Intriguing sets of points of $Q(2n, 2) \setminus Q^+(2n-1, 2)$

Bart De Bruyn

Ghent University, Department of Mathematics, Krijgslaan 281 (S22), B-9000 Gent, Belgium,
E-mail: bdb@cage.ugent.be

Abstract

Intriguing sets of vertices have been studied for several classes of strongly regular graphs. In the present paper, we study intriguing sets for the graphs Γ_n, $n \geq 2$, which are defined as follows. Suppose $Q(2n, 2)$, $n \geq 2$, is a nonsingular parabolic quadric of $\mathrm{PG}(2n, 2)$ and $Q^+(2n-1, 2)$ is a nonsingular hyperbolic quadric obtained by intersecting $Q(2n, 2)$ with a suitable nontangent hyperplane. Then the collinearity relation of $Q(2n, 2)$ defines a strongly regular graph Γ_n on the set $Q(2n, 2) \setminus Q^+(2n-1, 2)$. We describe some classes of intriguing sets of Γ_n and classify all intriguing sets of Γ_2 and Γ_3.

Keywords: intriguing set, tight set, parabolic quadric
MSC2000: 05B25, 51A50, 51E20

1 Introduction

Suppose $\Gamma = (V, E)$ is a finite connected regular graph of valency k and diameter $d \geq 2$, and let $k = \theta_0 > \theta_1 > \cdots > \theta_s$ denote the $s + 1 \geq d + 1 \geq 3$ distinct eigenvalues of Γ. Let $X \subseteq V$ be a set of vertices of Γ and let N denote the total number of ordered pairs of adjacent vertices of X.

The set X is said to be *intriguing* if there exist constants h_1 and h_2 such that every vertex $x \in X$ is adjacent to precisely h_1 vertices of X and every vertex $y \notin X$ is adjacent to precisely h_2 vertices of X. Clearly, $\emptyset$ and V are examples of intriguing sets. An intriguing set is said to be *nontrivial* if it is a nonempty proper subset of V. If X is a intriguing set of vertices of Γ, then by Proposition 3.7 of [8], there exists an $i \in \{1, \ldots, s\}$ such that $N = \theta_i \cdot |X| + \frac{k-\theta_i}{|V|} \cdot |X|^2$. We say that X is an *intriguing set of index* i. The index is uniquely determined if the intriguing set is nontrivial and can take any value of $\{1, \ldots, s\}$ in case of a trivial intriguing set.

By Proposition 3.8 of [8], we have

$$\theta_s \cdot |X| + \frac{k - \theta_s}{|V|} \cdot |X|^2 \leq N \leq \theta_1 \cdot |X| + \frac{k - \theta_1}{|V|} \cdot |X|^2. \tag{1}$$

If the lower [resp. upper] bound in equation (1) is attained, then X is called a *tight set of Type I* [resp. *tight set of Type II*]. The set X is called *tight* if it is either a tight set of Type I or a tight set of Type II. Clearly, $\emptyset$ and V are examples of tight sets. Since $\theta_s \cdot |X| + \frac{k-\theta_s}{|V|} \cdot |X|^2 = \theta_1 \cdot |X| + \frac{k-\theta_1}{|V|} \cdot |X|^2$ if and only if $|X| \in \{0, |V|\}$, $\emptyset$ and V are the only tight sets of Γ which are both of Type I and II. A tight set is said to be *nontrivial* if it is a nonempty proper subset of V.

For strongly regular graphs, the notions of tight sets and intriguing sets are equivalent. Intriguing sets of vertices have been studied for several classes of strongly regular graphs which arise as collinearity graphs of point-line geometries, see [3], [11] and [12] for generalized quadrangles, [2] for polar spaces, [7] for half-spin geometries and [1] for partial quadrangles. In the present paper, we study the intriguing sets of another class of strongly regular graphs.

If $\mathcal{Q}$ is a nonsingular quadric or Hermitian variety of a finite projective space Σ and Π is a hyperplane of Σ, then the graph Γ induced on the set $\mathcal{Q} \setminus \Pi$ by the collinearity relation of $\mathcal{Q}$ is usually not strongly regular. However, this graph Γ is strongly regular in the case that $\Sigma = \mathrm{PG}(2n, 2)$, $\mathcal{Q} = Q(2n, 2)$ and $\Pi \cap \mathcal{Q} = Q^+(2n-1, 2)$ $(n \geq 2)$. In the present paper we will deal with this case.

We refer to the respective books [9] and [10] for the basic properties of (strongly) regular graphs and quadrics which we will use throughout this paper.

Let $Q(2n, 2)$, $n \geq 2$, denote a nonsingular parabolic quadric in $\mathrm{PG}(2n, 2)$ and let Π be a hyperplane of $\mathrm{PG}(2n, 2)$ intersecting $Q(2n, 2)$ in a nonsingular hyperbolic quadric $Q^+(2n-1, 2)$. Put $\widetilde{Q}(2n, 2) := Q(2n, 2) \setminus Q^+(2n-1, 2)$. A maximal singular subspace of $Q(2n, 2)$ is also called a *generator* of $Q(2n, 2)$. If α is a generator of $Q(2n, 2)$ not contained in $Q^+(2n-1, 2)$, then $\alpha \setminus Q^+(2n-1, 2)$ is also called a *truncated generator* of $\widetilde{Q}(2n, 2)$. Two points of $\widetilde{Q}(2n, 2)$ are said to be *collinear* if the line of $\mathrm{PG}(2n, 2)$ joining them is contained in $Q(2n, 2)$. The collinearity relation of $\widetilde{Q}(2n, 2)$ defines a graph Γ_n on the set $\widetilde{Q}(2n, 2)$. We will prove in Lemma 3.1 that Γ_n is strongly regular. In Section 4, we describe the following classes of tight sets of Type II of Γ_n.

(1) Any union of mutually disjoint truncated generators of $\widetilde{Q}(2n, 2)$ is a tight set of Type II of Γ_n.

(2) Let $\Pi' \neq \Pi$ be a hyperplane of $\mathrm{PG}(2n, 2)$ intersecting $Q(2n, 2)$ in a nonsingular hyperbolic quadric $Q'^+(2n-1, 2)$. Then $Q'^+(2n-1, 2) \setminus Q^+(2n-1, 2)$ is a tight set of Type II of Γ_n.

