azumaya algebra over S by [43, (III.5.1.5)]. Let σ be an R–linear involution of the first or second kind on $\mathcal{A}$, and let σ' be an R–linear involution of the same kind on $\mathcal{A}'$. Then the map $\sigma \otimes_S \sigma'$ is an R–linear involution on $\mathcal{A} \otimes_S \mathcal{A}'$ of the same kind as σ (and σ'). We also denote the R–algebra with involution $(\mathcal{A} \otimes_S \mathcal{A}', \sigma \otimes_S \sigma')$ by $(\mathcal{A}, \sigma) \otimes_S (\mathcal{A}', \sigma')$. Let R' be a domain that is also an R–algebra. We write $(\mathcal{A}, \sigma)_{R'} = (\mathcal{A}_{R'}, \sigma_{R'}) = (\mathcal{A} \otimes_R R', \sigma \otimes_R \mathrm{id}_{R'})$.

1.16 Proposition. *Let $(\mathcal{A}, \sigma)$ be an R–algebra with involution and let R' be a domain that is also an R–algebra. Then $(\mathcal{A}, \sigma)_{R'}$ is an R'–algebra with involution.*

Proof. We have that $\mathcal{A}_{R'} \cong \mathcal{A} \otimes_{Z(\mathcal{A})} (Z(\mathcal{A}) \otimes_R R')$ is an Azumaya algebra over $Z(\mathcal{A}) \otimes_R R'$ by Proposition 1.11 (a). The statement is then clear if σ is of the first kind. Suppose that σ is of the second kind. It is clear that $Z(\mathcal{A}) \otimes_R R' \cong R'[z]$ is a separable quadratic R'–algebra. Furthermore, $\sigma_{R'}$ is an R'–linear involution on $\mathcal{A}_{R'}$, which restricts to the nontrivial R'–automorphism of $Z(\mathcal{A}) \otimes_R R'$. □

1.17 Examples. Let L be a field. A 4–dimensional central simple L–algebra Q is called *a quaternion algebra over L*. Suppose that $\mathrm{char}(L) \neq 2$. The subset $Q' = \{x \in Q \mid x \notin L \text{ and } x^2 \in L\} \cup \{0\}$ of Q is the set of *pure quaternions of Q*.

(a) There is a unique involution of the first kind on Q that restricts to the identity on L, and to minus the identity on Q'. This involution is called *the canonical (symplectic) involution on Q*. We will encounter this involution later in the thesis. It can be shown (see e.g. [66, §8.11]) that Q has an L–basis $\{1, i, j, ij\}$ such that $i^2, j^2 \in L^\times$ and $ij = -ji$. In terms of this basis, we have that $Q' = \{a_1 i + a_2 j + a_3 ij \mid a_1, a_2, a_3 \in L\}$. We use the standard notation $(a, b)_L$ for Q in the case where $i^2 = a \in L^\times$ and $j^2 = b \in L^\times$.

(b) Let $\tilde{L}/L$ be a quadratic separable field extension, and denote its nontrivial L–automorphism by $\tilde{\iota}$. Let γ be as in (a) the canonical involution on Q. Then $\tilde{Q} = Q \otimes_L \tilde{L}$ is a quaternion algebra over $\tilde{L}$ and $\gamma \otimes_L \tilde{\iota}$ is an involution of the second kind on $\tilde{Q}$.

Let $(\mathcal{A}, \sigma)$ and $(\mathcal{A}', \sigma')$ be R–algebras with involution. By an *isomorphism of R–algebras with involution* $(\mathcal{A}, \sigma) \to (\mathcal{A}', \sigma')$, we mean an isomorphism $\varphi : \mathcal{A} \to \mathcal{A}'$ of R–algebras such that $\varphi \circ \sigma = \sigma' \circ \varphi$. We call $(\mathcal{A}, \sigma)$ and $(\mathcal{A}', \sigma')$ *R–isomorphic* if there exists an isomorphism of R–algebras with involution $(\mathcal{A}\ \sigma) \to (\mathcal{A}', \sigma')$, and we denote this by $(\mathcal{A}, \sigma) \cong_R (\mathcal{A}', \sigma')$. Suppose moreover that there exist R–isomorphisms $f : S \to Z(\mathcal{A})$ and $f' : S \to Z(\mathcal{A}')$, and an isomorphism of R–algebras with involution $(\mathcal{A}, \sigma) \to (\mathcal{A}', \sigma')$ that is S–linear with respect to f and f'. Then we say that $(\mathcal{A}, \sigma)$ and $(\mathcal{A}', \sigma')$ are *S–isomorphic* and we write $(\mathcal{A}, \sigma) \cong_S (\mathcal{A}', \sigma')$.

Let $\mathcal{B}$ be an Azumaya algebra over R. Define a new multiplication on $\mathcal{B}$ by $a * b = ba$, for all $a, b \in \mathcal{B}$. The R–module $\mathcal{B}$ with the new operation $*$ as multiplication is also an R–algebra, called *the opposite algebra of $\mathcal{B}$*, and we denote it by $\mathcal{B}^{\mathrm{op}}$. The map $\mathrm{sw}_{\mathcal{B}} : \mathcal{B} \times \mathcal{B}^{\mathrm{op}} \to \mathcal{B} \times \mathcal{B}^{\mathrm{op}} : (a, b) \mapsto (b, a)$ defines an involution of the second kind on $\mathcal{B} \times \mathcal{B}^{\mathrm{op}}$, called *the switch involution*.

1.18 Proposition. *Let $\mathcal{A}$ be an Azumaya algebra over $R \times R$. Then there exist Azumaya algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ over R such that $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$. Furthermore, if σ is an involution of the second kind on $\mathcal{A}$, then $\mathcal{A}_2 \cong \mathcal{A}_1^{\mathrm{op}}$ and $(\mathcal{A}, \sigma) \cong_R (\mathcal{A}_1 \times \mathcal{A}_1^{\mathrm{op}}, \mathrm{sw}_{\mathcal{A}_1})$.*

Proof. Let $\mathcal{A}_1 = \mathcal{A}(1, 0)$ and $\mathcal{A}_2 = \mathcal{A}(0, 1)$. These are R–algebras and it is clear that $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$. By Proposition 1.11 (c), for each maximal ideal $\mathfrak{m}$ of R, we have that $\mathcal{A}/\mathcal{A}(\mathfrak{m} \times R)$ is a central simple algebra over $(R \times R)/(\mathfrak{m} \times R)$. Using the natural isomorphisms, this gives $\mathcal{A}_1/\mathfrak{m}\mathcal{A}_1$ the structure of a central simple $R/\mathfrak{m}$–algebra. Proposition 1.11 (c) yields that $\mathcal{A}_1$ is an Azumaya algebra over R. Similarly, by considering $\mathcal{A}/\mathcal{A}(R \times \mathfrak{m})$, we get that $\mathcal{A}_2$ is an Azumaya algebra over R.

Let σ be an involution of the second kind on $\mathcal{A}$. Since σ restricts to the switch involution on $R \times R$, the map $g : \mathcal{A}_1^{\mathrm{op}} \to \mathcal{A}_2$ defined by $\sigma(x, 0) = (0, g(x))$ is an isomorphism of R–algebras. Under the induced isomorphism $\mathcal{A}_1 \times \mathcal{A}_2 \cong \mathcal{A}_1 \times \mathcal{A}_1^{\mathrm{op}}$, the involution σ corresponds to the switch involution $\mathrm{sw}_{\mathcal{A}_1}$. □

1.19 Proposition. *Assume that R is semilocal. Let $\mathcal{A}$ be an Azumaya algebra over S. Then $\mathcal{A}$ is free as an R–module. If S is a domain, assume that it is semilocal. Then $\mathcal{A}$ is also free as an S–module.*

Proof. Since $\mathcal{A}$ is an Azumaya algebra over S, $\mathcal{A}$ is finitely generated, projective as an S–module. Since S is a finite–dimensional R–module, it follows that $\mathcal{A}$ is also finitely generated, projective as an R–module. By the main theorem of [31], finitely generated, projective modules over a semilocal domain are free, proving the statement. □

1.20 Proposition. *Assume that S is a semilocal domain, or that $S \cong R \times R$. Let $\mathcal{A}$ be an Azumaya algebra over S. Then every S–automorphism of $\mathcal{A}$ is inner.*

Proof. If S is a domain this follows from [4, (3.6)], since finitely generated, projective modules over S are free by the main theorem of [31]. Suppose that $S \cong R \times R$. By Proposition 1.18, there exist Azumaya algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ over R such that $\mathcal{A} \cong \mathcal{A}_1 \times \mathcal{A}_2$. Let $\varphi \in \mathrm{Aut}_R(\mathcal{A}_1 \times \mathcal{A}_2)$. One easily checks that the restriction of φ to $\mathcal{A}_1$ (resp. $\mathcal{A}_2$) is an R–automorphism of $\mathcal{A}_1$ (resp. $\mathcal{A}_2$). By the first part of the proof, it follows that there exist $u \in \mathcal{A}_1^\times$ and $v \in \mathcal{A}_2^\times$ such that $\varphi|_{\mathcal{A}_1} = \mathrm{Int}(u)$ and $\varphi|_{\mathcal{A}_2} = \mathrm{Int}(v)$. This implies that $\varphi = \mathrm{Int}(u, v)$. □

In the rest of this section, we are interested in the case where R is a semilocal Bézout domain.

We use the following terminology. Let L be a field and let C be a subring of L. Let L'/L a field extension. We say that a valuation ring $\mathcal{O}'$ of L' with maximal ideal $\mathcal{M}'$ is *lying over C* if $C \subset \mathcal{O}'$ and $\mathcal{M}' \cap C$ is a maximal ideal of C. In that case, we also say that $\mathcal{M}'$ is lying over $\mathcal{M}' \cap C$. If C is a valuation ring of L, then we also say that $\mathcal{O}'$ is *an extension* of C to L'.

1.21 Proposition. *Suppose that there exist different valuation rings $\mathcal{O}_1, \dots, \mathcal{O}_m$ of F such that $R = \mathcal{O}_1 \cap \dots \cap \mathcal{O}_m$ (i.e. R is a semilocal Bézout domain). If S is a domain, then it is the intersection of the valuation rings of K lying over some $\mathcal{O}_i$. In particular, S is a semilocal Bézout domain.*

Proof. Since $\mathcal{O}_1, \dots, \mathcal{O}_m$ are integrally closed in F, it follows that R is integrally closed in F. Proposition 1.14 (b) then yields that S is the integral closure of R in F. By [21, (3.1.3)], it follows that S is the intersection of the valuation rings of K lying over R. Let S' be the intersection of the valuation rings of K lying over some $\mathcal{O}_i$. Since K/F is a finite field extension, there are only finitely many such valuation rings by [21, (3.2.9)]. Hence, S' is a semilocal Bézout domain by Proposition 1.3. Furthermore, it is clear that $S \subset S'$. For the other inclusion, note first that S' is integral over $\mathcal{O}_1, \dots, \mathcal{O}_m$, since

S' is contained in the integral closure of each $\mathcal{O}_i$ in K. Let $x \in S'$ be arbitrary. Since $\mathcal{O}_1, \ldots, \mathcal{O}_m$ are integrally closed in F and x is integral over $\mathcal{O}_1, \ldots, \mathcal{O}_m$, it follows that the minimal polynomial of x over F has its coefficients in $R = \mathcal{O}_1 \cap \ldots \cap \mathcal{O}_m$, and hence, x is integral over R. It follows that $S' \subset S$. □

1.22 Proposition. *Suppose that R is a valuation ring. Denote its maximal ideal by $\mathfrak{m}$ and its residue field by κ. Suppose that S is a domain. Then one of the following cases occurs:*

(a) *There is a unique valuation ring of K extending R. Then S is equal to this extension. Furthermore, S has the same value group as R, has maximal ideal $\mathfrak{m}S$ and its residue field is a separable quadratic extension of κ.*

(b) *There are two valuation rings of K extending R. These both have the same value group as R and residue field κ. Furthermore, S is equal to their intersection, has Jacobson radical $\mathfrak{m}S$, and $S/\mathfrak{m}S \cong \kappa \times \kappa$.*

Proof. By Proposition 1.21, S is the intersection of the valuation rings of $K = S \otimes_R F$ lying over R. By [21, (3.2.9)], since K/F is a quadratic extension, S is either a valuation ring or the intersection of two (incomparable) valuation rings. Let $\bar{f}(x) = x^2 - \bar{a}x - \bar{b} \in \kappa[x]$. Note that the discriminant of $\bar{f}(x)$ is nonzero, since $a^2 + 4b \in R^\times$, and hence, $\bar{f}(x)$ is separable. It is easy to see that S is a valuation ring if and only if $\bar{f}(x)$ is irreducible. If this is the case, then $\mathfrak{m}S$ is the unique maximal ideal of S, and hence, $J(S) = \mathfrak{m}S$. This also implies that the residue field of S is a separable quadratic extension of κ. By [21, (3.2.3)], it follows that R and S have the same value group.
Suppose that S is the intersection of two incomparable valuation rings. Both of these valuation rings have the same value group and residue field as R by [21, (3.3.4)]. We have that S has two maximal ideals $\mathcal{M}_1$ and $\mathcal{M}_2$ by Proposition 1.4. Now $\bar{f}(x)$ is reducible over κ, and hence, $S/\mathfrak{m}S \cong \kappa \times \kappa$ since $\bar{f}(x)$ is separable. It follows that $\mathfrak{m}S = \mathcal{M}_1 \cap \mathcal{M}_2 = J(S)$. □

1.23 Proposition. *Suppose that R is a semilocal Bézout domain, and assume that S is a domain. Let $\mathcal{A}$ be an Azumaya algebra over S and let σ and σ' be two R−linear involutions of the same kind on $\mathcal{A}$.*

(a) *If σ and σ' are of the first kind then there is an element $s \in \mathcal{A}^\times$ such that $\sigma(s) = \pm s$ and $\sigma' = \mathrm{Int}(s) \circ \sigma$.*

(b) *If σ and σ' are of the second kind then there is an element $s \in \mathcal{A}^\times$ such that $\sigma(s) = s$ and $\sigma' = \mathrm{Int}(s) \circ \sigma$.*

Proof. Since $\sigma' \circ \sigma$ is an S-automorphism of $\mathcal{A}$, Corollary 1.20 yields that there exists $s \in \mathcal{A}^\times$ such that $\sigma' = \mathrm{Int}(s) \circ \sigma$. It follows that $\mathrm{id}_{\mathcal{A}} = \sigma'^2 = \mathrm{Int}(s\sigma(s)^{-1})$. This implies that $s\sigma(s)^{-1} = \lambda^{-1}$, for some $\lambda \in S^\times$. In other words, $\sigma(s) = \lambda s$. It follows that $s = \sigma^2(s) = \sigma(\lambda)\lambda s$. Since $s \in \mathcal{A}^\times$, we get that $\sigma(\lambda)\lambda = 1$. If σ and σ' are of the first kind, this implies that $\lambda^2 = 1$ and hence, $\lambda = \pm 1$.
Suppose that σ and σ' are of the second kind. Then $1 = \sigma(\lambda)\lambda = \iota(\lambda)\lambda$. In order to prove the statement, it suffices to show a Hilbert 90 type statement for S, namely that there exists $\mu \in S^\times$ such that $\mu = \lambda\iota(\mu)$. For then $\mathrm{Int}(s) = \mathrm{Int}(\mu s)$ and $\sigma(\mu s) = \sigma(s)\iota(\mu) = \lambda\iota(\mu)s = \mu s$.

By Proposition 1.2, there exist pairwise incomparable valuation rings $\mathcal{O}_1, \ldots, \mathcal{O}_m$ of F such that $R = \mathcal{O}_1 \cap \ldots \cap \mathcal{O}_m$. For $i = 1, \ldots, m$, let $\mathfrak{m}_i$ be the maximal ideal of $\mathcal{O}_i$. After renumbering if necessary, we may assume that there is an index $\ell \in \{1, \ldots, m\}$ such that $\mathcal{O}_1, \ldots, \mathcal{O}_\ell$ extend uniquely to K and $\mathcal{O}_{\ell+1}, \ldots, \mathcal{O}_m$ have two extensions to K. For $i = 1, \ldots, m$, let S_i be the integral closure of $\mathcal{O}_i$ in K. By [21, (3.1.3)] and Proposition 1.21, $S = S_1 \cap \ldots \cap S_m$. For $i = 1, \ldots, \ell$, we have that S_i is a valuation ring, and S has therefore a unique maximal ideal $\mathcal{M}_i$ lying over $\mathfrak{m}_i$. For $i = \ell + 1, \ldots, m$, S_i is the intersection of the two (incomparable) extensions of $\mathcal{O}_i$ to K and S has two maximal ideals $\mathcal{M}_{i1}$ and $\mathcal{M}_{i2}$ lying over $\mathfrak{m}_i$. By [49, (IX.2.1)], we have that $\mathcal{M}_{i2} = \iota(\mathcal{M}_{i1})$.

Let $\tilde{S} \in \{S_1, \ldots, S_m\}$ be arbitrary, and let $\tilde{\mathcal{O}} = \tilde{S} \cap F$. Using the Hilbert 90 theorem for K/F, there exists $\tilde{\mu} \in K$ such that $\lambda\iota(\tilde{\mu}) = \tilde{\mu}$. Suppose first that $\tilde{S}$ is a valuation ring. Since, by Proposition 1.22, the value groups of $\tilde{S}$ and $\tilde{\mathcal{O}}$ are equal, there exists $a \in F$ such that $a\tilde{\mu} \in \tilde{S}^\times$, and furthermore $\lambda\iota(a\tilde{\mu}) = a\lambda\iota(\tilde{\mu}) = a\tilde{\mu}$. Suppose that $\tilde{S} = V_1 \cap V_2$, with V_1 and V_2 the extensions of $\tilde{\mathcal{O}}$ to K. Then V_1 and V_2 have the same value groups as $\tilde{\mathcal{O}}$ by Proposition 1.22, and $V_2 = \iota(V_1)$ by [21, (3.2.14)]. By the previous case, we may assume that $\tilde{\mu} \in V_1^\times$. Then $\iota(\tilde{\mu}) \in V_2^\times$, and since $\lambda \in R^\times \subset \tilde{\mathcal{O}}^\times$, it follows that $\tilde{\mu} \in V_2^\times$ as well.

So, for $i = 1, \ldots, m$, there exist $\mu_i \in S_i^\times$ such that $\lambda\iota(\mu_i) = \mu_i$. By the Chinese Remainder Theorem for R, there exist elements $\alpha_1 \ldots, \alpha_\ell \in R$ such that $\alpha_i \equiv 1 \bmod \mathfrak{m}_i$ and $\alpha_i \equiv 0 \bmod \mathfrak{m}_j$ for $j \in \{1, \ldots, m\} \setminus \{i\}$. By the Chinese Remainder Theorem for S, there exist elements $\alpha_{\ell+1}, \ldots, \alpha_m \in S$ such that $\alpha_i \equiv 1 \bmod \mathcal{M}_{i1}$ and α_i is contained in all other maximal ideals of S, i.e. $\alpha_i \in \mathcal{M}_j$ for $j = 1, \ldots, \ell$, $\alpha_i \in \mathcal{M}_{i2}$ and $\alpha_i \in \mathcal{M}_{j1} \cap \mathcal{M}_{j2}$ for $j \neq i$.
For $i = 1, \ldots, \ell$, let $\mu_i' = \alpha_i\mu_i$, and for $i = \ell + 1, \ldots, m$, let $\mu_i' = \alpha_i + \lambda\iota(\alpha_i)$. Since $\iota(\lambda)\lambda = 1$, it follows that $\lambda\iota(\mu_i') = \mu_i'$ for $i = 1, \ldots, m$. For $i = 1, \ldots, \ell$, μ_i' is contained in all maximal ideals of S except $\mathcal{M}_i$, and for $i = \ell + 1, \ldots, m$, μ_i' is contained in all maximal ideals of S except $\mathcal{M}_{i1}$ and $\mathcal{M}_{i2}$. It follows that $\mu = \sum_{i=1}^m \mu_i' \in S^\times$, and, furthermore, $\lambda\iota(\mu) = \mu$. □

1.24 Proposition. *Let T be a semilocal Bézout domain, and let Δ be an Azumaya algebra over T. Then Δ is a left and right Bézout ring.*

Proof. We denote the fraction field of T by L. By Proposition 1.2, there exist pairwise incomparable valuation rings $\mathcal{O}_1, \ldots, \mathcal{O}_r$ of L such that $T = \mathcal{O}_1 \cap \ldots \cap \mathcal{O}_r$. By Proposition 1.4, T has r different maximal ideals, say $\mathcal{M}_1, \ldots, \mathcal{M}_r$, and $\mathcal{O}_1 = T_{\mathcal{M}_1}, \ldots, \mathcal{O}_r = T_{\mathcal{M}_r}$. Let $\Delta_i = \Delta \otimes_T \mathcal{O}_i$ for $i = 1, \ldots, r$. Then Δ_i is an Azumaya algebra over $\mathcal{O}_i$ by Proposition 1.11 (a), and therefore integral over $\mathcal{O}_i$. Furthermore, $\Delta = \cap_{i=1}^r \Delta_i$. The inclusion $\Delta \subset \cap_{i=1}^r \Delta_i$ is clear. For the converse, let $x \in \cap_{i=1}^r \Delta_i$. Then there exist $d_1, \ldots, d_r \in \Delta$ and $a_i \in T \setminus \mathcal{M}_i$ for $i = 1, \ldots, r$, such that $x = d_1/a_1 = \ldots = d_r/a_r$. Using that $a_1 T + \ldots + a_r T = T$, it follows that $x \in \Delta$.

By [50, (7.13)], Δ and $\Delta_1, \ldots, \Delta_r$ are so–called Dubrovin valuation rings of the division algebra Δ_L. If we show that Δ_i and Δ_j are pairwise incomparable if $i \neq j$, i.e. $\Delta_i \subsetneq \Delta_j$, then it follows from [50, (15.5) and (15.7)] that Δ is a left and right Bézout ring, using that Δ_i is integral over $\mathcal{O}_i$ for $i = 1, \ldots, r$, and that $\Delta_i = \Delta\, \mathcal{O}_i$ inside Δ_L.
So, suppose for the sake of contradiction that there exist $i \neq j$ such that $\Delta_i \subset \Delta_j$. Then $\mathcal{O}_i \subset \Delta_j$. Since Δ_j is an Azumaya algebra over $\mathcal{O}_j$, it is free over $\mathcal{O}_j$ by Corollary 1.19, and an $\mathcal{O}_j$–basis of Δ_j is an L–basis of $\Delta_j \otimes_{\mathcal{O}_j} L$. It follows that $\mathcal{O}_j = Z(\Delta_j) = Z(\Delta_j \otimes_{\mathcal{O}_j} L) \cap \Delta_j = L \cap \Delta_j \supset \mathcal{O}_i$, but this contradicts the fact that $\mathcal{O}_i$ and $\mathcal{O}_j$ are pairwise incomparable. This proves the statement. □

1.3 The Brauer group of a ring

The concept of the Brauer group of a field has been extended to commutative rings in [4]. Let T be a commutative ring. A T–module P is called *faithful* if whenever $t \in T$ is such that $tP = 0$, then $t = 0$. Let $\mathfrak{a}(T)$ be the set of isomorphism classes of all Azumaya algebras with center T, and let $\mathfrak{a}_0(T)$ be the subset of $\mathfrak{a}(T)$ consisting of the $\mathrm{End}_T(P)$, with P a finitely generated, faithfully projective T–module. One can prove that $\mathfrak{a}(T)$ and $\mathfrak{a}_0(T)$ are closed under tensor products. Consider the following equivalence relation on $\mathfrak{a}(T)$. Let $\mathcal{A}_1, \mathcal{A}_2 \in \mathfrak{a}(T)$. Then we say that $\mathcal{A}_1$ and $\mathcal{A}_2$ are *Brauer equivalent*, denoted by $\mathcal{A}_1 \sim \mathcal{A}_2$, if there exist algebras $\Omega_1, \Omega_2 \in \mathfrak{a}_0(T)$ such that $\mathcal{A}_1 \otimes_T \Omega_1 \cong \mathcal{A}_2 \otimes_T \Omega_2$. We denote $\mathfrak{a}(T)$ modulo this equivalence relation by $\mathrm{Br}(T)$. This is called *the Brauer group of T*. In [28, (1.5)], A. Grothendieck showed that every element of $\mathrm{Br}(T)$ has finite order. Given an Azumaya algebra $\mathcal{A}$ over T, we denote its Brauer class in $\mathrm{Br}(T)$ by $[\mathcal{A}]$. We use the term *exponent* for the order of $[\mathcal{A}]$ in $\mathrm{Br}(T)$. As for fields, $\mathrm{Br}(T)$ has the structure of an abelian group, in which the equivalence class of T is the identity. For more details we refer to [4].

We say that T *has the Wedderburn property* if the following holds:

Let $\mathcal{A}$ be an Azumaya algebra over T. Then there exists an up to T–isomorphism unique Azumaya algebra Δ over T without zero divisors, and a finite–dimensional right Δ–module M such that $\mathcal{A} \cong \mathrm{End}_\Delta(M)$ as T–algebras.

We thank M. Ojanguren for providing the main ideas for some of the proofs in this section.

1.25 Proposition. *Let T be a domain with fraction field L. Suppose that finitely generated, torsion–free T–modules are free. Let Δ be an Azumaya algebra over T without nontrivial idempotents. Then Δ does not have zero divisors, and $\Delta \otimes_T L$ is a division algebra.*

Proof. We have that $D = \Delta \otimes_T L$ is a central simple L–algebra by Proposition 1.11 (a) and (b). It is clear that if D is a division algebra, then Δ does not have zero divisors, and vice versa. In order to show that D is a division algebra, it suffices to show that D does not have nontrivial idempotents. So, suppose for the sake of contradiction that D contains a nontrivial idempotent x. Consider the right ideal xD of D and let $I = xD \cap \Delta$. This is a right ideal of Δ different from Δ itself. If we can show that Δ/I is projective as a Δ–module, then the exact sequence of right Δ–modules

$$0 \longrightarrow I \longrightarrow \Delta \longrightarrow \Delta/I \longrightarrow 0$$

splits, which implies that $\Delta \cong I \oplus \Delta/I$. The projection from Δ to I then yields a nontrivial idempotent in $\mathrm{End}_\Delta(I \oplus \Delta/I) \cong \Delta$, where the T–algebra isomorphism $\Delta \to \mathrm{End}_\Delta(\Delta)$ is given by left multiplication (where Δ is considered as a right module over itself in $\mathrm{End}_\Delta(\Delta)$).
Clearly, Δ/I is finitely generated over Δ, and then also over T. In order to show that Δ/I is projective over Δ, it suffices to show that Δ/I is projective over T by [43, (VII.8.2.6)]. First of all, we have that Δ/I is torsion–free over T. For suppose that there are elements $a \in \Delta \setminus I$ and $0 \neq r \in T$ such that $ar \in I$, then $a = (ar)r^{-1} \in xD \cap \Delta = I$, a contradiction. The hypothesis now yields that Δ/I is a free T–module, and hence, in particular a projective T–module. □

1.26 Proposition. *Let T be a semilocal Bézout domain or a polynomial ring in one variable over a perfect field. Then T has the Wedderburn property.*

Proof. This follows from [16, Corollary 1], combined with Proposition 1.25. □

The condition "perfect" in Proposition 1.26 may seem to come out of nowhere. It is used in the proof of [16, Corollary 1] to guarantee that Azumaya algebras with center $k[t]$ are up to Brauer equivalence extended from k. If k is not perfect then this is no longer true, but the Brauer classes that are extended from k have been characterised in [4]. The result of Proposition 1.26 can then be made more precise, as we show below.

1.27 Proposition. *Let k be a field and let $T = k[t]$ be the polynomial ring in one variable. Then T has the Wedderburn property for Azumaya algebras over T of exponent not divisible by $\mathrm{char}(k)$, i.e. let $\mathcal{A}$ be an Azumaya algebra over T of exponent r, where $\mathrm{char}(k) \nmid r$, then there exists an up to T–isomorphism unique Azumaya algebra Δ over T without zero divisors and a finite–dimensional right Δ–module M such that $\mathcal{A} \cong \mathrm{End}_\Delta(M)$.*

Proof. In order to obtain the result of [16, Corollary 1], F.R. Demeyer used the fact from [4, (7.5)] that, in the case where T is a polynomial ring over a perfect field, an Azumaya algebra over T is up to Brauer equivalence obtained by scalar extension from a central simple algebra over k, i.e. the map $\mathrm{Br}(k) \to \mathrm{Br}(k[t])$ is surjective. In [4, p. 389], the authors show that $\mathrm{Br}(k[t]) \cong \mathrm{Br}(k) \times \mathrm{Br}'(k[t])$, where $\mathrm{Br}'(k[t])$ is the kernel of the homomorphism $\mathrm{Br}(k[t]) \to \mathrm{Br}(k)$, induced by the residue homomorphism $k[t] \to k$. In [4, (7.6)], it is then shown that an element of $\mathrm{Br}'(k[t])$ has exponent a power of $\mathrm{char}(k)$. Hence, it follows that Azumaya algebras over $k[t]$ of exponent not divisible by $\mathrm{char}(k)$ are up to Brauer equivalence extended from k. From this point on, one can follow the proof of [16, Corollary 1] in order to obtain the statement. □

1.28 Proposition. *Let T be a semilocal domain. Brauer equivalent Azumaya algebras over T of the same T–dimension are isomorphic.*

Proof. Let $\mathcal{A}, \mathcal{A}'$ be Azumaya algebras over T. Assume that $\dim_T(\mathcal{A}) = \dim_T(\mathcal{A}')$ and that $[\mathcal{A}] = [\mathcal{A}'] \in \mathrm{Br}(T)$. Then there exist finitely generated, faithfully projective T–modules P_1 and P_2 such that

$$\mathcal{A} \otimes_T \mathrm{End}_T(P_1) \cong \mathcal{A}' \otimes \mathrm{End}_T(P_2).$$

By the main theorem of [31], P_1 and P_2 are free as T–modules, and hence, there exist $n_1, n_2 \in \mathbb{N}$ such that $\mathrm{End}_T(P_1) \cong \mathrm{M}_{n_1}(T)$ and $\mathrm{End}_T(P_2) \cong \mathrm{M}_{n_2}(T)$. Since $\dim_T(\mathcal{A}) = \dim_T(\mathcal{A}')$ by assumption, it follows that $\dim_T(\mathcal{A} \otimes_T \mathrm{M}_{n_1}(T)) = \dim_T(\mathcal{A}' \otimes \mathrm{M}_{n_2}(T))$, and hence, $n_1 = n_2$. The cancellation law for Azumaya algebras over semilocal rings (see [43, (III.5.2.3)] then yields that $\mathcal{A} \cong \mathcal{A}'$. □

1.29 Proposition. *Let T be a semilocal domain with fraction field L. Suppose that the natural map $\mathrm{Br}(T) \to \mathrm{Br}(L)$ is injective. Let $\mathcal{A}$ and $\mathcal{A}'$ be Azumaya algebras over T. If $\mathcal{A}_L \cong \mathcal{A}'_L$ then $\mathcal{A} \cong \mathcal{A}'$.*

Proof. By Corollary 1.19, $\mathcal{A}$ and $\mathcal{A}'$ are free over T and hence,

$$\dim_T \mathcal{A} = \dim_L \mathcal{A}_L = \dim_L \mathcal{A}'_L = \dim_T \mathcal{A}'.$$

Furthermore, since $[\mathcal{A}_L] = [\mathcal{A}'_L] \in \mathrm{Br}(L)$ and $\mathrm{Br}(T) \to \mathrm{Br}(L)$ is injective by assumption, it follows that $[\mathcal{A}] = [\mathcal{A}'] \in \mathrm{Br}(T)$. Proposition 1.28 then yields that $\mathcal{A} \cong \mathcal{A}'$. □

1.30 Proposition. *Let T be a domain and denote its fraction field by L. Suppose that finitely generated, torsion–free T–modules are free. Then the natural map $\mathrm{Br}(T) \to \mathrm{Br}(L)$ is injective.*

Proof. Let $\mathcal{A}$ be an Azumaya algebra over T such that $\mathcal{A}_L$ is split, i.e. there exist a simple right $\mathcal{A}_L$–module V such that $\mathcal{A}_L \cong \mathrm{End}_L(V)$. Note that $\dim_L(V)^2 = \dim_L(\mathcal{A}_L) = \dim_T(\mathcal{A})$. Let $u \in V \setminus \{0\}$. Consider the right $\mathcal{A}$–module $M = u\mathcal{A}$. Since V is a simple right $\mathcal{A}_L$–module, it follows that $ML = V$. Since $\mathcal{A}$ is finitely generated as a T–module, it follows that M is finitely generated as a T–module as well. Since V is free over L, it follows that M is torsion–free as a T–module. The hypothesis yields that M is then free as a T–module. We now have a T–algebra homomorphism $\varphi : \mathcal{A} \to \mathrm{End}_T(M)$ defined by mapping $a \in \mathcal{A}$ to the T–endomorphism of M given by right multiplication by a. This induces a L–algebra homomorphism $\varphi_L : \mathcal{A}_L \to \mathrm{End}_L(V)$, which is an L–isomorphism. It follows that φ is injective and hence, $\varphi(\mathcal{A}) \cong \mathcal{A}$. By [4, (3.3)], there is an Azumaya algebra $\mathcal{B}$ over T such that $\mathrm{End}_T(M) \cong \varphi(A) \otimes_T \mathcal{B}$. Since $\dim_T \mathcal{A} = \dim_L(V)^2 = \dim_T(M)^2 = \dim_T(\mathrm{End}_T(M))$, it follows that $\mathcal{B} \cong T$, and hence $\mathcal{A} \cong \mathrm{End}_T(M)$. This yields the injectivity. □

1.31 Corollary. *Let T be a Bézout domain, and denote its fraction field by L. Then the natural map $\mathrm{Br}(T) \to \mathrm{Br}(L)$ is injective.*

Proof. By Proposition 1.30, all we need to show is that finitely generated, torsion–free T–modules are free. Since T is a Bézout domain this follows from Proposition 1.9. □

1.32 Remark. If, in Proposition 1.30, one replaces the assumption that finitely generated, torsion–free T–modules are free, by the weaker assumption that finitely generated, projective T–modules are free, then the map $\mathrm{Br}(T) \to \mathrm{Br}(L)$ need not be injective anymore. An illustration of this fact is given in [4, p. 388], taking for T the local domain $\mathbb{R}[x, y]/(x^2 + y^2)$.

1.33 Corollary. *Let T be a semilocal Bézout domain with fraction field L. Let $\mathcal{A}$ and $\mathcal{A}'$ be Azumaya algebras over $T \times T$. If $\mathcal{A}_{L\times L} \cong \mathcal{A}'_{L\times L}$ then $\mathcal{A} \cong \mathcal{A}'$.*

Proof. By Proposition 1.18, there exist Azumaya algebras $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}'_1, \mathcal{A}'_2$ over T such that $\mathcal{A} \cong \mathcal{A}_1 \times \mathcal{A}_2$ and $\mathcal{A}' \cong \mathcal{A}'_1 \times \mathcal{A}'_2$. Suppose that

$$(\mathcal{A}_1)_L \times (\mathcal{A}_2)_L \cong (\mathcal{A}'_1)_L \times (\mathcal{A}'_2)_L.$$

Since the simple components are unique up to isomorphism, we may assume without loss of generality that $(\mathcal{A}_1)_L \cong (\mathcal{A}'_1)_L$ and $(\mathcal{A}_2)_L \cong (\mathcal{A}'_2)_L$. Invoking Proposition 1.29 and Corollary 1.31, it follows that $\mathcal{A}_1 \cong \mathcal{A}'_1$ and $\mathcal{A}_2 \cong \mathcal{A}'_2$, and hence $\mathcal{A} \cong \mathcal{A}'$. □

Corollary 1.33 still holds for a finite number of copies of T. The proof in the case of two copies goes through completely. We formulate the statement for two copies of T since we will use the result only in that situation.

1.34 Remarks.

(a) In the sequel, we will only need the isomorphism result from Proposition 1.29 for domains T that have the Wedderburn property and for which the map from the Brauer group of T to the Brauer group of its fraction field L is injective. The isomorphism statement can then also be proved differently. For let Δ and Δ' be Azumaya algebras over T without zero divisors, and let $n, n' \in \mathbb{N}$ be uniquely determined such that $\mathcal{A} \cong \mathrm{M}_n(\Delta)$ and $\mathcal{A}' \cong \mathrm{M}_{n'}(\Delta')$. The hypothesis implies that $[\Delta_L] = [\mathcal{A}_L] = [\mathcal{A}'_L] = [\Delta'_L] \in \mathrm{Br}(L)$. The injectivity of $\mathrm{Br}(T) \to \mathrm{Br}(L)$ then yields that $[\Delta] = [\Delta'] \in \mathrm{Br}(T)$, and hence $\Delta \cong \Delta'$ by the Wedderburn property. Since $\mathcal{A}_L \cong \mathcal{A}'_L$, this implies that $n = n'$ and hence, we get that $\mathcal{A} \cong \mathcal{A}'$. So, by Proposition 1.26 and Corollary 1.31, if T is a polynomial ring in one variable over a perfect field, then the statements of Proposition 1.29 and Corollary 1.33 also hold.

(b) The injectivity of the Brauer group map also holds in the case where T is a regular domain (see [4, (7.2)]), and hence, the isomorphism result from Proposition 1.29 holds for a regular local ring. It is in general not known whether regular local rings have the Wedderburn property. By [16, Corollary 1], they do satisfy the weaker condition that each Azumaya algebra over a regular local ring T is isomorphic to a matrix algebra over an Azumaya algebra without nontrivial idempotents over T, and the latter Azumaya algebra is uniquely determined up to isomorphism.

(c) Apart from the results in [16], there are other (earlier) Wedderburn type results for Azumaya algebras in the literature. For example, in [3, (3.9)], it is shown that if T is a Dedekind domain and $\mathcal{A}$ an Azumaya algebra over T, then there exists an Azumaya algebra Δ over T without zero divisors, and a finitely generated, projective Δ−module P such that $\mathcal{A} \cong \mathrm{End}_\Delta(P)$. However, in general this P need not be free over Δ, and Δ need not be unique up to isomorphism.

1.4 Algebras with involution over fields

In this section we recall some concepts and results for algebras with involution over fields, the standard reference being [45]. We also extend some notions for central simple algebras to the semisimple algebras we consider in the context of involutions of the second kind. In particular, we introduce the notion of a "balanced ideal" in order to treat involutions of the first and second kind in a more uniform way in arguments here and later in the thesis. This condition balanced is trivial in the central simple case, but restricts the set of ideals in the semisimple case.

Throughout this section F is a field and (B, τ) is an F–algebra with involution.

Let V be a finite–dimensional F–vector space. A bilinear form $b : V \times V \to F$ is called *symmetric* if $b(x, y) = b(y, x)$ for all $x, y \in V$, and it is called *alternating* if $b(x, x) = 0$ for all $x \in V$. For the concept of adjoint involutions of non–singular symmetric and alternating bilinear forms, we refer to [45, p.1]. It is also contained in the study of adjoint involutions of (skew–)hermitian spaces in section 2.2. Suppose that τ is of the first kind. Then τ is called *symplectic* if it becomes adjoint to an alternating bilinear form over any splitting field of B, and *orthogonal* otherwise. This is called *the type of* τ.

Discriminant and Clifford algebra

In this section we assume that $\mathrm{char}(F) \neq 2$. Suppose that τ is orthogonal and $\deg(B)$ is even. Then there is a *discriminant* $\mathrm{disc}(\tau)$ associated to τ, and a *Clifford algebra* $C(B, \tau)$. These generalise the corresponding notions for non–singular quadratic forms over F. We will recall some basics on quadratic forms in section 6.1. We have that

$$\mathrm{disc}(\tau) = (-1)^{\deg(B)/2}\,\mathrm{Nrd}_B(a)F^{\times 2} \in F^\times/F^{\times 2},$$

for every $a \in B^\times$ such that $\tau(a) = -a$. For more details on the discriminant we refer to [45, §7]. For the definition of the Clifford algebra $C(B, \tau)$ we refer to [45, §8]. For the convenience of the reader we list some properties of the Clifford algebra, which we will use frequently later on. In [45], they are stated more generally for quadratic pairs, but for orthogonal involutions (since $\mathrm{char}(F) \neq 2$), they translate to the versions we give below.

1.35 Proposition. *Suppose* $\deg(B) = 2m$. *Let* $\mathrm{disc}(\tau) = \delta \in F^\times/F^{\times 2}$. *Then the center* Z *of* $C(B, \tau)$ *is isomorphic to* $F[X]/(X^2 - \delta)$. *If* $\delta \notin F^{\times 2}$ *then* Z *is a field and* $C(B, \tau)$ *is a central simple* Z*–algebra of degree* 2^{m-1}. *If* $\delta \in F^{\times 2}$ *then* $Z \cong F \times F$ *and* $C(B, \tau) = C_+ \times C_-$, *with* C_+ *and* C_- *central simple* F*–algebras of degree* 2^{m-1}.

Proof. See [45, (8.10)]. □

Let L/F be a separable quadratic extension. Let φ be the nontrivial F−automorphism of L. For any central simple L−algebra C, *the conjugate algebra* is defined by ${}^{\varphi}C = \{{}^{\varphi}a \mid a \in C\}$, with the following operations: ${}^{\varphi}a + {}^{\varphi}b = {}^{\varphi}(a+b), {}^{\varphi}a{}^{\varphi}b = {}^{\varphi}(ab), {}^{\varphi}(\alpha a) = \varphi(\alpha){}^{\varphi}a$, for all $a, b \in C$ and all $\alpha \in L$. It follows from [18, §8] that ${}^{\varphi}C$ is also a central simple L−algebra. Let $s : {}^{\varphi}C \otimes_L C \to {}^{\varphi}C \otimes_L C$ be the map defined by $s({}^{\varphi}a \otimes b) = {}^{\varphi}b \otimes a$. The *Norm or Corestriction of* C is by definition

$$N_{L/F}(C) = \{u \in {}^{\varphi}C \otimes_L C \mid s(u) = u\}.$$

It is shown in [45, (3.13)] that $N_{L/F}(C)$ is a central simple F−algebra, and that the corestriction induces a map $N_{L/F} : \mathrm{Br}(L) \to \mathrm{Br}(F)$.

1.36 Proposition.

(a) *Suppose that* $\mathrm{disc}(\tau)$ *is trivial and let* $C(B,\tau) = C_+ \times C_-$.

1. *Suppose that* $\deg(B) \equiv 0 \bmod 4$. *Then* $[C_+] + [C_-] = [B]$ *and* $2[C_+] = 2[C_-] = 0$ *in* $\mathrm{Br}(F)$.
2. *Suppose that* $\deg(B) \equiv 2 \bmod 4$. *Then* $[C_+] + [C_-] = 0$ *and* $2[C_+] = 2[C_-] = [B]$ *in* $\mathrm{Br}(F)$.

(b) *Suppose that* $\mathrm{disc}(\tau) = \delta \in F^{\times}/F^{\times 2}$ *is nontrivial.*

1. *Suppose that* $\deg(B) \equiv 0 \bmod 4$. *Then* $N_{F(\sqrt{\delta})/F}([C(B,\tau)]) = [B]$ *in* $\mathrm{Br}(F)$ *and* $2[C(B,\tau)] = 0$ *in* $\mathrm{Br}(F(\sqrt{\delta}))$.
2. *Suppose that* $\deg(B) \equiv 2 \bmod 4$. *Then* $2[C(B,\tau)] = [B_{F(\sqrt{\delta})}]$ *in* $\mathrm{Br}(F(\sqrt{\delta}))$ *and* $N_{F(\sqrt{\delta})/F}([C(B,\tau)]) = 0$ *in* $\mathrm{Br}(F)$.

Proof. See [45, (9.12)]. □

In [70], D. Tao obtained the following result concerning the nature of the Clifford algebra of a tensor product of two algebras of even degree with involution of the same type. Note that [45, (7.3)] implies that the discriminant of such an involution is trivial. Hence, the Clifford algebra has two components in that case.

1.37 Theorem (Tao). *Let* (B_1, ρ_1) *and* (B_2, ρ_2) *be* F*−algebras of even degree with involution of the first kind, and of the same type. Let* $(B,\tau) = (B_1,\rho_1) \otimes_F (B_2,\rho_2)$.

(a) *Suppose that* ρ_1 *and* ρ_2 *are both orthogonal. Denote by* Q *the quaternion algebra* $(\mathrm{disc}(\rho_1), \mathrm{disc}(\rho_2))_F$. *If at least one of* $\deg(B_1)$ *and* $\deg(B_2)$ *is divisible by 4, then one of the components of* $C(B,\tau)$ *is Brauer equivalent to* $B \otimes_F Q$ *and the other one to* Q. *If* $\deg(B_1) \equiv \deg(B_2) \equiv 2 \bmod 4$, *then one of the components of* $C(B,\tau)$ *is Brauer equivalent to* $B_1 \otimes_F Q$ *and the other to* $B_2 \otimes_F Q$.

(b) *Suppose that ρ_1 and ρ_2 are both symplectic. If at least one of $\deg(B_1)$ and $\deg(B_2)$ is divisible by 4, then one of the components of $C(B,\tau)$ is split, and the other is Brauer equivalent to B. If $\deg(B_1) \equiv \deg(B_2) \equiv 2 \bmod 4$, then one of the components of $C(B,\tau)$ is Brauer equivalent to B_1 and the other one to B_2.*

Proof. See [70, (4.12), (4.14), (4.16)]. The result is also stated in [45, p. 150]. □

Balanced ideals

In this section, F is of arbitrary characteristic. We extend the notions of degree and Schur index of a central simple algebra to B. We denote by $\deg(B)$ the square root of $\dim_F(B)/\dim_F(Z(B))$. If B is simple, we let $\operatorname{ind}(B)$ be the usual Schur index of B as a central simple $Z(B)$−algebra. If B is not simple then $Z(B) \cong F \times F$, and Proposition 1.18 yields that there exists a central simple F−algebra E such that $B \cong E \times E^{\mathrm{op}}$. In that case, we set $\operatorname{ind}(B) = \operatorname{ind}(E)$.
Suppose that there exists a central simple F−algebra E such that $B \cong E \times E^{\mathrm{op}}$. Let I be a right ideal of B. Then I corresponds to a right ideal $I_1 \times I_2^{\mathrm{op}}$ of $E \times E^{\mathrm{op}}$, with I_1 a right ideal of E and I_2 a left ideal of E, and we identify I and $I_1 \times I_2^{\mathrm{op}}$ under the isomorphism $B \cong E \times E^{\mathrm{op}}$.

We call a right ideal I of B *balanced* if it is free as a module over $Z(B)$. For such an ideal, $\dim_{Z(B)}(I)$ is divisible by $\deg(B)\operatorname{ind}(B)$, by Proposition 1.38 and [45, pp. 5–6]. We call

$$\operatorname{rdim}(I) = \frac{\dim_{Z(B)}(I)}{\deg(B)}$$

the *reduced dimension of I*. It is clear that $\operatorname{ind}(B) \mid \operatorname{rdim}(I)$ and that $\operatorname{rdim}(I) \leqslant \deg(B)$. It extends the notion of reduced dimension for right ideals of central simple algebras from [45] to cover the semisimple case as well.

1.38 Lemma. *Let T be a domain and let M and N be finite–dimensional T−modules. Then $M \times N$ is free as a $(T \times T)$−module if and only if $\dim_T(M) = \dim_T(N)$, and in that case $\dim_{T\times T}(M \times N) = \dim_T(M) = \dim_T(N)$.*

Proof. This follows from the fact that $(T \times T)^n \cong T^n \times T^n$ as $(T \times T)$−modules for all $n \in \mathbb{N}$, and that $T^n \times T^m$ is not a free $(T \times T)$−module if $n \neq m$. □

If $B \cong E \times E^{\mathrm{op}}$ for a central simple F−algebra E, and I is a balanced right ideal of B that corresponds to the ideal $I_1 \times I_2^{\mathrm{op}}$ of $E \times E^{\mathrm{op}}$, then $\operatorname{rdim}(I) = \frac{\dim_F I_1}{\deg(E)} = \operatorname{rdim}(I_1)$ by Lemma 1.38.

Replacing 'right' by 'left' in the above, we get analogous results for left ideals of B.

For a right ideal I of B, the left ideal

$$I^0 = \{\, x \in B \mid xI = 0 \,\}$$

I^0 is *the annihilator of* I. Similarly, for a left ideal J of B, the right ideal

$$J^0 = \{\, x \in B \mid Jx = 0 \,\}$$

is *the annihilator of* J.

1.39 Proposition. *Let I be a balanced right (resp. left) ideal of B. Then I^0 is a balanced left (resp. right) ideal of B and* $\mathrm{rdim}(I) + \mathrm{rdim}(I^0) = \deg(B)$.

Proof. If $Z(B)$ is a field this is the statement of [45, (1.14)]. Assume that $Z(B) \cong F \times F$ and $B \cong E \times E^{\mathrm{op}}$, for some central simple F−algebra E. Let I be a right ideal of B. The proof for left ideals is analogous. We identify I with a right ideal $I_1 \times I_2^{\mathrm{op}}$ of $E \times E^{\mathrm{op}}$, where I_1 is a right ideal of E and I_2 is a left ideal of E, under the isomorphism $B \cong E \times E^{\mathrm{op}}$. One can check that $I^0 \cong I_1^0 \times (I_2^0)^{\mathrm{op}}$. Since I is balanced, we have that $\mathrm{rdim}(I_1) = \mathrm{rdim}(I_2)$ and hence, $\mathrm{rdim}(I_1^0) = \mathrm{rdim}(I_2^0)$, by the first part of the proof, since I_1 and I_2 are ideals of the central simple F−algebra E. So, I^0 is also balanced, and the first part of the proof yields that

$$\mathrm{rdim}(I) + \mathrm{rdim}(I^0) = \mathrm{rdim}(I_1) + \mathrm{rdim}(I_1^0) = \deg(E) = \deg(B).$$

□

A right (resp. left) ideal I of B is called *isotropic* with respect to τ if $I \subset \tau(I)^0$. We also use the standard notation $I^\perp$ for $\tau(I)^0$. The algebra with involution (B, τ), or τ itself, is called *isotropic* if B contains a nonzero isotropic right ideal, and *anisotropic* otherwise. Note that τ is isotropic if and only if there is a nonzero element $x \in B$ such that $\tau(x)x = 0$. The algebra with involution (B, τ), or τ itself, is called *hyperbolic* if there exists an idempotent $x \in B$ such that $\tau(x) = 1 - x$, and is called *metabolic* if (B, τ) contains an isotropic balanced right ideal I of reduced dimension $\deg(B)/2$. By Proposition 1.39, the latter is equivalent to $I = I^\perp$.

1.40 Proposition. *Suppose that, if* $\mathrm{char}(F) = 2$, *τ is not orthogonal. Then (B, τ) is metabolic if and only if it is hyperbolic.*

Proof. See [45, (6.7)]. □

1.41 Proposition. *Let I be an isotropic balanced right ideal of (B, τ). Then* $\mathrm{rdim}(I) \leqslant \deg(B)/2$.

Proof. If $Z(B)$ is a field this is stated in [45, (6.3)]. Let us consider the case $(B, \tau) \cong (E \times E^{\mathrm{op}}, \mathrm{sw}_E)$, with E a central simple F−algebra. We identify I with a right ideal $I_1 \times I_2^{\mathrm{op}}$ of $E \times E^{\mathrm{op}}$, where I_1 is a right ideal of E and I_2 is a left ideal of E, under the isomorphism $B \cong E \times E^{\mathrm{op}}$. Since I is balanced, we have that $\mathrm{rdim}(I) = \mathrm{rdim}(I_2) = \mathrm{rdim}(I_1) = \ell \cdot \mathrm{ind}(E) = \ell \cdot \mathrm{ind}(B)$ for some $\ell \leqslant \deg(B)/\mathrm{ind}(B)$. Since I is isotropic, it follows that for any $a, c \in I_1$ and any $b, d \in I_2$, we have

$$0 = \mathrm{sw}_E(a, b)(c, d) = (b, a)(c, d) = (bc, a * d),$$

and hence, $I_2 \subset I_1^0$. By Proposition 1.39, we have that

$$\ell \cdot \mathrm{ind}(B) + \ell \cdot \mathrm{ind}(B) = \mathrm{rdim}(I_1) + \mathrm{rdim}(I_2) \leqslant \mathrm{rdim}(I_1) + \mathrm{rdim}(I_1^0) = \deg(B),$$

whence the statement. □

1.42 Proposition. *Suppose that (B, τ) is degenerate. Then τ is hyperbolic, and for any $\ell \in \mathbb{N}$ such that $0 \leqslant \ell \leqslant \frac{\deg(B)}{2\,\mathrm{ind}(B)}$, there exists an isotropic balanced right ideal of (B, τ) of reduced dimension $\ell \cdot \mathrm{ind}(B)$.*

Proof. We have that $(B, \tau) \cong (E \times E^{\mathrm{op}}, \mathrm{sw}_E)$, for some central simple F−algebra E. The element $(1, 0) \in E \times E^{\mathrm{op}}$ is idempotent and $\mathrm{sw}_E(1, 0) = (0, 1) = (1, 1) - (1, 0)$. Hence, τ is hyperbolic. Let $\ell \in \mathbb{N}$ such that $0 \leqslant \ell \leqslant \frac{\deg(B)}{2\,\mathrm{ind}(B)}$. By the characterisation of left and right ideals of E in [45, (1.12)] (see also Proposition 2.12 for the case of right ideals), there exists a right ideal I_1 of E with $\mathrm{rdim}(I_1) = \ell \cdot \mathrm{ind}(B) = \ell \cdot \mathrm{ind}(E) \leqslant \deg(B)/2$ and a left ideal I_2 of E inside I_1^0 of the same reduced dimension. Then $I_1 \times I_2^{\mathrm{op}}$ is an isotropic balanced right ideal of $E \times E^{\mathrm{op}}$ of reduced dimension $\ell \cdot \mathrm{ind}(B)$. □

Using the concept of balanced ideals, we can now give a uniform definition of the concept of the index of an F−algebra with involution. Namely, we define *the index of* (B, τ) to be

$$\mathrm{ind}(B, \tau) = \{\ \mathrm{rdim}(I) \mid I \text{ an isotropic balanced right ideal of } (B, \tau)\ \}.$$

This definition coincides with the one given in [45, p. 73]. This is clear if $Z(B)$ is a field and the degenerate case follows from Proposition 1.42.

2

Hermitian and skew–hermitian spaces

> We could use up two eternities in learning all that is to be learned about our own world and the thousands of nations that have arisen and flourished and vanished from it. Mathematics alone would occupy me eight million years.
>
> Mark Twain

In this chapter we study hermitian and skew–hermitian spaces over Azumaya algebras with involution with center a semilocal Bézout domain, and in most cases we assume that the Azumaya algebra does not have zero divisors. We will see that some statements for (skew–)hermitian spaces over division algebras with involution carry over to the (skew–)hermitian spaces we study here. For instance, there is a Witt cancellation result (which follows from a more general Witt cancellation theorem by B. Keller for semilocal algebras, see [43, (VI.6.7.2)]), and a Witt decomposition result. Furthermore, any Azumaya algebra with involution with center a semilocal Bézout domain, can be obtained as the adjoint algebra with involution of some (skew–)hermitian space over an Azumaya algebra with involution without zero divisors, with the same center. We will use this correspondence between (skew–)hermitian spaces and algebras with involution extensively in chapters 4 and 5, in particular the relation between similarity of (skew–)hermitian spaces and isomorphism of their adjoint algebras with involution (see

section 2.3). We also recall the concept of the anisotropic part of a simple algebra with involution over a field of characteristic not 2, from [15].
The last sections of this chapter deal with representation theorems, on the one hand for (skew–)hermitian spaces and on the other hand for Azumaya algebras with involution, and the relations between them. We first present a representation result in the Noetherian case, and then consider the non–Noetherian case in sections 2.6 and 2.7.

Throughout this section F denotes a field, and R denotes a domain with fraction field F.

2.1 Preliminaries

Let C be a (not necessarily commutative) ring. Let θ be an involution on C. Let V be a finitely generated, projective right C–module. A *sesquilinear form on V (with respect to θ)* is a bi–additive map $h : V \times V \to C$ such that for all $x, y \in V$ and all $\alpha, \beta \in C$, we have that $h(x\alpha, y\beta) = \theta(\alpha)h(x,y)\beta$. Let $\varepsilon = \pm 1$. Then h is called *an ε–hermitian form* if in addition $h(y,x) = \varepsilon\theta(h(x,y))$ for all $x, y \in V$. We call the pair (V,h) *an ε–hermitian module*. Furthermore, h is called *hermitian* if $\varepsilon = 1$ and *skew–hermitian* if $\varepsilon = -1$. If $\theta = \mathrm{id}_C$ then h is a bilinear form.

Let (V,h) be an ε–hermitian module. Let $V^* = \mathrm{Hom}_C(V,C)$. This is a left C–module. Define the right C–module ${}^{\theta}V^*$ by ${}^{\theta}V^* = \{{}^{\theta}\varphi \mid \varphi \in V^*\}$ with the operations ${}^{\theta}\varphi + {}^{\theta}\psi = {}^{\theta}(\varphi + \psi)$, $({}^{\theta}\varphi)\alpha = {}^{\theta}(\theta(\alpha)\varphi)$ for all $\varphi, \psi \in V^*$ and all $\alpha \in C$. Then h is called *non–singular* if the adjoint transformation

$$\hat{h} : V \to {}^{\theta}V^* : x \mapsto {}^{\theta}\varphi, \quad \text{where } \varphi(y) = h(x,y) \text{ for all } y \in V,$$

is an isomorphism of right C–modules. We call (V,h) *an ε–hermitian space* if h is non–singular.
Let $\varphi : (C,\theta) \to (C',\theta')$ be a homomorphism of rings with involution, i.e. φ is a ring homomorphism from C to C' such that $\theta' \circ \varphi = \varphi \circ \theta$. Consider C' as a left C–module via φ. Let (V,h) be an ε–hermitian module over (C,θ). Then the map $h_{C'} : V_{C'} \times V_{C'} \to C'$ defined by

$$h_{C'}(x \otimes a', y \otimes b') = \theta'(a')\varphi(h(x,y))b' \quad \text{for all } x, y \in V \text{ and all } a', b' \in C',$$

is an ε–hermitian form on $V_{C'}$ with respect to θ'. The ε–hermitian module $(V_{C'}, h_{C'})$ is called *the ε–hermitian module induced from (V,h) by scalar extension from C to C'*.

2.1 Notation. Let $(\mathcal{C},\theta)$ be an R–algebra with involution and (V,h) an ε–hermitian module over $(\mathcal{C},\theta)$, with $\varepsilon = \pm 1$. Let R' be a domain that is also an R–algebra. Consider the induced involution $\theta_{R'}$ on $\mathcal{C}_{R'}$. Then we denote the ε–hermitian module $(V_{\mathcal{C}_{R'}}, h_{\mathcal{C}_{R'}})$

by $(V, h)_{R'}$. In the sequel, we will in particular consider the case where R' is the fraction field of R, or where R' is the quotient of R by a maximal ideal.

Let $\tilde{h}$ be a sesquilinear form over V with respect to θ. Define a sesquilinear form $\tilde{h}^*$ on V by $\tilde{h}^*(x, y) = \theta(\tilde{h}(y, x))$ for all $x, y \in V$. An ε–hermitian form h over V is called *even* if there exists a sesquilinear form $\tilde{h}$ over V such that $h = \tilde{h} + \varepsilon\tilde{h}^*$. We then call (V, h) an even ε–hermitian module. If $2 \in C^\times$, by taking $\tilde{h} = \frac{1}{2}h$, one easily sees that every ε–hermitian form over V with respect to θ is even.

Suppose that V is free over C and let $\mathfrak{B} = (e_1, \ldots, e_n)$ be a C–basis for V. Then h defines a matrix $C_h = (h(e_i, e_j)_{i,j}) \in \mathrm{M}_n(C)$. The *dual basis* $\mathfrak{B}^\# = (e_1^\#, \ldots, e_n^\#)$ is defined by the property $e_i^\#(e_j) = \delta_{ij}$ for $i, j = 1, \ldots, n$. Then $({}^\theta e_1^\#, \ldots, {}^\theta e_n^\#)$ is a C–basis for ${}^\theta V^*$. The matrix of $\hat{h}$ with respect to the bases $(\mathfrak{B}, \mathfrak{B}^\#)$ is given by εC_h. Hence, $\hat{h}$ is an isomorphism if and only C_h is invertible. If h is non–singular, we may consider the elements ${}^\theta e_1^\#, \ldots, {}^\theta e_n^\#$ as elements of V.

Let $\alpha_1, \ldots, \alpha_n \in C^\times$ be elements such that $\theta(\alpha_i) = \varepsilon\alpha_i$. Then the matrix $\mathrm{diag}(\alpha_1, \ldots, \alpha_n)$ defines a non–singular ε–hermitian form on C^n with respect to θ. We denote the corresponding ε–hermitian space by $\langle\alpha_1, \ldots, \alpha_n\rangle_\theta$.

Let U be a C–submodule of V. The *orthogonal complement of* U, which is equal to $\{x \in V \mid h(x, y) = 0 \text{ for all } y \in U\}$, is denoted by $U^\perp$. The subspace U is called *totally isotropic* if $U \subset U^\perp$. The ε–hermitian module (V, h) is called *isotropic* if it contains a nonzero totally isotropic subspace U, and *anisotropic* otherwise. Equivalently, (V, h) is isotropic if there exists an element $0 \neq x \in V$ such that $h(x, x) = 0$. (V, h) is called *metabolic* if it contains a direct summand U such that $U^\perp = U$. There is also a notion of a *hyperbolic* ε–hermitian space (see [43, (I.3.5)]). The notions of hyperbolic and metabolic coincide if $2 \in C^\times$, and even in a more general situation, as the following proposition shows.

2.2 Proposition. *If (V, h) is an even ε–hermitian space, then (V, h) is metabolic if and only if it is hyperbolic.*

Proof. See [43, (I.3.7.3)]). □

Let (V, h) and (V', h') be two ε–hermitian modules over (C, θ). They are called *isometric*, denoted by $(V, h) \simeq (V', h')$, if there is a C–linear bijection $\varphi : V \to V'$ such that $h(x, y) = h'(\varphi(x), \varphi(y))$ for all $x, y \in V$. They are called *similar* if there exists $a \in C$ such that $(V, h) \simeq (V', ah')$. The *orthogonal sum* of (V, h) and (V', h') is the ε–hermitian space $(V \oplus V', h \perp h')$, where $(h \perp h')(x + x', y + y') = h(x, y) + h'(x', y')$, for all $x, y \in V$ and all $x', y' \in V'$. We call (V, h) and (V', h') *Witt equivalent* if there exist hyperbolic

ε–hermitian spaces $(\tilde{V},\tilde{h})$ and $(\tilde{V}',\tilde{h}')$ over (C,θ) such that $(V,h) \perp (\tilde{V},\tilde{h}) \simeq (V',h') \perp (\tilde{V}',\tilde{h}')$, and denote this by $(V,h) \sim (V',h')$. The set of Witt classes of ε–hermitian spaces over (C,θ) forms a group for the orthogonal sum, and this group is denoted by $W^{\varepsilon}(C,\theta)$.

2.3 Proposition. *Let $\varepsilon = \pm 1$. Let (V,h) be an ε–hermitian space over (C,θ). Let U be a C–submodule of V that is finitely generated, projective over C. If $h|_U$ is non–singular, then $(V,h) \simeq (U,h|_U) \perp (U^{\perp},h|_{U^{\perp}})$, and $(U^{\perp},h|_{U^{\perp}})$ is also an ε–hermitian space over (C,θ).*

Proof. See [43, (I.6.3.1), (I.3.6.2)]. □

In the rest of this section we work with (skew–)hermitian spaces over Azumaya algebras with involution without zero divisors.

2.4 Proposition. *Assume that R is a semilocal Bézout domain. Let (Δ,θ) be an R–algebra with involution without zero divisors. Let $\varepsilon = \pm 1$ and let (V,h) be an ε–hermitian space over (Δ,θ). The following hold:*

(a) V is free as a Δ–module.

(b) Let (V',h') be another ε–hermitian space over (Δ,θ). Assume that (V,h) and (V',h') are even. Let H be a hyperbolic plane over (Δ,θ), i.e. a 2–dimensional hyperbolic ε–hermitian space over (Δ,θ). If $(V,h) \perp H \simeq (V',h') \perp H$ then $(V,h) \simeq (V',h')$ (Witt cancellation).

Proof. By Propositions 1.21 and 1.24, Δ is a left and right Bézout ring. Since V is finitely generated, projective over Δ, it is torsion–free as a Δ–module, and Proposition 1.9 yields that V is free over Δ.
The isometry $(V,h) \perp H \simeq (V',h') \perp H$ yields that $\dim_{\Delta}(V) = \dim_{\Delta}(V')$. We denote the center of Δ by S. By Corollary 1.13, we have that $J(\Delta) = J(S)\Delta$. By Proposition 1.22, S is semilocal. Let $\mathcal{M}_1,\ldots,\mathcal{M}_r$ be the different maximal ideals of S. Then $J(S) = \mathcal{M}_1 \cap \ldots \cap \mathcal{M}_r = \mathcal{M}_1 \cdots \mathcal{M}_r$. By [49, (XVI.2.7)] and the Chinese Remainder Theorem, it follows that

$$\Delta/J(\Delta) \cong \Delta \otimes_S S/J(S) \cong \prod_{i=1}^{r} \Delta \otimes_S S/\mathcal{M}_i.$$

By Proposition 1.11 (c), $\Delta_i = \Delta \otimes_S S/\mathcal{M}_i$ is a central simple $S/\mathcal{M}_i$–algebra for $i = 1,\ldots,r$. Hence, Δ is semilocal. Let $\mathfrak{m}_i = \mathcal{M}_i \cap R$. Then $S/\mathcal{M}_i$ is either equal to $R/\mathfrak{m}_i$ or a separable quadratic extension of $R/\mathfrak{m}_i$. In the first case, let ι be the identity on $S/\mathcal{M}_i$, and in the second case, let ι be the nontrivial element of $\mathrm{Gal}((S/\mathcal{M}_i)/(R/\mathfrak{m}_i))$. The involution θ on Δ induces an involution $\theta \otimes_S \iota$ on Δ_i. Since h is non–singular, the induced ε–hermitian modules $(V_{\Delta_i},h_{\Delta_i})$ $(i = 1,\ldots,r)$ are nonzero. Since (V,h) and

(V', h') are even ε–hermitian spaces over (Δ, θ), they are in particular unitary spaces (see [43, Chapter I]). The statement now follows from a general Witt cancellation result of B. Keller (see [43, (VI.5.7.2)]). $\square$

2.5 Proposition. *Assume that R is a semilocal Bézout domain. Let (Δ, θ) be an R–algebra with involution without zero divisors. Let $\varepsilon = \pm 1$. Then hyperbolic ε–hermitian spaces over (Δ, θ) of the same Δ–dimension are isometric.*

Proof. This follows from the definition of a hyperbolic space in [43, (I.3.5)] and the fact that ε–hermitian spaces over (Δ, θ) are free over Δ by Proposition 2.4 (a). $\square$

2.6 Proposition. *Assume that R is a semilocal Bézout domain. Let (Δ, θ) be an R–algebra with involution without zero divisors. Let $\varepsilon = \pm 1$ and let (V, h) be an ε–hermitian space over (Δ, θ). Then (V, h) is the orthogonal sum of an anisotropic ε–hermitian space over (Δ, θ) and a metabolic ε–hermitian space over (Δ, θ). Furthermore, if (V, h) is an even ε–hermitian space, then this decomposition is unique up to isometry.*

Proof. If h is anisotropic there is nothing to prove. Suppose that h is isotropic. Let $0 \neq x \in V$ be such that $h(x, x) = 0$. Let $(m_1, \ldots, m_r)$ be a Δ–basis for V. Then $x = \sum_{i=1}^r m_i x_i$, with $x_1, \ldots, x_r \in \Delta$. Since Δ is a left Bézout ring, there exists $d \in \Delta$ such that $\Delta x_1 + \ldots + \Delta x_r = \Delta d$. It follows that there exist $b_1, \ldots, b_r, c_1, \ldots, c_r \in \Delta$ such that $x_i = b_i d$ and $\sum_{i=1}^r c_i x_i = d$. So, we have $x = (\sum_{i=1}^r m_i b_i) d$. Let $y = \sum_{i=1}^r m_i b_i$. Then $0 = h(x, x) = \theta(d) h(y, y) d$. Since Δ does not have zero divisors and $d \neq 0$, it follows that $h(y, y) = 0$. We have that $\sum_{i=1}^r c_i b_i = 1$. By Proposition 1.7, there exists an invertible matrix U over Δ with $(b_1, \ldots, b_r)$ as its first column. Let $(n_1, \ldots, n_r) = (m_1, \ldots, m_r) U$. Then $n_1 = y$. Since U is invertible, this means that $(y, n_2, \ldots, n_r)$ is a Δ–basis for V.
It follows that there exists $\varphi \in \mathrm{Hom}_\Delta(V, \Delta)$ such that $\varphi(y) = 1$ (take the element $y^\#$ of the dual basis). Since h is non–singular, there exists $y' \in V$ such that $1 = \varphi(y) = h(y', y)$. Consider the right Δ–subspace $U = y\Delta + y'\Delta$ of V, and note that in fact $U = y\Delta \oplus y'\Delta$. The matrix of $h|_U$ with respect to the basis (y, y') is given by

$$\begin{pmatrix} 0 & \varepsilon \\ 1 & h(y', y') \end{pmatrix}.$$

This matrix is invertible over Δ. Hence, $h|_U$ is non–singular and isotropic. It is a so–called metabolic plane. Proposition 2.3 yields that

$$(V, h) \simeq (U, h|_U) \perp (U^\perp, h|_{U^\perp}).$$

Furthermore, by Proposition 2.3, $(U^\perp, h|_{U^\perp})$ is also an ε–hermitian space over (Δ, θ) and hence, $U^\perp$ is free over Δ. If it is anisotropic we are done. If $h|_{U^\perp}$ is isotropic then we can repeat the above procedure. Eventually we obtain a decomposition of the desired form.

