Non-abelian representations of the slim dense near hexagons on 81 and 243 points

B. De Bruyn, B. K. Sahoo and N. S. N. Sastry

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Abstract

We prove that the near hexagon $Q(5,2) \times \mathbb{L}_3$ has a non-abelian representation in the extra-special 2-group 2_+^{1+12} and that the near hexagon $Q(5,2) \otimes Q(5,2)$ has a non-abelian representation in the extra-special 2-group 2_-^{1+18}. The description of the non-abelian representation of $Q(5,2) \otimes Q(5,2)$ makes use of a new combinatorial construction of this near hexagon.

Keywords: near hexagon, non-abelian representation, extra-special 2-group
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1 Introduction

Let $\mathcal{S} = (P, L)$ be a partial linear space with point set P and line set L. We suppose that $\mathcal{S}$ is *slim*, i.e., that every line of $\mathcal{S}$ is incident with precisely three points. For distinct points $x, y \in P$, we write $x \sim y$ if they are collinear. In that case, we denote by xy the unique line containing x and y and define $x * y$ by $xy = \{x, y, x * y\}$. For $x \in P$, we define $x^\perp := \{x\} \cup \{y \in P : y \sim x\}$. If $x, y \in P$, then $d(x, y)$ denotes the distance between x and y in the collinearity graph of $\mathcal{S}$.

A *representation* [9, p.525] of $\mathcal{S}$ is a pair (R, ψ), where R is a group and ψ is a mapping from P to the set of involutions of R, satisfying:

(R1) R is generated by the image of ψ;

(R2) ψ is one-one on each line $\{x, y, x * y\}$ of $\mathcal{S}$ and $\psi(x)\psi(y) = \psi(x * y)$.

Notice that if $x \sim y$, then $\psi(x)$ and $\psi(y)$ necessarily commute by condition (R2). The group R is called a *representation group* of $\mathcal{S}$. A representation (R, ψ) of $\mathcal{S}$ is *faithful* if ψ is injective and is *abelian* or *non-abelian* according as R is abelian or not. Note that, in [9], 'non-abelian representation' means that 'the representation group is not necessarily abelian'. Abelian representations are called embeddings in the literature. For an abelian representation, the representation group is an elementary abelian 2-group and hence can be considered as a vector space over the field $\mathbb{F}_2$ with two elements. We refer to [8] and [12, Sections 1 and 2] for more on representations of partial linear spaces with $p+1$ points per line, where p is a prime.

A finite 2-group G is called *extra-special* if its Frattini subgroup $\Phi(G)$, its commutator subgroup $G' = [G, G]$ and its center $Z(G)$ coincide and have order 2. We refer to [5, Section 20, pp.78–79] or [6, Chapter 5, Section 5] for the properties of extra-special 2-groups which we will mention now. An extra-special 2-group is of order 2^{1+2m} for some integer $m \geq 1$. Let D_8 and Q_8, respectively, denote the dihedral and the quaternion groups of order 8. A non-abelian 2-group of order 8 is extra-special and is isomorphic to either D_8 or Q_8. If G is an extra-special 2-group of order 2^{1+2m}, $m \geq 1$, then the exponent of G is 4 and either G is a central product of m copies of D_8, or G is a central product of $m-1$ copies of D_8 and one copy of Q_8. If the former (respectively, latter) case occurs, then the extra-special 2-group is denoted by 2_+^{1+2m} (respectively, 2_-^{1+2m}).

A partial linear space $\mathcal{S} = (P, L)$ is called a *near polygon* if for every point p and every line L, there exists a unique point on L nearest to p. If d is the maximal distance between two points of $\mathcal{S}$, then the near polygon is also called a *near $2d$-gon*. A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance 2 have at least two common neighbours. By [1], there are up to isomorphism 11 slim dense near hexagons. The paper [13] initiated the study of the non-abelian representations of these dense near hexagons.

Suppose (R, ψ) is a non-abelian representation of a slim dense near hexagon. Then by [13, Proposition 4.1, p.205], (R, ψ) necessarily is faithful and for $x, y \in P$, $[\psi(x), \psi(y)] \neq 1$ if and only if x and y are at maximal distance 3 from each other. If $\mathcal{S}$ is the (up to isomorphism) unique slim dense near hexagon on 81 points, which will be denoted by $Q(5,2) \times \mathbb{L}_3$ in the sequel, then it was shown in [13, Theorem 1.6, p.199] that R necessarily is isomorphic to the extra-special 2-group 2_+^{1+12}. If $\mathcal{S}$ is the (up to isomorphism) unique slim dense near hexagon on 243 points, which will be denoted by

$Q(5,2) \otimes Q(5,2)$ in the sequel, then it was shown in [13, Theorem 1.6, p.199], that R necessarily is isomorphic to the extra-special 2-group 2_-^{1+18}. The question whether such non-abelian representations exist remained however unanswered in [13]. The following theorem, which is the main result of this paper, deals with these existence problems.

Theorem 1.1. (1) *The slim dense near hexagon $Q(5,2) \times \mathbb{L}_3$ has a non-abelian representation in the extra-special 2-group 2_+^{1+12}.*

(2) *The slim dense near hexagon $Q(5,2) \otimes Q(5,2)$ has a non-abelian representation in the extra-special 2-group 2_-^{1+18}.*

The slim dense near hexagon $Q(5,2) \otimes Q(5,2)$ has many substructures isomorphic to $Q(5,2) \times \mathbb{L}_3$. We will describe a non-abelian representation of $Q(5,2) \times \mathbb{L}_3$ in Section 4. In Section 5, we will use this to construct a non-abelian representation of $Q(5,2) \otimes Q(5,2)$. To describe the non-abelian representation of $Q(5,2) \otimes Q(5,2)$, we make use of a model of $Q(5,2) \otimes Q(5,2)$ which we discuss in Sections 2 and 3.

Remark. Two other constructions of non-abelian representations of slim dense near polygons, in particular, of the slim dense near hexagons on 105 and 135 points, can be found in the paper [10].

2 The point-line geometry $\mathcal{S}_\theta$

Near quadrangles are usually called *generalized quadrangles* (GQ's). A GQ is said to be of *order* (s,t) if every line is incident with precisely $s+1$ points and if every point is incident with precisely $t+1$ lines. Up to isomorphism, there exist unique GQ's of order $(2,2)$ and $(2,4)$, see e.g. [11]. These GQ's are denoted by $W(2)$ and $Q(5,2)$, respectively. A *spread* of a point-line geometry is a set of lines partitioning its point set. A spread S of $Q(5,2)$ is called a *spread of symmetry* if for every line $L \in S$ and every two points $x_1, x_2 \in L$, there exists an automorphism of $Q(5,2)$ fixing each line of S and mapping x_1 to x_2. By [2, Section 7.1], $Q(5,2)$ has up to isomorphism a unique spread of symmetry.

Now, suppose S is a given spread of symmetry of $Q(5,2)$. If L_1 and L_2 are two distinct lines of S and if G denotes the unique (3×3)-subgrid of

$Q(5,2)$ containing L_1 and L_2, then the unique line L_3 of G disjoint from L_1 and L_2 is also contained in S.

