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Unprovability and phase transitions in Ramsey theory

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Preface

Two sorts of truth: trivialities, where
opposites are obviously absurd, and
profound truths, recognised by the fact
that the opposite is also a profound truth.

Niels H.D. Bohr

The first mathematically interesting, first-order arithmetical example of incompleteness was given in the late seventies and is know as the Paris-Harrington principle. It is a strengthened form of the finite Ramsey theorem which can not be proved, nor refuted in Peano Arithmetic. In this dissertation we will investigate several other unprovable statements of Ramseyan nature and determine the threshold functions for the related phase transitions. There are six chapters, followed by an appendix which provides a Dutch summary.

Chapter 1 sketches out the historical development of unprovability and phase transitions, and offers a little information on Ramsey theory. In addition, it introduces the necessary mathematical background by giving definitions and some useful lemmas.

Chapter 2 deals with the pigeonhole principle, presumably the most well-known, finite instance of the Ramsey theorem. Although straightforward in itself, the principle gives rise to unprovable statements. We investigate the related phase transitions and determine the threshold functions.

Chapter 3 explores a phase transition related to the so-called infinite subsequence principle, which is another instance of the Ramsey theorem.

Chapter 4 considers the Ramsey theorem without restrictions on the dimensions and colours. First, generalisations of results on partitioning α-large sets are proved, as they are needed later. Second, we show that an iteration of a finite version of the Ramsey theorem leads to unprovability.

Chapter 5 investigates the template "thin implies Ramsey", of which one of the theorems of Nash-Williams is an example. After proving a more universal instance, we study the strength of the original Nash-Williams theorem. We conclude this chapter by presenting an unprovable statement related to Schreier families.

Chapter 6 is intended as a vast introduction to the Atlas of prefixed polynomial equations. We begin with the necessary definitions, present some specific members of the Atlas, discuss several issues and give technical details.

A considerable part of this PhD dissertation consists of published and unpublished papers and drafts which were written together with Andreas Weiermann ([DSW08, DSW10a, DSW10b, DSW]) and Andrey Bovykin ([BDS10, BDS]). The author is grateful to them for their permission to insert those papers.

Acknowledgements

After completing my master's degree it seemed unthinkable to end my journey through the rich fields of mathematics. Luckily, I was given the opportunity to study logic and the foundations of mathematics for another four years. That research resulted in the doctoral dissertation you are holding right now. Of course, completing this endeavour would not have been within my reach without the advice and encouragement of so many people.

First of all, I would like to thank my supervisor Prof. Andreas Weiermann for his guidance. Without him this dissertation would never have been written. Whenever I had a question, I could turn to him for answers, advice and suggestions. Moreover, he gave me the liberty to pursue my own interests and develop my own projects. I will never forget our discussions in his office, which have shaped many of my ideas on research and academia.

My long-term research stay in Bristol was a great success, in large part thanks to my co-supervisor Dr. Andrey Bovykin. He taught me to dare look at mathematics and, particularly, logic from new and different perspectives. I have fond memories of the evenings when we discussed mathematics, philosophy and politics, with plenty of food and drinks. No doubt I have been influenced by his refreshing and substantial ideas.

Also, I am indebted to the Research Foundation - Flanders (FWO) and the Department of Mathematics of Ghent University for creating the perfect conditions for me to work and travel. I am quite sure this contributed to my research and the person I am today. I also express my gratitude to my Belgian and foreign colleagues, especially Frédéric and Thomas, for the friendly atmosphere and stimulating conversations.

Of course, I want to thank my girlfriend Lies, my family and my friends here and abroad, for supporting me throughout these years. Their presence in my life definitely means much more than can be described in a few lines. All of them, each in their own way, showed me how mathematics is only one of the many important and enjoyable aspects of life and how most of them, unlike mathematics, do not necessarily rely on logic.

Finally, I want to thank my parents for their never-ending support, for a warm and pleasant home, for providing a moral compass and for raising me with an open mind.

Thanks.

Michiel De Smet
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Chapter 1

INTRODUCTION

1.1 Historical background

As there have been written plenty of complete and correct pieces about foundations of mathematics, we will not give many details here and rather refer to the literature. So for thorough information, one might want to have a look at e.g. [Kle52, FBHL73, Bar77, HMSS85, Tak87, TvD88a, TvD88b, Bus98b, AR01, Sim09, Bov09a, BW09, Fri10a], among many other comprehensive survey articles and books. However, to give the reader the possibility of viewing this dissertation in context, we offer a brief account of how the subject of unprovability emerged. Next we introduce phase transitions in logic and in the last subsection we touch upon Ramsey theory.

1.1.1 How unprovability theory emerged

It is hard to give a full account of the development of mathematical logic. So the short survey below does by no means pretend to be complete. It rather aims at providing some general historical background by touching upon important events and introducing some of the great contributors.

Intentionally passing over many relevant ideas formulated by thinkers living as early as Classical Greece, we start our story at the end of the 19th century. By then the first steps towards a rigorous foundation for mathematics were taken by Georg Cantor, Richard Dedekind and Gottlob

Frege, among others. Such a steady basis became necessary as mathematical knowledge had increased rapidly and some of the new concepts turned out problematic. Worries were often related to antinomies, i.e. contradictions obtained by mere logical reasoning. A common example in this context is Russell's paradox, i.e.

"Does the set $\{X : X \notin X\}$ contain itself?".

Soon it became clear how difficult it was to get rid of those worries, which led to the *foundational crisis of mathematics* at the dawn of the 20th century (*Grundlagenkrise der Mathematik*, after [Wey21]). This term denotes the quest for a solid foundation for mathematics, which took place roughly during the first three decades of that century. Different approaches were proposed to find a way out of the crisis. An important contribution was the *Principia Mathematica* written by Bertrand Russell and Alfred North Whitehead, which consisted of three parts published between 1910 and 1913.

Out of several different views we mention but two prevailing ones: the axiomatic approach, forcefully defended by David Hilbert, and the intuitionistic way of thinking, as described by Luitzen Egbertus Jan Brouwer who, crudely speaking, proposed a constructive method to deal with the foundations. It can be said that the dispute was regularly acrimonious. Throughout those years Hilbert worked on a new proposal which he hoped would not be troubled with inconsistencies, as was the case with Cantor's set theory. His project, which was first presented in full in the early twenties, became known as *Hilbert's programme*. The idea was twofold. First, all of mathematics had to be formalised in an axiomatic way. That approach needed to ensure that concrete statements derived by means of abstract techniques, could be derived without them. Second, a proof of the consistency of this formalisation had to be provided by "finitary" methods. We are about to see why the programme could not be carried out in its original, rough form.

Indeed, on 7 September 1930 in a discussion session of a conference in Königsberg, Kurt Gödel announced the first of his two incompleteness theorems, which was published together with the second one some months afterwards in [Göd31]. The first theorem says that if T is a formal system (theory) containing basic arithmetic, then either T is inconsistent or there are arithmetical statements that cannot be proved or refuted in T. The second theorem states that "T is consistent" is such an independent statement. It is widely accepted that this second theorem makes Hilbert's programme as originally conceived unattainable. However, work on revised forms of Hilbert's programme has been fruitful for logic (see e.g. [Kre58, Sch60, Fef00, Zac06]). Here are some important continuations of Hilbert's ideas:

- The search for consistency proofs was an important objective during the early progress of proof theory. That study was initiated by Gerhard Gentzen and later Kurt Schütte as well as Gaisi Takeuti and resulted in *ordinal analysis of theories* ([Rat06]) and *proof mining* ([Koh08]).

- Logicians still investigate the proof-theoretic reduction of systems of classical mathematics to more restricted systems. *Relatitivized Hilbert programs* fit in exactly with those ideas ([Fef88]).

- *Reverse Mathematics* can be seen as a partial realisation of Hilbert's programme ([Sim09]). Stephen Simpson writes in [Sim88]: "*Any mathematical theorem which can be proved in* WKL_0 *is finitistically reducible in the sense of Hilbert's Program.*".

Clearly, much more material on these and related topics is available in the literature. Below we will encounter another branch of logic which is connected to Hilbert's original ideas.

The incompleteness theorems have astonishing consequences for logic, but some people questioned the relevance for those areas of mathematics not heavily related to pure logic. Gödel's original independent statement indeed was constructed by coding the syntax and using logical tricks, which led some mathematicians to the idea that their subject would not be affected. That situation lasted for almost half a century, until the late seventies, when Laurie Kirby and Jeff Paris studied models of PA ([KP77]).

Paris, building on their joint work, succeeded in giving the first examples of mathematical statements which are independent of Peano Arithmetic. Next Leo Harrington realised Paris's results could be adapted to a simple first-order extension of the finite Ramsey theorem, which resulted in the *Paris-Harrington principle*, or PH for short ([PH77]). Kirby recalls those momentous days as follows.

> *"I was a PhD student of Jeff Paris and we were studying combinatorial properties of initial segments of models of PA inspired by large cardinal properties in set theory e.g. weakly compact cardinals. Out of this came the notions of strong cut and of indicator as means of obtaining structural properties of cuts e.g. by modifying the MacDowell-Specker theorem. So our program was about structural properties of non-standard models and the application to independence was a byproduct. But I have a feeling such an idea was in the back of Jeff's mind all along. I was young and innocent and it hadn't occurred to me. I remember the day when he showed me how you could get a combinatorial independent statement (the earliest form involving the iterative notion of n-denseness) and it was only that night I realised how remarkable this was – I had to come back the next morning to say so. Soon he got a letter from Harrington (no emails in those days) with the simplified version."* (L.A.S. Kirby, personal communication, March 15, 2011)

The Paris-Harrington principle says the following:

"for all numbers e, m and n, there exists a number N, such that for every colouring f of n-element subsets of $\{0, 1, \ldots, N\}$ into e colours, there is an $H \subseteq \{0, 1, \ldots, N\}$ of at least size m, such that $|H| > \min H$ and f is constant on the set of n-element subsets of H".

So neither PH, nor its negation, is provable in PA, and at the same time the principle is of mathematical, rather than metamathematical, nature. In this context we should mention previous examples of independent statements. Indeed, it was shown before by Gödel and Paul Cohen that the continuum hypothesis and the axiom of choice are independent of ZFC and ZF, respectively. However, these examples belong to third-order arithmetic and so are less concrete than the first-order arithmetical statement of Paris and Harrington.

Notice that PA is a powerful theory which proves most of mathematics which is encodable in the language of first-order arithmetic produced by non-logicians so far. Moreover, even a small fragment of it, called Exponential Function Arithmetic, is believed to capture already most of today's finitary mathematics (see Subsection 6.1.1). Stronger extensions of PA, such as full second-order arithmetic Z_2, can prove PH.

Since then, many have contributed to the arising subject of unprovability by providing natural examples of independent statements. Investigating those unprovable assertions, one will probably stumble upon one of the following: the arboreal statement by George Mills ([Mil80]), the Hydra battle and the termination of Goodstein sequences by Kirby and Paris ([KP82]), the flipping principle of Kirby ([Kir82]), the kiralicity and regality principles by Peter Clote and Kenneth McAloon ([CM83]), the combinatorial principle concerning approximations of functions by Pavel Pudlák ([HP87]), combinatorial principles related to finite trees, and to Higman and Kruskal theorems by Harvey Friedman ([HMSS85]) and the regressive Ramseyan statement by Akihiro Kanamori and Kenneth McAloon ([KM87]). Although independent of PA (or even stronger theories, for some of the examples above), all of them are provable in Z_2. Moreover, depending on the assertion, specific subsystems of second order arithmetic will suffice. As the statements themselves are provable in some stronger natural theories, we usually ignore their negation and use the word unprovable, instead of independent. The study of the mechanisms behind unprovability phenomena is called *unprovability theory*.

Nowadays, unprovability theory has expanded substantially and offers a wide variety of research topics. In the beginning, one of the goals was finding mathematically interesting sentences which are unprovable in certain theories, often PA or fragments thereof. That objective turned into a quest for statements which are unprovable in strong, possibly incomparable, theories. A major branch is the search for concrete mathematical incompleteness as undertaken by Friedman in his Boolean Relation Theory ([Fri10a]) and Upper Shift Kernel Theory ([Fri10b]). The central statements in those studies are examples of *templates*. Also in this dissertation we will encounter templates, more precisely in Chapter 5 (*thin implies Ramsey*) and Chapter 6 (the *Atlas*).

From the 21st century on, unprovability theory diversified even more as Andreas Weiermann started the field of provability phase transitions for systems of arithmetic. These shifts between provability and unprovability revealed connections between logic and completely different areas of "concrete" mathematics, such as analytic number theory. This topic will be introduced in detail in the next subsection. Another part of the study of unprovability is dedicated to the investigation of (meta)mathematical reasons for incompleteness phenomena. Clearly, there is a connection with Reverse Mathematics and also philosophy of mathematics. Yet another new branch of research is the study of the Atlas of *prefixed polynomial equations* (see Chapter 6). Of course it is impossible to describe in just a few lines all features of this area at the intersection of mathematics and metamathematics. So the interested reader is advised to have a look at up to date literature.

1.1.2 Phase transitions

A *phase transition* is a general phenomenon which occurs in mathematics and several related areas. In particular, it is often clearly noticeable in physics. Informally, we could describe a phase transition as the behaviour observed when a small change of a certain parameter of a system causes an extreme transition in some other property of the system. Usually, one also detects a specific *threshold point*.

A common example of a phase transition would be the melting of ice, or the boiling of water by raising temperature a little. In this case, the related threshold point would be the ice melting, or water boiling temperature, respectively. Figure 1.1 gives a sketch of this daily life example from physics, namely a phase diagram for H_2O.

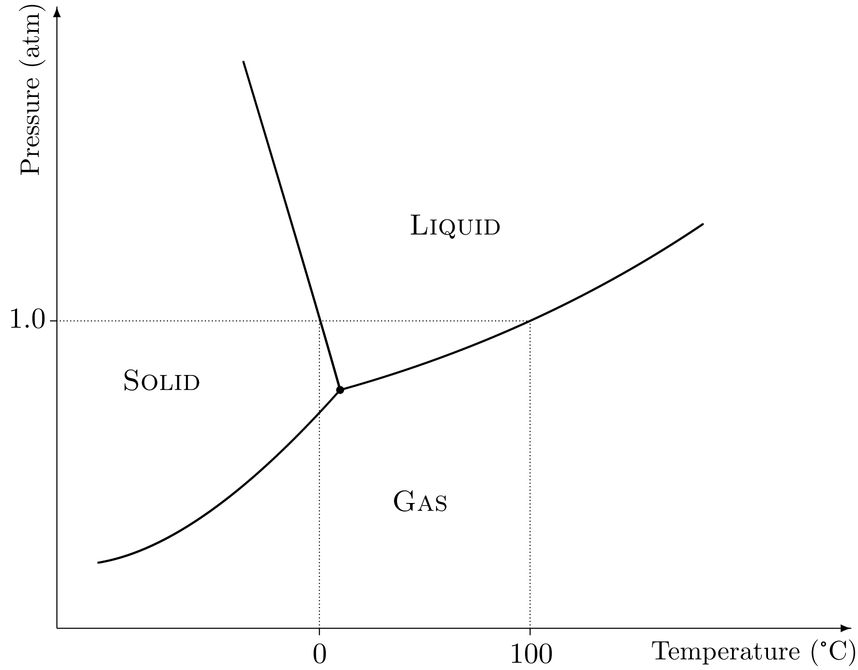


Figure 1.1: *Phase transitions in physics*

The same phenomenon happens throughout many mathematical and computational disciplines. We present a classical example from combinatorics, originally published in [ER60] where Paul Erdős and Alféd Rényi study the random graph model. Most of the following definitions are taken from [Spe01]. Let $G(n, p(n))$ be the random graph on n vertices with probability function $p(n)$. For two functions f and g, we write $f(x) \ll g(x)$ if for every real number $M > 0$, there exists a constant x_0, such that $|f(x)| \leq M \cdot |g(x)|$ for all $x > x_0$. Let B denote a monotone property of graphs, such as "containing a triangle", and $\text{Prob}[G(n, p(n)) \models B]$ denote the probability of $G(n, p(n))$ satisfying B. Then a function $p(n)$ is called a *threshold function* for B if the following holds. If $p'(n) \ll p(n)$, then $\text{Prob}[G(n, p'(n)) \models B] \to 0$ and if $p(n) \ll p'(n)$, then $\text{Prob}[G(n, p'(n)) \models$

$B] \to 1$. For example, in case B is the property "containing a triangle", then $p(n) = \frac{1}{n}$ is a threshold function for B. Thus, a small change of a certain parameter (the probability function) causes a clear transition from one phase to another completely different one.

Other areas in mathematics where phase transitions occur are statistical physics ([Cha87]), evolutionary graph theory ([Bol01]), percolation theory ([Gri99]) computational complexity ([CK02]), artificial intelligence ([MZK$^+$99]), *etc.*. For a mathematical description of phase transitions one is referred to [Gib60, LY52, YL52].

Here, the situation is as follows. We will investigate the shift from provability to unprovability of a statement with regard to well-known theories. This transition will be obtained by slightly varying a parameter or a parameter function. Let us give an example of such a phase transition in logic, which in addition is of historical significance. Consider the following statement A:

"for every natural number K, there is a number N, such that for every finite sequence $T_0, \ldots, T_N$ of finite trees such that for all $i \leq N$ the number of nodes of T_i is at most $K + i$, there are indices $i < j \leq N$ such that T_i is homeomorphically embeddable into T_j".

As one might have noticed, A is a finite version of Kruskal's theorem ([Kru60, Sim85]). Friedman showed that the statement A is independent of PA, and even of the much stronger second-order theory ATR_0 (see e.g. [Sim85]). We slightly change the original statement A into A_r by replacing $K + i$, by $K + r\log_2(i)$, where the parameter r is a non-negative rational number. Martin Loebl and Jiří Matoušek proved in [LM87] that A_r is independent of PA in case $r = 4$, and it is provable in PA if $r = \frac{1}{2}$. Evidently, one could ask whether it is possible to find an exact threshold point. In other words, does there exist a real number r_0 such that A_r is provable in PA if and only if $r < r_0$? (See Figure 1.2.)

It was Weiermann who solved this problem using analytic combinatorics ([Wei03]). He succeeded in finding such a particular r_0, namely $r_0 = \frac{\ln(2)}{\ln(\alpha)}$, where α is the so-called Otter's tree constant (which has numerical value $2.95576\ldots$). Moreover, he observed that this sort of phenomenon,

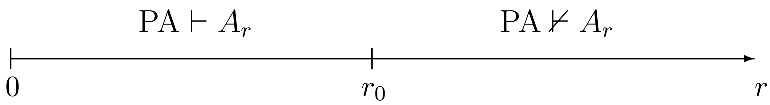


Figure 1.2: *Phase transition from provability to unprovability – Type I*

a phase transition for an incompleteness result, occurs in several other areas connecting "real" mathematics with logic. It was a surprise to find out that incompleteness is related to concrete areas like analytic number theory, combinatorial probability, Tauberian theory and finite combinatorics. Weiermann, and collaborators, obtained soon more results, which led to the research programme *phase transitions in logic and combinatorics* (see e.g. [Ara02, Wei03, Wei04, Wei05a, Wei05b, Wei05c, Wei06b, Wei06c, Wei07, KLOW08, Wei09, CLW11]).

In this dissertation we will deal with parameter functions, instead of parameter numbers as was the case in the example above. In general, we will consider a statement A_f which depends on a function $f\colon \mathbb{N} \to \mathbb{N}$. This sentence A_f will have the property that for certain slow growing functions f, it will be provable in a specified theory T, whereas for slightly faster growing functions A_f becomes unprovable in T. In this case, marking the threshold point becomes determining the threshold region as sharp as possible. This general pattern is illustrated in Figure 1.3.

Consider the following concrete example which was one of the first of this type of phase transitions. For a given parameter function $f\colon \mathbb{N} \to \mathbb{N}$, let PH_f be the assertion:

"for all numbers e, m and n, there exists a number N, such that for every colouring F of n-element subsets of $\{0, 1, \ldots, N\}$ into e colours, there is an $H \subseteq \{0, 1, \ldots, N\}$ of size at least m, such that $|H| > f(\min H)$ and F is constant on H".

This sentence is almost equal to the original Paris-Harrington statement, with the difference of changing $|H| > \min H$ into $|H| > f(\min H)$. Clearly, if f equals the identity function, then PH_f is independent of PA. Weiermann showed that if f is a fixed iteration of the binary length function,

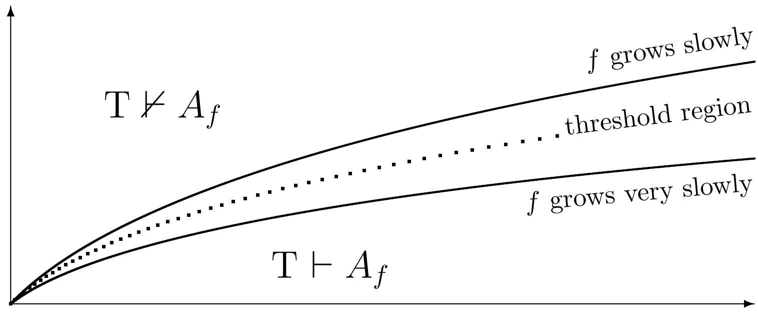


Figure 1.3: *Phase transition from provability to unprovability – Type II*

then PH_f is again unprovable in PA. On the other hand $\mathrm{PH}_{\log^*}$ turns out to be provable in PA. Moreover, in [Wei04] he improved those bounds significantly as follows. Let $f_\alpha(i) = |i|_{h_\alpha^{-1}(i)}$ where $|i|_d$ denotes the d times iterated binary length of i and h_α^{-1} denotes the inverse function (see Subsection 1.2.3) of the αth member h_α of the Hardy hierarchy (see Definition 1.13). Then PH_{f_α} is independent of PA (for $\alpha \leq \varepsilon_0$) if and only if $\alpha = \varepsilon_0$. Hence the upper and lower bounds for this phase transition are very precisely determined.

1.1.3 Ramsey theory

Ramsey theory is a branch of combinatorial analysis named after the British mathematician Frank Ramsey. He also contributed to philosophy and economics, but will presumably be best remembered for his paper "On a problem of formal logic" ([Ram30]). According to him, the principal goal of the paper was to solve a problem in logic, but at the same time he recognised that some theorems proven in the course of that investigation have independent interest. One of those theorems is known nowadays as the (infinite) Ramsey Theorem, which we will denote by RT. Surprisingly, the original formulation of RT (see [Ram30], Theorem A), would by modern standards not be labelled as being of "Ramsey type". Here, we prefer the more recent approach. Let n and k be natural numbers. Then RT^n_k stands for Ramsey's theorem for n dimensions and k colours, i.e.

$$\begin{aligned}\mathrm{RT}^n_k \leftrightarrow\ & \text{For every } G\colon [\mathbb{N}]^n \to k \text{ there exists an infinite set } H \\ & \text{such that } G \restriction [H]^n \text{ is constant.}\end{aligned}$$

In this definition we used two specific symbols: if A is a set and F a function acting on $B \supseteq A$, then $[A]^n$ denotes the set of all n-element subsets (i.e. subsets of size n) of A, and $F \restriction A$ stands for the restriction of F to A. Given that notation, RT equals $\forall n\, \forall k\, \mathrm{RT}^n_k$. Since the time of Johann Dirichlet, the beginning of the 19th century, several variations of Ramsey's theorem have been proved, but it was Ramsey who presented the first proof of the full version. It is difficult to describe the significance of RT and related statements in a few lines. It was the base for a whole new research domain and has many applications throughout mathematics. The following rather philosophical characterisation of Ramseyan statements by Harry Burkill and Leonid Mirsky sums it up nicely:

> *"There are numerous theorems in mathematics which assert, crudely speaking, that every system of a certain class possesses a large subsystem with a higher degree of organization than the original system."* ([BM73])

An classical monograph on Ramsey theory is [GRS90], providing an introduction, related problems, different forms and interpretations.

As mentioned in Subsection 1.1.1 many unprovable statements known so far are related to Ramsey theory to a greater or lesser extent. In this dissertation we encounter several more.

1.2 Preliminaries

1.2.1 Theories

Peano Arithmetic (denoted PA) is a first-order theory named after Giuseppe Peano and first presented in the nineteenth century. Its axioms postulate the properties of 0, successor, addition and multiplication. On top of that, PA contains a scheme of complete induction for all formulas in the language of arithmetic. When we restrict the induction schema of PA to formulas with at most n alternations of unbounded quantifiers, we obtain the subsystem $\mathrm{I}\Sigma_n$. The fragment $\mathrm{I}\Sigma_0$ is usually written as $\mathrm{I}\Delta_0$. The axioms of $\mathrm{I}\Delta_0 + \exp$ consist of a statement expressing the totality of the exponential function, in addition to all axioms of $\mathrm{I}\Delta_0$. In chapter 6 we will use another common variant of $\mathrm{I}\Delta_0 + \exp$, namely EFA which stands for Exponential Function Arithmetic. For rigorous definitions and properties we refer to [HP98].

Second-order theories we will use are RCA_0, WKL_0, ACA_0, ATR_0 and Π^1_1-CA_0, which are all subsystems of second-order arithmetic Z_2. We do not go into detail but refer to the detailed survey in [Sim09].

While exploring certain theories, we will deal with *consistency* and 1-*consistency* of a theory. Both are defined below. Let the symbol $\perp$ denote a logical contradiction, e.g. "$\exists x(x \neq x)$".

1.1 Definition. *Let* T *be a theory. The* consistency *of* T *(denoted* Con(T)*) is the statement*

$$\neg \mathrm{Pr}_{\mathrm{T}}(\ulcorner \perp \urcorner),$$

which one reads as "there is no proof in T *of a contradiction".*

For an accurate treatment of the provability predicate Pr_{T} for a theory T we refer to e.g. [Bek05].

1.2 Definition. *Let* T *be a theory. The* 1-consistency *of* T *(denoted* 1-Con(T)*) is the statement*

$$\forall \varphi \in \Pi^0_1 \; (\varphi \rightarrow \mathrm{Con}(\mathrm{T} + \varphi)),$$

which one reads as "for every Π^0_1*-statement* φ*, if* φ *holds, then* $\mathrm{T} + \varphi$ *is consistent".*

Remark that one needs to use the satisfaction predicate for Σ^0_1-formulas in order to write 1-Con(T) rigorously down as a formula. Also note that

1-Con(T) is a Π^0_2-statement, whereas the assertion claiming the consistency of T is Π^0_1.

Perhaps the reader has heard of the *uniform reflection principle* for Σ^0_1-formulas over T, i.e. the statement

$$\forall \varphi \in \Sigma^0_1\ (\mathrm{Pr}_{\mathrm{T}}(\ulcorner \varphi \urcorner) \to \varphi),$$

or $\mathrm{RFN}_{\Sigma^0_1}(\mathrm{T})$ for short. It is not too difficult to see that the 1-consistency of a theory T is equivalent to $\mathrm{RFN}_{\Sigma^0_1}(\mathrm{T})$. Moreover, their definitions are sometimes swapped in the literature, which clearly causes no problems as they are interchangeable. Further information on the reflection principles can be found in [Smo77, Bek05]. As $\mathrm{RFN}_{\Sigma^0_1}(\mathrm{T})$ (or, equivalently, 1-Con(T)) implies the consistency of T, it is unprovable in T by Gödel's second incompleteness theorem.

Many of the mathematically interesting examples of unprovable statements turned out to be equivalent to the 1-consistency of some theory. The Paris-Harrington principle we discussed above is actually equivalent to 1-Con(PA), and the finite combinatorial principle introduced by Friedman, McAloon and Simpson in [FMS82] is proven to be equivalent to 1-Con(ATR_0).

Two theories can be compared by means of their arithmetical strength.

1.3 Definition. *The* arithmetical strength *of a theory* T *is defined as the set of all first-order arithmetical consequences of* T.

The arithmetical strengths of two theories are compared by inclusion $\subseteq$. Unless stated differently, strength will always refer to arithmetical strength.

1.2.2 Ordinals

At the end of the 19th century, Cantor extended natural numbers into the transfinite by defining ordinal numbers, or *ordinals* for short. It enabled him to count into the transfinite and to study the order of such infinite numbers. Suppose we start counting $0, 1, 2, 3, \ldots$ and let ω denote the supremum of this sequence, i.e. the least infinite ordinal. Then Cantor's theory allows us to continue counting as follows

$$\omega, \omega + 1, \omega + 2, \ldots,$$

and even further

$$\omega + \omega, \omega + \omega + 1, \omega + \omega + 2, \ldots,$$

and so on.

A *well-order* is a pair $(X, <)$ such that the binary relation $<$ acting on the set X is irreflexive, transitive and linear and such that every non-empty subset of X has a least element with respect to $<$.

An *order type* is an equivalence class of well-orders under the equivalence relation of being order preserving isomorphic. Ordinals are defined as representatives of order types of well-orders. As we will be working in second-order arithmetic Z_2, it is interesting to mention the next result by Friedman. He showed that ATR_0 is equivalent, over RCA_0, to the comparability of countable well-orderings, i.e. the statement which asserts that for each pair of countable well-orderings, there is an isomorphism of one onto an initial segment of the other (see e.g. [Fri67, Fri75]).

Since the introduction of ordinals different implementations have been realised, see e.g. [Hal60, KM76, Poh89]. Studies on ordinal notations and constructive ordinals are carried out by Alonzo Church and Stephen Kleene ([Chu38, Kle38]). The original definition of an ordinal might give troubles when used in the framework of an arbitrary set theory, as the equivalence class of an ordering is not necessarily a set. In ZFC, one avoids this problem by defining an ordinal as a transitive set which is well-ordered by the relation $\in$. In the literature the reader may find more examples of introducing ordinals in different theories.

We briefly list some more definitions and properties. There is a least ordinal which will be denoted by 0. For every ordinal α there is a unique least ordinal β such that $\alpha < \beta$. We call this ordinal the *successor of* α and denote it by $\alpha + 1$. An ordinal λ which is neither 0 nor the successor of another ordinal is called a *limit ordinal*, or shortly a limit. If λ is a limit, then

$$\forall\beta(\beta < \lambda \to \beta + 1 < \lambda).$$

The class of limit ordinals is denoted by Lim. Define $\mathbb{N}$ as the smallest set of ordinals which contains 0 and is such that if α is in the subset then so is its successor $\alpha + 1$. Now define ω formally as $\sup \mathbb{N}$, which is the least limit ordinal.

One proceeds by introducing ordinal arithmetic, i.e. defining addition, multiplication and exponentiation for ordinals (see e.g. [Poh89]). Then it is possible to show that every ordinal α different from 0, can be written in *Cantor normal form*, i.e.

$$\alpha = \omega^{\alpha_0} \cdot a_0 + \ldots + \omega^{\alpha_s} \cdot a_s, \tag{1.1}$$

for some $\alpha_0 > \ldots > \alpha_s$ with $a_0, \ldots, a_s \in \mathbb{N} \setminus \{0\}$ (see e.g. [Sch77]). This form is also known as the Cantor normal form expansion. Given the form (1.1), we say α is written in *short Cantor normal form* if it is written as $\xi + \omega^{\alpha_s}$ (so with $\xi = \omega^{\alpha_0} \cdot a_0 + \ldots + \omega^{\alpha_s} \cdot (a_s - 1)$).

In this dissertation Greek letters always represent ordinals. Finite ordinals can also be denoted by Latin letters. λ is often, but not exclusively, reserved for representing a limit ordinal. The notation $\omega_k(\alpha)$ is defined for ordinals α with $\alpha < \varepsilon_\omega$, by $\omega_0(\alpha) = \alpha$ and $\omega_{k+1}(\alpha) = \omega^{\omega_k(\alpha)}$. We abbreviate $\omega_k(1)$ to ω_k.

Define an *epsilon number* as a fixpoint of the function acting on the class of ordinals by mapping ξ to ω^ξ. Then ε_α denotes the αth epsilon number, i.e. the αth ordinal ξ such that $\xi = \omega^\xi$. Remark that $\varepsilon_0 = \sup_{n\in\omega} \omega_n$. In this dissertation we will work with ordinals less than or equal to ε_ω.

One could verify that the ordinals below ω, the so-called *finite ordinals*, together with the corresponding addition and multiplication, satisfy the axioms of PA. We will identify finite ordinals and natural numbers. In particular, we will use ω as well as $\mathbb{N}$ to denote the set of natural numbers.

Let $\alpha \leq \varepsilon_\omega$ be written in Cantor normal form. The smallest (i.e. rightmost) exponent of α is denoted by $\mathrm{RM}(\alpha)$. Similarly, we write $\mathrm{LM}(\alpha)$ for the largest (i.e. leftmost) exponent of α in that particular form. We write[1] $\beta \gg \alpha$ if either $\alpha = 0$ or $\beta = 0$ or $\mathrm{RM}(\beta) \geq \mathrm{LM}(\alpha)$. The same situation but with strict inequality, is denoted by $\beta \ggg \alpha$. Observe that $\beta \gg \alpha$ does not imply $\beta \geq \alpha$, as one can verify in $\omega^3 \gg \omega^3 \cdot 2$.

Let us introduce *fundamental sequences* for ordinals up to ε_ω. Henryk Kotlarski gave such a treatment for ordinals below Γ_0, which we appreciate and will use here. In particular, the following definitions and several results can also be found in [Kot], often in a more general form. By a system of fundamental sequences for ordinals $\leq \varepsilon_\omega$, we mean a function

$$P\colon \mathrm{Lim} \cap (\leq \varepsilon_\omega) \to (\leq \varepsilon_\omega)^\omega,$$

such that for every limit $\lambda \leq \varepsilon_\omega$, $P(\lambda)$ is an increasing sequence of ordinals convergent to λ. We shall work with one fixed fundamental system, hence we shall simply write $\lambda[n]$, instead of $P(\lambda)(n)$. Moreover, we let the notion of $\alpha[n]$ be defined also for non-limit ordinals; we set $0[n] = 0$ and $(\alpha{+}1)[n] = \alpha$ for all $n \in \omega$. The particular fundamental system is determined by the following three conditions:

1. (additivity) If $\beta \gg \alpha$ then $(\beta + \alpha)[n] = \beta + \alpha[n]$;
2. (omega base) $\omega[n] = n$, $\omega^{\alpha+1}[n] = \omega^\alpha \cdot n$ and for λ limit but no epsilon number, we put $\omega^\lambda[n] = \omega^{\lambda[n]}$;
3. (epsilon numbers) $\varepsilon_0[n] = \omega_n$, $\varepsilon_{m+1}[n] = \omega_n(\varepsilon_m + 1)$ and $\varepsilon_\omega[n] = \varepsilon_n$.

The additivity property yields the following equality

$$\alpha[n] = \omega^{\alpha_0} \cdot a_0 + \ldots + \omega^{\alpha_s} \cdot (a_s - 1) + \omega^{\alpha_s}[n],$$

when α is written in its Cantor normal form (1.1). Using this fundamental system, we always work with the rightmost term in the Cantor normal form of the ordinal under consideration. Remark that one can find other definitions of fundamental sequences in the literature, most of them differing

[1] Do not confuse this $\gg$ defined for ordinals with the symbol $\ll$ defined for functions (see Subsection 1.1.2).

only slightly. Apart from being interesting in its own right, the concept of a fundamental sequence will be used to introduce other notions. The following small lemma turns out to be useful.

1.4 Lemma. *If $\beta \gg \lambda$ then $\beta \gg \lambda[n]$, for all n.*

Proof. See [Kot], Lemma 2.1. □

1.5 Definition. *For $\alpha, \beta \leq \varepsilon_\omega$ we write $\alpha \Rightarrow_n \beta$ if there exists a finite sequence $\alpha_0, \dots, \alpha_m$ such that $\alpha_0 = \alpha$, for every $j < m$, $\alpha_{j+1} = \alpha_j[n]$ and $\alpha_m = \beta$.*

The original study of $\Rightarrow_n$ for ordinals below ε_0 has been carried out by Jussi Ketonen and Robert Solovay in [KS81], even though they use a slightly different fundamental sequence. The treatment for ordinals up to ε_ω given below can be found in detail in [Kot]. In particular, the proofs of the following lemmas are all written down carefully in that paper (Lemma 3.1 – Lemma 3.4).

Let us start by looking more thoroughly at the definition of $\Rightarrow_n$. Given fixed m, the first two conditions determine the sequence $\alpha_0, \dots, \alpha_m$ uniquely. The interesting issue is whether there exists m such that β belongs to that sequence. It is also easy to see that $\alpha \Rightarrow_n \beta$ implies $\alpha \geq \beta$ and the sequence witnessing this relation is strictly decreasing. Clearly, in case this sequence has length strictly greater than 1 then $\alpha > \beta$. In addition, given fixed n, the relation $\alpha \Rightarrow_n \beta$ is transitive. It is also not too difficult to prove by transfinite induction that for every $\alpha \leq \varepsilon_\omega$ and every $n < \omega$, $\alpha \Rightarrow_n 0$. The following properties of the relation $\Rightarrow_n$ will turn out to be handy.

1.6 Lemma. *Let $\alpha \leq \varepsilon_\omega$ and $b, j \in \mathbb{N}$ such that $b \geq j > 0$. Then $\alpha[b] \Rightarrow_b \alpha[j]$ and hence $\alpha \Rightarrow_b \alpha[j]$.*

1.7 Lemma. *If $\alpha \Rightarrow_j \beta$ and $b \geq j > 0$ then $\alpha \Rightarrow_b \beta$.*

1.8 Lemma. *For all α, β, if $\beta > \alpha$ then there exists $b \in \mathbb{N}$ such that $\beta \Rightarrow_b \alpha$.*

1.9 Lemma. *For all $\alpha > 0$ and all $m > 0$, it holds that $\alpha \Rightarrow_m 1$.*

1.2.3 Hierarchies of functions

In this dissertation we will need several hierarchies of functions. Some specific ones will be defined just before they are needed. The more general ones are introduced below.

Let us start with some general remarks and definitions. Let $f\colon \mathbb{N} \to \mathbb{N}$ be an unbounded function. If k is a natural number, then f^k denotes the kth iteration of f, with the agreement that $f^0(x) = x$. The inverse of f is denoted by f^{-1} and defined by $f^{-1}(x) = \min\{y : f(y) \geq x\}$.

The following definitions can be found in [HP98]. Let $\mathcal{L}_0$ be the usual language of first order arithmetic, $\mathcal{L}$ an extension of $\mathcal{L}_0$ and T an $\mathcal{L}$-theory containing $\mathrm{I}\Delta_0$. A formula $\varphi(x, y)$ is a *definition* of a function $f\colon \mathbb{N} \to \mathbb{N}$ if $\varphi(x, y)$ defines the graph of f, that is to say

$$\{(x, y) \in \mathbb{N} \times \mathbb{N} : \varphi(x, y)\} = \{(x, y) \in \mathbb{N} \times \mathbb{N} : y = f(x)\}.$$

A formula $\varphi(x, y)$ *defines a total function in* T if[2]

$$\mathrm{T} \vdash (\forall x)(\exists! y)\varphi(x, y).$$

We may then extend T by defining a new function symbol f and the axiom $\varphi(x, f(x))$. A function $f\colon \mathbb{N} \to \mathbb{N}$ is T-*provably total* if it has a definition $\varphi(x, y)$ which defines a total function in T. The function f is T-*provably* Σ_n if it has a definition which is Σ_n in T. The function f is a T-*provably total* Σ_n *function* if it has a definition which is Σ_n in T and defines a total function in T. In particular we say f is T-*provably recursive* if it is T-provably total Σ_1. Using some recursion theory that name can be explained as follows. Suppose f is T-provably total Σ_1, i.e. it is T-provably total and it has a definition which is Σ_1. Then the graph of f, which is a subset of $\mathbb{N} \times \mathbb{N}$, is defined by a Σ_1-formula. Hence the graph of f is recursive enumerable, and so f is recursive. Thus T-provably recursive functions can be viewed as those recursive functions of which the totality is known to the theory T.

[2] The notation $(\exists! y)$ means "there exists a unique y such that...". This does not belong to the syntax of first-order logic, but is rather an abbreviation for a more complicated first-order formula.

We will freely use other formulations of the above concepts, as for example "T proves the totality of f", "f is a provably recursive function of T", *etc.*.

A theory T can be partially described in a natural way by classifying the T-provably recursive functions. There is, for instance, the following interesting property of $\mathrm{I}\Sigma_1$ proved by Charles Parsons ([Par70, Par71, Par72]), and independently by Grigori Mints ([Min71]) and Takeuti ([Tak87]).

1.10 Theorem. *The* $\mathrm{I}\Sigma_1$*-provably recursive functions are exactly the primitive recursive functions.*

The definitions of an *elementary function* and a *primitive recursive function* can be found in e.g. [Ros84]. The following *fast-growing hierarchy* turns out to be very useful for such investigations.

1.11 Definition. *Define the* fast-growing hierarchy $(F_\alpha)_{\alpha\leq\varepsilon_0}$ *as follows. For every* $x \in \mathbb{N}$*,*

$$\begin{aligned} F_0(x) &= x+1; \\ F_{\alpha+1}(x) &= \underbrace{F_\alpha(\ldots(F_\alpha}_{x\ times}(x))\ldots) = F_\alpha^x(x); \\ F_\lambda(x) &= F_{\lambda[x]}(x), \end{aligned}$$

where α *and* λ *are ordinals below* ε_0*, with* λ *a limit.*

This hierarchy is also known as the *Wainer hierarchy*, the *(transfinitely extended) Grzegorczyk hierarchy* and the *extended Ackermann hierarchy*. This last name is not a mere coincidence as the Ackermann hierarchy is given by $(F_\alpha)_{\alpha<\omega}$, with F_ω growing as fast as the Ackermann function A_ω. Moreover, we will usually write A_ω instead of F_ω. Also, the branches F_d of the Ackermann function (with $d \in \mathbb{N}$) will often be denoted by A_d.

Properties of the fast-growing hierarchy and its relation to provably recursive functions are studied in [Wai70, Sch71, Wai72, BW87, BCW94, Wei96, FW98, Bus98a]. We state some of the main results. Proofs are omitted, but can be found either in the aforementioned references or by combining results of [Par80, KS81, FS95].

1.12 Theorem. *We consider two similar cases.*

1. *Let $n > 0$. Then*
   (a) $\mathrm{I}\Sigma_n \vdash (\forall x)(\exists y)(F_\alpha(x) = y)$ *if and only if* $\alpha < \omega_n$.
   (b) *Let f be an $\mathrm{I}\Sigma_n$-provably Σ_1 function. Then f is $\mathrm{I}\Sigma_n$-provably total if and only if f is primitive recursive in F_α for some $\alpha < \omega_n$.*
   (c) *F_{ω_n} eventually dominates all $\mathrm{I}\Sigma_n$-provably total functions.*

2. (a) $\mathrm{PA} \vdash (\forall x)(\exists y)(F_\alpha(x) = y)$ *if and only if* $\alpha < \varepsilon_0$.
   (b) *Let f be a* PA*-provably Σ_1 function. Then f is* PA*-provably total if and only if f is primitive recursive in F_α for some $\alpha < \varepsilon_0$.*
   (c) *F_{ε_0} eventually dominates all* PA*-provably total functions.*

More on proof-theoretic characterisations of provably total functions can be found in e.g. [Tak87, Bus94] and [Wei06a] and in references cited therein. The following hierarchy is the prime example of an inner iteration hierarchy, which is based on the successor function (see e.g. [CW83, FW92, FW98] for more information).

1.13 Definition. *Define the* Hardy hierarchy $(h_\alpha)_{\alpha \le \varepsilon_\omega}$ *as follows. For every $x \in \mathbb{N}$,*

$$
\begin{aligned}
h_0(x) &= x; \\
h_{\alpha+1}(x) &= h_\alpha(x+1); \\
h_\lambda(x) &= h_{\lambda[x]}(x),
\end{aligned}
$$

where α is an ordinal below ε_ω, and λ a limit ordinal less than or equal to ε_ω.

In this dissertation we will use slightly different versions of the Hardy hierarchy, of which one is introduced in the next section.

1.2.4 α-Largeness

In [KS81] Ketonen and Solovay investigated by purely combinatorial means the growth rate of a function closely related to the Paris-Harrington principle. While doing so, they introduced and studied *α-largeness.* The concept turned out to be both useful for studying other mathematical objects and interesting in its own right, as we will see later on.

Let A be a subset of the natural numbers and define $h^A \colon A \backslash \{\max A\} \to A$, to be the successor function in the sense of A (i.e., h^A associates with every element in its domain the next element of A). In the next definition, $\simeq$ means that either both sides are defined and equal or both sides are undefined. As announced earlier we will need a hierarchy, slightly different from the Hardy hierarchy.

1.14 Definition. *Define the hierarchy of functions $(h^A_\alpha)_{\alpha \le \varepsilon_\omega}$ as follows. For every $x \in \mathbb{N}$,*

$$\begin{aligned} h^A_0(x) &\simeq x; \\ h^A_{\alpha+1}(x) &\simeq h^A_\alpha(h^A(x)); \\ h^A_\lambda(x) &\simeq h^A_{\lambda[x]}(x), \end{aligned}$$

where α is an ordinal below ε_ω, and λ a limit ordinal less than or equal to ε_ω. $(h^A_\alpha)_{\alpha \le \varepsilon_\omega}$ is called the Hardy hierarchy based on h^A*.*

Remark that in case A equals $\mathbb{N}$, then h^A becomes the normal successor function and $(h^A_\alpha)_{\alpha \le \varepsilon_\omega}$ is the standard Hardy hierarchy as given by Definition 1.13.

Now we are ready to give the definition of an α-large set.

