FUNCTIONS ON THE SPINGROUP

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Abstract

In this paper we interprete functions on the spingroup in terms of functions of several vector variables. We also evaluate the action of several invariant differential operators on Spin(m) which arise naturally within the Clifford algebra R_m.

Introduction

Let $\{e_1, \ldots, e_m\}$ be an orthonormal basis of R^m; then the Clifford algebra R_m is the 2^m-dimensional algebra determined by the defining relations $e_i e_j + e_j e_i = -2\delta_{ij}$.
The group Spin(m) is the set of elements of the form $s = w_1 \ldots w_{2l}$, whereby w_j belongs to the unit sphere. Two representations of the spingroup are readily defined. First of all for $s \in$ Spin(m) and $a \in R_m$ we simply put $l(s)[a] = s\,a$. In case a belongs to a minimal left ideal of R_m (a spinor space) this representation corresponds to the basic spin representation. Next, let $a \to \bar{a}$ denote the main anti-involution determined by $\overline{a\,b} = \bar{b}\,\bar{a}$, $\bar{e}_j = -e_j$, then for $a \in R_m$ we may consider the representation $h(s)[a] = s\ a\ \bar{s}$. It is easy to see that the spaces $R_{m,k}$ of k-vectors are invariant subspaces under this representation. Moreover for $k < \frac{m}{2}$ they are irreducible representations while for $k = \frac{m}{2}$, $m = 2n$, the representation space $R_{m,k}$ splits into two inequivalent irreducible representations, Moreover, the fundamental representations of Spin(m) are exactly the representation l on spinor spaces and h on spaces of k-vectors (see also [1], [2], [3]).
All the other finite dimensional irreducible representations are representable by means of suitable spaces of R_m-valued functions $a \to f(a)$ defined on special subsets of R_m (see also [1], [3], [5]). Hereby the spingroup may act in two different ways on functions, namely by means of the representation

$$H(s)f(a) = s\ f(\bar{s}\ a\ s)\ \bar{s},$$

or by means of the representation

$$L(s)f(a) = s\ f(\bar{s}\ a\ s).$$

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The representation $H(s)$ (resp. $L(s)$) is used for all representations with integer weight (resp. half integer weight). Note that the representation $H(s)$ satisfies $H(s) = H(-s)$ and may hence be regarded as a representation of the group SO(m). Moreover, its action on scalar valued functions coincides with the representation of SO(m)

$$Ho(s)\, f(a) = f(\bar{s}\; a\; s).$$

In section one we describe more in detail how to obtain these basic irreducible spin-representations. For more information we refer to [2] and [5].
In fact there is a general formula in which form all representations may be written and this form also leads to a canonical way of transforming functions on the spingroup into functions of several vector variables.
In section two we perform the explicit computation of invariant differential operators on Spin(m) in terms of vector variables $u_j = \sum e_k\, u_{jk}$ and vector derivatives (Dirac operators) $\partial_{u_j} = \sum e_k \partial_{u_{jk}}$. In particular we are interested in the explicit form of the Casimir operators $C(Ho)$ and $C(L)$ as well as the general "gamma operator" defined in the formula $C(L) = C(Ho) + \Gamma - \frac{1}{4}\binom{m}{2}$ (see also [2]).

1. Functions on the Spingroup

All finite dimensional representations of Spin(m) may be represented by means of functions $f(u_1, \ldots, u_k)$ of several vector variables. Indeed, we may start from Clifford tensors, i.e. multilinear functions $F(v_1, \ldots, v_k)$ taking values in R_m, on which the group Spin(m) acts by means of

$$H(s)\; F(v_1, \ldots, v_k) = s\; F(\bar{s}\, v_1 s, \ldots, \bar{s}\, v_k\, s)\bar{s}$$

or

$$L(s)\; F(v_1, \ldots, v_k) = s\; F(\bar{s}\, v_1 s, \ldots, \bar{s}\, v_k\, s).$$

