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THE SECOND-ORDER ESSCHER MARTINGALE DENSITIES FOR CONTINUOUS-TIME MARKET MODELS

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Abstract. In this paper, we introduce the second-order Esscher pricing notion for continuous-time models. Depending whether the stock price process S or its logarithm is the main driving noise/shock in the Esscher definition, we obtained two classes of second-order Esscher densities called linear class and exponential class. Using the semimartingale characteristics to parametrize S, we characterize the second-order Esscher densities (exponential and linear) using pointwise equations. The role of the second-order concept is highlighted in many manners and the relationship between the two classes is singled out for the one-dimensional case. Furthermore, when S is a compound Poisson model, we show how both classes are related to the Delbaen-Haenzendonck's risk-neutral measure. Afterwards, we restrict our model S to follow the jump-diffusion model, for simplicity only, and address the bounds of the stochastic Esscher pricing intervals. In particular, no matter what is the Esscher class, we prove that both bounds (upper and lower) are solutions to the same linear backward stochastic differential equation (BSDE hereafter for short) but with two different constraints. This shows that BSDEs with constraints appear also in a setting beyond the classical cases of constraints on gain-processes or constraints on portfolios. We prove that our resulting constrained BSDEs have solutions in our framework for a large class of claims' payoffs including bounded claims, in contrast to the literature, and we single out the monotonic sequence of BSDEs that "*naturally*" approximate them as well.

1. **Introduction.** The Esscher transform is a time-honoured concept in actuarial sciences, which was suggested by the Swedish actuary Esscher in [22], and it is also statistically known as *exponential tilting*. Then, the idea of changing the probability measure to obtain a consistent positive linear pricing rule, which appeared in the actuarial literature in the context of equilibrium reinsurance markets in [7], gave to the Esscher transform an important role in actuarial sciences. Afterwards, the Esscher transform was extended by Gerber and Shiu in [25] to the case where the logarithm of the stock is a stochastic process with stationary and independent increments. The impact of the latter extension was huge in both actuarial sciences and finance as well, see [26] and the references therein for more details and related

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discussions. The third key milestone in the evolution of the Esscher concept was elaborated by Bühlmann-Delbaen-Embrechts-Shiryaev in [8]. Here, the authors derive the *dynamical Esscher transform*, for discrete-time models, and coined it as the *conditional Esscher* transform. In fact, for any $t = 1, \ldots, T$, the Esscher parameter θ_t is obtained based on the information up to time $t-1$, and hence it is stochastic and the obtained density process Z takes the form of

$$Z_n = \exp\left(\sum_{k=1}^{n} \theta_k (X_k - X_{k-1}) + \sum_{i=1}^{n} K_i\right) \text{ with } K_i \text{ being observable at time } i-1,$$

where X is the logarithm of the stock price, and K is called the exponential cumulant. This interesting formulation of the dynamical Esscher led Kallsen and Shiryaev to extend *the dynamical Esscher* to the general continuous-time semimartingale setting in [31]. In this latter paper, which is the first to define the general continuous-time version of the Esscher, the authors singled out two classes, exponential and linear Esscher, depending whether X or S is used as the "underlying" process in the Esscher definition. For further developments about the continuous-time Esscher and its numerous applications, we refer the reader to [3, 21, 28, 29, 39] and references therein to cite a few. [31] gives various sufficient conditions on the cumulant process to guarantee the uniform integrability of the resulting density process, while it did not address the necessary and/or sufficient condition for the existence of those Esscher densities nor their relationship.

Certainly the Esscher risk-neutral measures, when they exist, give a nice and clear linear arbitrage-free pricing rule in incomplete markets besides other optimal neutral-risk measures. However, as noted in [5] for instance, the Esscher rule might not be as efficient as one can wish for the practical and the computational aspects. In this spirit, the second-order Esscher transform was introduced in [35], where they show that this two-parameters Esscher transform adds an important flexibility compared to the one parameter Esscher for pricing derivatives. For given historical dynamics which are estimated using stock return data and under no-arbitrage conditions, the two-parameters Esscher contrary to the one parameter Esscher still provides free parameters to match derivative prices.

What are our achievements? We introduce the Esscher martingale densities/deflators of order-two for any d-dimensional continuous-time semimartingale model S. We single out, as in [31], that there are exponential and linear classes of Esscher densities of order-two, and we characterize them using pointwise equations involving the characteristics of the underlying model S (Theorem 3.4 and Theorem 3.13-(a)). We elaborate the exact relationship between the first-order Esscher and our second-order Esscher for each class, where the change of priors plays a central role (Theorem 3.8 and Theorem 3.13-(b)). For the class of linear Esscher of order-two, we single out necessary and sufficient conditions for its existence in various manners. In fact, in an equivalent manner, we use some *local viability* of the market or equivalently a non-arbitrage condition, or the existence of a solution to an optimization problem dual to an optimal portfolio optimization (Theorem 3.6). For the one-dimensional case of S, we single out the exact relationship between the exponential Esscher and the linear Esscher (Theorem 3.15). For the case when S is one-dimensional and $\ln(S)$ follows a jump-diffusion model, we describe the upper and lower Esscher price processes, and find that they are the smallest solution to *constrained* linear backward stochastic differential equations (BSDE hereafter for short). The obtained constrained linear BSDEs are new as they are involved with Skorokhod conditions even though the value process does not appear in the constraints. These latter results and other related developments are summarized in Theorems 4.6 and 4.8.

The paper has four sections including this introductory section. The second section presents the mathematical model and gives some of its preliminary analysis that will be useful throughout the paper. The third section introduces the second-order Esscher deflators/densities for the most general d-dimensional continuous-time market models. The last section focuses on the class of one-dimensional jump-diffusion models, for simplicity, and addresses the bounds resulting from the second-order Esscher pricing for any claim that is "*nicely integrable*". The paper has three appendices, where the proofs for the technical lemmas are detailed therein, and some useful results are recalled.

2. The mathematical model and preliminaries.

Throughout the paper, we are given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. Here, the filtration is supposed to satisfy the usual condition, i.e. it is right-continuous and complete.

2.1. General notations.

Throughout the paper, for any probability Q on $(\Omega, \mathcal{F})$, we denote $\mathcal{A}(Q)$ (respectively $\mathcal{M}(Q)$) as the set of right-continuous with left limits (RCLL hereafter) and $\mathbb{F}$-adapted processes with Q-integrable variation (respectively uniformly Q-integrable martingales). When the probability measure is not mentioned, then by default we are using the probability measure P. The set $\mathcal{V}^+$ denotes the set of all RCLL, nondecreasing, and $\mathbb{F}$-adapted processes with finite values. For any process Y, we denote by ${}^{o,\mathbb{H}}(Y)$ (respectively ${}^{p,\mathbb{H}}(Y)$) the $\mathbb{F}$-optional (respectively $\mathbb{F}$-predictable) projection of Y. For an increasing process V, we denote $V^{o,\mathbb{F}}$ (respectively $V^{p,\mathbb{F}}$) as its dual $\mathbb{F}$-optional (respectively $\mathbb{F}$-predictable) projection. $\mathcal{O}$, $\mathcal{P}$, and Prog represent the $\mathbb{F}$-optional, the $\mathbb{F}$-predictable, and the $\mathbb{F}$-progressive σ-fields respectively on $\Omega \times [0, +\infty[$. For a semimartingale Y, we denote by $L(Y)$ the set of all Y-integrable processes in the Itô sense, and for $H \in L(Y)$, the resulting integral is a one-dimensional semimartingale denoted by $H \bullet Y := \int_0^{\cdot} H_u dY_u$. If $\mathcal{C}$ is a set of $\mathbb{F}$-adapted processes, then $\mathcal{C}_{loc}$ is the set of processes, Y, for which there exists a sequence of $\mathbb{F}$-stopping times, $(T_n)_{n\geq 1}$, that increases to infinity and Y^{T_n} belongs to $\mathcal{C}$ for each $n \geq 1$.

2.2. The model and its parametrization.

We consider a d-dimensional quasi-left-continuous semimartingale $X = \left(X^{(1)}, \ldots, X^{(d)}\right)^{tr}$ which is described mathematically using the *predictable characteristics* as follows:

$$X = X_0 + X^c + B + h_\epsilon(x) \star (\mu - \nu) + \big(x - h_\epsilon(x)\big) \star \mu. \tag{2.1}$$

Here, μ is the random measure associated with the jumps of X, and is defined by

$$\mu(dt, dx) := \sum_{s>0} \delta_{(s,\Delta X_s)}(dt, dx) I_{\{\Delta X_s \neq 0\}}, \tag{2.2}$$

X^c is the continuous local martingale part of X, B is a predictable process with finite variation, the random measure ν is the compensator of the random measure μ, $h_\epsilon(x) := x\mathbb{1}_{\{\|x\|\leq\epsilon\}}$ is the canonical truncation function with $\epsilon \in (0, \infty)$ fixed throughout the paper, and C is the matrix with entries $C^{ij} := \langle X^{c,i}, X^{c,j}\rangle$. Furthermore, we can find a version of the characteristics satisfying

$$\begin{aligned} &B = b \bullet A, \quad C = c \bullet A, \\ &\nu(\omega, dt, dx) = dA_t(\omega) F_t(\omega, dx), \quad F_t(\{0\}) = 0, \quad \int (|x|^2 \wedge 1) F_t(dx) \leq 1, \end{aligned} \tag{2.3}$$

where A is a right-continuous, nondecreasing, and predictable process, which we choose to be continuous because we assume that X is quasi-left-continuous, and b and c are predictable processes. The decomposition (2.1) is known as the canonical representation of X, while (b, c, F, A) is called the predictable characteristics of X. For more details about these and other related results and topics, we refer the reader to [27, 30].

Throughout the paper, the following notation will be useful

$$\begin{aligned} &\mathbf{e}(x) := (e^{x_1}, e^{x_2}, \dots, e^{x_d})^{tr} \in \mathbb{R}^d, \quad \text{For any} \quad x = (x_1, x_2, \dots, x_d)^{tr} \in \mathbb{R}^d, \\ &\mathbf{diag}(x) := \text{the diagonal matrix associated to the vector } x \in \mathbb{R}^d. \end{aligned}$$

For any d-dimensional semimartingale Y, we define the stochastic exponential $\mathcal{E}(Y)$ as the unique solution to the following SDE:

$$dZ = \mathbf{diag}(Z_-)dY, \quad Z_0 = \mathbb{I}_d := (1, 1, \dots, 1)^{tr} \in \mathbb{R}^d.$$

The d-dimensional process X, introduced in (2.1), represents the *driving shock* process for the d-risky assets S as follows

$$\mathbf{e}(X) = \mathbf{diag}(S_0)^{-1} S.$$

The following lemma expresses S as the stochastic exponential of a d-dimensional process, hereafter denoted $\widetilde{X}$, that is related to X.

Lemma 2.1. *Let X be given by (2.1), (2.2), and (2.3). Then, the following assertions hold.*
(a) *There exists a unique d-dimensional semimartingale $\widetilde{X} = (\widetilde{X}^{(1)}, \dots, \widetilde{X}^{(d)})^{tr}$ such that*

$$\mathbf{e}(X) = \mathcal{E}(\widetilde{X}) = \mathbf{diag}(S_0)^{-1} S = \left(\mathcal{E}(\widetilde{X}^{(1)}), \dots, \mathcal{E}(\widetilde{X}^{(d)})\right)^{tr}, \tag{2.4}$$

and is given by

$$\widetilde{X} = X + \frac{1}{2}\Delta \bullet A + (\mathbf{e}(x) - \mathbb{I}_d - x) \star \mu, \quad \text{where} \quad \Delta := (c_{11}, c_{22}, \dots, c_{dd})^{tr}. \tag{2.5}$$

Furthermore, $\widetilde{X}$ has the following canonical decomposition given by

$$\widetilde{X} = X_0 + X^c + h_\epsilon(x) \star (\widetilde{\mu} - \widetilde{\nu}) + \widetilde{b} \bullet A + (x - h_\epsilon(x)) \star \widetilde{\mu}, \tag{2.6}$$

where $(\widetilde{b}, \widetilde{\mu}, \widetilde{F}, \widetilde{\nu})$ is given (for any $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable W) by

$$\begin{aligned} &\widetilde{b} := b + \frac{\Delta}{2} + \int (h_\epsilon(\mathbf{e}(x) - \mathbb{I}_d) - h_\epsilon(x))\, F(dx), \; W \star \widetilde{\mu} := W(\cdot, \mathbf{e}(x) - \mathbb{I}_d) \star \mu, \\ &\int W(t, x) \widetilde{F}_t(dx) := \int W(t, \mathbf{e}(x) - \mathbb{I}_d) F_t(dx), \;\; \widetilde{\nu}(dt, dx) := \widetilde{F}_t(dx) dA_t. \end{aligned} \tag{2.7}$$

(b) *If X is locally bounded, then we have*

$$\begin{aligned} &X = X_0 + X^c + x \star (\mu - \nu) + b' \bullet A \text{ and } \widetilde{X} = X_0 + X^c + x \star (\widetilde{\mu} - \widetilde{\nu}) + \widetilde{b}' \bullet A, \\ &\text{where } b' := b + \int x I_{\{\|x\| > \epsilon\}} F(dx), \; \widetilde{b}' := b' + \frac{\Delta}{2} + \int (\mathbf{e}(x) - \mathbb{I}_d - x) F(dx). \end{aligned} \tag{2.8}$$

The proof of this lemma follows directly from Itô's formula, see [31, Lemmas 2.6 and 2.7] for the one-dimensional case, and hence it will be omitted hereafter. We end this preliminary section by defining some sets of local martingale densities/deflators, which appear naturally in our analysis.

Definition 2.2. Let Y be a d-dimensional semimartingale, Z be a process, and Q be a probability.
(a) Z is a local martingale density/deflator for (Y,Q) if $Z>0$ and both ZY and Z are Q-local martingales. The set of local martingale densities/deflators for (Y,Q) will be denoted by $\mathcal{Z}_{loc}(Y,Q)$, and $\mathcal{Z}_{loc}^{L\log L}(Y,Q)$ denotes the set of $Z\in\mathcal{Z}_{loc}(Y,Q)$ such that $Z\ln(Z)$ is a Q-local submartingale. When $Q=P$, we simply omit the probability and write $\mathcal{Z}_{loc}(Y)$ and $\mathcal{Z}_{loc}^{L\log L}(Y)$, respectively.
(b) Let L be a positive local martingale, and let $(T_n)_{n\geq 1}$ be the sequence of stopping times that increases to infinity such that $L^{T_n}\in\mathcal{M}$ and $Q_n:=L_{T\wedge T_n}\cdot P$. Then, we define

$$\mathcal{Z}_{loc}(Y,L):=\left\{Z:\ Z^{T_n}\in\mathcal{Z}_{loc}(Y^{T_n},Q_n),\quad n\geq 1\right\},$$
$$\mathcal{Z}_{loc}^{L\log L}(Y,L):=\left\{Z:\ Z^{T_n}\in\mathcal{Z}_{loc}^{L\log L}(Y^{T_n},Q_n),\quad n\geq 1\right\}.$$

3. **Esscher pricing densities of order-two for continuous-time.** The second-order Esscher transform was introduced in [35] as the measure

$$Q:=\exp(\theta^{tr}Y_1+Y_1^{tr}\psi Y_1)\left(E[\exp(\theta^{tr}Y_1+Y_1^{tr}\psi Y_1)]\right)^{-1}\cdot P,$$

where Y_1 is a d-dimensional random variable, $\theta\in\mathbb{R}^d$ and ψ is $d\times d$-matrix of real numbers. Thus, by extending this notion to the conditional Esscher setting *à la* Bühlmann-Delbaen-Embrechts-Shirayev, we obtain the following density Z:

$$Z_n:=\prod_{i=1}^n\frac{\exp(\theta_i^{tr}\Delta Y_i+\Delta Y_i^{tr}\psi_i\Delta Y_i)}{E[\exp(\theta^{tr}\Delta Y_i+\Delta Y_i^{tr}\psi_i\Delta Y_i)|\mathcal{F}_{i-1}]},\quad n\geq 1,$$

where θ_i and ψ_i are $\mathcal{F}_{i-1}$-measurable vectors for any $i\geq 1$. This extension seems to be tailor-made for the multi-period models. By considering the predictable process $K=(K_n)_n$ given by $K_n:=\sum_{i=1}^n\ln\left(E[\exp(\theta^{tr}\Delta Y_i+\Delta Y_i^{tr}\psi_i\Delta Y_i)|\mathcal{F}_{i-1}]\right)$, the above equality takes the form of

$$Z_n=\exp\left(\sum_{i=1}^n\theta_i^{tr}\Delta Y_i+\sum_{i=1}^n(\Delta Y_i)^{tr}\psi_i\Delta Y_i-K_n\right),\quad n=0,1,\ldots.$$

This allows us to introduce the notion of *second-order Esscher* in the continuous-time setting. To this end, for any d-semimartingale Y, we define

$$\Theta(Y):=\left\{(\theta,\psi)\ \ \mathbb{R}^d\times\mathbb{R}^{d\times d}\text{-valued predictable process}:\ \begin{array}{l}\theta\in L(Y)\quad\text{and}\\ \psi\text{ is locally bounded}\end{array}\right\}.$$

Definition 3.1. Let Z be a positive, RCLL, and adapted process.
(a) Z is called a second-order exponential Esscher density for $(S,\mathbb{F})$ if there exist $(\theta,\psi)\in\Theta(X)$ and an RCLL predictable process K with finite variation such that

$$Z=Z^{(\theta,\psi)}:=\exp\left(\theta\bullet X+\sum_{0<s\leq\cdot}\Delta X_s^{tr}\psi_s\Delta X_s-K\right)\in\mathcal{Z}_{loc}(S).\tag{3.1}$$

The pair (θ,ψ) is called the Esscher parameters of the density $Z^{(\theta,\psi)}$. If $Z^{(\theta,\psi)}$ is uniformly integrable, then $Q:=Z_\infty\cdot P$ is called a second-order exponential Esscher pricing measure for S. The set of these pricing densities (respectively measures) is denoted by $\mathcal{Z}^{EE}(S)$ (respectively $\mathcal{Q}^{EE}(S)$).

(b) Z is called a second-order linear Esscher density for S if there exist $(\theta,\psi)\in\Theta(S)$ and an RCLL predictable process $\widetilde{K}$ with finite variation such that

$$Z=Z^{(\theta,\psi)}:=\exp\left(\theta\bullet S+\sum_{0<s\le\cdot}\Delta S_s^{tr}\psi_s\Delta S_s-\widetilde{K}\right)\in\mathcal{Z}_{loc}(S). \tag{3.2}$$

The set of second-order linear Esscher pricing densities (respectively measures) for S is denoted by $\mathcal{Z}^{LE}(S)$ (respectively $\mathcal{Q}^{LE}(S)$).

In the definition of the set $\Theta(Y)$, we forced ψ to be locally bounded for the sake of simplifying the analysis and avoiding technicalities only. Indeed, this assumption on ψ can be replaced by the condition $\sum|(\Delta Y)^{tr}\psi\Delta Y|\in\mathcal{V}^+$. When Y is locally bounded, the local boundedness assumption of ψ tremendously simplifies the statements of the results and their proofs as well. Furthermore, when looking for the upper and lower prices, or when working in the practical framework, this assumption is not restrictive at all. In fact, in Section 4, we will allow ψ to span the set of all predictable and bounded processes.

Lemma 3.2. *Z is a second-order linear Esscher density for S if and only if there exist $(\overline{\theta},\overline{\psi})\in\Theta(\widetilde{X})$ and an RCLL predictable process $\widetilde{K}$ with finite variation such that*

$$Z=\exp\left(\overline{\theta}\bullet\widetilde{X}+\sum_{0<s\le\cdot}\Delta\widetilde{X}_s^{tr}\overline{\psi}_s\Delta\widetilde{X}_s-\widetilde{K}\right)\in\mathcal{Z}_{loc}(S).$$

Proof. As the two $d\times d$-dimensional processes $\mathbf{diag}(S_-)$ and $\mathbf{diag}(S_-)^{-1}$ are locally bounded, then it follows from relation (2.4) between $\widetilde{X}$ and S that $L(\widetilde{X})=L(S)$ and $\Theta(S)=\Theta(\widetilde{X})$. Furthermore, for any $(\theta,\psi)\in\Theta(S)$, we set $\overline{\theta}:=\mathbf{diag}(S_-)\theta$ and $\overline{\psi}:=\mathbf{diag}(S_-)\psi\mathbf{diag}(S_-)$, and derive

$$\theta\bullet S=\overline{\theta}\bullet\widetilde{X}\quad\text{and}\quad\Delta S^{tr}\psi\Delta S=\Delta\widetilde{X}^{tr}\overline{\psi}\Delta\widetilde{X}.$$

Thus, the proof of the lemma follows immediately from combining these facts and (3.2). ∎

Remark 3.3. 1) Clearly, our second-order Esscher pricing density generalizes the Esscher pricing density as the former reduces to the latter by setting $\psi\equiv 0$.
2) For $(\theta,\psi)\in\Theta(X)$ and an RCLL predictable process K' with finite variation, we consider the process

$$\overline{Z}^{(\theta,\psi)}:=\exp\left(\theta\bullet X+\sum_{i,j=1}^{d}[X^{(i)},\psi^{ij}\bullet X^{(j)}]-K'\right). \tag{3.3}$$

Then, we can also define a second-order Esscher pricing density by any process taking the form of $\overline{Z}^{(\theta,\psi)}$ such that $\overline{Z}^{(\theta,\psi)}\in\mathcal{Z}_{loc}(S)$. In fact, these two definitions, in (3.1) and (3.3) respectively, are equivalent. Indeed, it can be proven that we have $\overline{Z}^{(\theta,\psi)}=Z^{(\theta,\psi)}$ with $K=K'-\sum_{i,j=1}^d[(X^{(i)})^c,\psi^{ij}\bullet X^{(j)}]$, which is a predictable process with finite variation.
3) When X is a continuous process, then the second-order Esscher pricing density coincides with the Esscher pricing density. Indeed, in this case, it is enough to write $\exp(\theta\bullet X+\psi\bullet[X,X]-K'')=\exp(\theta\bullet X-K)$ with $K=K''-\psi\bullet[X,X]$ being a predictable process with finite variation.

3.1. **Linear Esscher martingale densities of order-two.** In this subsection, we address the second-order linear Esscher martingale densities, give their characterization, single out the necessary and sufficient conditions for their existence, and show how they can be connected to the classical (first-order) linear Esscher martingale densities as well.

Theorem 3.4. *Let Z be a positive, RCLL, and adapted process. Consider $\widetilde{X}$ given by (2.5) or (2.6), and $(\widetilde{b},\widetilde{F},\widetilde{\mu},\widetilde{\nu})$ defined in (2.7). Then, the following assertions are equivalent.*
(a) *Z is a second-order linear Esscher density for S.*
(b) *There exists a pair $(\theta,\psi)\in\Theta(S)$ satisfying*

$$\exp(\theta^{tr}x+x^{tr}\psi x)\Big(\|x\|I_{\{\|x\|>\epsilon\}}+I_{\{\epsilon<|\theta^{tr}x|\}}\Big)\star\widetilde{\mu}\in\mathcal{A}^+_{loc},\quad \textit{for some } \epsilon>0, \tag{3.4}$$

$$\widetilde{b}+c\theta+\int\Big(x\exp(\theta^{tr}x+x^{tr}\psi x)-h_\epsilon(x)\Big)\widetilde{F}(dx)=0,\quad P\otimes A-a.e., \tag{3.5}$$

and

$$Z=\mathcal{E}\Big(\theta\bullet X^c+\big(\exp(\theta^{tr}x+x^{tr}\psi x)-1\big)\star(\widetilde{\mu}-\widetilde{\nu})\Big). \tag{3.6}$$

(c) *There exists a pair $(\theta,\psi)\in\Theta(S)$ such that the condition (3.4) is fulfilled, and $Z=\mathcal{E}\Big(\theta\bullet X^c+(\exp(\theta^{tr}x+x^{tr}\psi x)-1)\star(\widetilde{\mu}-\widetilde{\nu})\Big)$ belongs to $\mathcal{Z}_{loc}(S)$.*

The proof will be given in Subsection 3.3.

Remark 3.5. Suppose S is locally bounded, and consider a predictable and locally bounded ψ. Then, the condition (3.4) is equivalent to

$$U_\epsilon:=e^{\theta^{tr}x}I_{\{|\theta^{tr}x|>\epsilon\}}\star\widetilde{\mu}\in\mathcal{A}^+_{loc},\quad \epsilon\in(0,\infty). \tag{3.7}$$

Indeed, due to the local boundedness of S and that of ψ, the process $|x^{tr}\psi x|\star\widetilde{\mu}$ is locally bounded.

Theorem 3.4 gives us a complete and explicit characterization for the second-order linear Esscher pricing densities in terms of pointwise equations. It is clear that when $\psi\equiv 0$, our second-order linear Esscher reduces to the classical first-order linear Esscher. This latter density coincides with the minimal entropy-Hellinger martingale density as noticed in [39] for the continuous-time setting and in [12, Theorem 5.1] for the discrete-time framework. Recall that the minimal entropy-Hellinger martingale density introduced in [13] is the local martingale deflator/density that minimizes the entropy-Hellinger process. Thus, in the spirit of [13], we assume S is locally bounded (for technical and exposition simplicity only) and derive various deep characterizations.

Theorem 3.6. *Suppose S is locally bounded, and consider $(\widetilde{b},\widetilde{F},\widetilde{\mu},,\widetilde{\nu})$ and $\widetilde{b}'$ defined by (2.6)-(2.7) and (2.8), respectively. Then, the following assertions are equivalent.*
(a) *$\mathcal{Z}^{LE}(S)\neq\emptyset$, i.e. S admits a second-order linear Esscher pricing density.*
(b) *$\mathcal{Z}^{L\log L}_{loc}(S)\neq\emptyset$, i.e. there exists $Z\in\mathcal{Z}_{loc}(S)$ such that $Z\ln(Z)$ is a local submartingale.*
(c) *For any locally bounded and predictable ψ, the pointwise minimization problem*

$$\min_\theta\Bigg(\theta^{tr}\widetilde{b}'+\frac{1}{2}\theta^{tr}c\theta+\int\Big(\exp(\theta^{tr}x+x^{tr}\psi x)-1-\theta^{tr}x\Big)\widetilde{F}(dx)\Bigg), \tag{3.8}$$

admits a solution $\widetilde{\theta} := \widetilde{\theta}(\psi)$ *satisfying*

$$\widetilde{\theta}^{tr} c \widetilde{\theta} \bullet A + f_{L\log L}(\exp(\widetilde{\theta}^{tr} x + x^{tr}\psi x) - 1) \star \widetilde{\mu} \in \mathcal{A}^+_{loc}, \tag{3.9}$$

where

$$f_{L\log L}(y) := (y+1)\ln(1+y) - y, \quad \forall\, y > -1.$$

(d) *There exists a locally bounded and predictable* ψ *such that the pointwise minimization problem (3.8) admits a solution* $\widetilde{\theta}$ *satisfying (3.9).*

The proof of the theorem is relegated to Subsection 3.3, while herein we discuss the well posedness of its ingredients and its meaning afterwards.

