Spectral Models for $1D$ Blood Flow Simulations

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***Abstract*— In this paper we introduce a new theoretical formulation for the description of the blood flow in the circulatory system. Starting from a linearized version of the Navier-Stokes equations, the Green's function of the propagation problem is computed in a rational form. As a consequence, the input-output transfer function relating the upstream and downstream pressure and blood flow is written in a rational form as well, leading to a time-domain state-space model suitable for transient analysis. The proposed theoretical formulation has been validated by pertinent numerical results.**

I. Introduction

The cardiovascular system can be considered as a wide hydraulic network working under the action of a pulsatile pump. This hydraulic network is a closed loop and shows a different behavior at various locations. For instance, the wave propagation in the arteria tree is of greater influence than in the capillary bed where the flow is almost steady [1]. The mathematical and numerical modeling of the human cardiovascular system has become a topic of great interest in the recent years (see e.g. [2], [3], [4], [5] and the references therein). The development of this research field arises from the medical community interested in scientifically rigorous and quantitative investigations of cardiovascular diseases, which are responsible today for about the 40 percent of death in industrialized societies [6].

Human arterial system can be mathematically described by different models with a different level of detail (see e.g. [7], [8], [9]). For instance, the Windkessel and similar lumped models are often used to represent blood flow and pressure in the arterial system [1]. These lumped models can be derived from electrical circuit analogies where current I represents arterial blood flow and voltage V represents arterial pressure. Resistors represent arterial and peripheral resistance that occur as a result of viscous dissipation inside the vessels, capacitors represent volume compliance of the vessels that allows them to store large amounts of blood, and inductors represent inertia of the blood [10]. The main advantage of lumped models is that they can be solved by ordinary differential equations.

One-dimensional models of the human arterial system were introduced for the first time by Euler [11] and more recently, in [12]. It has been shown numerically that the linearization of the one-dimensional model around a constant state matches the non-linear system itself. The first direct derivation of a linearized $1D$ model from axisymmetric Navier-Stokes equations was carried out in [13].

Over the last years, one-dimensional models of human arterial system have been intensively investigated [1], [14], [9], [2]. In [1], it has been shown that, for the particular case of blood flow, lumped networks can be regarded as first order discretization of one-dimensional linear systems. In this work, starting from a linearized version of the Navier-Stokes equations, we present a spectral $1D$ model which is suitable for transient blood flow analysis. Using the analogy of the blood flow propagation problem with transmission lines, the corresponding Green's function is developed in a spectral (rational) form [15]. Hence, rational models that relates downstream and upstream pressure and blood flow can be computed and used for time-domain simulations.

II. From non-linear conservation laws to Telegrapher's equations

A basic description of RLC networks as an approximation of non-linear conservation laws for blood flow can be found in [16]. In [1], a new approach is described to introduce RLC networks as approximants of non-linear equations. Starting from a simplified version of the Navier-Stokes equations in the axisymmetric form and integrating them over each cross-section $A(z,t)$ of the vessel, the following set of $1D$ equations is obtained for $0 \leq z \leq \ell$ (ℓ is the vessel length) and all $t > 0$:

$$\frac{\partial}{\partial t}A\left(z,t\right)+\frac{\partial}{\partial z}Q\left(z,t\right)=0 \tag{1a}$$

$$\frac{\partial}{\partial t}Q\left(z,t\right)+\frac{\partial}{\partial z}\frac{\alpha Q^2\left(z,t\right)}{A\left(z,t\right)}+\frac{A\left(z,t\right)}{\rho}\frac{\partial}{\partial z}P\left(z,t\right) = -K_r\frac{Q\left(z,t\right)}{A\left(z,t\right)} \tag{1b}$$

where α (the momentum-flux correction coefficient), ρ (the blood density) and K_r (the friction parameter) are supposed to be constant. A Poiseuille profile is assumed for velocity in the vessel. Under these hypotheses, the friction parameter can be assumed as $K_r = 8\pi\nu$, where ν is the blood viscosity. A, P and Q are the unknowns to be determined. The following

