Reverse Mathematics and Well-ordering Principles

Michael Rathjen and Andreas Weiermann [1]

Department of Pure Mathematics
University of Leeds
Leeds, LS2 9JT, UK

Department of Pure Mathematics and Computer Algebra
Ghent University
Krijgslaan 281 - Gebouw S22, B9000 Gent, Belgium

Abstract

The paper is concerned with generally Π^1_2 sentences of the form *"if X is well ordered then $f(X)$ is well ordered"*, where f is a standard proof theoretic function from ordinals to ordinals. It has turned out that a statement of this form is often equivalent to the existence of countable coded ω-models for a particular theory T_f whose consistency can be proved by means of a cut elimination theorem in infinitary logic which crucially involves the function f. To illustrate this theme, we shall focus on the well-known φ-function which figures prominently in so-called predicative proof theory. However, the approach taken here lends itself to generalization in that the techniques we employ can be applied to many other proof-theoretic functions associated with cut elimination theorems. In this paper we show that the statement *"if X is well ordered then $\varphi X0$ is well ordered"* is equivalent to $\mathbf{ATR}_0$. This was first proved by Friedman, Montalban and Weiermann [7] using recursion-theoretic and combinatorial methods. The proof given here is proof-theoretic, the main techniques being Schütte's method of proof search (deduction chains) [13], generalized to ω logic, and cut elimination for infinitary ramified analysis.

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Email address: `rathjen@maths.leeds.ac.uk,weiermann@cage.ugent.be` (Michael Rathjen and Andreas Weiermann).

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1 Introduction

The larger project broached in this paper is a form of reverse mathematics for Π^1_2 statements of the shape

$$\mathbf{WOP}(f) \qquad \text{“\textit{if } } X \text{ \textit{is well ordered then} } f(X) \text{ \textit{is well ordered}”} \tag{1}$$

where f is a standard proof theoretic function from ordinals to ordinals. There are by now several examples of functions f where the statement $\mathbf{WOP}(f)$ has turned out to be equivalent to one of the theories of reverse mathematics over a weak base theory (usually $\mathbf{RCA}_0$). The first example is due to Girard [8].

Theorem 1.1 (Girard 1987) *Let* $\mathbf{WO}(\mathfrak{X})$ *express that* $\mathfrak{X}$ *is a well ordering. Over* $\mathbf{RCA}_0$ *the following are equivalent:*

(i) *Arithmetic Comprehension*
(ii) $\forall \mathfrak{X}\, [\mathbf{WO}(\mathfrak{X}) \rightarrow \mathbf{WO}(2^{\mathfrak{X}})]$.

Recently two new results appeared in preprints [10,7]. These result give characterizations of the form (1) for the theories $\mathbf{ACA}_0^+$ and $\mathbf{ATR}_0$, respectively, in the form of familiar proof-theoretic functions. $\mathbf{ACA}_0^+$ denotes the theory $\mathbf{ACA}_0$ augmented by an axiom asserting that for any set X the ω-th jump in X exists while $\mathbf{ATR}_0$ asserts the existence of sets constructed by transfinite iterations of arithmetical comprehension. $\alpha \mapsto \varepsilon_\alpha$ denotes the usual ε function while φ stands for the two-place Veblen function familiar from predicative proof theory (cf. [13]). More detailed descriptions of $\mathbf{ATR}_0$ and the function $\mathfrak{X} \mapsto \varphi\mathfrak{X}0$ will be given shortly. Definitions of the familiar subsystems of reverse mathematics can be found in [15].

Theorem 1.2 (Montalban, Marcone, 2007) *Over* $\mathbf{RCA}_0$ *the following are equivalent:*

(i) $\mathbf{ACA}_0^+$

(ii) $\forall \mathfrak{X}\, [\mathbf{WO}(\mathfrak{X}) \rightarrow \mathbf{WO}(\varepsilon_{\mathfrak{X}})]$.

Theorem 1.3 (Friedman, Montalban, Weiermann 2007) *Over* $\mathbf{RCA}_0$ *the following are equivalent:*

(i) $\mathbf{ATR}_0$

(ii) $\forall \mathfrak{X}\, [\mathbf{WO}(\mathfrak{X}) \rightarrow \mathbf{WO}(\varphi\mathfrak{X}0)]$.

The proof of Theorem 1.3 uses rather sophisticated recursion-theoretic results about linear orderings and is quite combinatorial. Theorem 1.3 uses a result of Steel's [16] about descending sequences of degrees which states that if $Q \subseteq \mathrm{Pow}(\omega) \times \mathrm{Pow}(\omega)$ is arithmetic, then there is no sequence $\{A_n \mid n \in \omega\}$ such that (a) for every n, A_{n+1} is the unique set such that $Q(A_n, A_{n+1})$, (b) for every n, $A'_{n+1} \leq_T A_n$.

For a proof theorist, theorems 1.2 and 1.3 bear a striking resemblance to cut elimination theorems for infinitary logics. This prompted the first author of this paper to look for proof-theoretic ways of proving these results. The hope was that this would also unearth a common pattern behind them and possibly lead to more results of this kind. The project commenced with [2] where a purely proof-theoretic proof of Theorem 1.2 was presented. In this paper we shall give a new proof of Theorem 1.3. It is principally proof-theoretic, the main techniques being Schütte's method of proof search (deduction chains) [13] and cut elimination for ramified analysis. The general pattern, of which this paper provides a second example, is that a statement $\mathbf{WOP}(f)$ is often equivalent to a familiar cut elimination theorem for an infinitary logic which in turn is equivalent to the assertion that every set is contained in an ω-model of a certain theory T_f.

To guide the reader through the paper we shall briefly sketch the main parts of the proof of Theorem 1.3, i.e., that (ii) implies (i). We start with the observation that $\mathbf{ATR}_0$ can be be axiomatized over $\mathbf{ACA}_0$ via a single sentence of the form $\forall X(\mathbf{WO}(<_X) \to \forall Z \exists Y\, B_0(X, Y, Z))$ where $B_0(X, Y, Z)$ is an arithmetical formula (cf. Lemma 3.2). Thus to verify $\mathbf{ATR}_0$ it suffices to show that for every well-ordering $<_Q$ there exists an ω-model of $\mathbb{M}$ of $\mathbf{ACA}_0$ which contains Q such that $\mathbb{M} \models \forall Z \exists Y\, B_0(X, Y, Z)$. To find $\mathbb{M}$ we employ Schütte's method of proof search from [13, II§4], which he used to prove the completeness theorem for first order logic (cf. [13, Theorem 5.7]). The method has to be extended to ω-logic, though. Rather than working in the Schütte calculus of positive and negative forms we work in a Gentzen sequent calculus with finite sets of formulas called sequents. Let C be a sentence that axiomatizes arithmetic comprehension and let $D_Q(n)$ be the formula $n \in Q$ if the latter formula is true and $n \notin Q$ otherwise. The main idea is to start with the sequent $\{\neg\forall Z \exists Y\, B_0(Q, Y, Z), \neg C, \neg D_Q(0)\}$ and systematically apply the rules of ω-logic for the second order sequent calculus backwards, giving rise to a tree of sequents $\mathcal{D}_Q$. One also has to add the formula $\neg D_Q(n)$ to all sequents generated in this way after n steps.

