

Contributions to the study of shear-free and of purely radiative perfect fluids in general relativity

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And if the Wine you drink, the Lip you press,
End in the Nothing all Things end in - Yes -
Then fancy while Thou art, Thou art but what
Thou shalt be “Nothing” Thou shalt not be less.

A more literal translation could read:

If with wine you are drunk be happy,
If seated with a moon-faced (beautiful), be happy,
Since the end purpose of the universe is nothing-ness;
Hence picture your nothing-ness, then while you are, be happy!

Omar Khayyam (1048-1131)

Oh, come with old Khayyam, and leave the Wise
To talk; one thing is certain, that Life flies;
One thing is certain, and the Rest is Lies;
The Flower that once has blown for ever dies.

Omar Khayyam (FitzGerald's first edition 1859)

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Ghent, October 2012

I dedicate this thesis to the memory of my mother and my father, whose
lives were not enough to see the fruits of their inspiration.

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Samenvatting

Gravitatie wordt op een elegante en compacte manier beschreven door de Einstein veldvergelijkingen: ruimte en tijd worden weergegeven door een pseudo-Riemannse (Lorentz) variëteit en de Einstein vergelijkingen bepalen de interactie tussen materie en geometrie. Het construeren van exacte oplossingen van dit stelsel van partiële differentiaalvergelijkingen is echter niet eenvoudig, als gevolg van hun sterk niet-lineaire karakter. Exacte oplossingen worden dan ook doorgaans bekomen door de aanname van sterk vereenvoudigende hypothesen en ze beschrijven dikwijls onfysische situaties. Dat neemt niet weg dat een groot deel van onze kennis van gravitatie gebaseerd is op enkele belangrijke oplossingen, die wel degelijk overeenstemmen met fysisch relevante situaties, zoals het uitwendige veld van een bolvormige ster, of een homogeen en isotroop heelal, of vlakke of cilindrische gravitatiegolven.

Alhoewel de meest gangbare praktijk om de vergelijkingen te vereenvoudigen bestaat uit het opleggen van bepaalde symmetrievoorwaarden, is het ook mogelijk om exacte oplossingen te genereren aan de hand van vernuftig gekozen aannames over de meetkundige eigenschappen van de ruimtetijd (met name over de Weyl tensor, Ricci tensor, Riemann tensor of de metrische tensor zelf), alsook over de kinematische grootheden, gekoppeld aan bepaalde congruenties van wereldlijnen (zoals de rotatie, versnelling, expansie of afschuiving van geprefereerde families van waarnemers).

Het is deze laatste weg die in dit proefschrift bewandeld wordt. Het proefschrift bestaat uit vier hoofdstukken. In het eerste hoofdstuk worden de gebruikte notaties en basisgereedschappen (en concepten) toegelicht en wordt een overzicht gegeven van het formalisme dat we zullen hanteren. Het tweede hoofdstuk is gewijd aan afschuivingsvrije perfecte vloeistoffen, waar we tonen dat het vermoeden opgaat dat deze onder een aantal bijkomende voorwaarden (zoals het divergentie-vrij zijn van het elektrische of magnetis-

che deel van de Weyl tensor) rotatie- of expansie-vrij zijn.

Ervan uitgaande dat dit vermoeden —dat tot dusver bewezen werd onder een groot aantal uiteenlopende veronderstellingen— juist is, kunnen de afschuivingsvrije perfecte vloeistoffen worden opgedeeld in twee klassen: roterende en niet expanderende vloeistoffen en anderzijds expanderende maar niet roterende. We leggen hierbij de nadruk op de eerste familie, aangezien de laatste klasse reeds uitvoerig bestudeerd werd in de literatuur. We geven in het bijzonder een veralgemening van Collins' [16] classificatie van de vloeistoffen met zuiver elektrische Weyl tensor en identificeren enkele nieuwe exacte oplossingen waarin het magnetisch deel slechts divergentie-vrij is.

In het derde hoofdstuk bestuderen we zuiver stralende ruimte-tijden, waarin zowel het elektrisch deel als het magnetisch deel van de Weyl tensor (t.o.v. de geprefereerde vloeistofcongruentie) divergentie-vrij zijn. Beide voorwaarden zijn triviaal voldaan in homogene maar anisotrope kosmologische modellen en we tonen aan dat, omgekeerd en onder bepaalde technische voorwaarden (zoals het commuteren van afschuiving en van magnetische Weyl kromming), hypervlak-homogeniteit een gevolg is van het divergentie-vrij zijn.

Dit proefschrift eindigt met enkele slotbeschouwingen en een appendix met de formules en relaties die relevant zijn voor de in de verschillende hoofdstukken gevolgde redeneringen.

Summary

It is difficult to obtain explicit solutions of Einstein's gravitational field equations, when fully written out as a system of partial differential equations, on account of their complicated and nonlinear character. The differential equations express purely geometric requirements based on the idea that space and time can be represented by a pseudo-Riemannian (Lorentzian) manifold and that the interaction between matter and the geometry is described by Einstein's field equations. With sufficiently clever assumptions on geometric properties of the space-time (the Weyl tensor, Ricci tensor, Riemann tensor or the metric tensor itself) and kinematic quantities of some preferred time-like congruence, it is often possible to reduce the Einstein field equations to a much simpler system of equations and then obtain the corresponding exact solutions. Although most of the known exact solutions describe situations which are unphysical, some of the solutions can be interpreted as representing physically significant situations such as the exterior field of a spherical star, or a homogeneous and isotropic universe, or plane or cylindrical gravitational waves. This thesis deals with the study of some classes of exact solutions of the Einstein field equations (with or without cosmological constant) with a stress-energy source in the form of a perfect fluid. We attempt to contribute to the study of shear-free and of purely radiative perfect fluids where we assume that the shear tensor (a kinematic quantity) and the divergence of the magnetic part or the electric part of the Weyl tensor (a geometric variable) vanish.

Rotating cosmological models have attracted attention in relativistic and Newtonian theories of gravity with the view to avoid a singularity in the universe [74]. In contrast with the Newtonian theory, where shear-free, expanding and rotating models exist, there is considerable evidence in the literature supporting the conjecture that general relativistic, shear-free perfect fluids which obey a barotropic equation of state $p = p(\mu)$ such that $p + \mu \neq 0$, are either non-expanding ($\theta = 0$) or non-rotating ($\omega = 0$). This

conjecture has been established in many special cases (see subsection 2.1) but a general proof or counter-example is still lacking. It should be noted that the conjecture is not true in Newtonian theory and hence if true in general relativity, then it would highlight essential differences, like that of a well defined universal time, between the two theories [73].

We attempt to generalize the result by Collins [16], which established that the conjecture holds for purely electric perfect fluids and the result obtained by Lang [55] and Cyganowski and Carminati [24] on purely magnetic space-times. In our analysis, we shall make the weaker assumptions that the magnetic part of the Weyl tensor (\mathbf{H}) or the electric part of the Weyl tensor (\mathbf{E}) are solenoidal, that is, that their spatial divergence vanishes. Interestingly, it has been recently shown [3, 76] that the assumption of third order restrictions, such as $\text{div}\mathbf{H} = 0$ and/or $\text{div}\mathbf{E} = 0$, leads to physically relevant families of perfect fluid solutions. Also, in a classification attempt of these fluids the shear-free subfamily would appear to be a natural first candidate for further investigation. We also investigate the case $\mathbf{E} = 0$ of the conjecture [24] and present what we believe to be a more compact proof by using the orthonormal formalism.

Assuming the validity of the conjecture, the possible space-times can be classified into two classes according to whether they are expanding or not. We attempt to give a classification of rotating, non-expanding shear-free perfect fluids for which $\text{div}\mathbf{H} = 0$, thereby generalising the classification of Collins 1984 [16]. Herein we recover all the solutions obtained by Collins [16], which are Petrov type D and have the equation of state $p = \mu + \text{constant}$ (stiff fluids). We also find solutions under three equations of state where the magnetic part of the Weyl tensor is not zero. Substituting the γ -law condition, two of the equations of state reduce to two physically plausible solutions with $p = 7\mu/11 + C$ (when the acceleration and the vorticity are orthogonal) and $p = \mu/5 + C$ (when the acceleration and the vorticity are neither orthogonal nor parallel). For the remaining case (when the acceleration and the vorticity are parallel), γ -law solutions are not allowed.

If, on the other hand, the fluid is expanding, the shear-free conjecture would force the space-times to be irrotational. All such space-times are examined by Collins and Wainwright [22], and reviewed briefly herein.

Although gravitational waves in perfect fluid space-times are usually studied via transverse traceless tensor perturbations on a Friedman-Lemaître-

Robertson-Walker (FLRW) background, there is renewed interest in the so-called covariant approach [26, 36] in which gravitational radiation is described via the nonlocal fields \mathbf{E} , the tidal part of the curvature which generalizes the Newtonian tidal tensor, and by \mathbf{H} , which has no Newtonian analogue [30]. As such, \mathbf{H} may be considered as the true gravity wave tensor, since there is no gravitational radiation in Newtonian theory. We discuss purely radiative and geodesic space-times for which the magnetic part of the Weyl tensor is diagonal in the shear-electric eigen-frame, and we show that these uniquely characterize the Bianchi class A (non-tilted) perfect fluids.

The thesis consists of four chapters. The first chapter is supposed to define the notations used and to introduce some basic tools (and concepts) that are employed. The orthonormal and the $1+3$ covariant formalisms are described and the kinematic quantities and electric and magnetic parts of the Weyl tensor are defined. A brief summary of the Bianchi classification of the three dimensional groups is also given.

The second chapter is devoted to shear-free perfect fluids where we show that the shear-free conjecture holds true under either of the additional conditions $\mathbf{E} = 0$, $\text{div}\mathbf{H} = 0$ or $\text{div}\mathbf{E} = 0$. At the end of this chapter we attempt to give a classification of rotating, non-expanding shear-free perfect fluids for which $\text{div}\mathbf{H} = 0$, after which we review irrotational space-times briefly in section 2.7.

We devote the third chapter to purely radiative space-times. In the first section we show that when the shear tensor σ is degenerate, then $[\sigma, \mathbf{H}] = 0$ [48]. This implies that \mathbf{H} is diagonal in the (σ, \mathbf{E}) -eigen-frame (in the case of degenerate shear (3.12) implies that the eigen-planes of σ and \mathbf{E} coincide). We obtain then $\mu_\alpha = 0$ and $z_\alpha = 0$ (regardless of the degeneracy of σ), giving a first hint that the corresponding space-times might be spatially homogeneous indeed.

In the second section, using the result of the first section as a motivation to impose the extra condition $[\sigma, \mathbf{H}] = 0$ we show that the Bianchi class A perfect fluids can be uniquely characterized -modulo the class of purely electric and (pseudo-)spherically symmetric universes- as geodesic perfect fluid space-times which are purely radiative in the sense that the gravitational field satisfies $\text{div}\mathbf{E} = \text{div}\mathbf{H} = 0$. This thesis ends with concluding remarks and an appendix section containing mathematical relations relevant to derivations given in various chapters. It should be mentioned that all the calculations throughout this thesis have been done with the assis-

tance of the Maple symbolic packages Oframe [81] which is available at <http://users.ugent.be/~nvdbergh/rug/gr/>.

The author of the thesis has several publications related to the shear-free perfect fluids and purely radiative perfect fluids. In the shear-free part he has been collaborating as a co-author of the papers [10], [85] and [86]. He has improved [10] by adding the proof of the existence of a Killing vector along vorticity and proving the conjecture for a three-parameter family of equations of state, $G' = 0$ (generalizing the γ -law case) when a Killing vector exists and the vorticity is not a function of the matter density. The author of the thesis also proved the conjecture for the case when the Weyl tensor is purely magnetic by using the orthonormal formalism. He classified the non-expanding, rotating shear-free perfect fluids, a project which is not yet completely finished and which will be published in the future. In the case of purely radiative perfect fluids the author has published three papers [3], [48] and [84], where the paper "Purely radiative irrotational dust space-times" [84] is the product of a collaboration and the two other papers are largely based on the author's own work.

Contents

Samenvatting	i
Summary	iii
1 Generalities	3
1.1 Kinematic decomposition	7
1.2 The curvature tensor	10
1.3 Formalisms	12
1.3.1 The covariant propagation and constraint equations .	13
1.3.2 Orthonormal tetrads	15
1.4 Rotation in the (12)-plane	19
1.5 Space-time symmetries	21
1.6 Classification of Bianchi cosmologies	22
2 Shear-free perfect fluids	25
2.1 Literature review on the conjecture	27
2.2 Tetrad choice and relevant equations	33
2.3 Purely magnetic perfect fluids	39
2.3.1 $\omega = \omega(\mu)$	40
2.3.2 $\partial_{[a}\omega\partial_{b]}\mu \neq 0$	41
2.4 Solenoidal magnetic curvature	41
2.4.1 Perfect fluid with a γ -law equation of state	44
2.4.2 Perfect fluid with a equation of state $p = p(\mu)$	47
2.5 Solenoidal electric curvature	56
2.5.1 \dot{u}_A parallel to z_A	57
2.5.2 \dot{u}_A not parallel to z_A	57
2.6 Rotating, non-expanding perfect fluids	59
2.6.1 $r_3 + q_3 \neq 0$	62
2.6.2 $r_3 + q_3 = 0$	65

2.7	Expanding non-rotating perfect fluids	84
3	Purely radiative space-times	87
3.1	Geodesic perfect fluids	90
3.1.1	The case $n_2 + n_3 = 0$	93
3.1.2	The case $n_2 + n_3 \neq 0$	94
3.1.3	The case $r_1 = 0$	97
3.2	Geodesic perfect fluids with $[\boldsymbol{\sigma}, \boldsymbol{H}] = 0$	97
3.2.1	Non degenerate shear	99
3.2.2	Degenerate shear	100
4	General conclusion	105
A		109
B		113

Chapter 1

Generalities

The purpose of this chapter is to define the notations used in this thesis and to introduce some basic tools (and concepts) that are employed. Throughout, conventions used are such that Latin indices run from 0 to 3, Greek indices from 1 to 3. The upper-case Latin indices take the values 1, 2. The abstract index notation¹ [69] is used and the Einstein summation convention is adopted. Often appearing tensors like E_{ab} , H_{ab} or σ_{ab} are also written occasionally in boldface, as \mathbf{E} , \mathbf{H} and $\boldsymbol{\sigma}$. A comma denotes a partial derivative and a semi-colon is used to denote a covariant derivative of a tensor with respect to the Levi-Civita connection. We use geometrised units with $c = 1 = 8\pi G$, which means that all geometrical variables have physical dimensions that are integer powers of length. The round bracket and square bracket denote, respectively, symmetrization and skew-symmetrization; so

$$T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba}), \quad S_{[abc]} = \frac{1}{6}(S_{abc} + S_{cab} + S_{bca} - S_{acb} - S_{bac} - S_{cba}) \quad (1.1)$$

In geometrical treatments of general relativistic astrophysics and cosmology, on a space-time with a Lorentzian metric g_{ab} of signature $(-, +, +, +)$, one usually assumes the existence of a preferred time-like vector field \mathbf{u} . In a cosmological setting the integral curves of this vector field define the family of fundamental observers and they represent the average motion of matter at each point,

$$u^a = \frac{dx^a}{d\tau}, \quad u_a u^a = -1 \quad (1.2)$$

¹In the abstract index notation, one incorporates the nature of a tensor within its notation; the indices are abstract in the sense that they only serve to indicate the type of the tensor. For example, a vector \mathbf{v} is written v^a and a one-form $\boldsymbol{\tau}$ is written τ_a .

where τ is proper time measured along the world-lines. Given \mathbf{u} , the projection tensor into the instantaneous 3-space orthogonal to \mathbf{u} is denoted as h_{ab} , thus

$$h_{ab} = g_{ab} + u_a u_b \quad \Rightarrow \quad h^a_c h^c_b = h^a_b \quad h^a_a = 3 \quad h_{ab} u^b = 0.$$

In the tetrad formalism, a tetrad basis is chosen: a set of four independent vector fields \mathbf{e}_a that together span the tangent vector space at each point in space-time. Since the basis vector fields \mathbf{e}_a will not commute, the commutators $[\mathbf{e}_a, \mathbf{e}_b]$ of the vector fields defined by

$$[\mathbf{e}_a, \mathbf{e}_b]f = \mathbf{e}_a(\mathbf{e}_b f) - \mathbf{e}_b(\mathbf{e}_a f),$$

for a space-time scalar f will satisfy

$$[\mathbf{e}_a, \mathbf{e}_b] = D^c_{ab} \mathbf{e}_c \quad (1.3)$$

and thus $D^a_{(bc)} = 0$, where D^c_{ab} are the commutation coefficients.

In addition, we define the torsion tensor (sometimes called the Cartan (torsion) tensor) on basis vector fields \mathbf{e}_a and \mathbf{e}_b by

$$T(\mathbf{e}_a, \mathbf{e}_b) := 2 \nabla_{[\mathbf{e}_a} \mathbf{e}_{b]} - [\mathbf{e}_a, \mathbf{e}_b].$$

With

$$\nabla_a \mathbf{e}_b = \Gamma^c_{ab} \mathbf{e}_c, \quad (1.4)$$

where $\nabla_a \mathbf{e}_b$ is the covariant derivative of \mathbf{e}_a along \mathbf{e}_b and Γ^c_{ab} are the connection coefficients, we have for any tensor T^a_b

$$T^a_{b;c} = T^a_{b,c} + \Gamma^a_{dc} T^d_b - \Gamma^d_{bc} T^a_d. \quad (1.5)$$

The connection coefficients are related to the metric through $g_{bc;a} = 0$, such that we have

$$\Gamma_{(ab)c} = 0. \quad (1.6)$$

Now assuming that there is no torsion, one can easily derive that for a rigid tetrad (and in particular for an orthonormal tetrad)

$$2\Gamma^a_{[bc]} = -D^a_{bc} \quad \Rightarrow \quad \Gamma_{abc} = \frac{1}{2}(D_{cab} + D_{acb} - D_{bca}).$$

This shows that the connection coefficients and the commutation coefficients are each just linear combinations of the other. Hence, the 24 independent

commutation coefficients, D^a_{bc} are completely equivalent to the 24 independent connection coefficients.

Three vector fields X, Y and Z satisfy the relation

$$[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0, \quad (1.7)$$

called the Jacobi identity. If we write out the Jacobi identity for the basis vectors (e_a, e_b, e_c) we find

$$\partial_{[a} D^d_{bc]} - D^d_{e[c} D^e_{ab]} = 0 \quad (1.8)$$

and their contractions

$$\partial_a D^a_{bc} + \partial_c D^a_{ab} - \partial_b D^a_{ac} + D^a_{af} D^f_{bc} = 0. \quad (1.9)$$

where ∂_a is the directional derivative along the basis vector e_a .

We can also decompose the covariant derivative as defined by (1.4), (1.5) in a "1 + 3" covariant way as follows: define the covariant derivative along the fundamental world lines (the covariant time derivative) and the fully orthogonally projected covariant derivative (the covariant spatial derivative) for a generic tensor $S^{a...}_{b...}$ respectively by

$$\dot{S}^{a...}_{b...} = u^c S^{a...}_{b...;c}, \quad D_c S^{a...}_{b...} = h^f_c h^a_d \dots h^e_b \dots S^{d...}_{e...;f}.$$

The covariant time derivation is also called time propagation. Associated with the covariant spatial derivative (D_a), the covariant spatial divergence and curl of rank-2 tensors are defined by [60]

$$(\text{div} S)_a = D^b S_{ab}, \quad (\text{curl} S)_{ab} = \varepsilon_{cd(a} D^c S_{b)}^d,$$

generalizing the expressions for vectors

$$\text{div} V = D^a V_a, \quad (\text{curl} V)_a = \varepsilon_{abc} D^b S^c.$$

For the sake of simplicity, we shall henceforth omit brackets around "curl" unless this leads to ambiguity. We use angle brackets to denote orthogonal projections of vectors and the orthogonally projected symmetric trace-free part of tensors

$$v^{<a>} = h^a_b v^b, \quad S_{<ab>} = h^c_a h^d_b S_{(cd)} - \frac{1}{3} h_{ab} h^{cd} S_{cd}.$$

We also define in the instantaneous rest spaces of the co-moving observers, the 3-form

$$\varepsilon_{abc} = \eta_{abcd} u^d = \varepsilon_{[abc]},$$

where the totally-skew pseudo-tensor η (the 4-dimensional volume element) is defined by

$$\eta^{abcd} = \eta^{[abcd]} \quad \eta^{1234} = (-g)^{-1/2} \quad g = \det(g_{ab}).$$

The minus sign appears because the determinant g is negative. It follows that

$$\varepsilon_{abc} u^a = 0, \quad \eta_{abcd} = 2u_{[a} \varepsilon_{b]cd} - 2\varepsilon_{ab[c} u_{d]} \quad \text{and} \quad \varepsilon_{abc} \varepsilon^{def} = 3! h_{[a}^d h_b^e h_c^f].$$

Note that $D^c h_{ab} = 0 = D^d \varepsilon_{abc}$, while $\dot{h}_{ab} = 2u_{(a} \dot{u}_{b)}$ and $\dot{\varepsilon}_{abc} = 3u_{[a} \varepsilon_{bc]d} \dot{u}^d$. In general relativity, gravitation is understood as the curvature of space-time caused by energy and matter. The interaction of the geometry and the matter (how matter determines the geometry, which in turn determines the motion of matter) is mathematically described by the Einstein field equations,

$$R_{ab} - \frac{1}{2} R g_{ab} = T_{ab} - \Lambda g_{ab}, \quad (1.10)$$

where T_{ab} is the total energy-momentum tensor representing both the flux and density of energy and momentum of the matter fields and Λ is the cosmological constant. In attempting to find physically relevant solutions to Einstein's field equations, the energy-momentum tensor T_{ab} is assumed to take on a particular form. For simplicity it is often assumed that the anisotropic pressure and heat flux of the fluid are negligible and the energy-momentum tensor is then that of a perfect fluid [93],

$$T_{ab} = \mu_{fluid} u_a u_b + p_{fluid} h_{ab}, \quad (1.11)$$

where $\mu (= T_{ab} u^a u^b)$ and $p (= T_{ab} h^{ab}/3)$ are respectively the energy density and isotropic pressure of the fluid as measured by an observer moving with velocity \mathbf{u} . Moreover, the cosmological constant may be absorbed into the energy-momentum tensor by redefining μ and p as

$$\mu = \mu_{fluid} + \Lambda \quad p = p_{fluid} - \Lambda.$$

Hence the Einstein equation can be rewritten as

$$R_{ab} - \frac{1}{2} R g_{ab} = \mu u_a u_b + p h_{ab}. \quad (1.12)$$

1.1 Kinematic decomposition of a time-like congruence

For a gravitational field with perfect fluid source, the basic covariant variables are the energy density μ , the isotropic pressure p and finally the kinematic quantities associated to the fluid velocity. The latter completely describe how the integral curves in a congruence move with respect to one another. This can be made precise by finding the complete covariant decomposition of the 4-velocity covariant derivative and by giving the geometric interpretation of each part. The covariant derivative of the fluid 4-velocity may be covariantly split into the temporal and spatial derivatives as

$$u_{a;b} = -\dot{u}_a u_b + D_b u_a. \quad (1.13)$$

Additionally, we decompose the spatial derivative into its symmetric and antisymmetric parts as [30]

$$D_b u_a = \theta_{ab} + \omega_{ab}, \quad (1.14)$$

where θ_{ab} ($=\theta_{(ab)}$) and ω_{ab} ($=\omega_{[ab]}$) are known as the expansion tensor and vorticity tensor respectively. Furthermore we can decompose the expansion tensor into its traceless part σ_{ab} ($\sigma_{ab}u^b = 0$, $\sigma_a^a = 0$) and its trace θ . Hence, we get

$$u_{a;b} = -\dot{u}_a u_b + \sigma_{ab} + \frac{1}{3}\theta h_{ab} + \omega_{ab}, \quad (1.15)$$

where σ_{ab} is the shear tensor, determining the rate of distortion of the fluid flow, $\theta = u^a{}_{;a} = D^a u_a$ is the rate of volume expansion scalar of the fluid (with $H = \theta/3$ the Hubble scalar) and \dot{u}^a is the acceleration vector field which represents non-gravitational forces and vanishes when matter moves under influence of the gravity alone.

Because these tensors (θ_{ab} , ω_{ab} and σ_{ab}) live in the spatial hyperplane elements orthogonal to \mathbf{u} , we may think of them as three-dimensional second rank tensors. The spatial derivative of the fluid 4-velocity may be related to the evolution equation of a vector \mathbf{v} representing the separation of the world-lines of two infinitesimally neighboring observers O and N , moving along the integral curves of \mathbf{u} . Vector \mathbf{v} "joins" these two world-lines at some instant and therefore is called a connecting vector. More precisely one can define a curve $\gamma(s)$ connecting two points on the world-lines of O and N and then construct a family $\gamma(s, \tau)$ of curves by moving each point of the curve $\gamma(s)$ a distance τ along the integral curves of \mathbf{u} (see figure 1.1). If we

now define \mathbf{v} as $\partial\gamma(s, \tau)/\partial s$, it follows that $\mathcal{L}_{\mathbf{u}}\mathbf{v} = 0$ where $\mathcal{L}_{\mathbf{u}}$ is the Lie derivative with respect to \mathbf{u} :

$$\dot{v}^a \equiv u^b v^a{}_{;b} = u^a{}_{;b} v^b. \quad (1.16)$$

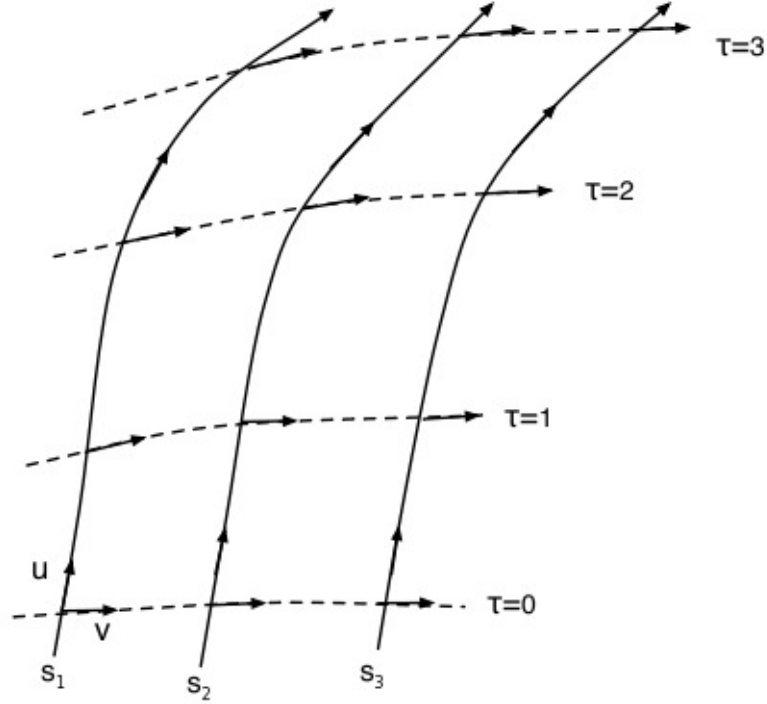


Figure 1.1: connecting vector

In support of the terminology following (1.15) we give the physical interpretation of vorticity tensor, expansion scalar and shear tensor for geodesic flow, separately (see figure 1.2). We start with the vorticity tensor. Using equations (1.15) and (1.16) and taking the time derivative of the squared length, we arrive at

$$\dot{v}_a = (\omega_{ab} + \theta_{ab})v^b, \quad |\dot{v}|^2 = u^a (v_b v^b)_{;a} = 2\theta_{ab} v^a v^b, \quad (1.17)$$

where the second equation represents the time derivative of the radial component of the relative velocity of the neighboring observers. Since ω_{ab} is anti-symmetric, $\omega_{ab}v^b$ is orthogonal to \mathbf{v} , $(\omega_{ab}v^b)v^a = 0$. In the case of

a non expanding congruence ($\theta_{ab} = 0$) and in view of (1.17), ω_{ab} is the instantaneous relative rotational velocity of N with respect to O . Using the orthogonally projected alternating tensor ε_{abc} , the vorticity tensor also defines a vector field ω^a , and vice versa, by

$$\omega_a = \frac{1}{2}\varepsilon_{abc}\omega^{bc} \quad \Leftrightarrow \quad \omega_{ab} = \varepsilon_{abc}\omega^c.$$

Let \mathbf{v}_α ($\alpha = 1, 2, 3$) and \mathbf{u} form an orthogonal basis at an arbitrary point of space-time. Hence $\dot{v}_\alpha^a = u^a{}_{;b}v_\alpha^b$. For a volume defined by $V = \eta^{abcd}u_a v_b^1 v_c^2 v_d^3$ one has then $\dot{V} = \theta V$: this means that θ gives the instantaneous rate of volume change, per unit volume, under the flow (see figure 1.2).

Now consider the shear tensor σ_{ab} . We choose three vector fields \mathbf{v}_α so that they also satisfy $\sigma^{ab} = L^\alpha v_\alpha^a v_\alpha^b$. That means \mathbf{v}_α is an eigenvector of σ^{ab} with eigenvalue L^α , $\sigma^a_b v_\nu^b = L^\nu v_\nu^a$. Since σ^{ab} is trace-free, the coefficients L^α sum to zero, $\sigma_a^a = L^\alpha v_\alpha^a v_a^a = L_1 + L_2 + L_3 = 0$. Now, suppose that ω_{ab} and θ are zero. Then, by (1.17) we get $\dot{v}_\nu^a = v_\nu^b u^a{}_{;b} = \sigma^a_b v_\nu^b = L^\nu v_\nu^a$. Thus each of the vectors \mathbf{v}_α is an axis of instantaneous expansion (or contraction) with associated magnitude L^α . Since the magnitudes sum to zero, expansion along one axis can only occur if there is contraction along other axes. Individual expansions and contractions compensate each other such that there is no net increase or decrease in volume (see figure 1.2).

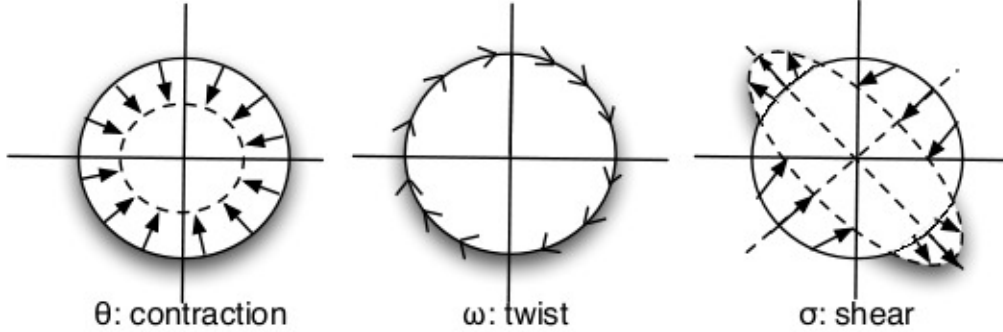


Figure 1.2: expansion/contraction, rotation, shear

Scalar quantities representing the local shear and rotation (or vorticity) of the fluid are given respectively by

$$2\sigma^2 = \sigma_{ab}\sigma^{ab} \quad 2\omega^2 = \omega_{ab}\omega^{ab}. \quad (1.18)$$

Furthermore we define z_α , μ_α and p' as the spatial gradients of θ , μ and the derivative of pressure with respect to matter density respectively:

$$z_\alpha = \partial_\alpha \theta \quad , \quad \mu_\alpha = \partial_\alpha \mu \quad \text{and} \quad p' = \frac{dp}{d\mu}. \quad (1.19)$$

1.2 The curvature tensor

The Riemann tensor measures the failure of commutativity of second order derivatives when applied to vector fields, by the Ricci identity:

$$v^a_{;cd} - v^a_{;dc} = v^b R^a_{bdc}. \quad (1.20)$$

It has the following properties

$$R_{(ab)cd} = 0 \quad , \quad R_{ab(cd)} = 0 \quad , \quad R_{abcd} = R_{cdab} \quad , \quad R^a_{[bcd]} = 0.$$

The covariant derivatives of the curvature tensor obey the Bianchi identities

$$R^a_{b[cd;e]} = 0. \quad (1.21)$$

Contracting once the equation (1.21) results in the identities

$$R^a_{bcd;a} + 2R_{b[c;d]} = 0, \quad (1.22)$$

a second contraction of which yields the contracted Bianchi identities

$$(R^{ab} - \frac{1}{2}g^{ab}R)_{;b} = 0, \quad (1.23)$$

where the Ricci tensor is obtained by contracting the first and third indices in the Riemann curvature tensor, $R_{ab} = R^c_{acb}$ and the Ricci scalar by the contraction $R = R^c_c = R^c_{acb}g^{ab}$.

The Riemann tensor can be expressed as follows

$$R_{abcd} = C_{abcd} + (g_{a[c}S_{d]b} + g_{b[d}S_{c]a}) + \frac{1}{6}Rg_{a[c}g_{d]b}, \quad (1.24)$$

where S_{ab} and C_{abcd} are defined by

$$S_{ab} = R_{ab} - \frac{1}{4}Rg_{ab},$$

$$C^{ab}_{cd} = R^{ab}_{cd} - 2g^{[a}_{[c}R^{b]}_{d]} + \frac{1}{3}g^a_{[c}g^b_{d]}R. \quad (1.25)$$

S_{ab} indicates the traceless part of the Ricci tensor R_{ab} and C_{abcd} is the Weyl tensor, which obeys the same symmetries as the Riemann tensor and furthermore is traceless. With respect to the time-like vector field \mathbf{u} the Weyl tensor can be decomposed into its electric E_{ab} and magnetic H_{ab} parts [65] respectively as

$$E_{ab} = E_{(ab)} = C_{acbd}u^cu^d, \quad E_{ab}u^b = 0, \quad E^a_a = 0, \quad (1.26)$$

$$H_{ab} = H_{(ab)} = \frac{1}{2}\eta_{ac}{}^{gh}C_{ghbd}u^cu^d, \quad H_{ab}u^b = 0, \quad H^a_a = 0, \quad (1.27)$$

where E_{ab} ($= E_{<ab>}$) and H_{ab} ($= H_{<ab>}$) are called the electric and magnetic parts of the Weyl tensor respectively. The nonlocal fields E_{ab} (the tidal part of the curvature), which generalizes the Newtonian tidal tensor, and H_{ab} , which has no Newtonian analogue, covariantly describe gravitational radiation [31, 33, 42]. As such, H_{ab} may be considered as the true gravitational wave tensor, since there is no gravitational radiation in Newtonian theory. However at least in the linear regime, as in electromagnetic theory, gravitational waves are characterized by H_{ab} and E_{ab} , where both are divergence-free but neither is curl-free [45].

The Weyl curvature tensor vanishes if its electric and magnetic parts vanish, and vice versa, $E_{ab} = 0 = H_{ab} \Leftrightarrow C_{abcd} = 0$. The definitions of E_{ab} and H_{ab} lead to [30]

$$C_{ab}{}^{cd} = 4(u_{[a}u^{[c} + h_{[a}{}^{[c}E_{b]}{}^{d]}) + 2\varepsilon_{abe}u^{[c}H^{d]e} + 2\varepsilon^{cde}u_{[a}H_{b]e}. \quad (1.28)$$

Applying the Ricci identity (1.20) to the velocity vector \mathbf{u} and using the decomposition (1.15) one finds

$$\begin{aligned} \frac{1}{2}R^d{}_{abc}u^d &= -\dot{u}_a(\omega_{bc} - \dot{u}_{[b}u_{c]}) - \dot{u}_{a;[c}u_{b]} + \omega_{a[b;c]} + \sigma_{a[b;c]} + \frac{1}{3}\theta_{[c}h_{b]a} \\ &\quad + \frac{1}{3}\theta(u_a\omega_{bc} - u_a\dot{u}_{[b}u_{c]} + \omega_{a[c}u_{b]} + \sigma_{a[c}u_{b]} + \frac{1}{3}\theta h_{a[c}u_{b]}) \end{aligned} \quad (1.29)$$

and from this, by contraction and/or multiplication with u^d , we get

$$\begin{aligned} R^a{}_b u_a &= \dot{u}^a\omega_{ab} - \dot{u}^a{}_{;a}u_b - \dot{u}^a\sigma_{ab} + \omega^a{}_{b;a} + \sigma^a{}_{b;a} \\ &\quad - \frac{2}{3}\theta_{;b} + \frac{1}{3}\dot{\theta}u_b + \frac{1}{3}\theta^2u_b, \end{aligned} \quad (1.30)$$

$$\begin{aligned} R^d{}_{abc}u^d u^b &= \dot{u}_a\dot{u}_c - \omega_{ab}\omega^b{}_c - \frac{1}{3}(\dot{\theta} + \frac{1}{3}\theta^2)h_{ca} - \sigma_{ab}\sigma^b{}_c - \frac{2}{3}\theta\sigma_{ac} \\ &\quad + h^d{}_a h^e{}_c (\dot{u}_{(d}u_{e)} - \dot{\sigma}_{de}), \end{aligned} \quad (1.31)$$

$$R^a{}_b u_a u^b = \dot{u}^a{}_{;a} + \omega_{ab}\omega^{ab} - \sigma_{ab}\sigma^{ab} - \dot{\theta} - \frac{\theta^2}{3}. \quad (1.32)$$

Furthermore, applying the Ricci identity to the basis vector fields e_a and using (1.4) and (1.3) we get an expression for the Riemann tensor in terms of the connection coefficients and commutation coefficients:

$$R^a_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{ec}\Gamma^e_{bd} - \Gamma^a_{ed}\Gamma^e_{bc} - \Gamma^a_{be}D^e_{cd}. \quad (1.33)$$

1.3 Formalisms

There exist several different approaches to the problem of decomposing the field equations in a way that allows us to obtain the subsequent evolution of the gravitational field. Specific formalisms differ in the way in which this decomposition is carried out. Here we concentrate on the 1 + 3 covariant and orthonormal tetrad formalisms which are based on projecting of objects and equations along the preferred time-like vector field \mathbf{u} and a set of four basis vector fields respectively.

The 1+3 covariant formalism is a good approach to use in obtaining solutions when there exists a preferred time-like vector field (the matter 4-velocity \mathbf{u}). In this formalism we split space-time into three-dimensional space on the one hand, and time on the other. Therefore, one can covariantly project the Ricci identities as applied to \mathbf{u} , and the Bianchi identities along parallel and orthogonal directions to \mathbf{u} in terms of the kinematic quantities $\{\theta, \dot{\mathbf{u}}, \boldsymbol{\sigma}, \boldsymbol{\omega}\}$ and the electric and magnetic parts of the Weyl tensor. The resulting parallel and orthogonal projected equations are called the evolution equations and constraint equations respectively.

In the orthonormal tetrad formalism, one projects objects and equations according to a tetrad which is a set of the four orthogonal unit vector fields e_a , $a = 0, 1, 2, 3$. Choosing the time-like vector field e_0 to be the 4-velocity of the matter fluid flow \mathbf{u} , the Einstein field equations and the Jacobi identities can be decomposed into evolution and constraint equations. The Jacobi identity is the integrability condition which the commutators of the tetrad e_a (1.53 and 1.54) must satisfy. There is a significant difference between the evolution and constraint equations in the 1+3 covariant formalism and the orthonormal tetrad formalism; in the former the equations are tensorial, and involve covariant derivatives, while in the latter the variables are scalars, and hence covariant derivatives are not required.

The electric and magnetic parts of Weyl tensor play important roles in our investigations (either in shear-free or purely radiative space-times). While the 1+3 covariant equations are transparent in terms of representing relations between kinematic quantities and the electric and magnetic parts of Weyl tensor, they do not form a complete set of equations which is equiv-

alent with the Einstein field equations and Jacobi identities (see subsection 1.3.2). That is why we prefer to use the orthonormal description and we then make use of the (second) Bianchi identities as equations for the Weyl tensor components. Such a set of variables and equations is called an extended orthonormal tetrad formalism.

1.3.1 The covariant propagation and constraint equations

In the 1+3 covariant approach there are three sets of equations, describing kinematic and dynamic evolution of the basic covariant variables. They result from identities satisfied by the Riemann tensor and the Einstein's field equations used algebraically to substitute for the Ricci tensor in terms of fluid matter variables. These three sets are as: 1) the propagation and constraint equations of the covariant variables, emerging from the Ricci identities, 2) the twice-contracted Bianchi identities and 3) Bianchi identities.

Ricci identity

The first set of the dynamical equations arises after applying the Ricci identity (1.20) to the fluid velocity \mathbf{u} ,

$$u^a{}_{;cd} - u^a{}_{;dc} = R^a{}_{bcd}u^b. \quad (1.34)$$

On substituting the covariant derivative of \mathbf{u} , the Riemann curvature tensor, the Weyl tensor from (1.15), (1.24), (1.28) respectively and substituting the Ricci curvature tensor in terms of fluid matter variables by the use of the Einstein field equations and then separating out the orthogonally and parallel projected parts into trace, symmetric trace-free, and skew symmetric parts, the expression (1.34) leads to a set of three propagation and three constraint equations, namely

$$\dot{\theta} = -\frac{1}{3}\theta^2 + D_a \dot{u}^a + \dot{u}_a \dot{u}^a - 2(\sigma^2 - \omega^2) - \frac{1}{2}(\mu + 3p), \quad (1.35)$$

$$\dot{\omega}_{\langle a} \rangle = -\frac{2}{3}\theta\omega_a + \sigma_{ab}\omega^b - \frac{1}{2}\text{curl}\dot{u}_a, \quad (1.36)$$

$$\dot{\sigma}_{\langle ab} \rangle = D_{\langle a}\dot{u}_{b\rangle} + \dot{u}_{\langle a}\dot{u}_{b\rangle} - \frac{2}{3}\theta\sigma_{ab} - \sigma_{c\langle a}\sigma_{b\rangle}^c - \omega_{\langle a}\omega_{b\rangle} - E_{ab}, \quad (1.37)$$

$$D^b\sigma_{ab} = \frac{2}{3}z_a + \text{curl}\omega_a - 2\varepsilon_{abc}\omega^b\dot{u}^c, \quad (1.38)$$

$$D_a\omega^a = \dot{u}_a\omega^a, \quad (1.39)$$

$$H_{ab} = 2\dot{u}_{\langle a}\omega_{b\rangle} + D_{\langle a}\omega_{b\rangle} + \text{curl}\sigma_{ab}. \quad (1.40)$$

Note that $\dot{u}_a = \dot{u}_{\langle a} \rangle$. Equation (1.35) is called the Raychaudhuri equation.

Twice-contracted Bianchi identities

The second set of equations emerges from the twice-contracted Bianchi identities which, by Einstein's field equations, imply the conservation of total energy-momentum $T^{ab}_{;b} = 0$. Projecting parallel and orthogonal to \mathbf{u} , for perfect fluids we obtain the propagation equation

$$\dot{\mu} = -\theta(\mu + p) \quad (1.41)$$

and constraint equation

$$D_a p = -(\mu + p)\dot{u}_a. \quad (1.42)$$

If $\mu + p$ is zero, the matter distribution is described by an effective cosmological constant ($T_{ab} = -\Lambda g_{ab}$ in 1.12).

Bianchi identities

In a cosmological context, where there is a preferred 4-velocity field \mathbf{u} , the electric and magnetic parts of the Weyl tensor, E_{ab} and H_{ab} , represent covariantly the locally free gravitational field, in the sense that they are not point-wise determined by the matter fields([30]-[90]). They do not arise explicitly in the Einstein equations, although they are not really independent of the matter fields, being constrained by integrability conditions in the form of the Bianchi identities. On using the splitting of R_{abcd} into R_{ab} and C_{abcd} and Einstein's field equations, the once-contracted Bianchi identities give two further propagation equations and two further constraint equations as

$$\begin{aligned} \dot{E}_{\langle ab \rangle} - \text{curl} H_{ab} = & -\frac{1}{2}(\mu + p)\sigma_{ab} - \theta E_{ab} + 3\sigma_{c\langle a} E_{b \rangle}^c - \omega^c \varepsilon_{cd(a} E_{b)}^d \\ & + 2\dot{u}^c \varepsilon_{cd(a} H_{b)}^d, \end{aligned} \quad (1.43)$$

$$\begin{aligned} \dot{H}_{\langle ab \rangle} + \text{curl} E_{ab} = & -\theta H_{ab} + 3\sigma_{c\langle a} H_{b \rangle}^c - 2\dot{u}^c \varepsilon_{cd(a} E_{b)}^d \\ & - \omega^c \varepsilon_{cd(a} H_{b)}^d, \end{aligned} \quad (1.44)$$

$$\text{div} E_a = \frac{1}{3} D_a \mu - 3\omega^b H_{ab} + [\sigma, H]_a, \quad (1.45)$$

$$\text{div} H_a = (\mu + p)\omega_a + 3\omega^b E_{ab} - [\sigma, E]_a. \quad (1.46)$$

where the spatial dual of the commutator of tensors, $[S, T]_a$ is defined as $[S, T]_a = \varepsilon_{abc} S^{bd} T_d^c$.