This work was supported by the Italian Ministry of University (MIUR) under a Program for the Development of Research of National Interest, (PRIN grant n. 20089J4SM9-002) and by the Research Foundation Flanders (FWO)

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relation between P and A is assumed [1]

$$P(A(z,t)) = \frac{\beta}{A_0}\left(\sqrt{A(z,t)} - \sqrt{A_0}\right) \tag{2}$$

where A_0 is the section area at rest and the coefficient β supposed constant along the whole vessel, is:

$$\beta = \frac{\sqrt{\pi} h E}{(1-\sigma^2)} \tag{3}$$

The constants E, h and σ are the Young modulus, the wall thickness and the Poisson ratio, respectively. Then, the system (1) can be rewritten in a non-conservative form by expressing (1) in (P,Q) variables and using the pressure law (2). Linearizing the system around the constant state $(A,u) = (A_0, 0)$ and setting the following parameters

$$C' = \frac{2A_0\sqrt{A_0}}{\beta}, L' = \frac{\rho}{A_0}, R' = \frac{\rho K_r}{A_0^2} \tag{4}$$

a simple linear model is obtained:

$$C'\frac{\partial}{\partial t}P(z,t) \quad + \quad \frac{\partial}{\partial z}Q(z,t) = 0 \tag{5a}$$
$$L'\frac{\partial}{\partial t}Q(z,t) \quad + \quad \frac{\partial}{\partial z}P(z,t) = -R'Q(z,t) \tag{5b}$$

Equations (5) can be regarded as Telegrapher's equations [17] whose per-unit-length parameters are given in (4). We propose a new approach to build rational models of linearized system (5), which are suitable for transient analysis of vessels.

III. The Telegrapher's equations as a Sturm-Liouville problem

The Laplace-transformation of the Telegrapher's equations (5) yields [17]:

$$\begin{aligned}\frac{d}{dz}P(z,s) &= -(R' + sL')\,Q(z,s)\\ &= -Z'(s)Q(z,s)\end{aligned} \tag{6a}$$
$$\begin{aligned}\frac{d}{dz}Q(z,s) &= -(sC')\,P(z,s) + Q_s(z,s)\\ &= -Y'(s)P + Q_s(z,s)\end{aligned} \tag{6b}$$

$Q_s(z,s)$ is a per-unit-length blood flow source located at abscissa z [15]. Taking the one-dimensional divergence of (6a) and replacing it into (6b), we obtain:

$$\frac{d^2}{dz^2}P(z,s) - \gamma^2(s)P(z,s) \quad = \quad -Z'(s)Q_s(z,s) \tag{7}$$

where $\gamma^2(s) = Z'(s)Y'(s)$. Equation (7) can be regarded as a Sturm-Liouville problem [18].

If we assume that the blood flow is specified only in correspondence of abscissa $z = 0$ and $z = \ell$, it can be described in terms of per-unit-length sources as:

$$Q_s(z,s) \quad = \quad Q_0(s)\delta(z) + Q_\ell(s)\delta(z-\ell) \tag{8}$$

where $\delta(z)$ is the one-dimensional Dirac distribution and $Q_0(s)$, $Q_\ell(s)$ represent the blood flow at the ends of the vessel. Since the differential problem (7) is self-adjoint [18], the corresponding Green's function can be developed in a series form.

The Green's function $G(z,z')$ satisfies the Sturm-Liouville problem corresponding to a unit source with homogenous boundary condition of the Neumann type:

$$[L + \lambda r(z)]\,G(z,z') = \delta(z,z') \tag{9a}$$
$$\frac{d}{dz}G(z,z')\,|_{z=0} = \frac{d}{dz}G(z,z')\,|_{z=\ell} = 0 \tag{9b}$$

where $\delta(z,z')$ is the one-dimensional Dirac distribution. Once $G(z,z')$ is computed, (7) can be solved as:

$$P(z,s) = \int_0^\ell G(z,z')\left(-Z'(s)Q_s(z,s)\right)dz' \tag{10}$$

The Green's function $G(z,z')$ can be expanded as a series of a complete set of orthonormal eigenfunctions:

$$G(z,z') \quad = \quad \sum_{n=0}^{\infty}\frac{\phi_n(z')}{\lambda - \lambda_n}\phi_n(z) \tag{11}$$