(3) Suppose $n \geq 3$. Let $p \in Q^+(2n-1, 2)$ and let T_p denote the hyperplane of $\mathrm{PG}(2n, 2)$ through p tangent to $Q(2n, 2)$. Let η be a hyperplane of T_p not containing p. Then $\eta \cap Q(2n, 2) = Q(2n-2, 2)$ and $\eta \cap Q^+(2n-1, 2) = Q^+(2n-3, 2)$. If X is a tight set of Type II of $\widetilde{Q}(2n-2, 2) := Q(2n-2, 2) \setminus Q^+(2n-3, 2)$, then $pX \setminus \{p\}$ is a tight set of Type II of Γ_n.

The following theorem, which we will also prove in Section 4, characterizes tight sets of Type I.

Theorem 1.1 *A set X of vertices of Γ_n is a tight set of Type I if and only if it intersects every truncated generator of $\widetilde{Q}(2n,2)$ in a constant number of points.*

An *ovoid* of $\widetilde{Q}(2n,2)$ is a set of points of $\widetilde{Q}(2n,2)$ which has a unique point in common with every truncated generator. Ovoids of $\widetilde{Q}(2n,2)$ are examples of tight sets of Type I of Γ_n. In Section 4 we will prove that ovoids can only exist if $n \in \{2,3\}$.

Theorem 1.2 *$\widetilde{Q}(2n,2)$ has no ovoids for every $n \geq 4$.*

The classification of the tight sets of vertices of Γ_2 is a trivial problem since Γ_2 is isomorphic to the complete bipartite graph $K_{3,3}$. In fact, every tight set of vertices of Γ_2 is either an ovoid or the union of mutually disjoint truncated generators of $\widetilde{Q}(4,2)$. In Section 6, we classify all tight sets of vertices of the graph Γ_3.

Theorem 1.3 (1) *Up to isomorphism, Γ_3 has 2 nontrivial tight sets of Type I. Every tight set of Type I of Γ_3 is either an ovoid of $\widetilde{Q}(6,2)$ or the complement of an ovoid of $\widetilde{Q}(6,2)$.*

(2) *Up to isomorphism, Γ_3 has 20 nontrivial tight sets of Type II. Up to isomorphism, there is a unique tight set of Type II of Γ_3 which does not arise as union of mutually disjoint truncated generators of $\widetilde{Q}(6,2)$.*

2 General properties of intriguing and tight sets

In this section, $\Gamma = (V,E)$ denotes a connected regular graph of diameter $d \geq 2$ and valency k with $s+1$ mutually distinct eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_s$. The following propositions are extracted from [8] (Propositions 3.4, 3.7 and Corollaries 3.9, 3.12). Proofs of Propositions 2.1, 2.2 and 2.3 are also more or less contained in the earlier literature on intriguing sets ([2], [3], [7], [11]).

Proposition 2.1 *Let X_1 and X_2 be two intriguing sets of vertices of Γ of the same index $i \in \{1,\ldots,s\}$. Then*

(1) *If $X_1 \subseteq X_2$, then $X_2 \setminus X_1$ is an intriguing set of index i.*

(2) *If $X_1 \cap X_2 = \emptyset$, then $X_1 \cup X_2$ is an intriguing set of index i.*

Proposition 2.2 *If X is an intriguing set of index $i \in \{1,\ldots,s\}$, then every vertex of X is adjacent to precisely $\theta_i + \frac{k-\theta_i}{|V|} \cdot |X|$ vertices of X and every vertex not contained in X is adjacent to precisely $\frac{k-\theta_i}{|V|} \cdot |X|$ vertices of X.*

Proposition 2.3 (1) *A set of vertices of Γ is a tight set of Type I if and only if it is an intriguing set of index s.*

(2) *A set of vertices of Γ is a tight set of Type II if and only if it is an intriguing set of index 1.*

(3) *Suppose* Γ *is a strongly regular graph. Then a set of vertices of* Γ *is tight if and only if it is intriguing.*

(4) *If* X_1 *is a tight set of Type I and* X_2 *is a tight set of Type II, then* $|X_1 \cap X_2| = \frac{|X_1| \cdot |X_2|}{|V|}$.

The graph Γ is called *strongly regular with parameters* (v, k, λ, μ) if $d = 2$ and if Γ has v vertices, k vertices adjacent to any given vertex x, λ vertices adjacent to any two given adjacent vertices y and z, and μ vertices adjacent to any two given distinct nonadjacent vertices y' and z'.

Proposition 2.4 *Suppose* Γ *is a connected strongly regular graph and put* $\{A, B\} = \{I, II\}$. *Let* X *be a set of vertices of* Γ *and let* $\mathcal{F}$ *be a nonempty family of nontrivial tight sets of Type* A *of* Γ *satisfying the following properties:* (i) *all elements of* $\mathcal{F}$ *have the same number of vertices;* (ii) *there exist constants* m_i, $i \in \{0, 1, 2\}$, *such that if* x *and* y *are two vertices of* Γ *at distance* i *from each other, then* $m_i = |\{Y \in \mathcal{F} \mid \{x, y\} \subset Y\}|$. *Then the following are equivalent:*

(a) X *is a tight set of Type* B*;*

(b) *every element of* $\mathcal{F}$ *intersects* X *in a constant number of vertices.*

3 Tight sets of vertices of Γ_n

As in Section 1, let $Q(2n, 2)$ be a nonsingular parabolic quadric in $\mathrm{PG}(2n, 2)$, $n \geq 2$, and let Π be a hyperplane of $\mathrm{PG}(2n, 2)$ intersecting $Q(2n, 2)$ in a nonsingular hyperbolic quadric $Q^+(2n - 1, 2)$.

Let Γ_n be the graph with vertex set $\widetilde{Q}(2n, 2) := Q(2n, 2) \setminus Q^+(2n - 1, 2)$, with two vertices adjacent if and only if they are collinear on $Q(2n, 2)$.

Lemma 3.1 *The graph* Γ_n *is strongly regular with parameters* $v = 2^{2n-1} - 2^{n-1}$, $k = 2^{2n-2} - 1$, $\lambda = 2^{2n-3} - 2$ *and* $\mu = 2^{2n-3} + 2^{n-2}$.

Proof. The number of vertices of Γ_n is equal to $v = |Q(2n, 2)| - |Q^+(2n - 1, 2)| = (2^{2n} - 1) - (2^{2n-1} - 1 + 2^{n-1}) = 2^{2n-1} - 2^{n-1}$.