1.15 Definition. *A set $A \subseteq \mathbb{N}$ is called* α-large *if $h^A_\alpha(\min A)$ is defined. A set $A \subseteq \mathbb{N}$ is called* α-small *if it is not α-large.*

Whenever it is clear which set A we are working with, we leave out the superscript and simply write h and h_α, instead of h^A and h^A_α. In the lemmas below, we will assume all functions h^A_α occurring are acting on their domain, so we can replace $\simeq$ by $=$. If not mentioned explicitly, A will denote an arbitrary subset of $\mathbb{N} \setminus \{0\}$. Notice that the previous definition is sometimes stated in a different form. Namely, a set A is 0-large if it is nonempty. A is $(\alpha + 1)$-large if $A \setminus \{\min A\}$ is α-large. Finally, with λ a limit ordinal, A is λ-large if it is $\lambda[\min A]$-large.

1.16 Definition. *A set $A \subseteq \mathbb{N}$ is called* exactly α-large *if it is α-large, but $A \setminus \{\max A\}$ is α-small.*

Proofs of the following lemmas can be found in [Kot] (Lemma 5.1 – Lemma 5.3 and Lemma 5.6).

1.17 Lemma. *For every α and every $\beta \gg \alpha$, $h_{\beta+\alpha} = h_\beta \circ h_\alpha$.*

For ordinals below ε_0, this result is often ascribed to Stanley Wainer. We can restate the lemma in the following manner.

1.18 Lemma. *Let A be a subset of the natural numbers and $\beta \gg \alpha$. Then A is $(\beta+\alpha)$-large if and only if there exists $u \in A$ such that $\{x \in A \,|\, x \leq u\}$ is α-large and $\{x \in A \,|\, u \leq x\}$ is β-large.*

The next lemmas will be convenient later.

1.19 Lemma. *For every $\alpha \leq \varepsilon_\omega$:*

1. *h_α is increasing;*
2. *for every β and b, if $\alpha \Rightarrow_b \beta$ and $h_\alpha(b)$ exists then $h_\beta(b)$ exists and $h_\alpha(b) \geq h_\beta(b)$.*

Whenever we write $A = \{a_0, a_1, \ldots, a_s\}$, we assume the elements of A are given in increasing order. In particular, a_0 will denote the minimum of A.

1.20 Lemma. *Let $A = \{a_0, \ldots, a_s\}$ and $B = \{b_0, \ldots, b_t\}$ be finite sets.*

1. *If $|A| = |B|$ and for every $i \leq s$, $b_i \leq a_i$, then for every $i \leq s$, if $h^A_\alpha(a_i)$ exists, then $h^B_\alpha(b_i)$ exists and $h^A_\alpha(a_i) \geq h^B_\alpha(b_i)$.*
2. *If A is α-large, $|A| = |B|$ and for every $i \leq s$, $b_i \leq a_i$, then B is α-large.*
3. *If $A \subseteq B$ and A is α-large, then B is α-large.*

Chapter 2

THE PIGEONHOLE PRINCIPLE

2.1 Introduction

The pigeonhole principle is one of the most well-known combinatorial principles, because of both its simplicity and usefulness. The principle is also known as the chest-of-drawers principle or Schubfachprinzip and is attributed to Dirichlet in 1834. The proof complexity of the propositional formulation of the principle and several variations thereof are studied widely (see e.g. [Kra95, BP98, MPW00, Raz02]). Although known and studied for a long time now, it still has interesting properties to reveal. So it is not surprising the principle gained the attention of several mathematicians lately. Terence Tao, for example, uses the example of the so called "finitary" infinite pigeonhole principle in an article on his blog ([Tao07]). Jaime Gaspar and Kohlenbach commented on his ideas and wrote a paper about it ([GK10]).

The pigeonhole principle is strongly related to Ramsey's theorem for 1-tuples, i.e. natural numbers. More precisely, it is a finite instance of $\mathrm{RT}^1_{<\infty}$, which stands for $\forall k\, \mathrm{RT}^1_k$, using the notation introduced in Subsection 1.1.3. In this chapter we will have a closer look at the statements $\mathrm{RT}^1_{<\infty}$ and RT^1_2. Let us start by mentioning but a few results from Reverse Mathematics: for any natural number k,

$$\mathrm{RCA}_0 \vdash \mathrm{RT}^1_k,$$

whereas

$$\mathrm{WKL}_0 \nvdash \mathrm{RT}^1_{<\infty}.$$

Both results are ascribed to Jeffry Hirst (see [Hir87], Theorem 6.3 and Theorem 6.5). In addition, it is also proved that $\mathrm{RT}^1_{<\infty}$ does not imply ACA_0 over RCA_0. As it does not tie in nicely with the programme of Reverse Mathematics, one might be tempted to think that $\mathrm{RT}^1_{<\infty}$ is of little importance. However, it pops up every now and then in the literature. It is, for instance, equivalent to Rado's Lemma over RCA_0 (see [Hir87], Theorem 6.6).

Since Paris introduced them in the late seventies ([Par78]), densities turned out to be fruitful for studying independence results, as they often generate strength. Motivated by their simplicity and Ramseyan nature we investigate the combinatorial complexity of two densities which are strongly related to the pigeonhole principle. More precisely, the aim is to miniaturise $\mathrm{RT}^1_{<\infty}$ and RT^1_2 by defining n-PHP-density and $(\alpha, 2)$-PHP-density. In addition, both densities depend on a parameter function $f\colon \mathbb{N} \to \mathbb{N}$. Then we define two first-order assertions and study their provability with respect to $\mathrm{I}\Sigma_1$, the first-order part of RCA_0.

We show for which f we obtain Ackermannian growth rate and determine the exact phase transition. In case of n-PHP-density Ackermannian growth is obtained for $f(i) = i^{\frac{1}{A_\omega^{-1}(i)}}$, whereas for $f(i) = i^{\frac{1}{A_d^{-1}(i)}}$ with d a natural number it is not. Here A_d denotes the dth branch of the Ackermann function A_ω (see Definition 1.11).

In the case of $(\alpha, 2)$-density we restrict ourselves to only two colours and strength disappears, as expected. However, iterating up to ω^2 suffices to gain strength again. It turns out that $f(i) = \frac{1}{A_d^{-1}(i)} \log(i)$, with d a natural number, gives rise to no more than primitive recursive growth, in contrast to $f(i) = \frac{1}{A_\omega^{-1}(i)} \log(i)$, which leads to Ackermannian growth.

Remark that the n-PHP-density threshold functions are exactly the same as those for the ISP-density (see Chapter 3) and the parameterised Kanamori-McAloon principle, whereas the $(\omega^2, 2)$-PHP-density functions equal those for the parameterised Paris-Harrington principle. Finally, the study of Ramseyan assertions of the kind considered here may contribute to the solution of the longstanding open problem of Ramsey's theorem for pairs.

2.2 n-PHP-Density

Let $X \subseteq \mathbb{N}$ and $f\colon \mathbb{N} \to \mathbb{N}$ be any function, such that $1 \leq f(x) \leq x$, for $x \in \mathbb{N}$. Let us define n-PHP-density, the first density notion related to the pigeonhole principle. In this case the number of colours depends on the minimum of X and the function f.

2.1 Definition. *Let $X \subseteq \mathbb{N}$. Then X is called* 0-PHP-dense(f) *if* $|X| \geq \max\{f(\min X), 3\}$, *and X is called* $(n+1)$-PHP-dense(f) *if for all $G\colon X \to f(\min X)$, there exists $Y \subseteq X$, such that Y is homogeneous for G and Y is* n-PHP-*dense(f).*

For the rest of this section we will leave out PHP and simply write n-density, as this will always refer to n-PHP-density.

2.2.1 Lower bound

We show that in case $f(i) = \lfloor i^{\frac{1}{A_d^{-1}(i)}} \rfloor$ for all $i \in \mathbb{N}$, with d a natural number, we obtain a lower bound. Thus, if the expression between the floor symbols $\lfloor$ and $\rfloor$ is not a natural number, we consider the greatest integer less than $i^{\frac{1}{A_d^{-1}(i)}}$, as f needs to be a number-theoretic function. Henceforth we leave out the floor symbols for the sake of clarity. We will do this for all parameter functions in this dissertation without mentioning explicitly.

2.2 Theorem. *Let $d \in \mathbb{N}$ and $f(i) = i^{\frac{1}{A_d^{-1}(i)}}$. Then*

$$\mathrm{I}\Sigma_1 \vdash (\forall n)(\forall a)(\exists b)([a,b] \text{ is } n\text{-dense}(f)).$$

Proof. Let n and a be given. Note that if $[a,b]$ is n-dense(f), then so is $[0, b-a]$. Hence without loss of generality, we can assume $a > 0$. Put

$b = 2^{A_d(a2^{n+1})2^{n+1}}$. We claim that any $Y \subseteq [a,b]$ with $|Y| > 2^{A_d(a2^{n+1})2^k}$ is k-dense(f). To prove the claim we proceed by induction on k.

Assume the claim holds for $k-1$, with $k > 0$, and consider $Y \subseteq [a,b]$ such that $|Y| > 2^{A_d(a2^{n+1})2^k}$. Since $2^{A_d(a2^{n+1})2^{n+1}} > A_d(2^{n+1})$, we have

$$\begin{aligned}
f(\min Y) &\leq f(b) \\
&= (2^{A_d(a2^{n+1})2^{n+1}})^{\frac{1}{A_d^{-1}(2^{A_d(a2^{n+1})2^{n+1}})}} \\
&\leq (2^{A_d(a2^{n+1})2^{n+1}})^{\frac{1}{A_d^{-1}(A_d(2^{n+1}))}} \\
&= (2^{A_d(a2^{n+1})2^{n+1}})^{\frac{1}{2^{n+1}}} \\
&= 2^{A_d(a2^{n+1})} \\
&\leq 2^{A_d(a2^{n+1})2^{k-1}}.
\end{aligned}$$

Let $c = f(\min Y)$ and $G\colon Y \to c$ be any function. Consider the partition of Y induced by G, i.e.

$$Y = \cup_{0 \leq i < c} Y_i,$$

with $Y_i = \{y \in Y : G(y) = i\}$. By contradiction, assume that $|Y_i| \leq 2^{A_d(a2^{n+1})2^{k-1}}$ for every $0 \leq i < c$. Then

$$\begin{aligned}
2^{A_d(a2^{n+1})2^k} &< |Y| \\
&\leq c \cdot 2^{A_d(a2^{n+1})2^{k-1}} \\
&= f(\min Y) \cdot 2^{A_d(a2^{n+1})2^{k-1}} \\
&\leq 2^{A_d(a2^{n+1})2^{k-1}} \cdot 2^{A_d(a2^{n+1})2^{k-1}} \\
&= 2^{A_d(a2^{n+1})2^{k-1} + A_d(a2^{n+1})2^{k-1}} \\
&= 2^{A_d(a2^{n+1})2^k},
\end{aligned}$$

a contradiction. Thus, there exists an index $i_0 \in \{0, \ldots, c-1\}$, such that $|Y_{i_0}| > 2^{A_d(a2^{n+1})2^{k-1}}$. The induction hypothesis yields that Y_{i_0} is $(k-1)$-dense(f) and by definition Y_{i_0} is homogeneous for G, so Y is k-dense(f).

If $k = 0$, then $|Y| > 2^{A_d(a2^{n+1})} \geq \max\{f(\min Y), 3\}$, which completes the induction argument and proves the claim.

Now return to $[a, b]$. Since

$$|[a, b]| \geq 2^{A_d(a2^{n+1})2^{n+1}} - a > 2^{A_d(a2^{n+1})2^n},$$

$[a, b]$ is n-dense(f). Remarking that the function $E\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, defined by $E(a, n) = 2^{A_d(a2^{n+1})2^{n+1}}$ is primitive recursive, completes the proof (see Theorem 1.10). $\square$

Let us connect the theorem above with the provably recursive functions of $\mathrm{I}\Sigma_1$. We define a function naturally related to our density notion.

2.3 Definition. *Define* $\mathrm{PHP}_f\colon \mathbb{N} \to \mathbb{N}$ *by* $\mathrm{PHP}_f(n) = \mathrm{PHP}_f(n, n)$*, where* $\mathrm{PHP}_f(n, a)$ *is the least natural number* b*, such that* $[a, b]$ *is* n*-dense(*f*).*

Let f be as above. Examining the statement $y = \mathrm{PHP}_f(x)$, one sees that PHP_f is $\mathrm{I}\Sigma_1$-provably Σ_1. Theorem 2.2 states that PHP_f is a provably total function of $\mathrm{I}\Sigma_1$, and thus $\mathrm{I}\Sigma_1$-provably recursive. Hence, Theorem 1.10 implies that PHP_f is primitive recursive.

2.2.2 Upper bound

Investigating an upper bound of the threshold region, we will use a hierarchy of functions which also depends on the parameter function $f\colon \mathbb{N} \to \mathbb{N}$.

2.4 Definition. *Define the hierarchy of functions* $(F_{f,k})_{k<\omega}$ *and the function* F_f *as follows. For every* $x \in \mathbb{N}$*,*

$$\begin{aligned} F_{f,0}(x) &= x + 1; \\ F_{f,k+1}(x) &= \underbrace{F_{f,k}(\ldots(F_{f,k}}_{f(x)\ \textit{times}}(x))\ldots) = F_{f,k}^{f(x)}(x); \\ F_f(x) &= F_{f,x}(x), \end{aligned}$$

for $k \in \mathbb{N}$*.*

Strictly speaking, the notation should indicate that $F_{f,k}$ depends on f, in order not to be confused with the fast-growing hiearchy (see Definition 1.11). In this chapter though, we omit stating this dependence to lighten the notation. So given a number-theoretic function f, we simply write write F_k and F, instead of $F_{f,k}$ and F_f respectively. To avoid ambiguity we use A_d and A_ω to denote members of the fast-growing hierarchy.

We start with the following general lemma.

2.5 Lemma. *Let $n \in \mathbb{N}$ and $X \subseteq \mathbb{N}$. If X is n-dense(f), then* $\max X \geq F_n(\min X)$.

Proof. Henceforth, let $x_0 = \min X$ and $c = f(x_0)$. The proof goes by induction on n.

If X is 0-dense(f), then $|X| \geq \max\{f(x_0), 3\}$. Thus, $\max X \geq x_0 + 2 \geq F_0(x_0)$.

Secondly, assume the statement is proven for n and X is $(n+1)$-dense(f). Consider the partition $\cup_{0 \leq i < c} Y_i$ of X, where Y_i is defined by

$$Y_i = \{x \in X \,:\, F_n^i(x_0) \leq x < F_n^{i+1}(x_0)\}$$

for $0 \leq i < c-1$ and $Y_{c-1} = \{x \in X \,:\, F_n^{c-1}(x_0) \leq x\}$. Now, define $G\colon X \to c$, as follows. If $x \in Y_i$, then $G(x) = i$. Since X is $(n+1)$-dense(f), there exists a subset Y of X, such that Y is n-dense(f) and homogeneous for G. By contradiction assume $Y \subseteq Y_{i_0}$ for some i_0 with $0 \leq i_0 < c-1$. The induction hypothesis yields $\max Y \geq F_n(\min Y)$, and so

$$\begin{aligned} F_n^{i_0+1}(x_0) - 1 &= \max Y_{i_0} \\ &\geq \max Y \\ &\geq F_n(\min Y) \\ &\geq F_n(\min Y_{i_0}) \\ &= F_n(F_n^{i_0}(x_0)) \\ &= F_n^{i_0+1}(x_0), \end{aligned}$$

a contradiction. So $Y \subseteq Y_{c-1}$, which implies

$$\begin{aligned}\max X &= \max Y_{c-1}\\ &\geq \max Y\\ &\geq F_n(\min Y)\\ &\geq F_n(\min Y_{c-1})\\ &= F_n(F_n^{c-1}(x_0))\\ &= F_n^c(x_0)\\ &= F_n^{f(x_0)}(x_0)\\ &= F_{n+1}(x_0),\end{aligned}$$

by the n-density of Y. This concludes the induction argument. □

Suppose f equals the identity function. Then for a given n, the function F_n would have the same growth rate as the nth branch of the Ackermann function A_ω (see Definition 1.11). In that case the statement

$$(\forall a)(\exists b)([a, b] \text{ is } n\text{-dense}(f))$$

implies the totality of the nth branch of the Ackermann function.

Now, let $f(i) = i^{\frac{1}{A_\omega^{-1}(i)}}$, for all $i \in \mathbb{N}$, and PHP_f as given by Definition 2.3. Then F is Ackermannian, because of Theorem 1 in [OW09]. We get the following unprovability result.

2.6 Theorem. *If* $f(i) = i^{\frac{1}{A_\omega^{-1}(i)}}$, *then*

$$\mathrm{I}\Sigma_1 \nvdash (\forall n)(\forall a)(\exists b)([a, b] \textit{ is } n\textit{-dense}(f)).$$

Proof. First recall that the provably recursive functions of $\mathrm{I}\Sigma_1$ are exactly the primitive recursive functions (Theorem 1.10). Assume by contradiction that

$$\mathrm{I}\Sigma_1 \vdash (\forall n)(\forall a)(\exists b)([a, b] \text{ is } n\text{-dense}(f)) \tag{2.1}$$

In other words, assume that PHP_f is a provably recursive function of $\mathrm{I}\Sigma_1$, and thus primitive recursive. Lemma 2.5 yields

$$\mathrm{PHP}_f(n) \geq F_n(n) = F(n),$$

for every $n \in \mathbb{N}$, and thus F is also primitive recursive. This contradicts the fact that F has Ackermannian growth rate. $\square$

Now let PHP_f stand for

$$(\forall n)(\forall a)(\exists b)([a, b] \text{ is } n\text{-dense}(f)).$$

Then the phase transition is described by Figure 2.1.

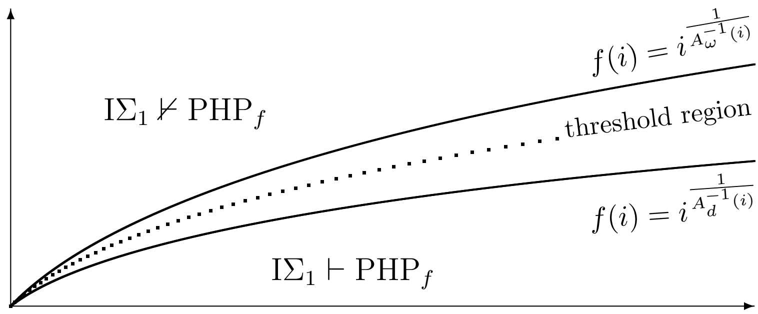


Figure 2.1: *Phase transition for* PHP_f.

2.3 $(\alpha, 2)$-PHP-Density

Contrary to the first density, here we work with a fixed number of colours, namely two. Looking for strength, we will allow transfinite iterations. Recall the definition of a fundamental sequence for a limit ordinal given in Subsection 1.2.2.

2.7 Definition. *Let X be a subset of $\mathbb{N}$. Then X is called* $(0,2)$-PHP-dense(f) *if* $|X| \geq \max\{f(\min X), 3\}$*. The set X is called* $(\alpha+1,2)$-PHP-dense(f) *if for all $G\colon X \to 2$ there exists $Y \subseteq X$, such that Y is* $(\alpha,2)$-PHP-*dense(f) and Y is homogeneous for G. If λ is a limit ordinal, then X is called* $(\lambda,2)$-PHP-dense(f) *if for all $G\colon X \to 2$ there exists $Y \subseteq X$, such that Y is* $(\lambda[f(\min X)],2)$-PHP-*dense(f) and Y is homogeneous for G.*

As before, for the rest of this section we will leave out PHP and simply write $(\alpha,2)$-density, since this will always refer to $(\alpha,2)$-PHP-density.

2.3.1 Lower bound

Let $f(i) = \frac{1}{A_d^{-1}(i)} \log_2(i)$, where we set $\log_2(0) = 0$ and where A_d denotes the dth branch of the Ackermann function A_ω. As the logarithm function always has basis two in this dissertation, we leave out the subscript 2.

2.8 Theorem. *If $f(i) = \frac{1}{A_d^{-1}(i)} \log(i)$, then*

$$\mathrm{I}\Sigma_1 \vdash (\forall a)(\exists b)([a,b] \text{ is } (\omega^2,2)\text{-dense}(f)).$$

Proof. Assume that a is given. Put $b = 2^{A_d(2^{a+2})2^{a+1}}$. We claim that any $Y \subseteq [a,b]$, with $|Y| > 2^{A_d(2^{a+2})2^k}$ is $(\omega \cdot k, 2)$-dense(f). The proof goes by induction on k.

Let k equal 0. Since $2^{A_d(2^{a+2})2^{a+1}} > A_d(2^{a+2})$, we have

$$\begin{aligned} f(\min Y) &\leq f(b) \\ &= \frac{1}{A_d^{-1}(2^{A_d(2^{a+2})2^{a+1}})} \log(2^{A_d(2^{a+2})2^{a+1}}) \\ &< \frac{1}{2^{a+2}} A_d(2^{a+2})2^{a+1} \\ &< A_d(2^{a+2}), \end{aligned}$$

and thus, $|Y| > 2^{A_d(2^{a+2})} > \max\{f(\min Y), 3\}$, i.e. Y is $(0,2)$-dense(f).

Assume the assertion holds for $k-1$, with $k > 0$, and consider $Y \subseteq [a,b]$ with $|Y| > 2^{A_d(2^{a+2})2^k}$. We claim that if $Z \subseteq Y$ and $|Z| > 2^{A_d(2^{a+2})2^{k-1}+l}$, then Z is $(\omega\cdot(k-1)+l, 2)$-dense(f). The proof goes by subsidiary induction on l.

If $l = 0$, then the claim follows by the main induction hypothesis. Assume the claim holds for $l-1$, with $l > 0$, and $|Z| > 2^{A_d(2^{a+2})2^{k-1}+l}$. Let $G\colon Z \to 2$ be any function. Consider the partition of Z induced by G, i.e.

$$Z = Z_0 \cup Z_1,$$

with $Z_i = \{z \in Z \,:\, G(z) = i\}$. By contradiction, assume that

$$|Z_i| \leq 2^{A_d(2^{a+2})2^{k-1}+l-1},$$

for $i = 0, 1$. Then

$$\begin{aligned} 2^{A_d(2^{a+2})2^{k-1}+l} &< |Z| \\ &\leq 2 \cdot 2^{A_d(2^{a+2})2^{k-1}+l-1} \\ &= 2^{A_d(2^{a+2})2^{k-1}+l}, \end{aligned}$$

a contradiction. Thus, there exists an index $i_0 \in \{0, 1\}$, such that $|Z_{i_0}| > 2^{A_d(2^{a+2})2^{k-1}+l-1}$. The induction hypothesis yields Z_{i_0} is $(\omega\cdot(k-1)+l-1, 2)$-dense$(f)$, and so Z is $(\omega\cdot(k-1)+l, 2)$-dense(f), since Z_{i_0} is homogeneous for G. This proves the latter claim.

Now return to Y. Let $G\colon Y \to 2$ be any function. Consider the partition of Y induced by G, i.e.

$$Y = Y_0 \cup Y_1,$$

with $Y_i = \{y \in Y \,:\, G(y) = i\}$. In the same way as above, one can prove by contradiction that there is an index $i_0 \in \{0, 1\}$, such that

$$|Y_{i_0}| > 2^{A_d(2^{a+2})2^{k}-1} = 2^{A_d(2^{a+2})2^{k-1}+A_d(2^{a+2})2^{k-1}-1}.$$

Since

$$A_d(2^{a+2})2^{k-1} \geq A_d(2^{a+2}) \geq f(\min Y) + 1,$$

we have $|Y_{i_0}| > 2^{A_d(2^{a+2})2^{k-1}+f(\min Y)}$. The latter claim yields Y_{i_0} is $(\omega\cdot(k-1)+f(\min Y), 2)$-dense$(f)$, i.e. $(\omega\cdot k[f(\min Y)], 2)$-dense$(f)$. Thus Y is

$(\omega \cdot k, 2)$-dense(f), since Y_{i_0} is homogeneous for G. So also the main claim is proven.

We finally prove that $[a, b]$ is $(\omega^2, 2)$-dense(f). Let $G\colon [a, b] \to 2$ be any function and consider the partition of $[a, b]$ induced by G, i.e.

$$[a, b] = Y_0 \cup Y_1,$$

with $Y_i = \{y \in [a, b] : G(y) = i\}$. Remark that $|[a, b]| > 2^{A_d(2^{a+2})2^{a+1}} - a \geq 2^{A_d(2^{a+2})2^a+1}$. Similarly as before, one can proof by contradiction that there is an index $i_0 \in \{0, 1\}$, such that

$$|Y_{i_0}| > 2^{A_d(2^{a+2})2^a} \geq 2^{A_d(2^{a+2})2^{f(a)}}.$$

The main claim yields Y_{i_0} is $(\omega \cdot f(a), 2)$-dense(f), i.e. $(\omega^2[f(a)], 2)$-dense(f). In combination with Y_{i_0} being homogeneous for G, this implies $[a, b]$ is $(\omega^2, 2)$-dense(f). Remarking that the function $E\colon \mathbb{N} \to \mathbb{N}$, defined by $E(a) = 2^{A_d(2^{a+2})2^{a+1}}$ is primitive recursive, completes the proof (see Theorem 1.10). $\square$

Let us again relate our results above with the provably recursive functions of $\mathrm{I}\Sigma_1$. In the same way as done for the case of n-PHP-density, we define a function naturally related to our density notion.

2.9 Definition. *Define* $\mathrm{PHP2}_f\colon \mathbb{N} \to \mathbb{N}$ *as follows. Given* $a \in \mathbb{N}$*, then* $\mathrm{PHP2}_f(a)$*, is the least natural number* b*, such that* $[a, b]$ *is* $(\omega^2, 2)$*-dense*(f)*.*

By inspecting the statement $y = \mathrm{PHP2}_f(x)$, one learns that $\mathrm{PHP2}_f$ is $\mathrm{I}\Sigma_1$-provably Σ_1. Now let f be as in Theorem 2.8. The same theorem states that $\mathrm{PHP2}_f$ is a $\mathrm{I}\Sigma_1$-provably total function, and thus $\mathrm{I}\Sigma_1$-provably recursive. Hence, Theorem 1.10 implies that $\mathrm{PHP2}_f$ is primitive recursive.

2.3.2 Upper bound

In this section we will use another hierarchy which we call $(B_{f,\alpha})_{\alpha<\varepsilon_0}$ and which turns out to be related to $F_{f,k}$ (see Definition 2.4).

2.10 Definition. *Define the hierarchy of functions $(B_{f,\alpha})_{\alpha<\varepsilon_0}$ as follows. For every $x \in \mathbb{N}$,*

$$\begin{aligned} B_{f,0}(x) &= x+1; \\ B_{f,\alpha+1}(x) &= B_{f,\alpha}(B_{f,\alpha}(x)) = B^2_{f,\alpha}(x); \\ B_{f,\lambda}(x) &= B_{f,\lambda[f(x)]}(x), \end{aligned}$$

for all ordinals α and λ, with the latter a limit ordinal.

As for $F_{f,k}$, we leave out the subscript f and write B_α if it is clear which f we are working with. However, in the following lemma, which shows the relation between the two hierarchies, it is of crucial importance to indicate which functions are used.

2.11 Lemma. *Let k, l and x be natural numbers. Then $B_{f,\omega\cdot k+l}(x) = F^{2^l}_{2^f,k}(x)$.*

Proof. We proceed by main induction on k and subsidiary induction on l.

If k equals l equals zero, we have $B_{f,0}(x) = x+1 = F_{2^f,0}(x)$.

Assume the statement is proven for $k-1$, with $k>0$, we will prove it for k by subsidiary induction on l.

If $l=0$, then the main induction hypothesis yields

$$B_{f,\omega\cdot(k-1)}(x) = F_{2^f,k-1}(x).$$

Assume the claim is proven for $l-1$, with $l>0$. We have

$$\begin{aligned} B_{f,\omega\cdot(k-1)+l}(x) &= B_{f,\omega\cdot(k-1)+l-1}(B_{f,\omega\cdot(k-1)+l-1}(x)) \\ &= F^{2^{l-1}}_{2^f,k-1}(F^{2^{l-1}}_{2^f,k-1}(x)) \\ &= F^{2^l}_{2^f,k-1}(x), \end{aligned}$$

which concludes the subsidiary induction. In other words, the statement is proven for $k-1$ and every l. Using this fact, we obtain

$$\begin{aligned} B_{f,\omega\cdot k}(x) &= B_{f,\omega\cdot(k-1)+\omega[f(x)]}(x)) \\ &= B_{f,\omega\cdot(k-1)+f(x)}(x)) \\ &= F_{2^f,k-1}^{2^{f(x)}}(x) \\ &= F_{2^f,k}(x), \end{aligned}$$

which concludes the main induction and proves the statement. □

In the following lemma we show in which way $(\alpha, 2)$-density is related to the function $B_{f,\alpha}$.

2.12 Lemma. *Let α be an ordinal. If $X \subseteq \mathbb{N}$ is $(\alpha, 2)$-dense(f), then* $\max X \geq B_{f,\alpha}(\min X)$.

Proof. Being of no concrete importance for the proof itself, we leave out the subscript f. Henceforth, let $x_0 = \min X$. The proof goes by transfinite induction on α.

If X is $(0, 2)$-dense(f), then $|X| \geq \max\{f(x_0), 3\}$. Thus, $\max X \geq x_0 + 2 > x_0 + 1 = B_0(x_0)$.

Assume the statement is proven for α and X is $(\alpha + 1, 2)$-dense(f). Define $G\colon X \to 2$ as follows

$$G(x) = \begin{cases} 0 & \text{if } x_0 \leq x < B_\alpha(x_0) \\ 1 & \text{if } B_\alpha(x_0) \leq x \end{cases},$$

for all $x \in X$. Since X is $(\alpha + 1, 2)$-dense(f), there exists a subset Y of X, such that Y is $(\alpha, 2)$-dense(f) and Y is homogeneous with respect to G. By contradiction, assume G takes colour 0 on Y. Then, by the induction hypothesis,

$$B_\alpha(x_0) - 1 \geq \max Y \geq B_\alpha(\min Y) \geq B_\alpha(x_0),$$

a contradiction. So, the colour needs to be 1, which implies

$$Y \subseteq \{x \in X : B_\alpha(x_0) \leq x\}.$$

The induction hypothesis yields

$$\max X \geq \max Y \geq B_\alpha(\min Y) = B_\alpha(B_\alpha(x_0)) = B_{\alpha+1}(x_0).$$

Finally, assume the statement is proven for all $\alpha < \lambda$, with λ a limit ordinal, and X is $(\lambda, 2)$-dense(f). There exists a subset Y which is $(\lambda[f(x_0)], 2)$-dense(f). We obtain by the induction hypothesis

$$\max X \geq \max Y \geq B_{\lambda[f(x_0)]}(\min Y) \geq B_{\lambda[f(x_0)]}(x_0) = B_\lambda(x_0).$$

This completes the proof. $\square$

Fix the function $f(i) = \frac{1}{A_\omega^{-1}(i)} \log(i)$ for all $i \in \mathbb{N}$ for the rest of this subsection.

2.13 Lemma. $\mathrm{PHP2}_f(2^{n^2}) \geq F_{2^f}(n)$, *for every* $n \in \mathbb{N}$.

Proof. Let $X = [2^{n^2}, \mathrm{PHP2}_f(2^{n^2})]$. Define $G\colon X \to 2$ by $G(x) = 0$ for every $x \in X$. Since X is $(\omega^2, 2)$-dense(f) there exists $Y \subseteq X$, such that Y is $(\omega^2[f(\min X)], 2)$-PHP2-dense(f), i.e. $(\omega \cdot f(2^{n^2}), 2)$-dense$(f)$. Lemma 2.11 and Lemma 2.12 yield

$$\begin{aligned}
\mathrm{PHP2}(2^{n^2}) &\geq \max Y \\
&\geq B_{f,\omega\cdot f(\min Y)}(\min Y) \\
&\geq B_{f,\omega\cdot f(2^{n^2})}(2^{n^2}) \\
&= F_{2^f, f(2^{n^2})}(2^{n^2}) \\
&\geq F_{2^f, n}(n) \\
&= F_{2^f}(n),
\end{aligned}$$

since $f(2^{n^2}) = \frac{1}{A_\omega^{-1}(2^{n^2})} \log(2^{n^2}) \geq n$. $\square$

Given the specific f we are working with, we get $2^{f(i)} = i^{\frac{1}{A_\omega^{-1}(i)}}$. Then F_{2^f} is Ackermannian because of Theorem 1 in [OW09]. We obtain the following unprovability result.

2.14 Theorem. *If* $f(i) = \frac{1}{A_\omega^{-1}(i)} \log(i)$, *then*

$$I\Sigma_1 \nvdash (\forall a)(\exists b)([a,b] \textit{ is } (\omega^2, 2)\textit{-dense}(f)).$$

Proof. Recall that the provably recursive functions of $I\Sigma_1$ are exactly the primitive recursive functions (Theorem 1.10). Assume by contradiction that

$$I\Sigma_1 \vdash (\forall a)(\exists b)([a,b] \text{ is } (\omega^2, 2)\text{-dense}(f)). \tag{2.2}$$

In other words, assume that $\mathrm{PHP2}_f$ is a provably recursive function of $I\Sigma_1$, and thus primitive recursive. Then so is $\mathrm{PHP2}_f(2^{n^2})$, as a composition of two primitive recursive functions. By Lemma 2.13 we know that $\mathrm{PHP2}_f(2^{n^2}) \geq F_{2^f}(n)$, for every $n \in \mathbb{N}$, which yields that F_{2^f} is also primitive recursive. This contradicts the fact that F_{2^f} has Ackermannian growth rate. □

Now let $\mathrm{PHP2}_f$ stand for

$$(\forall a)(\exists b)([a,b] \text{ is } (\omega^2, 2)\text{-dense}(f)).$$

Then the phase transition is described by Figure 2.2.

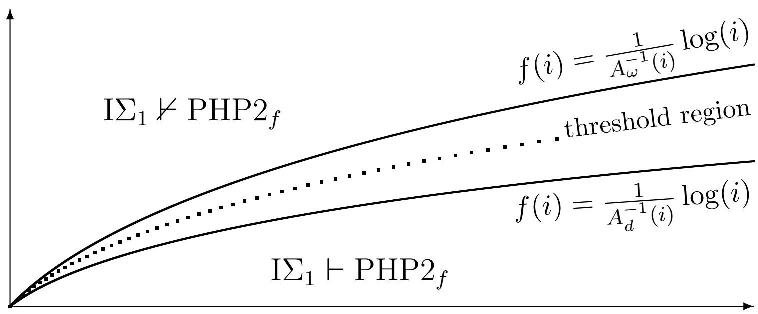


Figure 2.2: *Phase transition for* $\mathrm{PHP2}_f$.

Chapter 3

WEAKLY INCREASING SUBSEQUENCES

3.1 Introduction

It is well-known that every infinite sequence of natural numbers contains an infinite subsequence which is weakly increasing. Given its Ramseyan nature, we explore the strength generated from this *infinite subsequence principle*, or ISP for short. So given a theory T (one may think of RCA_0 for example), we wonder how much strength the theory T + ISP has. In this chapter we investigate a first-order approximation of ISP. To carry out such an investigation, we consider a miniaturisation of ISP in terms of densities, as done in the Chapter 2. The density statement under consideration depends on a parameter function $f\colon \mathbb{N} \to \mathbb{N}$ and turns out to be unprovable in $\mathrm{I}\Sigma_1$ for certain f. However, all those different instances can be proven by applying König's Lemma to ISP. The independent assertion will give rise to a phase transition, as explained from a broader perspective in Subsection 1.1.2.

In order to introduce the unprovable statement related to ISP we need some extra definitions. Let f be a weakly increasing number-theoretic function such that $1 \leq f(x) \leq x$, for $x \in \mathbb{N}$.

3.1 Definition. *A function $G\colon X \to \mathbb{N}$ is called* f-regressive *if $G(x) \leq f(x)$, for all $x \in X$.*

3.2 Definition. *Let* $X \subseteq \mathbb{N}$. *Then* X *is called* 0-ISP-dense(f) *if* $|X| > \max\{f(\min(X)), 3\}$, *and* X *is said to be* $(n+1)$-ISP-dense(f) *if for all* f*-regressive* $G\colon X \to \mathbb{N}$, *there exists* $Y \subseteq X$ *such that* Y *is* n-ISP*-dense*(f) *and*

$$y < y' \to G(y) \leq G(y'),$$

for all $y, y' \in Y$, *i.e. the restriction of* G *to* Y *is weakly increasing.*

For the rest of this chapter we will leave out ISP and simply write n-density, as this will always refer to n-ISP-density. Let us define a function closely related to the property of being n-dense(f).

3.3 Definition. *Define* $\mathrm{ISP}_f\colon \mathbb{N} \to \mathbb{N}$ *by* $\mathrm{ISP}_f(n) = \mathrm{ISP}_f(n,n)$, *where* $\mathrm{ISP}_f(n,a)$ *is the least natural number* b, *such that* $[a,b]$ *is* n*-dense*(f).

It is not too difficult to see that for a constant function f the function ISP_f is primitive recursive. Moreover, one could prove that ISP_f is Ackermannian for $f(i) = i$. So in between constant functions and the identity function there will be a threshold region for f where ISP_f switches from being primitive recursive to being Ackermannian. In this chapter we will show that for $f(i) = i^{\frac{1}{A_d^{-1}(i)}}$ the function ISP_f is a primitive recursive for every fixed $d \in \mathbb{N}$, whereas for $f(i) = i^{\frac{1}{A_\omega^{-1}(i)}}$ it becomes Ackermannian. Figure 3.1 illustrates this phenomenon. Remark that in the figure, abusing notation, ISP_f is short for the statement expressing the totality of the function ISP_f, i.e. $\forall x\, \exists y\, (y = \mathrm{ISP}_f(x))$.

As explained below, our results are intended to contribute partly to the investigation of the strength of RT^2_2, a classical problem in Reverse Mathematics. One can find information on this problem in, for example, [CJS01] and [Sim09].

We first introduce three infinitary principles which are related to RT^2_2: the Erdős-Szekeres principle (ES), the chain-antichain principle (CAC) and the Erdős-Moser principle (EM). The first principle is the infinitary counterpart of the finitary Erdős-Szekeres Theorem which states that a given sequence $a_0, \ldots, a_{n^2}$ of real numbers contains a weakly increasing subsequence of length $n+1$ or a strictly decreasing subsequence of length $n+1$

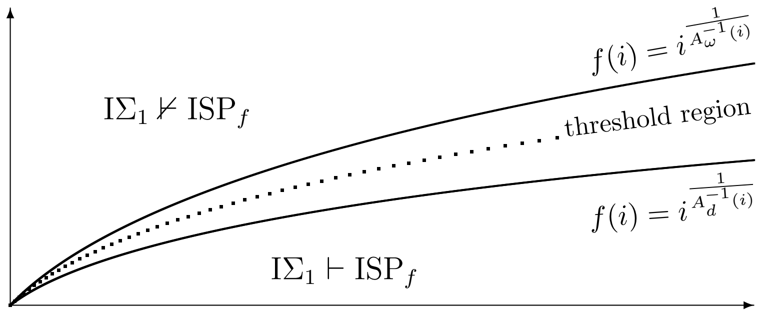


Figure 3.1: *Phase transition for* ISP_f*.*

([ES35]). So, ES states that a given infinite sequence $a_0, a_1, \ldots$ of real numbers contains an infinite weakly increasing subsequence or an infinite strictly decreasing subsequence. We will use the Erdős-Szekeres Theorem in some proofs later on. The principle CAC is the assertion that a given infinite partial order has either an infinite chain or an infinite antichain. It is the infinitary counterpart of the finitary Dilworth Theorem which states that a given partial order with distinct elements $a_0, \ldots, a_{n^2}$ contains a chain of length $n+1$ or an antichain of length $n+1$ ([Dil50]). Finally, let a tournament be a complete directed simple graph. Then EM says that every infinite tournament contains an infinite transitive subtournament. Erdős and Leo Moser studied a finitary version of EM in [EM64].

It is not difficult to see that, over RCA_0, RT^2_2 yields ISP, ES, CAC and EM. Well, we also have a reversal, namely $\mathrm{RCA}_0 + \mathrm{EM} + \mathrm{CAC}$ proves RT^2_2, which makes EM and CAC particularly interesting for studying RT^2_2. Classifying the strength of EM and CAC may yield progress in the investigation of the strength of RT^2_2. Notice that EM cannot be proved within RCA_0, since there exists a recursive tournament without an infinite recursive transitive subtournament ([BW05], Theorem 9). It is somewhat surprising that even ISP generates all primitive recursive functions with its miniaturisation. However, this should not be seen as an indication that $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ proves the totality of the Ackermann function.

In the following section we give an upper and a lower bound for the threshold function. In the third section we improve those results, and give sharper bounds which were obtained later.

3.2 Classifying the phase transition

3.2.1 Lower bound

Recall that the phase transitions we study, will be triggered by modifying a parameter function f. We have a look at the notion of n-density in the case of $f(i) = \log(i)$. In the following proof we need the Erdős-Szekeres Theorem, which is given in the introduction.

3.4 Theorem. *If* $f(i) = \log(i)$*, then*

$$\mathrm{I}\Sigma_1 \vdash (\forall n)(\forall a)(\exists b)([a,b] \text{ is } n\text{-dense}(f)).$$

Proof. Let a and n be given and note that if $[a,b]$ is n-dense(f), then so is $[0, b-a]$. Hence without loss of generality, we can assume $a > 0$. Put $b = 2^{2^{(n+1)\cdot(n+a+1)}}$. We claim that any $Y \subseteq [a,b]$ which satisfies $|Y| > 2^{2^{(k+1)\cdot(n+a+1)}}$ is k-dense(log). The proof goes by induction on k. Assume the assertion holds for $k-1$, with $k > 0$. Assume that $G\colon Y \to \mathbb{N}$ with $G(y) \leq \log(y)$ for every $y \in Y$. Then the Erdős-Szekeres Theorem yields the existence of a set $Z \subseteq Y$ such that $G \restriction Z$ is weakly increasing or strictly decreasing and $|Z| > 2^{2^{(k+1-1)\cdot(n+a+1)}}$. Most of the next inequalities are straightforward:

$$\begin{aligned} G(\min(Z)) &\leq \log(\min(Z)) \\ &\leq \log(b) \\ &\leq \log(2^{2^{(n+1)\cdot(n+a+1)}}) \\ &\leq 2^{(n+1)\cdot(n+a+1)} \\ &\leq 2^{2^{k\cdot(n+a+1)}} \\ &< |Z|, \end{aligned}$$

where the fifth inequality holds because of $k > 0$ and the inequality

$$(n+1) \cdot (n+a+1) \leq 2^{(n+a+1)},$$

for all natural numbers n and $a > 0$. So in case G is strictly decreasing on Z, then $G(\max(Z)) < 0$, a contradiction. Thus G is weakly increasing on Z. Since $|Z| > 2^{2^{(k+1-1)\cdot(n+a+1)}}$ the induction hypothesis yields that Z is $(k-1)$-dense(log), which implies that Y is k-dense(log). If $k = 0$ then $|Y| > 2^{2^{(n+a+1)}} \geq \log(b) \geq \log(\min(Y))$. □

Thus $f(i) = \log(i)$ yields a lower bound for the phase transition. We consider this result with regard to the function ISP_f, defined in the introduction (Definition 3.3). Theorem 3.4 states that $\mathrm{ISP}_{\log}$ is a provably total function of $\mathrm{I}\Sigma_1$. Scrutinising the assertion $y = \mathrm{ISP}_{\log}(x)$, it becomes clear $\mathrm{ISP}_{\log}$ is $\mathrm{I}\Sigma_1$-provably Σ_1. Then Theorem 1.10 implies that $\mathrm{ISP}_{\log}$ is primitive recursive. More precisely, the proof of Theorem 3.4 reveals that $\mathrm{ISP}_{\log}$ is an elementary function.

3.2.2 Upper bound

Investigating the upper bound, we will use the functions $F_{f,k}$ and F_f introduced in Definition 2.4. As done in Chapter 2 we do not mention the dependence on f explicitly to lighten the notation. Similarly, we continue using A_d and A_ω to denote members of the fast-growing hierarchy to avoid ambiguity.

We consider an arbitrary natural number d and the number-theoretic function $f(i) = \sqrt[d]{i}$. Using this f, the function F_f, shortly F, is Ackermannian, which is demonstrated in [OW09].

The next two lemmas will be used in the proof of Lemma 3.7.

3.5 Lemma. *Let $[a, b]$ be n-dense($\sqrt[d]{\ }$). Then there exists $Y \subseteq [a, b]$ such that Y is $(n-1)$-dense($\sqrt[d]{\ }$) and such that for all i with $F_1^i(a) \leq b$, we have that $Y \cap [F_1^i(a), F_1^{i+1}(a)[$ contains exactly one element.*

Proof. Define $G_0\colon [a,b] \to \mathbb{N}$ as follows. Let $x \in [a,b]$ and $i,j \in \mathbb{N}$ such that $x = F_1^i(a) + j$ and $F_1^i(a) \leq F_1^i(a) + j < F_1^{i+1}(a)$. Then G_0 is defined by

$$G_0(x) = \lfloor \sqrt[d]{F_1^i(a)} \rfloor - j.$$

In order to be accurate, we will write down the floor symbols of the parameter function in the calculation below. Notice that

$$\begin{aligned} F_1^i(a) + j &< F_1^{i+1}(a) \\ &= F_1(F_1^i(a)) \\ &= F_0^{\lfloor \sqrt[d]{F_1^i(a)} \rfloor}(F_1^i(a)) \\ &= F_1^i(a) + \lfloor \sqrt[d]{F_1^i(a)} \rfloor, \end{aligned}$$

which implies $j < \lfloor \sqrt[d]{F_1^i(a)} \rfloor$, and so $G_0(x) > 0$, for every $x \in [a,b]$. From now on, we leave out the floor symbols again. Then $G_0(x) \leq \sqrt[d]{x}$ for every $x \in [a,b]$. Since $[a,b]$ is n-dense($\sqrt[d]{\ }$), there exists $Y \subseteq [a,b]$ which is $(n-1)$-dense($\sqrt[d]{\ }$) and on which G_0 is weakly increasing. On every interval $[F_1^i(a), F_1^{i+1}(a)[\cap [a,b]$ the function G_0 is strictly decreasing. Hence $Y \cap [F_1^i(a), F_1^{i+1}(a)[$ contains at most one element.

