The inverse Fueter mapping theorem

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Abstract

In a recent paper the authors have shown how to give an integral representation of the Fueter mapping theorem using the Cauchy formula for slice monogenic functions. Specifically, given a slice monogenic function f of the form $f = \alpha + \underline{\omega}\beta$ (where α, β satisfy the Cauchy-Riemann equations) we represent in integral form the axially monogenic function $f = A + \underline{\omega}B$ (where A, B satisfy the Vekua's system) $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$ where Δ is the Laplace operator in dimension $n+1$. In this paper we solve the inverse problem: given $\breve{f}$ determine a slice monogenic function f (called Fueter's primitive of $\breve{f}$) such that $\breve{f} = \Delta^{\frac{n-1}{2}} f(x)$. We prove an integral representation theorem for f in terms of $\breve{f}$ which we call the inverse Fueter mapping theorem (in integral form). Such a result is obtained also for regular functions of a quaternionic variable of axial type. The solution f of the equation $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$ in the Clifford analysis setting, i.e. the inversion of the classical Fueter mapping theorem, is new in literature and has some consequences that are now under investigation.

Key words: Cauchy-Riemann equations, Vekua's system, axially monogenic function, slice monogenic functions, Fueter's primitive, Fueter mapping theorem in integral form, inverse Fueter mapping theorem in integral form.
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1 Introduction

The theory of slice monogenic functions has been developed in a series of papers see [5], [6], [10], [11] and [13] and it has been further generalized in [21]. Its main importance is that it allows a natural definition of a functional calculus for n-tuples of non necessarily commuting operators, see [6] and [12] and the book [14] in which it is also developed the theory of the quaternionic functional calculus see also [7]. In the paper [9] the authors, using the Cauchy formula for slice monogenic functions, show an integral relation between monogenic functions, see [1], [8], [19], [23] and slice monogenic functions. More precisely, given a slice monogenic function f we can generate a monogenic function $\breve{f}$ by the Fueter mapping theorem, that is $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$, where Δ is the Laplace operator in dimension $n+1$ and n is an odd number. The Fueter mapping theorem can be stated in differential form in various versions, see [24], [25], [26], [27], [28], [29], [30] and the literature therein. Let U be a suitable open set in $\mathbb{R}^{n+1}$, we denote by $\mathscr{H}(U)$ and $\mathcal{M}(U)$ the set of slice monogenic functions of the form $f = \alpha + \underline{\omega}\beta$ and the set of axially monogenic functions (see [15]) on U, respectively.

In [9] we have shown that $\breve{f}$ can be represented by the integral transform

$$\breve{f}(x) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} \mathcal{F}_n(s,x) ds_I f(s), \quad ds_I = ds/I, \tag{1}$$

where U is a bounded axially symmetric open set that contains the singularities of the kernel $\mathcal{F}_n(s,x)$ and $\mathbb{C}_I$ is the complex plane with imaginary unit I. The kernel $\mathcal{F}_n$ has a particular simple form given by

$$\mathcal{F}_n(s,x) := \gamma_n (s-\bar{x})(s^2 - 2\mathrm{Re}[x]s + |x|^2)^{-\frac{n+1}{2}},$$

where γ_n are known constants and x and s are paravectors in $\mathbb{R}^{n+1}$. From the integral transform (1) it follows that there exists a map of $\mathbb{R}_n$-modules

$$\mathscr{H}(U) \to \mathcal{M}(U), \qquad f \mapsto \breve{f},$$

where $\breve{f}(x)$ is an axially monogenic function. The main goal of this paper is to show that the map is surjective and that it is possible solve the inverse problem: given an axially monogenic function $\breve{f}$ it is possible to determine a slice monogenic function f such that $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$. By its definition, $\breve{f}(x)$ is of the form

$$\breve{f}(x) = A(x_0, |\underline{x}|) + \underline{\omega} B(x_0, |\underline{x}|), \tag{2}$$

where $\underline{\omega}^2 = -1$ and $A(x_0, |\underline{x}|)$, $B(x_0, |\underline{x}|)$ do not depend on $\underline{\omega}$ and satisfy the Vekua's system. Furthermore, every slice monogenic function f (see [10]) defined on a suitable domain, can be written in the form

$$f(x_0 + I|\underline{x}|) = \alpha(x_0, |\underline{x}|) + I\beta(x_0, |\underline{x}|),$$

where $I^2 = -1$ and α and β do not depend on I and satisfy the Cauchy-Riemann system. The goal of this paper is the solution of the following:

Problem 1.1. *Let n be an odd number and let U be an axially symmetric open set in $\mathbb{R}^{n+1}$. Suppose that $\breve{f}$ is an axially monogenic function and f is a slice monogenic function such that $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$. Find an explicit description of the map*

$$\mathcal{M}(U) \to \mathscr{H}(U), \qquad \breve{f} \mapsto f.$$

The solution of the problem requires the introduction of the kernels

$$\mathcal{N}_n^+(x) = \int_{\mathbb{S}^{n-1}} \mathcal{G}(x-\underline{\omega})\, dS(\underline{\omega}), \qquad \mathcal{N}_n^-(x) = \int_{\mathbb{S}^{n-1}} \mathcal{G}(x-\underline{\omega})\, \underline{\omega}\, dS(\underline{\omega}).$$

where $\mathcal{G}$ is the monogenic Cauchy kernel, $\mathbb{S}^{n-1}$ is the unit $(n-1)$-dimensional sphere in $\mathbb{R}^{n+1}$, while $dS(\underline{\omega})$ is a scalar element of area of $\mathbb{S}^{n-1}$ and $x = x_0 + \underline{x}$ is a paravector in $\mathbb{R}^{n+1}$. We will say that f is a Fueter's primitive of $\breve{f}$ if $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$. Every axially symmetric monogenic function $\breve{f}$ can be written as (2) and the inverse integral transform is of the form

$$f(x) = f_1(x) + f_2(x)$$

where

$$f_1(x) := \int_\Gamma \mathcal{W}_n^-\Big(\frac{1}{\rho}(x-y_0)\Big)\rho^{n-2}(dy_0\, A(y_0,\rho) - d\rho\, B(y_0,\rho))$$
$$f_2(x) := -\int_\Gamma \mathcal{W}_n^+\Big(\frac{1}{\rho}(x-y_0)\Big)\rho^{n-2}(dy_0 B(y_0,\rho) - d\rho A(y_0,\rho)).$$

where Γ is a suitable simple regular curve and the kernels $\mathcal{W}_n^+$ and $\mathcal{W}_n^-$ (which are Fueter's primitives of $\mathcal{N}_n^+$ and $\mathcal{N}_n^-$, respectively) can be explicitly computed for every given odd number n. For example for $n=3$ we have

$$\mathcal{W}_3^+(x) = \frac{1}{2\pi}\ \arctan x, \qquad\qquad \mathcal{W}_3^-(x) = -\frac{1}{2\pi}\ x\arctan x.$$

An analogous result is also given for regular functions of a quaternionic variable.
The authors point out the recent interesting developments in the field of Clifford Analysis can be found in [2] and [3].

The plan of the paper. In Section 2 we state the preliminary results, in Section 3 we introduce the kernels $\mathcal{N}_n^+(x)$ and $\mathcal{N}_n^-(x)$ for functions with values in a Clifford algebra $\mathbb{R}_n$, in Section 4 we state and prove the inverse Fueter mapping theorems for $\mathbb{R}_n$–valued functions. Finally, in Section 5, we study the case of regular functions of a quaternionic variable.