Suppose that (V, h) is an even ε–hermitian space. Then all ε–hermitian submodules of (V, h) are also even by [43, (I.3.1.1)]. Hence, the decomposition obtained above consists of even spaces. The uniqueness of the decomposition now follows from Propositions 2.2, 2.4 and 2.5. □

2.7 Remark. Note that Proposition 2.6 yields that an ε–hermitian space over (Δ, θ) is isotropic if and only if it contains a "unimodular" isotropic vector.

2.8 Proposition. *Assume that R is a semilocal Bézout domain. Let (Δ, θ) be an R–algebra with involution without zero divisors and let $\varepsilon = \pm 1$. The following hold:*

(a) Let (V, h) be an ε–hermitian space over (Δ, θ). If $(V, h)_F$ is isotropic (resp. metabolic), then (V, h) is already isotropic (resp. metabolic).

(b) Let (V, h) and (V', h') be even ε–hermitian spaces over (Δ, θ) of the same dimension over Δ. Then $(V', h') \perp (V, -h)$ is hyperbolic if and only if $(V', h') \simeq (V, h)$.

(c) Let (V, h) and (V', h') be even ε–hermitian spaces over (Δ, θ). If $(V, h)_F \simeq (V', h')_F$ then $(V, h) \simeq (V', h')$.

Proof. By Propositions 1.21 and 1.24, Δ is a left and right Bézout ring without zero divisors. We denote the center of Δ by S.
Suppose that $(V, h)_F$ is isotropic. Let $0 \neq x \in V_F$ be such that $h_F(x, x) = 0$. Then there exists $r \in R$ such that $rx \in V$. Then $rx \neq 0$ and $h(rx, rx) = 0$. Hence, (V, h) is isotropic. Suppose that $(V, h)_F$ is metabolic, but (V, h) non–metabolic. By Proposition 2.6, we can decompose $(V, h) \simeq (V_1, h_1) \perp (V_2, h_2)$, with (V_1, h_1) anisotropic and (V_2, h_2) metabolic. Then (V_1, h_1) remains anisotropic over F by the first part of the proof. But this means that $(V, h)_F$ is not metabolic, a contradiction. This yields (a).

Let (V', h') be an even ε–hermitian space over (Δ, θ). If $(V', h') \simeq (V, h)$ then $(V', h') \perp (V, -h)$ is metabolic by [43, (I.3.7.8)], and hence hyperbolic by Proposition 2.2, since both spaces are even. Suppose conversely that $(V', h') \perp (V, -h)$ is hyperbolic. Again invoking [43, (I.3.7.8)] and Proposition 2.2, this implies that there exist hyperbolic ε–hermitian spaces (V_1, h_1) and (V_1', h_1') over (Δ, θ) such that

$$(V, h) \perp (V_1, h_1) \simeq (V', h') \perp (V_1', h_1'). \tag{2.1.1}$$

By assumption, $\dim_\Delta(V) = \dim_\Delta(V')$ and hence, $\dim_\Delta(V_1) = \dim_\Delta(V_1')$. By the proof of Proposition 2.6, (V_1, h_1) and (V_1', h_1') are an orthogonal sum of hyperbolic planes, which are all isometric by Proposition 2.5. By Proposition 2.4, we may then cancel the hyperbolic parts in (2.1.1) in order to obtain the desired isometry $(V, h) \simeq (V', h')$. This proves (b). The statement in (c) now follows immediately from (a) and (b) since the isometry over F implies that $\dim_\Delta(V) = \dim_\Delta(V')$. □

In the situation of Proposition 2.6, suppose that R is a field. Then Δ is a division algebra. The *Witt index of* (V,h) is defined as the dimension over Δ of a maximal totally isotropic subspace of (V,h), and is denoted by $i_w(h)$. If (V,h) is even then the Witt index of (V,h) is equal to half the dimension of the hyperbolic part of (V,h) over Δ by [66, (7.9.2)]. The latter result is formulated for what W. Scharlau calls "regular quadratic spaces", but even ε−hermitian spaces are regular quadratic spaces by [66, (7.3.4)].

2.2 Adjoint involutions

(Skew–)hermitian spaces have a so–called "adjoint algebra with involution". We show below that Azumaya algebras with involution with center a semilocal Bézout domain are up to isomorphism obtained as the adjoint algebra with involution of some (skew–) hermitian space over an Azumaya algebra with involution without zero divisors.

2.9 Proposition. *Assume that R is a semilocal Bézout domain and let $(\mathcal{C},\theta)$ an R−algebra with involution with center a domain. Let $\varepsilon = \pm 1$ and let (V,h) be an ε−hermitian space over $(\mathcal{C},\theta)$. There exists a unique involution σ on $\mathrm{End}_{\mathcal{C}}(V)$ such that $\sigma(a) = \theta(a)$ for all $a \in Z(\mathcal{C})$, and for all $x, y \in V$ and all $f \in \mathrm{End}_{\mathcal{C}}(V)$ we have that*

$$h(x, f(y)) = h(\sigma(f)(x), y).$$

We denote this involution by ad_h. Then $(\mathrm{End}_{\mathcal{C}}(V), \mathrm{ad}_h)$ is an R−algebra with involution with center $Z(\mathcal{C})$, called the adjoint algebra with involution of h, *and denoted by* $\mathrm{Ad}(h)$. *If θ is of the first kind (resp. of the second kind), then ad_h is of the first kind (resp. of the second kind).*

Proof. We write $S = Z(\mathcal{C})$. Since S is a domain, we have that V is faithful as an S−module. Furthermore, by Proposition 1.21, S is a semilocal Bézout domain. Let $\mathfrak{m}$ be a maximal ideal of S. Then $\mathrm{End}_{\mathcal{C}}(V)/\mathfrak{m}\,\mathrm{End}_{\mathcal{C}}(V) \cong \mathrm{End}_{\mathcal{C}/\mathfrak{m}\mathcal{C}}(V/\mathfrak{m}V)$ by [43, (III.5.1.8)]. Since V is a finitely generated, projective $\mathcal{C}$−module and $\mathcal{C}$ is of finite dimension over S by Corollary 1.19, V is finitely generated and faithful as an S−module, and hence, $V/\mathfrak{m}V$ is nonzero. This implies that $\mathrm{End}_{\mathcal{C}/\mathfrak{m}\mathcal{C}}(V/\mathfrak{m}V)$ is a central simple algebra over $S/\mathfrak{m}$. The fact that $\mathrm{End}_{\mathcal{C}}(V)$ is an Azumaya algebra over $Z(\mathcal{C})$ now follows from Proposition 1.11 (c). One easily checks σ is an involution of the first or second kind on $\mathrm{End}_{\mathcal{C}}(V)$ and the uniqueness of ad_h follows from the fact that h is non–singular. □

The converse of Proposition 2.9 also holds.

2.10 Proposition. *Assume that R is a semilocal Bézout domain. Let $(\mathcal{A},\sigma)$ be an R−algebra with involution with center a domain. The following hold:*

(a) Every Azumaya algebra over $Z(\mathcal{A})$ Brauer equivalent to $\mathcal{A}$ carries an involution of the same kind as $\mathcal{A}$.

(b) *There exists an Azumaya algebra Δ over $Z(\mathcal{A})$ without zero divisors such that for every involution θ on Δ of the same kind as σ, there exists an ε–hermitian space (V, h) over (Δ, θ), with $\varepsilon = \pm 1$, such that $(\mathcal{A}, \sigma) \cong_{Z(\Delta)} \mathrm{Ad}(h)$.*

Proof. We denote $Z(\mathcal{A})$ by S and the fraction field of S by K. By Proposition 1.21, S is a semilocal Bézout domain. Let $\mathcal{C}$ be an Azumaya algebra over S Brauer equivalent to $\mathcal{A}$. If σ is of the first kind, then $\mathcal{A}_F$, and hence also $\mathcal{C}_F$, is of exponent 2 in $\mathrm{Br}(K)$ by [45, (3.1) (1)]. By Corollary 1.31, this implies that $\mathcal{C}$ is of exponent 2 in $\mathrm{Br}(S)$. Suppose that σ is of the second kind. Then the corestriction of $\mathcal{A}_F$ is split by [45, (3.1) (2)] and hence, so is the corestriction of $\mathcal{C}_F$. Invoking Corollary 1.31 once more, we obtain that the corestriction of $\mathcal{C}$ is also split. Since S is a semilocal domain, it is connected and [65, (4.4)] yields that there exists an involution θ on $\mathcal{C}$ of the same kind as σ. This proves (a).

Proposition 1.26 yields that there exists an Azumaya algebra Δ over S without zero divisors, and a finite–dimensional right Δ–module V such that $\mathcal{A} \cong \mathrm{End}_\Delta(V) \cong \mathrm{M}_n(\Delta)$ as S–algebras, with $n = \dim_\Delta(V)$. In the rest of the proof, we identify $\mathcal{A}$ with $\mathrm{M}_n(\Delta)$ through this isomorphism. By (a), there exists an involution θ on Δ of the same kind as σ. Let $(e_1, \ldots, e_n)$ be a Δ–basis of V. We define an involution * on $\mathcal{A}$ by $(d_{ij})^\star_{ij} = (\theta(d_{ij}))^t_{ij}$, where t denotes the transpose involution. By Proposition 1.23, there exists $a \in \mathcal{A}^\times$ with $a^\star = \pm a$ if * is of the first kind and $a^\star = a$ if * is of the second kind, and such that $\sigma = \mathrm{Int}(a) \circ {}^\star$. We define $h_\star : V \times V \to \Delta$ with respect to the basis $(e_1, \ldots, e_n)$ by $h_\star(x, y) = \sum_{i=1}^n \theta(x_i) y_i$. This is a hermitian form over V with respect to θ, and $^\star = \mathrm{ad}_{h_\star}$. Furthermore, since the matrix of $h_\star$ is given by the identity matrix, $h_\star$ is non–singular. Now define $h : V \times V \to \Delta$ by $h(x, y) = h_\star(a^{-1}(x), y)$. For all $x, y \in V$, we have that

$$\begin{aligned} h(y, x) &= h_\star(a^{-1}(y), x) = \varepsilon\theta(h_\star(x, a^{-1}(y))) = \varepsilon\theta(h_\star(a^{-\star}(x), y)) \\ &= \pm\varepsilon\theta(h_\star(u^{-1}(x), y) = \pm\varepsilon\theta(h(x, y)), \end{aligned}$$

and it is clear that $h(x\alpha, y\beta) = \theta(\alpha) h(x, y)\beta$ for all $\alpha, \beta \in C$. Hence, h is a hermitian form over (Δ, θ) if $a^\star = a$ and a skew–hermitian form if $a^\star = -a$. Furthermore, h is non–singular since $a \in \mathcal{A}^\times$. So, (V, h) is an ε–hermitian space. We have that

$$\begin{aligned} h(\sigma(f)(x), y) &= h_\star(a^{-1}\sigma(f)(x), y) = h_\star(f^\star(a^{-1}(x)), y) = h_\star(a^{-1}(x), f(y)) \\ &= h(x, f(y)), \end{aligned}$$

which means that $\sigma = \mathrm{ad}_h$. □

In the rest of this section we study the relation between isotropy (resp. metabolicity) of (skew–)hermitian spaces over central simple algebras with involution, and isotropy (resp. metabolicity) of their adjoint algebras with involution. These results can be found in the literature for (skew–)hermitian spaces over division algebras with involution. We thank J.–P. Tignol and T. Unger for their help with some of the proofs.

2.11 Lemma. *Let K be a field and E a central simple $K-$algebra.*

(a) All simple right (resp. left) $E-$modules are isomorphic and are projective as an $E-$module.

(b) Every finitely generated right (resp. left) $E-$module is a direct sum of simple $E-$modules.

Proof. See [66, (8.1.8)]. □

2.12 Proposition. *Let K be a field and E a central simple $K-$algebra. Let furthermore V be a finitely generated, projective right $E-$module. Then $\mathrm{End}_E(V)$ is a central simple $K-$algebra, and the following hold:*

(a) For every right $E-$subspace W of V, $\mathrm{Hom}_E(V,W)$ is a right ideal of $\mathrm{End}_E(V)$. Conversely, every right ideal of $\mathrm{End}_E(V)$ is of the form $\mathrm{Hom}_E(V, x(V))$, for some idempotent $x \in \mathrm{End}_E(V)$.

(b) Let W be any right $E-$subspace of V. Then

$$\mathrm{rdim}(\mathrm{Hom}_E(V,W)) = \frac{\dim_K(W)}{\deg(E)}.$$

Proof. Let W be any $E-$subspace of V. Since V is finite–dimensional over K, it follows that W is finitely generated as an $E-$module. Let N be a simple right $E-$module. By Lemma 2.11, we may identify V with N^r and W with N^s, for appropriate $r, s \in \mathbb{N}$. We write $D = \mathrm{End}_E(N)$. This is a division algebra with center K by [66, (8.1.3)], which is Brauer equivalent to E. We have that

$$\mathrm{End}_E(V) \cong D^{r^2} \quad \text{and} \quad \mathrm{Hom}_E(V,W) \cong D^{rs}.$$

This implies that $\mathrm{Hom}_E(V,W) \cong \mathrm{M}_{r,s}(D)$ and $\mathrm{End}_E(V) \cong \mathrm{M}_r(D)$ as $K-$vector spaces. In particular, $\mathrm{End}_E(V)$ is a central simple $K-$algebra. By Wedderburn's theorem, it follows that there exists $t \in \mathbb{N}$ such that $E \cong \mathrm{M}_t(D)$ as $K-$algebras. By [66, (8.1.8)], $N \cong D^t$, and hence, $\dim_K(N) = t \dim_K(D) = \frac{\deg(E)}{\mathrm{ind}(E)} \dim_K(D) = \deg(E)\,\mathrm{ind}(E)$.

It is clear that $\mathrm{Hom}_E(V,W)$ is a right ideal of $\mathrm{End}_E(V)$. Conversely, let I be a right ideal of $\mathrm{End}_E(V)$. By [45, (1.13)], there exists an idempotent $x \in \mathrm{End}_E(V)$ such that $I = x\,\mathrm{End}_E(V)$. It is clear that $x\,\mathrm{End}_E(V) \subset \mathrm{Hom}_E(V, x(V))$, and the other inclusion follows by using that x is idempotent. This proves (a).
We obtain

$$\mathrm{rdim}(\mathrm{Hom}_E(V,W)) = \frac{\dim_K[\mathrm{Hom}_E(V,W)]}{\deg(\mathrm{End}_E(V))} = \frac{rs\dim_K(D)}{r\deg(D)} = s\deg(D) = s\,\mathrm{ind}(E),$$

and since
$$s = \frac{\dim_K(W)}{\dim_K(N)} = \frac{\dim_K(W)}{\deg(E)\,\mathrm{ind}(E)},$$
this proves (b). □

2.13 Lemma. *Let K be a field and E a central simple K−algebra. Let W_1 and W_2 be right E−subspaces of V. Suppose that $\mathrm{Hom}_E(V, W_1) = \mathrm{Hom}_E(V, W_2)$. Then $W_1 = W_2$.*

Proof. By Lemma 2.11, finitely generated right E−modules are projective. Let $v \in V$ be a nonzero element and consider the exact sequence of right E−modules
$$0 \longrightarrow vE \longrightarrow V \xrightarrow{p} V/vE \longrightarrow 0.$$
By Lemma 2.11, since V/vE is a finitely generated right E−module, V/vE is projective as an E−module. This implies that there exists a section $s : V/vE \to V$ such that $ps = \mathrm{id}_{V/vE}$. Then $y = \mathrm{id}_V - sp \in \mathrm{End}_E(V)$ is an idempotent such that $y(V) = vE$. Using this, it is clear that if $\mathrm{Hom}_E(V, W_1) = \mathrm{Hom}_E(V, W_2)$, then $W_1 = W_2$. □

The next statement can be found in the literature (see [6, (1.3)] and [17, (4.8)]) in the case of (skew–)hermitian spaces over division algebras with involution.

2.14 Proposition. *Let (E, θ) be an F−algebra with involution where E is simple. Let $\varepsilon = \pm 1$ and let (V, h) be an ε−hermitian space over (E, θ). The following hold:*

(a) Let W be a right E−subspace of V. The right ideal $\mathrm{Hom}_E(V, W)$ of $\mathrm{End}_E(V)$ is isotropic for ad_h if and only if W is totally isotropic for h.

(b) ad_h is isotropic (resp. metabolic) if and only if h is isotropic (resp. metabolic).

Proof. By Proposition 2.12, there exists an idempotent $x \in \mathrm{End}_E(V)$ such that the right ideal $\mathrm{Hom}_E(V, W) = \mathrm{Hom}_E(V, x(V))$, and by Lemma 2.13, it follows that $x(V) = W$. Furthermore, we have that
$$\mathrm{Hom}_E(V, x(V))^\perp = \mathrm{Hom}_E(V, x(V)^\perp).$$
An argument is given in [45, (6.2)], where the statement assumes that E is a division algebra, whereas the proof does not. Suppose that W is totally isotropic. Then $\mathrm{Hom}_E(V, W) \subset \mathrm{Hom}_E(V, W^\perp) = \mathrm{Hom}_E(V, W)^\perp$. Hence, $\mathrm{Hom}_E(V, W)$ is an isotropic right ideal of $\mathrm{End}_E(V)$. Suppose conversely that $\mathrm{Hom}_E(V, W)$ is isotropic. Then we have that $\mathrm{Hom}_E(V, W) \subset \mathrm{Hom}_E(V, W)^\perp = \mathrm{Hom}_E(V, W^\perp)$. Since $W = x(V)$, it follows that $x \in \mathrm{Hom}_E(V, W^\perp)$, and hence $W = x(V) \subset W^\perp$. This proves (a).

It now follows directly from (a) that ad_h is isotropic if and only if h is isotropic. So, suppose that h is metabolic. Then there exists a right E−subspace W of V that is a direct summand of V, and such that $W = W^\perp$. It follows that $\mathrm{Hom}(V,W)^\perp = \mathrm{Hom}(V,W^\perp) = \mathrm{Hom}(V,W)$, and hence ad_h is metabolic. Suppose conversely that ad_h is metabolic. Then there exists a right ideal I of $\mathrm{End}_E(V)$ such that $I = I^\perp$. By Proposition 2.12, there exists an idempotent $x \in \mathrm{End}_E(V)$ such that $I = \mathrm{Hom}_E(V, x(V))$. By the above, it follows that $\mathrm{Hom}_E(V,x(V)) = \mathrm{Hom}_E(V,x(V)^\perp)$, and Lemma 2.12 then yields that $x(V) = x(V)^\perp$. Since x is idempotent, we have that $x(V)$ is a direct summand of V. This shows that h is metabolic. □

2.15 Corollary. *Let (B,τ) be an F−algebra with involution where B is simple. Let E be any central simple $Z(B)$−algebra Brauer equivalent to B, and let θ be an involution on E of the same kind as τ. Let $\varepsilon = \pm 1$ and let (V,h) be an ε−hermitian space over (E,θ) such that $(B,\tau) \cong_F \mathrm{Ad}(h)$. Then*

$$\mathrm{ind}(B,\tau) = \left\{ \frac{\dim_{Z(B)}(W)}{\deg(E)} \;\middle|\; W \textit{ is a totally isotropic subspace of } (V,h) \right\}.$$

Proof. Since $\mathrm{ind}(B,\tau)$ is the set of the reduced dimensions of isotropic right ideals of (B,τ), the statement follows immediately from Propositions 2.12 and 2.14. □

2.16 Corollary. *Let (D,θ) be an F−algebra with involution and assume that D is a division algebra. Let $\varepsilon = \pm 1$ and let (V,h) be an ε−hermitian space over (D,θ). Then*

$$\mathrm{ind}(\mathrm{Ad}(h)) = \{ j \cdot \mathrm{ind}(D) \mid 0 \leqslant j \leqslant i_w(h) \}.$$

Proof. Let W be a totally isotropic subspace of (V,h). Then

$$\dim_K(W) = \dim_D(W)\deg(D)^2 = \dim_D(W)\,\mathrm{ind}(D)^2.$$

Since $i_w(h)$ is the D−dimension of a maximal totally isotropic subspace of (V,h), the statement now follows immediately from Corollary 2.15. It can also be found in [45, p. 73]. □

Let (V,b) be a symmetric bilinear space over F. The Witt index of b is then the largest integer in $\mathrm{ind}(\mathrm{Ad}(b))$. In this respect, we define *the Witt index* of an F−algebra with involution (B,τ) as $\max\{i \in \mathbb{N} \mid i \in \mathrm{ind}(B,\tau)\}$, and we denote it by $i_w(\tau)$.

The following result is now immediate from Corollary 2.16.

2.17 Corollary. *Let (D,θ) be an F−algebra with involution and assume that D is a division algebra. Let $\varepsilon = \pm 1$ and let (V,h) be an ε−hermitian space over (D,θ). Then $i_w(\mathrm{ad}_h) = \mathrm{ind}(D)i_w(h)$.*

2.3 Similarity

In this section, we study properties of isomorphic Azumaya algebras with involution over a semilocal Bézout domain in terms of (skew–)hermitian spaces they are adjoint to. We also express some of these results in terms of multipliers of involutions. The latter formulation will be used when we consider the "rational isomorphism implies isomorphism" problem for Azumaya algebras with involution over a semilocal Bézout domain in later chapters.

2.18 Proposition. *Assume that R is a semilocal Bézout domain. Let $(\mathcal{A},\sigma)$ be an $R-$algebra with involution with center a domain. Let furthermore $s \in \mathcal{A}^\times$ be such that $\sigma(s) = s$ and let $\sigma' = \operatorname{Int}(s) \circ \sigma$. The following are equivalent:*

(i) $(\mathcal{A},\sigma) \cong_{Z(\mathcal{A})} (\mathcal{A},\sigma')$.

(ii) There exist elements $u \in R^\times$ and $g \in \mathcal{A}^\times$ such that $us = \sigma(g)g$.

Proof. Suppose that (i) holds. Let $\varphi : \mathcal{A} \to \mathcal{A}$ be a $Z(\mathcal{A})-$automorphism such that $\sigma' \circ \varphi = \varphi \circ \sigma$. By Propositions 1.21 and 1.20, there exists an element $g \in \mathcal{A}^\times$ such that $\varphi = \operatorname{Int}(\sigma(g))$. We get that

$$\operatorname{Int}(\sigma(g)) \circ \sigma = \operatorname{Int}(s) \circ \sigma \circ \operatorname{Int}(\sigma(g)) = \operatorname{Int}(sg^{-1}) \circ \sigma.$$

This implies that $\operatorname{Int}(\sigma(g)) = \operatorname{Int}(sg^{-1})$ and hence there is an element $u \in Z(\mathcal{A})^\times$ such that $\sigma(g) = usg^{-1}$. In other words

$$us = \sigma(g)g.$$

It follows that $\sigma(u)s = \sigma(s)\sigma(u) = \sigma(us) = us$. Since $s \in \mathcal{A}^\times$, it follows that $u \in R$. Since $u \in Z(\mathcal{A})^\times$, we have that u is in fact an element of $R^\times$. This proves (ii). For the converse, we can just go backwards through the proof of (i) ⇒ (ii). □

2.19 Proposition. *Assume that R is a semilocal Bézout domain. Let $(\mathcal{C},\theta)$ be an $R-$algebra with involution with center a domain. Let $\varepsilon = \pm 1$ and let (V,h) be an $\varepsilon-$hermitian space over $(\mathcal{C},\theta)$. Let $(\mathcal{A},\sigma) = \operatorname{Ad}(h)$. Let furthermore $s \in \mathcal{A}^\times$ be such that $\sigma(s) = s$ and let $\sigma' = \operatorname{Int}(s)\circ\sigma$. Define $h' : V \times V \to \Delta$ by $h'(x,y) = h(s^{-1}(x),y)$ for all $x,y,\in V$. Then (V,h') is an $\varepsilon-$hermitian space over $(\mathcal{C},\theta)$ such that $(\mathcal{A},\sigma') = \operatorname{Ad}(h')$. Furthermore, the following are equivalent:*

(i) $(\mathcal{A},\sigma) \cong_{Z(\mathcal{A})} (\mathcal{A},\sigma')$.

(ii) There exists elements $e \in R^\times$ and $g \in \mathcal{A}^\times$ such that $es = \sigma(g)g$.

(iii) There exists $u \in R^\times$ such that $(V,h') \simeq (V,uh)$.

Moreover, given $u \in R^\times$, we have that $(V, h') \simeq (V, uh)$ if and only if there exists $g \in \mathcal{A}^\times$ such that $us = \sigma(g)g$.

Proof. The equivalence of (i) and (ii) is given by Proposition 2.18. Suppose that (ii) holds. Using the equality $es = \sigma(g)g$, we get that

$$h'(x, y) = h(s^{-1}(x), y) = eh(g^{-1}\sigma(g^{-1})(x), y) = eh(\sigma(g^{-1})(x), \sigma(g^{-1})(y)).$$

This yields that $(V, h') \simeq (V, eh)$, whence (iii). Suppose conversely that $(V, h') \simeq (V, uh)$ for some $u \in R^\times$. Then there exists a $\mathcal{C}$–linear bijection $\varphi : V \to V$ such that $h'(x, y) = uh(\varphi(x), \varphi(y))$. Then $\varphi \in \mathrm{End}_{\mathcal{C}}(V)^\times = \mathcal{A}^\times$ and it follows that

$$h(s^{-1}(x), y) = h'(x, y) = h(u\sigma(\varphi)\varphi(x), y)$$

for all $x, y \in V$. The non–singularity of h yields that $s^{-1} = u\sigma(\varphi)\varphi$, i.e. $us = \varphi^{-1}\sigma(\varphi^{-1})$. □

Proposition 2.19 yields the equivalence between similarity of specific (skew–)hermitian spaces and isomorphism of their adjoint algebras with involution. This result holds in fact without constraints on the (skew–)hermitian spaces, as we show below. We start with a preliminary result.

2.20 Proposition. *Assume that R is a semilocal Bézout domain. Let $\mathcal{A}$ be an Azumaya algebra over R or a separable quadratic R–algebra that is a domain. Let σ and σ' be two R–linear involutions of the first or second kind on $\mathcal{A}$. Suppose that $(\mathcal{A}, \sigma)_F \cong_F (\mathcal{A}, \sigma')_F$. Then there exists $s \in \mathcal{A}^\times$ such that $\sigma(s) = s$ and $\sigma' = \mathrm{Int}(s) \circ \sigma$.*

Proof. By Proposition 1.23, there exists an element $s \in \mathcal{A}^\times$ such that $\sigma(s) = \pm s$ and $\sigma' = \mathrm{Int}(s) \circ \sigma$. If $\mathrm{char}(F) = 2$, we have automatically that $\sigma(s) = s$. Suppose that $\mathrm{char}(F) \neq 2$. Since σ_F and σ'_F are isomorphic, they must be of the same kind and type, and [45, (2.7) (3)] then yields that $\sigma(s) = \sigma_F(s) = s$. □

2.21 Proposition. *Assume that R is a semilocal Bézout domain. Let (Δ, θ) be an R–algebra with involution without zero divisors. Let $\varepsilon = \pm 1$ and let (V, h) and (V', h') be two ε–hermitian spaces over (Δ, θ). Then there exists $u \in R^\times$ such that $(V', h') \simeq (V, uh)$ if and only if $\mathrm{Ad}(h) \cong_{Z(\Delta)} \mathrm{Ad}(h')$.*

Proof. Suppose first that there exists $u \in R^\times$ and a Δ–linear bijection $\varphi : V \to V'$ such that $uh(x, y) = h'(\varphi(x), \varphi(y))$, for all $x, y \in V$. Then one easily checks that

$$\mathrm{End}_\Delta(V) \to \mathrm{End}_\Delta(V'); f \mapsto \varphi \circ f \circ \varphi^{-1}$$

defines a $Z(\Delta)$–isomorphism from $\mathrm{Ad}(h)$ to $\mathrm{Ad}(h')$.

Suppose conversely that there exists a $Z(\Delta)$–isomorphism $\beta : \mathrm{Ad}(h) \to \mathrm{Ad}(h')$. Then $\mathrm{End}_\Delta(V) \cong \mathrm{End}_\Delta(V')$ as $Z(\Delta)$–algebras, and hence V and V' have the same dimension over Δ. Hence, there is a Δ–linear bijection $\psi : V \to V'$. Define an ε–hermitian form $\tilde{h} : V' \times V' \to \Delta$ by $\tilde{h}(\psi(x), \psi(y)) = h(x, y)$, for all $x, y \in V$. Then $(V', \tilde{h}) \simeq (V, h)$ and therefore, $\mathrm{Ad}(\tilde{h}) \cong_{Z(\Delta)} \mathrm{Ad}(h) \cong_{Z(\Delta)} \mathrm{Ad}(h')$. Since $\mathrm{ad}_{h'}$ and $\mathrm{ad}_{\tilde{h}}$ are involutions on $\mathrm{End}_\Delta(V')$, it follows from Proposition 2.20 that there exists $s \in \mathrm{End}_\Delta(V')^\times$ such that $\mathrm{ad}_{\tilde{h}} = \mathrm{Int}(s) \circ \mathrm{ad}_{h'}$ and $\mathrm{ad}_{h'}(s) = s$. Define an ε–hermitian form $h'' : V' \times V' \to \Delta$ by

$$h''(x', y') = h'(s^{-1}(x'), y'),$$

for all $x', y' \in V'$. Then $\mathrm{ad}_{h''} = \mathrm{ad}_{\tilde{h}}$. Furthermore, we have that $v = \widehat{h''}^{-1} \circ \widehat{\tilde{h}} \in \mathrm{End}_\Delta(V')^\times$ and, by definition of v, $h''(v(x'), y') = \tilde{h}(x', y')$ for all $x', y' \in V'$. It follows that $\mathrm{ad}_{h''} = \mathrm{ad}_{\tilde{h}} = \mathrm{Int}(v^{-1}) \circ \mathrm{ad}_{h''}$. Hence, $v \in Z(\Delta)$, and we get $\tilde{h} = \theta(v)h''$. Since $\tilde{h}$ and h'' are both ε–hermitian, it follows that $\theta(v) = v$ and hence $v \in R^\times$. This implies that

$$(V', h'') \simeq (V', v^{-1}\tilde{h}) \simeq (V, v^{-1}h).$$

It follows that $\mathrm{Ad}(h'')$ and $\mathrm{Ad}(h)$, and hence $\mathrm{Ad}(h'')$ and $\mathrm{Ad}(h')$ are isomorphic via a $Z(\Delta)$–isomorphism. By Proposition 2.19, this means there exists $e \in R^\times, g \in \mathrm{End}_\Delta(V')^\times$ such that

$$es = \mathrm{ad}_{h'}(g)g.$$

For all $x', y' \in V'$, we get that

$$\begin{aligned} h''(x', y') = h'(s^{-1}(x'), y') &= eh'(g^{-1}\,\mathrm{ad}_{h'}(g^{-1})(x'), y') \\ &= eh'(\mathrm{ad}_{h'}(g^{-1})(x'), \mathrm{ad}_{h'}(g^{-1})(y')). \end{aligned}$$

This means that $(V', h'') \simeq (V', eh')$. So, putting everything together, we obtain $(V', h') \simeq (V', e^{-1}h'') \simeq (V, e^{-1}v^{-1}h)$, and $e^{-1}v^{-1} \in R^\times$. This yields the statement. □

2.22 Remark. In the case where R is a field, the statement of Proposition 2.21 is shown in [45, (12.34)].

Let (B, τ) be an F–algebra with involution. An element $f \in B$ is called *a similitude of* (B, τ) if $\tau(f)f \in F^\times$. The similitudes of (B, τ) form a group, denoted by $\mathrm{Sim}(B, \tau)$. If $f \in \mathrm{Sim}(B, \tau)$ then $\mu(f) = \sigma(f)f \in F^\times$ is called *a multiplier of* (B, τ). The multipliers of (B, τ) form a subgroup of $F^\times$, denoted by $G(B, \tau)$.

2.23 Lemma. *Let (C, θ) be an F–algebra with involution of any kind with center a domain. Let $\varepsilon = \pm 1$ and let (V, h) be an ε–hermitian space over (C, θ). Then*

$$G(\mathrm{Ad}(h)) = \{a \in F^\times \mid (V, h) \simeq (V, ah)\}.$$

Proof. Let $a \in F^\times$. We have that $(V,h) \simeq (V,ah)$ if and only if there exists a invertible C−linear map $\varphi : V \to V$ such that

$$h(x,y) = ah(\varphi(x),\varphi(y)) = ah(\mathrm{ad}_h(\varphi)\varphi(x),y), \quad \text{for all } x,y \in V.$$

The non–singularity of h implies that this equality holds if and only if $\mathrm{ad}_h(\varphi)\varphi = a^{-1}$. Since $\mathrm{ad}_h(a)a = a^2$, this yields the desired equality. □

2.24 Proposition. *Let (B,τ) be an F−algebra with involution of any kind.*

(a) If (B,τ) is hyperbolic, then $G(B,\tau) = F^\times$.

(b) Let L/F be a finite field extension. Then

$$N_{L/F}(G((B,\tau)_L)) \subset G(B,\tau).$$

Proof. If $Z(B)$ is a domain then (a) follows using the characterisation of $G(B,\tau)$ in terms of ε−hermitian spaces from Proposition 2.23 combined with Proposition 2.5. The statement in (b) then holds by [45, (12.21)]. Suppose that $Z(B)$ is not a domain. By Proposition 1.18, there exists a central simple F−algebra E such that $(B,\tau) \cong_F (E \times E^{\mathrm{op}}, \mathrm{sw}_E)$. Identifying F with $\{(a,a) \in F \times F\}$, it follows that for any $a \in F^\times$, we have that $\mathrm{sw}_E(a,1)(a,1) = a$. In this case, (b) also holds since $G(B,\tau) = F^\times$ by (a). □

Using the language of multipliers, property (ii) in Proposition 2.18 can be replaced by another (seemingly weaker) property. We will use this version in Chapter 5.

2.25 Corollary. *Assume that R is a semilocal Bézout domain and that $2 \in R^\times$. Let $(\mathcal{A},\sigma)$ be an R−algebra with involution with center a domain. Let furthermore $s \in \mathcal{A}^\times$ be such that $\sigma(s) = s$ and let $\sigma' = \mathrm{Int}(s) \circ \sigma$. The following are equivalent:*

(i) $(\mathcal{A},\sigma) \cong_{Z(\mathcal{A})} (\mathcal{A},\sigma')$.

(ii') There exist elements $e \in G((\mathcal{A},\sigma)_F)R^\times$ and $g \in \mathcal{A}_F^\times$ such that $es = \sigma_F(g)g$.

Proof. By Proposition 2.18, it is clear that (i) implies (ii'). Assume that (ii') holds. By Proposition 2.10, there exists an R−algebra with involution without zero divisors (Δ,θ), with $Z(\Delta) = Z(\mathcal{A})$ and θ of the same kind as σ, and an ε−hermitian space (V,h) over (Δ,θ), with $\varepsilon = \pm 1$, such that $(\mathcal{A},\sigma) \cong_{Z(\mathcal{A})} \mathrm{Ad}(h)$. Identifying $(\mathcal{A},\sigma)$ and $\mathrm{Ad}(h)$ through this isomorphism, we consider s as element of $\mathrm{End}_\Delta(V)^\times$ and g as element of $\mathrm{End}_{\Delta_F}(V_F)^\times$. Then $(\mathcal{A},\sigma') = \mathrm{Ad}(h')$, where $h' : V \times V \to \Delta$ is defined by $h'(x,y) = h(s^{-1}(x),y)$ for all $x,y \in V$, by Proposition 2.19. By the same proposition, $es = \sigma_F(g)g$ yields that $(V,h')_F \simeq (V,eh)_F$. Write $e = ua$ with $u \in R^\times$ and $a \in G((\mathcal{A},\sigma)_F)$. By Lemma 2.23, we have that $(V,h)_F \simeq (V,ah)_F$ and hence, it follows that $(V,h')_F \simeq (V,uh)_F$. Since $2 \in R^\times$, (V,uh) is an even ε−hermitian space over (Δ,θ). Proposition 2.8 (c) yields that $(V,h') \simeq (V,uh)$. This implies (i) by Proposition 2.19. □

2.4 Anisotropic parts

In this section, we assume that the characteristic of F is different from 2. In [15], the notions of "Witt equivalence" and "anisotropic part" of simple F−algebras with involution were defined. The latter can be expressed using the notion of an "orthogonal sum" of simple F−algebras with involution, defined in [13]. We explain this in this section.

Let (B,τ) be a simple F−algebra with involution. By Proposition 2.10, $(B,\tau) \cong_{Z(B)} (\mathrm{End}_\Delta(V), \mathrm{ad}_h)$, with (V,h) an ε−hermitian space over an F−algebra with involution (D,θ) where D is division, with $\varepsilon = \pm 1$, and where θ is of the same kind as τ. By Proposition 2.6, there is a decomposition into ε−hermitian spaces:

$$(V,h) \simeq (V_1,h_1) \perp (V_2,h_2)$$

with (V_1,h_1) anisotropic and (V_2,h_2) hyperbolic. Furthermore, this decomposition is unique up to isometry since $\mathrm{char}(F) \neq 2$. We call $\mathrm{Ad}(h_1)$ *the anisotropic part of* $\mathrm{Ad}(h)$.

2.26 Remark. Let K be a field and D a division algebra over K of exponent 2. Then D carries an involution of the first kind by [45, (3.1) (1)]. Furthermore, D carries orthogonal and symplectic involutions, if and only if D is non–split (see [45, (2.8)]). Hence, in the above, if D is non–split, and τ is of the first kind, we can choose θ of the same type as τ, and hence, we can take $\varepsilon = 1$.

2.27 Lemma. *Let (C,θ) be a ring with involution. Let $u \in C^\times$ be such that $\theta(u) = \varepsilon_0 u$, with $\varepsilon_0 = \pm 1$. Let $\varepsilon = \pm 1$ and let (V,h) be an ε−hermitian space over (C,θ). Then (V,uh) is an $\varepsilon\varepsilon_0$−hermitian space over $(C, \mathrm{Int}(u) \circ \theta)$.*

Proof. We set $\theta' = \mathrm{Int}(u) \circ \theta$. Let $x,y \in V$ and $a,b \in C$ be arbitrary. We have that

$$uh(xa,yb) = u\theta(a)h(x,y)b = \theta'(a)uh(x,y)b \qquad \text{and}$$

$$\begin{aligned} uh(y,x) &= u\varepsilon\theta(h(x,y)) = u\varepsilon u^{-1}\theta'(h(x,y))u = \varepsilon\theta(h(x,y))u \\ &= \varepsilon\theta'(h(x,y))\varepsilon_0\theta'(u) = \varepsilon\varepsilon_0\theta'(uh(x,y)). \end{aligned}$$

Since h is non–singular by assumption, and $u \in C^\times$, it is clear that uh is also non–singular. □

2.28 Proposition. *Let (D,θ) be an F−algebra with involution, and assume that D is division. Let θ' be an F−linear involution on D of the same kind as θ. Let $\varepsilon, \varepsilon' \in \{+1,-1\}$. Let (V,h) be an ε−hermitian space over (D,θ), and (V',h') an ε'−hermitian space over (D,θ') such that $\mathrm{Ad}(h) \cong_{Z(D)} \mathrm{Ad}(h')$. Then $\mathrm{Ad}(h)$ and $\mathrm{Ad}(h')$ have isomorphic anisotropic parts.*

Proof. Since θ' and θ are of the same kind, Proposition 1.23 yields that there exists $u \in D^\times$ such that $\theta' = \mathrm{Int}(u) \circ \theta$ and $\theta(u) = \varepsilon_0 u$, with $\varepsilon_0 = \pm 1$. By Proposition 2.6, we can write

$$(V, h) \simeq (V_1, h_1) \perp (V_2, h_2) \quad \text{and} \quad (V', h') \simeq (V_1', h_1') \perp (V_2', h_2'),$$

with (V_1, h_1) and (V_1', h_1') anisotropic, and (V_2, h_2) and (V_2', h_2') hyperbolic. By Lemma 2.27, uh is an $\varepsilon\varepsilon_0$–hermitian form over (D, θ'). Furthermore, uh and h define the same adjoint involution on $\mathrm{End}_D(V)$, since $u \in D^\times$. By [45, (4.2)], we have that $\varepsilon' = \varepsilon\varepsilon_0$, and $\mathrm{Ad}(uh) \cong_{Z(D)} \mathrm{Ad}(h')$. Proposition 2.21 yields that there exists $\lambda \in F^\times$ such that $(V', h') \simeq (V, \lambda uh)$. Proposition 2.6 yields that $(V_1', h_1') \simeq (V_1, \lambda uh_1)$. Hence, $\mathrm{Ad}(h_1') \cong_{Z(D)} \mathrm{Ad}(h_1)$, which proves the claim. □

Let (B, τ) be a simple F–algebra with involution. Let D be a division algebra over $Z(D)$ Brauer equivalent to B, and let θ be an F–linear involution on D of the same kind as τ. Let $\varepsilon = \pm 1$ and let (V, h) be an ε–hermitian space over (D, θ) such that $(B, \tau) \cong_{Z(D)} \mathrm{Ad}(h)$. We define *the anisotropic part of* (B, τ) to be the anisotropic part of $\mathrm{Ad}(h)$, and we denote it by $(B, \tau)_{\mathrm{an}}$. By Proposition 2.28, $(B, \tau)_{\mathrm{an}}$ is well–defined up to an isomorphism of simple F–algebras with involution.

Let (B_1, τ_1) and (B_2, τ_2) be Brauer equivalent simple F–algebras with involution of the same kind and type. A simple F–algebra with involution (B, τ) is called *an orthogonal sum* of (B_1, τ_1) and (B_2, τ_2) if there exist idempotents $e_1, e_2 \in B$ with $\tau(e_1) = e_1$, $\tau(e_2) = e_2$, and $e_1 + e_2 = 1$, and such that $(B_1, \tau_1) \cong (e_1 B e_1, \tau_{|e_1 B e_1})$ and $(B_2, \tau_2) \cong (e_2 B e_2, \tau_{|e_2 B e_2})$. In that case, B is Brauer equivalent to B_1 and B_2, and τ is of the same kind and type as τ_1 and τ_2. This notion of orthogonal sum was introduced in [13].

2.29 Proposition. *Let (D, θ) an F–algebra with involution and assume that D is a division algebra. Let $\varepsilon = \pm 1$ and let (V_1, h_1) and (V_2, h_2) be ε–hermitian spaces over (D, θ). For each $\lambda \in F^\times$, the F–algebra with involution $(\mathrm{End}_D(V_1 \oplus V_2), \mathrm{ad}_{h_1 \perp \lambda h_2})$ is an orthogonal sum of $\mathrm{Ad}(h_1)$ and $\mathrm{Ad}(h_2)$.*

Proof. See [25, p. 327]. □

In the situation of Proposition 2.29, different values of λ might give rise to non–isomorphic orthogonal sums. In the case where h_2 is hyperbolic, $(V_2, \lambda h_2) \simeq (V_2, h_2)$ for all $\lambda \in F^\times$ and it follows that $\mathrm{Ad}(h_1 \perp \lambda h_2) \cong \mathrm{Ad}(h_1 \perp h_2)$ for all $\lambda \in F^\times$. Therefore, if (B_1, τ_1) and (B_2, τ_2) are Brauer equivalent simple F–algebras with involution of the same type, and one of τ_1 and τ_2 is hyperbolic, then we can talk about *<u>the</u> orthogonal sum of* (B_1, τ_1) *and* (B_2, τ_2), and we denote this by

$$(B_1, \tau_1) \boxplus (B_2, \tau_2).$$

Let (B,τ) be a simple F−algebra with involution. Note that, by Proposition 2.29 and Proposition 2.14, we get

$$(B,\tau) \cong (B,\tau)_{\text{an}} \boxplus (H,\mu),$$

with (H,μ) a hyperbolic simple F−algebra with involution.

Two simple F−algebras with involution (B_1,τ_1) and (B_2,τ_2) are said to be *Witt equivalent* (see [15]) if there exist hyperbolic simple F−algebras with involution (H_1,μ_1) and (H_2,μ_2) such that

$$(B_1,\tau_1) \boxplus (H_1,\mu_1) \cong (B_2,\tau_2) \boxplus (H_2,\mu_2).$$

We denote this by $(B_1,\tau_1) \sim (B_2,\tau_2)$. In particular, Witt equivalent F−algebras with involution are Brauer equivalent and the involutions are of the same kind and type.

2.30 Proposition. *Let (D,θ) be an F−algebra with involution, and assume that D is division. Let $\varepsilon = \pm 1$ and let (V,h) and (V',h') be ε−hermitian spaces over (D,θ). If (V,h) is Witt equivalent to (V',h') in $W^{\varepsilon}(D,\theta)$, then* $\mathrm{Ad}(h)$ *is Witt equivalent to* $\mathrm{Ad}(h')$.

Proof. By assumption, there exist hyperbolic ε−hermitian spaces (V_1,h_1) and (V_1',h_1') over (D,θ) such that

$$(V,h) \perp (V_1,h_1) \simeq (V',h') \perp (V_1',h_1').$$

By Corollary 2.29, it follows that $\mathrm{Ad}(h) \boxplus \mathrm{Ad}(h_1) \cong \mathrm{Ad}(h \perp h_1) \cong \mathrm{Ad}(h' \perp h_1') \cong \mathrm{Ad}(h') \perp \mathrm{Ad}(h_1')$, and furthermore, $\mathrm{Ad}(h_1)$ and $\mathrm{Ad}(h_1')$ are hyperbolic by Proposition 2.14. This proves the statement. □

2.31 Proposition. *Witt equivalent simple F−algebras with involution of the same degree are isomorphic.*

Proof. Let (B,τ) and (B',τ') be Witt equivalent simple F−algebras with involution. Then B is Brauer equivalent to B' and τ and τ' are of the same kind and type. Let D be an F−algebra with center $Z(B)$ Brauer equivalent to B and B'. By Proposition 2.10, there exists an involution θ on D of the same kind as τ and ε−hermitian spaces (V,h) and (V',h') over (D,θ), with $\varepsilon = \pm 1$, such that $(B,\tau) \cong_{Z(D)} \mathrm{Ad}(h)$ and $(B',\tau') \cong_{Z(D)} \mathrm{Ad}(h')$. Furthermore, by assumption, there exist hyperbolic ε−hermitian spaces (V_1,h_1) and (V_1',h_1') over (D,θ), such that

$$\mathrm{Ad}(h \perp h_1) \cong_{Z(D)} \mathrm{Ad}(h) \boxplus \mathrm{Ad}(h_1) \cong_{Z(D)} \mathrm{Ad}(h') \boxplus \mathrm{Ad}(h_1') \cong_{Z(D)} \mathrm{Ad}(h' \perp h_1').$$

By Proposition 2.21, it follows that there exists $u \in F^{\times}$ such that $(V',h') \perp (V_1',h_1') \simeq (V,uh) \perp (V_1,uh_1)$. Since $\deg(B_1) = \deg(B_2)$ by assumption, it follows that $\dim_D(V_1') = \dim_D(V_1)$. Since h_1 and h_1' are hyperbolic, it follows from Propositions 2.5 and 2.4 (b) that $(V',h') \simeq (V,uh)$. This implies that $\mathrm{Ad}(h') \cong_{Z(D)} \mathrm{Ad}(uh) = \mathrm{Ad}(h)$, which proves the statement. □

2.5 Representation theorems

The classical Cassels–Pfister theorem for quadratic forms says the following.

2.32 Theorem (Cassels – Pfister). *Assume that* $\mathrm{char}(F) \neq 2$. *Let* $q : V \to F$ *be a quadratic form. Let* $F[t]$ *be the polynomial ring in one variable and* $F(t)$ *the rational function field. If* q *represents a polynomial over* $F(t)$*, then it already represents this polynomial over* $F[t]$.

Proof. See [66, (4.3.2)]. □

An analogous representation result for non–singular quadratic forms over valuation rings was proved independently by M. Kneser (see [10, (4.5)]) and J.-L. Colliot–Thélène (see [12]).

2.33 Theorem (Kneser, Colliot–Thélène). *Let* $\mathcal{O}$ *be a valuation ring of* F *and assume that* $2 \in \mathcal{O}^\times$. *Let* $q : \mathcal{V} \to \mathcal{O}$ *be a regular quadratic form. If* q *represents an element of* $\mathcal{O}$ *over* F*, then it already represents this element over* $\mathcal{O}$.

In [71], J.–P. Tignol considered an involution analogue of Theorem 2.32. In this chapter we consider analogues of Theorem 2.33, on the one hand for algebras with involution and on the other hand for (skew–)hermitian spaces, and the relations between them.

In this section, we assume that R is a semilocal Bézout domain with $2 \in R^\times$.

From here on until Proposition 2.38, we fix an R–algebra with involution (Δ, θ) without zero divisors, and an ε–hermitian space (V, h) over (Δ, θ), where $\varepsilon = \pm 1$. Then $Z(\Delta)$ is a domain and $D = \Delta \otimes_R F$ is a division algebra. A $(\Delta-)$*lattice in* $(V, h)_F$ is a finitely generated, right Δ–submodule of V_F containing a D–basis of V_F. Lattices in $(V, h)_F$ are torsion–free Δ–modules, and hence free as Δ–modules by Propositions 1.9 and 1.24.

Let $\mathcal{L}$ be a lattice in $(V, h)_F$. The *dual of* $\mathcal{L}$ is defined as

$$\mathcal{L}^\# = \{v \in V_F \mid h_F(v, \mathcal{L}) \subset \Delta\}.$$

$\mathcal{L}$ is called *integral (with respect to* h_F*)* if $\mathcal{L} \subset \mathcal{L}^\#$ and *self–dual* or *unimodular* if $\mathcal{L} = \mathcal{L}^\#$.

2.34 Proposition. *Let* $\mathcal{L}$ *be an integral lattice in* $(V, h)_F$ *and denote the restriction of* h_F *to* $\mathcal{L}$ *by* $h_{\mathcal{L}}$. *The following are equivalent:*

(i) $(\mathcal{L}, h_{\mathcal{L}})$ *is an* ε*–hermitian space over* (Δ, θ).

(ii) $\mathcal{L}^\# = \mathcal{L}$.

Proof. Let $\mathfrak{B} = (f_1, \dots, f_n)$ be a Δ–basis for $\mathcal{L}$. Then $\mathfrak{B}$ is a D–basis for V_F. Consider the elements of the dual basis ${}^{\theta}\mathfrak{B}^{\#} = ({}^{\theta}f_1^{\#}, \dots, {}^{\theta}f_n^{\#})$ as elements of V_F. Suppose that (i) holds. Then the elements of ${}^{\theta}\mathfrak{B}^{\#}$ belong to $\mathcal{L}$. Suppose that $h_F(v, \mathcal{L}) \subset \Delta$ for some $v \in V_F$. We can write $v = \sum_{i=1}^{n} {}^{\theta}f_i^{\#} x_i$, with $x_1, \dots, x_n \in D$. Then $h_F(v, f_j) = \sum_{i=1}^{n} \theta_F(x_i)\delta_{ij} = \theta_F(x_j) \in \Delta$ and hence $v \in \mathcal{L}$. So, $\mathcal{L}$ is unimodular.
Suppose that $\mathcal{L} = \mathcal{L}^{\#}$. Since $h_F({}^{\theta}f_i^{\#}, f_j) = \delta_{ij} \in \Delta$ and $\mathcal{L}$ is unimodular, we have that ${}^{\theta}\mathfrak{B}^{\#}$ belongs to $\mathcal{L}$. The matrix of $\hat{h}_{\mathcal{L}}$ is the matrix of base change from $\mathfrak{B}$ to ${}^{\theta}\mathfrak{B}^{\#}$, and hence invertible over Δ. This yields (i). □

2.35 Corollary. *V is a unimodular lattice in $(V, h)_F$.*

Proof. It is clear that V is an integral lattice in $(V, h)_F$. Since (V, h) is a ε–hermitian space, the statement follows from Proposition 2.34. □

We refer to [71] for the proofs of the following known facts on lattices.

2.36 Proposition.

(a) *Let $\mathcal{L}$ be a lattice in $(V, h)_F$. Then $\mathcal{L}^{\#}$ is a lattice in $(V, h)_F$ and $\mathcal{L}^{\#\#} = \mathcal{L}$.*

(b) *Let $\mathcal{L}_1, \mathcal{L}_2$ be lattices in $(V, h)_F$. Then*

$$(\mathcal{L}_1 + \mathcal{L}_2)^{\#} = \mathcal{L}_1^{\#} \cap \mathcal{L}_2^{\#} \quad \text{and} \quad (\mathcal{L}_1 \cap \mathcal{L}_2)^{\#} = \mathcal{L}_1^{\#} + \mathcal{L}_2^{\#}.$$

In particular, the intersection of two lattices in $(V, h)_F$ is again a lattice.

(c) *Let $\mathcal{L}_1, \mathcal{L}_2$ be lattices in $(V, h)_F$. If $\mathcal{L}_1 \subset \mathcal{L}_2$, then $\mathcal{L}_2^{\#} \subset \mathcal{L}_1^{\#}$.*

2.37 Lemma. *The self–dual lattices in $(V, h)_F$ are exactly the maximal integral lattices in $(V, h)_F$.*

2.38 Proposition. *Let $\mathcal{L}$ be a unimodular lattice in $(V, h)_F$. Then there is an isometry u of $(V, h)_F$ such that $u(V) = \mathcal{L}$.*

Proof. Let $h_{\mathcal{L}}$ denote the restriction of h_F to $\mathcal{L}$. The ε–hermitian spaces $(\mathcal{L}, h_{\mathcal{L}})$ and (V, h) become isometric over F since they become two representations of the same form. Proposition 2.8 (c) yields that $(\mathcal{L}, h_{\mathcal{L}}) \simeq (V, h)$. This means that there is a bijective Δ–linear map $u : V \to \mathcal{L}$ such that $h_{\mathcal{L}}(u(x), u(y)) = h(x, y)$. Extending scalars to F, u defines an isometry of $(V, h)_F$ with itself, with $u(V) = \mathcal{L}$. □

We can now prove a representation theorem for algebras with involution over semilocal principal ideal domains.

2.39 Theorem. *Assume that R is a semilocal principal ideal domain. Let $(\mathcal{A},\sigma)$ be an $R-$algebra with involution with center a domain. Let $f \in \mathcal{A}_F$ be such that $\sigma_F(f)f \in \mathcal{A}$. Then there exists an element $u \in \mathcal{A}_F^\times$ such that $\sigma_F(u)u = 1$ and $uf \in \mathcal{A}$.*

Proof. We follow the proof of the main theorem of [71].

A semilocal principal ideal is of course a semilocal Bézout domain, and so we can use the above results on lattices, in particular Proposition 2.38.
By Proposition 1.26, there exists an $R-$algebra with involution (Δ,θ) without zero divisors and an $\varepsilon-$hermitian space (V,h) over (Δ,θ), with $\varepsilon = \pm 1$ such that $(\mathcal{A},\sigma) \cong_R \mathrm{Ad}(h)$. Let $D = \Delta \otimes_R F$. Then $(\mathcal{A},\sigma)_F \cong_F (\mathrm{End}_D(V_F), \mathrm{ad}_{h_F})$. Let $f \in \mathrm{End}_D(V_F)$ be such that $\sigma_F(f)f \in \mathrm{End}_\Delta(V)$. Then we can write $f = d^{-1}\tilde{f}$ for some $\tilde{f} \in \mathrm{End}_\Delta(V)$ and $d \in R$. For all $m,m' \in V$, we have that

$$h_F(f(m), f(m')) = h_F(\mathrm{ad}_{h_F}(f)f(m), m') \in \Delta,$$

since $\mathrm{ad}_{h_F}(f)f(m) \in V$ and $V = V^\#$.

Note that $f(V)$ is not necessarily a lattice in $(V,h)_F$. However, $f(V)+dV$ is a lattice and it is also integral since $df \in \mathrm{End}_\Delta(V)$. Since R is a principal ideal domain, every integral lattice in $(V,h)_F$ is contained in a unimodular lattice. This can be seen as follows. Let $\mathcal{L}_1 \subset \mathcal{L}_2 \subset \ldots$ be a chain of integral lattices, we have that $\mathcal{L}_i \subset \mathcal{L}_1^\#$, for all i. Since $\mathcal{L}_1^\#$ is a lattice over Δ, Δ is finitely generated over R, and R is a Noetherian ring, it follows that $\mathcal{L}_1^\#$ is a Noetherian $R-$module. Since all $\mathcal{L}_i$ are $R-$submodules of $\mathcal{L}_1^\#$ by Proposition 2.36 (c), it follows that the chain $\mathcal{L}_1 \subset \mathcal{L}_2 \subset \ldots$ must stop. Hence, any integral lattice in $(V,h)_F$ is contained in a maximal integral lattice.

Let $\mathcal{L}$ be a maximal integral lattice in $(V,h)_F$ containing $f(V) + dV$. Then $\mathcal{L}$ is unimodular by Lemma 2.37. Proposition 2.38 implies that there is an isometry u of $(V,h)_F$ such that $u(V) = \mathcal{L}$. It follows that $f(V) \subset \mathcal{L} = u(V)$, so $u^{-1}f(V) \subset V$ and hence $u^{-1}f \in \mathrm{End}_\Delta(V)$. Since u is an isometry of $(V,h)_F$, we have that

$$h_F((\mathrm{ad}_{h_F}(u^{-1})u^{-1})(x), y) = h_F(u^{-1}(x), u^{-1}(y)) = h_F(x,y),$$

for all $x,y \in V_F$. Since h_F is non–singular, it follows that $\mathrm{ad}_{h_F}(u^{-1})u^{-1} = 1$. This proves the statement. □

2.40 Corollary. *Assume that R is a semilocal principal ideal domain. Let (Δ,θ) be an $R-$algebra with involution without zero divisors. Let $\varepsilon = \pm 1$ and let (V,h) be an $\varepsilon-$hermitian space over (Δ,θ). If there exists $x \in V_F$ such that $h_F(x,x) \in \Delta$, then there exists $x' \in V$ such that $h(x',x') = h_F(x,x)$.*

Proof. Let $(e_1, \ldots, e_n)$ be a Δ–basis for V. Let $\delta = h_F(x, x)$, with $x \in V_F$. Consider the element $f \in \mathrm{End}_D(V_F)$ defined by $f(e_1) = x$ and $f(e_i) = 0$, for $i = 2, \ldots, n$. Since $h_F((\mathrm{ad}_h(f)f)(e_j), e_i) = h_F(f(e_j), f(e_i)) = 0$ for $i \neq 1$ or $j \neq 1$, and

$$h_F((\mathrm{ad}_h(f)f)(e_1), e_1) = h_F(f(e_1), f(e_1)) = h_F(x, x) \in \Delta,$$

it follows that $h_F((\mathrm{ad}_h(f)f)(y), y) \in \Delta$ for all $y \in V$. Therefore, $(\mathrm{ad}_h(f)f)(y) \in V^{\#} = V$, and hence $\mathrm{ad}_h(f)f \in \mathrm{End}_\Delta(V)$.
By Theorem 2.39 there exists an element $u \in \mathrm{End}_D(V_F)^\times$ with $\mathrm{ad}_{h_F}(u)u = 1$ such that $uf \in \mathrm{End}_\Delta(V)$. Since $\mathrm{ad}_{h_F}(u)u = 1$, we have $h_F(u(y), u(y)) = h_F((\mathrm{ad}_h(u)u)(y), y) = h_F(y, y)$ for all $y \in V_F$ and therefore in particular

$$h_F(u(x), u(x)) = h_F(x, x) = \delta.$$

So, we have found an element $z = u(x) = u(f(e_1)) \in V$ representing δ. □

When we started looking at these representation questions for algebras with involution, we were interested in the case where R is a valuation ring. Theorem 2.39 only covers the case of discrete valuation rings. The fact that those valuation rings, and more generally semilocal principal ideal domains, are Noetherian, yields directly that an integral lattice in $(V, h)_F$ is contained in a maximal integral, and hence unimodular, lattice. The question then naturally arises whether this remains true in the case where R is a nondiscrete valuation ring. This is the only obstruction in order for the proof of Theorem 2.39 to go through in the case where R is a general valuation ring. We were not able to give a positive answer in general, but we did manage to show that any integral lattice in $(V, h)_F$ is contained in a unimodular lattice, in the case where Δ is not just an Azumaya algebra, but moreover a valuation ring of a division algebra. Mimicking the proof of Theorem 2.39 then yields the following result.

2.41 Theorem. *Let $\mathcal{O}$ be a valuation ring with fraction field F. Assume that $2 \in \mathcal{O}^\times$. Let $(\mathcal{A}, \sigma)$ be an $\mathcal{O}$–algebra with involution with center a domain. Assume that $\mathcal{A}$ is Brauer equivalent to an Azumaya algebra over $Z(\mathcal{A})$ that is moreover a valuation ring of a division algebra. Let $f \in \mathcal{A}_F$ be such that $\sigma_F(f)f \in \mathcal{A}$. Then there exists $u \in \mathcal{A}_F$ such that $\sigma_F(u)u = 1$ and $uf \in \mathcal{A}$.*

From Theorem 2.41, one can then derive a similar statement as in Corollary 2.40, in the case where R is a valuation ring and Δ is an Azumaya algebra that is moreover a valuation ring of a division algebra. We can also prove this statement directly without passing via adjoint involutions. We do this in section 2.6, where we also apply this result in order to show that, up to some exceptions, (skew–)hermitian spaces over an Azumaya algebra with involution that is moreover a valuation ring, have an orthogonal basis.
For the interested reader, we present the (fairly technical) proof of the fact that an integral lattice in $(V, h)_F$ is contained in a unimodular lattice, in the case where Δ is a valuation

ring of a division algebra, in section 2.7, as an appendix to this chapter. In the argument, we use the fact that Δ, being a valuation ring, is an elementary divisor domain by [35], and some properties of the Dieudonné determinant.
The remaining results in chapter 2 are not used later in the thesis, apart from Proposition 2.42, Lemma 2.44, Proposition 2.46, Proposition 2.47 and Corollary 2.51.

2.6 Noncommutative valuation rings

Let K be a field and let D be a K–division algebra. A *valuation ring* of D is a subring Λ of D such that for all $x \in D$, we have that $x \in \Lambda$ or $x^{-1} \in \Lambda$, and furthermore, Λ is invariant under conjugation with elements of D. (In the literature, Λ is sometimes called an invariant valuation ring of D.) Let Γ be a totally ordered abelian group and let ∞ a symbol of a set strictly containing Γ, and satisfying $\gamma < \infty$ and $\infty + \infty = \infty + \gamma = \gamma + \infty$ for all $\gamma \in \Gamma$. A map $w : D \to \Gamma \cup \{\infty\}$ is called *a valuation on D* if $w^{-1}(\{\infty\}) = \{0\}$, $w(a+b) \geqslant \min(w(a), w(b))$ and $w(ab) = w(a) + w(b)$, for all $a, b \in D$. We call the pair (D, w) *a valued K–division algebra*. The ring $\mathcal{O}_D = \{a \in D \mid w(a) \geqslant 0\}$ is a valuation ring of D. The valuation ring $\mathcal{O}_D$ has a unique maximal left (and right) ideal M_D, which is equal to $\{a \in D \mid w(a) > 0\}$. One calls $\Gamma_D = w(D^\times)$ *the value group of w or $\mathcal{O}_D$*, and one can show that $\Gamma_D \cong D^\times/{\mathcal{O}_D}^\times$ (see e.g, [75, p. 4]). Given a valuation ring Λ of D, there exists a valuation on D with valuation ring precisely Λ (see e.g. [75, p. 4]). So, there is a one-to-one correspondence between valuations on D and valuation rings of D.

For the rest section 2.6, we start from the following set–up. Let $\mathcal{O}$ be a valuation ring of F with $2 \in \mathcal{O}^\times$. Let v be a valuation on F with valuation ring $\mathcal{O}$, and denote the maximal ideal of $\mathcal{O}$ by $\mathfrak{m}$.

2.42 Proposition. *Let Δ be an Azumaya algebra without zero divisors with center either $\mathcal{O}$ or a separable quadratic $\mathcal{O}$–algebra. We write $S = Z(\Delta)$ and denote the fraction field of S by K. Assume that Δ is a valuation ring of the division algebra $D = \Delta \otimes_{\mathcal{O}} F$. Then S is a valuation ring of K. Let v_S be a valuation on K with valuation ring S. Then the map*

$$w(x) = \frac{1}{\deg(D)} v_S(\mathrm{Nrd}_D(x)) \quad \textit{for all } x \in D, \tag{2.6.1}$$

is the unique valuation on D with valuation ring Δ. Furthermore, $J(\Delta) = \mathfrak{m}\Delta$, and $\Delta^\times = \Delta \setminus J(\Delta)$.

Proof. Since Δ is a valuation ring, S is necessarily a valuation ring as well. This means that v extends uniquely to K by Proposition 1.22. The formula for w is then stated in [75, (2.2)]. By Proposition 1.13, it follows that $J(\Delta) = \mathfrak{m}_S\Delta$, where $\mathfrak{m}_S$ is the maximal ideal of S, and the latter is equal to $\mathfrak{m}S$ by Proposition 1.22 (a). Since Δ is the valuation ring of a valuation on D by [75, p. 4], it follows that $\Delta^\times = \Delta \setminus J(\Delta)$. $\square$

In Lemma 2.43 – Lemma 2.45, we let (Δ, θ) be an $\mathcal{O}$-algebra with involution without zero divisors, and we assume that Δ is a valuation ring of the division algebra $D = \Delta \otimes_{\mathcal{O}} F$. We let S, K and w be as in Proposition 2.42.

