INVERSE STOCHASTIC PROPAGATION WITH POLYNOMIAL CHAOS

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Abstract. Monte Carlo techniques have been widely used for investigating the impact of stochastic parameters on the response in forward models. These methods can also be employed for assessing the influence of the stochastic parameters available in the forward model on the inverse solutions. They may however lead to prohibitively long overall computational times, especially when the (numerical) forward model is time demanding. Alternatively, polynomial chaos methods can determine the impact of stochastic parameters in forward models. This paper presents a method based on polynomial chaos so to quantify the impact of stochastic parameters on the inverse solutions. We provide a mathematical analysis of the method and a validation onto analytical models. The presented method is valid for any given measurement or objective, resulting in a time efficient method for the solution of inverse problems with stochastic parameters in the forward model.

Keywords: stochastic, polynomial chaos, accuracy, reconstruction methods

INTRODUCTION

In many fields in engineering and science, models may contain parameters whose values are not exactly known. It may be of importance to evaluate to what extent these uncertain parameters propagate towards the responses in the (forward) models. When solving optimization or inverse problems, these uncertain parameters may have a large influence on the reconstructed parameters. It is in this way possible to evaluate the accuracy or robustness of a certain optimization or reconstruction process. In electromagnetism, uncertainties can either be the geometry, material sources or the sources that act as input to the Maxwell's equations. These uncertain values can follow a certain probability density function. Stochastic methods for quantifying the propagation of uncertainties can be classified into intrusive and non-intrusive methods. Intrusive methods implement the stochastic propagation in the forward solver itself. The stochastic finite element method is a widely used stochastic solver [1]. These methods can however be time and memory consuming. Non-intrusive methods such as Monte Carlo consider the forward model as a black box model where the input parameters are sampled and each evaluated in the forward model. Polynomial chaos is a computationally efficient method [2].

Polynomial Chaos (PC) is a technique to discretize the aleatory dimension if we assume it can be described by a finite number of uncorrelated random variables. Depending on the distribution of the variable, one aims to expand in a series, there exists an optimal base of orthogonal polynomials in the sense of convergence. Gaussian variables correspond to Hermite polynomials, uniform variables to Legendre polynomials. After describing the stochastic input, one uses the completeness of the PC-basis to also represent the output of the model as a (truncated) series, under the assumption the output has a finite variance.

We present a method for quantifying the impact of stochastic parameters onto the reconstructed parameters based on polynomial chaos. When solving optimization problems, the minimization of a certain cost function needs to be performed only once while for inverse problems the measurements can change. It is thus important to construct a numerical method that can solve inverse problems in a stochastic way for different sets of measurement data.

METHODS

Definitions

We define the problem as the reconstruction of parameters $\mathbf{x} \in X \subset R^n$ with feasible domain X, for given uncertainty parameters (denoted as $\mathbf{p} \in U \subset R^p$ with domain U) and objective or measurement $\mathbf{y} \in R^m$. The forward model is denoted by the function $\mathbf{f}(\mathbf{x},\mathbf{p}): X \times U \to Y$ and needs to approximate as close as possible the objective or measurements $\mathbf{y}$. Let us assume that the uncertainty parameter $\mathbf{p}(\omega)$ is a random variable of finite variance defined in a probability space (Θ, F, P) with random coordinate ω. It can be expanded as a truncated series of order q_p of Hermite polynomials ψ_i: $\mathbf{p}(\omega) = \sum_{i=0}^{q_p} \mathbf{p}_i \psi_i(\xi(\omega))$ with $\xi(\omega)$ as Gaussian argument. The sum

(1) $$\mathbf{f}(\mathbf{x},\omega) = \sum_{i=0}^{q_f} \mathbf{f}_i(\mathbf{x}) \psi_i(\xi(\omega))$$

is defined as the forward uncertainty propagator. The L_2-norm between $\mathbf{y}$ and $\mathbf{f}(\mathbf{x},\mathbf{p})$ is mostly used as the objective or cost function and we define here the following reconstruction function $\mathbf{g}$ under uncertainty

(2) $$\mathbf{g}(\mathbf{y},\omega) = \arg\min_{\mathbf{x}\in X} \left\| \mathbf{f}(\mathbf{x},\omega) - \mathbf{y} \right\|_2$$

for given $\mathbf{y}$. Note that we only take the norm in $L_2(X)$ and not in the full tensor space $L_2(X) \otimes L_2(\Theta)$.