Two important types of operations commute with these representations namely the Young symmetry operations and the projection operators involved in the monogenic decomposition of tensors (see [5]). Hereby a tensor F is called monogenic if it satisfies the system

$$\partial_{v_1} F(v_1, \ldots, v_k) = \ldots = \partial_{v_k} F(v_1, \ldots, v_k) = 0.$$

Moreover, a Young symmetrized tensor may be written into the form

$$F(v_1 \wedge \ldots \wedge v_{k_1},\; v_{k_1+1} \wedge \ldots \wedge v_{k_1+k_2}, \ldots, v_{k-k_{n+1}} \wedge \ldots \wedge v_k),$$

whereby $k = k_1 + \ldots + k_n,\ k_1 \geq k_2 \ldots \geq k_n$ and this expression is invariant under permutations of the sets of indices

$$\{1,\ k_1+1, \ldots, k-k_n+1\}, \quad \{2,\ k_1+2, \ldots, k-k_n+2\},\ \ldots$$

This means that a Young symmetrized tensor is representable by a polynomial of the form $F(u_1 \wedge \ldots \wedge u_{k_1}, u_1 \wedge \ldots u_{k_2}, \ldots, u_1 \wedge \ldots \wedge u_{k_n})$. The consideration of both Young symmetrization and monogenicity brings us to the consideration of polynomials of the above form satisfying the monogenicity condition

$$\partial_{u_j}\ F(u_1 \wedge \ldots \wedge u_{k_1}, u_1 \wedge \ldots u_{k_2}, \ldots, u_1 \wedge \ldots \wedge u_{k_n}) = 0$$

and it is by using spaces of such polynomials that one arrives at the models for finite dimensional representation spaces of Spin(m). Now any such polynomial may be written in a canonical way as a function of the special variable $a \in R_m$ given by

$$a = a(u_1, \ldots, u_m) = u_1 + u_1 \wedge u_2 + \ldots + u_1 \wedge \ldots \wedge u_m.$$

Moreover, we have that

$$\begin{aligned} s\,a\,\bar{s} &= s\,u_1\,\bar{s} + s\,u_1 \wedge u_2\,\bar{s} + \ \ldots\ + s\,u_1 \wedge \ldots \wedge u_m\,\bar{s} \\ &= a(s\,u_1\,\bar{s},\ s\,u_2\,\bar{s}, \ldots,\ s\,u_m\,\bar{s}) \end{aligned}$$

so that the action of $h(s)$ on this type of element $a \in R_m$ is equivalent to the action of $h(s)$ on the frame $\{u_1, \ldots, u_m\}$, which in general may be regarded as an element of $Gl(m)$. It is also clear that the application on the set of frames

$$(u_1, \ldots, u_m) \rightarrow a(u_1, \ldots, u_m) = u_1 + u_1 \wedge u_2 + \ldots + u_1 \wedge \ldots \wedge u_m$$

becomes an isomorphism on the set of right handed orthonormal frames i.e. on the group SO(m). Following this idea it is not hard to prove

Lemma There is a one to one correspondence between functions $f(s)$ on the group Spin(m) and pairs of functions (F, G) of the variable $a(u_1, \ldots, u_m)$.

Proof. Clearly any function $f(s)$ on Spin(m) may be written into the form $f(s) = F(s) + s\,G(s)$ whereby $F(s)$ and $G(s)$ are even functions on Spin(m), i.e. they satisfy $F(-s) = F(s),\ G(-s) = G(s)$ and may hence be regarded as functions on SO(m). This means that $F(s)$ and $G(s)$ may in fact always be thought of as functions of the form

$$f(s) = f(\bar{s}\,e_1\,s, \ldots, \bar{s}\,e_m\,s) = f(\bar{s}\,a(e_1, \ldots, e_m)\,s)$$

and there is a unique way of writing $\bar{s}\; a(e_1,\ldots,e_m)\; s$ as an element of the form $a(u_1,\ldots,u_m)$ with $(u_1,\ldots,u_m)\in SO(m)$. Moreover, if one fixes a certain degree of homogeneity in the variables u_j, which is always done, a function of the form $f(\bar{s}\; a(e_1,\ldots,e_m)\; s)$ is in fact a function of the form $f(a(u_1,\ldots,u_m))$ which is homogeneous in the variables u_j.
In fact, speaking of functions on Spin(m) means speaking of functions of the form

$$R(s)\; F(a(u_1,\ldots,u_m)),\quad R=H_o,\; H\quad or\quad R=L,$$

which are homogenenous in the variables $u_1,\ldots,u_m$.