The orthonormal functions $\phi_n(z)$ and eigenvalues λ_n are obtained by solving the corresponding eigenvalue problem with homogenous boundary conditions of Neumann type. They are found to be:

$$\lambda_n = \left(\frac{n\pi}{\ell}\right)^2 \qquad n = 0,1,2,\cdots \tag{12}$$
$$\phi_n(z) = A_n\cos\left(\frac{n\pi}{\ell}z\right) \tag{13}$$

where

$$A_0 = \sqrt{\frac{1}{\ell}} \qquad A_n = \sqrt{\frac{2}{\ell}}, \quad n = 1,\cdots,\infty \tag{14}$$

Finally, the Green's function for the one-dimensional wave propagation is

$$G(z,z') \quad = \quad -\sum_{n=0}^{\infty}A_n^2\frac{\cos\left(\frac{n\pi}{\ell}z\right)\cos\left(\frac{n\pi}{\ell}z'\right)}{\gamma^2(s) + \left(\frac{n\pi}{\ell}\right)^2} \tag{15}$$

The general solution for the downstream and upstream pressure (voltage) of the vessel (transmission line) is computed by (10) leading to the input/output matrix representation of the vessel:

$$\begin{bmatrix} P_0(s) \\ P_\ell(s)\end{bmatrix} = \begin{bmatrix}\mathcal{H}_{11}(s) & \mathcal{H}_{12}(s)\\ \mathcal{H}_{21}(s) & \mathcal{H}_{22}(s)\end{bmatrix}\cdot\begin{bmatrix}Q_0(s)\\ Q_\ell(s)\end{bmatrix} \tag{16}$$

where

$$\mathcal{H}_{11}(s) \quad = \quad \mathcal{H}_{22}(s) = \sum_{n=0}^{\infty}\frac{A_n^2 Z'(s)}{\gamma^2(s) + \left(\frac{n\pi}{\ell}\right)^2} \tag{17a}$$
$$\mathcal{H}_{12}(s) \quad = \quad \mathcal{H}_{21}(s) = \sum_{n=0}^{\infty}\frac{A_n^2 Z'(s)\cos(n\pi)}{\gamma^2(s) + \left(\frac{n\pi}{\ell}\right)^2} \tag{17b}$$

Assuming that the infinite summation in (17) is truncated to n_{modes}, the transfer function matrix $\boldsymbol{\mathcal{H}}(s)$ can be rewritten in a pole/residue form as:

$$\boldsymbol{\mathcal{H}}(s) = \sum_{k=1}^{n_{poles}}\frac{\boldsymbol{R}_k}{s - p_k} \tag{18}$$

where $n_{poles} = 2(n_{modes}+1)$ represents the total number of poles and the residues $\boldsymbol{R}_k, k = 1, \cdots, n_{poles}$ and poles $p_k, k = 1, \cdots, n_{poles}$ can be computed by standard techniques [19].

The rational representation (18) is well-suited for both circuit synthesis [20] and state-space realization [21]. The state-space equivalent form can be written as:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{\mathcal{A}}\boldsymbol{x}(t) + \boldsymbol{\mathcal{B}}\boldsymbol{Q}(t) \quad (19a)$$
$$\boldsymbol{P}(t) = \boldsymbol{\mathcal{C}}\boldsymbol{x}(t) + \boldsymbol{\mathcal{D}}\boldsymbol{Q}(t) \quad (19b)$$

where $\boldsymbol{\mathcal{A}} \in \Re^{p\times p}$, $\boldsymbol{\mathcal{B}} \in \Re^{p\times q}$, $\boldsymbol{\mathcal{C}} \in \Re^{q\times p}$, $\boldsymbol{\mathcal{D}} \in \Re^{q\times q}$, p is the number of states and $q = 2$ is the number of ports. $\boldsymbol{Q}(t) = [Q_0(t)\ Q_\ell(t)]^T$ and $\boldsymbol{P}(t) = [P_0(t)\ P_\ell(t)]^T$ correspond to upstream and downstream blood flow and pressure, respectively.