1.3.2 Orthonormal tetrads

In the orthonormal tetrad approach one chooses a set of four orthogonal unit basis vector fields $\{\mathbf{e}_0, \mathbf{e}_\alpha\}$ such that the vector \mathbf{e}_0 is time-like and \mathbf{e}_α are space-like. The tetrads correspond to a family of ideal observers; the integral curves of the time-like unit vector field are the world-lines of these observers, and at each event along a given world-line, the three space-like unit vector fields specify the spatial triad carried by the observer. All the following considerations are purely local.

Let x^i denote a local coordinate system. Assuming the first half $\{a, b, c, \dots\}$ and the second half $\{i, j, k, \dots\}$ of the latin alphabet respectively as tetrad indices and coordinate indices, the tetrad \mathbf{e}_a can be written in terms of a local coordinate basis $\partial/\partial x^i$ by means of the tetrad components $e_a^i = \mathbf{e}_a(x^i)$, as follows

$$\mathbf{e}_a = e_a^i \partial/\partial x^i$$

Thus the tetrad components of the metric are given by

$$g_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b = (e_a^i \partial/\partial x^i) \cdot (e_b^j \partial/\partial x^j) = e_a^i e_b^j (\partial/\partial x^i) \cdot (\partial/\partial x^j) = g_{ij} e_a^i e_b^j.$$

We also define the dual components e^a_j by

$$e_a^i e^a_j = \delta^i_j \quad \Leftrightarrow \quad e_a^i e^b_i = \delta_a^b.$$

As the tetrad is orthonormal,

$$g_{ab} = \eta_{ab} = \text{diag}[-1, +1, +1, +1], \quad (1.47)$$

the metric components g^{ab} defined by $g^{ab} g_{bc} = \delta^a_c$, are numerically equal to the g_{ab} ; tetrad indices are raised and lowered by the metric components g^{ab} and g_{ab} .

From (1.4), the connection coefficients Γ_{abc} are given by

$$\Gamma_{bc}^a = e^a_j e^i_c e^j_{b;i}.$$

Aligning the time-like basis vector of the orthonormal frame with the tangent of the preferred time-like congruence, $\mathbf{e}_0 = \mathbf{u}$, the commutation coefficients D^c_{ab} (see 1.3) with one or two indices equal to zero can be expressed in terms of the kinematic quantities associated with the time-like congruence as

$$D^0_{0\alpha} = \dot{u}_\alpha, \quad D^0_{\alpha\beta} = -2\varepsilon^\gamma_{\alpha\beta} \omega_\gamma, \quad (1.48)$$

$$D^\alpha_{0\beta} = -\theta^\alpha_\beta - \varepsilon^\alpha_{\beta\gamma} (\omega^\gamma + \Omega^\gamma), \quad (1.49)$$

where Ω^a is the local angular velocity, in the rest-frame of an observer with four-velocity \mathbf{u} , of a set of Fermi-propagated axes with respect to the triad \mathbf{e}_α , is given as

$$\Omega^a := \frac{1}{2} \eta^{abcd} u_b \mathbf{e}_c \cdot \dot{\mathbf{e}}_d. \quad (1.50)$$

The purely spatial components, $D^\alpha_{\beta\delta}$, can be decomposed, following Schücking, Kundt and Behr [35], as follows

$$D^\alpha_{\beta\gamma} = \varepsilon_{\beta\gamma\zeta} n^{\zeta\alpha} + 2a_{[\beta} \delta^\alpha_{\gamma]}, \quad (1.51)$$

where δ^α_β is the Kronecker delta and we may refer to $n^{\alpha\beta}$ ($n^{[\alpha\beta]} = 0$) and a_α as the purely spatial connection variables. Because of some computational advantages, we will be replacing in chapters 2 and 3, $n_{\alpha\beta}$ ($\alpha \neq \beta$) and a_α with the new variables q_α and r_α defined by

$$n_{\alpha-1 \alpha+1} = (r_\alpha + q_\alpha)/2, \quad a_\alpha = (r_\alpha - q_\alpha)/2, \quad (1.52)$$

expressions which must be read modulo 3.

Commutators

The commutators are described by the expressions for D^a_{bc} . Their 1+3 decomposition leads to

$$[\mathbf{e}_0, \mathbf{e}_\alpha] = \dot{u}^\alpha \mathbf{e}_0 - \left[\frac{1}{3} \theta \delta_\alpha^\beta + \sigma_\alpha^\beta + \varepsilon^\beta_{\alpha\gamma} (\omega^\gamma + \Omega^\gamma) \right] \mathbf{e}_\beta, \quad (1.53)$$

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = -2\varepsilon_{\alpha\beta\gamma} \omega^\gamma \mathbf{e}_0 + [2a_{[\alpha} \delta^\gamma_{\beta]} + \varepsilon_{\alpha\beta\delta} n^{\delta\gamma}] \mathbf{e}_\gamma. \quad (1.54)$$

Einstein Field Equations

Inserting the contraction of (1.33) for the Ricci tensor R_{ab} and (1.11) for T_{ab} into the equation (1.10) yields the 10 Einstein field equations:

the (0 0) equation

$$\dot{\theta} + \theta^{\alpha\beta} \theta_{\alpha\beta} - 2\omega^2 - \partial_\alpha \dot{u}^\alpha - \dot{u}_\alpha \dot{u}^\alpha + 2a_\alpha \dot{u}^\alpha + \frac{1}{2}(\mu + 3p) = 0, \quad (1.55)$$

the (0 α) equations

$$\frac{2}{3} \partial_\alpha \theta - \partial_\beta \sigma^\beta_\alpha - \varepsilon^\beta_{\alpha\delta} \partial_\beta \omega^\delta + 3\sigma^\beta_\alpha a_\beta - n_{\alpha\beta} \omega^\beta + \varepsilon_{\alpha\beta\gamma} \omega^\beta (a^\gamma - 2\dot{u}^\gamma) - \varepsilon_{\alpha\beta\gamma} n^{\gamma\delta} \sigma^\beta_\delta = 0 \quad (1.56)$$

and the $(\alpha\beta)$ equations

$$\begin{aligned}
-R_{\alpha\beta}^* &\equiv -\partial_{(\alpha}a_{\beta)} + \varepsilon_{\delta\gamma(\alpha}\partial^\delta n_{\beta)}^\gamma + 2\varepsilon_{\gamma\delta(\alpha}n_{\beta)}^\gamma a^\delta - 2n_{(\alpha}^\gamma n_{\beta)\gamma} + n_{\gamma}^\gamma n_{\alpha\beta} \\
&\quad + \delta_{\alpha\beta}(2a_{\gamma}a^{\gamma} + n^{\gamma\delta}n_{\gamma\delta} - (n_{\alpha}^{\alpha})^2/2 - \partial_{\gamma}a^{\gamma}) \\
&= \partial_0\theta_{\alpha\beta} - \partial_{(\alpha}\dot{u}_{\beta)} - \dot{u}_{\alpha}\dot{u}_{\beta} - \dot{u}_{(\alpha}a_{\beta)} + n_{(\alpha}^\gamma\varepsilon_{\beta)\gamma\delta}\dot{u}^\delta + 2\Omega_{(\alpha}\omega_{\beta)} + \\
&\quad \theta\theta_{\alpha\beta} + 2\theta_{(\alpha}^\gamma\varepsilon_{\beta)\gamma\delta}\Omega^\delta + \delta_{\alpha\beta}(\dot{u}^\gamma a_{\gamma} - 2\Omega^\gamma\omega_{\gamma} - \frac{1}{2}(\mu - p)). \quad (1.57)
\end{aligned}$$

Here, R_{ab}^* is a quantity which is so defined that it would be the Ricci curvature of a hyper-surface orthogonal to e_0 , if the fluid was non-rotating. Using equation (1.55), the contraction of (1.57) implies that the corresponding Ricci scalar R^* is

$$R^* = -\frac{2\theta^2}{3} + 2(\sigma^2 - \omega^2) + 4\omega_{\gamma}\Omega^{\gamma} + 2\mu. \quad (1.58)$$

Jacobi Identity

Finally, substituting (1.48), (1.49) and (1.51) in the Jacobi identity (1.8), in tetrad components, we obtain

$$\partial_{\alpha}\omega^{\alpha} = \omega^{\alpha}(\dot{u}_{\alpha} + 2a_{\alpha}), \quad (1.59)$$

$$\partial_{\alpha}n^{\alpha\delta} + \varepsilon^{\delta\alpha\beta}\partial_{\alpha}a_{\beta} - 2\theta^{\delta}_{\alpha}\omega^{\alpha} - 2n^{\delta}_{\beta}a^{\beta} - 2\varepsilon^{\delta\alpha\beta}\omega_{\alpha}\Omega_{\beta} = 0, \quad (1.60)$$

$$\begin{aligned}
2\partial_0\omega^{\gamma} + \varepsilon^{\gamma\alpha\beta}\partial_{\alpha}\dot{u}_{\beta} - n^{\gamma\alpha}\dot{u}_{\alpha} - \varepsilon^{\gamma\alpha\beta}a_{\alpha}\dot{u}_{\beta} + 2\theta\omega^{\gamma} \\
- 2\varepsilon^{\gamma\alpha\beta}\omega_{\alpha}\Omega_{\beta} - 2\theta^{\gamma}_{\alpha}\omega^{\alpha} = 0, \quad (1.61)
\end{aligned}$$

$$\begin{aligned}
2\partial_0a_{\alpha} - \partial_{\delta}\theta^{\delta}_{\alpha} + \partial_{\alpha}\theta + \varepsilon_{\alpha}^{\delta\beta}\partial_{\delta}(\Omega_{\beta} + \omega_{\beta}) + \theta\dot{u}_{\alpha} + \theta^{\beta}_{\alpha}(2a_{\beta} - \dot{u}_{\beta}) \\
- \varepsilon_{\alpha\beta\gamma}(2a^{\beta} - \dot{u}^{\beta})(\omega^{\gamma} + \Omega^{\gamma}) = 0, \quad (1.62)
\end{aligned}$$

$$\begin{aligned}
\partial_0n^{\alpha\beta} - \varepsilon^{\delta\gamma(\alpha}\partial_{\gamma}\theta^{\beta)}_{\delta} + \partial^{(\alpha}(\Omega^{\beta)} + \omega^{\beta)}) - 2n^{\gamma(\alpha}\varepsilon^{\beta)}_{\gamma\delta}(\Omega^{\delta} + \omega^{\delta}) \\
+ \theta_{\gamma}^{(\alpha}\varepsilon^{\beta)\delta\gamma}\dot{u}_{\delta} + \dot{u}^{(\alpha}(\omega^{\beta)} + \Omega^{\beta)}) - 2n^{\alpha\beta\gamma}\theta_{\gamma} + n^{\alpha\beta}\theta \\
- \delta^{\alpha\beta}(\partial_{\gamma}\Omega^{\gamma} + 2\omega^{\gamma}\dot{u}_{\gamma} + 2\omega^{\gamma}a_{\gamma} + \dot{u}^{\gamma}\Omega_{\gamma}) = 0. \quad (1.63)
\end{aligned}$$

Using (1.25), (1.26), (1.27), (1.33), (1.12), (1.48), (1.49) and (1.51) one can express the electric and magnetic components of the Weyl tensor in terms

of the orthonormal frame variables:

$$\begin{aligned}
E_{\alpha\beta} = & -\partial_0\sigma_{\alpha\beta} + \partial_\beta\dot{u}_\alpha - \varepsilon_{\alpha\beta}^\gamma\partial_0\omega_\gamma + a_\alpha\dot{u}_\beta + 2\omega_{[\alpha}\Omega_{\beta]} - \omega_\alpha\omega_\beta + \dot{u}_\alpha\dot{u}_\beta \\
& -\sigma_{\alpha\gamma}\sigma_{\beta}^\gamma - \frac{2}{3}\theta\sigma_{\alpha\beta} + \frac{1}{2}\varepsilon_{\alpha\beta}^\gamma n_{\gamma}^\delta\dot{u}_\delta + \varepsilon_{(\alpha}^{\gamma\delta}n_{\beta)\delta}\dot{u}_\gamma + 2\varepsilon_{[\alpha}^{\gamma\delta}\sigma_{\beta]\delta}\omega_\gamma \\
& -\frac{2}{3}\varepsilon_{\alpha\beta}^\gamma\omega_\gamma\theta - 2\varepsilon_{(\alpha}^{\gamma\delta}\sigma_{\beta)\delta}\Omega_\gamma - \frac{1}{3}\delta_{\alpha\beta}[\partial_0\theta + \frac{1}{3}\theta^2 - 3\omega^2 \\
& + 3a^\gamma\dot{u}_\gamma + \frac{1}{2}(\mu + 3p)]
\end{aligned} \tag{1.64}$$

$$\begin{aligned}
H_{\alpha\beta} = & -\partial_0n_{\alpha\beta} - \partial_{(\alpha}\Omega_{\beta)} + a_{(\alpha}\omega_{\beta)} + \dot{u}_{(\alpha}(\omega_{\beta)} - \Omega_{\beta)}) - n_{\gamma(\alpha}\sigma_{\beta)\gamma} - \frac{1}{3}n_{\alpha\beta}\theta \\
& + \frac{1}{2}\sigma_{\alpha\beta}n_{\gamma}^\gamma + \varepsilon_{\delta(\alpha}^{\gamma}\sigma_{\beta)\gamma}(a^\delta + \dot{u}^\delta) + \varepsilon_{\delta(\alpha}^{\gamma}n_{\beta)\gamma}(2\Omega^\delta + \omega^\delta) \\
& + \frac{1}{6}\delta_{\alpha\beta}(3\partial_0n_{\gamma}^\gamma + n_{\gamma}^\gamma\theta - a^\gamma\omega_\gamma - \omega^\gamma\dot{u}_\gamma).
\end{aligned} \tag{1.65}$$

As $E_{\alpha\beta}$ and $H_{\alpha\beta}$ are symmetric, from (1.64) and (1.65), one obtains:

$$\begin{aligned}
\varepsilon_{\alpha\beta}^\gamma\partial_0\omega_\gamma = & 2\omega_{[\alpha}\Omega_{\beta]} + a_{[\alpha}\dot{u}_{\beta]} - \partial_{[\alpha}\dot{u}_{\beta]} - 2\varepsilon_{[\alpha}^{\gamma\delta}\sigma_{\beta]\gamma}\omega_\delta - \frac{2}{3}\varepsilon_{\alpha\beta}^\gamma\omega_\gamma\theta \\
& - \varepsilon_{[\alpha}^{\gamma\delta}n_{\beta]\gamma}\dot{u}_\delta + \frac{1}{2}\varepsilon_{\alpha\beta}^\gamma\dot{u}_\gamma n_{\delta}^\delta,
\end{aligned} \tag{1.66}$$

$$\begin{aligned}
E_{(\alpha\beta)} = & -\partial_0\sigma_{\alpha\beta} + \partial_{(\alpha}\dot{u}_{\beta)} + a_{(\alpha}\dot{u}_{\beta)} - \omega_\alpha\omega_\beta - \sigma_{\alpha\gamma}\sigma_{\beta}^\gamma + \dot{u}_\alpha\dot{u}_\beta - \frac{2}{3}\theta\sigma_{\alpha\beta} \\
& - 2\varepsilon_{(\alpha}^{\gamma\delta}\sigma_{\beta)\delta}\Omega_\gamma - \varepsilon_{(\alpha}^{\gamma\delta}n_{\beta)\gamma}\dot{u}_\delta - \frac{2}{3}\delta_{\alpha\beta}[\partial_0\theta + \frac{1}{3}\theta^2 - 3\omega^2 + 3a^\gamma\dot{u}_\gamma \\
& + \frac{1}{2}(\mu + 3p)],
\end{aligned} \tag{1.67}$$

$$\begin{aligned}
\varepsilon_{\alpha\beta}^\gamma\partial_0a_\gamma = & -2\partial_{[\alpha}\Omega_{\beta]} + 2(2a_{[\alpha} - \dot{u}_{[\alpha]}\Omega_{\beta]} + 2(a_{[\alpha} + \dot{u}_{[\alpha]}\omega_{\beta]}) \\
& - 2\varepsilon_{[\alpha}^{\gamma\delta}\sigma_{\beta]\gamma}(a_\delta + \dot{u}_\delta) - \frac{2}{3}\theta\varepsilon_{\alpha\beta}^\gamma(a_\gamma + \dot{u}_\gamma) + 2\varepsilon_{[\alpha}^{\gamma\delta}n_{\beta]\gamma}\omega_\delta \\
& - \varepsilon_{\alpha\beta}^\gamma\omega_\gamma n_{\delta}^\delta + 2n_{\gamma[\alpha}\sigma_{\beta]}^\gamma
\end{aligned} \tag{1.68}$$

and

$$\begin{aligned}
H_{(\alpha\beta)} = & -\partial_{(\alpha}\Omega_{\beta)} - \partial_0n_{\alpha\beta} + (a_{(\alpha} + \dot{u}_{(\alpha}\omega_{\beta)} - \dot{u}_{(\alpha}\Omega_{\beta)} + \frac{1}{6}\sigma_{\alpha\beta}(3n_{\gamma}^\gamma - 2\theta) \\
& - n_{\gamma(\alpha}\sigma_{\beta)}^\gamma + \varepsilon_{(\alpha}^{\gamma\delta}\sigma_{\beta)\gamma}(\dot{u}_\delta + a_\delta) + \varepsilon_{(\alpha}^{\gamma\delta}n_{\beta)\gamma}(\omega_\delta + 2\Omega_\delta) \\
& + \frac{1}{3}\delta_{\alpha\beta}(3\partial_0n_{\gamma}^\gamma + n_{\gamma}^\gamma\theta - a^\gamma\omega_\gamma - \omega^\gamma\dot{u}_\gamma),
\end{aligned} \tag{1.69}$$

where (1.66) and (1.68) are respectively corresponding to (1.61) and the difference of (1.62) and (1.56).

Notice that using the expressions (1.67) and (1.69) for $E_{\alpha\beta}$ and $H_{\alpha\beta}$, the Ricci identities, (1.35)-(1.40), respectively results in (1.55), (1.61), (1.57), (1.56), (1.59) and (1.63). One can also obtain the Bianchi identities in the form of the orthonormal tetrad components (see Appendix B).

So far, there are still two more equations (1.60) and (1.62) of the Jacobi identities which cannot be derived from the covariant equations (1.35)-(1.46). Alternatively one can show that the $(\alpha\beta)$ Einstein field equations $(\alpha, \beta = 1, 2, 3)$ in general cannot be recovered from the Ricci identities (as applied to \mathbf{u}), the Bianchi identities and the commutators, together with the Jacobi identities. Hence the covariant equations do not form a complete set of equations which guarantee the existence of corresponding solutions.

1.4 Rotation in the (12)-plane

When using orthonormal frames, it is customary to fix \mathbf{e}_0 in some invariant way. One then chooses the spatial axes such that the tetrad alignment is well suited for the physical problem at hand, for example aligning the \mathbf{e}_3 -axis with the vorticity or the acceleration. The remaining tetrad freedom is then given by rotations about the \mathbf{e}_3 -axis, the transformation of the tetrad being given by

$$\begin{aligned} \mathbf{e}'_1 &= \mathbf{e}_1 \cos \alpha + \mathbf{e}_2 \sin \alpha \\ \mathbf{e}'_2 &= -\mathbf{e}_1 \sin \alpha + \mathbf{e}_2 \cos \alpha, \end{aligned} \quad (1.70)$$

where α is possibly depending on the position in space-time.

We can calculate then the new spin coefficients as functions of the old ones, after which we will be able to deduce the transformation formulae between the components of the kinematic variables. Indicating the components w.r.t. (\mathbf{e}'_α) with a dash, we obtain from (A.27)-(A.32)

$$\begin{aligned} n'_{11} &= \cos(\alpha)^2 n_{11} + \sin(\alpha)^2 n_{22} - \sin(2\alpha) n_{12} - \partial_3 \alpha, \\ n'_{22} &= \sin(\alpha)^2 n_{11} + \cos(\alpha)^2 n_{22} + \sin(2\alpha) n_{12} - \partial_3 \alpha, \\ n'_{12} &= \frac{1}{2} \sin(2\alpha) n_{11} - \frac{1}{2} \sin(2\alpha) n_{22} + \cos(2\alpha) n_{12}, \\ n'_{13} &= \cos(\alpha) n_{13} - \sin(\alpha) n_{23} + \frac{1}{2} \cos(\alpha) \partial_1 \alpha - \frac{1}{2} \sin(\alpha) \partial_2 \alpha, \\ n'_{23} &= \sin(\alpha) n_{13} + \cos(\alpha) n_{23} + \frac{1}{2} \sin(\alpha) \partial_1 \alpha + \frac{1}{2} \cos(\alpha) \partial_2 \alpha, \\ n'_{33} &= n_{33} \end{aligned} \quad (1.71)$$

and

$$\begin{aligned} a'_1 &= \cos(\alpha)a_1 - \sin(\alpha)a_2 + \frac{1}{2}\sin(\alpha)\partial_1\alpha + \frac{1}{2}\cos(\alpha)\partial_2\alpha, \\ a'_2 &= \sin(\alpha)a_1 + \cos(\alpha)a_2 - \frac{1}{2}\cos(\alpha)\partial_1\alpha + \frac{1}{2}\sin(\alpha)\partial_2\alpha, \\ a'_3 &= a_3. \end{aligned} \quad (1.72)$$

For the acceleration vector, vorticity vector and shear tensor we obtain

$$\dot{u}'_1 = \cos(\alpha)\dot{u}_1 - \sin(\alpha)\dot{u}_2, \quad (1.73)$$

$$\dot{u}'_2 = \sin(\alpha)\dot{u}_1 + \cos(\alpha)\dot{u}_2, \quad (1.74)$$

$$\dot{u}'_3 = \dot{u}_3, \quad (1.75)$$

$$\Omega'_1 = -\sin(\alpha)\Omega_2 + \cos(\alpha)\Omega_1, \quad (1.76)$$

$$\Omega'_2 = \cos(\alpha)\Omega_2 + \sin(\alpha)\Omega_1, \quad (1.77)$$

$$\Omega'_3 = \Omega_3 - \partial_0\alpha, \quad (1.78)$$

$$\omega'_1 = \cos(\alpha)\omega_1 - \sin(\alpha)\omega_2, \quad (1.79)$$

$$\omega'_2 = \cos(\alpha)\omega_2 + \sin(\alpha)\omega_1, \quad (1.80)$$

$$\omega'_3 = \omega_3 \quad (1.81)$$

and

$$\begin{aligned} \sigma'_{11} &= \cos(\alpha)^2\sigma_{11} + \sin(\alpha)^2\sigma_{22} - \sin(2\alpha)\sigma_{12}, \\ \sigma'_{22} &= \sin(\alpha)^2\sigma_{11} + \cos(\alpha)^2\sigma_{22} + \sin(2\alpha)\sigma_{12}, \\ \sigma'_{12} &= \frac{1}{2}\sin(2\alpha)\sigma_{11} - \frac{1}{2}\sin(2\alpha)\sigma_{22} + \cos(2\alpha)\sigma_{12}, \\ \sigma'_{13} &= \cos(\alpha)\sigma_{13} - \sin(\alpha)\sigma_{23}, \\ \sigma'_{23} &= \sin(\alpha)\sigma_{13} + \cos(\alpha)\sigma_{23}, \\ \sigma'_{33} &= \sigma_{33}. \end{aligned} \quad (1.82)$$

Note that, introducing quantities q_α and r_α by (1.52) we obtain from (1.71) and (1.72) that

$$\begin{aligned} q'_1 &= q_1 \cos \alpha + r_2 \sin \alpha \\ r'_2 &= -q_1 \sin \alpha + r_2 \cos \alpha, \end{aligned}$$

showing that q_1 and $-r_2$ transform as the components of a vector under the above transformation.

1.5 Space-time symmetries

An isometry of a manifold (\mathcal{M}, g) is a mapping of \mathcal{M} into itself that leaves the metric g and hence all geometrical properties invariant. As we deal here only with continuous symmetries, the metric tensor is mapped into itself by the transformations generated by a vector field ξ if and only if

$$\mathcal{L}_\xi g = 0, \quad (1.83)$$

which is usually written in the expanded form as:

$$\xi_{a;b} + \xi_{b;a} = 0. \quad (1.84)$$

This is known as Killing's equation and any solution of it is known as a Killing vector field. Therefore if there exists a solution of (1.84) for a given g_{ab} , then the corresponding ξ_a represents an infinitesimal isometry of the metric g_{ab} .

Space-time admits at most a finite number of linearly independent Killing vector fields. In 4-dimensions, the maximum number of independent symmetries is $r = 10$; this maximum is attained for space-times of constant curvature, but most exact solutions admit significantly fewer.

If ξ_1 and ξ_2 are Killing vectors, then also $[\xi_1, \xi_2]$ is a Killing vector. Indeed one has

$$\mathcal{L}_{[\xi_1, \xi_2]} g_{ab} = \mathcal{L}_{\xi_1} \mathcal{L}_{\xi_2} g_{ab} - \mathcal{L}_{\xi_2} \mathcal{L}_{\xi_1} g_{ab} = 0. \quad (1.85)$$

The commutator $[\xi_\alpha, \xi_\beta]$ can be expanded relative to the basis, giving the identity

$$[\xi_\alpha, \xi_\beta] = C^\mu_{\alpha\beta} \xi_\mu, \quad (1.86)$$

where $C^\mu_{\alpha\beta} = -C^\mu_{\beta\alpha}$ are the structure constants of the Lie algebra generated by the Killing vectors. By the Jacobi identity (1.7) for the ξ_α , these structure constants must satisfy

$$C^\mu_{\nu[\gamma} C^\nu_{\alpha\beta]} = 0.$$

The set of all Killing vector fields thus forms a Lie algebra, with a basis ξ_α , $\alpha = 1, 2, \dots, r$. The finite transformations generated by $\{\xi_\alpha\}$ form the corresponding Lie group G_r of symmetries of the space-time.

The orbit of a point $p \in \mathcal{M}$ is the set of all points into which p is mapped when all elements of the group G_r act on p . The Killing vector fields ξ_α at each point are tangent to the orbit through that point. If the dimension of the orbit equals the dimension of the group G_r , the group is said to act simply transitively on an orbit, otherwise it acts multiply transitively on the orbit. When the group acts multiply transitively, there are Killing

vectors that are zero at p . These Killing vectors will generate a sub-group of isometries which leave p fixed, called the isotropy group of p , of dimension $s = r - d$, where d is the dimension of the orbits of the maximal group of motion.

1.6 Classification of Bianchi cosmologies

A space-time is called spatially homogeneous, if there are space-like hyper-surfaces in which any point can be moved to any other point by an isometry. All physical quantities are then the same at every point of this three space. Spatially homogeneous cosmological models in general are anisotropic. Such cosmologies provide interesting generalizations of the standard Friedmann-Lemaître Robertson-Walker models of cosmology, with spatial sections of constant, but time-dependent curvature. To be of cosmological interest, the space-times must be nonempty ($T_{ab} \neq 0$). More specifically, a Bianchi cosmology is a space-time whose metric admits a three-dimensional group of isometries acting simply transitively on space-like hyper-surfaces, which are surfaces of homogeneity.

Bianchi cosmologies can be classified by classifying their Lie algebra of Killing vector fields, and hence the associated isometry group G_3 . The group G_3 might be a sub-group of a larger multiply transitive symmetry group G_4 or G_6 , in which case in general there will be several different simply transitive sub-groups G_3 . The only spatially homogeneous cosmological models that are not Bianchi universes are the Kantowski-Sachs locally rotationally symmetric family with $K = +1$, which are invariant under a group of symmetries G_4 with no simply transitive subgroup [29].

Acting simply transitively on a hyper-surface, the Killing vector fields can be taken as basis vectors. However one can choose a new basis ξ'_α by making an arbitrary constant linear transformation of the original basis vectors, hence changing the form of the structure constants; so the same Lie algebra can be represented in many different ways. Bianchi gave a complete solution to this problem in the case of 3-dimensional Lie algebras, determining nine different group types and giving canonical forms for the structure constants in each case. Under a constant change of basis $\xi'_\alpha = A_\alpha^\beta \xi_\beta$, $C^\mu_{\alpha\beta}$ transforms as a (1,2) tensor which has nine components. One may decompose these according to Schücking, Kundt and Behr [35] as follows

$$C^\mu_{\alpha\beta} = \varepsilon_{\alpha\beta\nu} \hat{n}^{\mu\nu} + \hat{a}_\alpha \delta_\beta^\mu - \hat{a}_\beta \delta_\alpha^\mu,$$

where $\hat{n}^{\mu\nu} = \hat{n}^{\nu\mu}$ and \hat{a}_α are constants. The hats distinguish these quantities from the corresponding quantities in (1.51), which are not constant in general. The identity $C^\mu_{\alpha\beta} = -C^\mu_{\beta\alpha}$ is satisfied and $C^\mu_{\nu[\gamma} C^\nu_{\alpha\beta]} = 0$ reduces to

$$\hat{n}^{\alpha\beta} \hat{a}_\beta = 0. \quad (1.87)$$

This results in a simple classification of the Lie algebras on using the obvious basis diagonalising the symmetric tensor $\hat{n}^{\alpha\beta}$ and choosing \hat{a}_α in the \mathbf{e}_1 -direction when it is non-zero. Following Ellis and MacCallum [35], one first divides the Lie algebras into class A ($\hat{a}_\alpha = 0$) and class B ($\hat{a}_\alpha \neq 0$). One then classifies further by the sign of the eigenvalues of $\hat{n}^{\alpha\beta}$. In class B one may introduce a scalar h by the following formula [20]

$$\hat{a}_\alpha \hat{a}_\beta = \frac{1}{2} h \varepsilon_{\alpha\mu\nu} \varepsilon_{\beta\sigma\tau} \hat{n}^{\mu\sigma} \hat{n}^{\nu\tau} \quad (1.88)$$

In the eigen-basis of $\hat{n}^{\alpha\beta}$, with $\hat{a}_1 \neq 0$,

$$(\hat{n}^{\alpha\beta}) = \text{diag}(\hat{n}_1, \hat{n}_2, \hat{n}_3), \quad (\hat{a}_\alpha) = (\hat{a}, 0, 0)$$

the equation (1.88) reduces to $\hat{a}^2 = h \hat{n}_2 \hat{n}_3$, so that h is well defined if and only if $\hat{n}_2 \hat{n}_3 \neq 0$ in class B. The labels given to the ten equivalence classes, as listed in Table 1.1, are based on the classification by Bianchi, as modified by Schücking and Behr [35]. The missing Bianchi type III corresponds to VI_{-1} in this classification.

Group class	Group type	\hat{n}_1	\hat{n}_2	\hat{n}_3
$A(\hat{a} = 0)$	I	0	0	0
	II	+	0	0
	VI_0	0	+	-
	VII_0	0	+	+
	VIII	-	+	+
	IX	+	+	+
$B(\hat{a} \neq 0)$	V	0	0	0
	IV	0	0	+
	VI_h	0	+	-
	VII_h	0	+	+

Table 1.1: Classification of Bianchi cosmologies into ten group types using the eigenvalues of $\hat{n}^{\alpha\beta}$

Chapter 2

Shear-free perfect fluids

The highly isotropic microwave background (interpreted as a relic of an early hot stage of the universe) is an indication that space-time, on a sufficiently large scale, is close to a FLRW universe. In fact, it was shown by Ehlers, Geren and Sachs [27] that if, in a given universe, all freely falling observers measure the cosmic background radiation to have exactly the same properties in all directions (that is, they measure the background radiation to be isotropic), then that universe is an isotropic and homogeneous FLRW space-time.

FLRW space-times still serve today as the most commonly used models of the universe. Given a barotropic equation of state, these models can be completely characterized by $\sigma_{ab} = 0 = \omega_{ab}$, $\dot{\mathbf{u}} = 0$, with the additional condition $\theta = 0$ singling out the Einstein static solution. On the other hand there is the Gödel solution [38] with closed, time-like world lines which can be characterized kinematically by $\sigma_{ab} = 0$, $\dot{\mathbf{u}} = 0$ and $\theta = 0$, with $\omega_{ab} \neq 0$ and with the additional condition that the vorticity vector be covariantly constant ($\omega_{a;b} = 0$). The first motivation for this study comes from the fact that the shear-free condition appears as a common link between these two cosmological models. As the FLRW and Gödel models have played an important role in the development of general relativity, it seems that a uniform treatment of shear-free solutions is highly desirable.

Another reason for investigating shear-free fluids is that, in the standard approach of dealing with the difficulties of non-linearities in general relativity, it is customary to postulate the existence of one or more symmetries in the form of Killing vector fields. An interesting alternative approach is to impose kinematic restrictions (such as $\sigma_{ab} = 0$). In this approach no geometrical symmetries are imposed a priori, but any allowed exact solutions

become strongly restricted by the assumption of vanishing shear.

In this chapter we are considering perfect fluids in which the fluid flow is shear-free and the energy density and the pressure satisfy a barotropic equation of state, $p = p(\mu)$. There is considerable evidence in the literature which supports the conjecture that general relativistic, shear-free perfect fluids which obey a barotropic equation of state $p = p(\mu)$ such that $p + \mu \neq 0$, are either non-expanding ($\theta = 0$) or non-rotating ($\omega = 0$). We note that there are Newtonian perfect fluids, with a barotropic equation of state, which are rotating, expanding but non-shearing (as examples can be explicitly exhibited [44]). Two specific Newtonian examples, are the models discussed by Narlikar [67] and Senovilla et al [73].

Senovilla et al. [73] consider all Newtonian pressure-free, shear-free universe models which are homogeneous. This means p and μ have no spatial dependence, $\mu = \mu(t)$, $p = p(t)$ and the velocity vector field can be written in terms of the spatial coordinates as

$$v_i = V_{ij}(t)x_j.$$

Decomposing $v_{i,j}$ into its irreducible parts

$$v_{i,j} = \frac{1}{3}h_{ij}\theta + \sigma_{ij} + \omega_{ij}$$

and setting $\sigma_{ij} = 0$, one gets

$$\dot{\mu} = -\theta\mu, \tag{2.1}$$

$$\dot{\omega}_{ij} = -\frac{2}{3}\theta\omega_{ij}, \tag{2.2}$$

$$\dot{\theta} = -\frac{1}{3}\theta^2 + 2\omega^2 - 4\pi G\mu, \tag{2.3}$$

$$f_{ij} = \omega_{ik}\omega_{kj} + \frac{2}{3}(\omega^2 - \pi G\mu)\delta_{ij} \tag{2.4}$$

where G is the gravitational constant, $\omega^2 = \frac{1}{2}\omega_{ij}\omega_{ij}$ and $f_{ij} = \dot{V}_{ij} + V_{ik}V_{jk}$. Defining a variable $R = R(t)$ such that $\theta = 3\dot{R}/R$ and $R(t_0) = 1$, from (2.1) and (2.2) one gets

$$\mu = \mu(t_0)R^{-3} \quad , \quad \omega_{ij} = \omega_{ij}(t_0)R^{-2}$$

and then (2.3) results in the Heckmann-Schücking equation [44]

$$\dot{R}^2 = \frac{8}{3}\pi G\mu(t_0)R^{-1} - \frac{2}{3}\omega^2(t_0)R^{-2} + C \tag{2.5}$$

where C is an arbitrary constant. Solutions of the equation (2.5) define shear-free Newtonian cosmologies and in general are both expanding and rotating. As a consequence the centrifugal forces arising due to the rotation will prevent a collapse of the universe. Hence singularity-free Newtonian cosmological solutions are possible in contrast with general relativity where a singularity is unavoidable, as shown by Hawking and Penrose [43]. Hence, if the shear-free conjecture is true, such behavior of fluids would be a purely relativistic effect.

Knowing whether or not the conjecture is true, or at least to what extent it is valid, might be useful in seeking and studying perfect-fluid solutions of Einstein's field equations with a shear-free velocity vector field. Shear-free perfect-fluid models which are either rotation-free or expansion-free, have attracted some interest in the literature. Barnes [2] showed that shear-free irrotational perfect fluids ($\sigma_{ab} = 0 = \omega_{ab}$) are either of Petrov type I and static or of Petrov types D or O . We review shear-free irrotational perfect fluids briefly in section 2.7. The non-expanding case ($\sigma_{ab} = 0 = \theta$) has been investigated by Collins [18, 22, 23] in the case where the vorticity and acceleration of the fluid are parallel and by Krasinski [52, 53, 54], in which it is assumed that there is a Killing vector parallel to the vorticity vector. Non-expanding and rotating shear-free perfect fluids are investigated in general in section 2.6.

2.1 Literature review on the conjecture

As mentioned above, the shear-free fluid conjecture claims that for any general relativistic perfect fluid, in which the energy density μ and the pressure p satisfy a barotropic equation of state $p = p(\mu)$ with $p + \mu \neq 0$, necessarily the expansion θ or the vorticity ω vanishes. Note that when $p + \mu = 0$, the Einstein field equations describe an Einstein space, with an arbitrary time-like congruence \mathbf{u} , the kinematic quantities of which are not longer coupled to the geometry of the space-time.

The shear-free conjecture appears to have been first stated formally by Treiokas [78] and first alluded to in the literature by King [49]. It has been established in a number of particular cases but a general proof or counter-example is still lacking. An interesting case for which the conjecture is known to hold is that of dust, that is, when $p = 0$. This begins with the work of Gödel [38, 39], with the investigation of a space-time which is spatially homogeneous and rotating as well, with non-constant energy density. He

argues that the space-time must be a tilted spatially homogeneous universe of Bianchi type *IX* and introduces the concept of the expansion quadric (nowadays called the expansion tensor). He claims that the latter cannot be isotropic (hence the shear cannot be zero), but he does not give any details of the proof. The result of Gödel is generalized to spatially homogeneous dust space-times of arbitrary Bianchi type and demonstrated explicitly by Schücking [72] who derives the Gödel result in detail and gives the metric g_{ab} for a spatially homogeneous space-time with dust and vanishing shear.

Schücking's result [72] was generalised by Banerji [1]. He investigates spatially homogeneous rotating cosmological models with non-vanishing pressure, using the same approach as Schücking. Banerji shows that such models must in general be associated with shear if there is a γ -law equation of state $p = (\gamma - 1)\mu$, provided $\gamma \neq 10/9$ (the case $p = \mu/9$ was treated in [83] see below).

In 1967, Ellis [29] considers a tetrad and coordinate system to describe any dust-filled space-time and discusses the ones in which there exist multiply transitive groups of motion. He also investigates the ones which contain shear-free dust. In the latter case he shows that the restriction of spatial homogeneity is unnecessary and that space-time is either conformally flat and non-rotating, in which case the model is locally a FLRW universe, or non-expanding and stationary. He characterizes three simple classes of solutions with $\sigma = 0$, $\omega \neq 0$. Class A solutions are those in which the energy density is constant on the hyper-surfaces orthogonal to vorticity such that the space-time is locally homogeneous on these surfaces. In all class A solutions, space-time is locally either a FLRW universe or a Gödel Universe. The class B solutions are those in which there is a Killing vector in the 2-surfaces spanned by the fluid flow and the vorticity vector: space-time is then locally invariant under either an abelian G_3 of motions simply transitive on time-like hyper-surfaces or a G_5 of motions multiply transitive on space-time. In class C solutions, there is a Killing vector in the 3-surfaces orthogonal to the vorticity vector. Ellis shows that this class contains cases A and B as sub-classes and proves that non of these cases have both non-zero expansion and rotation.

In most papers, the conjecture is proved by first assuming that $\theta\omega \neq 0$ and then deriving a contradiction. Treciokas and Ellis [79] show that if there is a shear-free perfect fluid with the equation of state $p = \mu/3$, then either the flow is non-expanding or non-rotating. This case is physically interesting since it corresponds to a cosmological model with thermal radiation. Such a model is a good approximation to the universe at the early stages of its evolution.

White and Collins [95] show that the conjecture holds when the fluid's vorticity and acceleration are parallel. They describe the dynamics using an orthonormal tetrad in which the time-like axis is aligned along the fluid flow and where one of the spatial axes is aligned with the vorticity and acceleration. They notice that the latter can be chosen such that $\omega + \Omega = 0$: this choice simplifies the discussion considerably, as the derivatives of Ω don't appear explicitly in the field equations and it will be used in most subsequent investigation (see page 40). They also introduce the function

$$G \equiv \frac{p''}{p'} \mathcal{E} - p' + \frac{1}{3}, \quad G' \equiv \frac{dG}{d\mu}. \quad (2.6)$$

the use of which leads to further significant simplifications in the study of the integrability conditions.

Rotating, shear-free perfect fluids with a barotropic equation of state and in which the magnetic part H_{ab} of the Weyl tensor vanishes were investigated by Collins [16]. He showed that solutions are non-expanding, contradicting a result by Glass [37], who claimed that a necessary and sufficient condition for a shear-free perfect fluid to be irrotational is that the Weyl tensor be purely electric and hence the fluid is either static, type D or conformally flat. Collins observes that in the most general case of the solutions, acceleration and vorticity vectors are not parallel and there is a G_2 isometry group the orbits of which are time-like two-surfaces orthogonal to both the acceleration and vorticity vectors. He notices that irrespective of the relative orientation of the acceleration and vorticity vectors, there is a conformal Killing vector parallel to the vorticity vector. In the case where the acceleration is orthogonal to the vorticity vector and $p = \mu + 2\Lambda$ there is a third Killing vector parallel to the vorticity vector.

A different approach to the conjecture is based on the assumption of certain functional relationships between kinematic scalars. This appears somewhat ad hoc but actually is related to work by Bondi [4] on *observational (spatial) homogeneity*. A perfect fluid model is called observational homogeneous, when the thermodynamic histories of all fluid particles (individual fundamental observers) are identical. Bonnor and Ellis [7] show that observational homogeneity, which they rephrase as the Postulate of Uniform Thermal Histories (PUTH), is equivalent to the condition that $\mu = \mu(\tau)$, where τ represents the proper time along the fluid flow and also requires that $\theta = \theta(\tau)$ because of the energy conservation equation. When $\theta \neq 0$, as would be the case in a cosmological setting, the requirement that $\mu = \mu(\tau)$ and $\theta = \theta(\tau)$ is equivalent to the condition that $\theta = \theta(\mu)$. Shear-free perfect fluids with a barotropic equation of state, such that $\mu + p \neq 0$, and $\theta = \theta(\mu)$

were studied by Lang [55], and Lang and Collins [56]. They showed that space-time is locally either

- a spatially homogeneous and isotropic Friedmann-Robertson-Walker model,
- a tilted spatially homogeneous but anisotropic solution possessing planar symmetry, or its temporally homogeneous counterpart,
- or a spatially inhomogeneous and anisotropic solution possessing spherical symmetry.

In all three cases the vorticity is zero.

The proofs given in [55, 56] for the case $\theta = \theta(\mu)$ generalise the earlier investigations by Gödel, Schücking and Banerji (cf. above) which were all very much coordinate based. In [55] Lang also showed that the conjecture holds true for the case $p' = -1/3$.

Although in most of the above-mentioned results a tetrad formalism was used, recently it has been shown that a covariant approach can also be successfully employed. Sopuerta [75] gives a covariant proof of the conjecture in two particular cases: i) when the expansion and the energy density are functionally dependent, which is the case studied by Lang and Collins [56] using the tetrad formalism; ii) when the expansion and the rotation scalar are functionally dependent. In the proof, Sopuerta defines two more vector fields, v^a and A^a constructed from the acceleration and the vorticity by

$$v^a = \omega^a_b \dot{u}^b, \quad A^a = \omega_b \omega^b \dot{u}^a - \dot{u}^b \omega_b \omega^a,$$

such that v^a and A^a satisfy the orthogonality properties

$$v^a u_a = v^a \dot{u}_a = v^a \omega_a = 0, \quad A^a u_a = A^a v_a = A^a \omega_a = 0.$$

In general, the four vector fields $\mathbf{u}, \boldsymbol{\omega}, \mathbf{v}$ and \mathbf{A} form a set of four orthogonal directions and an orthogonal basis can be constructed from them. It is not clear therefore that this covariant approach, (which involves the projections of all relevant quantities on the $\mathbf{u}, \boldsymbol{\omega}, \mathbf{v}$ and \mathbf{A} directions) gives any significant advantage over and above an orthonormal tetrad approach in which the \mathbf{e}_α are chosen such that for example $\boldsymbol{\omega}$ is parallel to \mathbf{e}_3 and \dot{u}_1 or \dot{u}_2 are zero. In fact, on closer inspection the two approaches are completely identical.