Remark 3.7. (a) If S is locally bounded, then the integral in (3.8) is well defined for any pair (θ, ψ). In fact, by stopping, we can assume without loss of generality that S is bounded, and in this case we have $\|x\| \leq C\ d\widetilde{F}$-a.e. for some $C \in (0,\infty)$. Hence, we obtain

$$\begin{aligned}&\int |e^{\theta^{tr}x + x^{tr}\psi x} - 1 - \theta^{tr}x| \widetilde{F}(dx)\\ &\leq \exp(C\|\theta\| + C^2\|\psi\|)(\|\psi\| + C^2\|\psi\|^2 + \|\theta\|^2)\int \|x\|^2 \widetilde{F}(dx) < \infty, \quad P\text{-a.s..}\end{aligned}$$

This proves the claim.

(b) Suppose S is locally bounded, and let (θ, ψ) be a pair of predictable processes such that ψ is locally bounded and $\Big((\exp(\theta^{tr}x + x^{tr}\psi x) - 1)^2 \star \widetilde{\mu}\Big)^{1/2} \in \mathcal{A}^+_{loc}$. Then, we deduce that

$$\begin{aligned}&|x^{tr}\psi x|\left(1 + \exp(\theta^{tr}x + x^{tr}\psi x)\right) \star \widetilde{\mu} \in \mathcal{A}^+_{loc},\\ &\text{or equivalently } |x^{tr}\psi x|\left(1 + \exp(\theta^{tr}x + x^{tr}\psi x)\right) \star \widetilde{\nu} \in \mathcal{A}^+_{loc}.\end{aligned}$$

Theorem 3.6 gives various necessary and sufficient conditions for the existence of a second-order linear Esscher martingale density (i.e. $\mathcal{Z}^{LE}(S) \neq \emptyset$) when S is locally bounded. On the one hand, the condition $\mathcal{Z}^{LE}(S) \neq \emptyset$ is completely characterized via the arbitrage condition $\mathcal{Z}^{L\log L}_{loc}(S) \neq \emptyset$, which conveys the existence of a deflator for S belonging locally to the space of LlogL-integrable martingales. This arbitrage condition is equivalent to the existence of a "local" solution to the exponential utility maximization from terminal wealth, or equivalently to the "local" viability of the market whose agents are endowed with exponential utility. For further discussions and details about these claims, we refer the reader to [10, Lemma 3.3]. The second characterization is via a minimization problem which has a striking similarity with the minimization problem intrinsic to the minimization of Hellinger processes in [13]. This conveys the existence of possible intimate link between second-order linear Esscher and the classical linear Esscher densities somehow. This is the aim of the following theorem.

Theorem 3.8. *Suppose* S *is locally bounded, and for any locally bounded and predictable* ψ *we denote*

$$Z^{(\psi)} := \mathcal{E}\Big(\big(\exp(x^{tr}\psi x) - 1\big)\star(\widetilde{\mu} - \widetilde{\nu})\Big),\ \widetilde{\nu}^{(\psi)}(dt,dx) := \exp(x^{tr}\psi x)\widetilde{\nu}(dt,dx). \tag{3.10}$$

(a) *If Z is an RCLL and adapted process, then $Z \in \mathcal{Z}^{LE}(S)$ if and only if there exists a pair (θ, ψ) of predictable processes such that*

$$\begin{aligned} &\psi \text{ is locally bounded, } (\theta,\psi) \text{ fulfills (3.9), and}\\ &\frac{Z}{Z^{(\psi)}} = \mathcal{E}\left(\theta \bullet X^c + (e^{\theta^{tr}x} - 1) \star (\widetilde{\mu} - \widetilde{\nu}^{(\psi)})\right) \text{ and belongs to } \mathcal{Z}_{loc}(S, Z^{(\psi)}). \end{aligned} \tag{3.11}$$

(b) *$\mathcal{Z}^{LE}(S) \neq \emptyset$ if and only if $\mathcal{Z}^{L\log L}_{loc}(S, Z^{(\psi)}) \neq \emptyset$ for any locally bounded and predictable ψ.*

The proof of the theorem is relegated to Subsection 3.3. The theorem simultaneously conveys two main ideas which are intimately related. On the one hand, it says that any second-order linear Esscher martingale density, when it exists, coincides with the "classical" (first-order) linear Esscher martingale density *under a specific local change of measure.* In fact, consider any predictable locally bounded ψ such that $Z^{(\psi)}$ given in (3.10) belongs to $\mathcal{M}$. Then, put $Q^{(\psi)} := Z^{(\psi)}_\infty \cdot P$, and hence Z is a linear-Esscher density of-order-two with parameters (θ, ψ) if and only if there exists a "classical" linear-Esscher density $\widetilde{Z}$ (given by $\widetilde{Z} := Z/Z^{(\psi)}$) for the model $(S, Q^{(\psi)})$ (i.e. first-order linear Esscher for S under the probability $Q^{(\psi)}$). On the other hand, by virtue of (3.11), one can interpret the second-order linear Esscher as a linear Esscher under a class of equivalent priors, and this connects the second-order Esscher to the first-order Esscher via uncertainty modeling.

Corollary 3.9. *Consider $Z > 0$ and let (b', σ, γ) be a triplet of predictable and bounded processes such that $1/|\gamma|$ is bounded as well. Suppose S is one-dimensional, given by*

$$S = S_0 \exp(X), \quad X = \int_0^\cdot b'_s ds + \sigma \bullet W + \gamma \bullet \widetilde{N}, \quad \text{and} \quad \widetilde{N}_t := N_t - \lambda t. \tag{3.12}$$

Here, S_0 is a positive constant, W is a Brownian motion and N is a Poisson process with rate λ which is independent of W.

Then, $\mathcal{Z}^{LE}(S) \neq \emptyset$, and $Z \in \mathcal{Z}^{LE}(S)$ if and only if there exists $(\theta, \psi) \in \Theta(\widetilde{X})$ such that

$$\begin{aligned} &\int_0^\cdot \exp(\theta_t \widetilde{\gamma}_t) dt \in \mathcal{A}^+_{loc}, \quad \widetilde{b'} + \theta\sigma^2 + \lambda\widetilde{\gamma}\left(e^{\theta\widetilde{\gamma} + \psi\widetilde{\gamma}^2} - 1\right) = 0, \ P \otimes dt\text{-a.e.},\\ &\text{and} \quad Z = \mathcal{E}\left(\theta\widetilde{\sigma} \bullet W + (e^{\theta\widetilde{\gamma} + \psi\widetilde{\gamma}^2} - 1) \bullet \widetilde{N}\right). \end{aligned}$$

Here, the triplet $(\widetilde{b'}, \widetilde{\sigma}, \widetilde{\gamma})$ is given by

$$\widetilde{b'} := b' + \frac{\sigma^2}{2} + \lambda\left(e^\gamma - 1 - \gamma\right), \quad \widetilde{\sigma} := \sigma \quad \text{and} \quad \widetilde{\gamma} := e^\gamma - 1. \tag{3.13}$$

Proof. Remark that in this case, as γ is bounded, we have

$$\begin{aligned} \widetilde{X} &= X + \frac{1}{2}\int_0^\cdot \sigma_s^2 ds + (e^\gamma - 1 - \gamma) \bullet N = \int_0^\cdot \widetilde{b}'_s ds + \sigma \bullet W + (e^\gamma - 1) \bullet \widetilde{N}\\ &= \int_0^\cdot \widetilde{b}'_s ds + \sigma \bullet W + \widetilde{\gamma} \bullet \widetilde{N}. \end{aligned}$$

Thus, by using the canonical decomposition of X and that of $\widetilde{X}$, we derive

$$\begin{aligned} &c = \sigma^2, \quad F_t(dx) := \lambda\delta_{\gamma_t}(dx), \quad A_t = t,\\ &\widetilde{b'} = b' + \frac{\sigma^2}{2} + \lambda(e^\gamma - 1 - \gamma), \ \widetilde{\mu}(dt, dx) = dN_t \delta_{\widetilde{\gamma}_t}(dx), \ \widetilde{F}_t(dx) = \lambda\delta_{\widetilde{\gamma}_t}(dx). \end{aligned}$$

Therefore, by combining these with Theorem 3.4, the corollary follows immediately. This ends the proof of the corollary. □

Example 3.10. Let $(b', \sigma, \gamma) \in \mathbb{R}^3$ be such that $\sigma > 0 \neq \gamma$, and suppose that $S_t = S_0 \exp\left(b't + \sigma W_t + \gamma \widetilde{N}_t\right)$, $t \in [0, T]$, where $\widetilde{N}$ is given by (3.12) and $S_0 > 0$.

Then, for any $\psi \in \mathbb{R}$, there exists a unique $\theta \in \mathbb{R}$ solution to

$$\widetilde{b}' + \theta\sigma^2 + \lambda\widetilde{\gamma}\left(e^{\theta\widetilde{\gamma}+\psi\widetilde{\gamma}^2} - 1\right) = 0,$$

where the triplet $(\widetilde{b}', \widetilde{\sigma}, \widetilde{\gamma})$ is constant in (ω, t) and is given by (3.13).

Furthermore,

$$Z^\psi := \mathcal{E}\left(\sigma\theta W + (e^{\theta\widetilde{\gamma}+\psi\widetilde{\gamma}^2} - 1)\widetilde{N}\right)^T \in \mathcal{Z}^{LE}(S) \cap \mathcal{M}.$$

For more discussions about this case, especially the applications of the second-order Esscher pricing method for this case, we refer the reader to [11]. We end this subsection by illustrating the linear Esscher pricing measure of order-two on the compound Poisson process and its variants. It is worth mentioning that these latter models are often used in modeling risks in actuarial sciences.

Corollary 3.11. *Suppose S is a one-dimensional compound Poisson process given by*

$$S := S_0 \exp(X) \quad \text{and} \quad X := \sum_{k=1}^{N} J_k. \tag{3.14}$$

Here, S_0 is a positive constant, and $(J_k)_{k\geq 1}$ is a sequence of independent and identically distributed random variables satisfying $E[\exp(J_1)] < \infty$, and are independent of the Poisson process N with rate λ. Let $\psi \in \mathbb{R}$ (i.e. real number) such that

$$E\left[|\widetilde{J}_1| \exp(\psi \widetilde{J}_1^2)\right] < \infty, \quad \text{where} \quad \widetilde{J}_1 := e^{J_1} - 1. \tag{3.15}$$

Then, Z is a second-order linear Esscher martingale density for S with parameter ψ iff there exists a unique $\theta \in \mathbb{R}$ satisfying

$$\widetilde{\kappa}(\theta, \psi) + 1 := E\left[\exp\left(\theta\widetilde{J}_1 + \psi\widetilde{J}_1^2\right)\right] < \infty, \quad E\left[\widetilde{J}_1 \exp\left(\theta\widetilde{J}_1 + \psi\widetilde{J}_1^2\right)\right] = 0, \tag{3.16}$$

and

$$Z_t = \exp\left(\sum_{k=1}^{N_t}\left(\theta\widetilde{J}_k + \psi\widetilde{J}_k^2\right) - \lambda\widetilde{\kappa}(\theta, \psi)t\right), \text{ where } \widetilde{J}_k := e^{J_k} - 1,\ k \in \mathbb{N}\setminus\{0\}. \tag{3.17}$$

The filtration $\mathbb{F}$ for this model is the completed filtration generated by the two processes N and $J_N := (J_{N_s})_{s\geq 0}$, where $J_0 = 0$, and by convention $\sum_\emptyset = 0$, and $\prod_\emptyset = 1$. Then, the proof of the corollary is a direct consequence of Theorem 3.4 and the easy fact that $J_{N_{t-}+1}$ is independent of $\mathcal{F}_{t-}$ for any $t > 0$, and hence it will be omitted. It is worth mentioning that the equation in (3.16) has a root if and only if $\min(P(J_1 > 0), P(J_1 < 0)) > 0$.

Example 3.12. Consider the model given by (3.14). Suppose that there exists a finite sequence $(a_i)_{i=1}^n$ in $\mathbb{R}$ such that

$$\{a_1, \ldots, a_n\} \cap (0, \infty) \neq \emptyset,\ \{a_1, \ldots, a_n\} \cap (-\infty, 0) \neq \emptyset, \text{ and}$$

$$P(J_1 = a_i) = p_i \in (0, 1), \quad \text{for} \quad i = 1, \ldots, n, \quad \text{with} \quad \sum_{i=1}^n p_i = 1.$$

Then, for any $\psi \in \mathbb{R}$, (3.15) is fulfilled, and for any $(\theta, \psi) \in \mathbb{R}^2$, the integrability condition in (3.16) is also satisfied automatically. Furthermore, for any $\psi \in \mathbb{R}$, there exists a unique $\theta(\psi) \in \mathbb{R}$ solution to

$$0 = \sum_{i=1}^{n} p_i (e^{a_i} - 1) \exp\left(\theta(e^{a_i} - 1) + \psi(e^{a_i} - 1)^2\right),$$

and Z given by (3.17) is a second-order linear Esscher martingale density for S with parameter ψ.

It is clear that Example 3.12 gives us a concrete model for S where the integrability conditions in Corollary 3.11, namely the integrability conditions (3.15) and (3.16), are fully satisfied without any further assumptions. In this spirit, it is worth mentioning that without any assumptions on the jump sizes $(J_k)_{k\geq 1}$, for any $\psi \in (-\infty, 0)$, those integrability conditions are also fully satisfied.

Herein, for the general model (3.14), we will discuss the connection between linear Esscher measures and the existing pricing measures. In fact, it is clear that the linear Esscher pricing measure differs tremendously from the Gerber-Shiu risk-neutral measure, which is more related to the exponential Esscher addressed in the next subsection. However, the second-order linear Esscher measure falls into the family of Delbaen-Haezendonck's pricing measures, see [17] for more details about this family of measures and its application in pricing. Indeed, for model (3.14), Delbaen and Haezendonck introduced the risk-neutral measure

$$Z_t^{DH} := \exp\left(\sum_{n=1}^{N_t} \beta(J_n) - \lambda t E\left[e^{\beta(J_1)} - 1\right]\right), \quad t \geq 0, \tag{3.18}$$

where β is a Borel-measurable function. Thus, the linear Esscher measure of order-two is a particular case of the Delbaen-Haezendonck measure by setting the function β to $\beta(x) = \theta(e^x - 1) + \psi(e^x - 1)^2$, $x \in \mathbb{R}$.

3.2. **Exponential Esscher martingale densities of order-two.** This subsection treats the exponential Esscher pricing densities.

Theorem 3.13. *Suppose S is locally bounded, and let $\widetilde{b}'$ be given by (2.8). Then, the following hold.*

(a) *Let Z be a positive RCLL and adapted process. Then, Z is a second-order exponential Esscher density for S if and only if there exists $(\theta, \psi) \in \Theta(X)$ satisfying*

$$\exp(\theta^{tr} x) I_{\{\theta^{tr} x > \alpha\}} \star \mu \in \mathcal{A}^+_{loc}, \quad \textit{for some } \alpha > 0, \tag{3.19}$$

$$0 = \widetilde{b}' + c\theta + \int \Big(\exp(\theta^{tr} x + x^{tr} \psi x) - 1\Big)(\mathbf{e}(x) - \mathbb{I}_d) F(dx), \quad P \otimes dA - a.e., \tag{3.20}$$

and

$$Z = \mathcal{E}\Big(\theta \bullet X^c + \big(\exp(\theta^{tr} x + x^{tr} \psi x) - 1\big) \star (\mu - \nu)\Big). \tag{3.21}$$

(b) *$Z \in \mathcal{Z}^{EE}(S)$ if and only if there exists a predictable and locally bounded ψ such that $Z/\overline{Z}^{(\psi)}$ is an exponential Esscher density (i.e. of order-one) for S under $\overline{Z}^{(\psi)}$ defined by*

$$\overline{Z}^{(\psi)} := \mathcal{E}\left((\exp(x^{tr} \psi x) - 1) \star (\mu - \nu)\right). \tag{3.22}$$

Furthermore, in this case, we have

$$Z = \overline{Z}^{(\psi)} \mathcal{E}\left(\theta \bullet X^c + (e^{\theta^{tr} x} - 1) \star (\mu - \nu^{(\psi)})\right) \textit{ with } \nu^{(\psi)}(dt, dx) := \exp(x^{tr} \psi x) \nu(dt, dx).$$

For the case of one-dimensional models, the expressions and notation in Theorem 3.13 simplify, as illustrated in the following example.

Example 3.14. Suppose that the assumptions of Theorem 3.13 hold and $d = 1$. Then, $Z \in \mathcal{Z}^{EE}(S)$ if and only if there exists a pair $(\theta, \psi) \in \Theta(X)$ satisfying

$$e^{\theta x} I_{\{\theta x > 1/2\}} \star \mu \in \mathcal{A}^+_{loc},\ 0 = \widetilde{b}' + c\theta + \int \Big(\exp(\theta x + \psi x^2) - 1 \Big)(e^x - 1) F(dx),\ P \otimes dA - a.e.,$$

and

$$Z = \mathcal{E}\Big(\theta \bullet X^c + \big(\exp(\theta x + \psi x^2) - 1\big) \star (\mu - \nu)\Big).$$

The proof of Theorem 3.13 is relegated to Subsubsection 3.3.2. Similarly as for the linear Esscher case, the connection between the first-order and second-order exponential Esscher is fully described via a "local" change of probability in Theorem 3.13-(b). However, for the other aspects of our analysis on the exponential Esscher deflators (such as the necessary and sufficient condition(s) for its existence or its relationship to an optimal portfolio problem), the higher dimension (i.e. $d \geq 2$) brings serious difficulties in treating this exponential Esscher case. In fact when $d \geq 2$, in contrast to the linear Esscher case, there are difficulties in connecting the condition $\mathcal{Z}^{EE}_{loc}(S) \neq \emptyset$ to some known non-arbitrage condition. Again, when $d \geq 2$, the equations (3.5) and (3.20) sound non-comparable at all. The one-dimensional case, however, allows us to overcome all these obstacles.

Theorem 3.15. *Suppose X is a one-dimensional (i.e. $d = 1$) locally bounded semimartingale, and let $(\widehat{S}, \widehat{Z}, \widehat{\nu})$ be given by*

$$\begin{aligned} &\widehat{S} := \mathcal{E}(\widehat{X}), \quad \widehat{X} := \frac{1}{2} c \bullet A + X, \\ &\widehat{Z} := \mathcal{E}((f - 1) \star (\mu - \nu)), \quad f(x) := \frac{e^x - 1}{x} I_{\{x \neq 0\}} + I_{\{x = 0\}}. \end{aligned} \tag{3.23}$$

Then, the following assertions hold.
(a) *Let Z be an RCLL and positive adapted process. Then, $Z \in \mathcal{Z}^{EE}(S)$ with parameters $(\theta, \psi) \in \Theta(X)$ if and only if the process*

$$Z' := \frac{Z}{\widehat{Z}} \in \mathcal{Z}^{LE}(\widehat{S}, \widehat{Z}) \quad \textit{with the same pair of parameters } (\theta, \psi).$$

Furthermore, we have

$$Z' = \mathcal{E}\Big(\theta \bullet X^c + (e^{x\theta + x^2\psi} - 1) \star (\mu - \widehat{\nu})\Big) \quad \textit{where} \quad \widehat{\nu}(dt, dx) := f(x)\nu(dt, dx). \tag{3.24}$$

(b) $\mathcal{Z}^{EE}(S) \neq \emptyset$ *if and only if* $\mathcal{Z}^{L\log L}_{loc}(S) \neq \emptyset$ *if and only if* $\mathcal{Z}^{LE}(\widehat{S}, \widehat{Z}) \neq \emptyset$.

The proof of the theorem is relegated to Subsubsection 3.3.2. It is worth mentioning that Theorem 3.15 has two important messages, and we strongly believe (we conjecture) that these messages hold true in general, while up to now the dimensionality of S has posed serious challenges towards this goal. The first message conveys that the exponential Esscher densities for S are linear Esscher for "equivalent" transformed models $(\mathcal{T}(S), Q)$ with equivalent prior Q. The second message lies in claiming that the "arbitrage" condition of $Z^{L\log L}(S) \neq \emptyset$ is equivalent to the existence of an exponential Esscher density. We end this subsection by illustrating the above theorem on the particular cases of jump-diffusion models and compound Poisson models in two corollaries.

Corollary 3.16. *Let $Z > 0$ and be an RCLL process, and suppose S is one-dimensional and given by (3.12).*

Then, Z is a second-order exponential Esscher density for S if and only if there exists $(\theta, \psi) \in \Theta(X)$ such that

$$\int_0^\cdot \exp(\theta_t \gamma_t) dt \in \mathcal{A}^+_{loc}, \quad \widetilde{b'} + \theta\sigma^2 + \lambda\widetilde{\gamma}\Big(\exp(\theta\gamma + \psi\gamma^2) - 1\Big) = 0,$$

and

$$Z = \mathcal{E}\Big(\theta\sigma \bullet W + (\exp(\theta\gamma + \psi\gamma^2) - 1) \bullet \widetilde{N}\Big), \quad \textit{where} \quad \widetilde{N}_t = N_t - \lambda t.$$

The proof of the corollary follows directly from Theorem 3.13, and hence it will be omitted.

Example 3.17. Consider the model in (3.12) and assume that $(b', \sigma, \gamma) \in \mathbb{R}^3$, i.e., they are constant in (ω, t) such that $\sigma > 0 \neq \gamma$. Then, for any $\psi \in \mathbb{R}$, there exists a unique $\theta \in \mathbb{R}$ solution to

$$\widetilde{b'} + \theta\sigma^2 + \lambda\widetilde{\gamma}\Big(\exp(\theta\gamma + \psi\gamma^2) - 1\Big) = 0,$$

and

$$\begin{aligned} Z &= \mathcal{E}\Big(\theta\sigma W + (\exp(\theta\gamma + \psi\gamma^2) - 1)\widetilde{N}\Big) \\ &= \exp\left(\theta\sigma W - (\frac{\theta^2\sigma^2}{2} + \frac{\widetilde{b'} + \sigma^2\theta}{\widetilde{\gamma}})t + (\theta\gamma + \psi\gamma^2)N\right) \in \mathcal{Z}^{EE}(S). \end{aligned}$$

For more details and illustrations of this model, we refer the reader to [11].

Corollary 3.18. *Suppose S is one-dimensional and follows the model defined in (3.14) such that $E[\exp(J_1)] < \infty$, and let $\psi \in \mathbb{R}$ such that*

$$E\left[|\exp(J_1) - 1| \exp(\psi J_1^2)\right] < \infty. \tag{3.25}$$

Then, Z is a second-order exponential Esscher density for S associated with ψ if and only if there exists a real number θ satisfying

$$\kappa(\theta, \psi) + 1 := E\left[\exp(\theta J_1 + \psi J_1^2)\right] < \infty, \ E\Big[(e^{J_1} - 1)\exp\left(\theta J_1 + \psi J_1^2\right)\Big] = 0, \tag{3.26}$$

and

$$Z_t = \exp\left(\sum_{n=1}^{N_t}(\theta J_n + \psi J_n^2) - \lambda\kappa(\theta, \psi)t\right).$$

The proof of the corollary is a direct consequence of Theorem 3.13. A model similar to that in Example 3.12 can be easily treated herein. For this latter model, it is clear that the integrability conditions in (3.25) and (3.26) are always fulfilled, and equation (3.26) always has a solution for any given value for $\psi \in \mathbb{R}$.

On the one hand, it is clear that for model (3.14), our exponential Esscher pricing measure extends Gerber-Shiu's measure. This measure was introduced in [24], see also [25, 26] for related works, and can be deducted from ours by putting $\psi = 0$. On the other hand, the exponential Esscher measure is a particular case of Delbaen-Haenzendonck's pricing measure introduced in [17], see (3.18). In fact, by choosing the quadratic form $\beta(x) = \theta x + \psi x^2$ instead, Delbaen-Haenzendonck's measure coincides with the exponential Esscher.

We end this section by the proof of its main theorems.

3.3. **Proof of Theorems 3.4, 3.6, 3.8, 3.13, and 3.15.** The proof of these theorems relies on the following lemmas which are interesting in their own right, and for which the proof can be found in Appendix B.

Lemma 3.19. *Let Y be a d-dimensional semimartingale, (b^Y, c^Y, F^Y) be its predictable characteristics, μ_Y be the random measure of its jumps with the compensator ν_Y, and $\theta \in L(Y)$. Consider the following RCLL processes with finite variation:*

$$\begin{aligned} &U^{\theta}(Y) := \sum \Delta Y I_{\{|\theta^{tr}\Delta Y|>\epsilon \ or \ \|\Delta Y\|>\epsilon\}}, \quad Z^{\theta}(Y) := Y - U^{\theta}(Y),\\ &U^{\theta,1}(Y) := \sum \Delta Y I_{\{\|\Delta Y\|>\epsilon\geq|\theta^{tr}\Delta Y|\}}, \quad U^{\theta,2}(Y) := \sum \Delta Y I_{\{|\theta^{tr}\Delta Y|>\epsilon\}}. \end{aligned} \tag{3.27}$$

Then, the following assertions hold.
(a) *We always have*

$$\begin{aligned} &\int \|h_\epsilon(x)\| I_{\{\epsilon<|\theta^{tr}x|\}} F^Y(dx) < \infty, \ \ P\otimes A^Y\text{-a.e.},\\ &b^{\theta}(Y) := b^Y - \int h_\epsilon(x) I_{\{\epsilon<|\theta^{tr}x|\}} F^Y(dx) \in L(A^Y), \end{aligned} \tag{3.28}$$

and $Z^{\theta}(Y)$ admits the following canonical decomposition:

$$Z^{\theta}(Y) = Y^c + \underbrace{xI_{\{\max(\|x\|,|\theta^{tr}x|)\leq\epsilon\}} \star (\mu^Y - \nu^Y)}_{=:M^{\theta}(Y)} + b^{\theta}(Y) \cdot A^Y. \tag{3.29}$$

(b) *$\theta \in L(U^{\theta,1}(Y)) \cap L(U^{\theta,2}(Y)) \cap L(Y^c) \cap L(M^{\theta}(Y))$, and $\theta \cdot Z^{\theta}(Y)$ is a special semimartingale having the following canonical decomposition:*

$$\begin{aligned} \theta \cdot Z^{\theta}(Y) = \ &\theta \cdot Y^c + h_\epsilon(\theta^{tr}x) I_{\{\|x\|\leq\epsilon\}} \star (\mu^Y - \nu^Y)\\ &+ \left(\theta^{tr} b^Y - \int \theta^{tr} x I_{\{\|x\|\leq\epsilon<|\theta^{tr}x|\}} F^Y(dx)\right) \cdot A^Y. \end{aligned} \tag{3.30}$$

(c) *$\theta \cdot Y$ has the following canonical decomposition:*

$$\begin{aligned} \theta \cdot Y = \ &\theta \cdot Y^c + h_\epsilon(\theta^{tr}x) \star (\mu^Y - \nu^Y)\\ &+ \left(\theta^{tr} b^{\theta}(Y) + \int h_\epsilon(\theta^{tr}x) I_{\{\epsilon<\|x\|\}} F^Y(dx)\right) \cdot A^Y + \theta \cdot U^{\theta,2}(Y). \end{aligned} \tag{3.31}$$

The following lemma gives equivalent integrability conditions to (3.4) and (3.19), respectively. These equivalences simplify the proofs of the main theorems.