Suppose θ is a map from $S \times S$ to $\mathbb{Z}_3$ (the additive group of order three) satisfying the following property:

($*$) If L_1, L_2, L_3 are three lines of S contained in a grid of $Q(5,2)$, then $\theta(L_1, L_2) + \theta(L_2, L_3) = \theta(L_1, L_3)$.

Notice that $\theta(L, L) = 0$ and $\theta(M, L) = -\theta(L, M)$ for all $L, M \in S$. With θ, there is associated a point-line geometry $\mathcal{S}_\theta$. The points of $\mathcal{S}_\theta$ are of four types:

($P1$) The points x of $Q(5,2)$.

($P2$) The symbols $\bar{x}$, where x is a point of $Q(5,2)$.

($P3$) The symbols $\bar{\bar{x}}$, where x is a point of $Q(5,2)$.

($P4$) The triples (x, y, i), where $i \in \mathbb{Z}_3$ and x, y are distinct collinear points of $Q(5,2)$ satisfying $xy \in S$.

The lines of $\mathcal{S}_\theta$ are of nine types:

($L1$) The lines $\{x, y, z\}$ of $Q(5,2)$.

($L2$) The sets $\{\bar{x}, \bar{y}, \bar{z}\}$, where $\{x, y, z\}$ is a line of $Q(5,2)$.

($L3$) The sets $\{\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{z}}\}$, where $\{x, y, z\}$ is a line of $Q(5,2)$.

($L4$) The sets $\{x, \bar{x}, \bar{\bar{x}}\}$, where x is a point of $Q(5,2)$.

($L5$) The sets $\{a, (a, b, i), (a, c, i)\}$, where $i \in \mathbb{Z}_3$ and $\{a, b, c\} \in S$.

($L6$) The sets $\{\bar{a}, (b, a, i), (c, a, i)\}$, where $i \in \mathbb{Z}_3$ and $\{a, b, c\} \in S$.

($L7$) The sets $\{\bar{\bar{a}}, (b, c, i), (c, b, i)\}$, where $i \in \mathbb{Z}_3$ and $\{a, b, c\} \in S$.

($L8$) The sets $\{(a, b, i), (b, c, j), (c, a, k)\}$, where $\{i, j, k\} = \mathbb{Z}_3$ and $\{a, b, c\}$ is a line belonging to S.

($L9$) The sets $\{(a, u, i), (b, v, j), (c, w, k)\}$, where ($i$) $\{a, b, c\}$ and $\{u, v, w\}$ are two disjoint lines of $Q(5,2)$; (ii) $d(a, u) = d(b, v) = d(c, w) = 1$; ($iii$) $au, bv, cw \in S$; (iv) $j = i + \theta(au, bv)$, $k = i + \theta(au, cw)$.

Incidence is containment. One can easily show that $\mathcal{S}_\theta$ is a partial linear space. In order to show that two distinct points of $\mathcal{S}_\theta$ are contained in at most one line of Type ($L9$), one has to make use of Property ($*$).

3 An isomorphism $Q(5,2) \otimes Q(5,2) \cong \mathcal{S}_\theta$

The aim of this section is to show that the slim dense near hexagon $Q(5,2) \otimes Q(5,2)$ is isomorphic to a point-line geometry $\mathcal{S}_\theta$ for a suitable spread of symmetry S of $Q(5,2)$ and a suitable map $\theta : S \times S \to \mathbb{Z}_3$ satisfying Property $(*)$. We start with recalling some known properties of the near hexagon $Q(5,2) \otimes Q(5,2)$.

(1) Every two points x and y of $Q(5,2) \otimes Q(5,2)$ are contained in a unique convex subspace of diameter 2, called a *quad*. The points and lines which are contained in a given quad define a GQ which is isomorphic to either the (3×3)-grid or $Q(5,2)$.

(2) If Q is a $Q(5,2)$-quad and $x \notin Q$, then x is collinear with a unique point $\pi_Q(x) \in Q$ and we denote by $\mathcal{R}_Q(x)$ the unique point of $x\pi_Q(x)$ distinct from x and $\pi_Q(x)$. If $x \in Q$, then we define $\pi_Q(x) = \mathcal{R}_Q(x) := x$. The map $x \mapsto \mathcal{R}_Q(x)$ defines an automorphism of $Q(5,2) \otimes Q(5,2)$. If Q_1 and Q_2 are two disjoint $Q(5,2)$-quads, then the map $Q_1 \to Q_2; x \mapsto \pi_{Q_2}(x)$ defines an isomorphism between Q_1 and Q_2.

(3) There exist two partitions T_1 and T_2 of the point set of $Q(5,2) \otimes Q(5,2)$ into $Q(5,2)$-quads.

(4) Every element of T_1 intersects every element of T_2 in a line. As a consequence, $S^\otimes := \{Q_1 \cap Q_2 : Q_1 \in T_1 \text{ and } Q_2 \in T_2\}$ is a spread of $Q(5,2) \otimes Q(5,2)$.

(5) For every $Q \in T_i$, $i \in \{1,2\}$, the set $\{Q \cap R : R \in T_{3-i}\}$ is a spread of symmetry of Q.

(6) Every line L of $Q(5,2) \otimes Q(5,2)$ not belonging to $S^\otimes$ is contained in a unique quad of $T_1 \cup T_2$.

Now, let Q and $\overline{Q}$ be two disjoint $Q(5,2)$-quads belonging to T_1 and put $\overline{\overline{Q}} := \mathcal{R}_Q(\overline{Q}) = \mathcal{R}_{\overline{Q}}(Q)$. For every point x of Q, put $\bar{x} := \pi_{\overline{Q}}(x)$ and $\bar{\bar{x}} := \pi_{\overline{\overline{Q}}}(x)$.

Put $S = \{Q \cap Q_2 : Q_2 \in T_2\}$. Then S is a spread of symmetry of Q. For every $L \in S$, let R_L denote the unique element of T_2 containing L. Let L^* denote a specific line of S and put $R^* = R_{L^*}$. For every $L \in S$, $R_L \cap (Q \cup \overline{Q} \cup \overline{\overline{Q}})$ is a (3×3)-subgrid σ_L of R_L. This (3×3)-subgrid σ_L is contained in precisely three $W(2)$-subquadrangles of R_L. We denote by W^0, W^1, W^2 the three $W(2)$-subquadrangles of R^* containing $R^* \cap (Q \cup \overline{Q} \cup \overline{\overline{Q}})$. For every $L \in S$ and $i \in \mathbb{Z}_3$, put $W_L^i := \pi_{R_L}(W^i)$.

For every $i \in \mathbb{Z}_3$, for every $L \in S$ and for all $x, y \in L$ with $x \neq y$, we denote by (x, y, i) the unique point μ of $R_L \setminus (Q \cup \overline{Q} \cup \overline{\overline{Q}})$ such that $\pi_Q(\mu) = x, \pi_{\overline{Q}}(\mu) = \bar{y}$ and $\mu \in W_L^i$. The point (x, y, i) is the unique point of W_L^i collinear with x and $\bar{y}$, but not contained in σ_L.