Let x be an arbitrary vertex of Γ_n, i.e. a point of $Q(2n, 2) \setminus Q^+(2n - 1, 2)$. The tangent hyperplane T_x at x intersects $Q(2n, 2)$ in a cone $xQ(2n-2, 2)$ with top x and basis $Q(2n-2, 2) := T_x \cap \Pi \cap Q(2n, 2)$. So, the vertices of Γ_n adjacent to x are precisely the points of $xQ(2n-2, 2) \setminus \{x\}$ not contained in $Q(2n-2, 2)$. Hence, $k = |Q(2n-2, 2)| = 2^{2n-2} - 1$.

Now, let x and y be two adjacent vertices of Γ_n, i.e. x and y are two distinct points of $Q(2n, 2) \setminus Q^+(2n - 1, 2)$ which are collinear on $Q(2n, 2)$. The tangent subspace T_L at the line $L := xy$ intersects $Q(2n, 2)$ in a cone $LQ(2n-4, 2)$ with top L and basis a nonsingular parabolic quadric $Q(2n - 4, 2)$ in a $(2n - 4)$-dimensional subspace of T_L disjoint from L. (If $n = 2$, then we take the convention that $LQ(2n - 4, 2) = L$.) Hence, the number of planes through L contained in $Q(2n, 2)$ is equal to $|Q(2n - 4, 2)| = 2^{2n-4} - 1$. Each such plane contributes 2 to the number of vertices of Γ_n adjacent to x and y. Obviously, every vertex of Γ_n adjacent to x and y is counted precisely once in this way. Hence, $\lambda = 2 \cdot (2^{2n-4} - 1) = 2^{2n-3} - 2$.

Let x and y be two nonadjacent vertices of Γ_n. Let T_x, respectively T_y, denote the tangent hyperplane at the point x, respectively y. Then $x \notin T_y$ and $y \notin T_x$. Moreover, there exists a nonsingular quadric $Q(2n-2,2)$ in $T_x \cap T_y$ such that $T_x \cap Q(2n,2) = xQ(2n-2,2)$ and $T_y \cap Q(2n,2) = yQ(2n-2,2)$. Now, let α denote the unique hyperplane of $\mathrm{PG}(2n,2)$ through $T_x \cap T_y$ distinct from T_x and T_y. Let k^* denote the kernel of the quadric $Q(2n,2)$ and let β be a hyperplane of $\mathrm{PG}(2n,2)$ not containing k^*. For every point u of $Q(2n,2)$, let $\phi(u)$ denote the unique point of β on the line k^*u. It is well-known that there exists a unique symplectic polarity ζ in β such that two points x_1 and x_2 of $Q(2n,2)$ are collinear on $Q(2n,2)$ if and only if $\phi(x_2) \in \phi(x_1)^\zeta$. It is now clear that $\phi(T_x \cap Q(2n,2)) = \phi(x)^\zeta$, $\phi(T_y \cap Q(2n,2)) = \phi(y)^\zeta$ and $\phi(\alpha \cap Q(2n,2)) = \eta$, where η is the unique hyperplane of β through $\phi(x)^\zeta \cap \phi(y)^\zeta$ distinct from $\phi(x)^\zeta$ and $\phi(y)^\zeta$. It follows that $\phi(\alpha \cap Q(2n,2)) = z'^\zeta$, where z' is the unique point of the line $\phi(x)\phi(y)$ distinct from $\phi(x)$ and $\phi(y)$. Now, let z denote the unique point of the line k^*z' belonging to $Q(2n,2)$. Then $z' = \phi(z)$ and $\alpha \cap Q(2n,2)$ is equal to $T_z \cap Q(2n,2)$, where T_z denotes the tangent hyperplane at the point z. We now prove that z is not contained in Π. Notice that z belongs to the plane $\langle k^*, x, y\rangle$. So, z belongs to the line $k^*\tilde{z}$, where $\tilde{z}$ is the unique point of the line xy distinct from x and y. Since $x, y \notin \Pi$, $\tilde{z} \in \Pi$. Since x and y are not collinear on $Q(2n,2)$, $\tilde{z} \notin Q(2n,2)$. Hence, $z \neq \tilde{z}$ and $z \notin \Pi$ as claimed. Now, put $A_x := T_x \cap Q^+(2n-1,2)$, $A_y := T_y \cap Q^+(2n-1,2)$ and $A_z := T_z \cap Q^+(2n-1,2)$. Then $A_x \cup A_y \cup A_z = Q^+(2n-1,2)$ and $A_x \cap A_y = A_x \cap A_z = A_y \cap A_z$ and $A_x \cong A_y \cong A_z \cong Q(2n-2,2)$. Hence,

$$
\begin{aligned}
|A_x \cap A_y| &= \frac{3 \cdot |Q(2n-2,2)| - |Q^+(2n-1,2)|}{2} \\
&= \frac{3 \cdot (2^{2n-2} - 1) - (2^{2n-1} - 1 + 2^{n-1})}{2} = 2^{2n-3} - 1 - 2^{n-2}.
\end{aligned}
$$

It follows that x and y have

$$
\begin{aligned}
&|(T_x \cap T_y \cap Q(2n,2)) \setminus Q^+(2n-1,2)| = |T_x \cap T_y \cap Q(2n,2)| - |A_x \cap A_y| \\
&= |Q(2n-2,2)| - |A_x \cap A_y| = 2^{2n-2} - 1 - 2^{2n-3} + 1 + 2^{n-2} = 2^{2n-3} + 2^{n-2}
\end{aligned}
$$

common neighbors. ∎

From the theory of strongly regular graphs, it is well-known (see e.g. [9, Chapter 10]) that a strongly regular graph with parameters (v, k, λ, μ) has three distinct eigenvalues $\theta_0 = k$, $\theta_1 \geq 0$ and $\theta_2 < -1$, where θ_1 and θ_2 are the roots of the quadratic polynomial $X^2 + (\mu - \lambda)X + \mu - k$.

