Fourier transforms of hypercomplex signals

Hendrik De Bie[1] and Nele De Schepper[1]

[1]Department of Mathematical Analysis
Faculty of Engineering and Architecture
Ghent University
Galglaan 2, 9000 Ghent, Belgium
{hendrik.debie,nele.deschepper}@ugent.be

Abstract — *An overview is given to a new approach for obtaining generalized Fourier transforms in the context of hypercomplex analysis (or Clifford analysis). These transforms are applicable to higher-dimensional signals with several components and are different from the classical Fourier transform in that they mix the components of the signal.*
Subsequently, attention is focused on the special case of the so-called Clifford-Fourier transform where recently a lot of progress has been made. A fractional version of this transform is introduced and a series expansion for its integral kernel is obtained.

1 Introduction

Recently, there has been a lot of interest in the theory of hypercomplex signals (i.e. functions taking values in a Clifford algebra) and the possibility of defining and using Fourier transforms that interact with the Clifford algebra structure. This has been investigated from a practical engineering point of view (see e.g. [1, 2, 3, 4, 5]) but also from a purely mathematical point of view (see e.g. [6, 7, 8]) using the function theory of Clifford analysis established in a.o. [9]. Also in applications, there is an increasing interest in having available a good hypercomplex Fourier transform (e.g. in GIS research, see [10]).

In the context of this paper, a hypercomplex signal is a function of m variables $x_1, \ldots, x_m$ taking values in a Clifford algebra $\mathcal{C}l_{0,m}$. This means that our signals can equivalently be represented by a vector

$$[f_1(x_1, \ldots, x_m), \ldots, f_{2^m}(x_1, \ldots, x_m)]$$

with at most 2^m components $f_i(x_1, \ldots, x_m)$ which are real-valued signals.

There are quite a few drawbacks to most of the kernels proposed so far in the literature. First, several authors work only in low dimensions (dimension 3 or 4, enabling them to use quaternions instead of a full Clifford algebra) which is usually because they have a specific application in mind in these dimensions. Second, most authors use ad hoc formulations for the kernel function of their transforms, after which the properties of the related transform are studied in detail.

In recent work (see [11, 12, 13, 14, 15]) we have developed a different methodology: we start from a list of properties or general mathematical principles we want a hypercomplex Fourier transform (hypercomplex FT) to have, and then determine all kernels that satisfy these properties.

In the present paper, after a short introduction to Clifford algebras and analysis, we give an overview of the results obtained so far. In particular we devote our attention to the so-called Clifford-Fourier transform (as studied in [15, 11]). Finally, we introduce a new fractional version of the Clifford-Fourier transform and obtain an expression for its integral kernel.

2 Preliminaries

Clifford analysis (see e.g. [9]) is a theory that offers a natural generalization of complex analysis to higher dimensions. To $\mathbb{R}^m$, the Euclidean space in m dimensions, we first associate the Clifford algebra $\mathcal{C}l_{0,m}$, generated by the canonical basis e_i, $i = 1, \ldots, m$. These generators satisfy the multiplication rules $e_i e_j + e_j e_i = -2\delta_{ij}$.

The Clifford algebra $\mathcal{C}l_{0,m}$ can be decomposed as $\mathcal{C}l_{0,m} = \oplus_{k=0}^{m} \mathcal{C}l_{0,m}^k$ with $\mathcal{C}l_{0,m}^k$ the space of k-vectors defined by

$$\mathcal{C}l_{0,m}^k = \text{span}\{e_{i_1 \ldots i_k} = e_{i_1} \ldots e_{i_k}, i_1 < \ldots < i_k\}.$$

More precisely, we have that the space of 1-vectors is given by $\mathcal{C}l_{0,m}^1 = \text{span}\{e_i, i = 1, \ldots, m\}$ and it is obvious that this space is isomorphic with $\mathbb{R}^m$. The space of so-called bivectors is given explicitly by $\mathcal{C}l_{0,m}^2 = \text{span}\{e_{ij} = e_i e_j, i < j\}$.