Lemma 3.20. *Let $(\theta,\psi) \in \Theta(S)$. Then, the following assertions hold.*
(a) *Condition (3.4) is equivalent to*

$$\|x\exp(\theta^{tr}x + x^{tr}\psi x) - h_\epsilon(x)\| \star \widetilde{\mu} + \left|\exp(\theta^{tr}x + x^{tr}\psi x) - 1 - h_\epsilon(\theta^{tr}x)\right| \star \widetilde{\mu} \in \mathcal{A}^+_{loc}. \tag{3.32}$$

(b) *Condition (3.4) implies $\sqrt{(\exp(\theta^{tr}x + x^{tr}\psi x) - 1)^2 \star \widetilde{\mu}} \in \mathcal{A}^+_{loc}$.*
(c) *If (3.4) and (3.5) hold and $|e^{x^{tr}\psi x} - 1| \star \widetilde{\mu} \in \mathcal{A}^+_{loc}$, then*

$$|\theta^{tr}xe^{\theta^{tr}x+x^{tr}\psi x} - \theta^{tr}h_\epsilon(x)| \star \widetilde{\mu} + |\theta^{tr}xe^{\theta^{tr}x+x^{tr}\psi x} - e^{\theta^{tr}x+x^{tr}\psi x} + 1| \star \widetilde{\mu} \in \mathcal{A}^+_{loc}. \tag{3.33}$$

(d) *Suppose S is locally bounded. Then, condition (3.19) is equivalent to*

$$\begin{aligned}&\| \left(\exp(\theta^{tr}x + x^{tr}\psi x) - 1\right) (\mathbf{e}(x) - \mathbb{I}_d)\| \star \mu \in \mathcal{A}^+_{loc}\\ &\text{and} \left|\exp(\theta^{tr}x + x^{tr}\psi x) - 1 - h_\epsilon(\theta^{tr}x)\right| \star \mu \in \mathcal{A}^+_{loc}.\end{aligned} \tag{3.34}$$

The rest of this section is divided into two subsections. The first subsection proves Theorems 3.4, 3.6, and 3.8, while the second subsection deals with the proofs of Theorems 3.13 and 3.15.

3.3.1. *Proof of Theorems 3.4, 3.6, and 3.8.* The proofs of these three theorems require the following third lemma, besides Lemmas 3.19 and 3.20.

Lemma 3.21. *Suppose S is locally bounded, and let $\widehat{\theta}$ be a predictable process and ψ a predictable and locally bounded process. Then, the following assertions hold.*
(a) *Let Q be a probability and $(\widetilde{b}^Q, c^Q, \widetilde{F}^Q, A^Q)$ be the predictable characteristics of $\widetilde{X}$ under Q. Then, $Q\otimes dA^Q$-a.e. $\widehat{\theta}$ is a pointwise solution to*

$$\min_\theta \left(\theta^{tr}\widetilde{b}^Q + \frac{1}{2}\theta^{tr}c^Q\theta + \int \left(\exp(\theta^{tr}x) - 1 - \theta^{tr}x\right) \widetilde{F}^Q(dx)\right) \tag{3.35}$$

if and only if $\widehat{\theta}$ satisfies

$$\widetilde{b}^Q + c^Q\widehat{\theta} + \int \left(x\exp(\widehat{\theta}^{tr}x) - x\right) \widetilde{F}^Q(dx) \equiv 0, \quad Q\otimes dA^Q - a.e.. \tag{3.36}$$

(b) *Suppose that $\widehat{\theta}^{tr}c\widehat{\theta}\bullet A + \sqrt{(\exp(\widehat{\theta}^{tr}x + x^{tr}\psi x) - 1)^2 \star \widetilde{\mu}} \in \mathcal{A}^+_{loc}$. Then,*

$$Z := \mathcal{E}\left(\widehat{\theta}\bullet X^c + (\exp(\widehat{\theta}^{tr}x + x^{tr}\psi x) - 1)\star(\widetilde{\mu} - \widetilde{\nu})\right) \in \mathcal{Z}_{loc}(S)$$

if and only if $\widehat{\theta}$ solves (3.5).

The proof of this lemma is relegated to Appendix B, while here we outline the proof of Theorem 3.4.

Proof of Theorem 3.4. Remark that (b) $\Longleftrightarrow$ (c) becomes obvious due to Lemma 3.20 and Lemma 3.21-(b). Thus, the remaining part of this proof focuses on proving (a) $\Longleftrightarrow$ (b) in two parts, where we prove (a) $\Longrightarrow$ (b) and (b) $\Longrightarrow$ (a) respectively.
Part 1. Suppose assertion (a) holds. Then, there exists a triplet of predictable processes (θ, ψ, K) satisfying the following:

$$(\theta,\psi)\in\Theta(S),\ K\in\mathcal{A}_{loc},\ \text{and}\ Z = \exp\left(\theta\bullet\widetilde{X} + \sum(\Delta\widetilde{X})^{tr}\psi\Delta\widetilde{X} - K\right) \in \mathcal{Z}_{loc}(S).$$

Then, we set $\widetilde{Y} := \theta\bullet\widetilde{X} + \sum(\Delta\widetilde{X})^{tr}\psi\Delta\widetilde{X} - \widetilde{K}$, and we use Itô's formula to derive the following:

$$\begin{aligned}Z_-^{-1}\bullet Z &= \widetilde{Y} + \frac{1}{2}\langle\widetilde{Y}^c\rangle + \sum\left(e^{\Delta\widetilde{Y}} - 1 - \Delta\widetilde{Y}\right)\\ &= \theta\bullet\widetilde{X} + \sum(\Delta\widetilde{X})^{tr}\psi\Delta\widetilde{X} - \widetilde{K} + \frac{\theta^{tr}c\theta}{2}\bullet A\\ &\quad + \left(\exp(\theta^{tr}x + x^{tr}\psi x) - 1 - \theta^{tr}x - x^{tr}\psi x\right)\star\widetilde{\mu}\\ &= \theta\bullet\widetilde{X} + \left(\exp(\theta^{tr}x + x^{tr}\psi x) - 1 - \theta^{tr}x\right)\star\widetilde{\mu} + \frac{\theta^{tr}c\theta}{2}\bullet A - \widetilde{K}.\end{aligned}$$

We apply (3.31) of Lemma 3.19 to $Y=\widetilde{X}$ and set $\widetilde{b}^{\theta}:=\widetilde{b}-\int h_{\epsilon}(x)I_{\{|\theta^{tr}x|>\epsilon\}}\widetilde{F}(dx)$, then we get

$$\begin{aligned}
&Z_{-}^{-1}\bullet Z\\
&=\theta\bullet X^{c}+h_{\epsilon}(\theta^{tr}x)\star(\widetilde{\mu}-\widetilde{\nu})+\left(\theta^{tr}\widetilde{b}^{\theta}+\int h_{\epsilon}(\theta^{tr}x)I_{\{\epsilon<\|x\|\}}\widetilde{F}(dx)\right)\bullet A\\
&\quad+\theta^{tr}xI_{\{|\theta^{tr}x|>\epsilon\}}\star\widetilde{\mu}+\left(\exp(\theta^{tr}x+x^{tr}\psi x)-1-\theta^{tr}x\right)\star\widetilde{\mu}+\frac{\theta^{tr}c\theta}{2}\bullet A-\widetilde{K}\\
&=\theta\bullet X^{c}+h_{\epsilon}(\theta^{tr}x)\star(\widetilde{\mu}-\widetilde{\nu})+\left(\exp(\theta^{tr}x+x^{tr}\psi x)-1-h_{\epsilon}(\theta^{tr}x)\right)\star\widetilde{\mu}\\
&\quad+\left(\frac{\theta^{tr}c\theta}{2}+\theta^{tr}\widetilde{b}^{\theta}+\int h_{\epsilon}(\theta^{tr}x)I_{\{\epsilon<\|x\|\}}\widetilde{F}(dx)\right)\bullet A-\widetilde{K}.
\end{aligned}$$

Then, on the one hand, Z is a local martingale if and only if

$$\left(\exp(\theta^{tr}x+x^{tr}\psi x)-1-h_{\epsilon}(\theta^{tr}x)\right)\star\widetilde{\mu}\in\mathcal{A}_{loc},\tag{3.37}$$

and

$$\begin{aligned}
\widetilde{K}&=\left(\frac{\theta^{tr}c\theta}{2}+\theta^{tr}\widetilde{b}^{\theta}+\int h_{\epsilon}(\theta^{tr}x)I_{\{\epsilon<\|x\|\}}\widetilde{F}(dx)\right)\bullet A\\
&\quad+\left(\exp(\theta^{tr}x+x^{tr}\psi x)-1-h_{\epsilon}(\theta^{tr}x)\right)\star\widetilde{\nu}\\
&=\left(\frac{\theta^{tr}c\theta}{2}+\theta^{tr}\widetilde{b}\right)\bullet A+\left(\exp(\theta^{tr}x+x^{tr}\psi x)-1-\theta^{tr}h_{\epsilon}(x)\right)\star\widetilde{\nu},\\
Z&=\mathcal{E}\left(\theta\bullet X^{c}+\left(\exp(\theta^{tr}x+x^{tr}\psi x)-1\right)\star(\widetilde{\mu}-\widetilde{\nu})\right)=:\mathcal{E}(\widetilde{L}).
\end{aligned}$$

This proves (3.6). On the other hand, thanks to (2.6), we derive

$$\begin{aligned}
&\widetilde{X}+[\widetilde{X},\widetilde{L}]\\
&=X^{c}+h_{\epsilon}(x)\star(\widetilde{\mu}-\widetilde{\nu})+\widetilde{b}\bullet A+(x-h_{\epsilon}(x))\star\widetilde{\mu}+c\theta\bullet A\\
&\quad+\left(\exp(\theta^{tr}x+x^{tr}\psi x)-1\right)x\star\widetilde{\mu}\\
&=X^{c}+h_{\epsilon}(x)\star(\widetilde{\mu}-\widetilde{\nu})+(\widetilde{b}+c\theta)\bullet A+\left(\exp(\theta^{tr}x+x^{tr}\psi x)x-h_{\epsilon}(x)\right)\star\widetilde{\mu}.
\end{aligned}$$

Thus, ZS is a local martingale if and only if $\widetilde{X}+[\widetilde{X},\widetilde{L}]$ is a local martingale, or equivalently

$$\left(\exp(\theta^{tr}x+x^{tr}\psi x)x-h_{\epsilon}(x)\right)\star\widetilde{\mu}\in\mathcal{A}_{loc},\tag{3.38}$$

and (3.5) holds. Furthermore, by combining (3.38), (3.37), and Lemma 3.20-(a), we obtain condition (3.4). This proves that $Z\in\mathcal{Z}^{LE}(S)$ implies the existence of $(\theta,\psi)\in\Theta(S)$ satisfying (3.4)-(3.5)-(3.6), and the proof of (a) $\Longrightarrow$ (b) is complete.
Part 2. Suppose that assertion (b) holds. Thus, there exists a pair $(\theta,\psi)\in\Theta(S)$ satisfying (3.4)-(3.5)-(3.6). First of all, thanks to Lemma 3.20-(b), we remark that the process $\widetilde{M}:=\theta\bullet X^{c}+\left(e^{\theta^{tr}x+x^{tr}\psi x}-1\right)\star(\widetilde{\mu}-\widetilde{\nu})$ is a well defined local martingale.

Furthermore, thanks to Itô's formula, we have $Z = \mathcal{E}(\widetilde{M}) = \exp(L)$, where

$$\begin{aligned}
L &= \widetilde{M} - \frac{1}{2}\langle \widetilde{M}^c\rangle + \sum\Big(\ln(1+\Delta\widetilde{M}) - \Delta\widetilde{M}\Big)\\
&= \theta\bullet X^c + \Big(e^{\theta^{tr}x + x^{tr}\psi x} - 1\Big)\star(\widetilde{\mu}-\widetilde{\nu})\\
&\quad - \frac{1}{2}\theta^{tr}c\theta\bullet A + \Big(\theta^{tr}x + x^{tr}\psi x - e^{\theta^{tr}x+x^{tr}\psi x} + 1\Big)\star\widetilde{\mu}\\
&= \theta\bullet X^c + \Big(e^{\theta^{tr}x + x^{tr}\psi x} - 1\Big)\star(\widetilde{\mu}-\widetilde{\nu})\\
&\quad + \Big(\theta^{tr}h_\epsilon(x) - e^{\theta^{tr}x+x^{tr}\psi x} + 1\Big)\star\widetilde{\mu} - \frac{1}{2}\theta^{tr}c\theta\bullet A\\
&\quad + \big(\theta^{tr}xI_{\{\|x\|>\epsilon\}} + x^{tr}\psi x\big)\star\widetilde{\mu}.
\end{aligned}\tag{3.39}$$

Thanks to Lemma 3.20-(a,b), the condition (3.4) implies that for any $n\geq 1$ $I_{\{\|\theta\|\leq n\}}\Big(\theta^{tr}h_\epsilon(x) - e^{\theta^{tr}x+x^{tr}\psi x}+1\Big)\star\widetilde{\mu}\in\mathcal{A}_{loc}$. Thus, by inserting the compensator of this process in (3.39), arranging terms, and inserting (3.5) in the resulting equation afterwards, we obtain

$$\begin{aligned}
&I_{\{\|\theta\|\leq n\}}\bullet L\\
&= I_{\{\|\theta\|\leq n\}}\theta\bullet X^c\\
&\quad + I_{\{\|\theta\|\leq n\}}\theta^{tr}h_\epsilon(x)\star(\widetilde{\mu}-\widetilde{\nu}) + I_{\{\|\theta\|\leq n\}}\Big(\theta^{tr}h_\epsilon(x) - e^{\theta^{tr}x+x^{tr}\psi x}+1\Big)\star\widetilde{\nu}\\
&\quad + I_{\{\|\theta\|\leq n\}}\big(\theta^{tr}xI_{\{\|x\|>\epsilon\}} + x^{tr}\psi x\big)\star\widetilde{\mu} - \frac{1}{2}\theta^{tr}c\theta I_{\{\|\theta\|\leq n\}}\bullet A\\
&= I_{\{\|\theta\|\leq n\}}\theta\bullet X^c + I_{\{\|\theta\|\leq n\}}\theta^{tr}h_\epsilon(x)\star(\widetilde{\mu}-\widetilde{\nu})\\
&\quad + I_{\{\|\theta\|\leq n\}}\theta^{tr}\widetilde{b}\bullet A + I_{\{\|\theta\|\leq n\}}\theta^{tr}xI_{\{\|x\|>\epsilon\}}\star\widetilde{\mu}\\
&\quad + \frac{1}{2}\theta^{tr}c\theta I_{\{\|\theta\|\leq n\}}\bullet A\\
&\quad + I_{\{\|\theta\|\leq n\}}\big(\theta^{tr}x\exp(\theta^{tr}x+x^{tr}\psi x) - \exp(\theta^{tr}x+x^{tr}\psi x)+1\big)\star\widetilde{\nu}\\
&\quad + I_{\{\|\theta\|\leq n\}}\bullet\sum(\Delta\widetilde{X})^{tr}\psi\Delta\widetilde{X}\\
&= I_{\{\|\theta\|\leq n\}}\theta\bullet\widetilde{X} + I_{\{\|\theta\|\leq n\}}\bullet\sum(\Delta\widetilde{X})^{tr}\psi\Delta\widetilde{X} - I_{\{\|\theta\|\leq n\}}\bullet\widetilde{K},
\end{aligned}$$

where $-\widetilde{K} := \frac{1}{2}\theta^{tr}c\theta\bullet A + (\theta^{tr}x\exp(\theta^{tr}x+x^{tr}\psi x) - \exp(\theta^{tr}x+x^{tr}\psi x)+1)\star\widetilde{\nu}$ is predictable and belongs to $\mathcal{A}_{loc}$. Hence, by taking the limits on both sides of the latter equality above and using the facts that $\theta\in L(X)$ and the three processes L, $\sum(\Delta\widetilde{X})^{tr}\psi\Delta\widetilde{X}$, and $\widetilde{K}$ are semimartingales, assertion (a) follows immediately. This ends the proof of the theorem. $\square$

Proof of Theorem 3.6. Remark that (c) $\Longrightarrow$ (d) is obvious. Hence, the rest of this proof is divided into three parts, where we prove (a) $\Longrightarrow$ (b), (b) $\Longrightarrow$ (c) and (d) $\Longrightarrow$ (a) respectively.

Part 1. Here we prove the implication (a) $\Longrightarrow$ (b). Thus, we suppose that assertion (a) holds, and by virtue of Theorem 3.4, we obtain the existence of $(\theta,\psi)\in\Theta(S)$ such that (3.4) holds and

$$Z = \mathcal{E}\Big(\theta\bullet X^c + \big(\exp(\theta^{tr}x + x^{tr}\psi x) - 1\big)\star(\widetilde{\mu}-\widetilde{\nu})\Big)\in\mathcal{Z}_{loc}(S).$$

Thus, assertion (b) follows as soon as we prove that $Z\ln(Z)$ is a special semimartingale. Thanks to [13, Lemma 3.2 and Proposition 3.5], $Z\ln(Z)$ is a special semimartingale if and only if

$$H^E(Z,P) := \frac{1}{2}\theta^{tr}c\theta \bullet A + \left((\theta^{tr}x + x^{tr}\psi x)e^{\theta^{tr}x+x^{tr}\psi x} - e^{\theta^{tr}x+x^{tr}\psi x} + 1\right) \star \widetilde{\mu} \in \mathcal{A}^+_{loc}.$$

On the one hand, we already have $\theta^{tr}c\theta \bullet A \in \mathcal{A}^+_{loc}$ due to $\theta \in L(\widetilde{X})$, and thanks to Lemma 3.20-(c), we deduce that

$$|\theta^{tr}x \exp\left(\theta^{tr}x + x^{tr}\psi x\right) - \exp\left(\theta^{tr}x + x^{tr}\psi x\right) + 1| \star \widetilde{\mu} \in \mathcal{A}^+_{loc}.$$

This latter holds due to (3.4), (3.5), and the local boundedness of $\widetilde{X}$, which implies that $|1 - e^{x^{tr}\psi x}| \star \widetilde{\mu} \in \mathcal{A}^+_{loc}$. On the other hand, by virtue of Remark 3.5 and the local boundedness of $\widetilde{X}$ again, we derive

$$\begin{aligned}
&|x^{tr}\psi x|e^{\theta^{tr}x+x^{tr}\psi x} \star \widetilde{\mu} \\
&= |x^{tr}\psi x|e^{\theta^{tr}x+x^{tr}\psi x} I_{\{|\theta^{tr}x|\leq\epsilon\}} \star \widetilde{\mu} + |x^{tr}\psi x|e^{\theta^{tr}x+x^{tr}\psi x} I_{\{|\theta^{tr}x|>\epsilon\}} \star \widetilde{\mu} \\
&\leq e^{\epsilon}\sum |(\Delta\widetilde{X})^{tr}\psi\Delta\widetilde{X}|e^{(\Delta\widetilde{X})^{tr}\psi\Delta\widetilde{X}} + |(\Delta\widetilde{X})^{tr}\psi\Delta\widetilde{X}|e^{(\Delta\widetilde{X})^{tr}\psi\Delta\widetilde{X}} \bullet U_\epsilon \in \mathcal{A}^+_{loc},
\end{aligned}$$

where U_ϵ is defined in (3.7). This proves that $Z \in \mathcal{Z}^{L\log L}_{loc}(S)$, and assertion (b) follows immediately.

Part 2. Now we prove (b) $\Longrightarrow$ (c). To this end, we suppose that assertion (b) holds. Consider a predictable and locally bounded process ψ, and take

$$\begin{aligned}
f(\theta) &:= \theta^{tr}\widetilde{b}' + \frac{1}{2}\theta^{tr}c\theta + \int \left(\exp\left(\theta^{tr}x + x^{tr}\psi x\right) - 1 - \theta^{tr}x\right)\widetilde{F}(dx), \\
f_\psi(\theta) &:= \theta^{tr}b^{(\psi)} + \frac{1}{2}\theta^{tr}c\theta + \int \left(e^{\theta^{tr}x} - 1 - \theta^{tr}x\right)\widetilde{F}^{(\psi)}(dx), \\
b^{(\psi)} &:= \widetilde{b}' + \int x\left(e^{x^{tr}\psi x} - 1\right)\widetilde{F}(dx), \quad \widetilde{F}^{(\psi)}(dx) := e^{x^{tr}\psi x}\widetilde{F}(dx).
\end{aligned}$$

Then, direct calculation shows that

$$f(\theta) = f_\psi(\theta) + \widetilde{\Gamma}^{(\psi)}, \text{ for any } \theta, \text{ where } \widetilde{\Gamma}^{(\psi)} := \int \left(e^{x^{tr}\psi x} - 1\right)\widetilde{F}(dx). \tag{3.40}$$

By stopping, without loss of generality we assume that

$$Z^{(\psi)} \in \mathcal{M} \quad \text{and} \quad Q^{(\psi)} := Z^{(\psi)}_T \cdot P \quad \text{is a well defined probability measure.}$$

Hence, under $Q^{(\psi)}$, the process $\widetilde{X}$ admits the following canonical decomposition

$$\widetilde{X} = b^{(\psi)} \bullet A + x \star (\widetilde{\mu} - \widetilde{\nu}^{(\psi)}) + X^c,\ \widetilde{\nu}^{(\psi)}(dt,dx) := e^{x^{tr}\psi x}\widetilde{\nu}(dt,dx) = \widetilde{F}^{(\psi)}_t(dx)dA_t.$$

Thanks to the local boundedness of $Z^{(\psi)}$ and $1/Z^{(\psi)}$, it is clear that

$$\mathcal{Z}^{L\log L}_{loc}(S) \neq \emptyset \iff \mathcal{Z}^{L\log L}_{loc}(S, Q^{(\psi)}) \neq \emptyset, \tag{3.41}$$

for any ψ predictable and locally bounded. Therefore, by virtue of (3.41) and (3.40), our minimization problem (3.8) has a solution if and only if

$$\min_\theta f_\psi(\theta) \quad \text{has a solution under the condition } \mathcal{Z}^{L\log L}_{loc}(S, Q^{(\psi)}) \neq \emptyset. \tag{3.42}$$

Thus, the resulting minimization problem is exactly the minimization problem considered in [13, Lemma 4.4] for the model $(\widetilde{X}, Q^{(\psi)})$ instead. Hence, the existence of a unique solution $\widehat{\theta}^\psi =: \widehat{\theta}$ to (3.42) follows immediately from [13, Lemma 4.4].

Thus, assertion (c) will follow as soon as we prove that (3.9) holds. To this end, it is enough to prove that $\widehat{\theta}$ satisfies

$$\widehat{V} := \widehat{\theta}^{tr} c\widehat{\theta} \bullet A + f_{L\log L}(\exp(\widehat{\theta}^{tr}x) - 1) \star \widetilde{\nu}^{(\psi)} \in \mathcal{A}^+_{loc}(Q^{(\psi)}). \tag{3.43}$$

In fact, on the one hand, it is clear that $V \in \mathcal{A}^+_{loc}(Q^{(\psi)})$ if and only if $V \in \mathcal{A}^+_{loc}$ for any process V due to the local boundedness of $\widetilde{X}$. On the other hand, due to $f_{L\log L}(y) \geq a(y-1)$ for any $y \geq e^a$ and $a \in (0,\infty)$, the condition (3.43) implies that $|\exp(\widehat{\theta}^{tr}x) - 1|I_{\{\widehat{\theta}^{tr}x > \ln(e^a+1)+1\}} \star \widetilde{\nu}^{(\psi)} \in \mathcal{A}^+_{loc}$. Therefore, this easily implies that $|x^{tr}\psi x(e^{\widehat{\theta}^{tr}x} - 1)|e^{x^{tr}\psi x} \star \widetilde{\nu} \in \mathcal{A}^+_{loc}$. By combining this latter fact with

$$\begin{aligned}\widehat{V} =& \widehat{\theta}^{tr} c\widehat{\theta} \bullet A + f_{L\log L}(\exp(\widehat{\theta}^{tr}x + x^{tr}\psi x) - 1) \star \widetilde{\nu} - x^{tr}\psi x(e^{\widehat{\theta}^{tr}x} - 1)e^{x^{tr}\psi x} \star \widetilde{\nu}\\ &- f_{L\log L}(e^{x^{tr}\psi x} - 1) \star \widetilde{\nu},\end{aligned}$$

and $f_{L\log L}(e^{x^{tr}\psi x} - 1) \star \widetilde{\nu} + |e^{x^{tr}\psi x} - 1| \star \widetilde{\nu} \in \mathcal{A}^+_{loc}$, we conclude that (3.43) implies (3.9). Thus, the rest of this part focuses on proving (3.43). To this end, we use Lemma 3.21-(a) for $Q = Q^{(\psi)}$ and conclude that $\widehat{\theta}$, the solution to (3.42), satisfies

$$b^{(\psi)} + c\widehat{\theta} + \int \left(x\exp(\widehat{\theta}^{tr}x) - x\right) \widetilde{F}^{(\psi)}(dx) = 0. \tag{3.44}$$

Besides this, a combination of $\mathcal{Z}^{L\log L}_{loc}(S, Q^{(\psi)}) \neq \emptyset$ and Theorem A.1 yields the existence of a pair (β, f) such that

$$\begin{aligned}&b^{(\psi)} + c\beta + \int x\,(f(x) - 1))\, \widetilde{F}^{(\psi)}(dx) = 0, \text{ and}\\ &\beta^{tr}c\beta \bullet A + (f\ln(f) - f + 1) \star \widetilde{\nu}^{(\psi)} \in \mathcal{A}^+_{loc}(Q^{(\psi)}).\end{aligned}$$

Thus, by combining the first equality above with (3.44), we get

$$c(\beta - \widehat{\theta}) + \int x\left(f(x) - \exp(\widehat{\theta}^{tr}x)\right) \widetilde{F}^{(\psi)}(dx) = 0. \tag{3.45}$$

Thanks to the convexity of $\theta^{tr}c\theta$ and $g(y) := y\ln(y) - y + 1$ for $y > 0$, we get

$$\beta^{tr}c\beta \geq \widehat{\theta}^{tr}c\widehat{\theta} + (\beta - \widehat{\theta})^{tr}c\widehat{\theta} \text{ and } g(f(x)) - g(\exp(\widehat{\theta}^{tr}x)) \geq \widehat{\theta}^{tr}x(f(x) - \exp(\widehat{\theta}^{tr}x)).$$

By integrating the two inequalities above with respect to A and $\widetilde{\nu}^{(\psi)}$ respectively, adding them, and using (3.45) afterwards, we get

$$\beta^{tr}c\beta \bullet A + (f\ln(f) - f + 1) \star \widetilde{\nu}^{(\psi)} \geq \widehat{\theta}^{tr}c\widehat{\theta} \bullet A + f_{L\log L}(\exp(\widehat{\theta}^{tr}x) - 1) \star \widetilde{\nu}^{(\psi)},$$

and the claim (3.43) follows immediately. This proves assertion (c).