2 Preliminary material

The setting in which we work is the real Clifford algebra $\mathbb{R}_n$ over n imaginary units $e_1,\dots,e_n$ satisfying the relations $e_ie_j+e_je_i=-2\delta_{ij}$. An element in the Clifford algebra will be denoted by $\sum_A e_Ax_A$ where $A=i_1\dots i_r$, $i_\ell\in\{1,2,\dots,n\}$, $i_1<\dots<i_r$, is a multi-index, $e_A=e_{i_1}e_{i_2}\dots e_{i_r}$ and $e_\emptyset=1$. As it is well known, $n=1$, we have that $\mathbb{R}_1$ is the algebra of complex numbers $\mathbb{C}$ (the only case in which the Clifford algebra is commutative), while for $n=2$ we obtain the division algebra of real quaternions $\mathbb{H}$. for $n>2$, the Clifford algebras $\mathbb{R}_n$ have zero divisors. In the Clifford algebra $\mathbb{R}_n$, we can identify some specific elements with the vectors in the Euclidean space $\mathbb{R}^n$: an element $(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$ can be identified with a so called 1-vector in the Clifford algebra through the map $(x_1,x_2,\dots,x_n)\mapsto \underline{x}=x_1e_1+\dots+x_ne_n$.
An element $(x_0,x_1,\dots,x_n)\in\mathbb{R}^{n+1}$ will be identified with the element $x=x_0+\underline{x}$ called, in short, paravector. The norm of $x\in\mathbb{R}^{n+1}$ is defined as $|x|^2=x_0^2+x_1^2+\dots+x_n^2$. The real part x_0 of x will be also denoted by $\mathrm{Re}[x]$. A function $f:\ U\subseteq\mathbb{R}^{n+1}\to\mathbb{R}_n$ is seen as a function $f(x)$ of the paravector x.

Definition 2.1. *We denote by $\mathbb{S}^{n-1}$ the sphere of unit 1-vectors in $\mathbb{R}^{n+1}$, i.e.*

$$\mathbb{S}^{n-1}=\{x=e_1x_1+\dots+e_nx_n\ :\ x_1^2+\dots+x_n^2=1\}.$$

Note that $\mathbb{S}^{n-1}$ is an $(n-1)$-dimensional sphere in $\mathbb{R}^{n+1}$. The vector space $\mathbb{R}+I\mathbb{R}$ passing through 1 and $I\in\mathbb{S}^{n-1}$ will be denoted by $\mathbb{C}_I$, while an element belonging to $\mathbb{C}_I$ will be denoted by $u+Iv$, for u, $v\in\mathbb{R}$. Observe that $\mathbb{C}_I$, for every $I\in\mathbb{S}^{n-1}$, is a 2-dimensional real subspace of $\mathbb{R}^{n+1}$ isomorphic to the complex plane. The isomorphism turns out to be an algebra isomorphism.
Given a paravector $x=x_0+\underline{x}\in\mathbb{R}^{n+1}$ let us set

$$I_x=\begin{cases}\dfrac{\underline{x}}{|\underline{x}|} & \text{if } \underline{x}\neq 0,\\ \text{any element of } \mathbb{S}^{n-1} & \text{otherwise,}\end{cases}$$

so, by definition we have $x\in\mathbb{C}_{I_x}$ and $x=\mathrm{Re}[x]+I_x|\underline{x}|$.

Definition 2.2. *Given an element $x \in \mathbb{R}^{n+1}$, we define*

$$[x] = \{y \in \mathbb{R}^{n+1} \ : \ y = \mathrm{Re}[x] + I|\underline{x}|, \quad I \in \mathbb{S}^{n-1}\}.$$

Remark 2.3. The set $[x]$ is a $(n-1)$-dimensional sphere in $\mathbb{R}^{n+1}$. When $x \in \mathbb{R}$, then $[x]$ contains x only. In this case, the sphere has radius equal to zero.

We recall the definitions of slice monogenic functions and of axially monogenic functions and we mention the properties we are going to use in this paper.

Definition 2.4 (Slice monogenic functions). *Let $U \subseteq \mathbb{R}^{n+1}$ be an open set and let $f : U \to \mathbb{R}_n$ be a real differentiable function. Let $I \in \mathbb{S}^{n-1}$ and let f_I be the restriction of f to the complex plane $\mathbb{C}_I := \mathbb{R} + I\mathbb{R}$ and denote by $u + Iv$ an element on $\mathbb{C}_I$. We say that f is a left slice monogenic (for short* s-monogenic*) function if, for every $I \in \mathbb{S}^{n-1}$, we have*

$$\frac{1}{2}\left(\frac{\partial}{\partial u} + I\frac{\partial}{\partial v}\right) f_I(u + Iv) = 0$$

on $U \cap \mathbb{C}_I$.

Slice monogenic functions have good properties when they are defined on the following domains:

Definition 2.5 (Slice domains). *Let $U \subseteq \mathbb{R}^{n+1}$ be a domain. We say that U is a* slice domain (s-domain for short) *if $U \cap \mathbb{R}$ is non empty and if $\mathbb{C}_I \cap U$ is a domain in $\mathbb{C}_I$ for all $I \in \mathbb{S}^{n-1}$.*

Definition 2.6 (Axially symmetric open sets). *Let $U \subseteq \mathbb{R}^{n+1}$ be an open set. We say that U is* axially symmetric *if, for all $u + Iv \in U$, the whole $(n-1)$-sphere $[u + Iv]$ is contained in U.*

Theorem 2.7 (Representation Formula, see [5]). *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain and let f be an s-monogenic function. For any vector $\mathbf{x} = u + I_{\mathbf{x}}v \in U$ the following formulas hold:*

$$f(\mathbf{x}) = \frac{1}{2}\Big[1 - I_{\mathbf{x}}I\Big] f(u + Iv) + \frac{1}{2}\Big[1 + I_{\mathbf{x}}I\Big] f(u - Iv) \tag{3}$$

$$= \frac{1}{2}[f(u + Iv) + f(u - Iv)] + I_x I\frac{1}{2}[f(u - Iv) - f(u + Iv)].$$

Moreover, the two quantities

$$\alpha(u, v) := \frac{1}{2}[f(u + Iv) + f(u - Iv)], \qquad \beta(u, v) := I\,\frac{1}{2}[f(u - Iv) - f(u + Iv)],$$

do not depend on $I \in \mathbb{S}$.

We now define the following set of functions which is related to slice monogenic functions, see the remark after the definition.

Definition 2.8. *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric open set and $\mathcal{U} \subseteq \mathbb{R} \times \mathbb{R}$ be such that $x = u + Iv \in U$ for all $(u, v) \in \mathcal{U}$. We denote by $\mathscr{H}(U)$ the set of functions on U of the form*

$$f(x) = \alpha(u, v) + I\beta(u, v)$$

where α, β are $\mathbb{R}_n$-valued differentiable functions, $\alpha(u, v) = \alpha(u, -v)$, $\beta(u, v) = \beta(u, -v)$ for all $(u, v) \in \mathcal{U}$ and moreover satisfy the Cauchy-Riemann system

$$\begin{cases} \partial_u\alpha - \partial_v\beta = 0, \\ \partial_v\alpha + \partial_u\beta = 0. \end{cases} \tag{4}$$

Remark 2.9. A function in $\mathscr{H}(U)$ is an s-monogenic function; however, the set $\mathscr{H}(U)$ coincides with the set of slice monogenic functions only on axially symmetric s-domains U, by virtue of the Representation Formula. In the sequel, when we consider an s-monogenic function f we will always refer to a function belonging to $\mathscr{H}(U)$.