2. Casimir Operator, Gamma Operator.

Let E be an abstract representation space of Spin(m) corresponding to a representation R; then the infinitesimal representation dR of the Lie algebra $R_{m,2}$ of Spin(m) (i.e. the space of bivectors) is given by

$$dR(w)\; f=\lim_{\epsilon\to 0}\frac{1}{\epsilon}(R(\exp(\epsilon w))-1)f.$$

In particular we may let this act on the function $F(s)=R(s)\,f$ on Spin(m) for which we have the action

$$dR(w)\; F(s)=\lim_{\epsilon\to 0}\frac{1}{\epsilon}(F(\exp(\epsilon w)\,s)-F(s))=D_w\,F(s).$$

D_w being the right invariant vector field (first order differential operator) corresponding to the bivector w.
The Casimir operator is generally given by

$$C(R)=\frac{1}{4}\sum_{i<j}dR(e_{ij})^2$$

and we do have that

$$C(R)\; F(s)=\Delta\,F(s),$$

Δ the Laplacian on Spin(m). In case R is the representation on functions in R^m

$$R(s)\; f(x)=H_o(s)\; f(x)=f(\bar{s}\,x\,s),$$

the operators $dH_o(e_{ij})$ are given by

$$dH_0(e_{ij})=2\,L_{ij}=2\;(x_j\,\partial_{x_i}-x_i\,\partial_{x_j}).$$

Moreover, for the representation L we find that

$$dL(e_{ij}) = dH_o(e_{ij}) + e_{ij}$$

Hence the Casimir operator $C(L)$ of L is given by

$$C(L) = C(H_o) + \Gamma - \frac{1}{4}\binom{m}{2},$$

whereby

$$\Gamma = \frac{1}{2}\sum e_{ij}\, dH_o(e_{ij})$$

is called the "general gamma operator". In case of the representation on functions $f(x)$ the gamma operator is given by

$$\Gamma = \sum e_{ij}\; L_{ij} = -x \wedge \partial_x .$$

In general we may also define a "Dirac-like" operator on Spin(m) as in [2] by means of

$$\partial = \sum e_{ij}\, D_{ij}, \quad D_{ij} = D_{e_{ij}}$$

and we do have that

$$\partial\; H_o(s)f = \sum e_{ij}\, dH_o(e_{ij})\; H_o(s)f = 2\Gamma\, H_o(s)f,$$

$$\partial\; L(s)f = \sum e_{ij}\, dL(e_{ij})\, L(s)f = (2\Gamma - (m2))\, L(s)f.$$

The operator ∂ is not what would correspond to the Dirac operator on a manifold and its square is not the Laplacian. But we have the identities (see also [2])

$$[\partial^2]_0 = -\Delta, \quad [\partial^2]_2 = (m-2)\partial.$$

Here we have to consider functions of the form

$$f(\bar{s}\, a\, s), \quad sf(\bar{s}\, a\, s), \quad a = u_1 + u_1 \wedge u_2 + \ldots + u_1 \wedge \ldots \wedge u_m$$

which may be rewritten as functions of the form

$$f(\bar{s}\, u_1\, s, \ldots, \bar{s}\, u_m\, s), \quad s\, f(\bar{s}\, u_1\, s, \ldots, \bar{s}\, u_m s).$$