Applying this here, we see that Γ_n has eigenvalues $\theta_0 = k = 2^{2n-2} - 1$, $\theta_1 = 2^{n-2} - 1$ and $\theta_2 = -2^{n-1} - 1$. By equation (1), we then have:

Corollary 3.2 *Let X be a set of points of $\widetilde{Q}(2n,2)$ and let N denote the total number of ordered pairs of distinct collinear points of $\widetilde{Q}(2n,2)$ belonging to X. Then*

$$-(2^{n-1}+1)\cdot|X| + \frac{2^{n-1}+1}{2^n-1}\cdot|X|^2 \leq N \leq (2^{n-2}-1)\cdot|X| + \frac{1}{2}|X|^2.$$

If the lower bound is achieved, then X is a tight set of Type I. If the upper bound is achieved, then X is a tight set of Type II.

4 Some examples of tight sets of vertices of Γ_n

We continue with the notation introduced in Section 3.

Proposition 4.1 *Every truncated generator of $\widetilde{Q}(2n,2)$ is a tight set of Type II of Γ_n.*

Proof. If X is a truncated generator, then $|X| = 2^{n-1}$ and the number of ordered pairs of distinct collinear points of $\widetilde{Q}(2n,2)$ belonging to X is equal to $N = 2^{n-1}(2^{n-1}-1)$. Since $N = (2^{n-2}-1)\cdot|X| + \frac{1}{2}|X|^2$, X is a tight set of Type II. ■

The following is an immediate corollary of Propositions 2.1 and 4.1.

Corollary 4.2 *Any union of mutually disjoint truncated generators of $\widetilde{Q}(2n,2)$ is a tight set of Type II of Γ_n.*

The following is an immediate corollary of Propositions 2.4 and 4.1.

Corollary 4.3 *A set of points of $\widetilde{Q}(2n,2)$ is a tight set of type I if and only if it intersects every truncated generator in a constant number of points.*

Definition. Recall that an *ovoid* of $\widetilde{Q}(2n,2)$ is a set of points of $\widetilde{Q}(2n,2)$ having a unique point in common with every truncated generator of $\widetilde{Q}(2n,2)$. The number of truncated generators is equal to (# number of generators of $Q(2n,2)$) - (# generators of $Q^+(2n-1,2)$) $= (2+1)(2^2+1)\cdots(2^n+1) - 2\cdot(2+1)(2^2+1)\cdot(2^{n-1}+1) = (2+1)(2^2+1)\cdot(2^{n-1}+1)(2^n-1)$. Notice also that the number of truncated generators of $\widetilde{Q}(2n,2)$ through a given point $x \in \widetilde{Q}(2n,2)$ is equal to $(2+1)(2^2+1)\cdot(2^{n-1}+1)$. It readily follows that an ovoid of $\widetilde{Q}(2n,2)$ contains precisely 2^n-1 mutually noncollinear points. Conversely, every set of 2^n-1 mutually noncollinear points of $\widetilde{Q}(2n,2)$ necessarily is an ovoid of $\widetilde{Q}(2n,2)$.

Proposition 4.4 *An ovoid of $\widetilde{Q}(2n,2)$ is a tight set of Type I.*

Proof. If X is an ovoid of $\widetilde{Q}(2n,2)$, then $|X| = 2^n - 1$ and the number of ordered pairs of distinct collinear points of $\widetilde{Q}(2n,2)$ belonging to X equals $N = 0$. Since $-(2^{n-1}+1)\cdot|X| + \frac{2^{n-1}+1}{2^n-1}\cdot|X|^2 = N$, X is a tight set of Type I. Notice also that this proposition immediately follows from Corollary 4.3. ■

The following proposition tells us that almost no examples of ovoids exist.

Proposition 4.5 *If $n \geq 4$, then $\widetilde{Q}(2n,2)$ has no ovoids.*

Proof. By Blokhuis and Moorhouse [4, Theorem 1.6], a set of mutually noncollinear points of the quadric $Q(2n,p^h)$, p prime, contains at most $\binom{p+2n-1}{2n}^h + 1$ points. Hence, a set of mutually noncollinear points of $\widetilde{Q}(2n,2)$ contains at most $2n+2$ points. If $\widetilde{Q}(2n,2)$ has an ovoid, we must have $2n+2 \geq 2^n - 1$ and hence $n \leq 3$. ■

We will see in Section 6 that $\widetilde{Q}(4,2)$ and $\widetilde{Q}(6,2)$ have ovoids.

Proposition 4.6 *Let $\Pi' \neq \Pi$ be a hyperplane of* $\mathrm{PG}(2n,2)$ *intersecting $Q(2n,2)$ in a nonsingular hyperbolic quadric ${Q'}^+(2n-1,2)$. Then $X := {Q'}^+(2n-1,2) \setminus Q^+(2n-1,2)$ is a tight set of Type II.*

Proof. The subspace $\Pi \cap \Pi'$ is a hyperplane of Π. There are two possibilities:

(i) $\Pi \cap \Pi' \cap Q(2n,2)$ is a nonsingular parabolic quadric $Q(2n-2,2)$ of $\Pi \cap \Pi$';

(ii) $\Pi \cap \Pi' \cap Q(2n,2)$ is a singular quadric of $\Pi \cap \Pi'$. In this case, the top of the singular quadric is a point p and the basis is a nonsingular hyperbolic quadric $Q^+(2n-3,2)$ in a hyperplane of $\Pi \cap \Pi'$ not containing p.

We will show that case (i) cannot occur. Suppose the contrary and let Π'' denote the unique hyperplane through $\Pi' \cap \Pi$ distinct from Π' and Π. Then $|\Pi'' \cap Q(2n,2)| = |Q(2n,2)| - 2\cdot|Q^+(2n-1,2)| + 2\cdot|Q(2n-2,2)| = (2^{2n}-1) - 2\cdot(2^{2n-1}-1+2^{n-1}) + 2\cdot(2^{2n-2}-1) = 2^{2n-1} - 1 - 2^n$. On the other hand, we know that there are 3 possibilities for $\Pi'' \cap Q(2n,2)$.

(1) $\Pi'' \cap Q(2n,2)$ is a nonsingular elliptic quadric of Π''. Then $|\Pi'' \cap Q(2n,2)| = 2^{2n-1} - 1 - 2^{n-1}$, a contradiction.

(2) $\Pi'' \cap Q(2n,2)$ is a nonsingular hyperbolic quadric of Π''. Then $|\Pi'' \cap Q(2n,2)| = 2^{2n-1} - 1 + 2^{n-1}$, a contradiction.