Definition 2.10 (Axially monogenic function)**.** *Let U be an axially symmetric open set in $\mathbb{R}^{n+1}$, and let $x = x_0 + \underline{x} = x_0 + r\underline{\omega} \in U$. Assume that $\breve{f} : U \to \mathbb{R}_n$ is a monogenic functions, that is it is in the kernel of the Dirac operator*

$$\Big(\partial_{x_0} + \sum_{j=1}^{n} e_j \partial_{x_j}\Big)\breve{f}(x) = 0.$$

We sat that $\breve{f}$ is an axially monogenic function if there exist two functions $A = A(x_0, r)$ and $B = B(x_0, r)$, independent of $\underline{\omega} \in \mathbb{S}^{n-1}$ and with values in $\mathbb{R}_n$, such that

$$\breve{f}(x) = A(x_0, r) + \underline{\omega} B(x_0, r).$$

We denote by $\mathcal{M}(U)$ the set of left axially monogenic functions on the open set U.

Theorem 2.11 (See [15])**.** *Let U be an axially symmetric open set in $\mathbb{R}^{n+1}$. Then the functions $A = A(x_0, r)$ and $B = B(x_0, r)$ satisfy the Vekua's system, i.e.*

$$\begin{cases} \partial_{x_0} A(x_0, r) - \partial_r B(x_0, r) = \frac{n-1}{r} B(x_0, r), \\ \partial_{x_0} B(x_0, r) + \partial_r A(x_0, r) = 0. \end{cases} \tag{5}$$

3 The kernels $\mathcal{N}_n^+$ and $\mathcal{N}_n^-$ and their Fueter's primitives

The main tool to solve Problem 1.1 are the kernels $\mathcal{N}_n^+$ and $\mathcal{N}_n^-$ that we determine in this section. To reformulate Problem 1.1 we give the definition of Fueter's primitive.

Definition 3.1. *Let n be an odd number and let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric domain. Suppose that $f \in \mathscr{H}(U)$. We say that a function f is a Fueter primitive of $\breve{f} \in \mathcal{M}(U)$ if $\Delta^{\frac{n-1}{2}} f(x) = \breve{f}(x)$ on U.*

Remark 3.2. The definition of Fueter primitive of $\breve{f}$ is well posed since all the function $f \in \mathscr{H}(U)$ are infinitely differentiable as it was proved in [11].

So Problem 1.1 can be, more precisely, reformulated as follows

Problem 3.3. *Let n be an odd number and let U be an axially symmetric domain in $\mathbb{R}^{n+1}$. Given an axially monogenic function $\breve{f}$ defined on U determine a Fueter primitive f.*

First of all let us recall that slice monogenic functions admit power series expansions only at points on the real axis, see [10].

Remark 3.4. Let n be an odd number and $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain. Suppose that $W \in \mathscr{H}(U)$. So the function W admits the power series expansion

$$W(x) = \sum_{\ell \geq 0} \frac{1}{\ell!} \underline{x}^{\ell} V^{(\ell)}(x_0), \tag{6}$$

where $V(x_0) := W(x)|_{\underline{x}=0}$. The series converges is in a suitable ball $B(x_0, r)$ centered at $x_0 \in U \cap \mathbb{R}$ and with radius $r > 0$.

We have the following results who will be crucial in the sequel. We reason in the ball $B(x_0, r)$ of convergence and, for the Identity Principle for slice monogenic functions, see [10], the result can be extended to the whole axially symmetric s-domain U.

Proposition 3.5. *Let U be an axially symmetric s-domain in $\mathbb{R}^{n+1}$ and suppose that $W \in \mathscr{H}(U)$. Let $x_0 \in U \cap \mathbb{R}$ and suppose that (6) is the power series expansion of W in $B(x_0, r)$. Then, for every odd number n, there exists a positive constant $\mathcal{K}_n$, independent of x_0, such that*

$$\Delta_x^{(n-1)/2} W(x)|_{\underline{x}=0} = \mathcal{K}_n V^{(n-1)}(x_0),$$

where

$$\mathcal{K}_n = \sum_{k=0}^{(n-1)/2} \frac{(-1)^k}{(2k)!} \binom{(n-1)/2}{k} \Pi_{i=1}^k (2i)\ \Pi_{j=0}^{k-1} (n+2j). \tag{7}$$

Proof. We set for simplicity $m = (n-1)/2$ and we calculate $\Delta_x^m W(x)$ keeping in mind that we have to restrict it to $\underline{x} = 0$. Since $\Delta_x = \partial_{x_0}^2 + \Delta_{\underline{x}}$, we can write

$$\Delta_x^m W(x) = \sum_{k=0}^m \sum_{\ell \geq 0} \binom{m}{k} \frac{1}{\ell!} \partial_{x_0}^{2(m-k)} \Delta_{\underline{x}}^k \left(\underline{x}^\ell\, V^{(\ell)}(x_0) \right) = \sum_{k=0}^m \sum_{\ell \geq 0} \binom{m}{k} \frac{1}{\ell!} \Delta_{\underline{x}}^k \left(\underline{x}^\ell \right) \partial_{x_0}^{2(m-k)} V^{(\ell)}(x_0)$$

and we observe that

$$\Delta_{\underline{x}}^k \left(\underline{x}^\ell \right) = \begin{cases} 0 & \text{if } 2k > \ell, \\ C_\ell & \text{if } 2k = \ell, \\ \mathcal{E}(\underline{x}) & \text{if } 2k < \ell, \end{cases}$$

where C_ℓ are constants depending on ℓ and $\mathcal{E}$ is a continuous function such that $\mathcal{E}(\underline{x}) \to 0$ for $\underline{x} \to 0$. Moreover, it is easy to see that all the terms corresponding to $2k = \ell$ contain

$$\partial_{x_0}^{2(m-k)} V^{(2k)}(x_0) = V^{(2m)}(x_0).$$

So we have

$$\begin{aligned} \Delta_x^m W(x)|_{\underline{x}=0} &= \sum_{k=0}^m \sum_{\ell \geq 0} \binom{m}{k} \frac{1}{\ell!} \Delta_{\underline{x}}^k \left(\underline{x}^\ell \right) \partial_{x_0}^{2(m-k)} V^{(\ell)}(x_0) \Big|_{\underline{x}=0} \\ &= \sum_{k=0}^m \frac{1}{(2k)!} \binom{m}{k} \Delta_{\underline{x}}^k (\underline{x}^{2k})\ \ V^{(2m)}(x_0). \end{aligned}$$

We define

$$\mathcal{K}_n := \sum_{k=0}^m \frac{1}{(2k)!} \binom{m}{k} \Delta_{\underline{x}}^k (\underline{x}^{2k}), \qquad m = (n-1)/2,$$

we recall that $\Delta_{\underline{x}} = -\partial_{\underline{x}}^2$ where $\partial_{\underline{x}}$ is the Dirac operator in dimension n and that

$$\partial_{\underline{x}} (\underline{x}^p) = \begin{cases} -p\ \underline{x}^{p-1} & \text{if } p \ \text{ is even}, \\ -(p+n-1)\ \underline{x}^{p-1} & \text{if } p \ \text{ is odd}. \end{cases}$$