Part 3. This part proves (d) $\Longrightarrow$ (a). Thus, we suppose assertion (d) holds, and remark that $f_{L\log L}(y) := (y+1)\ln(1+y) - y \geq |y|I_{\{y>2e-1\}} + (y^2/(4e))I_{\{|y|\leq 2e-1\}}$ for any $y > -1$. Thus, by combining this with condition (3.9) and [1, Proposition B.2-(a)], which states that $\sqrt{(\exp(\widehat{\theta}^{tr}x + x^{tr}\psi x) - 1)^2 \star \widetilde{\mu}} \in \mathcal{A}^+_{loc}$ if and only if for any $\alpha > 0$

$$(e^{\widehat{\theta}^{tr}x + x^{tr}\psi x} - 1)^2 I_{\{|e^{\widehat{\theta}^{tr}x + x^{tr}\psi x} - 1| \leq \alpha\}} \star \widetilde{\mu} + |e^{\widehat{\theta}^{tr}x + x^{tr}\psi x} - 1| I_{\{|e^{\widehat{\theta}^{tr}x + x^{tr}\psi x} - 1| > \alpha\}} \star \widetilde{\mu} \in \mathcal{A}^+_{loc},$$

we deduce that

$$\sqrt{(\exp(\widehat{\theta}^{tr}x + x^{tr}\psi x) - 1)^2 \star \widetilde{\mu}} \in \mathcal{A}^+_{loc} \quad \text{and} \quad \widehat{\theta} \in L(X^c).$$

Thus, as a result, by combining this with Lemma 3.21, we deduce that the following process:

$$Z := \mathcal{E}\left(\widehat{\theta}\bullet X^c + (\exp(\widehat{\theta}^{tr}x + x^{tr}\psi x) - 1)\star(\widetilde{\mu} - \widetilde{\nu})\right) \in \mathcal{Z}_{loc}(S). \tag{3.46}$$

Now, thanks to Itô's formula, we calculate the semimartingale $\ln(Z)$ as follows:

$$\begin{aligned}\ln(Z) &= \widehat{\theta}\bullet X^c + (\exp(\widehat{\theta}^{tr}x + x^{tr}\psi x) - 1)\star(\widetilde{\mu} - \widetilde{\nu}) - \frac{1}{2}\widehat{\theta}^{tr}c\widehat{\theta}\bullet A\\ &\quad + \left(\widehat{\theta}^{tr}x + x^{tr}\psi x - \exp(\widehat{\theta}^{tr}x + x^{tr}\psi x) + 1\right)\star\widetilde{\mu}\\ &= \widehat{\theta}\bullet X^c + (\exp(\widehat{\theta}^{tr}x + x^{tr}\psi x) - 1)\star(\widetilde{\mu} - \widetilde{\nu}) - \frac{1}{2}\widehat{\theta}^{tr}c\widehat{\theta}\bullet A\\ &\quad + \left(\widehat{\theta}^{tr}x - \exp(\widehat{\theta}^{tr}x + x^{tr}\psi x) + 1\right)\star\widetilde{\mu} + \sum(\Delta\widetilde{X})^{tr}\psi\Delta\widetilde{X}.\end{aligned} \tag{3.47}$$

Remark that for any n it is clear that $I_{\{\|\widehat{\theta}\|\leq n\}}|\widehat{\theta}^{tr}x - \exp(\widehat{\theta}^{tr}x + x^{tr}\psi x) + 1|\star\widetilde{\mu} \in \mathcal{A}^+_{loc}$. Thus, by compensating this process in (3.47) and using (3.44) afterwards, we derive

$$\begin{aligned}&I_{\{\|\widehat{\theta}\|\leq n\}}\bullet\ln(Z)\\ &= I_{\{\|\widehat{\theta}\|\leq n\}}\widehat{\theta}\bullet X^c + I_{\{\|\widehat{\theta}\|\leq n\}}\widehat{\theta}^{tr}x\star(\widetilde{\mu} - \widetilde{\nu}) - \frac{1}{2}I_{\{\|\widehat{\theta}\|\leq n\}}\widehat{\theta}^{tr}c\widehat{\theta}\bullet A\\ &\quad + I_{\{\|\widehat{\theta}\|\leq n\}}\left(\widehat{\theta}^{tr}x - e^{\widehat{\theta}^{tr}x + x^{tr}\psi x} + 1\right)\star\widetilde{\nu} + \sum I_{\{\|\widehat{\theta}\|\leq n\}}(\Delta\widetilde{X})^{tr}\psi\Delta\widetilde{X}\\ &= I_{\{\|\widehat{\theta}\|\leq n\}}\widehat{\theta}\bullet X^c + I_{\{\|\widehat{\theta}\|\leq n\}}\widehat{\theta}^{tr}x\star(\widetilde{\mu} - \widetilde{\nu}) - \frac{1}{2}I_{\{\|\widehat{\theta}\|\leq n\}}\widehat{\theta}^{tr}c\widehat{\theta}\bullet A\\ &\quad + \sum I_{\{\|\widehat{\theta}\|\leq n\}}(\Delta\widetilde{X})^{tr}\psi\Delta\widetilde{X} + I_{\{\|\widehat{\theta}\|\leq n\}}\left(\widehat{\theta}^{tr}x(1 - e^{\widehat{\theta}^{tr}x + x^{tr}\psi x})\right)\star\widetilde{\nu}\\ &\quad + I_{\{\|\widehat{\theta}\|\leq n\}}\left((\widehat{\theta}^{tr}x)e^{\widehat{\theta}^{tr}x + x^{tr}\psi x} - e^{\widehat{\theta}^{tr}x + x^{tr}\psi x} + 1\right)\star\widetilde{\nu}\\ &= I_{\{\|\widehat{\theta}\|\leq n\}}\widehat{\theta}\bullet X^c + I_{\{\|\widehat{\theta}\|\leq n\}}\widehat{\theta}^{tr}x\star(\widetilde{\mu} - \widetilde{\nu}) + I_{\{\|\widehat{\theta}\|\leq n\}}\widehat{\theta}^{tr}\widetilde{b}'\bullet A\\ &\quad + \sum I_{\{\|\widehat{\theta}\|\leq n\}}(\Delta\widetilde{X})^{tr}\psi\Delta\widetilde{X} + \frac{1}{2}I_{\{\|\widehat{\theta}\|\leq n\}}\widehat{\theta}^{tr}c\widehat{\theta}\bullet A\\ &\quad + I_{\{\|\widehat{\theta}\|\leq n\}}\left((\widehat{\theta}^{tr}x)e^{\widehat{\theta}^{tr}x + x^{tr}\psi x} - e^{\widehat{\theta}^{tr}x + x^{tr}\psi x} + 1\right)\star\widetilde{\nu}\\ &= I_{\{\|\widehat{\theta}\|\leq n\}}\widehat{\theta}\bullet\widetilde{X} + \sum I_{\{\|\widehat{\theta}\|\leq n\}}(\Delta\widetilde{X})^{tr}\psi\Delta\widetilde{X} + I_{\{\|\widehat{\theta}\|\leq n\}}\bullet\widetilde{K},\end{aligned} \tag{3.48}$$

where $\widetilde{K}$ is given by

$$\widetilde{K} := \left((\widehat{\theta}^{tr}x)\exp(\widehat{\theta}^{tr}x + x^{tr}\psi x) - \exp(\widehat{\theta}^{tr}x + x^{tr}\psi x) + 1\right)\star\widetilde{\nu} + \frac{1}{2}\widehat{\theta}^{tr}c\widehat{\theta}\bullet A,$$

and it is a well-defined predictable with finite variation process due to (3.9) and Remark 3.7-(b).

Now, as the three processes $\ln(Z)$, $\widetilde{K}$ and $V := \sum(\Delta\widetilde{X})^{tr}\psi\Delta\widetilde{X}$ are semimartingales, thanks to [41, Proposition 1.7], it is clear that the processes $I_{\{\|\widehat{\theta}\|\leq n\}}\bullet\ln(Z)$, $I_{\{\|\widehat{\theta}\|\leq n\}}\bullet\widetilde{K}$, and $I_{\{\|\widehat{\theta}\|\leq n\}}\bullet V$ converge in the semimartingale topology to $\ln(Z)$, $\widetilde{K}$, and V, respectively. This implies that $I_{\{\|\widehat{\theta}\|\leq n\}}\widehat{\theta}\bullet\widetilde{X}$ converges in the semimartingale topology. Thus, by virtue of [41, Definition 2.1], we deduce that $\widehat{\theta} \in L(\widetilde{X})$, and its limit is the semimartingale $\widehat{\theta}\bullet\widetilde{X}$. Furthermore, by combining these latter remarks and (3.48), we obtain

$$\ln(Z) = \widehat{\theta}\bullet\widetilde{X} + \sum(\Delta\widetilde{X})^{tr}\psi\Delta\widetilde{X} + \widetilde{K}.$$

Therefore, assertion (a) follows immediately (i.e. $Z \in \mathcal{Z}^{LE}(S)$) from combining the above equality with (3.46). This ends the proof of the theorem. □

Remark 3.22. The last part of the proof above contains an important statement. This confirms the following claim: If S is locally bounded and there exists a pair (θ, ψ) of predictable processes such that

$$\psi \text{ is locally bounded, (3.9) holds, and } \theta \text{ is the solution to (3.5),}$$

then $(\theta, \psi) \in \Theta(S)$.

Proof of Theorem 3.8. Thanks to Theorem 3.6 (assertions (a) and (d)), we deduce that $Z \in \mathcal{Z}^{LE}(S)$ if and only if there exists a pair (θ, ψ) of predictable processes such that

$$\begin{aligned}&\psi \text{ is locally bounded, (3.9) holds, } \theta \text{ is the solution to (3.8),}\\ &\text{and } Z = \mathcal{E}\left(\theta \bullet X^c + (\exp(\theta^{tr}x + x^{tr}\psi x) - 1) \star (\widetilde{\mu} - \widetilde{\nu})\right).\end{aligned}$$

By virtue of Lemma 3.21, we have that θ is the solution to (3.8) if and only if θ is the solution to (3.5), or equivalently $Z \in \mathcal{Z}_{loc}(S)$. Furthermore, direct calculations show that $Z \in \mathcal{Z}_{loc}(S)$ if and only if $Z/Z^{(\psi)} \in \mathcal{Z}_{loc}(S, Z^{(\psi)})$. Thus, by combining all these remarks, we conclude that $Z \in \mathcal{Z}^{LE}(S)$ if and only if there exists a pair (θ, ψ) of predictable processes satisfying

$$\psi \text{ is locally bounded, (3.9) holds and } Z/Z^{(\psi)} \in \mathcal{Z}_{loc}(S, Z^{(\psi)}).$$

Thus, assertion (a) follows as soon as we prove the equality in (3.11). To this end, as ψ and S are locally bounded, we remark that $(\exp(x^{tr}\psi x) - 1) \star (\widetilde{\mu} - \widetilde{\nu})$ (or equivalently $Z^{(\psi)}$) is a well defined local martingale. Hence, using Yor's formula (i.e. $\mathcal{E}(Y_1)\mathcal{E}(Y_2) = \mathcal{E}(Y_1 + Y_2 + [Y_1, Y_2])$ for any pair of semimartingales Y_1 and Y_2), we derive

$$\begin{aligned}
&Z^{(\psi)} Z^{(\theta,\psi)}\\
&:= Z^{(\psi)}\mathcal{E}\left(\theta \bullet X^c + (e^{\theta^{tr}x} - 1) \star (\widetilde{\mu} - \widetilde{\nu}^{(\psi)})\right)\\
&= \mathcal{E}\left((e^{x^{tr}\psi x} - 1) \star (\widetilde{\mu} - \widetilde{\nu}) + \theta \bullet X^c + (e^{\theta^{tr}x} - 1) \star (\widetilde{\mu} - \widetilde{\nu}^{(\psi)}) + (e^{x^{tr}\psi x} - 1)(e^{\theta^{tr}x} - 1) \star \widetilde{\mu}\right)\\
&= \mathcal{E}\left(\theta \bullet X^c + (e^{x^{tr}\psi x} - 1) \star (\widetilde{\mu} - \widetilde{\nu}) + (e^{\theta^{tr}x} - 1) \star (\widetilde{\mu} - \widetilde{\nu}^{(\psi)}) + (e^{x^{tr}\psi x} - 1)(e^{\theta^{tr}x} - 1) \star \widetilde{\mu}\right)\\
&= \mathcal{E}\left(\theta \bullet X^c + (\exp(\theta^{tr}x + x^{tr}\psi x) - 1) \star (\widetilde{\mu} - \widetilde{\nu})\right) = Z.
\end{aligned}$$

The last equality follows from the fact that

$$(e^{\theta^{tr}x} - 1) \star (\widetilde{\mu} - \widetilde{\nu}^{(\psi)}) = (e^{\theta^{tr}x} - 1) \star (\widetilde{\mu} - \widetilde{\nu}) - (e^{x^{tr}\psi x} - 1)(e^{\theta^{tr}x} - 1) \star \widetilde{\nu}.$$

This completes the proof of assertion (a). To prove assertion (b), we remark that by virtue of Theorem 3.6, we have $\mathcal{Z}^{L\log L}_{loc}(S) \neq \emptyset$ if and only $\mathcal{Z}^{LE}(S) \neq \emptyset$, and this is equivalent to S admitting the first-order Esscher density under $Z^{(\psi')}$ for any predictable locally bounded ψ' due to assertion (a). Finally, this latter property is equivalent to $\mathcal{Z}^{L\log L}_{loc}(S, Z^{(\psi')}) \neq \emptyset$; for any predictable locally bounded ψ', see [13] for details. This ends the proof of the theorem. □

3.3.2. *Proof of Theorems 3.13 and 3.15.* We start with the proof of Theorem 3.13.

Proof of Theorem 3.13. The proof of the theorem is given in two parts, where we prove assertions (a) and (b), respectively.

Part 1. This part proves assertion (a). To this end, we consider $(\theta,\psi)\in\Theta(X)$, and we take

$$\begin{aligned}&\overline{X}:=\theta\bullet X+\sum(\Delta X)^{tr}\psi\Delta X-K,\quad Z:=\exp(\overline{X}),\quad \text{and}\\ &\overline{K}:=K-\sum\left(e^{-\Delta K}-1+\Delta K\right).\end{aligned}$$

Then, by combining Itô's formula and the simple fact that

$$e^{\Delta\overline{X}}-1-\Delta\overline{X}=(e^{\Delta Y}-1-\Delta Y)+(e^{-\Delta K}-1+\Delta\overline{K}),\quad \text{with}\quad Y:=\overline{X}+K,$$

which is a direct consequence of $\Delta X\Delta K\equiv 0$, we derive

$$\begin{aligned}Z_{-}^{-1}\bullet Z&=\overline{X}+\frac{1}{2}\langle\overline{X}^{c}\rangle+\sum\left(e^{\Delta\overline{X}}-1-\Delta\overline{X}\right)\\ &=\theta\bullet X+\sum(\Delta X)^{tr}\psi\Delta X-\overline{K}+\frac{\theta^{tr}c\theta}{2}\bullet A\\ &\quad+\left(\exp(\theta^{tr}x+x^{tr}\psi x)-1-\theta^{tr}x-x^{tr}\psi x\right)\star\mu\\ &=\theta\bullet X-\overline{K}+\frac{\theta^{tr}c\theta}{2}\bullet A+\left(\exp(\theta^{tr}x+x^{tr}\psi x)-1-\theta^{tr}x\right)\star\mu.\end{aligned}$$

Thus, by applying (3.31) to $Y=X$ (Lemma 3.19-(c)) and inserting it in the above equality, we get

$$\begin{aligned}&Z_{-}^{-1}\bullet Z\\ &=\theta\bullet X^{c}+\theta^{tr}xI_{\{|\theta^{tr}x|\leq\epsilon\}}\star(\mu-\nu)+\left(\theta^{tr}b^{\theta}+\int\theta^{tr}xI_{\{|\theta^{tr}x|\leq\epsilon<\|x\|\}}F(dx)\right)\bullet A\\ &\quad+\theta\bullet U_{2}^{\theta}-\overline{K}+\frac{\theta^{tr}c\theta}{2}\bullet A+\left(\exp(\theta^{tr}x+x^{tr}\psi x)-1-\theta^{tr}x\right)\star\mu\\ &=\theta\bullet X^{c}+\theta^{tr}xI_{\{|\theta^{tr}x|\leq\epsilon\}}\star(\mu-\nu)-\overline{K}\\ &\quad+\left(\frac{\theta^{tr}c\theta}{2}+\theta^{tr}b^{\theta}+\int\theta^{tr}xI_{\{|\theta^{tr}x|\leq\epsilon<\|x\|\}}F(dx)\right)\bullet A\\ &\quad+\left(\exp(\theta^{tr}x+x^{tr}\psi x)-1-\theta^{tr}xI_{\{|\theta^{tr}x|\leq\epsilon\}}\right)\star\mu.\end{aligned}$$

Therefore, on the one hand, Z is a local martingale if and only if

$$\left(\exp(\theta^{tr}x+x^{tr}\psi x)-1-h_{\epsilon}(\theta^{tr}x)\right)\star\mu\in\mathcal{A}_{loc},\tag{3.49}$$

and

$$\begin{aligned}\overline{K}&=\left(\frac{\theta^{tr}c\theta}{2}+\theta^{tr}b^{\theta}+\int\theta^{tr}xI_{\{|\theta^{tr}x|\leq\epsilon<\|x\|\}}F(dx)\right)\bullet A\\ &\quad+\left(\exp(\theta^{tr}x+x^{tr}\psi x)-1-h_{\epsilon}(\theta^{tr}x)\right)\star\nu\\ Z&=\mathcal{E}\left(\theta\bullet X^{c}+\left(\exp(\theta^{tr}x+x^{tr}\psi x)-1\right)\star(\mu-\nu)\right):=\mathcal{E}(L).\end{aligned}$$

Hence, $\overline{K}=K$ is continuous, and (3.21) follows immediately. On the other hand, due to (2.8), we get

$$\begin{aligned}&\widetilde{X}+[\widetilde{X},L]\\ &=X^{c}+x\star(\widetilde{\mu}-\widetilde{\nu})+\widetilde{b}'\bullet A+c\theta\bullet A+\left(\exp(\theta^{tr}x+x^{tr}\psi x)-1\right)(\mathbf{e}(x)-\mathbb{I}_{d})\star\mu\\ &=X^{c}+x\star(\widetilde{\mu}-\widetilde{\nu})+(\widetilde{b}'+c\theta)\bullet A+\left(\exp(\theta^{tr}x+x^{tr}\psi x)-1\right)(\mathbf{e}(x)-\mathbb{I}_{d})\star\mu.\end{aligned}$$

Thus, ZS is a local martingale if and only if $\widetilde{X}+[\widetilde{X},N]$ is a local martingale, and this is equivalent to

$$\big(\exp(\theta^{tr}x+x^{tr}\psi x)-1\big)\left(\mathbf{e}(x)-\mathbb{I}_d\right)\star\mu\in\mathcal{A}_{loc},\tag{3.50}$$

and (3.20) holds. Furthermore, by virtue of Lemma 3.20-(d), the conditions (3.49) and (3.50) are equivalent to (3.19). This ends the proof of assertion (a).

Part 2. This part proves assertion (b). To this end, we consider $(\theta,\psi)\in\Theta(S)$. Then, we associate with ψ the process $\overline{Z}^{(\psi)}$ defined in (3.22), which is a well defined and locally bounded local martingale. Thus, by stopping it, there is no loss of generality in assuming that $\overline{Z}^{(\psi)}$ is a uniformly integrable martingale. Hence $Q^{(\psi)}:=\overline{Z}^{(\psi)}_T\cdot P$ is a well defined probability. Furthermore, the $Q^{(\psi)}$-compensators of μ and $\widetilde{\mu}$, denoted by $\nu^{(\psi)}$ and $\widetilde{\nu}^{(\psi)}$, respectively, and the canonical decomposition of $\widetilde{X}$ under $Q^{(\psi)}$ are

$$\begin{aligned}
&\nu^{(\psi)}(dt,dx):=\exp(x^{tr}\psi x)\nu(dt,dx)=:F^{(\psi)}_t(dx)dA_t,\\
&F^{(\psi)}_t(dx):=\exp(x^{tr}\psi x)F_t(dx),\\
&W\star\widetilde{\nu}^{(\psi)}=W(\cdot,\Phi(x))\star\nu^{(\psi)},\quad \Phi(x):=\mathbf{e}(x)-\mathbb{I}_d,\quad\forall\ W,\\
&\widetilde{X}=X_0+X^c+x\star(\widetilde{\mu}-\widetilde{\nu}^{(\psi)})+\left(\widetilde{b}'+\int(e^{x^{tr}\psi x}-1)(\mathbf{e}(x)-\mathbb{I}_d)F(dx)\right)\bullet A.
\end{aligned}$$

Then, via direct calculations, it is easy to check that $(\theta,\psi)\in\Theta(S)$ satisfies (3.19) and

$$Z:=\mathcal{E}\left(\theta\bullet X^c+(\exp(\theta^{tr}x+x^{tr}\psi x)-1)\star(\mu-\nu)\right)\in\mathcal{Z}_{loc}(S),$$

if and only if

$$\begin{aligned}
&\exp(\theta^{tr}x)I_{\{\theta^{tr}x>\alpha\}}\star\mu\in\mathcal{A}^+_{loc}(Q^{\psi}),\quad\text{for some }\alpha>0,\\
&\frac{Z}{\overline{Z}^{(\psi)}}=\mathcal{E}\left(\theta\bullet X^c+(\exp(\theta^{tr}x)-1)\star(\mu-\nu^{(\psi)})\right)\in\mathcal{Z}_{loc}(S,Q^{(\psi)}).
\end{aligned}$$

This proves assertion (b), and the proof of this theorem is complete. $\square$

The proof of Theorem 3.15 relies on the following lemma that is true in general.

Lemma 3.23. *Suppose that the assumptions of Theorem 3.15 hold. Let $(\widehat{S},\widehat{\nu},\widehat{Z})$ be given by (3.23) and consider a pair (β,g) such that $\beta\in L^1_{loc}(X^c)$, $g(t,\omega,x)$ is positive, $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable and $\sqrt{(g-1)^2\star\mu}\in\mathcal{A}^+_{loc}$.*

Then, the condition $Z:=\mathcal{E}(\beta\bullet X^c+(g-1)\star(\mu-\nu))\in\mathcal{Z}^{L\log L}_{loc}(S)$ is equivalent to $Z':=\mathcal{E}(\beta\bullet X^c+(g-1)\star(\mu-\widehat{\nu}))\in\mathcal{Z}^{L\log L}_{loc}(\widehat{S},\widehat{Z})$.

The proof of this lemma is relegated to Appendix B, and we instead prove Theorem 3.15 next.