In [83] Van den Bergh gave a tetrad-based approach for the two elusive particular cases of $\mu + 3p = \text{constant}$, which was done also in Lang's thesis [55], and $\mu - 9p = \text{constant}$ ($\gamma = 10/9$) which includes Banerji's exceptional

case. Van den Bergh proves the conjecture for the two cases and establishes a set of equations which in principle would allow one to deal with the more general case of a γ -law equation of state $p = (\gamma - 1)\mu + \text{constant}$.

In the case where the Weyl tensor is purely magnetic with respect to the fluid flow \mathbf{u} , Lang [55] provides a modern-coordinate independent proof, involving Cartan's equivalence method and differential forms. In the same paper [55] Lang also shows that the conjecture holds true for the case $p' = -1/3$ (or $p = -\mu/3 + \text{cons}$). Later, Cyganowski and Carminati [24] also provide a proof of the conjecture for these two cases: when the electric part of Weyl tensor with respect to the fluid flow vanishes (purely magnetic space-time) and the space-time is not conformally flat and when $p = -\mu/3$. The framework used is based on the Newman-Penrose formalism and they specialised the null tetrad $\{\mathbf{l}, \mathbf{n}, \mathbf{m}, \overline{\mathbf{m}}\}$ such that \mathbf{l} and \mathbf{n} lie in the same plane as that defined by the 4-velocity of the fluid, \mathbf{u} and the vorticity vector $\boldsymbol{\omega}$ (In section 2.3, we present what we believe to be a more compact proof for purely magnetic space-times by using the orthonormal formalism). Cyganowski and Carminati also proved the conjecture for Petrov types *III* [11, 12] and *N* [9].

In all the above cases, the conjecture has been proved for the full non-linear Einstein field equations, without any approximations having been made. Recently Ellis [32] showed that shear-free solutions don't have the same implications in Newtonian theory and linearised gravity. For example, in 2011, Nzioki et.al [68] considered shear-free fluid flows in the case of linearised perturbations of FLRW universe models. They deduce that these solutions must be either expansion or rotation free except for the case of a specific equation of state

$$p'G - \left(\frac{7}{6} - \frac{1}{2}p'\right)p' - \frac{{}^3R}{2\mathcal{E}}(1 - p')p'^2 = 0, \quad (2.7)$$

where $p' = \frac{dp}{d\mu}$ is assumed to be equivalent to the local isentropic sound speed ($0 \leq c_s^2 \leq 1$) and with the spatial curvature given by

$${}^3R = \frac{k}{a(t)^2} = k \exp \left\{ \frac{2}{3} \int \frac{d\mu}{p + \mu} \right\},$$

$k = 1, 0, +1$ denoting open, flat and closed universes respectively. This equation of state does not appear to be a known, physically realistic, barotropic equation of state.

In summary, as far as we know, the conjecture is known to hold in the following special circumstances:

1. Spatially homogeneous dust of Bianchi type IX; Gödel [38, 39]
2. Spatially homogeneous dust; Schücking [72].
3. Spatially homogeneous space-times with a γ -law equation of state $p = (\gamma - 1)\mu$, where $\gamma \neq 10/9$; Banerji [1]. All these cases (with tilted spatially homogeneous perfect fluids) are generalised by (10) below [56].
4. General relativistic pressure-free matter; Ellis [29]. As White and Collins [95] observed, the proof of Ellis also holds for the more general situation when the pressure is constant.
5. Conformally flat space-times, i.e. space-times of Petrov type O by Ellis [30].
6. Perfect fluid with $p = \mu/3$ and claim of a proof for PUTH by Treciokas and Ellis [79] and $p = -\mu/3$ by Lang [55]
7. Perfect fluids with $p = \mu/9 + \text{constant}$ ($\gamma = 10/9$) or $p = -\mu/3 + \text{constant}$; Van den Bergh [83].
8. All spatially homogeneous space-times with $p + \mu > 0$; King and Ellis [50]. The condition $p + \mu > 0$ was replaced in White [94] by the more general condition $p + \mu \neq 0$.
9. Perfect fluids with acceleration parallel to the vorticity and with $p + \mu \neq 0$; this includes the case of constant pressure by White and Collins [95]. This generalizes the dust case by Ellis.
10. Perfect fluids that obey PUTH and with $p + \mu \neq 0$ ($\theta = \theta(\mu)$); Lang and Collins [56], Lang [55].
11. $\theta = \theta(\omega)$; Sopuerta [75].
12. Perfect fluids with a Weyl tensor which is purely electric; Collins [16].
13. Perfect fluids with a Weyl tensor which is purely magnetic; Lang [55].
14. Petrov type N space-times; Carminati [9].
15. Petrov type III space-times; Carminati and Cyganowski [11, 12].
16. Fluids with a conformal Killing vector parallel to the velocity, together with the extension of Treciokas and Ellis [79] to cover $p = \mu/3 + \text{constant}$ by Coley [14].

17. Shear-free and barotropic perfect fluids, perturbations of FLRW Universes; Nzioki et al [68]

2.2 Tetrad choice and relevant equations

Crucial in the successful investigation of any problem with a tetrad formalism is the choice of the tetrad alignment as it can dramatically alter the appearance and complexity of the resulting complete set of equations. We begin the analysis by choosing, what we believe to be, a well suited tetrad alignment for the physical problem at hand. First, \mathbf{e}_0 and \mathbf{e}_3 are aligned with \mathbf{u} and $\boldsymbol{\omega}$, respectively, such that $\boldsymbol{\omega} = \omega \mathbf{e}_3 \neq 0$. The relevant variables then become $\mu, p, \omega, \theta, \dot{u}_\alpha$ and the quantities Ω_α (which determine the rotation of the triad \mathbf{e}_α with respect to a set of Fermi-propagated axes, see (1.50)), together with the quantities $n_{\alpha\beta}$ and a_α . The sum of matter density and pressure will be written as

$$\mathcal{E} = \mu + p. \quad (2.8)$$

Apart from $\mu\omega\theta\mathcal{E} \neq 0$ we shall also assume that $\dot{u}_1^2 + \dot{u}_2^2 \neq 0$, $E_{ab} \neq 0$, $H_{ab} \neq 0$, θ not just a function of either μ or ω , so as to avoid duplication of known results. Also, it turns out to be computationally advantageous to replace p'' , the second derivative of the pressure with respect to the matter density, with the function G , (see 2.6).

As neither the Jacobi identities nor the field equations contain expressions for the evolution of Ω_α , it is good practice to choose the triad \mathbf{e}_α such that $\boldsymbol{\Omega} + \boldsymbol{\omega} = 0$. The fact that this is always possible follows from acting with the commutators $[\partial_3, \partial_1]$ and $[\partial_2, \partial_3]$ (A.31, A.32) on p and using Jacobi identity (1.61). One finds that $\Omega_1 = \Omega_2 = 0$, after which a rotation in the (12)-plane (see section 1.4) can be chosen such that $\Omega_3 + \omega = 0$. This is similar to the procedure followed by White and Collins in [95]. Herewith, the tetrad is determined up to rotations in the (12)-plane by an angle α satisfying $\partial_0 \alpha = 0$ (see section 1.4 or [83]). Noting that $[n_{11} - n_{22}, 2n_{12}]$ transforms as a vector under rotations in the (12)-plane (see section 1.4),

$$\begin{aligned} n'_{11} - n'_{22} &= \cos(2\alpha)(n_{11} - n_{22}) - 2\sin(2\alpha)n_{12}, \\ 2n'_{12} &= \sin(2\alpha)(n_{11} - n_{22}) + 2\cos(2\alpha)n_{12}, \end{aligned} \quad (2.9)$$

and that the evolution equations (A.11), (A.12) and (A.14) for the quantities $n_{11} - n_{22}$ and n_{12} give

$$\partial_0 \left(\frac{n_{11} - n_{22}}{n_{12}} \right) = 0,$$

which allows one to further fix the tetrad by making either $n_{11} - n_{22} = 0$ or $n_{12} = 0$. Henceforth, our choice will be

$$n_{11} = n_{22} = n.$$

We will also use extension variables, z_α and j , which are related to the components of the gradient of the expansion, by

$$\begin{aligned}\partial_0\theta &= -\theta^2/3 + 2\omega^2 - (\mu + 3p)/2 + j, \\ \partial_\alpha\theta &= z_\alpha,\end{aligned}\tag{2.10}$$

j being the (3+1) covariant divergence of the acceleration,

$$j \equiv \dot{u}^\alpha{}_{;\alpha} = \partial_\alpha \dot{u}^\alpha + \dot{u}^\alpha \dot{u}_\alpha - \dot{u}^\alpha (r_\alpha - q_\alpha).\tag{2.11}$$

In our chosen tetrad the components of the (trace-free) magnetic part of the Weyl curvature (see appendix B or 1.69) are given by

$$\begin{aligned}H_{11} &= -\omega(\dot{u}_3 + r_3), \quad H_{22} = -\omega(\dot{u}_3 - q_3) \\ H_{13} &= z_2/3 - \omega q_1, \quad H_{23} = -z_1/3 + \omega r_2, \quad H_{12} = 0.\end{aligned}\tag{2.12}$$

In order to relate the components of E_{ab} with the spatial gradient of the acceleration, we will make use of the Ricci identity (1.37)

$$E_{ab} = D_{\langle a} \dot{u}_{b\rangle} - \omega_{\langle a} \omega_{b\rangle} + \dot{u}_{\langle a} \dot{u}_{b\rangle}.\tag{2.13}$$

The complete set of initial equations of the formalism are now the Einstein field equations and the Jacobi equations, which we present, using the simplifications above, in the appendix B. First notice that the equations (B.8, B.11) immediately lead to evolution equations for the variables r_α and q_α ,

$$3\partial_0 r_\alpha = -z_\alpha - \theta(\dot{u}_\alpha + r_\alpha),\tag{2.14}$$

$$3\partial_0 q_\alpha = z_\alpha + \theta(\dot{u}_\alpha + q_\alpha),\tag{2.15}$$

while (B.1) and the (0α) field equations (B.13-B.15) give us the spatial derivatives of ω ,

$$\partial_1\omega = \frac{2}{3}z_2 - \omega(q_1 + 2\dot{u}_1),\tag{2.16}$$

$$\partial_2\omega = -\frac{2}{3}z_1 + \omega(r_2 - 2\dot{u}_2),\tag{2.17}$$

$$\partial_3\omega = \omega(\dot{u}_3 + r_3 - q_3),\tag{2.18}$$

together with the algebraic restriction

$$n_{33} = \frac{2}{3\omega} z_3. \quad (2.19)$$

The evolution equation for n follows from (B.9),

$$\partial_0 n = -\frac{\theta}{3} n. \quad (2.20)$$

Acting with the commutators $[\partial_0, \partial_\alpha]$ and $[\partial_1, \partial_2]$ on the pressure and using (B.7) together with the conservation laws, (1.41) and (1.42) leads to a first set of evolution equations for the acceleration and vorticity:

$$\partial_0 \dot{u}_\alpha = p' z_\alpha - G \theta \dot{u}_\alpha, \quad (2.21)$$

$$\partial_0 \omega = \frac{1}{3} \omega \theta (-2 + 3p'). \quad (2.22)$$

The spatial derivatives of the acceleration can be obtained from (2.13), using (2.11):

$$\partial_A \dot{u}_A = -\frac{1}{3} \omega^2 + \frac{1}{3} j - \dot{u}_{A+1} q_{A+1} + \dot{u}_{A-1} r_{A-1} - \dot{u}_A^2 + E_{AA} \quad (2.23)$$

$$\partial_3 \dot{u}_3 = \frac{2}{3} \omega^2 + \frac{1}{3} j - \dot{u}_1 q_1 + \dot{u}_2 r_2 - \dot{u}_3^2 + E_{33} \quad (2.24)$$

$$\partial_1 \dot{u}_2 = -p' \omega \theta + q_2 \dot{u}_1 + \frac{1}{2} n_{33} \dot{u}_3 - \dot{u}_1 \dot{u}_2 + E_{12} \quad (2.25)$$

$$\partial_2 \dot{u}_1 = p' \omega \theta - r_1 \dot{u}_2 - \frac{1}{2} n_{33} \dot{u}_3 - \dot{u}_1 \dot{u}_2 + E_{12} \quad (2.26)$$

$$\partial_1 \dot{u}_3 = -\frac{1}{2} \dot{u}_2 n_{33} - r_3 \dot{u}_1 - \dot{u}_1 \dot{u}_3 + E_{13} \quad (2.27)$$

$$\partial_2 \dot{u}_3 = \frac{1}{2} \dot{u}_1 n_{33} + q_3 \dot{u}_2 - \dot{u}_2 \dot{u}_3 + E_{23} \quad (2.28)$$

$$\partial_3 \dot{u}_1 = -\frac{1}{2} \dot{u}_2 n_{33} + n \dot{u}_2 + q_1 \dot{u}_3 - \dot{u}_1 \dot{u}_3 + E_{13} \quad (2.29)$$

$$\partial_3 \dot{u}_2 = \frac{1}{2} \dot{u}_1 n_{33} - n \dot{u}_1 - r_2 \dot{u}_3 - \dot{u}_2 \dot{u}_3 + E_{23}. \quad (2.30)$$

Next we act with the $[\partial_0, \partial_\alpha]$ commutators on ω and θ and use the propagation of (2.19) along \mathbf{u} in order to obtain expressions for the evolution of z_α along e_0 and for the spatial gradient of j :

$$\partial_0 z_1 = \theta(-1 + p') z_1 - \frac{1}{2} \omega(-1 + 9p') z_2 + \frac{1}{2} \theta \omega(9G - 2) \dot{u}_2 \quad (2.31)$$

$$\partial_0 z_2 = \theta(-1 + p') z_2 + \frac{1}{2} \omega(-1 + 9p') z_1 - \frac{1}{2} \theta \omega(9G - 2) \dot{u}_1 \quad (2.32)$$

$$\partial_0 z_3 = \theta(-1 + p') z_3, \quad (2.33)$$

and

$$\begin{aligned}\partial_1 j &= p'\theta z_1 - \frac{1}{6}\omega(27p' + 13)z_2 + \frac{1}{3}\dot{u}_1(18\omega^2 + \theta^2 - 3j - 3\mu) - \frac{\mathcal{E}}{2p'}\dot{u}_1 \\ &+ \frac{1}{2}\theta\omega(9G - 2)\dot{u}_2 + 4\omega^2 q_1,\end{aligned}\quad (2.34)$$

$$\begin{aligned}\partial_2 j &= p'\theta z_2 + \frac{1}{6}\omega(27p' + 13)z_1 + \frac{1}{3}\dot{u}_2(18\omega^2 + \theta^2 - 3j - 3\mu) - \frac{\mathcal{E}}{2p'}\dot{u}_2 \\ &- \frac{1}{2}\theta\omega(9G - 2)\dot{u}_1 - 4\omega^2 r_2,\end{aligned}\quad (2.35)$$

$$\partial_3 j = p'\theta z_3 + \frac{1}{3}\dot{u}_3(\theta^2 - 18\omega^2 - 3j - 3\mu) - \frac{\mathcal{E}}{2p'}\dot{u}_3 - 4(r_3 - q_3)\omega^2. \quad (2.36)$$

Now we may evaluate $\sum_\alpha [\partial_0, \partial_\alpha] \dot{u}_\alpha$ by using (B.8) and (2.23-2.24), which leads to an equation for the evolution of j ,

$$\begin{aligned}\partial_0 j &= p'\partial_\alpha z^\alpha - \theta(G + \frac{1}{3})j - \theta(2G - \frac{G'}{p'}\mathcal{E} - 1)\dot{u}_\alpha \dot{u}^\alpha - (2G - 2p' - 1)\dot{u}_\alpha z^\alpha \\ &+ p'(q_\alpha - r_\alpha)z^\alpha.\end{aligned}\quad (2.37)$$

From the expressions

$$\begin{aligned}\partial_0[(B.19)+(B.21)/2] + \theta[2(B.19)+(B.21)]/3 - [\partial_0, \partial_2]r_2 + [\partial_0, \partial_3]q_3, \\ \partial_0[(B.20)+(B.21)/2] + \theta[2(B.20)+(B.21)]/3 + [\partial_0, \partial_1]q_1 - [\partial_0, \partial_3]r_3, \\ [\partial_1, \partial_2]\omega - \omega(B.4) \text{ and the } [\partial_0, \partial_\alpha] \text{ commutators on } \dot{u}_\alpha, \text{ one can obtain}\end{aligned}$$

a) the evolution of $E_{\alpha\alpha}$:

$$\begin{aligned}\partial_0 E_{AA} &= \frac{1}{3(1+3p')}[(18p'^2 - 3p' + G - 1)\theta\omega^2 - (3G + 9p' + 1)\theta E_{AA} \\ &- 2G(3\dot{u}_A z_A - \dot{u}_\alpha z^\alpha) - (2p'G - G'\mathcal{E})\theta(3\dot{u}_A^2 - \dot{u}_\alpha \dot{u}^\alpha)],\end{aligned}\quad (2.38)$$

$$\begin{aligned}\partial_0 E_{33} &= -\frac{1}{3(1+3p')}[2(18p'^2 - 3p' + G - 1)\theta\omega^2 + (3G + 9p' + 1)\theta E_{33} \\ &+ 2G(3\dot{u}_3 z_3 - \dot{u}_\alpha z^\alpha) + (2p'G - G'\mathcal{E})\theta(3\dot{u}_3^2 - \dot{u}_\alpha \dot{u}^\alpha)],\end{aligned}\quad (2.39)$$

b) the spatial derivatives, $\partial_\alpha z_\alpha$:

$$\begin{aligned}\partial_1 z_1 &= \frac{1}{p'(1+3p')} [p'\theta(-2+3G)E_{11} + (2p'G - G'\mathcal{E})\theta(2\dot{u}_1^2 - \dot{u}_2^2 - \dot{u}_3^2) \\ &\quad + 2(-3p' + 2G - 1)\dot{u}_1 z_1 - 2p'G(\dot{u}_2 z_2 + \dot{u}_3 z_3) - p'(1+3p')(q_2 z_2 - r_3 z_3) \\ &\quad - \theta\omega^2 p'(1+G-9p'^2-6p')],\end{aligned}\quad (2.40)$$

$$\begin{aligned}\partial_2 z_2 &= \frac{1}{p'(1+3p')} [p'\theta(-2+3G)E_{22} + (2p'G - G'\mathcal{E})\theta(2\dot{u}_2^2 - \dot{u}_1^2 - \dot{u}_3^2) \\ &\quad + 2(-3p' + 2G - 1)\dot{u}_2 z_2 - 2p'G(\dot{u}_1 z_1 + \dot{u}_3 z_3) + p'(1+3p')(r_1 z_1 - q_3 z_3) \\ &\quad - \theta\omega^2 p'(1+G-9p'^2-6p')],\end{aligned}\quad (2.41)$$

$$\begin{aligned}\partial_3 z_3 &= \frac{1}{p'(1+3p')} [p'\theta(-2+3G)E_{33} + (2p'G - G'\mathcal{E})\theta(2\dot{u}_3^2 - \dot{u}_1^2 - \dot{u}_2^2) \\ &\quad + 2(-3p' + 2G - 1)\dot{u}_3 z_3 - 2p'G(\dot{u}_1 z_1 + \dot{u}_2 z_2) - p'(1+3p')(q_1 z_1 - r_2 z_2) \\ &\quad - \theta\omega^2 p'(1+G-9p'^2-6p')].\end{aligned}\quad (2.42)$$

The expressions

$2\partial_0(\text{B.16}) + [\partial_0, \partial_1]r_2 - [\partial_0, \partial_2]q_1 + 4\theta(\text{B.16})/3$,
 $2\partial_0[2(\text{B.18}) + (\text{B.3})] + 2\theta[2(\text{B.18}) + (\text{B.3})]/3 - 2[\partial_0, \partial_1]q_3 - [\partial_0, \partial_2]n_{33}$, in combination with the $[\partial_\alpha, \partial_\beta]$ commutators on θ together with the commutators $[\partial_0, \partial_2]$, $[\partial_0, \partial_3]$ on \dot{u}_1 and \dot{u}_2 give then

c) the evolution of $E_{\alpha\beta}$, ($\alpha \neq \beta$)

$$\begin{aligned}\partial_0 E_{\alpha\beta} &= -\frac{1}{1+3p'} [(G+3p' + \frac{1}{3})\theta E_{\alpha\beta} + G(\dot{u}_\alpha z_\beta + \dot{u}_\beta z_\alpha) \\ &\quad + (2G - \frac{G'}{p'}\mathcal{E})\theta \dot{u}_\alpha \dot{u}_\beta],\end{aligned}\quad (2.43)$$

d) and the remaining spatial derivatives of z_α ,

$$\begin{aligned}\partial_1 z_2 &= q_2 z_1 + \frac{1}{2}n_{33}z_3 + \omega(\frac{\mu+3p}{2} - j + \frac{1}{3}\theta^2 - 2\omega^2) + \\ &\quad \frac{1}{p'(1+3p')} [(-1-3p'+3G)(\dot{u}_2 z_1 + \dot{u}_1 z_2) + 3(2p'G - G'\mathcal{E})\theta \dot{u}_1 \dot{u}_2 \\ &\quad + \theta p'(-2+3G)E_{12}],\end{aligned}\quad (2.44)$$

$$\begin{aligned}\partial_2 z_1 &= -r_1 z_2 - \frac{1}{2}n_{33}z_3 - \omega(\frac{\mu+3p}{2} - j + \frac{1}{3}\theta^2 - 2\omega^2) + \\ &\quad \frac{1}{p'(1+3p')} [(-1-3p'+3G)(\dot{u}_2 z_1 + \dot{u}_1 z_2) + 3(2p'G - G'\mathcal{E})\theta \dot{u}_1 \dot{u}_2 \\ &\quad + \theta p'(-2+3G)E_{12}],\end{aligned}\quad (2.45)$$

$$\begin{aligned}\partial_3 z_1 &= q_1 z_3 - \frac{1}{2}(n_{33} - 2n)z_2 + \frac{1}{1+3p'}[(-1-3p'+3G)(\dot{u}_3 z_1 + \dot{u}_1 z_3) + \\ &\quad 3\theta(2G - \frac{G'}{p'}\mathcal{E})\dot{u}_1 \dot{u}_3 + \theta(-2+3G)E_{13}],\end{aligned}\quad (2.46)$$

$$\begin{aligned}\partial_3 z_2 &= -r_2 z_3 + \frac{1}{2}(n_{33} - 2n)z_1 + \frac{1}{1+3p'}[(-1-3p'+3G)(\dot{u}_3 z_2 + \dot{u}_2 z_3) + \\ &\quad 3\theta(2G - \frac{G'}{p'}\mathcal{E})\dot{u}_2 \dot{u}_3 + \theta(-2+3G)E_{23}]\end{aligned}\quad (2.47)$$

$$\begin{aligned}\partial_1 z_3 &= -r_3 z_1 - \frac{1}{2}n_{33}z_2 + \frac{1}{1+3p'}[(-1-3p'+3G)(\dot{u}_3 z_1 + \dot{u}_1 z_3) + \\ &\quad 3\theta(2G - \frac{G'}{p'}\mathcal{E})\dot{u}_1 \dot{u}_3 + \theta(-2+3G)E_{13}],\end{aligned}\quad (2.48)$$

$$\begin{aligned}\partial_2 z_3 &= q_3 z_2 + \frac{1}{2}n_{33}z_1 + \frac{1}{1+3p'}[(-1-3p'+3G)(\dot{u}_3 z_2 + \dot{u}_2 z_3) + \\ &\quad 3\theta(2G - \frac{G'}{p'}\mathcal{E})\dot{u}_2 \dot{u}_3 + \theta(-2+3G)E_{23}].\end{aligned}\quad (2.49)$$

The evolution equations for j (2.37) can then be rewritten as

$$\begin{aligned}\partial_0 j &= (\frac{G'}{p'}\mathcal{E} - 2G + 1)\theta \dot{u}_\alpha \dot{u}^\alpha + (1 - 2G)z^\alpha \dot{u}_\alpha - \frac{1}{3}(G + 1)\theta j \\ &\quad - p'(1 - 9p')\omega^2 \theta.\end{aligned}\quad (2.50)$$

Also equations (B.4) and (B.16) give

$$\begin{aligned}\partial_2 q_1 &= \frac{1}{3p'(1+3p')\omega}[\theta(2Gp' - G'\mathcal{E})(\dot{u}_\alpha \dot{u}^\alpha - 3\dot{u}_3^2) + 2p'G(z^\alpha \dot{u}_\alpha - 3z_3 \dot{u}_3) \\ &\quad - \theta p'(3G - 2)E_{33} - (2G + 9p'^2 - 9p')\theta\omega^2] + \frac{1}{3\omega}(3\dot{u}_3 - 3q_3 + r_3)z_3 \\ &\quad - \frac{1}{3\omega}(r_2 z_2 - q_1 z_1) + (r_1 + q_1)r_2 + n(q_3 + r_3) - E_{12}\end{aligned}\quad (2.51)$$

and

$$\begin{aligned}\partial_1 r_2 &= \frac{1}{3p'(1+3p')\omega}[\theta(2Gp' - G'\mathcal{E})(\dot{u}_\alpha \dot{u}^\alpha - 3\dot{u}_3^2) + 2p'G(z^\alpha \dot{u}_\alpha - 3z_3 \dot{u}_3) \\ &\quad - \theta p'(3G - 2)E_{33} - (2G + 9p'^2 - 9p')\theta\omega^2] + \frac{1}{3\omega}(3\dot{u}_3 - q_3 + 3r_3)z_3 \\ &\quad - \frac{1}{3\omega}(r_2 z_2 - q_1 z_1) - (r_2 + q_2)q_1 - n(q_3 + r_3) + E_{12}.\end{aligned}\quad (2.52)$$

The above expressions can be simplified by the introduction of the following

non-linear combinations of kinematic quantities, each having a clear geometric meaning,

$$U = \dot{u}_1^2 + \dot{u}_2^2, V = \dot{u}_1 z_1 + \dot{u}_2 z_2, W = \dot{u}_1 z_2 - \dot{u}_2 z_1, Z = z_1^2 + z_2^2. \quad (2.53)$$

The evolution of the new variables U , V , W and Z is given by

$$\partial_0 U = -2\theta GU + 2p'V, \quad (2.54)$$

$$\partial_0 V = p'Z - \theta(G - p' + 1)V + \frac{1}{2}\omega(1 - 9p')W, \quad (2.55)$$

$$\partial_0 W = -\frac{\omega\theta}{2}(9G - 2)U - \frac{\omega}{2}(1 - 9p')V - \theta(G - p' + 1)W, \quad (2.56)$$

$$\partial_0 Z = 2\theta(p' - 1)Z - \theta\omega(9G - 2)W. \quad (2.57)$$

2.3 Purely magnetic perfect fluids

Non-conformally flat, non-vacuum perfect fluid models for which the electric/resp. magnetic part of the Weyl tensor w.r.t. the fluid congruence vanishes, are called purely magnetic/resp. purely electric perfect fluids. While the purely electric case of the conjecture was treated succinctly by Collins [16] using the orthonormal tetrad formalism, the purely magnetic case was treated by Lang [55] using an approach involving Cartan's equivalence Method, and Cyganowski and Carminati [24] using the NP formalism. In order to facilitate comparison with the remainder of this work, I re-derive below the Cyganowski and Carminati result using the orthonormal tetrad formalism.

Imposing the condition $E_{\alpha\beta} = 0$ and using the expression (2.12) for $H_{\alpha\beta}$, Bianchi identities (B.47)-(B.49) lead to

$$\mathcal{E}\dot{u}_1 + 3p'\omega(z_2 - 3\omega q_1) = 0, \quad (2.58)$$

$$\mathcal{E}\dot{u}_2 - 3p'\omega(z_1 - 3\omega r_2) = 0, \quad (2.59)$$

$$(\mathcal{E} + 18p'\omega^2)\dot{u}_3 - 9p'\omega^2(-r_3 + q_3) = 0. \quad (2.60)$$

Eliminating q_1 and r_2 from equations (2.58), (2.59) and their time evolutions one obtains

$$\frac{1}{2}p'(2\mathcal{E} - 9\omega^2 + 27p'\omega^2)z_1 - \frac{1}{6}\theta(-2\mathcal{E} + 12p'\mathcal{E} + 81Gp'\omega^2)\dot{u}_1 = 0, \quad (2.61)$$

$$\frac{1}{2}p'(2\mathcal{E} - 9\omega^2 + 27p'\omega^2)z_2 - \frac{1}{6}\theta(-2\mathcal{E} + 12p'\mathcal{E} + 81Gp'\omega^2)\dot{u}_2 = 0. \quad (2.62)$$

I now give a separate discussion for the cases where the vorticity is a function of the matter density ($\partial_{[a}\omega\partial_{b]}\mu = 0$), or where $\partial_{[a}\omega\partial_{b]}\mu \neq 0$.

2.3.1 $\omega = \omega(\mu)$

When $\omega = \omega(\mu)$, the expressions $\partial_{[0}\omega\partial_{\alpha]}\mu = 0$ lead to

$$V = 0 \tag{2.63}$$

$$(6\omega^2 - 27p'\omega^2 - \mathcal{E})\dot{u}_2 - 3p'\omega z_1 = 0, \tag{2.64}$$

$$(6\omega^2 - 27p'\omega^2 - \mathcal{E})\dot{u}_1 + 3p'\omega z_2 = 0. \tag{2.65}$$

Hereby use was made of equations (2.58) and (2.59) in order to eliminate q_1 and r_2 . Propagating equation (2.63) along \mathbf{u} leads to

$$2p'Z + \omega(-1 + 9p')W = 0,$$

which by (2.64) and (2.65) simplifies to

$$(2\mathcal{E} - 9\omega^2 + 27p'\omega^2)(\mathcal{E} + 27p'\omega^2 - 6\omega^2)U = 0. \tag{2.66}$$

Now we have two cases to be looked at,

$$\mathcal{E} + 27p'\omega^2 - 6\omega^2 = 0 \tag{2.67}$$

and

$$2\mathcal{E} - 9\omega^2 + 27p'\omega^2 = 0, \tag{2.68}$$

as the case $U = 0$ (the acceleration parallel to the vorticity) was treated in White and Collins [95].

In the case (2.67), equations (2.64) and (2.65) lead to $z_1 = z_2 = 0$. Herewith, equations (2.61) and (2.62) reduce to

$$\omega^2\theta\dot{u}_A(-1 + 9p')(-1 + 3p') = 0,$$

leading us back to cases in which the conjecture is known to hold.

In the case(2.68), propagating $2\mathcal{E} - 9\omega^2 + 27p'\omega^2 = 0$ along \mathbf{u} (or 2.61, 2.62) gives

$$9p'G - 1 + 9p' - 18p'^2 = 0, \tag{2.69}$$

while propagation along \mathbf{e}_3 , using equation (2.69), yields

$$\dot{u}_3\omega^2(3p' - 1)^2 = 0.$$

When $\dot{u}_3 \neq 0$ it follows that $p = \frac{1}{3}\mu + \text{const}$ in which case the conjecture is known to hold. When $\dot{u}_3 = 0$, from (2.19), (2.21) and (2.24) together with

equations (2.58), (2.59), (2.64), (2.64) and (2.68), we deduce $z_3 = 0$, $n_{33} = 0$ and an expression for j ,

$$U - 2p'\omega^2 - p'j = 0 \quad (2.70)$$

Propagating the latter expression along \mathbf{u} and using equations (2.58), (2.59), (2.64), (2.64) and (2.68), one obtains

$$(-4p'G - 27p'G'\omega^2 + 9G'\omega^2)U + 2p'\omega^2(-2 + 3p' + 9p'^2 + 2G) = 0. \quad (2.71)$$

Evaluation of $[\partial_0, \partial_1]\dot{u}_1 + [\partial_0, \partial_2]\dot{u}_2$, using equations (2.71) and (2.58), (2.59), (2.64), (2.64), (2.68), leads then to

$$p'^2\omega^2(3p' - 1) = 0,$$

and hence again $p = \frac{1}{3}\mu + \text{const.}$

2.3.2 $\partial_{[a}\omega\partial_{b]}\mu \neq 0$

Equations (2.61) and (2.62) then show that z_A is parallel to \dot{u}_A . Propagating $W = 0$ along \mathbf{u} , one finds

$$\theta(9G - 2)U^2 - (9p' - 1)V = 0 \quad (2.72)$$

and hence, using (2.61) and (2.62) one has

$$27p'(3G + 3p' - 1)\omega^2 + [54p'^2 - 9(3G + 1)p' + 1]\mathcal{E} = 0, \quad (2.73)$$

where $3G + 3p' - 1 \neq 0$, otherwise equation (2.73) reduces to $\mathcal{E}(9p' - 1)^2 = 0$ in which case the conjecture holds true. Hence, the vorticity is a function of the matter density, which is a contradiction.

2.4 Perfect fluids with solenoidal magnetic curvature

In the present section we investigate a direct generalization of Collins' 1984 result that $\omega\theta = 0$ holds where the magnetic part \mathbf{H} of the Weyl curvature vanishes. Instead, we shall make the weaker assumption that \mathbf{H} is solenoidal, that is, that its spatial divergence vanishes. Interestingly, we show in chapter 3 that the assumption of third order restrictions, such as $\text{div}\mathbf{H} = 0$ and/or $\text{div}\mathbf{E} = 0$, leads to physically relevant families of perfect fluid solutions with non-vanishing shear. In a classification of these fluids

the shear-free subfamily would appear to be a natural first candidate for further investigation, thereby further motivating the present section.

The equations given in section (2.2) describe the behavior of a general shear-free perfect fluid. We now introduce the condition that the magnetic part of the Weyl tensor is divergence-free. The Bianchi identity (1.46)

$$\text{div} H_a = (\mu + p)\omega_a + 3\omega^b E_{ab}$$

with $\text{div} \mathbf{H} = 0$, shows that the vorticity is an eigenvector of the electric part of the Weyl tensor \mathbf{E} with the eigenvalue $-(\mu + p)/3$. As $\omega = \omega e_3$ this implies

$$E_{13} = E_{23} = 0 \quad (2.74)$$

$$E_{33} = -\frac{\mu + p}{3}. \quad (2.75)$$

Imposing (2.74) we deduce from (2.43) and (2.39) the following

$$G(\dot{u}_1 z_3 + \dot{u}_3 z_1) + (2G - \frac{G'}{p'}\mathcal{E})\theta \dot{u}_1 \dot{u}_3 = 0, \quad (2.76)$$

$$G(\dot{u}_2 z_3 + \dot{u}_3 z_2) + (2G - \frac{G'}{p'}\mathcal{E})\theta \dot{u}_2 \dot{u}_3 = 0 \quad (2.77)$$

and

$$\begin{aligned} & [-6\theta p' \omega^2 + 3p'(\theta \mathcal{E} + 2V + 2\theta U - 4\dot{u}_3 z_3 - 4\theta \dot{u}_3^2)]G - 3\theta(U - 2\dot{u}_3^2)\mathcal{E}G' \\ & - 6\theta p'(6p' + 1)(3p' - 1)\omega^2 - p'\theta(9p'^2 + 3p' + 2)\mathcal{E} = 0. \end{aligned} \quad (2.78)$$

Using equations (2.76), (2.77) and (2.78), the spatial derivatives of z_3 simplify to

$$\partial_1 z_3 = (-\dot{u}_3 - r_3)z_1 - \frac{z_2 + 3\omega \dot{u}_1}{3\omega} z_3, \quad (2.79)$$

$$\partial_2 z_3 = (-\dot{u}_3 + q_3)z_2 + \frac{z_1 - 3\omega \dot{u}_2}{3\omega} z_3, \quad (2.80)$$

$$\partial_3 z_3 = -2\dot{u}_3 z_3 - 3(3p' - 1)\theta \omega^2 - \theta p' \mathcal{E} - q_1 z_1 + r_2 z_2. \quad (2.81)$$

Acting with the commutators $[\partial_0, \partial_1]$ and $[\partial_0, \partial_2]$ on z_3 and using again equations (2.76) and (2.77) gives two more polynomials,

$$\begin{aligned} & 2\omega(-1 + 9p')(-\dot{u}_3 + q_3)z_1 + 2(3z_2 - 9p'z_2 + 7G\theta \dot{u}_2)z_3 - \\ & 2(-2 + 9G)(-\dot{u}_3 + q_3)\omega \theta \dot{u}_1 + 4G\theta z_2 \dot{u}_3 = 0, \end{aligned} \quad (2.82)$$

$$\begin{aligned} & 2\omega(-1 + 9p')(\dot{u}_3 + r_3)z_2 + 2(3z_1 - 9p'z_1 + 7G\theta \dot{u}_1)z_3 - \\ & 2(-2 + 9G)(\dot{u}_3 + r_3)\omega \theta \dot{u}_2 + 4G\theta z_1 \dot{u}_3 = 0. \end{aligned} \quad (2.83)$$

We shall argue by contradiction in order to establish that a shear-free fluid under the given conditions satisfies $\omega\theta = 0$. This argument is split into the γ -law case, $p = (\gamma - 1)\mu$, and the general case, $p = p(\mu)$, in the two following sub-sections. Before proceeding with the proof of the conjecture, we present the following lemma, which leads in the case $\dot{u}_3 = 0$ to a great simplification of the governing equations and, although this sub-case still contains all the intricacies of the ‘general’ problem, it allows us to complete the proof afterwards.

Lemma. Consider any shear-free perfect fluid solution of the Einstein field equations where the fluid pressure satisfies a barotropic equation of state, the magnetic part of the Weyl tensor is divergence-free, the acceleration is orthogonal to the vorticity and $\omega\theta \neq 0$. Then a Killing vector parallel to the vorticity exists.

Proof

If $\dot{u}_3 = 0$, from (2.21) and (2.19) one has $z_3 = n_{33} = 0$. Furthermore, with (2.27), (2.28), (2.48) and (2.49) the conditions $\dot{u}_3 = z_3 = 0$ yield

$$r_3 z_1 = q_3 z_2 = 0, \quad r_3 \dot{u}_1 = q_3 \dot{u}_2 = 0. \quad (2.84)$$

Since we have assumed $U \neq 0$, equation (2.84) implies that $r_3 = q_3 = 0$. If, for example $r_3 = 0$ and $q_3 \neq 0$ then $\dot{u}_2 = z_2 = 0$. The expressions for $\partial_2 r_3$ and $\partial_3 \dot{u}_2$ would imply

$$q_2 = n = 0, \quad (2.85)$$

whereas from equations (2.102) and (2.105) we would have

$$E_{12} = p' \omega \theta, \quad (2.86)$$

$$E_{11} = \frac{2\mathcal{E}}{3} + 3\omega^2. \quad (2.87)$$

Applying ∂_3 to equation (2.87) and using (2.85)-(2.87) yields $\omega^2 p' q_3 = 0$, and we are done. The case when $q_3 = 0$ and $r_3 \neq 0$ is dealt with in a similar way. This leaves us with the case $r_3 = q_3 = 0$ to consider.

One can explicitly show that, under the conditions $\dot{u}_3 = z_3 = r_3 = q_3 = 0$,

the Killing equations (1.84) reduce to

$$\begin{aligned}
\partial_0 K_0 &= K_1 \dot{u}_1 + K_2 \dot{u}_2, \\
\partial_1 K_1 &= -q_2 K_2 + \frac{1}{3} K_0 \theta, \\
\partial_2 K_2 &= r_1 K_1 + \frac{1}{3} K_0 \theta, \\
\partial_3 K_3 &= -q_1 K_1 + r_2 K_2 + \frac{1}{3} \theta K_0, \\
\partial_1 K_0 + \partial_0 K_1 &= \frac{1}{3} \theta K_1 - 2\omega K_2 + \dot{u}_1 K_0, \\
\partial_2 K_0 + \partial_0 K_2 &= 2\omega K_1 + \frac{1}{3} \theta K_2 + \dot{u}_2 K_0, \\
\partial_3 K_0 + \partial_0 K_3 &= \frac{1}{3} K_3 \theta, \\
\partial_1 K_2 + \partial_2 K_1 &= q_2 K_1 - r_1 K_2, \\
\partial_1 K_3 + \partial_3 K_1 &= n K_2 + q_1 K_3, \\
\partial_2 K_3 + \partial_3 K_2 &= -n K_1 - r_2 K_3.
\end{aligned} \tag{2.88}$$

Substituting $K_1 = 0$, $K_2 = 0$ and $K_0 = 0$ into the Killing equations (2.88) one obtains the following system for the function K_3 :

$$\begin{aligned}
\partial_0 K_3 &= \frac{1}{3} \theta K_3, \\
\partial_1 K_3 &= q_1 K_3, \\
\partial_2 K_3 &= -r_2 K_3, \\
\partial_3 K_3 &= 0,
\end{aligned} \tag{2.89}$$

Using the equations (2.14), (2.15), (2.42), (2.51), (2.52) and acting with the commutators $[\partial_A, \partial_3]$ on ω which imply $\partial_3 q_1 = -nr_2$ and $\partial_3 r_2 = nq_1$, the integrability conditions for the differential system (2.89) are identically satisfied. It hence follows that there is a Killing vector along \mathbf{e}_3 . This completes the proof of the lemma.

2.4.1 Perfect fluid with a γ -law equation of state

We will assume $\gamma \neq 1$ [29, 38, 72], $\gamma \neq \frac{4}{3}$ [79], $\gamma \neq \frac{2}{3}$ [83], $\gamma \neq \frac{10}{9}$ [83] and, of course $\gamma \neq 0$ (cf. section 2.1). Imposing the γ -law condition ($p' = \gamma - 1$ and hence $G = \gamma - 4/3$, $G' = 0$), we deduce from (2.76) and (2.77), dividing out

a factor¹ $\gamma - 4/3$, the following

$$(z_3 + 2\theta\dot{u}_3)\dot{u}_1 + \dot{u}_3 z_1 = 0, \quad (2.90)$$

$$(z_3 + 2\theta\dot{u}_3)\dot{u}_2 + \dot{u}_3 z_2 = 0. \quad (2.91)$$

We now continue the argument in terms of whether $z_3 + 2\theta\dot{u}_3$ is zero or not:

a) $z_3 + 2\theta\dot{u}_3 \neq 0$

Propagating equations (2.90) and (2.91) along u results in a linear system in z_1, z_2 ,

$$\alpha z_1 + \beta z_2 = 0, \quad \beta z_1 - \alpha z_2 = 0 \quad (2.92)$$

with

$$\begin{aligned} \alpha &= (\gamma - 1)(z_3 + \dot{u}_3\theta)^2 + \left(\frac{\mu + 3p}{2} - j + \left(\gamma - \frac{4}{3}\right)\theta^2 - 2\omega^2 \right) \dot{u}_3^2, \\ \beta &= -\frac{9\gamma - 10}{4}(z_3 + \theta\dot{u}_3)\omega\dot{u}_3. \end{aligned}$$

If $\beta \neq 0$ the system (2.92) has only the 0-solution, and consequently (2.90) and (2.91) would imply $\dot{u}_1 = \dot{u}_2 = 0$. In this case, the acceleration would be parallel to the vorticity and the conjecture $\omega\theta = 0$ follows by [95]. When $\beta = 0$ we have the following possibilities:

- $\gamma = \frac{10}{9}$: in this case the conjecture $\omega\theta = 0$ has been demonstrated [83].
- $\dot{u}_3 = 0$: (2.21) implies $z_3 = 0$ which is a contradiction.
- $z_3 + \theta\dot{u}_3 = 0$ and $\dot{u}_3 \neq 0$: by (2.90) and (2.91) one has then $z_1 + \theta\dot{u}_1 = z_2 + \theta\dot{u}_2 = 0$, which implies that the spatial gradient of $\phi = \log \theta - \int \frac{dp}{\mu+p}$ vanishes, and hence the fluid flow is irrotational² [73] (unless of course $\phi = \log \theta - \int \frac{dp}{\mu+p} = \text{constant}$, in which case $\omega\theta = 0$ would follow from $\theta = \theta(\mu)$ and [56]).