There are two possible outcomes. If the tree $\mathcal{D}_Q$ is not well-founded then it contains an infinite path $\mathbb{P}$. Now define a set M via

$$(M)_i = \{n \mid n \notin U_i \text{ occurs in } \mathbb{P}\}$$

and let $\mathbb{M} = (\mathbb{N}; \{(M)_i \mid i \in \mathbb{N}\}, +, \cdot, 0, 1, <)$. For a formula F, let $F \in \mathbb{P}$ mean that F occurs in $\mathbb{P}$, i.e. $F \in \Gamma$ for some $\Gamma \in \mathbb{P}$. Let $U_0, U_1, U_2, \ldots$ be an enumeration of the free set variables. For the assignment $U_i \mapsto (M)_i$ one can then show that $F \in \mathbb{P} \;\; \Rightarrow \;\; \mathbb{M} \models \neg F$. Whence $\mathbb{M}$ is an ω-model of **ACA** and $\mathbb{M} \models \forall Z \exists Y\, B_0(Q, Y, Z)$. Also note that $(M)_0 = Q$, thus Q is in $\mathbb{M}$.

The other conceivable outcome is that $\mathcal{D}_Q$ is well-founded, i.e. all paths in $\mathcal{D}_Q$ are finite, and thus every maximal path ends in a sequent which contains a basic axiom. In other words $\mathcal{D}_Q$ is proof tree and the Kleene-Brouwer ordering of this tree is some well-ordering τ. The crucial step to perform next is viewing $\mathcal{D}_Q$ as a skeleton of proof tree in infinitary ramified analysis, dubbed $\mathbf{RA}^*$ in [13]. In actuality $\mathcal{D}_Q$ can be viewed as the skeleton of a proof of the empty sequent in $\mathbf{RA}^*$. As we can remove all cuts in this proof we end up with a cut free proof of the empty sequent. But this is impossible, and therefore $\mathcal{D}_Q$ cannot be well-founded. To be able to carry out the removal of all cuts we require crucial help from arithmetical transfinite induction, roughly up to the ordinal $\varphi\tau 0$, hence this is where the principle $\forall \mathfrak{X}\; [\mathbf{WO}(\mathfrak{X}) \rightarrow \mathbf{WO}(\varphi\mathfrak{X}0)]$ enters the stage in showing Theorem 1.3(i).

2 The ordering $\varphi\mathfrak{X}0$

Via simple coding procedures, countable well-orderings and functions on them can be expressed in the language of second order arithmetic, L_2. Variables $X, Y, Z, \ldots$ are supposed to range over subsets of $\mathbb{N}$. Using an elementary injective pairing function $\langle , \rangle$ (e.g. $\langle n, m\rangle := (n+m)^2 + n + 1$), every set X encodes a sequence of sets $(X)_i$, where $(X)_i := \{m \mid \langle i, m\rangle \in X\}$. We also adopt from [15], II.2 the method of encoding a finite sequence $(n_0, \ldots, n_{k-1})$ of natural numbers as a single number $\langle n_0, \ldots, n_{k-1}\rangle$.

Definition 2.1 Every set of natural numbers Q can be viewed as encoding a binary relation $<_Q$ on $\mathbb{N}$ via $n <_Q m$ iff $\langle n, m\rangle \in Q$. The **field** of Q, $\mathrm{fld}(Q)$ is the set $\{n \mid \exists m\, [n <_Q m \vee m <_Q n]\}$.

We say that Q is a **well-ordering** if $<_Q$ is a well-ordering, that is $<_Q$ is a linear ordering of its field and every non-empty subset U of $\mathrm{fld}(Q)$ has a $<_Q$-least element.

Definition 2.2 Let Q be a linear ordering with least element 0_Q. Let $\varphi ua := \langle 0, \langle u, a\rangle\rangle$, $\mathbf{H} := \{\varphi ua \mid u, a \in \mathbb{N}\}$, $\mathbf{h}(\varphi ua) = u$ and $\mathbf{h}(b) = 0_Q$ if $b \notin \mathbf{H}$.

We introduce the ordering $\varphi Q0$ by inductively defining its field $\mathrm{fld}(\varphi Q0)$ and the ordering $<_{\varphi Q0}$:

(1) $0 \in \mathrm{fld}(Q)$.

(2) $0 <_{\varphi Q0} \alpha$ if $\alpha \in \mathrm{fld}(\varphi Q0)$ and $\alpha \neq 0$.

(3) $\varphi u\alpha \in \mathrm{fld}(\varphi Q0)$ if $u \in \mathrm{fld}(Q)$, $\alpha \in \mathrm{fld}(\varphi Q0)$ and $\mathbf{h}(\alpha) \leq_Q u$.

(4) If $\alpha_1, \ldots, \alpha_n \in \mathrm{fld}(\varphi Q0) \cap \mathbf{H}$, $n > 1$ and $\alpha_n \leq_{\varphi Q0} \ldots \leq_{\varphi Q0} \alpha_1$, then

$$\alpha_1 + \ldots + \alpha_n \in \mathrm{fld}(\varphi Q0)$$

where $\alpha_1 + \ldots + \alpha_n := \langle 1, \langle \alpha_1, \ldots, \alpha_n \rangle \rangle$.

(5) If $\alpha_1 + \ldots + \alpha_n, \beta_1 + \ldots + \beta_m \in \mathrm{fld}(\varphi Q0)$, then

$$\begin{aligned} &\alpha_1 + \ldots + \alpha_n <_{\varphi Q0} \beta_1 + \ldots + \beta_m \text{ iff} \\ &n < m \wedge \forall i \leq n\, \alpha_i = \beta_i \quad \text{or} \\ &\exists i \leq \min(n, m)[\alpha_i <_{\varphi Q0} \beta_i \wedge \forall j < i\, \alpha_j = \beta_j]. \end{aligned}$$

(6) If $\alpha_1 + \ldots + \alpha_n \in \mathrm{fld}(\varphi Q0)$, $\varphi u\beta \in \mathrm{fld}(\varphi Q0)$ and $\varphi u\beta \leq_{\varphi Q0} \alpha_1$ then $\varphi u\beta <_{\varphi Q0} \alpha_1 + \ldots + \alpha_n$.

(7) If $\alpha_1 + \ldots + \alpha_n \in \mathrm{fld}(\varphi Q0)$, $\varphi u\beta \in \mathrm{fld}(\varphi Q0)$ and $\alpha_1 <_{\varphi Q0} \varphi u\beta$ then $\alpha_1 + \ldots + \alpha_n <_{\varphi Q0} \varphi u\beta$.