(3) $\Pi'' \cap Q(2n,2)$ is a cone with top a point x and with basis a nonsingular parabolic quadric in a hyperplane of Π'' not containing x. In this case, $|\Pi'' \cap Q(2n,2)| = 1 + 2\cdot(2^{2n-2}-1) = 2^{2n-1} - 1$, again a contradiction.

Hence, case (ii) occurs. So, $|X| = |Q^+(2n-1,2)| - |pQ^+(2n-3,2)| = (2^{2n-1}-1+2^{n-1}) - 1 - 2\cdot(2^{2n-3}-1+2^{n-2}) = 2^{2n-2}$. For every point $x \in X = {Q'}^+(2n-1,2) \setminus Q^+(2n-1,2)$, the tangent hyperplane of Π' at the point x intersects Π in a nonsingular hyperbolic quadric of type $Q^+(2n-3,2)$. Hence, the total number of ordered pairs of distinct collinear points of $\widetilde{Q}(2n,2)$ contained in X is equal to $N = 2^{2n-2} \cdot |Q^+(2n-3,2)| = 2^{2n-2} \cdot (2^{2n-3} - 1 + 2^{n-2})$. Since $|N| = (2^{n-2} - 1) \cdot |X| + \frac{1}{2}|X|^2$, X is a tight set of Type II. ∎

Tight sets of Type II can be constructed in a recursive way.

Proposition 4.7 *Suppose $n \geq 3$. Let $p \in Q^+(2n-1,2)$ and let T_p denote the hyperplane of* $\mathrm{PG}(2n,2)$ *through p tangent to $Q(2n,2)$. Let η be a hyperplane of T_p not containing x. Then $\eta \cap Q(2n,2) = Q(2n-2,2)$ and $\eta \cap Q^+(2n-1,2) = Q^+(2n-3,2)$. If X is a tight set of Type II of $\widetilde{Q}(2n-2,2) := Q(2n-2,2) \setminus Q^+(2n-3,2)$, then $pX \setminus \{p\}$ is a tight set of Type II of $\widetilde{Q}(2n,2) := Q(2n,2) \setminus Q^+(2n-1,2)$.*

Proof. Let N denote the total number of ordered pairs of distinct collinear points of X and let N' denote the total number of ordered pairs of distinct collinear points of $pX \setminus \{p\}$. Then

$$N' = 2 \cdot |X| + 4N.$$

Now, suppose X is a tight set of Type II of $\widetilde{Q}(2n-2,2)$. Then

$$N = (2^{n-3} - 1) \cdot |X| + \frac{1}{2}|X|^2,$$

and hence

$$N' = (2^{n-1} - 2) \cdot |X| + 2 \cdot |X|^2 = (2^{n-2} - 1) \cdot |pX \setminus \{p\}| + \frac{1}{2}|pX \setminus \{p\}|^2,$$

proving that $pX \setminus \{p\}$ is a tight set of Type II of $\widetilde{Q}(2n,2)$. ∎

5 Tight sets of vertices of the complement of the line graph of a complete graph

Let A be a set of size $n \geq 5$. Consider the graph Γ'_n whose vertices are the subsets of size 2 of A, with two vertices adjacent whenever they are disjoint. Then Γ'_n is the complement of the line graph of the complete graph on the set A. It is straightforward to verify that Γ'_n is a strongly regular graph with parameters $v = \frac{n(n-1)}{2}$, $k = \frac{(n-2)(n-3)}{2}$, $\lambda = \frac{(n-4)(n-5)}{2}$, $\mu = \frac{(n-3)(n-4)}{2}$, and that the automorphism group of Γ'_n is isomorphic to the symmetric group S_n. Γ'_n has three distinct eigenvalues $\theta_0 = k = \frac{(n-2)(n-3)}{2}$, $\theta_1 = 1$ and $\theta_2 = -(n-3)$. If X is a set of vertices of Γ'_n, then

$$-(n-3) \cdot |X| + \frac{n-3}{n-1}|X|^2 \leq N \leq |X| + \frac{n-4}{n}|X|^2,$$

where N denotes the total number of ordered pairs of adjacent vertices of Γ'_n. If the lower bound in achieved, then X is a tight set of Type I. If the upper bound is achieved, then X is a tight set of Type II.

Examples. (I) Let $i \in A$ and suppose X consists of all pairs $\{i, j\}$ where $j \in A \setminus \{i\}$. Then $|X| = n - 1$ and $N = 0$. So, X is a tight set of Type I.

(II) Suppose $i \in A$ and X consists of all pairs $\{j_1, j_2\}$, where j_1 and j_2 are two distinct elements of $A \setminus \{i\}$. Then $|X| = \frac{(n-1)(n-2)}{2}$ and $N = \frac{(n-1)(n-2)(n-3)(n-4)}{4}$. Hence, X is a tight set of Type I. In fact, it is the complement of the example mentioned in (I).

(III) Suppose G is a regular graph of valency k' with vertex set A. Let X consist of all edges of G. Then $|X| = \frac{1}{2}nk'$ and $N = \frac{1}{2}nk'\left(\frac{1}{2}nk' - 2k' + 1\right)$. So, X is a tight set of Type II.

Theorem 5.1 *If X is a tight set of Γ'_n, then X is obtained as described in (I), (II) or (III) above.*

Proof. Let G be the graph with vertex set A, with two distinct vertices i and j adjacent whenever $\{i, j\} \in X$. For every vertex x of G, let k_x denote its valency. Since X is an intriguing set, there exist constants N_1 and N_2 such that the following holds:

(1) if x and y are two distinct adjacent vertices of G, then $k_x + k_y = N_1$;

(2) if x and y are two distinct nonadjacent vertices of G, then $k_x + k_y = N_2$.

If G is regular, then clearly X is obtained as described in (III) above. So, in the sequel we will suppose that G is not regular.