¹ $\gamma = 4/3$ or $p = \mu/3$ corresponds to a pure radiation perfect fluid, in which case the conjecture $\omega\theta = 0$ is known to hold, see section 2.1

²As then \mathbf{u} is parallel to $\nabla\phi$ and therefore the fluid flow is hypersurface orthogonal.

b) $z_3 + 2\theta\dot{u}_3 = 0$

Now (2.90) implies that $\dot{u}_3 z_1 = \dot{u}_3 z_2 = 0$. If $\dot{u}_3 \neq 0$ propagation of the conditions $z_1 = z_2 = 0$ along \mathbf{u} gives, using (2.31) and (2.32),

$$\dot{u}_1(9\gamma - 10)\omega\theta = \dot{u}_2(9\gamma - 10)\omega\theta = 0,$$

which again would imply that the vorticity and acceleration are parallel. We therefore conclude that $\dot{u}_3 = 0$ and hence also $z_3 = 0$. Using the result of the lemma of page 43, it follows that there is a Killing vector parallel to \mathbf{e}_3 . This result however is not essential for the continuation of the proof in the γ -law case: (2.24), (2.42) imply

$$j + 2\omega^2 - 3\dot{u}_1 q_1 + 3\dot{u}_2 r_2 + 3E_{33} = 0, \quad (2.93)$$

$$(3\gamma - 2)(-3q_1 z_1 + 3r_2 z_2 - 3\theta E_{33}) + (3\gamma - 4)[2\theta U + 2V + (9\gamma - 14)\omega^2\theta] = 0. \quad (2.94)$$

Propagating (2.75) along \mathbf{u} gives a third algebraic relation among the same variables, which allows us to express q_1 and r_2 as

$$q_1 = ((\gamma - 1)(p + \mu) + 3(3\gamma - 4)\omega^2)\theta\dot{u}_2 W^{-1} + (j - p - \mu + 2\omega^2)z_2(3W)^{-1}$$

$$r_2 = ((\gamma - 1)(p + \mu) + 3(3\gamma - 4)\omega^2)\theta\dot{u}_1 W^{-1} + (j - p - \mu + 2\omega^2)z_1(3W)^{-1}$$

and by which we can rewrite (2.94) as

$$(3\gamma - 2)^2(p + \mu)\theta + (3\gamma - 4)[4(9\gamma - 8)\theta\omega^2 + 2V + 2\theta U] = 0. \quad (2.95)$$

Note that $W \neq 0$.

We now focus on equations (2.34) and (2.35). Applying the $[\partial_1, \partial_2]$ commutator on j and using (2.16, 2.17, 2.23, 2.25, 2.26, 2.40, 2.41, 2.44, 2.45, 2.51, 2.52) we obtain

$$2(4\gamma - 5)W + (7\gamma - 10)\omega j + (3\gamma - 2)(11\gamma - 8)(p + \mu)\omega + (459\gamma^2 - 1048\gamma + 580)\omega^3 = 0. \quad (2.96)$$

Propagating this relation twice along \mathbf{u} and simplifying the result using (2.95) and (2.96), yields a homogeneous linear system

$$\begin{bmatrix} 6\gamma(21\gamma - 20)(3\gamma - 4) & (3\gamma - 2)(47\gamma^2 - 76\gamma + 30) \\ 4(5 - 3\gamma)(21\gamma - 20)(3\gamma - 4) & (3\gamma - 2)(47\gamma^2 - 76\gamma + 30) \end{bmatrix} \begin{bmatrix} \omega^2 \\ p + \mu \end{bmatrix} = 0, \quad (2.97)$$

The determinant of the coefficient matrix being zero results in

$$2(9\gamma - 10)(3\gamma - 4)(3\gamma - 2)(21\gamma - 20)(47\gamma^2 - 76\gamma + 30) = 0.$$

The first three cases $9\gamma - 10 = 0$, $3\gamma - 4 = 0$ and $3\gamma - 2 = 0$ correspond to $p = \frac{1}{9}\mu$, $p = \frac{1}{3}\mu$ and $p = -\frac{1}{3}\mu$ (cf. section 2.1). From the linear system (2.97), the two cases $21\gamma - 20 = 0$ and $47\gamma^2 - 76\gamma + 30 = 0$ imply $\omega = 0$ and $p + \mu = 0$ respectively.

This completes the proof that a shear-free perfect fluid, obeying an equation of state $p = (\gamma - 1)\mu$ and satisfying the ‘solenoidal’ condition $\text{div} H = 0$, is either expansion free or vorticity free. In the next section we try to generalise this result by allowing an arbitrary barotropic equation of state $p = p(\mu)$.

2.4.2 Perfect fluid with a equation of state $p = p(\mu)$

We begin by first proving $\omega\theta = 0$, assuming that there is no Killing vector parallel to the vorticity ($\dot{u}_3 \neq 0$). Again we argue by contradiction and specifically show that the assumption $\theta\omega \neq 0$ will enforce a γ -law equation of state, for which the proof was given in the previous paragraph. For the barotropic case with a Killing vector along the vorticity, the equations require substantially deeper analysis. In fact, the increased difficulty for this special alignment case was pointed out already by Collins [17] in his study of the conjecture. In a second step we show that the conjecture still holds when there exists a Killing vector parallel to the vorticity, provided that the magnitude of the vorticity and the matter density are functionally related. In a last step we demonstrate that the latter property is ‘generically’ enforced by the solenoidal condition, in the sense that, when the fluid is non-rotating and non-expanding, the pressure and its derivatives w.r.t. the matter density have to satisfy a largely over-determined set of ordinary differential equations. The inconsistency of this set is demonstrated explicitly for a three-parameter family of equations of state generalizing the γ -law case (see pages 54-55).

Using Bianchi identities (B.40)-(B.42), (B.44) and considering the ex-

pressions (2.12), (2.74) and (2.75), one can obtain

$$\begin{aligned}\partial_2 r_3 &= \frac{1+3p'}{3\omega p'(1+3p')}(\dot{u}_3 z_1 - z_3 \dot{u}_1) - \frac{G}{\omega p'(1+3p')}(\dot{u}_3 z_1 + z_3 \dot{u}_1) \\ &\quad + \frac{1}{3\omega^2}[\omega(r_3 z_1 - 3q_1 z_3) + z_2 z_3] - \frac{-\mu G' + 2Gp' - pG'}{\omega p'(1+3p')}\theta \dot{u}_3 \dot{u}_1 \\ &\quad - (r_3 + q_3)q_2, \tag{2.98}\end{aligned}$$

$$\begin{aligned}\partial_1 q_3 &= \frac{1+3p'}{3\omega p'(1+3p')}(\dot{u}_3 z_2 - z_3 \dot{u}_2) - \frac{G}{\omega p'(1+3p')}(\dot{u}_3 z_2 + z_3 \dot{u}_2) \\ &\quad - \frac{1}{3\omega^2}[\omega(q_3 z_2 - 3r_2 z_3) - z_1 z_3] - \frac{-\mu G' + 2Gp' - pG'}{\omega p'(1+3p')}\theta \dot{u}_3 \dot{u}_2 \\ &\quad + (r_3 + q_3)r_1, \tag{2.99}\end{aligned}$$

$$\begin{aligned}\partial_3 E_{12} &= \left(\frac{z_3}{3\omega} - 2n\right)E_{11} + (-2\dot{u}_3 - q_3)E_{12} - \frac{1}{9\omega}[2\mathcal{E} - 3\omega^2(1+3p')]z_3 \\ &\quad + \frac{1}{3}\mathcal{E}n + \omega\theta(-p' + G)\dot{u}_3 - p'\omega\theta r_3, \tag{2.100}\end{aligned}$$

$$\begin{aligned}\partial_3 E_{11} &= (-2\dot{u}_3 - q_3)E_{11} - \left(\frac{z_3}{3\omega} - 2n\right)E_{12} - \frac{1}{2p'}(\mathcal{E} + 6\omega^2 p')\dot{u}_3 \\ &\quad + \left(\frac{2}{3}\mathcal{E} + \omega^2\right)q_3 - 2\omega^2 r_3 \tag{2.101}\end{aligned}$$

Furthermore, Bianchi identities (B.43), (B.45), (B.46) and (B.49) result in

$$\begin{aligned}3(r_3 + q_3)\omega E_{12} + [\mathcal{E} + 2\omega^2(3p' - 1)]z_3 - 6\omega^2\theta(-p' + G)\dot{u}_3 \\ + 3\omega^2 p'\theta(r_3 - q_3) = 0, \tag{2.102}\end{aligned}$$

$$\begin{aligned}6p'(q_1 - \dot{u}_2)E_{11} - 6p'(\dot{u}_2 + r_2)E_{12} + p'\omega(-1 + 9p')z_2 - 2\theta p'^2 z_1 + \\ 2p'(2\mathcal{E} - 3\omega^2)q_1 - 9\omega\theta p'G\dot{u}_2 + \mathcal{E}(2p' - 1)\dot{u}_1 + 6p'^2\theta\omega r_2 = 0, \tag{2.103}\end{aligned}$$

$$\begin{aligned}-6p'(\dot{u}_2 + r_2)E_{11} + 6p'(\dot{u}_1 - q_1)E_{12} + p'\omega(-1 + 9p')z_1 + 2\theta p'^2 z_2 + \\ 2p'(2\mathcal{E} - 3\omega^2)r_2 - 9\omega\theta p'G\dot{u}_1 + \mathcal{E}\dot{u}_2 - 6p'^2\theta\omega q_1 = 0 \tag{2.104}\end{aligned}$$

and

$$\begin{aligned}3p'(r_3 + q_3)E_{11} - p'(2\mathcal{E} + 9\omega^2)q_3 + [(p' + 2)\mathcal{E} + 18p'\omega^2]\dot{u}_3 \\ + p'(\mathcal{E} + 9\omega^2)r_3 = 0. \tag{2.105}\end{aligned}$$

At this stage, we may already conclude that G cannot vanish, by the following argument. Substituting $G = G' = 0$ into equation (2.78) and dividing out the factor $\theta p'$, gives

$$6(6p' + 1)(3p' - 1)\omega^2 + (9p'^2 + 3p' + 2)\mathcal{E} = 0, \quad (2.106)$$

Propagating this equation along \mathbf{u} and using (2.22), we obtain

$$2(15p' + 4)(3p' - 1)\omega^2 + (4p' + 2 + 9p'^2 + 27p'^3)\mathcal{E} = 0. \quad (2.107)$$

Eliminating ω^2 from (2.106) and (2.107) results in

$$\mathcal{E}^2(3p' - 1)(3p' + 1)(81p'^3 - 9p'^2 + 6p' - 1) = 0,$$

implying a γ -law equation of state and we are done. So henceforth we shall assume $G \neq 0$.

Combining equations (2.76) and (2.77) gives the important branching condition

$$\dot{u}_3 W = 0. \quad (2.108)$$

When $\dot{u}_3 = 0$ the lemma of page 43 guarantees the existence of a Killing vector along the vorticity: the discussion of this case will be deferred to sub-section 2.4.2.3. The cases $\dot{u}_3 \neq 0 = \dot{u}_1 \dot{u}_2$ and $\dot{u}_1 \dot{u}_2 \dot{u}_3 \neq 0$ will first be studied separately.

2.4.2.1 $\dot{u}_3 z_3 \neq 0, \dot{u}_1 = 0$ (and hence $\dot{u}_2 \neq 0$)

Condition (2.108) implies $z_1 = 0$ and from the $\partial_0 z_1$ and $\partial_3 \dot{u}_1$ derivatives we obtain

$$n = \frac{1}{3\omega} z_3 - \frac{\dot{u}_3}{3\dot{u}_2} q_1, \quad (2.109)$$

$$z_2 = \frac{2 - 9G}{9p' - 1} \theta \dot{u}_2. \quad (2.110)$$

The $\partial_3 z_1$ derivative gives

$$q_1 [z_3(1 - 9p') - \theta \dot{u}_3(2 - 9G)] = 0.$$

If $q_1 \neq 0$ substitution of the resulting expressions for z_3 and n into (2.83) implies

$$\theta^2 \omega \dot{u}_2 \dot{u}_3 (2 - 9G)(3G - 1 + 3p') = 0 \quad (2.111)$$

and hence $2 - 9G = 0$ (the last factor being 0 would imply $p'' = 0$). Substitution of $G = 2/9$ into (2.76) leads to $p' = 0$. On the other hand, if $q_1 = 0$, substitution of (2.109, 2.110) into equation (2.83) implies

$$\theta G(2 - 9G) \dot{u}_3 - (G(10 - 9p') - 3 + 9p') z_3 = 0, \quad (2.112)$$

Applying the ∂_1 derivative to this equation results in

$$\theta(2 - 9G)(3G - 1 + 3p')\dot{u}_2 z_3 = 0.$$

Again if $2 - 9G = 0$, then combined with (2.112) we have $(10 - 9p')z_3 = 0$ and we are done.

The analysis of the case $\dot{u}_2 = 0, \dot{u}_1 \neq 0$ proceeds in a similar way.

2.4.2.2 $\dot{u}_1 \dot{u}_2 \dot{u}_3 z_3 \neq 0$

Using $W = 0$ and $\partial_0 W = 0$ gives

$$z_1 = \dot{u}_1 \theta(9G - 2)/(9P' - 1), \quad (2.113)$$

$$z_2 = \dot{u}_2 \theta(9G - 2)/(9P' - 1). \quad (2.114)$$

Next, substituting (2.113) and (2.114) into (2.76) and (2.82) results in

$$2(3 - 10G - 9p' + 9Gp')z_3 + 2G\theta(2 - 9G)\dot{u}_3 = 0, \quad (2.115)$$

$$2\mathcal{E}\theta(9p' - 1)\dot{u}_3 G' - 2p'\theta G(9G + 18p' - 4)\dot{u}_3 - 2p'G(9p' - 1)z_3 = 0. \quad (2.116)$$

Applying ∂_0 to (2.114) and (2.116) gives, respectively,

$$54\theta^2 \mathcal{E}(9p' - 1)G' + 2(2 - 9G)(3G + 27Gp' - 2 + 6p')\theta^2 + 12(2 - 9G)(9p' - 1)\omega^2 - 3(2j - \mu - 3p)(9G - 2)(9p' - 1) = 0, \quad (2.117)$$

and

$$\begin{aligned} & \mathcal{E}[12G'(9G - 1)\dot{u}_3\theta^2 - 6z_3G'(9p' - 10)\theta + 9G(9G - 2)\dot{u}_3] + \\ & 2G(3G + 1)(9G - 2)\dot{u}_3\theta^2 - 6z_3(3 + 5Gp' - 9p' - 10G + 18G^2p')\theta - \\ & 6G(j + 2\omega^2 + \mu)(9G - 2)\dot{u}_3 = 0. \end{aligned} \quad (2.118)$$

Eliminating z_3 between equations (2.116) and (2.115) yields

$$\begin{aligned} & \mathcal{E}G'(9p' - 1)(9p' - 10)G - 3(9p' - 1)(3p' - 1)] - 9p'(-11 + 18p')G^3 - \\ & 3p'(-105p' + 54p'^2 + 23)G^2 + 6p'(9p' - 2)(3p' - 1)G = 0. \end{aligned} \quad (2.119)$$

Finally, eliminating z_3 between equations (2.116) and (2.118) and then combining the result with (2.117) to eliminate j , eventually gives

$$(9G - 2)^2[(9p' - 1)(3p' - 1)\mathcal{E}G' + 54G^3p' + 6p'(9p' - 5)G^2 - 4p'(3p' - 1)G] = 0. \quad (2.120)$$

When $9G - 2 \neq 0$ we find

$$(9p' - 1)(3p' - 1)\mathcal{E}G' + 54G^3p' + 6p'(9p' - 5)G^2 - 4p'(3p' - 1)G = 0. \quad (2.121)$$

We have now two third order differential equations for the pressure, namely (2.119) and (2.121). By means of reduction of the order, these eventually lead to

$$(9p' - 1)(2187p'^4 + 891p'^3 - 2412p'^2 + 839p' - 53)(3p' - 1)^2(9p' - 10)^2 = 0 \quad (2.122)$$

and hence we have a γ -law equation of state.

When $9G - 2 = 0$, then (2.119) immediately yields $p'(9p' - 1)^2 = 0$, and the proof of the conjecture, when there is no Killing vector parallel to the vorticity, is now complete. This allows us to formulate already the following theorem

Theorem: *Shear-free rotating perfect fluid solution of the Einstein field equations where*

- *the fluid pressure satisfies a barotropic equation of state with $p + \mu \neq 0$,*
- *the spatial divergence of the magnetic part of the Weyl tensor is zero,*
- *the acceleration is not orthogonal to the vorticity,*

are non-expanding.

2.4.2.3 Existence of a Killing vector along the vorticity

We now consider the case $\dot{u}_3 = 0$, in which, as a consequence of the lemma of page 43, a Killing vector along the vorticity exists. As mentioned in the beginning of section, the conjecture is now considerably more difficult to establish; the vanishing of ∂_3 on all quantities has the welcomed effect of simplifying the field equations considerably but unfortunately at the cost of a major loss of information, such that the integrability conditions become more deeply imbedded in the remaining field equations.

First notice that $W = z_1\dot{u}_2 - z_2\dot{u}_1 \neq 0$ as otherwise $\partial_0 W = 0$ would result in

$$\theta(9G - 2)U - (9p' - 1)V = 0. \quad (2.123)$$

Simplifying (2.123) by using $z_1\dot{u}_2 - z_2\dot{u}_1 = 0$ it would follow that (as $\dot{u}_3 = z_3 = 0$)

$$\frac{9G - 2}{9p' - 1}\dot{u}_\alpha - \frac{z_\alpha}{\theta} = 0, \quad (2.124)$$

implying the existence of a function $\phi = \log \theta + f(\mu)$ such that the spatial gradient of ϕ vanishes: if ϕ is constant then $\theta = \theta(\mu)$ and Lang's result [56] applies, whereas if ϕ is not constant automatically the vorticity vanishes as the flow is then orthogonal to the surfaces $\phi = \text{constant}$ [73].

As we have assumed that the acceleration is not parallel to the vorticity, it is possible to fix the tetrad completely. In the resulting tetrad, all the remaining variables are invariants and, as there exists a Killing vector along \mathbf{e}_3 , from $\partial_3 \dot{u}_1 = \partial_3 \dot{u}_2 = 0$, it follows that $n = 0$ in any invariantly defined tetrad.

The main equations now become (2.24) and (2.42), which allow one to solve for q_1 and r_2 (Note that the corresponding system determinant is not 0). Using equations (2.34, 2.35) to evaluate the commutator $[\partial_1, \partial_2]j$ leads now to the following algebraic relation:

$$\begin{aligned} & 2Gp'^2(1 - 12G)W + [\mathcal{E}(13G - 54p'^2 + 18p')G' + 36Gp'^2(3G + 3p' - 1)]\omega U \\ & - G\omega p'^2(2 + 21G)j - p'^2[42G^2 + (567p'^2 + 108p' - 95)G + 36(6p' \\ & + 1)(3p' - 1)^2]\omega^3 + p'^2[(21p + 30\mu - 9)G^2 - (63\mathcal{E}p'^2 - 27\mathcal{E} + 34p \\ & + 36\mu - 2)G - 6\mathcal{E}(3p' - 1)(9p'^2 + 3p' + 2)]\omega = 0, \end{aligned} \quad (2.125)$$

We note that (2.50) can be rewritten as:

$$\partial_0 j = -\frac{1}{3}\theta(3G+1)j + (1-2G)V + \left(\frac{G'}{p'}\mathcal{E} - 2G+1\right)\theta U - \theta p'(1-9p')\omega^2; \quad (2.126)$$

while (2.78) becomes

$$\begin{aligned} & 3\theta U(2Gp' - \mathcal{E}G') - 6\theta p'(18p'^2 - 3p' + G - 1)\omega^2 + \\ & 6Gp'V - p'\theta\mathcal{E}(3G - 9p'^2 - 3p' - 2) = 0. \end{aligned} \quad (2.127)$$

Propagating (2.127) along \mathbf{e}_0 results in an equation for Z ,

$$Z + \theta^2 \phi_1(U, \omega, \mu) + \phi_2(U, W, \omega, j, \mu) = 0, \quad (2.128)$$

in which the μ dependence occurs through p, p', G, G', G'' and in which ϕ_1, ϕ_2 are linear in j, U and W . Together with (2.125) and (2.127) this allows one to express V, W and Z as polynomials in U, j, ω, θ , with coefficients depending on the matter density only. If one now propagates equation (2.125) along \mathbf{e}_0 , simplifies the results by using the previous expressions for V, W, Z and iterates this procedure twice, one obtains a sequence of relations *in which*

the expansion scalar no longer appears:

$$F_1 U + F_2 \omega^2 + F_3 j + F_4 = 0 \quad (2.129)$$

$$G_1 U + G_2 \omega^2 + G_3 = 0 \quad (2.130)$$

$$H_1 \omega^2 + H_2 = 0 \quad (2.131)$$

Herein the functions F_i, G_i, H_i are polynomials in μ, p, p', G and higher order derivatives of G up to orders 2, 3 and 4 respectively. It is clear then that for a *generic* equation of state ω, U and j will be functions of the matter density only. From (2.127) and (2.125) it follows that also V/θ and W depend then on μ only. By means of the conservation law $\partial_0 \mu = -\mathcal{E}\theta$ it follows that $\partial_0(V/\theta)/\theta$ is a function of μ as well, after which (2.55) and (2.128) imply that $\theta = \theta(\mu)$ and we are done. The emphasis above is on the word ‘generic’, as the reasoning may break down for specific equations of state (when for example the coefficients H_i are identically 0). However, the argument can be made precise, provided we *assume* that ω and μ are functionally related. The reason is that, instead of calculating q_1, r_2 from (2.24) and (2.42), one can use now also the *spatial information* coming from the relations $\partial_{[\alpha} \omega \partial_{\beta]} \mu = 0$. This leads to

$$q_1 = \frac{2}{3} \frac{z_2}{\omega} - \frac{9p' - 2}{3p'} \dot{u}_1, \quad r_2 = \frac{2}{3} \frac{z_1}{\omega} + \frac{9p' - 2}{3p'} \dot{u}_2, \quad (2.132)$$

allowing one to rewrite (2.24) and (2.42) as

$$V = \frac{3\theta p'}{9p' - 2} (\mathcal{E} p' + 3(3p' - 1)\omega^2) \quad (2.133)$$

$$W = \frac{\omega}{2} (j + 2\omega^2 - \mathcal{E} + \frac{9p' - 2}{p'} U). \quad (2.134)$$

Together with (2.125) and (2.127) this yields explicit expressions for U, V, W and j as functions of θ and μ (with the dependence on μ occurring through ω, p, p', G and G'), while (2.128) gives an expression for Z involving G'' as well. The evolution equations (2.54), (2.55), (2.56) and (2.126) for U, V, W and j result then in two relations which can be combined to give the polynomial equations

$$B_1(\omega, G', G, p', \mu) = 0, \quad (2.135)$$

$$B_2(\omega, G'', G', G, p', \mu) = 0. \quad (2.136)$$

Propagating (2.135) allows one to obtain a first ‘equation of state’

$$B_3(G', G, p', \mu) = 0. \quad (2.137)$$

A second 'equation of state' could in principle be obtained by propagating (2.136), but this turns out to be very impractical. It is better to use (2.135) for eliminating the G'' term occurring in the expression for Z . Again using (2.135) to eliminate ω from the evolution equation (2.57) for Z results then in a polynomial equation

$$B_4(G', G, p', \mu)\theta^2 + B_5(G'', G', G, p', \mu)\theta + B_6(G'', G', G, p', \mu) = 0. \quad (2.138)$$

The structure of the (huge) polynomials B_5 and B_6 is irrelevant: what matter is B_4 , which factorizes as a product of two polynomials in Q', Q, p', μ of length respectively 64 and 164 terms. Together with (2.137) this enables one, via a reduction of the order procedure, to show that $B_4 = 0$ necessarily implies a γ -law. When on the other hand $B_4 \neq 0$ we have $\theta = \theta(\mu)$ and so we have proved the following

Theorem: *if a rotating perfect fluid solution of the Einstein field equations satisfies the following conditions,*

- *the shear vanishes, $\sigma_{ab} = 0$*
- *there exists a barotropic equation of state, $p = p(\mu)$ with $p + \mu \neq 0$*
- *the magnetic part of the Weyl tensor is divergence free, $\text{div}\mathbf{H} = 0$*
- *there exists a Killing vector along the vorticity,*
- *the vorticity is a function of the matter density only, $\omega = \omega(\mu)$*

then the fluid expansion vanishes, $\theta = 0$

Because of this the proof of the general case really hinges on excluding the possibility that equation (2.131) is an identity. When H_1, H_2 are identically 0, an over-determined set of differential equations for the pressure results, which *in principle* would allow one to deduce an inconsistency. However, while F_i and G_i are polynomials of modest length (30 to 150 terms), the polynomials H_i have about 3000 terms each and contain fourth-order derivatives of the function G . Carrying out a reduction-of-order procedure is therefore not feasible, unless one restricts the analysis to the study of certain subclasses, such as the case $G' = 0$, a special case for which a separate analysis of the whole procedure has to be carried out anyhow and which gives rise to a 3-parameter family of barotropic equations of state, generalizing the γ -law

family. When $G' = 0$ one has

$$\begin{aligned} H_1 \equiv & (81648p'^3 - 11664p'^2 + 19926p' - 1080)G^3 \\ & + (326592p'^4 - 222588p'^3 + 76464p'^2 - 42822p' + 2268)G^2 \\ & - 6(3p' - 1)(20304p'^3 - 1629p'^2 + 4059p' - 250)G \\ & - 6(1584p'^2 + 681p' - 52)(3p' - 1)^2. \end{aligned} \quad (2.139)$$

Taking the derivative with respect to μ one obtains

$$p'(3G + 3p' - 1)G_1 = 0 \quad (2.140)$$

with

$$\begin{aligned} G_1 \equiv & (13608p'^2 - 1296p' + 1107)G^3 + (72576p'^3 - 37098p'^2 + \\ & 8496p' - 2379)G^2 - (81216p'^3 - 25191p'^2 + 9204p' - 1603)G \\ & - (3p' - 1)(6336p'^2 + 987p' - 331). \end{aligned} \quad (2.141)$$

As $3G + 3p' - 1 = 0$ is incompatible with $G' = 0$ (unless one has a γ -law equation of state), equation (2.140) implies $G_1 = 0$. Then eliminating G from $G_1 = 0$ and (2.139) results in

$$\begin{aligned} & 18887232875212800p'^{12} - 53792062797957120p'^{11} - 4298152909001472p'^{10} \\ & - 12255122940708096p'^9 - 11301347108473920p'^8 + 6983472331592316p'^7 \\ & - 1474242102536805p'^6 + 165841422134787p'^5 - 11187921314241p'^4 \\ & + 446052852675p'^3 - 9164770128p'^2 + 82845372p' - 232900 = 0, \end{aligned} \quad (2.142)$$

implying a γ -law equation of state and we are done. So henceforth we shall assume $G' \neq 0$.

In order to make progress with the general case, one could try to use also extra spatial information, as in the case $\omega = \omega(\mu)$. First notice that one can explicitly demonstrate, starting from equations (2.129-2.131) that, in the worst case scenario, j, U, V, W, Z and θ are all functions of μ and ω . From this one derives that *in a fixed tetrad* only two functionally independent functions remain among the components of the Riemann tensor and its covariant derivatives. Hence any rotating and expanding solution admits a 2-dimensional isometry group G_2 (which moreover is maximal, as otherwise ω would depend on μ) and therefore, next to the ∂_3 operator, there should exist a new operator $D = \alpha\partial_0 + \beta\partial_1 + \gamma\partial_2$ which is identically 0 on all variables. Using the fact that $D(\mu) = D(\theta) = 0$ determines α, β, γ up to a proportionality factor, we find that

$$D = -6W\partial_0 + (2\theta^2 + 3\mu + 9p - 6j - 12\omega^2)(\dot{u}_2\partial_1 - \dot{u}_1\partial_2) + 6\theta p'(z_2\partial_2 - z_1\partial_2) \quad (2.143)$$

Next, noting that for any variable x both $D(\partial_0(x)) = \partial_0(D(x)) = 0$ and using the commutator relations to simplify this expression, we obtain a new operator $\tilde{D} = \lambda\partial_1 + \mu\partial_2$, which also identically vanishes on all variables. This contradicts the maximality of the group, unless $\lambda = \mu = 0$. Elimination of z_1 or z_2 from these equations gives rise to a single new relation, which is only of second order in G ,

$$\begin{aligned} & 12\theta(-\mathcal{E}G' + 2Gp' - p')(V^2 + W^2)W + 12p'(-1 + 2G)VWZ + \\ & p'\theta\omega(-12\omega^2 + 2\theta^2 - 6\mu + 9\mathcal{E})(9G - 2)(V^2 + W^2) - 6p'^2\omega\theta Z^2(-1 + \\ & 9p') - 2p'\theta(9p'\mathcal{E} + 18\omega^2p' + 3\mathcal{E} - 16\omega^2 + 54p'^2\omega^2 - 2j - 6Gj)WZ + \\ & \omega p'(108\omega^2p' + 54p'\mu + 30p'\theta^2 - 81p'\mathcal{E} + 54p'\theta^2G - 12\omega^2 + 2\theta^2 - 6\mu + \\ & 9\mathcal{E} + 6j - 54p'j)VZ - 6\omega\theta p'(9G - 2)(V^2 + W^2)j, \end{aligned}$$

but which, after elimination of the functions j, V, W, Z , gives rise to a new polynomial relation between U, ω and μ of about 18570 terms. Combining this with equation (2.130) should allow one to deduce that $\omega = \omega(\mu)$, but so far we have been unable to work out the details of this procedure: the analysis appears to be well beyond current computer capabilities. This finishes the proof of the shear-free conjecture for all solenoidal magnetic curvature space-times, with as possible exception certain quite restricted special cases within the class of solutions in which there exists a Killing vector aligned with the vorticity and for which the magnitude of the vorticity ω and μ are functionally independent.

2.5 Perfect fluids with solenoidal electric curvature

Newtonian homogeneous cosmologies exist [73] which provide counter-examples to the *Newtonian* shear-free fluid conjecture. In these models the spatial divergence of the tidal field (the Newtonian analogue of \mathbf{E}) automatically vanishes (because of homogeneity), such that, when searching for possible counter-examples to the *relativistic* shear-free fluid conjecture, it is tempting to look for candidates within the class of perfect fluids having a solenoidal electric part of the Weyl tensor. This section therefore also generalises the result obtained by Lang [55] and Cyganowski and Carminati [24] on purely magnetic space-times (see also section 2.3). Specifically, we prove

Theorem: Consider any shear-free perfect fluid solution of the Einstein

field equations where the fluid pressure satisfies a barotropic equation of state and the spatial divergence of the electric part of the Weyl tensor is zero. Then either the fluid is non-rotating or non-expanding.

With the components of H_{ab} given by (2.12), the Bianchi identity (B.47)-(B.49), with $\text{div} \mathbf{E} = 0$, reduces to

$$\mathcal{E}\dot{u}_1 + 3p'\omega(z_2 - 3\omega q_1) = 0, \quad (2.144)$$

$$\mathcal{E}\dot{u}_2 - 3p'\omega(z_1 - 3\omega r_2) = 0, \quad (2.145)$$

$$(\mathcal{E} + 18\omega^2 p')u_3 - 9\omega^2 p'(-r_3 + q_3) = 0. \quad (2.146)$$

As before we try to obtain a contradiction from $\omega\theta \neq 0$, assuming also $\mathcal{E} \neq 0$ and $p' \neq 0, \pm 1/3, 1/9$, $\dot{u}_1^2 + \dot{u}_2^2 \neq 0$, $E_{ab} \neq 0$, $H_{ab} \neq 0$, θ not just a function of either μ or ω , so as to avoid duplication of known results. Propagating equations (2.144) and (2.145) again results in the equations (2.61) and (2.62), showing that z_A is parallel to \dot{u}_A , unless

$$2\mathcal{E} - 9(1 - 3p')\omega^2 = 2\mathcal{E}(6p' - 2) + 81Gp'\omega^2 = 0. \quad (2.147)$$

2.5.1 \dot{u}_A parallel to z_A

Propagating $W = 0$ along \mathbf{u} , one finds

$$27p'(3G + 3p' - 1)\omega^2 - \mathcal{E}(-1 + 27Gp' + 9p' - 54p'^2) = 0.$$

If the coefficient of ω^2 were zero, the latter equation would give

$$\mathcal{E}(9p' - 1)^2 = 0.$$

Hence, the vorticity is a function of the matter density. From $\partial_{[1}\mu\partial_{2]}\omega = 0$, one deduces then $\mathcal{E}V = 0$, implying $V = W = 0$ and hence the vorticity is parallel to the acceleration.

2.5.2 \dot{u}_A not parallel to z_A

Eliminating ω from (2.147) fixes the equation of state to be

$$9Gp' - 18p'^2 + 9p' - 1 = 0. \quad (2.148)$$

Note that a γ -law equation of state is then only possible when $p' = 1/3$ or $1/9$, so henceforth we will exclude all sub-cases in which p' is constant. By (2.147) the vorticity is again a function of matter density. It follows that

$$\partial_A \mu \partial_3 \omega - \partial_A \omega \partial_3 \mu = 0,$$

which after eliminating q_1 , r_2 and q_3 using equations (2.144), (2.145) and (2.146) yield

$$\mathcal{E}(3\omega\dot{u}_A - z_{A+1})\dot{u}_3 = 0.$$

It follows that \dot{u}_3 is zero, as otherwise propagating equation (2.147) along \mathbf{e}_A , using $3\omega\dot{u}_A - z_{A+1} = 0$, would imply

$$\mathcal{E}(3p' - 1)\dot{u}_A = 0 \quad (2.149)$$

and again $\dot{u}_A = 0$.

With $\dot{u}_3 = 0$, the expressions (2.21), (2.27), (2.28), (2.24) and (2.19) show that

$$\begin{aligned} z_3 &= 0, & n_{33} &= 0, & E_{13} &= \dot{u}_1 r_3, & E_{23} &= -\dot{u}_2 q_3, \\ E_{33} &= -\frac{1}{3}(2\omega^2 + j) + \dot{u}_1 q_1 - \dot{u}_2 r_2. \end{aligned}$$

Moreover, the condition $\text{div} E = 0$, (2.144-2.146) can now be rewritten as follows

$$\begin{aligned} \mathcal{E}\dot{u}_1 + 3\omega p'(z_2 - 3\omega q_1) &= 0, \\ \mathcal{E}\dot{u}_2 - 3\omega p'(z_1 - 3\omega r_2) &= 0, \\ p'(q_3 - r_3)\omega^2 &= 0. \end{aligned} \quad (2.150)$$

Propagating equation (2.147)(a) along \mathbf{e}_1 and \mathbf{e}_2 , using equations (2.150), yields

$$\mathcal{E}(\dot{u}_1 - 3p'q_1) = 0, \quad (2.151)$$

$$\mathcal{E}(\dot{u}_2 + 3p'r_2) = 0. \quad (2.152)$$

Propagating the latter equations again along \mathbf{e}_1 and \mathbf{e}_2 and eliminating E_{12} yields

$$\begin{aligned} 9\theta p'^2(3p' - 1)(18p'^2 - 15p' + 1)j + 9p'^2(3p' - 1)(36p'^2 - 9p' + 5)V \\ - 9\theta(3p' - 1)(162p'^4 + 108p'^3 - 162p'^2 + 21p' - 1)U \\ + 6p'^3\theta(27p'^2 - 18p' + 7)\mathcal{E} = 0. \end{aligned} \quad (2.153)$$

Now we use $z_3 = 0$ and equations (2.147-2.151), (2.152) and (2.150) to simplify equation (2.42) to

$$\begin{aligned} (3p' - 1)(162p'^4 + 108p'^3 - 162p'^2 + 21p' - 1)U - 6(27p'^2 - 18p' \\ + 7)\mathcal{E}p'^3 - 9p'^2(3p' - 1)(18p'^2 - 15p' + 1)j = 0. \end{aligned} \quad (2.154)$$

Equations (2.153) and (2.154) can then be combined to give

$$p'^2(3p' - 1)(36p'^2 - 9p' + 5)V = 0.$$

It follows that \dot{u}_A is orthogonal to z_A ($V = 0$), after which propagation of equation (2.153) along \mathbf{u} gives

$$\begin{aligned} & 2(1296p'^4 - 2106p'^3 + 522p'^2 + 9p' - 5)(3p' - 1)^2U \\ & - 24p'^2\mathcal{E}(p' + 1)(81p'^3 - 207p'^2 + 93p' - 7) = 0. \end{aligned} \quad (2.155)$$

Taking the time derivative of the latter equation, elimination of j and z_1 from the resulting equation and (2.153) gives again an equation between p , p' and U . Eliminating the acceleration between the latter equation and equation (2.155) eventually results in

$$\begin{aligned} & \frac{\mathcal{E}p'^3(3p' - 1)(18p'^2 - 15p' + 1)}{1296p'^4 - 2106p'^3 + 522p'^2 + 9p' - 5} (26244p'^6 - 59778p'^5 - 106515p'^4 \\ & + 156249p'^3 - 43065p'^2 + 3267p' - 26) = 0, \end{aligned}$$

showing that p' has to be a constant, in contradiction with (2.148) and our assumptions $p \neq 1/3, 1/9$.

2.6 Rotating, non-expanding perfect fluids with a solenoidal magnetic field

Assuming the validity of the conjecture $\omega\theta = 0$ and noticing that all non-rotating models [2], as explained in the next section 2.7 automatically have a purely electric Weyl tensor, we attempt to give a classification of rotating, non-expanding shear-free perfect fluids for which $\text{div}\mathbf{H} = 0$, thereby generalizing the classification of Collins 1984 [16].

We choose our tetrad as in section 2.2. As there is an equation of state and the fluid flow is both shear-free and expansion free, there is a Killing vector parallel to the fluid flow: $\theta = 0$ implies $z_\alpha = 0$ and hence all \mathbf{e}_0 derivatives appearing in section 2.2 vanish.

First note that we can assume that the acceleration does not vanish and hence in particular $p' \neq 0$. With vanishing acceleration the Raychaudhuri equation (2.10) and equation (2.13) for E_{ab} respectively result in $\omega^2 = (\mu + 3p)/4$ and $E_{ab} = \frac{\omega^2}{3}\text{diag}(1, 1, -2)$ after which the $\text{div}\mathbf{H} = 0$ condition (cf 2.157) yields then $p = \mu = \omega^2$. Then (1.41) and (1.42) gives rise to the Gödel-like models with $p = \mu = \text{constant}$. These shear-free solutions belong

to sub-class \mathcal{E}_+ studied by Wyllleman [98].

We express the vanishing of the spatial divergence of the magnetic part of the Weyl tensor by (1.46) :

$$E_{13} = E_{23} = 0, \quad (2.156)$$

$$E_{33} = -\frac{\mu + p}{3}. \quad (2.157)$$

On the other hand (2.10) with $\theta = 0$ implies

$$j = \frac{1}{2}(\mu + 3p) - 2\omega^2. \quad (2.158)$$

The spatial derivatives (2.16)-(2.18) simplify then to

$$\begin{aligned} \partial_1 \omega &= -\omega(q_1 + 2\dot{u}_1) \\ \partial_2 \omega &= \omega(r_2 - 2\dot{u}_2) \\ \partial_3 \omega &= \omega(\dot{u}_3 + r_3 - q_3) \end{aligned}$$

while (2.19) gives $n_{33} = 0$.

The spatial derivatives of the acceleration are given by (2.23)-(2.30) and the components of the magnetic part of the Weyl curvature, by (2.12).

Next eliminating $\partial_1 n + \partial_3 q_2$ and $\partial_2 n - \partial_1 q_3$ from Jacobi identities (B.2)-(B.4) and Einstein field equations (B.16)-(B.18) leads one to

$$\begin{aligned} \partial_2 q_1 &= n(r_3 + q_3) + r_2(r_1 + q_1) - E_{12}, \\ \partial_1 r_2 &= -n(r_3 + q_3) - q_1(r_2 + q_2) + E_{12}, \\ \partial_1 q_3 &= r_1(r_3 + q_3), \\ \partial_2 r_3 &= -q_2(r_3 + q_3). \end{aligned} \quad (2.159)$$

From Bianchi identities (B.42) and (B.43) we deduce the derivative of E_{12} along \mathbf{e}_3 ,

$$\partial_3 E_{12} = \frac{1}{3}n(6E_{22} - \mathcal{E}) - (2\dot{u}_3 + q_3)E_{12} \quad (2.160)$$

and the following significant branching condition

$$E_{12}(r_3 + q_3) = 0 \quad (2.161)$$

If $n_{12} = q_3 + r_3 \neq 0$ then the frame with $n_{11} = n_{22} (= n)$ is fixed and one has $E_{12} = 0$. If $n_{12} = q_3 + r_3 = 0$ then there is a remaining rotational freedom, which, as

$$\partial_0 \left(\frac{E_{11} - E_{22}}{E_{12}} \right) = 0 \quad \text{and} \quad n_{11} - n_{22} = 0$$

allows us to perform an extra rotation in the (12)-plane, preserving $\Omega_1 = \Omega_2 = 0$ and $n_{11} - n_{22} = n_{12} = 0$, such that E_{12} becomes zero (see section 1.4). So henceforth we assume $E_{12} = 0$.

If $E_{12} = 0$ using (2.160) one obtains

$$n(E_{11} - E_{22}) = 0. \quad (2.162)$$

Assuming now $n = 0$ ($n \neq 0$ remains as future work) and Einstein field equations (B.18) and (B.17), the following derivatives are found

$$\partial_3 q_2 = (q_2 + r_2)r_3, \quad (2.163)$$

$$\partial_3 r_1 = -(r_1 + q_1)q_3. \quad (2.164)$$

Bianchi identities (B.47) and (B.45) give an expression for the derivative of E_{22} along e_3 and two constraint equations as follows

$$\partial_3 E_{22} = \frac{1}{6p'}(\mathcal{E} + 2p'\mathcal{E} + 18p'\omega^2)\dot{u}_3 - (2\dot{u}_3 + q_3)E_{22} - \left(\frac{1}{3}\mathcal{E} + \omega^2\right)q_3 + 2\omega^2 r_3 \quad (2.165)$$

and

$$6p'(\dot{u}_1 - q_1)E_{22} - \mathcal{E}\dot{u}_1 - 2p'q_1(3\omega^2 - 2\mathcal{E}) = 0, \quad (2.166)$$

$$6p'(\dot{u}_2 + r_2)E_{11} - \mathcal{E}\dot{u}_2 + 2p'r_2(3\omega^2 - 2\mathcal{E}) = 0. \quad (2.167)$$

Acting with the commutators $[\partial_1, \partial_2]$ on \dot{u}_1, \dot{u}_2 and with $[\partial_1, \partial_3]$ on \dot{u}_1 and using the Einstein field equations (B.19)-(B.21) gives then

$$\begin{aligned} \partial_3 E_{11} = & \frac{1}{6p'}(18p'\omega^2 + \mathcal{E})\dot{u}_3 - (\dot{u}_3 - r_3)E_{11} + \dot{u}_3 E_{22} + \\ & \frac{1}{3}(3\omega^2 + \mathcal{E})r_3 - 2\omega^2 q_3, \end{aligned} \quad (2.168)$$

$$\partial_2 E_{11} = \frac{1}{6p'}\mathcal{E}\dot{u}_2 - (2\dot{u}_2 + q_2)E_{11} + (q_2 - \dot{u}_2)E_{22} + 2\omega^2 r_2, \quad (2.169)$$

$$\begin{aligned} \partial_1 E_{11} = & -\frac{1}{6p'}\mathcal{E}(2p' + 3)\dot{u}_1 + (r_1 + \dot{u}_1)E_{11} + (2\dot{u}_1 - r_1)E_{22} + \\ & 2\omega^2 q_1. \end{aligned} \quad (2.170)$$

The Bianchi identities (B.50)-(B.52) together with the Einstein field equations (B.19), (B.21) result in

$$\partial_3 r_3 = \frac{1}{2}(E_{11} - E_{22}) - \frac{1}{6}(p - \mu) - r_2 q_2 + r_3^2 + q_1^2 + \partial_1 q_1, \quad (2.171)$$

$$\partial_3 q_3 = \frac{1}{2}(E_{11} - E_{22}) + \frac{1}{6}(p - \mu) + r_1 q_1 - q_3^2 - r_2^2 + \partial_2 r_2, \quad (2.172)$$

$$\partial_1 r_3 = (r_1 + q_1)q_3 + (r_1 + 3\dot{u}_1)r_3 + (-3q_1 + 3\dot{u}_1)\dot{u}_3 - \partial_3 q_1, \quad (2.173)$$

$$\partial_2 q_3 = (3\dot{u}_2 - q_2)q_3 - (q_2 + r_2)r_3 - (3r_2 + 3\dot{u}_2)\dot{u}_3 - \partial_3 r_2, \quad (2.174)$$

$$\partial_1 r_1 = \frac{1}{2}(E_{11} + E_{22}) + \frac{1}{6}(p + 3\mu) - 3\omega^2 + q_2^2 - r_3 q_3 + r_1^2 + \partial_2 q_2. \quad (2.175)$$

The evolution of $E_{11} + E_{22} - \frac{1}{3}\mathcal{E} = 0$, along \mathbf{e}_3 leads to

$$(2\mathcal{E} + p'\mathcal{E} + 18p'\omega^2)\dot{u}_3 - p'(\mathcal{E} + 3E_{22} + 9\omega^2)q_3 + p'(2\mathcal{E} - 3E_{22} - 9\omega^2)r_3 = 0 \quad (2.176)$$

and from (2.169) and (2.170) we obtain

$$\partial_1 E_{11} = \frac{2p' - 3}{6p'}\mathcal{E}\dot{u}_1 - \frac{\mathcal{E}}{3}r_1 + (2r_1 - \dot{u}_1)E_{11} + 2q_1\omega^2, \quad (2.177)$$

$$\partial_2 E_{11} = -\frac{2p' - 1}{6p'}\mathcal{E}\dot{u}_2 + \frac{\mathcal{E}}{3}q_2 - (2q_2 + \dot{u}_2)E_{11} + 2r_2\omega^2. \quad (2.178)$$

Now we continue with the cases $r_3 + q_3 \neq 0$ and $r_3 + q_3 = 0$ in the following sections separately.