Lemma 3.1. *Every point of $Q(5,2) \otimes Q(5,2)$ not contained in $Q \cup \overline{Q} \cup \overline{\overline{Q}}$ has received a unique label.*

Proof. Let μ be a point of $Q(5,2) \otimes Q(5,2)$ not contained in $Q \cup \overline{Q} \cup \overline{\overline{Q}}$, let R denote the unique element of T_2 containing μ and put $L := R \cap Q$. Then $R = R_L$. There exists a unique $W(2)$-subquadrangle of R containing μ and σ_L. Let $i \in \mathbb{Z}_3$ such that $\mu \in W_L^i$. Let x and y be the points of L such that $x = \pi_Q(\mu)$ and $\bar{y} = \pi_{\overline{Q}}(\mu)$. If $x = y$, then $\{x, \bar{y}, \mu\}$ is a set of mutually collinear points, implying that $\mu = \bar{\bar{x}}$, contradicting $\mu \notin \overline{\overline{Q}}$. Hence $x \neq y$ and the point μ has label (x, y, i). It is also clear that μ cannot be labeled in different ways. $\square$

We will now define a map $\theta : S \times S \to \mathbb{Z}_3$. For each ordered pair (L_1, L_2) of lines of S, the map $R^* \to R^*; x \mapsto \pi_{R^*} \circ \pi_{R_{L_2}} \circ \pi_{R_{L_1}}(x)$ determines an automorphism of R^* fixing each line of the spread $\{R^* \cap Q_1 : Q_1 \in T_1\}$ of R^*. By [2, Theorem 4.1], such an automorphism either is trivial or acts on any line of the form $R^* \cap Q_1, Q_1 \in T_1$, as a cycle. Since every line $R^* \cap Q_1, Q_1 \in T_1 \setminus \{Q, \overline{Q}, \overline{\overline{Q}}\}$, intersects each $W(2)$-subquadrangle W^i, $i \in \mathbb{Z}_3$, in a unique point, the map $R^* \to R^*; x \mapsto \pi_{R^*} \circ \pi_{R_{L_2}} \circ \pi_{R_{L_1}}(x)$ is either trivial or permutes the elements of $\{W^0, W^1, W^2\}$ in one of the following ways:

$$W^0 \to W^1 \to W^2 \to W^0, \; W^0 \to W^2 \to W^1 \to W^0.$$

Hence, there exists a unique $\theta(L_1, L_2) \in \mathbb{Z}_3$ such that

$$\pi_{R^*} \circ \pi_{R_{L_2}} \circ \pi_{R_{L_1}}(W^i) = W^{i+\theta(L_1,L_2)}$$

for every $i \in \mathbb{Z}_3$.

Lemma 3.2. *The following holds:*

(*i*) *For every $L \in S$, $\theta(L, L) = 0$.*

(*ii*) *For any two lines L_1 and L_2 of S, $\theta(L_2, L_1) = -\theta(L_1, L_2)$.*

(*iii*) *If L_1, L_2, L_3 are three lines of S which are contained in a grid, then* $\theta(L_1, L_2) + \theta(L_2, L_3) = \theta(L_1, L_3)$.

(*iv*) *If L_1, L_2, L_3 are three lines of S which are not contained in a grid, then* $\theta(L_1, L_2) + \theta(L_2, L_3) \neq \theta(L_1, L_3)$.

Proof. (*i*) For every $i \in \mathbb{Z}_3$, we have $\pi_{R^*} \circ \pi_{R_L} \circ \pi_{R_L}(W^i) = \pi_{R^*} \circ \pi_{R_L}(W^i) = W^i$. Hence, $\theta(L, L) = 0$.

(*ii*) If $\pi_{R^*} \circ \pi_{R_{L_2}} \circ \pi_{R_{L_1}}(W^i) = W^{i+\theta(L_1,L_2)}$ for every $i \in \mathbb{Z}_3$, then $W^i = \pi_{R^*} \circ \pi_{R_{L_1}} \circ \pi_{R_{L_2}}(W^{i+\theta(L_1,L_2)})$ for every $i \in \mathbb{Z}_3$. It follows that $\theta(L_2, L_1) = -\theta(L_1, L_2)$.

(*iii*) Let L_1, L_2, L_3 be three lines of S which are contained in a grid. Then $\pi_{R^*} \circ \pi_{R_{L_3}} \circ \pi_{R_{L_1}}(W^i) = \pi_{R^*} \circ \pi_{R_{L_3}} \circ \pi_{R_{L_2}} \circ \pi_{R_{L_1}}(W^i) = \pi_{R^*} \circ \pi_{R_{L_3}} \circ \pi_{R_{L_2}} \circ \pi_{R^*} \circ \pi_{R_{L_2}} \circ \pi_{R_{L_1}}(W^i) = \pi_{R^*} \circ \pi_{R_{L_3}} \circ \pi_{R_{L_2}}(W^{i+\theta(L_1,L_2)}) = W^{i+\theta(L_1,L_2)+\theta(L_2,L_3)}$. Hence, $\theta(L_1, L_3) = \theta(L_1, L_2) + \theta(L_2, L_3)$.

(*iv*) Let L_1, L_2, L_3 be three lines of S which are not contained in a grid. Suppose that $\theta(L_1, L_3) = \theta(L_1, L_2) + \theta(L_2, L_3)$. Then for every $y \in R^*$, $\pi_{R^*} \circ \pi_{R_{L_3}} \circ \pi_{R_{L_1}}(y) = (\pi_{R^*} \circ \pi_{R_{L_3}} \circ \pi_{R_{L_2}}) \circ (\pi_{R^*} \circ \pi_{R_{L_2}} \circ \pi_{R_{L_1}})(y)$, that is, $\pi_{R^*} \circ \pi_{R_{L_3}} \circ \pi_{R_{L_1}}(y) = \pi_{R^*} \circ \pi_{R_{L_3}} \circ \pi_{R_{L_2}} \circ \pi_{R_{L_1}}(y)$. Hence, the map $R_{L_3} \to R_{L_3}$ defined by $x \mapsto \pi_{R_{L_3}} \circ \pi_{R_{L_2}} \circ \pi_{R_{L_1}}(x)$ is the identity map on R_{L_3}. This implies that the points $x, \pi_{R_{L_1}}(x), \pi_{R_{L_2}}(x)$ are mutually collinear for every $x \in R_{L_3}$, that is, $\{x, \pi_{R_{L_1}}(x), \pi_{R_{L_2}}(x)\}$ is a line for every $x \in R_{L_3}$. This contradicts the fact that L_1, L_2, L_3 are not contained in a grid. Hence, $\theta(L_1, L_3) \neq \theta(L_1, L_2) + \theta(L_2, L_3)$. $\square$

Proposition 3.3. *$Q(5,2) \otimes Q(5,2) \cong \mathcal{S}_\theta$, where θ is as defined above.*

Proof. We must show that the set of lines of $Q(5,2) \otimes Q(5,2)$ are in bijective correspondence with the sets of Type $(L1), (L2), \cdots, (L9)$ defined in Section 2. Obviously,

- the set of lines of $Q(5,2) \otimes Q(5,2)$ contained in Q correspond to the sets of Type $(L1)$;
- the set of lines of $Q(5,2) \otimes Q(5,2)$ contained in $\overline{Q}$ correspond to the sets of Type $(L2)$;
- the set of lines of $Q(5,2) \otimes Q(5,2)$ contained in $\overline{\overline{Q}}$ correspond to the sets of Type $(L3)$;

- the set of lines of $Q(5,2) \otimes Q(5,2)$ meeting $Q, \overline{Q}$ and $\overline{\overline{Q}}$ correspond to the sets of Type $(L4)$.