In case there are intervals which contain no element of Y, we add to Y an element of each such interval as follows. Let i be the smallest index such that

$$Y \cap [F_1^i(a), F_1^{i+1}(a)[= \emptyset.$$

We distinguish three different cases depending on the position of $F_1^i(a)$. If $i = 0$, then add $F_1^1(a) - 1$ to Y and consider the next interval without an element of Y. In case $0 < i$ and $F_1^{i+1}(a) \leq b$, search for the element in $[F_1^i(a), F_1^{i+1}(a)[$ which has the same value under G_0 as $Y \cap [F_1^{i-1}(a), F_1^i(a)[$ has. A quick look at G_0 shows this element exists. Now add that element to Y. Continue in this way until all intervals $[F_1^i(a), F_1^{i+1}(a)[$ with $0 < i$ and $F_1^{i+1}(a) \leq b$ contain exactly one element of the enlarged Y. Finally, if $0 < i$ and $F_1^i(a) \leq b < F_1^{i+1}(a)$, then add $F_1^i(a)$ to Y. The way in which we extended Y results in G_0 being still weakly increasing on Y. Since supersets of $(n-1)$-dense($\sqrt[d]{\ }$) sets are again $(n-1)$-dense($\sqrt[d]{\ }$), we are done. $\square$

The next lemma, as well as the idea behind its proof, is similar to the previous one.

3.6 Lemma. *Let n and k be natural numbers with $n > k > 0$. If $Y \subseteq [a, b]$ is $(n-k)$-dense($\sqrt[d]{\ }$) and $Y \cap [F_k^i(a), F_k^{i+1}(a)[$ contains exactly one element for all i with $F_k^i(a) \leq b$. Then there exists a nonempty $Z \subseteq Y$ such that Z is $(n-k-1)$-dense($\sqrt[d]{\ }$) and such that for all i with $F_{k+1}^i(a) \leq b$, we have that $Z \cap [F_{k+1}^i(a), F_{k+1}^{i+1}(a)[$ is a singleton.*

Proof. Define $G_k \colon Y \to \mathbb{N}$ by $G_k(y_j^i) = \lfloor \sqrt[d]{F_{k+1}^i(a)} \rfloor - j$ for $y_j^i \in Y$ such that

$$y_j^i \in [F_{k+1}^i(a), F_{k+1}^{i+1}(a)[$$

and

$$y_j^i \in [F_k^j(F_{k+1}^i(a)), F_k^{j+1}(F_{k+1}^i(a))[.$$

As before we will write down the floor symbols of the parameter function in the calculation below to be accurate. And once again, we will leave them out afterwards. Remark that

$$\begin{aligned} F_k^j(F_{k+1}^i(a)) &< F_{k+1}^{i+1}(a) \\ &= F_{k+1}(F_{k+1}^i(a)) \\ &= F_k^{\lfloor \sqrt[d]{F_{k+1}^i(a)} \rfloor}(F_{k+1}^i(a)), \end{aligned}$$

which yields $j < \lfloor \sqrt[d]{F_{k+1}^i(a)} \rfloor$, and so $G_k(y) > 0$, for every $y \in Y$. Furthermore, it is obvious that $G_k(y) \leq \sqrt[d]{y}$, for every $y \in Y$. Since Y is $(n-k)$-dense($\sqrt[d]{\ }$), there exists a nonempty $Z \subseteq Y$ which is $(n-k-1)$-dense($\sqrt[d]{\ }$) and on which G_k is weakly increasing. Note that on every interval $Y \cap [F_{k+1}^i(a), F_{k+1}^{i+1}(a)[$ the function G_k is strictly decreasing. Hence $Z \cap [F_{k+1}^i(a), F_{k+1}^{i+1}(a)[$ contains at most one element.

In case there are intervals which contain no element of Z, we add to Z an element of each such interval as follows. Let i be the smallest index such that

$$Z \cap [F_{k+1}^i(a), F_{k+1}^{i+1}(a)[= \emptyset.$$

We distinguish three different cases depending on the position of $F^i_{k+1}(a)$. If $i = 0$, then add the maximal element of $Y \cap [a, F^1_{k+1}(a)[$ to Z and consider the next interval without an element of Z. In case $0 < i$ and $F^{i+1}_{k+1}(a) \leq b$, search for the element in $Y \cap [F^i_{k+1}(a), F^{i+1}_{k+1}(a)[$ which has the same value under G_k as $Z \cap [F^{i-1}_{k+1}(a), F^i_{k+1}(a)[$ has. A quick look at G_k shows this element exists. Now add that element to Z. Continue in this way until all intervals $[F^i_{k+1}(a), F^{i+1}_{k+1}(a)[$ with $0 < i$ and $F^{i+1}_{k+1}(a) \leq b$ contain exactly one element of the enlarged Z. Finally, if $0 < i$ and $F^i_{k+1}(a) \leq b < F^{i+1}_{k+1}(a)$, then add the minimal element of $Y \cap [F^i_{k+1}(a), F^{i+1}_{k+1}(a)[$ to Z. As a result of the speficic way in which we extended Z, the function G_k is still weakly increasing on Z. Since supersets of $(n-k-1)$-dense($\sqrt[d]{\ }$) sets are $(n-k-1)$-dense($\sqrt[d]{\ }$), we are done. □

3.7 Lemma. *Let a, b and n be natural numbers, with $a \geq 1$. If $[a, b]$ is n-dense($\sqrt[d]{\ }$), then $b \geq F_n(a)$.*

Proof. Assume $[a, b]$ is n-dense($\sqrt[d]{\ }$). Then applying Lemma 3.5 and Lemma 3.6 results in the existence of a nonempty $Z \subseteq [a, b]$ which is 0-dense($\sqrt[d]{\ }$) and such that $[F^i_n(a), F^{i+1}_n(a)[\cap Z$ is a singleton for all i with $F^i_n(a) \leq b$. Then

$$|Z| > \sqrt[d]{\min Z} \geq \sqrt[d]{a} \geq 1,$$

as $a \geq 1$ and Z is 0-dense($\sqrt[d]{\ }$). Hence Z contains at least two elements and the second one is greater than $F_n(a)$, so $b \geq F_n(a)$. □

We are now ready to state the unprovability result, which leads to the upper bound of the phase transition.

3.8 Theorem. *Let $d \in \mathbb{N}$ and $f(i) = \sqrt[d]{i}$. Then*

$$\mathrm{I}\Sigma_1 \nvdash (\forall n)(\forall a)(\exists b)([a, b] \textit{ is } n\textit{-dense}(f)).$$

Proof. As seen in the introductory chapter (Theorem 1.10), the provably recursive functions of $\mathrm{I}\Sigma_1$ are exactly the primitive recursive functions. Assume by contradiction that

$$\mathrm{I}\Sigma_1 \vdash (\forall n)(\forall a)(\exists b)([a, b] \text{ is } n\text{-dense}(f)). \tag{3.1}$$

Lemma 3.7, with $n = a \geq 1$, implies $\mathrm{ISP}_{\sqrt[d]{}}(n) \geq F(n)$, for every $n \in \mathbb{N}$. In combination with (3.1) this yields that F is a provably recursive function of $\mathrm{I}\Sigma_1$, and thus primitive recursive. This contradicts the fact that F has Ackermannian growth rate. $\square$

Combining Theorem 3.4 and Theorem 3.8, we can sharpen the range of the threshold region in comparison with our first estimation. Indeed, instead of claiming that the phase transition occurs for a function f between constant functions and the identity function, it is now justified to say that the threshold region for f is between log and $\sqrt[d]{}$. In the next section we will sharpen this region.

3.3 Improved bounds

As announced in the introduction we will improve the bounds for the threshold function we obtained in Section 3.2, and present a more elegant result.

3.3.1 Lower bound

We have a look at the notion of ISP-density(f) in the case of $f(i) = i^{\frac{1}{A_d^{-1}(i)}}$, with d a natural number.

3.9 Theorem. *Let $d \in \mathbb{N}$ and $f(i) = i^{\frac{1}{A_d^{-1}(i)}}$. Then*

$$\mathrm{I}\Sigma_1 \vdash (\forall n)(\forall a)(\exists b)([a,b] \textit{ is } n\textit{-dense}(f)).$$

Proof. Let a and n be given and note that if $[a,b]$ is n-dense(f), then so is $[0, b-a]$. Hence without loss of generality, we can assume $a > 0$. Put $b = 2^{A_d(a \cdot 2^n) \cdot 2^n}$. We claim that any $Y \subseteq [a,b]$ with $|Y| > 2^{A_d(a \cdot 2^n) \cdot 2^k}$ is k-dense(f). The proof goes by induction on k. Assume the assertion holds for $k-1$, with $k > 0$, and let $Y \subseteq [a,b]$ with $|Y| > 2^{A_d(a \cdot 2^n) \cdot 2^k}$. Assume that $G \colon Y \to \mathbb{N}$, such that $G(y) \leq f(y)$ for every $y \in Y$. Then by the Erdős-Szekeres Theorem there is a set $Z \subseteq Y$ such that $G \restriction Z$ is weakly increasing or strictly decreasing and $|Z| > 2^{A_d(a\cdot 2^n)\cdot 2^{(k-1)}}$. Since $2^{A_d(a\cdot 2^n)\cdot 2^n} \geq A_d(2^n)$, the following holds:

$$
\begin{aligned}
G(\min(Z)) &\leq f(\min(Z)) \\
&\leq f(b) \\
&= \left(2^{A_d(a\cdot 2^n)\cdot 2^n}\right)^{\frac{1}{A_d^{-1}(2^{A_d(a\cdot 2^n)\cdot 2^n})}} \\
&\leq \left(2^{A_d(a\cdot 2^n)\cdot 2^n}\right)^{\frac{1}{2^n}} \\
&= 2^{A_d(a\cdot 2^n)} \\
&\leq 2^{A_d(a\cdot 2^n)\cdot 2^{(k-1)}} \\
&< |Z|.
\end{aligned}
$$

So if G is strictly decreasing on Z, then $G(\max(Z)) < 0$, a contradiction. Thus F is weakly increasing on Z. Since $|Z| > 2^{A_d(a\cdot 2^n)\cdot 2^{(k-1)}}$ the induction hypothesis yields that Z is $(k-1)$-dense(f), which implies that Y is k-dense(f). If $k = 0$, then $|Y| > 2^{A_d(a\cdot 2^n)} \geq f(b) \geq f(\min(Y))$. Hence, Y is 0-dense(f), which completes the proof by induction. □

The previous theorem shows that $f(i) = i^{\frac{1}{A_d^{-1}(i)}}$ yields a lower bound, for every fixed $d \in \mathbb{N}$. If we use the Ackermann function A_ω instead of a dth branch A_d, we will get an upper bound, as demonstrated in the second part of this section.

3.3.2 Upper bound

We will use the number theoretic function $f(i) = i^{\frac{1}{A_\omega^{-1}(i)}}$. Once again the hierarchy of functions $(F_n)_{n<\omega}$ as introduced in Definition 2.4 turns out very useful.

One can prove the following three lemmata in the same way as done in Section 3.2 (Lemma 3.5 – Lemma 3.7). It suffices to replace the dth root function $\sqrt[d]{i}$ by $f(i)$.

3.10 Lemma. *Let $[a, b]$ be n-dense(f). Then there exists $Y \subseteq [a, b]$ such that Y is $(n-1)$-dense(f) and such that for all i with $F_1^i(a) \leq b$, we have that $Y \cap [F_1^i(a), F_1^{i+1}(a)[$ contains exactly one element.*

3.11 Lemma. *Let n and k be natural numbers with $n > k > 0$. If $Y \subseteq [a, b]$ is $(n-k)$-dense(f) and $Y \cap [F_k^i(a), F_k^{i+1}(a)[$ contains exactly one element for all i with $F_k^i(a) \leq b$. Then there exists a nonempty $Z \subseteq Y$ such that Z is $(n-k-1)$-dense(f) and such that for all i with $F_{k+1}^i(a) \leq b$, we have that $Z \cap [F_{k+1}^i(a), F_{k+1}^{i+1}(a)[$ is a singleton.*

3.12 Lemma. *Let a, b and n be natural numbers, with $a \geq 1$. If $[a, b]$ is n-dense(f), then $b \geq F_n(a)$.*

Finally, to prove that $f(i) = i^{\frac{1}{A_\omega^{-1}(i)}}$ yields an upper bound for the density statement, we use the function ISP_f, introduced in section 3.1. Due to Theorem 1 in [OW09], F has Ackermannian growth rate. Lemma 3.12 yields for every $n \geq 1$,

$$\mathrm{ISP}_f(n) \geq F_n(n) = F(n),$$

so ISP_f is Ackermannian. Since the provably recursive functions of $\mathrm{I}\Sigma_1$ are exactly the primitive recursive functions, we immediately obtain the following result.

3.13 Theorem. *If $f(i) = i^{\frac{1}{A_\omega^{-1}(i)}}$, then*

$$\mathrm{I}\Sigma_1 \nvdash (\forall n)(\forall a)(\exists b)([a, b] \text{ is } n\text{-dense}(f)).$$

Combining Theorem 3.9 and Theorem 3.13 we have obtained a threshold for a phase transition related to the ISP principle, illustrated by Figure 3.1. It is clear that, by regarding A_ω as the limit of the dth branches A_d for d going to infinity, this result is sharp as well as elegant.

We conclude this chapter with some remarks about possible future research. In a similar way as we have done for ISP, we could connect ES and CAC with their finitary versions via an appropriate density notion. Moreover, one could consider so-called ES-density and CAC-density depending on a parameter function $f\colon \mathbb{N} \to \mathbb{N}$ and carry out a similar study of unprovability and phase transitions. We conjecture that one would obtain results which are similar to the ones above. The following problem then arises. Suppose we combine $\mathrm{I}\Sigma_1$ with CAC-density in an appropriate way, do we obtain the same provably recursive functions as given by $\mathrm{RCA}_0 + \mathrm{CAC}$? The idea is again to approximate Ramsey for pairs here and CAC would be a first step. The case of EM-density has already been studied in [BW05], which revealed a certain weakness of EM.

Chapter 4

INFINITE RAMSEY THEOREM

4.1 Introduction

4.1.1 Historical background

In 1985 McAloon writes in [McA85]:

> *It would be interesting to develop proofs of these results with the "direct" method of α-large sets of Ketonen-Solovay.*

The results mentioned in this quote concern concrete examples of incompleteness using finite versions of the Ramsey Theorem. In that paper he gives a first-order axiomatisation of the first-order consequences of $\mathrm{ACA}_0 + \mathrm{RT}$, where RT stands for the infinite version of Ramsey's theorem, i.e. RT equals $\forall n \, \forall k \, \mathrm{RT}^n_k$, using the notation of Subsection 1.1.3.

Ketonen and Solovay used α-largeness in their paper on rapidly growing Ramsey functions ([KS81]), in which they extended the famous result of Paris and Harrington. They established sharp upper and lower bounds on the Ramsey function by purely combinatorial means. In this context it is appropriate to mention [Kun95] in which Kenneth Kunen uses the Boyer-Moore Prover, Nqthm, to verify the Paris-Harrington version of the Ramsey Theorem. Teresa Bigorajska and Kotlarski generalised the ideas of Ketonen and Solovay to ordinals below ε_0 and obtained several results on partitioning α-large sets (see [BK99, BK02, BK06]). Here, we will generalise some of their results in order to allow ordinals up to ε_ω.

Since Paris introduced them in the late seventies densities proved to be of interest for studying independence results (see e.g. [Par78, FMS82, BW05] and Chapter 2 and Chapter 3). We will define such a density as a finitary version of the Ramsey Theorem. This time, we consider the full infinite version with no restrictions on the dimensions or colours (RT). It turns out that our miniaturisation gives rise to transfinite induction up to ε_ω. Notice that McAloon proved that $\mathrm{ACA}_0 + \mathrm{RT}$ can be axiomatised by $\mathrm{ACA}_0 + \forall X \forall n$ ($\mathrm{TJ}(n, X)$, the nth Turing jump of X, exists). Analysing the latter theory one obtains that the proof-theoretic ordinal of $\mathrm{ACA}_0 + \mathrm{RT}$ is exactly ε_ω. For a more recent approach we refer to the PhD dissertation of Bahareh Afshari ([Afs09]).

We would like to remark that an alternative way to miniaturise RT has been given by Zygmunt Ratajczyk in [Rat93]. Ratajczyk's approach is based on relating iterated Ramseyan principles to iterated reflection principles.

4.1.2 Definitions and preliminaries

In this subsection we start by defining the *pseudonorm*, and proving some useful properties about it. Afterwards we estimate the number of ordinals having a specific bound on their size and pseudonorm, and we finish the preliminaries by giving some lower bounds for Hardy functions.

The pseudonorm of an ordinal α is introduced exactly as done by Kotlarski in [Kot].

4.1 Definition. *We define the* pseudonorm $\mathrm{psn}\colon (\leq \varepsilon_\omega) \to \mathbb{N}$ *by the following conditions:*

1. $\mathrm{psn}(n) = n$ *for* $n < \omega$*;*
2. $\mathrm{psn}(\alpha) = \max\{b_0, \ldots, b_s, a_0, \ldots, a_s\}$ *where* α *is written in its Cantor normal form (1.1) and* $b_j = 1 + \mathrm{psn}(\alpha_j)$ *if* α_j *is not an epsilon number and* $b_j = \mathrm{psn}(\alpha_j)$ *otherwise.*

3. *If α is an epsilon number $\varepsilon_\beta \leq \varepsilon_\omega$, then* $\mathrm{psn}(\alpha) = 1 + \max\{1, \mathrm{psn}(\beta)\}$.

Note that this pseudonorm differs from the one in [BK06] and [KPW07]. In those papers the pseudonorm of α is defined as the greatest natural number which occurs in its Cantor normal form. That definition is appropriate for ordinals below ε_0, but will lead to difficulties in our case. Therefore we use a slightly different notion. Roughly, going through the Cantor normal form we add 1 to the pseudonorm whenever we jump to an exponent. A close look at the definition, in particular the second item, will make this clear.

The following properties will be useful. Proofs of the next two lemmas can be found in [Kot] (Lemma 4.1 and Lemma 4.5).

4.2 Lemma. *Let $\lambda \leq \varepsilon_\omega$ be a limit ordinal, $\alpha < \lambda$ and $m \in \mathbb{N}$. If* $\mathrm{psn}(\alpha) < m$, *then $\lambda[m] \Rightarrow_m \alpha$.*

4.3 Lemma. *For every limit $\lambda \leq \varepsilon_\omega$,* $\mathrm{psn}(\lambda[m]) \geq m$.

The pseudonorm of some ordinals having a specific shape is simple. Moreover, they turn out to be handy to know later on, which is the reason why we have a look at them in the next two lemma's. Recall the definition of ω_k in Subsection 1.2.2.

4.4 Lemma. *Let $k \geq 1$, then* $\mathrm{psn}(\omega_k) = k + 1$.

Proof. Straightforwardly by induction on k. □

4.5 Lemma. *If $k \geq 2$, then*

1. $\mathrm{psn}(\omega_k(\varepsilon_0 + 1)) = k + 2$;

2. $\mathrm{psn}(\omega_k(\varepsilon_m + 1)) = m + k + 1$, *for $m \geq 1$.*

Proof. 1. By induction on k. If $k = 0$, then

$$\mathrm{psn}(\omega_0(\varepsilon_0 + 1)) = \mathrm{psn}((\varepsilon_0 + 1)) = 2.$$

Assume the statement for k. Then

$$\begin{aligned}\operatorname{psn}(\omega_{k+1}(\varepsilon_0+1)) &= \operatorname{psn}(\omega^{\omega_k(\varepsilon_0+1)}) \\ &= 1+\operatorname{psn}(\omega_k(\varepsilon_0+1)) \\ &= k+3.\end{aligned}$$

2. By induction on k. If $k=0$, then

$$\operatorname{psn}(\omega_0(\varepsilon_m+1)) = \operatorname{psn}((\varepsilon_m+1)) = m+1,$$

since $m \geq 1$. Assume the statement for k. Then

$$\begin{aligned}\operatorname{psn}(\omega_{k+1}(\varepsilon_m+1)) &= \operatorname{psn}(\omega^{\omega_k(\varepsilon_m+1)}) \\ &= 1+\operatorname{psn}(\omega_k(\varepsilon_m+1)) \\ &= m+k+2.\end{aligned}$$

□

Given an ordinal α and a natural number n, we would like to be able to estimate the number of ordinals below α having a pseudonorm smaller than n. Therefore, we introduce the following notations:

$$N_k(a) = |\{\alpha < \omega_k \,:\, \operatorname{psn}(\alpha) \leq a\}|$$

and

$$M_{k,m}(a) = |\{\alpha < \omega_k(\varepsilon_m+1) \,:\, \operatorname{psn}(\alpha) \leq a\}|,$$

and study those sets. We also need the function $\operatorname{tow}_n(\alpha)$ which is defined for ordinals α with $0 < \alpha < \varepsilon_\omega$, by $\operatorname{tow}_0(\alpha) = 1$ and $\operatorname{tow}_{k+1}(\alpha) = \alpha^{\operatorname{tow}_k(\alpha)}$.

We will give estimations for $N_k(a)$ and $M_{k,m}(a)$ for a which are at least 1, as $\operatorname{psn}(\alpha) \leq 0$ implies the trivial case $\alpha = 0$.

4.6 Lemma. *Let $a, k \in \mathbb{N}$, with $1 \leq a$ and $1 \leq k$. Then*

$$N_k(a) < \operatorname{tow}_k(a+2).$$

Proof. By induction on k. If k equals 1, then

$$\begin{aligned} N_1(a) &= |\{\alpha < \omega \,:\, \mathrm{psn}(\alpha) \leq a\}| \\ &= |\{0, \ldots a\}| \\ &= a + 1 \\ &< a + 2. \end{aligned}$$

Assume the inequality for k, we show it for $k + 1$.

$$\begin{aligned} N_{k+1}(a) &= |\{\alpha < \omega_{k+1} \,:\, \mathrm{psn}(\alpha) \leq a\}| \\ &= |\{\alpha < \omega^{\omega_k} \,:\, \mathrm{psn}(\alpha) \leq a\}| \\ &\leq (a+1)^{N_k(a-1)} \\ &< (a+1)^{N_k(a)} \\ &< (a+2)^{\mathrm{tow}_k(a+2)} \\ &= \mathrm{tow}_{k+1}(a+2). \end{aligned}$$

Let us explain how we obtained the first inequality. Let $\{\alpha < \omega_k \,:\, \mathrm{psn}(\alpha) \leq a - 1\} = \{\alpha_0, \ldots, \alpha_s\}$ such that the elements are presented in decreasing order. Every ordinal $\alpha < \omega^{\omega_k}$ with $\mathrm{psn}(\alpha) \leq a$ can be written in a unique way as follows:

$$\alpha = \omega^{\alpha_0} \cdot a_0 + \ldots + \omega^{\alpha_s} \cdot a_s,$$

where some a_i, $i \leq s$, could be equal to zero. This form is indeed very similar to the Cantor normal form (1.1), except for the possibility of having some coefficients equal to zero. It is easy to see that each such α can be identified with a sequence of coefficients which has length $N_k(a-1)$. Now it suffices to notice that the coefficients must be strictly smaller than $a+1$. $\square$

4.7 Lemma. *Let $a, k, m \in \mathbb{N}$, with $1 \leq a$. Then*

$$M_{k,m}(a) < \mathrm{tow}_{(m+1)2a+k+1}(a+2).$$

Proof. Take any $a \in \mathbb{N}$, with $1 \leq a$. The proof proceeds by main induction on m and subsidiary induction on k.

1. $m = 0$.

   (a) $k = 0$. Using the fact that $\mathrm{psn}(\omega_n) = n+1$ (Lemma 4.4) and the estimation in Lemma 4.6, we get

$$\begin{aligned} M_{0,0}(a) &= |\{\alpha < \omega_0(\varepsilon_0 + 1) : \mathrm{psn}(\alpha) \le a\}| \\ &= |\{\alpha < \varepsilon_0 + 1 : \mathrm{psn}(\alpha) \le a\}| \\ &\le 1 + |\{\alpha < \omega_a : \mathrm{psn}(\alpha) \le a\}| \\ &< 1 + \mathrm{tow}_a(a+2) \\ &< \mathrm{tow}_{2a+1}(a+2). \end{aligned}$$

   (b) $k \to k+1$. Assume the statement for k. Using the subsidiary induction hypothesis, we obtain

$$\begin{aligned} M_{k+1,0}(a) &= |\{\alpha < \omega_{k+1}(\varepsilon_0 + 1) : \mathrm{psn}(\alpha) \le a\}| \\ &= |\{\alpha < \omega^{\omega_k(\varepsilon_0+1)} : \mathrm{psn}(\alpha) \le a\}| \\ &\le (a+1)^{M_{k,0}(a-1)} \\ &< (a+2)^{M_{k,0}(a)} \\ &< (a+2)^{\mathrm{tow}_{2a+k+1}(a+2)} \\ &< \mathrm{tow}_{2a+k+2}(a+2). \end{aligned}$$

   The first inequality is obtain in a similar way as in Lemma 4.6.

2. $m \to m+1$. Assume the statement for m.

   (a) $k = 0$. Using Lemma 4.5 and the main induction hypothesis, we obtain

$$\begin{aligned} M_{0,m+1}(a) &= |\{\alpha < \omega_0(\varepsilon_{m+1} + 1) : \mathrm{psn}(\alpha) \le a\}| \\ &= |\{\alpha < \varepsilon_{m+1} + 1 : \mathrm{psn}(\alpha) \le a\}| \\ &\le 1 + |\{\alpha < \omega_a(\varepsilon_m + 1) : \mathrm{psn}(\alpha) \le a\}| \\ &< 1 + \mathrm{tow}_{(m+1)2a+a+1}(a+2) \\ &< \mathrm{tow}_{(m+2)2a+1}(a+2). \end{aligned}$$

(b) $k \to k+1$. Assume the statement for k. Using the subsidiary induction hypothesis, we obtain

$$\begin{aligned}
M_{k+1,m+1}(a) &= |\{\alpha < \omega_{k+1}(\varepsilon_{m+1}+1) \,:\, \mathrm{psn}(\alpha) \le a\}| \\
&= |\{\alpha < \omega^{\omega_k(\varepsilon_{m+1}+1)} \,:\, \mathrm{psn}(\alpha) \le a\}| \\
&\le (a+1)^{M_{k,m+1}(a-1)} \\
&< (a+2)^{M_{k,m+1}(a)} \\
&< (a+2)^{\mathrm{tow}_{(m+2)2a+k+1}(a+2)} \\
&= \mathrm{tow}_{(m+2)2a+k+2}(a+2).
\end{aligned}$$

Also in this case we obtained the first inequality by proceeding as explained in Lemma 4.6.

□

The following lemmas are about lower bounds for the values of the functions h_α of the Hardy hierarchy $(h_\alpha)_{\alpha \le \varepsilon_\omega}$ (see Definition 1.13).

4.8 Lemma. *If $m \ge 1$ and $a \ge 1$, then we have for all $x > 0$,*

$$h_{\omega^2 \cdot 2am}(x) \ge \underbrace{\mathrm{tow}_a(\ldots(\mathrm{tow}_a}_{m \text{ times}}(x+1))\ldots).$$

Proof. Fix $a \ge 1$. By induction on $m \ge 1$.

If $m = 1$, then $h_{\omega^2 \cdot 2a}(x) \ge \mathrm{tow}_a(x+1)$, by Lemma 4.5 in [BK06].

Assume the statement is proven for m, we prove it for $m+1$. The case $m = 1$ and the induction hypothesis imply

$$\begin{aligned}
h_{\omega^2 \cdot 2a(m+1)}(x) &\ge h_{\omega^2 \cdot 2am}(h_{\omega^2 \cdot 2a}(x)) \\
&\ge h_{\omega^2 \cdot 2am}(\mathrm{tow}_a(x+1)) \\
&\ge \underbrace{\mathrm{tow}_a(\ldots(\mathrm{tow}_a}_{m \text{ times}}(\mathrm{tow}_a(x+1)+1))\ldots) \\
&\ge \underbrace{\mathrm{tow}_a(\ldots(\mathrm{tow}_a}_{m+1 \text{ times}}(x+1))\ldots),
\end{aligned}$$

which completes the proof. □

We will need the following small lemma to prove Lemma 4.10.

4.9 Lemma. *Let k, m and n be natural numbers with $k \geq 1$ and $n \geq 2$. Then $\omega_k(\varepsilon_m + 1) \Rightarrow_n \varepsilon_m \cdot 2$.*

Proof. Because of Lemma 1.7, it suffices to prove the statement for $n = 2$. The proof goes by induction on k. If $k = 1$, then

$$\omega^{\varepsilon_m+1}[2] = \omega^{\varepsilon_m} \cdot 2 = \varepsilon_m \cdot 2,$$

so $\omega_1(\varepsilon_m + 1) \Rightarrow_2 \varepsilon_m \cdot 2$. Assume the statement for k. We will prove it for $k + 1$. First notice that

$$\omega_{k+1}(\varepsilon_m + 1)[2] = \omega_k(\omega^{\varepsilon_m+1}[2]) = \omega_k(\omega^{\varepsilon_m} \cdot 2) = \omega_k(\varepsilon_m \cdot 2).$$

Also, $\varepsilon_m \Rightarrow_2 1$ by Lemma 1.9, and thus $\varepsilon_m \cdot 2 \Rightarrow_2 \varepsilon_m + 1$ because of additivity of the fundamental system. None of the elements of the sequence witnessing this last relation is an epsilon number, so we can apply k times Lemma 3.6 of [Kot] to get $\omega_k(\varepsilon_m \cdot 2) \Rightarrow_2 \omega_k(\varepsilon_m + 1)$. The induction hypothesis yields $\omega_k(\varepsilon_m + 1) \Rightarrow_2 \varepsilon_m \cdot 2$. Transitivity of the $\Rightarrow_2$ relation now implies $\omega_{k+1}(\varepsilon_m + 1) \Rightarrow_2 \varepsilon_m \cdot 2$, which concludes the proof. □

4.10 Lemma. *For all $m \geq 0$ and $a > 4$, we have*

$$h_{\varepsilon_m}(a) > h_{\omega^2 \cdot 2a(m+3)}(2a).$$

Proof. By induction on m.

If $m = 0$, then

$$\begin{aligned} h_{\varepsilon_0}(a) &= h_{\omega_a}(a) \\ &\geq h_{\omega^3 + \omega \cdot 3}(a) \\ &= h_{\omega^2 \cdot 8a}(8a) \\ &= h_{\omega^2 \cdot 2a}(h_{\omega^2 \cdot 6a}(8a)) \\ &> h_{\omega^2 \cdot 6a}(8a) \\ &\geq h_{\omega^2 \cdot 6a}(2a), \end{aligned}$$

where the last inequality holds because h_α is an increasing function. Due to Lemma 4.2, we have $\omega_a \Rightarrow_a \omega^3 + \omega \cdot 3$, since $\mathrm{psn}(\omega^3 + \omega \cdot 3) = 4 < a$ and $\omega^3 + \omega \cdot 3 < \omega_a$. Then the first inequality is implied by Lemma 1.19.

Assume the statement holds for m, we prove it for $m + 1$. We have

$$\begin{aligned} h_{\varepsilon_{m+1}}(a) &= h_{\omega_a(\varepsilon_m+1)}(a) \\ &\geq h_{\varepsilon_m \cdot 2}(a) \\ &= h_{\varepsilon_m}(h_{\varepsilon_m}(a)) \\ &> h_{\varepsilon_m}(h_{\omega^2 \cdot 2a(m+3)}(2a)) \\ &> h_{\omega^2 \cdot 2a(m+3)}(h_{\omega^2 \cdot 2a(m+3)}(2a)) \\ &= h_{\omega^2 \cdot 2a(m+4)}(h_{\omega^2 \cdot 2a(m+2)}(2a)) \\ &\geq h_{\omega^2 \cdot 2a(m+4)}(2a). \end{aligned}$$

The first inequality holds because of Lemma 1.19, as $\omega_a(\varepsilon_m+1) \Rightarrow_a \varepsilon_m \cdot 2$ by Lemma 4.9. The second inequality is caused by the induction hypothesis. The third inequality holds because of the induction hypothesis and the fact that h_α is increasing (Lemma 1.19). The last inequality is because h_α is an increasing function. □

4.11 Lemma. *For all $m \geq 0$ and $a > 4$, we have*

$$h_{\varepsilon_m}(a) > 2\,\underbrace{\mathrm{tow}_a(\ldots(\mathrm{tow}_a}_{m+2 \text{ times}}(2a+1))\ldots).$$

Proof. Combine Lemma 4.8 and Lemma 4.10 with

$$\underbrace{\mathrm{tow}_a(\ldots(\mathrm{tow}_a}_{m+3 \text{ times}}(2a+1))\ldots) > 2\,\underbrace{\mathrm{tow}_a(\ldots(\mathrm{tow}_a}_{m+2 \text{ times}}(2a+1))\ldots).$$

□

4.12 Lemma. *Let n, m and x be natural numbers such that $n \geq 2$, and $x \geq \max\{2, m\}$. Then*

1. $h_{\omega^n \cdot (m+1)}(x) \geq n(m+2)x$;

2. $h_{\omega^\omega \cdot (m+1)}(x) \geq (m+1)^2 x^2$.

Proof. 1. By main induction on n and subsidiary induction on m.

(a) $n = 2$.
$m = 0$. Then

$$\begin{aligned} h_{\omega^2}(x) &\geq h_{\omega \cdot x}(x) \\ &\geq 2^x x \\ &\geq 4x \\ &\geq 2(0+2)x. \end{aligned}$$

$m \to m+1$. Then

$$\begin{aligned} h_{\omega^2 \cdot (m+2)}(x) &= h_{\omega^2 \cdot (m+1)}(h_{\omega^2}(x)) \\ &\geq h_{\omega^2 \cdot (m+1)}(2^x x) \\ &\geq 2(m+2)2^x x \\ &\geq 2(m+3)x. \end{aligned}$$

(b) $n \to n+1$.
$m = 0$. Then, using the main induction hypothesis twice, we obtain

$$\begin{aligned} h_{\omega^{n+1}}(x) &= h_{\omega^n \cdot x}(x) \\ &\geq h_{\omega^n \cdot (x-1)}(h_{\omega^n}(x)) \\ &\geq h_{\omega^n \cdot (x-1)}(2nx) \\ &\geq h_{\omega^n}(2nx) \\ &\geq 2n \cdot 2nx \\ &\geq 4n^2 x \\ &\geq (n+1)(0+2)x. \end{aligned}$$

$m \to m+1$. Then, using the subsidiary induction hypothesis and one of the equalities above,

$$\begin{aligned} h_{\omega^{n+1}\cdot(m+2)}(x) &= h_{\omega^{n+1}\cdot(m+1)}(h_{\omega^{n+1}}(x)) \\ &\geq h_{\omega^{n+1}\cdot(m+1)}(2^x x) \\ &\geq (n+1)(m+2)2^x x \\ &\geq (n+1)(m+3)x. \end{aligned}$$

2. By induction on m.
   $m = 0$. Then

$$\begin{aligned} h_{\omega^\omega}(x) &\geq h_{\omega^x}(x) \\ &\geq h_{\omega^2}(x) \\ &\geq 2^x x \\ &\geq x^2. \end{aligned}$$

$m \to m+1$. Then, using the induction hypothesis,

$$\begin{aligned} h_{\omega^\omega\cdot(m+2)}(x) &= h_{\omega^\omega\cdot(m+1)}(h_{\omega^\omega}(x)) \\ &\geq h_{\omega^\omega\cdot(m+1)}(2^x x) \\ &\geq (m+1)^2(2^x x)^2 \\ &\geq (m+1)^2 2^{2x} x^2 \\ &\geq (m+2)^2 x^2, \end{aligned}$$

which completes the proof.

□

4.2 The Estimation Lemma

As mentioned in the introduction, we will generalise results of Bigorajska and Kotlarski about partitioning α-large sets. Many of the results below were already obtained in [BK06] for ordinals below ε_0. Some of the lemmas and proofs can be generalised to ordinals below ε_ω in a rather straightforward manner. However, other results need much more reworking, partly as a result of the slightly different pseudonorm.

Crucial for the study will be generalising the *Estimation Lemma*, which we will do in this section. Let us start by defining the *natural sum*. The idea remains the same as for ordinals below ε_0.

4.13 Definition. *If α and β are ordinals below ε_ω, then we define their* natural sum *in the following manner. We write both α and β in their Cantor normal forms, and permute items of both of these expansions so that we obtain a nonincreasing sequence of exponents. Then we write this sequence as the Cantor normal form of some ordinal which we denote $\alpha \oplus \beta$ (to be more precise we join items which have the same exponents by removing the coefficients behind parentheses).*

The natural sum will always be greater than or equal to the usual sum.

4.14 Lemma. *If $h_{\beta\oplus\alpha}(a)\downarrow$, then $h_\beta \circ h_\alpha(a)\downarrow$ and $h_\beta \circ h_\alpha(a) \leq h_{\beta\oplus\alpha}(a)$. In other words, if a set A is $\beta \oplus \alpha$-large, then there exists $u \in A$, such that $\{a \in A : a \leq u\}$ is α-large and $\{a \in A : u \leq a\}$ is β-large.*

Proof. The proof given in [BK06] (Lemma 3.3) generalises to ordinals below ε_ω (use Lemma 1.17 and Lemma 1.19). □

Given a natural number a and an ordinal α, let us determine the greatest ordinal below α whose pseudonorm is less than or equal to a. We define the symbol $\mathrm{GO}(a, \alpha)$ for $a > 0$ and $\alpha > 0$ by induction on α. We let $\mathrm{GO}(a, 1) = 0$, $\mathrm{GO}(a, \omega) = a$, $\mathrm{GO}(a, \varepsilon_0) = \mathrm{GO}(a, \omega_a)$ and $\mathrm{GO}(a, \varepsilon_{m+1}) = \mathrm{GO}(a, \omega_a(\varepsilon_m + 1))$, for $m \geq 0$. Other cases are as follows. The successor step becomes:

$$\mathrm{GO}(a, \alpha+1) = \begin{cases} \alpha & \text{if} \quad \mathrm{psn}(\alpha) \leq a, \\ \mathrm{GO}(a, \alpha) & \text{if} \quad \mathrm{psn}(\alpha) > a. \end{cases}$$

Before giving the limit step we put for ν not an epsilon number:

$$\mathrm{GO}(a, \omega^\nu) = \omega^{\mathrm{GO}(a-1,\nu)} \cdot a + \mathrm{GO}(a, \omega^{\mathrm{GO}(a-1,\nu)}).$$

Finally, if $\alpha = \xi + \omega^\nu$ in short Cantor normal form, with $\xi \neq 0$, then

$$\mathrm{GO}(a,\alpha) = \begin{cases} \xi + \mathrm{GO}(a,\omega^\nu) & \text{if} \quad \mathrm{psn}(\xi) \leq a, \\ \mathrm{GO}(a,\xi) & \text{if} \quad \mathrm{psn}(\xi) > a. \end{cases}$$

We extend Lemma 3.6 of [BK06]. Recall the definition of $\ggg$ given in Subsection 1.2.2.

4.15 Lemma. *For every $a > 0$ and every $\alpha > 0$ we have: for all γ if $\gamma < \alpha$ and $\mathrm{psn}(\gamma) \leq a$, then $\gamma \leq \mathrm{GO}(a,\alpha)$.*

Proof. By induction on α. We consider only those cases which differ significantly from the ones proven in [BK06].

For $\alpha = \varepsilon_0$ and for $\alpha = \varepsilon_{m+1}$ with $m \geq 0$ the statement is obvious, having Lemma 4.4 and Lemma 4.5 in mind. As we changed the definition of the pseudonorm, we need to reconsider the case $\alpha = \omega^\nu$, with ν not an epsilon number. So, assume the statement holds for each $\beta < \alpha = \omega^\nu$. Let $\gamma < \omega^\nu$ and $\mathrm{psn}(\gamma) \leq a$. Write $\gamma = \omega^\xi \cdot g + \psi$ with $\omega^\xi \ggg \psi$. Then $\xi < \nu$, because $\gamma < \alpha$, so $\xi \leq \mathrm{GO}(a-1,\nu)$ because $1 + \mathrm{psn}(\xi) \leq \mathrm{psn}(\gamma) \leq a$, so we may apply the inductive assumption to ν. Also $g \leq a$ because $\mathrm{psn}(\gamma) \leq a$. Moreover $\mathrm{psn}(\psi) \leq \mathrm{psn}(\gamma) \leq a$ and $\psi < \omega^\xi$, so $\psi < \omega^{\mathrm{GO}(a-1,\nu)}$, so by the inductive assumption applied to this ordinal, $\psi \leq \mathrm{GO}(a, \omega^{\mathrm{GO}(a-1,\nu)})$. Combining those facts we get

$$\begin{aligned} \gamma &= \omega^\xi \cdot g + \psi \\ &\leq \omega^{\mathrm{GO}(a-1,\nu)} \cdot a + \mathrm{GO}(a, \omega^{\mathrm{GO}(a-1,\nu)}) \\ &= \mathrm{GO}(a,\omega^\nu), \end{aligned}$$

as required.

The case $\alpha = \xi + \omega^\nu$, where $\xi \gg \omega^\nu$ and $\xi \neq 0$, is dealt with as in [BK06]. □

In the next definition we encounter the new notation $(< \alpha)$ which denotes the set of all ordinals strictly below α, for any ordinal α. The set $(\leq \alpha)$ is defined likewise.

4.16 Definition. *Define $F\colon (<\varepsilon_\omega) \to (<\varepsilon_\omega)$ by the following conditions:*

1. $F(0) = 0$;
2. $F(\alpha+1) = F(\alpha)+1$;
3. $\beta \gg \alpha \Rightarrow F(\beta+\alpha) = F(\beta) \oplus F(\alpha)$;
4. $F(\omega^n) = \omega^n + \omega^{n-1} + \ldots + \omega^0$ *for* $n < \omega$;
5. $F(\omega^\alpha) = \omega^\alpha \cdot 2 + 1$ *for* $\alpha \geq \omega$.

Let us write an explicit formula for F. Suppose α is written in Cantor normal form with the difference that we allow some m_i's to be zero, i.e.

$$\alpha = \omega^{\alpha_0} \cdot a_0 + \ldots + \omega^{\alpha_s} \cdot a_s + \omega^n \cdot m_n + \ldots + \omega^0 \cdot m_0,$$

where $\alpha_0 > \ldots > \alpha_s \geq \omega$ and $n < \omega$. Then $F(\alpha)$ is equal to

$$\begin{aligned} \omega^{\alpha_0} \cdot 2a_0 + \ldots + \omega^{\alpha_s} \cdot 2a_s \\ +\omega^n \cdot m_n + \omega^{n-1} \cdot (m_n + m_{n-1}) + \ldots + \omega^0 \cdot (m_n + \ldots + m_0) \\ +(a_0 + \ldots + a_s). \end{aligned} \tag{4.1}$$

Remark that for $\omega^\omega \leq \alpha < \varepsilon_\omega$, $F(\alpha)$ is approximatively equal to $\alpha \cdot 2$. Above we defined the natural sum for two ordinals. We will also use the *natural product* of an ordinal and a natural number. The aim is similar to the one for introducing the natural sum, namely, ensuring that no terms disappear while multiplying.

4.17 Definition. *Let $\alpha \leq \varepsilon_\omega$ be an ordinal and n a natural number. The* natural product *$\alpha \odot n$ is defined by*

$$\alpha \odot n = \underbrace{\alpha \oplus \ldots \oplus \alpha}_{n \text{ times}}.$$

As one can grasp looking at some simple examples, the natural product is always greater than or equal to the usual product. We will sometimes write down ε_{-1} instead of ω in order to make notation uniform. In the proof of the next lemma, for example, it will be handy.

4.18 Lemma. *If c and m are natural numbers, with $c \geq 4$, and $\nu > 1$ an ordinal, then*

$$\operatorname{psn}(F(\mathrm{GO}(c, \omega^\nu)) \odot (m+3)) \leq c \cdot \operatorname{tow}_{2c^2+c}(c+2) \cdot (m+3).$$

Proof. The statement is a consequence of the explicit formula (4.1) for F. Suppose $\omega_k(\varepsilon_l+1) \leq \mathrm{GO}(c, \omega^\nu) < \omega_{k+1}(\varepsilon_l+1)$, with $k \in \mathbb{N}$ and $l \in \mathbb{N} \cup \{-1\}$. Then clearly $k + l < c$, because of Lemma 4.5. If $\mathrm{GO}(c, \omega^\nu)$ is written in its Cantor normal form, then each summand $\omega^\rho \cdot c$ adds c to the coefficient of ω^0, which is the largest coefficient occurring. The number of such terms in the expansion is no more than $|\{\alpha < \omega_k(\varepsilon_l + 1) : \operatorname{psn}(\alpha) \leq c\}|$, which at its turn is less than $\operatorname{tow}_{(l+1)2c+k+2}(c + 2)$ by Lemma 4.7 in case $l \geq 0$, and by Lemma 4.6 if $l = -1$ since $\omega_k(\omega + 1) < \omega_{k+2}$. Due to $k + l < c$, the latter number is less than $\operatorname{tow}_{2c^2+c}(c + 2)$.