2.43 Lemma. *The value groups of w and v are equal.*

Proof. Let $a \in D^\times$. Then there exists a nonzero element $r \in \mathcal{O}$ such that $ra \in \Delta$. Since Δ is a valuation ring of D, we have that $x^{-1}\Delta x = \Delta$, for all nonzero $x \in D$. This implies that $(ra)\Delta = \Delta(ra)$ and hence, $(ra)\Delta$ is a two–sided ideal of Δ. By Proposition 1.12, we have that $(ra)\Delta = I\Delta$, with I an ideal of S, and we may assume that I is finitely generated. Since S is a valuation ring, it follows that $I = cS$, for some $c \in S$, and hence $(ra)\Delta = c\Delta$. It follows that $v(r) + w(a) = w(ra) = v_S(c)$, which implies that $w(a) \in v_S(K^\times) = v(F^\times)$. $\square$

2.44 Lemma. *We have that $w = w \circ \theta_F$.*

Proof. Let $x \in D$. Suppose that θ is of the first kind. We have that $\mathrm{Nrd}_D(\theta_F(x)) = \mathrm{Nrd}_D(x)$, by [45, (2.2)]. Using the formula (2.6.1) for w, it is clear that $w(x) = w(\theta_F(x))$. Suppose that θ is of the second kind. Then [45, (2.16)] says that $\mathrm{Nrd}_D(\theta_F(x)) = \iota(\mathrm{Nrd}_D(x))$, with ι the nontrivial F-automorphism of K. Then $w = w \circ \theta_F$ if and only if $v_S = v_S \circ \iota$, which is the case if and only if v extends uniquely to K by [21, (3.2.14)], and the latter is satisfied. $\square$

2.45 Lemma. *Let $\varepsilon = \pm 1$ and let (V, h) be an ε-hermitian space over (Δ, θ). Let $x \in V_F$ be nonzero.*

(a) *There exists an element $y \in V$ such that $h_F(x, y) \in F$ and $h_F(x, V) = h_F(x, y)\Delta$.*

(b) *If $\theta \neq \mathrm{id}_\Delta$ then exists an element $y' \in V$ such that $\theta_F(h_F(x, y')) = -h_F(x, y')$ and $h_F(x, V) = h_F(x, y')\Delta$.*

Proof. We have that $h_F(x, V)$ is a right Δ-module in D. Since V is finitely generated over Δ, it follows that $h_F(x, V)$ is finitely generated over Δ as well. Since Δ is a valuation ring, it is in particular a right Bézout domain. Hence, there exists $z \in V$ such that $h_F(x, V) = h_F(x, z)\Delta$. By Lemma 2.43, there exists $\alpha \in F^\times$ such that $v(\alpha) = -w(h_F(x, z))$. Then $\alpha h_F(x, z)\Delta = \Delta$, and hence $h_F(x, z)\Delta = \alpha^{-1}\Delta$. It follows that there exists $\delta \in \Delta^\times$ such that $h_F(x, z\delta) = h_F(x, z)\delta = \alpha^{-1}$. The element $y = z\delta$ then satisfies $h_F(x, y) \in F$ and $h_F(x, y)\Delta = h_F(x, V)$.
Suppose that $\theta \neq \mathrm{id}_\Delta$. Then there exists a nonzero element $\tilde{d} \in D$ be such that $\theta_F(\tilde{d}) = -\tilde{d}$. Let $a \in F$ be such that $v(a) = -w(\tilde{d})$ and let $d = \tilde{d}a$. Then $d \in \Delta^\times$ and $\theta(d) = \theta_F(a)\theta_F(\tilde{d}) = -a\tilde{d} = -d$. Let $y' = yd$. Since $d \in \Delta^\times$, it follows that $h_F(x, y')\Delta = h_F(x, y)d\Delta = h_F(x, y)\Delta = h_F(x, V)$. $\square$

2.46 Lemma. *Let $(\mathcal{A},\sigma)$ be an $\mathcal{O}$–algebra with involution. For every nonzero $x \in \mathcal{A}_F$ there exists $a \in F$ such that $ax \in \mathcal{A} \smallsetminus J(\mathcal{A})$.*

Proof. By Proposition 1.13, we have that $J(\mathcal{A}) = J(S)\,\mathcal{A}$. Furthermore, it follows from Proposition 1.22 that $J(S) = \mathfrak{m}S$. We have that $\mathcal{A}$ is free as an $\mathcal{O}$–module by Corollary 1.19. Let $(e_1,\ldots,e_n)$ be an $\mathcal{O}$–basis for $\mathcal{A}$. Then it is an F–basis for $\mathcal{A}_F$. Let $x \in \mathcal{A}_F$ be nonzero. There exist $x_1,\ldots,x_n \in F$ such that $x = \sum_{i=1}^n e_i x_i$. Without loss of generality, we may assume that $v(x_1) = \min_{1\leqslant i\leqslant n}(v(x_i))$. It follows that $xx_1^{-1} \in \mathcal{A} \smallsetminus \mathfrak{m}\mathcal{A}$. □

2.47 Proposition. *Let (Δ,θ) be an $\mathcal{O}$–algebra with involution and assume that $Z(\Delta)$ is a valuation ring. Let $\varepsilon = \pm 1$ and let (V,h) be an ε–hermitian space over (Δ,θ). Excluding the case $\theta = \mathrm{id}_\Delta$ and $\varepsilon = -1$, there exists $x \in V$ such that $h(x,x) \in \Delta \smallsetminus J(\Delta)$.*

Proof. By Proposition 2.4 (a), V is free as a Δ–module. Let $\mathfrak{B} = (e_1,\ldots,e_n)$ be a Δ–basis for V. If one of $h(e_1,e_1),\ldots,h(e_n,e_n) \in \Delta \smallsetminus J(\Delta)$, then we are done. So, suppose that $h(e_1,e_1),\ldots,h(e_n,e_n) \in J(\Delta)$. We first show there exists $d \in \Delta \smallsetminus J(\Delta)$ such that $\theta(d) = \varepsilon d$. Since the case where $\theta = \mathrm{id}_\Delta$ and $\varepsilon = -1$ is excluded, there exists a nonzero element $\tilde{d} \in D = \Delta \otimes_{\mathcal{O}} F$ such that $\theta_F(\tilde{d}) = \varepsilon\tilde{d}$. By Lemma 2.46, there exists $a \in F$ such that $d = a\tilde{d} \in \Delta \smallsetminus J(\Delta)$, and we have that $\theta(d) = a\theta_F(\tilde{d}) = \varepsilon d$. Let $C = (h(e_i,e_j)_{ij})$ be the matrix of h with respect to $\mathfrak{B}$. Since h is non–singular, every row and every column of C contains at least one element of $\Delta^\times$. This means that each row and column is unimodular. Hence, there exist $\lambda_1,\ldots,\lambda_n \in \Delta$ such that $(d\lambda_1,\ldots,d\lambda_n)C$ is a matrix with the element d in the upper left corner. It follows that $d = \sum_{i=1}^n d\lambda_i h(e_i,e_1) = h(\sum_{i=1}^n e_i\theta(d\lambda_i),e_1)$. Let $x = \sum_{i=1}^n e_i(d\lambda_i)$. Then $h(x,e_1) = d$. If $h(x,x) \in \Delta \smallsetminus J(\Delta)$ then we are done. So, suppose $h(x,x) \in J(\Delta)$. It follows that

$$h(e_1+x,e_1+x) = h(e_1,e_1)+h(x,x)+h(e_1,x)+h(x,e_1) = h(e_1,e_1)+h(x,x)+2d \in \Delta\smallsetminus J(\Delta),$$

since $2 \in \Delta^\times$ and $d \in \Delta \smallsetminus J(\Delta)$. This proves the statement. □

2.48 Corollary. *Let $\mathcal{O}_1,\ldots,\mathcal{O}_\ell$ be valuation rings of F and let $R = \mathcal{O}_1 \cap \ldots \cap \mathcal{O}_\ell$. Assume that $2 \in R^\times$. Let (Δ,θ) be an R–algebra with involution without zero divisors. For $i = 1,\ldots,\ell$, suppose that $\Delta_i = \Delta_{\mathcal{O}_i}$ is a valuation ring of $(\Delta_i)_F$. Let $\varepsilon = \pm 1$ and let (V,h) be an ε–hermitian space over (Δ,θ). Excluding the case $\theta = \mathrm{id}_\Delta$ and $\varepsilon = -1$, there exists an element $x \in V$ such that $h(x,x) \in \Delta^\times$.*

Proof. For $i = 1,\ldots,\ell$, we denote the maximal ideal of $\mathcal{O}_i$ by $\mathfrak{M}_i$. By Proposition 1.11 (a), we have that Δ_i is an Azumaya algebra over $\mathcal{O}_i$. We write (V_i,h_i) for $(V,h)_{\mathcal{O}_i}$. Since $\Delta_1,\ldots,\Delta_\ell$ are valuation rings, we have that $\Delta_i^\times = \Delta_i \smallsetminus J(\Delta_i)$ for $i = 1,\ldots,\ell$. By Proposition 2.47, there exist elements $x_i \in V_i$ such that $h_i(x_i,x_i) \in \Delta_i^\times$. Since the natural map $V \to \prod_{i=1}^\ell V_i/\mathfrak{M}_i V_i$ is surjective by the Chinese Remainder Theorem, there exists

$x \in V$ such that $x - x_i \in \mathfrak{M}_i V_i$ for $i = 1, \ldots, \ell$. It follows that for $i = 1, \ldots, \ell$, $h(x, x) = h_i(x_i, x_i) + h(x - x_i, x_i) + h(x_i, x - x_i) + h(x - x_i, x - x_i)$. The first term is a unit in Δ_i and each of the other terms is contained in $J(\Delta_i) = \mathfrak{M}_i \Delta_i$. Hence, $h(x, x) \in \Delta_1^\times \cap \ldots \cap \Delta_\ell^\times = \Delta^\times$. This proves the statement. □

2.49 Corollary. *Let $\mathcal{O}_1, \ldots, \mathcal{O}_\ell$ be valuation rings of F and let $R = \mathcal{O}_1 \cap \ldots \cap \mathcal{O}_\ell$. Assume that $2 \in R^\times$. Let (Δ, θ) be an R–algebra with involution without zero divisors. For $i = 1, \ldots, \ell$, suppose that $\Delta_i = \Delta_{\mathcal{O}_i}$ is a valuation ring of $(\Delta_i)_F$. For $\varepsilon = \pm 1$, every ε–hermitian space over (Δ, θ) has an orthogonal basis, excluding the case where $\theta = \mathrm{id}_\Delta$ and $\varepsilon = -1$.*

Proof. Let (V, h) be an ε–hermitian space over (Δ, θ) such that $\theta \neq \mathrm{id}_\Delta$ if $\varepsilon = -1$. By Corollary 2.48, there exists an element $x \in V$ such that $h(x, x) \in \Delta^\times$. Then $(x\Delta, h|_{x\Delta})$ is an ε–hermitian space over (Δ, θ) and by Proposition 2.3, we have that

$$(V, h) \simeq (x\Delta, h|_{x\Delta}) \perp ((x\Delta)^\perp, h|_{(x\Delta)^\perp}),$$

and $((x\Delta)^\perp, h|_{(x\Delta)^\perp})$ is an ε–hermitian space over (Δ, θ). The statement now follows by induction on $\dim_\Delta(V)$. □

Using Lemma 2.45, we can prove the following representation theorem for ε–hermitian spaces over (Δ, θ). We mimick M. Kneser's proof of Theorem 2.33 given in [10, (4.5)].

2.50 Theorem. *Let (Δ, θ) be an $\mathcal{O}$–algebra with involution without zero divisors, and assume that Δ is a valuation ring of the division algebra $D = \Delta \otimes_{\mathcal{O}} F$. Let $\varepsilon = \pm 1$ and let (V, h) be an ε–hermitian space over (Δ, θ). If there exists $x \in V_F$ such that $h_F(x, x) \in \Delta$, then there exists $x' \in V$ such that $h(x', x') = h_F(x, x)$.*

Proof. Suppose that $\theta = \mathrm{id}_\Delta$ and $\varepsilon = -1$. Then $h_F(x, x) = 0$ for all $x \in V_F$ and $h(x', x') = 0$ for all $x' \in V$. So, in this case the statement trivially holds. From now on we suppose that $\varepsilon = 1$ if $\theta = \mathrm{id}_\Delta$. Let $x \in V_F$ be such that $h_F(x, x) \in \Delta$ and suppose that $x \notin V$. We show that there exists an element $x' \in V$ such that $h(x', x') = h_F(x, x)$.

Let $a, b \in D$ with $b \neq 0$. We will denote $b^{-1}a$ by $\frac{a}{b}$.

We define an element $\tilde{y} \in V$ as follows. If h is hermitian, then we set $\tilde{y} = y \in V$, where y is as in Lemma 2.45 such that $h_F(x, V) = h_F(x, y)\Delta$ and $h_F(x, y) \in F$. If h is skew–hermitian, then we set $\tilde{y} = y' \in V$, where y' is as in Lemma 2.45 such that

$h_F(x,V) = h_F(x,y')\Delta$ and $\theta_F(h_F(x,y')) = -h_F(x,y')$. We get that

$$\begin{aligned}
\frac{h_F(x-\tilde{y},x-\tilde{y})}{h_F(x,\tilde{y})} &= \frac{h_F(x,x) - h_F(x,\tilde{y}) - h_F(\tilde{y},x) + h_F(\tilde{y},\tilde{y})}{h_F(x,\tilde{y})} \\
&= \frac{h_F(x,x) - h_F(x,\tilde{y}) - \varepsilon\theta_F(h_F(x,\tilde{y})) + h_F(\tilde{y},\tilde{y})}{h_F(x,\tilde{y})} \\
&= \frac{h_F(x,x) + h_F(\tilde{y},\tilde{y})}{h_F(x,\tilde{y})} - 2,
\end{aligned}$$

by the choice of $\tilde{y}$. Since $x \notin V = V^{\#}$, it follows that $h_F(x,\tilde{y}) \notin \Delta$. Since Δ is a valuation ring of D, this implies that $h_F(x,\tilde{y})^{-1} \in \Delta$ and hence $h_F(x,\tilde{y})^{-1} \in \mathfrak{m}\Delta$. We have that $h_F(x,x) \in \Delta$ by assumption and $h_F(\tilde{y},\tilde{y}) \in \Delta$ since $\tilde{y} \in V$. Since $2 \in S^\times \subset \Delta^\times$, it follows that

$$\frac{h_F(x-\tilde{y},x-\tilde{y})}{h_F(x,\tilde{y})} \in \Delta^\times$$

It follows that $h_F(x,V) = h_F(x,\tilde{y})\Delta = h_F(x-\tilde{y},x-\tilde{y})\Delta$. Note that $h_F(x-\tilde{y},x-\tilde{y}) \notin \Delta$. Hence, $h_F(x-\tilde{y},x-\tilde{y})^{-1} \in \Delta$, which implies that $\Delta \subset h_F(x-\tilde{y},x-\tilde{y})\Delta$.

We consider the hyperplane reflection with respect to $x-\tilde{y}$.

$$\begin{aligned}
x' &= x - 2(x-\tilde{y})\frac{h_F(x-\tilde{y},x)}{h_F(x-\tilde{y},x-\tilde{y})} \\
&= x + 2(x-\tilde{y})\frac{h_F(\tilde{y}-x,x)}{h_F(x-\tilde{y},x-\tilde{y})} \\
&= x + 2(x-\tilde{y})\frac{h_F(\tilde{y}-x,x) + \frac{1}{2}h_F(\tilde{y}-x,\tilde{y}-x) - \frac{1}{2}h_F(\tilde{y}-x,\tilde{y}-x)}{h_F(x-\tilde{y},x-\tilde{y})} \\
&= \tilde{y} + (x-\tilde{y})\frac{h_F(\tilde{y}-x,\tilde{y}+x)}{h_F(x-\tilde{y},x-\tilde{y})} \\
&= \tilde{y} + (x-\tilde{y})\frac{h_F(\tilde{y},\tilde{y}) - h_F(x,x)}{h_F(x-\tilde{y},x-\tilde{y})},
\end{aligned}$$

by the choice of $\tilde{y}$. Since reflections are isometries, we have that $h_F(x',x') = h_F(x,x)$. We show that $x' \in V$. We have that

$$h_F(x-\tilde{y},V) = h_F(x,V) - h_F(\tilde{y},V) \subset h_F(x,V) + \Delta = h_F(x-\tilde{y},x-\tilde{y})\Delta = \theta_F(h_F(x-\tilde{y},x-\tilde{y}))\Delta,$$

since $w \circ \theta_F = w$ by Lemma 2.44. It follows that

$$\theta_F(h_F(x-\tilde{y},x-\tilde{y}))^{-1}h_F(x-\tilde{y},V) = h_F((x-\tilde{y})h_F(x-\tilde{y},x-\tilde{y})^{-1},V) \subset \Delta.$$

Since $h_F(\tilde{y},\tilde{y}), h_F(x,x) \in \Delta$ and again invoking Lemma 2.44, it follows that

$$h_F\left((x-\tilde{y})\frac{h_F(\tilde{y},\tilde{y})-h_F(x,x)}{h_F(x-\tilde{y},x-\tilde{y})}, V\right) \subset \Delta.$$

Since $h_F(\tilde{y},V) \subset \Delta$, it follows that $h_F(x',V) \subset \Delta$. This implies that $V + x'\Delta$ is an integral lattice containing V. Since V is unimodular, it is a maximal integral lattice by Proposition 2.37, and hence, $x' \in V$. □

Let (Δ,θ) be an $\mathcal{O}$−algebra with involution without zero divisors, and we assume that Δ is a valuation ring of the division algebra $D = \Delta \otimes_{\mathcal{O}} F$. Using the previous result, we obtain another proof of Proposition 2.8 (c), using that the cancellation holds for $(\Delta,\theta)_F$, the proof of which (see [43, (I.6.3.4)]) is considerably easier than the one of B. Keller's cancellation result, which is used to prove Proposition 2.4 (b).

2.51 Corollary. *Let (Δ,θ) be an $\mathcal{O}$−algebra with involution without zero divisors, and assume that Δ is a valuation ring of the division algebra $D = \Delta \otimes_{\mathcal{O}} F$. Let (V,h) and (V',h') be ε−hermitian spaces over (Δ,θ). If $\theta = \mathrm{id}_\Delta$, assume that $\varepsilon = 1$. If $(V,h)_F \simeq (V',h')_F$ then $(V,h) \simeq (V',h')$.*

Proof. By Proposition 2.49, we have that $h \simeq \langle \alpha_1,\dots,\alpha_n\rangle_\theta$ and $h' \simeq \langle \alpha'_1,\dots,\alpha'_n\rangle_\theta$, for certain $\alpha_1,\dots,\alpha_n,\alpha'_1,\dots,\alpha'_n \in \Delta^\times$. By assumption, $\langle \alpha_1,\dots,\alpha_n\rangle_{\theta_F} \simeq \langle \alpha'_1,\dots,\alpha'_n\rangle_{\theta_F}$. Hence, there exists $x \in V'_F$ such that $h'_F(x,x) = \alpha_1$. By Theorem 2.50, there exists $y \in V'$ such that $h'(y,y) = h'_F(x,x)$. This implies that there exist $\beta'_2,\dots,\beta'_n \in \Delta^\times$ such that $h' \simeq \langle \alpha_1,\beta'_2,\dots,\beta'_n\rangle_\theta$. Since Δ_F is a division algebra, the Witt cancellation property of [43, (I.6.3.4)] yields that

$$\langle \alpha_2,\dots,\alpha_n\rangle_{\theta_F} \simeq \langle \beta'_2,\dots,\beta'_n\rangle_{\theta_F}.$$

The statement now follows by induction on $\dim_\Delta(V)$. □

2.7 Integral lattices and unimodular lattices

In this section, we give the proof of the lattice result announced at the end of section 2.5. The set–up is the following. Let $\mathcal{O}$ be a valuation ring with fraction field F. We denote its maximal ideal by $\mathfrak{m}$. Let (Δ,θ) be an $\mathcal{O}$−algebra with involution without zero divisors and assume that Δ is a valuation ring of the division algebra $D = \Delta_F$. Let w be the valuation on D given by (2.6.1) with valuation ring Δ. By Proposition 2.42, $J(\Delta) = \mathfrak{m}\Delta$. Note that $\Delta/J(\Delta)$ is a division ring. Let $n \in \mathbb{N}$. Then $\mathrm{M}_n(\Delta)$ is an Azumaya algebra over $Z(\Delta)$, and Corollary 1.13 yields that $J(\mathrm{M}_n(\Delta)) = J(Z(\Delta))\,\mathrm{M}_n(\Delta) = \mathfrak{m}\,\mathrm{M}_n(\Delta) = \mathrm{M}_n(\mathfrak{m}\Delta)$, by Proposition 1.22. In the sequel, we will denote $J(\mathrm{M}_n(\Delta))$ simply by J.

Let $\varepsilon = \pm 1$. We fix an ε–hermitian space (V, h) over (Δ, θ).

Proposition. Let $\mathcal{L}$ be an integral lattice in $(V, h)_F$. Then $\mathcal{L}$ is contained in a unimodular lattice of $(V, h)_F$.

Before we prove the above proposition, we prove some preliminary results on lattices and the Dieudonné determinant. We start by recalling the latter concept.

The abelianisation of $D^\times$ is equal to $D^\times/[D^\times, D^\times]$ and we denote it by $\widetilde{D^\times}$. We adjoin a zero element 0 with obvious multiplication and call the resulting semi–group $\widetilde{D}$. To every $C \in \mathrm{M}_n(D)$, one can associate a determinant $\delta(C) \in \widetilde{D}$, called *the Dieudonné determinant*. We say that C is *non–singular* if and only if $C \in \mathrm{M}_n(D)^\times$. The Dieudonné determinant is defined inductively as follows. If $C = (a)$, then $\delta(C) = a \in \widetilde{D}$. If C is singular, then $\delta(C) = 0 \in \widetilde{D}$. Suppose that $C \in \mathrm{M}_n(D)$ is non–singular. Then the row vectors of C, which we denote by R_i, are left linearly independent. Hence, there exist $\lambda_1, \ldots, \lambda_n \in D$ such that $\sum_{i=1}^n \lambda_i R_i = (1, 0, \ldots, 0)$. Let $i \in \{1, \ldots, n\}$ such that $\lambda_i \neq 0$. Let C_i be the $(n-1) \times (n-1)$ matrix obtained by sweeping the first column of C and the i–th row of C. Then $\delta(C) = (-1)^{i+1}\lambda_i^{-1}\delta(C_i) \in \widetilde{D}$. One can show that this definition is independent of the chosen $\lambda_i \neq 0$.

2.52 Proposition.

(a) The unit matrix has Dieudonné determinant equal to 1.

(b) If C' is obtained from C by multiplying a row on the left with an element $\mu \in D$, then $\delta(C') = \mu\delta(C)$. If C' is obtained from C by multiplying a column on the right with an element $\lambda \in D$, then $\delta(C') = \delta(C)\lambda$.

(c) $\delta(CC') = \delta(C)\delta(C')$.

(d) If C' is obtained from C by multiplying a row (resp. column) on the left (resp. right) with an element and adding it to another row (resp. column), then $\delta(C) = \delta(C')$.

(e) If C' is obtained from C by interchanging two rows, then $\delta(C') = -\delta(C)$.

Proof. See [2, IV.1]. □

2.53 Proposition. *Let $C \in \mathrm{M}_n(D)$. Then there exist elements $d_1, \ldots, d_n \in D$ with $w(d_1) \leqslant \ldots \leqslant w(d_n)$ and $P, Q \in \mathrm{M}_n(\Delta)^\times$ such that $PCQ = \mathrm{diag}(d_1, \ldots, d_n)$. Furthermore, $d_1, \ldots, d_n$ are uniquely determined up to units in Δ.*

Proof. There exists $r \in \mathcal{O}$ such that $rC \in \mathrm{M}_n(\Delta)$. By [35], Δ is an elementary divisor ring. Hence, there exist $x_1, \ldots, x_n \in \Delta$ with $w(x_1) \leqslant \ldots \leqslant w(x_n)$ and invertible matrices P and Q over Δ such that $P(rC)Q = \mathrm{diag}(x_1, \ldots, x_n)$. Then $PCQ = \mathrm{diag}(x_1 r^{-1}, \ldots, x_n r^{-1})$ and $w(x_1 r^{-1}) \leqslant \ldots \leqslant w(x_n r^{-1})$. It follows from a uniqueness statement for invariant factors of modules in [35, (9.3)] that the invariant factors $x_1, \ldots, x_n$ are unique up to units in Δ. Hence, $x_1 r^{-1}, \ldots, x_n r^{-1}$ are also unique up to units in Δ. □

2.54 Remark. In [35], the author mentions how the diagonalisation of rC can be obtained, namely by putting an element of smallest valuation in the upper left corner and sweeping the other elements of the first row and column, and so on. So, the valuation of d_1 is the minimum of the valuations of the entries of C.

2.55 Proposition. *Let $C \in \mathrm{M}_n(\Delta)$. Then $C \in \mathrm{M}_n(\Delta)^\times$ if and only if $\overline{C} \in \mathrm{M}_n(\Delta)/J$ is invertible. Furthermore, we have that $\delta(C) \in \Delta/[D^\times, D^\times]$, and $\delta(C) \in \Delta^\times/[D^\times, D^\times]$ if and only if $C \in \mathrm{M}_n(\Delta)^\times$.*

Proof. Note first of all that indeed $[D^\times, D^\times] \subset \Delta$ since $w(aba^{-1}b^{-1}) = w(a) + w(b) - w(a) - w(b) = 0$. So, in fact $[D^\times, D^\times] \subset \Delta^\times$.

We show that C is invertible in $\mathrm{M}_n(\Delta)$ if and only if $\overline{C}$ is invertible in $\mathrm{M}_n(\Delta)/J$. Suppose that C is invertible over Δ. Then there exists $A \in \mathrm{M}_n(\Delta)$ such that $CA = AC = I_n$. Then $\overline{CA} = \overline{AC} = I_n$ and hence $\overline{C}$ is invertible in $\mathrm{M}_n(\Delta)/J$. Suppose that C is not invertible in $M_n(\Delta)$. Then $C\,\mathrm{M}_n(\Delta) \neq \mathrm{M}_n(\Delta)$ and hence C is contained in a proper maximal right ideal M of $\mathrm{M}_n(\Delta)$. Then $M/J \neq \mathrm{M}_n(\Delta)/J$, since if they would be equal, then for any $x \in \mathrm{M}_n(\Delta)$, there would exist a $y \in M$ such that $x - y \in J$, but since J is the intersection of all right maximal ideals of $\mathrm{M}_n(\Delta)$, we have that $J \subset M$ and hence $x - y \in M$ and therefore also $x \in M$. Hence, $M = \mathrm{M}_n(\Delta)$, a contradiction. We conclude that $\overline{C}$ is not invertible in $\mathrm{M}_n(\Delta)/J$.

We now show inductively that $\delta(C) \in \Delta/[D^\times, D^\times]$. If $C = (a)$, then $\delta(C) = a \in D/[D^\times, D^\times]$, and since $a \in \Delta$, clearly $\delta(C) \in \Delta/[D^\times, D^\times]$. Suppose that the statement is true for $n-1$. We show the statement for n. Since interchanging two rows or two columns multiplies the Dieudonné determinant by $-1 \in \widetilde{D}$, and $-1 \in \Delta$, we may assume that the entry of C of smallest valuation is in the top left corner. Using row operations over Δ, we can make all entries of the first column except the first one equal to 0. Row operations and interchanging rows or columns are obtained by multiplying C on the left or right with invertible matrices, so of Dieudonné determinant 1. So, it suffices to show that the Dieudonné determinant of the modified C is an element of $\Delta/[D^\times, D^\times]$. We have that

$$\begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c_{nn} \end{pmatrix} = \mathrm{diag}(c_{11}, 1, \dots, 1) \begin{pmatrix} 1 & \cdots & c_{11}^{-1} c_{1n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c_{nn} \end{pmatrix} = \mathrm{diag}(c_{11}, 1, \dots, 1) C'.$$

Using the formulas above, we get that $\delta(C) = \delta(\mathrm{diag}(c_{11}, 1, \dots, 1))\delta(C') = c_{11}\delta(C')$. Note that C' is a matrix over Δ, since c_{11} is an entry of minimal valuation of C. Denote the $(n-1) \times (n-1)$ matrix in the lower right corner of C' by C''. This is a matrix over Δ and by induction we have that $\delta(C'') \in \Delta/[D^\times, D^\times]$. Furthermore, $\delta(C') = \delta(C'')$ and hence $\delta(C) = c_{11}\delta(C') \in \Delta/[D^\times, D^\times]$.

We now show the last equivalence in the statement. Suppose that $C \in \mathrm{M}_n(\Delta)^\times$. Then $\delta(C) \in \Delta^\times/[D^\times, D^\times]$. Conversely, suppose that $\delta(C) \in \Delta^\times/[D^\times, D^\times]$. Then the calculations above show that the Dieudonné determinant of $\overline{C} \in \mathrm{M}_n(\Delta)/J \cong \mathrm{M}_n(\Delta/\mathfrak{m}\Delta)$ is the reduction of $\delta(C)$, and hence nonzero. Since $\Delta/\mathfrak{m}\Delta$ is a division ring, it follows that $\overline{C}$ is invertible, and hence, by the above, we get that $C \in \mathrm{M}_n(\Delta)^\times$, as desired. □

2.56 Lemma. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two Δ–lattices in $(V,h)_F$. Then there exists a Δ–basis $(e_1, \dots, e_n)$ for $\mathcal{L}_1$ and elements $d_1, \dots, d_n \in D$ with $w(d_1) \leqslant \dots \leqslant w(d_n)$ such that $(e_1 d_1, \dots, e_n d_n)$ is a Δ–basis for $\mathcal{L}_2$.*

Proof. Recall that $\mathcal{L}_1$ and $\mathcal{L}_2$ are free over Δ. Let $(f_1, \dots, f_n)$ be a Δ–basis for $\mathcal{L}_1$ and $(g_1, \dots, g_n)$ a Δ–basis for $\mathcal{L}_2$. Since $(f_1, \dots, f_n)$ and $(g_1, \dots, g_n)$ are D–bases for V_F, we can write $f_i = \sum g_j a_{ji}$. Let $A = (a_{ij})_{1 \leqslant i,j \leqslant n} \in \mathrm{M}_n(D)$. By Proposition 2.53, there are invertible matrices P, Q over Δ such that PAQ is a diagonal matrix $\mathrm{diag}(d_1, \dots, d_n)$ with $d_1, \dots, d_n \in D$ and $w(d_1) \leqslant \dots \leqslant w(d_n)$. We have that $(f_1, \dots, f_n)Q = (f'_1, \dots, f'_n)$ is a Δ–basis for $\mathcal{L}_1$ and $(g_1, \dots, g_n)P^{-1} = (g'_1, \dots, g'_n)$ is a Δ–basis for $\mathcal{L}_2$. It follows that

$$(g'_1 d_1, \dots, g'_n d_n) = (g'_1, \dots, g'_n)PAQ = (f'_1, \dots, f'_n).$$

□

2.57 Proposition. *Let $\varepsilon = \pm 1$ and let (V, h) be an ε–hermitian space over (Δ, θ). Every integral lattice of $(V, h)_F$ is contained in a unimodular lattice.*

Proof. Let $\mathcal{L}$ be an integral lattice in $(V, h)_F$. By Lemma 2.56, there exists a Δ–basis $(e_1, \dots, e_n)$ for V such that $(e_1 d_1, \dots, e_n d_n)$ is a Δ–basis for $\mathcal{L}$ for certain $d_1, \dots, d_n \in D$ such that

$$w(d_1) \leqslant \dots \leqslant w(d_n).$$

Let C be the matrix of h with respect to $(e_1, \dots, e_n)$ and $C_{\mathcal{L}}$ the matrix of h_F with respect to the basis $(e_1 d_1, \dots, e_n d_n)$. Since V is unimodular, Propositions 2.34 and 2.55 yield that $\delta(C) \in \Delta^\times/[D^\times, D^\times]$, and, furthermore, due to the choice of basis and since $\mathcal{L}$ is integral, we have that

$$\delta(C_{\mathcal{L}}) = \theta_F(d_1)\cdots\theta_F(d_n)\delta(C)d_1\cdots d_n \in \Delta/[D^\times, D^\times].$$

It follows that $\theta_F(d_1)\cdots\theta_F(d_n)d_1\cdots d_n \in \Delta$. Expressing this using w and using Lemma 2.44 yields

$$w(d_1) + \ldots + w(d_n) \geqslant 0.$$

If we have equality, then $\delta(C_{\mathcal{L}}) \in \Delta^\times/[D^\times, D^\times]$ and $\mathcal{L}$ is unimodular by Propositions 2.34 and 2.55.

So, suppose that $w(d_1) + \ldots + w(d_n) > 0$. It is clear that the d_i cannot all have negative valuation. If they all have positive valuation, then $\mathcal{L} \subset V$ and we are done. So, suppose that at least one of the d_i has negative valuation. Then there is an $r \in \{1, \ldots, n-1\}$ such that $w(d_1), \ldots, w(d_r) < 0$ and $w(d_{r+1}), \ldots, w(d_n) \geqslant 0$. By Propositions 2.55 and 2.34, it suffices to show that $\mathcal{L}$ is contained in a lattice whose Dieudonné determinant is in $\Delta^\times/[D^\times, D^\times]$.

Step 1

Since $\delta(C) \in \Delta^\times/[D^\times, D^\times]$, Proposition 2.55 yields that every row of C contains at least one entry that is a unit in Δ. Since $\theta_F(d_i)h_{ij}d_j \in \Delta$ for all i, j, by the definition of r, it follows that $w(h_{11}), \ldots, w(h_{1r}) > 0$ and hence that at least one of $h_{1,r+1}, \ldots, h_{1n} \in \Delta^\times$. Suppose that $h_{1i} \in \Delta^\times$ with $i < n$. Consider the following base change for V: $e_n \mapsto e'_n = e_n + e_i x$ with $x \in \Delta$ such that $w(h_{1n} + h_{1i}x) = 0$, and the following base change for $\mathcal{L}$: $e_n d_n \mapsto e'_n = (e_n + e_i x)d_n$. This is possible since $h_{1i} \in \Delta^\times$ and defines a proper base change for $\mathcal{L}$ since $d_n d_i^{-1} \in \Delta$ by assumption. Then $h(e_1, e'_n) \in \Delta^\times$. Hence, without loss of generality we may assume that $h_{1n} \in \Delta^\times$. We then define the following base change for V: $e_i \mapsto e'_i = e_i + e_n x_i$ for $i = r+1, \ldots, n-1$, where the $x_i \in \Delta$ are chosen such that $w(h(e_1, e'_i)) = w(h_{1i} + h_{1n}x_i)$ is large (we can make it as large as we want since $h_{1n} \in \Delta^\times$), and the base change $e_i d_i \mapsto e'_i = (e_i + e_n x_i)d_i$ for $i = r+1, \ldots, n-1$ for $\mathcal{L}$. We can make sure the x_i are such that $d_n^{-1}x_i d_i \in \Delta$ to make this a proper base change for $\mathcal{L}$. To simplify notation, we denote e'_i again by e_i.

Since $h_{1n} \in \Delta^\times$, it follows that $d_1 d_n \in \Delta$ and hence

$$\mathcal{L} \subset \mathcal{L}_1 = e_1 d_1 \Delta \oplus \ldots \oplus e_r d_r \Delta \oplus e_{r+1} d_{r+1} \Delta \oplus \ldots \oplus e_n d_1^{-1} \Delta.$$

Since $w(d_i d_1^{-1}) \geqslant 0$ for $i = 1, \ldots, n$, $\mathcal{L}_1$ is an integral lattice in $(V, h)_F$. If $r = 1$, then

$$\mathcal{L}_1 \subset e_1 d_1 \Delta \oplus e_2 \Delta \oplus \ldots \oplus e_{n-1} \Delta \oplus e_n d_1^{-1} \Delta,$$

which is an integral lattice, since $h_{12}, \ldots, h_{1,n-1}$ have large valuation (more specifically, they can be chosen of valuation larger than $-w(d_1)$). Note that $h_{21}, \ldots h_{n-1,1}$ then also have large valuation. Furthermore, $e_1d_1\Delta \oplus e_2\Delta \oplus \ldots \oplus e_{n-1}\Delta \oplus e_nd_1^{-1}\Delta$ has determinant a unit. Hence, it is a unimodular lattice and we are done in this case.

Step 2

Suppose that $r > 1$. Let C' be the matrix

$$C' = \begin{pmatrix} h_{1n} & h_{11} & \cdots & h_{1,n-1} \\ h_{nn} & h_{n1} & \cdots & h_{n,n-1} \\ h_{2n} & h_{21} & \cdots & h_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n-1,n} & h_{n-1,1} & \cdots & h_{n-1,n-1} \end{pmatrix}$$

and let

$$A = \begin{pmatrix} h_{22} & \cdots & h_{2,n-1} \\ \vdots & \ddots & \vdots \\ h_{n-1,2} & \cdots & h_{n-1,n-1} \end{pmatrix}.$$

We want to show that $A \in \mathrm{M}_n(\Delta)^\times$. By Proposition 2.55, it suffices to show that $\overline{A}$ is invertible in $\mathrm{M}_n(\Delta)/J$. Note that C' is obtained from C by permuting rows and columns, which is achieved by multiplying C on the left and right with invertible matrices. Since these have determinant a unit, we have that $\delta(C') \in \Delta^\times/[D^\times, D^\times]$. Proposition 2.55 yields that $\overline{C'}$ is invertible in $\mathrm{M}_n(\Delta)/J$. Since $w(h_{11}), \ldots, w(h_{1,n-1}), w(h_{21}), \ldots, w(h_{n-1,1}) > 0$, we have that

$$\overline{C'} = \begin{pmatrix} \overline{h_{1n}} & 0 & \cdots & 0 \\ \overline{h_{nn}} & \overline{h_{n1}} & \cdots & \overline{h_{n,n-1}} \\ \overline{h_{2n}} & 0 & \cdots & \overline{h_{2,n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{h_{n-1,n}} & 0 & \cdots & \overline{h_{n-1,n-1}} \end{pmatrix}$$

Since $h_{1n}, h_{n1} \in \Delta^\times$, we have that $\overline{h_{1n}}, \overline{h_{n1}} \neq 0$ in $\Delta/\mathfrak{m}\Delta$ and hence invertible since $\Delta/\mathfrak{m}\Delta$ is a division ring. It follows that

$$\delta(\overline{C'}) = \overline{h_{1n}h_{n1}}\delta(\overline{A}) \in \widetilde{\Delta/\mathfrak{m}\Delta}.$$

It follows that $\delta(\overline{A})$ is non–zero in $\widetilde{\Delta/\mathfrak{m}\Delta}$, which means that $\overline{A}$ is non–singular and hence invertible in $\mathrm{M}_n(\Delta)/J$. Proposition 2.55 yields that $A \in \mathrm{M}_n(\Delta)^\times$.

Let $V = e_2\Delta \oplus \ldots \oplus e_{n-1}\Delta$. Then A is the matrix of the $h|_V$. Since $r > 1$, we have that $w(h_{22}), \ldots, w(h_{2r}) > 0$. Hence, we have that one of $h_{2,r+1}, \ldots, h_{2,n-1} \in \Delta^\times$. By the same reasoning as before, we may assume that $h_{2,n-1} \in \Delta^\times$ and that the valuation of $h_{2,r+1}, \ldots, h_{2,n-2}$ is large. It follows that $d_2d_{n-1} \in \Delta$ and hence

$$\mathcal{L}_1 \subset \mathcal{L}_2 = e_1d_1\Delta \oplus e_2d_2\Delta \oplus \ldots \oplus e_rd_r\Delta \oplus e_{r+1}d_{r+1}\Delta \oplus \ldots \oplus e_{n-1}d_2^{-1}\Delta \oplus e_nd_1^{-1}\Delta.$$

Since $w(h_{1,n-1})$ is large (larger than $-w(d_2^{-1}d_1)$ is sufficient), this is an integral lattice. If $r = 2$, it follows that

$$\mathcal{L}_2 \subset e_1 d_1 \Delta \oplus e_2 d_2 \Delta \oplus e_3 \Delta \ldots \oplus e_{n-2} \oplus e_{n-1} d_2^{-1} \Delta \oplus e_n d_1^{-1} \Delta,$$

which is an integral lattice since $h_{13}, \ldots, h_{1,n-1}, h_{23}, \ldots, h_{2,n-2}$ have large enough valuation (if they were not large enough before, we can enlarge them again using that $h_{1n}, h_{2,n-1} \in \Delta^\times$).

Next steps

We can repeat this procedure, in each step "taking care of" one positive and one negative coefficient (symmetric with respect to the left and right end of the lattice $\mathcal{L}$). In **Step** i, we find that the determinant of $e_{i+1}\Delta \oplus \ldots \oplus e_{n-i}\Delta$ is a unit. Suppose that $n - r < r$, then in **Step** $n - r$ we have that the determinant of $e_{n-r+1}\Delta \oplus \ldots \oplus e_r \Delta$ is a unit. By assumption, $e_{n-r+1} d_{n-r+1} \Delta \oplus \ldots \oplus e_r d_r \Delta$ is integral. This implies that $w(d_{n-r+1}) + \ldots + w(d_r) \geqslant 0$, which contradicts the fact that $d_{n-r+1}, \ldots, d_r$ all have negative valuation. Therefore, we have that $r \leqslant n - r$. Then we eventually obtain an integral lattice

$$\mathcal{L}_r = e_1 d_1 \Delta \oplus \ldots \oplus e_r d_r \Delta \oplus e_{r+1} d_{r+1} \Delta \oplus \ldots \oplus e_{n-r} d_{n-r} \Delta \oplus e_{n-r+1} d_r^{-1} \Delta \ldots \oplus e_{n-1} d_2^{-1} \Delta \oplus e_n d_1^{-1} \Delta,$$

which is contained in the unimodular lattice

$$e_1 d_1 \Delta \oplus \ldots \oplus e_r d_r \Delta \oplus e_{r+1} \Delta \oplus \ldots \oplus e_{n-r} \Delta \oplus e_{n-r+1} d_r^{-1} \Delta \ldots \oplus e_{n-1} d_2^{-1} \Delta \oplus e_n d_1^{-1} \Delta.$$

(Again, the integrality of the lattice follows from the fact that we can make the valuation of $h_{1,r+1}, \ldots, h_{1,n-1}, h_{2,r+1}, \ldots, h_{2,n-2}, \ldots, h_{r,r+1}, \ldots, h_{r,n-r}$ large enough.) $\square$

3

Varieties associated to algebras with involution over fields

Let no one ignorant of geometry enter here.

Plato

In this chapter we collect properties of certain varieties associated to algebras (with involution) over fields. These varieties have been studied in [52, 53] in relation with algebraic groups that arise from algebras with involution. We give an overview of Schur index reduction formulas for the function fields of these varieties, which were proved in the aforementioned papers. We use these varieties in order to show the following result. Let k be a field of characteristic different from 2 and let (B,τ) be a $k-$algebra with involution of degree at least 3. Let F/k and L/k be field extensions and $\lambda : F \to L^{\infty}$ a $k-$place. If τ_F is isotropic (resp. hyperbolic), then τ_L is isotropic (resp. hyperbolic). In fact, we prove something stronger, namely that $\operatorname{ind}((B,\tau)_F) \subset \operatorname{ind}((B,\tau)_L)$. We will come back to this result in the next chapter, where we give a variety–free argument, which works in a more general setting.

We first recall the concept of a place from one field to another, and the relation with valuation rings of fields. Let L be a field and let $L^{\infty} = L \cup \{\infty\}$, with the field operations of L extended to L^{∞} by $\infty + x = x + \infty = \infty$ for any $x \in L$, $x \cdot \infty = \infty \cdot x = \infty$ for any $0 \neq x \in L^{\infty}$, whereas $\infty + \infty, 0 \cdot \infty$ and $\infty \cdot 0$ are not defined. Let F be a field.

A *place from F to L* is a map $\lambda : F \to L^\infty$ such that $\lambda(1) = 1$, $\lambda(xy) = \lambda(x)\lambda(y)$ and $\lambda(x+y) = \lambda(x) + \lambda(y)$ for all $x, y \in F$ whenever the right hand sides are defined. If there are places in both directions between F and L, then we say that F and L are *place equivalent*. If F and L are both field extensions of a field k, and $\lambda : F \to L^\infty$ is a place such that $\lambda|_k = \mathrm{id}_k$, then we say that λ is *a k-place*.
Given a place $\lambda : F \to L^\infty$, the set $\mathcal{O}_\lambda = \{x \in F \mid \lambda(x) \neq \infty\}$ is a valuation ring of F with maximal ideal $\mathfrak{m}_\lambda = \{x \in F \mid \lambda(x) = 0\}$. The place λ identifies the residue field $\mathcal{O}_\lambda / \mathfrak{m}_\lambda$ with a subfield of L. Conversely, let $\mathcal{O}$ be a valuation ring of F with residue field κ. Setting $\lambda(a) = \overline{a} \in \kappa$ for all $a \in \mathcal{O}$ and $\lambda(a) = \infty$ for all $a \in F \setminus O$, defines a place $\lambda : F \to \kappa^\infty$. The concepts of places, valuation rings and valuations are equivalent in the sense that any one of those objects gives rise to the two others.

Let k be a field. By *a variety over k*, we mean a k-scheme that is separated and of finite type over k. Projective, geometrically irreducible varieties have several nice properties, especially with respect to k-places from one field extension of k to another.

3.1 Proposition. *Let X be a projective, geometrically irreducible variety over k, and let $k(X)$ be its function field.*

(a) $k(X)/k$ is a regular extension, i.e. k is algebraically closed in $k(X)$ and $k(X)$ is separably generated over k.

(b) Let M_1 and M_2 be two field extensions of k such that there exists a k-place $\lambda : M_1 \to M_2^\infty$. If X has a rational point over M_1, then it also has a rational point over M_2.

(c) Let M/k be a field extension. Then there exists a k-place $k(X) \to M^\infty$ if and only if X has an M-rational point.

Proof. See [39, (1.1), (3.1)] for (a) and (b) respectively. Since X has a rational point over $k(X)$, the necessary condition in (c) follows directly from (b). The converse follows from the fact that X is projective, and hence complete (i.e. proper over $\mathrm{Spec}(k)$), see [19, (103.)]. □

We recall from [52, 53] certain twisted flag varieties associated to algebras (with involution). We introduce these varieties as functors. That is, given an algebra (with involution) over k, we consider certain functors $\mathcal{F}$ from the category of field extensions of k to the category of sets. In some cases $\mathcal{F}$ will be represented by a projective, geometrically irreducible variety over k, by which we mean that for any field extension M/k, there is a one-to-one correspondence between the set of M-rational points of that variety and the set $\mathcal{F}(M)$.

Let M/k be an arbitrary field extension and let $i \in \mathbb{N}$. Let B be a central simple k–algebra. We define a functor $\mathrm{SB}_i(B)$ as follows. We let

$$\mathrm{SB}_i(B)(M) = \{I \text{ right ideal of } B_M \mid \mathrm{rdim}(I) = i\}.$$

If $i = 1$ one usually writes $\mathrm{SB}(B)$ instead of $\mathrm{SB}_1(B)$. It is clear that $\mathrm{SB}_i(B)(M) = \varnothing$ if $i > \deg(B)$.

Let (B, τ) be a k–algebra with involution of degree at least 3. We define functors $\mathrm{IV}_i(B, \tau)$ associated to (B, τ). For the definition of the functors we distinguish between three cases: τ is of the first kind, τ is of the second kind and B is simple, τ is of the second kind and B is not simple.

Assume that τ is of the first kind. We set

$$\mathrm{IV}_i(B, \tau)(M) = \{I \text{ right ideal of } B_M \mid \tau_M(I)I = 0 \text{ and } \mathrm{rdim}(I) = i\}.$$

Suppose that $\deg(B)$ is odd and τ is orthogonal. Then B is split by [45, (2.8)], and hence, τ is adjoint to a non–singular symmetric non–alternating bilinear form over F by Proposition 2.10. By Proposition 2.14, if $\mathrm{char}(F) \neq 2$, the functors defined here for (B, τ) correspond to the ones defined for b in [52].

Suppose that τ is of the second kind and that $Z(B)$ is a field. We let $\mathrm{IV}_i(B, \tau)(M)$ be the set

$$\{(I, J) \mid I \subset J \text{ balanced right ideals of } B_M, \tau_M(J)I = 0, \mathrm{rdim}(I) = i = \deg(B) - \mathrm{rdim}(J)\}.$$

For (I, J) to belong to $\mathrm{IV}_i(B, \tau)(M)$, we need that $J = \tau_M(I)^0$, since clearly $J \subset \tau_M(I)^0$ and $\mathrm{rdim}(I) + \mathrm{rdim}(J) = \mathrm{rdim}(I) + \mathrm{rdim}(\tau_M(I)^0) = \mathrm{rdim}(\tau_M(I)) + \mathrm{rdim}(\tau_M(I)^0) = \deg(B)$, by Proposition 1.39. It is then clear that a balanced right ideal I of B_M of reduced dimension i is isotropic if and only if $(I, \tau_M(I)^0) \in \mathrm{IV}_i(B, \tau)(M)$. If $\deg(B)$ is even and $i = \deg(B)/2$, then $J = I$ by dimension reasons.
Note furthermore that even though B is simple, B_M need not be simple!

Suppose that τ is of the second kind and that $B \cong E \times E^{\mathrm{op}}$, with E a central simple k–algebra. We set

$$\mathrm{IV}_i(B, \tau)(M) = \mathrm{SB}_i(E)(M).$$

3.2 Remark. Suppose that τ is of the second kind and B is simple. Suppose furthermore that B_M is not simple. Then $B_M \cong (B \otimes_{Z(B)} M) \times (B \otimes_{Z(B)} M)^{\mathrm{op}}$. Let $0 \leqslant i \leqslant \deg(B)/2$ and let us compare the functors $\mathrm{IV}_i(B, \tau)$ and $\mathrm{IV}_i(B_M, \tau_M)$. We have that $\mathrm{IV}_i(B_M, \tau_M)(M)$ and $\mathrm{IV}_i(B, \tau)(M)$ are either both empty, or both non–empty. Namely, let I_1 (resp. I_2) be a right (resp. left) ideal of $B \otimes_{Z(B)} M$ of reduced dimension i. If $(I_1 \times I_2^{\mathrm{op}}, I_2^0 \times (I_1^0)^{\mathrm{op}}) \in \mathrm{IV}_i(B,\tau)(M)$ then $I_1 \in \mathrm{IV}_i(B_M,\tau_M)(M)$. Conversely, if $I_1 \in \mathrm{IV}_i(B_M,\tau_M)(M)$, then by the characterisation of left ideals in [45, (1.12)], since $i \leqslant \deg(B)/2$, there exists a left ideal $I_2' \subset I_1^0$ such that $\mathrm{rdim}(I_2') = i$. It follows that $(I_1 \times I_2'^{\mathrm{op}}, I_2'^0 \times (I_1^0)^{\mathrm{op}}) \in \mathrm{IV}_i(B,\tau)(M)$. However, in general this I_2' is not unique, so there is no one-to-one correspondence between $\mathrm{IV}_i(B_M,\tau_M)(M)$ and $\mathrm{IV}_i(B,\tau)(M)$.

Note that by the reasoning in the previous remark, it follows that in the case where B is degenerate, that for $1 \leqslant i \leqslant \deg(B)/2$, B_M has an isotropic balanced right ideal of reduced dimension i if and only if $\mathrm{IV}_i(B,\tau)(M) \neq \varnothing$.

We can summarise the above observations on the relation between the functors $\mathrm{IV}_i(B,\tau)$ and isotropy of τ_M as follows:

$$\mathrm{ind}((B,\tau)_M) = \{0\} \cup \{i \in \{1,\dots,\deg(B)/2\} \mid \mathrm{IV}_i(B,\tau)(M) \neq \varnothing\}.$$

Note that, by Proposition 1.41, in the nondegenerate case, the sets $\mathrm{IV}_i(B,\tau)(M)$ will be empty if $i > \deg(B)/2$.

3.3 Proposition.

(a) Let B be a central simple k−algebra. For $1 \leqslant i \leqslant n$, $\mathrm{SB}_i(B)$ is represented by a smooth, projective, geometrically irreducible k−variety. These are called the generalised Severi–Brauer varieties associated to B.

(b) Let (B,τ) be a k−algebra with involution of degree at least 3. If

1. *τ is symplectic and $1 \leqslant i \leqslant \deg(B)/2$, or*
2. *τ is of the second kind, $\mathrm{char}(k) \neq 2$ and $1 \leqslant i \leqslant \deg(B)/2$, or*
3. *τ is orthogonal, $\mathrm{char}(k) \neq 2$ and $i < \deg(B)/2$,*

then the functor $\mathrm{IV}_i(B,\tau)$ is represented by a smooth, projective, geometrically irreducible k−variety. If $\deg(B)$ is even and greater than 2, and τ is orthogonal of trivial discriminant, then the functor $\mathrm{IV}_{\deg B/2}(B,\tau)$ is represented by a projective k−variety having two smooth, projective, geometrically irreducible components.

Proof. See [52, Section 5] and [53, Section 9]. □

3.4 Remarks.

(a) Let B be a k–quaternion algebra. Then the Severi–Brauer variety $\mathrm{SB}(B)$ is a smooth, projective conic, namely the one associated to the pure part of the reduced norm of B. Furthermore, let τ be an orthogonal involution on B. The fact that the functor $\mathrm{IV}_1(B,\tau)$ does not yield a variety of the right type reflects the fact that the projective quadric associated to a binary quadratic form over k consists of two points over an algebraic closure of k. So, this quadric is not geometrically irreducible.

(b) Suppose that $\mathrm{char}(k) \neq 2$ and let (V,b) be a symmetric bilinear space over k of dimension at least 3. Then the variety $\mathrm{IV}_1(\mathrm{Ad}(b))$ corresponds to the projective quadric defined by b (see [69]).

(c) Let (B,τ) be an F–algebra with orthogonal involution of trivial discriminant. The fact that $\mathrm{IV}_{\deg(B)/2}(B,\tau)$ is not irreducible is related to the fact that $C(B,\tau)$ is not simple. Once a labeling of the two simple components of $C(B,\tau)$ has been chosen, one can also label the two irreducible components of $\mathrm{IV}_{\deg(B)/2}(B,\tau)$ accordingly (see [52] for the relation between the components of $C(B,\tau)$ and $\mathrm{IV}_{\deg(B)/2}(B,\tau)$). So, if we denote the components of $C(B,\tau)$ by C_+ and C_-, we denote the irreducible component of $\mathrm{IV}_{\deg(B)/2}(B,\tau)$ corresponding to C_+ by $\mathrm{IV}_+(B,\tau)$, and the one corresponding to C_- by $\mathrm{IV}_-(B,\tau)$. In the sequel, we will often use a subindex $\varepsilon \in \{+,-\}$ and write $\mathrm{IV}_\varepsilon(B,\tau)$, with the convention that if $\varepsilon = +$ (resp. $\varepsilon = -$) then $-\varepsilon = -$ (resp. $-\varepsilon = +$).

(d) In the situation of Proposition 3.3, there are certain algebraic groups acting on the varieties SB_i and IV_i (with i in the right range), and with respect to these groups, the varieties SB_i and IV_i are so–called *twisted flag varieties over k*.

One can also define functors by considering flags of isotropic ideals of specified reduced dimensions. When the reduced dimensions are in the right range, one also obtains twisted flag varieties. These are studied in [52, 53].

3.5 Proposition. *Let X be a twisted flag variety over k and let $k(X)$ be its function field. If X has a k–rational point then $k(X)/k$ is a purely transcendental extension.*

Proof. See [39, (3.10)]. □

3.6 Notation. Let (B,τ) be a k–algebra with involution. If $i \in \mathbb{N}$ is such that $\mathrm{IV}_i(B,\tau)$ is an integral variety then we denote its function field by $k_i(\tau)$. If $\deg(B)$ is even and greater than 2, and $i = \deg(B)/2$, then we denote the function field of $\mathrm{IV}_+(B,\tau)$ by $k_+(\tau)$ and the function field of $\mathrm{IV}_-(B,\tau)$ by $k_-(\tau)$. If B is simple then we denote the function field of $\mathrm{SB}(B)$ by $k(B)$.

The generalised Severi–Brauer varieties and the involution varieties have useful properties.

3.7 Proposition. *Let B be a central simple k−algebra.*

(a) Let M/k be a field extension and let $1 \leqslant i \leqslant \deg(B)$. The variety $\mathrm{SB}_i(B)$ has an M−rational point if and only if $\mathrm{ind}(B_M) \mid i$. In particular, $\mathrm{SB}(B)$ has an M−rational point if and only if B_M is split.

(b) Let M_1/k and M_2/k be field extensions. Let $\lambda : M_1 \to M_2^\infty$ be a k−place. Then $\mathrm{ind}(B_{M_2}) \mid \mathrm{ind}(B_{M_1})$.

Proof. see [45, (1.17)] for (a). For (b), let $j = \mathrm{ind}(B_{M_1})$. Then $\mathrm{SB}_j(B)$ has an M_1−rational point by (a). By Proposition 3.1 (b), $\mathrm{SB}_j(B)$ then also has an M_2−rational point. By (a), it follows that $\mathrm{ind}(B_{M_2}) \mid j$, proving the statement. □

3.8 Proposition. *Let (B, τ) be a k−algebra with involution. Let M/k be a field extension and let $i \in \mathbb{N}$ be such that $\mathrm{IV}_i(B, \tau)(M) \neq \varnothing$. Then $\mathrm{ind}(B_M) \mid i$, and for all $d \in \mathbb{N}$ such that $d\,\mathrm{ind}(B_M) \leqslant i$, we have that $\mathrm{IV}_{d\,\mathrm{ind}(B_M)}(B, \tau)(M) \neq \varnothing$ as well.*

Proof. This follows immediately from Proposition 2.14. □

Given a k−algebra with involution (B, τ), we can use the varieties $\mathrm{IV}_i(B, \tau)$ to study the isotropy behaviour of (B, τ) under k−places.

3.9 Proposition. *Let F/k and L/k be field extensions and $\lambda : F \to L^\infty$ a k−place. Let (B, τ) be a k−algebra with involution of degree at least 3. If τ is orthogonal or of the second kind, assume that $\mathrm{char}(k) \neq 2$. Then*

$$\mathrm{ind}((B, \tau)_F) \subset \mathrm{ind}((B, \tau)_L).$$

Proof. If τ_F is anisotropic then the inclusion of the indices is trivial. So, suppose that τ_F is isotropic. Recall that for every field extension M/F, we have that

$$\mathrm{ind}((B, \tau)_M) = \{0\} \cup \{i \in \{1, \ldots, \deg(B)/2\} \mid \mathrm{IV}_i(B, \tau)(M) \neq \varnothing\}.$$

Suppose first that if $\deg(B)/2 \in \mathrm{ind}((B, \tau)_F)$, that we are not in the case where τ is orthogonal and $\mathrm{disc}(\tau)$ is nontrivial. Then it follows from Proposition 3.1 (b) applied to $\mathrm{IV}_i(B, \tau)$ or one of its irreducible components, that $\mathrm{ind}((B, \tau)_F) \subset \mathrm{ind}((B, \tau)_L)$.

Consider now the case where $\deg(B)/2 \in \mathrm{ind}((B, \tau)_F)$ and τ is orthogonal of nontrivial discriminant, say $\mathrm{disc}(\tau) = d \in k^\times/k^{\times 2}$. Then τ_F is hyperbolic. We have that $\tau_{k(\sqrt{d})}$ has trivial discriminant and hence, we can consider the variety $\mathrm{IV}_{\deg(B)/2}((B, \tau)_{k(\sqrt{d})})$.

This variety has two irreducible components, which we denote by Y_+ and Y_-. Since τ_F is hyperbolic, $\mathrm{disc}(\tau_F)$ is trivial. Let δ be a square root of d in F. Since δ is algebraic over k, it follows that $\delta \in \mathcal{O}_\lambda$, the valuation ring of F corresponding to λ, and hence d has a square root in L as well. We fix k−embeddings of $k(\sqrt{d})$ in F (resp. L), that map $\sqrt{d}$ to δ (resp. $\lambda(\delta)$). With respect to these embeddings, we may consider λ as a $k(\sqrt{d})$−place. By Proposition 3.1 (b), it follows that one of Y_+ and Y_- has a L−rational point. This means that $\deg(B)/2 \in \mathrm{ind}((B,\tau)_L)$. This proves the statement. □

3.10 Remark. When we consider surjective k−places whose corresponding valuation ring is Henselian, the converse of Proposition 3.9 also holds. Let F/k and L/k be field extensions and let $\lambda : F \to L^\infty$ be a k−place. Assume that the valuation ring $\mathcal{O}_\lambda$ of F corresponding to λ is Henselian, and that $\lambda(\mathcal{O}_\lambda) = L$. Let (B,τ) be a k−algebra with involution. If τ is orthogonal or of the second kind, assume that $\mathrm{char}(k) \neq 2$. Since $\mathcal{O}_\lambda$ is Henselian, [9, (2.3.5)] yields that an L−rational point of a smooth k−variety can be lifted to an F−rational point. Using this, the inclusion $\mathrm{ind}((B,\tau)_L) \subset \mathrm{ind}((B,\tau)_F)$ can then be shown in the same way as the reverse inclusion was shown in the proof of Proposition 3.9.

We will come back to the statement in Proposition 3.9 in chapter 4, where we will show more generally that, given fields F and L (without restrictions on the characteristic), a place $\lambda : F \to L^\infty$ with associated valuation ring $\mathcal{O}$, and an $\mathcal{O}$−algebra with involution $(\mathcal{A},\sigma)$, that $\mathrm{ind}((\mathcal{A},\sigma)_F) \subset \mathrm{ind}((\mathcal{A},\sigma)_L)$. In fact, we will show more precisely how one passes from an isotropic balanced right ideal of $(\mathcal{A},\sigma)_F$ of a certain reduced dimension, to an isotropic balanced right ideal of $(\mathcal{A},\sigma)_L$ of the same reduced dimension (see Theorem 4.9).
In later chapters we will study the behaviour of algebras with involution of the first kind after passing to the function field of the varieties IV_i, if $i \in \mathbb{N}$ is such that this function field exists. It will then be important to know how the Schur index of the algebra changes. In the literature, there are Schur index reduction formulas for the varieties IV_i and SB_1. We collect them below.

3.11 Theorem. *Suppose that* $\mathrm{char}(k) \neq 2$. *Let B be a central simple k−algebra of degree $2n$. Let D be a central simple k−algebra. Then*

$$\mathrm{ind}(D \otimes_k k(B)) = \min_{1 \leqslant j \leqslant 2n} \mathrm{ind}(D \otimes_k B^{\otimes j}).$$

Assume that $n \geqslant 2$. For $1 \leqslant i < n$ we set $d_i = v_2(\gcd(i,2n))$. Let τ be an involution on B of the first or second kind. If τ is orthogonal of trivial discriminant, let $C(B,\tau) = C_+ \times C_-$. We define

$$r_i = \begin{cases} d_i & \text{if } n \text{ is even or } i < n-1 \\ 0 & \text{if } n \text{ is odd and } i = n-1. \end{cases}$$

(a) *Assume that τ is orthogonal of trivial discriminant. Then*

$$\begin{aligned}\operatorname{ind}(D\otimes_k k_i(\tau)) &= \min(\operatorname{ind}(D), 2^{r_i}\operatorname{ind}(D\otimes_k B), 2^{n-i-1}\operatorname{ind}(D\otimes_k C_+),\\ &\quad 2^{n-i-1}\operatorname{ind}(D\otimes_k C_-))\\ \operatorname{ind}(D\otimes_k k_\varepsilon(\tau)) &= \min(\operatorname{ind}(D), 2^{d}\operatorname{ind}(D\otimes_k B), \operatorname{ind}(D\otimes_k C_\varepsilon),\\ &\quad 2^{d}\operatorname{ind}(D\otimes_k C_{-\varepsilon})),\end{aligned}$$

with $\varepsilon = \{+,-\}$ and $d = v_2(n)$.

(b) *Assume that τ is orthogonal of nontrivial discriminant, say* $\operatorname{disc}(\tau) = e \in k^\times/k^{\times 2}$. *Let $L = k(\sqrt{e})$. If $i < n-1$, then*

$$\operatorname{ind}(D\otimes_k k_i(\tau)) = \gcd(\operatorname{ind}(D), 2^{d_i}\operatorname{ind}(D\otimes_k B), 2^{n-i}\operatorname{ind}(D\otimes_k C(B,\tau))).$$

If $i = n-1$, then

$$\begin{aligned}\operatorname{ind}(D\otimes_k k_i(\tau)) &= \gcd(\operatorname{ind}(D), 2\operatorname{ind}(D\otimes_k B_L), 2^{d_i}\operatorname{ind}(D\otimes_k B),\\ &\quad 2^{n-i}\operatorname{ind}(D\otimes_k C(B,\tau))).\end{aligned}$$

(c) *Assume that τ is symplectic. Let $1 \leqslant i \leqslant n$. Then*

$$\operatorname{ind}(D\otimes_k k_i(\tau)) = \min(\operatorname{ind}(D), 2^{d_i}\operatorname{ind}(D\otimes_k B)).$$

Proof. For the proof of the first three formulas, and the symplectic case, we refer to the summary at the end of [52]. The proof of the remaining two formulas can be found in [53, p. 190]. □

3.12 Lemma. *Suppose that* $\operatorname{char}(k) \neq 2$. *Let (B,τ) be a $k-$algebra with orthogonal involution of even degree. Suppose that* $\operatorname{disc}(\tau)$ *is trivial. We write $C(B,\tau) = C_+ \times C_-$. Let $\varepsilon \in \{+,-\}$. Then C_ε splits over $k_\varepsilon(\tau)$. If* $\deg(B) \equiv 0 \bmod 4$ *and C_ε splits over $k_{-\varepsilon}(\tau)$, then B or C_ε already splits over k. If* $\deg(B) \equiv 2 \bmod 4$ *then C_ε always splits over $k_{-\varepsilon}(\tau)$.*

Proof. The formulas of Theorem 3.11 yield

$$\begin{aligned}\operatorname{ind}(C_\varepsilon\otimes_k k_\varepsilon(\tau)) &= \min(\operatorname{ind}(C_\varepsilon), 2^{d}\operatorname{ind}(C_\varepsilon\otimes_k B), \operatorname{ind}(C_\varepsilon\otimes_k C_\varepsilon),\\ &\quad 2^{d}\operatorname{ind}(C_\varepsilon\otimes_k C_{-\varepsilon})),\\ \operatorname{ind}(C_{-\varepsilon}\otimes_k k_\varepsilon(\tau)) &= \min(\operatorname{ind}(C_{-\varepsilon}), 2^{d}\operatorname{ind}(C_{-\varepsilon}\otimes_k B), \operatorname{ind}(C_{-\varepsilon}\otimes_k C_\varepsilon),\\ &\quad 2^{d}\operatorname{ind}(C_{-\varepsilon}\otimes_k C_{-\varepsilon})).\end{aligned}$$

with $d = v_2(n)$. The statement now follows using the properties of C_+ and C_- in Proposition 1.36. □

3.13 Corollary. *Suppose that* $\mathrm{char}(k) \neq 2$ *and let* (B, τ) *be a* k*−algebra with orthogonal involution of even degree. Suppose that* τ *is hyperbolic. Then* $\mathrm{disc}(\tau)$ *is trivial and one of the simple components of* $C(B, \tau)$ *is split over* k*.*

Proof. That $\mathrm{disc}(\tau)$ is trivial is stated in [45, (7.3) (6)]. We write $C(B, \tau) = C_+ \times C_-$, and we denote the corresponding irreducible components of $\mathrm{IV}_{\deg(B)/2}(B, \tau)$ by $\mathrm{IV}_+(B, \tau)$ and $\mathrm{IV}_-(B, \tau)$. Since τ is hyperbolic, one of $\mathrm{IV}_+(B, \tau)$ and $\mathrm{IV}_-(B, \tau)$ has a k−rational point, and its function field is then a purely transcendental extension of k by Proposition 3.8. Since C_+ (resp. C_-) splits over $k_+(\tau)$ (resp. $k_-(\tau)$) by Lemma 3.12, and the Schur index of an algebra does not change after passing to a finitely generated purely transcendental extension, it follows that one of C_+ and C_- already splits over k. In the case $\deg(B) \equiv 0 \bmod 4$, another proof of the statement can be found in [45, (8.31)]. □

3.14 Proposition. *Assume that* $\mathrm{char}(k) \neq 2$*. Let* (B, τ) *be a* k*−algebra with orthogonal involution of even degree. Assume that* τ *is hyperbolic. Then* $\mathrm{disc}(\tau)$ *is trivial. Let* $C(B, \tau) = C_+ \times C_-$*, and denote the corresponding irreducible components of* $\mathrm{IV}_{\deg(B)/2}(B, \tau)$ *by* $\mathrm{IV}_+(B, \tau)$ *and* $\mathrm{IV}_-(B, \tau)$*. Then the following are equivalent:*

(i) B *is split over* k*.*

(ii) Both $\mathrm{IV}_+(B, \tau)$ *and* $\mathrm{IV}_-(B, \tau)$ *have a* k*−rational point.*

(iii) C_+ *and* C_- *are both split over* k*.*

Proof. That $\mathrm{disc}(\tau)$ is trivial is stated in [45, (7.3) (6)]. Let n be an integer such that $\deg(B) = 2n$. Assume that B is split. Then $(B, \tau) \cong (M_{2n}(k), \mathrm{ad}_\varphi)$, with φ a hyperbolic symmetric bilinear form of dimension $2n$. Let V_n be the variety over k such that for any field extension L/k, $V_n(L)$ is the set of n−dimensional totally φ_L−isotropic subspaces of L^{2n}. Through the correspondence between subspaces of k^{2n} and right ideals of $M_{2n}(k)$ (see Proposition 2.12), one obtains that $\mathrm{IV}_n(B, \tau) \cong V_n$ as varieties. The two irreducible components of $\mathrm{IV}_n(B, \tau)$ then correspond to the two irreducible components of V_n, which we denote by V_+ and V_-. Since φ is hyperbolic, at least one of V_+ and V_- has a k−rational point. Furthermore, by [52, (5.5)], the action of $\mathrm{PSO}_{2n}(k)$ on $V_n(k)$ has two orbits, and these are exactly $V_+(k)$ and $V_-(k)$ (see [52, p. 577]). Hence $V_+(k)$ and $V_-(k)$ must both be nonempty. This implies that $\mathrm{IV}_+(B, \tau)$ and $\mathrm{IV}_-(B, \tau)$ both have a k−rational point. Hence, (i) implies (ii).

Assume that (ii) holds. Then we have k−places from $k_+(\tau)$ to k and from $k_-(\tau)$ to k. Lemma 3.12 together with Proposition 3.1 then implies that C_+ and C_- are both split over k, so (ii) implies (iii). By Proposition 1.36, we also have that (iii) implies (i), both in the case $\deg(B) \equiv 0 \bmod 4$, as in the case $\deg(B) \equiv 2 \bmod 4$. □

4

Specialisation and good reduction for involutions

> Tout mathématicien digne de ce nom
> a ressenti, même si ce n'est que
> quelques fois, l'état d'exaltation
> lucide dans lequel une pensée succède
> à une autre comme par miracle...
>
> André Weil

In this chapter, we study algebras with involution over fields that are obtained by scalar extension from Azumaya algebras with involution over valuation rings, and we show how they behave under specialisation, i.e. with respect to a place from one field to another. In section 4.2, we prove the first main specialisation result on isotropy (see Theorem 4.9), which generalises Proposition 3.9. In section 4.3, we focus on Henselian valuation rings. Algebras with involution over such rings are closely related to their induced structures over the fraction field and residue field of the valuation ring. We illustrate this in Theorem 4.20 by proving a lifting result for isotropy and hyperbolicity of involutions, strengthening Theorem 4.9. In the second part of section 4.3, we show that isomorphism of Azumaya algebras with involution over a Henselian valuation ring in which 2 is a unit, can be detected rationally (Theorem 4.34). This implies that Azumaya algebras with involution over a general valuation ring that become isomorphic over its fraction field, are also isomorphic over its residue field, provided that 2 is invertible in the valuation ring. This allows us to define a notion of good reduction with respect to

places for algebras with involution (section 4.4).
The results for Azumaya algebras with involution over a Henselian valuation ring in section 4.3 will be crucial for the isomorphism results in the next chapter.

Throughout this chapter F denotes a field.

4.1 Value functions

In this section, we recall some concepts from the theory of value functions on vector spaces over valued $F-$division algebras and algebras over a valued field, developed in [62, 73]. We also present some new results, which will be used later in this chapter.

Let D be an $F-$division algebra. Let Γ be a totally ordered abelian group and $w : D \to \Gamma \cup \{\infty\}$ a valuation on D. For every $\gamma \in \Gamma$, let $D_w^{\geqslant\gamma} = \{a \in D \mid w(a) \geqslant \gamma\}$ and $D_w^{>\gamma} = \{a \in D \mid w(a) > \gamma\}$. We then set

$$\mathrm{gr}_w(D) = \oplus_{\gamma\in\Gamma} D_w^{\geqslant\gamma}/D_w^{>\gamma}.$$

Then $\mathrm{gr}_w(D)$ is *a graded division ring*, i.e. every nonzero homogeneous element of $\mathrm{gr}_w(D)$ is invertible.