2.6.1 $r_3 + q_3 \neq 0$

We split the discussion into three sub-cases, where the acceleration is orthogonal to vorticity ($\dot{u}_3 = 0$), the acceleration is parallel to vorticity ($\dot{u}_1 = \dot{u}_2 = 0$, $\dot{u}_3 \neq 0$), or where the divergence of the electric part of the Weyl tensor is parallel to vorticity. The general case with $U \neq 0$ and $\dot{u}_3 \neq 0$ remains for future work.

2.6.1.1 Acceleration orthogonal to vorticity

Propagating $\dot{u}_3 = 0$ along \mathbf{e}_1 and \mathbf{e}_2 shows that $r_3\dot{u}_1 = 0$ and $q_3\dot{u}_2 = 0$. At least one of \dot{u}_1 or \dot{u}_2 must then be zero, otherwise one would have $r_3 = 0$ or $q_3 = 0$. If for example $q_3 = 0$, then equation (2.176) yields

$$(\mathcal{E} + 9\omega^2 + 3E_{11})r_3 = 0.$$

If $\mathcal{E} + 9\omega^2 + 3E_{11} = 0$, then its propagation along \mathbf{e}_3 implies $\omega^2 r_3 = 0$ and hence $r_3 = 0$. Similarly $r_3 = 0$ implies $q_3 = 0$. As we are in the case $r_3 + q_3 \neq 0$, there are no solutions.

2.6.1.2 Acceleration parallel to vorticity

We are now in the sub-case investigated by White and Collins [95]. When $\dot{u}_1 = \dot{u}_2 = 0$ and $\dot{u}_3 \neq 0$, the equations (2.23), (2.25), (2.26), (2.29) and (2.30), imply

$$\begin{aligned} E_{11} &= \omega^2 - r_3 \dot{u}_3 - \frac{1}{6}(\mu + 3p), \\ E_{22} &= \omega^2 + q_3 \dot{u}_3 - \frac{1}{6}(\mu + 3p), \\ q_1 &= 0 = r_2. \end{aligned}$$

Hence we obtain from (2.157)

$$(r_3 - q_3)\dot{u}_3 - 2\omega^2 + \frac{2}{3}(\mu + 2p) = 0. \quad (2.179)$$

Acting with the commutators $[\partial_1, \partial_3]$ and $[\partial_2, \partial_3]$ on r_3 , one obtains

$$\begin{aligned} r_1(q_3 + r_3)^2 &= 0, \\ q_2(q_3 + r_3)^2 &= 0, \end{aligned}$$

such that, $r_1 = 0 = q_2$. From $\partial_1 r_1 = 0$ one gets then

$$9\omega^2 - p - 2\mu + 3r_3 q_3 = 0. \quad (2.180)$$

From equations (2.179) and (2.180) the following expressions for matter density and pressure can be obtained:

$$\begin{aligned} \mu &= \frac{1}{2}(r_3 - q_3)\dot{u}_3 + 2r_3 q_3 + 5\omega^2, \\ p &= (q_3 - r_3)\dot{u}_3 - r_3 q_3 - \omega^2. \end{aligned}$$

At this stage all Einstein field equations and Jacobi identities are satisfied identically and the magnetic part and electric part of the Weyl tensor become diagonal:

$$E_{\alpha\beta} = \begin{bmatrix} \frac{1}{6}[\mathcal{E} - 3(r_3 + q_3)] & 0 & 0 \\ 0 & \frac{1}{6}[\mathcal{E} + 3(r_3 + q_3)] & 0 \\ 0 & 0 & -\frac{1}{3}\mathcal{E} \end{bmatrix}$$

with the divergence parallel to vorticity:

$$\operatorname{div} E_\alpha = \left[0, 0, (6\omega^2 + \frac{\mathcal{E}}{3p'})\dot{u}_3 - 3\omega^2(q_3 - r_3) \right],$$

and

$$H_{\alpha\beta} = \begin{bmatrix} -\omega(r_3 + \dot{u}_3) & 0 & 0 \\ 0 & \omega(q_3 - \dot{u}_3) & 0 \\ 0 & 0 & \omega(2\dot{u}_3 + r_3 - q_3) \end{bmatrix}.$$

We also have $n_{11} = n_{22} = n_{33} = n_{13} = n_{23} = a_1 = a_2 = 0$. Hence, these solutions form a sub-case of the Collins and White [23] class IIIAGi. They are non-LRS as $n_{12} \neq 0$, are Petrov type I and admit a G_3 of Bianchi type II acting transitively on the surfaces orthogonal to the vorticity [23], as it is not obvious that type D is impossible. They form a genuine sub-case of the IIIAGi family, as the $\operatorname{div} \mathbf{H} = 0$ condition explicitly specifies the behavior of μ , in contrast to the general IIIAGi family.

2.6.1.3 $\operatorname{div} \mathbf{E}$ parallel to vorticity

As in the previous case $\operatorname{div} \mathbf{E}$ is parallel to the vorticity, one might be tempted to generalize the IIIAGi-solutions by imposing only that $\operatorname{div} \mathbf{E}$ is parallel to $\boldsymbol{\omega}$. However, assuming that the divergence of the electric part of Weyl is parallel to vorticity vector and using (2.12), Bianchi identities (B.47) and (B.48) can be solved for q_1 and r_2 :

$$q_1 = \frac{\mathcal{E}}{9p'\omega^2}\dot{u}_1, \quad (2.181)$$

$$r_2 = -\frac{\mathcal{E}}{9p'\omega^2}\dot{u}_2, \quad (2.182)$$

after which Bianchi identities (B.45) and (B.46) can be simplified to yield

$$\mathcal{E}[(3p' - 5)\omega^2 + \mathcal{E}]\dot{u}_A = 0. \quad (2.183)$$

Either the acceleration is then parallel to the vorticity vector, or $(3p' - 5)\omega^2 + \mathcal{E} = 0$. In the latter case the propagation of (2.182) along \mathbf{e}_1 results in

$$\dot{u}_1 \dot{u}_2 \omega^2 (3p' - 5)(3G + 6p' + 1) = 0 \quad (2.184)$$

where $3p' - 5 = 0$ together with $\mathcal{E} + (3p' - 5)\omega^2 = 0$ results in a contradiction ($\mathcal{E} = 0$). Using then $3G + 6p' + 1 = 0$ and the equations (2.181) and (2.182), (2.166) reduces to

$$(12p' - 5)(6E_{11} - \mathcal{E}) = 0, \quad (2.185)$$

where $12p' - 5 = 0$ is inconsistent with $3G + 6p' + 1 = 0$, therefore $6E_{11} - \mathcal{E} = 0$. The propagation of $6E_{11} - \mathcal{E} = 0$ along \mathbf{e}_3 by using (2.176), (2.181) and (2.182) results in

$$(p' - 1)(q_3 + r_3) = 0, \quad (2.186)$$

in which $p' - 1 = 0$ contradicts $3G + 6p' + 1 = 0$. Hence, from (2.183) it follows that again the acceleration is parallel to the vorticity.

2.6.2 $r_3 + q_3 = 0$

When $r_3 + q_3 = 0$, (2.159) and (2.171)-(2.174) imply

$$\begin{aligned} \partial_3 q_1 &= 3(\dot{u}_1 - q_1)\dot{u}_3 - (3\dot{u}_1 - q_1)q_3, \\ \partial_3 r_2 &= (3\dot{u}_2 + r_2)q_3 - 3(r_2 + \dot{u}_2)\dot{u}_3, \\ \partial_1 q_1 + \partial_2 r_2 &= \frac{1}{3}\mathcal{E} + r_2(r_2 + q_2) - q_1(r_1 + q_1) - 2E_{11}. \end{aligned} \quad (2.187)$$

We investigate the two sub-cases, $\dot{u}_3 = 0$ and $\dot{u}_3 \neq 0$ separately:

2.6.2.1 Acceleration orthogonal to vorticity

When $\dot{u}_3 = 0$, (2.27), (2.28) and (2.24) give

$$\begin{aligned} \dot{u}_1 q_3 &= 0 = \dot{u}_2 q_3, \\ 6(\dot{u}_1 q_1 - \dot{u}_2 r_2) + \mu - p &= 0, \end{aligned} \quad (2.188)$$

and hence $q_3 = r_3 = 0$, otherwise the acceleration becomes zero.

Now the investigation can be performed in the two sub-cases: $\dot{u}_2 + r_2 \neq 0$ and $\dot{u}_2 + r_2 = 0$ (also note that if $E_{11} = E_{22}$ then a rotation exists which makes $\dot{u}_2 + r_2 = 0$). In this thesis we consider the case $\dot{u}_2 + r_2 = 0$, and the former remains as future work. With $\dot{u}_2 + r_2 = 0$, equation (2.167) reduces to

$$\dot{u}_2 [(4p' - 1)\mathcal{E} - 6p'\omega^2] = 0. \quad (2.189)$$

When $\dot{u}_2 \neq 0$, (2.189) shows that $\omega = \omega(\mu)$, implying $\partial_{[1}\omega\partial_{2]}\mu = 0$, which gives

$$\mathcal{E}\omega\dot{u}_2(q_1 - \dot{u}_1) = 0$$

and hence $q_1 - \dot{u}_1 = 0$. Eliminating E_{22} between $\partial_1(q_1 - \dot{u}_1) = 0$ and $\partial_2(r_2 + \dot{u}_2) = 0$, together with the propagation of $(4p' - 1)\mathcal{E} - 6p'\omega^2 = 0$ along \mathbf{e}_2 , gives then

$$\mathcal{E}\dot{u}_2(30p' - 4 + 3G - 60p'^2) = 0 \quad \text{and} \quad p' - 1 = 0,$$

which are incompatible.

Henceforth we assume that $\dot{u}_2 = r_2 = 0$. Propagating this along \mathbf{e}_1 and \mathbf{e}_2 , one obtains $q_2 = 0$ and

$$E_{22} - r_1 q_1 - \frac{1}{3}p = 0, \quad (2.190)$$

$$E_{22} + r_1 \dot{u}_1 - \omega^2 + \frac{1}{6}(3p + \mu) = 0. \quad (2.191)$$

Elimination of E_{22} from (2.190) and (2.191) gives

$$\omega^2 - (\dot{u}_1 + q_1)r_1 - \frac{1}{6}(\mu + 5p) = 0, \quad (2.192)$$

while elimination of q_1 from equations (2.188) and (2.166) yields

$$3p'(6\dot{u}_1^2 + \mu - p)E_{22} - 3\mathcal{E}\dot{u}_1^2 + p'(\mu - p)(3\omega^2 - 2\mathcal{E}) = 0. \quad (2.193)$$

Let us first show the coefficient of E_{22} in (2.193) is non-zero. If $6\dot{u}_1^2 + \mu - p = 0$, then (2.193) reduces to the following relation between vorticity and matter density:

$$6p'\omega^2 - (4p' - 1)\mathcal{E} = 0. \quad (2.194)$$

Taking two successive derivatives of $6\dot{u}_1^2 + \mu - p = 0$ along \mathbf{e}_1 , eliminating \dot{u}_1 from the results and using equation (2.194) one deduces

$$3(p - \mu)G + 3(21p - 15\mu)p'^2 - 6(5p + \mu)p' + 13\mu + 5p = 0. \quad (2.195)$$

On the other hand, propagating equation (2.194) along \mathbf{e}_1 , and using equations (2.188), (2.194), (2.195) and $6\dot{u}_1^2 + \mu - p = 0$ to eliminate r_1 , \dot{u}_1 , ω , G , we get

$$(105\mu - 123p)p'^2 - 12(2\mu - 5p)p' + 9\mathcal{E} = 0. \quad (2.196)$$

Propagating (2.196) along \mathbf{e}_1 and simplifying the result by (2.196) and (2.195), one obtains

$$1259712(35\mu - 41p)(\mu - p)(607p^2 + 1012\mu p + 2989\mu^2)\mathcal{E}^2 = 0, \quad (2.197)$$

where $607p^2 + 1012\mu p + 2989\mu^2 > 0$. If $35\mu - 41p = 0$, (2.196) reduces to a contradiction ($\mu = 0$). In the case $\mu - p = 0$, by $6\dot{u}_1^2 + \mu - p = 0$ one has $\dot{u}_1 = 0$ which contradicts the condition $\dot{\mathbf{u}} \neq 0$. Therefore we conclude that $6\dot{u}_1^2 + \mu - p \neq 0$.

The propagation of equations (2.166) and (2.193) along \mathbf{e}_1 , using equations (2.193), (2.188) and (2.191), gives now rise to

$$\begin{aligned}
& 216\mathcal{E}(3G + 6p'^2 - 4)\dot{u}_1^8 - [216(p^2 - \mu^2)G + 648p'(\mathcal{E} - 2p'p)\omega^2 - \\
& 36(48p'^2 - 42p' + 7)\mu^2 + 36(36p'^2 - 23)p^2 - 216(2p'^2 - 7p' + \\
& 5)p\mu]\dot{u}_1^6 + 6(\mu - p)[3(\mu^2 - p^2)G - (7 + 54p'^2 - 81p')\mu^2 \\
& + (126p'^2 - 234p')\mu\omega^2 + 342p'^2p\omega^2 - (240p'^2 - 246p' + 6)p\mu \\
& - (186p'^2 - 165p' - 1)p^2]\dot{u}_1^4 + 3p'(\mu - p)^2[(64p' - 7)\mu^2 \\
& + 108p'p\mu - 54p'(5\mu + 3p)\omega^2 + 216p'\omega^4 + (44p' + 7)p^2]\dot{u}_1^2 \\
& - 12\omega^2p'^2(\mu - p)^4 = 0
\end{aligned} \tag{2.198}$$

and $\Sigma_1\Sigma_2 = 0$ with

$$\Sigma_1 = 2\mathcal{E}(1 - p')\dot{u}_1^2 + p'(\mu - p)(\mathcal{E} - 2\omega^2) \tag{2.199}$$

and

$$\begin{aligned}
\Sigma_2 = & 12p'(\mu - p)^2\omega^2 + 36(\mathcal{E} - 2p'p)\dot{u}_1^4 + 6(\mu - p)(3p'\mu - p'p - \mathcal{E})\dot{u}_1^2 \\
& - p'(5\mu + 7p)(\mu - p)^2.
\end{aligned} \tag{2.200}$$

We first consider $\Sigma_1 = 0$ and look at the special case $p' = 1$. This simplifies $\Sigma_1 = 0$ to

$$6\mathcal{E}(\mu - p)(2\omega^2 - \mu - p) = 0.$$

When $2\omega^2 - \mu - p = 0$, equation (2.198) becomes

$$6\mathcal{E}(\mu - p)^2(6\dot{u}_1^2 + \mu - p)^2 = 0,$$

hence $\mu = p = \omega^2$. The propagation however of $\mu = p = \omega^2$ along \mathbf{e}_1 together with equation (2.188) leads to $\dot{u}_1 = 0$, so that we can assume $p = \mu \neq \omega^2$. Equation (2.188) then gives $q_1 = 0$ and from the propagation of q_1 along \mathbf{e}_1 , using (2.187), one gets $E_{11} = \mu/3$. From equations (2.191) and (2.192) one finds then $p = \mu = \omega^2 - \dot{u}_1 r_1$. The surviving spatial curvature quantities are then $a_1 = n_{23} = r_1/2$ and the only non-zero remaining derivatives are those along \mathbf{e}_1 . The space-time is purely electric with $E_{\alpha\beta} = \text{diag}(\mu/3, \mu/3, -2\mu/3)$ and the Weyl tensor is of Petrov type D. The exterior derivative of the basis one-forms ω^a is given

$$d\omega^0 = 2\omega\omega^1 \wedge \omega^2 - \dot{u}_1\omega^0 \wedge \omega^1, \tag{2.201}$$

$$d\omega^1 = 0, \tag{2.202}$$

$$d\omega^2 = -r_1\omega^1 \wedge \omega^2, \tag{2.203}$$

$$d\omega^3 = 0. \tag{2.204}$$

In a similar way as described on page 72, the line element can be written as follows

$$ds^2 = -(\omega^{-\frac{1}{2}}dt + gdy)^2 + dx^2 + f^2dy^2 + dz^2, \quad (2.205)$$

where g and f are functions of x , and should satisfy

$$\frac{df}{dx} = -r_1 f, \quad (2.206)$$

$$\frac{dg}{dx} = 2\omega f + \dot{u}_1 g, \quad (2.207)$$

$$(2.208)$$

and \dot{u}_1 , r_1 and ω satisfying

$$\begin{aligned} \frac{d\dot{u}_1}{dx} &= -\dot{u}_1(r_1 + \dot{u}_1), \\ \frac{dr_1}{dx} &= r_1(r_1 - \dot{u}_1) - 2\omega^2, \\ \frac{d\omega}{dx} &= -2\omega\dot{u}_1 \end{aligned} \quad (2.209)$$

There is a G_3 isometry group of Bianchi type I acting transitively on time-like hypersurfaces orthogonal to the acceleration vector. The isometry group is abelian and there is a Killing vector parallel to the vorticity. Note that these solutions have some properties in common with the models investigated by Kransinki [52], they don't belong to Krasinski's class as there only the case was considered where ω^2 is proportional to \mathcal{E} and $H_{ab} \neq 0$. It can be also easily shown that these solutions are the only ones when $E_{11} = E_{22}$, as follows: the propagation of $E_{11} - E_{22} = 0$ along \mathbf{e}_1 and \mathbf{e}_2 results in

$$2\mathcal{E}(p' - 1)\dot{u}_1 + 12p'\omega^2 q_1 = 0, \quad (2.210)$$

$$[6p'\omega^2 + (p' - 1)\mathcal{E}]u_2 = 0 \quad (2.211)$$

where $6p'\omega^2 + (p' - 1)\mathcal{E} \neq 0$, as otherwise using (2.189), equation (2.210) would reduce to $\dot{u}_1 - q_1 = 0$ and then propagating $\dot{u}_1 - q_1 = 0$ along \mathbf{e}_1 would lead to a contradiction, $\mathcal{E} = 0$. Hence from (2.211), one has $\dot{u}_2 = 0$. Eliminating q_1 from (2.210) and (2.166) implies

$$\mathcal{E}(p' - 1)(4\omega^2 - \mathcal{E})\dot{u}_1 = 0. \quad (2.212)$$

where $\dot{u}_1 \neq 0$ (as assumed $U \neq 0$). If $4\omega^2 - \mathcal{E} = 0$, its propagation along \mathbf{e}_1 and (2.210) reduce to $p' = 2/5$ and $4\dot{u}_1 + q_1 = 0$ respectively. Propagating $4\dot{u}_1 + q_1 = 0$ along \mathbf{e}_1 gives then

$$20\dot{u}_1^2 - \frac{1}{6}(11p + \mu) = 0. \quad (2.213)$$

Substituting $q_1 = -4\dot{u}_1$ into the propagation of (2.213) along \mathbf{e}_1 results in $7p + 2\mu = 0$ which contradicts with $p' = 2/5$. Therefore, from (2.212), $p' - 1 = 0$.

Now, we look at the case $\Sigma_1 = 0$ and $p' \neq 1$: using (2.193) and (2.188), the propagation of $\Sigma_1 = 0$ along \mathbf{e}_1 gives

$$\begin{aligned} & 12\mathcal{E}(3p'G + 12p'^2 - 7p' - 3)\dot{u}_1^6 + [12p'(3(\mu - p)G + 6(3\mu - 2p)p' \\ & - 2(p + 2\mu)\omega^2 - 2\mathcal{E}(6p'(\mu - p)G + 15\mathcal{E}p'^2 - 2(\mu - 20p)p' \\ & + 3(3\mu + p))\dot{u}_1^4 - p'(\mu - p)[(6(p - \mu)G + 12(2p - \mu)p' \\ & - 4(\mu + 2p))\omega^2 + \mathcal{E}(3(\mu - p)G + (\mu - 7p)p' + 7\mu - p)]\dot{u}_1^2 \\ & - 2p'^2\omega^2(\mu - p)^3 = 0. \end{aligned} \quad (2.214)$$

Using $\Sigma_1 = 0$ to eliminate \dot{u}_1 from equations (2.214) and (2.198) one obtains

$$\begin{aligned} & [12(\mu - p)G - 24\mu p'^2 + 12(5\mu - p)p' + 4(p - 7\mu)]\omega^4 - 2\mathcal{E}[6(\mu - p)G \\ & + (\mu - 13p)p'^2 + 2(5\mu + 7p)p' - 7\mu - 5p]\omega^2 + \mathcal{E}^2[3(\mu - p)G \\ & + (5\mu - 11)p'^2 - (\mu - 13p)p' - 2(\mu + 2p)] = 0 \end{aligned} \quad (2.215)$$

and

$$\begin{aligned} & [12(\mu - p)G + 24(\mu - 2p)p'^2 + 12(3p + \mu)p' + 4(p - 7\mu)]\omega^4 \\ & - 2\mathcal{E}[6(\mu - p)G + (23\mu - 35p)p'^2 - 4(4\mu - 10)p' \\ & - 3(3p + \mu)]\omega^2 + \mathcal{E}^2[3(\mu - p)G + (13\mu - 19p)p'^2 \\ & - (11\mu - 23p)p' - 6p] = 0. \end{aligned} \quad (2.216)$$

Subtracting equations (2.215) and (2.216) results in

$$(\mu - p)(\mathcal{E} - 4\omega^2)(p' - 1)[(4p' - 1)\mathcal{E} - 6p'\omega^2] = 0. \quad (2.217)$$

Now $\mathcal{E} - 4\omega^2$ is non-zero, as its propagation along \mathbf{e}_1 gives $5p' + 1 = 0$ which simplifies equation (2.198) to

$$\mathcal{E}(\mu - 4p)(\mu - p)^3 = 0,$$

in contradiction with $5p' + 1 = 0$. Also notice that $(4p' - 1)\mathcal{E} - 6p'\omega^2 \neq 0$, as this would reduce equation (2.199) to

$$6\dot{u}_1^2 + \mu - p = 0. \quad (2.218)$$

which is a contradiction (see page 66).

Now we assume that $\Sigma_2 = 0$. Propagating twice $\Sigma_2 = 0$ along \mathbf{e}_1 , we deduce the two following polynomials in G , G' , p' , μ , p , \dot{u}_1 :

$$\begin{aligned} & 216(2p'p - \mathcal{E})^2 \dot{u}_1^6 + 6(\mu - p)[3(\mu^2 - p^2)G + 6(\mu^2 + 4p^2 - 7\mu p)p'^2 + \\ & 3(7\mu^2 - 3p^2 + 4\mu p)p' - 7\mu^2 + p^2 - 6\mu p]\dot{u}_1^4 - 9p'(\mu - p)^3[3\mathcal{E} - \\ & (5\mu + p)p']\dot{u}_1^2 - p'^2(5\mu + 7p)(\mu - p)^4 = 0 \end{aligned} \quad (2.219)$$

and

$$\begin{aligned} & 11664(2p'p - \mathcal{E})^3 \dot{u}_1^8 + 216(\mu - p)(2p'p - \mathcal{E})[3[4p(\mu - p)p' + \mu^2 - p^2]G + \\ & 6(3\mu^2 - 9p^2 - 8\mu p)p'^2 + 8(3\mu^2 + 8p^2 + 10\mu p)p' - 31\mu^2 - 11p^2 - \\ & 42p\mu]\dot{u}_1^6 + 18(\mu - p)^2[3(\mu^2 - p^2)G' + [12(\mu^3 - 4p^3 - 24p\mu^2 + 33\mu p^2)p'^2 \\ & + 12(\mu^3 + 3p^3 - 3p\mu^2 - \mu p^2)p' - 3(7\mu^3 - 5p^3 + 5p\mu^2 - 7\mu p^2)]G - (48\mu^3 \\ & + 426p^3 - 126p\mu^2 - 468\mu p^2)p'^3 - (37\mu^3 - 475p^3 + 607p\mu^2 + 191\mu p)p'^2 + \\ & (218\mu^3 - 132p^3 + 312p\mu^2 - 38\mu p^2)p' - 49\mu^3 + 11p^3 - 71p\mu^2 - 11p^2\mu]\dot{u}_1^4 \\ & + 3(\mu - p)^4 p'[[9(10\mu^2 - 2p^2 - 8p\mu)p' - 15(\mu^2 - p^2)]G - (57\mu^2 - 15p^2 + \\ & 546p\mu)p'^2 + (378\mu^2 + 186p^2 + 612p\mu)p' - 181\mu^2 - 113p^2 - 294p\mu]\dot{u}_1^2 - \\ & 2p'^2(\mu - p)^5[3[5\mu^2 - 7p^2 + 2p\mu]G - 3(9\mu^2 + 23p^2 + 16p\mu)p' + 46\mu^2 \\ & + 28p^2 + 70p\mu] = 0. \end{aligned} \quad (2.220)$$

The resultant of equations (2.219) and (2.220) with respect to \dot{u}_1 leads to a relation $(2p'p - \mathcal{E})F(G, G', p', \mu, p) = 0$ where F is a polynomial with 846 terms:

$$\begin{aligned} F \equiv & -2916(5\mu + 7p)^2 \mathcal{E}^3 (\mu - p)^6 (2pp' - \mathcal{E})G'^3 + 486\mathcal{E}^2 (\mu - p)^4 \\ & [9(5\mu + 7p)^2 (\mu - p)^2 (8pp' - 3\mathcal{E})G^2 + \psi_1 G + \psi_2]G'^2 - 27\mathcal{E}(\mu - p)^3 \\ & [648(5\mu + 7p)^2 (\mu - p)^3 (4pp' - \mathcal{E})G^4 + \psi_3 G^3 + \psi_4 G^2 + \psi_5 G \\ & + \psi_6]G' + 5832(5\mu + 7p)^2 (\mu - p)^6 (8pp' - \mathcal{E})G^6 + \psi_7 G^5 \\ & + \psi_8 G^4 + \psi_9 G^3 + \psi_{10} G^2 + \psi_{11} G + \psi_{12}. \end{aligned} \quad (2.221)$$

with ψ_1, \dots and ψ_{12} being functions of p' , p and μ .

First notice that $2p'p - \mathcal{E} = 0$ takes us back to the case $\Sigma_1 = 0$. Indeed, $2p'p - \mathcal{E} = 0$ implies $G = 1 - \frac{1}{2}(\frac{\mathcal{E}}{p})^2$ after which equation (2.219) implies

$$(\mu - p)^4(5\mu + 7p - 45\dot{u}_1^2) = 0. \quad (2.222)$$

When $5\mu + 7p - 45\dot{u}_1^2 = 0$, its propagation along \mathbf{e}_1 and equation (2.193) and (2.200) result in

$$\dot{u}_1(\mu - 13p) = 0,$$

which is clearly incompatible with $\mu + p - 2p'p = 0$. When on the other hand $\mu - p = 0$ (this is also valid in the case $F = 0$ when $p' - 1 = 0$), we are led back to the case $\Sigma_1 = 0$. This leaves us with the equation $F = 0$ to consider. We check whether it admits solutions with a γ -law equation of state. Substitution of $G = -p' + 1/3$, $G' = 0$, $p = p'\mu + c$ into the equation $F = 0$ results in

$$\begin{aligned} & -26244p'^2(11p' - 7)(2354p'^3 + 244p'^2 + 319p' - 1)(1 + 11p')^2 \\ & (2p' + 1)^3(-1 + p')^8\mu^9 + \dots(\dots)\mu^8 + \dots(\dots)\mu^7 + \dots(\dots)\mu^6 + \dots - \\ & 2916c^9(1876 - 43140p'^2 + 15474p' - 1894845p'^4 + 1485646p'^3 - \\ & 949025091p'^7 - 819796296p'^9 + 1225322374p'^8 - 66913777p'^5 + \\ & 386131891p'^6 + 225588528p'^{10}) = 0 \end{aligned}$$

When $p' = \text{constant}$ the latter equation leads to $\mu = \text{constant}$, unless all coefficients of μ vanish³.

It turns out that the only physically plausible solution is given by $p' = 7/11$ and $c = 0$. Herewith equation (2.219) becomes

$$128\mu^2(14\mu - 165\dot{u}_1^2)(728\mu^2 + 12540\dot{u}_1^2\mu + 136125\dot{u}_1^4) = 0. \quad (2.223)$$

As the propagation of $728\mu^2 + 12540\dot{u}_1^2\mu + 136125\dot{u}_1^4$ along \mathbf{e}_1 gives $\mu = 0$, one has then $14\mu - 165\dot{u}_1^2 = 0$ (or $\mu = 165\dot{u}_1^2/14$). The surviving kinematical quantities are then

$$\begin{aligned} \omega &= \frac{3\sqrt{154}}{14}\dot{u}_1, & a_1 &= n_{23} = -\frac{33}{14}\dot{u}_1, \\ E_{\alpha\beta} &= \begin{bmatrix} -45\dot{u}_1^2/7 & 0 & 0 & 0 \\ 0 & 15\dot{u}_1^2/14 & 0 & 0 \\ 0 & 0 & 75\dot{u}_1^2/14 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

and

$$H_{\alpha\beta} = \begin{bmatrix} 0 & 15\sqrt{154}\dot{u}_1^2/98 & 0 & 0 \\ 15\sqrt{154}\dot{u}_1^2/98 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

in which all derivatives of variables are zero except along \mathbf{e}_1 :

$$\partial_1\dot{u}_1 = -\frac{9}{7}\dot{u}_1^2. \quad (2.224)$$

³More general solutions might exist, but we haven't been able to arrive at a conclusive answer.

The exterior derivative of the basis one-forms ω^a is given by the first Cartan equations, which in the present case $p' = 7/11$ reduce to

$$d\omega^0 = \sqrt{\frac{198}{7}}\dot{u}_1\omega^1 \wedge \omega^2 - \dot{u}_1\omega^0 \wedge \omega^1, \quad (2.225)$$

$$d\omega^1 = 0, \quad (2.226)$$

$$d\omega^2 = 4\dot{u}_1\omega^1 \wedge \omega^2, \quad (2.227)$$

$$d\omega^3 = -\frac{5}{7}\dot{u}_1\omega^1 \wedge \omega^3. \quad (2.228)$$

As

$$\omega^2 \wedge d\omega^2 = 0 \quad \text{and} \quad \omega^3 \wedge d\omega^3 = 0$$

one has by Frobenius' theorem (see (2.44) in [51])

$$\omega^0 = h_t dt + h_x dx + h_y dy + h_z dz, \quad (2.229)$$

$$\omega^1 = dx, \quad (2.230)$$

$$\omega^2 = f dy, \quad (2.231)$$

$$\omega^3 = g dz, \quad (2.232)$$

where h_t, h_x, h_y, h_z, f and g are functions of x . We can always set one of the coefficients h_x, h_y, h_z of (2.229) to zero by an appropriate choice of the coordinate t . Since $\omega^1 \wedge d\omega^0 = 0$, the most convenient choice appears to be $h_x = 0$. Substituting (2.229)-(2.232) into (2.225)-(2.228), one gets the following system of differential equations

$$\frac{df}{dx} = 4\dot{u}_1 f, \quad (2.233)$$

$$\frac{dg}{dx} = -\frac{5}{7}\dot{u}_1 g, \quad (2.234)$$

$$\frac{dh_t}{dx} = \dot{u}_1 h_t, \quad (2.235)$$

$$\frac{dh_y}{dx} = \sqrt{\frac{198}{7}}\dot{u}_1 f + \dot{u}_1 h_y, \quad (2.236)$$

$$\frac{dh_z}{dx} = \dot{u}_1 h_z. \quad (2.237)$$

Solving these differential equations, by using (2.224), one can rewrite (2.229)-

(2.232) as follows

$$\omega^0 = \dot{u}_1^{-\frac{7}{9}}(dt + dz) + \sqrt{\frac{22}{7}}\dot{u}_1^{-\frac{28}{9}}dy, \quad (2.238)$$

$$\omega^1 = -\frac{7}{9}\frac{d\dot{u}_1}{\dot{u}_1^2}, \quad (2.239)$$

$$\omega^2 = \dot{u}_1^{-\frac{28}{9}}dy, \quad (2.240)$$

$$\omega^3 = \dot{u}_1^{\frac{5}{9}}dz, \quad (2.241)$$

We now simplify (2.238)-(2.241) by a change of variables such that $\dot{u}_1 = 1/x$ and $\tilde{t} = t + z$, Then the line element can be written as follows

$$ds^2 = -x^{\frac{14}{9}}(dt + \sqrt{\frac{22}{7}}x^{\frac{7}{3}}dy)^2 + (\frac{7}{9})^2dx^2 + x^{\frac{56}{9}}dy^2 + x^{-\frac{10}{9}}dz^2. \quad (2.242)$$

The solutions have Petrov type I and admit a G_3 isometry group of Bianchi type I acting transitively on time-like hypersurfaces orthogonal to the acceleration vector. The isometry group is abelian and there is a Killing vector parallel to the vorticity. Hence these solutions in general belong to the class studied by Krasinski [52].

2.6.2.2 Acceleration not orthogonal to vorticity ($\dot{u}_3 \neq 0$)

First notice that $\mathcal{E} + 6\omega^2$ is non-zero: if $\mathcal{E} + 6\omega^2 = 0$ then its propagation along e_1 , e_2 and e_3 together with equation (2.176) respectively results in $\dot{u}_1 + q_1 = 0$, $\dot{u}_2 - r_2 = 0$, $q_3 - \dot{u}_3 = 0$ and $p' - 1 = 0$. Herewith equations (2.166) and (2.167) respectively reduce to

$$\begin{aligned} 2\dot{u}_1(6E_{11} + \mathcal{E}) &= 0, \\ 6\dot{u}_2(2E_{11} - \mathcal{E}) &= 0. \end{aligned}$$

If $\dot{u}_1 \neq 0$ ($\dot{u}_2 \neq 0$) the propagation of $6E_{11} + \mathcal{E} = 0$ ($2E_{11} - \mathcal{E} = 0$) along e_3 implies $\dot{u}_3 = 0$. It follows that the acceleration is parallel to the vorticity: $\dot{u}_1 = \dot{u}_2 = 0$. Evaluating $\partial_1\dot{u}_1 - \partial_2\dot{u}_2 = 0$ gives $6E_{11} - \mathcal{E} = 0$ after which its propagation along e_3 would result in $\dot{u}_3 = 0$. Therefore we assume henceforth that $\mathcal{E} + 6\omega^2 \neq 0$.

We also first investigate the special case $p' = 1$ (hence $G = -2/3$) for which a separate analysis of the whole procedure has to be carried out:

Equation (2.176) results then in $q_3 = \dot{u}_3$ and the propagation of (2.166), (2.167) along \mathbf{e}_3 leads to

$$\begin{aligned}(6E_{11} - \mathcal{E})q_1 &= 0, \\ (6E_{11} - \mathcal{E})r_2 &= 0.\end{aligned}$$

When $6E_{11} - \mathcal{E} = 0$, its propagation along \mathbf{e}_1 and \mathbf{e}_2 gives $r_2 = q_1 = 0$, while if $r_2 = q_1 = 0$, the propagation of $q_3 - \dot{u}_3 = 0$ along \mathbf{e}_3 gives $6E_{11} - \mathcal{E} = 0$. We conclude that $q_1 = r_2 = 6E_{11} - \mathcal{E} = 0$. Note that $E_{33} = -\mathcal{E}/3$ then implies that also $E_{11} = E_{22} = \mathcal{E}/6$ and hence an extra rotation in the (1,2)-plane exists such that $\dot{u}_1 = 0$. Propagating $\dot{u}_1 = 0$ along \mathbf{e}_1 and \mathbf{e}_2 gives then

$$\dot{u}_3^2 + \omega^2 + \dot{u}_2 q_2 - \frac{1}{3}(\mu + 2p) = 0 \quad (2.243)$$

and $\dot{u}_2 r_1 = 0$. When $r_1 = 0$, equation (B.19), using (2.243) gives

$$\partial_2 q_2 = 4\omega^2 - q_2(q_2 - \dot{u}_2) - \mathcal{E}. \quad (2.244)$$

Then the propagation of (2.243) along \mathbf{e}_1 gives $\dot{u}_2 \partial_1 q_2 = 0$. The surviving rotation coefficients are

$$n_{13} = -a_2 = \frac{1}{2}q_2, \quad a_3 = -\dot{u}_3.$$

One furthermore has $H_{\alpha\beta} = 0$ and

$$E_{\alpha\beta} = \begin{bmatrix} \mathcal{E}/6 & 0 & 0 \\ 0 & \mathcal{E}/6 & 0 \\ 0 & 0 & -\mathcal{E}/3 \end{bmatrix},$$

with $\text{div} \mathbf{E} = 0$. The space-time is purely electric of Petrov type D and the only non-zero remaining derivatives are

$$\partial_3 \dot{u}_3 = \partial_3 q_3 = -\dot{u}_3^2 - \frac{1}{6}(\mu - p), \quad \partial_3 q_2 = -q_2 \dot{u}_3, \quad (2.245)$$

$$\partial_2 \dot{u}_2 = \dot{u}_2(q_2 - \dot{u}_2), \quad \partial_3 \dot{u}_2 = -\dot{u}_2 \dot{u}_3 \quad (2.246)$$

$$\partial_2 \omega = -2\omega \dot{u}_2, \quad \partial_3 \omega = -\omega \dot{u}_3. \quad (2.247)$$

These solutions have Petrov type D and admit a G_2 acting transitively on the time-like two-surfaces orthogonal to the vorticity and the acceleration vector. The commutators $[\mathbf{e}_0, \mathbf{e}_1] = 0$ and $[\mathbf{e}_2, \mathbf{e}_3] = \dot{u}_3 \mathbf{e}_2$ show that this group is abelian and orthogonally transitive.

When $\dot{u}_1 = \dot{u}_2 = 0$, from equation (2.243) we get

$$\dot{u}_3^2 + \omega^2 - \frac{1}{3}(\mu + 2p) = 0. \quad (2.248)$$

The equation (2.277) then reduces to

$$(\mu + 3\omega^2)(\mu - \omega^2 - \dot{u}_3^2) = 0.$$

When $\mu + 3\omega^2 = 0$, evaluating $\partial_3(\mu + 3\omega^2) = 0$ and using (2.248) results in $\dot{u}_3(4\omega^2 + \dot{u}_3^2) = 0$ which is impossible. It follows that $\mu - \omega^2 - \dot{u}_3^2 = 0$, which then simplifies equation (2.248) to $p = \mu$. Therefore $\mu = p = \omega^2 + \dot{u}_3^2$ and the other surviving rotation coefficients are

$$n_{13} = -a_2 = \frac{1}{2}q_2 \quad , \quad n_{23} = a_1 = \frac{1}{2}r_1 \quad , \quad a_3 = -\dot{u}_3.$$

One has $H_{\alpha\beta} = 0$ and

$$E_{\alpha\beta} = \begin{bmatrix} \mathcal{E}/6 & 0 & 0 \\ 0 & \mathcal{E}/6 & 0 \\ 0 & 0 & -\mathcal{E}/3 \end{bmatrix},$$

with $\text{div} E = 0$. The space-time is purely electric of Petrov type D and the only non-zero remaining derivatives are

$$\partial_3 \dot{u}_3 = \partial_3 q_3 = -\dot{u}_3^2, \quad \partial_3 r_1 = -r_1 \dot{u}_3, \quad \partial_3 q_2 = -q_2 \dot{u}_3, \quad (2.249)$$

$$\partial_1 r_1 - \partial_2 q_2 = 4\dot{u}_3^2 + r_1^2 + q_2^2 - 2\mu, \quad (2.250)$$

These solutions are locally rotationally symmetric and in general belong to type IIIAGii of Collins and White. They admit a G_4 isometry group acting multiply transitively on time-like hypersurfaces orthogonal to the vorticity (the acceleration).

So henceforth we assume that $p' \neq 1$ and $\mathcal{E} + 6\omega^2 \neq 0$. Propagating equation (2.176) along e_1 and e_2 gives

$$3p'[6(3G + 18p' - 1)\omega^2 + (3G + 7p' + 4)\mathcal{E}]\dot{u}_1 q_3 + 108p'^2 \omega^2 q_1 (q_3 - \dot{u}_3) - [18p'(3G + 18p' - 1)\omega^2 + (3p'G + 9p'^2 + 14p' + 6)\mathcal{E}]\dot{u}_1 \dot{u}_3 = 0 \quad (2.251)$$

and

$$3p'[6(3G + 18p' - 1)\omega^2 + (3G + 7p' + 4)\mathcal{E}]\dot{u}_2 q_3 - 108p'^2 \omega^2 r_2 (q_3 - \dot{u}_3) - [18p'(3G + 18p' - 1)\omega^2 + (3p'G + 9p'^2 + 14p' + 6)\mathcal{E}]\dot{u}_2 \dot{u}_3 = 0 \quad (2.252)$$

Now, we can simplify equations (2.251) and (2.252), using equation (2.176), to obtain

$$\begin{aligned} & [6(3G - 12p'^2 + 18p' - 4)\omega^2 + (3G - p'^2 + 2p' + 1)\mathcal{E}]\dot{u}_1 - \\ & 36(p' - 1)p'\omega^2 q_1 = 0 \end{aligned} \quad (2.253)$$

and

$$\begin{aligned} & [6(3G - 12p'^2 + 18p' - 4)\omega^2 + (3G - p'^2 + 2p' + 1)\mathcal{E}]\dot{u}_2 + \\ & 36(p' - 1)p'\omega^2 r_2 = 0. \end{aligned} \quad (2.254)$$

The previous equations (2.253) and (2.254) can be used to eliminate q_1 and r_2 from equations (2.166) and (2.167) such that we get

$$\begin{aligned} & \dot{u}_1[18(3G - 18p'^2 - 4 + 24p')\omega^2 E_{11} + 3\mathcal{E}(3G - p'^2 + 2p' + 1)E_{11} \\ & - 18(3G - 12p'^2 + 18p' - 4)\omega^4 + 3\mathcal{E}(3G + 16p' - 11p'^2 - 3)\omega^2 \\ & + \mathcal{E}^2(3G - p'^2 + 2p' + 1)] = 0 \end{aligned} \quad (2.255)$$

and

$$\begin{aligned} & \dot{u}_2[18(3G - 18p'^2 + 24p' - 4)\omega^2 E_{11} + 3\mathcal{E}(3G - p'^2 + 2p' + 1)E_{11} \\ & - 18(3G - 12p'^2 + 18p' - 4)\omega^4 - 3\mathcal{E}(9G - 47p'^2 + 64p' - 11)\omega^2 \\ & - 2\mathcal{E}^2(3G - p'^2 + 2p' + 1)] = 0. \end{aligned} \quad (2.256)$$

We now investigate the three sub-cases 1) $\dot{u}_1 \neq 0 \neq \dot{u}_2$, 2) $\dot{u}_1 = 0 \neq \dot{u}_2$ (or $\dot{u}_2 = 0 \neq \dot{u}_1$) and 3) $\dot{u}_1 = 0 = \dot{u}_2$ separately.

1) $\dot{u}_1 \neq 0 \neq \dot{u}_2$

Eliminating E_{11} from (2.255) and (2.256) one finds

$$\begin{aligned} & (108G - 432p'^2 + 648p' - 144)\omega^4 - 6\mathcal{E}(6G - 29p'^2 + 40p' - 7)\omega^2 \\ & - 3\mathcal{E}^2(3G - p'^2 + 2p' + 1) = 0 \end{aligned} \quad (2.257)$$

which shows that the vorticity is a function of the matter density. Evaluation of $\partial_{[1}\mu\partial_{2]}\omega = 0$ gives then

$$\begin{aligned} & 54p'(p' - 1)\omega^4 + 3\mathcal{E}(6G - 14p'^2 + 19p' - 1)\omega^2 \\ & + \mathcal{E}^2(3G - p'^2 + 2p' + 1) = 0 \end{aligned} \quad (2.258)$$

Eliminating G from equations (2.257) and (2.258), we arrive at

$$36p'\omega^4 + 2\mathcal{E}(p' - 7)\omega^2 - 3\mathcal{E}^2(2p' - 1) = 0, \quad (2.259)$$

the propagation of which along \mathbf{e}_3 results in

$$\mathcal{E}^{16}(p' - 1)^6(51p' - 11)^2(3p' + 1)^2 = 0.$$

With $51p' - 11 = 0$ (hence $G = 2/17$), equations (2.258) and (2.259) result in $\mathcal{E} = 0$. When $3p' + 1 = 0$ ($G = 2/3$), equation (2.259) gives $(3\mathcal{E} + 2\omega^2)(5\mathcal{E} - 18\omega^2) = 0$. Now $3\mathcal{E} + 2\omega^2$ cannot be zero, as (2.258) then simplifies to $\mathcal{E} = 0$. Therefore $5\mathcal{E} - 18\omega^2 = 0$, after which equations (2.253) and (2.253) simplify to $3\dot{u}_1 + q_1 = 0$ and $3\dot{u}_2 - r_2 = 0$ respectively. The propagation of $3\dot{u}_1 + q_1 = 0$ along \mathbf{e}_2 gives then $\dot{u}_1\dot{u}_2 = 0$ which is a contradiction.