(8) If $\varphi u\alpha, \varphi v\beta \in \mathrm{fld}(\varphi Q0)$, then

$$\begin{aligned} \varphi u\alpha <_{\varphi Q0} \varphi v\beta \text{ iff } & u <_Q v \wedge \alpha <_{\varphi Q0} \varphi v\beta \quad \text{or} \\ & u = v \wedge \alpha <_{\varphi Q0} \beta \quad \text{or} \\ & v <_Q u \wedge \varphi u\alpha <_{\varphi Q0} \beta. \end{aligned}$$

Lemma 2.3 ($\mathbf{RCA}_0$)

(i) If Q is a linear ordering then so is $\varphi Q0$.
(ii) $\varphi Q0$ is elementary recursive in Q.

3 The theory $\mathbf{ATR}_0$

Definition 3.1 Let $A(u, Y)$ be any formula. Define $H_A(X, Y)$ to be the formula which says that $<_X$ is a linear ordering and that Y is equal to the set of pairs $\langle n, j \rangle$ such that j is in the field of $<_X$ and $A(n, Y^j)$ where $Y^j = \{\langle m, i \rangle \mid i <_X j \wedge \langle m, i \rangle \in Y\}$. Intuitively $H_A(X, Y)$ says that Y is the result of iterating A along $<_X$.

$\mathbf{ATR}_0$ is the formal system in the language of second order arithmetic whose axioms consist of $\mathbf{ACA}_0$ plus all instances of

$$\forall X(\mathbf{WO}(<_X) \to \exists Y\, H_A(X,Y))$$

where A is arithmetical.

Lemma 3.2 $\mathbf{ATR}_0$ *can be axiomatized over* $\mathbf{ACA}_0$ *via a single sentence*

$$\forall X(\mathbf{WO}(<_X) \to \forall Z \exists Y\, B_0(X,Y,Z)) \tag{2}$$

where $B_0(X,Y,Z)$ *is of the form* $H_A(X,Y)$ *for some arithmetical formula* $A(u,Y,Z)$ *with all free variables exhibited.*

Proof: This is a standard result. One could for instance take $B_0(X,Y,Z)$ to mean that Y is obtained from Z by iterated the Turing jump operation along $<_X$ starting with Z; so $A(u,Y,Z)$ would actually be a Σ^0_1 (complete) formula. Another (shorter and citable) way of showing this is to use the fact that $\mathbf{ATR}_0$ is equivalent over $\mathbf{RCA}_0$ to the statement that every two well-orderings are comparable (see [15], Theorem V.6.8). The proof of the latter statement in $\mathbf{ATR}_0$ just requires an instance H_A of said form (see the proof of [15] Lemma V.2.9). □

Definition 3.3 Let T be a theory in the language of second order arithmetic, L_2. A *countable coded* ω*-model of* T is a set $W \subseteq \mathbb{N}$, viewed as encoding the L_2-model

$$\mathbb{M} = (\mathbb{N}, \mathcal{S}, +, \cdot, 0, 1, <)$$

with $\mathcal{S} = \{(W)_n \mid n \in \mathbb{N}\}$ such that $\mathbb{M} \models T$.

This definition can be made in $\mathbf{RCA}_0$ (see [15], Definition VII.2).

We write $X \in W$ if $\exists n\, X = (W)_n$.

4 Main Theorem

The main result we want to prove is the following.

Theorem 4.1 $\mathbf{RCA}_0 + \forall \mathfrak{X}\, [\mathbf{WO}(\mathfrak{X}) \to \mathbf{WO}(\varphi\mathfrak{X}0)]$ *proves* $\mathbf{ATR}_0$.

A central ingredient of the proof will be a method of proof search (deduction chains) pioneered by Schütte [13].

Definition 4.2(i) Let $U_0, U_1, U_2, \ldots$ be an enumeration of the free set variables of L_2. For a closed term t, let $t^{\mathbb{N}}$ be its numerical value. We shall assume that all predicate symbols of the language L_2 are symbols for primitive recursive relations. L_2 contains predicate symbols for the primitive recursive relations of equality and inequality and possibly more (or all) primitive recursive relations. If R is a predicate symbol in L_2 we denote by $R^{\mathbb{N}}$ the primitive recursive relation it stands for. If $t_1, \ldots, t_n$ are closed terms the formula $R(t_1, \ldots, t_n)$ $(\neg R(t_1, \ldots, t_n))$ is said to be *true* if $R^{\mathbb{N}}(t_1^{\mathbb{N}}, \ldots, t_n^{\mathbb{N}})$ is true (is false).

(ii) Henceforth a **sequent** will be a finite set of L_2-formulas *without free number variables.*

(iii) A sequent Γ is **axiomatic** if it satisfies at least one of the following conditions:

(1) Γ contains a true **literal**, i.e. a true formula of either form $R(t_1, \ldots, t_n)$ or $\neg R(t_1, \ldots, t_n)$, where R is a predicate symbol in L_2 for a primitive recursive relation and $t_1, \ldots, t_n$ are closed terms.

(2) Γ contains formulas $s \in U$ and $t \notin U$ for some set variable U and terms s, t with $s^{\mathbb{N}} = t^{\mathbb{N}}$.

(iv) A sequent is **reducible** or a **redex** if it is not axiomatic and contains a formula which is not a literal.

Definition 4.3 For $Q \subseteq \mathbb{N}$ define

$$D_Q(n) = \begin{cases} \bar{n} \in U_0 & \text{if } n \in Q \\ \bar{n} \notin U_0 & \text{otherwise} \end{cases}$$

For the proof of Theorem 4.1 it is convenient to have a finite axiomatization of arithmetic comprehension.

Lemma 4.4 $\mathbf{ACA}_0$ *can be axiomatized via a single* Π^1_2 *sentence* $\forall X\, C(X)$.

Proof: [15], Lemma VIII.1.5. □

Definition 4.5 Let $<_Q$ be a well-ordering. Let $B(U_i)$ be the formula $\exists Y\, B_0(U_0, Y, U_i)$ of Lemma 3.2.

A Q**-deduction chain** is a finite string

$$\Gamma_0, \Gamma_1, \ldots, \Gamma_k$$

of sequents Γ_i constructed according to the following rules:

(i) $\Gamma_0 \;=\; \neg D_Q(0), \neg B(U_0), \neg C(U_0)$.
(ii) Γ_i is not axiomatic for $i < k$.
(iii) If $i < k$ and Γ_i is not reducible then

$$\Gamma_{i+1} \;=\; \Gamma_i, \neg D_Q(i+1), \neg B(U_{i+1}), \neg C(U_{i+1}).$$

(iv) Every reducible Γ_i with $i < k$ is of the form

$$\Gamma_i', E, \Gamma_i''$$

where E is not a literal and Γ_i' contains only literals.
E is said to be the **redex** of Γ_i.