Let (D, w) be a valued $F-$division algebra. Let V be a finite–dimensional $D-$vector space. A map $\alpha : V \to \Gamma \cup \{\infty\}$ is called *a $w-$value function* on V if $\alpha^{-1}(\{\infty\}) = \{0\}$, $\alpha(xa) = \alpha(x) + w(a)$ and $\alpha(x + y) \geqslant \min(\alpha(x), \alpha(y))$ for all $x, y \in V$ and all $a \in D$. For every $\gamma \in \Gamma$ we let $V_\alpha^{\geqslant\gamma} = \{x \in V \mid \alpha(x) \geqslant \gamma\}$ and $V_\alpha^{>\gamma} = \{x \in V \mid \alpha(x) > \gamma\}$. We then set

$$\mathrm{gr}_\alpha(V) = \oplus_{\gamma\in\Gamma} V_\alpha^{\geqslant\gamma}/V_\alpha^{>\gamma}.$$

This is a graded $\mathrm{gr}_w(D)-$module.

Let v be a valuation on F and let B be a finite–dimensional $F-$algebra. A $v-$value function α on B is called *surmultiplicative* if $\alpha(1) = 0$ and $\alpha(ab) \geqslant \alpha(a) + \alpha(b)$ for all $a, b \in B$. In this case, $\mathrm{gr}_\alpha(B)$ has the structure of a graded $\mathrm{gr}_v(F)-$algebra.

For the rest of this section, we fix a valued $F-$division algebra (D, w), and denote its valuation ring (resp. its maximal ideal), by $\mathcal{O}_D$ (resp. M_D).

4.1 Example. Let V be a finite–dimensional $D-$vector space and let $\mathfrak{B} = (b_1, \dots, b_n)$ be a $D-$basis for V. Let $\gamma = (\gamma_1, \dots, \gamma_n) \in \Gamma^n$. It is an easy verification that the map

$$w_{\gamma,\mathfrak{B}} : \sum_{i=1}^{n} b_i x_i \mapsto \min_{1\leqslant i\leqslant n}(w(x_i) + \gamma_i), \qquad \text{for } x_1, \dots, x_n \in D,$$

is a w−value function on V. If $\gamma = (0,\dots,0) \in \Gamma^n$ then we write $w_{\mathfrak{B}}$ for $w_{\gamma,\mathfrak{B}}$.

A w−value function α on a finite–dimensional D−vector space V is called a *w−norm* if there exists a D−basis $\mathfrak{B}$ for V and an element $\gamma \in \Gamma^n$ such that $\alpha = w_{\gamma,\mathfrak{B}}$; the basis $\mathfrak{B}$ is then called *a splitting basis for α*.

4.2 Remark. Note that, if $\Gamma = w(D^\times)$, then any D−basis of V can be scaled by elements of D such that the basis elements have value zero with respect to a given w−value function. In that case, given a w−norm α on V, there always exists a splitting basis for α whose elements have value zero.

4.3 Proposition. *Let $\mathcal{V}$ be a finite–dimensional right $\mathcal{O}_D$−module and let $V = \mathcal{V} \otimes_{\mathcal{O}_D} D$. Let $\mathfrak{B}$ and $\mathfrak{B}'$ be different $\mathcal{O}_D$−bases for $\mathcal{V}$. Then $w_{\mathfrak{B}} = w_{\mathfrak{B}'}$.*

Proof. Since $V = \mathcal{V} \otimes_{\mathcal{O}_D} D$, any $\mathcal{O}_D$−basis of $\mathcal{V}$ is a right D−basis of V. It follows that $V^{\geqslant 0}_{w_{\mathfrak{B}}} = V^{\geqslant 0}_{w_{\mathfrak{B}'}} = \mathcal{V}$. Using the matrix of base change from $\mathfrak{B}$ to $\mathfrak{B}'$ (whose entries lie in $\mathcal{O}_D$), one easily obtains that $w_{\mathfrak{B}}(x) \geqslant w_{\mathfrak{B}'}(x)$ for all $x \in V$. Interchanging the roles of $\mathfrak{B}$ and $\mathfrak{B}'$ yields the other inequality. □

4.4 Proposition. *Let V be a finite–dimensional D−vector space.*

(a) Let W be a nonzero D−subspace of V. Any w−norm on V restricts to a w−norm on W.

(b) Let $\mathfrak{B} = (b_1,\dots,b_n)$ be a D−basis for V. Let $b = \sum_{i=1}^n b_i x_i$ be a nonzero element of V. Let $j \in \{1,\dots,n\}$ be such that $w_{\mathfrak{B}}(b) = w(x_j)$ and let $\mathfrak{B}'$ be the family obtained from $\mathfrak{B}$ by replacing b_j by b if $w_{\mathfrak{B}}(b) = 0$, and by bx_j^{-1} if $w_{\mathfrak{B}}(b) \neq 0$. Then $\mathfrak{B}'$ is also a splitting basis for $w_{\mathfrak{B}}$.

Proof. See [62, (2.5)] for (a) and [62, (2.3) (iii)] for (b). □

4.5 Corollary. *Let V be a finite–dimensional D−vector space. Let $\mathfrak{B} = (b_1,\dots,b_n)$ be a D−basis for V. Let $x \in V$ be such that there exists an index $i \in \{1,\dots,n\}$ such that $x - b_i \in V^{\geqslant 0}_{w_{\mathfrak{B}}} M_D$. Let $\mathfrak{B}'$ be the family obtained from $\mathfrak{B}$ by replacing b_i by x. Then $\mathfrak{B}'$ is also a splitting basis for $w_{\mathfrak{B}}$.*

Proof. We have that $w_{\mathfrak{B}}(x - b_i) > 0$. Since $\mathfrak{B}$ is a splitting basis for $w_{\mathfrak{B}}$, we can write $x - b_i = \sum_{j=1}^n b_j x_j$, with $x_1,\dots,x_n \in M_D$. It follows that $w_{\mathfrak{B}}(x) = 0 = w_{\mathfrak{B}}(b_i(1 + x_i))$. Hence, by Proposition 4.4 (b), $\mathfrak{B}'$ is a splitting basis for $w_{\mathfrak{B}}$. □

Using value functions we obtain a different proof of Proposition 1.10 in the semilocal case.

4.6 Proposition. *Let T be a semilocal Bézout domain with fraction field F. Let $\mathcal{V}$ be a finite–dimensional T–module and let $V = \mathcal{V} \otimes_T F$. Let W be a nonzero F–subspace of V. Then $W \cap \mathcal{V}$ is free as a T–module and*

$$\dim_F(W) = \dim_T(W \cap \mathcal{V}).$$

Proof. By Proposition 1.3, there exist valuation rings $\mathcal{O}_1, \dots, \mathcal{O}_\ell$ of F such that $T = \mathcal{O}_1 \cap \dots \cap \mathcal{O}_\ell$, and we may assume that they are pairwise incomparable. Let $v_1, \dots, v_\ell$ be corresponding valuations on F. For $i = 1, \dots, \ell$, let $\mathfrak{M}_i$ be the unique maximal ideal of $\mathcal{O}_i$ and $\mathcal{M}_i = \mathfrak{M}_i \cap T$. By Proposition 1.4, we have that $\mathcal{O}_i = T_{\mathcal{M}_i}$ for $i = 1, \dots, \ell$ and $\mathcal{M}_1, \dots, \mathcal{M}_\ell$ are the different maximal ideals of T. Note that, for $i = 1, \dots, \ell$, $T/\mathcal{M}_i$ is naturally isomorphic to $T_{\mathcal{M}_i}/\mathcal{M}_i T_{\mathcal{M}_i} = \mathcal{O}_i/\mathfrak{M}_i$ via $a \mod \mathcal{M}_i \mapsto \frac{a}{1} \mod \mathcal{M}_i T_{\mathcal{M}_i}$.

For $i = 1, \dots, \ell$, we have that $\mathcal{V}_i = \mathcal{V}\mathcal{O}_i \subset V$. Let $\mathfrak{B} = (e_1, \dots, e_n)$ be a T–basis for $\mathcal{V}$. Then $\mathfrak{B}$ is an $\mathcal{O}_i$–basis for $\mathcal{V}_i$ for $i = 1, \dots, \ell$. For $i = 1, \dots, \ell$, we consider the v_i–norm $\alpha_i = (v_i)_{\mathfrak{B}}$ on $V = \mathcal{V}_i F$. We have that $V_{\alpha_i}^{\geqslant 0} = \mathcal{V}_i$. Let W be a nonzero F–subspace of V and let $\mathcal{W} = W \cap \mathcal{V}$. By Proposition 4.4 (a), $\alpha_i|_W$ is a v_i–norm for $i = 1, \dots, \ell$. Let $(d_1^i, \dots, d_r^i)$ be a splitting basis for $\alpha_i|_W$. We prove that there is a common splitting basis for $\alpha_1|_W, \dots, \alpha_\ell|_W$.

Since $\mathcal{M}_1, \dots, \mathcal{M}_\ell$ are pairwise different maximal ideals of T, they are pairwise coprime. By the Chinese Remainder Theorem, the natural isomorphisms $T/\mathcal{M}_i \to \mathcal{O}_i/\mathfrak{M}_i$ for $i = 1, \dots, \ell$, and [49, (XVI.2.7)], the T–homomorphism

$$\varphi : \mathcal{W} \to \mathcal{W}\mathcal{O}_1/\mathcal{W}\mathfrak{M}_1 \times \dots \times \mathcal{W}\mathcal{O}_\ell/\mathcal{W}\mathfrak{M}_\ell$$

is surjective. We show that $\mathcal{W}\mathcal{O}_i = W_{\alpha_i}^{\geqslant 0}$ for $i = 1, \dots, \ell$. It is clear that $\mathcal{W}\mathcal{O}_i \subset W_{\alpha_i}^{\geqslant 0}$. Conversely, let $x \in W_{\alpha_i}^{\geqslant 0}$. Then there exists $t \in T \setminus \mathcal{M}_i$ such that $xt \in \mathcal{W}$. Since $1/t \in T_{\mathcal{M}_i} = \mathcal{O}_i$, it follows that $x \in \mathcal{W}\mathcal{O}_i$. Similarly, one obtains $\mathcal{W}\mathfrak{M}_i = W_{\alpha_i}^{>0}$. By the surjectivity of φ, there exist $f_1, \dots, f_r \in \mathcal{W}$ such that

$$\varphi(f_j) = (\overline{d_j^1}, \dots, \overline{d_j^\ell}) \in \prod_{i=1}^{\ell} W_{\alpha_i}^{\geqslant 0}/W_{\alpha_i}^{>0}.$$

By Corollary 4.5, $(f_1, \dots, f_r)$ is a splitting basis for $\alpha_1|_W, \dots, \alpha_\ell|_W$. It follows that

$$\mathcal{W} \subset W_{\alpha_1}^{\geqslant 0} \cap \dots \cap W_{\alpha_\ell}^{\geqslant 0} = \left\{ \sum_{j=1}^{r} f_j x_j \mid x_1, \dots, x_\ell \in \mathcal{O}_1 \cap \dots \cap \mathcal{O}_\ell \right\} \subset \mathcal{W},$$

since $T = \mathcal{O}_1 \cap \dots \cap \mathcal{O}_\ell$. Hence, $\mathcal{W}$ is free over T with $(f_1, \dots, f_r)$ as a basis, which yields the statement. □

In the previous results we only used value functions on vector spaces. In the last result of this section we prove the existence of a certain gauge on an algebra with involution over the fraction field of a valuation ring. We will use this result in section 4.3 when we consider Azumaya algebras with involution under specialisation with respect to a Henselian valuation ring.

Suppose that $D = F$. Then we write $v = w, \mathcal{O} = \mathcal{O}_D$ and $\mathfrak{m} = M_D$. Let B be a finite–dimensional F–algebra and α a surmultiplicative v–norm on B. Then α is said to be a *v–gauge* if $\mathrm{gr}_\alpha(B)$ is *graded semisimple*, i.e. $\mathrm{gr}_\alpha(B)$ does not contain any nonzero nilpotent homogeneous two–sided ideals.
Let (B, τ) be an F–algebra with involution. A v–gauge α on B is called *τ–invariant* if $\alpha(\tau(x)) = \alpha(x)$ for all $x \in B$.

4.7 Proposition. *Let $(\mathcal{A}, \sigma)$ be an $\mathcal{O}$–algebra with involution. Then there exists a unique σ_F–invariant v–gauge α on $\mathcal{A}_F$ such that $(\mathcal{A}_F)_\alpha^{\geqslant 0} = \mathcal{A}$ and $(\mathcal{A}_F)_\alpha^{>0} = \mathfrak{m}\mathcal{A}$.*

Proof. We denote the residue field of $\mathcal{O}$ by κ. We have that $\mathcal{A}_\kappa \cong \mathcal{A}/\mathfrak{m}\mathcal{A}$ and we will work with the residue algebra in the latter form. By Corollary 1.19, $\mathcal{A}$ is free over $\mathcal{O}$. Let $\mathfrak{B} = (e_1, \dots, e_n)$ be an $\mathcal{O}$–basis for $\mathcal{A}$. Then $\mathfrak{B}$ is an F–basis for $\mathcal{A}_F$. It is clear that $(\mathcal{A}_F)_{v_\mathfrak{B}}^{\geqslant 0} = \mathcal{A}$ and $(\mathcal{A}_F)_{v_\mathfrak{B}}^{>0} = \mathfrak{m}\mathcal{A}$. Furthermore, any other v–norm β on $\mathcal{A}_F$ such that $(\mathcal{A}_F)_\beta^{\geqslant 0} = \mathcal{A}$ and $(\mathcal{A}_F)_\beta^{>0} = \mathfrak{m}\mathcal{A}$ must be equal to $v_\mathfrak{B}$. This can be seen as follows. Let $x \in \mathcal{A}_F$ be a non–zero element. By Proposition 2.46, there exists $r \in F$ such that $xr \in \mathcal{A} \setminus \mathfrak{m}\mathcal{A}$. It follows that $\beta(xr) = 0$, and hence, $\beta(x) = -v(r) \in v(F^\times)$. So, β has the same value group as v, and it follows from Remark 4.2 and Proposition 4.3 that $\beta = v_\mathfrak{B}$.

We show that $v_\mathfrak{B}$ is a v–gauge and that it is σ_F–invariant. Since $1 \in \mathcal{A}$, we have that $v_\mathfrak{B}(1) \geqslant 0$ and hence $v_\mathfrak{B}(1) = 0$ since $\mathcal{A} \neq \mathfrak{m}\mathcal{A}$. In order to show that $v_\mathfrak{B}$ is surmultiplicative, by [73, (1.2)], it suffices to show that $v_\mathfrak{B}(e_i e_j) \geqslant v_\mathfrak{B}(e_i) + v_\mathfrak{B}(e_j) = 0$ for all $i, j \in \{1, \dots, n\}$. Since $\mathcal{A} = (\mathcal{A}_F)_{v_\mathfrak{B}}^{\geqslant 0}$ is multiplicatively closed, this is clearly satisfied.

We next verify that $v_\mathfrak{B}$ is σ_F–invariant. Let $i \in \{1, \dots, n\}$. There exist $d_{i1}, \dots, d_{in} \in \mathcal{O}$ such that $\sigma(e_i) = \sum_{k=1}^n e_k d_{ik}$. Let $(x_1, \dots, x_n) \in F^n$ be arbitrary. Then for $k = 1, \dots, n$ we have that $v(\sum_{i=1}^n x_i d_{ik}) \geqslant \min_{1 \leqslant i \leqslant n}(v(x_i))$. We have that $\sigma_F(\sum_{i=1}^n e_i x_i) = \sum_{i=1}^n \sigma(e_i) x_i = \sum_{k=1}^n e_k (\sum_{i=1}^n x_i d_{ik})$, and hence

$$v_\mathfrak{B}\left(\sigma_F\left(\sum_{i=1}^n e_i x_i\right)\right) = \min_{1 \leqslant k \leqslant n}\left(v\left(\sum_{i=1}^n x_i d_{ik}\right)\right) \geqslant \min_{1 \leqslant i \leqslant n}(v(x_i)) = v_\mathfrak{B}\left(\sum_{i=1}^n e_i x_i\right).$$

This yields that $v_\mathfrak{B}(x) = v_\mathfrak{B}(\sigma_F^2(x)) \geqslant v_\mathfrak{B}(\sigma_F(x)) \geqslant v_\mathfrak{B}(x)$, for all $x \in \mathcal{A}_F$. This proves the σ_F–invariance of $v_\mathfrak{B}$.

In order to have that $v_{\mathfrak{B}}$ is a v–gauge, all that remains to be shown is that the graded algebra $\mathrm{gr}_{v_{\mathfrak{B}}}(\mathcal{A}_F)$ is semisimple. Suppose for the sake of contradiction that $\mathrm{gr}_{v_{\mathfrak{B}}}(\mathcal{A}_F)$ contains a nonzero homogeneous two–sided nilpotent ideal I. Let $I_0 = I \cap \mathrm{gr}_{v_{\mathfrak{B}}}(\mathcal{A}_F)_0 = I \cap \mathcal{A}/\mathfrak{m}\mathcal{A}$. For a nonzero $x \in B$, we write $\tilde{x} = x + (\mathcal{A}_F)_{v_{\mathfrak{B}}}^{>v_{\mathfrak{B}}(x)}$. Let a be a nonzero element of $\mathcal{A}_F$ such that $\tilde{a} \in I$. Since $v_{\mathfrak{B}}$ has the same value group as v, there exists $u \in F$ such that $v_{\mathfrak{B}}(a) = -v(u)$. Then $v_{\mathfrak{B}}(au) = v_{\mathfrak{B}}(a) + v(u) = 0$ and since I is an ideal of $\mathrm{gr}_{v_{\mathfrak{B}}}(\mathcal{A}_F)$, we have that $\tilde{a}\tilde{u} \in I$. Furthermore, since $v_{\mathfrak{B}}(au) = v_{\mathfrak{B}}(a) + v_{\mathfrak{B}}(u)$, we have that $\tilde{a}\tilde{u} = \widetilde{au}$. Since $\widetilde{au} \neq 0$, this implies that $I_0 \neq 0$. We have that I_0 is a proper nilpotent two–sided ideal of the semisimple algebra $\mathcal{A}/\mathfrak{m}\mathcal{A}$, we get a contradiction. □

4.2 Specialisation and the index

In this section we fix a field L and a place $\lambda : F \to L^\infty$. Let $\mathcal{O}$ be the valuation ring corresponding to λ. Let (V, b) be a non–singular symmetric bilinear space over $\mathcal{O}$. If $2 \in \mathcal{O}^\times$ (i.e. $\mathrm{char}(L) \neq 2$) then the result in [66, (4.6.2)] says that if $(V, b)_F$ is isotropic, then (V, b) is isotropic, and hence, $(V, b)_L$ is also isotropic. We present an involution analogue of the result that isotropy is preserved under λ, without restrictions on $\mathrm{char}(L)$.

Let $(\mathcal{A}, \sigma)$ be an $\mathcal{O}$–algebra with involution. By Proposition 1.16 we have that $(\mathcal{A}, \sigma)_F$ is an F–algebra with involution and $(\mathcal{A}, \sigma)_L$ is an L–algebra with involution. We show below that balanced ideals of $(\mathcal{A}, \sigma)_F$ specialise in an appropriate way under λ. In the proof we apply the general results for modules from section 4.1 to right ideals in $(\mathcal{A}, \sigma)_F$.

4.8 Theorem. *Let $\mathcal{A}$ be an Azumaya algebra with center either $\mathcal{O}$ or a separable quadratic $\mathcal{O}$–algebra. Then $\deg(\mathcal{A}_F) = \deg(\mathcal{A}_L)$. Let I be a balanced right ideal of $\mathcal{A}_F$. Then $I \cap \mathcal{A}$ is free as a $Z(\mathcal{A})$–module and $(I \cap \mathcal{A}) \otimes_{\mathcal{O}} L$ is a balanced right ideal of $\mathcal{A}_L$ of the same reduced dimension as I.*

Proof. Since $\mathcal{A}$ is free as an $\mathcal{O}$–module by Corollary 1.19, we have that $\dim_F(\mathcal{A}_F) = \dim_{\mathcal{O}}(\mathcal{A}) = \dim_L(\mathcal{A}_L)$. This clearly implies that $\deg(\mathcal{A}_F) = \deg(\mathcal{A}_L)$.

Let I be a balanced right ideal of $\mathcal{A}_F$. If $I = 0$, then there is nothing to prove. So, in the rest of the proof, we may assume $I \neq 0$. We write $S = Z(\mathcal{A})$. It is clear that $(I \cap \mathcal{A}) \otimes_{\mathcal{O}} L$ is a right ideal of $\mathcal{A}_L$. Suppose first that $Z(\mathcal{A}_F) = S \otimes_{\mathcal{O}} F$ is a field. Then S is a domain and it is the integral closure of $\mathcal{O}$ in $Z(\mathcal{A}_F)$, by Proposition 1.14 (b), and by Proposition 1.22, S is either a valuation ring or the intersection of two valuation rings of $Z(\mathcal{A}_F)$. Furthermore, $\mathcal{A}$ is free as an S–module by Corollary 1.19. We have that $\mathcal{A} \otimes_S Z(\mathcal{A}_F) \cong \mathcal{A} \otimes_{\mathcal{O}} F$ as $Z(\mathcal{A}_F)$–modules. Since I is a $Z(\mathcal{A}_F)$–subspace of $\mathcal{A}_F$, we can apply Proposition 4.6 to obtain that $I \cap \mathcal{A}$ is free as an S-module and

$$\dim_S(I \cap \mathcal{A}) = \dim_{Z(\mathcal{A}_F)}(I).$$

Assume that $Z(\mathcal{A}_F)$ is not a field. By Proposition 1.18, there exists an Azumaya algebra $\mathcal{B}$ over $\mathcal{O}$ such that $\mathcal{A} \cong \mathcal{B} \times \mathcal{B}^{\mathrm{op}}$. Then $\mathcal{B}$ is free as an $\mathcal{O}$-module by Corollary 1.19. We have that $\mathcal{A}_F \cong (\mathcal{B} \otimes_{\mathcal{O}} F) \times (\mathcal{B} \otimes_{\mathcal{O}} F)^{\mathrm{op}}$. Under this isomorphism, I corresponds to a right ideal $I_1 \times I_2^{\mathrm{op}}$ of $(\mathcal{B} \otimes_{\mathcal{O}} F) \times (\mathcal{B} \otimes_{\mathcal{O}} F)^{\mathrm{op}}$, where I_1 is a right ideal of $\mathcal{B} \otimes_{\mathcal{O}} F$ and I_2 a left ideal of $\mathcal{B} \otimes_{\mathcal{O}} F$, and we identify I with $I_1 \times I_2^{\mathrm{op}}$ under this isomorphism. Then $I \cap \mathcal{A} = (I_1 \cap \mathcal{B}) \times (I_2 \cap \mathcal{B})^{\mathrm{op}}$. Since I is balanced, Lemma 1.38 yields that $\dim_F I_1 = \dim_F I_2$. By Proposition 4.6, we have that $I_1 \cap \mathcal{B}$ and $I_2 \cap \mathcal{B}$ are free as $\mathcal{O}$-modules and

$$\dim_{\mathcal{O}}(I_1 \cap \mathcal{B}) = \dim_F I_1 = \dim_F I_2 = \dim_{\mathcal{O}}(I_2 \cap \mathcal{B}).$$

Applying Lemma 1.38 to $T = \mathcal{O}$ yields that $I \cap \mathcal{A}$ is a free S-module and $\dim_S(I \cap \mathcal{A}) = \dim_{Z(\mathcal{A}_F)}(I)$.

We have the following isomorphisms of $Z(\mathcal{A}_L)$-modules:

$$(I \cap \mathcal{A}) \otimes_S Z(\mathcal{A}_L) \cong ((I \cap \mathcal{A}) \otimes_S S) \otimes_{\mathcal{O}} L \cong (I \cap \mathcal{A}) \otimes_{\mathcal{O}} L.$$

Since $I \cap \mathcal{A}$ is free as an S-module, it follows that $(I \cap \mathcal{A}) \otimes_{\mathcal{O}} L$ is free as a $Z(\mathcal{A}_L)$-module. In other words, $(I \cap \mathcal{A}) \otimes_{\mathcal{O}} L$ is a balanced right ideal of $\mathcal{A}_L$.
It remains to verify the claim about the reduced dimensions. It follows from the above that

$$\dim_{Z(\mathcal{A}_F)} I = \dim_S(I \cap \mathcal{A}) = \dim_{Z(\mathcal{A}_L)}[(I \cap \mathcal{A}) \otimes_{\mathcal{O}} L]$$

Dividing each term by $\deg(\mathcal{A}_F) = \deg(\mathcal{A}_L)$ yields the statement. □

Adding isotropy data to Theorem 4.8, we obtain the following result.

4.9 Theorem. *Let $(\mathcal{A}, \sigma)$ be an $\mathcal{O}$-algebra with involution. Let I be an isotropic balanced right ideal of $(\mathcal{A}, \sigma)_F$. Then $I \cap \mathcal{A}$ is free as a $Z(\mathcal{A})$-module and $(I \cap \mathcal{A}) \otimes_{\mathcal{O}} L$ is an isotropic balanced right ideal of $(\mathcal{A}, \sigma)_L$ of the same reduced dimension as I.*

Proof. It is clear that if I is an isotropic right ideal of $(\mathcal{A}, \sigma)_F$, then $(I \cap \mathcal{A}) \otimes_{\mathcal{O}} L$ is an isotropic right ideal of $(\mathcal{A}, \sigma)_L$. The rest of the statement follows from Theorem 4.8. □

The following corollary is now immediate.

4.10 Corollary. *Let $(\mathcal{A}, \sigma)$ be an $\mathcal{O}$-algebra with involution. If σ_F is isotropic (resp. metabolic) then σ_L is isotropic (resp. metabolic) as well.*

We recast the result of Theorem 4.9 in terms of the index, obtaining a generalisation of Proposition 3.9.

4.11 Corollary. *Let $(\mathcal{A},\sigma)$ be an $\mathcal{O}$ −algebra with involution. Then*

$$\operatorname{ind}((\mathcal{A},\sigma)_F) \subset \operatorname{ind}((\mathcal{A},\sigma)_L).$$

Proof. Let $0 \neq i \in \operatorname{ind}((\mathcal{A},\sigma)_F)$ and let I be an isotropic balanced right ideal of $(\mathcal{A},\sigma)_F$ of reduced dimension i. By Theorem 4.9, $(I \cap \mathcal{A}) \otimes_{\mathcal{O}} L$ is an isotropic balanded right ideal of $(\mathcal{A},\sigma)_L$ and $\operatorname{rdim}[(I \cap \mathcal{A}) \otimes_{\mathcal{O}} L] = i$. It follows that $i \in \operatorname{ind}((\mathcal{A},\sigma)_L)$. □

4.3 Henselian valuation rings

Throughout section 4.3, we fix a valuation ring $\mathcal{O}$ of F. We denote its maximal ideal by $\mathfrak{m}$, its residue field by κ, and its value group by Γ. We further fix a valuation v on F with valuation ring $\mathcal{O}$.
The valuation ring $\mathcal{O}$ is called *Henselian* if it extends uniquely to a valuation ring of any separable closure of F. We say that v is Henselian if $\mathcal{O}$ is Henselian.

Although this section is centered around Henselian valuation rings, we do not assume a priori that $\mathcal{O}$ is Henselian, since we also want formulate some results for general valuation rings. Wherever $\mathcal{O}$ is assumed to be Henselian, we will say this explicitly.

In Corollary 4.10, we obtained a "going down" result for $\mathcal{O}$ −algebras with involution, namely isotropy (resp. metabolicity) of the involution over F yields isotropy (resp. metabolicity) over κ, which is an analogue of a result for symmetric bilinear spaces. When working with objects defined over a Henselian valuation ring, it is known that some properties of the object can be lifted from κ to F. For example, if $\mathcal{O}$ is Henselian and $2 \in \mathcal{O}^\times$, there is a "going up" result for isotropy (resp. hyperbolicity) for symmetric bilinear spaces over $\mathcal{O}$. This can be found in [66, (6.2.4)], where the statement assumes that $\mathcal{O}$ is a discrete valuation ring, but the proof does not. As a consequence of the latter result, isometry can also be lifted from κ to F. In section 4.3.1, we consider analogues of these two results for $\mathcal{O}$ −algebras with involution. More precisely, we show in Theorem 4.20 that, given an $\mathcal{O}$ −algebra with involution $(\mathcal{A},\sigma)$ where $\mathcal{O}$ is Henselian and $2 \in \mathcal{O}^\times$, we have that $\operatorname{ind}((\mathcal{A},\sigma)_F) = \operatorname{ind}((\mathcal{A},\sigma)_\kappa)$. As a consequence, we obtain for $\mathcal{O}$ −algebras with involution a lifting result for isomorphism from κ to F. So, for Henselian valuation rings, the structures induced over F and κ by an $\mathcal{O}$ −algebra with involution behave similarly.

In section 4.3.2, using the equality of the indices of algebras with involution mentioned above, we will show that, if $\mathcal{O}$ is Henselian and $2 \in \mathcal{O}^\times$, then isomorphism of

$\mathcal{O}$−algebras with involution can be detected over F (Theorem 4.34). This will then imply a "going down" result for isomorphism of algebras with involution over general valuation rings in which 2 is a unit. This is an analogue of a result on isometry for symmetric billinear spaces.

We thank J.–P. Tignol for his suggestion to consider the result on the index of an algebra with involution in the Henselian case.

4.3.1 Lifting isotropy and hyperbolicity

4.12 Proposition. *Let $(\mathcal{A},\sigma)$ be an $\mathcal{O}$−algebra with involution with center a domain, which we denote by S. Denote the fraction field of S by K. Suppose that $\mathcal{O}$ extends uniquely to K. Then the following hold:*

- *(a) S is equal to the unique valuation ring of K extending $\mathcal{O}$, and its residue field is a separable quadratic extension of κ.*
- *(b) Suppose that $\mathcal{A}$ does not have zero divisors. Then $\mathcal{A}_F$ is a division algebra over K. Suppose that $\mathcal{O}$ extends to a valuation ring of $\mathcal{A}_F$. Then this valuation ring is equal to $\mathcal{A}$. Furthermore, the value groups of $\mathcal{O}$ and $\mathcal{A}$ are equal.*

Proof. Since $\mathcal{O}$ extends uniquely to K, Proposition 1.22 yields that S is equal to this valuation ring, and its residue field is a separable quadratic extension of κ. This proves (a).
Suppose that $\mathcal{A}$ does not have zero divisors. Then $\mathcal{A}_F$ does not have zero divisors either. Since $\mathcal{A}_F$ is a central simple K−algebra by Proposition 1.11 (a) and (b), $\mathcal{A}_F$ is a division algebra. By (a), S is a valuation ring and by assumption, $\mathcal{O}$ (and hence also S) extends to a valuation ring V_A of $\mathcal{A}_F$. We denote the residue field of S by κ_S. Note that the value group of S is equal to Γ by Proposition 1.22. It follows from [34, (2.5)] that $V_A = \mathcal{A}$ (and in this case, the proof of [34, (2.5)] in fact simplifies). We denote the value group of $\mathcal{A}$ by $\Gamma_{\mathcal{A}}$. The fact that $\Gamma = \Gamma_{\mathcal{A}}$ now follows from the fundamental inequality $\dim_K(\mathcal{A}_F) \geqslant \dim_{\kappa_S}(\mathcal{A}_\kappa)[\Gamma_{\mathcal{A}} : \Gamma]$ (see e.g. [75, p.9]). □

4.13 Corollary. *Suppose that $\mathcal{O}$ is Henselian. Let $(\mathcal{A},\sigma)$ be an $\mathcal{O}$−algebra with involution with center a domain, which we denote by S. Denote the fraction field of S by K. Then there exists an Azumaya algebra Δ over S without zero divisors, that is moreover a valuation ring of $D = \Delta \otimes_{\mathcal{O}} F$, an $\mathcal{O}$−linear involution θ on Δ of the same kind as σ, and an ε−hermitian space (V,h) over (Δ,θ), with $\varepsilon = \pm 1$, such $(\mathcal{A},\sigma) \cong_S \mathrm{Ad}(h)$. Furthermore, the value groups of Δ and $\mathcal{O}$ are equal.*

Proof. By Proposition 2.10, there exists an Azumaya algebra Δ over S without zero divisors, an involution θ of the same kind as σ, and an ε−hermitian space (V,h) over (Δ,θ), with $\varepsilon = \pm 1$, such that $(\mathcal{A}, \sigma) \cong_{\mathcal{O}} (\mathrm{End}_\Delta(V), \mathrm{ad}_h) = \mathrm{Ad}(h)$. Since $\mathcal{O}$ is Henselian, it extends to a valuation ring of D by [75, (2.1)]. By Proposition 4.12, this valuation ring is necessarily equal to Δ, and Δ and $\mathcal{O}$ have equal value groups. □

In order to prove the main result of this section (Theorem 4.20), we use the characterisation, given by Corollary 2.17, of the index of a non–degenerate algebra with involution over a field in terms of a (skew–)hermitian space to which it is adjoint. To this end, we need to prove equality of the Schur indices of the algebra over F and κ, and equality of the Witt indices of the (skew–)hermitian space over F and κ. So, we will jump back and forth a bit between results on involutions and results on (skew–)hermitian spaces. In this way, we also obtain a lifting result for isotropy (resp. hyperbolicity) for (skew–)hermitian spaces (see Corollary 4.18).

We first show the equality of the Schur indices, and this will already imply the desired lifting result for isotropy and hyperbolicity for degenerate $\mathcal{O}$–algebras with involution.

4.14 Proposition. *Suppose that $\mathcal{O}$ is Henselian. Let $\mathcal{A}$ be an Azumaya algebra with center $\mathcal{O}$ or a separable quadratic $\mathcal{O}$–algebra. Then $\mathcal{A}_F$ is simple if and only if $\mathcal{A}_\kappa$ is simple.*

Proof. We denote the center of $\mathcal{A}$ by S. We have that $\mathcal{A}_F$ (resp. $\mathcal{A}_\kappa$) is simple if and only if $Z(\mathcal{A}_F)$ (resp. $Z(\mathcal{A}_\kappa)$) is a field. If $S = \mathcal{O}$ then $\mathcal{A}_F$ and $\mathcal{A}_\kappa$ are both simple. So, suppose that $S \neq \mathcal{O}$. If S is not a domain then $S \cong \mathcal{O} \times \mathcal{O}$ by Proposition 1.14, and none of $\mathcal{A}_F, \mathcal{A}_\kappa$ is simple. Suppose that S is a domain different from $\mathcal{O}$. Then $Z(\mathcal{A}_F)$ is a field by Proposition 1.14 (b). Furthermore, S is a valuation ring of K by Proposition 4.12. Since $Z(\mathcal{A}_\kappa) = S \otimes_{\mathcal{O}} \kappa \cong S/\mathfrak{m}S$, it follows from Proposition 1.22 that $Z(\mathcal{A}_\kappa)$ is a field. Hence, $\mathcal{A}_F$ and $\mathcal{A}_\kappa$ are both simple. □

4.15 Proposition. *Suppose that $\mathcal{O}$ is Henselian. Let $\mathcal{A}$ be an Azumaya algebra with center $\mathcal{O}$ or a separable quadratic $\mathcal{O}$–algebra that is a domain. Then* $\mathrm{ind}(\mathcal{A}_F) = \mathrm{ind}(\mathcal{A}_\kappa)$.

Proof. Since $Z(\mathcal{A})$ is a domain, it follows from Proposition 4.14 that $\mathcal{A}_F$ and $\mathcal{A}_\kappa$ are both simple. The result can then be shown using that $\mathcal{A}$ is Brauer equivalent to a valuation ring Δ of a division algebra Brauer equivalent to $\mathcal{A}_F$, by Corollary 4.13, and that $\Delta_\kappa \cong \Delta/J(\Delta)$ is a division algebra. We also present a different argument, which does not use noncommutative valuation rings, but the results on right ideals we obtained earlier in this chapter.

The Schur index of a central simple algebra can be characterised as the reduced dimension of a minimal right ideal. Let I be a minimal right ideal of $\mathcal{A}_F$. Theorem 4.8 yields that $(I \cap \mathcal{A}) \otimes_{\mathcal{O}} \kappa$ is a right ideal of $\mathcal{A}_\kappa$ and $\mathrm{rdim}[(I \cap \mathcal{A}) \otimes_{\mathcal{O}} \kappa] = \mathrm{rdim}_F(I) = \mathrm{ind}(\mathcal{A}_F)$.

Since the reduced dimension of any right ideal of a central simple algebra is divisible by the Schur index of that algebra, we get that $\operatorname{ind}(\mathcal{A}_\kappa) \mid \operatorname{ind}(\mathcal{A}_F)$. Conversely, let $\overline{I}$ be a minimal right ideal of $\mathcal{A}_\kappa$. By [45, (1.13)]), there is an idempotent $\bar{x} \in \mathcal{A}_\kappa$ such that $\overline{I} = \overline{x}\mathcal{A}_\kappa$. By [50, (A.18)], we can lift the idempotent $\overline{x}$ to an idempotent $x \in \mathcal{A}$. Then $x\mathcal{A}_F$ is a right ideal of $\mathcal{A}_F$ and since x is idempotent, we have that $x\mathcal{A}_F \cap \mathcal{A} = x\mathcal{A}$ and $x\mathcal{A} \otimes_{\mathcal{O}} \kappa \cong \overline{x}\mathcal{A}_\kappa$. Again invoking Theorem 4.8, we get that $\operatorname{rdim}(x\mathcal{A}_F) = \operatorname{rdim}(\overline{x}\mathcal{A}_\kappa) = \operatorname{ind}(\mathcal{A}_\kappa)$, and hence $\operatorname{ind}(\mathcal{A}_F) \mid \operatorname{ind}(\mathcal{A}_\kappa)$. □

4.16 Corollary. *Suppose that $\mathcal{O}$ is Henselian. Let $(\mathcal{A},\sigma)$ be an $\mathcal{O}-$algebra with involution whose center is not a domain. Then* $\operatorname{ind}((\mathcal{A},\sigma)_F) = \operatorname{ind}((\mathcal{A},\sigma)_\kappa)$.

Proof. If $Z(\mathcal{A})$ is not a domain, then $Z(\mathcal{A}) \cong \mathcal{O}\times\mathcal{O}$ by Proposition 1.14 (b). By Proposition 1.18, there exists an Azumaya algebra $\mathcal{B}$ over $\mathcal{O}$ such that $(\mathcal{A},\sigma) \cong_{\mathcal{O}} (\mathcal{B}\times\mathcal{B}^{\mathrm{op}}, \mathrm{sw}_{\mathcal{B}})$. By Proposition 4.15, we have that $\operatorname{ind}(\mathcal{A}_F) = \operatorname{ind}(\mathcal{B}_F) = \operatorname{ind}(\mathcal{B}_\kappa) = \operatorname{ind}(\mathcal{A}_\kappa)$. Since $\deg(\mathcal{A}_F) = \deg(\mathcal{A}_\kappa)$ by Theorem 4.8, the equality of the indices now follows from Proposition 1.42. □

4.17 Proposition. *Suppose that $\mathcal{O}$ is Henselian. Let $(\mathcal{A},\sigma)$ be an $\mathcal{O}-$algebra with involution with center a domain. Furthermore, if* $\operatorname{char}(\kappa) = 2$, *suppose that σ_F is not orthogonal. Then σ_F is isotropic if and only σ_κ is isotropic.*

Proof. The statement follows from [74, (2.3)], provided that $\mathcal{A}_F$ is tame over F in the sense of [74, p. 121], and that there exists a σ_F-invariant $v-$gauge α on $\mathcal{A}_F$ such that $(\mathcal{A}_F)_\alpha^{\geqslant 0} = \mathcal{A}$ and $(\mathcal{A}_F)_\alpha^{>0} = \mathfrak{m}\mathcal{A}$. The tameness condition for $\mathcal{A}_F$ means that K/F is tame and $\mathcal{A}_F$ splits over the maximal tamely ramified extension of (K, v_S).

The existence of the gauge follows from Proposition 4.7. Let us prove the tameness condition. Let S and Δ be as in Corollary 4.13. Then Δ is a valuation ring of the division algebra $D = \Delta_F$. Let w be a valuation on D with valuation ring Δ and let v_S be the restriction of w to K. Then S is the valuation ring of v_S. By Proposition 4.12 (b), v, v_S and w have the same value group.
The fact that K/F is tame follows from the equality of the value groups of v and v_S, which even implies that (K, v_S) is an unramified extension of (F, v). Let L/K be maximal subfield of D. Then D splits over L. Furthermore, since the value groups of v and w are equal, the unique extension $\tilde{v}$ of v to L has the same value group as v. Hence, $(L,\tilde{v})$ is an unramified, and therefore also tamely ramified, extension of (K, v_S) splitting $\mathcal{A}_F$. It follows that $\mathcal{A}_F$ is tame over F. □

4.18 Corollary. *Suppose that $\mathcal{O}$ is Henselian. Let (Δ,θ) be an $\mathcal{O}-$algebra with involution without zero divisors. Let $\varepsilon = \pm 1$ and let (V, h) be an even $\varepsilon-$hermitian space over (Δ,θ). Then $i_w(h_F) = i_w(h_\kappa)$. In particular, h_F is isotropic (resp. hyperbolic) if and only if h_κ is isotropic (resp. hyperbolic).*

Proof. Let $(\mathcal{A},\sigma) = \mathrm{Ad}(h)$. Since (V,h) is an even ε−hermitian space, we have that σ_F is not orthogonal if $\mathrm{char}(\kappa) = 2$, by [45, (4.2)]. By Proposition 4.17, σ_F is isotropic if and only if σ_κ is isotropic. Proposition 2.14 yields that h_F is isotropic if and only if h_κ is isotropic. Suppose that h_F is not hyperbolic. Then, by Propositions 2.6 and 2.2, we may write

$$(V,h) \simeq (V_1,h_1) \perp (V_2,h_2),$$

with (V_1,h_1) (resp. (V_2,h_2)) an anisotropic (resp. hyperbolic) even ε−hermitian space over (Δ,θ). Since h_1 is anisotropic, Proposition 2.8 (a) yields that $(h_1)_F$ is also anisotropic. By the first part of the proof, we obtain that $(h_1)_\kappa$ is anisotropic. Therefore, $i_w(h_F) = i_w(h_\kappa)$ and the statement follows. □

4.19 Corollary. *Suppose that $\mathcal{O}$ is Henselian. Let $(\mathcal{A},\sigma)$ be an $\mathcal{O}$−algebra with involution with center a domain, and let σ' be an $\mathcal{O}$−linear involution on $\mathcal{A}$ of the same kind as σ. Assume that* $\mathrm{char}(\kappa) \neq 2$. *Then $(\mathcal{A},\sigma)_\kappa \cong_{Z(\mathcal{A}_\kappa)} (\mathcal{A},\sigma')_\kappa$ if and only if $(\mathcal{A},\sigma) \cong_{Z(\mathcal{A})} (\mathcal{A},\sigma')$.*

Proof. It is clear that $(\mathcal{A},\sigma) \cong_{Z(\mathcal{A})} (\mathcal{A},\sigma')$ implies $(\mathcal{A},\sigma)_\kappa \cong_{Z(\mathcal{A}_\kappa)} (\mathcal{A},\sigma')_\kappa$. Let us prove the converse. Suppose first that $Z(\mathcal{A})$ is a domain. Let (Δ,θ) and (V,h) be as in Corollary 4.13 such that $(\mathcal{A},\sigma) \cong \mathrm{Ad}(h)$. We identify $(\mathcal{A},\sigma)$ and $\mathrm{Ad}(h)$ through this isomorphism. Note that (V,h) is even, since $\mathrm{char}(\kappa) \neq 2$ by assumption. By Proposition 1.23, there exists $s \in \mathcal{A}^\times$ such that $\sigma' = \mathrm{Int}(s) \circ \sigma$ and $\sigma(s) = \pm s$. Since σ_κ and σ'_κ are isomorphic by assumption, and $\mathrm{char}(\kappa) \neq 2$, the result in [45, (2.7) (3)] says that in fact $\sigma(s) = s$. Then $(\mathcal{A},\sigma') = \mathrm{Ad}(h')$, where $h' : V \times V \to \Delta$ is defined by $h'(x,y) = h(s^{-1}(x),y)$ for all $x,y \in V$. By Proposition 2.19, since $\mathrm{Ad}(h_\kappa) \cong_{Z(\mathcal{A}_\kappa)} \mathrm{Ad}(h'_\kappa)$, there is an element $a \in \mathcal{O}^\times$ such that $(V,h')_\kappa \simeq (V,ah)_\kappa$. Then $h' \perp -ah$ is an ε−hermitian space over (Δ,θ) that becomes hyperbolic over κ, and hence becomes hyperbolic over F by Corollary 4.18. Then $h' \perp -ah$ is already hyperbolic by Proposition 2.8 (a), and hence, $(V,h') \simeq (V,ah)$, by Proposition 2.8 (b). Proposition 2.19 now yields that $(\mathcal{A},\sigma) \cong_{Z(\mathcal{A})} (\mathcal{A},\sigma')$ by □

4.20 Theorem. *Suppose that $\mathcal{O}$ is Henselian. Let $(\mathcal{A},\sigma)$ be an $\mathcal{O}$−algebra with involution. Assume that σ_F is not orthogonal if* $\mathrm{char}(\kappa) = 2$. *Then*

$$\mathrm{ind}((\mathcal{A},\sigma)_F) = \mathrm{ind}((\mathcal{A},\sigma)_\kappa).$$

In particular, σ_F is isotropic (resp. hyperbolic) if and only if σ_κ is isotropic (resp. hyperbolic).

Proof. If $Z(\mathcal{A})$ is not a domain, this is the statement of Corollary 4.16. So, for the rest of the proof we may assume that $Z(\mathcal{A})$ is a domain. Let (Δ,θ) and (V,h) be as in Corollary 4.13. Since $\mathrm{ind}(D) = \mathrm{ind}(\Delta_F) = \mathrm{ind}(\Delta_\kappa)$ by Proposition 4.15, we have that Δ_κ is a division algebra. In order to prove the statement, by Corollary 2.17, it suffices to show that $\operatorname{ind}(\mathcal{A}_F) = \operatorname{ind}(\mathcal{A}_\kappa)$ and $i_w(h_F) = i_w(h_\kappa)$. The equality of the Schur indices follows from Proposition 4.15 and the equality of the Witt indices from Corollary 4.18. □

4.21 Remark. In order to prove Theorem 4.20, we lift the numbers in the index over the residue field to the fraction field. One could also ask whether isotropic balanced right ideals can be lifted explicitly. As noted in the proof of Proposition 4.15, idempotents can be lifted from $\mathcal{A}_\kappa$ to $\mathcal{A}$. However, we don't see how one can lift an isotropic idempotent of $\mathcal{A}_\kappa$ to an isotropic idempotent of $\mathcal{A}$.

The lifting result for isotropy and hyperbolicity for symmetric bilinear spaces over Henselian valuation rings, mentioned in the beginning of section 4.3, holds in fact more generally for 2−Henselian valuation rings. The valuation ring $\mathcal{O}$ is called 2−*Henselian* if it extends uniquely to a valuation ring of $F(2)$, the maximal Galois 2−extension of F, by which we mean the compositum of all finite Galois extensions of F of 2−power degree inside a fixed (but arbitrary) separable closure of F. In view of the results for symmetric bilinear spaces in the 2−Henselian case, it is natural to ask the following question.

4.22 Question. Does the equality of the indices given in Theorem 4.20 still hold if $\mathcal{O}$ is 2−Henselian?

4.23 Remark. It is not clear whether the method used in the Henselian case could also work in the 2−Henselian case. The problem is that we don't know whether a 2−Henselian valuation ring of F extends to a division algebra of 2−power degree with center F. The 2−Henselian property only guarantees that the valuation ring extends uniquely to Galois extensions of the center. An arbitrary subfield of the division algebra containing F has degree a power of 2 over F, but need not be contained in $F(2)$. If $\operatorname{char}(F) \neq 2$ and the division algebra has degree 2, then the maximal subfields of the division algebra are Galois extensions of degree 2 and hence, the 2−Henselian valuation ring extends uniquely to each maximal subfield, and therefore also to the division algebra by [75, (2.1)]. Using this, we do obtain a positive answer for the above question on 2−Henselian valuation rings in a particular case. We sketch this below.

Suppose that $\mathcal{O}$ is 2−Henselian and that $\operatorname{char}(F) \neq 2$. Let $(\mathcal{A}, \sigma)$ be an $\mathcal{O}$−algebra with involution such that $\operatorname{ind}(\mathcal{A}_F) = 2$, and such that σ_F is not orthogonal if $\operatorname{char}(\kappa) = 2$. Let Δ be an Azumaya algebra over $Z(\mathcal{A})$ without zero divisors, Brauer equivalent to $\mathcal{A}$. Then Δ_F is a division algebra of degree 2 Brauer equivalent to $\mathcal{A}_F$. As explained above, $\mathcal{O}$ extends to a valuation ring of D, and this valuation ring is then necessarily equal to Δ by Proposition 4.12. We looked at the proofs of the results of [62, 74] for Henselian valuation rings that are used in order to obtain the result of Proposition 4.17. In the case of algebras of Schur index 2, the Henselian assumption is used there in order to lift zeroes of polynomials of degree 2 from the residue field to the valuation ring, and to have that the extension of a Henselian valuation on F to a separable quadratic field extension is again Henselian. Both of these facts still hold in the case where $\mathcal{O}$ is 2−Henselian. Therefore, the proofs of [62, (4.6)] and [74, (2.2)] go through in this case, and hence, the statement of Proposition 4.17 still holds. Moreover, since Δ is a valuation ring, it follows that Δ_κ is a division algebra, and hence $\operatorname{ind}(\mathcal{A}_F) = \operatorname{ind}(\Delta_F) = \operatorname{ind}(\Delta_\kappa) = \operatorname{ind}(\mathcal{A}_\kappa)$. The proof of Theorem 4.20 then also goes through.

We can use Theorem 4.20 to prove a result on hyperbolicity of ε−hermitian spaces over Azumaya algebras with involution, under the extra assumption that the Azumaya algebra is a valuation ring. This result will be important the next section and in chapter 5, in order to obtain isomorphism results for Azumaya algebras with involution. We present two proofs of this hyperbolicity result.

4.24 Proposition. *Let (Δ, θ) be an $\mathcal{O}$−algebra with involution without zero divisors. Let $\varepsilon = \pm 1$ and let (V, h) be an ε−hermitian space over (Δ, θ). Suppose that there exists $x \in V_F$ such that $h_F(x, x) \notin F^{\times 2}(\Delta \setminus \mathfrak{m}\Delta)$. Then h_κ is isotropic.*

Proof. By Propositions 1.22 and 2.42, we have that $J(\Delta) = \mathfrak{m}\Delta$. By Proposition 2.4 (a), V is a free Δ−module. Let $\mathfrak{B} = (e_1, \dots, e_n)$ be a Δ−basis for V. Then $\mathfrak{B}$ is a Δ_F−basis for V_F. We write $x = \sum_{i=1}^n e_i x_i$, with $x_1, \dots, x_n \in D$. By Lemma 2.46, there exist $a_1, \dots, a_n \in F$ such that $a_i x_i \in \Delta \setminus J(\Delta) = \Delta \setminus \mathfrak{m}\Delta$. Without loss of generality, we may assume that $v(a_1) = \max_{1 \leqslant i \leqslant n}(v(a_i))$. Then $y = x a_1 \in V$. Since $(\bar{e}_1, \dots, \bar{e}_n)$ is a Δ_κ−basis for $V_\kappa \cong V/\mathfrak{m}V$, and $a_1 x_1 \notin \mathfrak{m}\Delta$, we have that $\bar{y} \in V/\mathfrak{m}V$ is nonzero. Furthermore, since $h_F(x, x) \notin F^{\times 2}(\Delta \setminus \mathfrak{m}\Delta)$, it follows that $h(y, y) \notin F^{\times 2}(\Delta \setminus \mathfrak{m}\Delta)$, and since $h(y, y) \in \Delta$, we get that $h(y, y) \in \mathfrak{m}\Delta$. This implies that $h_\kappa(\bar{y}, \bar{y}) = 0$, and since $\bar{y} \neq 0$, this proves the statement. □

4.25 Corollary. *Suppose that $\mathcal{O}$ is Henselian. Let (Δ, θ) be an $\mathcal{O}$−algebra with involution without zero divisors. Let $\varepsilon = \pm 1$ and let (V, h) and (V', h') be two even ε−hermitian spaces over (Δ, θ). Suppose that there exists a scalar $e \in \mathcal{O}$ such that $(V, eh)_F \simeq (V', h')_F$. If $e \notin F^{\times 2}\, \mathcal{O}^\times$ then h and h' are hyperbolic.*

Proof. If $\theta = \mathrm{id}_\Delta$ and $\varepsilon = -1$, then h and h' are hyperbolic by Proposition 2.6, since a skew–hermitian space over $(\Delta, \mathrm{id}_\Delta)$ is necessarily isotropic. So, for the rest of the proof, we assume that $\varepsilon = 1$ if $\theta = \mathrm{id}_\Delta$. Since $\mathcal{O}$ is Henselian, it extends uniquely to a valuation ring of the division algebra Δ_F by [75, (2.1)]. It follows from Proposition 4.12 (b) that Δ is a valuation ring of Δ_F. By Proposition 2.4 (a), V' is a free Δ−module. Let

$\mathfrak{B}' = (e_1', \ldots, e_n')$ be a Δ–basis for V'. Then $\mathfrak{B}'$ is a Δ_F–basis for V_F'. By assumption, there exists a bijective Δ_F–linear map $\varphi : V_F \to V_F'$ such that for all $x \in V_F$, we have that

$$eh_F(x,x) = h_F'(\varphi(x), \varphi(x)).$$

By Proposition 2.47, there exists $x \in V$ such that $h(x,x) \in \Delta \setminus J(\Delta) = \Delta \setminus \mathfrak{m}\Delta$. Since $e \notin F^{\times 2}\mathcal{O}^\times$, it follows that $eh(x,x) \notin F^{\times 2}(\Delta \setminus \mathfrak{m}\Delta)$. Proposition 4.24 then implies that h_κ' is isotropic.
Since $\mathcal{O}$ is Henselian, and we work with even ε–hermitian spaces, it follows from Corollary 4.18 that h_F' is isotropic as well. Suppose that h_F' is non–hyperbolic. Then h_F is non–hyperbolic as well, and by Proposition 2.6, we can decompose $(V,h) \simeq (V_1,h_1) \perp (V_2,h_2)$ and $(V',h') \simeq (V_1',h_1') \perp (V_2',h_2')$, where (V_1,h_1) and (V_1',h_1') are anisotropic even ε–hermitian spaces over (Δ,θ), and (V_2,h_2) and (V_2',h_2') hyperbolic ε–hermitian spaces over (Δ,θ). It follows that

$$(V_1',h_1')_F \perp (V_2',h_2')_F \simeq (V_1,eh_1)_F \perp (V_2,eh_2)_F.$$

We have that $(h_1')_F$ and $e(h_1)_F$ anisotropic by Proposition 2.8 (a), and $(h_2')_F$ and $e(h_2)_F$ hyperbolic. The Witt cancellation property for even ε–hermitian spaces over division rings (see [43, (I.6.4.5)]) together with Proposition 2.5 yields that

$$(V_1',h_1')_F \simeq (V_1,eh_1)_F.$$

However, the reasoning above now yields that $(h_1')_F$ is isotropic, a contradiction. Therefore, h_F' is hyperbolic, and then h_F is clearly hyperbolic as well. Proposition 2.8 (a) yields that h' and h are already hyperbolic. $\square$

J.–P. Tignol has suggested a different proof of Corollary 4.25. We thank him for his permission to include his proof in this dissertation. Corollary 4.18 is not used in the proof we give below, but is replaced by hyperbolicity results involving value functions in [62]. Therefore, we start with some more preliminaries on value functions, following [62].

For the rest of this section, we fix an $\mathcal{O}$–algebra with involution (Δ,θ) without zero divisors such that Δ is a valuation ring of the division algebra $D = \Delta_F$. We denote the center of D by K. Let w be a valuation on D with valuation ring Δ. We denote the value group of w by Γ_D. We consider Γ_D as subgroup of a divisible totally ordered abelian group Γ (e.g. one can take for Γ the divisible hull of Γ_D). By Lemma 2.44, we have that $w \circ \theta_F = w$. Then θ_F induces a well–defined graded involution $\tilde{\tau}$ on $\mathrm{gr}_w(D)$. Let $\varepsilon = \pm 1$ and let (V,h) be an ε–hermitian space over (Δ,θ). Recall that V is a free Δ–module by Proposition 2.4 (a). Let $\alpha : V_F \to \Gamma \cup \{\infty\}$ be a w–norm on V_F. The *dual norm* $\alpha^\#$ is defined by $\alpha^\#(x) = \min\{w(h(x,y)) - \alpha(y) \mid y \in V_F\}$. We say that α *is compatible with* h_F if for all $x,y \in V_F$, we have that $\alpha(x) + \alpha(y) \leqslant w(h(x,y))$ and if for each $x \in V_F$, there is an element $y \in V_F$ such that $\alpha(x) + \alpha(y) = w(h(x,y))$. In that case, $\alpha = \alpha^\#$ by [62, (3.5) (ii)]. For $z \in V_F$, we write $\tilde{z} = z + (V_F)_\alpha^{>\alpha(z)}$. Suppose that α is compatible with h_F. Then h_F induces a graded ε−hermitian form $(\tilde{h}_F)_\alpha : \mathrm{gr}_\alpha(V_F) \times \mathrm{gr}_\alpha(V_F) \to \mathrm{gr}_w(D)$ with respect to $\tilde{\tau}$, defined by

$$(\tilde{h}_F)_\alpha(\tilde{x},\tilde{y}) = \begin{cases} \widetilde{h_F(x,y)} & \text{if } w(h_F(x,y)) = \alpha(x) + \alpha(y) \\ 0 & \text{if } w(h_F(x,y)) > \alpha(x) + \alpha(y), \end{cases}$$

for all $x, y \in V_F$.

4.26 Proposition. *Let $\varepsilon = \pm 1$ and let (V,h) be an ε−hermitian space over (Δ,θ). Assume that V is a free Δ−module. Then there exists a w−norm α on V_F compatible with h_F.*

Proof. Let $\mathfrak{B} = (e_1,\ldots,e_n)$ be a Δ−basis for V. Then $\mathfrak{B}$ is a D−basis for V_F. Let $\mathfrak{B}^\#$ be the dual basis of Δ with respect to h. Consider the w−norm $w_{\mathfrak{B}}$ on V_F as defined in Example 4.1. Since $\mathfrak{B}$ is a splitting basis for $w_{\mathfrak{B}}$, [62, (3.4) (i)] yields that $\mathfrak{B}^\#$ is a splitting basis for the dual norm $w_{\mathfrak{B}}^\#$, and each basis element of $\mathfrak{B}^\#$ has value zero under $w_{\mathfrak{B}}^\#$. It follows from Proposition 4.3 that $w_{\mathfrak{B}} = w_{\mathfrak{B}^\#} = w_{\mathfrak{B}}^\#$. It follows from [62, (3.5) (ii)] that $w_{\mathfrak{B}}$ is compatible with h_F. □

4.27 Lemma. *Let $\varepsilon = \pm 1$ and let (V,h) be an ε−hermitian space over (Δ,θ). Let α be as in Proposition 4.26. Let $e \in K^\times$ and let $w(e) = \gamma \in \Gamma_D$. The map $\beta : V_F \to \Gamma \cup \{\infty\}$ defined by $\beta(x) = \alpha(x) + \frac{\gamma}{2}$ is a w−norm on V_F compatible with eh_F.*

Proof. We leave the easy verification that β is a w−norm on V_F to the reader. The compatibility with eh_F follows directly from the fact that α is compatible with h_F. □

For the actual hyperbolicity theorem, we restrict to the case where $2 \in \mathcal{O}^\times$.

4.28 Theorem. *Suppose that $2 \in \mathcal{O}^\times$. Let $\varepsilon = \pm 1$. Let (V,h) and (V',h') be ε−hermitian spaces over (Δ,θ). Assume that $(V',h')_F \simeq (V,eh)_F$, with $e \in K^\times$ such that $\gamma = w(e) \notin 2\Gamma_D$. Let α be as in Proposition 4.26 and β as in Proposition 4.27. Then $(\tilde{h'}_F)_\alpha$ and $(\widetilde{eh_F})_\beta$ are hyperbolic. If moreover $Z(\Delta)$ is a Henselian valuation ring of K, then h' and h are hyperbolic.*

Proof. The assumptions imply that $\dim_\Delta(V) = \dim_\Delta(V')$. Hence, there exists a Δ−linear bijection $\varphi : V' \to V$. We identify (V',h') with its image induced by φ, and in this way we consider h' as an ε−hermitian form over V.
The norm α as in Proposition 4.26 is compatible with both h_F and h'_F. By Lemma 4.27, β is compatible with eh_F. By [62, (3.11)], $(\tilde{h'}_F)_\alpha$ and $(\widetilde{eh_F})_\beta$ are Witt equivalent. By [62, (1.4)], if they are not hyperbolic, then their anisotropic parts are isometric. Furthermore,

the degrees of the nonzero components of $(\tilde{h}'_F)_\alpha$ are in Γ_D, whereas the degrees of the nonzero components of $(\widetilde{eh_F})_\beta$ are in $\Gamma_D + \frac{\gamma}{2}$, and $\Gamma_D \cap (\Gamma_D + \frac{\gamma}{2}) = \varnothing$, since $\gamma \notin 2\Gamma_D$. Hence, $(\tilde{h}'_F)_\alpha$ and $(\widetilde{eh_F})_\beta$ must both be hyperbolic. If $Z(\Delta)$ is Henselian then [62, (4.6)] yields that h'_F and eh_F are already hyperbolic. By Proposition 2.8 (a), h' and h are already hyperbolic. □

4.3.2 Detecting isomorphism rationally

Let $(\mathcal{A}, \sigma)$ be an $\mathcal{O}$-algebra with involution. In this section, we further investigate the relation between the objects $(\mathcal{A}, \sigma)$, $(\mathcal{A}, \sigma)_F$ and $(\mathcal{A}, \sigma)_\kappa$, mainly in the case where $\mathcal{O}$ is Henselian. Whereas the focus was on isotropy results in the previous section, we focus on isomorphism results here. In view of Corollary 4.19, which relates isomorphism of algebras with involution over κ to isomorphism over $\mathcal{O}$, in the case where $\mathcal{O}$ is Henselian, it is natural to ask what the relation is between isomorphism over F and isomorphism over $\mathcal{O}$.

4.29 Question. Suppose that $\mathcal{O}$ is a Henselian valuation ring and let $(\mathcal{A}, \sigma)$ and $(\mathcal{A}', \sigma')$ be $\mathcal{O}$-algebras with involution. Assume that $(\mathcal{A}, \sigma)_F \cong_F (\mathcal{A}', \sigma')_F$. Does this imply that $(\mathcal{A}, \sigma) \cong_{\mathcal{O}} (\mathcal{A}', \sigma')$?

As announced at the beginning of section 4.3, we give an affirmative answer to this question in the case where 2 is invertible in $\mathcal{O}$ (Theorem 4.34). The lifting result for hyperbolicity given in Theorem 4.20 will play an important role in the proof. Since the residue structure of an $\mathcal{O}$-algebra with involution does not change by passing to a Henselisation, we obtain as a corollary a "going down" result for isomorphism of algebras with involution over a general valuation ring in which 2 is invertible.
We will extend the result of Theorem 4.34 in the next chapter, where we show that it also holds without the Henselian assumption on $\mathcal{O}$. More generally, Theorem 5.16 contains an affirmative answer to Question 4.29 when $\mathcal{O}$ is replaced by a semilocal Bézout domain in which 2 is a unit. The results for Henselian valuation rings will play an important role in the proof. In order to emphasise this fact and also to collect the different results for Henselian valuation rings in the same chapter, we choose to separate them from the results for general valuation rings in the next chapter.

The initial motivation to consider Question 4.29, also without the Henselian assumption on $\mathcal{O}$, was given by results for symmetric bilinear spaces over general valuation rings. Namely, it is shown in [66, (4.6.3)] that if $2 \in \mathcal{O}^\times$, then symmetric bilinear spaces over $\mathcal{O}$ that become isometric over F, are already isometric over $\mathcal{O}$. We will refer to this result as *rational isometry implies isometry*. The latter result implies that symmetric bilinear spaces over $\mathcal{O}$ that become isometric over F, are also isometric over κ. The condition that $2 \in \mathcal{O}^\times$ is necessary, as the following example shows.

4.30 Example. Consider the field of rational numbers $\mathbb{Q}$ and the 2−adic valuation v_2 on $\mathbb{Q}$. This valuation has residue field $\mathbb{F}_2$. It is a standard fact that

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \simeq_{\mathbb{Q}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

However,

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \not\simeq_{\mathbb{F}_2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

since the second bilinear form only represents 0, whereas the first one doesn't.

The aforementioned results for symmetric bilinear spaces are connected to the notion of "good reduction with respect to a place" for such spaces. The isomorphism results for algebras with involution obtained in this section allow us to introduce a well–defined notion of good reduction with respect to a place for algebras with involution. This is done in section 4.4.

4.31 Remark. Suppose that $2 \in \mathcal{O}^\times$. From the fact that rational isometry implies isometry for symmetric bilinear spaces over $\mathcal{O}$, one cannot immediately conclude, given an Azumaya algebra $\mathcal{A}$ with center $\mathcal{O}$ and two $\mathcal{O}$−linear involutions σ and σ' on $\mathcal{A}$ such that σ_F and σ'_F are orthogonal, that $(\mathcal{A},\sigma)_F \cong_F (\mathcal{A},\sigma')_F$ implies $(\mathcal{A},\sigma) \cong_{\mathcal{O}} (\mathcal{A},\sigma')$. The reason is that isomorphism of algebras with involution corresponds to similarity of bilinear spaces, and in general not to isometry (see Proposition 2.19). As far as we can tell, it was not known whether symmetric bilinear spaces over $\mathcal{O}$ that become similar over F are already similar over $\mathcal{O}$, but we will prove that this holds in the next chapter.

Some of the results needed in order to give a positive answer to Question 4.29, also hold for semilocal Bézout domains, and will be used as such in the next chapter. Therefore, we formulate those results already in that generality here.

The following proposition shows that, in order to answer Question 4.29, we may reduce to the case of two involutions on one algebra. The arguments in the proof are standard, but we formulate the result as a proposition in order to be able to refer to it later on.

4.32 Proposition. *Let R be a semilocal Bézout domain with fraction field F. Suppose that for all R−algebras with involution $(\mathcal{A},\sigma)$ the following holds: if σ' is an R−linear involution on $\mathcal{A}$ such that there is a $Z(\mathcal{A}_F)$−linear isomorphism $(\mathcal{A},\sigma)_F \to (\mathcal{A},\sigma')_F$, then $(\mathcal{A},\sigma) \cong_R (\mathcal{A},\sigma')$. Then for all pairs $((\mathcal{A},\sigma),(\mathcal{A}',\sigma'))$ of R−algebras with involution, we have that $(\mathcal{A},\sigma)_F \cong_F (\mathcal{A}',\sigma')_F$ implies that $(\mathcal{A},\sigma) \cong_R (\mathcal{A}',\sigma')$.*

Proof. Let $\varphi : (\mathcal{A},\sigma)_F \to (\mathcal{A}',\sigma')_F$ be an isomorphism of F−algebras with involution. Then φ restricts to an F−isomorphism $Z(\mathcal{A}_F) \to Z(\mathcal{A}'_F)$. Since $Z(\mathcal{A})$ is the integral closure of R in $Z(\mathcal{A}_F)$, and $Z(\mathcal{A}')$ is the integral closure of R in $Z(\mathcal{A}'_F)$, by Proposition 1.14 (b), and since φ is an R−homomorphism, it follows that $Z(\varphi(\mathcal{A})) = \varphi(Z(\mathcal{A})) = Z(\mathcal{A}')$. If we consider $\mathcal{A}'$ as an Azumaya algebra over $Z(\mathcal{A})$ via φ, then $\varphi : \mathcal{A}_F \to \mathcal{A}'_F$ is a $Z(\mathcal{A}_F)$−isomorphism. By Proposition 1.29, Corollary 1.31 and Corollary 1.33, it follows that there exists an isomorphism of $Z(\mathcal{A})$−algebras $\psi : \mathcal{A} \to \mathcal{A}'$. Let $\tilde{\sigma} = \psi^{-1} \circ \sigma' \circ \psi$. Then ψ is an isomorphism of R−algebras with involution from $(\mathcal{A}, \tilde{\sigma})$ to $(\mathcal{A}', \sigma')$. We have that $\varphi^{-1} \circ \psi_F : (\mathcal{A}, \tilde{\sigma})_F \to (\mathcal{A}, \sigma)_F$ is an isomorphism of F−algebras with involution that is $Z(\mathcal{A}_F)$−linear. The hypothesis now yields that $(\mathcal{A}, \tilde{\sigma}) \cong_R (\mathcal{A}, \sigma)$, and hence, $(\mathcal{A}, \sigma) \cong_R (\mathcal{A}', \sigma')$. □

Using the reduction in Proposition 4.32, it is easily seen that degenerate rationally isomorphic algebras with involution over a semilocal Bézout domain are isomorphic.

4.33 Proposition. *Let R be a semilocal Bézout domain with fraction field F. Let $(\mathcal{A}, \sigma)$ and $(\mathcal{A}', \sigma')$ be R−algebras with involution, and assume that $(\mathcal{A}, \sigma)_F \cong_F (\mathcal{A}', \sigma')_F$. Assume moreover that $Z(\mathcal{A}) \cong R \times R$. Then $(\mathcal{A}, \sigma) \cong_R (\mathcal{A}', \sigma')$.*

Proof. By Proposition 4.32, in order to show the claim we may assume that $\mathcal{A}' = \mathcal{A}$. Since all involutions of the second kind on $\mathcal{A}$ are isomorphic over R by Proposition 1.18, the statement follows. □

In the rest of this section, we turn back to the specific setting of valuation rings. We start with the main result for Henselian valuation rings, an affirmative answer to Question 4.29 in the case where $2 \in \mathcal{O}^\times$.