Proof of Theorem 3.15. Due to assertion (a), it is clear that $\mathcal{Z}^{EE}(S)\neq\emptyset$ holds if and only if $\mathcal{Z}^{LE}(\widehat{S},\widehat{Z})\neq\emptyset$ holds, and thanks to Theorem 3.6, this is equivalent to $\mathcal{Z}^{L\log L}_{loc}(\widehat{S},\widehat{Z})\neq\emptyset$. Thus, by virtue of Lemma 3.23, the latter claim is equivalent to $\mathcal{Z}^{L\log L}_{loc}(S)\neq\emptyset$. This proves assertion (b), while the rest of the proof focuses on assertion (a). To this end, we consider an RCLL and positive process $\overline{Z}>0$, and remark that by stopping we can assume that $\widehat{Z}$ is a uniformly integrable martingale and hence $\widehat{Q} := \widehat{Z}_\infty \cdot P$ is a well-defined probability measure. Furthermore,

$$\begin{aligned}
\widehat{X} &= \left(b' + \frac{c}{2}\right) \bullet A + x \star (\mu - \nu) + X^c = \widehat{b} \bullet A + x \star (\mu - \widehat{\nu}) + X^c \\
\widehat{b} &:= b' + \frac{c}{2} + \int x(f(x) - 1)F(dx), \\
\widehat{F}_t(dx) &:= f(x)F_t(dx), \;\; \widehat{\nu}(dt, dx) := \widehat{F}_t(dx)dA_t, .
\end{aligned} \tag{3.51}$$

By virtue of Theorem 3.4 (which we apply to $\widehat{S}$ under $\widehat{Q}$ instead), we deduce that $\overline{Z} \in \mathcal{Z}^{LE}(\widehat{S}, \widehat{Z}) \neq \emptyset$ if and only if there exists a pair (θ, ψ) such that

$$\begin{aligned}
&(\theta, \psi) \in \Theta(\widehat{S}), \quad \exp(x\theta)I_{\{|\theta x| > \epsilon\}} \star \mu \in \mathcal{A}^+_{loc}(\widehat{Q}), \\
&\widehat{b} + c\theta + \int x(\exp(x\theta + x^2\psi) - 1)\widehat{F}(dx) = 0, \\
&\overline{Z} = \mathcal{E}\left(\theta \bullet X^c + (\exp(x\theta + x^2\psi) - 1) \star (\mu - \widehat{\nu})\right).
\end{aligned} \tag{3.52}$$

Note that, by virtue of the first equation in (3.51) and $|\theta c| \bullet A \leq \sqrt{\theta^2 c \bullet A} + \sqrt{c \bullet A}$, it is clear that $(\theta, \psi) \in \Theta(\widehat{S})$ iff $(\theta, \psi) \in \Theta(S)$. On the one hand, due to the boundedness of S (hence that of $\widehat{Z}$),

$$\text{the second condition in (3.52) is equivalent to } e^{x\theta}I_{\{|\theta x| > \epsilon\}} \star \mu \in \mathcal{A}^+_{loc}. \tag{3.53}$$

On the other hand, by using the notation in (3.51), we deduce that the third condition in (3.52) is equivalent to

$$\widetilde{b}' + c\theta + \int (e^x - 1)(\exp(x\theta + x^2\psi) - 1)F(dx) = 0. \tag{3.54}$$

Furthermore, using Yor's formula and direct calculations, we conclude that the third condition in (3.52) holds if and only if

$$Z := \overline{Z}\widehat{Z} = \mathcal{E}\left(\theta \bullet X^c + (\exp(x\theta + x^2\psi) - 1) \star (\mu - \nu)\right). \tag{3.55}$$

Thus, by combining (3.55), (3.54), (3.53), (3.52), and Theorem 3.13-(a), assertion (a) follows immediately. This also proves (3.24), and the proof of the theorem is complete. ☐

4. **Esscher pricing bounds and linear constraints BSDEs.**

Throughout this section, we suppose $d = 1$ and $\mathbb{F}$ is the filtration generated by a Brownian motion W and a non-homogeneous Poisson process N with intensity process λ. Here, W and N are independent, and the compensated Poisson process $\widetilde{N}$ is given by $\widetilde{N}_t := N_t - \int_0^t \lambda_s ds$. Our market model consists of one risky asset S and one non-risky asset $S^{(0)} := \exp(B)$ having the following dynamics

$$\begin{aligned}
&B := \int_0^\cdot r_s ds, \;\; r \geq 0, \quad S := S_0 e^X = S_0\mathcal{E}(\widetilde{X}), \\
&dX := bdt + \sigma dW + \gamma d\widetilde{N}, \;\; X_0 = \widetilde{X}_0 = 0, \\
&\widetilde{N} := N - \int_0^\cdot \lambda_s ds, \;\; d\widetilde{X} := \widetilde{b}dt + \sigma dW + \widetilde{\gamma}d\widetilde{N}, \\
&\widetilde{b} := b + \frac{\sigma^2}{2} + \lambda(e^\gamma - 1 - \gamma) \text{ and } \widetilde{\gamma} := e^\gamma - 1.
\end{aligned} \tag{4.1}$$

Throughout the rest of the paper, we assume that $P\otimes dt$-a.e., $\sigma>0$, $\lambda>0$, and for some $C\in(0,\infty)$

$$\sigma+\lambda+\sigma^{-1}+\lambda^{-1}+|r|+|b|+|\gamma|+|\gamma|^{-1}\leq C. \tag{4.2}$$

Furthermore, we consider the following sets:

$$\Psi:=\{\psi:\ \psi \text{ is predictable and bounded}\},\quad \Psi_n:=\{\psi\in\Psi:\ |\psi|\leq n\},\quad n\in\mathbb{N}.$$

Remark 4.1. For the rest of the paper, we will work with this simple model, defined in (4.1), and under the assumptions (4.2). Our unique reason for these restrictions on $(S,\mathbb{F})$ lies in avoiding technicalities that might overshadow the key ideas. It is important to mention that all those restrictions on $(S,\mathbb{F})$ can be extended to the most general case, as the theory of BSDEs is well developed nowadays.

We start this section by analyzing the solution to the two martingale equations (3.5) and (3.20) for our model above, and which can be written in a unified manner.

Lemma 4.2. *For any $\psi\in\Psi$ and $\zeta\in\{\gamma,\widetilde{\gamma}\}$, we denote by $\eta(\psi)$ the unique root for*

$$\widetilde{b}-r+\eta\sigma^2+\lambda\widetilde{\gamma}\Big(\exp(\eta\zeta+\psi\zeta^2)-1\Big)=0, \tag{4.3}$$

and consider the functional Φ given by

$$\Phi(x):=\Phi(t,\omega,x):=\sigma_t(\omega)^2x+\lambda_t(\omega)\widetilde{\gamma}_t(\omega)\zeta_t(\omega)e^x,\quad t\in[0,T],\ \omega\in\Omega,\ x\in\mathbb{R}.$$

Then, the following assertions hold.
(a) *The functional Φ is continuous, strictly increasing, $\Phi(\infty)=\infty$, $\Phi(-\infty)=-\infty$, and*

$$\eta(\psi)=\frac{1}{\zeta}\Phi^{-1}\Big(\zeta(r-\widetilde{b}+\lambda\widetilde{\gamma})+\sigma^2\zeta^2\psi\Big)-\zeta\psi. \tag{4.4}$$

As a result, for $\psi\in\Psi$, there exists $C_\psi\in(0,\infty)$ such that $|\eta(\psi)|\leq C_\psi$, $P\otimes dt$-a.e..
(b) *$\eta(\psi)\widetilde{\gamma}^{-1}$ is decreasing ($P\otimes dt$-a.e.) with respect to ψ.*
(c) *For any $\zeta\in\{\gamma,\widetilde{\gamma}\}$, we have $P\otimes dt$-a.e.*

$$\lim_{\psi\downarrow-\infty}\frac{\eta(\psi)}{\widetilde{\gamma}}=\frac{\eta(-\infty)}{\widetilde{\gamma}}=\frac{\lambda\widetilde{\gamma}+r-\widetilde{b}}{\widetilde{\gamma}\sigma^2}=:\frac{\eta^*}{\widetilde{\gamma}},\quad \text{and}\quad \lim_{\psi\uparrow\infty}\frac{\eta(\psi)}{\widetilde{\gamma}}=-\infty. \tag{4.5}$$

(d) *For any $\zeta\in\{\gamma,\widetilde{\gamma}\}$ and any $\psi\in\Psi$ such that $\psi\geq0$, we have*

$$\lim_{n\longrightarrow\infty}\frac{-\eta(n)}{n\widetilde{\gamma}}=\frac{\zeta}{\widetilde{\gamma}},\quad \text{and}\quad \frac{\eta(0)-\eta(\psi)}{\widetilde{\gamma}}\leq\frac{\zeta}{\widetilde{\gamma}}\psi,\quad P\otimes dt\text{-a.e.}. \tag{4.6}$$

The proof of the lemma is relegated to Appendix C, and hereafter we address the integrability and other properties of the Esscher densities.

Lemma 4.3. *For any $\psi\in\Psi$ and any $\zeta\in\{\gamma,\widetilde{\gamma}\}$, let $\eta(\psi)$ be the unique root to (4.3) and*

$$\overline{D}^{\psi}:=\mathcal{E}\Big(\eta(\psi)\sigma\bullet W+(\exp(\eta(\psi)\zeta+\zeta^2\psi)-1)\bullet\widetilde{N}\Big).$$

(a) *If (4.2) holds, then for any $\psi\in\Psi$ we have $\overline{D}^{\psi}\in\mathcal{M}(P)$, and*

$$R_\psi:=\overline{D}^{\psi}_T\cdot P\quad \text{is a well defined probability measure.}$$

In particular, for $\psi=0\in\Psi$, we get

$$\overline{D}^0\in\mathcal{M}\quad \text{and}\quad R_0:=\overline{D}^0_T\cdot P\quad \text{is a well defined probability measure.}$$

(b) *Suppose (4.2) holds. For $\psi \in \Psi$, we have $(W^{\psi}, \widetilde{N}^{\psi}) \in \mathcal{M}_{loc}(R_{\psi}) \times \mathcal{M}_{loc}(R_{\psi})$, where*

$$W^{\psi} := W - \int_0^{\cdot} \eta_s(\psi)\sigma_s ds \quad \text{and} \quad \widetilde{N}^{\psi} := N - \int_0^{\cdot} \lambda_s \exp(\eta_s(\psi)\zeta_s + \zeta_s^2\psi)ds.$$

(c) *Suppose (4.2) holds. Then, for any $(\psi, \psi') \in \Psi \times \Psi$, we have*

$$\begin{aligned}
D^{(\psi,\psi')} &:= \frac{\overline{D}^{\psi}}{\overline{D}^{\psi'}} = \mathcal{E}\left(M^{(\psi,\psi')}\right), \ \text{where} \\
M^{(\psi,\psi')} &:= (\eta(\psi) - \eta(\psi'))\sigma \bullet W^{\psi'} \\
&\quad + \left(\exp((\eta(\psi) - \eta(\psi'))\zeta + \zeta^2(\psi - \psi')) - 1\right) \bullet \widetilde{N}^{\psi'},
\end{aligned} \tag{4.7}$$

and for any $p > 1$ and $n \in \mathbb{N}$, there exists $C_n \in (0, \infty)$ such that

$$|\eta(\psi)| + |\eta(\psi')| + E_{R_{\psi'}}\left[\left(\frac{D_T^{(\psi,\psi')}}{D_t^{(\psi,\psi')}}\right)^p \Big| \mathcal{F}_t\right] \leq C_n, \ P\text{-a.s.}, \tag{4.8}$$

for any $(\psi, \psi') \in \Psi_n \times \Psi_n$.

The proof of this lemma is relegated to Appendix C, and instead we proceed by defining the Esscher pricing bounds for suitably integrable claims. To this end, we consider $\mathcal{Q}(\zeta)$ and $\mathcal{Z}(\zeta)$, for $\zeta \in \{\gamma, \widetilde{\gamma}\}$, given by

$$\mathcal{Q}(\zeta) := \left\{R_{\psi} := D_T^{\psi} \cdot R_0 : \ D^{\psi} \in \mathcal{Z}(\zeta)\right\} \quad \text{and} \quad \mathcal{Z}(\zeta) := \left\{D^{\psi} := \frac{\overline{D}^{\psi}}{\overline{D}^{0}} : \ \psi \in \Psi\right\}.$$

Definition 4.4. Let ξ be the payoff of an arbitrary claim satisfying

$$E_{R_0}[|\xi|] \leq \sup_{\psi \in \Psi:\, R_{\psi} \in \mathcal{Q}(\zeta)} E_{R_{\psi}}[|\xi|] = \sup_{\psi \in \Psi:\, D^{\psi} \in \mathcal{Z}(\zeta)} E_{R_0}\left[D_T^{\psi}|\xi|\right] < \infty, \quad \zeta \in \{\gamma, \widetilde{\gamma}\}.$$

We denote by $]\!]Y^{(\text{inf,exp})}, Y^{(\text{up,exp})}[\![$ (respectively $]\!]Y^{(\text{inf,lin})}, Y^{(\text{up,lin})}[\![$) the exponential Esscher (respectively the linear Esscher) pricing stochastic interval for the claim ξ, where

$$\begin{aligned}
&Y^{(\text{inf,exp})} := \operatorname*{ess\,inf}_{\psi \in \Psi:\ R_{\psi} \in \mathcal{Q}(\gamma)} (Y^{\psi}), \quad Y^{(\text{up,exp})} := \operatorname*{ess\,sup}_{\psi \in \Psi:\ R_{\psi} \in \mathcal{Q}(\gamma)} (Y^{\psi}), \\
&Y^{(\text{inf,lin})} := \operatorname*{ess\,inf}_{\psi \in \Psi:\ R_{\psi} \in \mathcal{Q}(\widetilde{\gamma})} (Y^{\psi}), \quad Y^{(\text{up,lin})} := \operatorname*{ess\,sup}_{\psi \in \Psi:\ R_{\psi} \in \mathcal{Q}(\widetilde{\gamma})} (Y^{\psi}), \\
&Y_t^{\psi} := E_{R_0}\left[\frac{D_T^{\psi}}{D_t^{\psi}} e^{-\int_t^T r_s ds} \xi \Big| \mathcal{F}_t\right], \quad D^{\psi} \in \mathcal{Z}(\zeta), \quad \zeta \in \{\gamma, \widetilde{\gamma}\}.
\end{aligned} \tag{4.9}$$

Our main goal of this section resides in describing as explicit as possible the four processes $Y^{(\text{inf,exp})}$, $Y^{(\text{up,exp})}$, $Y^{(\text{inf,lin})}$, and $Y^{(\text{up,lin})}$ and then singling out their precise relationship as well. This can be achieved through BSDEs, which are a "*natural*" stochastic control tool for non-Markovian models with complex dynamics. The rest of this section is divided into three subsections. The first subsection gives some definitions and notation regarding BSDEs with constraints that we *naturally* encounter in our analysis. The second subsection is devoted to our main results of this section, while the last subsection proves these main results.

4.1. **Constrained BSDEs: Definitions and notation.** Throughout this subsection, we consider a probability measure Q on $(\Omega, \mathcal{F})$, and we denote by W^Q and $\widetilde{N}^Q$ the Q-Brownian motion and the compensated Poisson process under Q, respectively. For any $p \in (1, \infty)$, throughout the rest of the paper we consider the spaces $\mathbb{S}^p(Q)$, $\mathbb{L}^p(W, Q)$, $\mathbb{L}^p(N, Q)$, and $\mathcal{A}^p(Q)$ and their norms given below. For any unexplained notation, we refer to Section 2.

$$\begin{aligned}
&\mathbb{S}^p(Q) := \left\{ X \text{ RCLL and } \mathbb{F}\text{-adapted process}: \ \|X\|_{\mathbb{S}^p(Q)} < \infty \right\},\\
&\|X\|^p_{\mathbb{S}^p(Q)} := E_Q\Big[\sup_{0\le t\le T} |X_t|^p\Big],\\
&\mathbb{L}^p(N, Q) := \left\{ \varphi \in L(\widetilde{N}^Q, Q): \ \|\varphi\|_{\mathbb{L}^p(N,Q)} < \infty \right\},\\
&\|\varphi\|^p_{\mathbb{L}^p(N,Q)} := E_Q\Big[(|\varphi|^2 \bullet N)_T^{p/2}\Big],\\
&\mathbb{L}^p(W, Q) := \left\{ \varphi \in L(W^Q, Q): \ \|\varphi\|_{\mathbb{L}^p(W,Q)} < \infty \right\},\\
&\|\varphi\|^p_{\mathbb{L}^p(W,Q)} := E_Q\Big[\Big(\int_0^T |\varphi_t|^2 dt\Big)^{p/2}\Big],\\
&\mathcal{A}^p(Q) := \left\{ V \in \mathcal{A}_{loc}(Q): \ \|V\|_{\mathcal{A}^p(Q)} < \infty \right\}, \quad \|V\|^p_{\mathcal{A}^p(Q)} := E_Q\left[(\mathrm{Var}_T(V))^p\right].
\end{aligned}$$

Below we introduce *constrained* BSDEs for models with jumps.

Definition 4.5. Let $\xi \in L^p(Q)$ and f and Φ be $\mathbb{F}$-optional functionals in the variables (t, ω, y, z, u) and Lipschitz in $(y, z, u) \in \mathbb{R}^3$ such that $\Phi(t, \omega, y, z, u) \ge 0$ $P \otimes dt$ for almost every (ω, t) and for any (y, z, u).
(a) We call (Y, Z, U, K) an $L^p(Q)$-solution to the constrained BSDE(f, ξ, Φ) (CBSDE(f, ξ, Φ) hereafter for short) if the following properties hold: (Y, Z, U, K) belongs to $\mathbb{S}^p(Q) \times \mathbb{L}^p(W, Q) \times \mathbb{L}^p(N, Q) \times \mathcal{A}^p(Q)$, $K \in \mathcal{V}^+$ and is predictable,

$$\begin{aligned}
&Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s)ds - \int_t^T Z_s dW^Q_s - \int_t^T U_s d\widetilde{N}^Q_s + \int_t^T K_s ds, \quad Q\text{-a.s.},\\
&\text{and} \quad \Phi(t, Y_t, Z_t, U_t) = 0, \quad Q \otimes dt\text{-a.e.}.
\end{aligned} \tag{4.10}$$

(b) $(\widehat{Y}, \widehat{Z}, \widehat{U}, \widehat{K})$ is called the smallest $L^p(Q)$-solution to the CBSDE(f, ξ, Φ), i.e. (4.10), if it is a solution to this CBSDE(f, ξ, Φ), and $\widehat{Y} \le Y$ for any other solution (Y, Z, U, K) to (4.10).

It is clear, from this definition, that a reflected BSDE is a particular case of a constrained BSDE, where the constraint factor Φ depends on the value process Y only (i.e. $\Phi(t, y, z, u) = \Phi(t, y)$). In this case, the constraint naturally generates the Skorokhod condition $\int_0^T \Phi(s, Y_s) dK_s = 0$ Q-a.s..

To our knowledge, CBSDEs appeared for the first time in [16], for the Brownian setting only. The obtained CBSDE, in these papers, was motivated by the problem of super-replication when there are constraints on the portfolio. Hence, the constraints in the original problem of super-replication naturally translate into constraints on the solution of the resulting BSDE. The main challenge that was noticed by the early days of the CBSDEs lies in the existence of a solution to those CBSDEs. In fact, in [16], the authors gave a counter-example showing that the existence might fail in general. Thus, still in the Brownian setting and *under the assumption that the CBSDE under consideration indeed has a solution*, Peng proved in [37] that the smallest solution exists. Many extensions were attempted afterwards, for these we refer the reader to [20, 32, 38, 42] and the references therein to cite a few. However, in all these extensions, the assumption that the constrained BSDE should admit a solution persists and was not overcome at all, and hence in our setting these are not applicable.

4.2. **Main results on Esscher pricing bounds.** This subsection states our main results on the bounds for the Esscher pricing intervals. In particular, we prove that the upper and lower bounds of the Esscher pricing interval are solutions to constrained BSDEs, with different constraints, but having the same driver which is linear in the solution's variables (y, z, u). Below we state our main theorem on the upper bound for the Esscher price processes.

Theorem 4.6. *Suppose that (4.2) holds, and $\mathcal{T}$ denotes the set of all stopping times τ such that $\tau \leq T$ P-a.s.. Let $p \in (1, \infty)$ and $\xi \in L^0(P)$, satisfying*

$$E_{R_0}\left[(\xi^-)^p\right] + \sup_{\tau\in\mathcal{T},\psi\in\Psi} E_{R_0}\left[\frac{D_T^\psi}{D_\tau^\psi}(\xi^+)^p\right] < \infty. \tag{4.11}$$

For $\psi \in \Psi$, let $(Y^{(up,exp)}, Y^{(up,lin)}, Y^\psi)$ be defined in (4.9), and $Y^{(n)}$ be given by

$$Y^{(n)} := \operatorname*{ess\,sup}_{\psi\in\Psi_n}(Y^\psi), \quad \text{for any} \quad n \in \mathbb{N}.$$

If $Y^{(up)} \in \{Y^{(up,exp)}, Y^{(up,lin)}\}$, then the following assertions hold.
(a) *There exists a triplet $(Z^{(up)}, U^{(up)}, K^{(up)})$ such that $(Y^{(up)}, Z^{(up)}, U^{(up)}, K^{(up)})$ belongs to $\mathbb{S}^p(R_0) \times \mathbb{L}^p(W, R_0) \times \mathbb{L}^p(N, R_0) \times \mathcal{A}^{+,p}(R_0)$ and is the smallest $L^p(R_0)$-solution to*

$$\begin{aligned}
&dY_s = \left(\frac{\widetilde{b}_s - r_s - \widetilde{\gamma}_s\lambda_s}{\sigma_s}Z_s + \lambda_s U_s + r_s Y_s\right)ds + Z_s dW_s + U_s d\widetilde{N}_s - dK_s,\\
&Y_T = \xi, \quad P-a.s.,\\
&\widetilde{\gamma}_s Z_s - \sigma_s U_s \geq 0, \quad P\otimes ds\text{-}a.e., \quad \int_0^T (\widetilde{\gamma}_s Z_s - \sigma_s U_s)dK_s = 0, \;\; P-a.s..
\end{aligned} \tag{4.12}$$

(b) *There exists a unique pair $(Z^{(n)}, U^{(n)}) \in \mathbb{L}^p(W, R_0) \times \mathbb{L}^p(N, R_0)$ such that the triplet $(Y^{(n)}, Z^{(n)}, U^{(n)})$ belongs to $\mathbb{S}^p(R_0)\times\mathbb{L}^p(W, R_0)\times\mathbb{L}^p(N, R_0)$ and is the unique $L^p(R_0)$-solution to the following BSDE:*

$$\begin{aligned}
Y_t = \xi + \int_t^T &\left(\frac{\eta_s(-n) - \eta_s(0)}{\widetilde{\gamma}_s}\sigma_s(\widetilde{\gamma}_s Z_s - \sigma_s U_s)^+\right.\\
&\left.+\frac{\eta_s(0) - \eta_s(n)}{\widetilde{\gamma}_s}\sigma_s(\widetilde{\gamma}_s Z_s - \sigma_s U_s)^- - r_s Y_s\right)ds\\
&- \int_t^T Z_s dW_s^0 - \int_t^T U_s d\widetilde{N}_s^0.
\end{aligned} \tag{4.13}$$

Furthermore, there exists $\psi_n \in \Psi_n$ such that $(Y^{(n)}, Z^{(n)}, U^{(n)}) = (Y^{\psi_n}, Z^{\psi_n}, U^{\psi_n})$.
(c) *For any $n, m \in \mathbb{N}$, $Y^{(n+m)} - Y^{(n)}$ is a nonnegative R_0-supermartingale,*

$$Y^{(n)} \leq Y^{(n+1)} \leq Y^{(up)}, \quad \text{and} \quad Y^{(n)} \text{ converges pointwise to } Y^{(up)}. \tag{4.14}$$

(d) *The triplet* $(Y^{(n)}, Z^{(n)}, U^{(n)})$ *converges to the triplet* $(Y^{(up)}, Z^{(up)}, U^{(up)})$ *in the space* $\mathbb{S}^p(R_0) \times \mathbb{L}^p(W^0, R_0) \times \mathbb{L}^p(\widetilde{N}^0, R_0)$. *Furthermore, the nondecreasing process* $K^{(n)} := \int_0^{\cdot} (\eta_s(0) - \eta_s(n))\widetilde{\gamma}_s^{-1}\sigma_s(\widetilde{\gamma}_s Z_s^{(n)} - \sigma_s U_s^{(n)})^- ds$ *converges to* $K^{(up)}$ *in the space* $\mathcal{A}^{+,p}(R_0)$. *As a result, there exists a nonnegative and predictable process* $k^{(up)}$ *such that* $K^{(up)} = \int_0^{\cdot} k_s^{(up)} ds$.

The proof of the theorem is relegated to Subsection 4.3. Theorem 4.6-(a) completely characterizes the upper bound $Y^{(up)}$ as the smallest solution to a *constrained* BSDE with Skorokhod condition. This latter condition makes our resulting constrained BSDE novel among the literature and is unique in its kind, as the constraint of the BSDE does not involve the value process $Y^{(up)}$ at all. Assertion (b) proves that the "truncated" process $Y^{(n)}$ is the unique solution of a BSDE, and this gives us the "*natural*" monotone sequence of BSDEs that converges to the constrained BSDE (4.12). Further, in contrast to the existing literature, we prove that the predictable with finite variation part of the solution to the BSDEs in (4.13) (i.e. the process $K^{(n)}$) converges in the $\mathcal{A}^p$-norm.

Remark 4.7. The discounted upper Esscher price process $\widetilde{Y}^{(up)} := e^{-B} Y^{(up)}$ is a nonnegative R_0-supermartingale, and satisfies

$$\widetilde{Y}_t^{(up)} = E_{R_0}\left[\xi e^{-B_T} + \int_t^T \frac{\eta_s^* - \eta_s(0)}{\widetilde{\gamma}_s}\sigma_s(\widetilde{\gamma}_s \widetilde{Z}_s^{(up)} - \sigma_s \widetilde{U}_s^{(up)})ds + \widetilde{K}_T^{(up)} - \widetilde{K}_t^{(up)} \Big| \mathcal{F}_t\right]$$

$$\widetilde{Y}_t^{(up)} = \xi e^{-B_T} + \int_t^T \frac{\eta_s^* - \eta_s(0)}{\widetilde{\gamma}_s}\sigma_s(\widetilde{\gamma}_s \widetilde{Z}_s^{(up)} - \sigma_s \widetilde{U}_s^{(up)})ds - \int_t^T \widetilde{Z}_s^{(up)} dW_s^0 - \int_t^T \widetilde{U}_s^{(up)} d\widetilde{N}_s^0 + \widetilde{K}_T^{(up)} - \widetilde{K}_t^{(up)},$$

where $\widetilde{Z}^{(up)} := e^{-B} Z^{(up)}$, $\widetilde{U}^{(up)} = e^{-B_T} U^{(up)}$, and $\widetilde{K}^{(up)} := e^{-B} \bullet K^{(up)}$.

Below, we address the lower Esscher bound process, and we prove that it is also fully characterized by a constrained BSDE. Both CBSDEs (i.e. CBSDEs for upper bound and lower bound) have the same driver, but different constraints and different martingale structures naturally.