Observe that $\Delta_{\underline{x}}^0 (\underline{x}^0) = 1$ and consider the terms:

$$\Delta_{\underline{x}}(\underline{x}^2) = -2n, \quad \Delta_{\underline{x}}^2(\underline{x}^4) = 2 \cdot 4 \cdot n(n+2), \quad \Delta_{\underline{x}}^3(\underline{x}^2) = -2 \cdot 4 \cdot 6 \cdot n(n+2)(n+4);$$

by recurrence, one can easily verify that

$$\Delta_{\underline{x}}^k (\underline{x}^{2k}) = (-1)^k \Pi_{i=1}^k (2i)\ \Pi_{j=0}^{k-1} (n+2j),$$

so we obtain

$$\Delta_{\underline{x}}^{(n-1)/2} W(x)|_{\underline{x}=0} = \mathcal{K}_n\, V^{(n-1)}(x_0)$$

where

$$\mathcal{K}_n = \sum_{k=0}^{(n-1)/2} \frac{1}{(2k)!} \binom{(n-1)/2}{k} (-1)^k \Pi_{i=1}^{k}(2i)\ \Pi_{j=0}^{k-1}(n+2j).$$

This concludes the proof. $\square$

Remark 3.6. A direct computation shows that $\mathcal{K}_3 = -2$; the value of $\mathcal{K}_3$ which will be useful in Section 5.

In the sequel, we will denote by $P_m(t)$ the Legendre polynomials and by A_{n-1} the area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$, i.e.

$$A_{n-1} = \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})}.$$

Note that, by the Rodriguez formula, $P_m(t)$ can be expressed by

$$P_m(t) = \Big(-\frac{1}{2}\Big)^m \frac{\Gamma((n-1)/2)}{\Gamma(m+(n-1)/2)} (1-t^2)^{(3-n)/2} D^m (1-t^2)^{m+(n-3)/2}.$$

A crucial tool will be the following:

Theorem 3.7 (Funk-Hecke (see [20]))**.** *Denote by $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$ and by A_{n-1} its area. Let ξ and η be two unit vectors in $\mathbb{R}^n$. Let ψ be a real-valued function whose domain contains $[-1,1]$ and let $\mathcal{S}_m(\xi)$ be spherical harmonics, of degree m. Then we have*

$$\int_{\mathbb{S}^{n-1}} \psi(\langle \xi, \eta\rangle) \mathcal{S}_m(\eta)\, dS(\eta) = A_{n-1} \mathcal{S}_m(\xi) \int_{-1}^{1} \psi(t) P_m(t) (1-t^2)^{(n-3)/2}\, dt,$$

where $dS(\eta)$ is the scalar element of surface area on $\mathbb{S}^{n-1}$, by $\langle \xi, \eta\rangle$ we denote the scalar product of ξ, η.

Definition 3.8 (The monogenic Cauchy kernel)**.** We denote by $\mathcal{G}$ the monogenic Cauchy kernel on $\mathbb{R}^{n+1}$:

$$\mathcal{G}(x) = \frac{1}{A_{n+1}} \frac{\overline{x}}{|x|^{n+1}}, \qquad x \in \mathbb{R}^{n+1} \setminus \{0\}, \tag{8}$$

where

$$A_{n+1} = \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})}$$

is the area of the unit sphere in $\mathbb{R}^{n+1}$.

Definition 3.9 (The kernels $\mathcal{N}_n^+(x)$ and $\mathcal{N}_n^-(x)$)**.** Let $\mathcal{G}(x-\underline{y})$ be the monogenic Cauchy kernel defined in (8) with $x = x_0 + \underline{x} \in \mathbb{R}^{n+1}$ and for $\underline{y} = r\underline{\omega} \in \mathbb{R}^n$ we assume $r = 1$ and $\underline{\omega} \in \mathbb{S}^{n-1}$. We define the kernels

$$\mathcal{N}_n^+(x) = \int_{\mathbb{S}^{n-1}} \mathcal{G}(x-\underline{\omega})\, dS(\underline{\omega}), \qquad \mathcal{N}_n^-(x) = \int_{\mathbb{S}^{n-1}} \mathcal{G}(x-\underline{\omega})\, \underline{\omega}\, dS(\underline{\omega}). \tag{9}$$

where $dS(\omega)$ is the scalar element of surface area of $\mathbb{S}^{n-1}$.

Theorem 3.10 (The restrictions of the kernels $\mathcal{N}_n^+(x)$ and $\mathcal{N}_n^-(x)$ to $\underline{x}=0$)**.** *Let n be an odd number. Let $\mathcal{N}_n^+$ and $\mathcal{N}_n^-$ be the kernels defined in (9), respectively. Then their restrictions to $\underline{x}=0$ are given by*

$$\mathcal{N}_n^+(x)|_{\underline{x}=0} = \mathcal{C}_n \ \frac{x_0}{(x_0^2+1)^{(n+1)/2}}, \qquad \mathcal{N}_n^-(x)|_{\underline{x}=0} = -\mathcal{C}_n \ \frac{1}{(x_0^2+1)^{(n+1)/2}},$$

where

$$\mathcal{C}_n := \frac{1}{\sqrt{\pi}} \ \frac{\Gamma((n+1)/2)}{\Gamma(n/2)}. \tag{10}$$

Proof. Let us consider the function $\mathcal{N}_n^+(x)$, for all n odd number, and recalling the monogenic Cauchy kernel we write $\mathcal{N}_n^+(x)$ as:

$$\mathcal{N}_n^+(x) = \frac{1}{A_{n+1}} \int_{\mathbb{S}^{n-1}} \frac{x_0 - \underline{x} + \underline{\omega}}{(x_0^2 + \langle \underline{x}-\underline{\omega}, \underline{x}-\underline{\omega}\rangle)^{(n+1)/2}} \, dS(\underline{\omega}).$$

By setting $r = |\underline{x}|$, $I = \underline{x}/r$, we split it as

$$\mathcal{N}_n^+(x) = (x_0 - \underline{x}) J_n(x_0, r) + L_n(x_0, r)$$

where we set

$$J_n(x_0, r) := \frac{1}{A_{n+1}} \int_{\mathbb{S}^{n-1}} \frac{1}{(x_0^2 + 1 + r^2 - 2r\langle I, \underline{\omega}\rangle)^{(n+1)/2}} \, dS(\underline{\omega}),$$

$$L_n(x_0, r) := \frac{1}{A_{n+1}} \int_{\mathbb{S}^{n-1}} \frac{\underline{\omega}}{(x_0^2 + 1 + r^2 - 2r\langle I, \underline{\omega}\rangle)^{(n+1)/2}} \, dS(\underline{\omega}).$$