We identify the point $(x_1, \ldots, x_m)$ in $\mathbb{R}^m$ with the vector variable $\underline{x}$ given by $\underline{x} = \sum_{j=1}^{m} x_j e_j$. The Clifford product of two vectors splits into a scalar part and a bivector part:

$$\underline{x}\underline{y} = \underline{x}.\underline{y} + \underline{x} \wedge \underline{y},$$

with

$$\underline{x}.\underline{y} = -\langle \underline{x}, \underline{y} \rangle = -\sum_{j=1}^{m} x_j y_j = \frac{1}{2}(\underline{x}\underline{y} + \underline{y}\underline{x})$$

and

$$\underline{x} \wedge \underline{y} = \sum_{j<k} e_{jk}(x_j y_k - x_k y_j) = \frac{1}{2}(\underline{x}\underline{y} - \underline{y}\underline{x}).$$

Note that the square of a vector variable $\underline{x}$ is scalar-valued and equals the norm squared up to a minus sign: $\underline{x}^2 = -\langle \underline{x}, \underline{x} \rangle = -|\underline{x}|^2$. We also introduce a first order vector differential operator by

$$\partial_{\underline{x}} = \sum_{j=1}^{m} \partial_{x_j} e_j.$$

This operator is the so-called Dirac operator. Its square equals, up to a minus sign, the Laplace operator in $\mathbb{R}^m$: $\partial_{\underline{x}}^2 = -\Delta$.

The Dirac operator $\partial_{\underline{x}}$ acts on functions f taking values in $\mathcal{C}l_{0,m}$. Such functions can always be decomposed as

$$f(x) = f_0(x) + \sum_{i=1}^{m} e_i f_i(x) + \sum_{i<j} e_i e_j f_{ij}(x) + \ldots + e_1 \ldots e_m f_{1 \ldots m}(x)$$

with $f_0, f_i, f_{ij}, \ldots, f_{1\ldots m}$ all real-valued functions. As f has in total 2^m components (which equals the dimension of $\mathcal{C}l_{0,m}$), this is the maximal number of components of the signals that can be studied using the transforms in the next section.

Finally, another important operator in Clifford analysis is the so-called Gamma operator, defined by

$$\Gamma_{\underline{x}} = -\underline{x} \wedge \partial_{\underline{x}} = -\sum_{j<k} e_{jk}(x_j \partial_{x_k} - x_k \partial_{x_j}).$$

This operator is bivector-valued.

3 Overview of recent results

In the paper [11] we started our analysis from 4 different, yet equivalent, definitions of the classical Fourier transform in $\mathbb{R}^m$. Each of these formulations allows for generalization to hypercomplex FTs. We summarize them briefly. The first and most basic formulation is given by the integral transform

$$\textbf{F1} \quad \mathcal{F}[f](\underline{y}) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{-i\langle \underline{x}, \underline{y} \rangle} f(\underline{x})\, dx$$

with $\langle \underline{x}, \underline{y} \rangle$ the standard inner product and dx the Lebesgue measure on $\mathbb{R}^m$. Alternatively, one can rewrite the transform as

$$\textbf{F2} \quad \mathcal{F}[f](\underline{y}) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} K(\underline{x}, \underline{y})\, f(\underline{x})\, dx$$

where $K(\underline{x}, \underline{y})$ is, up to a multiplicative constant, the unique solution of the system of PDEs

$$\partial_{y_j} K(\underline{x}, \underline{y}) = -i x_j K(\underline{x}, \underline{y}), \quad j = 1, \ldots, m.$$

Yet another formulation is given by

$$\textbf{F3} \quad \mathcal{F} = e^{\frac{i\pi m}{4}} e^{\frac{i\pi}{4}(\Delta - |\underline{x}|^2)}$$

with Δ the Laplacian in $\mathbb{R}^m$. This expression connects the Fourier transform with the Lie algebra $\mathfrak{sl}_2$ generated by Δ and $|\underline{x}|^2$ and with the theory of the quantum harmonic oscillator. Finally, the kernel can also be expressed as an infinite series in terms of special functions as

$$\textbf{F4} \quad K(\underline{x}, \underline{y}) = 2^\lambda \Gamma(\lambda) \sum_{k=0}^{\infty} (k+\lambda)(-i)^k (|\underline{x}||\underline{y}|)^{-\lambda} J_{k+\lambda}(|\underline{x}||\underline{y}|)\, C_k^\lambda(\langle \underline{\xi}, \underline{\eta} \rangle), \tag{1}$$

where $\underline{\xi} = \underline{x}/|\underline{x}|$, $\underline{\eta} = \underline{y}/|\underline{y}|$ and $\lambda = (m-2)/2$. Here, J_ν is the Bessel function and C_k^λ the Gegenbauer polynomial.