Suppose first that G is connected. Since G is not regular, there exist two adjacent vertices x^* and y^* for which $k_{x^*} \neq k_{y^*}$. Clearly, $N_1 = k_{x^*} + k_{y^*}$. By (1), every vertex adjacent to a vertex with valency k_{x^*} has valency k_{y^*} and every vertex adjacent to a vertex with valency k_{y^*} has valency k_{x^*}. By connectedness of G, we can then conclude that every vertex of G has valency k_{x^*} or k_{y^*}. Let D_1, respectively D_2, denote the set of vertices of G of valency k_{x^*}, respectively k_{y^*}. If there was an edge connecting two distinct vertices x and y of D_1, then we would have $k_x + k_y = 2k_{x^*} \neq k_{x^*} + k_{y^*} = N_1$, a contradiction. So, there are no edges between vertices of D_1. In a similar way, one proves that there are no edges between vertices of D_2. So, G is a bipartite graph with classes D_1 and D_2. If u_1, v_1, u_2, v_2 are four distinct vertices such that $u_1, v_1 \in D_1$ and $u_2, v_2 \in D_2$, then we would have $2k_{x^*} = k_{u_1} + k_{v_1} = N_2 = k_{u_2} + k_{v_2} = 2k_{y^*}$, a contradiction. Hence, either $|D_1| = 1$ or $|D_2| = 1$. We see that X is obtained as described in (I) above.

We suppose now that G is not connected. Let C be a connected component of G. If x_1 and x_2 are two vertices of C and y is a vertex not belonging to C, then $k_{x_1} + k_y = N_2 = k_{x_2} + k_y$ implies that $k_{x_1} = k_{x_2}$. So, C is regular with valency $k_C := k_{x_1} = k_{x_2}$. Now, there exist two connected components C_1 and C_2 for which $k_{C_1} \neq k_{C_2}$. If there were a third connected component C_3, then considering pairs of points of $C_i \times C_3$ ($i \in \{1, 2\}$), we see that $k_{C_1} + k_{C_3} = N_2 = k_{C_2} + k_{C_3}$, contradicting the fact that $k_{C_1} \neq k_{C_2}$. So, C_1 and C_2 are the only connected components of G. If u_1, v_1, u_2, v_2 are four distinct vertices of G such that $\{u_1, v_1\}$ is an edge of C_1 and $\{u_2, v_2\}$ is an edge of C_2, then we would have $2k_{C_1} = k_{u_1} + k_{v_1} = N_1 = k_{u_2} + k_{v_2} = 2k_{C_2}$, a contradiction. So, without loss of generality, we may suppose that $|C_1| = 1$ and $|C_2| = n - 1$. If x and y are two nonadjacent vertices of C_2 and if z denotes the unique point in C_1, then we would have

$k_{C_1} + k_{C_2} = k_z + k_x = N_2 = k_y + k_x = 2k_{C_2}$, in contradiction with $k_{C_1} \neq k_{C_2}$. Hence, C_2 is a complete graph and X is as described in (II) above. ■

6 Classification of the intriguing sets of Γ_2 and Γ_3

As before, let $Q(2n, 2)$ be a nonsingular parabolic quadric in PG$(2n, 2)$, $n \geq 2$, and let Π be a hyperplane of PG$(2n, 2)$ intersecting $Q(2n, 2)$ in a nonsingular hyperbolic quadric $Q^+(2n-1, 2)$.

Let Γ_n be the graph with vertex set $\widetilde{Q}(2n, 2) := Q(2n, 2) \setminus Q^+(2n-1, 2)$ with two vertices adjacent whenever they are collinear on $Q(2n, 2)$.

Suppose first that $n = 2$. The points and lines contained in $Q(4, 2)$ determine a generalized quadrangle isomorphic to the symplectic generalized quadrangle $W(2)$ (see [13]). $\widetilde{Q}(4, 2)$ is the complement of a (3×3)-subgrid and hence Γ_2 is isomorphic to the complete bipartite graph $K_{3,3}$. Let D_1 and D_2 denote the two maximal cocliques (of size 3) of Γ_2. The following are all the nontrivial intriguing sets of Γ_2:

- D_1 and D_2 are tight sets of Type I. D_1 and D_2 are the two ovoids of Γ_2.
- If X is a set such that $|D_1 \cap X| = |D_2 \cap X| \in \{1, 2\}$, then X is a tight set of Type II. In this case, X is the union of $|D_1 \cap X| = |D_2 \cap X|$ mutually disjoint truncated generators of $\widetilde{Q}(4, 2)$.

In the sequel, we will suppose that $n = 3$.

Lemma 6.1 (1) *There exists a bijective correspondence ϕ between the points of $\widetilde{Q}(6, 2)$ and the subsets of size 2 of $A = \{1, 2, \ldots, 8\}$ such that if $T = \{x_1, x_2, x_3, x_4\}$ is a truncated generator of $\widetilde{Q}(6, 2)$, then $\{\phi(x_1), \phi(x_2), \phi(x_3), \phi(x_4)\}$ is a partition of A.*

(2) *The graph Γ_3 is isomorphic to the graph Γ_8' whose vertices are the subsets of size 2 of A, with two vertices adjacent whenever they are disjoint.*

Proof. Let $\mathbb{I}_3$ be the following point-line geometry:
- The points of $\mathbb{I}_3$ are the planes of $Q(6, 2)$ which are not contained in $Q^+(5, 2)$.
- The lines of $\mathbb{I}_3$ are the lines of $Q(6, 2)$ which are not contained in $Q^+(5, 2)$.
- Incidence is reverse containment.

Let $\mathbb{H}_3$ be the following point-line geometry:
- The points of $\mathbb{H}_3$ are the partitions of A in 4 subsets of size 2.
- The lines of $\mathbb{H}_3$ are the partitions of A in 2 subsets of size 2 and 1 subset of size 4.
- A point is incident with a line if and only if the partition corresponding to the point is a refinement of the partition corresponding to the line.

By [5] or [6], $\mathbb{I}_3$ and $\mathbb{H}_3$ are isomorphic near hexagons. The automorphism group of $\mathbb{I}_3 \cong \mathbb{H}_3$ is isomorphic to the symmetric group S_8. We will now list some properties of the near hexagons $\mathbb{I}_3$ and $\mathbb{H}_3$ whose proofs can be found in the book [6].

Every two points of $\mathbb{I}_3 \cong \mathbb{H}_3$ at distance 2 from each other are contained in a unique convex subspace of diameter 2 which is called a quad. The points and lines which are contained in a quad define a generalized quadrangle which is isomorphic to either $W(2)$ or the (3×3)-grid.

There exists a bijective correspondence between the $W(2)$-quads Q of $\mathbb{I}_3$ and the points x of $\widetilde{Q}(6,2)$: the planes of $Q(6,2)$ belonging to Q are precisely the planes of $Q(6,2)$ through x. Two distinct points of $\widetilde{Q}(6,2)$ are collinear on $Q(6,2)$ if and only if their corresponding $W(2)$-quads of $\mathbb{I}_3$ meet.