Theorem 4.8. *Suppose (4.2) holds, and let* $p \in (1, \infty)$ *and* $\xi \in L^0(P)$ *satisfying*

$$E_{R_0}\left[(\xi^+)^p\right] + \sup_{\tau \in \mathcal{T}, \psi \in \Psi} E_{R_0}\left[\frac{D_T^\psi}{D_\tau^\psi}(\xi^-)^p\right] < \infty.$$

For $\psi \in \Psi$, *let* $(Y^{(inf,exp)}, Y^{(inf,lin)}, Y^\psi)$ *be defined in (4.9), and* $\overline{Y}^{(n)}$ *be given by*

$$\overline{Y}^{(n)} := \operatorname*{ess\,inf}_{\psi \in \Psi_n}(Y^\psi), \quad \textit{for any} \quad n \in \mathbb{N}.$$

If $Y^{(inf)} \in \{Y^{(inf,lin)}, Y^{(inf,exp)}\}$, *then the following assertions hold.*
(a) *There exists* $(Z^{(inf)}, U^{(inf)}, K^{(inf)})$ *such that* $(Y^{(inf)}, Z^{(inf)}, U^{(inf)}, K^{(inf)})$ *belongs to* $\mathbb{S}^p(R_0) \times \mathbb{L}^p(W, R_0) \times \mathbb{L}^p(N, R_0) \times \mathcal{A}^{+,p}(R_0)$ *and is the smallest solution to the constrained BSDE*

$$\begin{aligned}
&dY_s = \left(\frac{\widetilde{b}_s - r_s - \widetilde{\gamma}_s\lambda_s}{\sigma_s} Z_s + \lambda_s U_s + r_s Y_s\right) ds + Z_s dW_s + U_s d\widetilde{N}_s + dK_s,\\
&Y_T = \xi, \quad P-a.s.,\\
&\widetilde{\gamma}_s Z_s - \sigma_s U_s \le 0, \quad P \otimes ds\text{-}a.e., \qquad \int_0^T (\widetilde{\gamma}_s Z_s - \sigma_s U_s) dK_s = 0, \ P-a.s..
\end{aligned}$$

(b) *There exists a unique pair* $(\overline{Z}^{(n)}, \overline{U}^{(n)})$ *such that* $(\overline{Y}^{(n)}, \overline{Z}^{(n)}, \overline{U}^{(n)})$ *belongs to* $\mathbb{S}^p(R_0) \times \mathbb{L}^p(W, R_0) \times \mathbb{L}^p(N, R_0)$*, and this triplet is the unique solution to the following BSDE:*

$$\begin{aligned}Y_t = \xi - \int_t^T \Big(&\frac{\eta_s(0) - \eta_s(n)}{\widetilde{\gamma}_s}\sigma_s(\widetilde{\gamma}_s Z_s - \sigma_s U_s)^+ \\ &+ \frac{\eta_s(-n) - \eta_s(0)}{\widetilde{\gamma}_s}\sigma_s(\widetilde{\gamma}_s Z_s - \sigma_s U_s)^- - r_s Y_s\Big) ds \\ &- \int_t^T Z_s dW_s^0 - \int_t^T U_s d\widetilde{N}_s^0.\end{aligned}$$

(c) *For any* $n, m \in \mathbb{N}$, $\overline{Y}^{(n)} - \overline{Y}^{(n+m)}$ *is a nonnegative* R_0*-submartingale,*

$$\overline{Y}^{(n)} \geq \overline{Y}^{(n+1)} \geq Y^{(inf)} \quad \textit{and} \quad \overline{Y}^{(n)} \textit{ converges pointwise to } Y^{(inf)}. \tag{4.15}$$

(d) *The triplet* $(\overline{Y}^{(n)}, \overline{Z}^{(n)}, \overline{U}^{(n)})$ *converges to the triplet* $(Y^{(inf)}, Z^{(inf)}, U^{(inf)})$ *in the space* $\mathbb{S}^p(R_0) \times \mathbb{L}^p(W^0, R_0) \times \mathbb{L}^p(\widetilde{N}^0, R_0)$*. Furthermore, the nondecreasing process* $\overline{K}^{(n)} := \int_0^{\cdot} (\eta_s(0) - \eta_s(n))\, \widetilde{\gamma}_s^{-1}\sigma_s(\widetilde{\gamma}_s\overline{Z}_s^{(n)} - \sigma_s\overline{U}_s^{(n)})^+ ds$ *converges to* $K^{(inf)}$ *in* $\mathcal{A}^{+,p}(R_0)$*-norm. As a result, there exists a predictable process* $k^{(inf)}$ *such that* $K^{(inf)} := \int_0^{\cdot} k_s^{(inf)} ds \in \mathcal{A}^{+,p}(R_0)$*.*

Proof. Let ξ be a claim and $\psi \in \Psi$, and we denote by $Y^\psi(\xi)$ and $Y^\psi(-\xi)$ the process defined in (4.9) associated with the claims ξ and $-\xi$, respectively. On the one hand, we denote the unique solution to (4.17) for ξ (respectively $-\xi$), see Lemma 4.10 below, by $(Y^\psi(\xi), Z^\psi(\xi), U^\psi(\xi))$ (respectively $(Y^\psi(-\xi), Z^\psi(-\xi), U^\psi(-\xi))$), and deduce that

$$(Y^\psi(-\xi), Z^\psi(-\xi), U^\psi(-\xi)) = -(Y^\psi(\xi), Z^\psi(\xi), U^\psi(\xi)).$$

On the other hand, due to the simple fact that $\operatorname*{ess\,inf}_{i\in I}(X_i) = -\operatorname*{ess\,sup}_{i\in I}(-X_i)$ for any family of random variables $(X_i)_{i\in I}$, we deduce that

$$Y^{(inf)}(\xi) = \operatorname*{ess\,inf}_{\psi\in\Psi}(Y^\psi(\xi)) = -\operatorname*{ess\,sup}_{\psi\in\Psi}(Y^\psi(-\xi)) = -Y^{(up)}(-\xi).$$

Therefore, by applying Theorem 4.6 to the claim $-\xi$ instead and using these remarks afterwards, all assertions of the theorem follow immediately, and the proof of the theorem is complete. □

We end this subsection with the following remark.

Remark 4.9. (a) For any claim with payoff ξ and any $p \in (1, \infty)$, the condition that we require for both Esscher pricing bounds is

$$E_{R_0}\left[|\xi|^p\right] \leq \sup_{\tau\in\mathcal{T},\psi\in\Psi} E_{R_0}\left[\frac{D_T^\psi}{D_\tau^\psi}|\xi|^p\right] < \infty. \tag{4.16}$$

It is clear that any bounded claim ξ satisfies this assumption.
(b) For any $\zeta \in \{\gamma, \widetilde{\gamma}\}$ and any ξ fulfilling (4.16), the processes $Y^{(up)}$ and $Y^{(inf)}$ are R_0-supermartingale and R_0-submartingale respectively such that

$$Y^{(inf)} \leq Y^{(\psi)} \leq Y^{(up)}, \quad \forall\, \psi \in \Psi.$$

Furthermore, there exist $(\theta^{up},\theta^{inf})\in L(S)\times L(S)$ and a unique pair $(L^{(up)},-L^{(inf)})$ of strict R_0-supermartingales such that the following hold:

$$Y^{(inf)}=Y_0^{(inf)}+\theta^{inf}\bullet S-K^{(inf)}+L^{(inf)},$$
$$Y^{(up)}=Y_0^{(up)}+\theta^{up}\bullet S-K^{(up)}-L^{(up)},$$
$$\Big\{Q\text{ equivalent probability measure}:\ L^{(up)}\in\mathcal{M}_{loc}(Q)\text{ or }L^{(inf)}\in\mathcal{M}_{loc}(Q)\Big\}=\emptyset.$$

4.3. **Proof of Theorem 4.6.** This subsection proves Theorem 4.6. To this end, we start with intermediate results that we summarize in three lemmas, which are interesting in and of themselves besides conveying the main ideas of our results. The first lemma characterizes the process Y^{ψ}, defined in (4.9), in a unique manner by a BSDE, and elaborates some of its properties that will be useful throughout the rest of the paper.

Lemma 4.10. *Suppose that (4.2) holds, and let $p\in(1,\infty)$ and $\xi\in L^0(P)$ satisfying (4.11). For any $\psi\in\Psi$, $\eta(\psi)$ denotes the unique root to (4.3), and Y^{ψ} is defined in (4.9). Then, the following hold.*
(a) *For any $\psi\in\Psi$, there exists a unique pair $(Z^{\psi},U^{\psi})\in\mathbb{L}^p(W,R_0)\times\mathbb{L}^p(N,R_0)$ such that the triplet $(Y^{\psi},Z^{\psi},U^{\psi})$ belongs to $\mathbb{S}^p(R_0)\times\mathbb{L}^p(W,R_0)\times\mathbb{L}^p(N,R_0)$ and is the unique solution to*

$$\begin{aligned}Y_t&=\xi+\int_t^T f_{\psi}(s,Y_s,Z_s,U_s)ds-\int_t^T Z_sdW_s-\int_t^T U_sd\widetilde{N}_s,\\&=\xi+\int_t^T h_{\psi}(s,Y_s,Z_s,U_s)ds-\int_t^T Z_sdW^0_s-\int_t^T U_sd\widetilde{N}^0_s.\end{aligned}\tag{4.17}$$

Here, the functionals f_{ψ} and h_{ψ} are given for any $t\in[0,T]$ and $(y,z,u)\in\mathbb{R}^3$ by

$$\begin{aligned}f_{\psi}(t,y,z,u)&:=\frac{\eta_t(\psi)\sigma_t}{\widetilde{\gamma}_t}(\widetilde{\gamma}_tz-\sigma_tu)-r_ty-\big(\frac{\widetilde{b}_t-r_t}{\widetilde{\gamma}_t}\big)u,\\h_{\psi}(t,y,z,u)&:=\frac{\sigma_t}{\widetilde{\gamma}_t}\big(\eta_t(\psi)-\eta_s(0)\big)\left(\widetilde{\gamma}_tz-\sigma_tu\right)-r_ty.\end{aligned}\tag{4.18}$$

Furthermore, for any $n\geq1$, there exists $C_n\in(0,\infty)$ such that for any $\psi\in\Psi_n$,

$$\begin{aligned}&\Vert Y^{\psi}\Vert_{\mathbb{S}^p(R_0)}+\Vert Z^{\psi}\Vert_{\mathbb{L}^p(W,R_0)}+\Vert U^{\psi}\Vert_{\mathbb{L}^p(N,R_0)}+\Vert\int_0^T|h_{\psi}(s,Y^{\psi}_s,Z^{\psi}_s,U^{\psi}_s)|ds\Vert_{L^p(R_0)}\\&\leq C_n\Vert\xi\Vert_{L^p(R_0)}.\end{aligned}\tag{4.19}$$

(b) *Let B be defined in (4.1), $\psi\in\Psi$, and $(Y^{\psi},Z^{\psi},U^{\psi})$ be the solution to (4.17). If we denote*

$$g_{\psi}:=\frac{\eta(\psi)-\eta(0)}{\widetilde{\gamma}}\sigma(\widetilde{\gamma}Z^{\psi}-\sigma U^{\psi}),\tag{4.20}$$

then we have

$$e^{-B_t}Y^{\psi}_t=E_{R_0}\left[e^{-B_T}\xi+\int_t^T e^{-B_s}g_{\psi}(s)ds\big|\mathcal{F}_t\right],\quad t\geq0.\tag{4.21}$$

(c) *Let $\psi_i\in\Psi$, $i=1,2$, and Γ be a predictable set. If $\psi_3:=\psi_1I_{\Gamma}+\psi_2I_{\Gamma^c}$ ($\Gamma^c:=\Omega\times[0,\infty)\setminus\Gamma$), then*

$$\begin{aligned}&d(e^{-B}Y^{\psi_3})=I_{\Gamma}d(e^{-B}Y^{\psi_1})+I_{\Gamma^c}d(e^{-B}Y^{\psi_2}),\\&Z^{\psi_3}=Z^{\psi_1}I_{\Gamma}+Z^{\psi_2}I_{\Gamma^c},\quad\text{and}\quad U^{\psi_3}=U^{\psi_1}I_{\Gamma}+U^{\psi_2}I_{\Gamma^c}.\end{aligned}\tag{4.22}$$

(d) *For any $\psi_i \in \Psi$, $i = 1, 2$, there exists $\psi_3 \in \Psi$ such that*

$$|\psi_3| \leq \max\left(|\psi_1|, |\psi_2|\right), \quad Y^{\psi_3} \geq \max\left(Y^{\psi_1}, Y^{\psi_2}\right), \quad g_{\psi_3} = \max\left(g_{\psi_1}, g_{\psi_2}\right). \tag{4.23}$$

As a result $\{Y^{\psi} : \ \psi \in \Psi\}$, $\{g_{\psi} : \ \psi \in \Psi\}$, $\{Y^{\psi} : \ \psi \in \Psi_n\}$, and $\{g_{\psi} : \ \psi \in \Psi_n\}$ are all upward directed.

The proof of the lemma is relegated to Appendix C. Below, we address the process g_{ψ} (see (4.20)), which is associated with the BSDE (4.17), and we show that it can be controlled uniformly when ψ spans Ψ_n.

Lemma 4.11. *Suppose (4.2) and (4.11) hold. For any $(n, \psi) \in \mathbb{N} \times \Psi$, we consider g_{ψ} defined in (4.20) and $(\widetilde{g}_n, g_n, \widetilde{g})$ given by*

$$\widetilde{g}_n(t) := \operatorname*{ess\,sup}_{\psi \in \Psi_n}(g_{\psi}(t)) \geq 0, \quad \widetilde{g}(t) := \sup_{n \geq 0} \widetilde{g}_n(t) = \operatorname*{ess\,sup}_{\psi \in \Psi}(g_{\psi}(t)), \quad t \geq 0,$$

$$g_n(t) := \frac{\eta_t(-n) - \eta_t(0)}{\widetilde{\gamma}_t}\sigma_t(\widetilde{\gamma}_t Z_t^{(n)} - \sigma_t U_t^{(n)})^+ + \frac{\eta_t(0) - \eta_t(n)}{\widetilde{\gamma}_t}\sigma_t(\widetilde{\gamma}_t Z_t^{(n)} - \sigma_t U_t^{(n)})^-, \tag{4.24}$$

where $(Y^{(n)}, Z^{(n)}, U^{(n)})$ constitutes the solution to (4.13). Then, the following assertions hold.

(a) *Let $\psi \in \Psi$, and consider $\psi_n := -nI_{\Gamma_n} + nI_{\Gamma_n^c}$, where $\Gamma_n := \{\widetilde{\gamma} Z^{(n)} - \sigma U^{(n)} \geq 0\}$ and $\Gamma_n^c := \{\widetilde{\gamma} Z^{(n)} - \sigma U^{(n)} < 0\}$ for $n \in \mathbb{N}$. Then, we have*

$$e^{-B_t} Y_t^{\psi} = E_{R_0}\left[\xi e^{-B_T} + \int_t^T e^{-B_s} g_{\psi}(s) ds \big| \mathcal{F}_t\right],$$

$$e^{-B_t} Y_t^{(n)} = E_{R_0}\left[\xi e^{-B_T} + \int_t^T e^{-B_s} g_n(s) ds \big| \mathcal{F}_t\right] = e^{-B_t} Y_t^{\psi_n}.$$

(b) *For any $n \in \mathbb{N}$, we have*

$$E_{R_0}\left[\int_0^T \widetilde{g}_n(t) dt\right]$$
$$\leq E_{R_0}\left[\int_0^T \operatorname*{ess\,sup}_{\psi \in \Psi_n} |g_{\psi}(t)| dt\right] = \sup_{\psi \in \Psi_n} E_{R_0}\left[\int_0^T |g_{\psi}(t)| dt\right] =: \widetilde{C}_n < \infty.$$

(c) *For $n \in \mathbb{N}$, $\widetilde{g}_n = g_n$, $P \otimes dt$-a.e., and hence $(g_n)_n$ increases to $\widetilde{g}$, $P \otimes dt$-a.e..*

This lemma is proved in Appendix C, and the following lemma is our last main technical step for the proof of Theorem 4.6, and it elaborates some inequalities for the process $\widetilde{g}$.

Lemma 4.12. *Suppose that $r \equiv 0$, and consider $\widetilde{g}$ defined in (4.24). Then, the following hold.*

(a) *For any stopping time τ, we have*

$$E_{R_0}\left[\int_{\tau}^T \widetilde{g}(u) du \big| \mathcal{F}_{\tau}\right] \leq E_{R_0}\left[\Delta + \xi^- \big| \mathcal{F}_{\tau}\right],$$

$$\text{where} \quad \Delta := \operatorname*{ess\,sup}_{\psi \in \Psi} \sup_{0 \leq t \leq T} E_{R_0}[D_T^{\psi}(D_t^{\psi})^{-1}\xi^+ \big| \mathcal{F}_t].$$

(b) *For $p \in (1,\infty)$, there exists a universal constant $C_p \in (0,\infty)$ (i.e. it depends on p only), such that*

$$E_{R_0}\left[\left(\int_0^T \widetilde{g}(u)du\right)^p\right] \leq C_p E_{R_0}\left[(\xi^-)^p\right] + C_p \sup_{\tau\in\mathcal{T},\psi\in\Psi} E_{R_0}\left[\frac{D_T^\psi}{D_\tau^\psi}(\xi^+)^p\right].$$

The proof of this lemma is relegated to Appendix C, and instead we now prove Theorem 4.6.

Proof of Theorem 4.6. Remark that there is no loss of generality in assuming that $r\equiv 0$, which will be enforced throughout this proof. The proof of the theorem is divided into three parts. The first part proves assertions (b) and (c). The second part proves assertion (d) and the last two conditions in (4.12), while the third part addresses assertion (a).

Part 1. We now prove assertion (b). To this end, for $s\in[0,T]$, $(y,z,u)\in\mathbb{R}^3$ and $n\in\mathbb{N}$, we set

$$\kappa_n(s,y,z,u) := \frac{\eta_s(-n)-\eta_s(0)}{\widetilde{\gamma}_s}\sigma_s(\widetilde{\gamma}_s z-\sigma_s u)^+ + \frac{\eta_s(0)-\eta_s(n)}{\widetilde{\gamma}_s}\sigma_s(\widetilde{\gamma}_s z-\sigma_s u)^-.$$

It is clear that κ_n is the driver of BSDE (4.13). By virtue of (4.2) and Lemma 4.3, there exists a constant $C_n\in(0,\infty)$ such that

$$\begin{aligned}
&|\kappa_n(t,y_1,z_1,u_1)-\kappa_n(t,y_2,z_2,u_2)|\\
&\leq C_n|(\widetilde{\gamma}_s z_1-\sigma_s u_1)^+ - (\widetilde{\gamma}_s z_2-\sigma_s u_2)^+| + C_n|(\widetilde{\gamma}_s z_1-\sigma_s u_1)^- - (\widetilde{\gamma}_s z_2-\sigma_s u_2)^-|\\
&\leq 2C_n^2\left(|z_1-z_2|+|u_1-u_2|+|y_1-y_2|\right).
\end{aligned}$$

This proves that the driver κ_n is Lipschitz. Hence, we deduce that there exists a unique solution to BSDE (4.13), denoted by $(\widetilde{Y}^{(n)},\widetilde{Z}^{(n)},\widetilde{U}^{(n)})$ and belonging to the space $\mathbb{S}^p(R_0)\times\mathbb{L}^p(W,R_0)\times\mathbb{L}^p(N,R_0)$. Thus, assertion (b) will follow immediately as soon as we prove that

$$Y^{(n)}=\widetilde{Y}^{(n)},\ \text{and there exists } \psi_n\in\Psi_n \text{ such that }\ \widetilde{Y}^{(n)}=Y^{\psi_n}. \tag{4.25}$$

To this end, we denote $\widetilde{\Omega}:=\Omega\times[0,\infty)$, and set

$$\eta^{*,n} := \frac{\eta_s(-n)-\eta_s(0)}{\widetilde{\gamma}_s}I_{\Gamma_n^+} + \frac{\eta_s(n)-\eta_s(0)}{\widetilde{\gamma}_s}I_{\widetilde{\Omega}\setminus\Gamma_n^+},\quad \Gamma_n^+ := \left\{\widetilde{\gamma}\widetilde{Z}^{(n)}-\sigma\widetilde{U}^{(n)}\geq 0\right\}.$$

Thus, by virtue of equation (4.4), direct calculations show that

$$\eta^{*,n} = \frac{\eta_s(\psi_n)-\eta_s(0)}{\widetilde{\gamma}_s} = \eta(\psi_n),\ \text{where } \psi_n := -nI_{\Gamma_n^+}+nI_{\widetilde{\Omega}\setminus\Gamma_n^+},\ \psi_n\in\Psi_n, \tag{4.26}$$

and hence we obtain $\kappa_n(s,\widetilde{Y}_s^{(n)},\widetilde{Z}_s^{(n)},\widetilde{U}_s^{(n)}) = h_{\psi_n}(s,\widetilde{Y}_s^{(n)},\widetilde{Z}_s^{(n)},\widetilde{U}_s^{(n)})$ with h_{ψ_n} defined in (4.18). This implies that $(\widetilde{Y}^{(n)},\widetilde{Z}^{(n)},\widetilde{U}^{(n)})$ is a solution to the BSDE (4.17) when $\psi=\psi_n$. Therefore, the uniqueness of the solution of this latter BSDE is guaranteed by Lemma 4.10, this yields $\widetilde{Y}^{(n)}=Y^{\psi_n}$, and the second equality in (4.25) is proved. As a result of the latter equality, we get $\widetilde{Y}^{(n)}=Y^{\psi_n}\leq Y^{(n)}$, and the rest of this part will be focused on proving the reverse inequality. To this end, we emphasize the following two remarks:

1) Due to Lemma 4.2, for $t\in[0,T]$, $(y,z,u)\in\mathbb{R}^3$, and $\psi\in\Psi_n$, we have

$$\begin{aligned}h_\psi(t,y,z,u)&=\frac{\sigma_t}{\widetilde{\gamma}_t}\big(\eta_t(\psi)-\eta_s(0)\big)\,(\widetilde{\gamma}_t z-\sigma_t u)^+-\frac{\sigma_t}{\widetilde{\gamma}_t}\big(\eta_t(\psi)-\eta_s(0)\big)\,(\widetilde{\gamma}_t z-\sigma_t u)^-\\&\le\frac{\sigma_t}{\widetilde{\gamma}_t}\big(\eta_t(-n)-\eta_s(0)\big)\,(\widetilde{\gamma}_t z-\sigma_t u)^+-\frac{\sigma_t}{\widetilde{\gamma}_t}\big(\eta_t(n)-\eta_s(0)\big)\,(\widetilde{\gamma}_t z-\sigma_t u)^-\\&=\kappa_n(t,y,z,u).\end{aligned}$$

2) Due to Lemma 4.3, for any $\psi\in\Psi_n$, we have

$$\begin{aligned}&D^\psi\in\mathcal{M}(R_0),\quad W^\psi=W^0-\int_0^\cdot\sigma_s(\eta_s(\psi)-\eta_s(0))ds\\&\text{and}\quad \widetilde{N}^\psi=\widetilde{N}^0+\int_0^\cdot\frac{\sigma_s^2}{\widetilde{\gamma}_s}(\eta_s(\psi)-\eta_s(0))ds.\end{aligned}$$

Thanks to these two remarks, on the other hand, we derive

$$\begin{aligned}&\widetilde{Y}^{(n)}_t-Y^\psi_t\\&=\int_t^T\Big(\kappa_n(s,\widetilde{Z}^{(n)},\widetilde{U}^{(n)})-h_\psi(t,Z^\psi_s,U^\psi_s)\Big)\,ds-\int_t^T(\widetilde{Z}^{(n)}_s-Z^\psi_s)dW^0_s\\&\quad-\int_t^T(\widetilde{U}^{(n)}_s-U^\psi_s)d\widetilde{N}^0_s\\&\ge\int_t^T\Big(h_\psi(s,\widetilde{Z}^{(n)}_s,\widetilde{U}^{(n)}_s)-h_\psi(t,Z^\psi_s,U^\psi_s)\Big)\,ds-\int_t^T(\widetilde{Z}^{(n)}_s-Z^\psi_s)dW^0_s\\&\quad-\int_t^T(\widetilde{U}^{(n)}_s-U^\psi_s)d\widetilde{N}^0_s\\&=-\int_t^T(\widetilde{Z}^{(n)}_s-Z^\psi_s)dW^\psi_s-\int_t^T(\widetilde{U}^{(n)}_s-U^\psi_s)d\widetilde{N}^\psi_s.\end{aligned}\tag{4.27}$$

On the other hand, it is clear that $Y^\psi=Y^\psi_0+Z^\psi\bullet W^\psi+U^\psi\bullet\widetilde{N}^\psi\in\mathcal{M}^p(R_\psi)$ for any $\psi\in\Psi$, and

$$\begin{aligned}\widetilde{Y}^{(n)}_t&=\xi-\int_t^T\widetilde{Z}^{(n)}_s dW^\psi_s-\int_t^T\widetilde{U}^{(n)}_s d\widetilde{N}^\psi_s\\&\quad-\int_t^T\frac{\eta_s(\psi)-\eta_s(\psi_n)}{\widetilde{\gamma}_s}\sigma(\widetilde{\gamma}_s Z^{(n)}_s-\sigma_s U^{(n)}_s)ds,\\&\frac{\eta_s(\psi)-\eta_s(\psi_n)}{\widetilde{\gamma}_s}\sigma(\widetilde{\gamma}_s Z^{(n)}_s-\sigma_s U^{(n)}_s)\ge0,\quad P\otimes dt-a.e.,\end{aligned}$$

and the latter proves that $\widetilde{Y}^{(n)}$ is a local R_ψ-submartingale. Hence, by combining this with [41, Lemma 1.9] and using (4.8) afterwards, we deduce the existence of constants C_n and C'_n such that

$$\begin{aligned}\|Z^{(n)}\bullet W^\psi+U^{(n)}\bullet\widetilde{N}^\psi\|_{\mathcal{M}^{p_1}(R_\psi)}&\le C_n\|\sup_{0\le t\le T}|Y^{\psi_n}_t|\|_{L^{p_1}(R_{\psi_n})}\\&\le C'_n\|\sup_{0\le t\le T}|Y^{\psi_n}_t|\|_{L^{p}(R_{\psi_n})}<\infty,\end{aligned}$$

for any $p_1\in(1,p)$. This proves that both $Z^{(n)}\bullet W^\psi$ and $U^{(n)}\bullet\widetilde{N}^\psi$ belong to $\mathcal{M}^{p_1}(R_\psi)$ for any $p_1\in(1,p)$, and hence $(\widetilde{Z}^{(n)}-Z^\psi)\bullet W^\psi-(\widetilde{U}^{(n)}-U^\psi)\bullet\widetilde{N}^\psi$ belongs to $\mathcal{M}^{p_1}(R_\psi)$. By combining this fact with (4.27) where we take the conditional expectation under

R_ψ, we get $\widetilde{Y}^{(n)} \geq Y^\psi$ for any $\psi \in \Psi_n$, and hence this yields $Y^{(n)} \leq \widetilde{Y}^{(n)}$. This ends the proof of (4.25), and the proof of assertion (b) is complete.