2) $\dot{\mathbf{u}}_1 = \mathbf{0} \neq \dot{\mathbf{u}}_2$

Note that the case $\dot{u}_2 = 0 \neq \dot{u}_1$ leads to a similar analysis. The spatial derivatives of $\dot{u}_1 = 0$ give rise to $r_1 = 0$, $q_1 = 0$ and

$$E_{11} - \dot{u}_2 q_2 - \dot{u}_3 q_3 - \omega^2 + \frac{1}{6}(\mu + 3p) = 0. \quad (2.260)$$

With $r_1 = 0$ and $q_1 = 0$ equations (B.19) and (B.20) result in

$$\begin{aligned} \partial_2 r_2 &= \frac{1}{3}\mathcal{E} - 2E_{11} + r_2^2 + r_2 q_2, \\ \partial_2 q_2 &= -\frac{1}{3}(p - 2\mu) + 3\omega^2 - q_2^2 - q_3^2. \end{aligned}$$

Taking now the propagation of equation (2.254) along \mathbf{e}_3 and eliminating r_2 and q_3 by using equations (2.254) and (2.176) we get

$$\begin{aligned} &[3888\mathcal{E}(p' - 1)G' - 1944(4p' - 1)G^2 - (3888p'^2 + 2592p' + 1296)G - \\ &15552p'^4 + 46656p'^3 - 49248p'^2 + 17280p' - 1728]\omega^6 + \\ &36\mathcal{E}[36\mathcal{E}(p' - 1)G' - 9(8p' - 1)G^2 - 3(p'^2 + 12p' + 6)G + 36p'^4 - \\ &132p'^3 + 160p'^2 - 100p' + 8]\omega^4 + 6\mathcal{E}^2[18\mathcal{E}(p' - 1)G' - 9(4p' + 1)G^2 + \\ &6(8p'^2 - 22p' + 5)G - 25p'^4 + 88p'^3 - 86p'^2 - 8p' + 11]\omega^2 - \\ &\mathcal{E}^3(3G - p'^2 + 2p' + 1)^2 = 0, \end{aligned} \quad (2.261)$$

showing that the vorticity is a function of the matter density (the polynomial (2.261) cannot be identically zero). Again expressing $\partial_{[3}\mu\partial_{2]}\omega = 0$, it follows that

$$(108G - 216p'^2 + 432p' - 144)\omega^4 + 6\mathcal{E}(6G + p'^2 - 2p' + 5)\omega^2 + \mathcal{E}^2(3G - p'^2 + 2p' + 1) = 0. \quad (2.262)$$

Eliminating ω from equations (2.261) and (2.262) results then in

$$(3G - p'^2 + 2p' + 1)P_2 = 0$$

with

$$\begin{aligned} P_2 \equiv & 2592\mathcal{E}^2G'^2 - 9\mathcal{E}[1152G^2 + (1332p' + 1356)G + 23p'^3 - 69p'^2 + \\ & 957p' + 369]G' + 10368G^4 + (23976p' - 6804)G^3 + (414p'^3 + \\ & 45495p'^2 - 20304p' - 8541)G^2 - (10773p'^4 - 44058p'^3 + 5220p'^2 + \\ & 14706p' + 1215)G + 1242p'^6 - 7452p'^5 + 11448p'^4 + 4348p'^3 - \\ & 5070p'^2 - 1128p' + 164. \end{aligned} \quad (2.263)$$

Notice that $3G - p'^2 + 2p' + 1 \neq 0$ as otherwise this would simplify (2.262) to

$$18\omega^2(p' - 1)^2(10\omega^2 - \mathcal{E}) = 0, \quad (2.264)$$

and then propagating $10\omega^2 - \mathcal{E} = 0$ (from 2.264) along \mathbf{e}_3 and using (2.176), $10\omega^2 - \mathcal{E} = 0$ would lead to $\mathcal{E}(p' - 1)\dot{u}_3 = 0$, which is a contradiction.

We suppose that the coefficient of E_{11} in (2.256),

$$6(3G - 18p'^2 + 24p' - 4)\omega^2 + \mathcal{E}(3G - p'^2 + 2p' + 1), \quad (2.265)$$

is non-zero, as otherwise elimination of G from the expression (2.265) and (2.262) and then propagating the result along \mathbf{e}_3 would lead to $p' - 1 = 0$.

Hence, propagating equation (2.176) along \mathbf{e}_3 and simplifying by the use of equations (2.254), (2.260), (2.256), (2.176) and (2.262) we arrive at

$$\begin{aligned} & (7p'\mathcal{E} - 4\mathcal{E} + 18p'\omega^2)(2\mathcal{E} + 18p'\omega^2 + p'\mathcal{E})(5p'\mathcal{E} - 2\mathcal{E} + 18p'\omega^2)\Xi \\ & 5832p'^3\omega^8 + 324p'^2(5p'p + 11p'\mu - 8\mathcal{E})\omega^6 - 18p'\mathcal{E}(44p'\mu - 22\mathcal{E} + 5p'^2\mu \\ & - 4p'p + 107p'^2p)\omega^4 - 3p'\mathcal{E}^2(8\mu + 125p'^2\mu - 88p'\mu + 299p'^2p \\ & - 232p'p + 32p)\omega^2 - 3p\mathcal{E}^3(5\mu - 18p'\mu + 16p'^2\mu \\ & + 9p - 34p'p + 31p'^2p) = 0, \end{aligned} \quad (2.266)$$

where $\Xi = \dot{u}_2^2 + \dot{u}_3^2$. First we notice that the coefficient of Ξ in equation (2.266) cannot be zero: propagating its respective factors along \mathbf{e}_3 results

in inconsistencies.

Using (2.176) and (2.262), the propagation of (2.266) along \mathbf{e}_3 gives then

$$\begin{aligned} & 3(18p'\omega^2 + p'\mathcal{E} + 2\mathcal{E})[11664p'^2(6p'^2 - p' + 3)\omega^8 + 648p'(69p'^3 - 5p'^2 \\ & + 50p' - 18)\mathcal{E}\omega^6 + 36\mathcal{E}^2(183p'^4 + 329p'^3 + 86p'^2 - 190p' + 24)\omega^4 \\ & + 6\mathcal{E}^3(p'^4 + 691p'^3 - 338p'^2 - 114p' + 48)\omega^2 + \mathcal{E}^4(35p'^4 + 164p'^3 \\ & - 109p'^2 - 42p' + 24)]\Xi + p'(\mathcal{E} + 6\omega^2)[104976p'^2(6p'^2 - p' + 3)\omega^{10} \\ & + \Psi_8(p', p, \mu)\omega^8 + \Psi_6(p', p, \mu)\omega^6 + \Psi_4(p', p, \mu)\omega^4 + \Psi_2(p', p, \mu)\omega^2 \\ & + \Psi_0(p', p, \mu)] = 0 \end{aligned} \quad (2.267)$$

The resultant of equation (2.266) and (2.267) with respect to Ξ gives

$$\begin{aligned} & 629856p'^4\omega^{10} + 104976p'^3(7p'p + 3p'\mu - 4\mathcal{E})\omega^8 - 5832p'^2\mathcal{E}(47p'^2\mu + 5\mathcal{E} \\ & - 10p'\mu + 38p'p - 49p'^2p)\omega^6 + 108p'\mathcal{E}^2(194\mathcal{E} - 1847p'^3\mu + 1836p'^2\mu \\ & - 885p'\mu - 405p'p - 420p'^2p + 577p'^3p)\omega^4 - 18\mathcal{E}^3(2116p'^4\mu + 56\mathcal{E} \\ & + 1443p'^2\mu - 436p'\mu - 2666p'^3\mu + 1126p'^3p - 764p'^4p - 93p'^2p \\ & - 244p'p)\omega^2 + \mathcal{E}^4(86p'\mu - 2078p'^4\mu - 849p'^2\mu - 8\mu + 2444p'^3\mu \\ & - 3716p'^3p + 56p - 746p'p + 2703p'^2p + 1622p'^4p) = 0. \end{aligned} \quad (2.268)$$

Again eliminating ω between (2.268) and its propagation along \mathbf{e}_3 one arrives at

$$p'^{20}(5\mu + 7p)(17p' - 5)(p' - 1)^{24}(p'p + 3p'\mu - 2\mathcal{E})\Phi = 0$$

with

$$\begin{aligned} \Phi \equiv & (940197969816p^7\mu - 5534090517592p^8 - 102117486403704p^5\mu^3 \\ & + 34637777964700p^6\mu^2 + 37785613447724\mu^8 + 28420846551192p\mu^7 \\ & - 287289143287436p^2\mu^6 - 655656901297080p^3\mu^5 - 508052155492836p^4\mu^4)p'^7 \\ & - 4\mathcal{E}(-3271802477671p^7 - 3207374831408p^6\mu - 21607676710506p^5\mu^2 \\ & - 139930409354697p^4\mu^3 - 254266111725675p^3\mu^4 - 163402053224274p^2\mu^5 \\ & - 11308851798212p\mu^6 + 16054742724683\mu^7)p'^6 + 3\mathcal{E}^2(15579723214641\mu^6 \\ & - 34371627299784\mu^5p - 184345672669575\mu^4p^2 - 213840606059208p^3\mu^3 \\ & - 96210427542365\mu^2p^4 - 23533571329648\mu p^5 - 6771983857485p^6)p'^5 \\ & - \mathcal{E}^3(20290571616083\mu^5 - 73620707380287p\mu^4 - 259364035349138p^2\mu^3 \\ & - 247461171692654p^3\mu^2 - 103661834838465p^4\mu - 21634617457747p^5)p'^4 \end{aligned}$$

$$\begin{aligned}
& +2\mathcal{E}^4(2779300700893\mu^4 - 15604559735122p\mu^3 - 39765220321590p^2\mu^2 \\
& - 29668710630526p^3\mu - 8066199518615p^4)p'^3 - 6\mathcal{E}^5(144168432939\mu^3 \\
& - 1359444348575p\mu^2 - 2318005711897p^2\mu - 952011206643p^3)p'^2 \\
& + 27\mathcal{E}^6(2556569781\mu^2 - 41187233852p\mu - 36275669465p^2)p' \\
& - 1547620101(3\mu - 35p)\mathcal{E}^7,
\end{aligned} \tag{2.269}$$

in which $5\mu + 7p$ and $17p' - 5$ are non-zero, as otherwise equation (2.263) leads to a contradiction.

Propagating twice $p'p + 3p'\mu - 2\mathcal{E} = 0$ along \mathbf{e}_3 gives

$$\begin{aligned}
& [18\mathcal{E} + 36(p + 2\mu)p'^2 + 6(4p + 3\mu)p']G + 18\mathcal{E}^2G' + 9(10\mu + 6p)p'^3 \\
& + 36(2p + \mu)p'^2 - 2(10p + 9\mu)p' - 18\mathcal{E} = 0,
\end{aligned} \tag{2.270}$$

$$(3p' + 1)p' + 6p'^2\mu - 3(1 - G)\mathcal{E} = 0. \tag{2.271}$$

The equations (2.271) and (2.270) simplify expression (2.263) to the following

$$(2\mu + p)(793\mu + 383p)(\mu - p)^2 = 0.$$

Substituting $2\mu + p$ and $793\mu + 383p$ separately in equation (2.262) and then propagating along \mathbf{e}_3 in both cases leads to $\mu = p = 0$. Hence, the expression Φ should be zero.

We first check whether the equation $\Phi = 0$ admits solutions with a γ -law equation of state. Substitution of $p = (\gamma - 1)\mu$, $G = 4/3 - \gamma$ and $G' = 0$ into the equation $\Phi = 0$ and (2.263) results in

$$\begin{aligned}
& 5534090517592\gamma^{14} - 85970584609196\gamma^{13} + 565283509827207\gamma^{12} - \\
& 2048382039891998\gamma^{11} + 4474791992492452\gamma^{10} - 5993162165353368\gamma^9 + \\
& 4611332900173616\gamma^8 - 1404570995726752\gamma^7 - 613252185698112\gamma^6 + \\
& 646996837159808\gamma^5 - 157439928845824\gamma^4 - 2843373502464\gamma^3 + \\
& 2060827754496\gamma^2 - 402854510592\gamma + 19327352832 = 0
\end{aligned} \tag{2.272}$$

and

$$(23\gamma + 50)(2\gamma - 3)(3\gamma - 2)(\gamma + 1)(\gamma - 2)^2 = 0 \tag{2.273}$$

which contradict each other. Hence solutions with a γ -law equation of state is not possible.

Taking a derivative of $\Phi = 0$ with respect to μ and using (2.262) and (2.268) one arrives at a polynomial in terms of μ , p , p' (where p' is of degree 45) and with 1978 terms such that each term contains a digit of order 10^{75} for which we were not able to verify whether the resulting polynomial equation

is consistent with $\Phi = 0$ in the general case. If so, the surviving rotation coefficients are given by

$$n_{13} = \frac{1}{2}(r_2 + q_2) \quad , \quad a_2 = \frac{1}{2}(r_2 - q_2) \quad , \quad a_3 = -\dot{u}_3.$$

One has

$$H_{\alpha\beta} = \begin{bmatrix} \omega(q_3 - \dot{u}_3) & 0 & 0 \\ 0 & \omega(q_3 - \dot{u}_3) & r_2\omega \\ 0 & r_2\omega & -2\omega(q_3 - \dot{u}_3) \end{bmatrix},$$

and

$$E_{\alpha\beta} = \begin{bmatrix} \omega^2 + q_3\dot{u}_3 + q_2\dot{u}_2 - \frac{1}{6}(\mu + 3p) & 0 & 0 \\ 0 & -\omega^2 - q_3\dot{u}_3 - q_2\dot{u}_2 + \frac{1}{6}(3\mu + 5p) & 0 \\ 0 & 0 & -\mathcal{E}/3 \end{bmatrix}.$$

The only non-zero remaining derivatives are

$$\begin{aligned} \partial_2\dot{u}_3 &= \dot{u}_2(q_3 - \dot{u}_3), \quad \partial_3\dot{u}_3 = -\dot{u}_3^2 + r_2\dot{u}_2 - \frac{1}{6}(\mu - p), \\ \partial_2\dot{u}_2 &= -\dot{u}_2^2 - 2\omega^2 - q_2\dot{u}_2 - 2q_3\dot{u}_3 + \frac{2}{3}(\mu + 2p), \\ \partial_3\dot{u}_2 &= -\dot{u}_3(r_2 + \dot{u}_2), \\ \partial_3q_3 &= -\omega^2 - q_3(\dot{u}_3 + q_3) - q_2(\dot{u}_2 - r_2) + \frac{1}{6}(\mu + 5p) \\ \partial_2r_2 &= r_2^2 - 2\omega^2 - q_2(2\dot{u}_2 - r_2) - 2q_3\dot{u}_3 + \frac{2}{3}(\mu + 2p), \\ \partial_3r_2 &= q_3(3\dot{u}_2 + r_2) - 3\dot{u}_3(\dot{u}_2 + r_2), \\ \partial_2q_2 &= 3\omega^2 - q_2^2 - q_3^2 - \frac{1}{3}(2\mu + p), \quad \partial_3q_2 = -q_3(q_2 + r_2), \\ \partial_2\omega &= \omega(r_2 - 2\dot{u}_2), \quad \partial_3\omega = \omega(\dot{u}_3 - 2q_3). \end{aligned} \tag{2.274}$$

These solutions admit a G_2 acting transitively on the time-like two-surfaces orthogonal to the vorticity and the acceleration vector.

3) $\dot{u}_1 = \dot{u}_2 = 0$

With $\dot{u}_1 = \dot{u}_2 = 0$, from (2.23), (2.29) and (2.30) one obtains $q_1 = r_2 = 0$ and

$$6E_{11} - \mathcal{E} = 0 \tag{2.275}$$

$$3q_3\dot{u}_3 + 3\omega^2 - \mu - 2p = 0. \tag{2.276}$$

Using (2.276), the equation (2.176) reduces to

$$Q\dot{u}_3^2 + p'(\mathcal{E} + 6\omega^2)(3\omega^2 - \mu - 2p) = 0. \quad (2.277)$$

with

$$Q \equiv 18p'\omega^2 + \mathcal{E}(p' + 2)$$

Note that $Q \neq 0$, as otherwise equation (2.277) reduces to $3\omega^2 - \mu - 2p = 0$, the propagation of which along \mathbf{e}_3 together with $Q = 0$ leads to $\mu = 0$.

Now, propagating (2.277) along \mathbf{e}_3 , using (2.176) and (2.277), we arrive at

$$\begin{aligned} & (648G - 1296p'^2 + 2592p' - 864)\omega^6 - [216pG - (1008p + 360\mu)p'^2 + \\ & (1908p + 828\mu)p' - 756p - 468\mu]\omega^4 - 6\mathcal{E}[(9\mu + 21p)G + (p + 7\mu)p'^2 - \\ & 8\mathcal{E}p' + 21p + 7\mu]\omega^2 - \mathcal{E}^2[6(2p + \mu)G - (5p + \mu)p'^2 + \\ & (5\mu + 7p)p' + 6p] = 0. \end{aligned} \quad (2.278)$$

One can easily show that (2.278) is not identically zero. Propagating (2.278) along \mathbf{e}_3 , using (2.176) and (2.277), gives

$$\begin{aligned} & [(11664p + 11664\mu)G' - 23328p'(2p' - 5)G - 15552p'(12p'^2 - 22p' + 7)]\omega^8 \\ & + [1944(\mu - p)\mathcal{E}G' + (15552\mathcal{E} + 28512pp'^2 + 5184\mu p'^2 - 49896pp' \\ & - 14904\mu p')G - 12312\mathcal{E} + 57024\mu p'^3 + 127008pp'^3 - 247968pp'^2 \\ & - 142992\mu p'^2 + 129384pp' + 102168\mu p']\omega^6 + [-972\mathcal{E}^2(\mu + 3p)G' \\ & + 108\mathcal{E}(54pp'^2 + 6\mu p'^2 - 69\mu p' - 127pp' - 62p - 18\mu)G - 36\mathcal{E}(4\mu p'^3 \\ & - 122pp'^3 - 414pp'^2 - 126\mu p'^2 + 489\mu p' + 1097pp' - 205\mu - 291p)]\omega^4 \\ & - [54\mathcal{E}^3(5\mu + 11p)G' + 18\mathcal{E}^2(12\mu p'^2 - 8pp'^2 + 63pp' + 45\mu p' + 116p \\ & + 60\mu)G + 6\mathcal{E}^2(25\mu p'^3 - 65pp'^3 + 87\mu p'^2 + 35pp'^2 + 24\mu p' + 84pp' \\ & + 288p + 98\mu)]\omega^2 - 18\mathcal{E}^4(\mu + 2p)G' + 3\mathcal{E}^3(10pp'^2 + 2\mu p'^2 - 29\mu p' \\ & - 43pp' - 18\mu - 30p)G + \mathcal{E}^3(75pp'^3 - 58\mu p' + 27\mu p'^3 - 59\mu p'^2 \\ & - 104pp' - 36p - 61pp'^2) = 0. \end{aligned} \quad (2.279)$$

The resultant of (2.278) and (2.279) with respect to ω gives

$$(3\mu + 5p)(p' - 1)^6(7\mu p' + 13pp' + 2\mathcal{E})\Psi(G, G', p', \mu, p) = 0, \quad (2.280)$$

where Ψ is a polynomial with 443 terms. If either $3\mu + 5p = 0$ or $7\mu p' + 13pp' + 2\mathcal{E} = 0$, their propagation along \mathbf{e}_3 and (2.278) leads to a contradiction. Now we check whether the equation $\Psi(G, G', p', \mu, p) = 0$ admits solutions with a

γ -law equation of state. Substitution of $G = -p' + 1/3$, $G' = 0$, $p = p'\mu + c$ into the equation $\Psi(G, G', p', \mu, p) = 0$ results in

$$\begin{aligned} & 36(5p' - 1)(2p' - 1)(5p' + 1)(3p' + 1)(2 + p')(p' - 1)^5(1721p'^4 + 3294p'^3 \\ & - 1312p'^2 + 594p' + 311)\mu^4 + 144(2p' - 1)(3p' + 1)(p' + 2)(\dots)c\mu^3 \\ & + 216(2p' - 1)(3p' + 1)(2 + p')(p' - 1)^4(\dots)c^2\mu^2 \\ & + 144(2p' - 1)(3p' + 1)(2 + p')(p' - 1)^4(\dots)c^3\mu \\ & + 36(2p' - 1)(3p' + 1)(2 + p')(p' - 1)^4(\dots)c^4 = 0 \end{aligned} \quad (2.281)$$

When $p' = \text{constant}$ the latter equation leads to $\mu = \text{constant}$, unless all coefficients of μ vanish⁴. In the case $c = 0$ the polynomial $\Psi(G, G', p', \mu, p)$ leads to

$$\begin{aligned} & (3 + 5p')(p' + 1)^6(p' - 1)^{11}(9p' + 13p'^2 + 2) \\ & (5p' - 1)(2p' - 1)(5p' + 1)(3p' + 1)(2 + p') \\ & (1721p'^4 + 3294p'^3 - 1312p'^2 + 594p' + 311) = 0, \end{aligned} \quad (2.282)$$

It turns out that the only remaining physically plausible solution is given by $p' = 1/5$. Herewith equation (2.278) becomes

$$(\mu - 3\omega^2)(11\mu^2 + 210\mu\omega^2 - 225\omega^4) = 0, \quad (2.283)$$

where $11\mu^2 + 210\mu\omega^2 - 225\omega^4 = 0$ together with its propagation along e_3 and (2.176) leads to $\mu = 0$. Therefore, $\mu - 3\omega^2 = 0$, which simplifies (2.277) to $15\dot{u}_3^2 - \mu = 0$. Hence $\omega = \sqrt{5}\dot{u}_3$ and $p = 3\dot{u}_3^2$ by (2.276). The remaining kinematical surviving quantities are then

$$n_{13} = -a_2 = \frac{1}{2}q_2 \quad , \quad n_{23} = a_1 = \frac{1}{2}r_1 \quad , \quad a_3 = -2\dot{u}_3,$$

$$E_{\alpha\beta} = \begin{bmatrix} 3\dot{u}_3^2 & 0 & 0 \\ 0 & 3\dot{u}_3^2 & 0 \\ 0 & 0 & -6\dot{u}_3^2 \end{bmatrix},$$

with $\text{div } \mathbf{E} = 0$ and

$$H_{\alpha\beta} = \begin{bmatrix} \sqrt{5}\dot{u}_3^2 & 0 & 0 \\ 0 & \sqrt{5}\dot{u}_3^2 & 0 \\ 0 & 0 & -2\sqrt{5}\dot{u}_3^2 \end{bmatrix},$$

⁴More general solutions might exist, but we haven't been able to arrive at a conclusive answer

with \dot{u}_3 , q_2 and r_1 obeying the following set of differential equations

$$\begin{aligned} \partial_1 \dot{u}_3 &= 0 & \partial_2 \dot{u}_3 &= 0 & \partial_3 \dot{u}_3 &= -3\dot{u}_3^2 \\ \partial_1 r_1 - \partial_2 q_2 &= r_1^2 + q_2^2 & \partial_3 r_1 &= -2r_1 \dot{u}_3, \\ \partial_3 q_2 &= -2q_2 \dot{u}_3. \end{aligned} \quad (2.284)$$

Hence space-time is of Petrov type D. The exterior derivative of the basis one-forms ω^a is given:

$$d\omega^0 = 2\sqrt{5}\dot{u}_3\omega^1 \wedge \omega^2 - \dot{u}_3\omega^0 \wedge \omega^3, \quad (2.285)$$

$$d\omega^1 = -q_2\omega^1 \wedge \omega^2 - 2\dot{u}_3\omega^1 \wedge \omega^3, \quad (2.286)$$

$$d\omega^2 = -r_1\omega^1 \wedge \omega^2 - 2\dot{u}_3\omega^2 \wedge \omega^3, \quad (2.287)$$

$$d\omega^3 = 0. \quad (2.288)$$

In a similar way as described on page 72, the line element can be written as follows

$$ds^2 = -z^{\frac{2}{3}}(dt + Cdy)^2 + z^{\frac{4}{3}}(A^2dx^2 + B^2dy^2) + \frac{1}{9}dz^2 \quad (2.289)$$

where A , B and C are functions of x and y , and should satisfy

$$\frac{\partial A}{\partial y} = QAB, \quad (2.290)$$

$$\frac{\partial B}{\partial x} = RAB, \quad (2.291)$$

$$\frac{\partial C}{\partial x} = 2\sqrt{5}AB \quad (2.292)$$

such that Q and R are functions of x and y , and satisfy

$$R\frac{\partial A}{\partial y} - Q\frac{\partial B}{\partial x} = 0, \quad B\frac{\partial R}{\partial x} + A\frac{\partial Q}{\partial y} = -AB(R^2 + Q^2). \quad (2.293)$$

These solutions are locally rotationally symmetric and in general belong to type IIIAGii of Collins and White. They admit a G_4 isometry group acting multiply transitively on time-like hypersurfaces orthogonal to the vorticity (the acceleration).

2.7 Expanding non-rotating perfect fluids

If the shear and vorticity tensors vanish the field equations and Bianchi identities are greatly simplified. Putting $\sigma = 0 = \omega$ in the equation (1.40),

one obtains immediately $\mathbf{H} = 0$, thus the Weyl tensor is purely electric. This result, due to Trümper [80], does not depend on the field equations (1.10) and (1.11) and is valid for any space-time admitting a shear-free non-twisting time-like vector field. These fields have been investigated extensively by Barnes [2] who has shown that the Weyl tensor is then either algebraically general or algebraically special of Petrov type D or O. For fields of Petrov I, \mathbf{u} is parallel to a Killing vector field and consequently the space-times are static. In the same paper it was argued that vacuum fields admitting a shear-free non-twisting time-like vector field \mathbf{u} were also static. Recently, Wyllie and Beke [99] re-examined Petrov type D gravitational fields with shear-free normal four-velocity in a perfect fluid where the energy density is spatially homogeneous. They present a new class of such fluid solutions, for which the fluid's expansion can be nonzero, contrary to the conclusion in the earlier investigation by Barnes [2].

Now, if a barotropic equation of state is imposed, Collins and Wainwright [22] show that the only solutions are locally i) a spatially homogeneous and isotropic FLRW model ($\dot{\mathbf{u}} = 0$), ii) a tilted spatially homogeneous but anisotropic solution possessing planar symmetry or its temporally homogeneous counterpart and iii) a spatially inhomogeneous and anisotropic solution possessing spherical symmetry and in which the fluid flow is non-geodesic. In the case (ii), the line element can be written in terms of co-moving coordinates $\{t, x, y, z\}$ as follows

$$ds^2 = \frac{C^2}{U^2} \left[-\frac{U'^2}{m^2} dt^2 + dx^2 + e^{-2x}(dy^2 + dz^2) \right], \quad (2.294)$$

where

$$U = U(v) \neq 0, \quad v = t + x \quad \text{and} \quad U'' + U' = -U^2, \quad (2.295)$$

so that a prime denotes differentiation with respect to v and the quantities C and m are nonzero constants. The energy density and pressure are given by

$$\mu = \frac{1}{C^2} [3m^2 - 2U^3 - 3(U' + U)^2] \quad \text{and} \quad \mu + p = \frac{2U^4}{C^2 U'}. \quad (2.296)$$

Three of the Killing vector fields of the space-time generate a three-parameter isometry group of Bianchi type V, with three-dimensional orbits ($t + x = \text{constant}$) and the fourth Killing vector gives rise to a local rotational symmetry about a space-like axis parallel to the acceleration. These solutions

contradict the results of King and Ellis [50] who claimed that the only shear-free spatially homogeneous perfect fluid models are FLRW.

In the case (iii) the space-time metric in terms of $\{t, r, \theta, \phi\}$ is

$$ds^2 = \frac{1}{U^2} \left[-\frac{U'^2}{At+B} dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right], \quad (2.297)$$

where

$$U = U(v) \neq 0, \quad v = t+r \quad \text{and} \quad U'' = U^2 \Leftrightarrow U'^2 = \frac{2}{3}U^3 - \frac{1}{4}A. \quad (2.298)$$

The quantities A and B are constants satisfying $A^2 + B^2 \neq 0$. The energy density and pressure are given by

$$\mu = 3(Av + B) + 12UU' \quad \text{and} \quad \mu + p = \frac{20U^4}{3U'}. \quad (2.299)$$

Wyman [101] proved that the solutions (2.297) are the only spherically symmetric solutions of the Einstein field equations with perfect fluid source with a equation of state, in which the shear is zero but the expansion and acceleration are nonzero.

Chapter 3

Purely radiative space-times

Perfect fluid space-times in which both the covariant transverse conditions $\text{div}\mathbf{E} = \text{div}\mathbf{H} = 0$ hold at the non-perturbative level have been called purely radiative [76] and were shown to be very restricted in the sense that they would have to obey two non-terminating chains of integrability conditions (see, for example [64], [90]). In the present chapter, we study this purely radiative condition under a number of additional assumptions and we classify the resulting space-times. All the evidence so far seems to indicate [60] that these conditions only hold in models with special symmetries, such as the spatially homogeneous space-times of Bianchi class A [35, 92]. In models with realistic inhomogeneity, the gravito-magnetic field would therefore necessarily be non-transverse at second and higher orders.

In a paper [84], we showed that all the non-rotating dust models belong to Bianchi class A, provided we assume that they are not purely electric. In the purely electric case, the exceptions were given by the Petrov type D spatially homogeneous and locally rotationally symmetric (LRS) metrics which are pseudo-spherically symmetric (and hence of Bianchi type III), or which are spherically symmetric and belong to the Kantowski-Sachs family of dust models and hence do not admit a three-dimensional isometry group acting simply transitively on the hypersurfaces of homogeneity. For technical reasons we also assumed that $[\boldsymbol{\sigma}, \mathbf{H}] = 0$. Together with the fact that $[\boldsymbol{\sigma}, \mathbf{E}] = 0$, which is a consequence of $\text{div}\mathbf{H} = 0$, this implies that $\boldsymbol{\sigma}$, \mathbf{E} and \mathbf{H} can be simultaneously diagonalized (it is not necessary to assume that $[\mathbf{E}, \mathbf{H}] = 0$, as a degenerate shear eigen-plane also must be a degenerate \mathbf{E} eigen-plane, see section 3.2.2). In [84], we also showed that this analysis can be generalized to purely radiative and geodesic perfect fluids [3]. When one considers fluids with non-constant pressure (but still with geodesic flow), the

spatial gradient of p vanishes and the velocity becomes orthogonal to the $p = \text{constant}$ hyper-surfaces, such that the flow is automatically irrotational [71, 77].

For irrotational and geodesic space-times, the Ricci and Bianchi equations (1.35)-(1.46) then simplify to

Ricci equations

$$\text{div} \sigma_a - \frac{2}{3} D_a \theta = 0, \quad (3.1)$$

$$\text{curl} \sigma_{ab} - H_{ab} = 0, \quad (3.2)$$

$$\dot{\sigma}_{\langle ab \rangle} + \frac{2}{3} \theta \sigma_{ab} + \sigma_{c \langle a} \sigma_{b \rangle}^c + E_{ab} = 0, \quad (3.3)$$

$$\dot{\theta} + \frac{1}{3} \theta^2 + \sigma_{ab} \sigma^{ab} + \frac{1}{2} (\mu + 3p) = 0, \quad (3.4)$$

Bianchi equations

$$\dot{\mu} = -(\mu + p)\theta, \quad (3.5)$$

$$\text{div} E_a = [\sigma, H]_a + \frac{1}{3} D_a \mu, \quad (3.6)$$

$$\text{div} H_a = -[\sigma, E]_a, \quad (3.7)$$

$$\dot{E}_{\langle ab \rangle} - \text{curl} H_{ab} = -\theta E_{ab} + 3\sigma_{c \langle a} E_{b \rangle}^c - \frac{1}{2} (\mu + p) \sigma_{ab}, \quad (3.8)$$

$$\dot{H}_{\langle ab \rangle} - \text{curl} E_{ab} = -\theta H_{ab} + 3\sigma_{c \langle a} H_{b \rangle}^c. \quad (3.9)$$

Clearly $\text{div} \mathbf{H} = 0$ implies $[\sigma, \mathbf{E}] = 0$ which guarantees the existence of a frame wherein \mathbf{E} and σ are both diagonal and which will be used henceforth. From (3.3) with $\alpha \neq \beta$ we see then that,

$$\Omega_1(\sigma_{22} - \sigma_{33}) = \Omega_2(\sigma_{33} - \sigma_{11}) = \Omega_3(\sigma_{11} - \sigma_{22}) = 0. \quad (3.10)$$

As a consequence Ω can be assumed to be 0: either all components automatically vanish, or, when e.g. the $(\mathbf{e}_2, \mathbf{e}_3)$ plane is a shear eigen-plane, then $\Omega_2 = \Omega_3 = 0$ and we can choose an extra rotation (see section 1.4) in this plane to make $\Omega_1 = 0$. The frame is then fixed up to rotations in the $(\mathbf{e}_2, \mathbf{e}_3)$ -plane for which the rotation angle φ satisfies $\partial_0 \varphi = 0$.

From the diagonal components of (3.2) we obtain the following algebraic relations between $H_{\alpha\alpha}$, $\sigma_{\alpha\alpha}$ and n_α :

$$\begin{aligned} H_{11} &= n_3(\sigma_{22} - \sigma_{11}) + n_2(\sigma_{33} - \sigma_{11}), \\ H_{22} &= n_1(\sigma_{33} - \sigma_{22}) + n_3(\sigma_{11} - \sigma_{22}), \\ H_{33} &= n_2(\sigma_{11} - \sigma_{33}) + n_1(\sigma_{22} - \sigma_{33}). \end{aligned} \quad (3.11)$$

The Raychaudhuri equation (3.4) and the conservation law (3.5) give us the evolution of respectively the expansion and the density, while (3.3) gives the evolution of the components $\sigma_{\alpha\alpha}$:

$$\begin{aligned} 3\partial_0\sigma_{11} &= (\sigma_{22}^2 + \sigma_{33}^2 - 2\sigma_{11}^2 - 2\sigma_{11}\theta) - 3E_{11}, \\ 3\partial_0\sigma_{22} &= (\sigma_{33}^2 + \sigma_{11}^2 - 2\sigma_{22}^2 - 2\sigma_{22}\theta) - 3E_{22}, \\ 3\partial_0\sigma_{33} &= (\sigma_{11}^2 + \sigma_{22}^2 - 2\sigma_{33}^2 - 2\sigma_{33}\theta) - 3E_{33}. \end{aligned} \quad (3.12)$$

The spatial derivatives of $\sigma_{\alpha\alpha}$ are obtained by (3.1) and by the off-diagonal components of (3.2) :

$$\begin{aligned} \partial_1\sigma_{11} &= \frac{2}{3}z_1 + r_1(\sigma_{11} - \sigma_{22}) - q_1(\sigma_{11} - \sigma_{33}), \\ \partial_2\sigma_{11} &= -\frac{1}{3}z_2 + q_2(\sigma_{22} - \sigma_{11}) - H_{13}, \\ \partial_3\sigma_{11} &= -\frac{1}{3}z_3 + r_3(\sigma_{11} - \sigma_{33}) + H_{12} \end{aligned} \quad (3.13)$$

and similar expressions obtained by cyclic permutation of the indices. Note that we can assume that the shear is non-zero, as otherwise solutions would be conformally flat and hence determine a Friedman-Lemaître-Robertson-Walker(FLRW) universe.

Note that the $\text{div}\mathbf{E} = 0$ condition relates the off-diagonal parts of $H_{\alpha\beta}$ to μ_α by equation (3.6)

$$\begin{aligned} \mu_1 &= 3H_{23}(\sigma_{33} - \sigma_{22}), \\ \mu_2 &= 3H_{13}(\sigma_{11} - \sigma_{33}), \\ \mu_3 &= 3H_{12}(\sigma_{22} - \sigma_{11}). \end{aligned} \quad (3.14)$$

Using equation (3.5), we can now act with the commutator $[\partial_0, \partial_\alpha] \equiv -(\sigma_{\alpha\alpha} + \frac{1}{3}\theta)\partial_\alpha$ on μ to obtain the relation

$$\partial_0\mu_\alpha = -\mu_\alpha(\sigma_{\alpha\alpha} + \frac{4}{3}\theta) - \mathcal{E}z_\alpha, \quad (3.15)$$

We also need the spatial derivatives of $E_{\alpha\alpha}$, which we can deduce from (3.6) and the off-diagonal part of (3.9) :

$$\begin{aligned} \partial_1E_{11} &= E_{11}(r_1 - 2q_1) - E_{22}(r_1 + q_1), \\ \partial_2E_{11} &= (E_{22} - E_{11})q_2, \\ \partial_3E_{11} &= (E_{11} - E_{33})r_3. \end{aligned} \quad (3.16)$$

Propagating (3.14) along the fluid flow and substituting (3.12) and (3.15)

gives then the following expressions:

$$3(\sigma_{11} - \sigma_{22})\partial_0 H_{12} = (6\sigma_{11}^2 - 6\sigma_{22}^2 - 6E_{22} - 3E_{33} + 2\sigma_{22}\theta - 2\sigma_{11}\theta)H_{12} + \mathcal{E}z_3 \quad (3.17)$$

$$3(\sigma_{22} - \sigma_{33})\partial_0 H_{23} = (6\sigma_{22}^2 - 6\sigma_{33}^2 - 6E_{33} - 3E_{11} + 2\sigma_{33}\theta - 2\sigma_{22}\theta)H_{23} + \mathcal{E}z_1 \quad (3.18)$$

$$3(\sigma_{33} - \sigma_{11})\partial_0 H_{13} = (6\sigma_{33}^2 - 6\sigma_{11}^2 - 6E_{11} - 3E_{22} + 2\sigma_{11}\theta - 2\sigma_{33}\theta)H_{13} + \mathcal{E}z_2. \quad (3.19)$$

In the following section 3.1 we will first show that when the shear tensor σ is degenerate, then $[\sigma, \mathbf{H}] = 0$ [48]. This implies that \mathbf{H} is diagonal in the (σ, \mathbf{E}) -eigen-frame (in the case of degenerate shear (3.3) or (3.12) imply that the eigen-planes of σ and \mathbf{E} coincide). From (3.14) and (3.15) or (3.17-3.19) one obtains then $\mu_\alpha = 0$ and $z_\alpha = 0$ (regardless of the degeneracy of σ), giving a first hint that the corresponding space-times might be spatially homogeneous indeed.

In section 3.2 we will use the previous result as a motivation to impose the extra condition $[\sigma, \mathbf{H}] = 0$ and treat the cases of degenerate and non-degenerate shear simultaneously. We show that the Bianchi class A perfect fluids can be uniquely characterized -modulo the class of purely electric and (pseudo-)spherically symmetric universes- as geodesic perfect fluid space-times which are purely radiative in the sense that the gravitational field satisfies $\text{div}\mathbf{E} = \text{div}\mathbf{H} = 0$.

3.1 Geodesic perfect fluids with degenerate shear tensor

We consider non-rotating geodesic perfect fluid space-times which are purely radiative in the sense that the gravitational field satisfies the covariant transverse conditions $\text{div}\mathbf{E} = \text{div}\mathbf{H} = 0$. We show that when the shear tensor σ is degenerate, \mathbf{H} , \mathbf{E} and σ necessarily commute. This can be expressed as the following

Theorem: *Purely radiative geodesic perfect fluids with non-commuting degenerate shear and magnetic part of the Weyl tensor do not exist.*

We now assume that the shear tensor is degenerate:

$$\sigma_{\alpha\beta} = \text{diag}(-2\sigma, \sigma, \sigma)$$

A similar, but altogether different, assumption was made in [66], where it was assumed that all kinematic variables, together with the electric and magnetic parts of the Weyl curvature, were rotationally symmetric about a common spatial axis. This assumption gives rise to the so-called partially locally rotationally symmetric (PLRS) cosmologies. The intersection of the PLRS cosmologies with our present class of models (i.e. those cosmologies having $\text{div} \mathbf{H} = \text{div} \mathbf{E} = 0$ and a degenerate shear tensor), precisely consists of the LRS Bianchi cosmologies and the Kantowski-Sachs perfect fluids discussed in the next section.

With $\dot{\mathbf{u}} = \boldsymbol{\omega} = 0$ and $E_{\alpha\beta} = \text{diag}(-2E, E, E)$, the equations (3.4), (3.5) and (3.11)-(3.16) simplify to the following:

$$\begin{aligned}\partial_0 \theta &= -\frac{1}{3}\theta^2 - \sigma^2 - \frac{1}{2}(\mu + 3p) \\ \partial_0 \mu &= -(\mu + p)\theta \\ \partial_0 \sigma &= \sigma\left(\sigma - \frac{2}{3}\theta\right) - E\end{aligned}\tag{3.20}$$

$$\begin{aligned}\partial_1 \sigma &= -\frac{1}{3}z_1 + \frac{3}{2}\sigma(r_1 - q_1) \\ \partial_2 \sigma &= \frac{2}{3}z_2 - 3\sigma q_2 \\ \partial_3 \sigma &= \frac{2}{3}z_3 + 3\sigma r_3\end{aligned}\tag{3.21}$$

$$\begin{aligned}\partial_1 E &= \frac{3}{2}(r_1 - q_1)E \\ \partial_2 E &= 3q_2 E \\ \partial_3 E &= -3r_3 E\end{aligned}\tag{3.22}$$

$$\begin{aligned}\partial_1 \mu &= 0 \\ \partial_2 \mu &= -9(z_2 - 3\sigma q_2)\sigma \\ \partial_3 \mu &= -9(z_3 + 3\sigma r_3)\sigma\end{aligned}\tag{3.23}$$

$$\begin{aligned}H_{11} &= 3(n_3 + n_2)\sigma \\ H_{22} &= -3n_3\sigma \\ H_{33} &= -3n_2\sigma\end{aligned}\tag{3.24}$$

$$\begin{aligned}
H_{12} &= -(z_3 + 3\sigma r_3) \\
H_{23} &= -\frac{3}{2}\sigma(r_1 + q_1) \\
H_{13} &= z_2 - 3\sigma q_2
\end{aligned} \tag{3.25}$$

while the commutators simplify to:

$$[\partial_0, \partial_1] \equiv (2\sigma - \frac{1}{3}\theta)\partial_1 \tag{3.26}$$

$$[\partial_0, \partial_2] \equiv -\frac{1}{3}(3\sigma + \theta)\partial_2 \tag{3.27}$$

$$[\partial_0, \partial_3] \equiv -\frac{1}{3}(3\sigma + \theta)\partial_3 \tag{3.28}$$

$$[\partial_2, \partial_3] \equiv (n_2 + n_3)\partial_1 + q_3\partial_2 + r_2\partial_3 \tag{3.29}$$

$$[\partial_3, \partial_1] \equiv r_3\partial_1 + (n_1 + n_3)\partial_2 + q_1\partial_3 \tag{3.30}$$

$$[\partial_1, \partial_2] \equiv q_2\partial_1 + r_1\partial_2 + (n_1 + n_2)\partial_3. \tag{3.31}$$

$$\tag{3.32}$$

Acting with $[\partial_0, \partial_1]$ on θ and μ implies $q_1 = r_1$ and $z_1 = 0$, while evaluation of the following combination of commutators, $9\sigma[\partial_2, \partial_3]\sigma + [\partial_2, \partial_3]\mu + 3\sigma[\partial_2, \partial_3]\theta$, leads to

$$q_2 z_3 + r_3 z_2 = 0. \tag{3.33}$$

Furthermore, acting with $[\partial_0, \partial_\alpha]$ ($\alpha = 2, 3$) on θ gives us the time evolution of z_2 and z_3 :

$$\begin{aligned}
\partial_0 z_2 &= -\frac{9}{2}\sigma z_2 + \frac{45}{2}q_2\sigma^2 - \theta z_2 \\
\partial_0 z_3 &= -\frac{9}{2}\sigma z_3 - \frac{45}{2}r_3\sigma^2 - \theta z_3,
\end{aligned} \tag{3.34}$$

which guarantees that we can rotate the (e_2, e_3) basis vectors such that for example $z_2 = z_3$ (as $\partial_0(z_2/z_3) = 0$). Note that z_2 and z_3 cannot both be zero, as then equation (3.34) would imply $q_2 = r_3 = 0$, such that by (3.23) the spatial gradient of μ would vanish. The tetrad could then be further specified by rotating such that $n_{23} = \frac{1}{2}(r_1 + q_1) = 0$, which by (3.25) would take us back to the case $[\sigma, \mathbf{E}] = [\sigma, \mathbf{H}] = [\mathbf{E}, \mathbf{H}] = 0$ (see sub-section 3.2.2). So henceforth we work in a fixed frame with $z_2 = z_3 \neq 0$ which by (3.34) implies $r_3 = -q_2$. Together with (3.21) the Jacobi equations imply then

$$\partial_0 r_1 = (2\sigma - \frac{1}{3}\theta)r_1, \tag{3.35}$$

$$\partial_0 q_2 = -z_2 + (5\sigma - \frac{1}{3}\theta)q_2. \tag{3.36}$$

Applying now the commutators $[\partial_1, \partial_2]$ and $[\partial_3, \partial_1]$ to θ and combining the resulting equations, we obtain:

$$2r_1 + 2n_1 + n_2 + n_3 = 0. \quad (3.37)$$

Propagating $z_2 = z_3$ along \mathbf{e}_2 and \mathbf{e}_3 results in

$$z_2(r_2 + q_3) + 3\sigma(n_3^2 - n_2^2) = 0 \quad (3.38)$$

which together with (3.37) and the ‘dot \mathbf{H} ’ equations (3.9) leads to the key equation:

$$(n_2^2 - n_3^2)\chi = 0 \quad (3.39)$$

where $\chi \equiv z_2(2E - 9\sigma^2) + 45q_2\sigma^3$. If we propagate $\chi = 0$ twice along the fluid flow lines, we re-obtain $z_2 = z_3 = 0$. From (3.39) we conclude therefore that $n_2^2 - n_3^2 = 0$ and hence, by (3.38), $q_3 = -r_2$. The resulting cases $n_2 + n_3 = 0$ and $n_2 + n_3 \neq 0$ (and hence $n_2 = n_3 \neq 0$) will be investigated in the following paragraphs.