There exists a bijective correspondence between the $W(2)$-quads Q of $\mathbb{H}_3$ and the subsets U of size 2 of A. A partition $\mathcal{P}$ of A in 4 subsets of size 2 belongs to Q if and only if $U \in \mathcal{P}$. Two subsets of size 2 of A are disjoint if and only if their corresponding $W(2)$-quads of $\mathbb{H}_3$ meet.

The claims (1) and (2) of the lemma now easily follow from the discussion above. The bijection ϕ relates the points of $\widetilde{Q}(6,2)$ with the subsets of size 2 of A via their connection with the $W(2)$-quads of $\mathbb{I}_3 \cong \mathbb{H}_3$. ■

By Lemma 6.1, the classification of all intriguing sets of vertices of the graph Γ_3 is equivalent to the classification of all intriguing sets of vertices of the graph Γ_8'. The latter problem was considered in Section 5.

By Theorem 5.1, there are up to isomorphism two nontrivial tight sets of Type I in Γ_8'.

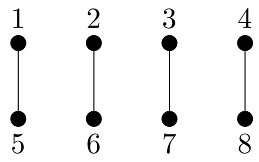
• Let $i \in A$ and suppose X consists of all pairs $\{i,j\}$ where $j \in A \setminus \{i\}$. Then X is a tight set of Type I. Since truncated generators of $\widetilde{Q}(6,2)$ correspond to partitions of A in four subsets of size 2, these tight sets of vertices correspond to ovoids of $\widetilde{Q}(6,2)$.

• Suppose $i \in A$ and X consists of all pairs $\{j_1, j_2\}$, where j_1 and j_2 are two distinct elements of $A \setminus \{i\}$. Then X is a tight set of Type I corresponding to the complement of an ovoid of $\widetilde{Q}(6,2)$.

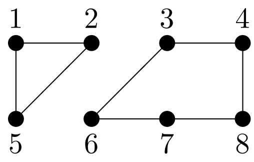
By Theorem 5.1, the nontrivial tight sets of Type II of Γ_8' bijectively correspond to the regular graphs of valency $k \in \{1,\ldots,6\}$ on the vertex set A. Every regular graph of valency 1 on the set A is isomorphic to the graph G_1 depicted below. Every regular graph of valency 2 on the set A is isomorphic to either G_2, G_3 or G_4. By [14, page 127], every regular graph of valency 3 on the set A is isomorphic to either G_5, G_6, G_7, G_8, G_9 or G_{10}. By [14, page 145], every regular graph of valency 4 on the set A is isomorphic to either $G_{11} \cong \overline{G_6}$, $G_{12} \cong \overline{G_7}$, $G_{13} \cong \overline{G_8}$, $G_{14} \cong \overline{G_9}$, $G_{15} \cong \overline{G_{10}}$ or $G_{16} \cong \overline{G_5}$. (Here, $\overline{G}$ denotes the complement of the graph G.) Every regular graph of valency 5 on the set A is isomorphic to either $G_{17} = \overline{G_2}$, $G_{18} = \overline{G_3}$ or $G_{19} = \overline{G_4}$ and every regular graph of valency 6 on the set A is isomorphic to $\overline{G_1}$.

So, up to isomorphism, there are 20 nontrivial tight sets of Type II in Γ_8' and hence also in Γ_3. With exception of the tight sets corresponding to G_2, all these tight sets of Γ_3 are the union of mutually disjoint truncated generators of $\widetilde{Q}(6,2)$. We illustrate this in Table 1. We explain the used notation. The decomposition 12/34/56/78 + 15/26/37/48 for the graph G_3 means that the tight set of Type II of Γ_3 corresponding to the regular graph G_3 is the union $\alpha_1 \cup \alpha_2$, where α_1 (respectively α_2) is the truncated generator of $\widetilde{Q}(6,2)$ corresponding to the partition $\{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\}$ (respectively $\{\{1,5\},\{2,6\},\{3,7\},\{4,8\}\}$) of A.