Let $i < k$ and Γ_i be reducible. Γ_{i+1} is obtained from $\Gamma_i = \Gamma_i', E, \Gamma_i''$ as follows:

(1) If $E \equiv E_0 \vee E_1$ then

$$\Gamma_{i+1} \;=\; \Gamma_i', E_0, E_1, \neg D_Q(i+1), \neg B(U_{i+1}), \neg C(U_{i+1}).$$

(2) If $E \equiv E_0 \wedge E_1$ then

$$\Gamma_{i+1} \;=\; \Gamma_i', E_j, \neg D_Q(i+1), \neg B(U_{i+1}), \neg C(U_{i+1})$$

where $j = 0$ or $j = 1$.

(3) If $E \equiv \exists x\, F(x)$ then

$$\Gamma_{i+1} \;=\; \Gamma_i', F(\bar{m}), \neg D_Q(i+1), \neg B(U_{i+1}), \neg C(U_{i+1}), E$$

where m is the first number such that $F(\bar{m})$ does not occur in $\Gamma_0, \ldots, \Gamma_i$.

(4) If $E \equiv \forall x\, F(x)$ then

$$\Gamma_{i+1} \;=\; \Gamma_i', F(\bar{m}), \neg D_Q(i+1), \neg B(U_{i+1}), \neg C(U_{i+1})$$

for some m.

(5) If $E \equiv \exists X\, F(X)$ then

$$\Gamma_{i+1} \;=\; \Gamma_i', F(U_m), \neg D_Q(i+1), \neg B(U_{i+1}), \neg C(U_{i+1}), E$$

where m is the first number such that $F(U_m)$ does not occur in $\Gamma_0, \ldots, \Gamma_i$.

(6) If $E \equiv \forall X\, F(X)$ then

$$\Gamma_{i+1} \;=\; \Gamma_i', F(U_m), \neg D_Q(i+1), \neg B(U_{i+1}), \neg C(U_{i+1})$$

where m is the first number such that $m \neq i+1$ and U_m does not occur in Γ_i.

The set of Q-deduction chains forms a tree $\mathcal{D}_Q$ labeled with strings of sequents. We will now consider two cases.

Case I: $\mathcal{D}_Q$ is not well-founded. Then $\mathcal{D}_Q$ contains an infinite path $\mathbb{P}$. Now define a set M via

$$(M)_i = \{t^{\mathbb{N}} \mid t \notin U_i \text{ occurs in } \mathbb{P}\}.$$

Set $\mathbb{M} = (\mathbb{N}; \{(M)_i \mid i \in \mathbb{N}\}, +, \cdot, 0, 1, <)$.

For a formula F, let $F \in \mathbb{P}$ mean that F occurs in $\mathbb{P}$, i.e. $F \in \Gamma$ for some $\Gamma \in \mathbb{P}$.

Claim: Under the assignment $U_i \mapsto (M)_i$ we have

$$F \in \mathbb{P} \quad \Rightarrow \quad \mathbb{M} \models \neg F. \tag{3}$$

The Claim will imply that $\mathbb{M}$ is an ω-model of **ACA**. Also note that $(M)_0 = Q$, thus Q is in $\mathbb{M}$. The proof of (3) follows by induction on F using Lemma 4.6 below. The upshot of the foregoing is that we can prove Theorem 4.1 under the assumption that $\mathcal{D}_Q$ is ill-founded for all sets $Q \subseteq \mathbb{N}$.

Lemma 4.6 *Let Q be an arbitrary subset of $\mathbb{N}$ and $\mathcal{D}_Q$ be the corresponding deduction tree. Moreover, suppose $\mathcal{D}_Q$ is not well-founded. Then $\mathcal{D}_Q$ has an infinite path $\mathbb{P}$. $\mathbb{P}$ has the following properties:*

(1) $\mathbb{P}$ does not contain literals which are true in $\mathbb{N}$.
(2) $\mathbb{P}$ does not contain formulas $s \in U_i$ and $t \notin U_i$ for constant terms s and t such that $s^{\mathbb{N}} = t^{\mathbb{N}}$.
(3) If $\mathbb{P}$ contains $E_0 \vee E_1$ then $\mathbb{P}$ contains E_0 and E_1.
(4) If $\mathbb{P}$ contains $E_0 \wedge E_1$ then $\mathbb{P}$ contains E_0 or E_1.
(5) If $\mathbb{P}$ contains $\exists x F(x)$ then $\mathbb{P}$ contains $F(\bar{n})$ for all n.
(6) If $\mathbb{P}$ contains $\forall x F(x)$ then $\mathbb{P}$ contains $F(\bar{n})$ for some n.
(7) If $\mathbb{P}$ contains $\exists X F(X)$ then $\mathbb{P}$ contains $F(U_m)$ for all m.
(8) If $\mathbb{P}$ contains $\forall X F(X)$ then $\mathbb{P}$ contains $F(U_m)$ for some m.
(9) $\mathbb{P}$ contains $\neg B(U_m)$ for all m.
(10) $\mathbb{P}$ contains $\neg C(U_m)$ for all m.
(11) $\mathbb{P}$ contains $\neg D_Q(m)$ for all m.

Proof: Standard. □

Corollary 4.7 *If $\mathcal{D}_Q$ is ill-founded then there exists a countable coded ω-model of $\mathbf{ACA}_0$ containing Q which satisfies $\forall Z \exists Y\, B_0(Q, Y, Z)$.*

The remainder of the paper will be devoted to ruling out the possibility that, whenever Q is a well-ordering, $\mathcal{D}_Q$ can be a well-founded tree. This is the place were cut elimination for the infinitary proof system of ramified analysis, $\mathbf{RA}^*$ (see [13], part C), enters the stage. In a nutshell the idea is that a well-founded $\mathcal{D}_Q$ gives rise to a derivation of the empty sequent (contradiction) in $\mathbf{RA}^*$ which can be ruled by showing cut elimination for $\mathbf{RA}^*$ using transfinite induction up to $\varphi\mathfrak{X}0$, where $\mathfrak{X}$ is a well ordering not much longer than Q. However, to simplify the technical treatment we first introduce an intermediate system Δ^1_1-$\mathbf{CR}^Q_\infty$ based on the Δ^1_1-comprehension rule and the ω-rule. This theory basically coincides with Schütte's system $\mathbf{DA}^*$ (see [13], part C). It is not difficult to see that a well-founded $\mathcal{D}_Q$ can be viewed as a derivation of the empty sequent in Δ^1_1-$\mathbf{CR}^Q_\infty$. The last step towards reaching a contradiction consists in embedding Δ^1_1-$\mathbf{CR}^Q_\infty$ into $\mathbf{RA}^*$. Here we can basically follow [13] Theorem 22.14.