Moreover we have that

$$\begin{aligned} D_{ij} f(\bar{s}\, u_1\, s, ..., \bar{s}\, u_m\, s) &= dH_o(e_{ij}) f(\bar{s}\, u_1 s, ..., \bar{s}\, u_m s) \\ &= 2(L_{u_1,ij} + ... + L_{u_m,ij})\, f(\bar{s}\, u_1 s, ..., \bar{s}\, u_m s), \\ D_{ij}\, s\, f(\bar{s}\, u_1\, s, ..., \bar{s}\, u_m\, s) &= dL(e_{ij})\, s\, f(\bar{s}\, u_1\, s, ..., \bar{s}\, u_m s) \\ &= (2(L_{u_1,ij} + ... + L_{u_m,ij}) + e_{ij})\, s\, f(\bar{s}\, u_1 s, ..., \bar{s}\, u_m s), \end{aligned}$$

so that

$$\begin{aligned} \partial f(\bar{s}\, u_1\, s, ..., \bar{s}\, u_m\, s) &= 2(\Gamma_{u_1} + ... + \Gamma_{u_m})\, f(\bar{s}\, u_1\, s, ..., \bar{s}\, u_m\, s), \\ \partial s f(\bar{s}\, u_1\, s, ..., \bar{s}\, u_m\, s) &= (2(\Gamma_{u_1} + ... + \Gamma_{u_m}) - \binom{m}{2}) s f(\bar{s}\, u_1\, s, ..., \bar{s}\, u_m s), \end{aligned}$$

which means that the operator ∂ is evaluable in terms of the angular momentum (gamma) operators Γ_{u_j}. Indeed, any function on Spin(m) is expressable in terms of functions of the above form.
The same does not hold for the operator $\frac{1}{4}\Delta$ which is given by

$$\begin{aligned} \frac{1}{4}\Delta_s\, f(\bar{s}\, u_1\, s, ...\bar{s}\, u_m\, s) =& \ (\Delta_{u_1} + ... + \Delta_{u_m} + \sum \Delta_{u_k u_l})\, f(s\, u_1\, s, ..., \bar{s}\, u_m\, s), \\ \frac{1}{4}\Delta_s\, s\, f(\bar{s}\, u_1\, s, ...\bar{s}\, u_m\, s) =& \ (\Delta_{u_1} + ... + \Delta_{u_m} + \sum \Delta_{u_k u_l} \\ & + \Gamma_{u_1} + ... + \Gamma_{u_m} - \frac{1}{4}\binom{m}{2})\, s\, f(\bar{s}\, u_1\, s, ..., \bar{s}\, u_m\, s), \end{aligned}$$

Hereby the operator Δ_u is the Laplace-Beltrami operator

$$\Delta_u = \sum L_{u,ij}^2 = \Gamma_u(m - 2 - \Gamma_u), \quad \Gamma_u = -u \wedge \partial_u$$

and the operator Δ_{uv} is a "mixed Laplace-Beltrami operator" given by

$$\Delta_{uv} = \sum L_{u,ij} L_{v,ij}.$$

The operator Δ_u is algebraically expressable in terms of the operators u and ∂_u. It is important to see whether the operators Δ_{uv} are also algebraically expressable in terms of u, v, ∂_u and ∂_v. This is to be expected in fact because the decomposition of the functions $f(\bar{s}\, u_1\, s, \ldots, \bar{s}\, u_m\, s)$ into irreducible pieces corresponds to a decomposition of this function into eigenfunctions of Δ. From [3], [5] we also know that irreducible representation spaces may be represented by functions of the above form which are monogenic in each of the variables u_j.

Hence it must be possible to let the operator Δ_{uv} act on monogenic functions and this can only be done in case Δ_{uv} is indeed expressable in terms of u, v, ∂_u and ∂_v.
Now first we have that

$$\begin{aligned}\Delta_{uv} &= \sum (u_i\,\partial_{u_j} - u_j\,\partial_{u_i})(v_i\,\partial_{v_j} - v_j\,\partial_{v_i}) \\ &= \sum (u_i\,v_i\,\partial_{u_j}\,\partial_{v_j} + u_j\,v_j\,\partial_{u_i}\,\partial_{v_i} - u_i\,v_j\,\partial_{u_j}\,\partial_{v_i} - u_j\,v_i\,\partial_{u_i}\,\partial_{v_j})\end{aligned}$$

