Upper bounds for Fourier transforms of exponential functions

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Abstract

Meaningful upper bounds for the Fourier transform of polynomial exponential functions are often hard to come by. Regarding Fourier transforms of rational exponential functions, which are of importance e.g. in Campbell's sampling theorem, the purpose of finding significant upper bounds is an even more demanding exercise. In this paper we propose a new approach in order to obtain significant upper bounds for Fourier transforms of general exponential functions. The technique is shown to allow further generalization in order to deal with Fourier-like integrals and rational exponential integrals.
Keywords : Fourier analysis; Upper bounds; Exponential functions; Sampling theorem;

1 Introduction

Obtaining significant upper bounds for the Fourier transform of exponential functions, even when the exponent is a mere polynomial, is an arduous exercise. To that effect, promising results were obtained in [1, 2], where judicious use of the Legendre-Fenchel transform [3] led to meaningful upper bounds. Bounds for Fourier transforms of even more complex exponential functions, the so-called rational exponential integrals [4], where the exponent is a rational function, are still more difficult to obtain. Nonetheless, it happens that Fourier transforms of rational exponential integrals are of importance in establishing sampling theorems for Fourier transforms of distributions with compact support [5, 6]. For instance, the Campbell sampling theorem can be stated as follows [5] :

Theorem 2 [5] : Let $g(\omega)$ *be a distribution with support contained in the open interval* $\{\omega : |\omega| < (1-q)\Omega\}$, *where* $0 < q < 1$. *Let* $f(t)$ *be the Fourier transform of* $g(\omega)$. *Then*

$$f(t) = \sum f\left(\frac{n\pi}{\Omega}\right) \frac{\sin(\Omega t - n\pi)}{\Omega t - n\pi} \, \mathcal{S}(q[\Omega t - n\pi]) \tag{1}$$

where the function $\mathcal{S}(\cdot)^1$ *is given by*

$$\mathcal{S}(y) = \frac{\int_{-1}^{1} \exp[1/(x^2-1) + ixy] \, dx}{\int_{-1}^{1} \exp[1/(x^2-1)] \, dx} \tag{2}$$

[1] Referred to as the Campbell function in the sequel

Note that $\int_{-1}^{1} \exp[1/(x^2-1)+ixy]\,dx$ can be interpreted as the Fourier transform of the $C_\infty[\mathbb{R}]$ function defined as the rational exponential function $\exp[1/(x^2-1)]$ over the compact support $[-1,1]$ and zero elsewhere.
In this paper we modify and generalize the approach adopted in [1, 2] in order to obtain significant upper bounds for Fourier integrals under quite general and simple conditions. Several examples illustrating the technique are given. The present approach is then further extended in order to deal with Fourier-like integrals with applications to rational exponential integrals including the Campbell function $\mathcal{S}(\cdot)$ as a pertinent example.

2 Fourier integrals

Theorem 1 : Consider the Fourier transform

$$F(\omega) = \int_{-\infty}^{\infty} e^{-f(x)+i\omega x}\,dx \tag{3}$$

where $f(z)$ is a function analytic in a strip $|\Im z| < b$ with $b > 0$. The function $f(x)$ satisfies $f(x) \geq 0$ on the real line and $\Re f(\pm\infty + iy) = +\infty$ for $|y| < b$. Then

$$\log|F(\omega)| \leq \inf_{\substack{-b<y<b \\ 0<a<1}} \{-\Phi(y,a) - \omega y - Z(a)\} \tag{4}$$

where the functions $\Phi(y,a)$ and $Z(a)$ are defined as

$$\Phi(y,a) = \inf_{x\in\mathbb{R}} \{\Re f(x+iy) - af(x)\} \quad 0 < a < 1 \quad |y| < b \tag{5}$$

and

$$Z(a) = -\log \int_{-\infty}^{\infty} e^{-af(x)}\,dx \tag{6}$$

where the last integral is supposed to be finite for $0 < a \leq 1$.[2]
Proof : From the premises it is seen that $\Phi(y,a) > -\infty$ for $|y| < b$. It is clear that $F(\omega)$ can be written as

$$F(\omega) = \int_{-\infty}^{\infty} e^{-f(x+iy)+i\omega(x+iy)}\,dx \quad \text{for} \quad |y| < b \tag{7}$$

and hence

$$|F(\omega)| \leq \int_{-\infty}^{\infty} e^{-\Re f(x+iy)-\omega y}\,dx \quad \text{for} \quad |y| < b \tag{8}$$