Each formulation has its specific advantages and uses. **F1** immediately yields a bound of the kernel and is hence ideal to study the transform on L_1 spaces or more general function spaces. **F2** gives the calculus properties of the transform. **F3** emphasizes the structural (Lie algebraic) properties of the Fourier transform and allows to compute its eigenfunctions and spectrum. Finally, **F4** connects the Fourier transform with the theory of special functions, and is the ideal formulation to obtain e.g. the Bochner identities.

These 4 definitions serve as guidance in defining hypercomplex Fourier transforms, as each definition gives access to a crucial piece of information about the transform. A suitable hypercomplex transform should hence be expressible in these 4 different ways.

So far, we have applied these ideas in 3 different directions of hypercomplex FTs, namely

- k-vector Fourier transforms ([12])
- radially deformed Fourier transforms ([13, 14])
- Clifford-Fourier transforms ([15, 11]).

We briefly discuss the first 2 directions, after which we focus on the Clifford-Fourier transform. k-vector Fourier transforms are transforms with integral kernel $K(\underline{x}, \underline{y})$ a k-vector valued function (i.e. taking values in $\mathcal{C}l^k_{0,m}$) which is moreover a solution to the system of PDEs (compare with **F2**)

$$\partial_{\underline{y}} K(\underline{x}, \underline{y}) = -iK(\underline{x}, \underline{y})\underline{x}. \tag{2}$$

Contrary to the classical case, the system (2) does not have a single unique solution but instead a whole family of suitable solutions.

Radially deformed Fourier transforms are obtained by adapting the formulation **F3** of the classical Fourier transform to

$$\mathcal{F}_{\mathbf{D}} = e^{i\frac{\pi}{2}\left(\frac{1}{2}+\frac{\mu-1}{2(1+c)}\right)} e^{\frac{-i\pi}{4(1+c)^2}\left(\mathbf{D}^2-(1+c)^2\underline{x}^2\right)}.$$

with

$$\mathbf{D} = \partial_{\underline{x}} + c|\underline{x}|^{-2}\underline{x}\sum_{j=1}^{m} x_j \partial_{x_j},$$

a radial deformation (depending on the numerical parameter $c \in \mathbb{R}$) of the Dirac operator $\partial_{\underline{x}}$ underlying the Laplace operator and $\mu = -\mathbf{D}(\underline{x}) \in \mathbb{R}$. So far, for this class of transforms also formulation **F2** and **F4** have been obtained, but the question to find an explicit analytical expression for its kernel (formulation **F1**) is extremely hard.

The situation is quite different for the Clifford-Fourier transform, where by now an almost complete treatment has been obtained. This transform was first introduced in [16] and further studied in [17, 18]. It was initially also defined by a generalization of **F3**, namely

$$\mathcal{F}_{\pm} = e^{\frac{i\pi m}{4}} e^{\mp\frac{i\pi}{2}\Gamma} e^{\frac{i\pi}{4}(\Delta-|\underline{x}|^2)} = e^{\frac{i\pi m}{4}} e^{\frac{i\pi}{4}(\Delta-|\underline{x}|^2\mp 2\Gamma)}.$$

The motivation behind this definition was to find a couple of transforms $\mathcal{F}_{\pm}$ such that

$$\mathcal{F}_{+}\mathcal{F}_{-} = \mathcal{F}_{-}\mathcal{F}_{+} = \mathcal{F}^2.$$

It turned out to be a difficult problem to find the other formulations for this particular transform. A breakthrough was recently obtained in [15]. As an analog of formulation **F4** the following result was obtained:

Theorem 3.1 *The Clifford-Fourier transform* $\mathcal{F}_- = e^{\frac{i\pi m}{4}} e^{\frac{i\pi}{2}\Gamma} e^{\frac{i\pi}{4}(\Delta - |\underline{x}|^2)}$ *is given by the integral transform*

$$\frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} K_-(\underline{x}, \underline{y})\, f(\underline{x})\, dx$$

with integral kernel

$$K_-(\underline{x}, \underline{y}) = A_\lambda + B_\lambda + \left(\underline{x} \wedge \underline{y}\right) C_\lambda$$

with

$$\begin{aligned}
A_\lambda(w, z) =& \, 2^{\lambda-1}\Gamma(\lambda+1) \sum_{k=0}^{\infty} (i^{2\lambda+2} + (-1)^k) z^{-\lambda} J_{k+\lambda}(z)\, C_k^\lambda(w), \\
B_\lambda(w, z) =& -2^{\lambda-1}\Gamma(\lambda) \sum_{k=0}^{\infty} (k+\lambda)(i^{2\lambda+2} - (-1)^k) z^{-\lambda} J_{k+\lambda}(z)\, C_k^\lambda(w), \\
C_\lambda(w, z) =& -2^{\lambda}\Gamma(\lambda+1) \sum_{k=1}^{\infty} (i^{2\lambda+2} + (-1)^k) z^{-\lambda-1} J_{k+\lambda}(z)\, C_{k-1}^{\lambda+1}(w),
\end{aligned}$$

where $z = |\underline{x}||\underline{y}|$, $w = \langle \underline{\xi}, \underline{\eta} \rangle$ *and* $\lambda = (m-2)/2$.

Note that it can be computed that the kernel $K_+(\underline{x}, \underline{y})$ of $\mathcal{F}_+$ is given by $K_+(\underline{x}, \underline{y}) = \overline{K_-(\underline{x}, -\underline{y})}$ where the bar denotes complex conjugation.

Moreover, it was then possible to obtain formulation **F1** in the case where the dimension m is even.

Theorem 3.2 *The kernel of the Clifford-Fourier transform in even dimension* $m > 2$ *is given by*

$$\begin{aligned}
K_-(\underline{x}, \underline{y}) &= e^{i\frac{\pi}{2}\Gamma_{\underline{y}}} e^{-i\langle \underline{x}, \underline{y} \rangle} \\
&= (-1)^{\frac{m}{2}} \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left(A^*_{(m-2)/2}(s,t) + B^*_{(m-2)/2}(s,t) + (\underline{x} \wedge \underline{y})\, C^*_{(m-2)/2}(s,t)\right)
\end{aligned}$$

where $s = \langle \underline{x}, \underline{y} \rangle$ *and* $t = |\underline{x} \wedge \underline{y}| = \sqrt{|\underline{x}|^2|\underline{y}|^2 - s^2}$ *and*

$$\begin{aligned}
A^*_{(m-2)/2}(s,t) &= \sum_{\ell=0}^{\lfloor \frac{m}{4} - \frac{3}{4} \rfloor} s^{m/2-2-2\ell} \frac{1}{2^\ell \ell!} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2} - 2\ell - 1\right)} \widetilde{J}_{(m-2\ell-3)/2}(t), \\
B^*_{(m-2)/2}(s,t) &= -\sum_{\ell=0}^{\lfloor \frac{m}{4} - \frac{1}{2} \rfloor} s^{m/2-1-2\ell} \frac{1}{2^\ell \ell!} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2} - 2\ell\right)} \widetilde{J}_{(m-2\ell-3)/2}(t), \\
C^*_{(m-2)/2}(s,t) &= -\sum_{\ell=0}^{\lfloor \frac{m}{4} - \frac{1}{2} \rfloor} s^{m/2-1-2\ell} \frac{1}{2^\ell \ell!} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2} - 2\ell\right)} \widetilde{J}_{(m-2\ell-1)/2}(t)
\end{aligned}$$

with $\widetilde{J}_\alpha(t) = t^{-\alpha} J_\alpha(t)$.