To prove assertion (c), we use the definition of the essential supremum, remark that $(Y^{(n)})_n$ is a nondecreasing sequence, and

$$Y^{(n)} \leq Y^{(n+1)} \leq Y^{(up)}, \quad \text{and} \quad \sup_n (Y^{(n)}) \leq Y^{(up)}. \tag{4.28}$$

Furthermore, thanks to Lemma 4.10-(d), there exists a sequence $(\psi_k)_k \subset \Psi$ such that

$$Y^{(up)} = \operatorname*{ess\,sup}_{\psi\in\Psi} Y^\psi = \lim_{k\longrightarrow\infty} Y^{\psi_k}.$$

Thus, as ψ_k is bounded, there exists $n_k \in \mathbb{N}$ such that $\psi_k \in \Psi_{n_k}$, and we deduce

$$Y^{(up)} = \lim_{k\longrightarrow\infty} Y^{\psi_k} \leq \sup_k Y^{(n_k)} \leq \sup_n Y^{(n)}.$$

A combination of this with (4.28) yields the convergence almost surely of $Y^{(n)}$ to $Y^{(up)}$. This proves (4.14). To prove the remaining claim of assertion (c), for any $n \geq 0$, we consider the unique solution to (4.13), denoted by $(Y^{(n)}, Z^{(n)}, U^{(n)})$, and thanks to Lemma 4.11-(a), for any $n, m \in \mathbb{N}$ we derive

$$\begin{aligned}
&Y_t^{(n+m)} - Y_t^{(n)} \\
&= E_{R_0}\left[\int_t^T (g_{n+m}(u) - g_n(u))\, du \Big| \mathcal{F}_t\right] \\
&= E_{R_0}\left[\int_0^T (g_{n+m}(u) - g_n(u))\, du \Big| \mathcal{F}_t\right] - \int_0^t (g_{n+m}(u) - g_n(u))\, du.
\end{aligned} \tag{4.29}$$

Thus, by virtue of Lemma 4.11-(c), we conclude that $Y^{(n+m)} - Y^{(n)}$ is in fact a nonnegative supermartingale under R_0. This ends the proof of assertion (c).

Part 2. In this part, we prove assertion (d) and the last two conditions in (4.12) (see assertion (a) of the theorem). To this end, we start by proving the following properties:

$$\begin{aligned}
&(i) \quad (Y^{(n)}, Z^{(n)}, U^{(n)}) \quad \text{converges in} \quad \mathbb{S}^p(R_0) \times \mathbb{L}^p(W, R_0) \times \mathbb{L}^p(N, R_0),\\
&(ii) \quad \text{the limit of } Y^{(n)} \text{ coincides with the process } Y^{(up)},\\
&(iii) \quad \text{there exists a subsequence of } (Y^{(n)}, Z^{(n)}, U^{(n)}) \text{ that converges } P\otimes dt\text{-a.e..}
\end{aligned} \tag{4.30}$$

On the one hand, by combining Lemma 4.12-(b), Lemma 4.11-(c), and the dominated convergence theorem, we deduce that under assumption (4.11), we have

$$\sup_m E_0\left[\left(\int_0^T (g_{n+m}(u) - g_n(u))du\right)^p\right] \longrightarrow 0 \quad \text{and} \quad E_0\left[\left(\int_0^T (\widetilde{g}(u) - g_n(u))du\right)^p\right] \longrightarrow 0.$$

On the other hand, thanks to (4.29), we get

$$\begin{aligned}
0 \leq Y_t^{(n+m)} - Y_t^{(n)} &= E_0\left[\int_t^T (g_{n+m}(u) - g_n(u))\, du \Big| \mathcal{F}_t\right]\\
&\leq E_0\left[\int_0^T (g_{n+m}(u) - g_n(u))\, du \Big| \mathcal{F}_t\right].
\end{aligned}$$

Then, by combining this latter inequality with Doob's inequality, we deduce the existence of a universal constant $C_p\in(0,\infty)$ such that

$$\sup_m \|Y^{(n+m)}-Y^{(n)}\|^p_{\mathbb{S}^p(R_0)}\leq C_p\sup_m E_0\left[\left(\int_0^T(g_{n+m}(u)-g_n(u))du\right)^p\right]\longrightarrow 0.$$

This implies that $(Y^{(n)})_n$ is a Cauchy sequence in $\mathbb{S}^p(R_0)$-norm, and hence it converges in this norm to its limit $\widetilde{Y}\in\mathbb{S}^p(R_0)$. By combining this with assertion (c) proved in Part 1, which states that $Y^{(n)}$ converges almost surely pointwise to $Y^{(up)}$, and the uniqueness of the limit, we obtain the convergence of $Y^{(n)}$ almost surely pointwise and in the $\mathbb{S}^p(R_0)$-norm to $Y^{(up)}$. This proves (ii) in (4.30).

Now we address the convergence of $(Z^{(n)},U^{(n)})$ in $\mathbb{L}^p(W,R_0)\times\mathbb{L}^p(N,R_0)$. Thus, by combining assertion (c) which guarantees that the process $Y^{(n)}-Y^{(n+m)}$ is an R_0-submartingale, and [41, Lemma 1.9] applied to $Y^{(n)}-Y^{(n+m)}$, we obtain a universal constant C_p such that

$$\|Z^{(n+m)}-Z^{(n)}\|^p_{\mathbb{L}^p(W,R_0)}+\|U^{(n+m)}-U^{(n)}\|^p_{\mathbb{L}^p(N,R_0)}\leq C_p\|Y^{(n+m)}-Y^{(n)}\|^p_{\mathbb{S}^p(R_0)}.$$

As a result, we conclude the existence of $(Z^{(up)},U^{(up)})\in\mathbb{L}^p(W,R_0)\times\mathbb{L}^p(N,R_0)$ such that the triplet $(Y^{(n)},Z^{(n)},U^{(n)})$ converges to $(Y^{(up)},Z^{(up)},U^{(up)})$ in the space $\mathbb{S}^p(R_0)\times\mathbb{L}^p(W,R_0)\times\mathbb{L}^p(N,R_0)$, and property (i) in (4.30) is proved. Furthermore, if $(T_k)_k$ is a sequence of stopping times that increases to infinity and $N^{T_k}\leq k$, then for any $k\geq 1$ we have

$$\begin{aligned}
&R_0\otimes\frac{dt}{T}\left(\lambda e^{\eta(0)\zeta}|U^{(up)}-U^{(n)}|I_{[\![0,T_k]\!]}>\epsilon\right)\\
&\leq (T\epsilon)^{-1}E_{R_0}\left[\int_0^{T_k}\lambda_s e^{\eta_s(0)\zeta_s}|U^{(up)}_s-U^{(n)}_s|ds\right]\\
&=(T\epsilon)^{-1}E_{R_0}\left[\int_0^{T_k}|U^{(up)}_s-U^{(n)}_s|dN_s\right]\\
&\leq (T\epsilon)^{-1}\|U^{(up)}-U^{(n)}\|^p_{\mathbb{L}^p(N,R_0)}\|\sqrt{N_{T_k}}\|_{\mathbb{L}^p(R_0)}.
\end{aligned}$$

This proves that $U^{(n)}$ converges to $U^{(up)}$ in probability under $R_0\otimes(dt/T)$. Then, there exists a subsequence of $(Y^{(n)},Z^{(n)},U^{(n)})$ that converges to $(Y^{(up)},Z^{(up)},U^{(up)})$ in both the space norm and $R_0\otimes dt$-a.e.. This proves property (iii) in (4.30), and the proof of (4.30) is complete.

Therefore, the rest of this part proves that $(Y^{(up)},Z^{(up)},U^{(up)})$ satisfies the constraints and the Skorokhod condition, i.e. the last two conditions in (4.12), and proves the last statement of assertion (d).

Thus, consider the subsequence $(Z^{(n)},U^{(n)})$ that converges to $(Z^{(up)},U^{(up)})$, $R_0\otimes dt$-a.e., which can be obtained due to (4.30). Thus, by combining this with the fact that $\eta_s(-n)-\eta_s(0)/\widetilde{\gamma}_s$ converges to $(\eta^*_s-\eta_s(0))/\widetilde{\gamma}_s$, $R_0\otimes dt$-a.e. due to Lemma 4.2 -(c), we deduce that $(\eta_s(-n)-\eta_s(0))\widetilde{\gamma}_s^{-1}\sigma_s(\widetilde{\gamma}_sZ^{(n)}_s-\sigma_sU^{(n)}_s)^+$ converges to $(\eta^*_s-\eta_s(0))\widetilde{\gamma}_s^{-1}\sigma_s(\widetilde{\gamma}_sZ^{(up)}_s-\sigma_sU^{(up)}_s)^+$, $R_0\otimes dt$-a.e.. Hence, as g_n increases to $\widetilde{g}$ in both $\mathcal{A}^p(R_0)$-norm and $R_0\otimes dt$-a.e. (see Lemmas 4.11 and 4.12), and

$$g_n(s)=\underbrace{\frac{\eta_s(-n)-\eta_s(0)}{\widetilde{\gamma}_s}\sigma_s(\widetilde{\gamma}_sZ^{(n)}_s-\sigma_sU^{(n)}_s)^+}_{=:g^{(1)}_n(s)}+\underbrace{\frac{\eta_s(0)-\eta_s(n)}{\widetilde{\gamma}_s}\sigma_s(\widetilde{\gamma}_sZ^{(n)}_s-\sigma_sU^{(n)}_s)^-}_{=:g^{(2)}_n(s)}$$

we obtain

$$E_{R_0}\left[\int_0^T \frac{g_n^{(2)}(t)}{n}dt\right] \leq \frac{1}{n}E_{R_0}\left[\int_0^T g_n(u)du\right] \leq \frac{1}{n}E_{R_0}\left[\int_0^T \widetilde{g}(u)du\right] \longrightarrow 0.$$

By combining this latter fact with Fatou's lemma and the convergence a.e. of $(\eta_s(0)-\eta_s(n))\sigma_s(\widetilde{\gamma}_s Z_s^{(n)} - \sigma_s U_s^{(n)})^-/n\widetilde{\gamma}_s$ to $\zeta_s\sigma_s(\widetilde{\gamma}_s Z_s^{(up)} - \sigma_s U_s^{(up)})^-/\widetilde{\gamma}_s$, which is due to (4.6) in Lemma 4.2-(d) and $\zeta\sigma/\widetilde{\gamma} > 0$, we deduce that

$$\widetilde{\gamma}_s Z_s^{(up)} - \sigma_s U_s^{(up)} \geq 0 \quad P\otimes dt - a.e..$$

This proves that the constraint equation in (4.12) is fulfilled. As both processes $\int_0^\cdot g_n(t)dt$ and $\int_0^\cdot g_n^{(1)}(t)dt$ converge in p-variation-norm (i.e. $\mathcal{A}^p(R_0)$-norm), we deduce that $K_n(t) := \int_0^t g_n^{(2)}(s)ds$ converges in p-variation-norm. Hence, there exists a nonnegative and adapted process k^{up} such that

$$K^{(n)} := \int_0^\cdot g_n^{(2)}(s)ds \quad \text{converges in } \mathcal{A}^p(R_0)\text{-norm to} \quad \int_0^\cdot k_s^{up}ds =: K^{up}.$$

Therefore, on the one hand, this proves the last statement of assertion (d), and hence the proof of assertion (d) is complete. On the other hand, as $I_{\{\widetilde{\gamma}Z^{(n)}-\sigma U^{(n)}\geq\epsilon\}}$ converges $R_0\otimes dt$-almost everywhere to $I_{\{\widetilde{\gamma}Z^{(up)}-\sigma U^{(up)}\geq\epsilon\}}$ for any $\epsilon > 0$, we deduce that for any $\epsilon > 0$, we have

$$0 = E_{R_0}\left[\left(I_{\{\widetilde{\gamma}Z^{(n)}-\sigma U^{(n)}\geq\epsilon\}}\bullet K^{(n)}\right)_T\right] \longrightarrow E_{R_0}\left[\left(I_{\{\widetilde{\gamma}Z^{(up)}-\sigma U^{(up)}\geq\epsilon\}}\bullet K^{up}\right)_T\right].$$

This implies, when taking ϵ to zero, that

$$\begin{aligned} &\int_0^T I_{\{\widetilde{\gamma}_s Z_s^{(up)}-\sigma_s U_s^{(up)}>0\}}dK_s^{up} = 0 \quad P\text{-.a.s.},\\ \text{or equivalently} \quad &\int_0^T (\widetilde{\gamma}_s Z_s^{(up)} - \sigma_s U_s^{(up)})dK_s^{up} = 0 \quad P\text{-.a.s..}\end{aligned}$$

This proves the Skorokhod condition (i.e. the last conditions in (4.12)), and Part 2 is complete.

Part 3. This part proves assertion (a). Thanks to Part 2, we consider a subsequence $(Y^{(n)}, Z^{(n)}, U^{(n)})$ that converges to $(Y^{(up)}, Z^{(up)}, U^{(up)})$ in both the space norm and $R_0\otimes dt$-a.e., and $(Z^{(n)}\bullet W_t^0, U^{(n)}\bullet \widetilde{N}_t^0)$ converges P-a.s. to $(Z^{(up)}\bullet W_t^0, U^{(up)}\bullet \widetilde{N}_t^0)$ for any $t\in[0,T]$. Thus, by taking the limit $R_0\otimes dt$-a.e. in the BSDE (4.13), or equivalently in

$$\begin{aligned} Y_t^{(n)} =&\xi + \int_t^T \frac{\eta_s(-n)-\eta_s(0)}{\widetilde{\gamma}_s}\sigma_s(\widetilde{\gamma}_s Z_s^{(n)} - \sigma_s U_s^{(n)})^+ds + K_T^{(n)} - K_t^{(n)}\\ &-\int_t^T Z_s^{(n)}dW_s^0 - \int_t^T U_s^{(n)}d\widetilde{N}_s^0,\end{aligned}$$

we deduce that the quadruplet $(Y^{(up)}, Z^{(up)}, U^{(up)}, K^{up})$, which belongs to the space $\mathbb{S}^p(R_0)\times\mathbb{L}^p(W,R_0)\times\mathbb{L}^p(N,R_0)\times\mathcal{A}^{+,p}(R_0)$, fulfills (4.12) fully. To prove that this solution is the smallest $L^p(R_0)$-solution, we consider any $L^p(R_0)$-solution (Y,Z,U,K) of the constrained BSDE (4.12). Note that (Y,Z,U,K) satisfies

$$dY = \frac{\sigma}{\widetilde{\gamma}}(\eta(0)-\eta^*)(\widetilde{\gamma}Z - \sigma U)dt - dK + ZdW^0 + Ud\widetilde{N}^0, \quad Y_T = \xi.$$

A combination of this with arguments similar to those used in (4.27) yields

$$\begin{aligned}
&d(Y-Y^{\psi})\\
&=\frac{\sigma}{\widetilde{\gamma}}(\eta(0)-\eta^*)(\widetilde{\gamma}Z-\sigma U)dt+\frac{\sigma}{\widetilde{\gamma}}(\eta(\psi)-\eta(0))(\widetilde{\gamma}Z^{\psi}-\sigma U^{\psi})dt\\
&\quad+(Z-Z^{\psi})dW^0+(U-U^{\psi})d\widetilde{N}^0-dK\\
&=\frac{\sigma}{\widetilde{\gamma}}(\eta(\psi)-\eta^*)(\widetilde{\gamma}Z-\sigma U)dt-dK+(Z-Z^{\psi})\Big(dW^0-\sigma(\eta(\psi)-\eta(0))dt\Big)\\
&\quad+(U-U^{\psi})\Big(d\widetilde{N}^0-\frac{\sigma^2}{\widetilde{\gamma}}(\eta(\psi)-\eta(0))dt\Big)\\
&=\frac{\sigma}{\widetilde{\gamma}}(\eta(\psi)-\eta^*)(\widetilde{\gamma}Z-\sigma U)dt+(Z-Z^{\psi})dW^{\psi}+(U-U^{\psi})d\widetilde{N}^{\psi},
\end{aligned}$$

for any $\psi\in\Psi$. By virtue of $\widetilde{\gamma}Z-\sigma U\geq 0$ and $\widetilde{\gamma}^{-1}(\eta(\psi)-\eta^*)\leq 0$, which is due to assertions (b) and (c) of Lemma 4.2, we deduce that $Y-Y^{\psi}$ is a local R_{ψ}-supermartingale. Then again, by following the same arguments after (4.27) of Part 1, which lead to claim that $\|Y-Y^{\psi}\|_{\mathbb{S}^p(R_{\psi})}<\infty$, we conclude that $Y-Y^{\psi}$ is in fact an R_{ψ}-supermartingale p-integrable with null terminal value. This implies that

$$Y_t-Y_t^{\psi}\geq E_{R_{\psi}}\Big[Y_T-Y_T^{\psi}\ \Big|\mathcal{F}_t\Big]=0.$$

Therefore, $Y\geq Y^{\psi}$ for any $\psi\in\Psi$, and hence we get $Y\geq \operatorname{ess\,sup}_{\psi\in\Psi}(Y^{\psi})=Y^{(up)}$. This proves that $Y^{(up)}$ is the smallest $L^p(R_0)$-solution to the constrained BSDE (4.12). Hence, assertion (a) follows immediately and the proof of the theorem is complete. ☐

Appendices.

Appendix A. Local martingale deflators via predictable characteristics.

Theorem A.1. *Suppose that S is locally bounded. Then, the following assertions hold.*
(a) $\mathcal{Z}_{loc}(S)\neq\emptyset$ *if and only if there exists a pair (β,f) such that β is a d-dimensional predictable process, f is a real-valued positive $\mathcal{P}\otimes\mathcal{B}(\mathbb{R})$-measurable functional (i.e. $f>0$),*

$$\beta^{tr}c\beta\bullet A+\sqrt{(f(x)-1)^2\star\widetilde{\mu}}\in\mathcal{A}^+_{loc},\ \ \textit{and}\ \ \widetilde{b}'+c\beta+\int(xf(x)-x)\,\widetilde{F}(dx)=0. \tag{A.1}$$

(b) $\mathcal{Z}^{L\log L}_{loc}(S)\neq\emptyset$ *if and only if there exists a pair (β,f) such that (A.1) holds and*

$$(f(x)\ln(f(x))-f(x)+1)\star\widetilde{\mu}\in\mathcal{A}^+_{loc}.$$

Appendix B. Proofs for Lemmas 3.19, 3.20, 3.21, and 3.23.

Proof of Lemma 3.19: Let $\theta\in L(Y)$, and consider the notation given in the lemma. The proof of the lemma will be achieved in three parts.
1) Here we prove assertion (a). To this end, on the one hand, we remark that $\theta\in L(Y)$ yields $I_{\{|\theta^{tr}x|>\epsilon\}}\star\mu^Y=\sum_{0<s\leq\cdot}I_{\{|\theta_s^{tr}\Delta Y_s|>\epsilon\}}\in\mathcal{A}^+_{loc}$. Hence, we deduce that

$$b_1^{\theta}:=F^Y\left(\{x:\ |\theta^{tr}x|>\epsilon\}\right)<\infty\quad P\otimes dA^Y\text{-a.e.},\quad\text{and}\quad b_1^{\theta}\in L(A^Y). \tag{B.1}$$

On the other hand, we have

$$\int \|x\| I_{\{|\theta^{tr}x|>\epsilon\geq\|x\|\}} F^Y(dx) \leq \epsilon F^Y\left(\{x:\ |\theta^{tr}x|>\epsilon\}\right).$$

By combining this inequality with (B.1) and $b^Y \in L(A^Y)$, we deduce that (3.28) follows immediately. To prove (3.29), we appeal to the canonical decomposition of Y (i.e. (2.1)) and use (3.27) and derive

$$\begin{aligned}
Z^\theta(Y) &= Y - U^\theta(Y)\\
&= Y_0 + Y^c + b^Y \cdot A^Y + h_\epsilon(x) \star (\mu^Y - \nu^Y)\\
&\quad + \big(x - h_\epsilon(x)\big) \star \mu^Y - xI_{\{|\theta^{tr}x|>\epsilon \text{ or } \|x\|>\epsilon\}} \star \mu^Y\\
&= Y_0 + Y^c + b^Y \cdot A^Y + h_\epsilon(x) \star (\mu^Y - \nu^Y) - xI_{\{\|x\|\leq\epsilon<|\theta^{tr}x|\}} \star \mu^Y.
\end{aligned}$$

Remark that $xI_{\{\|x\|\leq\epsilon<|\theta^{tr}x|\}} \star \mu^Y \in \mathcal{A}_{loc}$, and by compensating it and inserting the compensator process in the above equality, we derive

$$\begin{aligned}
Z^\theta(Y) &= Y_0 + Y^c + b^Y \cdot A^Y + h_\epsilon(x) \star (\mu^Y - \nu^Y) - xI_{\{\|x\|\leq\epsilon<|\theta^{tr}x|\}} \star (\mu^Y - \nu^Y)\\
&\quad - xI_{\{\|x\|\leq\epsilon<|\theta^{tr}x|\}} \star \nu^Y\\
&= Y_0 + Y^c + b^Y \cdot A^Y + xI_{\{\|x\|\leq\epsilon\ \&\ |\theta^{tr}x|\leq\epsilon\}} \star (\mu^Y - \nu^Y)\\
&\quad - xI_{\{\|x\|\leq\epsilon<|\theta^{tr}x|\}} \star \nu^Y.
\end{aligned}$$

This proves (3.29), and ends the proof of assertion (a).
2) Here we prove assertion (b). To this end, we remark that $\theta \in L(U^{\theta,1}) \cap L(U^{\theta,2})$. This obviously implies that $\theta \in L(U^\theta(Y))$, and hence we deduce that $\theta \in L(Z^\theta(Y))$. Therefore, $\theta \in L(Y^c) \cap L(M^\theta) \cap L(b^\theta(Y) \cdot A^Y)$ is a direct consequence of [41, Corollaire 2.6] and the fact that both local martingales Y^c and M^θ are orthogonal. Furthermore, by virtue of (3.29), we get

$$\theta \cdot Z^\theta(Y) = \theta \cdot Y^c + \theta^{tr} xI_{\{\|x\|\leq\epsilon\ \&\ |\theta^{tr}x|\leq\epsilon\}} \star (\mu^Y - \nu^Y) + \theta^{tr} b^\theta(Y) \cdot A^Y.$$

This proves (3.30), and ends the proof of assertion (b).
3) This part proves assertion (c). To this end, we use (3.30), and derive

$$\begin{aligned}
\theta \cdot Y &= \theta \cdot Z^\theta(Y) + \theta \cdot U^\theta(Y)\\
&= \theta \cdot Y^c + \theta^{tr} xI_{\{\|x\|\leq\epsilon\ \&\ |\theta^{tr}x|\leq\epsilon\}} \star (\mu^Y - \nu^Y) + \theta^{tr} b^\theta(Y) \cdot A^Y\\
&\quad + \theta \cdot U^{\theta,1}(Y) + \theta \cdot U^{\theta,2}(Y).
\end{aligned}$$

As $\theta \cdot U^{\theta,1}(Y) = \sum \theta^{tr} \Delta Y I_{\{|\theta^{tr}\Delta Y|\leq\epsilon<\|\Delta Y\|\}} = \theta^{tr} xI_{\{|\theta^{tr}x|\leq\epsilon<\|x\|\}} \star \mu^Y \in \mathcal{A}_{loc}$, and by compensating this process in the above equality, we obtain

$$\begin{aligned}
\theta \cdot Y &= \theta \cdot Y^c + \theta^{tr} xI_{\{\|x\|\leq\epsilon\ \&\ |\theta^{tr}x|\leq\epsilon\}} \star (\mu^Y - \nu^Y) + \theta^{tr} b^\theta(Y) \cdot A^Y\\
&\quad + \theta^{tr} xI_{\{|\theta^{tr}x|\leq\epsilon<\|x\|\}} \star (\mu^Y - \nu^Y)\\
&\quad + \theta^{tr} xI_{\{|\theta^{tr}x|\leq\epsilon<\|x\|\}} \star \nu^Y + \theta \cdot U^{\theta,2}(Y)\\
&= \theta \cdot Y^c + h_\epsilon(\theta^{tr}x) \star (\mu^Y - \nu^Y)\\
&\quad + \theta^{tr} b^\theta(Y) \cdot A^Y + \theta^{tr} xI_{\{|\theta^{tr}x|\leq\epsilon<\|x\|\}} \star \nu^Y + \theta \cdot U^{\theta,2}(Y).
\end{aligned}$$

Thus, this equality yields the decomposition (3.31), and the proof of the lemma is complete. $\square$

Proof of Lemma 3.20. 1) This part proves assertion (a). To this end, we derive

$$\begin{aligned}
&e^{\theta^{tr}x+x^{tr}\psi x}-1-h_\epsilon(\theta^{tr}x)\\
&=\left(e^{\theta^{tr}x+x^{tr}\psi x}-1-\theta^{tr}x\right)I_{\{|\theta^{tr}x|\leq\epsilon\}}+\left(e^{\theta^{tr}x+x^{tr}\psi x}-1\right)I_{\{|\theta^{tr}x|>\epsilon\}}\\
&=\left(e^{\theta^{tr}x+x^{tr}\psi x}-1-\theta^{tr}x\right)I_{\{|\theta^{tr}x|\leq\epsilon\ \&\ \|x\|\leq\epsilon\}}+e^{\theta^{tr}x+x^{tr}\psi x}I_{\{|\theta^{tr}x|\leq\epsilon<\|x\|\}}\\
&\quad-(1+\theta^{tr}x)I_{\{|\theta^{tr}x|\leq\epsilon<\|x\|\}}-I_{\{|\theta^{tr}x|>\epsilon\}}+e^{\theta^{tr}x+x^{tr}\psi x}I_{\{|\theta^{tr}x|>\epsilon\}}.
\end{aligned}$$

Remark that the three processes $I_{\{|\theta^{tr}x|>\epsilon\}}\star\widetilde{\mu}$, $|1+\theta^{tr}x|I_{\{|\theta^{tr}x|\leq\epsilon<\|x\|\}}\star\widetilde{\mu}$, and $|e^{\theta^{tr}x+x^{tr}\psi x}-1-\theta^{tr}x|I_{\{|\theta^{tr}x|\leq\epsilon\ \&\ \|x\|\leq\epsilon\}}\star\widetilde{\mu}$ belong to $\mathcal{A}^+_{loc}$. Therefore, we deduce that $\left|\exp(\theta^{tr}x+x^{tr}\psi x)-1-h_\epsilon(\theta^{tr}x)\right|\star\widetilde{\mu}\in\mathcal{A}^+_{loc}$ if and only if

$$e^{\theta^{tr}x+x^{tr}\psi x}\left(I_{\{|\theta^{tr}x|>\epsilon\}}+I_{\{|\theta^{tr}x|\leq\epsilon<\|x\|\}}\right)\star\widetilde{\mu}\in\mathcal{A}^+_{loc}.\tag{B.2}$$