[2]Note that $Z(1) = -\log F(0)$

From the premises this implies that

$$|F(\omega)| \leq \int_{-\infty}^{\infty} e^{-af(x)-\Phi(y,a)-\omega y}\, dx \quad \text{for} \quad |y| < b \quad 0 < a < 1 \tag{9}$$

Since this is valid for all admissible y and a, the result follows $\square$.
Remark 1: The minimization problem (4) can be written as

$$\log |F(\omega)| \leq - \sup_{0<a<1} \left\{ Z(a) + \sup_{-b<y<b} [\Phi(y,a) + \omega y] \right\} \tag{10}$$

When $b = \infty$ and $-\Phi(y, a)$ is a convex function of y, the maximization

$$\sup_{y\in\mathbb{R}} [\omega y + \Phi(y,a)] \tag{11}$$

is known as the Legendre-Fenchel transform [1, 2, 3] of $-\Phi(y, a)$.
Remark 2: A necessary condition for the minimum in (5) is

$$\Re f'(x+iy) - af'(x) = 0 \tag{12}$$

This defines the position of the minimum $x(y)$ implicitly and hence

$$\Phi(y,a) = \Re f(x(y)+iy) - af(x(y)) \tag{13}$$

Supposing $b = \infty$ the minimum with respect to y in (4) has as necessary condition

$$\begin{aligned} 0 &= \frac{\partial \Phi(y,a)}{\partial y} + \omega \\ &= \Re f'(x(y)+iy)(x'(y)+i) - af(x(y))x'(y) + \omega \\ &= -\Im f'(x(y)+iy) + \omega \end{aligned} \tag{14}$$

Equations (12) and (14) together can be neatly written as

$$f'(x+iy) = af'(x) + i\omega \tag{15}$$

This determines the values of x and y in terms of ω. Note that, if we tentatively put $a = 0$ in (15) we obtain the equation $-f'(z) + i\omega = 0$, which is the equation for the saddle point in the well-known steepest descent asymptotic method [7]. But in our approach the value $a = 0$ is of course not allowed since $Z(0) = -\infty$. Also, we endeavor to obtain upper bounds for $|F(\omega)|$, not asymptotic expressions for large values of $|\omega|$.

2.1 Examples

Let $f(x) = x^{2m}$ where m is a positive integer. $F(\omega)$ is an even entire function of ω given by the Taylor expansion

$$F(\omega) = \frac{1}{m} \sum_{k=0}^{\infty} (-1)^k \, \Gamma\left(\frac{k}{m} + \frac{1}{2m}\right) \frac{\omega^{2k}}{(2k)!} \tag{16}$$

Clearly $b = \infty$ and $\Phi(0, a) = 0$. For $y \neq 0$ we have

$$\Phi(y, a) = \beta_m(a) y^{2m} \tag{17}$$

where

$$\beta_m(a) = \inf_{t \in \mathbb{R}} \left\{ \Re \left\{ (t+i)^{2m} \right\} - a t^{2m} \right\} \tag{18}$$

Note that $\Re \left\{ (t+i)^{2m} \right\} - a t^{2m}$ is an even polynomial in t with leading term $(1-a)t^{2m}$ and hence the minimum $\beta_m(a)$ exists. It is an easy matter to show that

$$\beta_1(a) = -1, \qquad \beta_2(a) = -\frac{8+a}{1-a} \tag{19}$$

and in general $\beta_m(a) < 0$ for $0 < a < 1$. This follows from the fact that (18) can be written, with the change of variable $t = \cot\theta$, as

$$\beta_m(a) = \inf_{0<\theta<\pi} \frac{\cos(2m\theta) - a|\cos\theta|^{2m}}{|\sin\theta|^{2m}} \tag{20}$$

and considering that $\cos(2m\theta)$ vanishes $2m$ times in the open interval $(0, \pi)$. The function $Z(a)$ is given by

$$Z(a) = -\log\left[\frac{\Gamma(\frac{1}{2m})}{m}\right] + \frac{1}{2m}\log a \tag{21}$$