Coefficients of terms $\omega^\rho \cdot c$ with $\rho \geq \omega$ are multiplied by two, so they become $2c$ after applying F, which is still below $\operatorname{tow}_{2c^2+c}(c+2)$. Remark that exponents of $F(\mathrm{GO}(c, \omega^\nu))$ will not change the pseudonorm, as they do not increase. When we take the natural sum of the $m+3$ terms $F(\mathrm{GO}(c, \omega^\nu))$, it becomes clear that $\operatorname{psn}(F(\mathrm{GO}(c, \omega^\nu)) \odot (m + 3))$ is going to be less than or equal to $c \cdot \operatorname{tow}_{2c^2+c}(c + 2) \cdot (m + 3)$ (the coefficient of ω^0 after summation). □

4.19 Lemma. *Let $k, m \in \mathbb{N} \cup \{-1\}$, α and β ordinals with $\beta \gg \alpha$, such that either $\omega_k(\varepsilon_m + 1) \leq \beta + \alpha < \omega_{k+1}(\varepsilon_m + 1)$, or $\beta + \alpha = \varepsilon_m$ (k is set -1), or $\beta + \alpha$ is below ω (k and m are set -1), and $A \subseteq \mathbb{N}$ with $\min(A) \geq \max\{k, m + 2, 4, \operatorname{psn}(\alpha)\}$. If $G\colon A \to (\leq \beta + \alpha)$ is strictly decreasing and $\operatorname{psn}(G(a)) \leq \min A + (m + 2)a$, for all $a \in A$, then $\{a \in A : G(a) \geq \beta\}$ is at most $F(\alpha) \odot (m + 3)$-large.*

Proof. The proof has the same structure as the one of Lemma 3.10 in [BK06], but needs serious adaption to our situation. First remark that in case $\beta + \alpha < \omega$, the statement is easily proved. Let $T(\alpha)$ denote the statement of the lemma without the quantifier $\forall\alpha$. We prove $\forall\alpha T(\alpha)$ by induction on α.

For $\alpha = 0$ there is nothing to prove, for the set $\{a \in A : G(a) \geq \beta\}$ is either void or has only one element, so is at most 0-large. Note that $F(0) \odot (m+3) = 0$.

Assume $T(\alpha)$ for α; we prove it for $\alpha + 1$. So let $\beta \gg \alpha + 1$ and let $G\colon A \to (\leq \beta + \alpha + 1)$ satisfy the assumption. Let $A = \{a_0, \ldots, a_s\}$ with the elements given in increasing order. Thus, $G(a_0) \leq \beta + \alpha + 1$, hence $G(a_1) \leq \beta + \alpha$. We apply the inductive assumption to $G \restriction A'$, where $A' = A \setminus \{a_0\}$. Hence $\{a \in A' : G(a) \geq \beta\}$ is at most $F(\alpha) \odot (m+3)$-large. It follows that $\{a \in A : G(a) \geq \beta\}$ is at most $(F(\alpha) \odot (m+3) + 1)$-large. But $F(\alpha + 1) = F(\alpha) + 1$, so $\{a \in A : G(a) \geq \beta\}$ is a fortiori at most $F(\alpha + 1) \odot (m+3)$-large.

Let λ be the limit and assume $T(\alpha)$ for all $\alpha < \lambda$. We consider several cases because the definition of F depends on the form of λ. In each case let $G\colon A \to (\leq \beta + \lambda)$ satisfy the assumption, let $D = \{a \in A : G(a) \geq \beta\}$ be enumerated in increasing order as $\{d_0, \ldots, d_r\}$.

Case 1: $\lambda = \omega$. So let $\beta \gg \omega$ and D be as defined above. Suppose $G\colon A \to (\leq \beta + \omega)$ satisfies the assumption. Then $G(d_0) \leq \beta + \omega$, so $G(d_1) \leq \beta + a_0 + (m+2)d_1$. Let $D' = D \setminus \{d_0\}$. Then for $x \in D'$, $G(x)$ must be of the form $\beta + k_x$. The function $x \mapsto k_x$, being strictly decreasing, is one-to-one, so $|D'| \leq a_0 + (m+2)d_1 + 1 \leq (m+3)d_1$. Hence, D' is at most $(m+3)d_1$-large, which is a fortiori at most $\omega \odot (m+3)$-large. Thus D is at most $(\omega \odot (m+3) + 1)$-large, so at most $F(\omega) \odot (m+3)$-large, as $F(\omega) = \omega + 1$.

Case 2: $\lambda = \omega^n$ for some $1 < n < \omega$. Then $G(a_0) \leq \beta + \omega^n$, so

$$\begin{aligned} G(a_1) &\leq \mathrm{GO}(c, \beta + \omega^n) \\ &\leq \beta + \mathrm{GO}(c, \omega^n) \\ &\leq \beta + \omega^{n-1} \cdot c + \ldots + \omega^0 \cdot c, \end{aligned}$$

where c stands for $a_0 + (m+2)a_1$. By the inductive assumption applied to

$$\alpha = \omega^{n-1} \cdot c + \ldots + \omega^0 \cdot c,$$

the set $D \setminus \{a_0\}$ is at most $F(\alpha) \odot (m+3)$-large. Since

$$F(\alpha) = \omega^{n-1} \cdot c + \omega^{n-2} \cdot 2c + \ldots + \omega^1 \cdot (n-1)c + \omega^0 \cdot nc,$$

we get

$$F(\alpha)\odot(m+3) \\ = \omega^{n-1}\cdot c(m+3)+\omega^{n-2}\cdot 2c(m+3)+\ldots+\omega^0\cdot nc(m+3).$$

Assume that D is not at most $F(\omega^n)\odot(m+3)$-large, so the set $E = D\setminus\{\max D\}$ is still $F(\omega^n)\odot(m+3)$-large. Recall that $F(\omega^n)=\omega^n+\omega^{n-1}+\ldots+\omega^0$, so

$$F(\omega^n)\cdot(m+3)=\omega^n\cdot(m+3)+\omega^{n-1}\cdot(m+3)+\ldots+\omega^0\cdot(m+3).$$

Let $z = h^E_{\omega^n\cdot(m+2)+\omega^{n-1}\cdot(m+3)+\ldots+\omega^0\cdot(m+3)}(a_0)$. Having in mind that $n\geq 2$ and $a_1 > a_0 \geq m$ and using Lemma 1.19, Lemma 4.2 and Lemma 4.12, we obtain the following inequalities

$$\begin{aligned} z &\geq h^E_{\omega^n\cdot(m+2)+\omega^{n-1}\cdot(m+3)+\ldots+\omega\cdot(m+3)}(a_1+m+2) \\ &\geq h^E_{\omega^n\cdot(m+2)}(2^{m+3}\cdot a_1) \\ &\geq n(m+3)2^{m+3}\cdot a_1 \\ &> n(a_0+(m+2)a_1)(m+3) \\ &= nc(m+3) \\ &= \mathrm{psn}(F(\alpha)\odot(m+3)). \end{aligned}$$

From the inequality above we infer that $\omega^n \Rightarrow_z F(\alpha)\odot(m+3)$ by Lemma 4.2. Now Lemma 1.19 implies that as $h^E_{\omega^n}(z)$ exists, also $h^E_{F(\alpha)\odot(m+3)}(z)$ exists, and so $h^E_{F(\alpha)\odot(m+3)}(a_1)$ exists. This contradicts the fact that $D\setminus\{a_0\}$ is at most $F(\alpha)\odot(m+3)$-large.

CASE 3: $\lambda=\omega^\omega$. As usual, we have $G(a_0)\leq\beta+\omega^\omega$, so

$$\begin{aligned} G(a_1) &\leq \mathrm{GO}(c,\beta+\omega^\omega) \\ &\leq \beta+\mathrm{GO}(c,\omega^\omega) \\ &= \beta+\omega^{c-1}\cdot c+\omega^{c-2}\cdot c+\ldots+\omega^0\cdot c, \end{aligned}$$

where c stands for $a_0+(m+2)a_1$. By the inductive assumption applied to $\alpha=\omega^{c-1}\cdot c+\omega^{c-2}\cdot c+\ldots+\omega^0\cdot c$ we infer that $D\setminus\{a_0\}$ is at most

$F(\alpha) \odot (m+3)$-large, where $F(\alpha) \odot (m+3)$ equals

$$\omega^{c-1} \cdot c(m+3) + \omega^{c-2} \cdot 2c(m+3) + \ldots + \omega^0 \cdot c^2(m+3).$$

Assume that D is not at most $F(\omega^\omega) \odot (m+3)$-large, so $D \setminus \{a_0\}$ is not at most $(\omega^\omega \cdot 2(m+3) + m + 2)$-large. Let $E = D \setminus \{a_0, \max D\}$ and $z = h^E_{\omega^\omega \cdot (2m+5)+m+2}(a_1)$. Then, by Lemma 4.12,

$$\begin{aligned} z &\geq h^E_{\omega^\omega \cdot (m+3)}(h^E_{\omega^\omega \cdot (m+2)}(a_1 + m + 2)) \\ &\geq h^E_{\omega^\omega \cdot (m+3)}((m+2)^2(a_1 + m + 2)^2) \\ &\geq (m+3)^2((m+2)^2(a_1 + m + 2)^2)^2 \\ &> c^2(m+3) \\ &= \mathrm{psn}(F(\alpha) \odot (m+3)), \end{aligned}$$

so $\omega^\omega \Rightarrow_z F(\alpha) \odot (m+3)$ by Lemma 4.2. Again Lemma 1.19 implies the existence of $h^E_{F(\alpha)\odot(m+3)}(z)$, as $h^E_{\omega^\omega}(z)$ exists, and thus $h^E_{F(\alpha)\odot(m+3)}(a_1)$ exists. Hence $D \setminus \{a_0\}$ is not at most $F(\alpha) \odot (m+3)$-large, a contradiction.

Case 4: $\lambda = \omega^\nu$ with $\nu > \omega$ not an epsilon number. Using the same notation as above we see that $G(a_1) \leq \beta + \mathrm{GO}(c, \omega^\nu)$, where c stands for $a_0 + (m+2)a_1$. We apply the inductive assumption to $\alpha = \mathrm{GO}(c, \omega^\nu)$ and infer that $D \setminus \{a_0\}$ is at most $F(\mathrm{GO}(c, \omega^\nu)) \odot (m+3)$-large. Assume that D is not at most $F(\omega^\nu) \odot (m+3)$-large, so is not at most $(\omega^\nu \cdot 2(m+3) + m + 3)$-large. Then $D \setminus \{a_0\}$ is not at most $(\omega^\nu \cdot 2(m+3) + m + 2)$-large, so $E = D \setminus \{a_0, \max D\}$ is $(\omega^\nu \cdot 2(m+3) + m + 2)$-large. Let $z = h^E_{\omega^\nu \cdot (m+3)+m+2}(a_1)$ and recall that $a_1 > \max\{4, m+2, \mathrm{psn}(\omega^\nu)\}$. The following inequalities serve the purpose of showing $z > \mathrm{psn}(F(\mathrm{GO}(c, \omega^\nu)) \odot (m+3))$. The basic idea is to increase the argument in order to safely step down the ordinal and still end up with a sufficiently large upper bound. While doing so we apply Lemma 1.19, Lemma 4.2, Lemma 4.8 and Lemma 4.18, and use some of the inequalities already seen above. We get

$$\begin{aligned} z &\geq h^E_{\omega^\nu \cdot (m+3)}(a_1 + m + 2) \\ &= h^E_{\omega^\nu \cdot (m+2)}(h^E_{\omega^\nu}(a_1 + m + 2)) \\ &\geq h^E_{\omega^\nu \cdot (m+2)}(h^E_{\omega^2 \cdot 2 \cdot (m+2)}(a_1 + m + 2)) \end{aligned}$$

$$\begin{aligned}
&\geq h^E_{\omega^\nu \cdot (m+2)}(\mathrm{tow}_{m+2}(a_1+m+3)) \\
&\geq h^E_{\omega^\nu \cdot (m+1)}(h^E_{\omega^\nu}(\mathrm{tow}_{m+2}(a_1+m+3))) \\
&\geq h^E_{\omega^\nu \cdot (m+1)}(h^E_{\omega^2 \cdot 2 \cdot \mathrm{tow}_{m+1}(a_1+m+3)}(\mathrm{tow}_{m+2}(a_1+m+3))) \\
&\geq h^E_{\omega^\nu \cdot (m+1)}(\mathrm{tow}_{\mathrm{tow}_{m+1}(a_1+m+3)}(\mathrm{tow}_{m+2}(a_1+m+3)+1)) \\
&\geq h^E_{\omega^\nu \cdot (m+1)}(\mathrm{tow}_{a_1+m+3}((a_1+m+3)^{(a_1+m+3)})) \\
&\geq h^E_{\omega^2 \cdot 2 \cdot 2(a_1(m+3))^3}(\mathrm{tow}_{a_1+m+3}((a_1+m+3)^{(a_1+m+3)})) \\
&\geq \mathrm{tow}_{2(a_1(m+3))^3}(\mathrm{tow}_{a_1+m+3}((a_1+m+3)^{(a_1+m+3)})) \\
&> c \cdot \mathrm{tow}_{2c^2+c}(c+2) \cdot (m+3) \\
&\geq \mathrm{psn}(F(\mathrm{GO}(c,\omega^\nu)) \odot (m+3)),
\end{aligned}$$

so $\omega^\nu \Rightarrow_z F(\mathrm{GO}(a_1,\omega^\nu)) \odot (m+3)$ by Lemma 4.2. As $h^E_{\omega^\nu}(z)$ exists, Lemma 1.19 entails the existence of $h^E_{F(\mathrm{GO}(a_1,\omega^\nu))\odot(m+3)}(z)$, which implies that $h^E_{F(\mathrm{GO}(a_1,\omega^\nu))\odot(m+3)}(a_1)$ exists. This contradicts the fact that $D \setminus \{a_0\}$ is at most $F(\mathrm{GO}(a_1,\omega^\nu)) \odot (m+3)$-large.

Case 5: $\lambda = \varepsilon_0$. Remark that $\mathrm{GO}(c,\varepsilon_0) = \mathrm{GO}(c,\omega_c)$, where c stands for $a_0+(m+2)a_1$, and proceed as done in the previous case.

Case 6: $\lambda = \varepsilon_{m+1}$, with $m \geq 0$. Remark that

$$\mathrm{GO}(c,\varepsilon_{m+1}) = \mathrm{GO}(c,\omega_c(\varepsilon_m+1)),$$

where c stands for $a_0+(m+2)a_1$, and proceed as in case 4.

Case 7: The Cantor normal form of λ is nontrivial, that is, if λ is written in its Cantor normal form (1.1), then there are at least two summands or the coefficient is strictly greater than 1. Then $\lambda = \gamma + \delta$ for some γ and δ both different from zero with $\gamma \gg \delta$. Let β, G and A satisfy the assumptions. Then the set $\{a \in A : G(a) \geq \beta + \gamma\}$ is at most $F(\delta) \odot (m+3)$-large, indeed, every $G(x)$ for x in that set must be of the form $\beta+\gamma+\tau_x$, and the function $x \mapsto \tau_x$ satisfies the inductive assumption for $\delta < \lambda$. Similarly, the set $\{a \in A : \beta+\gamma \geq G(a) \geq \beta\}$ is at most $F(\gamma) \odot (m+3)$-large. Their union is just $\{a \in A : G(a) \geq \beta\}$. It is at most $(F(\gamma) \odot (m+3)) \oplus (F(\delta) \odot (m+3))$-large by Lemma 4.14. It suffices to note that

$$(F(\gamma) \odot (m+3)) \oplus (F(\delta) \odot (m+3)) = (F(\gamma) \oplus F(\delta)) \odot (m+3) \\ = F(\gamma + \delta) \odot (m+3)$$

by the definition of F the fact that the natural product distributes over the natural sum. □

Now we are finally ready to give the Estimation Lemma.

4.20 Lemma. *Let $k, m \in \mathbb{N} \cup \{-1\}$, α an ordinal, such that either $\omega_k(\varepsilon_m + 1) \leq \alpha < \omega_{k+1}(\varepsilon_m + 1)$, or $\alpha = \varepsilon_m$ (k is set -1), or $\alpha < \omega$ (k and m are set -1), and $A \subseteq \mathbb{N}$ with $\min(A) \geq \max\{k, m+2, 4, \mathrm{psn}(\alpha)\}$. If $G\colon A \to (\leq \alpha)$ is strictly decreasing and $\mathrm{psn}(G(a)) \leq \min A + (m+2)a$, for all $a \in A$, then A is at most $F(\alpha) \odot (m+3)$-large.*

Proof. Apply the previous lemma with $\beta = 0$. □

4.3 Partitioning α-large sets

Every ordinal α below ε_{m+1} can be written in *Cantor normal form to the base* $\omega_n(\varepsilon_m + 1)$, i.e.

$$\alpha = (\omega_n(\varepsilon_m + 1))^{\alpha_0} \cdot \beta_0 + \ldots + (\omega_n(\varepsilon_m + 1))^{\alpha_s} \cdot \beta_s, \tag{4.2}$$

for some $\alpha > \alpha_0 > \ldots > \alpha_s$ and $\beta_0, \ldots, \beta_s < \omega_n(\varepsilon_m + 1)$. More information on such expansions to another base can be found in Sections 7.5–7.7 of [KM76]. As before, ε_{-1} will denote ω which we will not always mention explicitly when working with epsilon numbers in this section.

4.21 Lemma. *Let $n > 0$ and $\alpha < \varepsilon_{m+1}$. Write α to the base $\omega_n(\varepsilon_m + 1)$ (i.e. in the form (4.2) above). Then*

$$\mathrm{psn}(\alpha) \geq \max\{\mathrm{psn}(\alpha_0), \ldots, \mathrm{psn}(\alpha_s), \mathrm{psn}(\beta_0), \ldots, \mathrm{psn}(\beta_s)\}.$$

Proof. The proof is similar to the one in [BK06], but with a different pseudonorm. In case $n = 1$, then the statement is obvious as a result of the definition of psn (Definition 4.1). So suppose $n \geq 2$ and recall that $\omega_0 = \omega_0(1) = 1$. To prove the statement it suffices to look at a single, arbitrary summand in the expansion of α. It is of the form $(\omega_n(\varepsilon_m + 1))^\gamma \cdot \delta$, which is equal to $\omega^{\omega_{n-1}(\varepsilon_m+1)\cdot\gamma} \cdot \delta$. Suppose γ and δ are written in the usual Cantor normal form to the base ω as follows:

$$\gamma = \omega^{\gamma_0} \cdot g_0 + \ldots + \omega^{\gamma_r} \cdot g_r$$
$$\delta = \omega^{\delta_0} \cdot d_0 + \ldots + \omega^{\delta_r} \cdot d_t.$$

Then the following equalities hold.

$$\begin{aligned}
&(\omega_n(\varepsilon_m + 1))^\gamma \cdot \delta \\
&\quad = (\omega_n(\varepsilon_m + 1))^{\omega^{\gamma_0}\cdot g_0+\ldots+\omega^{\gamma_r}\cdot g_r} \cdot (\omega^{\delta_0} \cdot d_0 + \ldots + \omega^{\delta_r} \cdot d_t) \\
&\quad = \omega^{\omega_{n-1}(\varepsilon_m+1)\cdot(\omega^{\gamma_0}\cdot g_0+\ldots+\omega^{\gamma_r}\cdot g_r)} \cdot (\omega^{\delta_0} \cdot d_0 + \ldots + \omega^{\delta_r} \cdot d_t) \\
&\quad = \omega^{\omega^{\omega_{n-2}(\varepsilon_m+1)+\gamma_0}\cdot g_0+\ldots+\omega^{\omega_{n-2}(\varepsilon_m+1)+\gamma_r}\cdot g_r} \cdot (\omega^{\delta_0} \cdot d_0 + \ldots + \omega^{\delta_r} \cdot d_t).
\end{aligned}$$

This expression is again a sum. As before, we consider a single summand, representing the general case. Such a term will be of the form

$$\omega^{\omega^{\omega_{n-2}(\varepsilon_m+1)+\gamma_0}\cdot g_0+\ldots+\omega^{\omega_{n-2}(\varepsilon_m+1)+\gamma_r}\cdot g_r+\delta_j} \cdot d_j,$$

with $0 \leq j \leq t$. Since the original α is written in the form (4.2), $\delta < \omega_n(\varepsilon_m + 1)$, and thus $\delta_j < \omega_{n-1}(\varepsilon_m + 1)$. This last inequality, together with the fact that γ was given in Cantor normal form, yields that none of the terms on the first level of exponents will be absorbed by the next one. So, returning to α, we have written the summand $(\omega_n(\varepsilon_m + 1))^\gamma \cdot \delta$ as a sum of ordinals, all expressed to the base ω, which allows us to determine their pseudonorm immediately. First, $\text{psn}(\alpha) \geq \text{psn}((\omega_n(\varepsilon_m + 1))^\gamma \cdot \delta) \geq \text{psn}(\gamma)$, as all exponents and coefficients in the Cantor normal form of γ, also occur on the first and second level of exponents in the expansion we obtained in the latest stage. Second, also $\text{psn}(\alpha) \geq \text{psn}(\delta_j)$ for all $0 \leq j \leq t$ since each δ_j occurs in this expansion on the first level of exponents. Combining that inequality with $\mathrm{psn}(\alpha) \geq d_j$, which is seen from the latest expansion, we get $\mathrm{psn}(\alpha) \geq \mathrm{psn}(\delta)$, which concludes the proof. $\square$

4.22 Definition. *Let α be an ordinal below ε_{m+1} written in Cantor normal form to the base $\omega_n(\varepsilon_m + 1)$, with $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{-1\}$. Then define $v(\omega_n(\varepsilon_m + 1); \alpha, \delta)$ as the coefficient of $(\omega_n(\varepsilon_m + 1))^\delta$ (and $v(\omega_n(\varepsilon_m + 1); \alpha, \delta) = 0$ if $(\omega_n(\varepsilon_m + 1))^\delta$ does not occur in the normal form). For ordinals α and β below ε_{m+1} define $\mathrm{LD}_{n,m}(\alpha, \beta)$ as the maximum of*

$$\{\delta < \varepsilon_{m+1} : v(\omega_n(\varepsilon_m + 1); \alpha, \delta) \neq v(\omega_n(\varepsilon_m + 1); \beta, \delta)\}.$$

As done in the original paper, we will construct a sequence of partitions of sets of ordinals. Let us start by looking at $\mathrm{LD}_{n,m}$. The main property of this partition is given in the following lemma, and will come back in Lemma 4.26.

4.23 Lemma. *Let Γ be a set of ordinals below ε_{m+1}, $m \in \mathbb{N} \cup \{-1\}$, which is homogeneous with respect to $\mathrm{LD}_{n,m}$. Then there exists a strictly increasing function $\Theta\colon \Gamma \to (< \omega_n(\varepsilon_m + 1))$ such that for all $\gamma \in \Gamma$, $\mathrm{psn}(\Theta(\gamma)) \leq \mathrm{psn}(\gamma)$.*

Proof. Let Γ satisfy the assumption. Thus, there exists δ such that for all α and β in Γ, $\mathrm{LD}_{n,m}(\alpha, \beta) = \delta$. It follows that every $\alpha \in \Gamma$ may be written as $\rho + (\omega_n(\varepsilon_m + 1))^\delta \cdot \xi_\alpha + \tau_\alpha$ to the base $\omega_n(\varepsilon_m + 1)$, where all exponents in ρ are strictly greater than δ and all exponents in τ_α are strictly smaller than δ. Clearly, ρ does not depend on α and the function $\Theta(\alpha) = \xi_\alpha$ has the required property about the pseudonorm by Lemma 4.21. The fact that Θ is strictly increasing is straightforward. $\square$

4.24 Definition. *Let $\Gamma \subseteq (< \varepsilon_\omega)$ and $\Theta\colon \Gamma \to (< \alpha)$. We say that Θ is an* ordinal *(or α-*ordinal*)* estimating function *if it is strictly increasing and for all $\gamma \in \Gamma$, $\mathrm{psn}(\Theta(\gamma)) \leq \mathrm{psn}(\gamma)$.*

We now proceed to the next partition, namely $L_{3,n,m}$. Let us make the convention that when working with finite sets of ordinals we write them down in decreasing order. Hence in the next definition $\alpha > \beta > \gamma$, in accordance with that convention.

4.25 Definition. *Let $\alpha, \beta, \gamma < \varepsilon_{m+1}$, $m \in \mathbb{N} \cup \{-1\}$ and $n \in \mathbb{N}$. Define*

$$L_{3,n,m}(\alpha, \beta, \gamma) = \begin{cases} 0 & \text{if } \mathrm{LD}_{n,m}(\alpha, \beta) < \mathrm{LD}_{n,m}(\beta, \gamma), \\ 1 & \text{if } \mathrm{LD}_{n,m}(\alpha, \beta) = \mathrm{LD}_{n,m}(\beta, \gamma), \\ 2 & \text{if } \mathrm{LD}_{n,m}(\alpha, \beta) > \mathrm{LD}_{n,m}(\beta, \gamma). \end{cases}$$

Strictly speaking, we abuse notation, as brackets are usually used for tuples. For the rest of the chapter, in case we really mean tuples it will be mentioned clearly.

4.26 Lemma. *Let $n, s \in \mathbb{N}$, $m \in \mathbb{N} \cup \{-1\}$ and Γ be a finite subset of $(< \varepsilon_{m+1})$.*

1. *If $L_{3,n,m}$ colours $[\Gamma]^3$ by 1, then there exists an ordinal estimating function Θ defined on Γ with values in $(< \omega_n(\varepsilon_m + 1))$.*

2. *If $L_{3,n,m}$ colours $[\Gamma]^3$ by 2 and $\max \Gamma < \omega_{n+s+1}(\varepsilon_m + 1)$, then there exists an ordinal estimating function Θ defined on $\Gamma \setminus \{\min \Gamma\}$ with values in $(< \omega_{n+s}(\varepsilon_m + 1))$.*

3. *If $L_{3,n,m}$ colours $[\Gamma]^3$ by 0 and $\max \Gamma < \omega_{n+s+1}(\varepsilon_m + 1)$, then*

$$|\Gamma| \leq \mathrm{tow}_{(m+1)2a+n+s+2}(a + 2),$$

where $a = \mathrm{psn}(\max \Gamma)$.

Proof. Let $\Gamma = \{\gamma_0, \ldots, \gamma_r\}$ be a decreasing enumeration of Γ.

1. Let $\gamma_i = \rho + (\omega_n(\varepsilon_m + 1))^{\delta_i} \cdot \xi_i + \tau_i$, be written in normal form to the base $\omega_n(\varepsilon_m + 1)$. In this case neither ρ nor δ_i depends on i. Put $\Theta(\gamma_i) = \xi_i$. Then Θ has the required property concerning the pseudonorm because of Lemma 4.21. That it is strictly increasing is not difficult to see.

2. Put $\delta_i = \mathrm{LD}_{n,m}(\gamma_i, \gamma_{i+1})$ and $\Theta(\gamma_i) = \delta_i$. Then Θ has the required properties (Lemma 4.21 and Definition 4.25). In particular, its values are smaller than $\omega_{n+s}(\varepsilon_m + 1)$.

3. Put $\delta_i = \mathrm{LD}_{n,m}(\gamma_i, \gamma_{i+1})$. We assert that $(\omega_n(\varepsilon_m+1))^{\delta_i}$ occurs with a nonzero coefficient in the Cantor normal form to the base $(\omega_n(\varepsilon_m+1))$ of γ_0 for every $i < r$. Indeed, fix $i < r$. We write $\gamma_i = \rho_i + (\omega_n(\varepsilon_m + 1))^{\delta_i} \cdot \xi_i + \tau_i$ and compare this with the expansion of γ_{i+1} to the base $\omega_n(\varepsilon_m + 1)$. We see that ξ_i must be greater than the coefficient at $(\omega_n(\varepsilon_m + 1))^{\delta_i}$ in the expansion of γ_{i+1}. In particular, $\xi_i > 0$. By the same argument as explained in the proof of Lemma 4.6 we obtain

$$\begin{aligned}|\Gamma| &= |\{\delta_i \,:\, i < r\}| + 1\\ &\leq |\{\alpha < \omega_{n+s}(\varepsilon_m + 1) \,:\, \mathrm{psn}(\alpha) \leq a\}| + 1\\ &< \mathrm{tow}_{(m+1)2a+n+s+2}(a + 2) + 1,\end{aligned}$$

with $a = \mathrm{psn}(\gamma_0)$, and where the last inequality is justified by Lemma 4.7 in case $m \geq 0$ and by Lemma 4.6 if $m = -1$ (since $\omega_{n+s}(\omega + 1) < \omega_{n+s+2}$). So, $|\Gamma| \leq \mathrm{tow}_{(m+1)2a+n+s+2}(a + 2)$, which concludes the proof.

□

The properties of $L_{3,n,m}$, proven in the previous lemma, will be partly used to prove characteristics of the partition $L_{k,m}$ which is introduced in the next lemma.

4.27 Lemma. *Let $k \in \mathbb{N}$, with $k \geq 3$, and $m \in \mathbb{N}\cup\{-1\}$. Then there existst a partition $L_{k,m}$ of $[(<\varepsilon_{m+1})]^k$ into 3^{k-2} parts such that for every $\Gamma \subseteq (<\varepsilon_{m+1})$ homogeneous for $L_{k,m}$, if $\max\Gamma < \omega_{k-2}(\varepsilon_m + 1)$, then letting Γ' be Γ without the last (i.e. smallest) $\frac{(k-2)(k-1)}{2}$ elements we have: there exists an ordinal estimating function $\Theta\colon \Gamma' \to (\leq \varepsilon_m)$ or $|\Gamma| \leq \mathrm{tow}_{(m+1)2a+k-1}(a + k - 1)$, where $a = \mathrm{psn}(\max\Gamma)$.*

Proof. By induction on k.

$k = 3$. Let $L_{3,m}$ equal $L_{3,0,m}$ and apply Lemma 4.26 above with n replaced by 0.

$k \to k+1$. In the same way as done in [BK06], but again adapted to our situation. Assume the result for k. We construct the partition for $k+1$. Let

$\alpha = \{\alpha_0, \ldots, \alpha_k\}$ be a $(k+1)$-set of ordinals below ε_{m+1} (as usual written down in decreasing order). We begin by putting $G(\alpha) = L_{3,0,m}(\alpha_0, \alpha_1, \alpha_2)$, so the image of the set α does not depend on its last $k-2$ coordinates. Thus if Γ is homogeneous for G, then Γ'' is homogeneous for $L_{3,0,m}$, where Γ'' denotes Γ without its last $k-2$ elements.

For α as above we let $\delta_i = \mathrm{LD}_{0,m}(\alpha_i, \alpha_{i+1})$. Notice that the k-tuple $(\delta_0, \ldots, \delta_{k-1})$ is not necessarily monotonic. Having that fact in mind, we define the function W as follows.

$$W(\alpha) = \begin{cases} L_{k,m}(\delta_0, \ldots, \delta_{k-1}) & \text{if } \delta_0 > \ldots > \delta_{k-1}, \\ L_{k,m}(\delta_{k-1}, \ldots, \delta_0) & \text{if } \delta_0 < \ldots < \delta_{k-1}, \\ 0 & \text{in other cases.} \end{cases}$$

Finally we put $L_{k+1,m}(\alpha) = \langle G(\alpha), W(\alpha) \rangle$. We assert that this partition has the right properties. So suppose Γ is homogeneous for $L_{k+1,m}$, where $\Gamma = \{\gamma_0, \ldots, \gamma_r\}$, written in decreasing order. Let Γ' and Γ'' denote Γ without its last (i.e. smallest) $\frac{(k-1)k}{2}$ and $k-2$ elements, respectively.

Case 1. G colours $[\Gamma]^{k+1}$ by 1. Then $L_{3,0,m}$ colours $[\Gamma'']^3$ by 1, so by part 1. of Lemma 4.26 there exists an ordinal estimating function $\Theta \colon \Gamma'' \to (\leq \varepsilon_m)$, where we replaced $(< \varepsilon_m + 1)$ by $(\leq \varepsilon_m)$. The domain of this function contains the required set Γ'.

Case 2. G colours $[\Gamma]^{k+1}$ by 2. Then $L_{3,0,m}$ colours $[\Gamma'']^3$ by 2. Let $\zeta_i = \mathrm{LD}_{0,m}(\gamma_i, \gamma_{i+1})$ for $i < r - (k-2)$. By the assumption of the case, the set $Z = \{\zeta_0, \ldots, \zeta_{r-k+1}\}$ is written in strictly decreasing order. By homogeneity of Γ with respect to W, the set Z is homogeneous with respect to $L_{k,m}$. The induction hypothesis yields the following two subcases:

(a) The case $|Z| \leq \mathrm{tow}_{(m+1)2a+k-1}(a+k-1)$, where $a = \mathrm{psn}(\max Z)$. Due to Lemma 4.21, $\mathrm{psn}(\max Z) \leq \mathrm{psn}(\max \Gamma'') = \mathrm{psn}(\max \Gamma)$, so

$$|\Gamma''| \leq \mathrm{tow}_{(m+1)2a'+k-1}(a'+k-1) + 1,$$

where $a' = \mathrm{psn}(\max \Gamma)$. Thus

$$\begin{aligned} |\Gamma| &\leq \mathrm{tow}_{(m+1)2a'+k-1}(a'+k-1) + 1 + k - 2 \\ &\leq \mathrm{tow}_{(m+1)2a'+k}(a'+k), \end{aligned}$$

which is the second case in the lemma.

(b) There exists an ordinal estimating function $\Theta\colon Z' \to (\leq \varepsilon_m)$, where Z' is Z without its last $\frac{(k-2)(k-1)}{2}$ elements. Consider the function $\Delta\colon \gamma_i \mapsto \Theta(\zeta_i)$. As Θ is strictly increasing and $\{\zeta_0, \ldots, \zeta_{r-k+1}\}$ and $\{\gamma_0, \ldots, \gamma_r\}$ are both written in decreasing order, Δ will also be strictly increasing. It also holds that

$$\begin{aligned} \mathrm{psn}(\Delta(\gamma_i)) &= \mathrm{psn}(\Theta(\zeta_i)) \\ &\leq \mathrm{psn}(\zeta_i) \\ &\leq \mathrm{psn}(\gamma_i), \end{aligned}$$

where the last inequality holds because of Lemma 4.21. So $\Delta\colon \Gamma' \to (\leq \varepsilon_m)$ is an ordinal estimating function, where Γ' is Γ without its last $\frac{(k-2)(k-1)}{2} + 1 + k - 2 = \frac{(k-1)k}{2}$ elements.

Case 3. G colours $[\Gamma]^{k+1}$ by 0 and so $L_{3,0,m}$ colours $[\Gamma'']^3$ by 0. It is given that $\max\Gamma < \omega_{k-1}(\varepsilon_m + 1)$. Clearly, $\max\Gamma'' = \max\Gamma$. Now part 3. of Lemma 4.26 yields

$$|\Gamma''| \leq \mathrm{tow}_{(m+1)2a+k}(a+2),$$

where $a = \mathrm{psn}(\max\Gamma)$ and so

$$\begin{aligned} |\Gamma| &\leq \mathrm{tow}_{(m+1)2a+k}(a+2) + k - 2 \\ &\leq \mathrm{tow}_{(m+1)2a+k}(a+k). \end{aligned}$$

This completes the induction step and consequently the proof. $\square$

To proceed our investigation, we will need some more machinery. These notions are introduced for ordinals below ε_0 in e.g. [BK06, KPW07]. First notice that it can happen that $h^A_\alpha(a) = h^A_\beta(a)$ even if $\alpha \neq \beta$. Think, for instance, of $h_\omega(a) = h_a(a)$. As one will see later, it is convenient to pick one specific ordinal out of several such ordinals in a unique way. This can be done as follows.

Let a set A be given. Let $\mu < \varepsilon_\omega$. We define a sequence $(\mu_j)_{j<\omega}$ of ordinals and a sequence $(b_j)_{j<\omega}$ of natural numbers by the following induction. We let $\mu_0 = \mu$ and $b_0 = a_0 = \min A$. Assume that μ_j and b_j are constructed. If $\mu_j = 0$, then the construction terminates. If $\mu_j > 0$ and μ_j is a limit ordinal, then we let $\mu_{j+1} = \mu_j[b_j]$ and $b_{j+1} = b_j$. If μ_j is nonlimit, then the construction terminates if $b_j = \max A$, otherwise we let $\mu_{j+1} = \mu_j - 1$ and $b_{j+1} = h^A(b_j)$, the next element of A. This completes the definition of the sequences $(\mu_j)_{j<\omega}$ and $(b_j)_{j<\omega}$.

Observe the following property.

4.28 Lemma. *Under the notation introduced above, a subset A is μ-large if and only if there exists $j \in \mathbb{N}$ such that $\mu_j = 0$.*

Proof. The proof of Proposition 2.7 in [BK06] immediately generalises to ordinals below ε_ω. □

Finally, this lemma allows us to associate a unique ordinal with every ordinal μ and every element $a \in A$, with A being at most μ-large. Let $\mathrm{KS}^A(\mu; a)$ denote the last μ_j such that $a = b_j$. Observe that if A is not at most μ-large, then $\mathrm{KS}^A(\mu; a)$ is not defined for all elements of A. We will need the following two lemmas.

4.29 Lemma. *If $k, m \in \mathbb{N}$, $A \subseteq \mathbb{N}$, with $\min A \geq 2$, and $a \in A$, then*

1. $\mathrm{psn}(\mathrm{KS}^A(\omega_k(\omega + 1); a)) \leq k + a$;
2. $\mathrm{psn}(\mathrm{KS}^A(\omega_k(\varepsilon_m + 1); a)) \leq k + (m + 2)a$.

Proof. 1. By induction on k.

$k = 0$. Then

$$\begin{aligned}\mathrm{psn}(\mathrm{KS}^A(\omega + 1; a)) &\leq \max\{\mathrm{psn}(\omega + 1), \mathrm{psn}(a)\}\\ &= a.\end{aligned}$$

$k \to k+1$. Assume the assertion for k, then, using the induction hypothesis,

$$\begin{aligned}
&\mathrm{psn}(\mathrm{KS}^A(\omega_{k+1}(\omega+1);a)) \\
&\quad = \mathrm{psn}(\mathrm{KS}^A(\omega^{\omega_k(\omega+1)};a)) \\
&\quad \leq 1+\max\{a,\mathrm{psn}(\mathrm{KS}^A(\omega_k(\omega+1);a))\} \\
&\quad \leq 1+k+a.
\end{aligned}$$

2. By main induction on m and subsidiary induction on k.

   (a) $m=0$.
   $k=0$. Then

$$\begin{aligned}
&\mathrm{psn}(\mathrm{KS}^A(\varepsilon_0+1;a)) \\
&\quad \leq \max\{\mathrm{psn}(\varepsilon_0+1),\mathrm{psn}(\omega_a)\} \\
&\quad \leq a+1 \\
&\quad \leq 2a.
\end{aligned}$$

   $k \to k+1$. Assume the assertion for k, then, using the subsidiary induction hypothesis,

$$\begin{aligned}
&\mathrm{psn}(\mathrm{KS}^A(\omega_{k+1}(\varepsilon_0+1);a)) \\
&\quad = \mathrm{psn}(\mathrm{KS}^A(\omega^{\omega_k(\varepsilon_0+1)};a)) \\
&\quad \leq 1+\max\{a,\mathrm{psn}(\mathrm{KS}^A(\omega_k(\varepsilon_0+1);a))\} \\
&\quad \leq 1+k+2a.
\end{aligned}$$

   (b) $m \to m+1$.
   $k=0$. Assume the assertion for m, then, using the main induction hypothesis,

$$\begin{aligned}
&\mathrm{psn}(\mathrm{KS}^A(\varepsilon_{m+1}+1;a)) \\
&\quad \leq \max\{\mathrm{psn}(\varepsilon_{m+1}+1),\mathrm{KS}^A(\omega_a(\varepsilon_m+1);a)\} \\
&\quad \leq a+(m+2)a \\
&\quad = (m+1+2)a.
\end{aligned}$$

$k \to k+1$. Assume the assertion for k, then, using the subsidiary induction hypothesis,

$$\begin{aligned}
&\mathrm{psn}(\mathrm{KS}^A(\omega_{k+1}(\varepsilon_{m+1}+1);a)) \\
&\qquad = \mathrm{psn}(\mathrm{KS}^A(\omega^{\omega_k(\varepsilon_{m+1}+1)};a)) \\
&\qquad \leq 1+\max\{a, \mathrm{psn}(\mathrm{KS}^A(\omega_k(\varepsilon_{m+1}+1);a))\} \\
&\qquad \leq 1+k+(m+1+2)a.
\end{aligned}$$

□

4.30 Lemma. *For all $m \geq 0$ and $x \geq \max\{m+2, 4\}$, we have*

$$h_{\varepsilon_m}(x) > \mathrm{tow}_{2(m+3)^2x}((m+4)x).$$

Proof. By induction on m. In both the base and the successor step will make use of Lemma 1.19, Lemma 4.2, Lemma 4.8, the definition of psn and the fact that $x \geq \max\{m+2, 4\}$.

$m = 0$. Then

$$\begin{aligned}
h_{\varepsilon_0}(x) &\geq h_{\omega^\omega+\omega^2\cdot 2}(x) \\
&\geq h_{\omega^\omega+\omega^2}(2^x x) \\
&\geq h_{\omega^\omega}(2^{2^x x}2^x x) \\
&\geq h_{\omega^2\cdot 2\cdot 2^{2^x x}2^x}(2^{2^x x}2^x x) \\
&\geq \mathrm{tow}_{2^{2^x x}2^x}(2^{2^x x}2^x x+1) \\
&> \mathrm{tow}_{18x}(4x),
\end{aligned}$$

which concludes the base step.

$m \to m+1$. Assume the assertion for m. We show it also holds for $m+1$. By using the induction hypothesis twice, we get

$$\begin{aligned}
h_{\varepsilon_{m+1}}(x) &\geq h_{\varepsilon_m\cdot 2}(x) \\
&\geq h_{\varepsilon_m}(\mathrm{tow}_{2(m+3)^2x}((m+4)x)) \\
&\geq \mathrm{tow}_{2(m+3)^2\mathrm{tow}_{2(m+3)^2x}((m+4)x)}((m+4)(\mathrm{tow}_{2(m+3)^2x}((m+4)x)) \\
&> \mathrm{tow}_{2(m+4)^2x}((m+5)x),
\end{aligned}$$

which concludes the proof. $\square$

We are finally ready to introduce the main result of this section, namely the existence of a certain partition R_k.

4.31 Theorem. *Let $k \in \mathbb{N}$, with $k \geq 3$, and $m \in \mathbb{N} \cup \{-1\}$. Let $A \subseteq \mathbb{N}$ be at most $\omega_{k-2}(\varepsilon_m + 1)$-large with $\min(A) \geq \max\{k, m+2, 4\}$. Then there exists a partition*

$$R_k \colon [A]^k \to 3^{k-2},$$

such that every $D \subseteq A$ homogeneous for R_k is at most $(F(\varepsilon_m) \odot (m+3) + \frac{(k-2)(k-1)}{2})$-large.

Proof. Let A satisfy the assumption. Let $L_{k,m}$ be a partition of $[(<\varepsilon_{m+1})]^k$ with the properties described in Lemma 4.27. For $a = \{a_0, \ldots, a_{k-1}\}$ we let

$$R_k(a) = L_{k,m}(\mathrm{KS}^A(\omega_{k-2}(\varepsilon_m + 1); a_0), \ldots, \mathrm{KS}^A(\omega_{k-2}(\varepsilon_m + 1); a_{k-1}))$$

and verify that this partition has the desired properties. So let D be a subset of A which is homogeneous for R_k. Then

$$\Gamma = \{\mathrm{KS}^A(\omega_{k-2}(\varepsilon_m + 1); d) \,:\, d \in D\}$$

is homogeneous for $L_{k,m}$. Recall that the elements of $\Gamma = \{\gamma_0, \ldots, \gamma_r\}$ are written in decreasing order, whereas the elements of $D = \{d_0 \ldots, d_r\}$ are given in increasing order. Let Γ' denote Γ without its last (i.e. smallest) $\frac{(k-2)(k-1)}{2}$ elements and let D' be D without its last (i.e. greatest) $\frac{(k-2)(k-1)}{2}$ elements.

Case 1. There exists an ordinal estimating function $\Theta \colon \Gamma' \to (\leq \varepsilon_m)$. Define $G \colon D' \to (\leq \varepsilon_m)$ by $G(d) = \Theta(\mathrm{KS}^A(\omega_{k-2}(\varepsilon_m + 1); d))$, for every $d \in D'$. Then G is strictly decreasing and for every $d \in D'$,

$$\begin{aligned}
\mathrm{psn}(G(d)) &= \mathrm{psn}(\Theta(\mathrm{KS}^A(\omega_{k-2}(\varepsilon_m + 1); d))) \\
&\leq \mathrm{psn}(\mathrm{KS}^A(\omega_{k-2}(\varepsilon_m + 1); d)) \\
&\leq k - 2 + (m+2)d \\
&\leq \min A + (m+2)d \\
&\leq \min D' + (m+2)d,
\end{aligned}$$

where the second inequality is justified by Lemma 4.29. Also notice that $\min D' \geq \min A \geq \max\{m+2, 4\} \geq \mathrm{psn}(\varepsilon_m)$ (which is also valid in case $m = -1$). Then D' is at most $F(\varepsilon_m) \odot (m+3)$-large by the Estimation Lemma (Lemma 4.20), and the result follows having Lemma 1.20 in mind.

CASE 2. Let $a = \mathrm{psn}(\max \Gamma)$. Then

$$\begin{aligned} a &= \mathrm{psn}(\mathrm{KS}^A(\omega_{k-2}(\varepsilon_m + 1); d_0)) \\ &\leq k - 2 + (m+2)d_0 \\ &\leq (m+3)d_0 \end{aligned}$$

by Lemma 4.29 and $d_0 = \min D' \geq \min(A) \geq \max\{k, m+2, 4\}$. As we are dealing with the second case, we get

$$\begin{aligned} |\Gamma| &\leq \mathrm{tow}_{(m+1)2a+k-1}(a + k - 1) \\ &\leq \mathrm{tow}_{(m+1)2(m+3)d_0+k-1}((m+3)d_0 + k - 1) \\ &\leq \mathrm{tow}_{2(m+3)^2 d_0}((m+4)d_0) \\ &< h_{\varepsilon_m}(d_0), \end{aligned}$$

where the last inequality is because of Lemma 4.30. Since $|D| = |\Gamma|$, D is ε_m-small, and so a fortiori $F(\varepsilon_m) \odot (m+3)$-small, as $m \geq 0$. ☐

The meaning of the relation $A \to (\alpha)^r_m$ is as usual in Ramsey theory, with the slight adaptation that we demand the homogeneous subset to be α-large. So $A \to (\alpha)^r_m$ will denote: for every colouring of r-element subsets of A into m colours there exists a subset of A which is α-large and is homogeneous for this colouring.