Consider a line M of R_L which is not contained in σ_L and which intersects σ_L in a point $a \in L$ of Q. Put $L = \{a,b,c\}$. There exists a unique $W(2)$-subquadrangle W_L^i containing σ_L and M. One readily sees that the points of M have labels a, (a,b,i) and (a,c,i). So, M corresponds to a set of Type $(L5)$. Conversely, every set of Type $(L5)$ corresponds to a (necessarily unique) line of $Q(5,2) \otimes Q(5,2)$.

Next, consider a line M of R_L which is not contained in σ_L and which intersects σ_L in a point $\bar{a}$ of $\overline{Q}$. Put $L = \{a,b,c\}$. Then, there exists a unique $W(2)$-subquadrangle W_L^i containing σ_L and M. One readily sees that the points of M have labels $\bar{a}$, (b,a,i) and (c,a,i). So, M corresponds to a set of Type $(L6)$. Conversely, every set of Type $(L6)$ corresponds to a (necessarily unique) line of $Q(5,2) \otimes Q(5,2)$.

Now, consider a line M of R_L which is not contained in σ_L and which intersects σ_L in a point $\bar{\bar{a}}$ of $\overline{\overline{Q}}$. Put $L = \{a,b,c\}$. Then, there exists a unique $W(2)$-subquadrangle W_L^i containing σ_L and M. One readily sees that the points of M have labels $\bar{\bar{a}}$, (b,c,i) and (c,b,i). So, M corresponds to a set of Type $(L7)$. Conversely, every set of Type $(L7)$ corresponds to a (necessarily unique) line of $Q(5,2) \otimes Q(5,2)$.

Consider next a line M of R_L which is disjoint from σ_L. Then M intersects each W_L^i, $i \in \mathbb{Z}_3$, in a unique point. Put $L = \{a,b,c\}$. The labels of the points of M are (u,u',i), (v,v',j), (w,w',k), where $\{i,j,k\} = \{0,1,2\}$, $\{u,v,w\} = \pi_Q(M) = \{a,b,c\}$, $\{u',v',w'\} = \pi_Q \circ \pi_{\overline{Q}}(M) = \{a,b,c\}$, $u \neq u'$, $v \neq v'$, $w \neq w'$. It readily follows that $\{(u,u',i),(v,v',j),(w,w',k)\}$ is a set of Type $(L8)$. Conversely, one can readily verify that every set of Type $(L8)$ corresponds to a line of $Q(5,2) \otimes Q(5,2)$.

Finally, let M be a line of $Q(5,2) \otimes Q(5,2)$ not belonging to $S^{\otimes}$ and contained in a quad of $T_1 \setminus \{Q, \overline{Q}, \overline{\overline{Q}}\}$. With M, there corresponds a set of the form $\{(a,u,i),(b,v,j),(c,w,k)\}$. We have that $\{a,b,c\} = \pi_Q(M)$ is a line of Q not belonging to S. Similarly, $\{u,v,w\} = \pi_Q \circ \pi_{\overline{Q}}(M)$ is a line of Q not belonging to S. Moreover, we have that $au, bv, cw \in S$ and $j = i + \theta(au,bv)$, $k = i + \theta(au,cw)$ by the definition of the map θ. So, M corresponds to a set of Type $(L9)$. Conversely, we show that every set $\{(a,u,i),(b,v,j),(c,w,k)\}$ of Type $(L9)$ corresponds to a line of $Q(5,2) \otimes Q(5,2)$ not belonging to $S^{\otimes}$ and contained in a quad of $T_1 \setminus \{Q, \overline{Q}, \overline{\overline{Q}}\}$. Let x denote the point of $Q(5,2) \otimes Q(5,2)$ corresponding to (a,u,i), let Q_1 denote the unique element of T_1 containing x and let $M = \pi_{Q_1}(\{a,b,c\})$. Then M corresponds to a set of the form $\{(a,u,i),(b,*,*),(c,*,*)\}$. Since v,w,j,k are uniquely determined by a,u,i,b,c, this set is equal to $\{(a,u,i),(b,v,j),(c,w,k)\}$.

By the above discussion, we indeed know that $Q(5,2)\otimes Q(5,2) \cong \mathcal{S}_\theta$. $\square$

Definitions. (1) An *admissible triple* is a triple $\Sigma = (\mathcal{L}, G, \Delta)$, where:

• G is a nontrivial additive group whose order $s+1$ is finite.

• $\mathcal{L}$ is a linear space, different from a point, in which each line is incident with exactly $s+1$ points. We denote the point set of $\mathcal{L}$ by P.

• Δ is a map from $P \times P$ to G such that the following holds for any three points x, y and z of $\mathcal{L}$: x, y and z are collinear $\Leftrightarrow \Delta(x,y)+\Delta(y,z) = \Delta(x,z)$.

(2) Suppose $\Sigma_1 = (\mathcal{L}_1, G_1, \Delta_1)$ and $\Sigma_2 = (\mathcal{L}_2, G_2, \Delta_2)$ are two admissible triples, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are not lines. Then Σ_1 and Σ_2 are called *equivalent* if there exists an isomorphism α from $\mathcal{L}_1$ to $\mathcal{L}_2$, an isomorphism β from G_1 to G_2 and a map f from the point set of $\mathcal{L}_1$ to G_1 satisfying $\Delta_2(\alpha(x),\alpha(y)) = (f(x)+\Delta_1(x,y)-f(y))^\beta$ for all points x and y of $\mathcal{L}_1$.

Let $\mathcal{L}_S$ denote the linear space whose points are the elements of S and whose lines are the unordered triples of lines of S which are contained in a grid, with incidence being containment. Then $\mathcal{L}_S$ is isomorphic to the affine plane $\mathrm{AG}(2,3)$ of order three. By Lemma 3.2, we know that $(\mathcal{L}_S, \mathbb{Z}_3, \theta)$ is an admissible triple.

Proposition 3.4. *Let θ_1 and θ_2 be two maps from $S \times S$ to $\mathbb{Z}_3$ such that $\Sigma_1 = (\mathcal{L}_S, \mathbb{Z}_3, \theta_1)$ and $\Sigma_2 = (\mathcal{L}_S, \mathbb{Z}_3, \theta_2)$ are admissible triples. If Σ_1 and Σ_2 are equivalent, then $\mathcal{S}_{\theta_1} \cong \mathcal{S}_{\theta_2}$.*

Proof. Since Σ_1 and Σ_2 are equivalent, there exists an automorphism α of $\mathcal{L}_S$, an automorphism β of $\mathbb{Z}_3$ and a map f from S to $\mathbb{Z}_3$ satisfying $\theta_2(\alpha(x),\alpha(y)) = (f(x)+\theta_1(x,y)-f(y))^\beta$ for all points x and y of $\mathcal{L}_S$. There exists an automorphism ϕ of Q such that $\alpha(L) = \phi(L)$ for every line L of S, see e.g. [3, Section 3, Example 1]. One readily verifies that the map $x \mapsto x^\phi; \bar{x} \mapsto \overline{x^\phi}; \bar{\bar{x}} \mapsto \overline{\overline{x^\phi}}; (a,b,i) \mapsto (a^\phi, b^\phi, (i-f(ab))^\beta)$ defines an isomorphism between $\mathcal{S}_{\theta_1}$ and $\mathcal{S}_{\theta_2}$. $\square$

It is known that the affine plane $\mathrm{AG}(2,3)$ admits, up to equivalence, a unique admissible triple. [This follows, for instance, from [4, Theorem 2.1] and the fact that there exists a unique generalized quadrangle of order $(2,4)$, namely

$Q(5,2)$, and a unique spread of symmetry in $Q(5,2)$.] If we coordinatize $\mathrm{AG}(2,3)$ in the standard way, then an admissible triple can be obtained by putting $\Delta[(x_1,y_1),(x_2,y_2)] := x_1y_2 - x_2y_1 \in \mathbb{Z}_3$.