The logarithmic bound is :

$$\log|F(\omega)| \leq \inf_{\substack{-\infty<y<\infty \\ 0<a<1}} \left\{ -\beta_m(a) y^{2m} - |\omega| y - Z(a) \right\} \tag{22}$$

Straightforward minimization with respect to y yields

$$\begin{aligned} \log|F(\omega)| \quad &\leq \quad \inf_{0<a<1} \left\{ -\beta_m(a)(1-2m)\left[\frac{|\omega|}{-2m\beta_m(a)}\right]^{2m/(2m-1)} \right. \\ &+ \quad \left. \log\left[\frac{\Gamma(\frac{1}{2m})}{m}\right] - \frac{1}{2m}\log a \right\} \end{aligned} \tag{23}$$

For $m = 1$ this leads to the bound

$$|F(\omega)| \leq \sqrt{\pi} e^{-\frac{\omega^2}{4}} \tag{24}$$

which is remarkable since for $m = 1$ the function $F(\omega)$ is actually equal to its bound $\sqrt{\pi} e^{-\frac{\omega^2}{4}}$. For $m = 2$ we have

$$\log|F(\omega)| \leq \inf_{0<a<1} \left\{ -3\left(\frac{8+a}{1-a}\right)^{-1/3} \left[\frac{|\omega|}{4}\right]^{4/3} - \frac{1}{4}\log a + \log\left[\frac{\Gamma(\frac{1}{4})}{2}\right] \right\} \tag{25}$$

The best bound is found by solving

$$\frac{(8+a)^{4/3}(1-a)^{2/3}}{36a} = \left[\frac{|\omega|}{4}\right]^{4/3} \tag{26}$$

for $a = \hat{a}(|\omega|)$ and inserting the result in (25). Note that $\hat{a}(0) = 1$ and $\hat{a}(\infty) = 0$ and $\hat{a}(|\omega|)$ is a strictly decreasing function of $|\omega|$. In Fig. 1 we plot the relative error bound, defined as

$$E(\omega) = \frac{B(\omega) - |F(\omega)|}{B(\omega)} \tag{27}$$

where $B(\omega)$ is the given upper bound, as a function of ω. It is seen that $0 \leq E(\omega) \leq 1$ in general, and $E(\omega) = 1$ at the locations of the real zeros of the function $F(\omega)$.

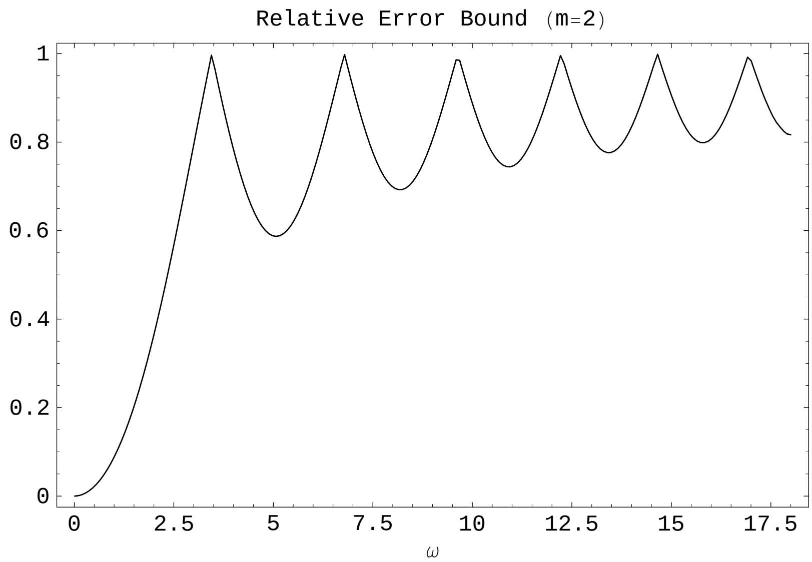

Figure 1: Relative error bound for the case $m = 2$

Of course, working by means of equation (26) is an implicit, not an explicit approach. For general m we have no explicit minimization results with respect to a, but of course (23) implies that