4.32 Corollary. *Let $k \in \mathbb{N}$, with $3 \leq k$, $m \in \mathbb{N} \cup \{-1\}$ and $\alpha = F(\varepsilon_m) \odot (m+3) + \frac{(k-2)(k-1)}{2} + 1$. Let A be such that $A \to (\alpha)^k_{3^{k-2}}$ and $\min(A) \geq \max\{k, m+2, 4\}$. Then A is $\omega_{k-2}(\varepsilon_m + 1)$-large.*

Proof. The proof of the following claim can be found in [BK06]. It goes by induction on α and is also valid for ordinals up to ε_ω.

CLAIM: For every B, if B is α-small, then there exists C such that $\max B < \min C$ and $B \cup C$ is exactly α-large.

Granted the claim we argue as follows. Assume that A is $\omega_{k-2}(\varepsilon_m+1)$-small. Let C be as in the claim, so $\max A < \min C$ and $A \cup C$ is exactly $\omega_{k-2}(\varepsilon_m + 1)$-large. By Theorem 4.31 there exists a partition L of $[A \cup C]^k$ into 3^{k-2} parts without an $(F(\varepsilon_m) \odot (m + 3) + \frac{(k-2)(k-1)}{2} + 1)$-large homogeneous set. We restrict L to $[A]^k$ and see that this restriction does not admit an $(F(\varepsilon_m) \odot (m+3) + \frac{(k-2)(k-1)}{2} + 1)$-large homogeneous set. □

This last corollary will be useful to prove a miniaturisation of the infinite Ramsey Theorem, as done in the next section.

4.4 Ramsey density

4.4.1 Introduction

We approximate the strength of the infinite full Ramsey Theorem by iterating a finitary version. Recall that by strength we mean arithmetical strength (see Definition 1.3). This finitary version is a density principle, which, together with PA will give rise to a first-order theory which achieves a lot of the strength of ACA_0 combined with the original infinitary version. Bovykin and Weiermann analysed in a similar way the Ramsey Theorem for pairs and two colours, the Canonical Ramsey Theorem for pairs and the Regressive Ramsey Theorem for pairs ([BW05]). To prove our result, we use the generalisation of the results by Bigorajska and Kotlarski about partitioning α-large sets obtained in the previous section.

Let us recall the infinite form of Ramsey's theorem (RT):

$$\begin{aligned} \mathrm{RT} &\leftrightarrow (\forall n)(\forall k)\mathrm{RT}^n_k \\ &\leftrightarrow (\forall n)(\forall k)(\text{For every } G\colon [\mathbb{N}]^n \to k \text{ there exists an infinite set } H \\ &\qquad \text{such that } G \restriction [H]^n \text{ is constant}), \end{aligned}$$

In this section we will give a miniaturisation of RT using the following density.

4.33 Definition. *Let $A \subseteq \mathbb{N}$, with* $\min A > 3$*. Then A is called* 0-RT-dense *if* $|A| > \min A$*. We say A is* $(n+1)$-RT-dense*, if* $\min A \geq n+1$ *and for every colouring*

$$G\colon [A]^{\min A+2} \to 3^{\min A},$$

there exists a subset $B \subseteq A$, such that B is homogeneous for G and B is n-RT*-dense.*

As done in the previous chapters, we will leave out RT, and simply write down n-density. We investigate the strength of this density after shortly considering the original statement.

Let RT^n denote[1] $(\forall k)\mathrm{RT}^n_k$. The following is known about the strength of RT^n.

4.34 Lemma. *1.* $\mathrm{ACA}_0 \vdash \mathrm{RT}^0$*;*

2. $\mathrm{ACA}_0 \vdash (\forall n)(\mathrm{RT}^n \to \mathrm{RT}^{n+1})$*.*

Proof. Proofs can be found in [Sim09] (Lemma III.7.4). □

Clearly, the previous lemma implies that for each fixed natural number n, the statement RT^n is provable in ACA_0. Moreover, it turns out that over RCA_0, RT^n is equivalent to ACA_0, for any $n \geq 3$ (Lemma III.7.6 in [Sim09]). Simpson also remarks that the Π^1_2-sentence $(\forall n)\mathrm{RT}^n$, shortly RT, is known to be unprovable in ACA_0. However, by Lemma 4.34, RT is provable in ACA_0 plus Π^1_2-induction.

We started with a theorem (RT) which is unprovable in ACA_0. Clearly, the question rises whether iterating our first-order density will be able to approximate the strength of RT (over RCA_0).

4.4.2 Investigating n-density

We estimate the Hardy functions using the following lemma.

4.35 Lemma. *Let $k \in \mathbb{N}$, $A \subseteq \mathbb{N}$, with* $\min A > 4$*, and $x \in A$. Then*

[1] In Chapter 2 the instance RT^1 was denoted by $\mathrm{RT}^1_{<\infty}$.

1. $h^A_{\omega\cdot 3^{x-1}}(x) \geq 2^{3^{x-1}}x$;

2. $h^A_{\varepsilon_k\cdot 3^{x-1}}(x) \geq 2^{3^{x-1}}x$.

Proof. 1. It is easy to see the following inequalities hold.

$$\begin{aligned}
h^A_{\omega\cdot 3^{x-1}}(x) &= h^A_{\omega\cdot(3^{x-1}-1)+\omega}(x)\\
&\geq h^A_{\omega\cdot(3^{x-1}-1)}(2x)\\
&= h^A_{\omega\cdot(3^{x-1}-2)+\omega}(2x)\\
&\geq h^A_{\omega\cdot(3^{x-1}-2)}(2^2x)\\
&\geq \dots\\
&\geq h^A_{\omega}(2^{(3^{x-1}-1)}x)\\
&\geq 2^{3^{x-1}}x.
\end{aligned}$$

2. Remark that

$$h_{\varepsilon_m}(x) > 2\underbrace{\mathrm{tow}_x(\dots(\mathrm{tow}_x}_{m+2 \text{ times}}(2x+1))\dots),$$

for $x > 4$, as stated in Lemma 4.11. So we get

$$\begin{aligned}
h^A_{\varepsilon_k\cdot 3^{x-1}}(x) &= h^A_{\varepsilon_k\cdot(3^{x-1}-1)+\varepsilon_k}(x)\\
&> h^A_{\varepsilon_k\cdot(3^{x-1}-1)}(2\underbrace{\mathrm{tow}_x(\dots(\mathrm{tow}_x}_{k+2 \text{ times}}(2x+1))\dots))\\
&\geq 2^{3^{x-1}}x,
\end{aligned}$$

where we used the induction hypothesis and Lemma 1.19.

□

Given fixed n, the n-density will not force us to leave the realm of $\mathrm{ACA}_0 + \mathrm{RT}$. We prove this in the next theorem via a standard method.

4.36 Theorem. *For every $n \in \mathbb{N}$,*

$$\mathrm{ACA}_0 + \mathrm{RT} \vdash (\forall a)(\exists b)([a,b] \textit{ is } n\textit{-dense}).$$

Proof. Let $n \in \mathbb{N}$. By applying König's lemma and the infinite version of Ramsey's theorem (RT), we get the finite version of Ramsey's theorem. Then apply the latter n times to obtain a sufficiently large b. $\square$

However, once we quantify over n, and thus the dimension of the colourings, we obtain an unprovable statement. Let us start with the following lemma, which already indicates the power of n-density.

4.37 Lemma. *Let $n \in \mathbb{N}$ and $A \subseteq \mathbb{N}$, such that $\min A > 4$. If A is $2n$-dense, then A is ε_{n-1}-large.*

Proof. By induction on n. Put $a_0 = \min A$.

If n equals zero, then A is 0-dense and $|A| > a_0$. Thus $h^A_{a_0}(a_0) = h^A_{\varepsilon_{-1}}(a_0)$ exists, so A is ε_{-1}-large.

Assume the statement for $n = k$ and let A be $(2k+2)$-dense. Let $G \colon [A]^{a_0+2} \to 3^{a_0}$ be any function and B a subset of A such that B is homogeneous for G and B is $(2k+1)$-dense. To invoke Lemma 4.32 we will show that B is $(F(\varepsilon_{k-1}) \odot (k+2) + \frac{a_0(a_0+1)}{2} + 1)$-large, with F as defined by Definition 4.16. Put $b_0 = \min B$ and

$$B_i = \{b \in B \,:\, h^B_{\varepsilon_{k-1}\cdot(i-1)}(b_0) \leq b < h^B_{\varepsilon_{k-1}\cdot i}(b_0)\},$$

for $i \in \{1, \ldots, 3^{b_0} - 1\}$. Define $H \colon [B]^{b_0+2} \to 3^{b_0}$ by

$$H(x_1, \ldots, x_{b_0+2}) = \begin{cases} 1 & \text{if } x_1 \in B_1 \\ 2 & \text{if } x_1 \in B_2 \\ \vdots & \qquad \vdots \\ 3^{b_0} & \text{if } h^B_{\varepsilon_{k-1}\cdot(3^{b_0}-1)}(b_0) \leq x_1 \end{cases},$$

for each $(x_1, \ldots, x_{b_0+2}) \in [B]^{b_0+2}$. Since B is $(2k+1)$-dense, there must exist $C \subseteq B$, such that C is homogeneous for H and C is $2k$-dense. The induction hypothesis yields that C is ε_{k-1}-large. As every B_i is ε_{k-1}-small, C cannot take any colour i with $i < 3^{b_0}$. Hence C must be a subset of $\{b \in B : h^B_{\varepsilon_{k-1}\cdot(3^{b_0}-1)}(b_0) \leq b\}$. Since C is ε_{k-1}-large, $h^B_{\varepsilon_{k-1}\cdot 3^{b_0}}(b_0)$ exists and B is $\varepsilon_{k-1}\cdot 3^{b_0}$-large. Remark that, as A is $(2k+2)$-dense, $a_0 \geq 2k+2$, and so $b_0 \geq 2k+2$. Then we obtain

$$\begin{aligned}
h^B_{\varepsilon_{k-1}\cdot 3^{b_0}}(b_0) &= h^B_{\varepsilon_{k-1}\cdot 3^{b_0-1}\cdot 2+\varepsilon_{k-1}\cdot 3^{b_0-1}}(b_0)\\
&= h^B_{\varepsilon_{k-1}\cdot 3^{b_0-1}\cdot 2}(h^B_{\varepsilon_{k-1}\cdot 3^{b_0-1}}(b_0))\\
&\geq h^B_{\varepsilon_{k-1}\cdot 3^{b_0-1}\cdot 2}(2^{3^{b_0-1}}b_0)\\
&\geq h^B_{\varepsilon_{k-1}\cdot 2(k+2)+\frac{b_0(b_0+1)}{2}+k+3}(2^{3^{b_0-1}}b_0)\\
&\geq h^B_{F(\varepsilon_{k-1})\odot(k+2)+\frac{b_0(b_0+1)}{2}+1}(2^{3^{b_0-1}}b_0)\\
&\geq h^B_{F(\varepsilon_{k-1})\odot(k+2)+\frac{b_0(b_0+1)}{2}+1}(b_0)\\
&\geq h^B_{F(\varepsilon_{k-1})\odot(k+2)+\frac{a_0(a_0+1)}{2}+1}(b_0).
\end{aligned}$$

The first and fourth inequality are because of Lemma 1.19 and the fact that

$$h^B_{\varepsilon_{k-1}\cdot 3^{b_0-1}\cdot 2}(b_0) \geq 2^{3^{b_0-1}}b_0 > \frac{b_0(b_0+1)}{2} + 3(k+2)+1 \geq b_0,$$

because of lemma 4.35 and $b_0 \geq \max\{2k+2, 5\}$. The second and the third inequality hold because of Lemma 4.2 and Lemma 1.19 and the last one is caused by $\min A \leq \min B$. Hence B is $(F(\varepsilon_{k-1})\odot(k+2)+\frac{a_0(a_0+1)}{2}+1)$-large. Lemma 4.32 yields A is $\omega_{a_0}(\varepsilon_{k-1}+1)$-large, i.e. ε_k-large. □

4.38 Corollary. *The following holds:*

$$\mathrm{ACA}_0 \vdash (\forall n)(\forall a)(\exists b)([a,b] \textit{ is } n\textit{-dense}) \rightarrow (\forall a)(\exists b)([a,b] \textit{ is } \varepsilon_\omega\textit{-large}).$$

4.39 Theorem. *The following holds:*

$$\mathrm{ACA}_0 + \mathrm{RT} \nvdash (\forall n)(\forall a)(\exists b)([a,b] \textit{ is } n\textit{-dense}).$$

Proof. Assume by contradiction that

$$\mathrm{ACA}_0 + \mathrm{RT} \vdash (\forall n)(\forall a)(\exists b)([a, b] \text{ is } n\text{-dense}).$$

Then, by Corollary 4.38,

$$\mathrm{ACA}_0 + \mathrm{RT} \vdash (\forall a)(\exists b)([a, b] \text{ is } \varepsilon_\omega\text{-large}),$$

which states that h_{ε_ω} is a provably total function of $\mathrm{ACA}_0 + \mathrm{RT}$. This contradicts the claim that for each provably total function of $\mathrm{ACA}_0 + \mathrm{RT}$ there exists $k \in \mathbb{N}$, such that h_{ε_k} eventually dominates this function. The validity of this last claim can be roughly seen as follows. First look at an ordinal analysis of $\mathrm{ACA}_0 + \mathrm{RT}$ (see e.g. [McA85] or [Afs09]) and notice that the proof-theoretic ordinal of this theory is ε_ω. Then combine these results with one of Buchholz mentioned in the introduction of [Buc97], to deduce that every arithmetical formula which is provable in $\mathrm{ACA}_0 + \mathrm{RT}$, is provable in PA plus the scheme of transfinite induction over all ordinals strictly below ε_ω. Then apply results of Friedman and Sheard on provably total functions of subsystems of first-order arithmetic (see Section 2 in [FS95]). □

Thus iterating the first-order notion of n-density gives rise to certain strength as it is unprovable in $\mathrm{ACA}_0 + \mathrm{RT}$. Now one could ask whether it is possible to weaken the finitary statement in order to really approximate the strength of RT (over RCA_0). To answer this question, we look at the first-order consequences of the n-density combined with PA. Let us start with the following result proved by McAloon in [McA85] (Theorem 4).

4.40 Theorem. *The theory* $\mathrm{ACA}_0 + \mathrm{RT}$ *has the same first-order consequences as the theory obtained from* PA *by iterating the uniform reflection principle arbitrarily often.*

Iterating our n-density, we end up with conservation for Π^0_2-statements.

4.41 Theorem. *The theories* $\mathrm{PA} + \{(\forall a)(\exists b)([a, b]$ *is* n*-dense*$)\}_{n\in\mathbb{N}}$ *and* $\mathrm{ACA}_0 + \mathrm{RT}$ *have the same* Π^0_2*-consequences.*

Proof. First remark that

$$\mathrm{PA} + \{(\forall a)(\exists b)([a,b] \text{ is } n\text{-dense})\}_{n\in\mathbb{N}} \subseteq \mathrm{ACA}_0 + \mathrm{RT},$$

because for every concrete $n \in \mathbb{N}$, Theorem 4.36 implies that

$$(\forall a)(\exists b)([a,b] \text{ is } n\text{-dense})$$

can be proved in $\mathrm{ACA}_0 + \mathrm{RT}$.

Now suppose $\mathrm{ACA}_0 + \mathrm{RT}$ proves $\Phi = \forall x \exists y \varphi(x,y)$, where φ is a Δ_0 formula. This means that the function defined by φ, i.e. $f(x) = \min\{y : \varphi(x,y)\}$, is provably total in $\mathrm{ACA}_0 + \mathrm{RT}$. Hence there exists $k \in \mathbb{N}$, such that h_{ε_k} eventually dominates this function (see the proof of Theorem 4.39). Note that $\mathrm{ACA}_0 + \mathrm{RT}$ is closed under primitive recursive definitions. Due to Lemma 4.37, we can pick $n > 2(k+1)$, such that $\mathrm{PA} + (\forall a)(\exists b)([a,b]$ is n-dense) proves the totality of h_{ε_k}, and so Φ. This completes the proof. □

Remark that many important mathematical and metamathematical theorems and conjectures are in Π^0_1- or Π^0_2-form. Among examples from the former category we find the quadratic reciprocity law, Fermat's Last Theorem, Goldbach's conjecture, the Riemann hypothesis and Con(T) for theories T. Examples belonging to the latter group are the twin prime conjecture, the Paris-Harrington principle, the finite version A of Kruskal's theorem which we presented in Subsection 1.1.2, the totality of several well-known functions (as e.g. the van der Waerden function or the Ackermann function) and 1-Con(T) for theories T.

4.4.3 Phase Transition

We also conjecture that the unprovability result above will give rise to a phase transition as described below. Half of the conjecture is proven. The other half needs a generalisation of Lemma 4.32, which looks plausible but remains unproven. Let us first introduce the parametrised version of n-density (see Definition 4.33). As in previous chapters, let f be the parameter function, such that $1 \leq f(x) \leq x$, for $x \in \mathbb{N}$.

4.42 Definition. *Let* $A \subseteq \mathbb{N}$, *with* $\min A > 3$. *Then* A *is called* 0-RT-*dense(f) if* $|A| > f(\min A)$. *We say* A *is* $(n+1)$-RT-*dense(f), if* $f(\min A) \geq n+1$ *and for every colouring*

$$G\colon [A]^{f(\min A)+2} \to 3^{f(\min A)},$$

there exists a subset $B \subseteq A$, *such that* B *is homogeneous for* G *and* B *is* n-RT-*dense(f).*

As before, we will leave out RT, and simply write down n-density(f). We need the following fast-growing hierarchy.

4.43 Definition. *Define the fast-growing hierarchy* $(D_k)_{k<\omega}$ *as follows. For every* $x \in \mathbb{N}$,

$$\begin{aligned} D_0(x) &= 2^x; \\ D_{k+1}(x) &= \underbrace{D_k(\dots(D_k}_{x \text{ times}}(x))\dots) = D_k^x(x), \end{aligned}$$

for all natural numbers k.

Recall that given a function $G\colon \mathbb{N} \to \mathbb{N}$, the inverse function $G^{-1}\colon \mathbb{N} \to \mathbb{N}$ is defined by $G^{-1}(x) = \min\{y : G(y) \geq x\}$. So D_k^{-1} will denote the inverse function of D_k, for all $k \in \mathbb{N}$. Sometimes brackets are used to avoid confusion. Let the notation $2_x(y)$ be defined by $2_0(y) = y$ and $2_{x+1}(y) = 2^{2_x(y)}$, for x and y natural numbers.

4.44 Theorem. *If* $f(x) = D_2^{-1}(x)$ *for every* $x \in \mathbb{N}$, *then*

$$\mathrm{RCA}_0 \vdash (\forall n)(\forall a)(\exists b)([a,b] \textit{ is } n\textit{-dense}(f)).$$

Proof. Let a and n be given and note that if $[x,b]$ is n-dense(f), with $x \geq a$, then so is $[a,b]$. The case $n = 0$ is dealt with easily, so suppose $n > 0$. For any natural number l, define $G_l\colon \mathbb{N} \to \mathbb{N}$ by $G_l(x) = 2_{4(l+2)}(3^l \cdot x)$. Ketonen and Solovay showed that with this definition $G_l(x)$ satisfies the following Ramseyan statement:

$$G_l(x) \to (x)^{l+2}_{3^l},$$

i.e. for any set X of size at least $G_l(x)$ and any colouring of the $(l+2)$-element subsets of X into 3^l colours, there exists $Y \subseteq X$ of size at least x which is homogeneous for the colouring (see Subsection 3.8 in [KS81]).

Choose $d \in \mathbb{N}$ large enough such that $D_d^{d-1}(d) \geq a$ and

$$|D_1^d(d) - D_1^{d-1}(d)| > \underbrace{G_d(\ldots(G_d}_{n \text{ times}}(d))\ldots) = G_d^n(d).$$

We now fix d and write G instead of G_d in order to lighten notation.

Claim: Any $X \subseteq [D_1^{d-1}(d), D_1^d(d)]$ with $|X| > G^k(d)$ is k-dense(f). We will prove the claim by induction k. Then, to conclude the proof of the theorem it suffices to notice that RCA_0 proves the totality of all D_l $(l \in \mathbb{N})$ and allows the simple combinatorial reasoning we use to prove the claim.

First remark that, for $x \in \mathbb{N}$

$$D_2^{-1}(x) = \min\{y \,:\, D_2(y) \geq x\} = \min\{y \,:\, D_1^y(y) \geq x\}.$$

Hence, for all x in the interval $[D_1^{d-1}(d), D_1^d(d)]$,

$$D_2^{-1}(x) = \min\{y \,:\, D_1^y(y) \geq x\} = d.$$

Now suppose $X \subseteq [D_1^{d-1}(d), D_1^d(d)]$ with $|X| \geq G^k(d)$. If $k = 0$, then

$$|X| > G^0(d) = d = D_2^{-1}(\min X) = f(\min X),$$

since $\min X \in [D_1^{d-1}(d), D_1^d(d)]$. Hence, X is 0-dense(f).

Now assume the claim for k and let X be a subset of $[D_1^{d-1}(d), D_1^d(d)]$ with $|X| > G^{k+1}(d) = G(G^k(d))$. Let

$$H\colon [X]^{f(\min X)+2} \to 3^{f(\min X)},$$

be any colouring and notice that $f(\min X) = d$, since $\min X$ is an element of $[D_1^{d-1}(d), D_1^d(d)]$. Because of

$$G(x) \to (x)_{3^d}^{d+2},$$

there exists $Y \subseteq X$ homogeneous for H, such that $|Y| \geq G^k(d)$. The induction hypothesis yields that Y is k-dense(f), so X is $(k+1)$-dense(f).

$\square$

For determining an upper bound, we will use α-f-largeness, which is a parameterised version of α-largeness. In other words, we want to make the notion of α-largeness dependent on a function f. Let $f\colon \mathbb{N} \to \mathbb{N}$ be a nondecreasing function, such that $1 \leq f(x) \leq x$, for $x \in \mathbb{N}$. We introduce another hierarchy of functions, which differs slightly from the original Hardy hierarchy and is handy for introducing α-f-largeness.

4.45 Definition. *Define the hierarchy of functions $(h^A_{f,\alpha})_{\alpha \leq \varepsilon_\omega}$ as follows. For every $x \in \mathbb{N}$,*

$$\begin{aligned} h^A_{f,0}(x) &\simeq x; \\ h^A_{f,\alpha+1}(x) &\simeq h^A_{f,\alpha}(h^A(x)); \\ h^A_{f,\lambda}(x) &\simeq h^A_{f,\lambda[f(x)]}(x), \end{aligned}$$

where α is an ordinal below ε_ω, and λ a limit ordinal less than or equal to ε_ω. The hierarchy $(h^A_{f,\alpha})_{\alpha \leq \varepsilon_\omega}$ is called the Hardy hierarchy based on h^A, relative to f.

Remark that in case A equals $\mathbb{N}$, then h^A becomes the normal successor function. If in addition f would equal the identity function, then $(h^A_{f,\alpha})_{\alpha \leq \varepsilon_\omega}$ is the standard Hardy hierarchy as given by Definition 1.13.

Now we are ready to give the definition of an α-f-large set.

4.46 Definition. *A set $A \subseteq \mathbb{N}$ is called α-f-*large *if $h^A_{f,\alpha}(\min A)$ is defined.*

Whenever it is clear which set A and which function f we are working with, we leave out the super- and subscript and simply write h and h_α, instead of h^A and $h^A_{f,\alpha}$. As before, in the lemmas below, we will assume all functions $h^A_{f,\alpha}$ occurring are acting on their domain, so we can replace $\simeq$ by $=$. If not mentioned explicitly, A will denote an arbitrary subset of $\mathbb{N} \setminus \{0\}$. The definitions of *$\alpha$-$f$-small* and *exactly α-f-large* are similar to Definition 1.15 and Definition 1.16.

The following lemmas are the analogues of Lemma 1.17, Lemma 1.18, Lemma 1.19 and Lemma 1.20, but with α-f-largeness instead of α-largeness. As one can see below, the original proofs generalise rather straightforwardly.

Make sure not to confuse the hierarchy h_α which we use in this chapter (Definition 4.45), with the ones we have seen earlier (Definition 1.13 and Definition 1.14). Remark that, if not mentioned clearly, all ordinals α and β in this chapter are smaller than ε_ω.

4.47 Lemma. *For every α and every $\beta \gg \alpha$, $h_{\beta+\alpha} = h_\beta \circ h_\alpha$.*

Proof. By induction on α. The base and successor step are exactly the same as in the original case. So, let λ be limit and assume the lemma for all $\alpha < \lambda$. By Lemma 1.4 we have if $\beta \gg \lambda$, then $\beta \gg \lambda[f(b)]$ for all b. It follows by the induction hypothesis that if $\beta \gg \lambda$, then for all b

$$h_{\beta+\lambda}(b) = h_{(\beta+\lambda)[f(b)]}(b) = h_{\beta+\lambda[f(b)]}(b) = h_\beta \circ h_{\lambda[f(b)]}(b) = h_\beta \circ h_\lambda(b),$$

as required. □

We can restate this fact in the following manner.

4.48 Lemma. *Let A be a finite set and let $\beta \gg \alpha$. Then A is $(\beta+\alpha)$-f-large if and only if there exists $u \in A$ such that $\{x \in A \mid x \leq u\}$ is α-f-large and $\{x \in A \mid u \leq x\}$ is β-f-large.*

4.49 Lemma. *For every $\alpha \leq \varepsilon_\omega$:*

1. *h_α is increasing;*
2. *for every $\beta < \varepsilon_\omega$ and $b \in \mathbb{N}$: if $\alpha \Rightarrow_{f(b)} \beta$ and $h_\alpha(b)$ exists, then $h_\beta(b)$ exists and $h_\alpha(b) \geq h_\beta(b)$.*

Proof. By simultaneous induction on α. The base and successor step are exactly the same as in the original case. So, assume both claims for each $\alpha < \lambda$, λ limit. We show the second part. Let β and b be such that $\lambda \Rightarrow_{f(b)} \beta$. If $\lambda = \beta$, then we are done. So suppose $\lambda > \beta$, then also $\lambda[f(b)] \Rightarrow_{f(b)} \beta$ and

$$h_\lambda(b) = h_{\lambda[f(b)]}(b) \geq h_\beta(b),$$

where the inequality holds because of the induction hypothesis. In particular the right hand side exists. Let us show the first claim for λ.

So let $x < y$ be elements of the domain of h_λ. Then by Lemma 1.6, $\lambda[f(y)] \Rightarrow_{f(y)} \lambda[f(x)]$, hence, by the second claim for $\alpha = \lambda[f(y)]$ and $\beta = \lambda[f(x)]$,

$$h_\lambda(y) = h_{\lambda[f(y)]}(y) \geq h_{\lambda[f(x)]}(y) \geq h_{\lambda[f(x)]}(x) = h_\lambda(x),$$

where the second inequality holds because of the induction hypothesis. $\square$

4.50 Lemma. *Let $A = \{a_0, a_1, \dots\}$ and $B = \{b_0, b_1, \dots\}$ be finite sets.*

1. *If $|A| = |B|$ and for every $i < |A|$, $b_i \leq a_i$, then for every $i < |A|$, if $h^A_\alpha(a_i)$ exists, then $h^B_\alpha(b_i)$ exists and $h^A_\alpha(a_i) \geq h^B_\alpha(b_i)$.*

2. *If A is α-f-large, $|A| = |B|$ and for every $i < |A|$, $b_i \leq a_i$, then B is α-f-large.*

3. *If $A \subseteq B$ and A is α-f-large, then B is α-f-large.*

Proof. The first statement is easy to prove by induction on α, having in mind that f is nondecreasing (so $x \leq y$ implies $f(x) \leq f(y)$). The second statement is a consequence of the first. The third part follows from the fact that if $A \subseteq B$ then B has an initial segment of size $|A|$. Apply the second statement to A and that initial segment. Clearly, if a set has an α-f-large initial segment it is α-f-large itself. $\square$

The following is the parametrised version of Lemma 4.32. Here $\xrightarrow{f}$ is the parametrised version $\rightarrow$, i.e. one uses α-f-largeness instead of α-largeness.

4.51 Conjecture. *Let $k \in \mathbb{N}$, with $3 \leq k$, $m \in \mathbb{N} \cup \{-1\}$ and $\alpha = F(\varepsilon_m) \odot (m+3) + \frac{(k-2)(k-1)}{2} + 1$. Let A be such that $A \xrightarrow{f} (\alpha)^k_{3^{k-2}}$ and $\min(A) \geq \max\{k, m+2, 4\}$. Then A is $\omega_{k-2}(\varepsilon_m + 1)$-$f$-large.*

Given the moderate parameter functions we deal with in this chapter ($(D^k_1)^{-1}$ and D_2), we do not expect troubles proving the conjecture. So far, we have not carried out the investigation though.

4.52 Lemma. *For all $k, x \in \mathbb{N}$, with $f(x) \geq 2k+2$,*

$$h^B_{\varepsilon_{k-1}\cdot 3^{f(x)-1}}(x) \geq h^B_{f(x)\cdot 3^{f(x)-1}}(x).$$

Proof. As $f(x) \geq 2$ and $\varepsilon_{k-1} \Rightarrow_1 \omega$, Lemma 1.7 implies $\varepsilon_{k-1} \Rightarrow_n \omega$, for all $k, n \in \mathbb{N}$ with $n \geq 1$. In particular, $\varepsilon_{k-1} \Rightarrow_{f(x)} \omega$. Then the following holds:

$$\begin{aligned}
h^B_{\varepsilon_{k-1}\cdot 3^{f(x)-1}}(x) &= \underbrace{h^B_{\varepsilon_{k-1}}(\dots(h^B_{\varepsilon_{k-1}}}_{3^{f(x)-1}\text{ times}}(x))\dots) \\
&= \underbrace{h^B_{\varepsilon_{k-1}}(\dots(h^B_{\varepsilon_{k-1}}}_{3^{f(x)-1}-1\text{ times}}(h^B_\omega(x)))\dots) \\
&\geq \dots \\
&\geq \underbrace{h^B_{\omega}(\dots(h^B_{\omega}}_{3^{f(x)-1}\text{ times}}(x))\dots) \\
&\geq \underbrace{h^B_{\omega}(\dots(h^B_{\omega}}_{3^{f(x)-1}-1\text{ times}}(h^B_{f(x)}(x)))\dots) \\
&\geq \underbrace{h^B_{f(x)}(\dots(h^B_{f(x)}}_{3^{f(x)-1}\text{ times}}(x))\dots) \\
&\geq h^B_{f(x)\cdot 3^{f(x)-1}}(x),
\end{aligned}$$

using Lemma 4.47 and Lemma 4.49 a couple of times. □

The next lemma, is the parameterised version of Lemma 4.37.

4.53 Lemma. *Let $n \in \mathbb{N}$ and $A \subseteq \mathbb{N}$, such that* $\min A > 4$, *and suppose Conjecture 4.51 is provable. If A is $2n$-dense(f), then A is ε_{n-1}-f-large.*

Proof. The proof is similar to the one of Lemma 4.37, but adapted to the situation of α-f-largeness. By induction on n. Put $a_0 = \min A$.

If n equals zero, then A is 0-dense(f) and $|A| > f(a_0)$. So $h^A_{f(a_0)}(a_0) = h^A_\omega(a_0) = h^A_{\varepsilon_{-1}}(a_0)$ exists, so A is ε_{-1}-f-large.

Assume the statement for $n = k$ and let A be $(2k+2)$-dense(f). Let $G : [A]^{f(a_0)+2} \to 3^{f(a_0)}$ be any function and B a subset of A such that B is homogeneous for G and B is $(2k+1)$-dense(f). To invoke Lemma 4.51 we will show that B is $(F(\varepsilon_{k-1}) \odot (k+2) + \frac{f(a_0)(f(a_0)+1)}{2} + 1)$-$f$-large, with F as defined by Definition 4.16. Put $b_0 = \min B$ and

$$B_i = \{b \in B \,:\, h^B_{\varepsilon_{k-1}\cdot(i-1)}(b_0) \leq b < h^B_{\varepsilon_{k-1}\cdot i}(b_0)\},$$

for $i \in \{1, \ldots, 3^{f(b_0)} - 1\}$. Define $H\colon [B]^{f(b_0)+2} \to 3^{f(b_0)}$ by

$$H(x_1, \ldots, x_{f(b_0)+2}) = \begin{cases} 1 & \text{if } x_1 \in B_1 \\ 2 & \text{if } x_1 \in B_2 \\ \vdots & \qquad\vdots \\ 3^{f(b_0)} & \text{if } h^B_{\varepsilon_{k-1}\cdot(3^{f(b_0)}-1)}(b_0) \leq x_1 \end{cases},$$

for each $(x_1, \ldots, x_{f(b_0)+2}) \in [B]^{f(b_0)+2}$. Since B is $(2k+1)$-dense(f), there must exist $C \subseteq B$, such that C is homogeneous for H and C is $2k$-dense(f). The induction hypothesis yields that C is ε_{k-1}-f-large. C cannot take any colour i with $i < 3^{f(b_0)}$, since every B_i is ε_{k-1}-f-small. Thus $C \subseteq \{b \in B :$ $h^B_{\varepsilon_{k-1}\cdot(3^{b_0}-1)}(b_0) \leq b\}$. Since C is ε_{k-1}-f-large, $h^B_{\varepsilon_{k-1}\cdot 3^{f(b_0)}}(b_0)$ exists and B is $\varepsilon_{k-1}\cdot 3^{f(b_0)}$-$f$-large. Remark that, as A is $(2k+2)$-dense, $f(a_0) \geq 2k+2$, and so $f(b_0) \geq 2k+2$. Then we obtain

$$\begin{aligned}
h^B_{\varepsilon_{k-1}\cdot 3^{f(b_0)}}(b_0) &= h^B_{\varepsilon_{k-1}\cdot 3^{f(b_0)-1}\cdot 2}(h_{\varepsilon_{k-1}\cdot 3^{f(b_0)-1}}(b_0)) \\
&\geq h^B_{\varepsilon_{k-1}\cdot 3^{f(b_0)-1}\cdot 2}(h^B_{f(b_0)\cdot 3^{f(b_0)-1}}(b_0)) \\
&\geq h^B_{\varepsilon_{k-1}\cdot 3^{f(b_0)-1}\cdot 2}(h^B_{\frac{f(b_0)(f(b_0)+1)}{2}+k+3}(b_0)) \\
&\geq h^B_{\varepsilon_{k-1}\cdot 2(k+2)}(h^B_{\frac{f(b_0)(f(b_0)+1)}{2}+k+3}(b_0)) \\
&= h^B_{\varepsilon_{k-1}\cdot 2(k+2)+\frac{f(b_0)(f(b_0)+1)}{2}+k+3}(b_0) \\
&= h^B_{F(\varepsilon_{k-1})\odot(k+2)+\frac{f(b_0)(f(b_0)+1)}{2}+1}(b_0) \\
&\geq h^B_{F(\varepsilon_{k-1})\odot(k+2)+\frac{f(a_0)(f(a_0)+1)}{2}+1}(b_0).
\end{aligned}$$

The first inequality holds because of Lemma 4.49 and Lemma 4.52. Notice that

$$3^{f(b_0)-1} \cdot f(b_0) \geq \frac{f(b_0)(f(b_0)+1)}{2} + k + 3,$$

and

$$3^{f(b_0)-1} \cdot 2 > f(b_0) + 2 = 2(k+2),$$

as $f(b_0) \geq 2k+2 \geq 2$, which results in the second and the third inequality. The last one is caused by $a_0 \leq b_0$.

Hence B is $(F(\varepsilon_{k-1}) \odot (k+2) + \frac{f(a_0)(f(a_0)+1)}{2} + 1)$-$f$-large. Lemma 4.51 yields A is $\mathrm{tow}_{f(a_0)}(\varepsilon_{k-1}+1)$-large, i.e. ε_k-f-large. □

Following earlier agreements we write $(D_1^k)^{-1}$ to denote the inverse of D_1^k, where brackets are used to avoid confusion.

4.54 Conjecture. *Suppose Conjecture 4.51 is provable in* $\mathrm{ACA}_0+\mathrm{RT}$*. Let* $k \in \mathbb{N}$*. If* $f(x) = (D_1^k)^{-1}(x)$ *for every* $x \in \mathbb{N}$*, then*

$$\mathrm{ACA}_0 + \mathrm{RT} \nvdash (\forall n)(\forall a)(\exists b)([a,b] \textit{ is } n\textit{-dense}(f)).$$

Proof sketch. Assume by contradiction that

$$\mathrm{ACA}_0 + \mathrm{RT} \vdash (\forall n)(\forall a)(\exists b)([a,b] \text{ is } n\text{-dense}(f)).$$

Then

$$\mathrm{ACA}_0 + \mathrm{RT} \vdash (\forall a)(\exists b)([a,b] \text{ is } \varepsilon_\omega\text{-}f\text{-large}),$$

which states that h_{f,ε_ω} is a provably total function of $\mathrm{ACA}_0 + \mathrm{RT}$. Notice that for fixed $k \in \mathbb{N}$,

$$f(x) = (D_1^k)^{-1}(x) \geq (D_1^{h_{\varepsilon_\omega}^{-1}(x)})^{-1}(x),$$

for x large enough. Proceed as in the proof of Theorem 4.39 to obtain a contradiction using a generalisation of Theorem 2 in [MW] up to ε_ω. □

Chapter 5

NASH-WILLIAMS RAMSEY THEORY

5.1 Original Nash-Williams Theorem and extensions

5.1.1 Introduction

Some of the many results Crispin Nash-William left us, have a clear connection with unprovability theory or related areas. He introduced, for instance, the *minimal bad sequence* argument and used it as an elegant, new approach to prove Higman's lemma and Kruskal's theorem ([NW63]). Both statements generate surprising strength and are classical examples in unprovability theory. The proofs are short and appealing thanks to the minimal bad sequence argument, which is a method that possess a lot of strength. More precisely, Alberto Marcone showed in [Mar96] that the general version of that argument has the strength of Π^1_1-CA_0.

Here we will study another result of Nash-Williams, namely a generalisation of Ramsey's theorem to families of finite subsets of $\mathbb{N}$ (Theorem 5.2). It was first presented in [NW65] and has gained quite some attention ever since (see e.g. [PR82, KT91, Far04, AT05]). The theorem is usually called Nash-Williams' partition theorem, but we will simply call it the Nash-Williams Theorem, or NWT for short.

Make sure not to confuse NWT with the Nash-Williams' theorem on transfinite sequences, which was originally presented in [NW68]. This is a different result of Nash-Williams which strength is conjectured to be equivalent to ATR_0 ([Mar96]).

Our original motivation was the study of unprovable statements on the level of ATR_0. We planned to investigate NWT and related first-order statements which might lead to unprovability phenomena. Soon other questions popped up, which also seemed worth having a closer look at. What is presented here is part of that still ongoing research. Some of the results are not mentioned, but can be found in [BDS]. In this chapter we briefly discuss the original NWT and present a generalisation called *relations Nash-Williams Theorem*. We also investigate the strength of NWT and end with an unprovable statement related to Schreier families.

While studying Nash-Williams Ramsey theory the book *Ramsey methods in analysis* by Spiros Argyros and Stevo Todorcevic has been very helpful. Among other results, we present generalisations of some definitions and proofs given in [AT05], namely Definition 5.4, Definition 5.6, Theorem 5.9 and Corollary 5.10.

5.1.2 Original Nash-Williams Theorem

Throughout this chapter, $\mathcal{F}$ will denote a family of finite structures. Depending on which finite structures we are working with, $s \sqsubseteq t$ (where $s, t \in \mathcal{F}$) will get its meaning.

Let $s = \{s_0, \ldots, s_n\}$ and $t = \{t_0, \ldots, t_m\}$ be finite subsets of $\mathbb{N}$. Unless stated differently we will assume the elements of sets of natural numbers are given in increasing order. Then we say that $s \sqsubseteq t$ if s is an initial segment of t, i.e. $n \leq m$ and $s_i = t_i$ for all $i \leq n$. The family $\mathcal{F}$ will be a family of finite subsets of $\mathbb{N}$. For $M \subseteq \mathbb{N}$, $\mathcal{F} \upharpoonright M$ will be a shorthand for $\mathcal{F} \cap \mathcal{P}(M)$[1].

The next definitions are given for the specific case of subsets of natural numbers described above. However, they easily generalise to other situations, depending on the meaning of $\mathcal{F}$ and $\sqsubseteq$.

[1] Notice that the symbol $\upharpoonright$ has a different meaning here than in Chapters 1–4

5.1 Definition. 1. *A family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ is called* thin *if $s \not\sqsubseteq t$ for every pair of distinct members s and t of $\mathcal{F}$.*

2. *A family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ is called* Ramsey *if for every infinite set $M \subseteq \mathbb{N}$ and every finite partition*

$$\mathcal{F} = \mathcal{F}_0 \cup \ldots \cup \mathcal{F}_k$$

there is an infinite set $N \subseteq M$ such that at most one of the restrictions

$$\mathcal{F}_0 \restriction N \,, \; \ldots , \; \mathcal{F}_k \restriction N$$

is non-empty.

5.2 Theorem (Nash-Williams)**.** *Every thin family of finite subsets of $\mathbb{N}$ is Ramsey.*

Proof. See [NW65] for the original proof. □

Instead of finite sets of natural numbers, one can also look at other structures as, for example, finite words in a finite alphabet. We elaborate for a moment on this rather trivial example. Suppose $(\Sigma, \leq)$ is a finite partial order, called the alphabet, and let Σ^* be the set of all finite sequences of elements of Σ, called the finite words. It is possible to order Σ^* by $\sqsubseteq$ as follows. Let $s = s_0 s_1 ... s_m$ and $t = t_0 t_1 ... t_n$ be finite words, i.e. elements of Σ^*. Then $s \sqsubseteq t$ if there exist $k_0 < k_1 < \ldots < k_m \leq n$ such that $s_i \leq t_{k_i}$ for all $i \leq m$.

Using infinite words, i.e. infinite sequences of elements of Σ, thinness and Ramseyness are defined for families of words in the same way as above. Remark that a subsequence of an infinite word, is again a word.

Also in this setup it turns out that thin families are Ramsey. Indeed, suppose a thin family $\mathcal{F}$ of words is given, as well as a finite partition of $\mathcal{F}$, and an infinite word W. As the alphabet Σ is finite, W contains at least one element of Σ infinitely many times. Call this element q and define $V = qqqqq\ldots$, i.e. an infinite word consisting of only the element q. Then clearly $V \subseteq W$. There are two possible cases.

Suppose that for every natural number $n > 0$, the finite word $w = \underbrace{qq\dots q}_{n \text{ times}}$ does not belong to $\mathcal{F}$. Then we are done, as no member of the partition will contain a subsequence of V. Now assume there exists a natural number $n_0 > 0$ such that the finite word $w = \underbrace{qq\dots q}_{n_0 \text{ times}}$ belongs to $\mathcal{F}$. Since $\mathcal{F}$ is thin, this n_0 is unique and once again we are done, as exactly one member of the partition will contain a subsequence of V.

It is possible to give similar examples of structures involving sequences of natural numbers, trees or graphs, such that thin families are Ramsey.

As one could suspect analysing the proof above, the example with finite words does not possess much strength. The original NWT, on the contrary, turns out to be much stronger as we will see in Section 5.2.

The examples above are instances of the *template*[2] "thin implies Ramsey". One of the aims of our study is to fully analyse this template, and the strength it gives rise to. In particular, we will look at another instance of the template by generalising the original NWT to labelled structures. In the next subsection we will explore that new setup.

5.1.3 Relational Nash-Williams Theorem

As announced above, we will generalise NWT and work with families of *finite labelled structures* instead of finite subsets of natural numbers. We start by introducing some new concepts.

5.3 Definition. *1. A* structure *s is a set with relations $R_1, R_2, \dots, R_n$, with $n \in \mathbb{N}$, on it. The relations have finite arity and the elements of the set will be called* vertices.

2. A labelled structure *s is a structure with natural number labels on the vertices, such that each label occurs only finitely many times. If the set of vertices is finite we call s a* finite labelled structure. *Otherwise s is called an* infinite labelled structure.

[2]We borrowed the word "template" from Harvey Friedman and his explanations presenting Boolean Relation Theory as the study of a template.

3. *We call two labelled structures* s_1 *and* s_2 isomorphic *if there exists a bijection between* s_1 *and* s_2 *which is label preserving, relation preserving and non-relation preserving. We write* $s_1 \simeq s_2$.

4. *A labelled structure* s_1 *is an* initial segment *of a labelled structure* s_2 *if there exists* $s_3 \subseteq s_2$ *such that* $s_1 \simeq s_3$ *and* $s_2 \setminus s_3$ *has only bigger labels than* s_3. *We write* $s_1 \sqsubseteq s_2$.

We will use lowercase letters $r, s, t, \ldots$ and uppercase letters $M, N, P, \ldots$ to denote finite and infinite labelled structures, respectively. Unless specially needed, we will not mention the relations and identify a finite or infinite labelled structure with its set of vertices.

We will work with classes of labelled structures in $R_1, \ldots, R_n$ which contain at least one infinite member and such that every subset of a member of the class is a member of the class, by inheriting labels and restricting relations. If in a statement several (families of) labelled structures are considered, it is implicitly assumed that exactly the same relations act on those structures.