3.1.1 The case $n_2 + n_3 = 0$

The ‘dot \mathbf{E} ’ equations (3.8) imply that also $\text{curl}\mathbf{H}$ is diagonal in the shear eigen-frame. Using the off-diagonal components of $\text{curl}\mathbf{H}$ we can propagate $n_2 + n_3 = 0$ along \mathbf{e}_2 and \mathbf{e}_3 respectively, which leads to

$$(z_2 - 4q_2\sigma)(n_2 - r_1) = 0 = (z_2 - 4q_2\sigma)(n_2 + r_1). \quad (3.40)$$

First notice that $z_2 - 4q_2\sigma = 0$ would lead to a vanishing spatial gradient of the matter density. In fact, propagating $z_2 - 4q_2\sigma = 0$ along the fluid flow lines gives $q_2(8E - 7\sigma^2) = 0$, after which the propagation of $8E - 7\sigma^2 = 0$ along \mathbf{e}_2 would yield $q_2\sigma^2 = 0$ and hence $z_2 = z_3 = 0$. From (3.40) we conclude therefore that $n_2 = n_3 = r_1 = 0$.

Propagating $n_2 = 0$ along \mathbf{e}_1 one obtains then

$$E - \frac{\mu}{3} - 2\sigma^2 - \frac{\sigma\theta}{3} + \frac{\theta^2}{9} + 2r_2q_2 = 0. \quad (3.41)$$

Acting with the commutator $[\partial_0, \partial_2]$ on z_2 and using (3.41) allows us to eliminate E , which implies

$$-12r_2q_2^3 + (2\mu + \frac{141}{2}\sigma^2 + 2\theta\sigma - \frac{2}{3}\theta^2)q_2^2 - \frac{69\sigma z_2}{2}q_2 + 5z_2^2 = 0 \quad (3.42)$$

and which also shows that $q_2 \neq 0$. Propagating the latter relation along \mathbf{e}_3 and using (3.42) to eliminate μ one finds:

$$(z_2 - 3q_2\sigma)(4z_2 - 21q_2\sigma) = 0. \quad (3.43)$$

When we propagate the first factor along the fluid flow lines, we obtain an expression for μ which, after substitution in (3.41), would lead to the “anti-Newtonian” result $\mathbf{E} = 0$, which is impossible [100]. On the other hand, if we apply the same procedure to the second factor of (3.43), we get $\sigma(4z_2 - 9q_2\sigma) = 0$ which gives a contradiction with $4z_2 - 21q_2\sigma = 0$.

3.1.2 The case $n_2 + n_3 \neq 0$

We now have $n_3 = n_2 \neq 0$, which, when propagated along \mathbf{e}_1 , implies

$$E = 2q_3q_2 - 2r_1n_2 + \frac{1}{3}\mu + 2\sigma^2 + \frac{1}{3}\sigma\theta - \frac{1}{9}\theta^2 - n_2^2 - r_1^2. \quad (3.44)$$

Acting with the commutator $[\partial_0, \partial_2]$ on μ , z_2 and r_1 we arrive at the following expressions involving the matter density and the pressure:

$$\begin{aligned} & -\frac{69}{2}q_2\sigma z_2 + 15n_2r_1\sigma^2 + 4n_2r_1\mu - \frac{4}{3}n_2r_1\theta^2 + 5z_2^2 + \frac{141}{2}q_2^2\sigma^2 \\ & + 12q_2^3q_3 - \frac{2}{3}\theta^2q_2^2 - 12n_2r_1q_2^2 + 2\mu q_2^2 - 6n_2^2q_2^2 - 6r_1^2q_2^2 + 4n_2r_1\sigma\theta \\ & + 24n_2r_1q_3q_2 + 2q_2^2\sigma\theta - 12r_1n_2^3 - 24n_2^2r_1^2 - 12n_2r_1^3 + 45\sigma^2r_1^2 = 0 \end{aligned} \quad (3.45)$$

$$\begin{aligned} & 30z_2n_2\mu - 18r_1z_2\mu + 54r_1^3z_2 - 90z_2n_2^3 - 18r_1z_2\sigma\theta - 108r_1z_2q_3q_2 \\ & + 30z_2n_2\sigma\theta + 180z_2n_2q_3q_2 + 2025\sigma^3q_2n_2 + 1377r_1q_2\sigma^3 - 405r_1z_2\sigma^2 \\ & + 6r_1z_2\theta^2 - 126r_1z_2n_2^2 + 18r_1^2z_2n_2 - 333z_2n_2\sigma^2 - 10z_2n_2\theta^2 = 0 \end{aligned} \quad (3.46)$$

and

$$\begin{aligned} p = & -\frac{27}{2}\sigma^2 - 6\sigma\theta - 4\mu - 18q_3q_2 + \theta^2 + 9n_2^2 + 18r_1n_2 + 9r_1^2 \\ & + \frac{189q_2\sigma^3 + 54q_2\sigma^2\theta + 36q_2\sigma\mu + 216q_2^2\sigma q_3 - 12q_2\sigma\theta^2}{2z_2} \\ & - \frac{108q_2\sigma n_2^2 + 216q_2\sigma r_1n_2 + 108q_2\sigma r_1^2}{2z_2}. \end{aligned} \quad (3.47)$$

Eliminating q_3 from (3.45) and (3.46) gives the relation

$$\begin{aligned} & -50z_2^3n_2 + 918r_1q_2^3\sigma^3 + 30z_2^3r_1 + 1350\sigma^3q_2^3n_2 + 1836r_1^2q_2\sigma^3n_2 \\ & + 2700\sigma^3q_2n_2^2r_1 + 153r_1z_2\sigma^2q_2^2 - 900r_1^2z_2\sigma^2n_2 - 927z_2n_2\sigma^2q_2^2 \\ & - 207q_2\sigma z_2^2r_1 + 345q_2\sigma z_2^2n_2 + 270\sigma^2r_1^3z_2 - 594z_2n_2^2\sigma^2r_1 = 0. \end{aligned} \quad (3.48)$$

Notice that r_1 has to be different from zero (see the section 3.1.3), while (3.36) guarantees that $q_2 \neq 0$. This enables us to introduce new dimensionless variables,

$$x = \frac{z_2}{q_2\sigma}, y = \frac{q_2}{r_1}, h = \frac{\theta}{\sigma}, n = \frac{n_2}{r_1}, r = \frac{r_1}{\sigma}, q = \frac{q_3}{\sigma} \text{ and } m = \frac{\mu}{\sigma^2},$$

which satisfy the following evolution equations

$$\begin{aligned} \sigma^{-1}\partial_0 x &= x^2 - [(n+1)^2 r^2 - 2qyr + (2h^2 - 6h - 6m + 153)/18]x + 45/2 \\ \sigma^{-1}\partial_0 n &= -6n \\ \sigma^{-1}\partial_0 r &= -(n+1)^2 r^3 + 2qyr^2 + -(h^2 - 6h - 3m - 27)/9r \\ \sigma^{-1}\partial_0 y &= -(x-3)y \end{aligned} \quad (3.49)$$

and

$$\begin{aligned} \sigma^{-1}\partial_2 x &= 2r[-\frac{1}{3}yx^2 + (n+1+2y^2)xy^{-1} - 6ny^{-1} - 3y] \\ \sigma^{-1}\partial_2 n &= ry[(\frac{1}{2} - \frac{1}{3}n - \frac{5}{6}n^2)x + 4n^2 - 2] + 2qn \\ \sigma^{-1}\partial_2 r &= (5 - 4n - \frac{7}{6}x + \frac{5}{6}xn)r^2y - 2qr \\ \sigma^{-1}\partial_2 y &= (\frac{1}{2}x - \frac{5}{6}xn + 4n - 3)y^2r + 3qy - 2rn - 2r. \end{aligned} \quad (3.50)$$

Note that (3.47) was used in order to substitute for all occurrences of the pressure in the right hand sides. In what follows we will show that the propagation of equations (3.46) and (3.48) implies that x is constant. Then however (3.50)(a) and $\partial_0\partial_2x = \partial_0^2\partial_2x = 0$ simplify to $(x-6)(x-3)^2y^2 + 9x = 0$ and $\sigma(x-6)(x-3)^3y^2 = 0$. This implies $x = 0$, which brings us back to the degenerate shear case of section 3.2.

First we rewrite equations (3.46)-(3.48) as follows

$$\begin{aligned} & -8nh^2 + 12y^2h + 24mn + 30y^2x^2 + 423y^2 - 207y^2x - 4h^2y^2 \\ & + 12my^2 - 36r^2n^2y^2 - 36r^2y^2 + 72ry^3q - 72r^2ny^2 - 72nr^2 \\ & - 144n^2r^2 - 72r^2n^3 + 270 + 24hn + 144rnqy + 90n = 0 \end{aligned} \quad (3.51)$$

$$\begin{aligned}
& -18mx + 30mxn - 18hx - 108qyrx + 30hxn + 180xyrnq \\
& + 54r^2x - 90xr^2n^3 + 2025n + 1377 - 405x + 6h^2x - 126r^2n^2x \\
& + 18r^2nx - 333nx - 10xnh^2 = 0
\end{aligned} \tag{3.52}$$

$$\begin{aligned}
& 1350y^2n + 918y^2 - 50x^3y^2n + 30x^3y^2 + 1836n + 2700n^2 \\
& + 153y^2x - 927xy^2n - 207y^2x^2 + 345y^2x^2n + 270x \\
& - 900nx - 594xn^2 = 0.
\end{aligned} \tag{3.53}$$

Propagation along \mathbf{e}_2 of (3.52) and (3.53) results in a linear homogeneous system in q and r , the coefficients of which are polynomials in n, x and y . Eliminating n from the determinant of this system and from (3.53) yields:

$$\begin{aligned}
& 5508rn - 6750y^2rn^2 + 2430rn^2x + 2607x^2y^2r - 1080rn^2 - 300x^3y^2r \\
& + 2250rn^3x - 540rny^2 - 5649x^2y^2rn^2 + 2880xyqn - 642x^2y^2nr \\
& + 660x^3y^2rn^2 + 104x^3y^2rn + 2754ry^2 - 13500rn^3 - 918xr \\
& + 13635xy^2rn^2 - 6093rxy^2 + 630xrny^2 - 738xrn - 672x^2yqn = 0
\end{aligned} \tag{3.54}$$

$$\begin{aligned}
& -19440rn + 22032yqn + 99792y^2rn^2 + 33750y^4xr + 64800y^3qn \\
& + 97200y^4rn^2 - 17928rn^2x - 3780x^2y^2r + 64800rn^2 + 5508y^3xq \\
& + 64800yn^2q + 129600y^2n^3r - 12339x^2y^4r + 360x^3y^2r + 1080x^3y^3q \\
& - 330x^4y^4r + 3129x^3y^4r - 86994y^4xrn^2 - 10152y^4xnr + 28836y^4nr \\
& - 7128rn^3x - 7452y^3x^2q - 54756y^4r + 33048y^3q - 8775x^3y^4rn^2 \\
& + 750x^4y^4rn^2 - 2400x^3y^3qn - 97848rny^2 + 20052x^2y^2rn^2 - 10800xyqn \\
& - 4644x^2y^2nr + 100x^4y^4rn - 600x^3y^2rn^2 - 240x^3y^2rn - 1926x^2y^4nr \\
& + 330x^3y^4rn - 55512y^2n^3xr + 5940x^2y^2n^3r - 14256xyn^2q - 53784ry^2 \\
& + 42768rn^3 + 3240xr - 94392xy^2rn^2 + 20952rxy^2 + 44280xrny^2 \\
& - 7560xrn + 38745y^4x^2rn^2 - 44496y^3xqn + 16560y^3x^2qn = 0.
\end{aligned} \tag{3.55}$$

Eliminating q and n from the latter pair of equations and (3.53) we obtain the following polynomial relation between x and y :

$$\begin{aligned}
& P_1(x, y) \equiv -545500000x^{14}y^{14} + 22883025000x^{13}y^{14} - 12001000000x^{14}y^{12} \\
& + \dots (76 \text{ terms}) \dots + 7746297620889600x^2 + 140466196858798080y^2 \\
& - 20656793655705600x = 0
\end{aligned} \tag{3.56}$$

We construct a second polynomial relation between x and y by propagating equation (3.53) along the fluid flow, which yields

$$\begin{aligned}
& -300nx^4y^2 + 250n^2x^4y^2 + 90x^4y^2 - 3600n^2x^3y^2 + 8460nx^3y^2 \\
& -3780x^3y^2 + 11925n^2x^2y^2 - 54900nx^2y^2 + 28647x^2y^2 \\
& +144018nxy^2 - 69714xy^2 - 8748n^2xy^2 + 2718n^2x^2 \\
& +2970n^3x^2 + 810x^2 - 4050nx^2 + 38556y^2 - 161028ny^2 \\
& -12960x - 49896n^3x + 1296n^2x + 65880nx + 162000n^3 \\
& -127656n^2 - 162648n + 38880 = 0.
\end{aligned} \tag{3.57}$$

Eliminating n from (3.53) and (3.57) results then in

$$\begin{aligned}
P_2(x, y) \equiv & 10721500x^{10}y^6 - 380191500x^9y^6 + 6061903695x^8y^6 \\
& + \dots (27 \text{ terms}) \dots - 1402072053120x - 1800380390400 = 0.
\end{aligned} \tag{3.58}$$

One now can calculate the resultant of (3.56) and (3.58) with respect to y : this yields a polynomial in x which is not identically 0 and thereby shows that x is a constant.

3.1.3 The case $r_1 = 0$

Propagation of $r_1 = 0$ along \mathbf{e}_2 gives $n_2(24q_2\sigma - 5z_2) = 0$. As the evolution of $24q_2\sigma - 5z_2 = 0$ along the fluid flow would lead to $\sigma = 0$ we have $n_2 = 0$. Propagating (3.45) along \mathbf{e}_2 and substituting $n_2 = 0$ gives $(z_2 - 3q_2\sigma)(4z_2 - 21q_2\sigma) = 0$. The first factor must be nonzero, as otherwise (with $r_1 = 0$) we would have a diagonal \mathbf{H} . When the second factor is 0 we obtain from (3.45)

$$\frac{435}{16}\sigma^2 + 12q_3q_2 - \frac{2}{3}\theta^2 + 2\mu + 2\sigma\theta = 0. \tag{3.59}$$

Eliminating μ from the time evolution of $4z_2 - 21q_2\sigma = 0$ and (3.59) gives $\sigma = 0$.

3.2 Geodesic perfect fluids with $[\sigma, \mathbf{H}] = 0$

In the previous section, we considered non-rotating geodesic perfect fluid space-times which are purely radiative in the sense that the gravitational field satisfies the covariant transverse conditions $\text{div}\mathbf{H} = \text{div}\mathbf{E} = 0$. We showed that when the shear tensor σ is degenerate, $[\sigma, \mathbf{H}] = 0$. This result motivates us to restrict to geodesic perfect fluids with $[\sigma, \mathbf{H}] = 0$,

also when the shear is non-degenerate. It is a straightforward consequence of the assumptions $\text{div} \mathbf{E} = 0$ and $[\boldsymbol{\sigma}, \mathbf{H}] = 0$ that the spatial gradient of the matter density μ vanishes too. As stated in [71] it follows that p , σ and θ are constants over the hypersurfaces of constant density, and hence the resulting space-times satisfy the so-called Postulate of Uniform Thermal Histories (PUTH) [7, 8, 15, 82]. This seems to suggest that a local group of isometries exists, mapping the flow lines into each other (so called observational homogeneity). However, not much progress has been made in this area and a detailed proof, or even a precise formulation of the conjecture that PUTH would lead to observational homogeneity, is still lacking. In fact it was suggested in [8] that any extra distinguishing mathematical property leading to observational homogeneity probably would have to be very complicated, involving at least third-order derivatives of the metric. An example of such an extra property is precisely the condition of being purely radiative, in the above sense of having $\text{div} \mathbf{E} = 0 = \text{div} \mathbf{H}$, which indeed enables us to show that the resulting space-times are spatially homogeneous and, more particularly, of Bianchi class A when we exclude the non-radiative purely electric sub-cases of section 3.2.2.

Together with the fact that $[\boldsymbol{\sigma}, \mathbf{E}] = 0$, which is a consequence of $\text{div} \mathbf{H} = 0$, the assumption $[\boldsymbol{\sigma}, \mathbf{H}] = 0$ implies that $\boldsymbol{\sigma}$, \mathbf{E} and \mathbf{H} can be simultaneously diagonalized (it is not necessary to separately assume that $[\mathbf{E}, \mathbf{H}] = 0$, in the non-degenerate case this is obvious, while for degenerate shear a shear eigen-plane also must be a degenerate \mathbf{E} eigen-plane: see (3.3)).

A second set of equations we use are Einstein equations (1.55-1.57) and the Jacobi equations (1.59-1.63) which I write out in an orthonormal eigen-frame of the shear and with the extra simplifications $\dot{u}_\alpha = \omega_\alpha = 0$ and $\Omega_\alpha = 0$ (except for the (00) Einstein equation these equations appear in triplets which can be obtained from each other by cyclic permutation of the indices) :

Jacobi equations

$$\partial_1(n_2 + n_3) + \partial_2 r_3 + \partial_3 q_2 = (r_1 - q_1)(n_2 + n_3) + r_3 r_2 - q_3 q_2, \quad (3.60)$$

$$\partial_0(r_1 - q_1) - \partial_1(\sigma_{11} - \frac{2}{3}\theta) = -(r_1 - q_1)(\sigma_{11} + \frac{1}{3}\theta), \quad (3.61)$$

$$\partial_0(n_2 + n_3) = (2\sigma_{11} - \frac{1}{3}\theta)(n_2 + n_3), \quad (3.62)$$

$$\partial_0(r_3 + q_3) + \partial_3(\sigma_{11} - \sigma_{22}) = -(\sigma_{33} + \frac{1}{3}\theta)(r_3 + q_3). \quad (3.63)$$

Einstein equations

$$\begin{aligned} \partial_0 \theta = & -[(\sigma_{11} + \frac{1}{3}\theta)^2 + (\sigma_{22} + \frac{1}{3}\theta)^2 + (\sigma_{33} + \frac{1}{3}\theta)^2] \\ & - \frac{1}{2}(\mu + 3p), \end{aligned} \quad (3.64)$$

$$\frac{2}{3}\partial_1 \theta - \partial_1 \sigma_{11} = \frac{1}{2}(r_1 + q_1)(\sigma_{22} - \sigma_{33}) - \frac{3}{2}(r_1 - q_1)\sigma_{11}, \quad (3.65)$$

$$\begin{aligned} -\partial_0(\sigma_{11} + \frac{1}{3}\theta) - \partial_1(r_1 - q_1) + \partial_2 q_2 - \partial_3 r_3 = & \frac{1}{2}(p - \mu) + q_2(r_2 - q_2) \\ & - r_3(r_3 - q_3) - r_1^2 - q_1^2 + 2n_2 n_3 + \theta(\sigma_{11} + \frac{1}{3}\theta), \end{aligned} \quad (3.66)$$

$$\begin{aligned} -\partial_1 r_2 + \partial_2 q_1 - \partial_3(n_1 - n_2) = & q_1(r_2 + q_2) + r_2(r_1 + 2q_1) \\ & - (r_3 - q_3)(n_1 - n_2) + 2n_3(r_3 + q_3). \end{aligned} \quad (3.67)$$

Using the $[\partial_0, \partial_\alpha]$ commutators on θ , we obtain from (3.3) and $z_\alpha = \mu_\alpha = \partial_\alpha p = 0$ that $\partial_\alpha S = 0$ ($S = \sigma_{\beta\gamma}\sigma^{\beta\gamma}$) and hence, using (3.13),

$$\begin{aligned} r_1(\sigma_{11} - \sigma_{22})^2 - q_1(\sigma_{11} - \sigma_{33})^2 &= 0 \\ r_2(\sigma_{22} - \sigma_{33})^2 - q_2(\sigma_{22} - \sigma_{11})^2 &= 0 \\ r_3(\sigma_{11} - \sigma_{33})^2 - q_3(\sigma_{33} - \sigma_{22})^2 &= 0. \end{aligned} \quad (3.68)$$

We will treat the two cases of non-degenerate and degenerate shear now separately.

3.2.1 Non degenerate shear

As the frame is invariantly defined, we first aim to proof that r_α and q_α are zero and that all spatial derivatives vanish. We first look at the evolution of S . From (3.3) it is easy to check that

$$\partial_0 S = -\frac{4}{3}\theta S - 2\Sigma + 2\mathbf{E} \cdot \boldsymbol{\sigma}, \quad (3.69)$$

where $\Sigma \equiv \sigma_{\gamma\alpha}\sigma_\beta{}^\gamma\sigma^{\alpha\beta}$ and $\mathbf{E} \cdot \boldsymbol{\sigma} \equiv E_{\alpha\beta}\sigma^{\alpha\beta}$.

Hence, acting with the commutator $[\partial_0, \partial_\alpha]$ on S we find:

$$\partial_0 S_\alpha = \frac{5}{3}\theta S_\alpha - \frac{4}{3}S z_\alpha - 2\partial_\alpha \Sigma - 2\partial_\alpha(\mathbf{E} \cdot \boldsymbol{\sigma}) - S_\alpha \sigma_{\alpha\alpha} \quad (3.70)$$

and hence, as $z_\alpha = S_\alpha = 0$, $\partial_\alpha \Sigma + \partial_\alpha (\mathbf{E} \cdot \boldsymbol{\sigma}) = 0$. With the aid of (3.13) and (3.16), this results in three algebraic expressions. For $\alpha = 1$ we get

$$\begin{aligned} r_1(\sigma_{22} - \sigma_{11})[4E_{22} + 2E_{33} - 3\sigma_{33}(\sigma_{22} - \sigma_{11})] - \\ q_1(\sigma_{33} - \sigma_{11})[2E_{22} + 4E_{33} - 3\sigma_{22}(\sigma_{33} - \sigma_{11})] = 0. \end{aligned} \quad (3.71)$$

Eliminating r_1 from (3.68) and (3.71), we find

$$q_1(\sigma_{33} - \sigma_{11})(\sigma_{22} - \sigma_{11})\chi = 0 \quad (3.72)$$

with

$$\begin{aligned} \chi \equiv \frac{2}{3}[E_{11}(\sigma_{22} - \sigma_{33}) + E_{22}(\sigma_{33} - \sigma_{11}) + E_{33}(\sigma_{11} - \sigma_{22})] \\ - (\sigma_{11} - \sigma_{22})(\sigma_{22} - \sigma_{33})(\sigma_{33} - \sigma_{11}). \end{aligned}$$

Repeating this calculation for the cases $\alpha = 2, 3$ we finally have (as the shear is non-degenerate)

$$q_1\chi = q_2\chi = q_3\chi = 0. \quad (3.73)$$

If $\chi \neq 0$ this implies with (3.68) that $r_\alpha = q_\alpha = 0$, while when $\chi = 0$ one has $\partial_1\chi = 0$ and hence, using (3.13) and (3.16),

$$q_1(\sigma_{33} - \sigma_{11})^3 + r_1(\sigma_{22} - \sigma_{11})^3 = 0. \quad (3.74)$$

Eliminating q_1 from (3.74) and (3.68.a) and using $\text{tr}(\sigma_{\alpha\beta}) = 0$ again leads to $r_\alpha = q_\alpha = 0$.

This means that $a_\alpha = 0$ and $n_{\alpha\beta}$ is diagonal. It is easy to check that all spatial derivatives of the rotation coefficients now vanish and hence [35] we obtain the non-degenerate spatially homogeneous Bianchi class A models, i.e. types *I*, *II*, *VI*₀, *VII*₀, *VIII* and *IX*.

3.2.2 Degenerate shear

From the degeneracy of the shear it is easily verified that \mathbf{E} must be degenerate too (3.12) and we may assume

$$\begin{aligned} \sigma_{\alpha\beta} &= \text{diag}(-2\sigma, \sigma, \sigma) \\ E_{\alpha\beta} &= \text{diag}(-2E, E, E). \end{aligned}$$

We can discard the case of purely magnetic ($\mathbf{E} = 0$) dust as a consequence of the non-existence of anti-Newtonian universes [97], but for non-constant pressure it is known that purely magnetic perfect fluids do exist.

These are necessarily of Petrov type I or D, (we don't consider the conformally flat cases, as these uniquely reduce under the present assumptions to the FLRW models, see [51]): for type I the only possibility is the Bianchi VI_0 metric discussed in [100], while for type D these space-times are locally rotationally symmetric class III [100]. As the fluid is non-rotating the solutions are then given precisely by the Bianchi type $VIII$ or IX Lozanovski-Aarons-Carminati (LAC) metrics [57, 58, 59], with a possible degeneracy to Bianchi type II (see below).

Henceforth we take \boldsymbol{E} non zero (both the type I and type D purely magnetic models above will reappear as $\boldsymbol{E} = 0$ special cases). Assuming that \boldsymbol{H} is diagonal in the $(\boldsymbol{\sigma}, \boldsymbol{E})$ -eigen-frame, $H_{\alpha\beta} = \text{diag}(-H_{22} - H_{33}, H_{22}, H_{33})$, (3.68) implies

$$r_1 - q_1 = r_3 = q_2 = 0, \quad (3.75)$$

while the (23)-component of (3.11) shows that $r_1 + q_1 = 0$ and hence $r_1 = q_1 = 0$.

At this point the only variables having possibly non-vanishing spatial derivatives are q_3, r_2 and n_1, n_2, n_3 . With the simplifications obtained so far, the (11)-Einstein equation (3.66) reduces to

$$0 = \frac{1}{3}\mu + \frac{1}{3}\theta(\sigma - \frac{1}{3}\theta) + \frac{2}{3}\sigma^2 - E - n_2 n_3. \quad (3.76)$$

Eliminating now $\dot{E}_{\alpha\beta}$ from (3.8) for $\alpha = \beta = 2$ and $\alpha = \beta = 3$, we obtain

$$0 = H_{11}(-2n_1 + n_2 - 2n_3) + H_{33}(-4n_1 - n_2 - n_3),$$

which in combination with (3.11) leads to

$$(n_1 + n_2 + n_3)(n_2 - n_3) = 0. \quad (3.77)$$

We will discuss the three cases that follow from (3.77) separately; i.e. $n_2 \neq n_3$, $n_2 = n_3 \neq 0$ and $n_2 = n_3 = 0$. Notice that when $n_2 = n_3$ the necessary and sufficient conditions for the fluid to be locally rotationally symmetric are automatically satisfied [40].

Degenerate shear, $n_2 - n_3 \neq 0$

By (3.77) we have $n_1 + n_2 + n_3 = 0$.

From (3.62) we get the evolution of n_α , namely

$$\begin{aligned} \partial_0 n_1 &= \sigma(n_2 + n_3) - \frac{1}{3}n_1\theta \\ \partial_0 n_2 &= -\sigma(n_2 + 3n_3) - \frac{1}{3}n_2\theta. \end{aligned} \quad (3.78)$$

Herewith we obtain, propagating $n_1 + n_2 + n_3$ along \mathbf{e}_0 ,

$$0 = \partial_0(n_1 + n_2 + n_3) = -3(n_2 + n_3)\sigma$$

and thus $n_1 = 0$ and $n_2 = -n_3 \neq 0$. Applying ∂_3 to (3.76) we find $\partial_3 n_2 = 0$. On the other hand, calculating $\partial_3 n_2$ from the (12)-Einstein equation (3.67) and the Jacobi equation (3.60), results in $\partial_3 n_2 = q_3(n_3 - n_2)$. As $n_2 \neq n_3$ by assumption, it follows that $q_3 = 0$. By (3.67) and (3.60) we have $\partial_1 q_3 = r_2 n_2$ whence also $r_2 = 0$. So r_α and q_α all become zero and we have

$$\begin{aligned} n_{\alpha\beta} &= \text{diag}(0, -n_2, n_2) \quad (n_2 \neq 0) \\ a_\alpha &= 0. \end{aligned}$$

This implies the vanishing of the spatial derivatives of all rotation coefficients ($\partial_\alpha \equiv 0$) and we obtain a spatially homogeneous universe of Bianchi class A, type VI_0 . Examples are the metrics without rotational symmetry in [19].

Degenerate shear, $n_2 = n_3 \neq 0$

From (3.76) and the (33)-Einstein equation (3.66) we can calculate

$$\partial_2 r_2 - \partial_3 q_3 \equiv \mu + 3S^2 - \frac{1}{3}\theta^2 - n_2^2 - 2n_2 n_1 + q_3^2 + r_2^2. \quad (3.79)$$

At this stage $n_{\alpha\beta}$ and a_α are given by

$$n_{\alpha\beta} = \begin{bmatrix} 2n_2 & q_3/2 & r_2/2 \\ q_3/2 & n_1 + n_2 & 0 \\ r_2/2 & 0 & n_1 + n_2 \end{bmatrix}, \quad a_\alpha = \begin{bmatrix} 0 \\ r_2/2 \\ -q_3/2 \end{bmatrix}.$$

Now $\partial_1 n_2 = 0$ by the Jacobi equation (3.60), while $\partial_2 n_2 = 0$ and $\partial_3 n_2 = 0$ follow by combining respectively the (31)-Einstein equation (3.67) and the (12)-Einstein equation (3.67) with the Jacobi equation (3.60), so the spatial derivatives of n_2 vanish. We now try to obtain the Bianchi A condition $a_\alpha = 0$: remembering that we still have a rotational degree of freedom left (namely rotations in the (23)-plane about an angle φ satisfying $\partial_0 \varphi = 0$), we use the following transformation formulas for the quantities $n_{\alpha\beta}$ and a_α : $n_{\alpha\beta} \rightarrow n_{\alpha\beta}^r$, $a_\alpha \rightarrow a_\alpha^r$ with

$$\begin{aligned} n_{11}^r &= n_{11}, \\ n_{22}^r &= n_{33}^r = n_{22} - \partial_1 \varphi, \\ n_{12}^r &= \frac{1}{2}[\cos(\varphi)(\partial_2 \varphi + q_3) - \sin(\varphi)(\partial_3 \varphi + r_2)], \\ n_{13}^r &= \frac{1}{2}[\cos(\varphi)(\partial_3 \varphi + r_2) + \sin(\varphi)(\partial_2 \varphi + q_3)] \end{aligned} \quad (3.80)$$

and

$$n_{23}^r = a_1^r = a_2^r - n_{13}^r = a_3^r + n_{12}^r = 0. \quad (3.81)$$

A rotation making $a_\alpha = 0$ and $n_{\alpha\beta}$ diagonal can then be obtained by $\partial_2\varphi = -q_3$ and $\partial_3\varphi = -r_2$, the integrability condition of which leads to $\partial_1\varphi = -\frac{K}{2n_2} + n_1 + n_2$, with

$$K \equiv \mu + 3\sigma^2 - \frac{1}{3}\theta^2 + n_2^2. \quad (3.82)$$

One can check that the integrability conditions for the resulting system of pde's

$$\begin{aligned} \partial_0\varphi &= 0, \\ \partial_1\varphi &= -\frac{K}{2n_2} + n_1 + n_2, \\ \partial_2\varphi &= -q_3, \\ \partial_3\varphi &= -r_2 \end{aligned} \quad (3.83)$$

are identically satisfied. For the rotated variable n_1 we find $n_1 + n_2 = \frac{K}{2n_2}$: all spatial derivatives now vanish and we obtain the spatially homogeneous LRS Bianchi class A space-times of types *II* ($K = 0$), *VIII* ($K < 0$) and *IX* ($K > 0$). An integrable sub-case arises when $\mu = 3n_2^2 - \theta(\sigma - \theta/3) - 2\sigma^2$: the electric part of the Weyl tensor is then 0 and we obtain the purely magnetic LAC metrics [57, 58, 59], for which the Bianchi type is *VIII* or *IX*, reducing to type *II* for the $p = \mu/5$ Collins-Stewart metric [21]. As $\boldsymbol{E} = 0$ the label ‘purely radiative’ is however not fully appropriate for these models.

Of course the above reasoning breaks down when $n_2 = n_3 = 0$, a case which will be dealt with below.

Degenerate shear, $n_2 = n_3 = 0$

Within the previous class the $n_2 = n_3 = 0$ models are exceptional in the sense that they are ‘purely electric’ (by (3.11) one has $\boldsymbol{H} = 0$) and again the terminology ‘purely radiative’ is somewhat inappropriate. For completeness however we present the full details of the resulting special cases.

As in the previous sub-section ($n_2 = n_3 \neq 0$) one shows that a rotation exists under which r_2 , n_1 and $\partial_3 q_3$ become zero. By (3.76) this implies $q_3^2 + K = 0$, where K is the same quantity as defined by (3.82). The rotation is now determined by a solution of the following system:

$$\begin{aligned}
\partial_0 \varphi &= 0, \\
\partial_1 \varphi &= n_1, \\
\partial_2 \varphi &= \cos(\varphi) \sqrt{-K} - q_3, \\
\partial_3 \varphi &= -\sin(\varphi) \sqrt{-K} - r_2
\end{aligned} \tag{3.84}$$

for which again the integrability conditions are identically satisfied.

When $K < 0$, one obtains

$$n_{\alpha\beta} = \begin{bmatrix} 0 & \sqrt{-K}/2 & 0 \\ \sqrt{-K}/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad a_\alpha = \begin{bmatrix} 0 \\ 0 \\ -\sqrt{-K}/2 \end{bmatrix},$$

which is a LRS spatially homogeneous Bianchi class B type *III* model [35].

When $K = 0$ the rotation (3.84) results in $a_\alpha = 0$, $n_{\alpha\beta} = 0$, which gives a LRS spatially homogeneous Bianchi class A type *I* model (equivalent to Bianchi class A type *VII*₀ [35]).

A problem arises when $K > 0$: q_3 , n_1 and n_2 can still be made zero by a rotation, but it is no longer possible to choose a frame in which all spatial derivatives identically vanish. However the Cartan equations can now be integrated easily and one finds the Kantowski-Sachs perfect fluids, with metric given by

$$ds^2 = -dt^2 + Q(t)^2 dr^2 + P(t)^2 [d\theta^2 + \sin^2(\theta) d\phi^2]$$

and $P(Q\ddot{P} - P\ddot{Q}) + \dot{P}(Q\dot{P} - P\dot{Q}) + Q = 0$. As is well known these metrics do not admit a 3-dimensional isometry group acting simply transitively on the $t = \text{constant}$ hyper-surfaces.

Chapter 4

General conclusion

Shear-free perfect fluids

Essentially, there are two fundamental obstacles associated with work on the shear-free perfect fluid conjecture. First, one is faced with, in the tetrad formalism, the "intelligent" manipulation of a large number of combined field equations, which incorporate many unknowns and their derivatives, with the aim of generating as many as possible polynomial constraint equations. These polynomials usually result from integrability conditions obtained from commutators or from spatial/temporal propagation of existing constraints. Secondly, one is then required to analyse these polynomial conditions, with the possible additional re-use of the remaining differential conditions, to establish the conjecture. In addition to the problem of numerous differential equations which can be quite lengthy, one is usually also faced with very large polynomials. Unless considerable care is exercised these can very quickly exceed the capacity of modern day computers running any one of the currently available algebraic computing systems.

By a careful selection of evolution equations and with the assistance of the Maple symbolic packages Oframe [81], we have been able to overcome these obstacles in the γ -law case [see section 2.4.1] with $\text{div}\mathbf{H} = 0$ and in the general barotropic case [see section 2.4.2] with the exception of certain quite restricted special cases when there also exists a Killing vector aligned with the vorticity and for which the magnitude of vorticity ω is not a function of the matter density μ alone. In these restricted cases the equation of state must satisfy an over-determined differential system, which we conjecture to admit no solutions, but which we so far have been unable to prove.

We have also generalised the result [12, 55] that the conjecture holds when

the electric part of the Weyl tensor vanishes, by considering space-times for which the electric part is solenoidal. Specifically, we have proved that the vorticity or the expansion vanishes for any shear-free perfect fluid solution of the Einstein field equations where the pressure satisfies a barotropic equation of state and the spatial divergence of the electric part of the Weyl tensor is zero.

Assuming the validity of the conjecture, we have given a partial classification of shear-free and non-expanding (hence stationary) rotating perfect fluids for which $\text{div} \mathbf{H} = 0$, thereby generalising the classification of Collins [16]. It is fairly straightforward to show then that in the canonical frame constructed (with $\Omega_A = \omega_A = 0$ and $n_{11} = n_{22} = n$) necessarily $E_{12}n_{12} = 0$. This is treated in two cases: i) $E_{12} = 0 \neq n_{12}$ and ii) $E_{12} = 0 = n_{12}$. Notice that if $n_{12} = 0$ there is a remaining rotational freedom as

$$\partial_0 \left(\frac{E_{11} - E_{22}}{E_{12}} \right) = 0$$

allows us to put $E_{12} = 0$. In the class (i) there exist solutions which form a genuine sub-case of the Collins and White [23] class IIIAGi and have Petrov type I. Most solutions we have found belong to the class (ii) where we have recovered all solutions for stiff fluids presented by Collins [16]. We have also found solutions under equations of state either $F = 0$ (when $\dot{u}_3 = 0$, see page 70), $\Phi = 0$ (when $\dot{u}_3 \neq 0 = \dot{u}_1$, see page 79) or $\Psi = 0$ (when $\dot{\mathbf{u}} \parallel \boldsymbol{\omega}$, see page 82) in which cases the magnetic part of Weyl tensor is not zero. Substituting the γ -law condition $G = -p' + 1/3$, $G' = 0$, $p = p'\mu + c$ into the expressions $F = 0$, $\Psi = 0$ gives two physically plausible solutions with $p' = 7/11$ and $p' = 1/5$ respectively. The tree 4.1 shows all possible solutions, together with some relevant properties.

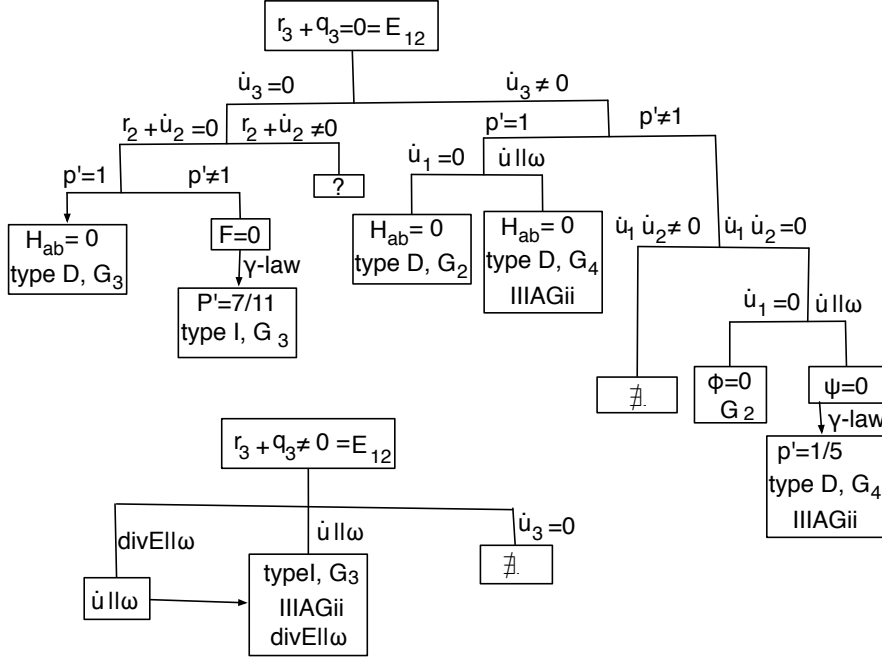


Figure 4.1: Non-expanding rotating perfect fluids; the symbol \nexists indicates that no solutions exist

Purely radiative space-times

We have shown that Bianchi class A perfect fluid models with non-constant pressure can be uniquely characterized as geodesic perfect fluid space-times which are purely radiative in the sense that the gravitational field satisfies $\text{div} \mathbf{E} = \text{div} \mathbf{H} = 0$, under the assumption that also the magnetic part of the Weyl tensor $H_{\alpha\beta}$ is diagonal in the shear-electric eigen-frame (i.e. $[\mathbf{H}, \boldsymbol{\sigma}] = [\mathbf{H}, \mathbf{E}] = 0$). The only possible exception arises in the purely electric case of degenerate shear and Petrov type D, when $K \equiv \mu + 3\sigma^2 - \frac{1}{3}\theta^2 > 0$, in which case the allowed solutions are the Kantowski-Sachs universes.

We have also proved that a geodesic and non-rotating perfect fluid (where ‘non-rotating’ can be dropped when we have non-constant pressure), with degenerate shear tensor $\boldsymbol{\sigma}$ and with $\text{div} \mathbf{E} = \text{div} \mathbf{H} = 0$ has commuting \mathbf{H} , \mathbf{E} and $\boldsymbol{\sigma}$. This implies that the resulting space-times are spatially homogeneous of Bianchi class A. More particularly, because of the degeneracy of $\boldsymbol{\sigma}$, they are of type VI₀ (in the non-LRS case), or, in the LRS case, of Bianchi types

I (VII₀), II, VIII or IX. The only exceptions arise in the purely electric case, where also the pseudo-spherically symmetric Bianchi class B type III space-times and the Kantowski-Sachs space-times are allowed.

The next logical step is to investigate the case where \mathbf{H} , \mathbf{E} and σ are not simultaneously diagonalizable: it is hoped that here new classes of cosmologically interesting solutions will turn up, or that one would be able to demonstrate the remarkable result that the Bianchi class A space-times are the unique purely radiative ones (work still in progress). Aside from the fact that such a characterization of the Bianchi A models would be quite neat, we feel that a further investigation of restrictions on $\text{div}\mathbf{E}$ and $\text{div}\mathbf{H}$ and of their role played in general relativistic perfect fluids might shed a new light on the outstanding problem on the issue of PUTH [7, 8, 15, 82] and observational homogeneity.

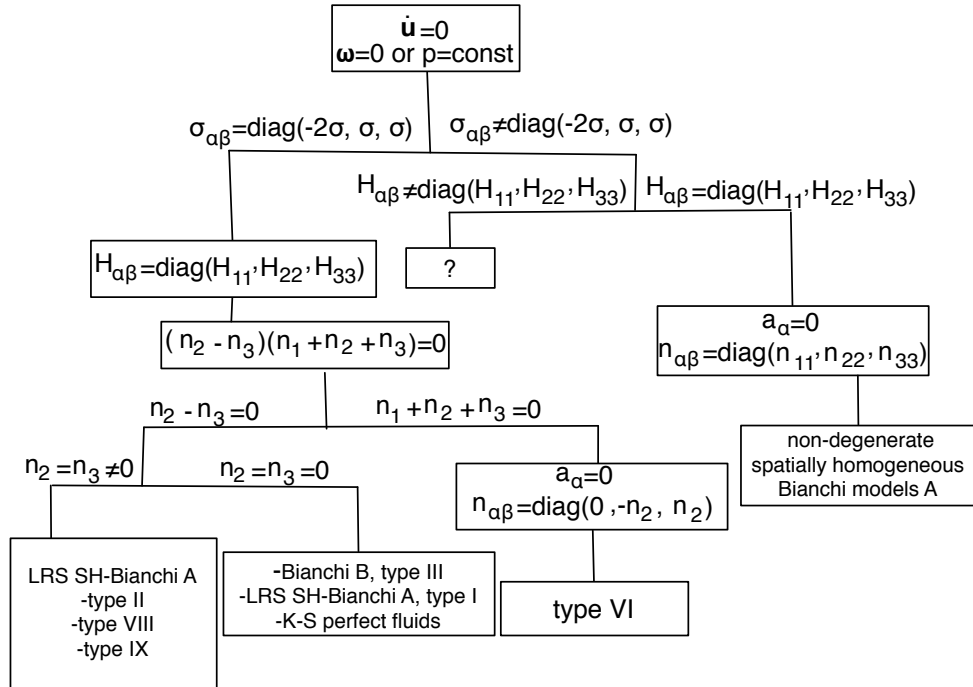


Figure 4.2: Purely Radiative Space-times (SH=spatially homogeneous and K-S=Kantowski-Sachs)

Appendix A

Jacobi equations and Einstein field equations in the orthonormal formalism without any restrictions on the tetrads or the kinematic quantities.