4.2 The infinitary calculus Δ^1_1-$\mathbf{CR}^Q_\infty$

In what follows we fix $Q \subseteq \mathbb{N}$ such that $<_Q$ is a well-ordering. In the main, the system Δ^1_1-$\mathbf{CR}^Q_\infty$ is obtained from $\mathbf{ACA}_0$ by adding the Δ^1_1-comprehension rule, the ω-rule and the basic diagram of Q. The language of Δ^1_1-$\mathbf{CR}^Q_\infty$ is the same as that of $\mathbf{ACA}_0$ but the notion of formula comes enriched with set terms. Formulas and **set terms** are defined simultaneously. Literals are formulas. Every set variable is a set term. If $A(x)$ is a formula without set quantifiers (i.e. arithmetical) then $\{x \mid A(x)\}$ is a set term. If P is a set term and t is a numerical term then $t \in P$ and $t \notin P$ are formulas. The other formation rules pertaining to $\wedge, \vee, \forall x, \exists x, \forall X, \exists X$ are as per usual.

We will be working in a Tait-style formalization of the second order arithmetic with formulas in negation normal form, i.e. negations only in front of atomic formulas. Due to the ω-rule there is no need for formulas with free numerical variables. Thus all sequents below are assumed to consist of formulas without free numerical variables.

Axioms of Δ^1_1-$\mathbf{CR}^Q_\infty$

(i) Γ, L where L is a true literal.

(ii) $\Gamma, s \in U, t \notin U$ where $s^{\mathbb{N}} = t^{\mathbb{N}}$.

(iii) $\Gamma, s \in U_0$ if $s^{\mathbb{N}} \in Q$.

(iv) $\Gamma, s \notin U_0$ if $s^{\mathbb{N}} \notin Q$.

Rules of Δ^1_1-$\mathbf{CR}^Q_\infty$

$(\wedge)\ \dfrac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}$

$(\vee)\ \dfrac{\Gamma, A_i}{\Gamma, A_0 \vee A_1}$ where $i \in \{0, 1\}$

$(\mathrm{Cut})\ \dfrac{\Gamma, A \qquad \Gamma, \neg A}{\Gamma}$

$(\omega)\ \dfrac{\Gamma, F(\bar{n}) \quad \text{for all } n}{\Gamma, \forall x F(x)}$

$(\exists_1)\ \dfrac{\Gamma, F(t)}{\Gamma, \exists x F(x)}$

$(\forall_2)\ \dfrac{\Gamma, F(P) \quad \text{for all set terms } P}{\Gamma, \forall X F(X)}$

$(\exists_2)\ \dfrac{\Gamma, F(P)}{\Gamma, \exists X F(X)}$ where P is a set term.

$(\Delta^1_1\text{-CR})\ \dfrac{\forall x[\forall Y A_0(x, Y) \leftrightarrow \exists Y A_1(x, Y)]}{\Gamma, \exists X \forall x[x \in X \leftrightarrow \forall Y A_0(x, Y)]}$ with A_0, A_1 arithmetical.

$(ST_1)\ \dfrac{\Gamma, A(t)}{\Gamma, t \in P}$ where P is the set term $\{x \mid A(x)\}$.

$(ST_2)\ \dfrac{\Gamma, \neg A(t)}{\Gamma, t \notin P}$ where P is the set term $\{x \mid A(x)\}$.

Δ^1_1-$\mathbf{CR}^Q_\infty$ is a sequent calculus version of the system DA* of [13, §20]. The language of DA*, though, is based on the connectives $\bot, \forall, \rightarrow$ while Δ^1_1-$\mathbf{CR}^Q_\infty$ has the connectives $\wedge, \vee, \forall, \exists, \neg$ and formulas are in negation normal form, i.e. the negation sign appears only in front of atomic formulas. The other main difference is that the deduction system of DA* is the Schütte calculus of positive and negative forms whereas Δ^1_1-$\mathbf{CR}^Q_\infty$'s is the Gentzen sequent calculus.

Lemma 4.8 *We shall use* Δ^1_1-$\mathbf{CR}^Q_\infty \vdash \Gamma$ *to convey that the sequent* Γ *is derivable in* Δ^1_1-$\mathbf{CR}^Q_\infty$*. Pivotal properties of* Δ^1_1-$\mathbf{CR}^Q_\infty$ *we shall exploit are the following:*

(a) $n \in Q \Rightarrow \Delta^1_1\text{-}\mathbf{CR}^Q_\infty \vdash \bar{n} \in U_0\,.$

(b) $n \notin Q \Rightarrow \Delta^1_1\text{-}\mathbf{CR}^Q_\infty \vdash \bar{n} \notin U_0\,.$

(c) $\Delta^1_1\text{-}\mathbf{CR}^Q_\infty \vdash \mathbf{WO}(U_0)\,.$

(d) Δ^1_1-$\mathbf{CR}^Q_\infty \mid\!\!- \exists Y\, H_A(U_0, Y)$ *for all arithmetical formulas* $A(u, Y)$ *having no other free numerical variables than* u*.*

Proof: (a) and (b) are immediate by the axioms (iii) and (iv) of Δ^1_1-$\mathbf{CR}^Q_\infty$.

(c) follows by (outer) transfinite induction on $<_Q$, crucially using the ω-rule. This is standard but it seems to be a challenge to find a reference. Via the axioms (iii) and (iv), the role of Q is played in Δ^1_1-$\mathbf{CR}^Q_\infty$ by the variable U_0. Writing $s \in Q$ and $s <_Q t$ for $s \in U_0$ and $\langle s, t\rangle \in U_0$, respectively, we would like to show that Δ^1_1-$\mathbf{CR}^Q_\infty \vdash \forall X(Prog_Q(U) \to \forall x\, x \in X)$, where $Prog_Q(U)$ stands for $\forall x[\forall y(y <_Q x \to y \in U) \to x \in U]$. It suffices to show

$$\Delta^1_1\text{-}\mathbf{CR}^Q_\infty \vdash \neg Prog_Q(U), \bar{n} \in U \tag{4}$$

for all n for an arbitrary set variable U. To this end we proceed by induction on Q. Inductively assume that Δ^1_1-$\mathbf{CR}^Q_\infty \vdash \neg Prog_Q(U), \bar{m} \in U$ holds for all $m <_Q n$. If $m <_Q n$ is false then $\langle m, n\rangle \notin Q$ and hence Δ^1_1-$\mathbf{CR}^Q_\infty \vdash \neg \bar{m} <_Q \bar{n}$. As a result, Δ^1_1-$\mathbf{CR}^Q_\infty \vdash \neg Prog_Q(U), \neg \bar{m} <_Q \bar{n}, \bar{m} \in U$ holds for all m. Using $(\vee)$ inferences followed by an application of the ω-rule, we get Δ^1_1-$\mathbf{CR}^Q_\infty \vdash \neg Prog_Q(U), \forall y(y <_Q \bar{n} \to u \in U)$. As Δ^1_1-$\mathbf{CR}^Q_\infty \vdash \bar{n} \notin Q, \bar{n} \in Q$, an inference $(\vee)$ (and weakening) yields

$$\Delta^1_1\text{-}\mathbf{CR}^Q_\infty \vdash \neg Prog_Q(U), \forall y(y <_Q \bar{n} \to u \in U) \wedge \bar{n} \notin Q, \bar{n} \in Q.$$