4 A non-abelian representation of the near hexagon $Q(5,2)\times\mathbb{L}_3$

The slim dense near hexagon $Q(5,2)\times\mathbb{L}_3$ is obtained by taking three isomorphic copies of $Q(5,2)$ and joining the corresponding points to form lines of size 3. In this section we prove that there exists a non-abelian representation of $Q(5,2)\times\mathbb{L}_3$.

Let Q and B, respectively, be the point and line set of $Q(5,2)$. Set $\overline{Q} = \{\bar{x} : x \in Q\}$, $\overline{\overline{Q}} = \{\bar{\bar{x}} : x \in Q\}$, $\overline{B} = \{\{\bar{x},\bar{y},\bar{z}\} : \{x,y,z\} \in B\}$ and $\overline{\overline{B}} = \{\{\bar{\bar{x}},\bar{\bar{y}},\bar{\bar{z}}\} : \{x,y,z\} \in B\}$. Then $(\overline{Q},\overline{B})$ and $(\overline{\overline{Q}},\overline{\overline{B}})$ are isomorphic to $Q(5,2)$. The near hexagon $Q(5,2)\times\mathbb{L}_3$ is isomorphic to the geometry whose point set P is $Q\cup\overline{Q}\cup\overline{\overline{Q}}$ and whose line set L is $B\cup\overline{B}\cup\overline{\overline{B}}\cup\{\{x,\bar{x},\bar{\bar{x}}\} : x \in Q\}$.

It is known that if $Q(5,2)\times\mathbb{L}_3$ admits a non-abelian representation, then the representation group must be the extra-special 2-group 2^{1+12}_{+} [13, Theorem 1.6, p.199]. Let $R = 2^{1+12}_{+}$ with $R' = \{1,\lambda\}$. Set $V = R/R'$. Consider V as a vector space over $\mathbb{F}_2$. The map $f : V\times V \to \mathbb{F}_2$ defined by

$$f(xR',yR') = \begin{cases} 0 & \text{if } [x,y]=1 \\ 1 & \text{if } [x,y]=\lambda \end{cases}$$

for $x,y \in R$, is a non-degenerate symplectic bilinear form on V [5, Theorem 20.4, p.78]. Write V as an orthogonal direct sum of six hyperbolic planes K_i ($1 \le i \le 6$) in V and let H_i be the inverse image of K_i in R (under the canonical homomorphism $R \to R/R'$). Then each H_i is generated by two involutions x_i and y_i such that $[x_i,y_i] = \lambda$. Let $M = \langle x_i : 1 \le i \le 6\rangle$ and $\overline{M} = \langle y_i : 1 \le i \le 6\rangle$. Then M and $\overline{M}$ are elementary abelian 2-subgroups of R each of order 2^6. Further, $M, \overline{M}$ and $Z(R)$ pairwise intersect trivially and $R = M\overline{M}Z(R)$. Also, $C_M(\overline{M})$ and $C_{\overline{M}}(M)$ are trivial.

We regard the points and lines of Q as the points and lines of a nonsingular elliptic quadric of the projective space $\mathrm{PG}(M)$, where M is regarded as a 6-dimensional vector space over $\mathbb{F}_2$. Let (M,τ) be the natural abelian representation of (Q,B) associated with this embedding of Q in $\mathrm{PG}(M)$. For every point x of Q, put $m_x = \tau(x)$. There exists a unique non-degenerate symplectic bilinear form g on M such that $m_x^\perp = \langle m_y : y \in x^\perp \rangle$ for every point x of Q, see e.g. [7, Section 22.3]. Here, the following notational convention has been used: for every $m \in M$, $m^\perp$ denotes the set of all $m' \in M$ for which $g(m, m') = 0$.

Now, let m be an arbitrary element of M. If $m = 1$, then we define $\overline{m} := 1$. Suppose now that $m \neq 1$. Then $m^\perp$ is maximal in M, that is, of index 2 in M. So, the centralizer of $m^\perp$ in $\overline{M}$ is a subgroup $\langle \overline{m} \rangle$ of order 2. Since $m^\perp$ is maximal in M, $\langle m^\perp, m' \rangle = M$ for every $m' \in M \setminus m^\perp$. The triviality of $C_{\overline{M}}(M)$ then implies that $[\overline{m}, m'] = \lambda$ for every $m' \in M \setminus m^\perp$.

We prove that the map $M \to \overline{M}; m \mapsto \overline{m}$ is an isomorphism. This map is easily seen to be bijective. (Notice that $C_M(\overline{m}) = m^\perp$.) So, it suffices to prove that $\overline{m_1 m_2} = \overline{m_1}\ \overline{m_2}$ for all $m_1, m_2 \in M$. Clearly, this holds if $1 \in \{m_1, m_2\}$ or $m_1 = m_2$. So, we may suppose that $m_1 \neq 1 \neq m_2 \neq m_1$. The set $\{m_1, m_2, m_1 m_2\}$ corresponds to a line of $\mathrm{PG}(M)$. So, for every $m \in (m_1 m_2)^\perp$, $([\overline{m_1}, m], [\overline{m_2}, m])$ is equal to either $(1, 1)$ or (λ, λ). Then

$$[\overline{m_1}\ \overline{m_2}, m] = [\overline{m_1}, m][\overline{m_2}, m] = 1.$$

The first equality holds since R has nilpotency class 2. Thus $\overline{m_1}\ \overline{m_2} \in C_{\overline{M}}((m_1 m_2)^\perp) = \langle \overline{m_1 m_2} \rangle$. Since $\overline{m_1}\ \overline{m_2} \neq 1$, we have $\overline{m_1}\ \overline{m_2} = \overline{m_1 m_2}$.

We conclude that if we define $\overline{\tau} : \overline{Q} \to \overline{M}; \bar{x} \mapsto \overline{m_x}$ for every $x \in Q$, then $(\overline{M}, \overline{\tau})$ is a faithful abelian representation of $(\overline{Q}, \overline{B})$.