To compute $J_n(x_0, r)$ and $L_n(x_0, r)$ we use Theorem 3.7 (Funk-Hecke) setting

$$\psi_r(t) := \frac{1}{(x_0^2 + 1 + r^2 - 2rt)^{(n+1)/2}}, \quad \text{where} \quad t := \langle I, \underline{\omega}\rangle;$$

the term $J_n(x_0, r)$ can be written as

$$J_n(x_0, r) = \frac{1}{A_{n+1}} A_{n-1} \int_{-1}^{1} \psi_r(t)(1-t^2)^{(n-3)/2} \, dt$$

since $\mathcal{S}_0(\underline{\omega}) = 1$ and $P_0(t) = 1$; analogously we have

$$L_n(x_0, r) = I \, \frac{1}{A_{n+1}} \, A_{n-1} \int_{-1}^{1} \psi_r(t)(1-t^2)^{(n-3)/2} \, t \, dt. \tag{11}$$

since $\mathcal{S}_1(\underline{\omega}) = \underline{\omega}$ and $P_1(t) = t$. With the position

$$\mathcal{Q}_p(r, n) := \int_{-1}^{1} \psi_r(t)(1-t^2)^{(n-3)/2} \ t^p \, dt, \qquad p = 0, 1, \tag{12}$$

we can write $J_n(x_0, r)$ and $L_n(x_0, r)$ as

$$J_n(x_0, r) = \frac{1}{A_{n+1}} A_{n-1} \mathcal{Q}_0(r, n),$$

$$L_n(x_0, r) = I A_{n-1} \, \frac{1}{A_{n+1}} \, \mathcal{Q}_1(r, n).$$

One can easily verify that $\mathcal{N}_n^+(x)$ is an axially monogenic function, since it satisfy the Vekua system which is an elliptic system. So $\mathcal{N}_n^+(x)$ is determined by its restriction to $\underline{x}=0$. This fact allows us to simplify the computation of the integrals $\mathcal{Q}_p(r,n)$. As one can see $\psi_0(t)$ is independent of t:

$$\psi_0(t) := \psi_0 = \frac{1}{(x_0^2+1)^{(n+1)/2}}.$$

Moreover, observe that

$$sup_{t\in[-1,1]} |\psi_0(t) - \psi_r(t)| \to 0 \quad \text{as} \quad r \to 0$$

so we can pass to the limit under the integral and we have

$$\mathcal{Q}_p(0,n) = \int_{-1}^{1} \psi_0(t)(1-t^2)^{(n-3)/2}\ t^p\, dt, = \psi_0 \int_{-1}^{1} (1-t^2)^{(n-3)/2}\ t^p\, dt, \qquad p=0,1.$$

Now we recall that

$$\int_{-1}^{1} (1-t^2)^q\, dt = \sqrt{\pi}\ \frac{\Gamma(q+1)}{\Gamma(q+\frac{3}{2})} \quad \text{for} \quad q\in\mathbb{R}^+.$$

so

$$\mathcal{Q}_0(0,n) = \psi_0\sqrt{\pi}\ \frac{\Gamma((n-1)/2)}{\Gamma(n/2)}$$

while, as it can be easily seen, we have

$$\mathcal{Q}_1(0,n) = \psi_0 \int_{-1}^{1} (1-t^2)^{(n-3)/2}\ t\, dt = 0.$$

With the above considerations we obtain

$$J_n(x_0,r) = \psi_0\sqrt{\pi}\frac{1}{A_{n+1}}A_{n-1}\ \frac{\Gamma((n-1)/2)}{\Gamma(n/2)}$$

and

$$L_n(x_0,0) = 0$$

so we obtain

$$\mathcal{N}_n^+(x)|_{\underline{x}=0} = x_0 J_n(x_0,0) = \frac{1}{\sqrt{\pi}}\ \frac{\Gamma((n+1)/2)}{\Gamma(n/2)}\ \frac{x_0}{(x_0^2+1)^{(n+1)/2}}.$$

Now we consider $\mathcal{N}_n^-(x)$. We have

$$\begin{aligned}
\mathcal{N}_n^-(x) &= \frac{1}{A_{n+1}} \int_{\mathbb{S}^{n-1}} \frac{x_0 - \underline{x} + \underline{\omega}}{(x_0^2 + \langle \underline{x}-\underline{\omega}, \underline{x}-\underline{\omega}\rangle)^{(n+1)/2}}\, \underline{\omega}\, dS(\underline{\omega}) \\
&= \frac{1}{A_{n+1}} \int_{\mathbb{S}^{n-1}} \frac{(x_0 - \underline{x})\underline{\omega} - 1}{(x_0^2 + \langle \underline{x}-\underline{\omega}, \underline{x}-\underline{\omega}\rangle)^{(n+1)/2}}\ dS(\underline{\omega}) \\
&= (x_0-\underline{x})L_n(x_0,r) - J_n(x_0,r).
\end{aligned}$$

So we finally get

$$\mathcal{N}_n^-(x)|_{\underline{x}=0} = -J_n(x_0,0) = -\frac{1}{\sqrt{\pi}}\ \frac{\Gamma((n+1)/2)}{\Gamma(n/2)}\ \frac{1}{(x_0^2+1)^{(n+1)/2}}.$$

$\square$

Remark 3.11. Recalling the well known property of the Gamma function $n\Gamma(n) = \Gamma(n+1)$ and $\Gamma(n+1) = n!$, $\Gamma(1/2) = \sqrt{\pi}$ (see [20]) a direct computation shows that $\mathcal{C}_3 = 2/\pi$, which will be useful in Section 5.

The Fueter primitives of $\mathcal{N}_n^+$ and $\mathcal{N}_n^-$ can be expressed by a multiple integral. We explicitly write the case $n = 3$ and for any n odd we show the procedure to construct them.

Theorem 3.12 (The structure of the Fueter primitives of $\mathcal{N}_n^+$ and $\mathcal{N}_n^-$)**.** *Let n be an odd number and denote by $\mathcal{W}_n^+$ and $\mathcal{W}_n^-$ the Fueter primitives of $\mathcal{N}_n^+$ and $\mathcal{N}_n^-$, respectively. Consider the functions:*

$$\mathcal{W}_n^+(x_0) := \frac{\mathcal{C}_n}{\mathcal{K}_n}\ D^{-(n-1)} \frac{x_0}{(x_0^2+1)^{(n+1)/2}},$$

$$\mathcal{W}_n^-(x_0) := -\frac{\mathcal{C}_n}{\mathcal{K}_n}\ D^{-(n-1)} \frac{1}{(x_0^2+1)^{(n+1)/2}},$$

where the symbol $D^{-(n-1)}$ stands for the $(n-1)$ integrations with respect to x_0. Then replacing x_0 by x in $\mathcal{W}_n^+(x_0)$ and in $\mathcal{W}_n^-(x_0)$ we get $\mathcal{W}_n^+(x)$ and $\mathcal{W}_n^-(x)$, respectively. Moreover, the functions $\mathcal{W}_n^+(x)$ and $\mathcal{W}_n^-(x)$ are extendable to slice monogenic functions defined for all $x \in \{x_0 + r\underline{\omega} \mid (x_0, r) \neq (0,1)\}$.

Proof. Let us observe that the kernels $\mathcal{N}_n^+$ and $\mathcal{N}_n^-$defined in (9) are axially monogenic, they satisfy the Vekua system which is elliptic so $\mathcal{N}_n^+$ and $\mathcal{N}_n^-$ are determined by their restrictions $\mathcal{N}_n^+(x)|_{\underline{x}=0}$ and $\mathcal{N}_n^-(x)|_{\underline{x}=0}$, respectively. From Theorem 3.10 we have that

$$\mathcal{N}_n^+(x)|_{\underline{x}=0} = \mathcal{C}_n\ \frac{x_0}{(x_0^2+1)^{(n+1)/2}}, \qquad \mathcal{N}_n^-(x)|_{\underline{x}=0} = -\mathcal{C}_n\ \frac{1}{(x_0^2+1)^{(n+1)/2}},$$

where the constants $\mathcal{C}_n$ are explicitly determined by (10). Now recall that if W is a slice monogenic function, defined on an axially symmetric s-domain in $\mathbb{R}^{n+1}$, from Proposition 3.5 we have that