Next consider the product $\Gamma_u\,\Gamma_v$; it consists of a scalar a bivector and a four vector part and it is easy to see that in fact

$$\frac{1}{2}(\Gamma_u\,\Gamma_v + \Gamma_v\,\Gamma_u) = [\Gamma_u\,\Gamma_v]_0 + [\Gamma_u\,\Gamma_v]_4$$

whereby

$$-[\Gamma_u\,\Gamma_v]_0 = \sum L_{u,ij}\,L_{v,ij} = \Delta_{uv}$$

$$[\Gamma_u\,\Gamma_v]_4 = \sum e_{ijkl}\,L_{u,ij}\,L_{v,kl}.$$

Hence the scalar part of $\frac{1}{2}(\Gamma_u\,\Gamma_v + \Gamma_v\,\Gamma_u)$ is already equal to the operator $-\Delta_{uv}$. But this isn't good enough because we have to cancel the four vector part. To that end we evaluate

$$\begin{aligned}&-\frac{1}{2}[(u\wedge\partial_v)(v\wedge\partial_u) + (v\wedge\partial_u)(u\wedge\partial_v)]_0 \\ &= \frac{1}{2}\sum (u_i\,\partial_{v_j} - u_j\partial_{v_i})(v_i\,\partial_{u_j} - v_j\,\partial_{u_i}) + \frac{1}{2}\ \text{ibid}\,\{u <> v\} \\ &= \sum (u_i\,v_i\,\partial_{u_j}\,\partial_{v_j} + u_j\,v_j\,\partial_{u_i}\,\partial_{v_i} - u_i\,v_j\,\partial_{u_i}\,\partial_{v_j} - u_j\,v_i\,\partial_{u_j}\,\partial_{v_i}) \\ &\qquad - (m-1)(\sum u_i\,\partial_{u_i} + \sum v_i\,\partial_{v_i}).\end{aligned}$$

We also need the evaluation of

$$\begin{aligned}-[(u\wedge v)(\partial_u\wedge\partial_v)]_0 &= \sum (u_i\,v_j - u_j\,v_i)(\partial_{u_i}\,\partial_{v_j} - \partial_{u_j}\,\partial_{v_i}) \\ &= \sum (u_i\,v_j\,\partial_{u_i}\,\partial_{v_j} + u_j\,v_i\,\partial_{u_j}\,\partial_{v_i} - u_i\,v_j\,\partial_{u_j}\,\partial_{v_i} \\ &\qquad - u_j\,v_i\,\partial_{u_i}\,\partial_{v_j})\end{aligned}$$

so that we clearly have that

$$-[\frac{1}{2}[((u \wedge \partial_v)(v \wedge \partial_u) + (v \wedge \partial_u)(u \wedge \partial_v)) + (u \wedge v)(\partial_u \wedge \partial_v)]_0$$

$$= \Delta_{uv} - (m-1)(\sum u_i \, \partial_{u_i} + \sum v_i \, \partial_{v_i}).$$

Hence it is worthwhile to consider the four-vector part of the right hand side and see whether it is proportional to that of the operator $\Gamma_u \, \Gamma_v$. We have that

$$\begin{aligned}[(u \wedge \partial_v)(v \wedge \partial_u)]_4 &= \sum e_{ijkl}(u_i \, \partial_{v_j} - u_j \, \partial_{v_i})(v_k \, \partial_{u_l} - v_l \, \partial_{u_k}) \\ &= \sum e_{ijkl}(u_i \, v_k \, \partial_{v_j} \, \partial_{u_l} + u_j \, v_l \, \partial_{v_i} \, \partial_{u_k} - u_i \, v_l \, \partial_{v_j} \, \partial_{u_k} \\ &\qquad - u_j \, v_k \, \partial_{v_i} \, \partial_{u_l}) \\ &= \sum e_{ijkl}(u_i \, v_j \, \partial_{v_l} \, \partial_{u_k} + u_j \, v_i \, \partial_{v_k} \, \partial_{u_l} - u_i \, v_j \, \partial_{v_k} \, \partial_{u_l} \\ &\qquad - u_j \, v_i \, \partial_{v_l} \, \partial_{u_k})\end{aligned}$$