Now, let m be an arbitrary element of M. If $m = 1$, then we define $\overline{\overline{m}} := 1$. If $m = m_x$ for some $x \in Q$, then we define $\overline{\overline{m}} := m\overline{m}$. If $m \neq 1$ and $m \neq m_x$, $\forall x \in Q$, then we define $\overline{\overline{m}} := m\overline{m}\lambda$. Since $m^2 = \overline{m}^2 = \lambda^2 = [m, \overline{m}] = 1$, $\overline{\overline{m}}$ is an involution. We prove that the map $m \mapsto \overline{\overline{m}}$ defines an isomorphism between M and an elementary abelian 2-group $\overline{\overline{M}}$ of order 2^6. Since $R = M\overline{M}Z(R)$, this map is injective and hence it suffices to prove that $\overline{\overline{m_1 m_2}} = \overline{\overline{m_1}}\ \overline{\overline{m_2}}$ for all $m_1, m_2 \in M$. Obviously, this holds if $1 \in \{m_1, m_2\}$ or $m_1 = m_2$. So, we may suppose that $m_1 \neq 1 \neq m_2 \neq m_1$. The set $\{m_1, m_2, m_1 m_2\}$ corresponds to a line of $\mathrm{PG}(M)$. Suppose $3 - N$ elements of $\{m_1, m_2, m_1 m_2\}$ correspond to points of Q. Then $m_1 \in m_2^\perp$ if and only if N is even[1]. So, $[\overline{m_1}, m_2] = \lambda^N$. If N' is the number of elements of $\{m_1, m_2\}$ corresponding

[1] Perhaps the case $N = 3$ needs more explanation. Suppose $N = 3$ and $m_1 \in m_2^\perp$. Then the hyperplane π of $\mathrm{PG}(M)$ corresponding to $m_2^\perp$ intersects Q in a nonsingular parabolic quadric $Q(4, 2)$ of π. Since the point of $\mathrm{PG}(M)$ corresponding to m_2 is the kernel of $Q(4, 2)$, the line of $\mathrm{PG}(M)$ corresponding to $\{m_1, m_2, m_1 m_2\} \subset \pi$ must meet $Q(4, 2)$, in contradiction with $N = 3$.

to points of Q, then $2-N'-N \in \{-1,0\}$ and $2-N'-N=0$ if and only if m_1m_2 corresponds to a point of Q. Hence, $\overline{\overline{m_1}}\ \overline{\overline{m_2}} = m_1\ \overline{m_1}\ m_2\ \overline{m_2}\ \lambda^{2-N'} = m_1\ m_2\ \overline{m_1}\ \overline{m_2}\ \lambda^{2-N'-N} = m_1m_2\ \overline{m_1m_2}\ \lambda^{2-N'-N} = \overline{\overline{m_1m_2}}$.

So, if we define $\overline{\overline{\tau}} : \overline{\overline{Q}} \to \overline{\overline{M}}$ by putting $\overline{\overline{\tau}}(\bar{\bar{x}}) := \overline{\overline{m_x}} = m_x\overline{m_x}$ for all $x \in Q$, then $(\overline{\overline{M}}, \overline{\overline{\tau}})$ is a faithful abelian representation of $(\overline{\overline{Q}}, \overline{\overline{B}})$.

Now, define a map $\psi : P \to R$ which coincides with τ on Q, $\overline{\tau}$ on $\overline{Q}$ and $\overline{\overline{\tau}}$ on $\overline{\overline{Q}}$. Since $R = \langle M, \overline{M}\rangle$, $R = \langle \psi(P)\rangle$. By construction, (R,ψ) also satisfies Property (R2) in the definition of representation. Hence, (R,ψ) is a non-abelian representation of $Q(5,2) \times \mathbb{L}_3$.

5 A non-abelian representation of the near hexagon $Q(5,2) \otimes Q(5,2)$

In this section, we prove that the slim dense near hexagon $Q(5,2) \otimes Q(5,2)$ has a non-abelian representation. By Proposition 3.3, this is equivalent with showing that the partial linear space $\mathcal{S}_\theta$ has a non-abelian representation, where θ is as defined in Section 3.

We continue with the notation introduced in Section 3. Let M^* be a line of $S^{\otimes}$ contained in R^* but distinct from $R^* \cap Q$, $R^* \cap \overline{Q}$ and $R^* \cap \overline{\overline{Q}}$. Then M^* intersects each W^i, $i \in \{1,2,3\}$, in a unique point. For every point x of L^*, put $\epsilon(x) := i$ if the unique point of M^* collinear with x belongs to W^i. If $y \in Q \setminus L^*$, then we define $\epsilon(y) := \epsilon(x)$, where x is the unique point of L^* collinear with y.

Lemma 5.1. *Let L_1 and L_2 be two distinct lines in S and let $\alpha_i \in L_i$, $i \in \{1,2\}$. Then $\alpha_1 \sim \alpha_2$ if and only if $\epsilon(\alpha_2) - \epsilon(\alpha_1) = \theta(L_1,L_2)$.*

Proof. Let α_2' be the unique point of L_2 collinear with α_1, let x_1 and x_2 be the unique points of L^* nearest to α_1 and α_2', respectively, and let z_i, $i \in \{1,2\}$, denote the unique point of M^* collinear with x_i. The automorphism $R^* \to R^*; x \mapsto \pi_{R^*} \circ \pi_{R_{L_2}} \circ \pi_{R_{L_1}}(x)$ of R^* maps x_1 to x_2 and hence z_1 to z_2. This implies that $W^{\epsilon(x_1)+\theta(L_1,L_2)} = W^{\epsilon(x_2)}$, i.e. $\theta(L_1,L_2) = \epsilon(x_2) - \epsilon(x_1) = \epsilon(\alpha_2') - \epsilon(\alpha_1)$. Hence, $\alpha_1 \sim \alpha_2$ if and only if $\alpha_2 = \alpha_2'$, i.e. if and only if $\epsilon(\alpha_2) - \epsilon(\alpha_1) = \theta(L_1,L_2)$. □

Lemma 5.2. *Let $N = 2_-^{1+6}$ with $N' = \{1,\lambda\}$ and let $I_2(N)$ be the set of involutions in N. Then there exists a map δ from Q to $I_2(N)$ satisfying the following:*

(i) *δ is one-one.*

(ii) *For $x, y \in Q$, $[\delta(x), \delta(y)] = 1$ if and only if $y \in x^{\perp}$.*

(iii) *If $x, y \in Q$ with $x \sim y$, then*

$$\delta(x * y) = \begin{cases} \delta(x)\delta(y) & \text{if } xy \in S \\ \delta(x)\delta(y)\lambda & \text{if } xy \notin S \end{cases}.$$

(iv) *The image of δ generates N.*

Proof. We use a model for the generalized quadrangle $Q \cong Q(5, 2)$ which is described in [11, Section 6.1, pp.101–102]. Put $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\Omega' = \{1', 2', 3', 4', 5', 6'\}$. Let $\mathcal{E}$ be the set of all 2-subsets of Ω and let $\mathcal{F}$ be the set of all partitions of Ω in three 2-subsets of Ω. Then the point set of Q can be identified with the set $\mathcal{E} \cup \Omega \cup \Omega'$ and the line set of Q can be identified with the set $\mathcal{F} \cup \{\{i, \{i, j\}, j'\} : 1 \leq i, j \leq 6,\ i \neq j\}$. Now, consider the following nine lines of Q:

$$\begin{aligned} &L_1 = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\};\ L_2 = \{\{1, 4\}, 1, 4'\};\ L_3 = \{\{2, 6\}, 2, 6'\}; \\ &L_4 = \{\{1, 6\}, \{2, 4\}, \{3, 5\}\};\ L_5 = \{\{1, 5\}, 1', 5\};\ L_6 = \{\{2, 3\}, 2', 3\}; \\ &L_7 = \{\{1, 3\}, \{2, 5\}, \{4, 6\}\};\ L_8 = \{\{3, 6\}, 3', 6\};\ L_9 = \{\{4, 5\}, 4, 5'\}. \end{aligned}$$

These 9 lines are mutually disjoint and hence determine a spread S' of Q. Any two distinct lines L_i and L_j of S' are contained in a unique (3×3)-subgrid and the unique line of this subgrid disjoint from L_i and L_j also belongs to S'. A spread of $Q(5, 2)$ having this property is called *regular*. Since any regular spread of $Q(5, 2)$ is also a spread of symmetry [2, Section 7.1], and there exists up to isomorphism a unique spread of symmetry in $Q(5, 2)$, we may without loss of generality suppose that $S = S'$.