The original idea was that this setup could result in a new model-theoretic approach for proving unprovability theorems. The relations of a labelled structure could play the role of formulas, and some new kind of indiscernibles would emerge. So far, this has not been achieved. So the reader should not expect many purely logical results in this section.

What is presented here is intended to ensure that the setup works well, and that the template indeed has instances of more general nature. In addition, it allows us to introduce in a general context a method called *combinatorial forcing*, which has been used by Nash-Williams for proving his original theorem.

Given a finite labelled structure s and an infinite labelled structure M, we define M/s as the set of all vertices in M whose labels are larger than the maximum label of s. By a previous remark, M/s is an infinite labelled structure, by inheriting labels and restricting relations from M. Finally we put

$$M^{[<\infty]} = \{s \subseteq M \, : \, s \text{ is finite}\}.$$

Before stating the main theorem of this section, we give some more definitions and two lemmas which will be used in the proof. Our approach is a generalisation of the one employed in Chapter 2 of [AT05], where Todorcevic investigates Nash-Williams' theory of fronts and barriers.

5.4 Definition. *Let $\mathcal{F}$ be a family of finite labelled structures and M an infinite labelled structure.*

1. *Define the* extensor $E_{\mathcal{F},M}\colon M^{[<\infty]} \to \mathcal{P}(M^{[<\infty]})$ *such that for every* $s \in M^{[<\infty]}$,

$$E_{\mathcal{F},M}(s) = \{t \in M^{[<\infty]} \,:\, s \sqsubseteq t \,\&\, (\exists u \in \mathcal{F})(u \sqsubseteq t)\}.$$

2. *Let $N \subseteq M$. Then $s \in M^{[<\infty]}$ is called:*

   - inextensible *in N if $N^{[<\infty]}$ contains no member of $E_{\mathcal{F},M}(s)$;*
   - extensible *in N if $N^{[<\infty]}$ contains a member of $E_{\mathcal{F},M}(s)$;*
   - strongly extensible *in N if for every infinite labelled structure $P \subseteq N$, s is extensible in P.*

If it is clear which family $\mathcal{F}$ and which infinite labelled structure M we are working with, we simply write $E(s)$, instead of $E_{\mathcal{F},M}(s)$.

5.5 Lemma. *Let M be an infinite labelled structure and $\mathcal{F}$ a family of finite labelled structures. Then there exists $N \subseteq M$, such that every finite subset of N is either inextensible in N or strongly extensible in N.*

Proof. We will define a sequence $n_1, n_2, \ldots \in M$ in stages.

Stage 1: Let s_0 be a finite labelled structure such that the set of vertices of s_0 is empty. If there is an infinite $A \subseteq M$ such that s_0 is inextensible in A, put $N_1^0 = A$. Otherwise, put $N_1^0 = M$. Let n_1 be a vertex of N_1^0 with the smallest label in N_1^0.

Stage $k+1$ ($k > 0$): So far, we have defined $\{n_1, \ldots, n_k\}$ and the infinite labelled structure $N_k^0 \subseteq M$ such that every finite subset of $\{n_1, \ldots, n_k\}$ is either inextensible or strongly extensible in N_k^0. If possible, consider a finite labelled structure s_k^0 by taking a subset of $\{n_1, \ldots, n_k\}$, in such a way that we have not dealt yet before with a finite labelled structure $t \subseteq \{n_1, \ldots, n_k\}$ with $s_k^0 \simeq t$. If there is an infinite $A \subseteq N_k^0$ such that s_k^0 is inextensible in A, put $N_k^1 = A$. Otherwise (i.e. s_k^0 is strongly extensible in N_k^0), put $N_k^1 = N_k^0$. As long as there is a finite labelled structure $s_k^i \subseteq \{n_1, \ldots, n_k\}$ and we have not dealt yet previously with a finite labelled structure $t \subseteq \{n_1, \ldots, n_k\}$ with $s_k^i \simeq t$, we continue as before in order to define $N_k^{i+1} \subseteq N_k^i$. At a certain moment we will have considered all possible finite labelled structures $s \subseteq \{n_1, \ldots, n_k\}$, up to isomorphism. Suppose the last infinite labelled structure we have defined was N_k^l. Then define N_{k+1}^0 as N_k^l and n_{k+1} as a vertex of $N_{k+1}^0 \setminus \{n_1, \ldots, n_k\}$ with lowest label. Proceed by moving to the next stage.

Define N as the infinite labelled structure with set of vertices $\{n_i \in M : 1 \leq i\} \subseteq M$ by inheriting labels and restricting relations. One easily verifies that N satisfies the conditions mentioned in the statement. □

5.6 Definition. *Let $\mathcal{D}$ be a set of infinite labelled structures.*

1. *The set $\mathcal{D}$ is called* open *if for all infinite labelled structures N and M, $N \subseteq M \in \mathcal{D}$ implies $N \in \mathcal{D}$.*

2. *The set $\mathcal{D}$ is called* dense below M *if for every infinite labelled structure N with $N \subseteq M$ there exists $P \subseteq N$ such that $P \in \mathcal{D}$.*

3. *A* dense-open-set assignment *on M is a family $\mathcal{D}_s$ ($s \in M^{[<\infty]}$) such that for all $s \in M^{[<\infty]}$, $\mathcal{D}_s$ is a set of infinite labelled structures and*

$$\mathcal{D}_s \textit{ is open and dense below } M/s.$$

5.7 Lemma. *For every infinite labelled structure M and for every dense-open-set assignment $\mathcal{D}_s$ ($s \in M^{[<\infty]}$) on M, there exists an infinite labelled structure $N \subseteq M$, such that $N/s \in \mathcal{D}_s$ for all $s \in N^{[<\infty]}$.*

Proof. We will define a sequence $n_1, n_2, \ldots \in M$ in stages.

Stage 1: Let s_0 be a finite labelled structure such that the set of vertices of s_0 is empty. Due to density there exists an infinite labelled structure

$N_1^0 \subseteq M$, such that $N_1^0/s_0 \in \mathcal{D}_{s_0}$. Define n_1 as a vertex in N_1^0 with lowest label in N_1^0. Move to stage 2.

Stage $k+1$ ($k > 0$): So far, the vertices $n_1, \ldots, n_k$ and the infinite labelled structure $N_k^0 \subseteq M$ have been defined in such a way that $N_k^0/s \in \mathcal{D}_s$ for every finite labelled structure $s \subseteq \{n_1, \ldots, n_k\}$. If possible, consider a finite labelled structure s_k^0 by taking a subset of $\{n_1, \ldots, n_k\}$, in such a way that we have not dealt yet before with a finite labelled structure $t \subseteq \{n_1, \ldots, n_k\}$ with $s_k^0 \simeq t$. Since $\mathcal{D}_{s_k^0}$ is dense below M/s_k^0 there exists an infinite labelled structure $N_k^1 \subseteq N_k^0/s_k^0$ such that $N_k^1 \in \mathcal{D}_{s_k^0}$. As long as there is a finite labelled structure $s_k^i \subseteq \{n_1, \ldots, n_k\}$ and we have not dealt yet previously with a finite labelled structure $t \subseteq \{n_1, \ldots, n_k\}$ with $s_k^i \simeq t$, we continue as before in order to define $N_k^{i+1} \subseteq N_k^i$. At a certain moment we will have considered all possible finite labelled structures $s \subseteq \{n_1, \ldots, n_k\}$, up to ismorphism. Suppose the last infinite labelled structure we have defined was N_k^l. Then define N_{k+1}^0 as N_k^l and n_{k+1} as a vertex of $N_{k+1}^0 \setminus \{n_1, \ldots, n_k\}$ with lowest label. Proceed by moving to the next stage.

Define N as the infinite labelled structure with set of vertices $\{n_i \in M : 1 \leq i\} \subseteq M$ by inheriting labels and restricting relations. One easily verifies that N satisfies the conditions mentioned in the statement. $\square$

Thinness and Ramseyness can also be defined in the case of families of finite labelled structures, which is done in the next definition.

5.8 Definition. *1. A family $\mathcal{F}$ of finite labelled structures is called* thin *if $s \not\sqsubseteq t$ for every pair of distinct members s and t of $\mathcal{F}$.*

2. A family $\mathcal{F}$ of finite labelled structures is called Ramsey *if for every infinite labelled structure M and every finite partition*

$$\mathcal{F} = \mathcal{F}_0 \cup \ldots \cup \mathcal{F}_k$$

there is an infinite labelled structure $N \subseteq M$ such that at most one of

$$\mathcal{F}_0 \cap N^{[<\infty]}, \ \ldots, \ \mathcal{F}_k \cap N^{[<\infty]}$$

is non-empty.

Remark that we relativise the notion of Ramseyness to a certain infinite labelled structure M in order to avoid trivial cases. If we did not require $N \subseteq M$, then it would suffice to take any infinite labelled structure N which has no vertices in common with previously considered structures.

5.9 Theorem. *For any infinite labelled structure M, any thin family $\mathcal{F}$ of finite labelled structures and any partition $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$, there is an infinite labelled structure $N \subseteq M$, such that at most one of $\mathcal{F}_0 \cap N^{[<\infty]}$ and $\mathcal{F}_1 \cap N^{[<\infty]}$ is non-empty.*

Proof. Whenever a family of finite labelled structures is required in the proof (e.g. in definitions or for applying lemmas), we will work with $\mathcal{F}_0$. Apply Lemma 5.5 (with $\mathcal{F}_0$ as the family of finite labelled structures) to obtain an infinite labelled structure $N \subseteq M$, such that for any finite labelled structure $s \subseteq N$ either s is inextensible in N or s is strongly extensible in N. So, we could as well have shrunk M to have the desired properties. Let us rename N by M.

For any $s \in M^{[<\infty]}$, define $\mathcal{D}_s$ as the collection of all infinite labelled structures P of M/s such that either

(1) the finite labelled structure $s \cup \{v\} \subseteq M$ is inextensible in M, for all vertices $v \in P/s$;

or

(2) the finite labelled structure $s \cup \{v\} \subseteq M$ is strongly extensible in M, for all vertices $v \in P/s$.

One can verify that $\mathcal{D}_s$ ($s \in M^{[<\infty]}$) is a dense-open-set assignment on M, so by Lemma 5.7 (with $\mathcal{F}_0$) there exists an infinite labelled structure $N \subseteq M$, such that $N/s \in \mathcal{D}_s$ for all $s \in N^{[<\infty]}$. This will be the N we are looking for.

Let s_0 be a finite labelled structure such that the set of vertices of s_0 is empty. We distinguish two cases.

Case 1: s_0 is inextensible in M and therefore in N. So N contains no member of $E(s_0) = E_{\mathcal{F}_0}(s_0)$ (recall Definition 5.4 of the extensor $E_{\mathcal{F}}$). Since the set of vertices of s_0 is empty, $\mathcal{F}_0 \subseteq E(s_0)$. Thus N contains no member of $\mathcal{F}_0$, which settles this case.

Case 2: s_0 is strongly extensible in M and therefore in N. Note that by the definition of $E(s)$, a finite labelled structure s is extensible in N if and only if there exists a vertex $v \in N/s$ such that the finite labelled structure $s \cup \{v\} \subseteq N$ is extensible in N. So by the choice of N, a finite labelled structure s is extensible in N if and only if for all vertices $v \in N/s$ the finite labelled structure $s \cup \{v\} \subseteq N$ is extensible in N. It follows that every finite labelled structure $s \subseteq N$ is (strongly) extensible in N. Combining this statement with the fact that $\mathcal{F}$ is thin and the definition of $E(s)$, one can conclude that N cannot contain a member of $\mathcal{F} \setminus \mathcal{F}_0$. $\square$

Scrutinising the proof above, one might notice that it is unnecessary to require to whole family to be thin. It suffices that one of both members of the partition is thin.

5.10 Corollary. *Any thin family $\mathcal{F}$ of finite labelled structures is Ramsey.*

Proof. It suffices to show that for every pair $\mathcal{F}$ and $\mathcal{F}_0 \subseteq \mathcal{F}$ of thin families of finite labelled structures and for every infinite labelled structure M, there exists an infinite labelled structure $N \subseteq M$ such that either $\mathcal{F}_0 \cap N^{[<\infty]} = \emptyset$, or $\mathcal{F} \cap N^{[<\infty]} \subseteq \mathcal{F}_0$. This is exactly what Theorem 5.9 says. $\square$

Clearly, if we restrict ourselves to families of finite subsets of natural numbers and define $\sqsubseteq$ accordingly, then Corollary 5.10 is exactly NWT. In [BDS] one can also find a version with labelled graphs. Now why would one study those generalisations? As explained after Definition 5.3, the original motivation was its possible relation to new unprovability phenomena. One could investigate whether this template gives rise to statements having strengths that differ. In other words, suppose we look for the weakest theories needed to prove specific instances of "thin implies Ramsey", will we encounter different ones while going through all such statements?

In the next section we take a first step towards the answer to that question by analysing an obvious example of the template, namely NWT. Clearly, part of that answer also depends on the potential of the relational NWT to yield a new kind of model-theoretic proofs.

Another benefit is of different nature. Namely, in [KP07] the authors present a topological view on Ramsey families in the original setup. It could be interesting to investigate what would happen in the case of the relational NWT.

5.2 Strength of the Nash-Williams Theorem

We investigate the strength of Nash-Williams' theorem by giving an upper and a lower bound. It will turn out that NWT is equivalent to ATR_0 over RCA_0. These results were already known before, so we do not claim to present new theorems (see e.g. [AT05, Sim09]). Instead, we intend to give a clear overview of the current knowledge and also provide rigorous proofs, as some of those seem to be missing from the literature (Theorem 5.15 and Theorem 5.18). Igor Kříž and Robin Thomas gave an analysis of NWT by means of ordinal types ([KT91]).

We start by linking NWT to RT, Ramsey's Theorem, which is investigated in more detail in Chapter 4. We will use RCA_0 as a base theory.

5.11 Theorem. NWT *implies* RT.

Proof. Let n and e be natural numbers and $f\colon [\mathbb{N}]^n \to e$ a colouring of n-element subsets of $\mathbb{N}$ into e colours. Define $\mathcal{F}$ as the set of all n-element subsets of $\mathbb{N}$, and consider the following partition of $\mathcal{F}$. For every $X = \{x_1, \ldots, x_n\}$ in $\mathcal{F}$, put

$$X \in \mathcal{F}_i \text{ if and only if } f(x_1, \ldots, x_n) = i,$$

where $i \in \{0, \ldots, e-1\}$. Since $\mathcal{F}$ is a thin family of subsets of $\mathbb{N}$, Theorem 5.1 yields the existence of an infinite $M \subseteq \mathbb{N}$, such that at most one of the restrictions $\mathcal{F}_0 \restriction M, \ldots, \mathcal{F}_{e-1} \restriction M$ is non-empty. Moreover, since $\mathcal{F}$ equals the set of all n-element subsets of $\mathbb{N}$ at least one of the restrictions $\mathcal{F}_0 \restriction M, \ldots, \mathcal{F}_{e-1} \restriction M$ is non-empty. Thus, exactly one restriction is non-empty, say $\mathcal{F}_i \restriction M$, for some i in $\{0, \ldots, e-1\}$. Then for every n-element set $\{x_1, \ldots, x_n\} \subseteq M$, $f(x_1, \ldots, x_n) = i$, i.e. M is homogeneous for f. $\square$

After reading the previous proof, the reader might have the impression we only used a fraction of NWT's actual strength. That feeling is indeed justified, as we will show that NWT implies ATR_0. In order to do so, let us introduce some useful definitions. Recall that we use capital letters to denote infinite structures, and define for $M \subseteq \mathbb{N}$,

$$M^{[\infty]} = \{P \subseteq M \,:\, P \text{ is infinite}\}.$$

In other words, $M^{[\infty]}$ denotes the set of all infinite subsets of M.

5.12 Definition. *An* open set $P \subseteq \mathbb{N}^{[\infty]}$ *is defined by a family* $\mathcal{F}_P \subseteq \mathbb{N}^{[<\infty]}$*, such that for every* $N \subseteq \mathbb{N}$*,* $N \in P$ *if and only if there exists a member of* $\mathcal{F}_P$ *which is an initial segment of* N*.*

Open sets as defined above are exactly the open sets of the Baire space with the product topology. By taking the minimal elements, it is always possible to define an open set of $\mathbb{N}^{[\infty]}$ by a thin family. Open sets are used to define extensions of RT.

5.13 Definition. *The* open Ramsey theorem *is defined to be the statement that for every open set* $P \subseteq \mathbb{N}^{[\infty]}$ *there exists* $N \subseteq \mathbb{N}$ *such that either for all* $M \subseteq N$*,* $M \in P$*, or for all* $M \subseteq N$*,* $M \notin P$*.*

The clopen Ramsey theorem *is defined to be the statement that for all open sets* $P, Q \subseteq \mathbb{N}^{[\infty]}$*, if for all* $L \subseteq \mathbb{N}$ $(L \in P \leftrightarrow L \notin Q)$ *then there exists an* $N \subseteq \mathbb{N}$ *such that either for all* $M \subseteq N$*,* $M \in P$*, or for all* $M \subseteq N$*,* $M \notin P$*.*

The next theorem connects both Ramsey theorems introduced above.

5.14 Theorem (Friedman–McAloon–Simpson)**.** *The following are pairwise equivalent over* RCA_0*:*

1. ATR_0*;*
2. *the open Ramsey theorem;*
3. *the clopen Ramsey theorem.*

Proof. See [Sim09], Theorem V.9.7. □

We will show that NWT is equivalent to ATR_0, over RCA_0. To do so, we start by proving that it implies the clopen Ramsey theorem.

5.15 Theorem. NWT *implies the clopen Ramsey theorem.*

Proof. Let P and Q be open sets in $\mathbb{N}^{[\infty]}$ such that for all $L \subseteq \mathbb{N}$ $(L \in P \leftrightarrow L \notin Q)$. Let $\mathcal{F}_P, \mathcal{F}_Q \subseteq \mathbb{N}^{[<\infty]}$ be the thin families defining P and Q, and put $\mathcal{F} = \mathcal{F}_P \cup \mathcal{F}_Q$. Suppose $\mathcal{F}$ is not thin, then there exist $s, t \in \mathcal{F}$ such that $s \sqsubseteq t$. Since $\mathcal{F}_P$ and $\mathcal{F}_Q$ are thin, s and t must belong to different members of the partition, say $s \in \mathcal{F}_P$ and $t \in \mathcal{F}_Q$. Let $L \subseteq \mathbb{N}$ be any set with initial segment t. Then $L \in Q$. Moreover, also s is an initial segment of L, so $L \in P$, a contradiction. Hence, $\mathcal{F}$ is thin.

Apply NWT to $\mathcal{F}$ and obtain $N \subseteq \mathbb{N}$, such that at most one of

$$\mathcal{F}_P \cap N^{[<\infty]} \text{ and } \mathcal{F}_Q \cap N^{[<\infty]}$$

is non-empty. First notice that since for all $L \subseteq \mathbb{N}$ $(L \in P \leftrightarrow L \notin Q)$, it is impossible that both intersections are empty at the same time. Suppose $\mathcal{F}_P \cap N^{[<\infty]}$ is empty. Then whichever $M \subseteq N$ you take, there is no initial segment of M that belongs to $\mathcal{F}_P$. Hence $M \notin P$ for all $M \subseteq N$.

Now suppose $\mathcal{F}_Q \cap N^{[<\infty]}$ is empty. Then whichever $M \subseteq N$ you take, there is no initial segment of M that belongs to $\mathcal{F}_Q$. Thus for all $M \subseteq N$, $M \notin Q$, which is equivalent to $M \in P$. Combining the two previous cases, we have found $N \subseteq \mathbb{N}$ such that either for all $M \subseteq N$, $M \in P$, or for all $M \subseteq N$, $M \notin P$, which concludes the proof of the clopen Ramsey theorem. $\square$

As a result of the two previous theorems, we have found a lower bound for the strength of NWT, namely ATR_0. The next step is to look for upper bounds. We will give two of them, namely the open Ramsey theorem and the Σ^0_1 Ramsey theorem, which is defined as follows.

5.16 Definition. *The* Σ^0_1 Ramsey theorem, *denoted* Σ^0_1-RT, *is the scheme*

$$\exists f\, (\forall g\, \varphi(f \circ g) \ \vee\ \forall g\, \neg\varphi(f \circ g)),$$

where φ is any Σ^0_1 formula. Here f and g range over the Ramsey space*, i.e. the space of all total functions $h\colon \mathbb{N} \to \mathbb{N}$ such that h is strictly increasing.*

It is rather straightforward to see that Σ^0_1 Ramsey theorem is equivalent to the open Ramsey theorem. Nevertheless we will show separately how each principle implies NWT, as both short proofs have their own flavour. In both theorems we will consider a slightly different version of NWT than we used before:

"for every thin family $\mathcal{F} \subseteq \mathbb{N}^{[<\infty]}$ and every partition $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$, there is $N \subseteq \mathbb{N}$ such that either $\mathcal{F}_0 \cap N^{[<\infty]} = \emptyset$ or $\mathcal{F}_1 \cap N^{[<\infty]} = \emptyset$".

One can verify that the statement above is equivalent to NWT.

5.17 Theorem. Σ^0_1-RT *implies* NWT.

Proof. Let $\mathcal{F} \subseteq \mathbb{N}^{[<\infty]}$ be a thin family and $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$ a partition. Let $\varphi(X)$ be the following Σ^0_1 statement:

$$\varphi(X) \leftrightarrow [\exists s \in \mathcal{F}_0(s \sqsubseteq X) \wedge \exists t \in \mathcal{F}_1(t \subseteq X)],$$

where X is a set variable. Apply Σ^0_1-RT to obtain an $N \subseteq \mathbb{N}$ such that either for all $M \subseteq N$, $\varphi(M)$ holds, or for all $M \subseteq N$, $\varphi(M)$ does not hold. Suppose the former case holds, then there exists $t \in \mathcal{F}_1$ such that $t \subseteq N$. Let M_t be any subset of N such that $t \sqsubseteq M_t$. Since $\varphi(M_t)$ holds, there must exist $s \in \mathcal{F}_0$ such that $s \sqsubseteq M_t$. As both s and t are initial segments of M_t, either $s \sqsubseteq t$ or $t \sqsubseteq s$, which contradicts the thinness of $\mathcal{F}$.

Suppose the latter case holds, then for all $M \subseteq N$, $\forall s \in \mathcal{F}_0(s \not\sqsubseteq M)$ or $\forall t \in \mathcal{F}_1(t \not\subseteq M)$. If there exists $M \subseteq N$, such that the latter one holds, then we are done, since $\mathcal{F}_1 \cap M^{[<\infty]} = \emptyset$. Otherwise, $\mathcal{F}_0 \cap N^{[<\infty]} = \emptyset$ which completes the proof. $\square$

The next statement can also be found without proof in e.g. [AT05] (Part 2, Corollary II.6.5).

5.18 Theorem. *The open Ramsey theorem implies* NWT.

Proof. Let $\mathcal{F} \subseteq \mathbb{N}^{[<\infty]}$ be a thin family and $\mathcal{F} = \mathcal{F}_P \cup \mathcal{F}_Q$ a partition. Then $\mathcal{F}_P$ and $\mathcal{F}_Q$ define open sets $P \subseteq \mathbb{N}^{[\infty]}$ and $Q \subseteq \mathbb{N}^{[\infty]}$, respectively. Remark that since $\mathcal{F}$ is thin and $\mathcal{F}_P$ and $\mathcal{F}_Q$ are disjoint no set can belong to both P and Q. Apply the open Ramsey theorem to P to obtain $N \subseteq \mathbb{N}$ such that either for all $M \subseteq N$, $M \in P$, or for all $M \subseteq N$, $N \notin P$. In case the former one holds, then for all $M \subseteq N$, $M \notin Q$, hence $\mathcal{F}_Q \cap N^{[<\infty]} = \emptyset$. If the latter case holds, $\mathcal{F}_P \cap N^{[<\infty]} = \emptyset$. $\square$

Now, Theorem 5.14 implies that ATR_0 is an upper bound for the strength of NWT. Hence, we have obtained the following result.

5.19 Theorem. *The following are equivalent over* RCA_0*:*

1. ATR_0*;*
2. NWT*.*

Proof. Combine Theorem 5.14, Theorem 5.15 and Theorem 5.18. $\square$

This concludes the investigation of the strength of one specific instance of the template "thin implies Ramsey". So what about other instances? To our knowledge, the strength of the relational NWT (see Corollary 5.10) has not been determined so far. Clearly, as it implies NWT, it will be at least as strong as ATR_0. The question arises whether the use of labelled structures would lead to stronger theories. Future research will hopefully settle this.

5.3 About Schreier families

While studying Nash-Williams Ramsey theory, or more precisely while looking at fronts and barriers, we also encountered Schreier families. These families have been used to give transfinite extensions of RT and NWT (see e.g. Theorem 5.21). As is often the case, the connection with Ramsey theory gives rise to unprovability phenomena, which explains our interest in those families. In this section we present a first-order PA-unprovable statement, called $\mathrm{FRO}(\omega)$.

Let us first introduce Schreier systems as done by Vassiliki Farmaki and Stylianos Negrepontis in [FN08]. The definition and some of the lemmas below can be given for any countable ordinal, but we will just consider ordinals below ε_ω.

5.20 Definition. *The* Schreier system $(A_\xi)_{\xi<\varepsilon_\omega}$ *is defined as follows. Let* $\alpha, \beta, \zeta, \lambda$ *and* ξ *be ordinals below* ε_ω. *Then*

1. $A_0 = \{\emptyset\}$;
2. $A_{\zeta+1} = \{s \in \mathbb{N}^{[<\infty]} : s = \{n\} \cup s_1$, *where* $n \in \mathbb{N}, \{n\} < s_1$ *and* $s_1 \in A_\zeta\}$;
3. $A_{\omega^{\beta+1}} = \{s \in \mathbb{N}^{[<\infty]} : s = \cup_{i=1}^n s_i$, *where* $n = \min s_1, s_1 < s_2 < \ldots < s_n$ *and* $s_1, \ldots, s_n \in A_{\omega^\beta}\}$;
4. *for a limit ordinal* λ, $A_{\omega^\lambda} = \{s \in \mathbb{N}^{[<\infty]} : s \in A_{\omega^{\lambda[n]}}$ *with* $n = \min s\}$;
5. *for a limit ordinal* ξ *such that* $\xi = \omega^\alpha p + \omega^{\alpha_1} p_1 + \ldots + \omega^{\alpha_m} p_m$ *in Cantor normal form, then* $A_\xi = \{s \in \mathbb{N}^{[<\infty]} : s = s_0 \cup (\cup_{i=1}^m s_i)$ *with* $s_m < \ldots < s_1 < s_0, s_0 = s_1^0 \cup \ldots \cup s_p^0$ *with* $s_1^0 < \ldots < s_p^0 \in A_{\omega^\alpha}$, *and* $s_i = s_1^i \cup \ldots \cup s_{p_i}^i$ *with* $s_1^i < \ldots < s_{p_i}^i \in A_{\omega^{\alpha_i}}, \forall 1 \leq i \leq m\}$.

For any $\xi < \varepsilon_\omega$, *the member* A_ξ *of the system is called a* Schreier family.

Even though the definition above depends on the particular choice of fundamental sequences, it turns out the complexity of the family, as measured by its Cantor-Bendixson index, is independent of that choice. See [FN08] for more information on this. Given the fundamental system defined in Chapter 1, a set s belongs to A_α if and only if $s \cup \{\max s + 1\}$ is exactly α-large, for every $\alpha < \varepsilon_\omega$.

The next theorem is called the "Ramsey partition theorem extended to countable ordinals". In [Far98] Farmaki gave a proof directly from the definitions involved. Another proof, using the combinatorial theorems of Nash-Williams in [NW65] is given in [PR82].

5.21 Theorem. *Let M be an infinite subset of $\mathbb{N}$, $\xi < \varepsilon_\omega$ and $\mathcal{F}$ a family of finite subsets of $\mathbb{N}$. Then there exists an infinite subset L of M such that either $A_\xi \cap L^{[<\infty]} \subseteq \mathcal{F}$, or $A_\xi \cap L^{[<\infty]} \subseteq \mathbb{N}^{[<\infty]} \setminus \mathcal{F}$.*

Proof. See Theorem 2.2 in [Far98]. □

If we replace ξ by a natural number n, then we obtain the original Ramsey Theorem for dimension n and two colours, since A_n contains exactly all possible n-tuples of natural numbers. Let us have a look at the case $\xi = \omega$.

$$\begin{aligned} A_\omega &= \{s \in \mathbb{N}^{[<\infty]} : s = \cup_{i=1}^n s_i, \text{ where } n = \min s_1, s_1 < s_2 < \ldots < s_n \\ &\quad \text{and } s_1, \ldots, s_n \in A_1\} \\ &= \{s \in \mathbb{N}^{[<\infty]} : s = \cup_{i=1}^n \{k_i\}, \text{ where } n = k_1, k_1 < k_2 < \ldots < k_n \\ &\quad \text{and } k_1, \ldots, k_n \in \mathbb{N}\} \\ &= \{s \in \mathbb{N}^{[<\infty]} : \min s = |s|\}. \end{aligned}$$

A subset s of the natural numbers for which $|s| \geq \min s$, is called large. Hence, all elements of A_ω are large. There are several examples of independent statements (e.g. PH), in which the largeness condition is crucial for demonstrating the unprovability. So, not surprisingly, A_ω is closely connected to unprovability phenomena, which is also noticed in [FN08] as follows:

> "*It is also noteworthy that the hereditary family*
>
> $$(A_\omega)_* = \{t \in \mathbb{N}^{[<\infty]} \,|\, t \subseteq s, \text{ for some } s \in A_\omega\} \cup \{\emptyset\},$$
>
> *generated by A_ω figures prominently (under the name of the family of "not large" sets) in questions of mathematical logic related to concrete realisations of Gödels incompleteness theorem, specifically in the (Ramsey type) Paris-Harrington statements* [...] *The higher order hereditary Schreier families $(A_\xi)_*$, and specifically a suitable finitary form of Theorem 1.5 involving sets in these families, might well be useful in forming and proving statements true but unprovable in certain systems endowed with induction stronger than that in Peano arithmetic.*".

We will show how unprovability indeed arises in this context by modifying Theorem 5.21. We call elements s and t of $\mathbb{N}^{[<\infty]}$ successive if $\min s \neq \min t$. Fix any ordinal $\xi < \varepsilon_\omega$ and let FRO(ξ) (Finite Ramsey partition theorem extended to countable Ordinals) stand for the following first-order statement:

"for all m, there exists N, such that for all families $\mathcal{F}$ of subsets of $[0, 2N]$ and all $M \subseteq [0, 2N]$, with $|M| \geq N$, there exists $L \subseteq M$, such that L contains $\max\{m, \min L\}$ successive elements of A_ξ and either $A_\xi \cap L^{[<\infty]} \subseteq \mathcal{F}$, or $A_\xi \cap L^{[<\infty]} \subseteq [0, 2N]^{[<\infty]} \setminus \mathcal{F}$".

Clearly, if for some m such an N exists, then $N \geq m$. We start our investigation of FRO(ξ) by proving the statement for every $\xi < \varepsilon_\omega$.

5.22 Theorem. *For every $\xi < \varepsilon_\omega$,* FRO(ξ) *holds.*

Proof. The proof utilises a typical compactness argument in the shape of König's lemma. Let $\xi < \varepsilon_\omega$ and assume FRO(ξ) fails, for the sake of contradiction. Then

"there exists m, such that for all N, there exists a family $\mathcal{F}$ of subsets of $[0, 2N]$ and $M \subseteq [0, 2N]$, such that $|M| \geq N$ and for all $L \subseteq M$ containing $\max\{m, \min L\}$ successive elements of A_ξ, we have $A_\xi \cap L^{[<\infty]} \nsubseteq \mathcal{F}$ and $A_\xi \cap L^{[<\infty]} \nsubseteq [0, 2N]^{[<\infty]} \setminus \mathcal{F}$"

holds. (Note that $A_\xi \cap L^{[<\infty]} \subseteq \mathcal{F}$ and $A_\xi \cap L^{[<\infty]} \subseteq [0, 2N]^{[<\infty]} \setminus \mathcal{F}$ implies $A_\xi \cap L^{[<\infty]} = \emptyset$, which is impossible.) Fix such m and consider the set T containing all triples $\langle N, \mathcal{F}_N, M_N\rangle$, with N a natural number and $\mathcal{F}_N \subseteq [0, 2N]^{[<\infty]}$ and $M_N \subseteq [0, 2N]$ such that $|M| \geq N$ and for all $L \subseteq M$ containing $\max\{m, \min L\}$ successive elements of A_ξ, we have $A_\xi \cap L^{[<\infty]} \nsubseteq \mathcal{F}$ and $A_\xi \cap L^{[<\infty]} \nsubseteq [0, 2N]^{[<\infty]} \setminus \mathcal{F}$. By the statement above there exists at least one such a triple for every N.

Now define a partial order relation $\prec$ on T as follows. For every two elements $\langle N_1, \mathcal{F}_{N_1}, M_{N_1}\rangle$ and $\langle N_2, \mathcal{F}_{N_2}, M_{N_2}\rangle$ in T, we write

$$\langle N_1, \mathcal{F}_{N_1}, M_{N_1}\rangle \prec \langle N_2, \mathcal{F}_{N_2}, M_{N_2}\rangle$$

if and only if

$$N_1 < N_2,\ \mathcal{F}_{N_1} \subseteq \mathcal{F}_{N_2} \text{ and } M_{N_1} \subseteq M_{N_2}.$$

The set T together with $\prec$ forms a tree, as for every $\langle N_1, \mathcal{F}_{N_1}, M_{N_1}\rangle$ in T, the set

$$\{\langle N_2, \mathcal{F}_{N_2}, M_{N_2}\rangle \in T\ :\ \langle N_2, \mathcal{F}_{N_2}, M_{N_2}\rangle \prec \langle N_1, \mathcal{F}_{N_1}, M_{N_1}\rangle\}$$

is well-ordered by $\prec$, as the standard ordering of the natural numbers is a well-order. Clearly, since there are only finitely many combinations to extend $\mathcal{F}_N$ to $\mathcal{F}_{N+1}$ and M_N to M_{N+1}, the tree T is finitely branching. On the other hand, T is infinite, so we can apply König's lemma to obtain an infinite branch $B = \langle \mathbb{N}, \mathcal{F}_{\mathbb{N}}, M_{\mathbb{N}}\rangle$, where $\mathcal{F}_{\mathbb{N}}$ will be the union of all $\mathcal{F}_N$ on B and $M_{\mathbb{N}}$ the union of all M_N on B. Now apply Theorem 5.21 with $\mathcal{F}_{\mathbb{N}}$, ξ and $M_{\mathbb{N}}$. Then there exists an infinite $L \subseteq M_{\mathbb{N}}$ such that either $A_\xi \cap L^{[<\infty]} \subseteq \mathcal{F}_{\mathbb{N}}$, or $A_\xi \cap L^{[<\infty]} \subseteq \mathbb{N}^{[<\infty]} \setminus \mathcal{F}_{\mathbb{N}}$.

Take a finite initial segment $L_0 = \{l_0, \ldots, l_s\}$ of L, such that L_0 contains $\max\{m, \min L\}$ successive elements of A_ξ. Each $l \in L_0$ appeared in M_{N_l} for for some N_l, thus at some level in the tree T. Put $N = \max\{N_{l_0}, \ldots, N_{l_s}\}$ and look at the vertex $\langle N, \mathcal{F}_N, M_N\rangle$. L_0 will contradict the fact that "... for all $L \subseteq M_N$ containing $\max\{m, \min L\}$ successive elements of A_ξ, we have $A_\xi \cap L^{[<\infty]} \not\subseteq \mathcal{F}_N$ and $A_\xi \cap L^{[<\infty]} \not\subseteq [0, 2N]^{[<\infty]} \setminus \mathcal{F}_N$". $\square$

Notice that the proof of the theorem above can be carried out for any countable ordinal ξ.

5.23 Theorem. FRO(ω) *is unprovable in* PA.

Proof. We show that FRO(ω) implies PH for two colours, which yields its unprovability in PA (see the corollary in [LN92]).

Suppose FRO(ω) holds. Let m and n be two given natural numbers. Remark that if $m \leq n$, it is easy to see that $N = 2n-1$ suffices. Indeed, take any colouring f of n-subsets of $[0, 2n-1]$ into 2 colours and put $H = [n, 2n-1]$. So we can assume that $m \geq n$. Put $p = 2^m$. Apply FRO(ω) (with m replaced by p) to obtain N. We will show that $2N$ satisfies the conditions of PH_2. Therefore, take any $f\colon [0,2N]^{[n]} \to 2$ and define $\mathcal{F} \subseteq [0,2N]^{[<\infty]}$ as follows. For any $X \in A_\omega \cap [m,2N]^{[<\infty]}$,

$$X \in \mathcal{F} \text{ if and only if } f(x_1,\ldots,x_n) = 0,$$

where $x_1,\ldots,x_n$ are the first n elements of X. Thus for all other $X \subseteq [0,2N]^{[<\infty]}$, $X \notin \mathcal{F}$. Put $M = [m,2N]$. Remark that $M \subseteq [0,2N]$ and $|M| \geq N$, since $N \geq m$. Both $\mathcal{F}$ and M satisfy the conditions of $\mathrm{FRO}(\omega)$, so there exists a subset L of M, such that L contains $\max\{p, \min L\}$ successive elements of A_ω and either $A_\omega \cap L^{[<\infty]} \subseteq \mathcal{F}$, or $A_\omega \cap L^{[<\infty]} \subseteq [0,2N]^{[<\infty]} \setminus \mathcal{F}$. Since $p = 2^m \geq 1$, L contains at least one element of A_ω. So we can define H as the member of $A_\omega \cap L^{[<\infty]}$ which has the smallest elements, i.e. H consists of the first $\min H = \min L$ elements of L. We show that H satisfies all necessary conditions in PH_2.

First, by definition of H, $|H| \geq \min H$. Second, $H \subseteq L \subseteq M = [m,2N]$ implies $|H| \geq m$. Finally, take any n elements $x_1,\ldots,x_n$ in H. Because L contains $\max\{p, \min H\}$ successive elements of A_ω, there exists $X \in A_\omega \cap L^{[<\infty]}$, such that $x_1,\ldots,x_n$ are the first n elements of X. Depending on whether $A_\omega \cap L^{[<\infty]} \subseteq \mathcal{F}$ or $A_\omega \cap L^{[<\infty]} \subseteq [0,2N]^{[<\infty]} \setminus \mathcal{F}$ is the case, we have one fixed value $f(x_1,\ldots,x_n)$, either 0 or 1, for all $x_1,\ldots,x_n$ in H. Hence, H is f-homogeneous. $\square$

Chapter 6

THE ATLAS OF PREFIXED POLYNOMIAL EQUATIONS

This chapter is intended as a rather vast introduction to what we will call the Atlas of prefixed polynomial equations. It deals with many of the main issues of this new subject. We will present members of the Atlas and give explanatory words and some technical details. We believe that this alternative way for looking at first-order arithmetical statements can produce new mathematical and metamathematical insights. For a thorough treatment with more representatives, a wide range of observations and several other features of the Atlas, we refer to [BDS10].

6.1 Definitions and explanations

6.1.1 Prefixed polynomial equations and the Atlas

We start with the prime definitions.

6.1 Definition. *A* prefixed polynomial equation *is an expression of the form*

$$Q_1x_1\ Q_2x_2\ \dots\ Q_nx_n\ P(x_1, x_2, \dots, x_n) = 0,$$

where P is a polynomial with integer coefficients whose variables $x_1, x_2, \dots,$ x_n range over natural numbers, that is preceded by a block of quantifiers

Q_1, Q_2,…, Q_n *over its variables* $x_1, x_2, \ldots, x_n$.

6.2 Definition. *The* Atlas *is the collection of all prefixed polynomial equations.*

We shall often refer to this Atlas as a *template* in the sense that it is the set of all substitution instances of a concrete polynomial P and a quantifier-block into one fixed pattern. Throughout this chapter we will also use other names such as "polynomial expression with a quantifier-prefix", or simply "polynomial expression" or "polynomial equation". Let us give a typical generic example of such a polynomial expression.

$$\forall\ m\ \exists\ N\ \forall\ ab\ \exists\ cd\ A\ X\ \forall\ xy\ \exists\ BCF\ \forall\ fg\ \exists\ hilnrpq$$

$$x \cdot (y+B-x) \cdot (A+m+B-y) \cdot [(((f-A)^2+(g-1)^2) \cdot ((f-B)^2+(g-x)^2) \cdot ((f-C)^2+ \\ +(g-y)^2)-h-1) \cdot ((dgi+i-c+f)^2+(f+h-dg)^2)+(B+l+1-C)^2+(C+n-N)^2+ \\ +(F+r-b(B+C^2))^2+(bp(B+C^2)+p-a+F)^2+((F-X)^2-qe)^2] = 0.$$

We will denote this polynomial expression by Φ.

To be able to compare different prefixed polynomial equations, we need a base theory. In [BDS10] it is explained in detail why we choose Exponential Function Arithmetic, shortly EFA, instead of possible weaker or stronger candidates. For a rigorous definition of this theory we refer to [Avi03], where it is called Elementary Arithmetic (EA). More information on EFA can also be found in the introduction of [Fri10a]. The theory is a fragment of PA which cannot prove the totality of the iterated exponential function, but is surprisingly solid. Exactly that robustness motivated Friedman to make the following conjecture (see [Avi03], Section 1).

> **Conjecture.** *Every theorem published in the Annals of Mathematics whose statement involves only finitary mathematical objects (i.e., what logicians call an arithmetical statement) can be proved in elementary arithmetic.*

Studying the Atlas, we investigate how prefixed polynomial equations relate to each other with regard to the equivalence relation of "being EFA-provably equivalent". Hence, two polynomial expressions φ and ψ are equivalent if there exists a proof of $\varphi \leftrightarrow \psi$ in EFA. The equivalence classes of the Atlas are partially ordered by the following relation: the class of an expression A is smaller than the class of an expression B if EFA proves that B implies A.

We choose to let the variables range over natural numbers. Instead we could have chosen the integers or rational numbers, which are equally interesting from a mathematical point of view. Each of those templates will have advantages and disadvantages. We decide on natural numbers as it is comfortable and usual for logicians.

Notice that the set of prefixed polynomial equations is *arithmetically complete* (i.e. every first-order arithmetical formula is EFA-equivalent to a prefixed polynomial expression). In this sense, the Atlas is just another way of talking about first-order arithmetical statements. The polynomial equation Φ given above, for example, is EFA-equivalent to the 1-consistency of $\mathrm{I}\Sigma_1$ (see Definition 1.2). We will often somewhat abuse notation by writing sentences as "the equivalence class of 1-Con($\mathrm{I}\Sigma_1$)" instead of "the equivalence class of a polynomial equation which is EFA-equivalent to 1-Con($\mathrm{I}\Sigma_1$)", since 1-Con($\mathrm{I}\Sigma_1$) is hardly ever presented in the form of a polynomial expression.

6.1.2 Size and seeds

We will compare members of the same or different equivalence classes by looking at their size. Therefore, we need to fix a way of counting the length of prefixed polynomial expressions.

6.3 Definition. *The* size *of a prefixed polynomial equation is defined as the length of the polynomial, which is counted as follows: every occurrence of a variable or multiplication or addition operation contributes* 1 *to the total size, a coefficient n contributes* $(n-1)$ *to the total size,* $+n$ *or* $-n$ *both contribute n, and the power n contributes* $(n-1)$ *to the total size.*

Remark that we ignore the quantifier-prefix and the final "$= 0$" in defining the size. Sometimes *length* is used instead of size. Notice how *deep* this prefixed polynomial equations template is: there are relatively short members in many non-trivial equivalence classes as we will see later. The representatives given in this chapter are still very rough, so some polynomial equations appear to be rather long and complicated at first sight. However, significant simplifications, leading to shorter, comprehensible polynomial expressions, are expected in the future.

Given our method for counting the size of a polynomial expression, we can introduce the following important notion.

6.4 Definition. *A* seed *is a prefixed polynomial equation that is of minimal length in its* EFA*-provable equivalence class. If a seed belongs to the* EFA*-provable equivalence class of an arithmetical formula φ, we shall say it is* a seed of φ.

Remark that we speak about "a" instead of "the" seed, because an EFA-provable equivalence class may have several different seeds. Given an equivalence class, possibly having a specific member in mind, we face the quest for a seed of that class. Figure 6.1 on the next page gives an idea of how the Atlas might look like.

6.2 Some important members

Let us give a few basic representatives of EFA-equivalence classes, to get the first view of the complexity of the Atlas. Recall Definition 1.2 of the 1-consistency of a theory T.