Subsequently, in [11] we studied the formulation **F2** for the Clifford-Fourier transform, which determines the kernel of this transform as a solution of the following system of PDEs:

$$\begin{aligned} \partial_{\underline{y}}[K_+(\underline{x},\underline{y})] &= (-i)^m \, K_-(\underline{x},\underline{y}) \, \underline{x} \\ [K_+(\underline{x},\underline{y})]\partial_{\underline{x}} &= (-i)^m \, \underline{y} \, K_-(\underline{x},\underline{y}). \end{aligned} \tag{3}$$

Indeed, it is e.g. easy to check that the result of Theorem 3.2 satisfies (3). However, a new phenomenon appears here, as the solution given by Theorem 3.2 is not the unique solution of the system (contrary to the case of the ordinary Fourier transform or the Dunkl transform, see [19]).

To obtain more insight in this phenomenon, we determined all solutions of (3) that are of the form

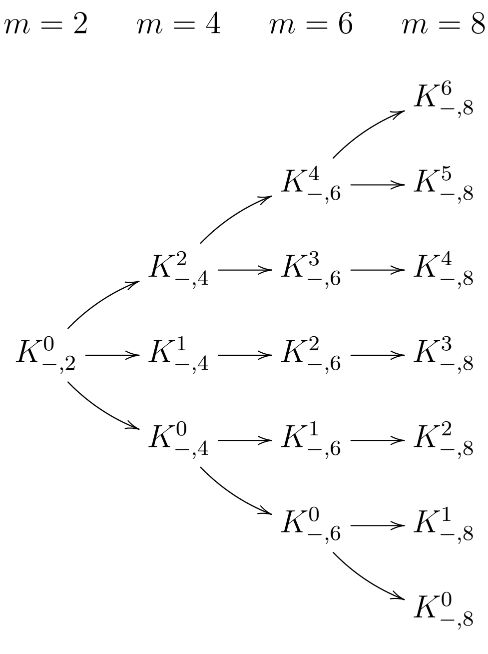
$$K_-(\underline{x},\underline{y}) = f(s,t) + (\underline{x} \wedge \underline{y}) \, g(s,t)$$

with $s = \langle \underline{x}, \underline{y} \rangle$, $t = |\underline{x} \wedge \underline{y}|$ and f and g real-valued functions. In this way, we aimed to find an explanation for the precise form of the kernel given in Theorem 3.2.

As result, we obtained, as solutions of (3) in even dimension m, a set of $m-1$ kernels

$$K^j_{-,m}(\underline{x},\underline{y}), \qquad j = 0, \dots, m-2$$

that can be organized in a diagram as follows



Here the arrows describe certain recursion relations (given explicitly in [11]).

The middle line, namely the kernels $K^{(m-2)/2}_{-,m}(\underline{x},\underline{y})$ correspond to the kernels obtained in Theorem 3.2. The other kernels yield new integral transforms that have the same calculus properties as the original Clifford-Fourier transform but with different spectrum. Of all these transforms, only the transform given by Theorem 3.2 is unitary. We observe that with each step in the dimension, 2 new kernels appear. In particular the kernels $K^0_{-,m}(\underline{x},\underline{y})$ on the lower diagonal are interesting, as they give us the Fourier-Bessel transform which was previously introduced in [20] for different reasons.

For explicit formulas for the kernels $K^j_{-,m}(\underline{x},\underline{y})$ we refer the reader to formula (4.6) and Theorem 4.2 (where series representations as in **F4** are obtained) in [11].