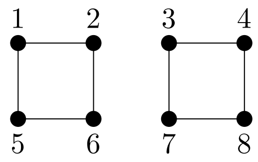
G_1



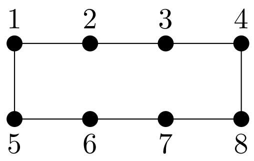
G_2



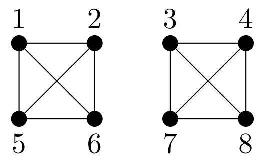
G_3



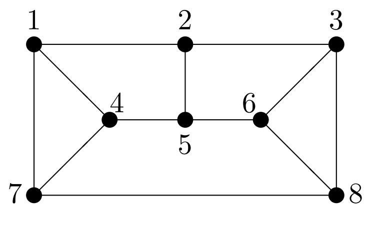
G_4



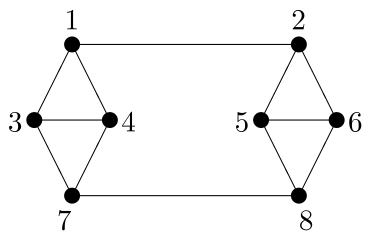
G_5



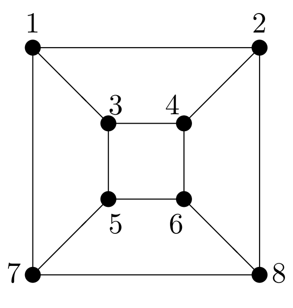
G_6



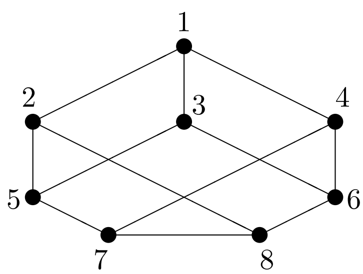
G_7



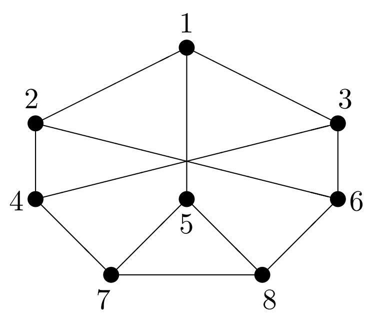
G_8



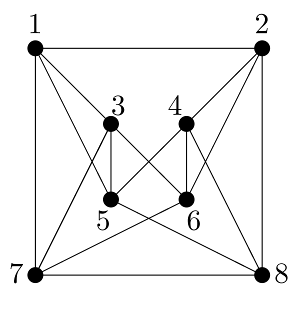
G_9



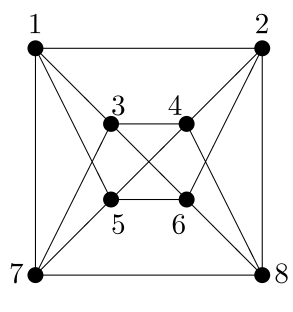
G_{10}



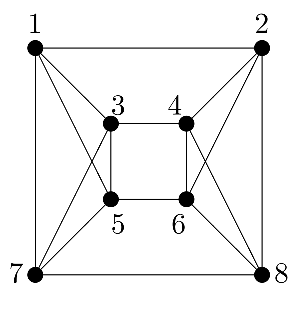
$G_{11} \cong \overline{G_6}$



$G_{12} \cong \overline{G_7}$



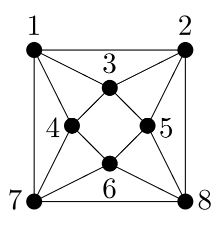
$G_{13} \cong \overline{G_8}$



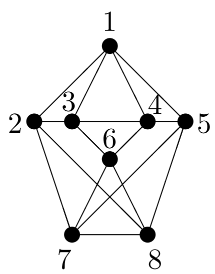
Graph	Decomposition
G_1	15/26/37/48
G_2	No decomposition
G_3	12/34/56/78 + 15/26/37/48
G_4	12/34/56/78 + 15/23/48/67
G_5	12/34/56/78 + 15/26/37/48 + 16/25/38/47
G_6	14/25/36/78 + 12/38/47/56 + 17/23/45/68
G_7	12/34/56/78 + 13/25/47/68 + 14/26/37/58
G_8	12/34/56/78 + 17/28/35/46 + 13/24/57/68
G_9	13/25/46/78 + 12/35/47/68 + 14/28/36/57
G_{10}	15/26/34/78 + 12/36/47/58 + 13/24/57/68
$G_{11} \cong \overline{G_6}$	12/35/46/78 + 17/28/36/45 + 13/24/58/67 + 15/26/37/48
$G_{12} \cong \overline{G_7}$	12/34/56/78 + 17/28/36/45 + 13/24/57/68 + 15/26/37/48
$G_{13} \cong \overline{G_8}$	12/34/56/78 + 17/28/35/46 + 13/24/57/68 + 15/26/37/48
$G_{14} \cong \overline{G_9}$	12/34/56/78 + 17/28/35/46 + 13/25/47/68 + 14/23/58/67
$G_{15} \cong \overline{G_{10}}$	12/34/58/67 + 13/27/45/68 + 15/23/46/78 + 14/28/36/57
$G_{16} \cong \overline{G_5}$	12/35/46/78 + 13/24/57/68 + 17/28/34/56 + 16/25/38/47
$G_{17} = \overline{G_2}$	16/27/38/45 + 13/24/57/68 + 14/26/37/58 + 17/28/35/46 + 18/23/47/56
$G_{18} = \overline{G_3}$	16/25/38/47 + 13/24/57/68 + 14/28/35/67 + 18/27/36/45 + 17/23/46/58
$G_{19} = \overline{G_4}$	18/26/37/45 + 16/25/38/47 + 17/28/35/46 + 13/24/57/68 + 14/27/36/58
$G_{20} = \overline{G_1}$	12/34/56/78 + 16/25/38/47 + 13/24/57/68 + 14/28/35/67 + 18/27/36/45 + 17/23/46/58

Table 1: The tight sets of Type II

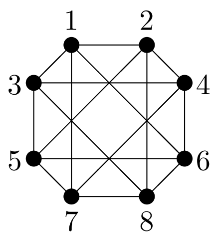
$G_{14} \cong \overline{G_9}$



$G_{15} \cong \overline{G_{10}}$



$G_{16} \cong \overline{G_5}$



Acknowledgment

At the time of the writing of this paper, the author was a Postdoctoral Fellow of the Research Foundation - Flanders (Belgium).

References

[1] J. Bamberg, F. De Clerck and N. Durante. Intriguing sets of partial quadrangles. *J. Combin. Des.* 19 (2011), 217–245.

[2] J. Bamberg, S. Kelly, M. Law and T. Penttila. Tight sets and m-ovoids of finite polar spaces. *J. Combin. Theory Ser. A* 114 (2007), 1293–1314.

[3] J. Bamberg, M. Law and T. Penttila. Tight sets and m-ovoids of generalised quadrangles. *Combinatorica* 29 (2009), 1–17.

[4] A. Blokhuis and G. E. Moorhouse. Some p-ranks related to orthogonal spaces. *J. Algebraic Combin.* 4 (1995), 295–316.

[5] A. E. Brouwer, A. M. Cohen, J. I. Hall and H. A. Wilbrink. Near polygons and Fischer spaces. *Geom. Dedicata* 49 (1994), 349–368.

[6] B. De Bruyn. *Near polygons.* Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.

[7] B. De Bruyn. Tight sets of points in the half-spin geometry related to $Q^+(9,q)$. *Linear Algebra Appl.* 424 (2007), 480–491.

[8] B. De Bruyn and H. Suzuki. Intriguing sets of vertices of regular graphs. *Graphs Combin.* 26 (2010), 629–646.

[9] C. Godsil and G. Royle. *Algebraic graph theory.* Graduate Texts in Mathematics 207. Springer-Verlag, New York, 2001.

[10] J. W. P. Hirschfeld and J. A. Thas. *General Galois Geometries.* Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1991.

[11] S. E. Payne. Tight pointsets in finite generalized quadrangles. *Congr. Numer.* 60 (1987), 243–260.

[12] S. E. Payne. Tight pointsets in finite generalized quadrangles. II. *Congr. Numer.* 77 (1990), 31–41.

[13] S. E. Payne and J. A. Thas. *Finite generalized quadrangles.* Second edition. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2009.

[14] R. C. Read and R. J. Wilson. *An atlas of graphs.* Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1998.