4.34 Theorem. *Suppose that $\mathcal{O}$ is Henselian and that $2 \in \mathcal{O}^\times$. Let $(\mathcal{A}, \sigma)$ and $(\mathcal{A}', \sigma')$ be $\mathcal{O}$−algebras with involution. If $(\mathcal{A}, \sigma)_F \cong_F (\mathcal{A}', \sigma')_F$ then $(\mathcal{A}, \sigma) \cong_{\mathcal{O}} (\mathcal{A}', \sigma')$.*

Proof. By Proposition 4.32, in order to show that $(\mathcal{A}, \sigma) \cong_{\mathcal{O}} (\mathcal{A}', \sigma')$, we may assume that $\mathcal{A}' = \mathcal{A}$ and that $(\mathcal{A}, \sigma)_F \cong_{Z(\mathcal{A}_F)} (\mathcal{A}, \sigma')_F$. If $Z(\mathcal{A})$ is not a domain, then we are done by Proposition 4.33. So, suppose that $Z(\mathcal{A})$ is a domain. By Proposition 2.20, there exists $s \in \mathcal{A}^\times$ such that $\sigma' = \mathrm{Int}(s) \circ \sigma$. By Proposition 2.18, there exist elements $e \in F^\times$ and $g \in \mathcal{A}_F^\times$ such that $es = \sigma_F(g)g$. Let (Δ, θ) and (V, h) be as in Corollary 4.13 such that $(\mathcal{A}, \sigma) \cong_{Z(\mathcal{A})} \mathrm{Ad}(h)$. Identifying $(\mathcal{A}, \sigma)$ and $\mathrm{Ad}(h)$ through this isomorphism, we consider s as element of $\mathrm{End}_\Delta(V)^\times$ and g as element of $\mathrm{End}_{\Delta_F}(V_F)^\times$. Then $(\mathcal{A}, \sigma') = \mathrm{Ad}(h')$, where $h' : V \times V \to \Delta$ is defined by $h'(x, y) = h(s^{-1}(x), y)$ for all $x, y \in V$. By Proposition 2.19, we have that $(V, h')_F \simeq (V, eh)_F$. Suppose that $e \notin F^{\times 2}\mathcal{O}^\times$. Since $2 \in \mathcal{O}^\times$, Corollary 4.25 yields that h_F and h'_F are hyperbolic. Since $(V, h)_F$ and $(V, h')_F$ have the same dimension, Proposition 2.5 yields that $(V, h)_F \simeq (V, h')_F$.
Suppose that $e \in F^{\times 2}\mathcal{O}^\times$. Since $(V, h')_F \simeq (V, eh)_F$ and elements of $F^{\times 2}$ are similarity factors of h_F, it follows that $(V, h')_F \simeq (V, uh)_F$ for some $u \in \mathcal{O}^\times$. So, in both cases, there exists an element $v \in \mathcal{O}^\times$ such that $(V, h')_F \simeq (V, vh)_F$. Proposition 2.8 (c) then yields that $(V, h') \simeq (V, vh)$, and therefore, $(\mathcal{A}, \sigma) \cong_{\mathcal{O}} (\mathcal{A}, \sigma')$, by Proposition 2.18. □

Theorem 4.34 implies in particular an affirmative answer to Question 4.29 in the case where $\mathcal{O}$ is a complete discrete valuation ring with $2 \in \mathcal{O}^\times$. This result is not new. It is an (unpublished) result of J. Tits, see [55]. There, it is used to show that in the case where $\mathcal{O}$ is a discrete valuation ring with $2 \in \mathcal{O}^\times$, isomorphism of $\mathcal{O}$-algebras with involution can be detected rationally. The result is formulated in the language of algebraic groups.

The method of proof of Theorem 4.34 yields an involution version of the hyperbolicity result of Corollary 4.25. It will be used in this form in chapter 5, and therefore, we already formulate it as such here.

4.35 Proposition. *Suppose that $\mathcal{O}$ is Henselian and that $2 \in \mathcal{O}^\times$. Let $(\mathcal{A}, \sigma)$ be an $\mathcal{O}$-algebra with involution. Suppose that there exist elements $e \in F^\times$, $s \in \mathcal{A}^\times$ and $g \in \mathcal{A}_F^\times$ such that $es = \sigma_F(g)g$. Suppose furthermore that $e \notin F^{\times 2}\,\mathcal{O}^\times$. Then $(\mathcal{A}, \sigma)_F$ and $(\mathcal{A}, \sigma)_\kappa$ are hyperbolic.*

Proof. If $Z(\mathcal{A})$ is not a domain, then $(\mathcal{A}, \sigma)_F$ is degenerate and hence automatically hyperbolic by Proposition 1.42 (that $\mathcal{O}$ is Henselian is not needed). So, suppose that $Z(\mathcal{A})$ is a domain. By going through the proof of Theorem 4.34, and combining this with Proposition 2.14, since $\mathrm{char}(F) \neq 2$, it follows that $(\mathcal{A}, \sigma)_F$ is hyperbolic. Then $(\mathcal{A}, \sigma)_\kappa$ is also hyperbolic by Corollary 4.10. □

From the above results for Henselian valuation rings, we can now derive results for the residue structure of algebras with involution over general valuation rings, since this structure does not change after passing to a Henselisation. These results will be used in section 5.1.

Let F^s be a separable closure of F and let $\mathcal{O}^s$ be an extension of $\mathcal{O}$ to F^s. Let $G = \{\rho \in \mathrm{Gal}(F^s/F) \mid \rho(\mathcal{O}^s) = \mathcal{O}^s\}$. Then $((F^s)^G, \mathcal{O}^s \cap (F^s)^G)$ is Henselian by [21, (3.2.15)]. It is called *a Henselisation* of $(F, \mathcal{O})$, and we denote it by $(F^h, \mathcal{O}^h)$. By [21, (5.2.5)], $(F, \mathcal{O}) \subset (F^h, \mathcal{O}^h)$ is *an immediate extension*, i.e. $(F, \mathcal{O})$ and $(F^h, \mathcal{O}^h)$ have the same value groups and residue fields.

4.36 Corollary. *Assume that $2 \in \mathcal{O}^\times$. Let $(\mathcal{A}, \sigma)$ be an $\mathcal{O}$-algebra with involution. Suppose that there exist elements $e \in F^\times$, $s \in \mathcal{A}^\times$ and $g \in \mathcal{A}_F^\times$ such that $es = \sigma_F(g)g$. Suppose furthermore that $e \notin F^{\times 2}\,\mathcal{O}^\times$. Then $(\mathcal{A}, \sigma)_\kappa$ is hyperbolic.*

Proof. Let $(F^h, \mathcal{O}^h)$ be a Henselisation of $(F, \mathcal{O})$. Since $\mathcal{O}$ and $\mathcal{O}^h$ have the same value groups, we have that $e \notin (F^h)^{\times 2}(\mathcal{O}^h)^\times$. Applying Proposition 4.35 to $\mathcal{O}^h$, and using that $\mathcal{O}^h$ still has residue field κ, now yields the result. □

4.37 Theorem. *Assume that $2 \in \mathcal{O}^\times$. Let $(\mathcal{A}, \sigma)$ and $(\mathcal{A}', \sigma')$ be $\mathcal{O}$-algebras with involution such that $(\mathcal{A}, \sigma)_F \cong_F (\mathcal{A}', \sigma')_F$. Then $(\mathcal{A}, \sigma)_\kappa \cong_\kappa (\mathcal{A}', \sigma')_\kappa$.*

Proof. Let $(F^h, \mathcal{O}^h)$ be a Henselisation of $(F, \mathcal{O})$. Since $(\mathcal{A}, \sigma)_F \cong_F (\mathcal{A}', \sigma')_F$ by assumption, it follows that $(\mathcal{A}, \sigma)_{F^h} \cong_{F^h} (\mathcal{A}', \sigma')_{F^h}$ as well. Theorem 4.34 then yields that $(\mathcal{A}, \sigma)_{\mathcal{O}^h} \cong_{\mathcal{O}^h} (\mathcal{A}', \sigma')_{\mathcal{O}^h}$. Since $(F, \mathcal{O}) \subset (F^h, \mathcal{O}^h)$ is an immediate extension, the residue field of $\mathcal{O}^h$ is still κ and hence, by scalar extension to κ, we get that $(\mathcal{A}, \sigma)_\kappa \cong_\kappa (\mathcal{A}', \sigma')_\kappa$. □

4.38 Remark. In the proof of Theorem 4.34, since Δ is a valuation ring, we don't need to invoke the general Witt cancellation result of B. Keller (used in the proof of Proposition 2.8 (c)) in order to have that $(V, h')_F \simeq (V, vh)_F$ implies $(V, h') \simeq (V, vh)$. If we are not in the case where $\varepsilon = -1$ and $\theta = \mathrm{id}_\Delta$, then $(V, h') \simeq (V, vh)$ by Corollary 2.51. In the excluded case, (V, h') and (V, vh) are hyperbolic and hence automatically isometric by Proposition 2.5. Note furthermore that, if one is only interested in Theorem 4.37, then at the end of of the proof of Theorem 4.34, one can pass directly from the isometry $(V, h')_F \simeq (V, vh)_F$ to $(V, h')_\kappa \simeq (V, vh)_\kappa$. In this way, the Witt cancellation result that is used can be further simplified, for we can then invoke the one for ε−hermitian spaces over division rings with involution from [43, (I.6.3.4)].

4.4 Good reduction

In this section, we recall the notion of good reduction for symmetric bilinear spaces over a valuation ring, and introduce an analogue for algebras with involution. We fix a field L and a place $\lambda : F \to L^\infty$. We denote the valuation ring of F associated to λ by $\mathcal{O}$, and its residue field by κ.

Let (V, b) be a symmetric bilinear space over F. Then (V, b) is said to *have good reduction with respect to* λ if (V, b) is obtained by scalar extension from a symmetric bilinear space over $\mathcal{O}$. This means that there exists an F−basis for V such that the matrix representation of b with respect to this basis consists of elements in $\mathcal{O}$, and the determinant is a unit in $\mathcal{O}$. Such a representation over $\mathcal{O}$ is called *a λ−unimodular representation*. As mentioned in the previous section, if $2 \in \mathcal{O}^\times$ (i.e. $\mathrm{char}(L) \neq 2$), then symmetric bilinear spaces over $\mathcal{O}$ that become isometric over F, are already isometric over $\mathcal{O}$ (see [66, (4.6.3)]). So, in that case, one can associate in a sensible way to a symmetric bilinear space (V, b) over F with good reduction with respect to λ, a residue symmetric bilinear space over L, denoted by $\lambda_*(V, b)$.

If a symmetric bilinear space over F has good reduction with respect to λ then its adjoint algebra with involution is obtained by scalar extension from an $\mathcal{O}$−algebra with involution. Therefore, it is natural to make the following definition. Let (B, τ) be an F−algebra with involution. Then we say that (B, τ) *has good reduction with respect to*

λ if there exists an $\mathcal{O}$–algebra with involution $(\mathcal{A},\sigma)$ such that $(B,\tau) \cong_F (\mathcal{A},\sigma)_F$. We call $(\mathcal{A},\sigma)$ *a λ–unimodular representation of* (B,τ).
This good reduction definition for algebras with involution does not completely generalise the good reduction definition for symmetric bilinear spaces, but it does so up to similarity, as we show below.

4.39 Proposition. *Let (V,b) be a symmetric bilinear space over F. Then $\mathrm{Ad}(b)$ has good reduction with respect to λ if and only if (V,b) has up to similarity good reduction with respect to λ.*

Proof. Suppose first that there exists $u \in F^\times$ such that (V,ub) has good reduction with respect to λ. Then V contains a free $\mathcal{O}$–module $\mathcal{V}$ such that $V = \mathcal{V}F$, $ub(\mathcal{V},\mathcal{V}) \subset \mathcal{O}$ and $ub|_{\mathcal{V}\times\mathcal{V}} : \mathcal{V}\times\mathcal{V} \to \mathcal{O}$ defines a non–singular bilinear form over $\mathcal{O}$. We denote $ub|_{\mathcal{V}\times\mathcal{V}}$ by φ. Then $(\mathrm{End}_{\mathcal{O}}(\mathcal{V}),\mathrm{ad}_\varphi)$ is an $\mathcal{O}$–algebra with involution by Proposition 2.9. Furthermore, we have that $\mathrm{Ad}(b) = \mathrm{Ad}(ub) \cong_F (\mathrm{End}_{\mathcal{O}}(\mathcal{V}),\mathrm{ad}_\varphi)_F$. Hence, $\mathrm{Ad}(b)$ has good reduction with respect to λ.

Suppose conversely that there exists an $\mathcal{O}$–algebra with involution $(\mathcal{A},\sigma)$ such that $(\mathcal{A},\sigma)_F \cong_F \mathrm{Ad}(b)$. Since $\mathcal{A}_F$ is split, Proposition 2.10 yields that $\mathcal{A} \cong \mathrm{M}_n(\mathcal{O})$, with $n = \dim_F(V)$. Since σ_F is orthogonal, there exists a symmetric bilinear space $(\mathcal{V},\varphi)$ over $\mathcal{O}$ such that $(\mathcal{A},\sigma) \cong_{\mathcal{O}} \mathrm{Ad}(\varphi)$. We get that $\mathrm{Ad}(\varphi_F) \cong_F \mathrm{Ad}(b)$. It follows from [45, (12.34)] that there exists $u \in F^\times$ such that $(V,b) \simeq (\mathcal{V},u\varphi)_F$. □

Let (B,τ) be an F–algebra with involution with good reduction with respect to λ, and let $(\mathcal{A},\sigma)$ be a λ–unimodular representation of (B,τ). We set $\lambda_*(B,\tau) = (\mathcal{A},\sigma)_L$. We show below that the results in the previous section imply directly that if $2 \in \mathcal{O}^\times$, then $\lambda_*(B,\tau)$ is well–defined up to L–isomorphism. We then call it *the residue algebra with involution of* (B,τ).

4.40 Proposition. *Assume that $2 \in \mathcal{O}^\times$. Let $(\mathcal{A},\sigma)$ and $(\mathcal{A}',\sigma')$ be $\mathcal{O}$–algebras with involution. If $(\mathcal{A},\sigma)_F \cong_F (\mathcal{A}',\sigma')_F$, then $(\mathcal{A},\sigma)_L \cong_L (\mathcal{A}',\sigma')_L$.*

Proof. We have that L contains up to isomorphism the residue field of $\mathcal{O}$. The statement now follows immediately from Corollary 4.37. □

We can now formulate Corollary 4.11 as follows.

4.41 Corollary. *Let (B,τ) be an F–algebra with involution with good reduction with respect to λ. If $2 \notin \mathcal{O}^\times$, assume that τ is not orthogonal. Then*

$$\mathrm{ind}(B,\tau) \subset \mathrm{ind}(\lambda_*(B,\tau)),$$

and the set on the right hand side is independent of the choice of a λ–unimodular representation for (B,τ).

Proof. The fact that $\operatorname{ind}(\lambda_*(B,\tau))$ is independent of the choice of a λ–unimodular representation for (B,τ) can be seen by passing to a Henselisation of $(F,\mathcal{O})$ and using Theorem 4.20. ◻

Let us take a look at the case where B is split and $2 \in \mathcal{O}^\times$. Then τ is adjoint to a symmetric or alternating bilinear space over F. We then retrieve the following specialisation statement for symmetric bilinear spaces, which follows from the result in [42, (1.20)], stating that λ_* behaves well with respect to orthogonal sums of bilinear spaces.

4.42 Corollary. *Suppose that $2 \in \mathcal{O}^\times$. Let (V,b) be a symmetric bilinear space over F with good reduction with respect to λ. Then the Witt index of $\lambda_*(V,b)$ is at least the Witt index of b. In particular, if b is isotropic, then $\lambda_*(V,b)$ is isotropic.*

Proof. By Proposition 4.39, since (V,b) has good reduction with respect to λ, it follows that $\operatorname{Ad}(b)$ has good reduction with respect to λ. Furthermore, we have that $\lambda_*(\operatorname{Ad}(b)) \simeq_L \operatorname{Ad}(\lambda_*(b))$. Since the Witt index of a symmetric bilinear space is the maximal element in the index of its adjoint algebra with involution, the statement follows from Corollary 4.41. ◻

5 Rational isomorphism versus isomorphism

> C'est dire que s'il y a une chose en mathématiques qui (depuis toujours sans doute) me fascine plus que toute autre, ce n'est ni "le nombre", ni "la grandeur", mais toujours la forme. Et parmi les mille-et-un visages que choisit la forme pour se révéler à nous, celui qui me fascine plus que toute autre et continue à me fasciner, c'est la structure cachée dans les choses mathématiques.
>
> Alexander Grothendieck

In this chapter, we now study the isomorphism question stated in Question 4.29 for general valuation rings, and the connection with multipliers of algebras with involution. We do this first in a local context and later in a (specific) global context. The results in this chapter were obtained in collaboration with J. Van Geel.

Throughout this chapter F denotes a field.

5.1 Question. Let R be a domain with fraction field F. Let $(\mathcal{A},\sigma)$ and $(\mathcal{A}',\sigma')$ be R−algebras with involution. Suppose that $(\mathcal{A},\sigma)_F \cong_F (\mathcal{A}',\sigma')_F$. Does this imply that $(\mathcal{A},\sigma) \cong_R (\mathcal{A}',\sigma')$?

If R is such that this question has a positive answer, then we say that *rational isomorphism implies isomorphism* for R, but we cannot expect a positive answer for a general domain R. On the level of the algebras alone, it does not always hold (see Remark 1.32). Furthermore, even though we showed in Theorem 4.34 that rationally isomorphic $R-$algebras are isomorphic, in the case where R is a Henselian valuation ring in which 2 is invertible, this is no longer true if one considers Henselian local domains. We give a counterexample below, inspired by our discussions on this topic with M. Ojanguren.

5.2 Example. Let $R = \mathbb{R}[[x, y]]/(x^2 + y^2)$. Since $\mathbb{R}[[x, y]]$ is a complete local domain, it is Henselian. It follows that R is also a local Henselian domain. The residue field of R is $\mathbb{R}$. We denote the fraction field of R by L. Consider the bilinear forms $b = \langle 1, 1\rangle$ and $b' = \langle 1, -1\rangle$ over $\mathbb{R}$. Then $\mathrm{Ad}(b_R)$ and $\mathrm{Ad}(b'_R)$ are $R-$algebras with involution of the first kind. Since $-1 \in F^2$, it follows that $\mathrm{Ad}(b_F) \cong_F \mathrm{Ad}(b'_F)$. However, since $-1 \notin \mathbb{R}^2$, b is not hyperbolic, and hence, $\mathrm{Ad}(b) \ncong_{\mathbb{R}} \mathrm{Ad}(b')$. This implies that $\mathrm{Ad}(b_R)$ and $\mathrm{Ad}(b'_R)$ are not isomorphic as $R-$algebras with involution.

In the rest of the chapter, we focus mainly on positive answers to Question 5.1. In the first section, which is a natural continuation of section 4.3.2, we show in Theorem 5.16 that rational isomorphism implies isomorphism in the case where R is a semilocal Bézout domain with $2 \in R^\times$, so in particular if R is a valuation ring with $2 \in R^\times$. We do this by studying multipliers of $F-$algebras with involution obtained by scalar extension from $R-$algebras with involution.

In the next sections, the focus shifts a bit from Question 5.1 to multiplier results for algebras with involution. In section 5.2, we consider the case where R is a discrete valuation ring. In that case, different (simpler and more direct) arguments can be given for some of the statements in section 5.1. We present some of the arguments not only for discrete valuation rings, but we make the jump to Dedekind domains in the next section (see Proposition 5.24). Since Dedekind domains are locally discrete valuation rings, this result forms a bridge between the local setting and the global setting. The conditions in Proposition 5.24 may look a bit artificial, but formulating the result in this way, we obtain a multiplier result simultaneously for two special kinds of principal ideal domains, namely semilocal principal ideal domains and polynomial rings in one variable over a field.

In section 5.4, we focus on the global setting in a very specific case for which the assumptions in Proposition 5.24 hold, namely coordinate rings of affine conics. We will see that we can apply Proposition 5.24 in order to decide, under certain conditions, whether an element of the function field of the conic, is a multiplier up to a unit in the Dedekind domain (Theorem 5.42). As an application, we obtain in a special case an answer to an isomorphism problem for algebras of Schur index 2 with orthogonal involution (Corollary 5.44).

5.1 Semilocal Bézout domains

In this section, we let R be a semilocal Bézout domain with fraction field F. In Theorem 5.16, we show that Question 5.1 has a positive answer if $2 \in R^\times$. We do this by giving, for an R–algebra with involution $(\mathcal{A}, \sigma)$, a local characterisation of the multipliers of $(\mathcal{A}, \sigma)_F$ up to units in R. We thereby use a norm argument based on an approximation theorem for valuations by P. Ribenboim ([63, Théorème 5']). This is done in Theorem 5.13. In order to complete the proof of Theorem 5.16, we use the characterisation of isomorphism of R–algebras with involution in terms of multipliers, given in Corollary 2.25. Together with the hyperbolicity result of Corollary 4.36 and Proposition 4.32, we obtain the desired result.

In view of the relation between (skew–)hermitian spaces over R–algebras with involution and their adjoint algebras with involution, in particular the correspondence between similarity of (skew–)hermitian spaces and isomorphism of their adjoint algebras with involution, it is natural to consider the following question in connection to Question 5.1.

5.3 Question. Let $(\mathcal{C}, \theta)$ be an R–algebra with involution with center a domain. Let (V, h) and (V', h') be hermitian or skew–hermitian spaces over $(\mathcal{C}, \theta)$. Suppose that $(V, h)_F$ and $(V', h')_F$ are similar. Does this imply that (V, h) and (V', h') are already similar?

We will give a positive answer to this question in the case where $2 \in R^\times$ and $\mathcal{C}$ does not have zero divisors.

In order to give the proof of the multiplier result, we start with some preliminary results purely on valuation rings, not involving any algebras with involution.

5.4 Theorem (Ribenboim). *Let L be a field and $v_1, \ldots, v_m$ valuations on L whose respective valuation rings $\mathcal{O}_1, \ldots, \mathcal{O}_m$ are pairwise incomparable. For $i = 1, \ldots, m$, let Γ_i be the value group of $\mathcal{O}_i$. For $i, j = 1, \ldots, m$, let V_{ij} be the smallest overring of $\mathcal{O}_i$ and $\mathcal{O}_j$ in L, and let Δ_{ij} be the convex subgroup of Γ_i such that Γ_i/Δ_{ij} is the value group of V_{ij}. Then $\Gamma_i/\Delta_{ij} \cong \Gamma_j/\Delta_{ji}$. Let θ_{ij} be the quotient map $\Gamma_i \to \Gamma_i/\Delta_{ij}$. Let $(\gamma_1, \ldots, \gamma_m) \in \Gamma_1 \times \ldots \times \Gamma_m$ be such that $\theta_{ij}(\gamma_i) = \theta_{ji}(\gamma_j)$ under the identification $\Gamma_i/\Delta_{ij} = \Gamma_j/\Delta_{ji}$. Then there exists an element $x \in L$ such that $v_i(x) = \gamma_i$ for $i = 1, \ldots, m$.*

Proof. See [63, Théorème 5']. □

In the situation of Theorem 5.4, if $\mathcal{O}_1, \ldots, \mathcal{O}_m$ are pairwise independent valuation rings, then $\Delta_{ij} = \Gamma_i$ and one gets the well–known classical approximation theorem. We will use the following consequence of Theorem 5.4.

5.5 Corollary. *Let L be a field and $\mathcal{O}_1,\dots,\mathcal{O}_m$ pairwise incomparable valuation rings of L. For $j = 2,\dots,m$, let V_{1j} be the smallest overring of $\mathcal{O}_1$ and $\mathcal{O}_j$ in L. Let $u \in L$ be such that $u \in V_{12}^\times \cap \dots \cap V_{1m}^\times$. Then there exists an element $x \in L$ such that $xu \in \mathcal{O}_1^\times$ and $x \in \mathcal{O}_2^\times \cap \dots \cap \mathcal{O}_m^\times$.*

Proof. Let $v_1,\dots,v_m$ be valuations on L with respective valuation rings $\mathcal{O}_1,\dots,\mathcal{O}_m$. For $i = 1,\dots,m$, let Γ_i be the value group of $\mathcal{O}_i$, and for $j = 1,\dots,m$, $j \neq i$, let Δ_{ij} be the convex subgroup of Γ_i such that Γ_i/Δ_{ij} is the value group of V_{ij}, and let furthermore v_{ij} be a valuation on L with valuation ring V_{ij}. Then the fact that $u \in V_{12}^\times \cap \dots \cap V_{1m}^\times$ is equivalent to $v_j(u) \in \Delta_{1j}$ for $j = 2,\dots,m$. Furthermore, in the notation of Theorem 5.4, we have that $\theta_{1j}(v_1(u^{-1})) = v_{1j}(u^{-1}) = 0$ for $j = 2,\dots,m$. Applying Theorem 5.4 with $\gamma_1 = v_1(u^{-1})$ and $\gamma_2 = \dots = \gamma_m = 0$ yields the statement. □

5.6 Lemma. *Let L/F be a finite field extension. Let $\mathcal{O}'$ be a valuation ring of L lying over $\mathcal{O}$. Let β be a prime ideal of $\mathcal{O}'$ that is not maximal. Then $\beta \cap \mathcal{O}$ is a prime ideal of $\mathcal{O}$ that is not maximal.*

Proof. Since β is a prime ideal of $\mathcal{O}'$, it is clear that $\beta \cap \mathcal{O}$ is a prime ideal of $\mathcal{O}$. Let T be the integral closure of $\mathcal{O}$ in L. By [21, (3.2.3), (3.2.9)], T is the intersection of the finitely many valuation rings of L lying over $\mathcal{O}$. Let $\mathcal{M}'$ be the maximal ideal of $\mathcal{O}'$. Then $\mathfrak{m}' = \mathcal{M}' \cap T$ is a maximal ideal of T and $\mathcal{O}' = T_{\mathfrak{m}'}$ by Proposition 1.4. By [49, (IX.1.11)], $\beta \cap T$ is a maximal ideal if and only if $\beta \cap \mathcal{O}$ is a maximal ideal. Suppose that $\beta \cap T$ is a maximal ideal. Since $\beta \cap T \subset \mathcal{M}' \cap T$, it follows that $\beta \cap T = \mathcal{M}' \cap T = \mathfrak{m}'$. This implies that β contains $\mathfrak{m}'\mathcal{O}' = \mathfrak{m}'T_{\mathfrak{m}'}$, which is equal to $\mathcal{M}'$, and hence, $\beta = \mathcal{M}'$, a contradiction. So, it follows that $\beta \cap \mathcal{O}$ is not maximal. □

5.7 Lemma. *Let $\mathcal{O}$ be a valuation ring of F and let $(F^h,\mathcal{O}^h)$ be a Henselisation of $(F,\mathcal{O})$. Let $F \subsetneq L \subset F^h$ be a finite subextension. Then there exists a finite subextension $L \subset M \subset F^h$ with the following properties:*

(a) $\mathcal{O}$ has more than one extension to M;

(b) let v be a valuation on F with valuation ring $\mathcal{O}$ and let $v_1 = v^h|_M, v_2,\dots,v_m$ be the different valuations on M extending v (i.e. such that $v_i(y) = v(y)$ for all $y \in F$). Then there exist $n_1,\dots,n_m \in \mathbb{N}$ such that for all $x \in M$, we have that

$$v(N_{M/F}(x)) = v_1(x) + \sum_{i=2}^{m} n_i v_i(x).$$

Proof. Let F^s be a separable closure of F and $\mathcal{O}^s$ an extension of $\mathcal{O}$ to F^s with respect to which $(F^h,\mathcal{O}^h)$ is defined. Let N/F be the Galois closure of L in F^s. Let v^s be a valuation on F^s with valuation ring $\mathcal{O}^s$, extending v. Let $v^h = v^s|_{F^h}$ and $w = v^s|_N$. Since

N/F is a Galois extension, all valuation rings of N lying over $\mathcal{O}$ are conjugate to $\mathcal{O}_w$ by [21, (3.2.15)]. Let $H = \{\tau \in \mathrm{Gal}(N/F) \mid \tau(\mathcal{O}_w) = \mathcal{O}_w\}$ and let M be the fixed field of H. Then $M = N \cap F^h$ (for an explicit argument see the proof of [21, (5.2.5)]). In particular, since $L \subset N \cap F^h$, this implies that $M \neq F$, and hence $H \neq \mathrm{Gal}(N/F)$.

Let $\{\rho_1 = \mathrm{id}_N, \rho_2, \ldots, \rho_t\}$ be a set of representatives for the right cosets of $\mathrm{Gal}(N/F)/H$. By Galois theory, the restrictions of the ρ_i to M are exactly the different F–embeddings of M in F^s. We have that $w \circ (\rho_i)|_M$ is a valuation on M with valuation ring $\rho_i^{-1}(\mathcal{O}_w) \cap M$. It is possible that $\rho_i^{-1}(\mathcal{O}_w) \cap M = \rho_j^{-1}(\mathcal{O}_w) \cap M$ for $i \neq j$, but by the proof of [21, (3.3.1)], it follows that $\mathcal{O}_w \cap M \neq \rho_i^{-1}(\mathcal{O}_w) \cap M$ if $i \neq 1$. This means that $w|_M \neq w \circ (\rho_i)|_M$ if $i \neq 1$, and hence, $\mathcal{O}$ has more than one extension to M, since $H \neq \mathrm{Gal}(N/F)$.

Let $x \in M$ be arbitrary. Then $N_{M/F}(x) = x\rho_2(x) \cdots \rho_t(x)$ by definition. It follows that

$$v(N_{M/F}(x)) = w(N_{M/F}(x)) = w(x\rho_2(x) \cdots \rho_t(x)) = w(x) + w(\rho_2(x)) + \ldots + w(\rho_t(x)).$$

Let $v_1 = v^h|_M, v_2, \ldots, v_m$ be the different valuations on M extending v. We have that $\{v_2, \ldots, v_m\} = \{w \circ (\rho_2)|_M, \ldots, w \circ (\rho_t)|_M\}$. By the reasoning above, it follows that there exist $n_1, \ldots, n_m \in \mathbb{N}$ such that

$$v(N_{M/F}(x)) = v_1(x) + \sum_{i=2}^{m} n_i v_i(x).$$

□

5.8 Lemma. *Let $\mathcal{O}_1, \ldots, \mathcal{O}_r$ be pairwise incomparable valuation rings of F and let $R = \mathcal{O}_1 \cap \ldots \cap \mathcal{O}_r$. Suppose that $x \in F$ is such that $x \in F^{\times 2}\mathcal{O}_i^\times$ for $i = 1, \ldots, r$. Then $x \in F^{\times 2}R^\times$.*

Proof. Let $w_1, \ldots, w_r$ be valuations on F with respective valuation rings $\mathcal{O}_1, \ldots, \mathcal{O}_r$. Denote their respective value groups by $\Gamma_1, \ldots, \Gamma_r$. The hypothesis implies that there exists a tuple $(\gamma_1, \ldots, \gamma_r) \in \Gamma_1 \times \ldots \times \Gamma_r$ such that $w_1(x) = 2\gamma_1, \ldots, w_r(x) = 2\gamma_r$. In the notations of Corollary 5.5, we have that $\theta_{ij}(2\gamma_i) = v_{ij}(x) = v_{ji}(x) = \theta_{ji}(2\gamma_j)$. Since the $\theta_{ij} : \Gamma_i \to \Gamma_i/\Delta_{ij}$ are group homomorphisms and since Γ_i/Δ_{ij} is an ordered abelian group and therefore torsion–free, it follows that $\theta_{ij}(\gamma_i) = \theta_{ji}(\gamma_j)$. Therefore, we can apply Corollary 5.5 to find an element $a \in F$ such that $w_1(a) = \gamma_1, \ldots, w_r(a) = \gamma_r$. Then $w_i(a^2x^{-1}) = 0$ for $i = 1, \ldots, r$, which means that $a^2x^{-1} \in R^\times$. This proves the claim. □

From the next result on, we bring in the algebras with involution. Let $(\mathcal{A}, \sigma)$ be an R–algebra with involution and let $e \in F^\times$. The aim is to show that if $2 \in R^\times$, then $e \in G((\mathcal{A}, \sigma)_F)R^\times$ if and only if for every valuation ring $\mathcal{O}$ of F containing R, and such that $e \notin F^{\times 2}\mathcal{O}^\times$, we have that σ is hyperbolic over the residue field of $\mathcal{O}$. The hardest part is to prove that the hyperbolicity conditions imply that $e \in G((\mathcal{A},\sigma)_F)R^\times$. We do this by looking at the overrings $R \subsetneq T \subset F$, and checking that $e \in G((\mathcal{A},\sigma)_F)T^\times$ for every such T.

We start with a lemma we will use in combination with Lemma 5.7.

5.9 Lemma. *Let (B,τ) an F–algebra with involution. Let F'/F be an algebraic field extension such that $\tau_{F'}$ is hyperbolic. Then there is a finite separable subextension L/F over which τ becomes hyperbolic.*

Proof. Let $(e_1,\dots,e_n)$ be an F–basis for B. Then it is an F'–basis for $B_{F'}$. Since $\tau_{F'}$ is hyperbolic, there is an idempotent $x \in B_{F'}$ such that $\tau_{F'}(x) = 1 - x$. Write $x = \sum_{i=1}^n e_i x_i$, with $x_1,\dots,x_n \in F'$. Then τ already becomes hyperbolic over $F(x_1,\dots,x_n)$. Since $x_1,\dots,x_n$ are algebraic over F, $F(x_1,\dots,x_n)$ is a finite extension of F. Since $\mathrm{char}(F) \neq 2$, we get that τ already becomes hyperbolic over the separable closure of F in $F(x_1,\dots,x_n)$, by [45, (9.16)]. □

5.10 Proposition. *Assume that $2 \in R^\times$. Let $(\mathcal{A},\sigma)$ be an R–algebra with involution and let $e \in F^\times$. Assume that for each maximal ideal $\mathfrak{m}$ of R such that $e \notin F^{\times 2}R_{\mathfrak{m}}^\times$, we have that σ becomes hyperbolic over $R/\mathfrak{m}$. Suppose furthermore that for every overring $R \subsetneq T \subset F$, we have that $e \in G((\mathcal{A},\sigma)_F)T^\times$. Then $e \in G((\mathcal{A},\sigma)_F)R^\times$.*

Proof. Since $F^{\times 2}R^\times \subset G((\mathcal{A},\sigma)_F)R^\times$, we may assume that $e \notin F^{\times 2}R^\times$. By Proposition 2.24 (a), this implies that σ is not hyperbolic over F. Let $\mathfrak{m}_1,\dots,\mathfrak{m}_\ell$ be the different maximal ideals of R. For $i = 1,\dots,\ell$, let $\mathcal{O}_i = R_{\mathfrak{m}_i}, \kappa_i = R/\mathfrak{m}_i$, and let $(F_i^h,\mathcal{O}_i^h)$ be a Henselisation of $(F,\mathcal{O}_i)$. Then $\mathcal{O}_1,\dots,\mathcal{O}_\ell$ are pairwise incomparable by Proposition 1.2. Since $e \notin F^{\times 2}R^\times$, it follows from Lemma 5.8 that $e \notin F^{\times 2}\mathcal{O}_i^\times$ for at least one $i \in \{1,\dots,\ell\}$. Without loss of generality, we may assume that $e \notin F^{\times 2}\mathcal{O}_1^\times$. By assumption σ is hyperbolic over κ_1. By Theorem 4.20, since $2 \in R^\times$, it follows that σ is hyperbolic over F_1^h. By Lemma 5.9, σ is already hyperbolic over a finite subextension $F \subsetneq L \subset F_1^h$. Let $L \subset M \subset F_1^h$ be a finite subextension as in Lemma 5.7 and let $\tilde{\mathcal{O}}_1 = \mathcal{O}_1^h \cap M$. Let $\mathcal{T}$ be the set of the rings $\tilde{\mathcal{O}}_1 W$, where W runs over the valuation rings of M lying over $\mathcal{O}_1$ and different from $\tilde{\mathcal{O}}_1$. Note that $\mathcal{T}$ is non–empty by Lemma 5.7 (a). Furthermore, $\mathcal{T}$ has finitely many elements by [21, (3.2.9)]. Since all overrings of $\tilde{\mathcal{O}}_1$ in M are linearly ordered, it follows that $\mathcal{T}$ has a smallest element, say $\tilde{V}$. Since the valuation rings of M lying over $\mathcal{O}_1$ are pairwise incomparable by [21, (3.2.8)], we have that $\tilde{\mathcal{O}}_1 \subsetneq \tilde{V}$.

We set $V = \tilde{V} \cap F$. Then V is a valuation ring of F containing $\mathcal{O}_1$. Let β be the maximal ideal of $\tilde{V}$. Then $\tilde{V} = (\tilde{\mathcal{O}}_1)_\beta \neq \tilde{\mathcal{O}}_1$ by [21, p. 43], and hence β is a prime ideal of $\tilde{\mathcal{O}}_1$ that is not maximal. We have that $\beta \cap V$ is the maximal ideal of V. Furthermore, it is clear that

$\beta \cap V = \beta \cap \mathcal{O}_1$. Hence, $V = (\mathcal{O}_1)_{\beta\cap\mathcal{O}_1}$ and since $\beta \cap \mathcal{O}_1$ is a prime ideal of $\mathcal{O}_1$ that is not maximal by Lemma 5.6, we have that $\mathcal{O}_1 \subsetneq V$. Let $T = V \cap \mathcal{O}_2 \cap \ldots \cap \mathcal{O}_\ell$. Then $R \subsetneq T$ by Lemma 1.6. By assumption, we have that $e = au$, with $a \in G((\mathcal{A},\sigma)_F)$ and $u \in T^\times$. Since $u \in T^\times \subset V^\times \subset \tilde{V}^\times$, it follows from the minimality of $\tilde{V}$ that u is a unit in every element of $\mathcal{T}$. Hence, Corollary 5.5 yields that there exists $x \in M$ such that $xu \in \tilde{\mathcal{O}}_1^\times$ and $x \in W^\times$, for every valuation ring W of M lying over $\mathcal{O}_1$ and different from $\tilde{\mathcal{O}}_1$. By Lemma 5.7, it follows that $N_{M/F}(x)u = N_{M/F}(x)x^{-1}xu \in \mathcal{O}_1^\times$, and $N_{M/F}(x)u \in \mathcal{O}_2^\times \cap \ldots \cap \mathcal{O}_\ell^\times$, since $u \in (\mathcal{O}_2 \cap \ldots \cap \mathcal{O}_\ell)^\times$. Hence, $N_{M/F}(x)u \in R^\times$. Furthermore, by Lemma 2.23 (a) and (b), we have that $N_{M/F}(x) \in G((\mathcal{A},\sigma)_F)$. Since $G((\mathcal{A},\sigma)_F)$ is a group, it follows that $eN_{M/F}(x) \in G((\mathcal{A},\sigma)_F)R^\times$, and hence $e \in G((\mathcal{A},\sigma)_F)R^\times$, as desired. □

As a consequence of the previous proposition, we already obtain the desired multiplier result in the case where R has Krull dimension 1.

5.11 Corollary. *Assume that $2 \in R^\times$ and that R has Krull dimension 1. Let $(\mathcal{A},\sigma)$ be an R-algebra with involution and let $e \in F^\times$. Assume that for each maximal ideal $\mathfrak{m}$ of R such that $e \notin F^{\times 2}R_{\mathfrak{m}}^\times$, we have that σ becomes hyperbolic over $R/\mathfrak{m}$. Then $e \in G((\mathcal{A},\sigma)_F)R^\times$.*

Proof. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_\ell$ be the different maximal ideals of R and let $\mathcal{O}_1 = R_{\mathfrak{m}_1}, \ldots, \mathcal{O}_\ell = R_{\mathfrak{m}_\ell}$. The map $\mathfrak{p} \mapsto \mathfrak{p}R_{\mathfrak{m}_i}$ defines a one-to-one correspondence between prime ideals of R contained in $\mathfrak{m}_i$ and prime ideals of $\mathcal{O}_i$ (cf. [36, Theorem 34]). So, it follows that $\mathcal{O}_1, \ldots, \mathcal{O}_\ell$ have Krull dimension 1. We now prove the statement by induction on ℓ. If $\ell = 1$ then F is the only overring of R inside F and clearly $e \in G((\mathcal{A},\sigma)_F)F^\times = F^\times$. Hence, by Proposition 5.10, it follows that $e \in G((\mathcal{A},\sigma)_F)R^\times$. Suppose that $\ell > 1$. Let $R \subsetneq T \subset F$ be an overring. By Lemma 1.6 (b), there exist valuation rings $V_1, \ldots, V_\ell$ such that $T = V_1 \cap \ldots \cap V_\ell$ and for each $i \in \{1, \ldots, \ell\}$, there exists $j \in \{1, \ldots, \ell\}$ such that $\mathcal{O}_j \subset V_i$, where at least one of these inclusions is strict. Since $\mathcal{O}_1, \ldots, \mathcal{O}_\ell$ have Krull dimension 1, they do not have overrings inside F that are different from F. Hence, without loss of generality, we may assume that there exists $r \in \{1, \ldots, \ell\}$ such that $V_1 = \ldots = V_r = F$ and $V_i = \mathcal{O}_i$ for $i = r+1, \ldots, \ell$. It follows that $\mathfrak{m}_{r+1}, \ldots, \mathfrak{m}_\ell$ are the maximal ideals of T. Hence, by the induction hypothesis, it follows that $e \in G((\mathcal{A},\sigma)_F)T^\times$. Since this holds for all $R \subsetneq T \subset F$, Proposition 5.10 yields that $e \in G((\mathcal{A},\sigma)_F)R^\times$. □

5.12 Theorem. *Assume that $2 \in R^\times$. Let $(\mathcal{A},\sigma)$ be an R-algebra with involution and $e \in F^\times$. Suppose that for every overring $R \subset T \subset F$, we have that for every maximal ideal $\mathcal{M}$ of T such that $e \notin F^{\times 2}T_{\mathcal{M}}^\times$, σ is hyperbolic over $T/\mathcal{M}$. Then $e \in G((\mathcal{A},\sigma)_F)R^\times$.*

Proof. Since $F^{\times 2}R^\times \subset G((\mathcal{A},\sigma)_F)R^\times$, we may assume that $e \notin F^{\times 2}R^\times$. Let $\mathcal{T}$ be the set of overrings $R \subset T \subset F$ such that $e \notin G((\mathcal{A},\sigma)_F)T^\times$. We need to show that $\mathcal{T} = \varnothing$. So, suppose for the sake of contradiction that $\mathcal{T} \neq \varnothing$. Then $\mathcal{T}$ is partially ordered by inclusion. We show that $\mathcal{T}$ contains a maximal element. By Zorn's Lemma, it suffices to show that every chain in $\mathcal{T}$ has an upper bound. So, let $(T_i)_{i\in I}$ be a chain in $\mathcal{T}$. Let $T = \bigcup_{i\in I} T_i$. We show that $T \in \mathcal{T}$. Let f be any element in $G((\mathcal{A},\sigma)_F)T^\times$, say $f = au$, with $a \in G((\mathcal{A},\sigma)_F)$ and $u \in T^\times$. Then there exist an index $j \in I$ such that $u, u^{-1} \in T_j$. This implies that $f \in G((\mathcal{A},\sigma)_F)T_j^\times$. Since $e \notin G((\mathcal{A},\sigma)_F)T_j^\times$ for all j, it follows that $e \notin G((\mathcal{A},\sigma)_F)T^\times$. Hence, $T \in \mathcal{T}$. It follows that $\mathcal{T}$ contains a maximal element, which we denote by $\tilde{T}$. Then for $\tilde{T} \subsetneq T' \subset F$, we have that $e \in G((\mathcal{A},\sigma)_F)T'^\times$. Applying Proposition 5.10 now yields that $e \in G((\mathcal{A},\sigma)_F)\tilde{T}^\times$, a contradiction. Hence, $\mathcal{T} = \varnothing$ and therefore $e \in G((\mathcal{A},\sigma)_F)R^\times$. □

5.13 Theorem. *Assume that $2 \in R^\times$. Let $(\mathcal{A},\sigma)$ be an R–algebra with involution and let $e \in F^\times$. Then the following are equivalent:*

(i) $e \in G((\mathcal{A},\sigma)_F)R^\times$.

(ii) For each valuation ring $\mathcal{O}$ of F containing R, such that $e \notin F^{\times 2}\mathcal{O}^\times$, we have that $(\mathcal{A},\sigma)$ is hyperbolic over the residue field of $\mathcal{O}$.

(iii) For every valuation ring $\mathcal{O}$ of F containing R that is maximal with the property that $e \notin F^{\times 2}\mathcal{O}^\times$, we have that $(\mathcal{A},\sigma)$ is hyperbolic over the residue field of $\mathcal{O}$.

Proof. That (i) implies (ii) follows from Corollary 4.36. The converse follows immediately from Theorem 5.12. It is clear that (ii) implies (iii). Suppose that (iii) holds. Let $\mathcal{O}$ be a valuation ring of F containing R such that $e \notin F^{\times 2}\mathcal{O}^\times$. Denote the residue field of $\mathcal{O}$ by κ. It is clear that the set $\mathcal{T}$ consisting of valuation rings V of F containing $\mathcal{O}$ such that $e \notin F^{\times 2}V^\times$ has a maximal element (namely the union of all such V), say $\tilde{V}$. Let $\tilde{\mathfrak{m}}$ be the maximal ideal of $\tilde{V}$. Then $\mathcal{O}/\tilde{\mathfrak{m}}$ defines a valuation ring of $\tilde{\kappa} = \tilde{V}/\tilde{\mathfrak{m}}$ with residue field κ. By assumption, σ becomes hyperbolic over $\tilde{\kappa}$. Corollary 4.10 then implies that σ also becomes hyperbolic over κ. □

5.14 Remark. If the center of $\mathcal{A}$ is not a domain then the properties (i) and (ii) in Theorem 5.13 both hold for trivial reasons. For if $(\mathcal{A},\sigma)$ is degenerate then $(\mathcal{A},\sigma)_F$ is also degenerate and hence hyperbolic. Proposition 2.24 (a) yields that $G((\mathcal{A},\sigma)_F) = F^\times$. Furthermore, $(\mathcal{A},\sigma)$ remains degenerate over the valuation rings of F containing R and hence, $(\mathcal{A},\sigma)$ is automatically hyperbolic over the residue fields of these valuation rings.

5.15 Proposition. *Assume that $2 \in R^\times$. Let S be a separable quadratic R–algebra that is a domain. Let $((\mathcal{A},\sigma),(\mathcal{A}',\sigma'))$ be a pair of R–algebras with involution. Assume that we are given R–isomorphisms $f : S \to Z(\mathcal{A})$ and $f' : S \to Z(\mathcal{A}')$ and an S_F–isomorphism $(\mathcal{A},\sigma)_F \to (\mathcal{A}',\sigma')_F$ with respect to f and f'. Then $(\mathcal{A},\sigma) \cong_S (\mathcal{A}',\sigma')$.*

Proof. By assumption, there is an S_F–isomorphism $\mathcal{A}'_F \to \mathcal{A}_F$ with respect to f' and f. By Propositions 1.29 and 1.31, this implies that there already is an S–isomorphism $\varphi : \mathcal{A}' \to \mathcal{A}$ with respect to f' and f. Then $\varphi : (\mathcal{A}', \sigma') \to (\mathcal{A}, \varphi \circ \sigma' \circ \varphi^{-1})$ is an S–linear isomorphism of R–algebras with involution. In order to prove the claim, it then suffices to show that $(\mathcal{A}, \sigma) \cong_S (\mathcal{A}, \varphi \circ \sigma' \circ \varphi^{-1})$ with respect to f. So, without loss of generality we may assume that $\mathcal{A}' = \mathcal{A}$. By Proposition 2.20, there exists an element $s \in \mathcal{A}^\times$ such that $\sigma(s) = s$ and $\sigma' = \operatorname{Int}(s) \circ \sigma$. By Proposition 2.18, there exist elements $e \in F^\times$ and $g \in \mathcal{A}_F^\times$ such that $es = \sigma_F(g)g$. Combining Corollary 4.36, Theorem 5.13 and Corollary 2.25 yields that $(\mathcal{A}, \sigma) \cong_S (\mathcal{A}, \sigma')$. □

Using Proposition 5.15, we can now give a positive answer to Question 5.1 in the case where $2 \in R^\times$. We thereby don't need the formulation of Proposition 5.15 with two different algebras, but we will use the more general formulation in order to obtain a result concerning Question 5.3 in Corollary 5.17.

5.16 Theorem. *Assume that $2 \in R^\times$. Let $(\mathcal{A}, \sigma)$ and $(\mathcal{A}', \sigma')$ be R–algebras with involution. If $(\mathcal{A}, \sigma)_F \cong_F (\mathcal{A}', \sigma')_F$ then $(\mathcal{A}, \sigma) \cong_R (\mathcal{A}', \sigma')$.*

Proof. By Proposition 4.32 we may assume that $\mathcal{A}' = \mathcal{A}$ and that $(\mathcal{A}, \sigma)_F \cong_{Z(\mathcal{A}_F)} (\mathcal{A}, \sigma')_F$. Furthermore, we may assume that $Z(\mathcal{A})$ is a domain by Proposition 4.33. The result now follows from Proposition 5.15. □

5.17 Corollary. *Assume that $2 \in R^\times$. Let (Δ, θ) be an R–algebra with involution without zero divisors. Let $\varepsilon = \pm 1$. Let (V, h) and (V', h') be ε–hermitian spaces over (Δ, θ). If $(V, h)_F$ and $(V', h')_F$ are similar then (V, h) and (V', h') are already similar.*

Proof. It follows from Proposition 2.21, applied in the case $R = F$, that $\operatorname{Ad}(h_F) \cong_{Z(\Delta_F)} \operatorname{Ad}(h'_F)$. Proposition 5.15 then yields that $\operatorname{Ad}(h) \cong_{Z(\Delta)} \operatorname{Ad}(h')$. Applying the other implication given by Proposition 2.21, we obtain that there exists $u \in R^\times$ such that $(V', h') \simeq_R (V, uh)$. □

5.18 Remark. Originally, we only had a proof of Theorem 5.13 in the case where R has finite Krull dimension. K.J. Becher then pointed out a way to obtain the statement of Theorem 5.16 also in the case of infinite Krull dimension, namely by showing that the "data" defining an R–algebra with involution can already be defined over a semilocal Bézout domain of finite Krull dimension. We then showed that the multiplier statement in Theorem 5.13 also holds for infinite Krull dimension, using the same technique. Inspired by the methods there, we eventually found a uniform proof of Theorem 5.13.

We finish this section with some history concerning Question 5.1. Let T be a domain and L its fraction field. The rational isomorphism implies isomorphism problem can be reformulated in terms of algebraic groups, and has also been studied from this point of view. Let $(\mathcal{A},\sigma)$ and $(\mathcal{A}',\sigma')$ be T−algebras with involution such that $(\mathcal{A},\sigma)_L \cong_L (\mathcal{A}',\sigma')_L$. Consider the linear algebraic group $\tilde{G} = \mathrm{Aut}_{Z(\mathcal{A})}(\mathcal{A},\sigma)$. The question whether $(\mathcal{A},\sigma) \cong_T (\mathcal{A}',\sigma')$ translates to the question whether principal $\tilde{G}$−homogeneous spaces that become isomorphic over L are already isomorphic over T. In the case where T is a regular local ring, this is related to a conjecture of A. Grothendieck, stating that for a reductive algebraic group G over T, principal G−homogeneous spaces that become trivial over L are already trivial (see [27, Remarque 3, pp. 26–27] and [29, Remarque 1.1 a]). If R is of geometric type over a field k and G is defined over k, then this conjecture is known as Serre's conjecture (see [67, Remarque p. 31]).

In [55], Y. Nisnevich proved Grothendieck's conjecture in the case where T is a regular local ring of dimension 1, or a Henselian regular local ring of arbitrary dimension. His work includes, in the case where T is a discrete valuation ring with $2 \in T^\times$, an affirmative answer to Question 5.1. Nisnevich's proof uses the fact that the principal homogeneous spaces considered are up to isomorphism classified by a pointed étale cohomology set. His proof is therefore of a very different nature than the one presented here in the case of discrete valuation rings.
Suppose that T is a regular local ring containing a field of characteristic different from 2. In [56], I. Panin proved a purity theorem on multipliers for $\mathrm{Aut}_{Z(\mathcal{A})}(\mathcal{A},\sigma)$, $(\mathcal{A},\sigma)$ a T−algebra with involution. Using Nisnevich's results for discrete valuation rings in [55], I. Panin obtained a positive answer to Question 5.1 for T, and hence, also confirmed Grothendieck's conjecture for $\mathrm{Aut}_{Z(\mathcal{A})}(\mathcal{A},\sigma)$.

In view of the important role of the Henselian case in the proof of Theorem 5.16, it would be interesting to know whether Grothendieck's conjecture holds in the case where T is a (general) Henselian valuation ring. Furthermore, in the spirit of this chapter, and keeping the results of Nisnevich in mind, one could ask the (more difficult) question whether for any reductive algebraic group G over T, principal G−homogeneous spaces that become isomorphic over L, are already isomorphic over T.

5.2 Discrete valuation rings

In this section, we present a different proof of Corollary 4.36 in the case of a discrete valuation ring, based on the arguments in the proof of [71, (2.4), (2.6)]. We still pass to a Henselisation in a preliminary lemma, but we don't need to invoke the results of section 4.3.2 for Henselian valuation rings. We complement the results obtained here in the next section with a different proof of the implication (ii) ⇒ (i) in Theorem 5.13, in the case of semilocal principal ideal domains and for polynomial rings in one variable over a field. The norm argument based on P. Ribenboim's approximation theorem is then replaced by a norm argument on ideals in a Dedekind domain, and is much more direct.

5.19 Lemma. *Assume that* $\mathrm{char}(F) \neq 2$. *Let* (B, τ) *be an* F*–algebra with involution. Then the following are equivalent:*

(i) (B, τ) *is hyperbolic.*

(ii) There is an element $b \in B$, *such that* $\tau(b)b = 0$ *and* $\dim_F(bB) \geqslant \frac{1}{2}\dim_F(B)$.

Proof. See [71, (2.1)]. □

5.20 Lemma. *Let* $\mathcal{O}$ *be a valuation ring of* F. *Let* $\mathcal{A}$ *be an Azumaya algebra with center* $\mathcal{O}$ *or a separable quadratic* $\mathcal{O}$*–algebra that is a domain. If* $Z(\mathcal{A}) \neq \mathcal{O}$, *assume that* $\mathcal{A}$ *carries an* $\mathcal{O}$*–linear involution of the second kind. Then there exists a field extension* L/F *splitting* $\mathcal{A}_F$, *such that* $\mathcal{O}$ *has an extension in* L *with the same value group as* $\mathcal{O}$.

Proof. We denote the center of $\mathcal{A}$ by S. Let $(F^h, \mathcal{O}^h)$ be a Henselisation of $(F, \mathcal{O})$. Suppose first that $S^h = S \otimes_{\mathcal{O}} \mathcal{O}^h$ is a domain. By Proposition 1.26, there exists an Azumaya algebra Δ^h over S^h without zero divisors, and an integer $r \in \mathbb{N}$ such that $\mathcal{A}_{\mathcal{O}^h} \cong M_r(\Delta^h)$. Let $D^h = \Delta^h_{F^h}$. Then $\mathcal{A}_{F^h} \cong M_r(D^h)$. Since $\mathcal{O}^h$ is Henselian, S^h is valuation ring by Proposition 1.22. By Proposition 4.12 and [75, (2.2)], S^h extends (uniquely) to a valuation ring of D^h, which is equal to Δ^h. Furthermore, Δ^h and S^h have the same value groups, and hence Δ^h and $\mathcal{O}^h$ have the same value groups. Let L be a maximal subfield of D^h and let $\mathcal{O}_L = \Delta^h \cap L$. Then L is a splitting field of $\mathcal{A}_F$ and the value groups of $\mathcal{O}_L$ and $\mathcal{O}$ are the same.

Suppose that S^h is not a domain. Since $\mathcal{A}$ carries an $\mathcal{O}$–linear involution of the second kind, by Proposition 1.18, there exists a central simple F^h–algebra B such that $\mathcal{A}_{F^h} \cong B \times B^{\mathrm{op}}$. By Proposition 1.26, there exists an F^h–division algebra D' and $n \in \mathbb{N}$ such that $B \cong \mathrm{M}_n(D')$. Since $\mathcal{O}^h$ is Henselian, it extends uniquely to a valuation ring Δ' of D', and the value groups of $\mathcal{O}^h$ and Δ' are equal. Taking, as in the reasoning above, a maximal subfield of D', we find a splitting field L of B containing a valuation ring extending $\mathcal{O}$ that has the same value group as $\mathcal{O}$. Then L is also a splitting field of $\mathcal{A}_F$. □

For the next result, we mimick the proof of [71, (2.4), (2.6)], where the author considered the case where F is the rational function field in one variable over a field.

5.21 Proposition. *Let* $\mathcal{O}$ *be a discrete valuation ring of* F *and assume that* $2 \in \mathcal{O}^\times$. *Denote the residue field of* $\mathcal{O}$ *by* κ. *Let* $(\mathcal{A}, \sigma)$ *be an* $\mathcal{O}$*–algebra with involution. Suppose that there exist elements* $e \in F^\times$, $s \in \mathcal{A}^\times$ *and* $g \in \mathcal{A}_F^\times$ *such that* $es = \sigma_F(g)g$. *If* $e \notin F^{\times 2}\mathcal{O}^\times$, *then* $(\mathcal{A}, \sigma)_\kappa$ *is hyperbolic.*

Proof. Note that, if $Z(\mathcal{A}) \cong \mathcal{O} \times \mathcal{O}$ then $(\mathcal{A}, \sigma)_\kappa$ is degenerate and hence automatically hyperbolic by Proposition 1.42. So, in the rest of the proof, we assume that $Z(\mathcal{A})$ is a domain.

Let v be a discrete valuation on F with valuation ring $\mathcal{O}$. Without loss of generality, we may assume that $v(e) = 1$. For let π be a uniformiser for v, then multiplying both sides of $\sigma_F(g)g = es$ with an appropriate even power of π, we obtain $\pi us = \sigma_F(g')g'$, with $u \in \mathcal{O}^\times, g' \in \mathcal{A}_F^\times$. By Theorem 2.39, there exists $\tilde{g} \in \mathcal{A}$ such that $\pi us = \sigma(\tilde{g})\tilde{g}$. By abuse of notation, we denote $\tilde{g}$ again by g in the rest of the proof.
Let $n \in \mathbb{N}$ be such that $\dim_{\mathcal{O}} \mathcal{A} = \dim_F \mathcal{A}_F = n^2$ if σ is of the first kind, and such that $\dim_{\mathcal{O}} \mathcal{A} = \dim_F \mathcal{A}_F = 2n^2$ if σ is of the second kind. Let L/F be as in Lemma 5.20 a splitting field of $\mathcal{A}_F$ such that there exists an extension $\tilde{\mathcal{O}}$ of $\mathcal{O}$ with the same value group as $\mathcal{O}$. Let w be a discrete valuation with valuation ring $\tilde{\mathcal{O}}$ and let Π be a uniformiser for w. Denote the residue field of $\tilde{\mathcal{O}}$ by $\tilde{\kappa}$. We denote the map $\mathcal{O} \to \kappa$ and the induced map $\mathcal{A} \to \mathcal{A}_\kappa$ both by $\overline{}$. We have that $\overline{\sigma(g)} = \sigma_\kappa(\overline{g})$, and since $v(e) = 1$, it follows that $0 = \sigma_\kappa(\overline{g})\overline{g}$. Since $2 \in \mathcal{O}^\times$ by assumption, i.e. $\mathrm{char}(\kappa) \neq 2$, in order to show that $(\mathcal{A}, \sigma)_\kappa$ is hyperbolic, by Lemma 5.19, it suffices to show that

$$\dim_\kappa(\overline{g}\mathcal{A}_\kappa) \geqslant \frac{1}{2}\dim_\kappa(\mathcal{A}_\kappa) = \frac{n^2}{2}.$$

We have a commutative diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\varphi"] \arrow[d, "-"'] & \mathcal{A} \otimes_{\mathcal{O}} \tilde{\mathcal{O}} \arrow[d, "-"] \\ \mathcal{A}_\kappa \arrow[r, "\psi"] & \mathcal{A}_\kappa \otimes_\kappa \tilde{\kappa}. \end{tikzcd}$$

We have that $\mathcal{A}_L \cong \mathrm{M}_n(L)$ if σ is of the first kind. Suppose that σ is of the second kind. Looking at the proof of Lemma 5.20, we may assume that $\mathcal{A}_F$ already becomes degenerate over a Henselisation of $(F, \mathcal{O})$, or that $Z(\mathcal{A}_F) \subset L$. In any case, it follows that $\mathcal{A}_L \cong \mathrm{M}_n(L) \times \mathrm{M}_n(L) \cong \mathrm{M}_n(L) \times \mathrm{M}_n(L)^{\mathrm{op}}$. It follows from Proposition 1.31 and Corollary 1.29 that $\mathcal{A} \otimes_{\mathcal{O}} \tilde{\mathcal{O}} \cong \mathrm{M}_n(\tilde{\mathcal{O}})$, and hence also $\mathcal{A} \otimes_{\tilde{\mathcal{O}}} \tilde{\kappa} \cong \mathrm{M}_n(\tilde{\kappa})$, if σ is of the first kind, and from Corollary 1.33 that $\mathcal{A} \otimes_{\mathcal{O}} \tilde{\mathcal{O}} \cong \mathrm{M}_n(\tilde{\mathcal{O}}) \times \mathrm{M}_n(\tilde{\mathcal{O}}) \cong \mathrm{M}_n(\tilde{\mathcal{O}}) \times \mathrm{M}_n(\tilde{\mathcal{O}})^{\mathrm{op}}$, and hence also $\mathcal{A} \otimes_{\tilde{\mathcal{O}}} \tilde{\kappa} \cong \mathrm{M}_n(\tilde{\kappa}) \times \mathrm{M}_n(\tilde{\kappa}) \cong \mathrm{M}_n(\tilde{\kappa}) \times \mathrm{M}_n(\tilde{\kappa})^{\mathrm{op}}$, if σ is of the second kind. Note that the involution $\sigma_{\tilde{\mathcal{O}}}$ on $\mathcal{A} \otimes_{\mathcal{O}} \tilde{\mathcal{O}}$ then corresponds to the switch involution on $\mathrm{M}_n(\tilde{\mathcal{O}}) \times \mathrm{M}_n(\tilde{\mathcal{O}})^{\mathrm{op}}$. Since

$$\dim_{\tilde{\kappa}} \psi(\overline{g})(\mathcal{A} \otimes_\kappa \tilde{\kappa}) = \dim_{\tilde{\kappa}}(\overline{g} \otimes 1)(\mathcal{A}_\kappa \otimes_\kappa \tilde{\kappa}) = \dim_{\tilde{\kappa}}(\overline{g}\mathcal{A}_\kappa \otimes_\kappa \tilde{\kappa}) = \dim_\kappa(\overline{g}\mathcal{A}_\kappa)$$

it suffices to show that

$$\dim_{\tilde{\kappa}} \psi(\overline{g})(\mathcal{A} \otimes_\kappa \tilde{\kappa}) = \dim_{\tilde{\kappa}} \overline{\varphi(g)}(\mathcal{A} \otimes_\kappa \tilde{\kappa}) \geqslant \frac{n^2}{2}.$$

Suppose that σ is of the first kind. It is well known that $\dim_{\tilde{\kappa}}(\overline{\varphi(g)}\mathcal{A}_{\tilde{\kappa}}) = \operatorname{rank}(\overline{\varphi(g)})\cdot n$. Since $\tilde{\mathcal{O}}$ is a valuation ring, it is an elementary divisor domain by Proposition 1.3. Hence, there are matrices $P, Q \in \mathrm{M}_n(\tilde{\mathcal{O}})^\times$ such that $\varphi(g) = P\operatorname{diag}(d_1,\dots,d_n)Q$, with $d_1,\dots,d_n \in \tilde{\mathcal{O}}$. Let $C = \operatorname{diag}(d_1,\dots,d_n)$. It follows that $\overline{\varphi(g)} = \overline{PCQ}$. Since $\overline{P}, \overline{Q} \in \mathrm{M}_n(\tilde{\kappa})^\times$, we have that $\operatorname{rank}(\overline{\varphi(g)}) = \operatorname{rank}(\overline{C})$. The latter is obtained by subtracting the number of $\overline{d_i} = 0$ from n. Let us denote this number by ℓ. Then ℓ is equal to the number of d_i that are divisible by Π. This number is at most $w(d_1) + \ldots + w(d_n) = w(\det(\varphi(g)))$, since $\det(P), \det(Q) \in \tilde{\mathcal{O}}^\times$. Taking determinants of the relation $e\varphi(s) = \varphi(\sigma(g))\varphi(g) = \sigma_{\tilde{\mathcal{O}}}(\varphi(g))\varphi(g)$ yields

$$e^n \det(\varphi(s)) = \det(\varphi(g))^2,$$

by [45, (2.2)]. Since $s \in \mathcal{A}^\times$, it follows that $\det(\varphi(s)) \in \tilde{\mathcal{O}}^\times$ and hence $w(\det(\varphi(g))) = \frac{n}{2}$. So, we get that

$$\dim_{\tilde{\kappa}}(\overline{\varphi(g)}(\mathcal{A}\otimes_\kappa \tilde{\kappa})) = n\cdot\operatorname{rank}(\overline{\varphi(g)}) = n(n-\ell) \geqslant n[n - w(\det(\varphi(g)))] = n^2/2.$$

Suppose that σ is of the second kind. Let $g' \in \mathrm{M}_n(\tilde{\mathcal{O}})$ and $g'' \in \mathrm{M}_n(\tilde{\mathcal{O}})^{\mathrm{op}}$ be such that $\varphi(g) = (g', g'') \in \mathcal{A}\otimes_{\mathcal{O}}\tilde{\mathcal{O}} \cong \mathrm{M}_n(\tilde{\mathcal{O}}) \times \mathrm{M}_n(\tilde{\mathcal{O}})^{\mathrm{op}}$. It follows that

$$\dim_{\tilde{\kappa}}(\overline{\varphi(g)}(\mathcal{A}\otimes_\kappa\tilde{\kappa})) = \dim_{\tilde{\kappa}}(\overline{g'}\,\mathrm{M}_n(\tilde{\kappa})) + \dim_{\tilde{\kappa}}(\overline{g''}\,\mathrm{M}_n(\tilde{\kappa})^{\mathrm{op}}).$$

Invoking the elementary divisor property of $\tilde{\mathcal{O}}$, we find matrices $P', Q', P'', Q'' \in \mathrm{M}_n(\tilde{\mathcal{O}})^\times$ and elements $d'_1,\dots,d'_n, d''_1,\dots,d''_n \in \tilde{\mathcal{O}}$ such that

$$g' = P'\operatorname{diag}(d'_1,\dots,d'_n)Q' \qquad \text{and} \qquad g'' = P''\operatorname{diag}(d''_1,\dots,d''_n)Q''.$$

We have that $\dim_{\tilde{\kappa}}(\overline{g'}\,\mathrm{M}_n(\tilde{\kappa})) = \operatorname{rank}(\overline{g'})\cdot n = n(n-\ell')$, where ℓ' is the number of indices in $i \in \{1,\dots,n\}$ such that $\overline{d'_i} = 0$, and $\dim_{\tilde{\kappa}}(\overline{g''}\,\mathrm{M}_n(\tilde{\kappa})) = \operatorname{rank}(\overline{g''})\cdot n = n(n-\ell'')$, where ℓ'' is the number of indices in $i \in \{1,\dots,n\}$ such that $\overline{d''_i} = 0$. We have that ℓ' is equal to the number of d'_i divisible by Π and ℓ'' is equal to the number of d''_i divisible by Π. As in the reasoning in the first kind case, we get that

$$\dim_{\tilde{\kappa}}(\overline{g'}\,\mathrm{M}_n(\tilde{\kappa})) \geqslant n(n - w(\det(g'))) \quad \text{and} \quad \dim_{\tilde{\kappa}}(\overline{g''}\,\mathrm{M}_n(\tilde{\kappa})^{\mathrm{op}}) \geqslant n(n - w(\det(g''))).$$

So, it follows that

$$\begin{aligned}\dim_{\tilde{\kappa}}(\overline{\varphi(g)}(\mathcal{A}\otimes_\kappa\tilde{\kappa})) &\geqslant n(n - w(\det(g'))) + n(n - w(\det(g''))) \\ &= 2n^2 - n(w(\det(g'g''))).\end{aligned} \tag{5.2.1}$$

Let $s', s'' \in \mathrm{M}_n(\tilde{\mathcal{O}})$ be such that $\varphi(s) = (s', s'')$. Since $s \in \mathcal{A}^\times$, it follows that $s', s'' \in \mathrm{M}_n(\tilde{\mathcal{O}})^\times$. Using the fact that $\sigma_{\tilde{\mathcal{O}}}$ acts as the switch involution on $\mathrm{M}_n(\tilde{\mathcal{O}}) \times \mathrm{M}_n(\tilde{\mathcal{O}})^{\mathrm{op}}$, we get $(es', es'') = (g'', g')(g', g'') = (g''g', g' * g'') = (g''g', g''g')$ in $\mathrm{M}_n(\tilde{\mathcal{O}}) \times \mathrm{M}_n(\tilde{\mathcal{O}})^{\mathrm{op}}$. It follows that $es' = g''g'$ in $\mathrm{M}_n(\tilde{\mathcal{O}})$, and hence, taking determinants, $e^n \det(s') =$

$\det(g''g')$. Applying w and using that $\det(s') \in \tilde{\mathcal{O}}^\times$, it follows that $w(\det(g''g')) = n$ (since $w(e) = v(e) = 1$). Plugging this in into (5.2.1) yields

$$\dim_{\tilde{\kappa}}(\overline{\varphi(g)}(\mathcal{A} \otimes_\kappa \tilde{\kappa})) \geqslant 2n^2 - n(w(\det(g'g''))) = n^2 \geqslant n^2/2.$$

□

5.22 Corollary. *Let $\mathcal{O}$ be a discrete valuation ring of F and assume that $2 \in \mathcal{O}^\times$. Denote its residue field by κ. Let $(\mathcal{A}, \sigma)$ be an $\mathcal{O}$−algebra with involution. Let $e \in G((\mathcal{A}, \sigma)_F)O^\times$. If $e \notin F^{\times 2}\mathcal{O}^\times$, then $(\mathcal{A}, \sigma)_\kappa$ is hyperbolic.*

Proof. By assumption, there exist $u \in \mathcal{O}^\times$ and $g \in \mathcal{A}_F^\times$ such that $eu = \sigma_F(g)g$. Proposition 5.21 yields the statement. □

The converse of Corollary 5.22, if $(\mathcal{A}, \sigma)_\kappa$ is hyperbolic then every element of $F^\times$ is a multiplier times a unit in $\mathcal{O}$, also holds. This will be shown in Corollary 5.27, in the more general setting of Dedekind domains.