Put $N = \langle a, b\rangle \circ \langle c, d\rangle \circ Q_8$, where a, b, c, d are involutions and $\langle a, b\rangle \cong \langle c, d\rangle \cong D_8$. So, $[a, b] = [c, d] = \lambda$. Take $Q_8 = \{1, \lambda, i, j, k, i\lambda, j\lambda, k\lambda\}$, where $i^2 = j^2 = k^2 = \lambda$, $ij = k$, $jk = i$, $ki = j$ and $[i, j] = [j, k] = [k, i] = \lambda$. We define $\delta : Q \to I_2(N)$ as follows:

$\delta(\{1, 2\}) = a$, $\delta(\{3, 4\}) = c$, $\delta(\{5, 6\}) = ac$,
$\delta(\{1, 4\}) = abdi$, $\delta(1) = cdj$, $\delta(4') = abck\lambda$,
$\delta(\{2, 6\}) = abi\lambda$, $\delta(2) = acdk$, $\delta(6') = bcdj\lambda$,
$\delta(\{1, 6\}) = b$, $\delta(\{2, 4\}) = bd$, $\delta(\{3, 5\}) = d$,

$\delta(\{1,5\}) = abci$, $\delta(1') = cdk\lambda$, $\delta(5) = abdj$,
$\delta(\{2,3\}) = bcdi\lambda$, $\delta(2') = acdj\lambda$, $\delta(3) = abk$,
$\delta(\{1,3\}) = abcd\lambda$, $\delta(\{2,5\}) = bc\lambda$, $\delta(\{4,6\}) = ad\lambda$,
$\delta(\{3,6\}) = acdi\lambda$, $\delta(3') = abj\lambda$, $\delta(6) = bcdk$,
$\delta(\{4,5\}) = cdi$, $\delta(4) = abcj$, $\delta(5') = abdk\lambda$.

Put $W = N/N'$. Suppose $\{x_1, x_2, \dots, x_6\}$ is a set of 6 points of Q such that the smallest subspace $[x_1, x_2, \dots, x_6]$ of Q containing $\{x_1, x_2, \dots, x_6\}$ coincides with Q. If τ is an abelian representation of Q in W, then by Property (R1) in the definition of representation, $W = \langle \tau(x_1), \dots, \tau(x_6) \rangle$ and hence $\{\tau(x_1), \dots, \tau(x_6)\}$ is a basis of W (regarded as $\mathbb{F}_2$-vector space). Conversely, if $\{w_1, \dots, w_6\}$ is a basis of W, then the map $x_i \mapsto w_i$, $i \in \{1, \dots, 6\}$, can be extended to a unique abelian representation τ of Q in W. (Since there exists an abelian representation of Q in W, there must exist an abelian representation τ for which $\tau(x_i) = w_i$, $i \in \{1, \dots, 6\}$. The uniqueness of τ follows from the fact that $\tau(y_1 * y_2) = \tau(y_1)\tau(y_2)$ for any two distinct collinear points y_1 and y_2 of Q.) Consider now the special case where $x_1 = \{1,2\}$, $x_2 = \{3,4\}$, $x_3 = \{3,5\}$, $x_4 = \{1,6\}$, $x_5 = \{4,5\}$, $x_6 = 1$, $w_1 = aN'$, $w_2 = cN'$, $w_3 = dN'$, $w_4 = bN'$, $w_5 = cdiN'$ and $w_6 = cdjN'$. One indeed readily verifies that $[x_1, \dots, x_6] = Q$ and that $\{w_1, \dots, w_6\}$ is a basis of the $\mathbb{F}_2$-vector space W. Let δ' denote the unique abelian representation of Q in W for which $\delta'(x_i) = w_i$, $i \in \{1, \dots, 6\}$. Then, using the fact that $\delta'(y_1 * y_2) = \delta'(y_1)\delta'(y_2)$ for any two distinct collinear points y_1 and y_2 of Q, one can verify that $\delta'(y) = \delta(y)N'$ for every $y \in Q$. This implies that $\delta(y_1 * y_2)$ is equal to either $\delta(y_1)\delta(y_2)$ or $\delta(y_1)\delta(y_2)\lambda$ for any two distinct collinear points y_1 and y_2 of Q.

Clearly, the map $\delta : Q \to I_2(N)$ satisfies the properties (i) and (iv) of the lemma. We will now prove that also property (ii) of the lemma is satisfied. So, if $\{y_1, y_2\}$ is one of the 351 unordered pairs of distinct points of Q, then we need to prove that $[\delta(y_1), \delta(y_2)] = 1$ if and only if $y_1 \in y_2^\perp$. Since $Q = [x_1, \dots, x_6]$, it suffices to prove the following three statements: (I) the above claim holds if $\{y_1, y_2\} \subseteq \{x_1, \dots, x_6\}$; (II) if $[\delta(y_1), \delta(y_2)] = 1$ for some distinct collinear points y_1 and y_2, then also $[\delta(y_1), \delta(y_1 * y_2)] = 1$; (III) if the above claim holds for unordered pairs $\{y_1, y_2\}$ and $\{y_1, y_3\}$ of points where $y_2 \sim y_3$ and $y_1 \notin y_2y_3$, then it also holds for the unordered pair $\{y_1, y_2 * y_3\}$. Statement (I) is easily verified by considering all 15 pairs $\{x_i, x_j\}$ where $i, j \in \{1, \dots, 6\}$ with $i \neq j$. As to Statement (II), notice that $[\delta(y_1), \delta(y_1 * y_2)]$ is equal to either $[\delta(y_1), \delta(y_1)\delta(y_2)]$ or $[\delta(y_1), \delta(y_1)\delta(y_2)\lambda]$

which is in any case equal to 1. We now prove Statement (III). Since $\delta(y_2 * y_3)$ is equal to either $\delta(y_2)\delta(y_3)$ or $\delta(y_2)\delta(y_3)\lambda$, we have $[\delta(y_1), \delta(y_2 * y_3)] = [\delta(y_1), \delta(y_2)\delta(y_3)] = [\delta(y_1), \delta(y_2)][\delta(y_1), \delta(y_3)]$. If y_1 is collinear with precisely one of y_2, y_3, then y_1 is not collinear with $y_2 * y_3$ and $[\delta(y_1), \delta(y_2 * y_3)] = [\delta(y_1), \delta(y_2)][\delta(y_1), \delta(y_3)] = 1 \cdot \lambda = \lambda$. If y_1 is collinear with $y_2 * y_3$, then $[\delta(y_1), \delta(y_2 * y_3)] = [\delta(y_1), \delta(y_2)][\delta(y_1), \delta(y_3)] = \lambda \cdot \lambda = 1$. So, this proves Statement (III) and finishes the proof of property (ii) of the lemma.

