Partial entrainment in the finite Kuramoto model

Dirk Aeyels and Filip De Smet
SYSTeMS Research Group,
Department of Electrical Energy, Systems and Automation,
Ghent University,
Technologiepark-Zwijnaarde 914, 9052 Zwijnaarde, Belgium,
Email: Dirk.Aeyels@UGent.be, Filip.DeSmet@UGent.be

1 Abstract

We consider the Kuramoto model with a finite number of oscillators. We provide a sufficient condition for the existence of a solution exhibiting entrainment of a given subset of oscillators.

2 The Kuramoto model

The Kuramoto model [2] was introduced to investigate synchronisation in systems of coupled oscillators. It is described by the following differential equations for the phases (θ_i) of the oscillators:

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \qquad \forall i \in \{1,\ldots,N\},$$

where $N > 1$ is the number of oscillators in the system, $K \geq 0$ is the coupling strength and the ω_i, which are drawn from a distribution g, represent the individual frequencies of the oscillators when they are isolated (i.e. when $K = 0$). Kuramoto considered the limit $N \to \infty$ and showed that, if g is unimodal and symmetric about some frequency Ω, there is a critical value K_c of the coupling strength above which a solution exists which exhibits partial synchronisation [3].

The assumptions Kuramoto made about the distribution to find this analytical solution do not help anymore when trying to analyse the model for finite N. Analytical results are hard to obtain and mostly refer to special cases such as identical individual frequencies [4] or the case of full synchronisation [1]. We consider the more general case where entrained subsets and drifting oscillators coexist.

3 Partial entrainment

We consider a solution to be entrained with respect to a given non-empty set $S_e \subset \{1,\ldots,N\}$ if the phase differences of all couples of oscillators from S_e are bounded, i.e.

$$\exists C > 0 : \left|\theta_i(t) - \theta_j(t)\right| < C, \quad \forall t \geq 0, \forall i,j \in S_e.$$

Let $\{S_1,S_2,S_3\}$ be a partition of $\{1,\ldots,N\}$ for which S_2 and S_3 are allowed to be empty and define $m,M \in \{1,\ldots,N\}$ by

$$\omega_m = \min{}_{i\in S_1}\omega_i, \qquad \omega_M = \max{}_{i\in S_1}\omega_i.$$

The set S_1 is the set of which the entrainment is investigated, S_2 contains all oscillators with frequencies that differ at least $2K$ from $(\omega_m + \omega_M)/2$, and S_3 contains the remaining oscillators. In our main result we prove that if $(\omega_M - \omega_m)/K$ is smaller than some value Ω then there exists a solution which is partially entrained with respect to S_1. The value of Ω is a function of the sizes of S_1, S_2 and S_3 and of the ratios $\frac{\omega_i}{K}$, with $i \in S_2$. (We will omit the mathematical formula since it is too long and does not provide any additional insight.) This result is only meaningful if $\Omega > 0$, which can be shown to be equivalent to

$$\sum_{i\in S_2}\frac{1}{1+\frac{\pi}{4}\left(\left|\frac{\omega_i}{K} - \frac{\omega_M+\omega_m}{2K}\right| - 2\right)} + |S_3| - |S_1| < 0.$$

This condition will hold if the oscillators that are not in S_1 have frequencies that are sufficiently different from those of the oscillators in S_1. A qualitatively similar condition for entrainment is also found in simulation results, but on a different scale.

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