$$\begin{aligned}\log|F(\omega)| \quad \leq \quad & \left\{ -\beta_m(a)(1-2m)\left[\frac{|\omega|}{-2m\beta_m(a)}\right]^{2m/(2m-1)} \right. \\ + \quad & \left. \log\left[\frac{\Gamma(\frac{1}{2m})}{m}\right] - \frac{1}{2m}\log a \right\}\end{aligned} \tag{28}$$

is valid for all $0 < a < 1$. A pertinent sub-optimal bound is found by taking $\theta = \frac{\pi}{4m}$ in equation (20), yielding

$$\beta_m(a) \approx -a \left|\cot\left(\frac{\pi}{4m}\right)\right|^{2m} \tag{29}$$

Inserting this value for $\beta_m(a)$ in equation (28), and letting a tend to 1, results in the bound

$$\log|F(\omega)| \leq \left\{-(2m-1)\left[\frac{|\omega|}{2m\cot(\pi/4m)}\right]^{2m/(2m-1)} + \log\left[\frac{\Gamma(\frac{1}{2m})}{m}\right]\right\} \tag{30}$$

This bound is exact for $\omega = 0$. Again, for $m = 1$ the function $F(\omega)$ is equal to its bound $\sqrt{\pi}e^{-\frac{\omega^2}{4}}$ in formula (30). Note that in [2] the cruder bound

$$\log|F(\omega)| \leq \left\{-(2m-1)2^{-4m^2}\left[\frac{|\omega|}{2m}\right]^{2m/(2m-1)} + \frac{1}{2}\log(2\pi e)\right\} \tag{31}$$

was obtained. In Fig. 2 we compare the relative errors for the fine bound (30) and crude bound (31) as a function of ω for $m = 3$.

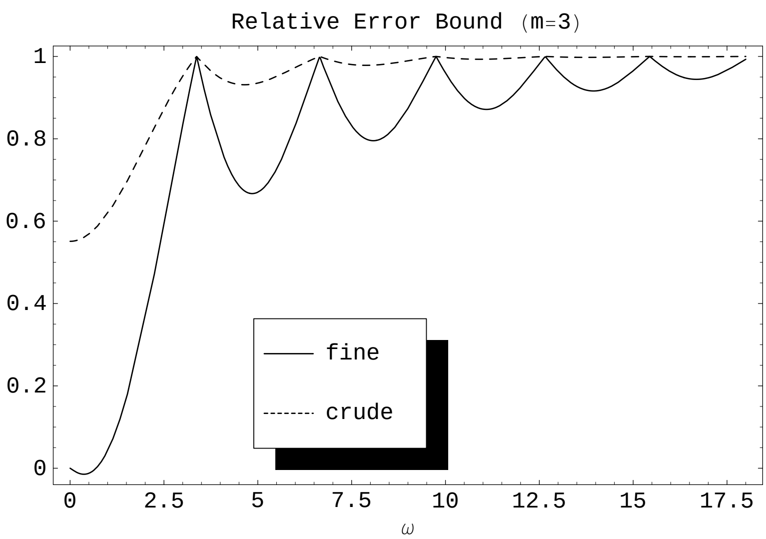

Figure 2: Relative errors for the fine resp. crude bounds in the case $m = 3$

3 Generalization

Theorem 2 : Consider the Fourier-like integral

$$F(\omega) = \int_{-\infty}^{\infty} e^{-f(x)+i\omega g(x)} w(x)\, dx \tag{32}$$

where $f(z), g(z), w(z)$ are functions analytic in a strip $|\Im z| < b$ with $b > 0$. We require $f(x) \geq 0$ and $w(x) > 0$ on the real line and $\Re f(\pm\infty + iy) = +\infty$ for $|y| < b$. The function $|\Im g(x + iy)|$ is required to be bounded for all $x \in \mathbb{R}$ and $|y| < b$. Define the function

$$\Psi(y, \omega, a) = \inf_{x \in \mathbb{R}} \{\Re f(x + iy) - af(x) + \omega \Im g(x + iy)\} \tag{33}$$

for $0 < a < 1$ and $|y| < b$. We suppose that b is chosen such that $\Psi(y, \omega, a) > -\infty$ for $|y| < b$. We further require that

$$\sup_{x \in \mathbb{R}} \frac{|w(x + iy)|}{w(x)} = \zeta(y) < \infty \quad \text{for} \quad |y| < b \tag{34}$$