Unprovability by primitive recursive means

6.5 Theorem. *The following polynomial expression is equivalent to* 1-Con($\mathrm{I}\Sigma_1$) *and hence unprovable in* $\mathrm{I}\Sigma_1$.

$$\forall\, em\ \exists\, N\ \forall\, ab\ \exists\, cd\ A\ X\ \forall\, xy\ \exists\, BCF\ \forall\, fg\ \exists\, hilnrpq$$

$$x\cdot(y+B-x)\cdot(A+m+B-y)\cdot[(((f-A)^2+(g-1)^2)\cdot((f-B)^2+(g-x)^2)\cdot((f-C)^2+ +(g-y)^2)-h-1)\cdot((dgi+i-c+f)^2+(f+h-dg)^2)+(B+l+1-C)^2+(C+n-N)^2+ +(F+r-b(B+C^2))^2+(bp(B+C^2)+p-a+F)^2+((F-X)^2-qe)^2]=0.$$

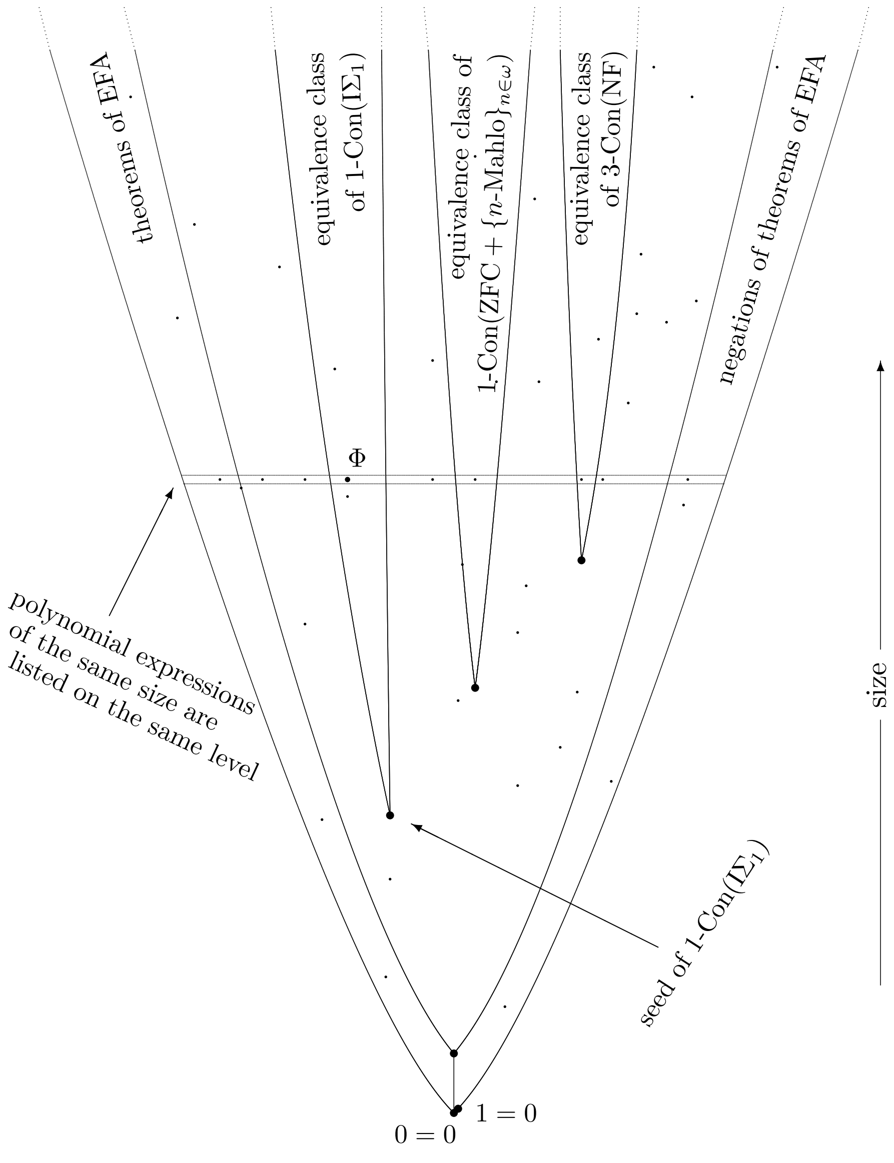


Figure 6.1: *Sketch of the Atlas regarding the size of polynomial expressions*

Proof. For a full proof see Section 6.5. The proof goes by demonstrating equivalence with $\forall e\, \mathrm{PH}^2_e$, the Paris-Harrington principle for pairs, wich is unprovable in $\mathrm{I}\Sigma_1$. $\square$

The prefixed polynomial equation of Theorem 6.5 is exactly the expression Φ which was presented in the introduction, and is also shown in Figure 6.1.

The polynomial equation Φ seems to have quantifier complexity Π^0_8. However, the last four blocks of quantifiers can be bounded by some exponential expressions, which can be struggled with and eliminated using the methods from [Mat93]. So the formula is equivalent to a Π^0_2 formula. We did not do any of it because it would blow up the size of the resulting polynomial.

With our method of counting length (see Definition 6.3), the polynomial has size 125. Hence, a seed of the equivalence class of 1-Con($\mathrm{I}\Sigma_1$) will have 125 as an upper bound for its size. The reader should be aware that the expression above is rough and lengthy. It could be reduced and simplified by reusing variables, applying ingenious coding tricks and using clever combinatorial equivalences during the proof. So we expect a seed of this class to be shorter and more simple.

Unprovability in two-quantifier-induction arithmetic

6.6 Theorem. *The following polynomial expression is equivalent to* 1-Con($\mathrm{I}\Sigma_2$)*, and thus unprovable in* $\mathrm{I}\Sigma_2$*.*

$$\forall\ em\ \exists\ N\ \forall\ ab\ \exists\ cd\ A\ X\ \forall\ xyz\ uvw\ \exists\ BCDGH\ \forall\ fg\ \exists\ hijklnpqrst\ FG$$

$$\begin{aligned}&x\cdot(y+B-x)\cdot(z+B-y)\cdot(A+m+B-z)\cdot[[((((f-A)^2+(g-1)^2)\cdot((f-B)^2+(g-x)^2)\cdot\\&\cdot((f-C)^2+(g-y)^2)\cdot((f-D)^2+(g-z)^2)-h-1)\cdot((dgi+i-c+f)^2+(f+h-dg)^2)+\\&+(D+l-dz)^2+(dzn+n-c+D)^2+(B+F+1-C)^2+(C+p+1-D)^2+(D+q-N)^2+\\&+(H+r-bt)^2+(bts+s-a+H)^2+(B+C^2+D^3-t)^2+((X-H)^2-Ge)^2]=0.\end{aligned}$$

Proof. For a full proof see Section 6.5. Again, we show that the statement above is equivalent to $\forall e\, \mathrm{PH}^3_e$, the Paris-Harrington principle for triples, which is equivalent to 1-Con($\mathrm{I}\Sigma_2$). $\square$

As before, the statement may look like Π^0_8 but is actually EFA-equivalent to a (much longer) Π^0_2 formula. With our method of counting, the polynomial has size 178. Again, this is a somewhat naive first attempt. We are convinced that by more delicate method we can find an example having much smaller size.

Unprovability in Peano Arithmetic

6.7 Theorem. *Consider the following polynomial equation with a parameter* $\mathbf{n}$*. For every* $\mathbf{n} > 1$ *the statement is equivalent to* 1-Con($\mathrm{I}\Sigma_{\mathbf{n}-1}$)*. So, with the quantifier prefix* $\forall\,\mathbf{n}$*, this statement is equivalent to* 1-Con(PA) *and hence is unprovable in* PA*.*

$$\forall\ em\ \exists\ N\ \forall\ ab\ \exists\ cd\ xy\ zw\ \forall\ it\ \alpha\beta\ \exists\ fg\ hklpqruv\ X\ ABCDEF\ \forall\ j\ GI\ \exists\ sH$$

$$\begin{aligned}&[i\cdot(\mathbf{n}+f+1-i)\cdot((g+f-yi)^2+(yih+h-x+g)^2+(g+l-wi)^2+(wik+k-z+g)^2)+\\&+((p+q-b(x^2+y))^2+(b(x^2+y)r+r-a+p)^2+((b(z^2+w)j+j-a+p)^2-s-1)\cdot\\&\cdot(p-b(z^2+w)-1-s))^2]\cdot[(z+x-d)^2+(yd+y-c+z)^2+(t\cdot(z+m+f-t)\cdot((g+l-dt)^2+\\&+(dtk+k-c+g)^2+(h+p-d(t+1))^2+(d(t+1)q+q-c+h)^2+(g+r+1-h)^2+\\&+(g+s-N)^2))+((u+1-X)^2+(X+v-\mathbf{n})^2+(A+C-\beta X)^2+(\beta XD+D-\alpha+A)^2+\\&+(B+E-\beta(X+1))^2+(\beta(X+1)F+F-\alpha+B)^2+(((dGI+I-c+A)^2-H-1)\cdot G\cdot\\&\cdot(A-dG-1-H)\cdot(z+m+H-G)\cdot(B+H+1-A))^2)\cdot((b(\alpha^2+\beta)B+B-a+u)^2+\\&+(u+A-b(\alpha^2+\beta))^2+((u-w)^2-ve)^2)]=0.\end{aligned}$$

Proof. For a full proof see Section 6.5. We prove that for all $\mathbf{n}$ the expression above is equivalent to the Paris-Harrington principle in dimension $\mathbf{n}$. □

Again this is a somewhat naive theorem, without fine-tuning or clever tricks, and again we expect much simpler polynomials, to be achieved using extra tricks. As before, all quantifiers after the first two blocks of quantifiers can be made bounded by some exponential functions, and the famous battle against the bounded quantifier (Chapter 6 of [Mat93]) can reduce the statement to its true Π^0_2 shape, although at the cost of losing the current small size. The polynomial expression is of size 353, where $\mathbf{n}$ is counted as a variable.

Varying the parameter $\mathbf{n}$ in the prefixed polynomial expression above will lead to different strengths, as stated in the theorem. With regard to seeds we get the following corollary.

6.8 Corollary. *For any natural number* $\mathbf{n} > 1$*, the size of a seed of* 1-Con($\mathrm{I}\Sigma_{\mathbf{n}-1}$) *is less than or equal to* $351 + 2\mathbf{n}$.

Proof. There are two free occurrences of the number $\mathbf{n}$ in the prefixed polynomial equation in Theorem 6.7, which results in adding $2\mathbf{n}$ to the total size of the polynomial. □

6.3 More about the Atlas

6.3.1 A plethora of equivalence classes

In Section 6.2 we explicitly wrote down members of different equivalence classes, namely 1-Con($\mathrm{I}\Sigma_1$), 1-Con($\mathrm{I}\Sigma_2$), 1-Con($\mathrm{I}\Sigma_n$) for $n \in \mathbb{N}$, and 1-Con(PA). Of course, this is just a glimpse of the vast range of the Atlas. The template of prefixed polynomial expressions allows many more different equivalence classes, as for example:

- theorems of EFA;
- negations of theorems of EFA;
- totality of the fifth branch of the Ackermann function;
- van der Waerden Theorem;
- 3-Con(ZFC);
- 17-Con($\mathrm{I}\Sigma_2$).

Remark that some of the equivalence classes are not comparable, hence the *partial* order we mentioned earlier. The statement 3-Con(ZFC) can, over EFA, imply only new Π^0_4-arithmetical formulas, so not the Π^0_{18}-formula 17-Con($\mathrm{I}\Sigma_2$). The statement 17-Con($\mathrm{I}\Sigma_2$) on the other hand implies all

Π^0_{18}-consequences of $\mathrm{I}\Sigma_2$, but not some simple consequences of ZFC, like the Paris-Harrington Principle.

We do not divide classes into "true" and "false" ones, but merely study the Atlas as a mathematical object without any philosophical presumptions. So far, we have not yet written down explicitly many members of equivalence classes with low strength, as we were eager to know whether we could give polynomial equations of reasonable size with high strength. It turned out we could. Moreover, other interesting properties emerged.

As it would lead us beyond the scope of this dissertation, we will not discuss all achieved results in detail. However, we want to give a taste of the possibilities of the template. So below we present a short overview of examples, all proved and discussed in [BDS10].

1. *A polynomial equation equivalent to the totality of the superexponential function.*

2. *A polynomial expression equivalent to the Finite Kruskal Theorem* (see Section 6.4).

3. *A polynomial equation which establishes a phase transition between* EFA-*provability and predicative unprovability* (see Section 6.4).

4. *A polynomial equation equivalent to the finite Graph Minor Theorem.* This theorem says that

   "for every positive integer n, there is an integer m so large that if $G_1, \ldots, G_m$ is a sequence of finite undirected graphs, where each G_i has size at most $n + i$, then G_j is a minor of G_k for some $j < k$",

   and is unprovable in at least Π^1_1-CA_0 (see [FRS87]).

5. *A polynomial expression that knows values of all polynomials on all inputs.* First we describe a way to code any polynomial by four variables a, b, c, d, and any input by two variables x, y. We then construct a prefixed polynomial equation $\varphi(a, b, c, d, x, y, w)$ with free

variables a, b, c, d, x, y, w, such that for any a, b, c, d, x, y, w, the polynomial coded by a, b, c, d assumes the value w on input coded by x, y, if and only if $\varphi(a, b, c, d, x, y, w)$.

6. *A polynomial equation for values of* BAF*-terms.* See Chapter 5 in [Fri10a] for a rigorous definition of a basic function (BAF). Roughly, a BAF is a function built by using $0, 1, +, -, \cdot, \exp, \log$ and variables. A BAF-term is a term defining a basic function.

   We start by explaining how a BAF-term can be coded by four variables a, b, c, d, and an input by two variables x, y. Then we give a prefixed polynomial equation $\psi(a, b, c, d, x, y, w)$ with free variables a, b, c, d, x, y, w, such that for any a, b, c, d, x, y, w, the BAF-term coded by a, b, c, d assumes the value w on input coded by x, y, if and only if $\psi(a, b, c, d, x, y, w)$.

7. *A polynomial equation equivalent to Friedman's Proposition E* (see [Fri10a], Section 6.1). Friedman showed that Proposition E is ACA_0'-equivalent[1] to 1-Con(ZFC + $\{$there exists an n-Mahlo cardinal$\}_{n\in\omega}$).

For a long time people thought it was not feasible to really write down many of the members above as short, comprehensible polynomial expressions. There is a sound historical explanation for that opinion. Unprovability theory, as we know it today, offers a wide range of unprovable first-order arithmetical statements, which were not available previously. So, when some people were dreaming in, say, the 1970s of finding a polynomial equation having considerable strength, they only knew one kind of unprovable first-order arithmetical statement, namely Con(T), i.e. the statement expressing the consistency of a theory T. Rigorously writing that assertion as a prefixed polynomial equation is indeed not really a feasible task, but luckily we do not have to think in terms of Con(T) any longer. Nowadays, with all the sophisticated unprovability machinery available, polynomial

[1] ACA_0' is the system $\mathrm{ACA}_0 + \forall n \in \mathbb{N}\ \forall X \subseteq \mathbb{N}$ (the nth Turing jump of X exists). See [Fri10a] for more information.

equations are ready to spring from different corners of the subject: Ramsey theory, well-partial-order theory, Nash-Williams theory and Friedman's Boolean Relation Theory.

6.3.2 Additional observations

While studying the Atlas, we discovered several interesting facets, ranging from down-to-earth algebraic tricks, to metamathematical insights. Some of those features have been discussed or touched upon already in the previous sections. In this subsection we present three more observations.

Nuclei of strength

Let us first expand a little on the idea of a seed. Recall that a seed is a prefixed polynomial equation that is of minimal length in its EFA-provable equivalence class. The length of the examples of the polynomial equations presented in this chapter clearly form an upper bound for the length of a seed of the corresponding equivalence class. So, while generating a list of all polynomial equations, starting from length 1 and increasing size, one would encounter a seed of, say, the equivalence class of 1-Con($\mathrm{I}\Sigma_1$). At the moment the upper bound for the length of a seed of that class is 125, but it could turn out to be surprisingly short, such as 27 or 38.

Given such short examples it would become even more clear that seeds really are nuclei of strength. In other words, seeds are the central parts which produce strength. By adding 1 and subtracting 1 from some variable, one could in a trivial way enlarge a seed and obtain a member of the same equivalence class. Although length would have been increased, the strength remains the same. Seeds, indeed, already contain all valuable information. Let us conclude the discussion on seeds by giving two trivial seeds:

1. $0 = 0$, the "seed of truth", the seed of all provable statements;
2. $1 = 0$, the "seed of lies", the seed of all refutable statements.

Richer language, shorter polynomial

Strictly speaking, our template does not allow the use of exp, log and other function symbols, apart from polynomials. By doing so we work with a pure, basic language, which is presumably more appealing to most readers in comparison with a language containing many, possibly unknown, function symbols. In addition, the language of polynomials has a universal flavour because of its simplicity.

However, suppose we allow some specific symbols, say exp, then we could reduce significantly the size of many polynomial equations. We could simplify the polynomial equation equivalent to the totality of the superexponential function, for example. In Section 6.4 we will see another polynomial expression which could be reduced notably by making exp part of our language.

Hopping

There exist prefixed polynomial expressions $P(n)$ having one free natural number parameter n, such that many different equivalence classes are visited as n varies. This phenomenon is called *hopping*. A specific example is given by James Jones in [Jon78]. He introduces a polynomial equation $F(x, n)$ which hops between all equivalence classes that contain a member Con(T) for some recursively axiomatized theory T. In his words:

> "*Given any of the usual axiomatic theories to which Gödel's Incompleteness Theorem applies, there exist a value of* n*, such that* $F(n, n)$ *is unprovable and irrefutable. Thus Gödel's Incompleteness Theorem can be "focused" into the formula* $F(n, n)$*. Thus some substitution instance of* $F(n, n)$ *is undecidable in Peano artihmetic, ZF set theory, etc.*". ([Jon78], p. 335).

Jones' polynomial expression $F(x, n)$ is the following.

$$\exists\ ab\ \forall\ i\ \exists\ swpq \forall\ jv\ \exists\ eg$$

$$(n+s+1-i)\cdot[((s+w)^2+3w+s-2i)^2+(((j-w)^2+(v-q)^2)\cdot((j-s)^2+(v-p)^2\cdot$$
$$\cdot((i-n)^2+(v-q-x)^2))\cdot((j-3i)^2+(v-p-q)^2)\cdot((j-3i-1)^2+(v-pq)^2)-e-1)^2\cdot$$

$\cdot((v+g-jb)^2+(v+e+ejb-a)^2)]=0.$

Another kind of hopping is given by Theorem 6.7. Changing $\mathbf{n_1}$ into $\mathbf{n_2}$ causes a jump between the equivalence classes of 1-Con($\mathrm{I}\Sigma_{\mathbf{n_1-1}}$) and 1-Con($\mathrm{I}\Sigma_{\mathbf{n_2-1}}$).

The previous examples are but two specific instances of hopping. We expect many more, and of different kind, to be found.

Connecting mathematics

It happens quite often that mathematicians prove each other's results in different setups. The results may seem to talk or prove lemmas about p-adics, or complex numbers or finite groups or about graph theory but after a certain period of time crucial connections are revealed. In many cases, it turns out they basically say the same, but are stated in a different language. What actually happened is that the connection between several members of the same EFA-provable equivalence class is established.

On the other hand, occasionally mathematicians stumble upon lemmas with a bit of strength which puzzles them: they sense the difference but do not know how to explain why Ramsey's theorem is not the same as the prime number theorem. Of course, this intuitive reason manifests itself in the fact that Ramsey's theorem is not provable in EFA, whereas the prime number theorem is. In other words, people are looking at statements belonging to different EFA-provable equivalence classes.

As a thought experiment let us assume it is fairly easy to create a reasonably sized polynomial expression equivalent to a given arithmetical statement from mathematical practice. Then many of the existing mathematical statements could be related to their EFA-provable equivalence class in the Atlas. More precisely, mathematicans could look for the EFA-provable equivalence class of their statement. This could help them obtaining better insights into associated theorems in their own subject, as well as linking their work to completely different fields of mathematics. The Atlas could offer a very neat overview of current mathematics.

6.4 Going beyond predicative mathematics

The reader could have thought for a moment that the three relatively compact polynomial equations in Section 6.2 result from pure luck and that it is much harder to reach high impredicative equivalence classes. We also thought that for a while until proving the theorems presented in this section. Afterwards, it turned out it is even feasible to write down still stronger polynomial equations, as explained in Section 6.3.

A coarse polynomial expression equivalent to the Finite Kruskal Theorem

6.9 Theorem. *The following prefixed polynomial equation is equivalent to Finite Kruskal Theorem and hence is unprovable in predicative mathematics, for example in the theory* ATR_0.

$$\forall\, K\ \exists\, M\ \forall\, ab\ \exists\, ijcdefhk\ \forall\, lmnpq\ A\ \exists\, grst\ BFGIJLOPQWXYZ$$

$$\forall \alpha\beta\gamma\delta\zeta\eta\theta\kappa\lambda\mu\nu\xi\pi\rho\sigma\tau\ \forall\, uvxyz\ CDHNT\ \exists ERS\ \forall\, U\ \exists\, V$$

$$\begin{aligned}
&[(i-c-1)^2+(i+d-M)^2+(w+1-t)^2+(t+X-q)^2+(g+1-s)^2+(s+Y+1-r)^2+\\
&+(r+Z-q)^2+((p+l^2-bi-1-B)\cdot(l+B-p)\cdot((biA+A-a+p+l^2)^2-B-1)\cdot\\
&\cdot((K+i-q)^2-B-1)\cdot(u-pr-1-E)\cdot((prC+C-l+u)^2-E-1)\cdot(v-ps-1-E)\cdot\\
&\cdot((psD+D-l+v)^2-E-1)\cdot(x-pt-1-E)\cdot((ptH+H-l+x)^2-E-1)\cdot(u+E-v)\cdot v\cdot\\
&\cdot(q+E+1-z)\cdot(((vN-u)^2-E-1)^2+(vR-x)^2+(uS-x)^2)\cdot(y-pz-1-E)\cdot\\
&\cdot((pzT+T-l+y)^2-E-1)\cdot((ER-u)^2+(ES-v)^2+((EU-y)^2-V-1)^2))^2]\cdot\\
&\cdot[mni(m-n)\cdot(K+i+r+1-m)\cdot(K+j+r+1-m)\cdot(j+r-i)\cdot(M+r+1-j)\cdot\\
&\cdot((f+e^2+r-bi)^2+(bis+s-a+f+e^2)^2+(k+h^2+t-bj)^2+(bjW+W-a+k+h^2)^2+\\
&+(k+X+1-h)^2+(F+Y-fm)^2+(fmZ+Z-e+F)^2+(F+F^2G^2+g-dm)^2+\\
&+(dmB+B-c+F+F^2G^2)^2+(kOR+R-h+G)^2+(S+1-OIPQ(e-f))^2+\\
&+(O+V-K-j)^2+(G+E-kO)^2+(J+\alpha-fn)^2+(dn\delta+\delta-c+J+J^2L^2)^2+\\
&+(fn\beta+\beta-e+J)^2+(J+J^2L^2+\gamma-dn)^2+((L-G)^2-\zeta-1)^2+(P+P^2Q^2+\eta-dI)^2+\\
&+(dI\theta+\theta-c+P+P^2Q^2)^2+(I+\kappa-K-i)^2+(P\lambda-F)^2+(P\mu-J)^2+(P-F\nu+J\xi)^2+\\
&+(Q\pi-G)^2+(Q\rho-L)^2+(Q-G\tau+L\sigma)^2)]=0.
\end{aligned}$$

Proof. For a full proof see Section 6.5. □

A phase transition polynomial between EFA-provability and predicative unprovability

The next example combines the template of prefixed polynomial expressions with the phenomenon of phase transitions. Let $A(\mathbf{m}, \mathbf{n})$ be the polynomial equation given by

$$\forall\, K\ \exists\, M\ \forall\, ab\ \exists\, ijcdefhk\ \phi\chi\ \forall\, lmnpq\ A\,\Gamma\Delta\ \exists\, grst\ BFGIJLOPQWXYZ$$

$$\forall\, \alpha\beta\gamma\delta\zeta\eta\theta\kappa\lambda\mu\nu\xi\pi\rho\sigma\tau\varphi\psi\omega\ \forall\, uvxyz\ CDHNT\ \Theta$$

$$\exists\, ERS\ \Lambda\Upsilon\Phi\Psi\Omega\ k_*l_*m_*n_*o_*p_*\ \forall\, U\ \exists\, V$$

$$\begin{aligned}
&[(((\Gamma-i)^2-\varphi-1)\cdot((\Delta-\phi)^2-\varphi-1))^2+(((\Gamma-j)^2-\psi-1)\cdot((\Delta-\chi)^2-\psi-1))^2]\cdot\\
&\cdot[(\Gamma^2+\Delta^2)\cdot((\omega\Phi+\Phi-\psi+1)^2+(\omega\Delta\Omega+\omega\Omega+\Omega-\psi+\varphi)^2+(\varphi+\Psi-\omega\Delta-\omega)^2+\\
&+(1+\Upsilon-\omega)^2+(\Delta+k_*+1-i)^2\cdot((2\Lambda+m_*-\omega\Theta-\omega)^2+(\omega\Theta n_*+\omega n_*+n_*-\psi+2\Lambda)^2+\\
&+(\Lambda+k_*-\omega\Theta)^2+(\omega\Theta l_*+l_*-\psi+\Lambda)^2)+(\varphi+o_*-\Gamma)^2+(\Gamma+p_*+1-2\varphi)^2))]+\\
&+[(i-c-1)^2+(i+d-M)^2+(w+1-t)^2+(t+X-q)^2+(g+1-s)^2+(s+Y+1-r)^2+\\
&+(r+Z-q)^2+((p+l^2-bi-1-B)\cdot(l+B-p)\cdot((biA+A-a+p+l^2)^2-B-1)\cdot\\
&\cdot(u-pr-1-E)\cdot((\mathbf{m}K+\mathbf{n}\phi-\mathbf{m}q)^2-B-1)\cdot((prC+C-l+u)^2-E-1)\cdot(v-ps-1-E)\cdot\\
&\cdot((psD+D-l+v)^2-E-1)\cdot(x-pt-1-E)\cdot((ptH+H-l+x)^2-E-1)\cdot(q+E+1-z)\cdot v\cdot\\
&\cdot(u+E-v)\cdot(((vN-u)^2-E-1)^2+(vR-x)^2+(uS-x)^2)\cdot((pzT+T-l+y)^2-E-1)\cdot\\
&\cdot(y-pz-1-E)\cdot((ER-u)^2+((EU-y)^2-V-1)^2+(ES-v)^2))^2]\cdot[mni(m-n)\cdot\\
&\cdot(\mathbf{m}K+\mathbf{n}\phi+r+1-\mathbf{m}m)\cdot(\mathbf{m}K+\mathbf{n}\chi+r+1-\mathbf{m}m)\cdot(j+r-i)\cdot(M+r+1-j)\cdot\\
&\cdot((f+e^2+r-bi)^2+(bis+s-a+f+e^2)^2+(k+h^2+t-bj)^2+(bjW+W-a+k+h^2)^2+\\
&+(k+X+1-h)^2+(fmZ+Z-e+F)^2+(F+Y-fm)^2+(dmB+B-c+F+F^2G^2)^2+\\
&+(F+F^2G^2+g-dm)^2+(kOR+R-h+G)^2+(G+E-kO)^2+(S+1-OIPQ(e-f))^2+\\
&+(\mathbf{m}O+V-\mathbf{m}K-\mathbf{n}\chi)^2+(J+\alpha-fn)^2+(fn\beta+\beta-e+J)^2+(dn\delta+\delta-c+J+J^2L^2)^2+\\
&+(J+J^2L^2+\gamma-dn)^2+(P+P^2Q^2+\eta-dI)^2+((L-G)^2-\zeta-1)^2+(P-F\nu+J\xi)^2+\\
&+(\mathbf{m}I+\kappa-\mathbf{m}K-\mathbf{n}\phi)^2(dI\theta+\theta-c+P+P^2Q^2)^2+(P\lambda-F)^2+(P\mu-J)^2+(Q\pi-G)^2+\\
&+(Q\rho-L)^2+(Q-G\tau+L\sigma)^2)]=0.
\end{aligned}$$

6.10 Theorem. *There exists a real number w such that:*

1. *if $\frac{\mathbf{n}}{\mathbf{m}} \le w$ then* EFA *proves $A(\mathbf{m}, \mathbf{n})$;*

2. *if $\frac{\mathbf{n}}{\mathbf{m}} > w$ then* ATR_0 *does not prove $A(\mathbf{m}, \mathbf{n})$.*

Proof. Notice that $A(\mathbf{m}, \mathbf{n})$ is equivalent to an adapted version of $\mathrm{KT}_{\frac{\mathbf{n}}{\mathbf{m}}\log}$. Then use Weierman's theorem on the phase transition of $\mathrm{KT}_{r\log}$ ([Wei03]). Weiermann's single compression might seem to provide only PA-unprovability, but it is easy to see (you can see this argument spelled out in [Bov09b]) that a second compression argument gives full finite Kuskal Theorem, and hence this statement is unprovable in ATR_0. □

The number w is the real number introduced by Andreas Weiermann in [Wei03] and is defined as $\frac{1}{\log(\alpha)}$, where α is Otter's tree constant (the inverse of the radius of convergence of the generating series for unordered trees), $w \approx 0.6395781750\ldots$.

Remark that if we would allow the use of the log symbol, then the size of the polynomial would shrink.

6.5 Technical details and proofs

In this final section we explain how we obtained the concrete members of EFA-provable equivalence classes of polynomial expressions. The general ideas as well as some historical background are given in the first subsection, which is highly influenced by the introductory sections in [Mat93]. The other subsections merely intend to ensure the reader of the correctness of the polynomial equations provided above. Even though they contain some nice coding tricks and tweaking of polynomials, some parts could be skipped by readers who are not interested in the technicalities.

6.5.1 Background information and basics

In 1900, at the second *International Congress of Mathematicians*, Hilbert presented 23 (groups of) problems which he thought were the most important unsolved mathematical problems left by the nineteenth century, to be solved by the twentieth century. Among other famous problems such as the Riemann hypothesis and Goldbach's conjecture, we find Hilbert's 10th problem:

> **Entscheidung der Lösbarkeit einer diophantischen Gleichung.** Eine diophantische Gleichung mit irgendwelchen Unbekannten und mit ganzen rationalen Zahlkoefficienten sei vorgelegt: *man soll ein Verfahren angeben, nach welchem sich mittels einer endlichen Anzahl von Operationen entscheiden läßt, ob die Gleichung in ganzen rationalen Zahlen lösbar ist.*[2]

A Diophantine equation is an equation of the form

$$D(x_1, \ldots, x_n) = 0,$$

where D is a polynomial with integer coefficients. The existence of solutions of this equation is expressed by

$$\exists\, x_1 \,\exists\, x_2 \,\ldots\, \exists\, x_n \; D(x_1, \ldots, x_n) = 0. \tag{6.1}$$

The solvability of Diophantine equations has been studied seriously in the 1950s–1970s, which resulted in a negative answer to Hilbert's 10th problem. More precisely, as the class of all Diophantine sets is proven to be identical to the class of all recursively enumerable sets, there cannot exist an algorithm ("*process*") to determine the solvability of a given Diophantine equation. This result is the combined work of Martin Davis, Yuri Matiyasevich, Hilary Putnam and Julia Robinson (see e.g. [Mat93]).

Solvability of Diophantine equations (the set of all sentences of the form (6.1)) is a Σ^0_1-*complete* set of sentences. It means that every Σ^0_1 formula in the language of first-order arithmetic is EFA-provably equivalent to a sentence from this set. However, the restriction of having only one block of quantifiers makes this template metamathematically boring: so far we have not encountered interesting, metamathematical phenomena with this restriction in place.

[2]**Determination of the solvability of a Diophantine equation.** Given a Diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: *To devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers.* (Cited from [Hil02], which is the English translation of [Hil00].)

It is not difficult to show that the problem of the existence of integer solutions is reducible to the problem of the existence of solutions in natural numbers, and vice versa (see [Mat93], Chapter 1). So after restricting the range of the variables to natural numbers, expression (6.1) is clearly an instance of the template of prefixed polynomial equations. Hence, the original template related to Hilbert's 10th problem is incorporated in our approach. Moreover, as the set of prefixed polynomial equations is arithmetically complete, we are able to do some proper metamathematics.

Throughout this section we use many big and small methods and tricks developed by the community of people who studied Hilbert's 10th problem and related topics in the 1950s–1980s.

We start by showing how our general template can contain basic mathematical properties, such as $a < b$ or $a \mid b$ (a divides b), where a and b are natural numbers. The following straightforward equivalences are quite often used.

$$
\begin{aligned}
a \neq b &\Leftrightarrow \exists x((a-b)^2 = x+1) \\
&\Leftrightarrow \exists x((a-b)^2 - x - 1 = 0) \\
a \leq b &\Leftrightarrow \exists x(a + x = b) \\
&\Leftrightarrow \exists x(a + x - b = 0) \\
a < b &\Leftrightarrow \exists x(a + x + 1 = b) \\
&\Leftrightarrow \exists x(a + x + 1 - b = 0) \\
a \mid b &\Leftrightarrow \exists x(ax = b) \\
&\Leftrightarrow \exists x(ax - b = 0) \\
a \nmid b &\Leftrightarrow \forall x(ax \neq b) \\
&\Leftrightarrow \forall x \exists y((ax-b)^2 = y+1) \\
&\Leftrightarrow \forall x \exists y((ax-b)^2 - y - 1 = 0)
\end{aligned}
$$

In the next basic equivalences rem and gcd stand for *remainder* and *greatest common divisor*, respectively.

$$\begin{aligned}
a = \mathrm{rem}(b,c) \quad &\Leftrightarrow \quad a < c \ \wedge \ c \mid b - a \\
&\Leftrightarrow \quad \exists x(a + x + 1 = c) \ \wedge \ \exists x(cx = b - a) \\
&\Leftrightarrow \quad \exists x(a + x + 1 - c = 0) \ \wedge \ \exists x(cx - b + a = 0) \\
a \neq \mathrm{rem}(b,c) \quad &\Leftrightarrow \quad a \geq c \ \vee \ c \nmid b - a \\
&\Leftrightarrow \quad \exists x(a = c + x) \ \vee \ \forall x \exists y((cx - b + a)^2 = y + 1) \\
a = \gcd(b,c) \quad &\Leftrightarrow \quad a \mid b \ \wedge \ a \mid c \ \wedge \ \exists xy(a = bx - cy) \\
&\Leftrightarrow \quad \exists x(ax = b) \ \wedge \ \exists x(ax = c) \ \wedge \ \exists xy(a = bx - cy) \\
&\Leftrightarrow \quad \exists x(ax - b = 0) \ \wedge \ \exists x(ax - c = 0) \ \wedge \ \exists xy(a - bx + cy = 0) \\
a \neq \gcd(b,c) \quad &\Leftrightarrow \quad \exists x(x \mid b \ \wedge \ x \mid c \ \wedge \ x \nmid a) \\
&\Leftrightarrow \quad \exists x(\exists y(xy = b) \ \wedge \ \exists y(xy = c) \ \wedge \ \forall y \exists z((xy - a)^2 = z + 1))) \\
&\Leftrightarrow \quad \exists x(\exists y(xy - b = 0) \ \wedge \ \exists y(xy - c = 0) \\
&\qquad\qquad\qquad \wedge \forall y \exists z((xy - a)^2 - z - 1 = 0)))
\end{aligned}$$

Notice that the expressions on the right-hand side of every last equivalence are not yet prefixed polynomial equations in the strict sense. We still need to get rid of the conjunctions and disjunctions. We deal with them as follows. Let $\overline{Q^1}(\overline{x})\ P^1(\overline{x}) = 0$, and $\overline{Q^2}(\overline{y})\ P^2(\overline{y}) = 0$, be shorthand for the polynomial equations

$$Q_1^1 x_1\ Q_2^1 x_2\ \ldots\ Q_n^1 x_n\ P^1(x_1, x_2, \ldots x_n) = 0,$$

and

$$Q_1^2 y_1\ Q_2^2 y_2\ \ldots\ Q_n^2 y_n\ P^2(y_1, y_2, \ldots y_n) = 0,$$

respectively. Then

$$\begin{aligned}
&\overline{Q^1}(\overline{x})\ P^1(\overline{x}) = 0 \ \wedge \ \overline{Q^2}(\overline{y})\ P^2(\overline{y}) = 0 \\
&\qquad\qquad \Leftrightarrow \ \overline{Q^1}(\overline{x})\ \overline{Q^2}(\overline{y})\ (P^1(\overline{x})^2 + P^2(\overline{y})^2 = 0)
\end{aligned}$$

and

$$\begin{aligned}
&\overline{Q^1}(\overline{x})\ P^1(\overline{x}) = 0 \ \vee \ \overline{Q^2}(\overline{y})\ P^2(\overline{y}) = 0 \\
&\qquad\qquad \Leftrightarrow \ \overline{Q^1}(\overline{x})\ \overline{Q^2}(\overline{y})\ (P^1(\overline{x}) \cdot P^2(\overline{y}) = 0).
\end{aligned}$$

Also remark that in case of a disjunction, we can reuse variables belonging to an existential quantifier, whereas in case of a conjunction, we will reuse variables belonging to a universal quantifier. The implication is dealt with as usual ($A \to B \Leftrightarrow \neg A \vee B$). When we apply these rules we obtain the following equivalences.

$$\begin{array}{rcl}
a = \mathrm{rem}(b,c) & \Leftrightarrow & \exists xy((a+x+1-c)^2 + (cy-b+a)^2 = 0) \\
a \neq \mathrm{rem}(b,c) & \Leftrightarrow & \forall x \exists y((a-c-y)\cdot((cx-b+a)^2 - y - 1) = 0) \\
a = \gcd(b,c) & \Leftrightarrow & \exists xyzu((ax-b)^2 + (ay-c)^2 + (a-bz+cu)^2 = 0) \\
a \neq \gcd(b,c) & \Leftrightarrow & \exists xyz\forall u \exists v((xy-b)^2 + (xz-c)^2 + ((xu-a)^2 - v - 1)^2 = 0).
\end{array}$$

It is well-known that, given the toolbox above, one could start trying to translate in a naive way known first-order arithmetical statements containing strength. This would give rise to prefixed polynomial expressions possessing certain strength, although done very rudimentarily. Thanks to our template, one could give up the restriction of having only one block of existential quantifiers, which results in already rather short expressions, in comparison to known examples. Moreover, clever coding and reuse of variables can shorten such a prefixed polynomial equation even further.

However, pure translation of known unprovable statements soon turns out to have its limits. As we reached this point, we needed to dig deeper in order to obtain more compact representatives of EFA-provable equivalence classes. First of all, we twisted the unprovable statements in such a way that coding became more neat. Clearly, knowledge of unprovability theory is very welcome at this stage. Second, we reduced the polynomials using small algebraic tricks. In the final result, one might not recognise the original statement, as it has been manipulated so often.

In the subsections below one can find a brief account of how we obtained the prefixed polynomial equations presented in Section 6.2 and Section 6.4. As explained, getting to the final polynomial is part of a process which develops in time, so it is impossible to write down every single detail. Nevertheless, enough detail is provided to consider the explanations below as proofs, in the traditional sense of the word.

6.5.2 Exponentiation and logarithm

As a way of demonstrating the possibilities of the toolbox introduced above, and because we will need parts of the expressions later, we show how to express the totality of the exponential and logarithm function as prefixed polynomial equations. Notice that the treatment of the superexponential function would be very similar to how we dealt with the exponentiation.

Exponentiation

So, we want to be able to express $x^y = z$, for natural numbers x, y and z. The idea is to construct a sequence $(1, x, x^2, \ldots, x^y)$. We will do so by defining a sequence $(a_1, a_2, \ldots, a_{y+1})$, such that $a_1 = 1$ and $a_{i+1} = x \cdot a_i$, for $0 < i \leq y$. Using Gödel coding we can code sequences $(a_1, \ldots, a_n)$ as pairs (a, b) of natural numbers in such a way that for $i = 1, \ldots, n$,

$$a_i = \mathrm{rem}(a, bi + 1).$$

We refer to Section 3.2 in [Mat93] for more information on this type of coding. Combining the previous information, we obtain the following equivalences.

$$\begin{aligned}
x^y = z \Leftrightarrow\ & \exists\, ab\, \forall\, i\ (1 = a_1\ \wedge\ z = a_{y+1}\ \wedge\ (0 < i \leq y \rightarrow a_{i+1} = xa_i)) \\
\Leftrightarrow\ & \exists\, ab\, \forall\, i\, \exists\, c \\
& \quad ((0 < i\ \wedge\ i \leq y) \rightarrow (1 = \mathrm{rem}(a, b+1)\ \wedge\ z = \mathrm{rem}(a, b(y+1)+1) \\
& \quad \wedge\ c = \mathrm{rem}(a, bi+1)\ \wedge\ xc = \mathrm{rem}(a, b(i+1)+1)) \\
\Leftrightarrow\ & \exists\, ab\, \forall\, i\, \exists\, c\, \forall\, rs \\
& \quad ((0 < i\ \wedge\ i \leq y\ \wedge\ ((r = 1\ \wedge\ s = 1)\ \vee\ (r = z\ \wedge\ s = y+1) \\
& \quad \vee\ (r = c\ \wedge\ s = i)\ \vee\ (r = xc\ \wedge\ s = i+1))) \rightarrow (r = \mathrm{rem}(a, bs+1)).
\end{aligned}$$

Using this last expression, we can present the totality of the exponential function $(\forall\ xy\ \exists\ z\ (x^y = z))$, after substituting the relevant prefixed polynomials, dealing with the implication, merging conjunctions and disjunctions, by the following prefixed polynomial equation:

$$\forall\, xy\, \exists\, z\, ab\, \forall\, i\, \exists\, c\, \forall\, rs\, \exists\, ef$$

$i \cdot (y+e+1-i) \cdot (((r-1)^2+(s-1)^2) \cdot ((r-z)^2+(s-y-1)^2) \cdot ((r-c)^2+(s-i)^2) \cdot ((r-xc)^2+$
$+(s-i-1)^2) - e - 1) \cdot ((r+e-bs)^2 + (bsf + f - a - r)^2) = 0$

Let us shortly comment on the general pattern of writing down the polynomial equations. We always start with a line (or several lines) containing the quantifiers and variables. Next, we begin a new line to write the actual polynomial. At the end of a line, we never break a term which is inside the most inner brackets. In case a power is redundant from a number-theoretic point of view, we leave it out in order to increase readability and reduce size. Finally, at the beginning of each line we repeat the last operation on the previous line for the sake of clarity.

Logarithm

Since we are working with natural numbers, we define $\log(x)$ as the floor (integer part) of the usual base-2 logarithm, with $\log(0)$ redefined as 0. Then we get the following.

$$\log(x) = y \quad \Leftrightarrow \quad \exists\, z\ ((x = 0\ \wedge\ y = 0)\ \vee\ (z = 2^y\ \wedge\ z \le x\ \wedge\ x < 2z)).$$

To express $z = 2^y$ we use the polynomial expression for exponentiation which is given above. Then we can present the totality of the logarithm function ($\forall\ x\ \exists\ y\ (\log(x) = y)$), after substituting the appropriate prefixed polynomials, merging conjunctions and disjunctions, by the following prefixed polynomial equation:

$$\forall\, x\ \exists\, yz\ ab\ \forall\, i\ \exists\, c\ \forall\, rs\ \exists\, ef\ gh$$

$(x^2+y^2) \cdot ((i \cdot (y+e+1-i) \cdot (((r-1)^2+(s-1)^2) \cdot ((r-z)^2+(s-y-1)^2) \cdot ((r-c)^2+(s-i)^2) \cdot$
$\cdot ((r-2c)^2+(s-i-1)^2)-e-1) \cdot ((r+e-bs)^2+(bsf+f-a-r)^2))^2+(z+g-x)^2+$
$+(x+h+1-2z)^2) = 0.$

6.5.3 Unprovability by primitive recursive means

Let $e > 2$ be any given natural number and consider the following statement:

"for every number m, there exists a number N, such that for every colouring f of 2-element subsets of $\{0, 1, \ldots, N\}$ into e colours, there is an f-homogeneous $H \subseteq \{0, 1, \ldots, N\}$ of size at least $\min H + m - 1$".

The assertion above is very similar to PH^2_e, the Paris-Harrington principle for pairs and e colours. In fact, after quantification over e it is equivalent to $\forall e\, \mathrm{PH}^2_e$, which is equivalent to 1-Con($\mathrm{I}\Sigma_1$) (see [Par80]).

Let us first fix notation and rewrite the previous statement using but mathematical symbols. As before, $[A]^n$ denotes the set of all n-element subsets of A, for every $n \geq 2$ and $A \subseteq \mathbb{N}$. If N is a natural number, then $[N]$ will denote the set $\{0, 1, \ldots, N-1\}$. $[[N]]^n$ will be simplified to $[N]^n$ and $f(\{x_1, \ldots, x_n\})$ is shortened to $f(x_1, \ldots, x_n)$, under the assumption that the x_i's are increasing. We obtain:

$$\begin{aligned} &\forall\, m\ \exists\, N\ \forall\, f\colon [N+1]^2 \to [e]\ \exists\, Hc \\ &\qquad (H \subseteq [N+1]\ \wedge\ |H| \geq \min H + m - 1\ \wedge\ f \restriction [H]^2 = \{c\}). \end{aligned} \tag{6.2}$$

The main idea is to represent colourings $f\colon [N+1]^2 \to e$ as sequences $(a_1, a_2, \ldots, a_n)$ of natural numbers, in such a way that, if $k < l \in [N+1]$ and $k + l^2 = i$, then

$$a_i \equiv f(k, l) \mod e.$$

Remark that if $k < l$ then the function which associates (k, l) with $k + l^2$ is injective. Using Gödel coding we can code sequences $(a_1, \ldots, a_n)$ as pairs (a, b), as done in the previous subsection.

If (a, b) codes a sequence $(a_1, \ldots, a_n)$ such that $n < N + (N+1)^2$, then not all values of possible 2-element subsets of $[N+1]$ will be covered. In that case the sequence is extended in a trivial way by adding a's at the end until the length of the sequence is at least $N + (N+1)^2$. This extended sequence defines a function $f\colon [N+1]^2 \to e$, as described above.

Remark that the equalities $a_i = \mathrm{rem}(a, bi + 1)$ and $a_i \equiv f(k, l) \mod e$ now hold for all $k < l \in [N+1]$ and $i = k + l^2$.

We will code the subset H as an increasing sequence $(c_1, \ldots, c_p)$, such that $c_i \in [N+1]$ for $i = 1, \ldots, p$. Using Gödel coding, this latter sequence is coded as a pair (c, d).