5.3 Dedekind domains

In this section, we complete the alternative proof of the multipler result for discrete valuation rings in which 2 is invertible, and in fact we give the proof more generally for semilocal principal ideal domains and polynomials in one variable over a field. For semilocal principal ideal domains, we can then also use a different argument to obtain the statement of Theorem 5.16, by using the representation results from section 2.5.
The polynomial rings we consider here are the first "global" domains that pop up in this thesis in the context of Question 5.1 and the related multiplier results. One could ask whether the multiplier results might hold for arbitrary principal ideal domains, or more generally Dedekind domains, in which 2 is invertible. Since they hold for each localisation of the Dedekind domain at a prime ideal (being a discrete valuation ring), this comes down to the following question.

5.23 Question. Let R be a Dedekind domain and $(\mathcal{A}, \sigma)$ an R−algebra with involution. Is $G((\mathcal{A}, \sigma)_F)R^\times = \cap_{\mathfrak{p}} G((\mathcal{A}, \sigma)_F)R_{\mathfrak{p}}^\times$, where $\mathfrak{p}$ runs over all prime ideals of R?

One cannot expect a positive answer to this question for general Dedekind domains, not even for principal ideal domains. We give a counterexample in Example 5.47. However, under certain conditions on R or $(\mathcal{A}, \sigma)$, we can give a positive answer. The first result of this kind is given in Proposition 5.24. We can then apply this result to simultaneously obtain the desired multiplier result for semilocal principal ideal domains and polynomial rings in one variable over a field. Moreover, we can also apply Proposition 5.24 in the next section, where we consider coordinate rings of affine conics, which are Dedekind domains but need not be principal ideal domains. This is the strongest motivation to formulate Proposition 5.24 as it is.

It is not clear whether a positive answer to Question 5.23 would also yield a positive answer to Question 5.1. In order to make the jump from the multiplier results to the actual isomorphism results in the previous section, we used results involving (skew–)hermitian spaces, based on the fact that there is a cancellation law for such spaces, and Proposition 2.10. We don't know in which generality these statements hold.

A *Dedekind domain* is a Noetherian, integrally closed domain of Krull dimension one. For the rest of this section, we fix a Dedekind domain R with fraction field F. Then the localisation of R at a nonzero prime ideal is a discrete valuation ring.

5.24 Proposition. *Let $(\mathcal{A},\sigma)$ be an R–algebra with involution. Let $e \in F^\times$ be such that for every prime ideal $\mathfrak{p}$ of R, we have that $e \in R_\mathfrak{p}^\times$ or $e \notin F^{\times 2}R_\mathfrak{p}^\times$. Assume furthermore that for all prime ideals $\mathfrak{p}$ of R such that $e \notin F^{\times 2}R_\mathfrak{p}^\times$, there exists a Henselisation $(F_\mathfrak{p}^h, R_\mathfrak{p}^h)$ of $(F, R_\mathfrak{p})$ and a finite subextension $F \subset L_\mathfrak{p} \subset F_\mathfrak{p}^h$ such that the integral closure of R in $L_\mathfrak{p}$ is a principal ideal domain and $\sigma_{L_\mathfrak{p}}$ is hyperbolic. Then $e \in G((\mathcal{A},\sigma)_F)R^\times$.*

Proof. Since R is a Dedekind domain, we can factor the fractional ideal eR into prime ideals. Let $\mathfrak{p}_1,\dots,\mathfrak{p}_r$ be prime ideals of R and $\alpha_1,\dots,\alpha_r \in \mathbb{Z}$ be such that $eR = u\mathfrak{p}_1^{\alpha_1}\cdots\mathfrak{p}_r^{\alpha_r}$. Then $\alpha_1,\dots,\alpha_r$ are odd by assumption. We will show that for all $i \in \{1,\dots,r\}$, the prime ideal $\mathfrak{p}_i$ is principal, and furthermore, that there exists a generator π_i for $\mathfrak{p}_i$ such that $\pi_i \in G((\mathcal{A},\sigma)_F)$.

Let $\mathfrak{p} \in \{\mathfrak{p}_1,\dots,\mathfrak{p}_r\}$ be arbitrary. Denote the residue field of $R_\mathfrak{p}$ by κ. By assumption, there exists a Henselisation $(F_\mathfrak{p}^h, R_\mathfrak{p}^h)$ of $(F, R_\mathfrak{p})$, and a finite subextension $F \subset L_\mathfrak{p} \subset F_\mathfrak{p}^h$ such that $\sigma_{L_\mathfrak{p}}$ is hyperbolic, and such that the integral closure of R in $L_\mathfrak{p}$ is a principal ideal domain. We denote this integral closure by R'. The norm map $N_{L_\mathfrak{p}/F} : L_\mathfrak{p} \to F$ induces a norm map from the set of ideals in R' to the set of ideals in R. Let $V = R_\mathfrak{p}^h \cap L_\mathfrak{p}$. Since $(F,R) \subset (F_\mathfrak{p}^h, R_\mathfrak{p}^h)$ is an immediate extension, the residue field of V is equal to κ. There is a principal prime ideal (Π) of R' such that $V = R'_{(\Pi)}$. Then $(\Pi) \cap R = \mathfrak{p}$, and taking norms of ideals, we have that $(N_{L_\mathfrak{p}/F}(\Pi)) = N_{L_\mathfrak{p}/F}((\Pi)) = \mathfrak{p}$ (cf. [48, (I.22)]), since the residue field of V is κ. Hence, $\pi = N_{L_\mathfrak{p}/F}(\Pi)$ is a generator for the prime ideal $\mathfrak{p}$. Since $(\mathcal{A},\sigma)_{L_\mathfrak{p}}$ is hyperbolic, $G((\mathcal{A},\sigma)_{L_\mathfrak{p}}) = L_\mathfrak{p}^\times$ by Proposition 2.24 (a). Invoking part (b) of the same proposition then implies that $\pi \in G((\mathcal{A},\sigma)_F)$.
It follows that $eR = (\pi_1^{\alpha_1}\cdots\pi_r^{\alpha_r})R$, and hence, there exists $u \in R^\times$ such that $e = \pi_1^{\alpha_1}\cdots\pi_r^{\alpha_r}u \in G((\mathcal{A},\sigma)_F)R^\times$. □

5.25 Remark. In Proposition 5.24, the assumptions can be weakened in the following way. Looking at the proof, one sees that it in fact suffices to assume that for each prime ideal $\mathfrak{p}$ such that $e \notin F^{\times 2}R_{\mathfrak{p}}^{\times}$, there is a finite subextension $F \subset L_{\mathfrak{p}} \subset F_{\mathfrak{p}}^h$ such that $\sigma_{L_{\mathfrak{p}}}$ is hyperbolic, and the prime ideals of the integral closure of R in $L_{\mathfrak{p}}$ lying over $\mathfrak{p}$ are principal.

We will only apply the following lemma in the case of function fields of conics, which are possibly split, but we formulate it for more general function fields since the proof works in that generality.

5.26 Lemma. *Let k be a field and let $k(X)$ be the function field of a projective, geometrically irreducible $k-$variety. Let w be a $k-$valuation on $k(X)$ and denote its residue field by k_w. Let furthermore L/k be a finite separable field extension such that there is a $k-$embedding $\psi : L \hookrightarrow k_w$. Then there exists a Henselisation of $k(X)$ at w containing $L(X_L)$.*

Proof. Since L/k is separable, we have that $L = k(\beta) \cong k[t]/(f_\beta)$, for some $\beta \in L$ and $f_\beta \in k[t]$ the minimal polynomial of β over k. Let $f_\beta = h_1 \cdots h_s \in k_w[t]$ be the factorisation of f_β in irreducible polynomials. Then the residue fields of the valuations on $L(X_L)$ lying over w, are given by $k_w[t]/(h_1), \ldots, k_w[t]/(h_s)$. Since there is an $k-$embedding $\psi : L \hookrightarrow k_w$, f_β has a root $\psi(\beta)$ in k_w. Hence, one of $h_1, \ldots, h_s$ is of degree 1, which means that one of $k_w[t]/(h_1), \ldots, k_w[t]/(h_s)$ is isomorphic to k_w.

We denote the valuation ring of w by $\mathcal{O}$. By the first part of the proof, there is a valuation ring $\tilde{\mathcal{O}}$ of $L(X_L)$ lying over $\mathcal{O}$ with residue field k_w. Let $k(X)^s$ be a separable closure of $k(X)$ containing L. Let $\mathcal{O}^s$ be an extension of $\tilde{\mathcal{O}}$ to $k(X)^s$ and let $\mathcal{M}^s$ be its maximal ideal. Let $(k(X)^h, \mathcal{O}^h)$ be the Henselisation of $(k(X), \mathcal{O})$ with respect to $\mathcal{O}^s$. We show that $L(X_L) \subset k(X)^h$. To this end, it suffices to show $L \subset k(X)^h$. Let $x \in L$ be artibrary. Then for all $\rho \in \mathrm{Gal}(k(X)^s/k(X))$, we have that x and $\rho(x)$ are algebraic over k, and hence, $\rho(x) - x$ is also algebraic over k. By [21, (3.2.11)], the residue field of $\mathcal{O}^s$ is an algebraic closure of k_w; we denote it by k_w^a. Let $\overline{\rho}$ be the k_w-automorphism of k_w^a induced by ρ. For all $y \in k(X)^s$, we write $\bar{y}$ for $y + \mathcal{M}^s$. Then $\overline{\rho}(\bar{y}) = \overline{\rho(y)}$. Since the residue field of $\mathcal{O}^s \cap L(X_L) = \tilde{\mathcal{O}}$ is equal to k_w, we have that $\overline{\rho}(\bar{x}) = \bar{x}$, and hence $\rho(x) - x \in \mathcal{M}^s$. Since $\rho(x) - x$ is algebraic over k, it follows that $\rho(x) = x$. Hence, we conclude that $L(X_L) \subset k(X)^h$. $\square$

The generality in which Proposition 5.24 was stated now allows us to obtain the multiplier result from Theorem 5.13 (in a slightly different formulation) both for semilocal principal ideal domains and for polynomial rings in one variable over a field in which 2 is invertible. In the latter case, we recover a result obtained by J.–P. Tignol in [71, (2.6)]. As a consequence, we obtain for such rings R a positive answer to Question 5.23, i.e. for any $R-$algebra with involution $(\mathcal{A}, \sigma)$, we have that $G((\mathcal{A}, \sigma)_F)R^\times = \cap_{\mathfrak{p}} G((\mathcal{A}, \sigma)_F)R_{\mathfrak{p}}^\times$, where $\mathfrak{p}$ runs over all prime ideals of R.

5.27 Corollary. *Suppose that R is semilocal or that R is a polynomial ring in one variable over a field, and assume that $2 \in R^\times$. Let $(\mathcal{A}, \sigma)$ be an R-algebra with involution. Let $e \in F^\times$ be such that for every prime ideal $\mathfrak{p}$ of R, we have that $e \in R_\mathfrak{p}^\times$ or $e \notin F^{\times 2} R_\mathfrak{p}^\times$. Then the following are equivalent:*

(i) For all $a \in F^{\times 2}$, $ae \in G((\mathcal{A}, \sigma)_F) R^\times$.

(ii) For each discrete valuation ring $\mathcal{O}$ of F containing R such that $e \notin F^{\times 2}\mathcal{O}^\times$, we have that $(\mathcal{A}, \sigma)$ becomes hyperbolic over the residue field of $\mathcal{O}$.

Proof. That (i) implies (ii) follows from Corollary 5.22. Assume that (ii) holds. Let L/F be a finite separable field extension. Since R is a Dedekind domain it follows from [23, (II.5)] that the integral closure of R in L is also a Dedekind domain.
Suppose that R is semilocal. By Proposition 1.3, there exist valuation rings $\mathcal{O}_1, \ldots, \mathcal{O}_r$ of F such that $R = \mathcal{O}_1 \cap \ldots \cap \mathcal{O}_r$. Then the integral closure R' of R in L is the intersection of the valuation rings of L lying over $\mathcal{O}_1, \ldots, \mathcal{O}_r$ by Proposition 1.21. There are only finitely many such valuation rings and therefore R' is a semilocal Dedekind domain, and hence a principal ideal domain by [48, (I.15)]. Let $\mathfrak{p}$ be a prime ideal of R such that $e \notin F^{\times 2} R_\mathfrak{p}$. Then σ is hyperbolic over the residue field of $R_\mathfrak{p}$, since we assume that (ii) holds. Let $(F_\mathfrak{p}^h, R_\mathfrak{p}^h)$ be a Henselisation of $(F, R_\mathfrak{p})$. By Theorem 4.20, $\sigma_{F_\mathfrak{p}^h}$ is hyperbolic. By Lemma 5.9, there exists a finite separable subextension $F \subset L_\mathfrak{p} \subset F_\mathfrak{p}^h$ such that $\sigma_{L_\mathfrak{p}}$ is hyperbolic. By the above, the integral closure of R in $L_\mathfrak{p}$ is a principal ideal domain, and Proposition 5.24 yields that $ae \in G((\mathcal{A}, \sigma)_F) R^\times$ for all $a \in F^{\times 2}$.

Suppose that $R = k[t]$ for a field k. Then $F = k(t)$. Let $f(t) \in k[t]$ be an irreducible polynomial. We denote the associated valuation ring $R_{(f(t))}$ of F by $\mathcal{O}$. Suppose that $e \notin F^{\times 2}\mathcal{O}^\times$. By assumption, σ is then hyperbolic over the residue field of $\mathcal{O}$, which we denote by κ_f. Let κ_f^s be the separable closure of k in κ_f. Since $\mathrm{char}(F) \neq 2$, κ_f/κ_f^s is of odd degree and hence, by [45, (6.16)], σ is already hyperbolic over κ_f^s. Then σ is also hyperbolic over $\kappa_f^s(t)$, and the integral closure of $k[t]$ in $\kappa_f^s(t)$ is $\kappa_f^s[t]$, which is clearly a principal ideal domain. By Lemma 5.26, there exists a Henselisation of $(F, \mathcal{O})$ containing $\kappa_f^s(t)$ as a subfield. Proposition 5.24 now yields that $ae \in G((\mathcal{A}, \sigma)_F) R^\times$, for all $a \in F^{\times 2}$. □

5.28 Remark. If the center of $\mathcal{A}$ is not a domain then the properties (i) and (ii) in Proposition 5.27 both hold for trivial reasons. For if $(\mathcal{A}, \sigma)$ is degenerate then $(\mathcal{A}, \sigma)_F$ is also degenerate and hence hyperbolic. Proposition 2.24 (a) yields that $G((\mathcal{A}, \sigma)_F) = F^\times$. Furthermore, $(\mathcal{A}, \sigma)$ remains degenerate over the valuation rings lying over R and hence, $(\mathcal{A}, \sigma)$ is automatically hyperbolic over the residue fields of these valuation rings.

5.29 Theorem. *Suppose that R is semilocal and that $2 \in R^\times$. Let $\mathcal{A}$ be an Azumaya algebra with center R or a separable quadratic R-algebra. Let σ and σ' be two R-linear* *involutions of the same kind on $\mathcal{A}$. If $(\mathcal{A},\sigma)_F \simeq_{Z(\mathcal{A}_F)} (\mathcal{A},\sigma')_F$ then $(\mathcal{A},\sigma) \simeq_{Z(\mathcal{A})} (\mathcal{A},\sigma')$.*

Proof. If $Z(\mathcal{A})$ is not a domain, then the involutions are of the second kind, and hence automatically isomorphic by Proposition 1.18. So, for the rest of the proof we may assume that $Z(\mathcal{A})$ is a domain. By Proposition 2.18, there exist elements $e \in F^\times$, $s \in \mathcal{A}^\times$ and $g \in \mathcal{A}_F^\times$ such that $es = \sigma_F(g)g$. Since R is semilocal, it is a principal ideal domain by [48, (I.15)]. In particular, R is a unique factorisation domain. This implies that we can write $e = a^2e'$ where $e' \in F^\times$ is such that for all prime ideals $\mathfrak{p}$ of R, either $e' \in R_\mathfrak{p}^\times$ or $e' \notin F^{\times 2}R_\mathfrak{p}^\times$. Proposition 5.21 and Corollary 5.27 yield that $e \in G((\mathcal{A},\sigma)_F)R^\times$. Let $u \in R^\times$ and $f \in \mathcal{A}_F$ be such that $e = \sigma_F(f)fu$. Then $us = \sigma_F(fg)fg$. By Theorem 2.39, there exists $\tilde{g} \in \mathcal{A}$ such that $us = \sigma(\tilde{g})\tilde{g}$. It follows that $\tilde{g} \in \mathcal{A}^\times$. Proposition 2.18 now yields that $(\mathcal{A},\sigma) \simeq_{Z(\mathcal{A})} (\mathcal{A},\sigma')$. □

5.30 Corollary. *Suppose that R is semilocal and that $2 \in R^\times$. Let $(\mathcal{A},\sigma)$ and $(\mathcal{A}',\sigma')$ be R−algebras with involution. If $(\mathcal{A},\sigma)_F \simeq_F (\mathcal{A}',\sigma')_F$ then $(\mathcal{A},\sigma) \simeq_R (\mathcal{A}',\sigma')$.*

Proof. Since R is a semilocal Dedekind domain, it is a principal ideal domain by [48, (I.15)], and hence, in particular, a Bézout domain. The statement then follows from Proposition 4.32 together with Theorem 5.29. □

5.31 Remark. In the case where $R = k[t]$, with k a field of characteristic different from 2, Theorem 5.29 also holds for R if we assume that $(\mathcal{A},\sigma)$ and $(\mathcal{A},\sigma')$ are defined over k. In that case, we can use the representation theorem of [71] to obtain the desired isomorphism statement. Note that this isomorphism result is not new. It appears for instance in [60, (4.1)].

In view of the above results for two specific types of principal ideal domains, one could ask whether Question 5.1 has an affirmative answer for general principal ideal domains in which 2 is invertible. This cannot be expected in general. We present a counterexample in the next section in the context of conics (see Example 5.47), which yields at the same time an example where Question 5.23 has a negative answer.

5.4 Coordinate rings of affine conics

Throughout this section k denotes a field of characteristic different from 2 and Q is a k−quaternion algebra. By Remarks 3.4 (a), $\mathrm{SB}(Q)$ is a projective conic over k, which we denote by C. It is well known that there is a one-to-one correspondence between the closed points of C and the (discrete) k−valuations on $k(Q)$. We will therefore sometimes also refer to the elements of $\mathbb{V}$ as points.

We fix some notation for the rest of section 5.4. For any field extension L/k, we denote the function field of $\mathrm{SB}(Q_L)$, i.e. of C_L, by $L(Q)$. We further denote by $\mathbb{V}$ the set of all (discrete) k−valuations on $k(Q)$. Let $v \in \mathbb{V}$. We denote the valuation ring of v by $\mathcal{O}_v$ and the ring $\cap_{w\in\mathbb{V}, w\neq v}\mathcal{O}_w$ by R^v. Then R^v is a Dedekind domain with fraction field $k(Q)$. We denote the residue field of v by k_v and the degree $[k_v : k]$ of the field extension k_v/k by f_v. We will refer to the latter as the *residue degree of* v. Let L/k be a finite field extension and let $\tilde{v}$ be an extension of the valuation v to $L(Q)$. Then $\tilde{v}$ is an L−valuation on $L(Q)$. We call the degree $[L_{\tilde{v}} : k_v]$ the *relative residue degree* of $\tilde{v}$ over v.

We started studying function fields of conics in the following context. In [60, (3.1), (5.3)], A. Quéguiner–Mathieu and J.–P. Tignol used cohomological invariants for algebras with involution in order to show the following. Let B be a central simple k−algebra of degree 8 Brauer equivalent to Q. Let τ and τ' be two orthogonal involutions on B such that (B,τ) and (B,τ') are isomorphic to a tensor product of k−quaternion algebras with involution. If $(B,\tau)_{k(Q)} \simeq_{k(Q)} (B,\tau')_{k(Q)}$ then $(B,\tau) \simeq_k (B,\tau')$. K.J. Becher and A. Quéguiner–Mathieu then raised the question whether this statement could hold more generally without restrictions on the degree of B.

5.32 Question. Let B be a central simple k−algebra Brauer equivalent to Q, and let τ and τ' be two orthogonal involutions on B. Suppose that $(B,\tau)_{k(Q)} \simeq_{k(Q)} (B,\tau')_{k(Q)}$. Does this imply that $(B,\tau) \simeq_k (B,\tau')$?

In the split case, it is easily seen that the above question has an affirmative answer.

5.33 Proposition. *Suppose that Q is split and let B be a split central simple k−algebra. Let τ and τ' be two orthogonal involutions on B. If $(B,\tau)_{k(Q)} \simeq_{k(Q)} (B,\tau')_{k(Q)}$ then $(B,\tau) \simeq_k (B,\tau')$.*

Proof. Since Q is split, $k(Q)$ is isomorphic to the rational function field in one variable over k, say $k(t)$. By Proposition 2.19, the statement comes down to showing that symmetric bilinear spaces over k that become similar over $k(t)$, are already similar over k. This can be seen by passing to the Laurent series field $k((t))$, and using the weak direction of Springer's theorem (see e.g. [47, (VI.1.6)]). □

As communicated to us by J.–P. Tignol, apart from the degree 8 case mentioned above, and the case where Q is split, Question 5.32 is only known to have a positive answer in the case where $\deg(B) = 2$, $\deg(B) = 4$ or $\deg(B) = 6$ and $\mathrm{disc}(\tau)$ is trivial. The results in degree 2 and 4 follow from [72, (3.6) (c), (3.10)] and in degree 6 from [45, (15.7)]. The proofs for these cases use (low) cohomological invariants for algebras with involution, namely the discriminant and the Clifford invariant. In order to obtain statements for higher degree, one might need higher cohomological invariants, but, for the moment, there are not many cases yet where such invariants are available.

Chronologically, we considered the question on isomorphism over $k(Q)$ versus isomorphism over k before the local isomorphism questions, treated in sections 4.3.2 and 5.1. However, after we obtained the results in the local case, we turned our attention back to $k(Q)$. Applying the results for algebras with involution over semilocal Bézout domains from section 5.1, we obtain the following result.

5.34 Proposition. *Let B be a central simple k–algebra and let τ and τ' be two orthogonal involutions on B. Suppose that $(B,\tau)_{k(Q)} \cong_{k(Q)} (B,\tau')_{k(Q)}$. Let R be an arbitrary intersection of finitely many valuation rings of $k(Q)$. Then $(B,\tau)_R \cong_R (B,\tau')_R$.*

Proof. This follows immediately from Theorem 5.16. □

In order to have an isomorphism over k in Proposition 5.34, we would need to consider an intersection of infinitely many valuation rings. We do not know how to pass from finitely many to infinitely many valuation rings in general. However, under certain conditions, there is a way to get around this.

5.35 Proposition. *Let B be a central simple k–algebra Brauer equivalent to Q. Let τ be an orthogonal involution on B and let $s \in B^\times$ be such that $\tau(s) = s$. Let $\tau' = \mathrm{Int}(s) \circ \tau$. Then $(B,\tau)_{k(Q)} \cong_{k(Q)} (B,\tau')_{k(Q)}$ if and only if there exist elements $e \in k(Q)^\times$ and $g \in B^\times_{k(Q)}$ such that $es = \tau_{k(Q)}(g)g$, and in that case $e \in \bigcap_{v\in\mathbb{V}} G((B,\tau)_{k(Q)})\,\mathcal{O}_v^\times$. Moreover, with such e and g, $(B,\tau) \cong_k (B,\tau')$ if and only if $e \in G((B,\tau)_{k(Q)})k^\times$.*

Proof. By Proposition 2.18, we have that $(B,\tau)_{k(Q)} \cong_{k(Q)} (B,\tau')_{k(Q)}$, if and only if there exist elements $e \in k(Q)^\times$ and $g \in B^\times_{k(Q)}$ such that $es = \tau_{k(Q)}(g)g$. If this is the case, then $e \in \bigcap_{v\in\mathbb{V}} G((B,\tau)_{k(Q)})\,\mathcal{O}_v^\times$, by Proposition 5.21 and Corollary 5.24. Suppose that $(B,\tau) \cong_k (B,\tau')$. Invoking Proposition 2.18 once more, it follows that there exist $u \in k^\times$ and $f \in B^\times$ such that $us = \tau(f)f$. So, we obtain that $\tau_{k(Q)}(g)ge^{-1} = \tau(f)fu^{-1}$, which yields $\tau_{k(Q)}(gf^{-1})gf^{-1} = eu^{-1} \in k(Q)^\times$. This means that $\tau_{k(Q)}(gf^{-1})gf^{-1} \in G((B,\tau)_{k(Q)})$, and hence, $e \in G((B,\tau)_{k(Q)})k^\times$.

Let us prove the converse. Let $u \in k^\times$ be such that $e \in G((B,\tau)_{k(Q)})u$. By [45, (4.2)], there exists a skew–hermitian space (V,h) over (Q,γ) such that $(B,\tau) \cong_k \mathrm{Ad}(h)$. Then $(B,\tau') \cong_k \mathrm{Ad}(h')$, where $h' : V \times V \to Q$ is the skew–hermitian form over (Q,γ) defined by $h'(x,y) = h(s^{-1}(x),y)$, for all $x,y \in V$. By Proposition 2.19, since $(B,\tau)_{k(Q)} \cong_{k(Q)} (B,\tau')_{k(Q)}$, it follows that $(V,h')_{k(Q)} \simeq (V,eh)_{k(Q)}$. By Lemma 2.23, we get that $(V,h')_{k(Q)} \simeq (V,uh)_{k(Q)}$. By [14] or [57, (3.3)], since $(V,h')_{k(Q)} \perp (V,-uh)_{k(Q)}$ is hyperbolic, it follows that $(V,h') \simeq (V,uh)$. Invoking Proposition 2.18 once more, we get $(B,\tau) \cong_k (B,\tau')$. □

In view of Proposition 5.35, we can now reformulate Question 5.32 as follows.

5.36 Question. Let B be a central simple k−algebra Brauer equivalent to Q and let τ be an orthogonal involution on B. Is $G((B,\tau)_{k(Q)})k^\times = \cap_{v\in\mathbb{V}} G((B,\tau)_{k(Q)})\,\mathcal{O}_v^\times$?

We were not able to give a complete answer to Question 5.36, but in the sequel we prove that under certain conditions, one can show that an element of $\cap_{v\in\mathbb{V}} G((B,\tau)_{k(Q)})\,\mathcal{O}_v^\times$ is contained in $G((B,\tau)_{k(Q)})k^\times$. In order to do this, we apply the general result of Proposition 5.24 to Dedekind domains R that are obtained as coordinate rings of an affine part of C (i.e. by putting one point of C at infinity), and make the result more explicit. This is done in Theorem 5.42. In the proof, we use the geometry of the conic, in particular the structure of its Picard group, or equivalently the structure of the class group of its associated coordinate ring.
Question 5.36 is of a similar nature as Question 5.23. However, if we consider Question 5.23 in the case where R is the coordinate ring of an affine part of C as above, then we only consider the "finite" k−valuations on $k(Q)$, whereas we consider all k−valuations on $k(Q)$ in Question 5.36. In Example 5.47, we give an example, under a certain assumption on k and the assumption that C is non–split, of a split quaternion algebra with orthogonal involution (B,τ), for which $G((B,\tau)_{k(Q)})k^\times = \cap_{v\in\mathbb{V}} G((B,\tau)_{k(Q)})\,\mathcal{O}_v^\times$, but $G((B,\tau)_{k(Q)})k^\times \neq \cap_{v\in\mathbb{V},v\neq v_\infty} G((B,\tau)_{k(Q)})\,\mathcal{O}_v^\times$, where v_∞ is a degree two point on C. Note however that this is not the context we consider in this section, since B is split, but Q is not, but it gives some indication that Questions 5.23 and 5.36 concern different problems, when we consider the latter question without the assumption that $B \sim Q$.

Let us also compare Question 5.36 with the results in [56]. There, I. Panin proved a purity result and using this result, he then obtained a positive answer to Question 5.1 for regular local rings containing a field of characteristic different from 2.

5.37 Theorem (Panin). *Let T be a regular local ring containing a field of characteristic different from 2, with fraction field L, and let $(\mathcal{A},\sigma)$ be a T−algebra with involution. Let $\mathbb{U}$ be the set of height one prime ideals of T. Then*

$$G((\mathcal{A},\sigma)_L)T^\times = \bigcap_{\mathfrak{p}\in\mathbb{U}} G((\mathcal{A},\sigma)_L)T_{\mathfrak{p}}^\times.$$

As mentioned in the introduction of this section, we will use the geometry of the conic C in order to give a partial answer to Question 5.36. Therefore, we start with some preliminaries on the Picard group of an affine conic and the correspondence with the class group of its associated coordinate ring R. We further describe how one can detect whether a prime ideal of R is principal. The arguments are mainly an exercise on the Riemann–Roch theorem. The reader who is familiar with the Picard group and the class group can skip the results from here on until Remark 5.40.

The *group of divisors* $\mathrm{Div}(C)$ of C is the free abelian group generated by the closed points of C, i.e.

$$\mathrm{Div}(C) := \left\{ \sum_{v\in\mathbb{V}} n_v v \;\middle|\; \text{almost all } n_v = 0 \right\}.$$

The group of divisors can be defined more generally than for conics, but we since we only use it in the case of conics, we define it only in this specific case. The homomorphism

$$\deg : \mathrm{Div}(C) \to \mathbb{Z};\ \sum_{v\in\mathbb{V}} n_v v \mapsto \sum_{v\in\mathbb{V}} f_v n_v,$$

is called *the degree homomorphism*. The subgroup of $\mathrm{Div}(C)$ of divisors of degree zero, the kernel of deg, is denoted by $\mathrm{Div}^0(C)$.

For every function $z \in k(Q)^\times$ one has that $v(z) = 0$ for almost all $v \in \mathbb{V}$. So, with every $z \in k(Q)$ one can associate an element of $\mathrm{Div}(C)$, namely $\mathrm{div}(z) = \sum_{v\in\mathbb{V}} v(z)v$. This is called *the divisor of z*. The set

$$\mathrm{Pr}(C) = \{\mathrm{div}(z) \mid z \in k(Q)^\times\},$$

is a subgroup of $\mathrm{Div}(C)$, called *the group of principal divisors*. It is well known that $\mathrm{Pr}(C)$ is a subgroup of $\mathrm{Div}^0(C)$, and it follows from the Riemann–Roch theorem (see [30, (IV.1.3)]) that $\mathrm{Pr}(C) = \mathrm{Div}^0(C)$. (This is no longer true for more general curves.)

We are interested in analogues of Corollary 5.27 for coordinate rings of affine parts of C. These coordinate rings are Dedekind domains. In particular, we are interested in the coordinate rings of smooth affine conics, obtained by considering one point v_0 of C as the point at infinity. For the rest of this section, we fix v_0 and we let $C^{\mathrm{aff}} = C \setminus \{v_0\}$ and $\mathbb{V}^{\mathrm{aff}} = \mathbb{V} \setminus \{v_0\}$. Then the coordinate ring of C^{aff} is R^{v_0}. One also defines the divisor group for C^{aff}, by considering only divisors with support in C^{aff},

$$\mathrm{Div}(C^{\mathrm{aff}}) := \{d \in \mathrm{Div}(C) \mid n_{v_0} = 0\}.$$

For $z \in k(Q)^\times$ we set $\mathrm{div}^{\mathrm{aff}}(z) = \sum_{v\in\mathbb{V}^{\mathrm{aff}}} v(z)v$. Note that the group homomorphism

$$\mathrm{Pr}(C) = \mathrm{Div}^0(C) \to \mathrm{Div}(C^{\mathrm{aff}});\ \sum_{v\in\mathbb{V}} n_v v \mapsto \sum_{v\in\mathbb{V}, v\neq v_0} n_v v,$$

is injective. Its image is $\mathrm{Pr}(C^{\mathrm{aff}}) := \{\mathrm{div}^{\mathrm{aff}}(z) \mid z \in k(Q)^\times\}$. These are the principal divisors on C^{aff}, and the quotient

$$\mathrm{Pic}(C^{\mathrm{aff}}) := \mathrm{Div}(C^{\mathrm{aff}})/\mathrm{Pr}(C^{\mathrm{aff}})$$

is called *the Picard group* of the affine curve C^{aff}.

Since $\mathrm{Div}(C^{\mathrm{aff}})$ is a subgroup of $\mathrm{Div}(C)$, the restriction of the degree map to $\mathrm{Div}(C^{\mathrm{aff}})$ defines a group homomorphism $\deg^{\mathrm{aff}} : \mathrm{Div}(C^{\mathrm{aff}}) \to \mathbb{Z}$. The following proposition says that the Picard group of C^{aff} is a cyclic group.

5.38 Proposition. *Let $d \in \mathbb{Z}$ be such that $\mathrm{im}(\deg^{\mathrm{aff}}) = d\mathbb{Z}$. Then the Picard group $Pic(C^{\mathrm{aff}})$ of C^{aff} is cyclic of order f_{v_0}/d. Furthermore, if C has a $k-$rational point then $d = 1$, otherwise $d = 2$.*

Proof. We have that d is the greatest common divisor of the f_v with $v \in \mathbb{V}^{\mathrm{aff}}$. If C has a $k-$rational point, then C^{aff} also has a $k-$rational point, and this point has degree 1. If C does not have a $k-$rational point, then it has points of degree 2, since Q splits over a quadratic extension of k. Then C^{aff} also has points of degree 2. Furthermore, C does not have points of odd residue degree by Springer's theorem (see e.g. [19, (71.3)]). Hence, $d = 2$ in that case. Let $z \in k(Q)^\times$. Then $\deg^{\mathrm{aff}}(z) = \sum_{v \neq v_0} f_v v(z) = -f_{v_0} v_0(z)$, since $\sum_{v \in \mathbb{V}} v(z) = 0$. It follows that the image of $\mathrm{Pr}(C^{\mathrm{aff}})$ under $\deg^{\mathrm{aff}}$ is a subgroup of $f_{v_0}\mathbb{Z}$. The homomorphism $\deg^{\mathrm{aff}}$ therefore induces a surjective homomorphism

$$\deg^{\mathrm{aff}} : \mathrm{Pic}(C^{\mathrm{aff}}) \to d\mathbb{Z}/f_{v_0}\mathbb{Z}.$$

Let $\omega \in \mathrm{Div}(C^{\mathrm{aff}})$ and suppose that $\deg^{\mathrm{aff}}(\omega) \in f_0\mathbb{Z}$, say $\deg^{\mathrm{aff}}(\omega) = f_{v_0} s$, for some $s \in \mathbb{Z}$. It follows that $\omega - sv_0$ is a divisor of degree zero on C, so a principal divisor. It follows that $\omega = \mathrm{div}^{\mathrm{aff}}(y) \in \mathrm{Pr}(C^{\mathrm{aff}})$, for some $y \in k(Q)^\times$. This shows that $\deg^{\mathrm{aff}} : \mathrm{Pic}(C^{\mathrm{aff}}) \to d\mathbb{Z}/f_{v_0}\mathbb{Z}$ is injective, and hence an isomorphism. □

There is a one-to-one correspondence between the elements of $\mathbb{V}^{\mathrm{aff}}$ and prime ideals of R^{v_0}. The valuation ring of a valuation in $\mathbb{V}^{\mathrm{aff}}$ is the localisation of R^{v_0} at a prime ideal. For each $v \in \mathbb{V}^{\mathrm{aff}}$, we denote the associated prime ideal of R^{v_0} by $\mathfrak{p}_v$. This correspondence yields that $\mathrm{Pic}(C^{\mathrm{aff}})$ is isomorphic to the class group of R^{v_0}. We briefly explain this below. A *fractional ideal of* R^{v_0} is a $R^{v_0}-$submodule I of $k(Q)$ such that there exists a nonzero element $t \in R^{v_0}$ such that $tI \subset R^{v_0}$. Two fractional ideals I and J of R^{v_0} are called *equivalent* if there exist $a, b \in R^{v_0}$ such that $aI = bJ$. The set of equivalence classes of fractional ideals of R^{v_0} forms a group for multiplication, called *the class group of* R^{v_0}. Furthermore, since R^{v_0} is a Dedekind domain, any fractional ideal of R^{v_0} can be factored into prime ideals of R^{v_0}. The correspondence $v \leftrightarrow \mathfrak{p}_v$ between valuations in $\mathbb{V}^{\mathrm{aff}}$ and prime ideals of R^{v_0} then induces a one-to-one correspondence between divisors in $\mathrm{Div}(C^{\mathrm{aff}})$ and fractional ideals of R^{v_0}. Namely, the divisor $\sum_{v \in \mathbb{V}^{\mathrm{aff}}} n_v v$ corresponds to the fractional ideal $\prod_{v \in \mathbb{V}^{\mathrm{aff}}} \mathfrak{p}_v^{n_v}$ of R^{v_0}. Principal divisors correspond to principal ideals. So the correspondence induces an isomorphism between the Picard group of C^{aff} and the class group of R^{v_0}. Furthermore, the class group of R^{v_0} is trivial if and only if R^{v_0} is a principal ideal domain, if and only if R^{v_0} is a unique factorisation domain. Proposition 5.38 yields the following result.

5.39 Corollary. *Let $v \in \mathbb{V}$. Then its corresponding prime ideal $\mathfrak{p}_v$ in R^{v_0} is principal if and only if $f_{v_0} \mid f_v$. Let $v' \neq v_0$ be a valuation on $k(Q)$ of minimal residue degree a ($a = 1$ if C has a rational point, and 2 otherwise). Then the class group of R^{v_0} is cyclic of order f_{v_0}/a. Furthermore, $(R^{v_0})^\times = k^\times$.*

Proof. The prime ideal $\mathfrak{p}_v$ is principal if and only if the divisor $v \in \mathrm{Div}(C^{\mathrm{aff}})$ is principal. By the proof of Proposition 5.38, this is the case if and only if $f_{v_0} \mid f_v$. The claim on the class group follows immediately from Proposition 5.38. We prove the claim on the units in R^{v_0}. Let $x \in (R^{v_0})^\times$. Then $x \in \mathcal{O}_w^\times$, for all $w \neq v_0$. Hence, $w(x) = 0$, for all $w \neq v_0$. Since $v_0(x) \leqslant 0$, it follows that necessarily $v_0(x) = 0$, since $0 = \sum_{w \in \mathbb{V}} f_w w(x) = f_{v_0} v_0(x)$. So, $x \in \cap_{w \in \mathbb{V}} \mathcal{O}_w = k$. □

5.40 Remark. The fact that R^{v_0} is a principal ideal domain if Q is split or if v_0 is a degree two point is well known. In the split case, R^{v_0} is isomorphic to the polynomial ring in one variable over k, and if Q is a division algebra and v_0 a degree two point, it was shown in [58] that R^{v_0} is a principal ideal domain, without using the Riemann–Roch theorem.

5.41 Lemma. *Let $w \in \mathbb{V}$. Let L/k be a finite field extension and let $w \in \mathbb{V}$. Suppose that k_w/k is a separable field extension and that there is a k−embedding $\varphi : k_w \hookrightarrow L$. Then there is a valuation on $L(Q)$ lying over w that is of residue degree 1 (over L).*

Proof. Since k_w/k is separable by assumption, we can write $k_w = k(\alpha) \cong k[t]/(f_\alpha)$, for some $\alpha \in k_w$, with $f_\alpha \in k[t]$ the minimal polynomial of α over k. The residue fields of the extensions of w to $L(Q) \cong k(Q) \otimes_k L$, are then obtained by considering the factorisation of $f_\alpha(t)$ over L. More precisely, let $f_\alpha = g_1 \cdots g_r \in L[t]$ be the factorisation of f_α in irreducible polynomials. Then $L[t]/(g_1), \ldots, L[t]/(g_r)$ are the residue fields of the valuations of $L(Q)$ lying over w. Since there is an k−embedding $\varphi : k_w \hookrightarrow L$, f_α has a root $\varphi(\alpha)$ in L. Hence, one of $g_1, \ldots, g_r$ is of degree 1, which means that one of $L[t]/(g_1), \ldots, L[t]/(g_r)$ is isomorphic to L. □

5.42 Theorem. *Let (B, τ) be an k−algebra with involution. Let $e \in k(Q)^\times$. Assume that for all $v \in \mathbb{V}$ such that $v(e)$ is odd, we have that τ_{k_v} is hyperbolic. Assume furthermore that there exists a finite field extension k_0/k splitting Q such that $k_0 \hookrightarrow k_v$ for all $v \in \mathbb{V}$ such that $v(e)$ is odd, and such that $[k_0 : k] \mid [k_v : k]$ for all $v \in \mathbb{V}$ such that $v(e)$ is even and nonzero. Then $e \in G((B, \tau)_{k(Q)})k^\times$.*

Proof. We write $\mathbb{W} = \{v \in \mathbb{V} \mid v(e) \neq 0\} = \mathbb{W}_{\mathrm{odd}} \cup \mathbb{W}_{\mathrm{even}}$, with $\mathbb{W}_{\mathrm{odd}} = \{v \in \mathbb{W} \mid v(e) \text{ is odd}\}$ and $\mathbb{W}_{\mathrm{even}} = \{v \in \mathbb{W} \mid v(e) \text{ is even}\}$.

Suppose first that k is a finite field. Then Q is split, and hence $k(Q)$ is isomorphic to the rational function field in one variable over k, say $k(t)$. Since τ_{k_v} is assumed to be hyperbolic for all $v \in \mathbb{W}_{\text{odd}}$, it follows from Corollary 5.27 that $e \in G((B,\tau)_{k(Q)})k[t]^\times$. Since $k[t]^\times = k^\times$, this proves the statement.

So, for the rest of the proof, we assume that k is an infinite field. Let k_0^s be the separable closure of k in k_0. Then k_0/k_0^s is a purely inseparable extension. Since $\operatorname{char}(k) \neq 2$, this means that k_0/k_0^s is an odd degree extension. It follows from Springer's theorem (see e.g. [47, (VII.2.7)]), it follows that Q already splits over k_0^s. Therefore, we may assume that k_0/k is a separable extension. Since Q splits over k_0, there is an $k-$place $\lambda : k(Q) \to k_0^\infty$. This means that k_0 contains, up to $k-$isomorphism, a residue field of an $k-$valuation of $k(Q)$. Without loss of generality, we may assume that k_0 is the residue field of an $k-$valuation v_0 on $k(Q)$. Suppose that $[k_{v_0} : k] = d$.

Since k is infinite, and Q_{k_0} is split, the conic C_{k_0} has infinitely many rational points. This means that there are infinitely many $k-$valuations of $k(Q)$ with residue field isomorphic to k_0. Since $\mathbb{W}$ is a finite set of valuations, we may therefore assume that $v_0 \notin \mathbb{W}$. This means that $v_0(e) = 0$. Consider the coordinate ring R^{v_0} of the affine conic $C^{\text{aff}} = C \setminus \{v_0\}$. Recall that $R^{v_0} = \cap_{v \neq v_0} \mathcal{O}_v$. By Lemma 5.39, R^{v_0} is a Dedekind domain with a cyclic class group of order $d/2$.

For any $v \in \mathbb{W}$, we let k_v^s be the separable closure of k in k_v and $\mathfrak{p}_v$ the prime ideal of R^{v_0} corresponding to v. Then $k_v^s(Q)/k(Q)$ is a separable extension of degree equal to $[k_v^s : k]$. Note that Q already splits over k_v^s. Note furthermore that even though R^{v_0} may not be a principal ideal domain, the assumption on k_0 yields that $d \mid [k_v : k]$ for all $v \in \mathbb{W}$, and hence $\mathfrak{p}_v$ is a principal ideal for every $v \in \mathbb{W}$ by Lemma 5.39. We write $\mathfrak{p}_v = (\pi_v)$. We can factor the principal fractional ideal eR^{v_0} in prime ideals:

$$eR^{v_0} = \prod_{v \in \mathbb{W}} \mathfrak{p}_v^{v(e)} = \left(\prod_{v \in \mathbb{W}_{\text{odd}}} \pi_v^{v(e)} \prod_{v' \in \mathbb{W}_{\text{even}}} \pi_{v'}^{v'(e)} \right).$$

Let $a = \prod_{v' \in \mathbb{W}_{\text{even}}} \pi_{v'}^{v'(e)}$ and $e' = \prod_{v \in \mathbb{W}_{\text{odd}}} \pi_v^{v(e)}$. Then $a \in k(Q)^{\times 2}$. By Lemma 5.39, there exists an element $u \in (R^{v_0})^\times = k^\times$ such that that $e = ue'a$. Clearly, we have for all $v \in \mathbb{V}^{\text{aff}}$ that $v(ue') = 0$ or $v(ue')$ is odd.

Let $v \in \mathbb{W}_{\text{odd}}$ and let S_v be the integral closure of R in $k_v^s(Q)$. This is a Dedekind domain. There are finitely many valuations $w_1, \dots, w_r$ of $k_v^s(Q)$ lying over v_0. We have that $S_v = \cap_{w \neq w_1, \dots, w_n} \mathcal{O}_w$. Since k_0/k is separable, the $k-$embedding $k_0 \hookrightarrow k_v$ already yields an $k-$embedding $k_0 \hookrightarrow k_v^s$. By Lemma 5.41 (i), one of the w_i has residue field k_v^s, say w_1. S_v contains the set $S'_v = \cap_{w \neq w_1} \mathcal{O}_w$, which is the coordinate ring of the affine conic $C \times_k k_v^s \setminus \{w_1\}$. Since w_1 is a rational point of $C \times_k k_v^s$, S'_v is a principal ideal domain by Lemma 5.39. Since S_v is obtained from S'_v by localisation, it is also a principal ideal domain.

By Lemma 5.26, $k_v^s(Q)$ is a subfield of a suitable Henselisation $k_v(Q)^h$ of $k(Q)$ at v. Furthermore, since τ_{k_v} is hyperbolic by assumption, and k_v/k_v^s is of odd degree since $\mathrm{char}(k) \neq 2$, [45, (6.16)] yields that τ is hyperbolic over k_v^s. It follows that τ is also hyperbolic over $k_v^s(Q)$. Hence, we can apply Proposition 5.24 to obtain that $e \in G((B,\tau)_{k(Q)})(R^{v_0})^\times = G((B,\tau)_{k(Q)})k^\times$. □

5.43 Corollary. *Let B be a central simple k−algebra Brauer equivalent to Q. Let τ be an orthogonal involution on B. Let $s \in B^\times$ and let $\tau' = \mathrm{Int}(s) \circ \tau$. Suppose that there exists elements $e \in k(Q)^\times$ and $g \in B_{k(Q)}^\times$ such that $es = \tau_{k(Q)}(g)g$. Suppose furthermore that there exists a finite field extension k_0/k splitting Q such that for all $v \in \mathbb{V}$ such that $v(e)$ is odd, we have that $k_0 \hookrightarrow k_v$, and for all $v \in \mathbb{V}$ such that $v(e)$ is even, we have that $[k_0 : k] \mid [k_v : k]$. Then $(B,\tau) \cong_k (B,\tau')$.*

Proof. Suppose first that k is a finite field. Then Q and B are necessarily split. In that case, the statement follows from Propositions 5.33 and 5.35. Suppose that k is an infinite field. Let $v \in \mathbb{V}$ be such that $v(e)$ is odd. Applying Proposition 5.21 to $(\mathcal{A},\sigma) = (B,\tau)_{\mathcal{O}_v}$, it follows that τ_{k_v} is hyperbolic. Hence, the assumptions of Theorem 5.42 are satisfied, and we obtain that $e \in G((B,\tau)_{k(Q)})k^\times$. □

Using Corollary 5.43, we obtain in a special case, a positive answer to Question 5.32.

5.44 Corollary. *Let B be a central simple k−algebra Brauer equivalent to Q. Let τ be an orthogonal involution on B. Suppose that $\mathrm{disc}(\tau) = \delta \in k^\times/k^{\times 2}$ is nontrivial, and that Q splits over $k(\sqrt{\delta})$. Let τ' be any orthogonal involution on B. If $(B,\tau)_{k(Q)} \cong_{k(Q)} (B,\tau')_{k(Q)}$ then $(B,\tau) \cong_k (B,\tau')$.*

Proof. If Q is split, then we are done by Proposition 5.33. By Proposition 2.20, there exists an element $s \in B^\times$ such that $\tau' = \mathrm{Int}(s) \circ \tau$. Suppose that $(B,\tau)_{k(Q)} \cong_{k(Q)} (B,\tau')_{k(Q)}$. By Proposition 2.18, there exists an element $e \in k(Q)^\times$ such that $es \in G((B,\tau)_{k(Q)})$. Let $v \in \mathbb{V}$ be such that $v(e)$ is odd. Since $\mathcal{O}_v$ is discrete, τ_{k_v} is hyperbolic by Proposition 5.21. Hence, $\mathrm{disc}(\tau_{k_v})$ is trivial. Therefore, there is a k−embedding $k(\sqrt{\delta}) \hookrightarrow k_v$. Furthermore, since Q is non–split, for any k−valuation v on $k(Q)$, we have that $[k_v : k]$ is even. Hence, $[k(\sqrt{\delta}) : k] \mid [k_v : k]$ for all $v \in \mathbb{V}$. Since Q splits over $k(\sqrt{\delta})$, there exists $v_0 \in \mathbb{V}$ such that $k_{v_0} \cong k(\sqrt{\delta})$. The statement now follows from Proposition 5.12. □

We thank J.–P. Tignol for pointing out the following observation.

5.45 Corollary. *Let B be a central simple k−algebra Brauer equivalent to Q. Let τ and τ' be two orthogonal involutions on B. Suppose that $\mathrm{disc}(\tau) = \delta \in k^\times/k^{\times 2}$ is nontrivial, and that (B,τ) has an improper similitude. If $(B,\tau)_{k(Q)} \cong_{k(Q)} (B,\tau')_{k(Q)}$ then $(B,\tau) \cong_k (B,\tau')$.*

Proof. Let f be an improper similitude of (B,τ) and $\mu = \tau(f)f$. By [45, (13.38)], B is Brauer equivalent to $(\delta,\mu)_F$. Hence, B splits over $F(\sqrt{\delta})$ and Corollary 5.44 applies. □

5.46 Remark. The method in the proof of Theorem 5.42 does not yield in general that local hyperbolicity conditions imply that the given element e is a multiplier of $(B,\tau)_{k(Q)}$ times a unit in k. However, the method does give some information on how far away e is from being a multiplier of $(B,\tau)_{k(Q)}$. Namely, adapting the proof of Theorem 5.42, one can show that the (finite) set of $k-$valuations on $k(Q)$ not vanishing in e, can be replaced by a set of $k-$valuations whose residue fields are minimal splitting fields for Q, and moreover such that these residue fields are all different. We think this is the maximum one can get out of these techniques using the Picard group and the class group.

We finish this section with the counterexample to Questions 5.1 and 5.23, announced at the end of section 5.3 and in the discussion after Question 5.36. We consider two non–singular 2−dimensional bilinear forms over the coordinate ring R of an affine part of C that is a principal ideal domain, that become similar over the fraction field of R, but are not similar over R. By taking the adjoint algebras with involution of these bilinear forms, we obtain an example of two involutions on the split Azumaya algebra $\mathrm{M}_2(R)$ that become isomorphic over the fraction field of R, but are not isomorphic over R. The fact that similarity of these bilinear forms over R is equivalent to $R-$isomorphism of their adjoint algebras with involution follows from the proof of Proposition 2.19. Although the statement there is formulated for semilocal Bézout domains, the only property that is used is that the $R-$automorphisms of Azumaya algebras over R are inner. By [4, (3.6)], this holds in the case where R is a principal ideal domain, since finitely generated, projective modules over a principal ideal domain are free.

In the example below, we will use results that are formulated for quadratic forms instead of bilinear forms, but since we work with rings in which 2 is invertible, we can switch between both concepts.

5.47 Example. Assume that k is such that C is non–split, and such that there exist degree two points v_0 and v_∞ of C with $k_{v_0} \ncong k_{v_\infty}$. Let $a \in k^\times$ be such that $k_{v_0} = k(\sqrt{a})$. By Corollary 5.39, the ring R^{v_∞} is a principal ideal domain. Let S be the integral closure of R^{v_∞} in $k_{v_0}(C)$. By Corollary 5.39, S is not a principal ideal domain, since v_∞ does not split in $k_{v_0}(C)$. Hence, there exists a prime ideal $\mathfrak{p}$ of S which is not principal. Let $\pi \in R^{v_\infty}$ be a prime element such that $\mathfrak{p} \cap R^{v_\infty} = \pi R^{v_\infty}$. Then $\pi S = \mathfrak{p}\overline{\mathfrak{p}}$, where $\mathfrak{p}$ is the conjugate of $\mathfrak{p}$. Since $\mathfrak{p}$ is not a principal ideal, it follows that for all $u \in (R^{v_\infty})^\times = k^\times$, $u\pi$ is not represented by $\langle 1, -a\rangle$ over R^{v_∞}.

We next show that the bilinear form $\langle \pi, -a\pi\rangle$ is non–singular over R^{v_∞}. To this end, by [54, (IV.3.1)], it suffices to show $\langle 1, -a\rangle$ is hyperbolic over the residue field of π. This is satisfied since π splits in S, since this implies that the residue field of π is equal to the one of $\mathfrak{p}$, and the latter contains $k_{v_0} = k(\sqrt{a})$. The reasoning in the first paragraph shows that for all $u \in (R^{v_\infty})^\times = k^\times$, we have that that $u\langle \pi, -a\pi \rangle \not\simeq_{R^{v_\infty}} \langle 1, -a \rangle$. So, $\langle 1, -a \rangle$ and $\langle \pi, -a\pi \rangle$ are not similar over R^{v_∞}. However, since $\pi \in k(Q)^\times$, $\langle 1, -a \rangle$ and $\langle \pi, -a\pi \rangle$ are clearly similar over $k(Q)$.

This example also provides a negative answer to Question 5.23. In order to show this, we make a particular choice of the prime element π. Namely, since R^{v_∞} is a principal ideal domain, we can choose π such that $v_0(\pi) = 1$ and $v(\pi) = 0$ for all $v \neq v_0, v_\infty$. Since v_0 and v_∞ are both points of degree 2, it follows that $v_\infty(\pi) = -1$. Since the residue field of π is k_{v_0}, πS factors into two different prime ideals. These prime ideals both have residue field k_{v_0} and hence, they are not principal by Corollary 5.39.
We have that $\pi \in \mathcal{O}_v^\times$ for all $v \neq v_0, v_\infty$, and since $\langle 1, -a \rangle$ is hyperbolic over k_{v_0}, Corollary 5.27 yields that $\pi \in \mathcal{O}_{v_0}^\times G(\langle 1, -a \rangle_{k(Q)})$. Hence, $\pi \in \cap_{v \in \mathbb{V}, v \neq v_\infty} \mathcal{O}_v^\times G(\langle 1, -a \rangle_{k(Q)})$. However, $\pi \notin \mathcal{O}_{v_\infty}^\times G(\langle 1, -a \rangle_{k(Q)})$. This can be seen using Corollary 5.22 and the fact that $\langle 1, -a \rangle$ is anisotropic over k_{v_∞}. It follows that $\pi \notin k^\times G(\langle 1, -a \rangle_{k(Q)})$.

Even though we showed above that $\cap_{v \in \mathbb{V}, v \neq v_\infty} \mathcal{O}_v^\times G(\langle 1, -a \rangle_{k(Q)}) \neq k^\times G(\langle 1, -a \rangle_{k(Q)})$, we now show that

$$\bigcap_{v \in \mathbb{V}} \mathcal{O}_v^\times G(\langle 1, -a \rangle_{k(Q)}) = k^\times G(\langle 1, -a \rangle_{k(Q)}).$$

The inclusion $\supseteq$ is clear. So, let $f \in \cap_{v \in \mathbb{V}} \mathcal{O}_v^\times G(\langle 1, -a \rangle_{k(Q)})$. If $f \in G(\langle 1, -a \rangle_{k(Q)})$ then there is nothing to prove. So, we may assume that $\langle 1, -a, -f, af \rangle$ is anisotropic. We show that $\langle 1, -a, -f, af \rangle$ is obtained by scalar extension from a non–singular bilinear form $\langle 1, -c, -d, cd \rangle$ over k. Since $k(Q)/k$ is excellent, by [20, (2.10)], it suffices to show that the Witt class of $\langle 1, -a, -f, af \rangle$ is contained in the image of the map $W(k) \to W(k(Q))$. By [58, Theorem 6b], this is the case if and only if for each $v \in \mathbb{V}$, the second residue form with respect to v of $\langle 1, -a, -f, af \rangle$ is trivial in $W(k_v)$. For each $v \in \mathbb{V}$, there exist by assumption $u_v \in \mathcal{O}_v^\times$ and $g_v \in G(\langle 1, -a \rangle_{k(Q)})$ such that $f = u_v g_v$. It follows that $\langle 1, -a, -f, af \rangle \simeq_{k(Q)} \langle 1, -a, -u_v, au_v \rangle$, and hence the second residue form with respect to each v is trivial. Therefore, by the above, there exist $c, d \in k^\times$ such that $\langle 1, -a, -f, af \rangle \simeq_{k(Q)} \langle 1, -c, -d, cd \rangle$. Since $\langle 1, -a, -f, af \rangle$ becomes hyperbolic over $k(\sqrt{a})(C)$, and $k(\sqrt{a})(C)$ is a purely transcendental extension of $k(\sqrt{a})$, it follows that $\langle 1, -c, -d, cd \rangle$ becomes hyperbolic over $k(\sqrt{a})$, and hence, we may assume that $c = a$ by [47, (VII.3.2)]. Consider the quaternion algebras $H = (a, f)_{k(Q)}$ and $H' = (a, d)_{k(Q)}$. Since $\langle 1, -a, -f, af \rangle \simeq_{k(Q)} \langle 1, -a, -d, ad \rangle$, we have that $H \cong_{k(Q)} H'$ and using [47, (III.2.11)], it follows that $H \otimes_{k(Q)} H' \sim (a, fd)$ is split. This implies that $\langle 1, -a, -fd, afd \rangle$ is hyperbolic over $k(Q)$. Since $d \in k^\times$, this means that $f \in k^\times G(\langle 1, -a \rangle_{k(Q)})$.

6

Generic isotropy and hyperbolicity

Description begins in the writer's imagination, but should finish in the reader's.

Stephen King

In [40], M. Knebusch viewed the function field of a non–singular quadratic form over a field of characteristic not 2, and different from the hyperbolic plane, as a field extension of F making the quadratic form generically isotropic. This function field is a purely transcendental extension of the function field of the projective quadric associated to the quadratic form. In this chapter we investigate whether algebras with involution of the first kind over fields also have such a generic isotropy field.
Throughout this chapter F denotes a field of characteristic different from 2.

6.1 Question. Let (B, τ) be an $F-$algebra with involution. When does there exist a field extension N/F such that τ_N is isotropic (resp. hyperbolic), and for every field extension L/F such that τ_L is isotropic (resp. hyperbolic), there is an $F-$place $\lambda : N \to L^\infty$? We call a field with these properties *a generic isotropy (resp. hyperbolicity) field for τ*.

In the study of the above question, the varieties associated to $F-$algebras with involution, studied in chapter 3, play in important role. By investigating the isotropy behaviour of the involution over the function fields of these varieties, we are able to give partial answers to Question 6.1.

We begin the chapter with some basics on quadratic forms, many of which are of course very similar to concepts we introduced in section 2.1. In the development of the algebraic theory of quadratic forms, forms that are either anisotropic or hyperbolic over any field extension of the ground field, so–called Pfister forms, play an important role. We consider involution analogues of Pfister forms, which have already been studied in the literature, and we give an overview of known, but sometimes unpublished, results relating both analogues. In section 6.2, we then use one of these analogues in order to obtain a weak analogue of a factorisation statement for Pfister forms.
The rest of the chapter is then concerned with the study of Question 6.1. We first investigate the existence of a generic hyperbolicity field in section 6.3. In the following sections, we turn to the generic isotropy question, and we mainly study the case of orthogonal involutions. We present some results relating the Anisotropic Splitting Conjecture for orthogonal involutions to Question 6.1, in particular in section 6.7, where we touch the question which involutions have a generic isotropy field that can be realised as the function field of a quadratic form. In the last section, we consider algebras with involution of low degree (4,6 and 8), and we give complete characterisations for the existence of a generic isotropy field, except in the case where the algebra has degree 8 and the involution is orthogonal of nontrivial discriminant. The advantage of these characterisations is that they contain conditions that are usually easier to check in practice than some of conditions we obtain in earlier sections, where we don't restrict the degree of the algebra.

6.2 Notation. We recall the notation we introduced in chapter 3 for the varieties associated to algebras with involution over fields, and their function fields (if applicable). Let (B,τ) be an F−algebra with involution of the first kind and let $i \in \mathbb{N}$. Let $\mathrm{IV}_i(B,\tau)$ be the F−variety described in chapter 3. If $\mathrm{IV}_i(B,\tau)$ is a projective, geometrically integral F−variety, then as in chapter 3, we denote its function field by $F_i(\tau)$. If τ is orthogonal of trivial discriminant, then we write $C(B,\tau) = C_+ \times C_-$ and we denote the corresponding irreducible components of $\mathrm{IV}_{\deg(B)/2}(B,\tau)$ by $\mathrm{IV}_+(B,\tau)$ and $\mathrm{IV}_-(B,\tau)$. As in chapter 3, we denote the function field of $\mathrm{IV}_+(B,\tau)$ (resp. $\mathrm{IV}_-(B,\tau)$) by $F_+(\tau)$ (resp. $F_-(\tau)$).

6.1 Involution analogues of Pfister forms

Let V be a finite–dimensional F−vector space. A *quadratic form on* V is a map $q : V \to F$ such that for all $a \in F$ and all $v \in V$, $q(av) = a^2q(v)$, and furthermore, the map $b_q : V \times V \to F$ defined by $b_q(v,w) = q(v+w) - q(v) - q(w)$ is a symmetric bilinear form over F. Since we assume that F does not have characteristic 2, there is a one-to-one correspondence between symmetric bilinear forms on $V \times V$ and quadratic forms on V. Given a symmetric bilinear form $b : V \times V \to F$, its associated quadratic form $q : V \to F$ is defined by $q(v) = \frac{1}{2}b(v,v)$.

Let $q : V \to F$ be a quadratic form. We write $\dim(q) = \dim_F(V)$ and call this *the dimension of* q. We call q *non–singular* if its associated bilinear form is non–singular. In that case, we call (V, q) a *quadratic space over* F. A quadratic space (V, q) is called *isotropic (resp. hyperbolic)* if its corresponding symmetric bilinear space is isotropic (resp. hyperbolic). Two quadratic spaces are called *isometric, similar or Witt equivalent* if their associated symmetric bilinear spaces are isometric, similar or Witt equivalent. We denote isometry by $\simeq$ and Witt equivalence by $\sim$. The adjoint algebra with involution of a quadratic space (V, q) over F is the adjoint algebra with involution of its corresponding symmetric bilinear space, and we denote it by $\mathrm{Ad}(q)$.

We recall some concepts such as the discriminant, Clifford algebra and Witt ring from the theory of quadratic forms, which we will use repeatedly. We refer to [47] for more details.
Quadratic spaces over F always have an orthogonal basis by [47, (I.2.4)]. Let $a_1, \ldots, a_m \in F^\times$. Then we denote by $\langle a_1, \ldots, a_m \rangle$ the quadratic space (F^m, q), where $q : F^m \to F; (x_1, \ldots, x_m) \mapsto \sum_{i=1}^m a_i x_i^2$. We also use the standard notation $\mathbb{H}$ for *the hyperbolic plane*, i.e. the isometry class of $\langle 1, -1 \rangle$. Let $(V, q) = \langle a_1, \ldots, a_m \rangle$. The *discriminant of* (V, q) *(or* q*)* is $(-1)^{m(m-1)/2} a_1 \cdots a_m \in F^\times / F^{\times 2}$, denoted by $\mathrm{disc}(q)$. For the definitions and structure theorems concerning the Clifford algebra $C(q)$ and the even Clifford algebra $C_0(q)$ of (V, q) (or q), we refer to [47, Chapter V]. In [45, (8.8)], it is shown that $C_0(q) \cong C(\mathrm{Ad}(q))$. Furthermore, by [47, (V.2.5)], if $\mathrm{disc}(q) = 1 \in F^\times / F^{\times 2}$, then the simple components of $C_0(q)$ are isomorphic, and Brauer equivalent to $C(q)$. The *Clifford invariant of* q is defined as the Brauer class of $C(q)$ in $\mathrm{Br}(F)$ if $\dim(q)$ is even, and as the Brauer class of $C_0(q)$ in $\mathrm{Br}(F)$ if $\dim(q)$ is odd, and denoted by $c(q)$.

For the formal definition of the orthogonal sum (resp. the tensor product) of quadratic spaces, denoted by $\perp$ (resp. $\otimes_F$), we refer to [47, p. 8, pp. 17–18]. Let $b_1, \ldots, b_n \in F^\times$. Then

$$\langle a_1, \ldots, a_m \rangle \perp \langle b_1, \ldots, b_n \rangle \simeq \langle a_1, \ldots, a_m, b_1, \ldots, b_n \rangle \quad \text{and}$$
$$\langle a_1, \ldots, a_m \rangle \otimes_F \langle b_1, \ldots, b_n \rangle \simeq \langle a_1 b_1, \ldots, a_1 b_n, a_2 b_1, \ldots, a_2 b_n, \ldots, a_m b_1, \ldots, a_m b_n \rangle.$$

In this chapter, we study involution analogues of the concept of "Pfister forms" in the theory of quadratic spaces over F. Let $n \in \mathbb{N}$. Let (V, q) be a quadratic space over F. Then q is called *an n–fold Pfister form* if (V, q) has an orthogonal basis such that with respect to this basis,

$$(V, q) \simeq \langle 1, -a_1 \rangle \otimes_F \ldots \otimes_F \langle 1, -a_n \rangle$$

for certain $a_1, \ldots, a_n \in F^\times$. We denote $\langle 1, -a_1 \rangle \otimes_F \ldots \otimes_F \langle 1, -a_n \rangle$ also by $\langle\langle a_1, \ldots, a_n \rangle\rangle$. Pfister forms over F have the property that they are either anisotropic or hyperbolic over any field extension of F (see [47, (X.1.7)]). Sometimes we will not specify the foldness of q, and just use the term *Pfister form*.

Let (V,q) be a quadratic space over F. Then there is the *Witt decomposition* $(V,q) \simeq (V,q)_{\mathrm{an}} \perp (V,q)_{\mathrm{hyp}}$, where $(V,q)_{\mathrm{an}}$ is anisotropic and $(V,q)_{\mathrm{hyp}}$ hyperbolic. The quadratic space $(V,q)_{\mathrm{an}}$ is called *the anisotropic part of* (V,q).
Together with the operations $\perp$ and $\otimes_F$, the set of Witt equivalence classes of quadratic spaces over F forms a ring, called *the Witt ring of* F, and denoted by $W(F)$. Let (V,q) be a quadratic space over F, then we denote its class in $W(F)$ by $[V,q]$ or sometimes just $[q]$. The classes of even–dimensional quadratic spaces over F form an ideal in $W(F)$, called *the fundamental ideal*, and denoted by $I(F)$. For $n \in \mathbb{N}$, the $n-$th power of $I(F)$ is denoted by $I^n(F)$. It is easy to show that $I^n(F)$ is additively generated by the classes of $n-$fold Pfister forms over F (see e.g. [47, (X.1.2)]). We will sometimes use the shorthand notation $q \in W(F)$ (resp. $q \in I^n(F)$) for $[V,q] \in W(F)$ (resp. $[V,q] \in I^n(F)$).

By abuse of notation, in the sequel, we will sometimes also use the notation for diagonal quadratic spaces, isometry, Witt equivalence, the orthogonal sum and the tensor product signs, for quadratic forms without mentioning the vector space they are defined on.

In the literature, two possible involution analogues of Pfister forms have been studied, on the one hand taking the decomposability property into binary forms as starting point, and on the other hand the fact that Pfister forms are either anisotropic or hyperbolic over every field extension of F.

Let (B,τ) be an $F-$algebra with involution of the first kind. Then (B,τ) is called *an $n-$fold totally decomposable algebra with involution* if there exists $n \in \mathbb{N}$ and quaternion algebras with involution of the first kind $(Q_1,\tau_1),\ldots,(Q_n,\tau_n)$ over F such that

$$(B,\tau) \cong_F (Q_1,\tau_1) \otimes_F \ldots \otimes_F (Q_n,\tau_n).$$

Sometimes we will not specify the foldness and just speak about *totally decomposable algebras with involution*. We say (B,τ) is *a Pfister algebra with involution* if for every field extension L/F, τ_L is either anisotropic or hyperbolic.