Property (iii) of the lemma is verified by considering all 45 lines L of Q and an ordered pair (x, y) of distinct points of L. Notice that by property (ii) of the lemma, we only need to consider one ordered pair (x, y) for each line L of Q. $\square$

It is known that if the near hexagon $Q(5,2) \otimes Q(5,2)$ admits a non-abelian representation, then the representation group must be the extraspecial 2-group 2_-^{1+18} [13, Theorem 1.6, p.199]. We next construct a non-abelian representation of $\mathcal{S}_\theta \cong Q(5,2) \otimes Q(5,2)$ in the group 2_-^{1+18}.

Let $R = 2_-^{1+18}$ with $R' = \{1, \lambda\}$. Write R as a central product $R = M \circ N$, where $M = 2_+^{1+12}$ and $N = 2_-^{1+6}$. Let $Y = Q \cup \overline{Q} \cup \overline{\overline{Q}}$. Then the subgeometry of $\mathcal{S}_\theta$ whose point set is Y together with the lines of types $(L1) - (L4)$ is isomorphic to $Q(5,2) \times \mathbb{L}_3$. Let P be the point set of $\mathcal{S}_\theta$ and let δ be a map from Q to $I_2(N)$ satisfying the conditions of Lemma 5.2. We extend δ to the set $P \setminus Y$ using the map $\epsilon : Q \to \mathbb{Z}_3$ which we defined in the beginning of this section:

> For $L_1 \in S$, distinct points $a, b \in L_1$ and $j \in \mathbb{Z}_3$, we define $\delta(a, b, j) := \delta(u)$, where u is the unique point of L_1 with $\epsilon(u) = j$.

Now, fix a non-abelian representation (M, ϕ) of Y. Such a representation exists by Section 4. Let ψ be the following map from P to R:

- if $q \in Y$, then $\psi(q) := \phi(q)$;
- if $q = (a, b, i) \in P \setminus Y$, then $\psi(q) = \psi(a, b, i) := \phi(b)\phi(\bar{a})\delta(a, b, i)$.

We prove the following.

Theorem 5.3. *(R, ψ) is a non-abelian representation of $\mathcal{S}_\theta$.*

Proof. Since the image of ϕ generates M and the image of δ generates N, we have $R = \langle \psi(P) \rangle$. For every line $L_1 \in S$ and distinct $a, b \in L_1$, we have $[\phi(a), \phi(\bar{b})] = 1$, since a and $\bar{b}$ are in distance two from each other. This implies that $\psi(q)$ is an involution for every $q \in P$. We need to verify condition $(R2)$ in the definition of representation. This is true for all lines of types $(L1) - (L4)$, since they are also lines of Y and ψ coincides with ϕ on Y.

Let $\{a, (a, b, i), (a, c, i)\}$ be a line of type $(L5)$. Since $\delta(a, b, i) = \delta(a, c, i)$, we have $\psi(a, b, i)\psi(a, c, i) = \phi(b)\phi(\bar{a})\phi(c)\phi(\bar{a}) = \phi(b)\phi(c) = \phi(a) = \psi(a)$. Similar argument holds for lines of types $(L6)$ and $(L7)$.

Next, consider a line $\{(a, b, i), (b, c, j), (c, a, k)\}$ of type $(L8)$. We have $\psi(a, b, i)\psi(b, c, j) = \phi(b)\phi(\bar{a})\phi(c)\phi(\bar{b})\delta(a, b, i)\delta(b, c, j)$. Since $\{i, j, k\} = \mathbb{Z}_3$, $\{\delta(a, b, i), \delta(b, c, j), \delta(c, a, k)\} = \{\delta(a), \delta(b), \delta(c)\}$. Since $\{a, b, c\} \in S$, Lemma 5.2(iii) implies that $\delta(a, b, i)\delta(b, c, j) = \delta(c, a, k)$. So, $\psi(a, b, i)\psi(b, c, j) = \phi(b)\phi(c)\phi(\bar{a})\phi(\bar{b})\delta(c, a, k) = \phi(a)\phi(\bar{c})\delta(c, a, k) = \psi(c, a, k)$. Notice that the second equality holds since $\{a, b, c\}$ and $\{\bar{a}, \bar{b}, \bar{c}\}$ are lines of Y.

Finally, consider a line $\{(a, u, i), (b, v, j), (c, w, k)\}$ of type $(L9)$. Here the lines au, bv, cw are in S, $j = i+\theta(au, bv)$ and $k = i+\theta(au, cw)$. Let $\delta(a, u, i) = \delta(\alpha), \delta(b, v, j) = \delta(\beta)$ and $\delta(c, w, k) = \delta(\gamma)$, where $\alpha \in au, \beta \in bv$ and $\gamma \in cw$. So $\epsilon(\alpha) = i, \epsilon(\beta) = j$ and $\epsilon(\gamma) = k$. Since $\epsilon(\beta) - \epsilon(\alpha) = j - i = \theta(au, bv)$, Lemma 5.1 implies that $\alpha \sim \beta$. Similarly, $\alpha \sim \gamma$. Thus $\{\alpha, \beta, \gamma\}$ is a line of Q not contained in S. Then by Lemma 5.2(iii), $\delta(a, u, i)\delta(b, v, j) = \delta(\alpha)\delta(\beta) = \delta(\gamma)\lambda = \delta(c, w, k)\lambda$. So $\psi(a, u, i)\psi(b, v, j) = \phi(u)\phi(\bar{a})\phi(v)\phi(\bar{b})\delta(c, w, k)\lambda$. Since v and $\bar{a}$ are in distance three from each other, $[\phi(\bar{a}), \phi(v)] = \lambda$. So $\phi(\bar{a})\phi(v) = \phi(v)\phi(\bar{a})[\phi(\bar{a}), \phi(v)] = \phi(v)\phi(\bar{a})\lambda$. Then $\psi(a, u, i)\psi(b, v, j) = \phi(u)\phi(v)\phi(\bar{a})\phi(\bar{b})\delta(c, w, k) = \phi(w)\phi(\bar{c})\delta(c, w, k) = \psi(c, w, k)$. The second equality holds since $\{\bar{a}, \bar{b}, \bar{c}\}$ and $\{u, v, w\}$ are lines of Y. This completes the proof. □

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Addresses:

B. De Bruyn. Department of Pure Mathematics and Computer Algebra, Ghent University, Krijglaan 281 (S22), B-9000 Gent, Belgium, `bdb@cage.ugent.be`

B. K. Sahoo. School of Mathematical Sciences, National Institute of Science Education and Research, Sainik School Post, Bhubaneswar-751005, Orissa, India, `bksahoo@niser.ac.in`

N. S. N. Sastry. Statistics and Mathematics Unit, Indian Statistical Institute, Mysore Road, R. V. College Post, Bangalore-560059, India, nsastry@isibang.ac.in