4 Fractional version of the Clifford-Fourier transform

In this section, we show how a fractional version of the Clifford-Fourier transform can be obtained. To that end, we adapt the definition **F3**

$$\mathcal{F}_\pm = e^{\frac{i\pi m}{4}} e^{\frac{i\pi}{4}(\Delta - |\underline{x}|^2 \mp 2\Gamma)}$$

to

$$\mathcal{F}_{\pm,\alpha} = e^{\frac{i\alpha m}{2}} e^{\frac{i\alpha}{2}(\Delta - |\underline{x}|^2 \mp 2\Gamma)}$$

where $\alpha \in [-\pi/2, \pi/2]$. Notice that we have immediately

$$\mathcal{F}_{+,\alpha}\mathcal{F}_{-,\alpha} = \mathcal{F}_\alpha^2$$

with $\mathcal{F}_\alpha = e^{\frac{i\alpha m}{2}} e^{\frac{i\alpha}{2}(\Delta - |\underline{x}|^2)}$ the fractional version of the ordinary Fourier transform, given explicitly by (see [21])

$$\mathcal{F}_\alpha(f)(\underline{y}) = \left(\pi(1 - e^{-2i\alpha})\right)^{-m/2} \int_{\mathbb{R}^m} e^{-i\langle \underline{x},\underline{y}\rangle/\sin\alpha} e^{\frac{i}{2}(\cot\alpha)(|\underline{x}|^2 + |\underline{y}|^2)} f(\underline{x}) dx$$

Our aim is to find an integral expression for $\mathcal{F}_{-,\alpha}$:

$$\mathcal{F}_{-,\alpha}(f)(\underline{y}) = \left(\pi(1 - e^{-2i\alpha})\right)^{-m/2} \int_{\mathbb{R}^m} K_{-,\alpha}(\underline{x},\underline{y})\, f(\underline{x})\, dx.$$

We compute formally

$$\begin{aligned}
\mathcal{F}_{-,\alpha} &= e^{\frac{i\alpha m}{2}} e^{\frac{i\alpha}{2}(\Delta - |\underline{x}|^2 + 2\Gamma)} \\
&= e^{\frac{i\alpha m}{2}} e^{i\alpha\Gamma} e^{\frac{i\alpha}{2}(\Delta - |\underline{x}|^2)} \\
&= \left(\pi(1 - e^{-2i\alpha})\right)^{-m/2} e^{i\alpha\Gamma_{\underline{y}}} \int_{\mathbb{R}^m} e^{-i\langle \underline{x},\underline{y}\rangle/\sin\alpha} e^{\frac{i}{2}(\cot\alpha)(|\underline{x}|^2 + |\underline{y}|^2)} (.) dx
\end{aligned}$$

hence the kernel is given by

$$\begin{aligned}
K_{-,\alpha}(\underline{x},\underline{y}) &= e^{i\alpha\Gamma_{\underline{y}}} \left(e^{-i\langle \underline{x},\underline{y}\rangle/\sin\alpha} e^{\frac{i}{2}(\cot\alpha)(|\underline{x}|^2 + |\underline{y}|^2)} \right) \\
&= e^{\frac{i}{2}(\cot\alpha)(|\underline{x}|^2 + |\underline{y}|^2)} e^{i\alpha\Gamma_{\underline{y}}} \left(e^{-i\langle \underline{x},\underline{y}\rangle/\sin\alpha} \right)
\end{aligned}$$

where the last line follows because $\Gamma_{\underline{y}}$ commutes with $|\underline{y}|$. Now using the series expansion (1), the term $e^{i\alpha\Gamma_{\underline{y}}} \left(e^{-i\langle \underline{x},\underline{y}\rangle/\sin\alpha} \right)$ can be computed. Indeed, we find

$$\begin{aligned}
& e^{i\alpha\Gamma_{\underline{y}}} \left(e^{-i\langle \underline{x},\underline{y}\rangle/\sin\alpha} \right) \\
&= e^{i\alpha\Gamma_{\underline{y}}} \Gamma(\lambda) \sum_{k=0}^{\infty} 2^\lambda (k+\lambda)(i\sin\alpha)^{-k} (|\underline{x}||\underline{y}|)^k \widetilde{J}_{k+\lambda}\left(\frac{|\underline{x}||\underline{y}|}{\sin\alpha} \right) \; C_k^\lambda(\langle \underline{\xi},\underline{\eta}\rangle) \\
&= \Gamma(\lambda) \sum_{k=0}^{\infty} 2^\lambda (k+\lambda)(i\sin\alpha)^{-k} \widetilde{J}_{k+\lambda}\left(\frac{|\underline{x}||\underline{y}|}{\sin\alpha} \right) \; e^{i\alpha\Gamma_{\underline{y}}} \left((|\underline{x}||\underline{y}|)^k C_k^\lambda(\langle \underline{\xi},\underline{\eta}\rangle) \right)
\end{aligned}$$

so we have reduced the problem to calculating $e^{i\alpha\Gamma_{\underline{y}}}\left((|\underline{x}||\underline{y}|)^k C_k^\lambda(\langle \underline{\xi}, \underline{\eta}\rangle)\right)$. This can be done in a manner analogous to Lemma 3.1 in [15]. The result is given in the following lemma.