$$\begin{aligned}[(u \wedge v)(\partial_u \wedge \partial_v)]_4 &= \sum e_{ijkl}(u_i \, v_j - u_j \, v_i)(\partial_{u_k} \, \partial_{v_l} - \partial_{u_l} \, \partial_{v_k}) \\ &= \sum e_{ijkl}(u_i \, v_j \, \partial_{u_k} \, \partial_{v_l} + u_j \, v_i \, \partial_{u_l} \, \partial_{v_k} - u_i \, v_j \, \partial_{u_l} \, \partial_{v_k} \\ &\qquad - u_j \, v_i \, \partial_{u_k} \, \partial_{v_l})\end{aligned}$$

while also

$$\begin{aligned}[\Gamma_u \, \Gamma_v]_4 &= \sum e_{ijkl}(u_i \, \partial_{u_j} - u_j \, \partial_{u_l})(v_k \partial_{v_l} - v_l \, \partial_{v_k}) \\ &= \sum e_{ijkl}(u_i \, v_k \, \partial_{u_j} \, \partial_{v_l} + u_j \, v_l \, \partial_{u_i} \, \partial_{v_k} - u_i \, v_l \, \partial_{u_j} \, \partial_{v_k} - u_j \, v_k \, \partial_{u_i} \, \partial_{v_l})\end{aligned}$$

so that the four vector part of

$$\Gamma_u \, \Gamma_v + \Gamma_v \, \Gamma_u + \frac{1}{2}[(u \wedge \partial_v)(v \wedge \partial_u) + (v \wedge \partial_u)(u \wedge \partial_v)] + (u \wedge v)(\partial_u \wedge \partial_v)$$

vanishes while the scalar part then again is given by

$$-2\Delta_{uv} - \Delta_{uv} + (m-1)(E_u + E_v) = -3\Delta_{uv} + (m-1)(E_u + E_v).$$

Question is still : what is the bivector part of the above object. The bivector part of $\Gamma_u \, \Gamma_v + \Gamma_v \, \Gamma_u$ vanishes. If we forget the action of ∂_{v_j} on v_j and ∂_{u_i} on

u_i, then also the bivector part of $(u \wedge \partial_v)(v \wedge \partial_u) + (v \wedge \partial_u)(u \wedge \partial_v)$ vanishes. The same holds for

$$\frac{1}{2}[(u \wedge v)(\partial_u \wedge \partial_v) + (\partial_u \wedge \partial_v)(u \wedge v)]$$

So we can now write

$$\{\Gamma_u,\, \Gamma_v\} + \frac{1}{2}[\{(u \wedge \partial_v),\ (v \wedge \partial_u)\} + \{(u \wedge v),\ (\partial_u \wedge \partial_v)\}] = -3\Delta_{uv} + \text{Part}$$

whereby "Part" is the part coming from the action of ∂_u on u or ∂_v on v and part of the scalar part of "Part" is already $-(m-1)(E_u + E_v)$. But is seems easier to at once calculate "Part" as a whole directly. Using the Hestenes overdot notation,

$$\begin{aligned} \text{Part} &= \text{Baard} + \text{Staart} \\ \text{Baard} &= \frac{1}{2}(u \wedge \dot{\partial}_v)(\dot{v} \wedge \partial_u) + \text{ibid}\{u <> v\} \\ &= \frac{1}{8}[u, \dot{\partial}_v][\dot{v}, \partial_u] + \text{ibid}\{u <> v\} \end{aligned}$$

and $[u,\ \dot{\partial}_v][\dot{v},\ \partial_u] =$

$$\begin{aligned} &= -m\, u\, \partial_u - \sum e_i\, u\, e_i\, \partial_u - u \sum e_i\, \partial_u\, e_i + \sum e_i\, u\, \partial_u\, e_i \\ &= -m\, u\, \partial_u - (m-2)u\, \partial_u - (m-2)u\, \partial_u + m\, E_u - (m-4)u \wedge \partial_u \\ &= 4E_u - (4m-8)u\, \partial_u \\ &= (4m-4)E_u + (4m-8)\Gamma_u \end{aligned}$$