It is to be pointed out that the proposed approach does not require any space discretization since the Green's function $G(z, z')$ is written in a closed-form in terms of the eigenfunctions of the differential problem (7). Furthermore, the proposed technique is also well-suited to incorporate frequency-dependent effects (e.g. the sleeve effect [13]) by means of a rational form of the $Z'(s)$ and $Y'(s)$ operators. The overall cardiovascular tree can be obtained by properly connecting the state-space models of each vessel.

IV. Numerical Example

The proposed modeling approach has been validated by a numerical simulation that computes the pressure $P_\ell(t)$ and the blood flow $Q_0(t)$ for a vessel of fixed length ($\ell = 60$ cm), while enforcing the pressure and blood flow at abscissa $z = 0$ and $z = \ell$, respectively. We have considered a vessel with a radius equal to 0.5 cm and thickness equal to 0.1 cm. The blood viscosity is set to ν=0.035 cm^2/s, its density is equal to 1 g/cm^3, the Young modulus is set to $3{\cdot}10^6$ dyne/cm^2, while the momentum-flux correction coefficient α is 1 [1]. The transfer function $\mathcal{H}(s)$ has been computed by using the proposed Green's function based method (GF) and compared with that of standard transmission line theory (TLT) [22]. The order of the spectral model is set to $n_{modes} = 60$. The algorithm to choose the number of modes is described in [23]. Fig. 1 shows the magnitude spectra of transfer functions $\mathcal{H}_{11}(s)$ and $\mathcal{H}_{12}(s)$.

Then, the state-space representation (19) has been generated. The pressure is imposed at the left boundary and the blood flow is enforced at the right boundary. The reference results are computed in the time-domain by using the exact formula based on Bessel functions [24]. The proposed state-space model has been integrated in the time-domain by using the Gear-Shichman scheme [25]. Figs. 2 and 3 compare the downstream and upstream pressure and blood flow obtained by the proposed approach (GF-Gear) with the reference results (TLT), showing a very good agreement. As clearly seen by the numerical results, the proposed approach is able to accurately capture the $1D$ blood flow propagation in a vessel structure.

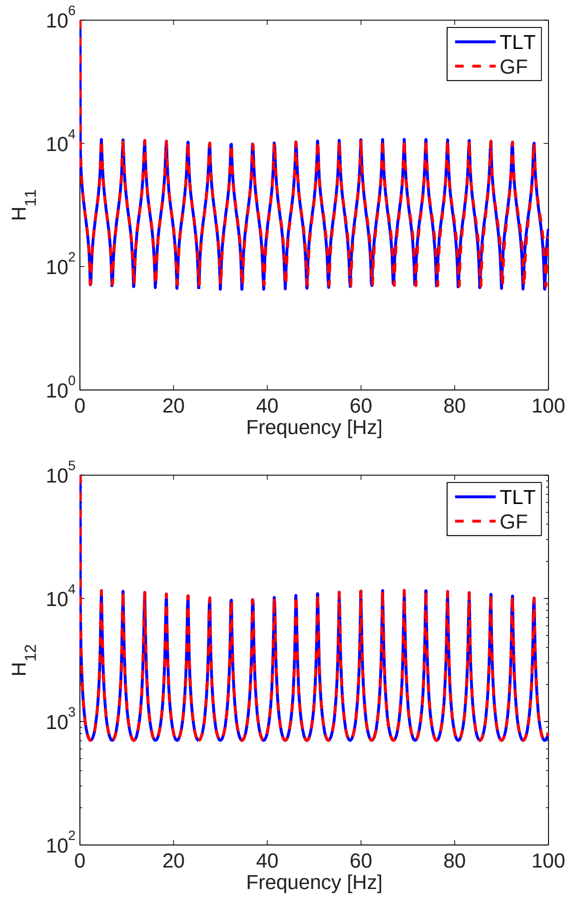

Fig. 1. Magnitude spectra of transfer functions $\mathcal{H}_{11}(s)$ and $\mathcal{H}_{12}(s)$.