To avoid ambiguity we use the letter Y instead of H to denote the homogeneous set. Also, X will denote the constant colour c. The intermediate statement, equivalent to (6.2) becomes:

$\forall\, m\, \exists\, N\, \forall\, ab\, \exists\, cd\ A\ X$	(a,b) codes f, (c,d) codes Y
$\forall\, xy\, \exists\, BCF$	more variables to express our needs
$((0 < x\ \wedge\ x < y$	x and y are indices of elements of Y
$\wedge\, y \leq A + m - 1)$	$\lvert Y\rvert \geq \min Y + m - 1$
$\rightarrow (A = \mathrm{rem}(c, d+1)$	A is the first element of Y
$\wedge\, B = \mathrm{rem}(c, dx+1)$	B is the x^{th} element of Y
$\wedge\, C = \mathrm{rem}(c, dy+1)$	C is the y^{th} element of Y
$\wedge\, B < C$	(c,d) codes elements of Y in strictly increasing order
$\wedge\, C < N+1$	$Y \subseteq [N+1]$
$\wedge\, F = \mathrm{rem}(a, b(B+C^2)+1)$	$F = f(B,C)$
$\wedge\, F \equiv X \mod e))$	$f(B,C)$ equals colour X

After substituting the relevant prefixed polynomials, merging conjunctions and disjunctions and tweaking the expressions, we end up with the following polynomial equation:

$$\forall\, m\, \exists\, N\, \forall\, ab\, \exists\, cd\ A\ X\, \forall\, xy\, \exists\, BCF\, \forall\, fg\, \exists\, hilnrpq$$

$$x\cdot(y+B-x)\cdot(A+m+B-y)\cdot[((((f-A)^2+(g-1)^2)\cdot((f-B)^2+(g-x)^2)\cdot((f-C)^2+ +(g-y)^2)-h-1)\cdot((dgi+i-c+f)^2+(f+h-dg)^2)+(B+l+1-C)^2+(C+n-N)^2+ +(F+r-b(B+C^2))^2+(bp(B+C^2)+p-a+F)^2+((F-X)^2-qe)^2]=0.$$

6.5.4 Unprovability in two-quantifier-induction arithmetic

Let $e > 1$ be any given natural number and consider the following statement:

"for every number m, there exists a number N, such that for every colouring f of 3-element subsets of $\{0,1,\dots,N\}$ into e colours, there is an f-homogeneous $H \subseteq \{0,1,\dots,N\}$ of size at least $\min H + m - 1$".

The statement above is very similar to PH^3_e, the Paris-Harrington principle for triples and e colours. In fact, after quantification over e it is equivalent to $\forall e\,\mathrm{PH}^3_e$, which is equivalent to 1-Con($\mathrm{I}\Sigma_2$) (see [Par80]). In purely mathematical language one would write it down as:

$$\begin{aligned}&\forall\, m\ \exists\, N\ \forall\, f\colon [N+1]^3 \to [e]\ \exists\, Hc \\ &\qquad (H \subseteq [N+1]\ \wedge\ |H| \geq \min H + m - 1\ \wedge\ f \restriction [H]^3 = \{c\}).\end{aligned} \tag{6.3}$$

The main idea is to represent colourings $f\colon [N+1]^3 \to [e]$ as sequences $(a_1, \ldots, a_n)$ of natural numbers, in such a way that, if $j < k < l \in [N+1]$ and $j + k^2 + l^3 = i$, then

$$a_i \equiv f(j,k,l) \mod e.$$

Remark that if $j < k < l$ then the function which associates (j,k,l) with $j + k^2 + l^3$ is injective. The intermediate translation of the statement (6.3) becomes (once again we use the letter Y instead of H to denote the homogeneous set, and X to denote the constant colour c):

$\forall\, m\ \exists\, N\ \forall\, ab\ \exists\, cd\ AX$	(a,b) codes f, (c,d) codes Y
$\forall\, xyz\ \exists\, BCDH$	more variables are needed
$[(0 < x\ \wedge\ x < y\ \wedge\ y < z$	x,y,z are indices of elements
$\wedge\ z \leq A + m - 1)$	$\lvert Y\rvert \geq \min Y + m - 1$
$\to (A = \mathrm{rem}(c, d+1)$	A is the first element of Y,
$\wedge\ B = \mathrm{rem}(c, dx+1)$	B is the x^{th} element of Y
$\wedge\ C = \mathrm{rem}(c, dy+1)$	C is the y^{th} element of Y
$\wedge\ D = \mathrm{rem}(c, dz+1)$	D is the z^{th} element of Y
$\wedge\ B < C$	(c,d) codes elements of Y in strictly increasing order
$\wedge\ C < D$	(c,d) codes elements of Y in strictly increasing order
$\wedge\ D < N+1$	$Y \subseteq [N+1]$
$\wedge\ H = \mathrm{rem}(a, b(B + C^2 + D^3) + 1)$	$H = f(B,C,D)$
$\wedge\ H \equiv X \mod e)]$	$f(B,C,D)$ equals colour X

After substituting the suitable prefixed polynomials, merging conjunctions and disjunctions and tweaking the expressions, we end up with the following polynomial equation:

$$\forall\, m\ \exists\, N\ \forall\, ab\ \exists\, cd\ A\ X\ \forall\, xyz\ uvw\ \exists\, BCDGH\ \forall\, fg\ \exists\, hijklnpqrst\ FG$$

$$x\cdot(y+B-x)\cdot(z+B-y)\cdot(A+m+B-z)\cdot[(((f-A)^2+(g-1)^2)\cdot((f-B)^2+(g-x)^2)\cdot \\ \cdot((f-C)^2+(g-y)^2)\cdot((f-D)^2+(g-z)^2)-h-1)\cdot((dgi+i-c+f)^2+(f+h-dg)^2)+ \\ +(D+l-dz)^2+(dzn+n-c+D)^2+(B+F+1-C)^2+(C+p+1-D)^2+(D+q-N)^2+ \\ +(H+r-bt)^2+(bts+s-a+H)^2+(B+C^2+D^3-t)^2+((X-H)^2-Ge)^2]=0.$$

6.5.5 Unprovability in Peano Arithmetic

Let e and n be any given natural numbers and consider the following statement:

"for every number m, there exists a number N, such that for every colouring f of n-element subsets of $\{0,1,\ldots,N\}$ into e colours, there is an f-homogeneous $H \subseteq \{0,1,\ldots,N\}$ of size at least $\min H + m - 1$".

The statement above is very similar to PH^n_e, the Paris-Harrington principle for n dimensions and e colours. In fact, after quantification over n, the statement above implies $\forall n\, \mathrm{PH}^n_e$, which is equivalent to 1-Con($\mathrm{I}\Sigma_n$) (see [Par80]).

In purely mathematical language one would write the assertion above as:

$$\begin{aligned} &\forall\, m\ \exists\, N\ \forall\, f\colon [N+1]^n \to [e]\ \exists\, Hc \\ &\qquad (H \subseteq [N+1]\ \wedge\ |H| \geq \min H + m - 1\ \wedge\ f \restriction [H]^n = \{c\}). \end{aligned} \tag{6.4}$$

The encoding of f and H will be similar to the previous cases. We need to express that every n-element subset of H is coloured in the same way. These n-element subsets will be represented by an increasing sequence of n elements of H.

The intermediate statement, equivalent to (6.4) becomes (once again we use the letter Y instead of H to denote the homogeneous set, and X to denote the constant colour c):

$$
\begin{array}{ll}
\forall\, m\ \exists\, N\ \forall\, ab\ \exists\, cd\ A\ X & (a,b) \text{ codes } f,\ (c,d) \text{ codes } Y, \\
[\forall xy\ zw\ i & (x,y) \text{ and } (z,w) \text{ code } n\text{-sets} \\
\quad ((0 < i\ \wedge\ i \leq n) & i \text{ is an index for elements} \\
\qquad \rightarrow\ \mathrm{rem}(x, yi+1) = \mathrm{rem}(z, wi+1)) & \text{the first } n \text{ elements of } (x,y) \\
 & \text{and } (z,w) \text{ are equal} \\
\qquad\quad \rightarrow \mathrm{rem}(a, b(x^2+y)+1) & \text{the } n\text{-sets represented by } (x,y) \text{ and} \\
\qquad\qquad = \mathrm{rem}(a, b(z^2+w)+1))] & (z,w) \text{ are coloured in the same way} \\
\rightarrow & \\
[\forall k\ \alpha\beta\ j\ FG\ \exists\ BCD\ f & \\
\quad (A = \mathrm{rem}(c, d+1) & A \text{ is the first element of } Y \\
\quad\ \wedge B = \mathrm{rem}(c, dk+1) & B \text{ is the } k^{\mathrm{th}} \text{ element of } Y \\
\quad\ \wedge C = \mathrm{rem}(c, d(k+1)+1) & C \text{ is the } (k+1)^{\mathrm{th}} \text{ element of } Y \\
\quad\ \wedge D = \mathrm{rem}(a, b(\alpha^2+\beta)+1) & D \text{ is the colour of the } n\text{-set } (\alpha,\beta) \\
\quad\ \wedge F = \mathrm{rem}(\alpha, \beta j+1) & F \text{ is the } j^{\mathrm{th}} \text{ element of } (\alpha,\beta) \\
\quad\ \wedge G = \mathrm{rem}(\alpha, \beta(j+1)+1) & G \text{ is the } (j+1)^{\mathrm{th}} \text{ element of } (\alpha,\beta) \\
\quad\ \wedge ((0 < k\ \wedge\ k \leq A+m-1) & k \text{ is an index for elements of } Y \\
\qquad \rightarrow\ (B < C\ \wedge\ C < N+1)) & \text{elements of } Y \subseteq [N+1] \text{ are given} \\
 & \text{in strictly increasing order} \\
\quad\ \wedge ((0 < j\ \wedge\ j \leq n) & j \text{ is an index for elements of } (\alpha,\beta) \\
\qquad \rightarrow\ (F = \mathrm{rem}(c, df+1) & F \text{ also belongs to } Y \\
\qquad\quad \wedge\, 0 < f\ \wedge\ f \leq A+m-1 & \\
\qquad\quad \wedge\, F < G) & \text{the elements of } (\alpha,\beta) \text{ are given} \\
 & \text{in strictly increasing order} \\
\qquad \rightarrow\ D \equiv X \mod e))] & \text{the colour of } (\alpha,\beta) \text{ equals } X
\end{array}
$$

After substituting the relevant prefixed polynomials, merging conjunctions and disjunctions and tweaking the expressions, we end up with the following polynomial equation. Remark that we reused several variables in order to reduce size.

$$\forall\, m\ \exists\, N\ \forall\, ab\ \exists\, cd\ xy\ zw \forall\ it\ \alpha\beta\ \exists\ fg\ hklpqruv\ X\ ABCDEF\ \forall j\ GI\ \exists\ sH$$

$$
\begin{aligned}
&[i\cdot(n+f+1-i)\cdot((g+f-yi)^2+(yih+h-x+g)^2+(g+l-wi)^2+(wik+k-z+g)^2)+\\
&+((p+q-b(x^2+y))^2+(b(x^2+y)r+r-a+p)^2+((b(z^2+w)j+j-a+p)^2-s-1)\cdot\\
&\cdot(p-b(z^2+w)-1-s))^2]\cdot[(z+x-d)^2+(yd+y-c+z)^2+(t\cdot(z+m+f-t)\cdot((g+l-dt)^2+\\
&+(dtk+k-c+g)^2+(h+p-d(t+1))^2+(d(t+1)q+q-c+h)^2+(g+r+1-h)^2+\\
&+(g+s-N)^2))+((u+1-X)^2+(X+v-n)^2+(A+C-\beta X)^2+(\beta XD+D-\alpha+A)^2+\\
&+(B+E-\beta(X+1))^2+(\beta(X+1)F+F-\alpha+B)^2+(((dGI+I-c+A)^2-H-1)\cdot G\cdot
\end{aligned}
$$

$\cdot(A-dG-1-H)\cdot(z+m+H-G)\cdot(B+H+1-A))^2)\cdot((b(\alpha^2+\beta)B+B-a+u)^2+$
$+(u+A-b(\alpha^2+\beta))^2+((u-w)^2-ve)^2)]=0$

6.5.6 Going beyond predicative mathematics

We start with the following slightly changed version of the finite Kruskal Theorem (proved ATR_0-unprovable by Friedman, see e.g. [Smi85]):

"for all K there is a number M such that whenever $T_1,\ldots,T_M$ are finite trees such that for all $i \leq M$, the number of vertices of T_i equals $K+i$ there are two indices $i<j\leq M$ such that the tree T_i inf-preservingly embeds into T_j".

In more mathematical language one could write this down as:

$$\begin{aligned}&\forall\, K\ \exists\, M\ \forall\, T_1,\ldots,T_M \\ &\qquad (\forall i(|T_i|=K+i)\rightarrow \exists ij(i<j\ \wedge\ T_i \text{ embeds inf-preservingly into } T_j)).\end{aligned} \tag{6.5}$$

The idea goes as follows. A tree T_i is coded by a pair (x,y), where x and y Gödel-code a sequence of $(K+i)$-many natural numbers, the vertices of the tree. The set is ordered by divisibility and, to make sure it is a tree, we demand that for every two natural numbers in the set, one not dividing another, there is no other number in the set divisible by them both.

So, in a first block first, we state that we are dealing with a sequence of trees as described above.

$$
\begin{array}{ll}
\forall\, K\, \exists\, M\, \forall\, ab & \\
[\forall\, i\, \exists\, \sigma pq\, \forall\, rst\, \exists\, uvxyz & \\
\quad q = K + i & q \text{ is the size of the tree } T_i \\
\quad \wedge\ (0 < i\ \wedge\ i \leq M & i \text{ is an index for trees} \\
\quad\quad \wedge\ 0 < t\ \wedge\ t \leq q & t \text{ is an index for nodes in } T_i \\
\quad\quad \wedge\ 0 < s\ \wedge\ s < r\ \wedge\ r \leq q) & s \text{ and } r \text{ are indices for nodes in } T_i \\
\quad\quad \rightarrow (p + \sigma^2 = \mathrm{rem}(a, bi + 1) & (p, \sigma) \text{ codes the tree } T_i \\
\quad\quad\quad \wedge\ p < \sigma & \text{needed for the coding} \\
\quad\quad\quad \wedge\ u = \mathrm{rem}(\sigma, pr + 1) & r \text{ is the index of the element } u \text{ of } T_i \\
\quad\quad\quad \wedge\ v = \mathrm{rem}(\sigma, ps + 1) & s \text{ is the index of the element } v \text{ of } T_i \\
\quad\quad\quad \wedge\ x = \mathrm{rem}(\sigma, pt + 1) & t \text{ is the index of the element } x \text{ of } T_i \\
\quad\quad\quad \wedge\ 0 < v\ \wedge\ v < u & \text{smaller indices code smaller elements} \\
\quad\quad\quad \wedge\ (v \nmid u \rightarrow (v \nmid x\ \vee\ u \nmid x)) & \text{two elements not on the same branch,} \\
 & \text{do not have a common supremum.} \\
\quad\quad\quad \wedge\ z \leq q & y \text{ actually is an element of } T_i \\
\quad\quad\quad \wedge\ y = \mathrm{rem}(\sigma, pz + 1) & z \text{ is the index of the element } y \text{ of } T_i \\
\quad\quad\quad \wedge\ y = \gcd(u, v)))] & y \text{ is the infimum of } u \text{ and } v
\end{array}
$$

Given this sequence of trees, we will claim the existence of two indices i and j $(i < j)$, such that there exists an embedding from T_i into T_j. This embedding is coded by the pair (c, d), which represents a sequence of pairs. Such a pair consists of a node of T_i and its corresponding node of T_j. So given the block of formulas above, we get the following implication:

$$
\begin{array}{ll}
\rightarrow & \\
[\exists\, ij\ cd\ ef\ hk\, \forall\, mn & (c, d) \text{ codes the embedding } T_i \rightarrow T_j \\
\ \exists\, FGp\ JL\ PQq & \\
\quad (0 < i\ \wedge\ i < j\ \wedge\ j \leq M & T_i \text{ and } T_j \text{ belong to our sequence} \\
\quad\quad \wedge\ 0 < m\ \wedge\ m \leq K + i & m \text{ is an index for nodes in } T_i \\
\quad\quad \wedge\ 0 < n\ \wedge\ n \leq K + j\ \wedge\ m \neq n) & n \text{ is an index for nodes in } T_j \\
\quad \rightarrow (e + f^2 = \mathrm{rem}(a, bi + 1) & (e, f) \text{ codes the tree } T_i \\
\quad\quad \wedge\ h + k^2 = \mathrm{rem}(a, bj + 1) & (h, k) \text{ codes the tree } T_j \\
\quad\quad \wedge\ F = \mathrm{rem}(e, fm + 1) & F \text{ belongs to } T_i \\
\quad\quad \wedge\ F + F^2G^2 = \mathrm{rem}(c, dm + 1) & (F, G) \text{ belongs to the embedding} \\
\quad\quad \wedge\ G = \mathrm{rem}(h, kp + 1) & G \text{ belongs to } T_j
\end{array}
$$

$\wedge\ 0 < p\ \wedge\ p \leq K + j$	
$\wedge\ J = \mathrm{rem}(e, fn + 1)$	J belongs to T_i
$\wedge\ J + J^2L^2 = \mathrm{rem}(c, dn + 1)$	(J, L) belongs to the embedding
$\wedge\ L \neq G$	the embedding is injective
$\wedge\ P + P^2Q^2 = \mathrm{rem}(c, dq + 1)$	(P, Q) belongs to the embedding
$\wedge\ 0 < q\ \wedge\ q \leq K + i$	
$\wedge\ 0 < P\ \wedge\ 0 < Q$	pairing function is injective
$\wedge\ \gcd(F, J) = P$	embedding preserves the infimum
$\wedge\ \gcd(G, L) = Q))]$	

Remark that some of the variables have already been merged, in order to make it easier to check the correctness of the polynomial afterwards.

Final prefixed polynomial equation for Kruskal's Theorem

The following prefixed polynomial equation is equivalent to statement (6.5) above.

$$\forall\ K\ \exists\ M\ \forall\ ab\ \exists\ ijcdefhk\ \forall\ lmnpq\ A\ \exists\ grst\ BFGIJLOPQWXYZ$$

$$\forall \alpha\beta\gamma\delta\zeta\eta\theta\kappa\lambda\mu\nu\xi\pi\rho\sigma\tau\ \forall\ uvxyz\ CDHNT\ \exists ERS\ \forall\ U\ \exists\ V$$

$$\begin{aligned}
&[(i-c-1)^2+(i+d-M)^2+(w+1-t)^2+(t+X-q)^2+(g+1-s)^2+(s+Y+1-r)^2+\\
&+(r+Z-q)^2+((p+l^2-bi-1-B)\cdot(l+B-p)\cdot((biA+A-a+p+l^2)^2-B-1)\cdot\\
&\cdot((K+i-q)^2-B-1)\cdot(u-pr-1-E)\cdot((prC+C-l+u)^2-E-1)\cdot(v-ps-1-E)\cdot\\
&\cdot((psD+D-l+v)^2-E-1)\cdot(x-pt-1-E)\cdot((ptH+H-l+x)^2-E-1)\cdot(u+E-v)\cdot v\cdot\\
&\cdot(q+E+1-z)\cdot(((vN-u)^2-E-1)^2+(vR-x)^2+(uS-x)^2)\cdot(y-pz-1-E)\cdot\\
&\cdot((pzT+T-l+y)^2-E-1)\cdot((ER-u)^2+(ES-v)^2+((EU-y)^2-V-1)^2))^2]\cdot\\
&\cdot[mni(m-n)\cdot(K+i+r+1-m)\cdot(K+j+r+1-m)\cdot(j+r-i)\cdot(M+r+1-j)\cdot\\
&\cdot((f+e^2+r-bi)^2+(bis+s-a+f+e^2)^2+(k+h^2+t-bj)^2+(bjW+W-a+k+h^2)^2+\\
&+(k+X+1-h)^2+(F+Y-fm)^2+(fmZ+Z-e+F)^2+(F+F^2G^2+g-dm)^2+\\
&+(dmB+B-c+F+F^2G^2)^2+(kOR+R-h+G)^2+(S+1-OIPQ(e-f))^2+\\
&+(O+V-K-j)^2+(G+E-kO)^2+(J+\alpha-fn)^2+(dn\delta+\delta-c+J+J^2L^2)^2+\\
&+(fn\beta+\beta-e+J)^2+(J+J^2L^2+\gamma-dn)^2+((L-G)^2-\zeta-1)^2+(P+P^2Q^2+\eta-dI)^2+\\
&+(dI\theta+\theta-c+P+P^2Q^2)^2+(I+\kappa-K-i)^2+(P\lambda-F)^2+(P\mu-J)^2+(P-F\nu+J\xi)^2+\\
&+(Q\pi-G)^2+(Q\rho-L)^2+(Q-G\tau+L\sigma)^2)]=0
\end{aligned}$$

Final polynomial expression for Weiermann's phase transition

Studying phase transitions, we parametrise the statement (6.5) above as follows:

$$\forall\, K\, \exists\, M\, \forall\, T_1, \ldots, T_M \qquad (6.6)$$
$$(\forall i(|T_i| = K + f(i)) \rightarrow \exists ij(i < j\ \wedge\ T_i \text{ embeds inf-preservingly into } T_j)).$$

Now we set, in the expresion (6.6) above, $f(i)$ to be $\frac{n}{m}\log(i)$ for every $i \in \mathbb{N}$. We modify the prefixed polynomial equation by introducing $\phi = \log(i)$ and $\chi = \log(j)$ and afterwards substituting it at the right places in our polynomial. We introduce ϕ and χ as follows.

$$\forall\, \Gamma\Delta\ ((\Gamma = i\ \wedge\ \Delta = \phi)\ \vee\ (\Gamma = j\ \wedge\ \Delta = \chi)) \rightarrow (\Delta = \log(\Gamma)).$$

We use the expression for the logarithm, which one can find above and modify it to fit the context. Then "$\log(\Gamma) = \Delta$" is equivalent to

$$\exists\, \varphi\, \psi\omega\, \forall\, \Theta\, \exists\, \Lambda\, \forall\, \Psi\Omega\, \exists\, \Upsilon\Phi\, k_* l_*$$

$$(\Gamma^2+\Delta^2)\cdot((\Theta\cdot(\Delta+\Upsilon+1-\Theta)\cdot(((r-1)^2+(s-1)^2)\cdot((r-\varphi)^2+(s-\Delta-1)^2)\cdot((r-\Lambda)^2+ +(s-\Theta)^2)\cdot((r-2\Lambda)^2+(s-\Theta-1)^2)-\Upsilon-1)\cdot((r+\Upsilon-\omega s)^2+(\omega s\Phi+\Phi-\psi-r)^2))^2+ +(\varphi+k_*-\Gamma)^2+(\Gamma+l_*+1-2\varphi)^2)=0.$$

So, after substituting the relevant prefixed polynomials, merging conjunctions and disjunctions and tweaking the expressions, we end up with the following polynomial equation:

$$\forall\, K\, \exists\, M\, \forall\, ab\, \exists\, ijcdefhk\ \phi\chi\, \forall\, lmnpq\ A\,\Gamma\Delta\, \exists\, grst\ BFGIJLOPQWXYZ$$

$$\forall\, \alpha\beta\gamma\delta\zeta\eta\theta\kappa\lambda\mu\nu\xi\pi\rho\sigma\tau\, \exists\, \varphi\psi\omega\, \forall\, uvxyz\ CDHNT\ \Theta$$

$$\exists\, ERS\ \Lambda\, \forall\, \Psi\Omega\, \exists\, \Upsilon\Phi\ k_* l_*\, \forall\, U\, \exists\, V$$

$$(((\Gamma-i)^2+(\Delta-\phi)^2)\cdot((\Gamma-j)^2+(\Delta-\chi)^2)-\varphi-1)\cdot[(\Gamma^2+\Delta^2)\cdot((\Theta\cdot(\Delta+\Upsilon+1-\Theta)\cdot \cdot(((r-1)^2+(s-1)^2)\cdot((r-\varphi)^2+(s-\Delta-1)^2)\cdot((r-\Lambda)^2+(s-\Theta)^2)\cdot((s-\Theta-1)^2+ +(r-2\Lambda)^2)-\Upsilon-1)\cdot((r+\Upsilon-\omega s)^2+(\omega s\Phi+\Phi-\psi-r)^2))^2+(\Gamma+l_*+1-2\varphi)^2+ +(\varphi+k_*-\Gamma)^2)]+[(i-c-1)^2+(i+d-M)^2+(w+1-t)^2+(t+X-q)^2+(s+Y+1-r)^2+ +(g+1-s)^2+(r+Z-q)^2+((p+l^2-bi-1-B)\cdot((biA+A-a+p+l^2)^2-B-1)\cdot(l+B-p)\cdot \cdot(u-pr-1-E)\cdot((\mathbf{m}K+\mathbf{n}\phi-\mathbf{m}q)^2-B-1)\cdot((prC+C-l+u)^2-E-1)\cdot(v-ps-1-E)\cdot$$

$$\cdot((psD+D-l+v)^2-E-1)\cdot(x-pt-1-E)\cdot((ptH+H-l+x)^2-E-1)\cdot(q+E+1-z)\cdot v\cdot$$
$$\cdot(u+E-v)\cdot(((vN-u)^2-E-1)^2+(vR-x)^2+(uS-x)^2)\cdot((pzT+T-l+y)^2-E-1)\cdot$$
$$\cdot(y-pz-1-E)\cdot((ER-u)^2+((EU-y)^2-V-1)^2+(ES-v)^2))^2]\cdot[mni(m-n)\cdot$$
$$\cdot(\mathbf{m}K+\mathbf{n}\phi+r+1-\mathbf{m}m)\cdot(\mathbf{m}K+\mathbf{n}\chi+r+1-\mathbf{m}m)\cdot(j+r-i)\cdot(M+r+1-j)\cdot$$
$$\cdot((f+e^2+r-bi)^2+(bis+s-a+f+e^2)^2+(k+h^2+t-bj)^2+(bjW+W-a+k+h^2)^2+$$
$$+(k+X+1-h)^2+(fmZ+Z-e+F)^2+(F+Y-fm)^2+(dmB+B-c+F+F^2G^2)^2+$$
$$+(F+F^2G^2+g-dm)^2+(kOR+R-h+G)^2+(G+E-kO)^2+(S+1-OIPQ(e-f))^2+$$
$$+(\mathbf{m}O+V-\mathbf{m}K-\mathbf{n}\chi)^2+(J+\alpha-fn)^2+(fn\beta+\beta-e+J)^2+(dn\delta+\delta-c+J+J^2L^2)^2+$$
$$+(J+J^2L^2+\gamma-dn)^2+(P+P^2Q^2+\eta-dI)^2+((L-G)^2-\zeta-1)^2+(P-F\nu+J\xi)^2+$$
$$+(\mathbf{m}I+\kappa-\mathbf{m}K-\mathbf{n}\phi)^2(dI\theta+\theta-c+P+P^2Q^2)^2+(P\lambda-F)^2+(P\mu-J)^2+(Q\pi-G)^2+$$
$$+(Q\rho-L)^2+(Q-G\tau+L\sigma)^2)]=0$$

Appendix A

DUTCH SUMMARY

Nederlandstalige samenvatting

In deze Nederlandstalige samenvatting beschrijven we kort en bondig welke onderwerpen er in deze doctoraatsthesis behandeld worden. De opbouw van deze appendix is vergelijkbaar met die van de Engelstalige tekst; we bespreken de zes hoofdstukken in dezelfde volgorde. Voor specifieke informatie zoals definities, stellingen, bewijzen, extra uitleg, details en verwijzingen naar de literatuur, verwijzen we naar het Engelstalige gedeelte.

A.1 Inleiding

Het is geen sinecure om een volledig overzicht van de ontwikkeling van de wiskundige logica te geven. Het zal dan ook geen verrassing zijn dat de bondige samenvatting hieronder onvolledig is. De bedoeling is veeleer enkele belangrijke personen te introduceren en betekenisvolle gebeurtenissen weer te geven.

Na decennia van grote expansie van wiskundige kennis, werd het in het prille begin van de twintigste eeuw duidelijk dat er dringend nood was aan stabiele grondslagen voor de wiskunde. Verschillende logici hadden hun eigen visie op hoe men uit deze grondslagencrisis kon komen. We vermelden hier slechts twee stromingen: het intuïtionisme met Luitzen Egbertus Jan

Brouwer als hoofdfiguur en de axiomatische benadering, sterk ontwikkeld door David Hilbert. Deze laatste stelde een allesomvattende aanpak voor: het *programma van Hilbert*. Later zou blijken dat dit programma, zoals oorspronkelijk geformuleerd, niet kon slagen.

Kurt Gödel presenteerde immers op 7 september 1930 tijdens een conferentie in Königsberg zijn eerste *onvolledigheidsstelling*, die later samen met de tweede gepubliceerd werd. Grofweg zeggen deze stellingen het volgende. Laat T een formeel systeem (theorie) zijn dat de elementaire rekenkunde bevat. Dan geldt er dat ofwel T inconsistent[1] is ofwel er rekenkundige uitspraken bestaan die bewezen noch ontkend kunnen worden in T. Bovendien is "T is consistent" zo'n zin. Omdat die onafhankelijke uitspraken wel bewezen kunnen worden in sterkere theorieën, zullen we dergelijke zinnen *onbewijsbaar* noemen, ten opzichte van een bepaalde (minder krachtige) theorie.

De onvolledigheidsstellingen hebben diepgaande gevolgen voor de logica en, zo bleek duidelijk een halve eeuw later, tevens voor andere takken in de wiskunde. Jeff Paris presenteerde immers in 1977, samen met Leo Harrington, het eerste voorbeeld van een uitspraak die natuurlijk oogde en relevant was voor wiskundigen, maar toch onbewijsbaar in Peano Aritmetica (dit is een theorie voor elementaire rekenkunde, kortweg PA). Deze uitspraak kwam uit het gebied van de combinatoriek, meer bepaald uit de Ramsey theorie. Het was een belangrijke doorbraak omdat de impact op wiskunde die niet sterk verwant is met logica, voordien ongekend was. Gödels oorspronkelijke, onafhankelijke uitspraak was immers geconstrueerd via codering van de syntax en via logische kneepjes.

Sindsdien zijn er verschillende voorbeelden van natuurlijke, onbewijsbare uitspraken gevonden en bestudeerd. We denken hierbij aan arboriciteit (George Mills), het Hydragevecht en het beëindigen van Goodsteinrijen (Laurie Kirby and Paris), flipbaarheid (Kirby), kiraliciteit en regaliteit (Peter Clote en Kenneth McAloon), het benaderen van functies (Pavel Pudlák),

[1] Een theorie is inconsistent als er een uitspraak A bestaat zodat T zowel A, als de negatie van A bewijst. Uiteraard is het wenselijk dat een theorie consistent (i.e. niet inconsistent) is.

principes gerelateerd aan eindige bomen, Higmans stelling en Kruskals stelling (Harvey Friedman), regressieve Ramsey stellingen (Akihiro Kanamori en McAloon), *etc.*.

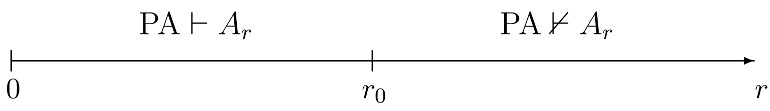
Onderzoek van onafhankelijke uitspraken leidde tot een expansie van het gebied dat vandaag bekend staat als *onbewijsbaarheidstheorie* en een waaier aan onderzoeksmogelijkheden aanbiedt. We gaan niet in detail, maar vermelden enkele essentiële onderwerpen. De studie van wiskundig relevante, onbewijsbare uitspraken is nog steeds belangrijk en wordt ondernomen door onder andere Friedman in zijn "Boolean Relation Theory" en "Upper Shift Kernel Theory". Beide domeinen bevatten voorbeelden van een *template*, hetgeen ook in deze thesis aan bod komt.

Een ander onderzoeksgebied bestudeert wiskundige redenen voor onbewijsbaarheidsfenomenen. Hier zijn er duidelijke verbanden met het "Reverse Mathematics"-programma, maar ook met de filosofie van de wiskunde. Een recent terrein van onbewijsbaarheidstheorie behandelt de *Atlas van veeltermvergelijkingen met een prefix* (zie Sectie A.6). We sluiten deze onvolledige opsomming af met een onderwerp dat uitermate belangrijk is in deze doctoraatsverhandeling: *faseovergangen*.

In het begin van de 21ste eeuw diversifieerde de onbewijsbaarheidstheorie verder toen Andreas Weiermann een nieuw studiedomein ontwikkelde: overgangen van onbewijsbaarheid naar bewijsbaarheid. Het is immers mogelijk een onbewijsbare zin van een parameter te voorzien, zodat bij een kleine verandering van de parameter de uitspraak plots bewijsbaar wordt. Men kan dit vergelijken met het verhogen van de temperatuur tot boven het smeltpunt van water, waardoor ijs verandert in vloeibaar water. Het fysisch systeem (of de wiskundige uitspraak) gaat van een bepaalde fase over in een andere.

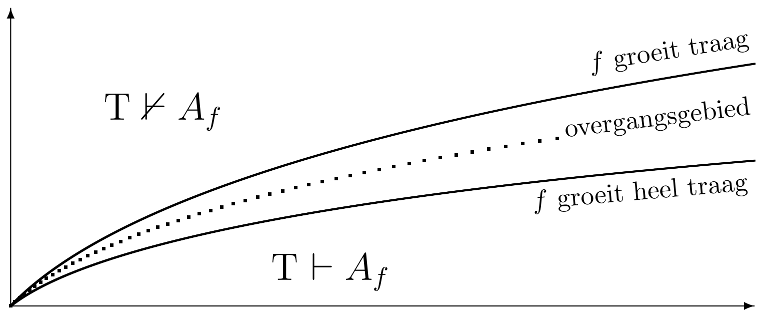
Weiermanns eerste resultaten in dit gebied waren gerelateerd aan een miniaturisatie van Kruskals stelling die we A_r noemen, waarbij de parameter r een rationaal getal is. Indien $r \leq r_0$ dan zal PA de uitspraak A_r kunnen bewijzen (in symbolen: $\mathrm{PA} \vdash A_r$). Als echter $r > r_0$, dan wordt A_r onbewijsbaar ten opzichte van PA (of dus $\mathrm{PA} \nvdash A_r$). (Zie Figuur A.1.)

De parameter hoeft echter geen getal te zijn, maar kan ook een functie zijn. Figuur A.2 toont een faseovergang voor een uitspraak A_f die afhan-



Figuur A.1: *Faseovergang van bewijsbaarheid naar onbewijsbaarheid – I*

kelijk is van zo'n parameterfunctie f. Zoals men kan zien, zal een theorie T (bijvoorbeeld PA) de uitspraak A_f kunnen bewijzen als f uiterst traag groeit. Indien f echter een beetje sneller groeit, dan wordt A_f onbewijsbaar ten opzichte van T. In deze doctoraatsverhandeling bestuderen we faseovergangen van de laatste soort.



Figuur A.2: *Faseovergang van bewijsbaarheid naar onbewijsbaarheid – II*

Zoals hierboven vermeld, behoort het eerste wiskundig natuurlijke voorbeeld van een onafhankelijke uitspraak (het Paris-Harrington principe) tot het domein van de Ramsey-theorie. Dit onderzoeksdomein is een tak van de combinatoriek die genoemd is naar de Britse wiskundige Frank Ramsey en die nog steeds nuttig is voor onderzoek naar onbewijsbaarheid. Ramsey's stelling wordt genoteerd als RT, waarbij $\mathrm{RT} = \forall n\, \forall k\, \mathrm{RT}^n_k$ en

$$\begin{aligned} \mathrm{RT}^n_k \leftrightarrow\ & \text{Voor elke } G\colon [\mathbb{N}]^n \to k \text{ bestaat er een oneindige } H \\ & \text{zodat } G \restriction [H]^n \text{ constant is.} \end{aligned}$$

De volgende uitspraak beschrijft op een eerder filosofische wijze de dieper liggende gedachte achter fenomenen uit de Ramsey theorie:

> *"There are numerous theorems in mathematics which assert, crudely speaking, that every system of a certain class possesses a large subsystem with a higher degree of organization than the original system."*[2] (H. Burkill and L. Mirsky)

In deze dissertatie onderzoeken we onbewijsbare uitspraken, waarvan de meeste uit het gebied van de Ramsey-theorie komen. We zullen ook de bijbehorende faseovergangen bepalen. Hiervoor hebben we de nodige wiskundige hulpmiddelen nodig. In het laatste deel van de inleiding worden belangrijke theorieën, ordinaalgetallen en hiërarchieën van functies ingevoerd. Bovendien worden standaardconcepten gedefinieerd alsook enkele nuttige eigenschappen vermeld.

A.2 Het duiventilprincipe

Het duiventilprincipe (of ladenprincipe, genoteerd PHP) zegt dat als er n duiven in een duiventil met m hokjes geplaatst worden met $n > m$, dat er dan minstens een hokje is dat meer dan één duif bevat. Dit principe is voor het eerst neergeschreven door Dirichlet in 1834 en is ondertussen alom gekend. Het is een specifiek geval van de eindige versie van Ramsey's stelling $\mathrm{RT}^1_{<\infty}$, hetgeen staat voor $\forall k\, \mathrm{RT}^1_k$, gebruik makende van de notatie ingevoerd in Sectie A.1.

Gezien het nauwe verband met de Ramsey-theorie die vaak aanleiding geeft tot onbewijsbaarheid, is het evident zich af te vragen welke de *logische sterkte* van deze uitspraak is (gegeven een basistheorie). Met "logische sterkte" van een theorie T bedoelen we "rekenkundige sterkte", hetgeen staat voor de verzameling van alle eerste orde, rekenkundige stellingen die bewijsbaar zijn in T.

[2] *"Er zijn talrijke stellingen in de wiskunde die poneren dat, grofweg gezegd, elk systeem van een bepaalde klasse een groot subsysteem bevat dat meer structuur bezit dan het originele systeem."*

Om die sterkte te bestuderen, introduceren we in dit hoofdstuk twee eigenschappen van deelverzamelingen van natuurlijke getallen. Deze eigenschappen (n-PHP-dichtheid en $(\alpha, 2)$-PHP-dichtheid) zijn gerelateerd aan $\mathrm{RT}^1_{<\infty}$ en RT^1_2 en geven aanleiding tot onbewijsbare uitspraken. Zo'n soort dichtheden zijn oorspronkelijk ingevoerd door Paris toen hij onbewijsbaarheid bestudeerde en deze blijken tot vandaag nog steeds handig.

Voor de onafhankelijke uitspraken die we zo verkrijgen bepalen we de faseovergangen. De parameterfuncties die de overgang bepalen voor de eerste PHP-dichtheid zijn dezelfde als deze voor de versie van het Kanamori-McAloon principe met parameter (zie Figuur 2.1). De functies die horen bij de tweede PHP-dichtheid zijn gelijk aan deze die de faseovergang bij het Paris-Harrington principe bepalen (zie Figuur 2.2).

A.3 Zwak stijgende deelrijen

We vertrekken van de volgende stelling die we noteren als ISP:

"elke oneindige rij van natuurlijke getallen bevat
een oneindige, zwak stijgende deelrij".

Men kan ISP direct afleiden uit een specifiek geval van de Ramsey stelling, namelijk de versie RT^2_2 voor dimensie twee en met twee kleuren. Zoals in Sectie A.2, doet het verband met de Ramsey-theorie ons vermoeden dat er logische sterkte kan gevonden worden.

Om die sterkte te bestuderen, introduceren we n-ISP-dichtheid, die nauw verbonden is met ISP. Eenmaal we dan ISP, dat over oneindige objecten spreekt, in een eindig vorm (ISP-dichtheid) hebben gegoten, tonen we aan dat we uitspraken verkrijgen die onbewijsbaar zijn ten opzichte van de theorie $\mathrm{I}\Sigma_1$, een fragment van PA.

We bepalen de bijbehorende faseovergang nauwkeurig (zie Figuur 3.1). Merk op dat de functies die behoren bij deze faseovergang dezelfde zijn als die voor de n-PHP-dichtheid, beschreven in Sectie A.2.

A.4 Oneindige Ramsey stelling

In dit hoofdstuk bestuderen we de volledige versie van de Ramsey stelling, dewelke we noteren als RT. Als we gebruik maken van de notatie zoals ingevoerd in Sectie A.1, dan is RT gedefinieerd als $\forall n \, \forall k \, \mathrm{RT}^n_k$. Analoog aan de werkwijze in Sectie A.2 en Sectie A.3, bestuderen we de logische sterkte van een eindige versie van RT en dit in de vorm van n-RT-dichtheid. Dit stelt ons in staat een deel van de sterkte van $\mathrm{RCA}_0 + \mathrm{RT}$ te omvatten, waarbij RCA_0 een deelsysteem van tweede-orde rekenkunde is (waarvan $\mathrm{I}\Sigma_1$ het eerste-orde deel is).

Om deze resultaten te verkrijgen, maken we gebruik van een veralgemening van een stelling van Teresa Bigorajska en Henryk Kotlarski die partities van α-grote verzamelingen behandelt. Nadat zij eerdere resultaten van Jussi Ketonen en Robert Solovay generaliseerden tot ordinaalgetallen kleiner dan ε_0, gaan wij verder en laten we ordinaalgetallen tot aan ε_ω toe. Een groot deel van dit vijfde hoofdstuk is daaraan gewijd.

We vermoeden dat de verkregen onbewijsbare uitspraak leidt tot een faseovergang. Deze is echter nog niet in detail uitgewerkt, zodat dit een mogelijk onderwerp is voor toekomstig onderzoek.

A.5 Nash-Williams Ramsey-theorie

Nash-Williams Ramsey-theorie is een tak in de Ramsey-theorie die ontstaan is rond een aantal stellingen van Nash-Williams, gepubliceerd in de jaren zestig. We bestuderen hier in detail één van deze stellingen, namelijk Nash-Williams' verdelingstelling, of kortweg NWT. Deze zegt het volgende:

> "elke dunne familie van eindige verzamelingen
> van natuurlijke getallen is Ramsey".

Vanuit een breder perspectief bekijken we NWT als een template, namelijk "dun impliceert Ramsey". In dit hoofdstuk bekijken we instanties van deze template. In een eerste deel geven we enkele zwakkere versies die vertrouwde concepten behandelen (woorden, bomen, grafen, *etc.*). Nadien veralgemenen we NWT voor families van eindige, gelabelde structuren. Hiervoor passen we een techniek toe die gekend is als “combinatorisch dwingen”, waarbij het bestaan van een verzameling met zekere eigenschap wordt “afgedwongen”. Deze methode werd reeds door Nash-Williams toegepast in zijn oorspronkelijke publicaties.

In het tweede deel bestuderen we in detail de sterkte van één van de instanties van de template, namelijk de oorspronkelijke NWT. Resultaten hieromtrent zijn reeds bekend, maar niet altijd even gemakkelijk terug te vinden in de literatuur. Daarom bewijzen we uitvoerig dat NWT equivalent is met ATR_0 over RCA_0. De sterkte van de veralgemening tot gelabelde structuren is nog niet gekend en vormt een mogelijk onderwerp voor toekomstig onderzoek.

Tenslotte gaan we even dieper in op Schreier families. Deze families geven aanleiding tot veralgemeningen van RT en NWT. Vanuit dat oogpunt formuleren we een eerste-orde uitspraak die onbewijsbaar is in PA.

A.6 De Atlas van veeltermvergelijkingen met een prefix

Dit hoofdstuk is bedoeld als een uitgebreide inleiding tot de studie van de *Atlas van veeltermvergelijkingen met een prefix*, hetgeen hierna wordt gedefinieerd. Een veeltermvergelijking met prefix is een uitdrukking met als algemene vorm

$$Q_1x_1\ Q_2x_2\ \dots\ Q_nx_n\ P(x_1, x_2, \dots, x_n) = 0,$$

waarbij P een veelterm is met gehele getallen als coëfficiënten, die is voorafgegaan door een blok van kwantoren Q_1, $Q_2, \dots, Q_n$ met als variabelen $x_1, x_2, \dots, x_n$ die de natuurlijke getallen bestrijken. Vaak spreken we kortweg over een veeltermvergelijking. Deze algemene vorm kunnen we bekijken als een template, waarvoor we specifieke instanties kunnen bestuderen. De collectie van alle veeltermvergelijkingen noemen we de Atlas.

Als basistheorie gebruiken we Exponentiële Functie Aritmetica (EFA), die ons toelaat veeltermvergelijkingen te vergelijken en te verdelen in equivalentieklassen, waarbij twee leden equivalent zijn als ze EFA-bewijsbaar equivalent zijn. Deze template is *rekenkundig compleet*. Dit wil zeggen dat elke eerste-orde, rekenkundige uitspraak equivalent is aan een veelterm-vergelijking met prefix. Na het definiëren van het begrip lengte van een veelterm, bestuderen we verschillende instanties uit de Atlas, en bepalen hun lengte en logische sterkte. We geven drie concrete voorbeelden van vrij beknopte veeltermvergelijkingen die verrassend genoeg reeds heel wat sterkte bezitten (aangezien ze onbewijsbaar zijn in $I\Sigma_1$, $I\Sigma_2$ en PA).

In het derde deel beschrijven we hoe de template aanleiding geeft tot een grote verscheidenheid aan mogelijkheden. In het bijzonder geven we uitleg bij enkele specifieke leden van de Atlas die relatief grote sterkte bezitten, maar die we hier niet in detail behandelen. Verder staan we even stil bij veeltermvergelijkingen met minimale lengte in hun equivalentieklasse (zogenaamde *zaden*), bekijken we hoe een uitbreiding van de wiskundige taal nieuwe mogelijkheden biedt en beschrijven we kort het fenomeen *hoppen*. Een laatste bedenking gaat over de Atlas als instrument om stellingen uit verschillende wiskundige disciplines te vergelijken.

Dat de Atlas ook leden bevat die heel wat logische sterkte impliceren, hoewel ze een aanvaardbare lengte hebben, wordt aangetoond in het voorlaatste deel. We geven hier ook een voorbeeld van een faseovergang gerelateerd aan een veeltermvergelijking.

Tenslotte geven we technische details en bewijzen.

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