so that

$$\text{Baard} = \frac{1}{2}(m-1)(E_u + E_v) + \frac{1}{2}(m-2)(\Gamma_u + \Gamma_v)$$

while also

$$\text{Staart} = \frac{1}{2}\{(\dot{\partial}_u \wedge \partial_v)(\dot{u} \wedge v) + \text{ib}(u <> v) + (\dot{\partial}_u \wedge \dot{\partial}_v)(\dot{u} \wedge \dot{v})\}$$

whereby clearly

$$\begin{aligned} \frac{1}{2}(\dot{\partial}_u \wedge \dot{\partial}_v)(\dot{u} \wedge \dot{v}) &= -\binom{m}{2} \\ \frac{1}{2}(\dot{\partial}_u \wedge \partial_v)(\dot{u} \wedge v) &= \frac{1}{8}[\dot{\partial}_u,\ \partial_v][\dot{u},\ v] \end{aligned}$$

and

$$\begin{aligned}[\dot{\partial}_u,\ \partial_v][\dot{u},\ v] = &\quad \sum e_i\,\partial_v\,e_i\,v + m\,\partial_v\,v - \sum e_i\,\partial_v\,v\,e_i + \partial_v \sum e_i\,v\,e_i \\ = &\quad 2(m-2)\partial_v\,v + m\,\partial_v\,v - m\,E_v - (m-4)v\wedge\partial_v \\ = &\quad -4(m-1)E_v + 4(m-2)\Gamma_v\end{aligned}$$

so that

$$\text{Staart} = -\frac{1}{2}(m-1)(E_u+E_v) + \frac{1}{2}(m-2)(\Gamma_u+\Gamma_v)$$

Hence we finally arrive at the fundamental identity

$$\{\Gamma_u,\,\Gamma_v\} + \frac{1}{2}[\{(u\wedge\partial_v),\ (v\wedge\partial_u)\} + \{(u\wedge v),\ (\partial_u\wedge\partial_v)\}] =$$

$$-3\Delta_{uv} + (m-2)(\Gamma_u+\Gamma_v) + \binom{m}{2}.$$

In spite of the clear importance of this result, it can also be used to evaluate the operator Δ_{uv} on functions $f(u,\,v)$ which are spherical monogenic in both variables u and v, i.e. $f(u,\,v)$ is homogeneous of degree (k,l) in (u,v) say and $\partial_u\,f(u,v) = \partial_v\,f(u,v) = 0$.
We know that $f(u,v)$ then satisfies

$$\Gamma_u\,f(u,v) = -k\,f(u,v), \quad \Gamma_v\,f(u,v) = -l\,f(u,v)$$

so that also

$$\{\Gamma_u,\,\Gamma_v\}\,f(u,v) = -2k\,l\,f(u,v)$$

and

$$(m-2)(\Gamma_u+\Gamma_v)\,f(u,v) = -(m-2)(k+l)\,f(u,v).$$

Next we have that

$$v\wedge\partial_u\,f(u,v) = \frac{1}{2}[v,\,\partial_u]\,f(u,v) = -\frac{1}{2}\partial_u\,v\,f(u,v) = < v,\partial_u > f(u,v)$$

so that $(u\wedge\partial_v)(v\wedge\partial_u)f(u,v) =$

$$\begin{aligned} = &\quad \frac{1}{2}[u,\partial_v] < v,\partial_u > f(u,v) \\ = &\quad \frac{1}{2}(u\partial_v < v,\partial_u > f(u,v) - \partial_v\,u < v,\partial_u > f(u,v)) \\ = &\quad \frac{1}{2}(2u\partial_v < v,\partial_u > f(u,v) + 2 < u,\partial_v >< v,\partial_u > f(u,v)) \\ = &\quad < u,\partial_v >< v,\partial_u > f(u,v) \\ = &\quad E_u\,f(u,v) + < u,\dot{\partial}_v >< v,\partial_u > \dot{f}(u,v)\end{aligned}$$

and so that also

$$\begin{aligned}\frac{1}{2}\{u\wedge\partial_v, v\wedge\partial_u\}f(u,v) = & \quad \frac{1}{2}(E_u+E_v)f(u,v) + < u,\dot{\partial}_v >< v,\partial_u > \dot{f}(u,v)\\ = & \quad \frac{1}{2}(k+l)f(u,v) + \text{kaart}\, f(u,v)\end{aligned}$$