Jacobi equations

$$\partial_1\omega_1 + \partial_2\omega_2 + \partial_3\omega_3 = \omega_1(\dot{u}_1 + 2a_1) + \omega_2(\dot{u}_2 + 2a_2) + \omega_3(\dot{u}_3 + 2a_3), \quad (\text{A.1})$$

$$\begin{aligned} &\partial_1 n_{11} + \partial_2(n_{12} + a_3) + \partial_3(n_{13} - a_2) - 2(\theta_{11}\omega_1 + \sigma_{12}\omega_2 + \sigma_{13}\omega_3) \\ &\quad - 2(n_{11}a_1 + n_{12}a_2 + n_{13}a_3) - 2(\omega_2\Omega_3 - \omega_3\Omega_2) = 0, \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned} &\partial_1(n_{12} - a_3) + \partial_2 n_{22} + \partial_3(n_{23} + a_1) - 2(\sigma_{12}\omega_1 + \theta_{22}\omega_2 + \sigma_{23}\omega_3) \\ &\quad - 2(n_{12}a_1 + n_{22}a_2 + n_{23}a_3) - 2(\omega_3\Omega_1 - \omega_1\Omega_3) = 0, \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned} &\partial_1(n_{13} + a_2) + \partial_2(n_{23} - a_1) + \partial_3 n_{33} - 2(\sigma_{13}\omega_1 + \sigma_{23}\omega_2 + \theta_{33}\omega_3) \\ &\quad - 2(n_{13}a_1 + n_{23}a_2 + n_{33}a_3) - 2(\omega_1\Omega_2 - \omega_2\Omega_1) = 0, \end{aligned} \quad (\text{A.4})$$

$$\begin{aligned} &2\partial_0\omega_1 + \partial_2\dot{u}_3 - \partial_3\dot{u}_2 - n_{11}\dot{u}_1 - n_{12}\dot{u}_2 - n_{13}\dot{u}_3 - a_2\dot{u}_3 + a_3\dot{u}_2 \\ &\quad + 2\theta\omega_1 - 2(\omega_2\Omega_3 - \omega_3\Omega_2) - 2(\theta_{11}\omega_1 + \sigma_{12}\omega_2 + \sigma_{13}\omega_3) = 0, \end{aligned} \quad (\text{A.5})$$

$$\begin{aligned} &2\partial_0\omega_2 + \partial_3\dot{u}_1 - \partial_1\dot{u}_3 - n_{12}\dot{u}_1 - n_{22}\dot{u}_2 - n_{23}\dot{u}_3 - a_3\dot{u}_1 + a_1\dot{u}_3 \\ &\quad + 2\theta\omega_2 - 2(\omega_3\Omega_1 - \omega_1\Omega_3) - 2(\sigma_{12}\omega_1 + \theta_{22}\omega_2 + \sigma_{23}\omega_3) = 0, \end{aligned} \quad (\text{A.6})$$

$$\begin{aligned} &2\partial_0\omega_3 + \partial_1\dot{u}_2 - \partial_2\dot{u}_1 - n_{13}\dot{u}_1 - n_{23}\dot{u}_2 - n_{33}\dot{u}_3 - a_1\dot{u}_2 + a_2\dot{u}_1 \\ &\quad + 2\theta\omega_3 - 2(\omega_1\Omega_2 - \omega_2\Omega_1) - 2(\sigma_{13}\omega_1 + \sigma_{23}\omega_2 + \theta_{33}\omega_3) = 0, \end{aligned} \quad (\text{A.7})$$

$$\begin{aligned} &2\partial_0 a_1 - \partial_1(\theta_{11} - \theta) - \partial_2(\sigma_{12} - \omega_3 - \Omega_3) - \partial_3(\sigma_{13} + \omega_2 + \Omega_2) + \theta\dot{u}_1 \\ &\quad + \theta_{11}(2a_1 - \dot{u}_1) + \sigma_{12}(2a_2 - \dot{u}_2) + \sigma_{13}(2a_3 - \dot{u}_3) - (2a_2 - \dot{u}_2)(\omega_3 + \Omega_3) \\ &\quad + (2a_3 - \dot{u}_3)(\omega_2 + \Omega_2) = 0, \end{aligned} \quad (\text{A.8})$$

$$\begin{aligned} &2\partial_0 a_2 - \partial_1(\sigma_{12} + \omega_3 + \Omega_3) - \partial_2(\theta_{22} - \theta) - \partial_3(\sigma_{23} - \omega_1 - \Omega_1) + \theta\dot{u}_2 \\ &\quad + \theta_{22}(2a_2 - \dot{u}_2) + \sigma_{12}(2a_1 - \dot{u}_1) + \sigma_{23}(2a_3 - \dot{u}_3) - (2a_3 - \dot{u}_3)(\omega_1 + \Omega_1) \\ &\quad + (2a_1 - \dot{u}_1)(\omega_3 + \Omega_3) = 0, \end{aligned} \quad (\text{A.9})$$

$$\begin{aligned}
& 2\partial_0 a_3 - \partial_1(\sigma_{13} - \omega_2 - \Omega_2) - \partial_2(\sigma_{23} + \omega_1 + \Omega_1) - \partial_3(\theta_{33} - \theta) + \theta \dot{u}_3 \\
& + \theta_{33}(2a_3 - \dot{u}_3) + \sigma_{13}(2a_1 - \dot{u}_1) + \sigma_{23}(2a_2 - \dot{u}_2) - (2a_1 - \dot{u}_1)(\omega_2 + \Omega_2) \\
& + (2a_2 - \dot{u}_2)(\omega_1 + \Omega_1) = 0, \tag{A.10}
\end{aligned}$$

$$\begin{aligned}
& \partial_0 n_{11} + \partial_1 \omega_1 + \partial_2(\sigma_{13} - \Omega_2) - \partial_3(\sigma_{12} + \Omega_3) + \dot{u}_2 \sigma_{13} - \dot{u}_3 \sigma_{12} \\
& - n_{11}(2\theta_{11} - \theta) - 2n_{12}(\sigma_{12} + \omega_3 + \Omega_3) - 2n_{13}(\sigma_{13} - \omega_2 - \Omega_2) + \dot{u}_1 \omega_1 \\
& - \dot{u}_2 \Omega_2 - \dot{u}_3 \Omega_3 - 2\omega_1(\dot{u}_1 + a_1) - 2\omega_2(\dot{u}_2 + a_2) - 2\omega_3(\dot{u}_3 + a_3) = 0, \tag{A.11}
\end{aligned}$$

$$\begin{aligned}
& \partial_0 n_{22} - \partial_1(\sigma_{23} + \Omega_1) + \partial_2 \omega_2 + \partial_3(\sigma_{12} - \Omega_3) + \dot{u}_3 \sigma_{12} - \dot{u}_1 \sigma_{23} \\
& - n_{22}(2\theta_{22} - \theta) - 2n_{12}(\sigma_{12} - \omega_3 - \Omega_3) - 2n_{23}(\sigma_{23} + \omega_1 + \Omega_1) - \dot{u}_1 \Omega_1 \\
& + \dot{u}_2 \omega_2 - \dot{u}_3 \Omega_3 - 2\omega_1(\dot{u}_1 + a_1) - 2\omega_2(\dot{u}_2 + a_2) - 2\omega_3(\dot{u}_3 + a_3) = 0, \tag{A.12}
\end{aligned}$$

$$\begin{aligned}
& \partial_0 n_{33} + \partial_1(\sigma_{23} - \Omega_1) - \partial_2(\sigma_{13} + \Omega_2) + \partial_3 \omega_3 + \dot{u}_1 \sigma_{23} - \dot{u}_2 \sigma_{13} \\
& - n_{33}(2\theta_{33} - \theta) - 2n_{13}(\sigma_{13} + \omega_2 + \Omega_2) - 2n_{23}(\sigma_{23} - \omega_1 - \Omega_1) - \dot{u}_1 \Omega_1 \\
& - \dot{u}_2 \Omega_2 + \dot{u}_3 \omega_3 - 2\omega_1(\dot{u}_1 + a_1) - 2\omega_2(\dot{u}_2 + a_2) - 2\omega_3(\dot{u}_3 + a_3) = 0, \tag{A.13}
\end{aligned}$$

$$\begin{aligned}
& \partial_0 n_{12} - \partial_1(\sigma_{13} - \omega_2 - \Omega_2)/2 + \partial_2(\sigma_{23} + \omega_1 + \Omega_1)/2 + \partial_3(\theta_{11} - \theta_{22})/2 \\
& - n_{13}(\omega_1 + \Omega_1) + n_{23}(\omega_2 + \Omega_2) + (n_{11} - n_{22})(\omega_3 + \Omega_3) + \dot{u}_3(\theta_{11} - \theta_{22})/2 \\
& - \dot{u}_1(\sigma_{13} - \omega_2 - \Omega_2)/2 + \dot{u}_2(\sigma_{23} + \omega_1 + \Omega_1)/2 - (n_{11} + n_{22})\sigma_{12} \\
& + n_{12}\theta_{33} - n_{13}\sigma_{23} - n_{23}\sigma_{13} = 0, \tag{A.14}
\end{aligned}$$

$$\begin{aligned}
& \partial_0 n_{13} + \partial_1(\sigma_{12} + \omega_3 + \Omega_3)/2 + \partial_2(\theta_{33} - \theta_{11})/2 - \partial_3(\sigma_{23} - \omega_1 - \Omega_1)/2 \\
& - n_{23}(\omega_3 + \Omega_3) + n_{12}(\omega_1 + \Omega_1) + (n_{33} - n_{11})(\omega_2 + \Omega_2) + \dot{u}_2(\theta_{33} - \theta_{11})/2 \\
& - \dot{u}_3(\sigma_{23} - \omega_1 - \Omega_1)/2 + \dot{u}_1(\sigma_{12} + \omega_3 + \Omega_3)/2 - (n_{11} + n_{33})\sigma_{13} \\
& + n_{13}\theta_{22} - n_{12}\sigma_{23} - n_{23}\sigma_{12} = 0, \tag{A.15}
\end{aligned}$$

$$\begin{aligned}
& \partial_0 n_{23} + \partial_1(\theta_{22} - \theta_{33})/2 - \partial_2(\sigma_{12} - \omega_3 - \Omega_3)/2 + \partial_3(\sigma_{13} + \omega_2 + \Omega_2)/2 \\
& - n_{12}(\omega_2 + \Omega_2) + n_{13}(\omega_3 + \Omega_3) + (n_{22} - n_{33})(\omega_1 + \Omega_1) + \dot{u}_1(\theta_{22} - \theta_{33})/2 \\
& - \dot{u}_2(\sigma_{12} - \omega_3 - \Omega_3)/2 + \dot{u}_3(\sigma_{13} + \omega_2 + \Omega_2)/2 - \sigma_{23}(n_{22} + n_{33}) \\
& + n_{23}\theta_{11} - n_{12}\sigma_{13} - n_{13}\sigma_{12} = 0. \tag{A.16}
\end{aligned}$$

The Einstein field equations

$$\begin{aligned}
& \partial_0 \theta - \partial_1 \dot{u}_1 - \partial_2 \dot{u}_2 - \partial_3 \dot{u}_3 + \theta_{11}^2 + \theta_{22}^2 + \theta_{33}^2 + 2(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2) \\
& - 2(\omega_1^2 + \omega_2^2 + \omega_3^2) + 2(a_1 \dot{u}_1 + a_2 \dot{u}_2 + a_3 \dot{u}_3) - \dot{u}_1^2 \\
& - \dot{u}_2^2 - \dot{u}_3^2 = -(\mu + 3p)/2, \tag{A.17}
\end{aligned}$$

$$\begin{aligned}
& \partial_1(\theta - \theta_{11}) - \partial_2(\sigma_{12} - \omega_3) - \partial_3(\sigma_{13} + \omega_2) + 3(a_1\theta_{11} + a_2\sigma_{12} + a_3\sigma_{13}) \\
& - a_1\theta - \omega_1n_{11} - \omega_2n_{12} - \omega_3n_{13} + \omega_2(a_3 - 2\dot{u}_3) - \omega_3(a_2 - 2\dot{u}_2) \\
& - n_{13}\sigma_{12} + n_{12}\sigma_{13} + \sigma_{23}(n_{22} - n_{33}) + n_{23}(\theta_{33} - \theta_{22}) = 0, \quad (\text{A.18})
\end{aligned}$$

$$\begin{aligned}
& \partial_2(\theta - \theta_{22}) - \partial_1(\sigma_{12} + \omega_3) - \partial_3(\sigma_{23} - \omega_1) + 3(a_1\sigma_{12} + a_2\theta_{22} + a_3\sigma_{23}) \\
& - a_2\theta - \omega_1n_{12} - \omega_2n_{22} - \omega_3n_{23} + \omega_3(a_1 - 2\dot{u}_1) - \omega_1(a_3 - 2\dot{u}_3) \\
& - n_{12}\sigma_{23} + n_{23}\sigma_{12} + \sigma_{13}(n_{33} - n_{11}) + n_{13}(\theta_{11} - \theta_{33}) = 0, \quad (\text{A.19})
\end{aligned}$$

$$\begin{aligned}
& \partial_3(\theta - \theta_{33}) - \partial_1(\sigma_{13} - \omega_2) - \partial_2(\sigma_{23} + \omega_1) + 3(a_1\sigma_{13} + a_2\sigma_{23} + a_3\theta_{33}) \\
& - a_3\theta - \omega_1n_{13} - \omega_2n_{23} - \omega_3n_{33} + \omega_1(a_2 - 2\dot{u}_2) - \omega_2(a_1 - 2\dot{u}_1) \\
& - n_{23}\sigma_{13} + n_{13}\sigma_{23} + \sigma_{12}(n_{11} - n_{22}) + n_{12}(\theta_{22} - \theta_{11}) = 0, \quad (\text{A.20})
\end{aligned}$$

$$\begin{aligned}
& -2\partial_1a_1 - \partial_2(a_2 - n_{13}) - \partial_3(a_3 + n_{12}) - 2(a_2n_{13} - a_3n_{12}) - (n_{11}^2 - n_{22}^2 - n_{33}^2)/2 \\
& - n_{22}n_{33} + 2n_{23}^2 + 2(a_1^2 + a_2^2 + a_3^2) = \partial_0\theta_{11} - \partial_1\dot{u}_1 - \dot{u}_1^2 - \dot{u}_1a_1 \\
& + n_{12}\dot{u}_3 - n_{13}\dot{u}_2 + \theta\theta_{11} + 2(\sigma_{13}\Omega_2 - \sigma_{12}\Omega_3) + 2\Omega_1\omega_1 + a_1\dot{u}_1 \\
& + a_2\dot{u}_2 + a_3\dot{u}_3 - 2(\omega_1\Omega_1 + \omega_2\Omega_2 + \omega_3\Omega_3) + (p - \mu)/2, \quad (\text{A.21})
\end{aligned}$$

$$\begin{aligned}
& -2\partial_2a_2 - \partial_1(n_{23} + a_1) - \partial_3(a_3 - n_{12}) - 2(a_3n_{12} - a_1n_{23}) - (n_{22}^2 - n_{11}^2 - n_{33}^2)/2 \\
& - n_{11}n_{33} + 2n_{13}^2 + 2(a_1^2 + a_2^2 + a_3^2) = \partial_0\theta_{22} - \partial_2\dot{u}_2 - \dot{u}_2^2 - \dot{u}_2a_2 \\
& + n_{23}\dot{u}_1 - n_{12}\dot{u}_3 + \theta\theta_{22} + 2(\sigma_{12}\Omega_3 - \sigma_{23}\Omega_1) + 2\Omega_2\omega_2 + a_1\dot{u}_1 \\
& + a_2\dot{u}_2 + a_3\dot{u}_3 - 2(\omega_1\Omega_1 + \omega_2\Omega_2 + \omega_3\Omega_3) + (p - \mu)/2, \quad (\text{A.22})
\end{aligned}$$

$$\begin{aligned}
& -\partial_3a_3 - \partial_1(a_1 - n_{23}) - \partial_2(a_2 + n_{13}) - 2(a_1n_{23} - a_2n_{13}) - (n_{33}^2 - n_{11}^2 - n_{22}^2)/2 \\
& - n_{11}n_{22} + 2n_{12}^2 + 2(a_1^2 + a_2^2 + a_3^2) = \partial_0\theta_{33} - \partial_3\dot{u}_3 - \dot{u}_3^2 - \dot{u}_3a_3 \\
& + n_{13}\dot{u}_2 - n_{23}\dot{u}_1 + \theta\theta_{33} + 2(\sigma_{23}\Omega_1 - \sigma_{13}\Omega_2) + 2\Omega_3\omega_3 + a_1\dot{u}_1 \\
& + a_2\dot{u}_2 + a_3\dot{u}_3 - 2(\omega_1\Omega_1 + \omega_2\Omega_2 + \omega_3\Omega_3) + (p - \mu)/2, \quad (\text{A.23})
\end{aligned}$$

$$\begin{aligned}
& -\partial_1(a_2 + n_{13})/2 - \partial_2(a_1 - n_{23})/2 + \partial_3(n_{11} - n_{22})/2 - a_3(n_{11} - n_{22}) \\
& - a_2n_{23} + a_1n_{13} - n_{12}(n_{11} + n_{22} - n_{33}) - 2n_{13}n_{23} = \partial_0\sigma_{12} - \partial_1\dot{u}_2/2 \\
& - \partial_2\dot{u}_1/2 - \dot{u}_1\dot{u}_2 - (\dot{u}_1a_2 + \dot{u}_2a_1)/2 + \theta\sigma_{12} + \Omega_1\omega_2 + \Omega_2\omega_1 \\
& + [n_{13}\dot{u}_1 - n_{23}\dot{u}_2 + (n_{22} - n_{11})\dot{u}_3]/2 - \sigma_{13}\Omega_1 + \sigma_{23}\Omega_2 \\
& + (\theta_{11} - \theta_{22})\Omega_3, \quad (\text{A.24})
\end{aligned}$$

$$\begin{aligned}
& -\partial_1(a_3 - n_{12})/2 - \partial_3(a_1 + n_{23})/2 - \partial_2(n_{11} - n_{33})/2 - a_2(n_{33} - n_{11}) \\
& - a_1n_{12} + a_3n_{23} - n_{13}(n_{11} - n_{22} + n_{33}) - 2n_{12}n_{23} = \partial_0\sigma_{13} - \partial_1\dot{u}_3/2 \\
& - \partial_3\dot{u}_1/2 - \dot{u}_1\dot{u}_3 - (\dot{u}_1a_3 + \dot{u}_3a_1)/2 + \theta\sigma_{13} + \Omega_3\omega_1 + \Omega_1\omega_3 \\
& + [n_{23}\dot{u}_3 - n_{12}\dot{u}_1 + (n_{11} - n_{33})\dot{u}_2]/2 - \sigma_{23}\Omega_3 + \sigma_{12}\Omega_1 \\
& + (\theta_{33} - \theta_{11})\Omega_2, \quad (\text{A.25})
\end{aligned}$$

$$\begin{aligned}
& -\partial_2(a_3 + n_{12})/2 - \partial_3(a_2 - n_{13})/2 - \partial_1(n_{33} - n_{22})/2 - a_1(n_{22} - n_{33}) \\
& -a_3n_{13} + a_2n_{12} - n_{23}(n_{22} - n_{11} + n_{33}) - 2n_{12}n_{13} = \partial_0\sigma_{23} - \partial_2\dot{u}_3/2 \\
& -\partial_3\dot{u}_2/2 - \dot{u}_2\dot{u}_3 - (\dot{u}_2a_3 + \dot{u}_3a_2)/2 + \theta\sigma_{23} + \Omega_2\omega_3 + \Omega_3\omega_2 \\
& + [n_{12}\dot{u}_2 - n_{13}\dot{u}_3 + (n_{33} - n_{22})\dot{u}_1]/2 - \sigma_12\Omega_2 + \sigma_{13}\Omega_3 \\
& + (\theta_{22} - \theta_{33})\Omega_1.
\end{aligned} \tag{A.26}$$

Commutators

The commutators are described by the expressions for γ^a_{bc} . Their 1+3 decomposition leads to

$$[\mathbf{e}_0, \mathbf{e}_1] = \dot{u}^1\mathbf{e}_0 - \theta_{11}\mathbf{e}_1 - (\sigma_{12} - \omega_3 - \Omega_3)\mathbf{e}_2 - (\sigma_{13} + \omega_2 + \Omega_2)\mathbf{e}_3, \tag{A.27}$$

$$[\mathbf{e}_0, \mathbf{e}_2] = \dot{u}^2\mathbf{e}_0 - (\sigma_{12} + \omega_3 + \Omega_3)\mathbf{e}_1 - \theta_{22}\mathbf{e}_2 - (\sigma_{23} - \omega_1 - \Omega_1)\mathbf{e}_3, \tag{A.28}$$

$$[\mathbf{e}_0, \mathbf{e}_3] = \dot{u}^3\mathbf{e}_0 - (\sigma_{13} - \omega_2 - \Omega_2)\mathbf{e}_1 - (\sigma_{23} + \omega_1 + \Omega_1)\mathbf{e}_2 - \theta_{33}\mathbf{e}_3, \tag{A.29}$$

$$[\mathbf{e}_1, \mathbf{e}_2] = -2\omega_3\mathbf{e}_0 + (n_{13} - a_2)\mathbf{e}_1 + (n_{23} + a_1)\mathbf{e}_2 + n_{33}\mathbf{e}_3, \tag{A.30}$$

$$[\mathbf{e}_2, \mathbf{e}_3] = -2\omega_1\mathbf{e}_0 + n_{11}\mathbf{e}_1 + (n_{12} - a_3)\mathbf{e}_2 + (n_{13} + a_2)\mathbf{e}_3, \tag{A.31}$$

$$[\mathbf{e}_3, \mathbf{e}_1] = -2\omega_2\mathbf{e}_0 + (n_{12} + a_3)\mathbf{e}_1 + n_{22}\mathbf{e}_2 + (n_{23} - a_1)\mathbf{e}_3. \tag{A.32}$$

Appendix B

Jacobi equations, Einstein field equations and Bianchi identities in the orthonormal formalism when the vorticity vector, $\boldsymbol{\omega} = \omega \mathbf{e}_3$, is aligned with \mathbf{e}_3 , the shear tensor is zero and (1.52) is used.

Jacobi equations:

$$\partial_3 \omega = \omega(\dot{u}_3 + r_3 - q_3), \quad (\text{B.1})$$

$$\partial_1 n + \partial_2 r_3 + \partial_3 q_2 - n(r_1 - q_1) - r_3 r_2 + q_3 q_2 = 0, \quad (\text{B.2})$$

$$\partial_2 n + \partial_3 r_1 + \partial_1 q_3 - n(r_2 - n q_2) - r_3 r_1 + q_3 q_1 = 0, \quad (\text{B.3})$$

$$\partial_1 r_2 + \partial_2 q_1 + \partial_3 n_{33} - \frac{2}{3} \theta \omega - r_2 r_1 + q_2 q_1 - n_{33}(r_3 - q_3) = 0, \quad (\text{B.4})$$

$$\partial_2 \dot{u}_3 - \partial_3 \dot{u}_2 - n \dot{u}_1 - q_3 \dot{u}_2 - r_2 \dot{u}_3 = 0, \quad (\text{B.5})$$

$$\partial_3 \dot{u}_1 - \partial_1 \dot{u}_3 - n \dot{u}_2 - r_3 \dot{u}_1 - q_1 \dot{u}_3 = 0, \quad (\text{B.6})$$

$$2\partial_0 \omega + \partial_1 \dot{u}_2 - \partial_2 \dot{u}_1 - \dot{u}_1 q_2 - \dot{u}_2 r_1 - \dot{u}_3 n_{33} + \frac{4}{3} \theta \omega = 0, \quad (\text{B.7})$$

$$\partial_0(r_\alpha - q_\alpha) + \frac{2}{3} z_\alpha + \frac{\theta}{3}(r_\alpha - q_\alpha + 2\dot{u}_\alpha) = 0, \quad (\text{B.8})$$

$$3\partial_0 n + 3\partial_3 \omega + n\theta - 3\omega(\dot{u}_3 + r_3 - q_3) = 0, \quad (\text{B.9})$$

$$3\partial_0 n_{33} + 3\partial_3 \omega + n_{33}\theta - 3\omega(\dot{u}_3 + r_3 - q_3) = 0, \quad (\text{B.10})$$

$$\partial_0(r_\alpha + q_\alpha) + \frac{1}{3} \theta(r_\alpha + q_\alpha) = 0. \quad (\text{B.11})$$

Einstein equations:

$$\partial_0 \theta = -\frac{1}{3} \theta^2 + 2\omega^2 - \frac{1}{2}(\mu + 3p) + j, \quad (\text{B.12})$$

$$2/3 z_1 + \partial_2 \omega - \omega(r_2 - 2\dot{u}_2) = 0, \quad (\text{B.13})$$

$$2/3 z_2 - \partial_1 \omega - \omega(q_1 + 2\dot{u}_1) = 0, \quad (\text{B.14})$$

$$2/3 z_3 - \omega n_{33} = 0, \quad (\text{B.15})$$

$$\begin{aligned} & \partial_1 r_2 - \partial_2 q_1 + r_1 r_2 + q_1 q_2 + 2r_2 q_1 + 2r_3 n - n_{33}(r_3 + q_3) + 2q_3 n = \\ & \partial_1 \dot{u}_2 + \partial_2 \dot{u}_1 + 2\dot{u}_1 \dot{u}_2 - \dot{u}_1 q_2 + \dot{u}_2 r_1, \end{aligned} \quad (\text{B.16})$$

$$\begin{aligned} & \partial_2 r_3 - \partial_3 q_2 - \partial_1 n + \partial_1 n_{33} + n(r_1 - q_1) + 2q_1 n_{33} + r_2 r_3 + q_2 q_3 + 2r_3 q_2 = \\ & \partial_2 \dot{u}_3 + \partial_3 \dot{u}_2 - \dot{u}_1(n_{33} - n) + 2\dot{u}_2 \dot{u}_3 - \dot{u}_2 q_3 + \dot{u}_3 r_2, \end{aligned} \quad (\text{B.17})$$

$$\begin{aligned} & \partial_1 q_3 - \partial_3 r_1 + \partial_2 n_{33} - \partial_2 n - r_2(2n_{33} - n) - nq_2 - r_1 r_3 - q_1 q_3 - 2r_1 q_3 = \\ & -\partial_1 \dot{u}_3 - \partial_3 \dot{u}_1 - \dot{u}_1(2\dot{u}_3 + r_3) + \dot{u}_2(n - n_{33}) + \dot{u}_3 q_1, \end{aligned} \quad (\text{B.18})$$

$$\begin{aligned} & \partial_1 r_1 - \partial_1 q_1 - \partial_2 q_2 + \partial_3 r_3 - q_2^2 - r_3^2 - \frac{1}{2}n_{33}(n_{33} - 2n) - r_1^2 - q_1^2 + r_2 q_2 + r_3 q_3 = \\ & -\frac{2}{9}\theta^2 - \frac{8}{3}\omega^2 + \frac{1}{3}(2\mu - j) + \partial_1 \dot{u}_1 + \dot{u}_1^2 - \dot{u}_3 r_3 + \dot{u}_2 q_2, \end{aligned} \quad (\text{B.19})$$

$$\begin{aligned} & \partial_2 r_2 - \partial_2 q_2 + \partial_1 r_1 - \partial_3 q_3 - q_3^2 - r_1^2 - \frac{1}{2}n_{33}(n_{33} - 2n) - r_2^2 - q_2^2 + r_1 q_1 + r_3 q_3 = \\ & -\frac{2}{9}\theta^2 - \frac{8}{3}\omega^2 + \frac{1}{3}(2\mu - j) + \partial_2 \dot{u}_2 + \dot{u}_2^2 - \dot{u}_1 r_1 + \dot{u}_3 q_3, \end{aligned} \quad (\text{B.20})$$

$$\begin{aligned} & \partial_3 r_3 - \partial_3 q_3 - \partial_1 q_1 + \partial_2 r_2 - q_1^2 + r_2^2 + \frac{1}{2}n_{33}^2 - r_3^2 - q_3^2 + r_1 q_1 + r_2 q_2 = \\ & -\frac{2}{9}\theta^2 - \frac{1}{3}(2\omega^2 - 2\mu + j) + \partial_3 \dot{u}_3 + \dot{u}_3^2 - \dot{u}_2 r_2 + \dot{u}_1 q_1. \end{aligned} \quad (\text{B.21})$$

Magnetic components of the Weyl tensor:

$$\begin{aligned} H_{11} &= -\partial_1 \Omega_1 - \frac{1}{2}\partial_0(n_{11} + n_{22} + n_{33}) - \Omega_1 \dot{u}_1 - (r_2 + q_2)\Omega_2 \\ &\quad + (r_3 + q_3)\Omega_3 - (\dot{u}_3 - q_3)\omega - \frac{1}{6}\theta(n_{11} + n_{22} + n_{33}), \end{aligned} \quad (\text{B.22})$$

$$\begin{aligned} H_{22} &= -\partial_2 \Omega_2 + \frac{1}{2}\partial_0(n_{11} - n_{22} + n_{33}) + (r_1 + q_1)\Omega_1 - \dot{u}_2 \Omega_2 \\ &\quad - (r_3 + q_3)\Omega_3 - (\dot{u}_3 + r_3)\omega + \frac{1}{6}\theta(n_{11} - n_{22} + n_{33}), \end{aligned} \quad (\text{B.23})$$

$$\begin{aligned} H_{33} &= -\partial_3 \Omega_3 + \frac{1}{2}\partial_0(n_{11} + n_{22} - n_{33}) - (r_1 + q_1)\Omega_1 + (r_2 + q_2)\Omega_2 \\ &\quad - \dot{u}_3 \Omega_3 + \frac{1}{6}\theta(n_{11} + n_{22} - n_{33}), \end{aligned} \quad (\text{B.24})$$

$$\begin{aligned} H_{12} &= -\partial_1 \Omega_2 - \partial_0 r_3 + q_2 \Omega_1 - (\dot{u}_1 + q_1)\Omega_2 - (n_{11} - n_{22})\Omega_3 \\ &\quad - \frac{1}{3}(\dot{u}_3 + r_3)\theta - \frac{1}{2}\omega(n_{11} - n_{22} + n_{33}), \end{aligned} \quad (\text{B.25})$$

$$\begin{aligned} H_{13} &= -\partial_1 \Omega_3 - \partial_0 q_2 - r_3 \Omega_1 + (n_{11} - n_{33})\Omega_2 - (\dot{u}_1 - r_1)\Omega_3 \\ &\quad + (\dot{u}_1 + r_1)\omega + \frac{1}{3}(\dot{u}_2 - q_2)\theta, \end{aligned} \quad (\text{B.26})$$

$$\begin{aligned}
H_{23} = & -\partial_2\Omega_3 - \partial_0r_1 - (n_{22} - n_{33})\Omega_1 + q_3\Omega_2 - (\dot{u}_2 + q_2)\Omega_3 \\
& + (\dot{u}_2 - q_2)\omega - \frac{1}{3}(\dot{u}_1 + r_1)\theta.
\end{aligned} \tag{B.27}$$

Electric components of the Weyl tensor:

$$E_{11} = \partial_1\dot{u}_1 - \frac{1}{3}\partial_0\theta + \dot{u}_1^2 + q_2\dot{u}_2 - r_3\dot{u}_3 + \omega^2 - \frac{1}{9}\theta^2 - \frac{1}{6}(\mu + 3p) \tag{B.28}$$

$$E_{22} = \partial_2\dot{u}_2 - \frac{1}{3}\partial_0\theta + \dot{u}_2^2 - r_1\dot{u}_1 + q_3\dot{u}_3 + \omega^2 - \frac{1}{9}\theta^2 - \frac{1}{6}(\mu + 3p) \tag{B.29}$$

$$E_{33} = \partial_3\dot{u}_3 - \frac{1}{3}\partial_0\theta + \dot{u}_3^2 + q_1\dot{u}_1 - r_2\dot{u}_2 - \frac{1}{9}\theta^2 - \frac{1}{6}(\mu + 3p), \tag{B.30}$$

$$E_{12} = \partial_2\dot{u}_1 - \partial_0\omega + (r_1 + \dot{u}_1)\dot{u}_2 + \frac{1}{2}(n_{11} - n_{22} + n_{33})\dot{u}_3 - \frac{2}{3}\omega\theta \tag{B.31}$$

$$E_{21} = \partial_1\dot{u}_2 + \partial_0\omega - (q_2 + \dot{u}_2)\dot{u}_1 + \frac{1}{2}(n_{11} - n_{22} - n_{33})\dot{u}_3 + \frac{2}{3}\omega\theta \tag{B.32}$$

$$E_{13} = \partial_3\dot{u}_1 - \frac{1}{2}(n_{11} + n_{22} - n_{33})\dot{u}_2 - (q_1 - \dot{u}_1)\dot{u}_3 - \omega\Omega_1, \tag{B.33}$$

$$E_{31} = \partial_1\dot{u}_3 - \frac{1}{2}(n_{11} - n_{22} - n_{33})\dot{u}_2 + (r_3 + \dot{u}_3)\dot{u}_1 + \omega\Omega_1, \tag{B.34}$$

$$E_{23} = \partial_3\dot{u}_2 + \frac{1}{2}(n_{11} + n_{22} - n_{33})\dot{u}_1 + (r_2 + \dot{u}_2)\dot{u}_3 - \omega\Omega_2, \tag{B.35}$$

$$E_{32} = \partial_2\dot{u}_3 - \frac{1}{2}(n_{11} - n_{22} + n_{33})\dot{u}_1 - (q_3 - \dot{u}_3)\dot{u}_2 + \omega\Omega_2, \tag{B.36}$$

Bianchi identities:

$$\begin{aligned}
\partial_0E_{11} + \partial_3M_{12} - \partial_2M_{13} = & 2\Omega_3E_{12} + 2\dot{u}_2M_{13} - 2\dot{u}_3M_{12} - \frac{1}{2}M_{11}(3n_{11} + n_{22} \\
& - n_{33})M_{11} - (n_{22} - n_{33})M_{33} - 2\Omega_2E_{13} + \omega E_{12} - r_2M_{13} - q_3M_{12} \\
& + M_{23}(r_1 + q_1) - \frac{1}{3}\theta(2E_{11} - E_{22} - E_{33}),
\end{aligned} \tag{B.37}$$

$$\begin{aligned}
\partial_0E_{22} + \partial_1H_{23} - \partial_3H_{12} = & -2\Omega_3E_{12} + 2\dot{u}_3H_{12} - 2\dot{u}_1H_{23} + \frac{1}{4}(n_{11} + 3n_{22} \\
& - 3n_{33})H_{11} + \frac{1}{4}(n_{22} + n_{33} - 3n_{11})H_{33} + 2\Omega_1E_{23} - \frac{1}{4}(3n_{22} + n_{33} + n_{11})H_{22} \\
& - \omega E_{12} + (r_2 + q_2)H_{13} - r_3H_{12} - q_1H_{23} + \frac{1}{3}\theta(E_{11} - 2E_{22} + E_{33}),
\end{aligned} \tag{B.38}$$

$$\begin{aligned}
\partial_0 E_{12} + \partial_1 H_{13} - \partial_3 H_{11} &= \Omega_1 E_{13} - \Omega_2 E_{23} + \Omega_3 (E_{22} - E_{11}) + \dot{u}_2 H_{23} + \dot{u}_3 (H_{11} \\
&\quad - H_{22}) - \dot{u}_1 H_{13} - \frac{1}{2} (3n_{22} + n_{11} - n_{33}) H_{12} + \frac{1}{2} \omega \mathcal{E} + \omega (E_{33} - E_{11}) \\
&\quad - \theta E_{12} - r_3 (2H_{11} + H_{22}) - q_2 H_{23} - 2q_1 H_{13}, \tag{B.39}
\end{aligned}$$

$$\begin{aligned}
\partial_0 E_{13} + \partial_2 H_{11} - \partial_1 H_{12} &= -\Omega_1 E_{12} + \Omega_2 (E_{11} - E_{33}) + \Omega_3 E_{23} - 2\dot{u}_2 (H_{11} + H_{22}) \\
&\quad + \dot{u}_1 H_{12} - \dot{u}_3 H_{23} - \frac{1}{2} (n_{11} - n_{22} + 3n_{33}) H_{13} - \omega E_{23} - \theta E_{13} - r_3 H_{23} \\
&\quad - q_2 (H_{11} - H_{22}) - 2r_1 H_{12}, \tag{B.40}
\end{aligned}$$

$$\begin{aligned}
\partial_0 E_{23} - \partial_1 H_{22} + \partial_2 H_{12} &= \Omega_1 (E_{33} - E_{22}) + \Omega_2 E_{12} - \Omega_3 E_{13} + \dot{u}_1 (H_{22} - H_{33}) \\
&\quad - \dot{u}_2 H_{12} + \dot{u}_3 H_{13} + \frac{1}{2} (n_{11} - n_{22} - 3n_{33}) H_{23} + \omega E_{13} - \theta E_{23} - 2r_1 H_{22} \\
&\quad - 2q_2 H_{12} - q_3 H_{13} - r_1 H_{33}, \tag{B.41}
\end{aligned}$$

$$\begin{aligned}
\partial_0 H_{11} - \partial_0 H_{22} - \partial_0 H_{33} + 2\partial_2 E_{13} - 2\partial_3 E_{12} &= -4\Omega_2 H_{13} + 4\Omega_3 H_{12} - 4\dot{u}_2 E_{13} \\
&\quad + 4\dot{u}_3 E_{12} + 2n_{11} E_{11} - (n_{11} - n_{22} + n_{33}) E_{33} - (n_{11} + n_{22} - n_{33}) E_{22} + 2\omega H_{12} \\
&\quad - \theta (H_{11} - H_{22} - H_{33}) + 2r_2 E_{13} - 2(r_1 + q_1) E_{23} + 2q_3 E_{12}, \tag{B.42}
\end{aligned}$$

$$\begin{aligned}
\partial_0 H_{22} - \partial_0 H_{11} - \partial_0 H_{33} + 2\partial_3 E_{12} - 2\partial_1 E_{23} &= 4\Omega_1 H_{23} - 4\Omega_3 H_{12} + 4\dot{u}_1 E_{23} \\
&\quad - 4\dot{u}_3 E_{12} - (n_{11} + n_{22} - n_{33}) E_{11} + 2n_{22} E_{22} + (n_{11} - n_{22} - n_{33}) E_{33} - 2\omega H_{12} \\
&\quad + \theta (H_{11} - H_{22} + H_{33}) - 2(r_2 + q_2) E_{13} + 2q_1 E_{23} + 2r_3 E_{12}, \tag{B.43}
\end{aligned}$$

$$\begin{aligned}
\partial_0 H_{12} + \partial_2 E_{23} - \partial_3 E_{22} &= \Omega_1 H_{13} + \Omega_3 (H_{22} - H_{11}) - \Omega_2 H_{23} + \dot{u}_1 E_{13} - \dot{u}_2 E_{23} \\
&\quad - \dot{u}_3 (E_{11} - E_{22}) + \frac{1}{2} (3n_{11} + n_{22} - n_{33}) E_{12} + 2r_2 E_{23} - \theta H_{12} + \frac{1}{4} \omega (3H_{11} \\
&\quad + 14H_{22} - H_{33}) + r_1 E_{13} + q_3 E_{22} - q_3 E_{33} - \frac{\dot{u}_3}{6p'} \mathcal{E}, \tag{B.44}
\end{aligned}$$

$$\begin{aligned}
\partial_0 H_{13} - \partial_3 E_{23} + \partial_2 E_{33} &= -\Omega_1 H_{12} + \Omega_2 (H_{11} - H_{33}) + \Omega_3 H_{23} - \dot{u}_1 E_{12} \\
&\quad + \dot{u}_2 (E_{11} - E_{33}) + \dot{u}_3 E_{23} + \frac{1}{2} E_{13} (3n_{11} - n_{22} + n_{33}) - r_2 (E_{22} - E_{33}) \\
&\quad + q_1 E_{12} + 2\omega H_{23} - \theta H_{13} + 2q_3 E_{23} + \frac{\dot{u}_2}{6p'} \mathcal{E}, \tag{B.45}
\end{aligned}$$

$$\begin{aligned}
\partial_0 H_{23} - \partial_1 E_{33} + \partial_3 E_{13} &= \Omega_1 (H_{33} - H_{22}) + \Omega_2 H_{12} - \Omega_3 H_{13} + \dot{u}_1 (E_{33} - E_{22}) \\
&\quad + \dot{u}_2 E_{12} - \dot{u}_3 E_{13} - \theta H_{23} + \frac{1}{2} (3n_{22} + n_{33} - n_{11}) E_{23} + 2r_3 E_{13} - q_1 (E_{11} - E_{33}) \\
&\quad + r_2 E_{12} + 2\omega H_{13} - \frac{\dot{u}_1}{6p'} \mathcal{E}, \tag{B.46}
\end{aligned}$$

$$\begin{aligned} \partial_1 E_{11} + \partial_2 E_{12} + \partial_3 E_{13} = & E_{23}(n_{22} - n_{33}) + (r_1 - 2q_1)E_{11} - (q_3 - 2r_3)E_{13} \\ & + (r_2 - 2q_2)E_{12} - (r_1 + q_1)E_{22} - 3\omega H_{13} - \frac{\dot{u}_1}{3p'}\mathcal{E}, \end{aligned} \quad (\text{B.47})$$

$$\begin{aligned} \partial_1 E_{12} + \partial_2 E_{22} + \partial_3 E_{23} = & (r_2 + q_2)E_{11} + (2r_1 - q_1)E_{12} - (n_{11} - n_{33})E_{13} \\ & + (2r_2 - q_2)E_{22} - (2q_3 - r_3)E_{23} - 3\omega H_{23} - \frac{\dot{u}_2}{3p'}\mathcal{E}, \end{aligned} \quad (\text{B.48})$$

$$\begin{aligned} \partial_1 E_{13} + \partial_2 E_{23} + \partial_3 E_{33} = & (r_3 - 2q_3)E_{33} + (r_1 - 2q_1)E_{13} + (2r_2 - q_2)E_{23} \\ & + (n_{11} - n_{22})E_{12} - (q_3 + r_3)E_{11} - 3\omega H_{33} - \frac{\dot{u}_3}{3p'}\mathcal{E}, \end{aligned} \quad (\text{B.49})$$

$$\begin{aligned} \partial_1 H_{11} + \partial_2 H_{12} + \partial_3 H_{13} = & (n_{22} - n_{33})H_{23} + (r_1 - 2q_1)H_{11} + (r_2 \\ & - 2q_2)H_{12} - (r_1 + q_1)H_{22} - (q_3 - 2r_3)H_{13} - 3\omega E_{13}, \end{aligned} \quad (\text{B.50})$$

$$\begin{aligned} \partial_1 H_{12} + \partial_2 H_{22} + \partial_3 H_{23} = & -(n_{11} - n_{33})H_{13} + (r_3 - 2q_3)H_{23} + (r_2 \\ & - 2q_2)H_{22} + (2r_1 - q_1)H_{12} - (r_2 + q_2)H_{33} + 3\omega E_{23}, \end{aligned} \quad (\text{B.51})$$

$$\begin{aligned} \partial_1 H_{13} + \partial_2 H_{23} + \partial_3 H_{33} = & (n_{11} - n_{22})H_{12} - \dot{u}_3(H_{11} + H_{22} + H_{33}) \\ & + \omega\mathcal{E} - \omega(E_{11} + E_{22} - 2E_{33}) - q_3(2H_{33} + H_{11}) - r_3(2H_{11} + H_{22}) \\ & + (2r_2 - q_2)H_{23} + (r_1 - 2q_1)H_{13}. \end{aligned} \quad (\text{B.52})$$

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