$$\Delta_x^{(n-1)/2} W(x)|_{\underline{x}=0} = \mathcal{K}_n V^{(n-1)}(x_0),$$

where $W(x)|_{\underline{x}=0} := V(x_0)$ and $\mathcal{K}_n$ are explicitly determined by (7). If we set

$$D^{(n-1)}\mathcal{W}_n^+(x_0) := \frac{\mathcal{C}_n}{\mathcal{K}_n}\ \frac{x_0}{(x_0^2+1)^{(n+1)/2}}$$

and we integrate the function $\frac{x_0}{(x_0^2+1)^{(n+1)/2}}$ for $(n-1)$ times we obtain

$$\mathcal{W}_n^+(x_0) := \frac{\mathcal{C}_n}{\mathcal{K}_n}\ D^{-(n-1)} \frac{x_0}{(x_0^2+1)^{(n+1)/2}}$$

where the symbol $D^{-(n-1)}$ stands for the $(n-1)$ integrations with respect to x_0, replacing now x_0 by x in $\mathcal{W}_n^+(x_0)$ we get $\mathcal{W}_n^+(x)$ which is a Fueter's primitive of $\mathcal{N}_n^+(x)$ since

$$\Delta_x^{(n-1)/2}\mathcal{W}_n^+(x)|_{\underline{x}=0} = \frac{\mathcal{C}_n}{\mathcal{K}_n}\mathcal{K}_n V^{(n-1)}(x_0) = \mathcal{C}_n\ \frac{x_0}{(x_0^2+1)^{(n+1)/2}} = \mathcal{N}_n^+(x)|_{\underline{x}=0}.$$

Analogously we set

$$D^{(n-1)}\mathcal{W}_n^-(x_0) := -\frac{\mathcal{C}_n}{\mathcal{K}_n}\ \frac{1}{(x_0^2+1)^{(n+1)/2}}$$

and we integrate the function $\frac{1}{(x_0^2+1)^{(n+1)/2}}$ for $(n-1)$ times to get

$$\mathcal{W}_n^-(x_0) := -\frac{\mathcal{C}_n}{\mathcal{K}_n}\ D^{-(n-1)}\frac{1}{(x_0^2+1)^{(n+1)/2}}$$

replacing now x_0 by x in $\mathcal{W}_n^-(x_0)$ we get $\mathcal{W}_n^-(x)$ which is a Fueter primitive of $\mathcal{N}_n^-(x)$. Finally we observe that the functions

$$x_0 \mapsto \frac{x_0}{(x_0^2+1)^{(n+1)/2}}, \qquad x_0 \mapsto \frac{1}{(x_0^2+1)^{(n+1)/2}},$$

can be integrated by parts in closed form an arbitrary number of times. Such primitives contain rational functions of x_0 and $\arctan x_0$ of the real variable x_0.

When we replace in such functions the real variable x_0 by the paravector variable x we obtain slice monogenic functions. So $\mathcal{W}_n^-$ and $\mathcal{W}_n^+$ are slice monogenic functions whenever they are defined, i.e. for all $x \in \{x_0 + r\underline{\omega} \mid (x_0, r) \neq (0, 1)\}$. □

Corollary 3.13 (Explicit Fueter's primitives of $\mathcal{N}_n^+$ and $\mathcal{N}_n^-$ for $n = 3$)**.**

$$\mathcal{W}_3^+(x) = \frac{1}{2\pi}\ \arctan x, \qquad \mathcal{W}_3^-(x) = -\frac{1}{2\pi}\ x \arctan x.$$

Proof. From Remarks 3.6 and 3.11 we determine the ratios $\frac{\mathcal{C}_n}{\mathcal{K}_n}$ for $n = 3$. Then integrating by parts we get the statement as a direct consequences of the previous theorem. We omit the simple but tedious computations. □

4 The inverse Fueter mapping theorem

We now recall the Cauchy's integral formula for monogenic functions that with the results of the previous section is the main tool to prove our main result.

Theorem 4.1 (Cauchy's integral representation theorem for monogenic functions)**.** *Let $\breve{f}$ be a left monogenic function in $U \subseteq \mathbb{R}^{n+1}$. Then, for every $M \subset U$ and for $x \in M$, we have*

$$\breve{f}(x) = \int_{\partial M} \mathcal{G}(y - x) d\sigma(y) \breve{f}(y), \tag{13}$$

where ∂M is supposed to be an n-dimensional compact smooth manifold in U, the differential form $d\sigma(y)$ is given by $d\sigma(y) = \eta(y)dS(y)$ where $\eta(y)$ is the outer unit normal to ∂M at point y and $dS(y)$ is the scalar element of surface area on ∂M.

We are now in position to state and prove the main result of this paper.

Theorem 4.2 (The inverse Fueter mapping theorem)**.** *Let $\breve{f}(x) = A(x_0, \rho) + \underline{\omega}B(x_0, \rho)$ be an axially monogenic function defined on an axially symmetric domain $U \subseteq \mathbb{R}^{n+1}$. Let Γ be the boundary of an open bounded subset of the half plane $\mathbb{R} + \underline{\omega}\mathbb{R}^+$. Moreover suppose that Γ is a regular curve whose parametric equations $y_0 = y_0(s)$, $\rho = \rho(s)$ are express in terms of the arc-length $s \in [0, L]$, $L > 0$. Then the function*

$$\begin{aligned} f(x) = &\int_\Gamma \mathcal{W}_n^-\Big(\frac{1}{\rho}(x - y_0)\Big)\rho^{n-2}(dy_0\, A(y_0, \rho) - d\rho\, B(y_0, \rho)) \\ -&\int_\Gamma \mathcal{W}_n^+\Big(\frac{1}{\rho}(x - y_0)\Big)\rho^{n-2}(dy_0 B(y_0, \rho) - d\rho A(y_0, \rho)). \end{aligned} \tag{14}$$

is a Fueter's primitive of $\breve{f}(x)$, where $\mathcal{W}_n^+$ and $\mathcal{W}_n^-$ are as in Theorem 3.12.

Proof. We represent axially monogenic functions $\breve{f}$ by the Cauchy formula (13) using the following manifold

$$\Sigma := \{y_0 + \underline{\omega}\rho \,|\, (y_0, \rho) \in \Gamma, \quad \underline{\omega} \in \mathbb{S}^{n-1}\}.$$

We specify the notations

- ds is the infinitesimal arc-length, $dS(\underline{\omega})$ is the infinitesimal element of surface area on $\mathbb{S}^{n-1}$
- $\mathbf{t} = \frac{d}{ds}(y_0 + \underline{\omega}\rho)$ is the unit tangent vector in a point of Γ, while the normal unit vector is given by
$$\mathbf{n} = -\underline{\omega}\mathbf{t} = \frac{d}{ds}[\rho(s) - \underline{\omega}y_0(s)].$$
- The scalar infinitesimal element of the manifold Σ, expressed in terms of ds and dS is given by
$$d\Sigma = \rho^{n-1}\, ds\, dS(\underline{\omega}).$$

  Finally, the oriented infinitesimal element of manifold $d\sigma(s, \underline{\omega})$ is given by
$$d\sigma(s, \underline{\omega}) = \mathbf{n} d\Sigma = \frac{d}{ds}[\rho(s) - \underline{\omega}y_0(s)]\, \rho^{n-1}\, ds\, dS(\underline{\omega})$$
  so finally we get
$$d\sigma(s, \underline{\omega}) = [d\rho(s) - \underline{\omega}dy_0(s)]\rho^{n-1}dS(\underline{\omega}).$$