Finally we have to evaluate $\frac{1}{2}\{u\wedge v,\partial_u\wedge\partial_v\}f(u,v)$.
Now for any four vectors a, b, c, d one has that

$$\frac{1}{2}\{a\wedge b, c\wedge d\} = [(a\wedge b)(c\wedge d)]_0 + [(a\wedge b)(c\wedge d)]_4$$

from which it follows that

$$\frac{1}{2}\{u\wedge v,\partial_u\wedge\partial_v\}f(u,v) = \text{staart}\, f - [(u\wedge v)(\partial_u\wedge\partial_v)]_2 f$$

and

$$[(a\wedge b)(c\wedge d)]_2 = (< b,c > a\wedge d- < b,d > a\wedge c+ < a,d > b\wedge c- < a,c > b\wedge d)$$

so that

$$\begin{aligned}[(u\wedge v)(\partial_u\wedge\partial_v)]_2 f &= (E_v\Gamma_u + E_u\Gamma_v + < v,\dot{\partial}_u > u\wedge\partial_v + < u,\dot{\partial}_v > v\wedge\partial_u)\dot{f}\\ &= (E_v\Gamma_u + E_u\Gamma_v) + 2\text{kaart}\, f\end{aligned}$$

Hence we get

$\frac{1}{2}\{u\wedge v,\partial_u\wedge\partial_v\}f$

$$\begin{aligned}= & \quad \frac{1}{2}((m-2)(\Gamma_u+\Gamma_v) - (m-1)(E_u+E_v))f - (E_v\Gamma_u + E_u\Gamma_v)f\\ & \quad - 2\text{kaart} f - \binom{m}{2} f\\ = & \quad [\frac{1}{2}((m-2)(-k-l) - (m-1)(k+l)) + 2kl - 2\text{kaart} - \binom{m}{2}]f\\ = & \quad -\frac{1}{2}(2m-3)(k+l)f + klf - 2\text{kaart} f - \binom{m}{2} f\end{aligned}$$

and so

$$(\{\Gamma_u,\Gamma_v\} + \frac{1}{2}(\{u\wedge\partial_v, v\wedge\partial_u\} + \{u\wedge v,\partial_u\wedge\partial_v\}))f(u,v)$$

$$= -\text{kaart} f - (m-2)(k+l)f - \binom{m}{2} f$$

As this should be equal to

$$-3\Delta_{uv} f - (m-2)(k+l)f - \binom{m}{2} f$$

we obtain that

$$\Delta_{uv} f(u,v) = \frac{1}{3} < v, \dot{\partial}_u >< u, \partial_v > \dot{f}(u,v).$$

The outcome of the whole calculation is hence simple. Yet, in case $f(u_1, \ldots, u_m)$ is spherical monogenic in the variables $u_1, \ldots, u_m$ it does not really follow that the functions

$$f(\bar{s}\, u_1\, s, \ldots, \bar{s}\, u_m\, s), \qquad s f(\bar{s}\, u_1\, s, \ldots, \bar{s}\, u_m\, s)$$

are eigenfunctions of the Laplace-Beltrami operator on the spingroup although they are eigenfunctions of the operator $\frac{1}{4}\Delta_s - \sum \Delta_{ui,uj}$. This is also to be expected because spherical monogenics of several vector variables $f(u_1, \ldots, u_m)$ do not give rise to models for irreducible representations of Spin(m); to arrive at such models one should consider spherical monogenic of the special form

$$f(u_1,\, u_1 \wedge u_2, \ldots, u_1 \wedge \ldots \wedge u_m).$$

Using the above formula for Δ_{uv}, it is not hard to show that the operator Δ_{uiuj} vanishes on spherical monogenics of this form. This proves that they are eigenfunctions of the Laplacian Δ_s as could be expected from theoretical considerations.

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