Then we have the logarithmic bound

$$\log |F(\omega)| \leq \inf_{\substack{-b<y<b \\ 0<a<1}} \{-\Psi(y, \omega, a) + \log \zeta(y) - Z(a)\} \tag{35}$$

where

$$Z(a) = -\log \int_{-\infty}^{\infty} e^{-af(x)} w(x)\, dx \tag{36}$$

Proof : It is clear that $F(\omega)$ can be written as

$$F(\omega) = \int_{-\infty}^{\infty} e^{-f(x+iy)+i\omega g(x+iy)} w(x + iy)\, dx \tag{37}$$

and hence

$$|F(\omega)| \leq \int_{-\infty}^{\infty} e^{-\Re f(x+iy)-\omega \Im g(x+iy)} |w(x + iy)|\, dx \tag{38}$$

implying that

$$|F(\omega)| \leq \int_{-\infty}^{\infty} e^{-af(x)-\Psi(y,\omega,a)}\, w(x)\zeta(y)\, dx \tag{39}$$

and the result follows $\Box$.

3.1 Application: Rational exponential integral

Consider the finite Fourier transform

$$F(\omega) = \int_{-1}^{1} e^{-Q(t)+i\omega t}\, dt \tag{40}$$

where $Q(t)$ is a non-negative rational function over the open interval $(-1, 1)$ with

$$\lim_{t \uparrow 1} Q(t) = \lim_{t \downarrow -1} Q(t) = +\infty \tag{41}$$

The change of variable $t = \tanh(x)$ transforms the finite Fourier transform into the infinite format

$$F(\omega) = \int_{-\infty}^{\infty} e^{-f(x)+i\omega g(x)} w(x)\, dx \tag{42}$$

where

$$f(x) = Q(\tanh(x)) \quad g(x) = \tanh(x) \quad w(x) = \frac{1}{\cosh^2 x} \tag{43}$$

We have

$$\zeta(y) = \sup_{x\in\mathbb{R}} \frac{|w(x+iy)|}{w(x)} = \frac{1}{\cos^2 y} \tag{44}$$

and hence we must require $0 < b < \frac{\pi}{2}$. The function $Z(a)$ is

$$Z(a) = -\log \int_{-1}^{1} e^{-aQ(t)}\, dt \tag{45}$$

The function $\Psi(y, \omega, a)$ is

$$\begin{aligned} \Psi(y,\omega,a) &= \inf_{x\in\mathbb{R}} \{\Re f(x+iy) - af(x) + \omega \Im \tanh(x+iy)\} & (46)\\ &= \inf_{-1<t<1} \left\{ \Re Q\left(\frac{t + i\tan y}{1 + it\tan y}\right) - aQ(t) + \omega \frac{(1-t^2)\tan y}{1+t^2\tan^2 y} \right\} & (47) \end{aligned}$$

The logarithmic bound is

$$\log|F(\omega)| \le \inf_{\substack{-b<y<b\\ 0<a<1}} \{-\Psi(y,\omega,a) - 2\log\cos y - Z(a)\} \tag{48}$$

3.2 Example : The Campbell function

Applied to the function $Q(t) = 1/(1-t^2)$ we have that $F(\omega)$ is an even entire function of ω given by the Taylor expansion

$$F(\omega) = \frac{1}{e} \sum_{k=0}^{\infty} (-1)^k\, \Gamma\left(\frac{1}{2}+k\right) U\left(\frac{1}{2}+k, 0, 1\right) \frac{\omega^{2k}}{(2k)!} \tag{49}$$

where $U(\cdot,\cdot,\cdot)$ is the Tricomi confluent hypergeometric function. The relationship with the Campbell function $\mathcal{S}(\omega)$ is the simple scaling $\mathcal{S}(\omega) = F(\omega)/F(0)$.