Hence via $(\exists_1)$ we arrive at

$$\Delta^1_1\text{-}\mathbf{CR}^Q_\infty \vdash \neg Prog_Q(U), \exists x[\forall y(y <_Q \bar{n} \to u \in U) \wedge \bar{n} \notin Q], \bar{n} \in Q,$$

which is the same as Δ^1_1-$\mathbf{CR}^Q_\infty \vdash \neg Prog_Q(U), \bar{n} \in Q$. Thus, by induction on $<_Q$, (4) follows.

(d) also follows by transfinite induction on $<_Q$ using Δ^1_1-CR. A reference will be provided in Lemma 4.10. $\square$

We shall need to measure the length of the previous derivations. For (c) and (d) the lengths of those derivations will be “longer” than Q, though not “much longer”. Let τ be the ordinal giving the order-type of Q. It is easy to cook up a new ordering Q^* in an elementary way from Q corresponding to the ordinal $\omega^2 + \omega \cdot \tau + \omega$ in such a way that $\mathbf{RCA}_0$ suffices to prove $\mathbf{WO}(Q) \to \mathbf{WO}(Q^*)$ (see [8]). The rationale for the choice of $\omega^2 + \omega \cdot \tau + \omega$ is that it gives us enough elbow room for calibrating the lengths of the foregoing derivations.

From the standing assumption that Q is a well-ordering we get that Q^* is a well-ordering, too.

Definition 4.9 If α is an element of the field of $<_{Q^*}$, we use the notation $\Delta^1_1\text{-}\mathbf{CR}^Q_\infty \overset{\alpha}{\vdash} \Gamma$ to convey that the sequent Γ is deducible in $\Delta^1_1\text{-}\mathbf{CR}^Q_\infty$ via a derivation of length $\leq \alpha$. More formally, this relation is defined by recursion on α as follows: $\Delta^1_1\text{-}\mathbf{CR}^Q_\infty \overset{\alpha}{\vdash} \Gamma$ holds if if either Γ is an axiom of $\Delta^1_1\text{-}\mathbf{CR}^Q_\infty$ or Γ is the conclusion of a $\Delta^1_1\text{-}\mathbf{CR}^Q_\infty$-inference with premisses $(\Gamma_i)_{i\in I}$ such that for every $i \in I$ there exists $\beta_i <_{Q^*} \alpha$ with $\Delta^1_1\text{-}\mathbf{CR}^Q_\infty \overset{\beta_i}{\vdash} \Gamma_i$.

Lemma 4.10 *(1) $\Delta^1_1\text{-}\mathbf{CR}^Q_\infty \overset{0}{\vdash} D_Q(n)$ for all n with 0 being the least element of Q.*

(2) $\Delta^1_1\text{-}\mathbf{CR}^Q_\infty \overset{\alpha}{\vdash} C(U)$ for some $\alpha \in$ field(Q^*) and all free set variables U.

(3) $\Delta^1_1\text{-}\mathbf{CR}^Q_\infty \overset{\beta}{\vdash} \mathbf{WO}(U_0)$ for some $\beta \in$ field(Q^*).

(4) $\Delta^1_1\text{-}\mathbf{CR}^Q_\infty \overset{\gamma}{\vdash} \exists Y\, H_A(U_0, Y)$ for some $\gamma \in$ field(Q^*) for all arithmetical formulas $A(u, Y)$ having no other free numerical variables than u.

(5) $\Delta^1_1\text{-}\mathbf{CR}^Q_\infty \overset{\delta}{\vdash} B(U)$ for some $\delta \in$ field(Q^*) and all free set variables U.

Proof: (1) is an immediate consequence of Lemma 4.8 (a) and (b). (2) follows since the rule $(\exists_2)$ gives arithmetical comprehension. (3) and (4) correspond to Lemma 4.8 (c) and (d), respectively. A detailed proof of (4) amounts to basically the same as that of [13, §21 Lemma 14] . (5) is an immediate consequence of (4). □

Recall that, by Corollary 4.7, there exists a countable coded ω-model of $\mathbf{ACA}_0$ containing Q and satisfying $\forall Z \exists Y\, B_0(Q, Y, Z)$ providing $\mathcal{D}_Q$ is ill-founded. Now let us assume that Q is a well-ordering and that $\mathcal{D}_Q$ is well-founded. Then $\mathcal{D}_Q$ can be viewed as a deduction with **hidden cuts** involving formulas of the shape $\neg B(U_{i+1})$, $\neg C(U_{i+1})$ and $\neg D_Q(i+1)$. Note that by Lemma 4.10, $\Delta^1_1\text{-}\mathbf{CR}^Q_\infty \overset{0}{\vdash} D_Q(n)$, $\Delta^1_1\text{-}\mathbf{CR}^Q_\infty \overset{\alpha}{\vdash} C(U)$, and $\Delta^1_1\text{-}\mathbf{CR}^Q_\infty \overset{\gamma}{\vdash} B(U)$ for some $\alpha, \gamma \in$ field(Q^*). Thus if Γ is the sequent attached to a node τ of $\mathcal{D}_Q$ and $(\Gamma_i)_{i\in I}$ is an enumeration of the sequents attached to the immediate successor nodes of τ in $\mathcal{D}_Q$ then the transition

$$\frac{(\Gamma_i)_{i\in I}}{\Gamma}$$

can be viewed as a combination of four inferences in $\Delta^1_1\text{-}\mathbf{CR}^Q_\infty$, the first one being a logical inferences and the other three being cuts. By interspersing $\mathcal{D}_Q$ with cuts and adding three cuts with cut formulas $\neg C(U_0)$, $\neg B(U_0)$ and $\neg D_Q(0)$ at the bottom we obtain a derivation $\tilde{\mathcal{D}}_Q$ in $\Delta^1_1\text{-}\mathbf{CR}^Q_\infty$ of the empty sequent. Since the preceding line of arguments can be done in $\mathbf{ACA}_0$ we arrive at the following:

Corollary 4.11 $(\mathbf{ACA}_0)$ *If Q is a well-ordering and $\mathcal{D}_Q$ is well-founded then there is a derivation $\tilde{\mathcal{D}}_Q$ in Δ^1_1-$\mathbf{CR}^Q_\infty$ of the empty sequent.*

To finish the paper we thus have to show that the latter is impossible. This we shall do by embedding Δ^1_1-$\mathbf{CR}^Q_\infty$ into a system $\mathbf{RA}^\infty$ defined below. Note that an upper bound for the length of $\tilde{\mathcal{D}}_Q$ is provided by $(\alpha+\gamma+\rho)\cdot 4$, where ρ corresponds to the Kleene-Brouwer ordering on $\mathcal{D}_Q$.