Thanks to the above considerations we have:

$$\breve{f}(x_0 + Ir) = \int_\Gamma \int_{\mathbb{S}^{n-1}} \mathcal{G}(y_0 + \underline{\omega}\rho - x_0 - rI)\, d\sigma(s, \underline{\omega})\, \breve{f}(y_0 + \underline{\omega}\rho),$$

so we get

$$\breve{f}(x_0 + Ir) = \int_\Gamma \int_{\mathbb{S}^{n-1}} \mathcal{G}(y_0 + \underline{\omega}\rho - x_0 - rI)[d\rho(s) - \underline{\omega}dy_0(s)]\rho^{n-1}dS(\underline{\omega})(A(y_0, \rho) + \underline{\omega}B(y_0, \rho)).$$

We can now split the integral in the following way

$$\breve{f}(x_0 + Ir) = -\int_\Gamma \Big[\int_{\mathbb{S}^{n-1}} \mathcal{G}(y_0 + \underline{\omega}\rho - x_0 - rI)\, \underline{\omega}\, dS(\underline{\omega})\Big]\, \rho^{n-1}[dy_0\, A(y_0, \rho) - d\rho\, B(y_0, \rho)]$$
$$+ \int_\Gamma \Big[\int_{\mathbb{S}^{n-1}} \mathcal{G}(y_0 + \underline{\omega}\rho - x_0 - rI)\, dS(\underline{\omega})\Big]\ \ \rho^{n-1}[dy_0 B(y_0, \rho) + d\rho A(y_0, \rho)]$$

and keeping in mind the property $\mathcal{G}(tx) = t^{-n}\mathcal{G}(tx)$ for $t > 0$, with a change of variables, we have

$$\breve{f}(x_0 + Ir) = \int_\Gamma \Big[\int_{\mathbb{S}^{n-1}} \rho^{-n}\mathcal{G}\Big(\frac{x_0 - y_0}{\rho} + \frac{r}{\rho}I - \underline{\omega}\Big)\, \underline{\omega}\ dS(\underline{\omega})\Big]\, \rho^{n-1}[dy_0\, A(y_0, \rho) - d\rho\, B(y_0, \rho)]$$
$$- \int_\Gamma \Big[\int_{\mathbb{S}^{n-1}} \rho^{-n}\mathcal{G}\Big(\frac{x_0 - y_0}{\rho} + \frac{r}{\rho}I - \underline{\omega}\Big)\, dS(\underline{\omega})\Big]\ \ \rho^{n-1}[dy_0 B(y_0, \rho) + d\rho A(y_0, \rho)].$$

Recalling the definitions of $\mathcal{N}_n^+$ and $\mathcal{N}_n^-$ we get

$$\breve{f}(x_0 + Ir) = \int_\Gamma \mathcal{N}_n^-\Big(\frac{x_0 - y_0}{\rho} + \frac{r}{\rho}I\Big)\, \rho^{-1}[dy_0\, A(y_0, \rho) - d\rho\, B(y_0, \rho)]$$

$$-\int_\Gamma \mathcal{N}_n^+\Big(\frac{x_0-y_0}{\rho}+\frac{r}{\rho}I\Big)\,\rho^{-1}[dy_0B(y_0,\rho)+d\rho A(y_0,\rho)].$$

Let us observe that, since $x=x_0+Ir$, we have

$$\breve{f}(x_0+Ir)=\int_\Gamma \mathcal{N}_n^-\Big(\frac{x-y_0}{\rho}\Big)\,\rho^{-1}[dy_0\,A(y_0,\rho)-d\rho\,B(y_0,\rho)]$$

$$-\int_\Gamma \mathcal{N}_n^+\Big(\frac{x-y_0}{\rho}\Big)\,\rho^{-1}[dy_0B(y_0,\rho)+d\rho A(y_0,\rho)].$$

By setting

$$x':=\frac{x-y_0}{\rho}$$

and using Theorem 3.12, we obtain

$$\breve{f}(x_0+Ir)=\frac{1}{A_n}\int_\Gamma \Delta_{x'}^{(n-1)/2}\mathcal{W}_n^-(x')\,\rho^{-1}[dy_0\,A(y_0,\rho)-d\rho\,B(y_0,\rho)]$$

$$-\frac{1}{A_n}\int_\Gamma \Delta_{x'}^{(n-1)/2}\mathcal{W}_n^+(x')\,\rho^{-1}[dy_0B(y_0,\rho)+d\rho A(y_0,\rho)]$$

and since $\Delta_{x'}^{(n-1)/2}=\rho^{n-1}\Delta_x^{(n-1)/2}$ we get:

$$\breve{f}(x_0+Ir)=\Delta_x^{(n-1)/2}\Big[\int_\Gamma \mathcal{W}_n^-\Big(\frac{x-y_0}{\rho}\Big)\rho^{n-2}[dy_0\,A(y_0,\rho)-d\rho\,B(y_0,\rho)]$$

$$-\int_\Gamma \mathcal{W}_n^+\Big(\frac{x-y_0}{\rho}\Big)\rho^{n-2}[dy_0B(y_0,\rho)+d\rho A(y_0,\rho)]\Big]$$

where

$$f(x):=\int_\Gamma \mathcal{W}_n^-\Big(\frac{x-y_0}{\rho}\Big)\rho^{n-2}[dy_0\,A(y_0,\rho)-d\rho\,B(y_0,\rho)]$$

$$-\int_\Gamma \mathcal{W}_n^+\Big(\frac{x-y_0}{\rho}\Big)\rho^{n-2}[dy_0B(y_0,\rho)+d\rho A(y_0,\rho)]. \tag{15}$$

The function $f(x)$ is slice monogenic because $\mathcal{W}_n^+$, $\mathcal{W}_n^-$ are slice monogenic by construction. So f is a Fueter primitive of $\breve{f}$. □

From the proof on the above theorem one can easily see that the following result holds.

Corollary 4.3. *Under the hypotheses of the above theorem the Cauchy integral formula for axially monogenic functions can be written in the form*

$$\breve{f}(x)=\int_\Gamma \mathcal{N}_n^-\Big(\frac{x-y_0}{\rho}\Big)\,\rho^{-1}[dy_0\,A(y_0,\rho)-d\rho\,B(y_0,\rho)]$$

$$-\int_\Gamma \mathcal{N}_n^+\Big(\frac{x-y_0}{\rho}\Big)\,\rho^{-1}[dy_0B(y_0,\rho)+d\rho A(y_0,\rho)].$$

Another consequence is the following:

Corollary 4.4. *Let n be an odd number. Let U be an axially symmetric open set in $\mathbb{R}^{n+1}$. There is a map of $\mathbb{R}_n$-modules*

$$\mathcal{M}(U)\to\mathscr{H}(U),\qquad \breve{f}\mapsto f.$$

For general monogenic functions in axially symmetric domains it is possible to construct the canonical projections onto the space of axially monogenic functions, see [17]. This result is expressed in the following theorem.

Theorem 4.5. *Let f be an axially monogenic function of the form $f(x) = A(x) + \underline{\omega}B(x)$, then A and B are given by*

$$A(x) = \frac{1}{A_{n-1}} \int_{\mathbb{S}^{n-1}} f(x_0 + r\underline{\omega})dS(\underline{\omega}), \qquad B(x) = \frac{1}{A_{n-1}} \int_{\mathbb{S}^{n-1}} \underline{\omega} f(x_0 + r\underline{\omega})dS(\underline{\omega}).$$

Moreover, for any monogenic function f the axially monogenic projection takes the same form $A(x) + \underline{\omega}B(x)$ where A, B are given by the above formulas.