The function $\Psi(y, \omega, a)$ is

$$\Psi(y,\omega,a) = \inf_{-1<t<1} \left\{ \frac{\cos^2 y - t^2 \sin^2 y - a}{1-t^2} + \omega \frac{(1-t^2)\tan y}{1+t^2\tan^2 y} \right\} \tag{50}$$

In order that $\Psi(y,\omega,a) > -\infty$ we must require $\cos 2y > a$. In other words b is a function of a given by

$$b = \frac{1}{2}\arccos(a) < \frac{\pi}{4} \tag{51}$$

Another requirement is that y must have the same sign as ω, but since $F(\omega)$ is even we may consider only non-negative values of ω and y. Tedious calculations result in the explicit expression

$$\begin{aligned}\Psi(y,\omega,a) &= \frac{1}{4}\Big((1-4a\sin^2(y)-\cos(4y)) + (3+4\cos(2y)+\cos(4y))\\ &\times \sqrt{\omega(\cos(2y)-a)\sec^6(y)\tan(y)}\Big)\end{aligned} \tag{52}$$

Also, after some algebraic manipulations, the function $Z(a)$ is found to be

$$Z(a) = -\log\left\{ae^{-a/2}\left[K_1(a/2) - K_0(a/2)\right]\right\} \tag{53}$$

were the $K.(\cdot)$ are the MacDonald Bessel functions. The logarithmic bound

$$\log|F(\omega)| \leq -\Psi(y,|\omega|,a) - 2\log\cos(y) - Z(a) \tag{54}$$

is valid for $0 < y < \frac{1}{2}\arccos(a)$ and $0 < a < 1$.This can be written as

$$\log|F(\omega)| \leq -A(y,a) - \sqrt{|\omega|}B(y,a) \tag{55}$$

where

$$A(y,a) = \frac{1}{4}(1-4a\sin^2(y)-\cos(4y)) + 2\log\cos(y) + Z(a) \tag{56}$$

$$B(y,a) = \frac{1}{4}(3+4\cos(2y)+\cos(4y))\sqrt{(\cos(2y)-a)\sec^6(y)\tan(y)} \tag{57}$$

Since $Z(0) = -\log 2 > -\infty$, the maximum of $B(y,a)$ is obtained for a tending to 0 and $y = \pi/8$. From the maximum value $B(\pi/8, 0) = 1$ we obtain the particular bound

$$\log|F(\omega)| \leq \min\left(-\frac{1}{4} - \log\left[\frac{2+\sqrt{2}}{8}\right] - \sqrt{|\omega|}, \kappa\right) \tag{58}$$

where

$$\kappa = -\frac{1}{2} + \log\left[K_1\left(\frac{1}{2}\right) - K_0\left(\frac{1}{2}\right)\right] \tag{59}$$

Of course, other bounds can also be obtained by inserting different allowable values of y and a in the expression (55). In Fig. 3 we plot the relative error bound as a function of ω.

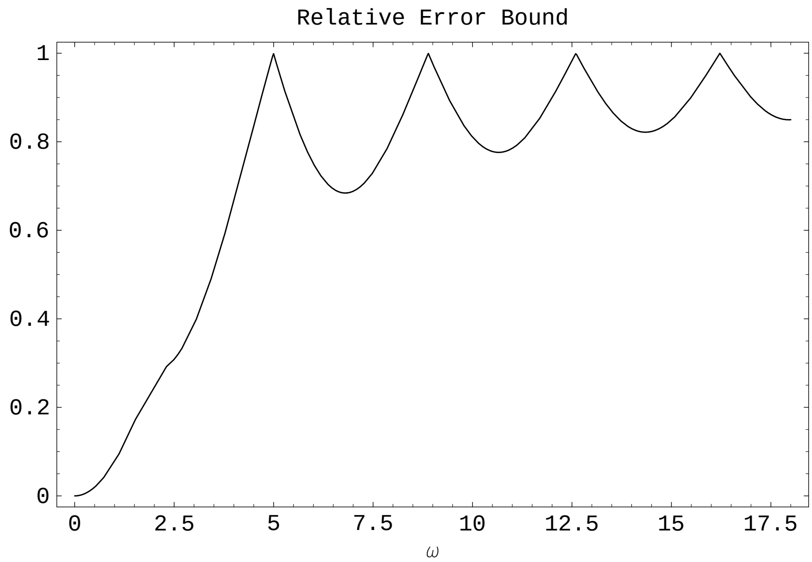

Figure 3: Relative error bound for the Campbell function

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