5 Ramified Analysis $\mathbf{RA}_\infty$

The theories $\mathbf{RA}_\rho$ are designed to capture Gödel's notion of *constructibility* restricted to sets of natural numbers. They use ordinal indexed variables $X^\alpha, Y^\alpha, Z^\alpha, \ldots$ for $\alpha < \rho$, with the intended meaning that level 0 variables range over sets definable by numerical quantification, and level $\alpha > 0$ variables range over sets definable by numerical quantification and level $< \alpha$ set quantification. The proof-theoretic ordinal of $\mathbf{RA}_\alpha$ is $\varphi\alpha 0$. We are interested in an infinitary version of ramified analysis.

Definition 5.1 $\mathbf{RA}^\infty$ is basically the same system as $\mathbf{RA}^*$ in [13, §22]. One difference is that the language of $\mathbf{RA}^*$ is based on the connectives $\bot, \forall, \rightarrow$ while $\mathbf{RA}^\infty$ has $\wedge, \vee, \forall, \exists, \neg$ and formulas are in negation normal form, i.e. the negation sign appears only in front of atomic formulas. The other difference is that the deduction system of $\mathbf{RA}^*$ is the Schütte calculus of positive and negative forms whereas $\mathbf{RA}^\infty$'s is the Gentzen sequent calculus.

The formulas of $\mathbf{RA}^\infty$ do not have free numerical variables. Literals are formulas of the form $R(t_1, \ldots, t_n)$ and $\neg R(t_1, \ldots, t_n)$ with R being a symbol for a primitive recursive relation and $t_1, \ldots, t_n$ being closed numerical terms.

$\mathbf{RA}^\infty$ uses ordinal indexed free set variables $U^\alpha, V^\alpha, W^\alpha, \ldots$ and bound set variables $X^\beta, Y^\beta, Z^\beta, \ldots$ with $\beta > 0$, where the ordinals are assumed to be elements of some countable well-ordering R.

The *set terms* and *formulas* together with their *levels* are generated as follows (cf. [13, §22]):

(1) Every literal is a formula of level 0.
(2) Every free set variable U^α is a **set term** of **level** α.
(3) If P is a set term of level α and t is a numerical term, then $t \in P$ and $t \notin P$ are **formulas of level** α.
(4) If A and B are formulas of levels α and β, then $A \vee B$ and $A \wedge B$ are formulas of level $\max(\alpha, \beta)$.
(5) If $F(0)$ is a formula of level α, then $\forall x F(x)$ and $\exists x F(x)$ are formulas of

level α and $\{x \mid F(x)\}$ is a **set term** of level α.

(6) If $F(U^\beta)$ is a formula of level α and $\beta > 0$, then $\forall X^\beta F(X^\beta)$ is a formula of level $\max(\alpha, \beta)$.

Definition 5.2 The calculus $\mathbf{RA}_Q^\infty$

Axioms

Γ, L where L is a true literal.

$\Gamma, s \in U^\alpha, t \notin U^\alpha$ where $s^{\mathbb{N}} = t^{\mathbb{N}}$.

$\Gamma, s \in U_0$ if $s^{\mathbb{N}} \in Q$.

$\Gamma, s \notin U_0$ if $s^{\mathbb{N}} \notin Q$.

Rules

$(\wedge), (\vee)$, (ω), numerical $(\exists)$ and (Cut) as per usual

$$(\exists^\alpha)\ \frac{\Gamma, F(P)}{\Gamma, \exists X^\alpha F(X^\alpha)} \qquad P \text{ set term of level} < \alpha.$$

$$(\forall^\alpha)\ \frac{\Gamma, F(P) \quad \text{for all set terms } P \text{ of level} < \alpha}{\Gamma, \forall X^\alpha F(X^\alpha)}$$

$$(ST_1)\ \frac{\Gamma, F(t)}{\Gamma, t \in \{x \mid F(x)\}}$$

$$(ST_2)\ \frac{\Gamma, \neg F(t)}{\Gamma, t \notin \{x \mid F(x)\}}$$

Definition 5.3 The **cut rank** of a formula A in $\mathbf{RA}_Q^\infty$, $|A|$, is defined as follows (cf. [13, §22]):

(1) $|L| = 0$ for arithmetical literals L.

(2) $|t \in U^\alpha| = |t \notin U^\alpha| = \omega \cdot \alpha$

(3) $|B_0 \wedge B_1| = |B_0 \vee B_1| = \max(|B_0|, |B_1|) + 1$

(4) $|\forall x B(x)| = |\exists x B(x)| = |t \in \{x \mid B(x)\}| = |t \notin \{x \mid B(x)\}| = |B(0)| + 1$

(5) $|\forall X^\alpha A(X^\alpha)| = |\exists X^\alpha A(X^\alpha)| = \max(\omega\cdot\gamma, |A(U^0)|+1)$
where γ is the level of $\forall X^\alpha A(X^\alpha)$.

By recursion on α we define the relation $\mathbf{RA}_Q^\infty \mid\frac{\alpha}{\rho}\, \Gamma$ as follows: $\mathbf{RA}_Q^\infty \mid\frac{\alpha}{\rho}\, \Gamma$ holds if either Γ is an axiom of $\mathbf{RA}_Q^\infty$ or Γ is the conclusion of an $\mathbf{RA}_Q^\infty$-inference with premisses $(\Gamma_i)_{i\in I}$ such that for every $i \in I$ there exists $\beta_i < \alpha$ with $\mathbf{RA}_Q^\infty \mid\frac{\beta_i}{\rho}\, \Gamma_i$ and, moreover, if this inference is a cut with cut formula A then $|A| < \rho$.

The following three statements are proved in [13] for the system $\mathbf{RA}^*$. It is routine to transfer them to $\mathbf{RA}_Q^\infty$ since cut elimination in a Schütte calculus of positive and negative is closely related to cut elimination in sequent calculi. Moreover, the additional axioms pertaining to Q do not impede the cut elimination process.