Pfister algebras with orthogonal involution and totally decomposable algebras with orthogonal involution behave similarly if we look at their discriminant and Clifford algebra. We show this below.

6.3 Proposition. *Let (B,τ) be a Pfister algebra with orthogonal involution such that* $\deg(B) > 4$. *Then* $\mathrm{disc}(\tau)$ *is trivial and one of the components of* $C(B,\tau)$ *is split.*

Proof. Since (B,τ) is a Pfister algebra with involution, by [66, (4.5.4)], there exists a Pfister form q over $F(B)$ such that $(B,\tau)_{F(B)} \cong \mathrm{Ad}(q)$. So, in particular, $\deg(B)$ is even, say $\deg(B) = 2n$. Since $\mathrm{disc}(\tau_{F(B)}) = \mathrm{disc}(q) = 1 \in F(B)^\times/F(B)^{\times 2}$ by [45, (7.3) (3)], and F is algebraically closed in $F(B)$ by Proposition 3.1 (a), it follows that $\mathrm{disc}(\tau) = 1 \in F^\times/F^{\times 2}$.
By hypothesis, τ is hyperbolic over $F_1(\tau)$. By Corollary 3.13, this implies that at least one of C_+ and C_- splits over $F_1(\tau)$. Suppose that C_+ splits over $F_1(\tau)$. Using the Schur index reduction formulas from Theorem 3.11, we obtain

$$\begin{aligned}\mathrm{ind}(C_+ \otimes_F F_1(\tau)) &= \min(\mathrm{ind}(C_+), \mathrm{ind}(C_-), 2^{n-2}\,\mathrm{ind}(C_+ \otimes_F C_+), 2^{n-2}\,\mathrm{ind}(C_+ \otimes_F C_-)) \\ &= 1.\end{aligned}$$

Since $n > 2$ by assumption, we get that C_+ or C_- already splits over F. □

Totally decomposable $F-$algebras with orthogonal involution of degree at least 8 also have trivial discriminant and their Clifford algebra has a split component. In fact, this already holds for more general decomposable algebras with involution, as we show below.

6.4 Proposition. *Let $(B_1,\tau_1),(B_2,\tau_2)$ and (B_3,τ_3) be $F-$algebras with involution of even degree. Let $(B,\tau) = (B_1,\tau_1)\otimes_F (B_2,\tau_2)\otimes_F (B_3,\tau_3)$ and assume that τ is orthogonal. Then $\mathrm{disc}(\tau)$ is trivial and one of the components of $C(B,\tau)$ is split.*

Proof. Let us write $(B,\tau) = (B_1,\tau_1)\otimes_F (\tilde{B},\tilde{\tau})$. Since τ is orthogonal, τ_1 and $\tilde{\tau}$ must have the same type, by [45, (2.23)]. Suppose that τ_1 and $\tilde{\tau}$ are both symplectic. Then $\mathrm{disc}(\tau)$ is trivial, by [45, (7.3) (5)], and Theorem 1.37 immediately yields that one of the components of $C(B,\tau)$ is split. Suppose that τ_1 and $\tilde{\tau}$ are both orthogonal. Since $\deg(\tilde{B})$ is even, [45, (7.3) (4)] yields that $\mathrm{disc}(\tau)$ is trivial. Similarly, $\mathrm{disc}(\tilde{\tau})$ is trivial. Since $\deg(\tilde{B}) \equiv 0 \bmod 4$, Theorem 1.37 yields that one of the components of $C(B,\tau)$ is Brauer equivalent to the quaternion algebra $(\mathrm{disc}(\tau_1), \mathrm{disc}(\tilde{\tau}))_F$, which is split since $\mathrm{disc}(\tilde{\tau})$ is trivial. □

The link between Pfister forms and totally decomposable $F-$algebras with orthogonal involution has been made more explicit by K.J. Becher, thereby confirming the Pfister Factor Conjecture.

6.5 Theorem (Becher). *Let (B,τ) be a split totally decomposable $F-$algebra with orthogonal involution. Then (B,τ) is adjoint to a Pfister form over F.*

Proof. See [8, Theorem 1]. □

This has inspired K.J. Becher to state the following conjecture, which he confirmed in the case of algebras of Schur index 2 (see [8, Corollary, Theorem 2]).

6.6 Conjecture. *Let (B,τ) be an F–algebra with involution of the first kind of degree a power of 2. Then (B,τ) is totally decomposable if and only if (B,τ) is a Pfister algebra with involution.*

One implication of this conjecture follows from the results in [8], together with recent work of N.A. Karpenko and J.–P. Tignol.

6.7 Theorem (Karpenko, Tignol).

(a) *Let (B,τ) be an F–algebra with orthogonal involution. If τ becomes hyperbolic over $F(B)$ then τ is already hyperbolic over F (Karpenko).*

(b) *Let (B,τ) be an F–algebra with symplectic involution. If τ becomes hyperbolic over the function field of $\mathrm{SB}_2(B)$, then τ is already hyperbolic (Tignol).*

Proof. See [37, Theorem 1.1, Theorem A.1]. Some special cases of (a) were already known before Karpenko proved it in the general case, for example the case $\mathrm{ind}(B) = 2$ follows from a result on skew–hermitian spaces shown independently in [14] and [57, (3.3)]. □

6.8 Theorem. *Totally decomposable F–algebras with involution are Pfister algebras with involution.*

Proof. This follows by combining [8, Corollary] and Theorem 6.5 with Theorem 6.7. □

As far as we know, the full conjecture has only been confirmed in low degree. We collect the known cases below.

6.9 Theorem. *Conjecture 6.6 is true for $\deg(B) \leqslant 8$ in the orthogonal case, and for $\deg(B) \leqslant 4$ in the symplectic case.*

Proof. See [5, (2.10)] for the orthogonal case and [64, Theorem B] for the symplectic case. □

There are nice characterisations of total decomposability in low degree. We collect them in the following proposition for convenience. By Theorem 6.9, in the cases treated in the next proposition, total decomposability is equivalent to being a Pfister algebra with involution.

6.10 Proposition. *Let (B,τ) be an F–algebra with involution of the first kind.*

(a) *Suppose that $\deg(B) = 4$ and τ is symplectic. Then (B,τ) is totally decomposable.*

(b) *Suppose that $\deg(B) = 4$ and τ is orthogonal. Then the following are equivalent:*

(i) (B,τ) *is totally decomposable.*

(ii) *For* $i = 1,2$*, there exist* F*–quaternion algebras* Q_i *with canonical involution* γ_i *such that*

$$(B,\tau) \cong (Q_1,\gamma_1) \otimes_F (Q_2,\gamma_2).$$

(iii) $\mathrm{disc}(\tau)$ *is trivial.*

(c) *Suppose that* $\deg(B) = 8$ *and* τ *is orthogonal. Then the following are equivalent:*

(i) (B,τ) *is totally decomposable.*

(ii) $\mathrm{disc}(\tau)$ *is trivial and one of the components of* $C(B,\tau)$ *is split.*

Proof. See [64, Theorem B], resp. [6, (3.4)], for (a) in the case where B is a division algebra, resp. $\mathrm{ind}(B) = 2$. In the case where B is split, τ is adjoint to a non–singular alternating bilinear form over F. Since such forms are classified up to isometry by their dimension by [49, (XIV.9.2)], all split degree 4 algebras with symplectic involution are isomorphic, and hence, in particular isomorphic to a split degree 4 totally decomposable F–algebra with symplectic involution. The equivalences in (b) are proved in [44] and (c) is shown in [45, (42.11)]. □

In the symplectic degree 8 case, Conjecture 6.6 also holds. This follows from results in [26] on the discriminant of a symplectic involution, as was communicated to us by J.–P. Tignol. We thank him for his permission to include this result here. Let (B,τ) be an F–algebra with symplectic involution such that $8 \mid \deg(B)$. The *discriminant of* (B,τ), denoted $\Delta(B,\tau)$ is an element of the cohomology group $H^3(F,\mu_2)$. For details on and properties of $\Delta(B,\tau)$ we refer to [26].

6.11 Proposition. *Let* (B,τ) *be an* F*–algebra with symplectic involution and assume that* $8 \mid \deg(B)$*. If* (B,τ) *is a Pfister algebra with involution then* $\Delta(B,\tau) = 0$.

Proof. If B is split, then τ is hyperbolic, and it follows from [26, Theorem B] that $\Delta(B,\tau) = 0$. If $\mathrm{ind}(B) = 2$, then it follows from [8, Corollary] that (B,τ) is totally decomposable and [26, (3.2)] implies that $\Delta(B,\tau) = 0$. So, assume that $\mathrm{ind}(B) > 2$. Since B carries an involution of the first kind, B has exponent 2 by [45, (3.1) (1)], and is therefore Brauer equivalent to a tensor product of F–quaternion algebras (e.g. see [47, p. 138]), say $B \sim Q_1 \otimes_F Q_2 \otimes_F \ldots \otimes_F Q_\ell$. Let q_{12} be the Albert form corresponding to $Q_1 \otimes_F Q_2$. Over $F(q_{12})$, we get

$$B \otimes_F F(q_{12}) \sim Q \otimes_F Q_3 \ldots \otimes_F Q_\ell,$$

for some F–quaternion algebra Q. Moreover, the restriction map

$$H^3(F,\mu_2) \to H^3(F(q_{12}),\mu_2)$$

is injective by [1, (5.6)] and [41, (8.2)]. If $\operatorname{ind}(B \otimes_F F(q_{12})) > 2$, we repeat the same process with the Albert form associated to two quaternion algebras out of $Q, Q_3, \ldots, Q_\ell$. We continue this process of taking function fields of Albert forms until we obtain, after a finite number of steps, a field extension F'/F such that $\operatorname{ind}(B_{F'}) = 2$, and such that

$$H^3(F,\mu_2) \to H^3(F',\mu_2)$$

is injective. By [8, Corollary], it follows that $(B,\tau)_{F'}$ is totally decomposable. Hence, $\Delta((B,\tau)_{F'}) = 0$. The injectivity of the map $H^3(F,\mu_2) \to H^3(F',\mu_2)$ then yields $\Delta(B,\tau) = 0 \in H^3(F,\mu_2)$. □

6.12 Theorem. *Let (B,τ) be an $F-$algebra of degree 8 with symplectic involution. Then the following are equivalent:*

(i) (B,τ) *is totally decomposable.*

(ii) (B,τ) *is a Pfister algebra with involution.*

(iii) $\Delta(B,\tau) = 0$.

Proof. In [26, Theorem B], the authors prove the equivalence of (i) and (iii). The implication (i) ⇒ (ii) follows from Theorem 6.8, and that (ii) implies (iii) follows from Proposition 6.11. □

6.2 A factorisation statement

In this section, we consider the following question.

6.13 Question. Let (B,τ) and (C,ρ) be Pfister algebras with orthogonal involution over F. Suppose that there exists an $F-$algebra with orthogonal involution (B',τ') such that $(C,\rho) \cong (B,\tau) \otimes_F (B',\tau')$. Does there exist a Pfister algebra with orthogonal involution $(\tilde{B},\tilde{\tau})$ over F such that $(C,\rho) \cong (B,\tau) \otimes_F (\tilde{B},\tilde{\tau})$?

This question was inspired by the following result for quadratic forms.

6.14 Theorem. *Let π and q be a Pfister forms over F. Then the following are equivalent:*

(i) $q \simeq \pi \otimes q'$, *for some quadratic form q' over F.*

(ii) $q \simeq \pi \otimes \pi'$, *for some Pfister form π' over F.*

Proof. See [47, (X.4.11), (X.4.13)]. □

6.15 Remarks.

(a) A positive answer to Question 6.13 cannot be expected in general. In [51, (1.1)], the authors showed that, given a degree 8 Pfister algebra with orthogonal involution (C,ρ) over F, if $\mathrm{ind}(C) \leqslant 4$, then there exists a biquaternion algebra with involution (D,θ) and $\lambda \in F^\times$ such that $(C,\rho) \cong (D,\theta) \otimes_F \mathrm{Ad}(\langle 1,-\lambda\rangle)$. In [68], the author showed that, if $\mathrm{ind}(C) = 4$, then there are examples where (D,θ) cannot be chosen to be a Pfister algebra with involution.

(b) As shown in [47, (X.4.11), (X.4.13)], conditions (i) and (ii) in Theorem 6.14 are also equivalent to the condition that π is a subform of q. One could ask whether there is an analogue of Theorem 6.14 for involutions starting from this seemingly weaker assumption. The notion "subform" could then be replaced by the notion "orthogonal summand". We do not consider this question in this thesis.

In this section, we consider Question 6.13 in a special case. We show that, given Pfister algebras with involution (B,τ) and (C,ρ) over F such that $(C,\rho) = (B,\tau) \otimes_F \mathrm{Ad}(\varphi)$ for some quadratic form φ over F satisfying certain conditions, then φ can be replaced by a Pfister form.

Let (B,τ) be an F−algebra with involution of the first kind and let (V,φ) be a quadratic space over F. We denote by φ_τ the hermitian form over (B,τ) determined by a matrix representation of φ, and we call φ_τ the *hermitian form associated to* φ.

6.16 Proposition. *Let (B,τ) an F−algebra with involution of the first kind. Let (V,φ) be a quadratic space over F and φ_τ its associated hermitian form over (B,τ). Then* $\mathrm{Ad}(\varphi_\tau) \cong (B,\tau) \otimes_F \mathrm{Ad}(\varphi)$.

Proof. Let $n = \dim(V)$ and let $(C,\rho) = (B,\tau) \otimes \mathrm{Ad}(\varphi)$. Let $\mathfrak{B} = (e_1,\dots,e_n)$ be an orthogonal basis for (V,φ) and let $a_1,\dots,a_n \in F^\times$ be such that $\varphi \simeq \langle a_1,\dots,a_n\rangle$ with respect to $\mathfrak{B}$. We have that $(e_1 \otimes 1,\dots,e_n \otimes 1)$ is a B−basis for $V \otimes_F B$. We define a map

$$\vartheta : A = \mathrm{End}_F(V) \otimes_F B \longrightarrow \mathrm{End}_B(V \otimes_F B)$$

by mapping a generator $f \otimes \alpha \in \mathrm{End}_F(V) \otimes_F B$ to the endomorphism defined on generators by $v \otimes d \mapsto f(v) \otimes \alpha d$. Then ϑ is in fact an F−algebra homomorphism. Since A is a central simple algebra, we get that ϑ is injective, and hence surjective by dimension reasons. So, ϑ is an F−algebra isomorphism. In order to prove the statement, we need to show that for all $x, y \in V \otimes_F B$ and all $g \in \mathrm{End}_B(V \otimes_F B)$

$$h(x, g(y)) = h(\mathrm{ad}_h(g)(x), y).$$

We have that $x = \sum_{i=1}^{n}(e_i \otimes 1)x_i$ and $y = \sum_{j=1}^{n}(e_j \otimes 1)y_j$, with $x_1, \dots, x_n, y_1, \dots, y_n \in B$. It follows that

$$h(x, g(y)) = h\left(\sum_{i=1}^{n}(e_i \otimes 1)x_i, g\left(\sum_{j=1}^{n}(e_j \otimes 1)y_j\right)\right) = \sum_{i,j=1}^{n} \tau(x_i) h\left(e_i \otimes 1, g(e_j \otimes 1)\right) y_j$$

and

$$\begin{aligned} h(\mathrm{ad}_h(g)(x), y) &= h\left(\mathrm{ad}_h(g)\left(\sum_{i=1}^{n}(e_i \otimes 1)x_i\right), \sum_{j=1}^{n}(e_j \otimes 1)y_j\right) \\ &= \sum_{i,j=1}^{n} \tau(x_i) h\left(\mathrm{ad}_h(g)(e_i \otimes 1), e_j \otimes 1\right) y_j. \end{aligned}$$

So, it is sufficient to prove for $i, j = 1, \dots, n$ that

$$h\left(e_i \otimes 1, g(e_j \otimes 1)\right) = h\left(\mathrm{ad}_h(g)(e_i \otimes 1), e_j \otimes 1\right).$$

Let $i, j \in \{1, \dots, n\}$ be arbitrary. Since ad_h is F–linear, we may assume that g is of the form $f \otimes \alpha \in \mathrm{End}_F\, V \otimes_F B$. For $k = 1, \dots, n$, let $\alpha_{kj}, \alpha'_{ki} \in F$ such that $f(e_j) = \sum_{k=1}^{n} e_k \alpha_{kj}$ and $\mathrm{ad}_\varphi(f)(e_i) = \sum_{k=1}^{n} e_k \alpha'_{ki}$. We have that

$$h\left(e_i \otimes 1, (f \otimes \alpha)(e_j \otimes 1)\right) = a_i (f(e_j) \otimes \alpha)_i = a_i \alpha_{ij} \alpha \qquad \text{and}$$

$$\begin{aligned} h\left(\mathrm{ad}_h(f \otimes \alpha)(e_i \otimes 1), e_j \otimes 1\right) &= h\left(\mathrm{ad}_\varphi(f)(e_i) \otimes \tau(\alpha), e_j \otimes 1\right) \\ &= a_j \tau\left((\mathrm{ad}_\varphi(f)(e_i) \otimes \tau(\alpha))_j\right) \\ &= a_j \tau(\alpha'_{ji} \tau(\alpha)) = a_j \tau(\tau(\alpha)\alpha'_{ji}) = a_j \alpha'_{ji} \alpha, \end{aligned}$$

where we used that B commutes with F and τ is trivial on F. If $\alpha \in B^\times$, then it remains to show that $a_i \alpha_{ij} = a_j \alpha'_{ji}$. This can be rewritten as the condition $b_\varphi(e_i, f(e_j)) = b_\varphi(\mathrm{ad}_\varphi(f)(e_i), e_j)$, which is satisfied by the properties of ad_φ. □

6.17 Lemma. *Let φ a 2^n–dimensional non–singular quadratic form over F. Let (B, τ) be a Pfister algebra with orthogonal involution. Let furthermore $(C, \rho) = (B, \tau) \otimes \mathrm{Ad}(\varphi)$ and assume that (C, ρ) is a Pfister algebra with involution. Let $t \leqslant n - 1$ and ψ a non–singular quadratic form with $\dim(\psi) < 2^{t+1}$, such that $\psi \perp \varphi \in I^{t+1}(F)$. Then $(B, \tau) \otimes \mathrm{Ad}(\psi)$ is hyperbolic.*

Proof. By Theorem 6.7 (a), it suffices to show that $(B, \tau) \otimes \mathrm{Ad}(\psi)$ is hyperbolic over $F(B)$. Let q be a quadratic form over $F(B)$ such that $(B, \tau) \cong \mathrm{Ad}(q)$. Since (B, τ) is a Pfister algebra with involution, q is anisotropic or hyperbolic over any field extension of $F(B)$. By [66, (4.5.4)], q is similar to a Pfister form, and since q is only determined up to similarity we may assume that q is a Pfister form. So, we have that $(C,\rho)_{F(B)} \cong \mathrm{Ad}(q \otimes \varphi_{F(B)})$. Let $L = F(B)(q \otimes \varphi_{F(B)})$. It is clear that ρ is isotropic over L, and hence hyperbolic since it is Pfister. In $W(L)$, we have that $q \otimes \psi \sim q \otimes (\psi \perp \varphi)$. By assumption, the latter is in $I^{r+t+1}(L)$, and hence, $q \otimes \psi \in I^{r+t+1}(L)$ as well. Since $\dim(q \otimes \psi) < 2^{r+t+1}$, the Arason–Pfister theorem (see [66, (4.5.6)]) implies that $q \otimes \psi$ is hyperbolic over L. Now, by assumption, $q \otimes \varphi$ is similar to a Pfister form. So $(q \otimes \psi)_{F(B)}$ becomes hyperbolic over the function field of a Pfister form over $F(B)$. Since $\dim(q \otimes \psi) < 2^{r+t+1} \leqslant \dim(q \otimes \varphi)$, this means that $q \otimes \psi$ is already hyperbolic over $F(B)$. Hence, $(B,\tau) \otimes \mathrm{Ad}(\psi)$ is hyperbolic over F. □

6.18 Corollary. *Let $n \geqslant 2$ and φ a 2^n−dimensional non–singular quadratic form over F. Let (B,τ) be a Pfister algebra with orthogonal involution. Let $(C,\rho) = (B,\tau) \otimes \mathrm{Ad}(\varphi)$ and assume that (C,ρ) is a Pfister algebra with involution. Then there exists a 2^n−dimensional form $\tilde{\varphi}$ over F of trivial discriminant such that $(C,\rho) \cong_F (B,\tau) \otimes \mathrm{Ad}(\tilde{\varphi})$.*

Proof. Let $\mathrm{disc}(\varphi) = d \in F^\times/F^{\times 2}$. If d is a square, there is nothing to prove, so let us assume that $\mathrm{disc}(\varphi)$ is nontrivial. Since φ is only determined up to a scalar, we may assume that $\varphi \simeq \langle d \rangle \perp \varphi'$, for some quadratic form φ' over F. Then $\tilde{\varphi} = \langle 1 \rangle \perp \varphi' \sim \varphi \perp \langle 1, -d \rangle \in I^2(F)$. By Lemma 6.17, $(B,\tau) \otimes \mathrm{Ad}(\langle 1,-d \rangle)$ is hyperbolic. By Lemma 6.16, $(C,\rho) \cong \mathrm{Ad}(\varphi_\tau)$ and $(B,\tau) \otimes \mathrm{Ad}(\langle 1,-d \rangle) \cong \mathrm{Ad}(\langle 1,-d \rangle_\tau)$. In $W(B,\tau)$, we now have that, $\varphi_\tau \sim \varphi_\tau \perp \langle 1,-d \rangle_\tau \sim \tilde{\varphi}_\tau$. This implies that $(B,\tau) \otimes \mathrm{Ad}(\varphi)$ is Witt equivalent to $(B,\tau) \otimes \mathrm{Ad}(\tilde{\varphi})$ by Proposition 2.30, and since the algebras have the same degree, Proposition 2.31 yields that $(B,\tau) \otimes \mathrm{Ad}(\varphi) \cong_F (B,\tau) \otimes \mathrm{Ad}(\tilde{\varphi})$. □

6.19 Proposition (Elman–Lam). *Let π_1 and π_2 be two Pfister forms over F. Suppose that $i_w(\pi_1 \perp -\pi_2) \geqslant 2^r$. Then there exists an r−fold Pfister form π over F, and Pfister forms q_1 and q_2 over F such that $\pi_1 \simeq \pi \otimes_F q_1$ and $\pi_2 \simeq \pi \otimes_F q_2$.*

Proof. This result is stated in [47, (X.5.13)] under the additional hypothesis that π_1 and π_2 have the same dimension, but this assumption is not used in the proof. □

6.20 Proposition. *Let $n \geqslant 2$ and let φ be a 2^n−dimensional non–singular quadratic form over F in $I^{n-1}(F)$. Assume that φ is Witt equivalent to an $(n-1)$−fold Pfister form in $I^{n-1}(F)/I^n(F)$. Then there exists an $(n-2)$−fold Pfister form θ over F and a 4−dimensional quadratic form θ' over F such that $\varphi \simeq \theta \otimes_F \theta'$.*

Proof. By assumption, there exists an $(n-1)$−fold Pfister form π over F such that $\varphi \equiv \pi$ mod $I^n(F)$. It follows that $\varphi \perp \pi \in I^n(F)$. Furthermore, we may scale φ such that it represents -1. Then $\dim((\varphi \perp \pi)_{\mathrm{an}}) < 2^n + 2^{n-1}$. By Vishik's Gap Theorem (see [47, (X.5.20)]), the dimension of $(\varphi \perp \pi)_{\mathrm{an}}$ is equal to $2^{n+1} - 2^{i+1}$, for some $i \in [0,n]$. Note that $2^n + 2^{n-1} - 2 < 2^{n+1} - 2^{n-1}$. Hence, $(\varphi \perp \pi)_{\mathrm{an}}$ is of dimension at most 2^n. Since this is a form in $I^n(F)$, by the Arason–Pfister theorem ([66, (4.5.6)]), this implies that either $\varphi \perp \pi$ is hyperbolic, or its anisotropic part has dimension 2^n. In the latter case, again invoking the Arason–Pfister theorem, we get that $(\varphi \perp \pi)_{\mathrm{an}}$ is either anisotropic or hyperbolic over every field extension of F, and hence similar to an n–fold Pfister form by [66, (4.5.4)]. In any case, there is an n–fold Pfister form $\tilde{\varphi}$ over F, possibly hyperbolic, such that $[\varphi \perp \pi] = [\tilde{\varphi}] \in W(F)$. It follows that $\dim((\tilde{\varphi} \perp -\pi)_{\mathrm{an}}) \leqslant 2^n$. This implies that $i_w(\tilde{\varphi} \perp -\pi) \geqslant 2^{n-2}$. It follows from Proposition 6.19 that there exists an $(n-2)$–fold Pfister form θ over F, and elements $a, b, c \in F^\times$ such that $\pi \simeq \theta \otimes_F \langle\langle a \rangle\rangle$, and $\tilde{\varphi} \simeq \theta \otimes_F \langle\langle b, c \rangle\rangle$. Because of dimension reasons, it follows that $\varphi \simeq \theta \otimes_F \langle -a, b, c, bc \rangle$, proving the statement. □

6.21 Proposition. *Let $n \geqslant 2$. Let φ be a 2^n–dimensional non–singular quadratic form over F in $I^{n-1}(F)$ Witt equivalent to an $(n-1)$–fold Pfister form in $I^{n-1}(F)/I^n(F)$. Let (B, τ) be a Pfister algebra with orthogonal involution. Let $(C, \rho) = (B, \tau) \otimes \mathrm{Ad}(\varphi)$ and assume that (C, ρ) is a Pfister algebra with involution. Then there is an n–fold Pfister form $\tilde{\varphi}$ over F such that $(C, \rho) \cong_F (B, \tau) \otimes \mathrm{Ad}(\tilde{\varphi})$.*

Proof. By Proposition 6.20, there exists an $(n-2)$–fold Pfister form θ over F and a quadratic form θ' over F of dimension 4 such that $\varphi \simeq \theta \otimes_F \theta'$. It follows that $(C, \rho) \cong (B, \tau) \otimes_F \mathrm{Ad}(\theta) \otimes_F \mathrm{Ad}(\theta')$. Applying Proposition 6.18, we get that there exists a 4–dimensional quadratic form θ'' over F of trivial discriminant, which is necessarily a scalar multiple of a Pfister form, such that $(C, \rho) \cong (B, \tau) \otimes_F \mathrm{Ad}(\theta) \otimes_F \mathrm{Ad}(\theta'') \cong_F (B, \tau) \otimes_F \mathrm{Ad}(\theta \otimes_F \theta'')$. Since $\theta \otimes_F \theta''$ is similar to an n–fold Pfister form, this proves the statement. □

Given a non–singular quadratic form $\varphi \in I^n(F)$, it is known that φ is Witt equivalent to an orthogonal sum of n–fold Pfister forms. We tried to use induction on the number of Pfister forms to prove Proposition 6.21 in greater generality, but did not succeed. We did obtain the following result if φ is an orthogonal sum of two scaled $(n-1)$–fold Pfister forms. This is the main result of this section, and using Proposition 6.20, it can be seen that Proposition 6.21 is a special case of this result.

6.22 Theorem. *Let $n \geqslant 3$ and let π_1, π_2 be two $(n-1)$–fold Pfister forms over F. Let $a, b \in F^\times$ and let $\varphi = a\pi_1 \perp b\pi_2$. Let (B, τ) be a Pfister algebra with orthogonal involution. Let $(C, \rho) = (B, \tau) \otimes \mathrm{Ad}(\varphi)$ and assume that (C, ρ) is a Pfister algebra with involution. Then there is an n–fold Pfister form $\tilde{\varphi}$ over F such that $(C, \rho) \cong_F (B, \tau) \otimes \mathrm{Ad}(\tilde{\varphi})$.*

Proof. Up to scaling, we may assume that φ represents 1. Let $\psi = (b\pi_1 \perp -b\pi_2)_{\mathrm{an}}$. It is clear that $\dim(\psi) \leqslant 2^n - 2$. Furthermore, since $\psi \perp \varphi$ is Witt equivalent to a scalar multiple of an n−fold Pfister form and $I^n(F)$ is an ideal, we have that $\psi \perp \varphi \in I^n(F)$. By Proposition 6.17, $(B,\tau) \otimes_F \mathrm{Ad}(\psi)$ is hyperbolic. It follows that

$$\varphi_\tau \sim \varphi_\tau \perp \psi_\tau \sim (\langle a,b\rangle \otimes \pi_1)_\tau.$$

By Propositions 2.30 and 2.31, it follows that $(B,\tau) \otimes \mathrm{Ad}(\varphi) \cong_F (B,\tau) \otimes \mathrm{Ad}(\langle a,b\rangle \otimes \pi_1) \cong_F (B,\tau) \otimes \mathrm{Ad}(\langle 1,ab\rangle \otimes \pi_1)$, proving the statement. □

We tried to use the technique of adding a "suitable" quadratic form ψ to φ also in the case where φ is a form in $I^{n-1}(F)$ Witt equivalent to an orthogonal sum of two $(n-1)$−fold Pfister forms modulo $I^n(F)$, but not isometric to an orthogonal sum of two scaled $(n-1)$−fold Pfister forms. By suitable, we mean that $(\varphi \perp \psi)_{\mathrm{an}}$ should have the right dimension. We tried this in the first place in the case where $n = 3$, but did not succeed. In the following proposition, we collect different characterisations of 8−dimensional forms in $I^2(F)$ isometric to an orthogonal sum of two 2−fold Pfister forms. The second characterisation suggests that one might need a different technique than finding this suitable form ψ, in order to find out whether Theorem 6.22 could hold for general forms in $I^2(F)$ in the case where $n = 3$.

6.23 Proposition. *Let φ be an 8−dimensional form in $I^2(F)$. Let ρ be the canonical involution on $C_0(\varphi)$. Then $(C_0(\varphi),\rho) = (A,\sigma) \times (A,\sigma)$, for some degree 8 F−algebra with orthogonal involution (A,σ). Assume that $\mathrm{ind}(A) = 4$. Let q_A be the Albert form associated to A. Then the following are equivalent:*

(i) φ is isometric to an orthogonal sum of two scaled 2−fold Pfister forms.

(ii) There is a scalar $b \in F^\times$ such that the anisotropic part of $b\varphi \perp q_A$ has dimension at most 8.

(iii) (A,σ) decomposes as a tensor product of three F−quaternion algebras with involution, with one factor being a split F−quaternion algebra with orthogonal involution.

Proof. Note that $c(\varphi) = [A] \in \mathrm{Br}(F)$. Suppose that $\varphi \simeq \beta_1\pi_1 \perp \beta_2\pi_2$, with π_1,π_2 2−fold Pfister forms over F and $\beta_1,\beta_2 \in F^\times$. By [33, (3.3)], this means that there is a quadratic extension $L = F(\sqrt{\delta})$ of F such that φ_L and $(q_A)_L$ are isotropic. This means that there exist $\alpha_1,\alpha_2 \in F^\times$ such that $\alpha_1\langle 1,-\delta\rangle \subset \varphi$ and $\alpha_2\langle 1,-\delta\rangle \subset q_A$. Then the dimension of the anisotropic part of $-\alpha_1\alpha_2\varphi \perp q_A$ is at most 10 and since this is a form in $I^3(F)$, that means that the dimension is at most 8, since forms of dimension 10 in $I^3(F)$ are isotropic (see e.g. [47, (XII.2.8)]). This shows that (i) implies (ii).
Suppose that there exists $b \in F^\times$ such that the anisotropic part of $b\varphi \perp q_A$ is of dimension at most 8. Then q_A and $-b\varphi$ contain a common 3−dimensional subform by [32, (3.11)]. That means that there is a quadratic extension of F making q_A and $-b\varphi$, and hence φ,

isotropic. By [33, (3.3)], this means that φ contains a subform similar to a 2−fold Pfister form. This implies that $\varphi \simeq \beta_1\varphi_1 \perp \beta_2\varphi_2$, with φ_1 and φ_2 Pfister forms, whence (i).
The equivalence of (i) and (iii) follows from [51, (4.1)]. □

6.24 Remark. Let φ be as in Proposition 6.23. If $F = F_0((t_1))\dots((t_i))$, with $i \geqslant 1$ and F_0 a local or global field, then the conditions (i)–(iii) are satisfied by [33, (5.1)]. In general, (i)–(iii) need not be satisfied, for example if $F = \mathbb{C}(x,y)((t_1))((t_2))$ then φ can be chosen such that (i)–(iii) do not hold, whereas they are always satisfied if $F = \mathbb{C}(x,y)((t_1))$ (see [33, (5.6)]).

6.3 Generic hyperbolicity

Let (V,q) be a quadratic space over F of dimension at least 3. Let X_q be the corresponding projective quadric in $\mathbb{P}(V)$. Let L/F be an arbitrary field extension. Then q_L is isotropic if and only if X_q has an L−rational point. By Proposition 3.1, this is the case if and only if there is an F−place $\lambda : F(X_q) \to L^\infty$. We denote $F(X_q)$ also by $F(q)$ and call it *a generic isotropy field for q* (in the literature, one also uses the term *generic zero field*). The field $F(q)$ is given by a purely transcendental extension of F of transcendence degree $\dim(q) - 2$, followed by a quadratic extension (see [47, (X.3.7)]). Furthermore, $F(q)/F$ is purely transcendental if and only if q is isotropic (see [47, (X.4.1)]).

6.25 Remark. Let (V,q) be a quadratic space. Then there are two kinds of function fields associated to (V,q). There is the so–called “small function field”; this is the function field of the projective quadric we considered above, denoted by $F(q)$. There we excluded the case where q is nonhyperbolic and $\dim(q) = 2$, since the corresponding projective quadric is not geometrically irreducible in that case, and hence, behaves differently than in the case $\dim(q) \geqslant 3$. The other function field is the so–called “big function field”. This is the field M. Knebusch works with in the generic splitting theory of quadratic forms. It is the function field of the affine quadric hypersurface $q(X) = 0$ in $F^{\dim(q)}$. We denote this hypersurface by X_q^a and its function field by $F(X_q^a)$. The big function field of q has transcendence degree $\dim(q) - 1$ over F, and is a purely transcendental extension in one variable over the small function field (see [47, p. 330]). Hence, $F(q)$ and $F(X_q^a)$ are place equivalent. This implies that for statements on isotropy of the quadratic form, $F(q)$ and $F(X_q^a)$ behave in the same way.

In the case of quadratic forms, generic isotropy fields can be realised as function fields of the quadrics associated to the forms. In the case of algebras with involution, we will also take the point of view of varieties. Let (B,τ) be an F−algebra with involution of the first kind. Suppose that there exists a projective, geometrically integral F−variety X such that for every field extension L/F, τ_L is isotropic (resp. hyperbolic) if and only if X has an L–rational point. Then the function field of X is a generic isotropy (resp. hyperbolicity) field for τ.

6.26 Remark. It is clear that if τ is isotropic then F is a generic isotropy field for τ.

We first treat the question on generic hyperbolicity, since there is then only one of the $\mathrm{IV}_i(B,\tau)$ that needs to be taken into account. From section 6.4 on, we treat the question on generic isotropy.

For the rest of this section, we fix an F–algebra with involution (B,τ) of the first kind. Since odd degree algebras cannot carry hyperbolic involutions, we assume that B has even degree, say $\deg(B) = 2n$. We exclude the case of F–quaternion algebras with involution, since $\mathrm{IV}_1(B,\tau)$ is not a projective, geometrically irreducible variety in that case. So, we assume that $n \geqslant 2$.

6.27 Proposition. *Suppose that τ is symplectic. Then there exists a generic hyperbolicity field for τ.*

Proof. Suppose that τ is symplectic. Then $\mathrm{IV}_n(B,\tau)$ is a projective, geometrically integral F–variety by Proposition 3.3. Hence, $F_n(\tau)$ is a generic hyperbolicity field for τ by Proposition 3.1. □

6.28 Proposition. *Suppose that τ is orthogonal of trivial discriminant. Then there exists a generic hyperbolicity field for τ if and only if $F_+(\tau)$ or $F_-(\tau)$ is a generic hyperbolicity field for τ.*

Proof. Suppose that there exists a generic hyperbolicity field for τ, say N. Then at least one of the varieties $\mathrm{IV}_+(B,\tau)$ and $\mathrm{IV}_-(B,\tau)$ has an N–rational point. Suppose that $\mathrm{IV}_+(B,\tau)$ has an N–rational point. By Proposition 3.1, there is an F–place $\lambda : F_+(\tau) \to N^\infty$. This implies that $F_+(\tau)$ is a generic hyperbolicity field for τ. □

In the situation of Proposition 6.28, we now study further under which conditions $F_+(\tau)$ or $F_-(\tau)$ is a generic hyperbolicity field for τ.

6.29 Proposition. *Suppose that B is split and that τ is orthogonal of trivial discriminant. Then $F_+(\tau)$ and $F_-(\tau)$ are both generic hyperbolicity fields for τ.*

Proof. Let L/F be a field extension such that τ_L is hyperbolic. Then by Proposition 3.14, $\mathrm{IV}_+(B,\tau)$ and $\mathrm{IV}_-(B,\tau)$ both have an L–rational point. By Proposition 3.1, it follows that there are places from $F_+(\tau)$ to L and from $F_-(\tau)$ to L. This proves the statement. □

6.30 Proposition. *Suppose that τ is orthogonal of trivial discriminant. Let $\varepsilon \in \{+,-\}$. If C_ε is split over F then $F_\varepsilon(\tau)$ is a generic hyperbolicity field for τ.*

Proof. Assume that C_+ is split over F and let L/F be an arbitrary field extension such that τ_L is hyperbolic. Then one of $\mathrm{IV}_+(B,\tau)$ and $\mathrm{IV}_-(B,\tau)$ has an L–rational point. If $\mathrm{IV}_-(B,\tau)$ has an L–rational point, then there is an F–place from $F_-(\tau)$ to L, and hence, C_- splits over L by Lemma 3.12 and Propositions 3.7 and 3.1 (b). Hence, B_L is split by Proposition 1.36. By Proposition 3.14, this implies that both components $\mathrm{IV}_+(B,\tau)$ and $\mathrm{IV}_-(B,\tau)$ have an L–rational point and hence, in particular, we have an F–place from $F_+(\tau)$ to L. It follows from the above that $\mathrm{IV}_+(B,\tau)$ has an L–rational point if B_L is not split. So, in any case, there is an F–place $F_+(\tau) \to L^\infty$. The reasoning in the case where C_- is split is completely analogous. □

6.31 Corollary. *Suppose that there exist F–algebras of even degree with involution of the first kind $(B_1,\tau_1),(B_2,\tau_2)$ and (B_3,τ_3) such that $(B,\tau) \cong (B_1,\tau_1) \otimes_F (B_2,\tau_2) \otimes_F (B_3,\tau_3)$. Then there exists a generic hyperbolicity field for τ.*

Proof. By Proposition 6.4, the case where τ is orthogonal follows from Proposition 6.30. The symplectic case follows from Proposition 6.27. □

6.32 Corollary. *Suppose that (B,τ) is a Pfister algebra with involution such that* $\deg(B) > 4$. *Then there exists a generic hyperbolicity field for τ.*

Proof. The orthogonal case follows from Propositions 6.3 and 6.30. The symplectic case follows from Proposition 6.27. □

As a special case of both Corollary 6.31 and Corollary 6.32, we obtain the following result.

6.33 Corollary. *Totally decomposable F–algebras with involution of degree at least 8 have a generic isotropy field and a generic hyperbolicity field.*

Proof. This follows from Corollary 6.31 and Theorem 6.8. □

If n is even and B is non–split, then the converse of Proposition 6.30 also holds.

6.34 Proposition. *Suppose that n is even, B is non–split and that τ is orthogonal of trivial discriminant. Let $\varepsilon \in \{+,-\}$. If $F_\varepsilon(\tau)$ is a generic hyperbolicity field for τ then C_ε is split over F.*

Proof. Suppose that $F_+(\tau)$ is a generic hyperbolicity field for τ. Since τ also becomes hyperbolic over $F_-(\tau)$, there is an F–place $F_+(\tau) \to F_-(\tau)^\infty$. It follows from Lemma 3.12 that C_+ splits over $F_+(\tau)$ and hence also over $F_-(\tau)$. Invoking the same proposition once more, we get that C_+ splits over F, since n is even and B is non–split. The reasoning in the case where $F_-(\tau)$ is a generic hyperbolicity field is completely analogous. □

6.35 Theorem. *Suppose that n is even, B is non–split and that τ is orthogonal of trivial discriminant. Then there exists a generic hyperbolicity field for τ if and only if one of C_+ and C_- is split over F.*

Proof. This follows from Proposition 6.28 together with Proposition 6.34. □

6.36 Proposition. *Suppose that n is even, B is non–split and that $\mathrm{disc}(\tau) = d \in F^\times/F^{\times 2}$ is nontrivial. Then there exists a generic hyperbolicity field for τ if and only if B splits over $F(\sqrt{d})$.*

Proof. Suppose that there exists a generic hyperbolicity field for τ, say N. Then $\mathrm{disc}(\tau_N)$ is trivial. Let δ be a square root of d in N. We denote the nontrivial F–automorphism of $F(\delta)$ by ι and we denote $C(B,\tau)$ by C. Consider the $F(\delta)$–variety $X = \mathrm{IV}_n((B,\tau)_{F(\delta)})$. This variety has two irreducible components, which we denote by X_+ and X_-.
Since τ_N is hyperbolic, one of the components X_+ and X_- has an N–rational point, and hence, by Proposition 3.1, there is an $F(\delta)$–place $\lambda : F(\delta)(X_\varepsilon) \to N^\infty$, for $\varepsilon = +$ or $\varepsilon = -$. Let $L/F(\delta)$ be a field extension such that τ_L is hyperbolic. Since N is a generic hyperbolicity field for τ, there is an F–place $\mu : N \to L^\infty$. Since δ is algebraic over F, it follows that $\delta \in \mathcal{O}_\mu$, the valuation ring of N corresponding to μ, and hence d has a square root in L as well. We fix an F–embedding of $F(\delta)$ in L mapping δ to $\mu(\delta)$. With respect to this embedding, we may consider μ as an $F(\delta)$–place. Hence, one of $F(\delta)(X_+)$ and $F(\delta)(X_-)$ is a generic hyperbolicity for $\tau_{F(\delta)}$. Suppose that $B_{F(\delta)}$ is non–split. Proposition 6.34 then yields that one of the components of $C((B,\tau)_{F(\delta)})$ is split over $F(\delta)$. Note that $C((B,\tau)_{F(\delta)}) \cong C \otimes_F F(\delta) \cong C \times {}^\iota C$. It follows from [18, §8] that $\mathrm{ind}(C) = \mathrm{ind}({}^\iota C)$. So, by the above, we have that C must be split over $F(\delta)$. By [45, (9.14)], it follows that $N_{F(\delta)/F}([C]) = [B] \in \mathrm{Br}(F)$. Hence, B is split over F, a contradiction. Hence, B is split over $F(\delta)$, and then also over $F(\sqrt{d})$.

Conversely, suppose that B splits over $F(\sqrt{d})$. By Proposition 6.29, $F(\sqrt{d})(X_+)$ and $F(\sqrt{d})(X_-)$ are both generic hyperbolicity fields for $\tau_{F(\sqrt{d})}$. Since the discriminant of τ becomes trivial over any field extension of F making τ hyperbolic, it follows that $F(\sqrt{d})(X_+)$ and $F(\sqrt{d})(X_-)$ are also generic hyperbolicity fields for τ. □

6.37 Example. Assume that C is a biquaternion division algebra over F and let ρ be an orthogonal involution on C of nontrivial discriminant. Let φ be a non–singular quadratic form over F of odd dimension. Let $(B,\tau) = (C,\rho) \otimes_F \mathrm{Ad}(\varphi)$. By [45, (7.3) (4)], $\mathrm{disc}(\tau) = \mathrm{disc}(\rho)$. Since $\mathrm{ind}(B) = 4$, Proposition 6.36 yields that there is no generic hyperbolicity field for τ.

6.4 Generic isotropy

We first consider F–algebras with involution of the first kind for which it can be shown in an elementary way that they have a generic isotropy field.

6.38 Lemma. *Let (Q,τ) be an F–quaternion algebra with orthogonal involution.*

(a) If $\operatorname{disc}(\tau)$ is trivial, then Q is split over F.

(b) Assume that $\operatorname{disc}(\tau) = d \bmod F^{*2}$. If Q is split then $(Q,\tau) \cong_F \operatorname{Ad}(\langle 1,-d\rangle)$.

Proof. The proof of (a) can be found in [45, (7.4)]. Let us prove (b). Since Q is split, by Proposition 2.10, there exists a quadratic form q over F such that $(Q,\tau) \cong \operatorname{Ad}(q)$. Since q is determined up to similarity by Proposition 2.21, we may assume that $q = \langle 1,c\rangle$, for some $c \in F$. Furthermore, by [45, (7.3)], $-c \bmod F^{*2} = \operatorname{disc}(q) = \operatorname{disc}(\tau)$, whence $\langle 1,c\rangle \simeq \langle 1,-d\rangle$. □

6.39 Proposition. *Let (B,τ) be an F–algebra with involution of the first kind. Assume one of the following:*

(a) $\deg(B) = 2$, or

(b) B is split and τ is orthogonal, or

(c) $\operatorname{ind}(B) = 2$ and τ is symplectic, or

(d) $\deg(B) = 4$ and τ is symplectic.

Then there exists a quadratic form φ over F such that for any field extension L/F, we have that φ_L is isotropic if and only if τ_L is isotropic.

Proof. Suppose that (B,τ) is a quaternion algebra with involution. Denote the norm form of B by N. Note that if τ is isotropic over a field extension L/F, then B_L is split, since division algebras do not carry isotropic involutions. Let us first treat the symplectic case. A split algebra with symplectic involution is hyperbolic since it is adjoint to an alternating bilinear form, which is hyperbolic. Let now L/F be any field extension. We have that τ_L is isotropic if and only if B_L is split, which is the case if and only if N_L is hyperbolic.
Assume that τ is orthogonal and that B is non–split. Then $\operatorname{disc}(\tau) = d \in F^\times/F^{\times 2}$ is nontrivial by Lemma 6.38. Let φ be the quadratic form $\langle 1,-d\rangle$ over F. Let L/F be a field extension. It follows from Lemma 6.38 that τ_L is isotropic if and only if $\operatorname{disc}(\tau_L)$ is trivial, which is the case if and only if φ_L is isotropic.

Suppose that (B,τ) is split orthogonal. Then there is a quadratic form q over F such that $(B,\tau) \cong \mathrm{Ad}(q)$ and for any field extension q_L is isotropic if and only if τ_L is isotropic.

Suppose that $\mathrm{ind}(B) = 2$ and τ is symplectic. Let Q be a quaternion division algebra over F Brauer equivalent to B, and let γ be the canonical involution on Q. Let V be a finitely generated right Q–module and $h : V \times V \to Q$ a hermitian form over (Q,γ) such that $(B,\tau) \cong \mathrm{Ad}(h)$. One easily checks that $h(v,v) \in F$ for all $v \in V$. Then $q_h : V \to F : v \mapsto h(v,v)$ defines a quadratic form over F (this is called the *trace form of* h – see [66, p. 352]). Clearly h is isotropic if and only if q_h is isotropic. Since h is a hermitian form over (Q,γ), it can be diagonalised by [43, (I.6.2.4)]. Let $a_1,\dots,a_m \in F$ be such that $h \simeq \langle a_1,\dots,a_m\rangle_\gamma$. One checks that this implies that $q_h \simeq \langle a_1,\cdots,a_m\rangle \otimes N_Q$, with N_Q the norm form of Q. Let L/F be an arbitrary field extension. Assume first that Q_L is non–split. Then $(q_h)_L$ is the trace form of h_L. Therefore, by Proposition 2.14 and [66, (10.1.7)], we get

$$\begin{aligned}\tau_L \text{ is isotropic (resp. hyperbolic)} &\iff h_L \text{ is isotropic (resp. hyperbolic)}\\ &\iff (q_h)_L \text{ is isotropic (resp. hyperbolic).}\end{aligned}$$

Consider the case in which Q_L is split. Then B_L is split as well and τ_L is a symplectic involution on B_L and therefore hyperbolic. Moreover, since Q_L is split, we have that N_Q is hyperbolic over L, and hence q_h is also hyperbolic over L. So, in this case we also have

$$\tau_L \text{ is hyperbolic} \iff (q_h)_L \text{ is hyperbolic.}$$

Suppose that $\deg(B) = 4$ and τ is symplectic. Let $V_\tau = \{a \in \mathrm{Sym}(B,\tau) \mid \mathrm{Trd}(a) = 0\}$ and $\varphi_\tau : V_\tau \to F$ defined by $\varphi_\tau(a) = a^2$, for all $a \in V_\tau$. Then V_τ is a 5–dimensional F–vector space and φ_τ is a non–singular quadratic form on V_τ (see [45, p. 216]). Furthermore, φ_τ is isotropic if and only if τ is hyperbolic, if and only if τ is isotropic, by [45, (15.20)]. Let L/F be an arbitrary field extension. Note that $\mathrm{Sym}((B,\tau)_L) \cong \mathrm{Sym}(B,\tau) \otimes_F L$ and $V_{\tau_L} \cong V_\tau \otimes_F L$. Therefore $\varphi_{\tau_L} \simeq (\varphi_\tau)_L$. By the first part, we get that $(\varphi_\tau)_L$ is isotropic if and only if τ_L is isotropic. Hence, $F(\varphi_\tau)$ is a generic isotropy field for τ. □

6.40 Corollary. *Let (B,τ) be an F–algebra with involution of the first kind and with $\deg(B)$ odd. Then there exists a generic isotropy field for τ.*

Proof. By [45, (2.8)], B is split and τ is orthogonal. The statement now follows from Proposition 6.39. □

Before studying the generic isotropy problem further, we first show that a generic isotropy field need not always exist.

6.41 Proposition. *Let (Q_1, γ_1) and (Q_2, γ_2) be two F–quaternion algebras endowed with their canonical involutions. Let $(B, \tau) = (Q_1, \gamma_1) \otimes_F (Q_2, \gamma_2)$. Let L/F be a field extension. Then τ_L is hyperbolic if and only if at least one of Q_1 and Q_2 splits over L.*

Proof. This follows from the result in [6, (2.5)]. □

6.42 Corollary. *Let (B, τ) be a totally decomposable F–algebra with involution of degree 4. Suppose that B is non–split and that τ is anisotropic. Then there does not exist a generic isotropy field for τ.*

Proof. It follows from Proposition 6.10 that there exist F–quaternion algebras Q_1 and Q_2, with respective canonical involutions γ_1 and γ_2, such that

$$(B, \tau) \cong_F (Q_1, \gamma_1) \otimes_F (Q_2, \gamma_2).$$

Suppose for the sake of contradiction that there exists a generic isotropy field N for τ. By Theorem 6.8, τ_N is hyperbolic. It follows from Proposition 6.41 that at least one of Q_1 and Q_2 splits over N. By Propositions 3.7 and 3.1, we get that there is an F–place from $F(Q_1)$ to N, or an F–place from $F(Q_2)$ to N. Since N is a generic isotropy field for τ, and τ is hyperbolic over $F(Q_1)$ and $F(Q_2)$, there are F–places from N to $F(Q_1)$ and to $F(Q_2)$. Composing F–places, we get an F–place between $F(Q_1)$ and $F(Q_2)$, or the other way around. This implies that Q_1 splits over $F(Q_2)$ or Q_2 splits over $F(Q_1)$. It is well known that the kernel of the restriction map $\mathrm{Br}(F) \to \mathrm{Br}(F(Q_i))$ is equal to $\{0, [Q_i]\}$. (This was first shown by E. Witt in [76], and can also be seen using the Schur index reduction formulas from Theorem 3.11.) It follows that one of Q_1 and Q_2 is split over F, or $Q_1 \cong_F Q_2$. In the first case, τ would be hyperbolic, a contradiction, and in the second case, B would be split, which also contradicts the hypothesis. □

6.43 Example. Let $x_1, \ldots, x_4$ be independent variables over F. Let $Q_1 = (x_1, x_2)_F$ and $Q_2 = (x_3, x_4)_F$. Let $(B, \tau) = (Q_1, \gamma_1) \otimes_F (Q_2, \gamma_2)$. Then B is a division algebra and hence τ is anisotropic, and τ is either anisotropic or hyperbolic over every field extension of F by Proposition 6.10. Proposition 6.42 then yields that (B, τ) does not have a generic isotropy field.

Let (B, τ) be an F–algebra with involution of the first kind. In order to study Question 6.1, we use the varieties $\mathrm{IV}_i(B, \tau)$ introduced in chapter 3. We will see that, if a generic isotropy field exists for τ, then this means that there are relations between the $\mathrm{IV}_i(B, \tau)$. We first recast the properties of the varieties IV_i in terms of their function fields.

6.44 Proposition. *Let (B, τ) be an F–algebra with involution of the first kind, of degree at least 3. Let $i \in \mathbb{N}$ be such that $\mathrm{IV}_i(B, \tau)$ is a projective, geometrically integral F–variety. Then the following are equivalent:*

(i) $F_i(\tau)$ *is a generic isotropy field for* τ.

(ii) For any field extension L/F *such that* τ_L *is isotropic, we have that* $i \in \operatorname{ind}((B,\tau)_L)$.

Proof. This easily follows from Proposition 3.1 (c). □

6.45 Proposition. *Let* (B,τ) *be an* F*−algebra with involution of the first kind. Assume that there exists a field extension* L/F *such that* τ_L *is isotropic and* $\deg(B) = 2\operatorname{ind}(B_L)$. *Assume moreover that there exists a generic isotropy field for* τ. *Then* (B,τ) *is a Pfister algebra with involution.*

Proof. If $\deg(B) = 2$, then clearly, (B,τ) is a Pfister algebra with involution. For the rest of the proof, we assume that $\deg(B) > 2$. Let N/F be a field extension such that N is a generic isotropy field for τ. Then there is an F−place $\lambda : N \to L^\infty$. By Proposition 3.7 (b) and since division algebras cannot carry isotropic involutions, it follows that $\operatorname{ind}(B_N) = \deg(B)/2$, and hence τ_N is hyperbolic. This implies, by Proposition 3.9, that for every field extension M/F such that τ_M is isotropic, τ_M is hyperbolic. Hence, (B,τ) is a Pfister algebra with involution. □

We complement Proposition 6.45 with the following result.

6.46 Proposition. *Let* (B,τ) *be an* F*−algebra with involution of the first kind, of degree at least 3. Let* m *be the maximal integer such that there exists a field extension* L/F *such that* τ_L *is isotropic and* $\operatorname{ind}(B_L) = m$. *Assume that* $m < \deg(B)/2$. *Then there exists a generic isotropy field for* τ *if and only if* $F_m(\tau)$ *is a generic isotropy field for* τ.

Proof. Let L/F be a field extension such that τ_L is isotropic and $\operatorname{ind}(B_L) = m$. Suppose that there exists a generic isotropy field N for τ. Then there is an F−place $\lambda : N \to L^\infty$. By Proposition 3.7 (b) and the maximality of m, it follows that $\operatorname{ind}(B_N) = \operatorname{ind}(B_L) = m$. Since $1 \leqslant m < \deg(B)/2$, the variety $\mathrm{IV}_m(B,\tau)$ is projective and geometrically integral, and since τ_N is isotropic, $\mathrm{IV}_m(B,\tau)$ has an N−rational point. By Proposition 3.1 (a), it follows that there is an F−place from $F_m(\tau)$ to N, and hence $F_m(\tau)$ is a generic isotropy field for τ. □

6.47 Corollary. *Let* (B,τ) *be an* F*−algebra with involution of the first kind, of degree at least 3. If there exists a generic isotropy field for* τ, *then there exists* $i \in \{1,\dots,\operatorname{ind}(B)\}$ *such that the function field of* $\mathrm{IV}_i(B,\tau)$ *or one of its irreducible components is a generic isotropy field for* τ.

Proof. Assume that there exists a field extension N/F such that N is a generic isotropy field for τ. Let $m = \operatorname{ind}(B_N)$. Suppose first that τ is symplectic. Since $m \leqslant \deg(B)/2$, the F−variety $\mathrm{IV}_m(B,\tau)$ is projective and geometrically irreducible, and there is an F−place

from $F_m(\tau)$ to N. It follows that $F_m(\tau)$ is a generic isotropy field for τ.
Suppose that τ is orthogonal. If $m = \deg(B)/2$ then (B,τ) is a Pfister algebra with involution by Proposition 6.45. Furthermore, N is a generic hyperbolicity field for τ, and since $\mathrm{disc}(\tau)$ is trivial by Proposition 6.3, Proposition 6.28 yields that the function field of one of the irreducible components of $\mathrm{IV}_m(B,\tau)$ is a generic hyperbolicity field, and hence, a generic isotropy field, for τ. Suppose that $m < \deg(B)/2$. Then it follows from Proposition 6.46 that $F_m(\tau)$ is a generic isotropy field for τ. □

6.48 Remark. Proposition 6.46 shows that, if $\mathrm{ind}(B_{F_i(\tau)}) = \mathrm{ind}(B_{F_j(\tau)}) = m$, for certain $i, j \in \mathbb{N}$, and there exists a generic isotropy field for τ, then $F_i(\tau)$ and $F_j(\tau)$ are both generic isotropy fields for τ. Conversely, if there are field extensions N/F and N'/F that are both generic isotropy fields for τ, then by Proposition 3.7 (b), it follows that $\mathrm{ind}(B_N) = \mathrm{ind}(B_{N'})$ and by Proposition 3.9, $\mathrm{ind}((B,\tau)_N) = \mathrm{ind}((B,\tau)_{N'})$.

6.49 Example. We give an example to illustrate Remark 6.48. Let (B,τ) be a Pfister algebra with involution over F with $\deg(B) \geqslant 8$ and $\mathrm{ind}(B) < \deg(B)/2$. By Corollary 6.32, there exists a generic hyperbolicity field for τ, and this is then also a generic isotropy field for τ, since (B,τ) is a Pfister algebra with involution. Proposition 6.28 shows that $F_-(\tau)$ or $F_+(\tau)$ is a generic hyperbolicity, and hence isotropy, field for τ. We show that $F_{\mathrm{ind}(B)}(\tau)$ is also a generic isotropy field for τ. Let L/F be a field extension making τ isotropic. Then τ_L is hyperbolic and hence $\deg(B)/2 \in \mathrm{ind}((B,\tau)_L)$. Since $\mathrm{ind}(B_L) \mid \mathrm{ind}(B)$ and $\mathrm{ind}(B) < \deg(B)/2$, we have that $\mathrm{ind}(B) \in \mathrm{ind}((B,\tau)_L)$. Furthermore, since $\mathrm{ind}(B) < \deg(B)/2$, we have that $\mathrm{IV}_{\mathrm{ind}(B)}(B,\tau)$ is a projective, geometrically integral F–variety, so its function field is defined. Proposition 6.44 now yields that $F_{\mathrm{ind}(B)}(\tau)$ is a generic isotropy field for τ.

The following lemma is easy but useful.

6.50 Lemma. *Let (B,τ) be an F–algebra with involution of the first kind for which $\deg(B) = 2\,\mathrm{ind}(B)$. Then τ is either anisotropic or hyperbolic.*

Proof. Assume that τ is isotropic. Then B contains a non–zero isotropic right ideal I. Furthermore, we have that $\deg(B)/2 = \mathrm{ind}(B) \mid \mathrm{rdim}(I) \leqslant \deg(B)/2$, by Proposition 1.41. It follows that $\mathrm{rdim}(I) = \deg(B)/2$ and hence τ is hyperbolic. □

Let (B,τ) be an F–algebra with involution of the first kind. We now investigate the different varieties $\mathrm{IV}_i(B,\tau)$ and look for conditions on (B,τ) which allow to decide whether a function field $F_i(\tau)$, for a certain $i \in \mathbb{N}$, is a generic isotropy field for τ.
We first consider the variety $\mathrm{IV}_1(B,\tau)$, since it takes up a special position. If $(B,\tau) \cong \mathrm{Ad}(q)$, then by [69], $\mathrm{IV}_1(B,\tau)$ is in fact the projective quadric defined by q. Proposition 6.39 then yields that for split algebras with orthogonal involution of degree at least 3, $F_1(\tau)$ is a generic isotropy field for τ. The next result is in fact already in [69] but we provide it here for convenience. It is a reformulation of Proposition 6.44.

6.51 Proposition. *Let (B,τ) be an F–algebra with involution of the first kind, of degree at least 3. Then the following are equivalent:*

(i) $F_1(\tau)$ is a generic isotropy field for τ.

(ii) For any field extension L/F such that τ_L is isotropic, we have that B_L is split.

Proof. It suffices to prove that for any field extension L/F, $\mathrm{IV}_1(B,\tau)$ has an L–rational point if and only if τ_L is isotropic and B_L is split.
If $\mathrm{IV}_1(B,\tau)$ has an L–rational point, then B_L contains an isotropic right ideal I with $\mathrm{rdim}(I) = 1$, and hence, τ_L is isotropic. Furthermore, since $\mathrm{ind}(B_L) \mid \mathrm{rdim}(I) = 1$, we get that B_L is split. For the converse, assume that L/F is a field extension such that τ_L is isotropic and B_L is split. It follows from Corollary 2.14 that $1 \in \mathrm{ind}((B,\tau)_L)$. Hence, $\mathrm{IV}_1(B,\tau)$ has an L–rational point. □

6.52 Remark. If we replace the variety $\mathrm{IV}_1(B,\tau)$ in Proposition 6.51 by $\mathrm{IV}_i(B,\tau)$, with i an odd number in $\{1,\ldots,\deg(B)/2-1\}$, then we still have that (i) implies (ii). The reason is that right ideals of an odd reduced dimension cannot exist if the algebra is non–split, since the reduced dimension is divisible by the Schur index of the algebra, which is a 2–power, by [45, (2.8) (2)].

The function field of $\mathrm{IV}_i(B,\tau)$, for i odd, can only be a generic isotropy field for (B,τ) in low degree, as the following proposition shows.

6.53 Proposition. *Let (B,τ) be an F–algebra with involution of the first kind. Suppose that B is non–split and $\deg(B) = 2n$, with $n \geqslant 2$. Let i be an odd number in $\{1,\ldots,n\}$. If $F_i(\tau)$ is a generic isotropy field, then τ is orthogonal and $n \leqslant 3$.*

Proof. Let us first look at the case in which τ is symplectic. Consider the variety $\mathrm{IV}_2(B,\tau)$. Using the Schur index reduction formulas of Theorem 3.11, we obtain, since B has exponent 2 by [45, (3.1) (1)], that

$$\mathrm{ind}(B \otimes_F F_2(\tau)) = \min(\mathrm{ind}(B), 2\,\mathrm{ind}(B \otimes_F B)) = 2.$$

Since τ becomes isotropic over $F_2(\tau)$, but B does not split over $F_2(\tau)$, by Remark 6.52, the field $F_i(\tau)$ cannot be a generic isotropy field for τ in this case.
Now consider the case in which τ is orthogonal and $n > 3$. If $\mathrm{disc}(\tau)$ is trivial, we obtain

$$\mathrm{ind}(B \otimes_F F_2(\tau)) = \min(\mathrm{ind}(B), 2, 2^{n-3}\,\mathrm{ind}(C_+), 2^{n-3}\,\mathrm{ind}(C_-)) > 1.$$

If $\mathrm{disc}(\tau)$ is nontrivial, then

$$\mathrm{ind}(B \otimes_F F_2(\tau)) = \gcd(\mathrm{ind}(B), 2, 2^{n-2}\,\mathrm{ind}(B \otimes_F C(B,\tau))) > 1.$$

So, in both cases $F_i(\tau)$ cannot be a generic isotropy field for τ by Remark 6.52. □

In the sequel we study symplectic and orthogonal involutions separately. The Schur index reduction formulas for the function fields of the varieties $\mathrm{IV}_i(B,\tau)$ are simpler in the symplectic case than in the orthogonal case. Therefore, we treat the symplectic case first.

6.54 Proposition. *Let (B,τ) be an F-algebra with symplectic involution of degree at least 4. Let $i=\min(\mathrm{ind}(B),\frac{1}{2}\deg(B))$. If there exists a generic isotropy field for τ, then $F_i(\tau)$ is a generic isotropy field for τ.*

Proof. By [45, (2.8) (2)], $\mathrm{ind}(B)$ is a power of 2, and hence i is also a power of 2. Since $i \leqslant \deg(B)/2$, we have that $\mathrm{IV}_i(B)$ is a projective, geometrically integral F-variety by Proposition 3.3. Using the formulas from Theorem 3.11, we get that

$$\mathrm{ind}(B\otimes_F F_i(\tau)) = \min(\mathrm{ind}(B), i\cdot \mathrm{ind}(B\otimes_F B)) = i,$$

since B has exponent 2 in $\mathrm{Br}(F)$ by [45, (3.1) (1)]. By Proposition 3.1, this implies that, if there exists a generic isotropy field N for τ, then $\mathrm{ind}(B_N)=i$, and hence, $F_i(\tau)$ is then a generic isotropy field for τ. □

6.55 Proposition. *Let (B,τ) be an F-division algebra with symplectic involution of degree at least 4. Let F' denote the function field of $\mathrm{SB}_2(B)$. Then the following are equivalent:*

(i) There exists a generic isotropy field for (B,τ).

(ii) (B,τ) is a Pfister algebra with involution.

(iii) $(B,\tau)_{F'}$ is totally decomposable.

Proof. That (i) implies (ii) follows from Proposition 6.54. The converse follows from Proposition 6.27. By [8, Corollary], (iii) holds if and only if $(B,\tau)_{F'}$ is a Pfister algebra with involution. In turn, this is equivalent to (ii), by [37, Theorem A.1]. □

6.5 The orthogonal case

In this section we fix an F-algebra with orthogonal involution (B,τ). In dealing with isotropy of orthogonal involutions, it is interesting to investigate how the isotropy of an orthogonal involution is affected by generically splitting the algebra, i.e. by passing to the function field of the Severi–Brauer variety of the algebra. From that point on the isotropy behaviour is completely determined by a quadratic form. This problem has been studied by N.A. Karpenko in [37, 38]. We already mentioned the main result of [37] (see Theorem 6.7 (a)). It is conjectured that Theorem 6.7 (a) also holds with “hyperbolic” replaced by “isotropic”.

Anisotropic Splitting Conjecture: If τ is isotropic over $F(B)$ then τ is already isotropic over F.

In [38], N.A. Karpenko has shown that, if τ is anisotropic over F, but isotropic over $F(B)$, then there exists an odd degree field extension L/F such that τ_L is isotropic. Furthermore, the Anisotropic Splitting Conjecture has been confirmed in some special cases.

6.56 Theorem. *Suppose that τ is anisotropic over F. If B is split,* $\operatorname{ind}(B) = 2$, $\frac{\deg(B)}{\operatorname{ind}(B)} = 2$, *or B is a division algebra, then τ remains anisotropic over $F(B)$.*

Proof. The case $\frac{\deg(B)}{\operatorname{ind}(B)} = 2$ is proved in [37], and for the other cases, N.A. Karpenko gives explicit references in [37]. □

6.57 Proposition. *Suppose that the anisotropic part of (B,τ) remains anisotropic over $F(B)$. Then* $\operatorname{ind}(B,\tau)$ *and* $\operatorname{ind}((B,\tau)_{F(B)})$ *contain the same multiples of* $\operatorname{ind}(B)$.

Proof. In order to prove the statement, by Corollary 2.16, it suffices to show that the largest multiple of $\operatorname{ind}(B)$ contained in $\operatorname{ind}((B,\tau)_{F(B)})$, is also contained in $\operatorname{ind}(B,\tau)$.

By Proposition 2.10, there exists an F−division algebra with involution (D,θ), θ of the same kind as σ, and an ε−hermitian space (V,h) over (D,θ), with $\varepsilon \in \{\pm 1\}$, such that $(B,\tau) \cong_F \operatorname{Ad}(h)$. Then $(B,\tau)_{F(B)} \cong_{F(B)} \operatorname{Ad}(h_{F(B)})$. By Proposition 2.6, we can decompose $(V,h) \simeq (V_1,h_1) \perp (V_2,h_2)$, with (V_1,h_1) (resp. (V_2,h_2)) an anisotropic (resp. hyperbolic) ε−hermitian space over (D,θ). Then $\operatorname{Ad}(h_1)$ is the anisotropic part of (B,τ). By assumption, $(\operatorname{ad}_{h_1})_{F(B)}$ is anisotropic. This implies that $(h_1)_{F(B)}$ is anisotropic by Proposition 2.14. Let W be a maximal totally isotropic subspace of (V_2,h_2). Then W is a maximal totally isotropic subspace of (V,h), and the above shows that $W_{F(B)}$ is a maximal totally isotropic subspace of $(V,h)_{F(B)}$. Since $\dim_F(W) = \dim_{F(B)}(W_{F(B)})$, Corollary 2.15 shows that

$$i_w(\tau) = \frac{\dim_F(W)}{\deg(D)} = \frac{\dim_{F(B)}(W_{F(B)})}{\deg(D_{F(B)})} = i_w(\tau_{F(B)}),$$

as desired. □

6.58 Remark. In Proposition 6.57, let (C,ρ) be the anisotropic part of (B,τ). Since C is Brauer equivalent to B, it follows from Propositions 3.7 and 3.1 that there are F−places between $F(C)$ and $F(B)$ in both directions, i.e. $F(C)$ and $F(B)$ are place equivalent.

Hence, saying that (C,ρ) remains anisotropic over $F(B)$ is the same as saying that (C,ρ) remains anisotropic over $F(C)$ (i.e. that the Anisotropic Splitting Conjecture holds for (C,ρ)).