Lemma 4.1 *One has*

$$\begin{aligned} e^{i\alpha\Gamma_{\underline{y}}}(|\underline{x}||\underline{y}|)^k C_k^\lambda(\langle \underline{\xi}, \underline{\eta}\rangle) = & \frac{1}{2}\left(e^{i\alpha(k+m-2)} + e^{-i\alpha k}\right)(|\underline{x}||\underline{y}|)^k C_k^\lambda(\langle \underline{\xi}, \underline{\eta}\rangle) \\ & - \frac{\lambda}{2(k+\lambda)}\left(e^{i\alpha(k+m-2)} - e^{-i\alpha k}\right)(|\underline{x}||\underline{y}|)^k C_k^\lambda(\langle \underline{\xi}, \underline{\eta}\rangle) \\ & + \frac{\lambda}{k+\lambda}\underline{x}\wedge\underline{y}(e^{i\alpha(k+m-2)} - e^{-i\alpha k})(|\underline{x}||\underline{y}|)^{k-1} C_{k-1}^{\lambda+1}(\langle \underline{\xi}, \underline{\eta}\rangle). \end{aligned}$$

Using this lemma and the previous computation, we arrive at the following series representation for the kernel of the fractional Clifford-Fourier transform.

Theorem 4.2 *The fractional Clifford-Fourier transform* $\mathcal{F}_{-,\alpha} = e^{\frac{i\alpha m}{2}} e^{\frac{i\alpha}{2}(\Delta - |\underline{x}|^2 + 2\Gamma)}$ *is given by the integral transform*

$$\left(\pi(1 - e^{-2i\alpha})\right)^{-m/2} \int_{\mathbb{R}^m} K_{-,\alpha}(\underline{x}, \underline{y})\, f(\underline{x})\, dx$$

with integral kernel

$$K_{-,\alpha}(\underline{x}, \underline{y}) = \left(A_\lambda + B_\lambda + \left(\underline{x}\wedge\underline{y}\right) C_\lambda\right) e^{\frac{i}{2}(\cot\alpha)(|\underline{x}|^2 + |\underline{y}|^2)}$$

with

$$\begin{aligned} A_\lambda(w,z) = & -2^{\lambda-1}\Gamma(\lambda+1)\sum_{k=0}^{\infty} i^{-k}(e^{i\alpha(k+2\lambda)} - e^{-i\alpha k}) z^{-\lambda} J_{k+\lambda}(z)\, C_k^\lambda(w), \\ B_\lambda(w,z) = & \, 2^{\lambda-1}\Gamma(\lambda)\sum_{k=0}^{\infty}(k+\lambda) i^{-k}(e^{i\alpha(k+2\lambda)} + e^{-i\alpha k}) z^{-\lambda} J_{k+\lambda}(z)\, C_k^\lambda(w), \\ C_\lambda(w,z) = & \frac{2^\lambda\Gamma(\lambda+1)}{\sin\alpha}\sum_{k=1}^{\infty} i^{-k}(e^{i\alpha(k+2\lambda)} - e^{-i\alpha k}) z^{-\lambda-1} J_{k+\lambda}(z)\, C_{k-1}^{\lambda+1}(w), \end{aligned}$$

where $z = (|\underline{x}||\underline{y}|)/\sin\alpha$, $w = \langle \underline{\xi}, \underline{\eta}\rangle$ *and* $\lambda = (m-2)/2$.

Finding an explicit analytic expression (as in **F1**) for this series representation is again a difficult problem. Hence we postpone its determination to a subsequent publication.

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