Theorem 5.4 (Cut-elimination I)

$$\mathbf{RA}_Q^\infty \mid\frac{\alpha}{\eta+1}\, \Gamma \quad\Rightarrow\quad \mathbf{RA}_Q^\infty \mid\frac{\omega^\alpha}{\eta}\, \Gamma$$

Proof: Similar to [13, §22 Lemma 4]. □

Theorem 5.5 (Cut-elimination II)

$$\mathbf{RA}_Q^\infty \mid\frac{\alpha}{\omega^\rho}\, \Gamma \quad\Rightarrow\quad \mathbf{RA}_Q^\infty \mid\frac{\varphi\rho\alpha}{0}\, \Gamma$$

Proof: Similar to [13, Theorem 22.7]. □

For a formula F of the language of Δ^1_1-$\mathbf{CR}^Q_\infty$ let F^σ be the result of replacing every bound variable X by X^σ and every free set variable by a set term of a level $< \sigma$. For $\Gamma = \{F_1, \ldots, F_n\}$ let $\Gamma^\sigma = \{F_1^\sigma, \ldots, F_n^\sigma\}$.

Theorem 5.6 (Interpretation Theorem)

$$\Delta^1_1\text{-}\mathbf{CR}^Q_\infty \mid\!\frac{\alpha}{}\, \Gamma \quad\Rightarrow\quad \mathbf{RA}_Q^\infty \mid\frac{\omega\cdot\sigma+\omega+\omega\cdot\alpha}{\omega\cdot\sigma}\, \Gamma^\sigma$$

for all σ of the form $\omega^\alpha\cdot\beta$ with $\beta \neq 0$.

Proof: This is basically the same as [13, Theorem 22.14]. □

There are different ways of formalizing infinite deductions in theories like **PA**. We just mention [14] and [6].

Recall that in order to finish the proof of Theorem 4.1 we want to show that $\mathcal{D}_Q$ is not well-founded whenever Q is a well-ordering. By Corollary 4.11, if Q is a well-ordering and $\mathcal{D}_Q$ is well-founded then there is a derivation $\tilde{\mathcal{D}}_Q$ in Δ^1_1-$\mathbf{CR}^Q_\infty$ of the empty sequent. By the Interpretation Theorem 5.6 we would then get a derivation in $\mathbf{RA}^\infty_Q$ of the empty sequent. Using the principle $\mathbf{WO}(\mathfrak{X}) \to \mathbf{WO}(\varphi\mathfrak{X}0)$ we can then employ the cut elimination Theorem 5.5 to obtain a cut-free derivation of the empty sequent in $\mathbf{RA}^\infty_Q$. But this is impossible.

From Corollary 4.7 we can thus conclude that for every well-ordering $\tilde{Q}$ there exists a countable coded ω-model of $\mathbf{ACA}_0$ containing $\tilde{Q}$ and satisfying $\forall Z \exists Y\, B_0(\tilde{Q}, Y, Z)$. From this we would like to infer that for every well-ordering Q and every set Z_0 there exists a set Y such that $B_0(\tilde{Q}, Y, Z_0)$. We can do this by encoding Q and Z_0 in a well-ordering $\tilde{Q}$ from which Q and Z_0 can be retrieved in any ω-model of $\mathbf{ACA}_0$ containing $\tilde{Q}$. One way of doing this is to define the new ordering $\tilde{Q}$ by letting

$$\begin{aligned}\langle n, m\rangle <_{\tilde{Q}} \langle n', m'\rangle \text{ iff } & [n = n' = 0 \wedge m <_{\tilde{Q}} m'] \vee \\ & [n = n' = 1 \wedge m, m' \in Z_0 \wedge m < m'] \vee \\ & [n = 0 \wedge n' = 1 \wedge m \in \text{field}(Q) \wedge m' \in Z_0].\end{aligned}$$

Obviously $\tilde{Q}$ is a well-ordering, too, and any ω-model $\mathbb{M}$ of $\mathbf{ACA}_0$ containing $\tilde{Q}$ will contain Z_0 as well. Moreover, $\mathbb{M} \models \exists Y\, B_0(\tilde{Q}, Z_0)$ implies $\mathbb{M} \models \exists Y\, B_0(Q, Z_0)$. Hence, in view of Lemma 3.2, we get $\mathbf{ATR}_0$, thereby finishing the proof of Theorem 4.1.

6 Finishing the proof of Theorem 1.3

One direction of Theorem 1.3 follows from Theorem 4.1. The other direction is implicit in the proof of [13] Theorem 21.6.

7 Prospectus

The methodology exemplified in the proof of Theorem 1.3 should have many more applications. Every cut elimination theorem in ordinal-theoretic proof theory potentially encapsulates a theorem of type 1.3. The first author has looked at two more examples and sketched proofs of the pertaining theorems.

A familiar function from proof theory is the Γ-function where $\alpha \mapsto \Gamma_\alpha$ enumerates the fixed points of the φ-function. Since the proof of the next result has only been sketched we classify it as a conjecture.

Conjecture 7.1 *Over* $\mathbf{RCA}_0$ *the following are equivalent:*

(i) $\mathbf{RCA}_0$ + *Every set* X *is contained in a countable coded* ω*-model of* $\mathbf{ATR}_0$.

(ii) $\forall \mathfrak{X}\, [\mathbf{WO}(\mathfrak{X}) \rightarrow \mathbf{WO}(\Gamma_{\mathfrak{X}})]$.

The direction (i)⇒(ii) follows from [11, 4.13,4.16].

For an example from impredicative proof theory one would perhaps first turn to the ordinal representation system used for the ordinal analysis of the theory $\mathbf{ID}_1$ of non-iterated inductive definitions, which can be expressed in terms of the θ-function (cf. [4]). $\mathbf{ID}_1$ has the same strength as the subsystem of second order arithmetic based on bar induction, **BI** (cf. [4,5,12]). In Simpson's book the acronym used for **BI** is Π^1_∞-$\mathbf{TI}_0$ (cf. [15, §VII.2]). In place of the function θ we prefer to work with simpler ordinal representations based on the ψ-function introduced in [3] or the ϑ-function of [12]. For definiteness we refer to [12]. Given a well-ordering $\mathfrak{X}$, the relativized versions $\vartheta_{\mathfrak{X}}$ and $\psi_{\mathfrak{X}}$ of the ϑ-function and the ψ-function, respectively, are obtained by adding all the ordinals from $\mathfrak{X}$ to the sets $C_n(\alpha, \beta)$ of [12, §1] and $C_n(\alpha)$ of [12, Definition 3.1] as initial segments, respectively. The resulting well-orderings $\vartheta_{\mathfrak{X}}(\varepsilon_{\Omega+1})$ and $\psi_{\mathfrak{X}}(\varepsilon_{\Omega+1})$ are equivalent owing to [12, Corollary 3.2].

Again, as the following statement has not been buttressed by a complete proof we formulate it as a conjecture.

Conjecture 7.2 *Over* $\mathbf{RCA}_0$ *the following are equivalent:*

(i) $\mathbf{RCA}_0$ + *Every set* X *is contained in a countable coded* ω*-model of* **BI**.

(ii) $\forall \mathfrak{X}\, [\mathbf{WO}(\mathfrak{X}) \rightarrow \mathbf{WO}(\psi_{\mathfrak{X}}(\varepsilon_{\Omega+1}))]$.

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