We say (B,τ) has *strong anisotropic splitting* if for any field extension L/F for which τ_L is isotropic and $\operatorname{ind}(B_L) < \operatorname{ind}(B)$, the anisotropic part of $(B,\tau)_L$ remains anisotropic over $L(B_L)$. If B is split, then trivially (B,τ) has strong anisotropic splitting.

Proposition 6.44 says that, in order to have that a certain $F_i(\tau)$ is a generic isotropy field for τ, we need to check that i is contained in the index of (B,τ) over any field extension L/F making τ isotropic. The following proposition shows that, if (B,τ) has strong anisotropic splitting and $\deg(B) \geqslant 3\operatorname{ind}(B)$, then we only need to consider the index of (B,τ) over one field extension of F in order to conclude that a particular $F_i(\tau)$ is a generic isotropy field for τ.

6.59 Theorem. *Suppose that (B,τ) has strong anisotropic splitting. If $\deg(B) \geqslant 3\operatorname{ind}(B)$ then the following are equivalent:*

(i) $\operatorname{ind}(B) \in \operatorname{ind}((B,\tau)_{F_1(\tau)})$.

(ii) $F_{\operatorname{ind}(B)}(\tau)$ *is a generic isotropy field for* τ.

Proof. Note that, since $\deg(B) \geqslant 3\operatorname{ind}(B)$, the variety $\mathrm{IV}_{\operatorname{ind}(B)}(B,\tau)$ is a projective, geometrically integral F–variety by Proposition 3.3. Since τ becomes isotropic over $F_1(\tau)$, it follows immediately from Proposition 6.44 that (ii) implies (i). Let us prove the converse. Let L/F be a field extension such that τ_L is isotropic. Then $\operatorname{ind}(B_L) \in \operatorname{ind}((B,\tau)_L)$. So, if $\operatorname{ind}(B_L) = \operatorname{ind}(B)$ we are done. Suppose that $\operatorname{ind}(B_L) < \operatorname{ind}(B)$. We first consider the case where B_L is split. Then there is an F–place $\lambda : F_1(\tau) \to L^\infty$, and we have that $\operatorname{ind}(B) \in \operatorname{ind}((B,\tau)_{F_1(\tau)}) \subset \operatorname{ind}((B,\tau)_L)$, by Proposition 3.9.
Assume that B_L is non–split. Then the above shows that $\operatorname{ind}(B) \in \operatorname{ind}((B,\tau)_{L(B_L)})$. If $\tau_{L(B_L)}$ is hyperbolic, then τ_L is already hyperbolic by Theorem 6.7 and it follows that $\operatorname{ind}(B) \in \operatorname{ind}((B,\tau)_L)$. So suppose that τ_L is non–hyperbolic. Since (B,τ) has strong anisotropic splitting and $\operatorname{ind}(B_L) \mid \operatorname{ind}(B)$, Proposition 6.57 implies that $\operatorname{ind}(B) \in \operatorname{ind}((B,\tau)_L)$. Proposition 6.44 now yields the statement. □

6.60 Proposition. *Suppose that* $\frac{\deg(B)}{2} - \operatorname{ind}(B) - 1 \geqslant v_2(\operatorname{ind}(B))$ *if* $\operatorname{disc}(\tau)$ *is trivial, and* $\frac{\deg(B)}{2} - \operatorname{ind}(B) \geqslant v_2(\operatorname{ind}(B))$ *if* $\operatorname{disc}(\tau)$ *is nontrivial. Then there exists a generic isotropy field for τ if and only if $F_{\operatorname{ind}(B)}(\tau)$ is a generic isotropy field for τ.*

Proof. Note that the assumptions imply that $\operatorname{ind}(B) < \deg(B)/2$. Suppose that $\deg(B)$ is odd. Then B is split by [45, (2.8)], and $F_1(\tau)$ is a generic isotropy field for τ by Proposition 6.51. So, for the rest of the proof, we assume that $\deg(B)$ is even, say

$\deg(B) = 2n$. By [45, (2.8) (2)], there exists $i \in \mathbb{N}$ such that $\operatorname{ind}(B) = 2^i$. If $\operatorname{disc}(\tau)$ is trivial, we write $C(B,\tau) = C_+ \times C_-$.
Suppose that there exists a generic isotropy field for τ. Let m be the maximal integer such that there exists a field extension L/F such that τ_L is isotropic and $\operatorname{ind}(B_L) = m$. Since $\operatorname{ind}(B) < \deg(B)/2$, it follows that $m < \deg(B)/2$. Then $F_m(\tau)$ is a generic isotropy field by Proposition 6.46. In order to prove the claim, it suffices to show that $m = \operatorname{ind}(B)$.
We use the Schur index reduction formulas from Theorem 3.11. In the case of trivial discriminant, we obtain

$$\operatorname{ind}(B \otimes_F F_{2^i}(\tau)) = \min(2^i, 2^i, 2^{n-2^i-1}\operatorname{ind}(C_-), 2^{n-2^i-1}\operatorname{ind}(C_+)).$$

In the case of nontrivial discriminant, we find

$$\operatorname{ind}(B \otimes_F F_{2^i}(\tau)) = \gcd(2^i, 2^i, 2^{n-2^i}\operatorname{ind}(B \otimes_F C(B,\tau))).$$

Since, by assumption, $n - 2^i - 1 \geqslant i$ if $\operatorname{disc}(\tau)$ is trivial, and $n - 2^i \geqslant i$ if $\operatorname{disc}(\tau)$ is nontrivial, it follows that $\operatorname{ind}(B_{F_{2^i}(\tau)}) = \operatorname{ind}(B)$ in all cases. Since $F_m(\tau)$ is a generic isotropy field for τ, it follows that $m = \operatorname{ind}(B_L) \in \operatorname{ind}((B,\tau)_{F_{2^i}(\tau)})$. This implies that $\operatorname{ind}(B_L) \geqslant \operatorname{ind}(B)$, and hence $m = \operatorname{ind}(B)$. □

6.61 Remark. In the situation of Proposition 6.60, the conditions on $\deg(B)$ and $\operatorname{ind}(B)$ were chosen such that one does not need information on $C(B,\tau)$ in order to conclude that $m = \operatorname{ind}(B)$ in the proof. If one does have information on $C(B,\tau)$ or its components in the case of trivial discriminant, then the conditions on $\deg(B)$ and $\operatorname{ind}(B)$ can be weakened.

6.62 Corollary. *Suppose that* $\operatorname{ind}(B) \leqslant 4$ *and* $\deg(B) \geqslant 3\operatorname{ind}(B)$, *where the last inequality is assumed to be strict if* $\operatorname{disc}(\tau)$ *is trivial. Then the following are equivalent:*

(i) *There is a generic isotropy field for* τ.

(ii) $F_{\operatorname{ind}(B)}(\tau)$ *is a generic isotropy field for* τ.

(iii) $\operatorname{ind}(B) \in \operatorname{ind}((B,\tau)_{F_1(\tau)})$.

Proof. By Theorem 6.56 and Remark 6.58, the assumptions imply that (B,τ) has strong anisotropic splitting. Hence, the equivalence of (ii) and (iii) follows immediately from Theorem 6.59. The assumptions on $\deg(B)$ and $\operatorname{ind}(B)$ yield that $\frac{\deg(B)}{2} - \operatorname{ind}(B) - 1 \geqslant v_2(\operatorname{ind}(B))$ if $\operatorname{disc}(\tau)$ is trivial, and $\frac{\deg(B)}{2} - \operatorname{ind}(B) \geqslant v_2(\operatorname{ind}(B))$ if $\operatorname{disc}(\tau)$ is nontrivial. Hence, Proposition 6.60 yields the equivalence of (i) and (ii). □

In the case where $\operatorname{ind}(B) = 2$, we look for more concrete conditions for the existence of a generic isotropy field.

6.63 Proposition. *Assume that* $\operatorname{ind}(B) = 2$ *and* $\deg(B) \geqslant 6$, *with a strict inequality if* $\operatorname{disc}(\tau)$ *is trivial. Assume moreover that* τ *becomes hyperbolic over a quadratic extension of* F. *Then* $F_2(\tau)$ *is a generic isotropy field for* τ.

Proof. We denote the degree of B by $2n$. By hypothesis, we have that τ is hyperbolic over a quadratic field extension of F, say over $F(\sqrt{\delta})$. By [6, (3.3)], there exists $x \in B$ such that $x^2 = \delta$ and $\tau(x) = -x$, so $F(\sqrt{\delta})$ is embedded in B as a subfield. By [66, (8.5.12)], it follows that $\operatorname{Nrd}_B(x) = N_{F(\sqrt{\delta})/F}(x)^n$, and hence, $\operatorname{disc}(\tau) = \delta \in F^\times/F^{\times 2}$ if n is odd, and $\operatorname{disc}(\tau)$ is trivial if n is even.

Let q be a quadratic form over $F_1(\tau)$ such that $(B,\tau)_{F_1(\tau)} \cong \operatorname{Ad}(q)$. In order to prove the statement, by Corollary 6.62, it suffices to show that $i_w(q) \geqslant 2$. So, assume for the sake of contradiction that $i_w(q) = 1$. Then $q \simeq \langle 1,-1\rangle \perp q'$, with q' an anisotropic quadratic form over $F_1(\tau)$. We have that $\operatorname{disc}(q') = \operatorname{disc}(q)$. Since F is algebraically closed in $F_1(\tau)$ by Proposition 3.1 (a), if $\operatorname{disc}(\tau)$ is nontrivial, then $\operatorname{disc}(q) = \operatorname{disc}(\tau_{F_1(\tau)})$ is also nontrivial.
Since τ is hyperbolic over $F_1(\tau)(\sqrt{\delta})$, we have that q is hyperbolic over $F_1(\tau)(\sqrt{\delta})$. This implies that q' becomes hyperbolic over $F_1(\tau)(\sqrt{\delta})$ and hence, by [66, (2.5.2)], $q' \simeq \langle 1,-\delta\rangle \otimes_{F_1(\tau)} \rho$, for some quadratic form ρ over $F_1(\tau)$. Since q is of dimension $2n$, it follows that q' is of dimension $2n-2$ and ρ of dimension $n-1$. Suppose that n is odd. Then $\operatorname{disc}(\tau)$ is nontrivial. Hence, $\operatorname{disc}(q')$ is nontrivial. However, we get $\langle 1,-\delta\rangle \otimes_{F_1(\tau)} \rho \in I^2(F_1(\tau))$ and hence, $\operatorname{disc}(q')$ is trivial, a contradiction. Suppose that n is even. Then $\operatorname{disc}(\tau)$ is trivial, and hence $\operatorname{disc}(q')$ is trivial as well. However, we get $\operatorname{disc}(q') = (-1)^{(n-1)(2n-3)}\det(\rho)\det(-\delta\rho) = (-1)(-\delta) \in F_1(\tau)^\times/F_1(\tau)^{\times 2}$, which is nontrivial, since δ is not a square in F and therefore not in $F_1(\tau)$ either. So, here we also get a contradiction. In both cases, we therefore get that $i_w(q) \geqslant 2$, and hence, $F_2(\tau)$ is a generic isotropy field for τ. □

6.64 Corollary. *Suppose that there exists an* F*–quaternion division algebra with orthogonal involution* (Q,ρ) *and a quadratic form* ψ *over* F *such that* $(B,\tau) \cong_F (Q,\rho) \otimes_F \operatorname{Ad}(\psi)$. *Then* $F_2(\tau)$ *is a generic isotropy field for* τ.

Proof. We write $\operatorname{disc}(\rho) = e \in F^\times/F^{\times 2}$. Since Q is non–split, Lemma 6.38 yields that $e \notin F^{\times 2}$. Furthermore, ρ becomes hyperbolic over $F(\sqrt{e})$ by Lemma 6.38 (b) and hence, τ also becomes hyperbolic over $F(\sqrt{e})$. The statement now follows from Proposition 6.63. □

6.65 Remark. In the situation of Proposition 6.63, if B splits over a quadratic extension making τ hyperbolic, then by [6, (3.4)], there exists an F–quaternion division algebra with orthogonal involution (Q,ρ), and a quadratic form ψ over F such that $(B,\tau) \cong_F (Q,\rho) \otimes_F \operatorname{Ad}(\psi)$.

6.66 Proposition. *Assume that B is a division algebra and suppose that there exist F−algebras with involution (B_1,τ_1) and (B_2,τ_2) such that $(B,\tau)\cong_F (B_1,\tau_1)\otimes_F(B_2,\tau_2)$. Suppose that (B,τ) is not a Pfister algebra with involution. Then there does not exist a generic isotropy field for τ.*

Proof. Since B is a division algebra, its degree is a power of 2 by [45, (2.8) (2)]. It follows that B_1 and B_2 both have degree a power of 2. Then [45, (7.3) (4)] yields that $\operatorname{disc}(\tau)$ is trivial. Let $C(B,\tau) = C_+\times C_-$. Using the Schur index reduction formulas from Theorem 3.11, we get for $\varepsilon\in\{+,-\}$

$$\begin{aligned}\operatorname{ind}(B\otimes_F F_\varepsilon(\tau)) &= \min(\operatorname{ind}(B),\deg(B)/2,\operatorname{ind}(C_{-\varepsilon}),\operatorname{ind}(C_\varepsilon)\cdot\deg(B)/2)\\ &= \min(\deg(B)/2,\operatorname{ind}(C_{-\varepsilon})).\end{aligned}$$

If τ_1,τ_2 are both symplectic, it follows from Theorem 1.37 (ii) that at least one of C_+ and C_- is split. Suppose that τ_1,τ_2 are both orthogonal. Let Q be the quaternion algebra $(\operatorname{disc}(\tau_1),\operatorname{disc}(\tau_2))_F$. Then one of C_+ and C_- is Brauer equivalent to Q, by Theorem 1.37 (i) and hence of Schur index 2. In any case, one of the components of $C(B,\tau)$ has Schur index at most 2. Since $C_+\otimes_F C_-$ is Brauer equivalent to B by Proposition 1.36 (a), this implies that one of C_+ and C_- has Schur index at least $\deg(B)/2$. Plugging this in into the formulas above yields that $\operatorname{ind}(B\otimes_F F_+(\tau))$ or $\operatorname{ind}(B\otimes_F F_-(\tau))$ is equal to $\deg(B)/2$. Without loss of generality, we may assume that $\operatorname{ind}(B\otimes_F F_+(\tau)) = \deg(B)/2$. If there would exist a generic isotropy field N for τ, then Proposition 3.1 would imply that $\operatorname{ind}(B_N) = \deg(B)/2$, and hence, (B,τ) would be a Pfister algebra with involution, a contradiction. □

6.6 Examples where no generic isotropy field exists

In this section we provide some examples of algebras with involution for which there is no generic isotropy field.

6.67 Lemma. *Let Q be an F−quaternion division algebra, endowed with its canonical involution γ. Let V be a 1−dimensional Q−vector space, $\alpha\in Q^\times$ a pure quaternion and $u\in F^\times$. Consider the skew−hermitian form $h=\langle u\alpha\rangle_\gamma : V\times V\to Q$ over (Q,γ). Let $a=-\operatorname{Nrd}(\alpha)$. Then ad_h and h are hyperbolic over $F(\sqrt{a})$.*

Proof. We have that αh is a hermitian form over $(Q,\operatorname{Int}(\alpha)\circ\gamma)$: $\alpha h = \langle ua\rangle_{\operatorname{Int}(\alpha)\circ\gamma}$. Since $\alpha\in Q^\times$, we have that $\operatorname{ad}_h=\operatorname{ad}_{\alpha h}$. It follows from [52, (5.7)] that $\operatorname{disc}(\operatorname{ad}_h) = -\operatorname{Nrd}(ua)\operatorname{Nrd}(\alpha) = a\in F^\times/F^{\times 2}$. Let $L=F(\sqrt{a})$. We get that $\operatorname{disc}((\operatorname{ad}_h)_L) = 1\in L^\times/L^{\times 2}$. Lemma 6.38 now implies that $(\operatorname{ad}_h)_L$ is hyperbolic. Then h_L is hyperbolic by Proposition 2.14. □

6.68 Proposition. *Let $Q = (d,e)_F$ a quaternion division algebra, and let $\{1, i, j, ij\}$ be an F−basis for Q with $i^2 = d$ and $j^2 = e$. Let γ be the canonical involution on Q. Let q_1 and q_2 be two non–singular quadratic forms over F. Let (B,τ) be the F−algebra with involution adjoint to the skew–hermitian form $\langle i\rangle_\gamma \perp q_1\langle j\rangle_\gamma \perp q_2\langle ij\rangle_\gamma$ over (Q,γ). If $\langle -1, e\rangle \otimes q_1 \perp \sqrt{d}\langle 1, e\rangle \otimes q_2$ is anisotropic over $F(\sqrt{d})$, and excluding the case where* $\deg(B) = 6$ *and* $\operatorname{disc}(\tau)$ *is trivial, then there is no generic isotropy field for τ.*

Proof. Let $q_1 = \langle\alpha_1,\ldots,\alpha_r\rangle$ and $q_2 = \langle\beta_1,\ldots,\beta_s\rangle$. Let V_1 be a 1−dimensional right Q−module and consider the skew–hermitian form $h_1 = \langle i\rangle_\gamma : V_1 \times V_1 \to Q$ over (Q,γ). Let V_2 be an $(r+s)$−dimensional right Q−module and consider the skew–hermitian form $h_2 = q_1\langle j\rangle_\gamma \perp q_2\langle ij\rangle_\gamma : V_2 \times V_2 \to Q$ over (Q,γ). Then $(B,\tau) = \operatorname{Ad}(h_1 \perp h_2)$. We have that $\deg(B) = 2(r + s + 1)$ and τ is an orthogonal involution on B.
It is clear that Q splits over $L = F(\sqrt{d})$. Let $(C,\rho) = \operatorname{Ad}(h_2)$. We have that $(C,\rho)_L \cong \operatorname{Ad}(q)$, for some quadratic form q over L. One can check that the quadratic form

$$\begin{aligned} q' = \langle 1, -e^{-1}, \alpha_2\alpha_1^{-1}, -e^{-1}\alpha_2\alpha_1^{-1}, \ldots, \alpha_r\alpha_1^{-1}, -e^{-1}\alpha_r\alpha_1^{-1}\rangle \perp \\ \sqrt{d}\,\langle \beta_1\alpha_1^{-1}, \beta_1\alpha_1^{-1}e^{-1}, \beta_2\alpha_1^{-1}, \beta_2\alpha_1^{-1}e^{-1}, \ldots, \beta_s\alpha_1^{-1}, \beta_s\alpha_1^{-1}e^{-1}\rangle \end{aligned}$$

over L gives rise to the same adjoint involution on C_L as $(h_2)_L$, and hence we may take $q = \alpha_1 eq' = -q_1 \perp eq_1 \perp \sqrt{d}q_2 \perp \sqrt{d}eq_2$, which is anisotropic over L by assumption. Using Lemma 6.67 and Corollary 2.29, we get

$$\begin{aligned} (B,\tau)_L &\cong_L \operatorname{Ad}((h_1)_L) \boxplus \operatorname{Ad}((h_2)_L) \\ &\cong_L \operatorname{Ad}(\langle 1,-1\rangle_L) \boxplus \operatorname{Ad}(q) \\ &\cong_L \operatorname{Ad}(\langle 1,-1\rangle_L \perp q). \end{aligned}$$

Since q is anisotropic, $i_w(\langle 1,-1\rangle_L \perp q) = 1$ and therefore, $i_w(\tau_L) = 1$ as well. Hence, $\operatorname{ind}((B,\tau)_L) = \{0,1\}$. This also implies that $i_w(\tau) \leqslant 1$. Furthermore, $2 = \operatorname{ind}(B) \mid i_w(\tau)$ and hence τ is anisotropic over F. By Corollary 6.62, there is no generic isotropy field for τ. □

6.69 Lemma. *Let r be a positive integer. Let furthermore $\alpha_1,\ldots,\alpha_r \in F^\times$ and $t_1,\ldots,t_r$ different variables over F. Then the quadratic form $\langle\alpha_1 t_1,\ldots,\alpha_r t_r\rangle$ is anisotropic over $F(t_1,\ldots,t_r)$.*

Proof. We prove that statement by induction. It is clear that it holds for $r = 1$. Assume $r > 1$ and consider $\psi = \langle\alpha_1 t_1,\ldots,\alpha_{r-1}t_{r-1}\rangle$ over $M = F(t_1,\ldots,t_{r-1})$. Then $\langle\alpha_1 t_1,\ldots,\alpha_r t_r\rangle = \psi \perp t_r\langle\alpha_r\rangle$. By induction, ψ is anisotropic over M and so is $\langle\alpha_r\rangle$. Then $\psi \perp t_r\langle\alpha_r\rangle$ is anisotropic over $M((t_r))$ by [47, (VI.1.9)], and hence, $\psi \perp t_r\langle\alpha_r\rangle$ is also anisotropic over $M(t_r)$. □

6.70 Example. We give an explicit example to show that the conditions in Proposition 6.68 can be satisfied, and that one can obtain examples of trivial and of nontrivial discriminant.
Let $x, x_1, \dots, x_{n-1}$ be different variables over F and $d \in F^\times \smallsetminus F^{\times 2}$. Let furthermore $F' = F(x, x_1, \dots, x_{n-1})$ and consider the quaternion algebra $Q = (d, x)_{F'}$, endowed with its canonical involution γ over F'. Then Q is a division algebra. Consider the quadratic forms $q_1 = \langle x_1, \dots, x_r\rangle$ and $q_2 = \langle x_{r+1}, \dots, x_{n-1}\rangle$ over $F(x_1, \dots, x_{n-1})$. By Lemma 6.69, $-q_1 \perp \sqrt{d}q_2$ and $q_1 \perp \sqrt{d}q_2$ are anisotropic over $F(\sqrt{d})(x_1, \dots, x_{n-1})$. By [47, (VI.1.9)], the quadratic form $-q_1 \perp \sqrt{d}q_2 \perp x(q_1 \perp \sqrt{d}q_2)$ is anisotropic over $F'(\sqrt{d})$. Let $(B, \tau) = \mathrm{Ad}(h)$, where h is the skew–hermitian form $\langle i\rangle_\gamma \perp q_1\langle j\rangle_\gamma \perp q_2\langle k\rangle_\gamma$ over (Q, γ). Then there does not exist a generic isotropy field for τ by Proposition 6.68. Scaling h by i yields a hermitian form $ih = \langle d\rangle_\gamma \perp q_1\langle k\rangle_\gamma \perp q_2\langle dj\rangle_\gamma$ over $(Q, \mathrm{Int}(i) \circ \gamma)$. Note that ih and h give rise to the same adjoint involution, since i is invertible in Q. Using [52, (5.7)] and [45, (7.3)], we get

$$\begin{aligned}
\operatorname{disc}(\tau) &= \mathrm{Nrd}(d)\,\mathrm{Nrd}(kx_1)\cdots\mathrm{Nrd}(kx_r)\,\mathrm{Nrd}(jdx_{r+1})\cdots\mathrm{Nrd}(jdx_{n-1})\,\operatorname{disc}(\mathrm{Int}(i)\circ\gamma)^n \\
&= \mathrm{Nrd}(k)^r\,\mathrm{Nrd}(j)^{n-1-r}(-\mathrm{Nrd}(i))^n \\
&= (-1)^n\,\mathrm{Nrd}(j)^{n-1}\,\mathrm{Nrd}(i)^{n+r} \in F'^\times/F'^{\times 2}.
\end{aligned}$$

Depending on the parity of n and r, we get the following:

- r, n both odd: $\operatorname{disc}(\tau) = -1 \in F'^\times/F'^{\times 2}$;
- r odd and n even: $\operatorname{disc}(\tau) = dx \in F'^\times/F'^{\times 2}$;
- r even and n odd: $\operatorname{disc}(\tau) = d \in F'^\times/F'^{\times 2}$;
- r, n both even: $\operatorname{disc}(\tau) = -x \in F'^\times/F'^{\times 2}$.

In the last three cases the discriminant is always nontrivial. In the first case, it can be both, depending on whether -1 is a square in F' or not.

6.71 Proposition. *Let $Q = (a, b)_F$ be a quaternion division algebra over F and let γ be its canonical involution. Let (C, ρ) be an F–division algebra with orthogonal involution of degree at least 4. Let $(B, \tau) = (Q, \gamma) \otimes_F (C, \rho)$. Suppose that B is either a division algebra or that $\mathrm{ind}(B) = \deg(B)/2$. Let $\operatorname{disc}(\rho) = d \in F^\times/F^{\times 2}$. If the Pfister form $\langle\langle a, b, d\rangle\rangle$ is anisotropic over F then there does not exist a generic isotropy field for τ.*

Proof. Suppose for the sake of contradiction that there exists a generic isotropy field for τ. Then Proposition 6.54 yields that $F_{\deg(B)/2}(\tau)$ is a generic isotropy field for τ. Hence, (B, τ) is a Pfister algebra with involution. By Proposition 6.11, it follows that $\Delta(B, \tau) = 0 \in H^3(F, \mu_2)$. By [26, (3.1)], we have that $\Delta(B, \tau) = [Q] \cdot (d) \in H^3(F, \mu_2)$, which corresponds to the 3–fold Pfister form $\varphi = \langle\langle a, b, d\rangle\rangle$ in $I^3(F)/I^4(F)$. So, we would have that φ is hyperbolic over F, but this contradicts the hypothesis. □

6.7 Function fields of quadratic forms as generic isotropy fields

Given a quadratic form φ over F, one can ask the question which F–algebras with involution become isotropic over $F(\varphi)$. In general, this is a hard question to answer, but we can give some constraints on φ and the algebras with involution. We already came across F–algebras with involution for which a generic isotropy field can be realised as the function field of a quadratic form (see Proposition 6.39). We now consider this problem in more detail. We will see that there is a relation with the Anisotropic Splitting Conjecture.

A first result uses the separation theorem for quadratic forms proved by D.W. Hoffmann (see [47, (X.4.34)]).

6.72 Theorem (Separation theorem). *Let (B, τ) be a F–algebra with anisotropic orthogonal involution. Let $n \in \mathbb{N}$ such that $\deg(B) \leqslant 2^n$. Let furthermore φ be a quadratic form over F with $\dim(\varphi) > 2^n$. Then τ is non–hyperbolic over $F(\varphi)$. If the Anisotropic Splitting Conjecture holds for (B, τ) then τ is anisotropic over $F(\varphi)$.*

Proof. Let q be a quadratic form over $F(B)$ such that $(B, \tau)_{F(B)} \cong \mathrm{Ad}(q)$. Since τ is anisotropic, it is in particular non–hyperbolic and hence τ remains non–hyperbolic over $F(B)$, by Theorem 6.7. Therefore, q is non–hyperbolic over $F(B)$. Let $q' = (q_{F(B)})_{\mathrm{an}}$. Since

$$\dim(q') \leqslant \dim(q) = \deg(B) \leqslant 2^n < \dim(\varphi),$$

it follows from [47, (X.4.34)] that q' remains anisotropic over $F(B)(\varphi)$. This implies that q remains non–hyperbolic over $F(B)(\varphi)$. Hence, τ is non–hyperbolic over $F(B)(\varphi)$ and then clearly also non–hyperbolic over $F(\varphi)$.
It is clear that if the Anisotropic Splitting Conjecture holds for (B, τ), then one may replace "non–hyperbolic" by "anisotropic" in the above. □

6.73 Proposition. *Let (B, τ) be a F–algebra with anisotropic orthogonal involution. If there exists a quadratic form φ over F such that $F(\varphi)$ is a generic isotropy field for τ then τ remains anisotropic over $F(B)$.*

Proof. Assume for the sake of contradiction that τ becomes isotropic over $F(B)$. By the main theorem of [38], there exists an odd degree field extension L/F such that τ_L is isotropic. Let φ be a quadratic form over F such that $F(\varphi)$ is a generic isotropy field for τ. Then φ is anisotropic over F, since τ is anisotropic over F. Since τ_L is isotropic, we have an F–place $\lambda : F(\varphi) \to L^\infty$. However, this yields that φ_L is isotropic, which is not possible by Springer's theorem (see e.g. [47, (VII.2.7)]). □

6.74 Proposition. *Let (B, τ) be an F–algebra with involution of the first kind. Assume that there exists a generic isotropy field N for τ such that $\mathrm{ind}(B_N) < \mathrm{ind}(B)/2$. Then there is no quadratic form φ over F such that τ is isotropic over $F(\varphi)$.*

Proof. Let φ be a quadratic form over F. If φ is isotropic then $F(\varphi)/F$ is purely transcendental and hence, $\mathrm{ind}(B_{F(\varphi)}) = \mathrm{ind}(B)$. Suppose that φ is anisotropic. Then $F(\varphi)$ is given by a purely transcendental extension of F followed by a quadratic extension. It follows that $\mathrm{ind}(B_{F(\varphi)}) \geqslant \mathrm{ind}(B)/2$. If τ would be isotropic over $F(\varphi)$ then there would exist an F–place $\lambda : N \to F(\varphi)^\infty$. By Proposition 3.7 (b), this would imply that $\mathrm{ind}(B_N) \geqslant \mathrm{ind}(B_{F(\varphi)}) \geqslant \mathrm{ind}(B)/2$, a contradiction. □

Let (B, τ) be a totally decomposable F–algebra with orthogonal involution. By Theorem 6.8, (B, τ) is a Pfister algebra with involution. If $\deg(B) \geqslant 8$ then we know by Corollary 6.32 that there exists a generic isotropy field for τ. We show below that, if B has Schur index 2, then such a generic isotropy field can also be realised as the function field of a quadratic form of dimension $\deg(B)$.

Suppose that $\deg(B) = 2^n$ and $\mathrm{ind}(B) = 2$. By [8, Theorem 2], $(B, \tau) \cong (Q, \sigma) \otimes \mathrm{Ad}(q)$, for some orthogonal involution τ on Q and some $(n-1)$–fold Pfister form q over F. We write $q = \langle 1 \rangle \perp q'$. Suppose that $\mathrm{disc}(\sigma) = d \in F^\times/F^{\times 2}$. Then there exists $c \in F^\times$ such that $Q \cong (c, d)_F$ by [47, (III.4.1)]. Let $N = \langle 1, -c, -d, cd \rangle$ be the norm form of Q and let $\pi = \langle 1, -d \rangle \otimes q$. Since $(B, \tau)_{F(Q)} \cong_{F(Q)} \mathrm{Ad}(\pi_{F(Q)})$, one might be tempted to think that $F(\pi)$ is a generic isotropy field for τ. We show below that this is not the case. However, by twisting π a little bit, we obtain a quadratic form over F whose function field is a generic isotropy field for τ.

6.75 Theorem. *Let $\varphi = \langle 1, -d \rangle \otimes (\langle c \rangle \perp q')$. If $n \geqslant 3$ then $F(\varphi)$ is a generic isotropy field for τ. Furthermore, there is no Pfister form over F whose function field is a generic isotropy field for τ.*

Proof. We first prove that $F(\varphi)$ is a generic isotropy field for τ. Note that $\pi \perp -\varphi$ is Witt equivalent to N. Hence, for any splitting field M of Q, we have that $\pi_M \simeq \varphi_M$, and furthermore, by Lemma 6.38, $(B, \tau)_M \cong \mathrm{Ad}(\langle 1, -d \rangle_M) \otimes_F \mathrm{Ad}(q_M) \cong \mathrm{Ad}(\langle 1, -d \rangle_M \otimes q_M)$. Let L/F be an arbitrary field extension. We show that φ_L is isotropic if and only if τ_L is hyperbolic.

Suppose first that φ_L is isotropic. Then $\pi_{L(Q_L)} \simeq \varphi_{L(Q_L)}$ is isotropic and hence hyperbolic since π is a Pfister form. Since $(B, \tau)_{L(Q_L)} \cong \mathrm{Ad}(\pi_{L(Q_L)})$, it follows that $\tau_{L(Q_L)}$ is hyperbolic by Proposition 2.14. By [14] or [57, (3.3)], $(B, \tau)_L$ is already hyperbolic.

Suppose conversely that τ_L is hyperbolic. Then $\tau_{L(Q_L)}$ is hyperbolic and hence $\pi_{L(Q_L)}$ is hyperbolic. Suppose that π_L is already hyperbolic. Since $\dim(\pi) = \deg(B) > 4$, the subform $\langle 1, -d \rangle_L \otimes q'_L$ of π_L has dimension bigger than $\dim(\pi)/2$, and hence, $\langle 1, -d \rangle_L \otimes q'_L$ is isotropic. It follows that φ_L is isotropic as well. Suppose that π_L is anisotropic. Then Q_L is necessarily non–split, since otherwise $L(Q_L)/L$ would be a purely transcendental extension and hence, $\pi_{L(Q_L)}$ would be anisotropic, a contradiction. Since π_L becomes hyperbolic over $L(Q_L)$, by [66, (4.5.4) (iv)], there exists a quadratic form $\tilde{\varphi}$ over L such that

$$\pi_L \simeq N_L \otimes \tilde{\varphi} \simeq \tilde{\varphi} \perp \dots$$

Since π_L is a Pfister form, any element of $L^\times$ represented by π_L is a similarity factor of π_L (see [66, (4.1.5)]). Therefore, scaling π_L with an element in $L^\times$ represented by $\tilde{\varphi}$ if necessary, we may assume that $\tilde{\varphi}$ represents 1. By [47, (I.2.3)], this implies that $\tilde{\varphi} \simeq \langle 1, \dots \rangle$, i.e.

$$\pi_L \simeq \langle 1, -c, -d, cd \rangle_L \perp \dots.$$

It follows that

$$\langle 1, -d \rangle_L \perp \langle 1, -d \rangle_L \otimes q'_L \simeq \langle 1, -c, -d, cd \rangle_L \perp \dots$$

Witt cancellation (see [66, (I.5.8)]) yields

$$\langle 1, -d \rangle_L \otimes q'_L \simeq -c\langle 1, -d \rangle_L \perp \dots$$

and hence

$$\varphi_L = c\langle 1, -d \rangle_L \perp \langle 1, -d \rangle_L \otimes q'_L \simeq c\langle 1, -d \rangle_L \perp -c\langle 1, -d \rangle_L \perp \dots$$

is isotropic, as desired. So. $F(\varphi)$ is a generic isotropy field for τ.

Before we prove the second statement, we first show that φ is not similar to a Pfister form. Suppose for the sake of contradiction that there exists a scalar $a \in F^\times$ and a Pfister form ψ over F such that $\varphi \simeq a\psi$. Since $(B, \tau)_{F(\pi)(Q)} \cong \mathrm{Ad}(\pi_{F(\pi)(Q)})$, and π is hyperbolic over $F(\pi)(Q)$, it follows that τ is hyperbolic over $F(\pi)(Q)$ by Proposition 2.14. Theorem 6.7 then yields that τ is already hyperbolic over $F(\pi)$. Since $F(\varphi)$ is a generic isotropy field for τ, it follows that $\varphi_{F(\pi)}$ is isotropic, and hence hyperbolic, since φ is similar to a Pfister form. By [47, (X.4.9)], π is up to a scalar a subform of φ. Because of dimension reasons, and since π and ψ are both Pfister forms of the same dimension, it follows that $\varphi \simeq a\pi$. Since π is a Pfister form of dimension at least 8, we have that $\pi \in I^3(F)$, and hence, since $I^3(F)$ is an ideal in $W(F)$, we also have that $\varphi \in I^3(F)$. Hence, $N \sim \pi \perp -\varphi \in I^3(F)$. The Arason–Pfister theorem (see [66, (4.5.6)]) then implies that N is hyperbolic, but this would imply that Q is split, a contradiction. So, φ is not similar to a Pfister form.
Suppose now for the sake of contradiction that there exists a Pfister form q' over F such that $F(q')$ is a generic isotropy field for τ. Then q' becomes hyperbolic over $F(\varphi)$, and by [47, (X.4.9)] and dimension reasons, it follows that φ and q' are similar, but this is not possible by the above. □

6.76 Corollary. *Let (B,τ) be a totally decomposable F–algebra with orthogonal involution. Assume that τ is anisotropic, B is non–split, and $\deg(B) \geqslant 4$. Then there is no Pfister form q over F such that $F(q)$ is a generic isotropy field.*

Proof. Suppose that $\deg(B) = 2^n$. Assume for the sake of contradiction that there exists a Pfister form q over F such that $F(q)$ is a generic isotropy field for τ. Note that q is anisotropic, since otherwise τ would be isotropic over F, which is not the case. Assume that $\dim(q) = 2^r$, for some $r \in \mathbb{N}$. Using Theorem 6.5, we get that $(B,\tau)_{F(B)} \cong \mathrm{Ad}(\psi)$, for some n–fold Pfister form ψ over $F(B)$. Since τ is hyperbolic over $F(q)$, Theorem 6.72 implies that $2^n = \deg(B) \geqslant \dim(q) = 2^r$. On the other hand, we have that τ becomes hyperbolic over $F(B)(\psi)$, and hence q is hyperbolic over $F(B)(\psi)$ as well. Note that $q_{F(B)}$ is non–hyperbolic since $\tau_{F(B)}$ is non–hyperbolic by Theorem 6.7. Since q is a Pfister form, it follows that $q_{F(B)}$ is anisotropic. The result in [47, (X.4.9)] yields that ψ is up to similarity a subform of $q_{F(B)}$, and hence $2^n = \dim(\psi) \leqslant \dim(q) = 2^r$. It follows that $q_{F(B)}$ and ψ are similar over $F(B)$, and hence, $(B,\tau)_{F(B)} \cong \mathrm{Ad}(q_{F(B)})$.

Let F' be the function field of the variety $\mathrm{SB}_2(B)$. Then $\mathrm{ind}(B_{F'}) = 2$. Since τ is anisotropic over F, Theorems 6.7 and 6.8 yield that τ is still anisotropic over $F(B)$. Since there is an F–place from F' to $F(B)$, Proposition 3.9 implies that τ is also anisotropic over F'. We have that $(B,\tau)_{F'(B_{F'})} \cong \mathrm{Ad}(q_{F'(B_{F'})})$. Let L'/F' be a field extension such that $\tau_{L'}$ is isotropic. Then $q_{F'}$ becomes isotropic over L'. This implies that there exists an F'–place $\lambda : F'(q_{F'}) \to L'$. Since τ clearly becomes isotropic over $F'(q_{F'})$ (which is a field extension of $F(q)$), we get that $F'(q_{F'})$ is a generic isotropy field for $\tau_{F'}$. If $\deg(B) \geqslant 8$ this contradicts Theorem 6.73. If $\deg(B) = 4$ then there does not exist a generic isotropy field for τ by Corollary 6.42, and hence, $F'(q_{F'})$ cannot be a generic isotropy field for τ. This implies that there does not exist a Pfister form over F whose function field is a generic isotropy field for τ. □

6.77 Proposition. *Let (B,τ) be an F–algebra with orthogonal involution. Assume that $\mathrm{ind}(B) = 2$ and that $F_1(\tau)$ is a generic isotropy field for τ. If τ is anisotropic, then there is no quadratic form φ over F such that $F(\varphi)$ is a generic isotropy field for τ.*

Proof. Note that by Proposition 6.53, $\deg(B) \leqslant 6$. Assume for the sake of contradiction that there exists a quadratic form φ over F such that $F(\varphi)$ is a generic isotropy field for τ. Let Q be a quaternion division F–algebra Brauer equivalent to B. Since $F_1(\tau)$ is a generic isotropy field for τ, we get that B splits over $F(\varphi)$. Hence, π_Q, the norm form of Q, becomes hyperbolic over $F(\varphi)$. Since Q is assumed to be non–split, π_Q is anisotropic and we get that a scalar multiple of φ is a subform of π_Q, by [47, (X.4.9)]. Since $\dim(\pi_Q) = 4$, this implies that $\dim(\varphi) \leqslant 4$.
If $\dim(\varphi) \in \{3,4\}$, then φ is a Pfister neighbour of π_Q and we get that φ is either similar to π_Q or to the pure part of π_Q, denoted by π'_Q. Since $F(\pi_Q)$ is place equivalent to $F(\pi'_Q)$,

we get in both cases that $F(\varphi)$ is place equivalent to $F(\pi_Q)$. In turn, $F(\pi_Q)$ is place equivalent to $F(Q)$. So, we get that $F(Q)$ is a generic isotropy field for τ. However, since τ is anisotropic over F, it stays anisotropic over $F(Q)$, by [57, (3.4)]. Hence, φ must be of dimension 2 and we may assume that $\varphi = \langle 1, -a\rangle$, for some $a \in F^\times$. Moreover, a is not a square in F, as this would mean that φ is hyperbolic and hence $F(\varphi)$ would be place equivalent to F. This would yield that τ is isotropic over F, which is not the case. Note that $F(\sqrt{a})$ is a generic isotropy field for φ and therefore place equivalent to $F(\varphi)$. Let q be a quadratic form over $F(Q)$ such that $(B,\tau)_{F(Q)} \cong \mathrm{Ad}(q)$. Then τ becomes isotropic over $F(Q)(q)$ and hence there exists an F−place $\lambda : F(\sqrt{a}) \to F(Q)(q)^\infty$. This implies that φ becomes hyperbolic over $F(Q)(q)$. However, this is not possible, since F is relatively algebraically closed in $F(Q)$ and $F(Q)$ is relatively algebraically closed in $F(Q)(q)$. We conclude that there does not exist a quadratic form φ over F such that $F(\varphi)$ is a generic isotropy field for τ. □

6.8 Characterisations in low degree

In this section, the closing section of this dissertation, we give complete characterisations for the existence of a generic isotropy field for some low degree algebras with involution. In those cases where a generic isotropy field exists, we also explore whether one can construct an analogue of the concept of a generic splitting tower for quadratic forms (see [40, §5]).

Let (V,q) be a quadratic space over F of dimension at least 2, and different form the hyperbolic plane. Suppose that q is anisotropic. Let $F_1 = F(X_q^a)$ be the big function field associated to q, as in Remark 6.25. Suppose furthermore that q_{F_1} is isotropic but not hyperbolic. Then we decompose $(V,q)_{F_1} \simeq (V_1,q_1) \perp (V_2,q_2)$, where (V_1,q_1) is an anisotropic quadratic space over F_1, and (V_2,q_2) a hyperbolic quadratic space over F_1. Then $F_2 = F_1(X_{q_1}^a)$ is a generic isotropy field for q_1. If $(q_1)_{F_2}$ is not hyperbolic, let F_3 be the function field of its anisotropic part. Continuing in this way, taking anisotropic parts in each step, one obtains a tower of field extensions $F_0 = F \subset F_1 \subset F_2 \subset \ldots \subset F_h$ such that q_{F_h} is split. This is called *a generic splitting tower for q*. It has the following property.

6.78 Proposition. *Let L/F be a field extension. Then there exists $i \in \{0,\ldots,h\}$ such that there exists an F−place $\lambda_i : F_i \to L^\infty$ that cannot be extended to an F−place from F_{i+1} to L. Furthermore, $i_w(q_L) = i_w(q_{F_i})$.*

Proof. See [66, (4.6.1)]. □

It follows from the previous proposition that if L/F is a field extension such that q_L is hyperbolic, then there exists an F−place $\lambda_h : F_h \to L^\infty$. Therefore, we call F_h *a generic* *hyperbolicity field for q.* Note that F_h is the function field of a variety over F_{h-1}.

Let (B,τ) be an F−algebra with involution. We call a tower of field extensions $F = F_0 \subset F_1 \subset \ldots \subset F_h$ *a generic isotropy tower for* (B,τ) if for every field extension L/F, there exists $i \in \{0,\ldots,h\}$ such that there is an F−place $\lambda_i : F_i \to L^\infty$ that cannot be extended to an F−place from F_{i+1} to L, and moreover, $i_w(\tau_L) = i_w(\tau_{F_i})$.

6.79 Question. Let (B,τ) be an F−algebra with involution of the first kind. When does there exist a generic isotropy tower for (B,τ)?

Degree 4 and 6

6.80 Theorem. *Let (B,τ) be an F−algebra with orthogonal involution of degree 4. Suppose that τ is anisotropic. Let* $\mathrm{disc}(\tau) = \delta \in F^\times/F^{\times 2}$. *Then the following are equivalent:*

(i) There exists a generic isotropy field for τ.

(ii) $F_1(\tau)$ is a generic isotropy field for τ.

(iii) For any field extension L/F such that τ_L is isotropic, B_L is split.

(iv) B splits over $F(\sqrt{\delta})$.

Proof. Obviously, (ii) implies (i). The equivalence between (ii) and (iii) follows from Proposition 6.51. We prove that (iv) implies (ii). Since B splits over $F(\sqrt{\delta})$, we have that either B already splits over F or $\mathrm{ind}(B) = 2$. If B is split over F, then $F_1(\tau)$ is a generic isotropy field for τ, by Proposition 6.51. Let us now consider the case in which $\mathrm{ind}(B) = 2$. Let N/F be an arbitrary field extension for which τ_N is isotropic. If τ_N is hyperbolic, then $\mathrm{disc}(\tau_N) = 1 \in N^\times/N^{\times 2}$ and therefore $d \in N^{\times 2}$. Hence, N contains up to isomorphism the field $F(\sqrt{\delta})$, which implies that B_N is split. Assume now that τ_N is non–hyperbolic. Using Lemma 6.50, we get that B_N must be split. Proposition 6.51 now implies that $F_1(\tau)$ is a generic isotropy field for τ.

We prove that (i) implies (iii). Assume that N/F is a generic isotropy field for τ. It suffices to show that B splits over N. If B already splits over F, then it also splits over N. If B is non–split, then Proposition 6.42 implies that there is a field extension M/F over which τ becomes isotropic, but not hyperbolic. It follows that τ_N is non–hyperbolic. Using Lemma 6.50, we get that B_N is split.

We now prove that (iii) implies (iv). If $\tau_{F(\sqrt{\delta})}$ is isotropic, then $B_{F(\sqrt{\delta})}$ is split since we assume that (iii) holds. So, suppose that $\tau_{F(\sqrt{\delta})}$ is anisotropic. By Proposition 6.10, there exist $F(\sqrt{\delta})$–quaternion algebras Q_1 and Q_2, with respective canonical involutions γ_1 and γ_2, such that

$$(B,\tau)_{F(\sqrt{\delta})} \cong (Q_1,\gamma_1) \otimes_{F(\sqrt{\delta})} (Q_2,\gamma_2).$$

Let $M = F(\sqrt{\delta})(Q_1)$, then τ_M is hyperbolic and hence, B_M is split, by (iii). This means that Q_2 splits over the function field of Q_1. Since we assume that $\tau_{F(\sqrt{\delta})}$ is anisotropic, it follows that $Q_1 \cong Q_2$ over $F(\sqrt{\delta})$. This implies that B splits over $F(\sqrt{\delta})$. □

6.81 Corollary. *Let (B,τ) be an F–algebra with orthogonal involution of degree 4. Suppose that τ is anisotropic and that there exists a generic isotropy field for τ. Then* $\mathrm{ind}(B) \leqslant 2$.

Proof. This follows immediately from Proposition 6.80. □

6.82 Examples.

(a) Consider the field of rational number $\mathbb{Q}$ and the division algebra $Q = (-1,-1)_{\mathbb{Q}}$. Let $\{1,i,j,ij\}$ be a $\mathbb{Q}$–basis for Q such that $i^2 = j^2 = -1$. Let τ be an orthogonal involution on Q. Let furthermore $\alpha = j, \beta = 2j + ij \in Q$ and consider the hermitian form $h = \langle\alpha,\beta\rangle_\tau$ over (Q,τ). Then ad_h is an orthogonal involution on $M_2(Q)$. According to [52, (5.7)], we have that $\mathrm{disc}(\mathrm{ad}_h) = \mathrm{Nrd}(\alpha)\,\mathrm{Nrd}(\beta)\,\mathrm{disc}(\tau)^2 = \mathrm{Nrd}(\alpha)\,\mathrm{Nrd}(\beta) = 5 \in \mathbb{Q}/\mathbb{Q}^{\times 2}$. Let $L = \mathbb{Q}(\sqrt{5})$. By [47, (III.2.7)], Q_L is still a division algebra, since -1 is not a sum of squares in L. Therefore, Theorem 6.80 implies that there is no generic isotropy field for ad_h. In fact, this is the case for the adjoint involution of any hermitian form $\langle\alpha,\beta\rangle_\tau$ over (Q,τ) such that $\mathrm{Nrd}(\alpha)\,\mathrm{Nrd}(\beta)$ is not a square in $\mathbb{Q}$.

(b) Consider the quaternion algebra $Q = (5,-3)_{\mathbb{Q}}$. By [47, (III.2.7)], Q splits over $F = \mathbb{Q}(\sqrt{5})$ since $\langle 5,-3\rangle$ represents 1 over $\mathbb{Q}(\sqrt{5})$, but Q is a division algebra. Let τ be an orthogonal involution on Q. Let furthermore $\alpha = j$ and $\beta = 5j + 2ij$. Consider the hermitian form $h = \langle\alpha,\beta\rangle_\tau$ over (Q,τ). Then ad_h is an orthogonal involution on $M_2(Q)$. By [52, (5.7)], $\mathrm{disc}(\mathrm{ad}_h) = \mathrm{Nrd}(\alpha)\,\mathrm{Nrd}(\beta)\,\mathrm{disc}(\tau)^2 = 5 \in \mathbb{Q}^\times/\mathbb{Q}^{\times 2}$. According to Theorem 6.80, $F_1(\tau)$ is a generic isotropy field for τ.

6.83 Theorem. *Let (B,τ) be an F–algebra with orthogonal involution. Suppose that B is of degree 6 and non–split. Let* $\mathrm{disc}(\tau) = \delta \in F^\times/F^{\times 2}$.

(a) If $\delta \in F^{\times 2}$ then $F_1(\tau)$ is a generic isotropy field for τ.

(b) If $\delta \notin F^{\times 2}$ then there exists a generic isotropy field for τ if and only if τ becomes hyperbolic over $F(\sqrt{\delta})$. If this is the case, then $F_2(\tau)$ is a generic isotropy field.

Proof. Assume first that $\delta \in F^{\times 2}$. Let L/F be a field extension such that τ_L is isotropic. It follows from [45, (15.38)] that $\operatorname{ind}((B,\tau)_L) = \{0,1\}$ or $\{0,1,2,3\}$. So, in any case B_L contains a right isotropic ideal of reduced dimension 1, which means that B_L is split. By Proposition 6.51, $F_1(\tau)$ is a generic isotropy field for τ.

Assume now that $\delta \notin F^{\times 2}$. Since B is non–split, we have that $\operatorname{ind}(B) = 2$. The equivalence between the existence of a generic isotropy field and $F_2(\tau)$ being a generic isotropy field follows from Corollary 6.62.
Assume that τ becomes hyperbolic over $F(\sqrt{\delta})$. By Proposition 6.63, $F_2(\tau)$ is a generic isotropy field for τ.
Conversely, suppose that $F_2(\tau)$ is a generic isotropy field for τ. Then by Corollary 6.62, it follows that $2 \in \operatorname{ind}((B,\tau)_{F_1(\tau)})$. Let Q be an F–quaternion algebra Brauer equivalent to B and q a quadratic form over $F_1(\tau)$ such that $(B,\tau)_{F_1(\tau)} \cong \operatorname{Ad}(q)$. We have that $i_1(q) \geqslant 2$. Since q is of dimension 6, it is not similar to a Pfister form and hence $i_1(q) = 2$. By [41, p. 10], this is equivalent to the existence of a quadratic form θ over $F_1(\tau)$ such that $q \simeq \langle 1, -\delta\rangle \otimes_{F_1(\tau)} \theta$. In particular, q becomes hyperbolic over $F_1(\tau)(\sqrt{\delta})$. Hence, τ is also hyperbolic over $F_1(\tau)(\sqrt{\delta})$. By part (a), we have that $F_1(\tau_{F(\sqrt{\delta})}) = F_1(\tau)(\sqrt{\delta})$ is a generic isotropy field for $\tau_{F(\sqrt{\delta})}$. Let Q be an F–quaternion algebra Brauer equivalent to B and let φ be a quadratic form over $F(\sqrt{\delta})(Q_{F(\sqrt{\delta})})$ such that $(B,\tau)_{F(\sqrt{\delta})(Q_{F(\sqrt{\delta})})} \cong \operatorname{Ad}(\varphi)$. Then φ is anisotropic or hyperbolic over any field extension of $F(\sqrt{\delta})(Q_{F(\sqrt{\delta})})$. Since φ cannot be a scalar multiple of a Pfister form, it follows from [66, (4.5.3)] that φ is hyperbolic, and hence τ is hyperbolic over $F(\sqrt{\delta})(Q_{F(\sqrt{\delta})})$. By [14] or [57, (3.3)], it follows that τ is hyperbolic over $F(\sqrt{\delta})$. □

Since an F–algebra with involution of degree 6 that doesn't have Schur index 2, is necessarily split by [45, (2.8) (2)], the study of the existence of a generic isotropy field for F–algebras with involution of degree 4 or 6 is now complete by combining Proposition 6.39 with the results above.

6.84 Corollary. *Let (B,τ) be a F–algebra of degree 6 with orthogonal involution for which there exists a generic isotropy field. Then there is a generic isotropy tower for τ.*

Proof. If B is split then the statement follows from the generic splitting theory for quadratic forms. Assume that B is Brauer equivalent to an F–quaternion division algebra Q. Let $\operatorname{disc}(\tau) = \delta \in F^{\times}/F^{\times 2}$. If $\delta \in F^{\times 2}$, then by Theorem 6.83, the field $F_1(\tau)$ is a generic isotropy field for τ. We have that $(B,\tau)_{F_1(\tau)} \cong \operatorname{Ad}(q)$, for some quadratic form q over $F_1(\tau)$. Extending $F \subset F_1(\tau)$ by a generic splitting tower for q_{an}, yields a generic splitting tower for τ.
Assume that $\delta \notin F^{\times 2}$. By Theorem 6.83, $F_2(\tau)$ is a generic isotropy field for τ. Let (V,h) be a skew–hermitian space over (Q,γ) such that $(B,\tau) \cong \operatorname{Ad}(h)$. Since h is

isotropic over $F_2(\tau)$, and Q is still division over $F_2(\tau)$ by Theorem 3.11, we have that the anisotropic part of $(B,\tau)_{F_2(\tau)}$ is a quaternion algebra with orthogonal involution of nontrivial discriminant. The latter has a generic isotropy field by Proposition 6.39, namely $F_2(\tau)(\sqrt{\delta})$. Then $F \subset F_2(\tau) \subset F_2(\tau)(\sqrt{\delta})$ is a generic isotropy tower for τ. □

6.85 Remark. It follows from Proposition 6.63 that there exist examples of algebras of degree 6 and Schur index 2 with orthogonal involution of nontrivial discriminant, such that the Witt index of the involution is equal to 1 over some field extension of the ground field. Theorem 6.83 implies that there is no generic isotropy field for such involutions.

Degree 8

In this section, (B,τ) denotes a F−algebra with involution of the first kind of degree 8. We first consider the orthogonal case. In the orthogonal trivial discriminant case, we have a complete characterisation of those F−algebras with involution of degree 8 for which there exists a generic isotropy field.

6.86 Theorem. *Suppose that τ is orthogonal of trivial discriminant. We write $C(B,\tau) = C_+ \times C_-$. Then there exists a generic isotropy field for τ if and only if one of the following cases occurs:*

(a) one of B, C_+, C_- is split over F;

(b) two of B, C_+, C_- have Schur index 2.

In case (a), $F_1(\tau)$ is a generic isotropy field for τ if B is split, and for $\varepsilon = \pm$, $F_\varepsilon(\tau)$ is a generic isotropy field for τ if C_ε is split. Furthermore, in case (a), there always exists a quadratic form whose function field is a generic isotropy field for τ. In case (b), $F_2(\tau)$ is a generic isotropy field for τ.

Proof. Suppose that we are in case (a). If B is split, then there exists a generic isotropy field for τ by Proposition 6.39, and this can be realised as the function field of a quadratic form. Suppose that one of C_+ and C_- is split. Then there exists a generic isotropy field for τ by Proposition 6.34. In the latter case, (B,τ) is totally decomposable by Proposition 6.10, and there is also a quadratic form whose function field is a generic isotropy field for τ by [59, (5.1)] or Theorem 6.75.
In the sequel we use the following formula from Theorem 3.11:

$$\begin{aligned}\operatorname{ind}(B \otimes_F F_\varepsilon(\tau)) &= \min(\operatorname{ind}(B), 4\operatorname{ind}(B \otimes_F B), \operatorname{ind}(B \otimes_F C_\varepsilon), 4\operatorname{ind}(B \otimes_F C_{-\varepsilon}))\\ &= \min(\operatorname{ind}(B), 4, \operatorname{ind}(C_{-\varepsilon}), 4\operatorname{ind}(C_\varepsilon)). \end{aligned} \tag{6.8.1}$$

Throughout the proof, let q be a quadratic form over $F_1(\tau)$ such that $(B,\tau)_{F_1(\tau)} \cong \mathrm{Ad}(q)$. Since $\tau_{F_1(\tau)}$ is isotropic, so is q, and we write $q \simeq \mathbb{H} \perp q'$.

Let us consider case (b). If two of B, C_+, C_- have Schur index 2, then the third one has Schur index at most 4, since $[B]+[C_+]+[C_-] = 0 \in \mathrm{Br}(F)$ by Proposition 1.36 (a). Let us assume that we are not in case (a). Then B is non–split, and hence, $\mathrm{ind}(B) = 2$ or 4. We show that $F_2(\tau)$ is a generic isotropy field for τ. Since B splits over $F_3(\tau)$, $(B,\tau)_{F_3(\tau)}$ contains an isotropic right ideal of reduced dimension 1, and hence, by Proposition 3.1, there is an F–place from $F_1(\tau)$ to $F_3(\tau)$. Using Proposition 3.9, it therefore suffices to prove that over $F_1(\tau)$ and $F_\varepsilon(\tau)$ ($\varepsilon = \pm$), (B,τ) contains an isotropic ideal of reduced dimension 2.
We have that q' is a 6–dimensional form of trivial discriminant, and hence q' is similar to an Albert form by [47, p. 70]. In particular q' has the same Clifford invariant as an Albert form. We have that $c(q') = c(q) = [(C_+)_{F_1(\tau)}] = [(C_-)_{F_1(\tau)}] \in \mathrm{Br}(F_1(\tau))$. Since at least one of C_+ and C_- has Schur index 2, we get that $c(q')$ has Schur index at most 2 and therefore, q' is isotropic. Hence, $2 \in \mathrm{ind}((B,\tau)_{F_1(\tau)})$.
For the function fields $F_\varepsilon(\tau)$, the formula (6.8.1) yields $\mathrm{ind}(B \otimes_F F_\varepsilon(\tau)) = 2$. By Corollary 2.16, it follows that $2 \in \mathrm{ind}((B,\tau)_{F_\varepsilon(\tau)})$, as desired. Hence, $F_2(\tau)$ is a generic isotropy field for τ in case (b).

Let us now assume that we are not in case (a) and that there exists a generic isotropy field for τ. Since $\deg(B) = 8 > 6$ and B is non–split, $F_1(\tau)$ cannot be a generic isotropy field for τ by Proposition 6.53. So, one of $F_2(\tau), F_+(\tau)$ and $F_-(\tau)$ is a generic isotropy field for τ. Since we are not in case (a), B, C_+ and C_- are all non–split. Suppose that B and one of C_+ and C_- have Schur index at least 4. Then one of $\mathrm{ind}(B \otimes_F F_+(\tau))$ or $\mathrm{ind}(B \otimes_F F_-(\tau))$ is equal to 4 and hence, the generic isotropy field would be a generic hyperbolicity field. However, this does not exist by Theorem 6.35. So we have that $\mathrm{ind}(C_+), \mathrm{ind}(C_-) \geqslant 4$. By Propositions 3.11 and 1.36 (a),

$$\begin{aligned}\mathrm{ind}(C_\varepsilon \otimes_F F_1(\tau)) &= \min(\mathrm{ind}(C_\varepsilon), \mathrm{ind}(C_\varepsilon \otimes_F B), 4\,\mathrm{ind}(C_\varepsilon \otimes_F C_+), 4\,\mathrm{ind}(C_\varepsilon \otimes_F C_-))\\ &= \min(\mathrm{ind}(C_\varepsilon), \mathrm{ind}(C_{-\varepsilon}), 4, 4\,\mathrm{ind}(B)) = 4.\end{aligned}$$

Moreover, as before, we have that $c(q') = c(q) = [(C_+)_{F_1(\tau)}] = [(C_-)_{F_1(\tau)}] \in \mathrm{Br}(F)$. Hence, $c(q')$ has Schur index 4 and therefore $i_w(q) = 1$. This contradicts the fact that one of $F_2(\tau), F_+(\tau)$ and $F_-(\tau)$ is a generic isotropy field for τ. So, we conclude that two of B, C_+, C_- have Schur index 2. □

6.87 Corollary. *Assume that τ is orthogonal of trivial discriminant. We write $C(B,\tau) = C_+ \times C_-$. Assume that one of B, C_+, C_- is split. Then there is a generic isotropy tower for (B,τ). Assume that two of B, C_+, C_- are of Schur index 2 and none of them is split. Then there is no generic isotropy tower for (B,τ).*

Proof. If B is split and q is a quadratic form over F such that $(B,\tau) \simeq \mathrm{Ad}(q)$, then a generic splitting tower for q is a generic splitting isotropy for τ. Suppose that B is non–split and one of C_+ and C_- is split. Without loss of generality, we may assume that C_+

is split. Then $F_+(\tau)$ is a generic isotropy field for τ, and in fact τ becomes hyperbolic over $F_+(\tau)$, since (B,τ) is a Pfister algebra with involution by Proposition 6.10. Hence, $F \subset F_+(\tau)$ is a generic isotropy tower for τ. Suppose that at least two of B, C_+, C_- have Schur index 2, and that none of them is split. Then $F_2(\tau)$ is a generic isotropy field for τ. However, by Proposition 6.34, there does not exist a generic hyperbolicity field for τ, and hence, there does not exist a generic isotropy tower for τ. This can also be seen by looking at the anisotropic part of $(B,\tau)_{F_2(\tau)}$. Since $2 = \operatorname{ind}(B_{F_2(\tau)}) \mid i_w(\tau_{F_2(\tau)})$ and (B,τ) is not a Pfister algebra with involution by Proposition 6.10, $\tau_{F_2(\tau)}$ is non–hyperbolic, and hence, $i_w(\tau_{F_2(\tau)}) = 2$. This means that $(B_1,\tau_1) = ((B,\tau)_{F_2(\tau)})_{\mathrm{an}}$ is an algebra of degree 4 with orthogonal involution of trivial discriminant, and furthermore B_1 is non–split. It follows from Theorem 6.80 that there is no generic isotropy field for τ_1. □

If the discriminant of τ is nontrivial then we do not have a full characterisation for the existence of a generic isotropy field. We only obtain some necessary conditions that follow from the trivial discriminant case, and some conditions which imply the non–existence of a generic isotropy field.

6.88 Proposition. *Assume that τ is orthogonal of nontrivial discriminant. Assume that there exists a generic isotropy field for τ. Then $F_2(\tau)$ is a generic isotropy field and we have that $C(B,\tau)$ is split (and hence B is split) or* $\operatorname{ind}(C(B,\tau)) = 2$.

Proof. Let $d \in F^\times \setminus F^{\times 2}$ be such that $\operatorname{disc}(\tau) = d \in F^\times/F^{\times 2}$. Assume that there exists a generic isotropy field for τ. By Proposition 6.53, $F_2(\tau)$ is a generic isotropy field for τ. Then there also exists a generic isotropy field for $\tau_{F(\sqrt{d})}$, namely the function field of the $F(\sqrt{d})$–variety $\mathrm{IV}_2((B,\tau)_{F(\sqrt{d})})$. We write $C = C(B,\tau)$. Since $C((B,\tau)_{F(\sqrt{d})}) \cong C_{F(\sqrt{d})} \cong C \times {}^{\varphi}C$, where φ is the nontrivial F–automorphism of $F(\sqrt{d})$, it follows from Theorem 6.86 that one of the following cases occurs:

(a) one of B, C is split over $F(\sqrt{d})$, or

(b) $\operatorname{ind}(C) = 2$ (and possibly $\operatorname{ind}(B) = 2$).

It follows from Proposition 1.36 that $N_{F(\sqrt{d})/F}([C]) = [B] \in \operatorname{Br}(F)$. In particular, if C is split over $F(\sqrt{d})$, then B is split over F. This proves the statement. □

6.89 Corollary. *Suppose that τ is orthogonal of nontrivial discriminant, and suppose that B is a division algebra. Then there does not exist a generic isotropy field for τ.*

Proof. Let $d \in F^\times \setminus F^{\times 2}$ be such that $\operatorname{disc}(\tau) = d \in F^\times/F^{\times 2}$. This follows from Proposition 6.88 since $N_{F(\sqrt{d})/F}([C]) = [B] \in \operatorname{Br}(F)$ implies that $\operatorname{ind}(B) \leqslant \operatorname{ind}(C(B,\tau))^2$, which is at most 4 if there would exist a generic isotropy field for τ, by Proposition 6.88. □

Examples 6.70 (c) yields for certain fields F examples of F−algebras with involution of nontrivial discriminant of degree 8 and Schur index 2, for which there is no generic isotropy field (e.g. take $r = 1, n = 4$ and the field k such that $-1 \in k^2$). These examples are such that B splits over $F(\sqrt{d})$, where $\mathrm{disc}(\tau) = d \in F^\times/F^{\times 2}$.
Below, we give an example of an F−algebra with orthogonal involution (B,τ) of degree 8 and Schur index 4 for which there does not exist a generic isotropy field.

6.90 Example. Let k be a field and u, v, x, y independent variables over k. We write $\tilde{k} = k(u,v)$ and $F = \tilde{k}(x,y)$. By the proof of [47, (XIII.2.8)], $D = (u,x) \otimes_F (v,y)$ is a division algebra over F. Let $\{1, i, j, ij\}$ (resp. $\{1, i', j', i'j'\}$) be a usual F−basis for (u,x) (resp. (v,y)) such that $i^2 = u$ and $j^2 = x$ (resp. $i'^2 = v$ and $j'^2 = y$). Let γ_1 (resp. γ_2) be the canonical involution on (u,x) (resp. (v,y)). Then $\rho = \gamma_1 \otimes_F \gamma_2$ is an orthogonal involution on D.
Let (B,τ) be the F−algebra with involution adjoint to the hermitian form $\langle j\otimes j', -yi\otimes i'\rangle_\rho$ over (D,ρ). One can check that $\mathrm{disc}(\tau)$ is trivial. Over $F(\sqrt{uv}, \sqrt{u})$, τ becomes adjoint to the quadratic form

$$\langle y^{-1}, x, -1, -xy, u, u, yu, yu\rangle \simeq \langle y, x, -1, -xy, 1, 1, y, y\rangle.$$

So, the above form is isometric to $\mathbb{H} \perp y\langle 1,1,1,-x\rangle \perp \langle 1, x\rangle$, where the second part is the Albert form of the biquaternion algebra $(-1,-y) \otimes (x,y)$.

Suppose that $\langle 1,1,1\rangle$ is anisotropic over $\tilde{k}(\sqrt{uv}, \sqrt{u})$. (This can for example be achieved by taking $k = \mathbb{Q}$.) Then $\langle 1, x\rangle$ is anisotropic over $\tilde{k}(\sqrt{uv}, \sqrt{u})(x)$, and by [47, (VI.1.9)], $\langle 1,1,1,-x\rangle$ is anisotropic over $\tilde{k}(\sqrt{uv}, \sqrt{u})(x)$. Invoking the same result once more, we obtain that that $y\langle 1,1,1,-x\rangle \perp \langle 1,x\rangle$ is anisotropic over $F(\sqrt{uv}, \sqrt{u})$.

So, we have that $(B,\tau)_{F(\sqrt{uv},\sqrt{u})}$ is an algebra of degree 8 with orthogonal involution of trivial discriminant and Witt index 1. Furthermore, the components of $C((B,\tau)_{F(\sqrt{uv},\sqrt{u})})$ are the components of $C_0(\mathbb{H} \perp y\langle 1,1,1,-x\rangle \perp \langle 1,x\rangle)$. Since $y\langle 1,1,1,-x\rangle \perp \langle 1,x\rangle$ is an anisotropic Albert form, we have that its Clifford invariant has Schur index 4. This means that the components of $C((B,\tau)_{F(\sqrt{uv},\sqrt{u})})$ have Schur index 4 and so over the ground field F, the components have Schur index at least 4. So, we see that (B,τ) does not satisfy the conditions of Theorem 6.86 for the existence of a generic isotropy field, and hence, there does not exist a generic isotropy field for τ.

Using the fact that Pfister algebras with involution are totally decomposable in the degree 8 symplectic case (see Theorem 6.12), we easily obtain a characterisation for the existence of a generic isotropy field in that case.

6.91 Theorem. *Suppose that τ is symplectic. Then there exists a generic isotropy field for τ if and only if one of the following cases occurs:*

(a) B has Schur index 2.

(b) (B,τ) is totally decomposable.

Proof. Corollary 6.33 and Proposition 6.39 yield that there exists a generic isotropy field in cases (a) and (b). Assume now that there exists a generic isotropy field N/F for τ, and that $\mathrm{ind}(B) \neq 2$. If B is split then τ is adjoint to a non–singular alternating bilinear form, and hence τ is hyperbolic. It follows that $\Delta(B,\tau) = 0$ by [26], and hence, (B,τ) is totally decomposable by Theorem 6.12. So, assume that $\mathrm{ind}(B) = 4$ or 8. Using the formulas of Theorem 3.11, we find

$$\mathrm{ind}(B \otimes_F F_4(\tau)) = \min(\mathrm{ind}(B), 4) = 4.$$

This means that $\mathrm{ind}(B_N) \geqslant 4$ and hence (B,τ) is a Pfister algebra with involution. Theorem 6.12 then yields that (B,τ) is totally decomposable. $\square$

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Tell me one last thing, said Harry. Is this real? Or has this been happening inside my head? Of course it's happening inside your head, Harry, but why on earth should that mean that it is not real?

*— **J.K. Rowling, Harry Potter and the deathly hallows***