Inverse stochastic propagator

We will now project this cost function on a finite dimensional PC-subspace of $L_2(\Theta)$. Using the above definitions, we define the inverse stochastic function:

(3) $$\mathbf{g}(\mathbf{y},\omega) = \sum_{i=0}^{q_g} \mathbf{g}_i(\mathbf{y}) \psi_i(\xi(\omega)) \text{ with } \mathbf{g}_i(\mathbf{y}) = \frac{\langle \mathbf{g}(\mathbf{y},\omega), \psi_i(\xi(\omega)) \rangle}{\langle \psi_i(\xi(\omega)), \psi_i(\xi(\omega)) \rangle}$$

and $\langle \cdot,\cdot \rangle$ is the inner product defined in the given probability space $L_2(\Theta)$. In order to determine the coefficients $\mathbf{g}_i(\mathbf{y})$, we first rewrite the objective function as:

(4) $$\sum_{l=0}^{m} ((f_{1,l}(\mathbf{x}) - y_l)\psi_1 + \sum_{j>1} (f_{j,l}(\mathbf{x})\psi_j))^2 = \sum_{l=0}^{m} \Psi \Gamma_l(\mathbf{x},\mathbf{y}) \Psi^T = \Psi \Gamma(\mathbf{x},\mathbf{y}) \Psi^T \; ; \quad \Psi = \left[\psi_1, \ldots, \psi_{q_f} \right]$$

where the matrix $\Gamma = \sum_l \Gamma_l$ contains the objective $\mathbf{y}$ and the forward propagator coefficients of every *l*-th component $f_{i,l}(\mathbf{x})$ of $\mathbf{f}_i(\mathbf{x})$ $(l = 1,\ldots,m)$. The measurement data $\mathbf{y}$ has an impact on the Γ-matrix:

(5) $$\mathbf{\Gamma}_{11} = (\mathbf{f}_1(\mathbf{x}) - \mathbf{y})(\mathbf{f}_1(\mathbf{x}) - \mathbf{y})^T \text{ and } \mathbf{\Gamma}_{1k} = (\mathbf{f}_1(\mathbf{x}) - \mathbf{y})(\mathbf{f}_k(\mathbf{x}))^T \text{ for } k > 1 .$$

For the calculation of the inverse stochastic propagator coefficients, we can rewrite the $\langle \mathbf{g}(\mathbf{y},\omega), \psi_i(\xi(\omega)) \rangle$ term in $\mathbf{g}_i(\mathbf{y})$ as $\left\langle \arg\min_{\mathbf{x}\in X} \Psi(\omega)\Gamma(\mathbf{x},\mathbf{y})\Psi(\omega)^T, \psi_i(\omega) \right\rangle$.

The presented methodology goes as follows: first the coefficients $\mathbf{f}_i(\mathbf{x})$ $(i = 1,\ldots,q_f)$ from equation (1) are determined from which it is possible to build the Γ-matrix using (5) for given objective $\mathbf{y}$. Using the previous, the coefficients $\mathbf{g}_i(\mathbf{y})$ $(i = 1,\ldots,q_g)$ can be calculated so that it is possible to construct the stochastic reconstruction function (2). In the full paper we present a detailed mathematical analysis of the method and a validation onto analytical models.

CONCLUSIONS

This paper presents theoretical work for evaluating the propagation of stochastic parameters in inverse solutions. We propose the use of polynomial chaos for evaluating in a computationally efficient way this effect. We are able to pinpoint the interactions between the uncertainty coefficients, forward propagation coefficients and the inverse propagation coefficients.

REFERENCES

[1] R.G. Ghanem, and P.D. Spanos. *Stochastic Finite Elements. A Spectral Approach.* Dover Publication, 2003.

[2] D. Xiu, and G.E. Karniodakis. The Wiener-Askey Polynomial Chaos for stochastic differential equations. SIAM J. Sci. Comput., vol. 24, pp. 619-644, 2002.