V. Conclusions

We have presented an innovative methodology for $1D$ blood flow transient analysis. The Green's function of the one-dimensional propagation is evaluated and used to generate a rational model whose poles and residues can be easily computed by means of standard techniques. The rational model can be directly incorporated into ordinary differential equations or SPICE-like circuit solvers. The numerical results have demonstrated the robustness of the proposed method in capturing the $1D$ blood flow propagation and, hence, the downstream and upstream pressure and blood flow waveforms. The proposed methodology will be experimentally validated in the next future.

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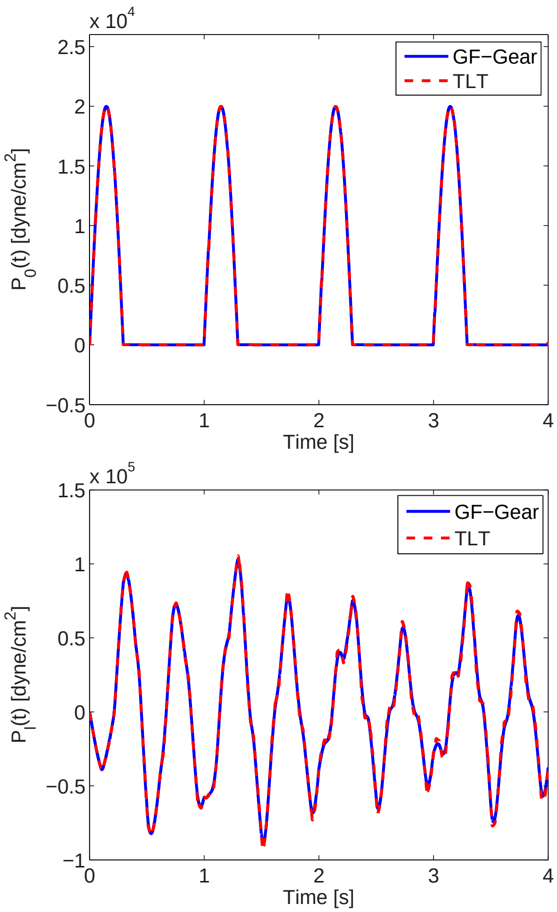

Fig. 2. Upstream (top) and downstream (bottom) pressure $P(t)$.

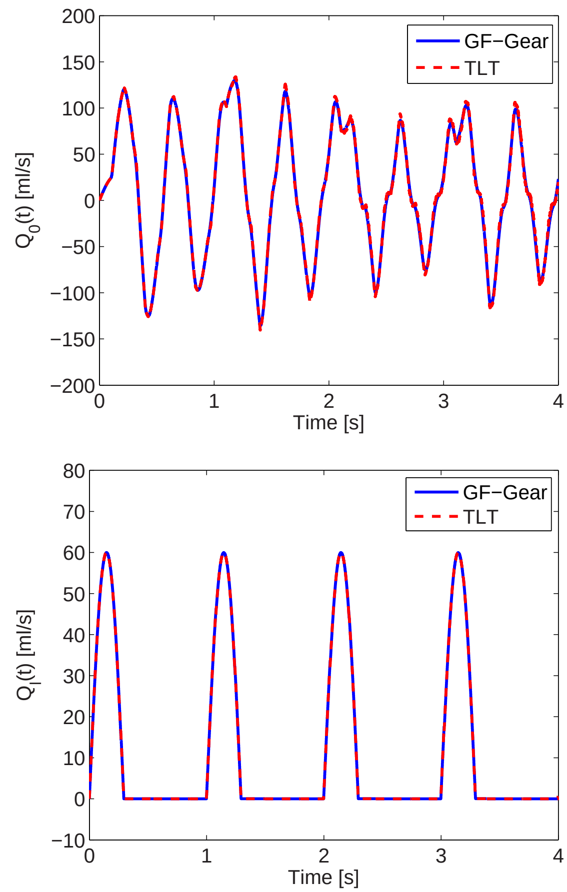

Fig. 3. Upstream (top) and downstream (bottom) blood flow $Q(t)$.

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