Proof. The first part of the theorem is immediately obtained by inserting f into the integrals. For the general case it suffices to prove that the axial part $A(x) + \underline{\omega}B(x)$ of the function f is still left monogenic. This has been proved in a more general context in [17], p. 237 where it has been proved also that f can be written in a unique way as a series of the form

$$f(x_0 + r\underline{\omega}) = \sum_{k=0}^{+\infty} (A_k(x_0, r, \underline{\omega}) + \underline{\omega}B_k(x_0, r, \underline{\omega}))$$

where $A_k(x_0, r, \underline{\omega})$, $B_k(x_0, r, \underline{\omega})$ are inner spherical monogenic of degree k in $\underline{\omega}$. □

5 The case of quaternionic valued functions

The quaternionic version of the theory is based on the notion of slice regularity which was inspired by the paper of Cullen [16] and its further advances are in the papers [4], [18]. We recall that the notion of slice regularity for quaternionic function is analogue to the one of slice monogenic functions with obvious modifications when we consider functions $f : U \subseteq \mathbb{H} \to \mathbb{H}$. Analogously the definitions of s-domain and of axially symmetric domain are the same. The representation formula is a particular case of the one for slice monogenic functions. Moreover, the notion of axially regular functions mimics the one of the monogenic case. In the following we denote by $\mathcal{SR}(U)$ the set of slice regular functions and by $\mathcal{R}(U)$ the set of Fueter regular functions of axial type on an open set U. The set U is assumed to be an axially symmetric domain. Let $U \subseteq \mathbb{H}$ be an axially symmetric domain. Suppose that $f \in \mathcal{SR}(U)$. We say that a function f is a Fueter primitive of $\breve{f} \in \mathcal{R}(U)$ if $\Delta f(q) = \breve{f}(q)$ on U. The analogue of the kernels $\mathcal{N}_n^+(x)$ and $\mathcal{N}_n^-(x)$ for the quaternionic case are $\mathcal{N}_3^+(x)$ and $\mathcal{N}_3^-(x)$ where x is replaced by a quaternion q. Precisely, let $\mathcal{G}(q - \underline{y})$ be the Cauchy-Fueter kernel with $q = q_0 + \underline{q} \in \mathbb{H}$ and for $\underline{y} = r\underline{\omega} \in \mathbb{H}$ we assume $r = 1$ and $\underline{\omega} \in \mathbb{S}^2$. We define the kernels

$$\mathcal{N}^+(q) = \int_{\mathbb{S}^2} \mathcal{G}(q - \underline{\omega})\, dS(\underline{\omega}), \qquad \mathcal{N}^-(x) = \int_{\mathbb{S}^2} \mathcal{G}(q - \underline{\omega})\, \underline{\omega}\, dS(\underline{\omega}). \tag{16}$$

where $dS(\omega)$ is the scalar element of surface area of $\mathbb{S}^2$.

Theorem 5.1. *Let $\mathcal{N}^+$ and $\mathcal{N}^-$ be the kernels defined in (16), respectively. Then their restrictions to $\underline{q} = 0$ are given by*

$$\mathcal{N}^+(q)|_{\underline{q}=0} = -\frac{1}{\pi}\, \frac{q_0}{(q_0^2 + 1)^2}, \qquad \mathcal{N}^-(q)|_{\underline{q}=0} = \frac{1}{\pi}\, \frac{1}{(q_0^2 + 1)^2}.$$

Moreover, the Fueter primitives of $\mathcal{N}^+$ and $\mathcal{N}^-$ are

$$\mathcal{W}^+(q) = \frac{1}{2\pi}\ \arctan q, \qquad\qquad \mathcal{W}^-(x) = -\frac{1}{2\pi}\ q\arctan q.$$

We can conclude with the main result.

Theorem 5.2 (The inverse Fueter mapping theorem for the quaternionic case)**.** *Let $\breve{f}(q) = A(q_0,\rho)+\underline{\omega}B(q_0,\rho)$ be an axially Fueter regular function defined on an axially symmetric domain $U \subseteq \mathbb{H}$. Let Γ be the boundary of an open bounded subset of the half plane $\mathbb{R}+\underline{\omega}\mathbb{R}^+$. Moreover suppose that Γ is a regular curve whose parametric equations $y_0 = y_0(s)$, $\rho = \rho(s)$ are expressed in terms of the arc-length $s \in [0, L]$, $L > 0$. Then the function*

$$\begin{aligned} f(q) &= \int_\Gamma \mathcal{W}^-\Big(\frac{1}{\rho}(q-y_0)\Big)\,\rho\,(dy_0\,A(y_0,\rho) - d\rho\,B(y_0,\rho)) \\ &- \int_\Gamma \mathcal{W}^+\Big(\frac{1}{\rho}(q-y_0)\Big)\,\rho\,(dy_0 B(y_0,\rho) - d\rho A(y_0,\rho)). \end{aligned} \tag{17}$$

is a Fueter primitive of $\breve{f}(q)$, where $\mathcal{W}^+$ and $\mathcal{W}^-$ are defined above.

Remark 5.3. The manifold chosen for the Cauchy-Fueter formula, to prove Theorem 5.2, is obviously given by:

$$\Sigma := \{y_0 + \underline{\omega}\rho \,|\, (y_0,\rho) \in \Gamma,\quad \underline{\omega} \in \mathbb{S}^2\}.$$

Remark 5.4. It is possible to compute directly the Laplacian of the slice regular function $\arctan q$. Let us work in spherical coordinates. Consider the paravector $q = q_0 + rI$ and the Laplacian

$$\Delta_q = \partial^2_{q_0} + \partial^2_r + \frac{2}{r}\partial_r + \frac{1}{r^2}\Delta_I$$

where Δ_I is the Laplace-Beltrami operator acting only on angular coordinates. As it is well known $\Delta_I I = -2I$. First of all we observe that, since $\arctan q$ is holomorphic on the plane $\mathbb{C}_I$, we have

$$(\partial^2_{x_0} + \partial^2_r)\arctan q = 0.$$

Next we have

$$\frac{2}{r}\partial_r \arctan q = \frac{2I}{r}\frac{1}{1+q^2},$$

so we can write

$$\arctan q = Re(\arctan q) + I[Im(\arctan q)].$$

Thus, the operator Δ_I acts only on I, and we have

$$\frac{1}{r^2}\Delta_I(\arctan q) = \frac{1}{r^2}\Delta_I I[Im(\arctan q)] = -\frac{2I}{r^2}[Im(\arctan q)]$$

but since

$$Im(\arctan q) = \frac{1}{4}[\ln(q_0^2 + (1+r)^2) - \ln(q_0^2 + (1-r)^2)]$$

we finally get

$$\Delta_q \arctan q = \frac{2I}{r}\frac{1}{1+q^2} - \frac{I}{2}[\ln(q_0^2 + (1+r)^2) - \ln(q_0^2 + (1-r)^2)].$$

Moreover, a direct computation of the integral $\int_{\mathbb{S}^2}\mathcal{G}(q-\underline{\omega})\,dS(\underline{\omega})$, using the table of integrals [22], can be done and one can verify directly that $\frac{1}{2\pi}\ \arctan q$ is its Fueter primitive. Analogous considerations can be done for $-\frac{1}{2\pi}\ q\arctan q$.

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