Similarly, we consider the following decomposition

$$\begin{aligned}
xe^{\theta^{tr}x+x^{tr}\psi x}-h_\epsilon(x)=&x\left(e^{\theta^{tr}x+x^{tr}\psi x}-1\right)I_{\{\|x\|\leq\epsilon\}}+xe^{\theta^{tr}x+x^{tr}\psi x}I_{\{\|x\|>\epsilon\}}\\
=&x\left(e^{\theta^{tr}x+x^{tr}\psi x}-1\right)I_{\{\|x\|\leq\epsilon\ \&\ |\theta^{tr}x|\leq\epsilon\}}-xI_{\{\|x\|\leq\epsilon<|\theta^{tr}x|\}}\\
&+xe^{\theta^{tr}x+x^{tr}\psi x}I_{\{\|x\|\leq\epsilon<|\theta^{tr}x|\}}+xe^{\theta^{tr}x+x^{tr}\psi x}I_{\{\|x\|>\epsilon\}}.
\end{aligned}$$

Thanks to the boundedness of ψ, we deduce that both $\|x\|I_{\{\|x\|\leq\epsilon<|\theta^{tr}x|\}}\star\widetilde{\mu}$ and $\|x\left(e^{\theta^{tr}x+x^{tr}\psi x}-1\right)\|I_{\{\|x\|\leq\epsilon\ \&\ |\theta^{tr}x|\leq\epsilon\}}\star\widetilde{\mu}$ belong to $\mathcal{A}^+_{loc}$. Hence, we conclude that $\|x\exp(\theta^{tr}x+x^{tr}\psi x)-h_\epsilon(x)\|\star\widetilde{\mu}\in\mathcal{A}^+_{loc}$ if and only if

$$\|x\|e^{\theta^{tr}x+x^{tr}\psi x}\left(I_{\{\|x\|>\epsilon\}}+I_{\{\|x\|\leq\epsilon<|\theta^{tr}x|\}}\right)\star\widetilde{\mu}\in\mathcal{A}^+_{loc}.\tag{B.3}$$

As a result, (3.32) holds if and only if (B.2) and (B.3) hold, or equivalently (3.4) is fulfilled. This ends the proof of assertion (a).
2) Here we prove assertion (b). To this end, throughout this part we suppose that (3.4) holds. On the one hand, we combine this latter assumption with the fact that $(I_{\{\|x\|>\epsilon\}}+I_{\{|\theta^{tr}x|>\epsilon\}})\star\widetilde{\mu}\in\mathcal{A}^+_{loc}$ and derive

$$\begin{aligned}
&|e^{\theta^{tr}x+x^{tr}\psi x}-1|I_{\{\|x\|>\epsilon\text{ or }|\theta^{tr}x|>\epsilon\}}\star\widetilde{\mu}\\
&\leq|e^{\theta^{tr}x+x^{tr}\psi x}-1|I_{\{\|x\|>\epsilon\}}\star\widetilde{\mu}+|e^{\theta^{tr}x+x^{tr}\psi x}-1|I_{\{|\theta^{tr}x|>\epsilon\}}\star\widetilde{\mu}\\
&\leq\left[\frac{\|x\|}{\epsilon}e^{\theta^{tr}x+x^{tr}\psi x}+1\right]I_{\{\|x\|>\epsilon\}}\star\widetilde{\mu}+\left[e^{\theta^{tr}x+x^{tr}\psi x}+1\right]I_{\{|\theta^{tr}x|>\epsilon\}}\star\widetilde{\mu}\in\mathcal{A}^+_{loc}.
\end{aligned}\tag{B.4}$$

On the other hand, as $((\theta^{tr}x)^2I_{\{|\theta^{tr}x|\leq\epsilon\}}\star\widetilde{\mu}\leq I_{\{|\theta^{tr}\Delta\widetilde{X}|\leq\epsilon\}}\bullet[\theta\bullet\widetilde{X},\theta\bullet\widetilde{X}]\in\mathcal{A}^+_{loc}$ due to $\theta\in L(\widetilde{X})$, we deduce that

$$\begin{aligned}
&(e^{\theta^{tr}x+x^{tr}\psi x}-1)^2I_{\{|\theta^{tr}x|\leq\epsilon\ \&\ \|x\|\leq\epsilon\}}\star\widetilde{\mu}\\
&\leq C_\epsilon\left((\theta^{tr}x)^2+|x^{tr}\psi x|\right)I_{\{|\theta^{tr}x|\leq\epsilon\ \&\ \|x\|\leq\epsilon\}}\star\widetilde{\mu}\in\mathcal{A}^+_{loc}.
\end{aligned}\tag{B.5}$$

Thus, $\sqrt{(e^{\theta^{tr}x+x^{tr}\psi x}-1)^2\star\widetilde{\mu}}\in\mathcal{A}^+_{loc}$ follows immediately from combining (B.5), (B.4), and the fact that

$$\begin{aligned}\sqrt{(e^{\theta^{tr}x+x^{tr}\psi x}-1)^2\star\widetilde{\mu}}\leq&\sqrt{(e^{\theta^{tr}x+x^{tr}\psi x}-1)^2 I_{\{\max(|\theta^{tr}x|,\|x\|)\leq\epsilon\}}\star\widetilde{\mu}}\\ &+|e^{\theta^{tr}x+x^{tr}\psi x}-1|I_{\{\|x\|>\epsilon\text{ or }|\theta^{tr}x|>\epsilon\}}\star\widetilde{\mu}.\end{aligned}$$

This proves assertion (b).
3) Here we prove assertion (c). Thus, we suppose that (3.4) and (3.5) hold and

$$|e^{x^{tr}\psi x}-1|\star\widetilde{\mu}\in\mathcal{A}^+_{loc}. \tag{B.6}$$

Then, we remark that it is easy to prove the following implication:

$$\text{If (3.4) holds, then } |\exp(\theta^{tr}x+x^{tr}\psi x)-1-\theta^{tr}h_\epsilon(x)|\star\widetilde{\mu}\in\mathcal{A}^+_{loc}. \tag{B.7}$$

In fact, this follows immediately from combining

$$\begin{aligned}&|e^{\theta^{tr}x+x^{tr}\psi x}-1-\theta^{tr}h_\epsilon(x)|\\ &=|e^{\theta^{tr}x+x^{tr}\psi x}-1-\theta^{tr}x|I_{\{\|x\|\leq\epsilon\}}+|e^{\theta^{tr}x+x^{tr}\psi x}-1|I_{\{\|x\|>\epsilon\}}\\ &\leq|x^{tr}\psi x|e^{\alpha\theta^{tr}x+\alpha x^{tr}\psi x}I_{\{\|x\|\leq\epsilon\}}+(e^{\theta^{tr}x+x^{tr}\psi x}+1)I_{\{\|x\|>\epsilon\}}\\ &\leq\epsilon^2\|\psi\|e^{\theta^{tr}x+x^{tr}\psi x}I_{\{|\theta^{tr}x|>\epsilon\}}+C_\epsilon|x^{tr}\psi x|I_{\{\|x\|\leq\epsilon\}}+(e^{\theta^{tr}x+x^{tr}\psi x}+1)I_{\{\|x\|>\epsilon\}},\end{aligned}$$

which holds for some $\alpha\in[0,1]$ and some $C_\epsilon>0$ due to Taylor's expansion with the fact $(e^{\theta^{tr}x+x^{tr}\psi x}+1)I_{\{\|x\|>\epsilon\}}\star\widetilde{\mu}+|x^{tr}\psi x|I_{\{\|x\|\leq\epsilon\}}\star\widetilde{\mu}\in\mathcal{A}^+_{loc}$ and (3.4).

Thus, by virtue of (B.7), it is clear that $|\theta^{tr}xe^{\theta^{tr}x+x^{tr}\psi x}-e^{\theta^{tr}x+x^{tr}\psi x}+1|\star\widetilde{\mu}$ belongs to $\mathcal{A}^+_{loc}$ if and only if $|\theta^{tr}xe^{\theta^{tr}x+x^{tr}\psi x}-\theta^{tr}h_\epsilon(x)|\star\widetilde{\mu}\in\mathcal{A}^+_{loc}$, and hence assertion (c) follows immediately as soon as we prove the latter property. To this end, on the one hand, we remark that we always have

$$\begin{aligned}&\theta^{tr}xe^{\theta^{tr}x+x^{tr}\psi x}-\theta^{tr}h_\epsilon(x)\\ &=e^{x^{tr}\psi x}\Big((\theta^{tr}x)e^{\theta^{tr}x}-e^{\theta^{tr}x}+1\Big)+\Big(e^{\theta^{tr}x+x^{tr}\psi x}-1-\theta^{tr}h_\epsilon(x)\Big)+(1-e^{x^{tr}\psi x}).\end{aligned}$$

Thus, a combination of this latter equality with (B.6) and (B.7) allows us to conclude that

$$\begin{aligned}&\int\Big(\theta^{tr}xe^{\theta^{tr}x+x^{tr}\psi x}-\theta^{tr}h_\epsilon(x)\Big)\widetilde{F}(dx)\\ &=\theta^{tr}\widetilde{b}+\theta^{tr}c\theta\in L(A)\quad\text{iff}\quad|e^{\theta^{tr}x+x^{tr}\psi x}-1-\theta^{tr}h_\epsilon(x)|\star\widetilde{\mu}\in\mathcal{A}^+_{loc},\end{aligned}$$

where the equality above is due to (3.5). On the other hand, by combining $\theta\in L(S)$, (3.4), and (B.7), we deduce the following properties:
a) $\theta^{tr}c\theta\in L(A)$ (due to $\theta^{tr}c\theta\bullet A\leq I_{\{|\theta^{tr}\Delta\widetilde{X}|\leq\epsilon\}}\bullet[\theta\bullet\widetilde{X},\theta\bullet\widetilde{X}]\in\mathcal{A}^+_{loc}$), or equivalently $\theta\in L^1(X^c)$,
b) $\theta\in L^1_{loc}(h_\epsilon(x)\star(\widetilde{\mu}-\widetilde{\nu}))\cap L((x-h_\epsilon(x))\star\widetilde{\mu})$. This follows directly from the fact that $|h_\epsilon(\theta^{tr}x)-\theta^{tr}h_\epsilon(x)|\star\widetilde{\mu}\in\mathcal{A}^+_{loc}$, which is a direct consequence of assertion (a) and (B.7). This ends the proof of assertion (c).

4) Here we prove assertion (d). First, we remark that

$$\begin{aligned}
&\left(e^{\theta^{tr}x+x^{tr}\psi x}-1\right)(\mathbf{e}(x)-\mathbb{I}_d)\\
&=\left(e^{\theta^{tr}x+x^{tr}\psi x}-1\right)(\mathbf{e}(x)-\mathbb{I}_d)I_{\{|\theta^{tr}x|>\alpha\}}\\
&\quad+\left(e^{\theta^{tr}x+x^{tr}\psi x}-1\right)(\mathbf{e}(x)-\mathbb{I}_d)I_{\{|\theta^{tr}x|\leq\alpha\}}\\
&=e^{\theta^{tr}x+x^{tr}\psi x}(\mathbf{e}(x)-\mathbb{I}_d)I_{\{|\theta^{tr}x|>\alpha\}}-(\mathbf{e}(x)-\mathbb{I}_d)I_{\{|\theta^{tr}x|>\alpha\}}\\
&\quad+\left(e^{\theta^{tr}x+x^{tr}\psi x}-1\right)(\mathbf{e}(x)-\mathbb{I}_d)I_{\{|\theta^{tr}x|\leq\alpha\}}.
\end{aligned}$$

As S and ψ are locally bounded, it is clear that

$$\|\mathbf{e}(x)-\mathbb{I}_d\|I_{\{|\theta^{tr}x|>\alpha\}}\star\mu+\|\left(e^{\theta^{tr}x+x^{tr}\psi x}-1\right)(\mathbf{e}(x)-\mathbb{I}_d)\|I_{\{|\theta^{tr}x|\leq\alpha\}}\star\mu\in\mathcal{A}^+_{loc}.$$

Furthermore, $e^{\theta^{tr}x+x^{tr}\psi x}\|\mathbf{e}(x)-\mathbb{I}_d\|I_{\{|\theta^{tr}x|>\alpha\}}\star\mu\in\mathcal{A}^+_{loc}$ if and only if

$$e^{\theta^{tr}x}\|\mathbf{e}(x)-\mathbb{I}_d\|I_{\{|\theta^{tr}x|>\alpha\}}\star\mu\in\mathcal{A}^+_{loc}. \tag{B.8}$$

Second, we derive

$$\begin{aligned}
&e^{\theta^{tr}x+x^{tr}\psi x}-1-h_\epsilon(\theta^{tr}x)\\
&=\left(e^{\theta^{tr}x+x^{tr}\psi x}-1-\theta^{tr}x\right)I_{\{|\theta^{tr}x|\leq\epsilon\}}+\left(e^{\theta^{tr}x+x^{tr}\psi x}-1\right)I_{\{|\theta^{tr}x|>\epsilon\}}\\
&=\left(e^{\theta^{tr}x+x^{tr}\psi x}-1-\theta^{tr}x\right)I_{\{|\theta^{tr}x|\leq\epsilon\}}-I_{\{|\theta^{tr}x|>\epsilon\}}+e^{\theta^{tr}x+x^{tr}\psi x}I_{\{|\theta^{tr}x|>\epsilon\}}.
\end{aligned}$$

On the one hand, it is clear that both $\left(e^{\theta^{tr}x+x^{tr}\psi x}-1-\theta^{tr}x\right)I_{\{|\theta^{tr}x|\leq\epsilon\}}\star\mu$ and $I_{\{|\theta^{tr}x|>\epsilon\}}\star\mu$ belong to $\mathcal{A}_{loc}$. On the other hand, $e^{\theta^{tr}x+x^{tr}\psi x}I_{\{|\theta^{tr}x|>\epsilon\}}\star\mu\in\mathcal{A}^+_{loc}$ if and only if

$$e^{\theta^{tr}x}I_{\{|\theta^{tr}x|>\epsilon\}}\star\mu\in\mathcal{A}^+_{loc}. \tag{B.9}$$

Thanks to the local boundedness of S, it is clear that (B.9) implies (B.8). Thus, this proves that (3.34) is equivalent to (3.19), and the proof of assertion (d) is complete. □

Proof of Lemma 3.21. Remark that the proof of assertion (b) follows the same footsteps of the end of part 1 in the proof of Theorem 3.4, while using the fact that (3.38) always holds when $\widetilde{X}$ is locally bounded and $\sqrt{(\exp(\widehat{\theta}^{tr}x+x^{tr}\psi x)-1)^2\star\widetilde{\mu}}\in\mathcal{A}^+_{loc}$ holds as well. Thus, the rest of the proof concerns assertion (a). To this end, for any θ and any $\epsilon\in\mathbb{R}$, we set $\overline{\theta}:=\widehat{\theta}+\epsilon(\theta-\widehat{\theta})=:\widehat{\theta}+\epsilon\Delta$, and

$$f(\theta):=\theta^{tr}\widetilde{b}^Q+\frac{1}{2}\theta^{tr}c^Q\theta+\int\left(\exp(\theta^{tr}x)-1-\theta^{tr}x\right)\widetilde{F}^Q(dx).$$

Then, we derive

$$
\begin{aligned}
&f(\overline{\theta})-f(\widehat{\theta})\\
&=\epsilon\Delta^{tr}c^Q\widehat{\theta}+\frac{\epsilon^2}{2}\Delta^{tr}c^Q\Delta+\epsilon\Delta^{tr}\widetilde{b}^Q+\int\left(e^{\widehat{\theta}^{tr}x}(e^{\epsilon\Delta^{tr}x}-1)-\epsilon\Delta^{tr}x\right)\widetilde{F}^Q(dx)\\
&=\epsilon\Delta^{tr}\left(\widetilde{b}^Q+c^Q\widehat{\theta}+\int\left(xe^{\widehat{\theta}^{tr}x}-x\right)\widetilde{F}^Q(dx)\right)+\frac{\epsilon^2}{2}\Delta^{tr}c^Q\Delta\\
&\quad+\int e^{\widehat{\theta}^{tr}x}(e^{\epsilon\Delta^{tr}x}-1-\epsilon\Delta^{tr}x)\widetilde{F}^Q(dx),
\end{aligned}
\tag{B.10}
$$

and hence when ϵ goes to zero, we get

$$
\frac{f(\overline{\theta})-f(\widehat{\theta})}{\epsilon}\longrightarrow\Delta^{tr}\left(\widetilde{b}^Q+c^Q\widehat{\theta}+\int\left(xe^{\widehat{\theta}^{tr}x}-x\right)\widetilde{F}^Q(dx)\right).
$$

Therefore, assertion (a) follows immediately from combining this, (B.10), and the fact that both terms $\frac{\epsilon^2}{2}\Delta^{tr}c\Delta$ and $\int e^{\widehat{\theta}^{tr}x}(e^{\epsilon\Delta^{tr}x}-1-\epsilon\Delta^{tr}x)\widetilde{F}^Q(dx)$ are nonnegative. ☐

Proof of Lemma 3.23. The proof of the lemma is achieved in two parts.
1) Here we prove $Z:=\mathcal{E}(\beta\bullet X^c+(g-1)\star(\mu-\nu))\in\mathcal{Z}_{loc}(S)$ if and only if $Z':=\mathcal{E}(\beta\bullet X^c+(g-1)\star(\mu-\widehat{\nu}))\in\mathcal{Z}_{loc}(\widehat{S},\widehat{Z})$. To this end, on the one hand thanks to [15, Lemma 2.4-(1)], we deduce that $Z\in\mathcal{Z}_{loc}(S)$ if and only if

$$
\begin{aligned}
&(e^x-1)(g-1)\star\mu\in\mathcal{A}_{loc}\ \text{ and}\\
&\widetilde{b}'+c\beta+\int xf(x)(g(x)-1)F(dx)\equiv 0\ P\otimes dA-a.e..
\end{aligned}
\tag{B.11}
$$

On the other hand, by stopping we can assume without loss of generality that $\widehat{Z}\in\mathcal{M}$, and we set $\widehat{Q}:=\widehat{Z}_T\cdot P$. Then, by virtue of [15, Lemma 2.4-(1)] again applied to $(\widehat{S},\widehat{Q})$, we conclude that $Z'\in\mathcal{Z}_{loc}(\widehat{S},\widehat{Q})$ if and only if $x(g-1)\star\mu\in\mathcal{A}_{loc}(\widehat{Q})$ and

$$
b'+\frac{c}{2}+\int(e^x-1-x)F(dx)+c\beta+\int x(g(x)-1)f(x)F(dx)\equiv 0,\ P\otimes dA-a.e.. \tag{B.12}
$$

On the one hand, thanks to $\widetilde{b}'=b'+c/2+\int(e^x-1-x)F(dx)$, it is clear that the second equation in (B.11) coincides with (B.12). On the other hand, it is easy to prove that $x(g-1)\star\mu\in\mathcal{A}_{loc}(\widehat{Q})$ if and only if $(e^x-1)(g-1)\star\mu\in\mathcal{A}_{loc}$, and both these conditions hold. In fact, we remark that

$$
\begin{aligned}
&|(e^x-1)(g-1)|\star\mu\\
&=\sum|(e^{\Delta X}-1)(g(\Delta X)-1)|\leq\sqrt{\sum(e^{\Delta X}-1)^2}\sqrt{\sum(g(\Delta X)-1)^2}\\
&\leq C\sqrt{[X,X]}\sqrt{(g-1)^2\star\mu}.
\end{aligned}
$$

As $[X,X]$ is locally bounded and $\sqrt{(g-1)^2\star\mu}$ belongs to $\mathcal{A}^+_{loc}$, we deduce that $\sqrt{[X,X]}\sqrt{(g-1)^2\star\mu}\in\mathcal{A}^+_{loc}$. Hence, we conclude that $|(e^x-1)(g-1)|\star\mu\in\mathcal{A}^+_{loc}$ and the first part is complete.
2) Here we prove that $Z\ln(Z)$ is a special semimartingale if and only if $\widehat{Z}Z'\ln(Z')$ is a special semimartingale. By stopping, we can assume without loss of generality that $\widehat{Z}\in\mathcal{M}$ and set $\widehat{Q}:=\widehat{Z}_\infty\cdot P$. Then, the problem reduces to proving that $Z\ln(Z)$ is a special semimartingale if and only if $Z'\ln(Z')$ is a special semimartingale under

$\widehat{Q}$. Thanks to the entropy-Hellinger process defined in [13, Definition 4.3] and [13, Lemma 3.2], the claim becomes $h^E(Z,P)\in\mathcal{A}^+_{loc}$ if and only if $h^E(Z',\widehat{Q})\in\mathcal{A}^+_{loc}$. Thus, [13, Proposition 3.5] yields

$$h^E(Z,P)=\frac{1}{2}\beta c\beta\bullet A+(g\ln(g)-g+1)\star\nu$$
$$\text{and}\quad h^E(Z',\widehat{Q})=\frac{1}{2}\beta c\beta\bullet A+(g\ln(g)-g+1)\star\widehat{\nu}.$$

Hence, as $\widehat{\nu}(dt,dx)=f(x)\nu(dt,dx)$ and f is locally bounded and locally bounded away from zero, we deduce that the claim holds. This ends the proof of the lemma. $\square$

Appendix C. Proofs for Lemmas 4.2, 4.3, 4.10, 4.11, and 4.12.

Proof of Lemma 4.2. 1) The continuity of Φ is obvious, while the facts $\Phi(\infty)=\infty$, $\Phi(-\infty)=-\infty$, and $\Phi'(x)=\sigma^2+\lambda\widetilde{\gamma}\zeta e^x>0$ follow from $\lambda>0$, $\sigma>0$, and $\zeta\widetilde{\gamma}>0$. Thus, Φ^{-1} (the inverse function) exists, and direct calculations yield (4.4). Furthermore, by virtue of (4.2), for any $\psi\in\Psi$ there exists $C'_\psi\in(0,\infty)$ such that $|\zeta\psi|+|\zeta(r-\widetilde{b}+\lambda\widetilde{\gamma})+\sigma^2\zeta^2\psi|\leq C'_\psi$ $P\otimes dt$-a.e., and for any $C_1>0$, there exists $C_2\in(0,\infty)$ such that $-C_2\leq\Phi^{-1}(C_1)\leq C_2$. Thus, by combining these, we get the existence of $C_\psi>0$ such that $|\eta(\psi)|\leq C_\psi$ $P\otimes dt$-a.e.. This proves assertion (a).
2) To prove assertion (b), we remark that due to $\zeta\widetilde{\gamma}>0$, we have

$$\frac{\partial\eta(\psi)}{\widetilde{\gamma}\partial\psi}=\frac{\zeta}{\widetilde{\gamma}}\left(\frac{\sigma^2}{\sigma^2+\lambda\widetilde{\gamma}\zeta\exp(\Phi^{-1}(\Delta_\psi))}-1\right)<0,\ \text{where}\ \Delta_\psi:=\zeta(r-\widetilde{b}+\lambda\widetilde{\gamma})+\sigma^2\zeta^2\psi.$$

This proves assertion (b).
3) Thanks to assertion (b), we know that $\eta(\psi)/\widetilde{\gamma}$ is decreasing with respect to ψ, thus both

$$l_\infty:=\lim_{x\longrightarrow-\infty}\frac{\eta(x)}{\widetilde{\gamma}}=\frac{\eta(-\infty)}{\widetilde{\gamma}}\quad\text{and}\quad\bar{l}_\infty:=\lim_{x\longrightarrow\infty}\frac{\eta(x)}{\widetilde{\gamma}}=\frac{\eta(\infty)}{\widetilde{\gamma}}$$

exist and belong to $[-\infty,\infty]$. On the one hand, remark that due to (4.3), we have

$$\eta(\psi)/\widetilde{\gamma}\leq\left(r-\widetilde{b}+\lambda\widetilde{\gamma}\right)/\widetilde{\gamma}\sigma^2=:\Delta^{up}<\infty,\quad\text{for any}\quad\psi.$$

A combination of this fact and assertion (b) yields $-\infty<\eta(0)/\widetilde{\gamma}\leq l_\infty\leq\Delta^{up}<\infty$. Then, by letting ψ to go to $-\infty$ in (4.3), we obtain the first equality in (4.5). To prove the second equality in (4.5), we note that $\bar{l}_\infty\leq\Delta^{up}<\infty$, and we use (4.3) to contradict the assumption $\bar{l}_\infty>-\infty$. This ends the proof of assertion (c).
4) Notice that the second inequality in (4.6) is a direct consequence of

$$\begin{aligned}\frac{\eta(0)-\eta(\psi)}{\widetilde{\gamma}}&=\frac{\zeta\psi}{\widetilde{\gamma}}\left(1-\frac{\sigma^2}{\sigma^2+\lambda\widetilde{\gamma}\zeta\exp(\Phi^{-1}(\Delta'_\psi))}\right)\\&\leq\frac{\zeta\psi}{\widetilde{\gamma}},\quad\text{where}\ \Delta'_\psi\in[\Delta_0,\Delta_\psi],\quad\text{for any}\quad\psi\geq0.\end{aligned}$$

The first equality in (4.6) follows from (4.4), which yields

$$\frac{\eta(n)}{n\widetilde{\gamma}}=-\frac{\zeta}{\widetilde{\gamma}}+\frac{\Phi^{-1}(\Delta_0+\zeta^2\sigma^2n)}{n\zeta\widetilde{\gamma}},$$

and the fact that $\Phi(x)/x$ goes to infinity with x (or equivalently $\Phi^{-1}(y)/y$ goes to zero when y goes to infinity. This ends the proof of the lemma. $\square$

Proof of Lemma 4.3 . For any $\psi \in \Psi$, we denote

$$\overline{M}^{\psi} := \eta(\psi)\sigma \bullet W + \big(\exp(\eta(\psi)\zeta + \zeta^2\psi) - 1\big) \bullet \widetilde{N} \in \mathcal{M}_{loc}.$$

Then, we calculate the predictable quadratic variation of $\overline{M}^{\psi}$, under P, as follows:

$$\langle \overline{M}^{\psi}, \overline{M}^{\psi}\rangle = \int_0^{\cdot} \left\{ \eta(\psi)_s^2\sigma_s^2 + \lambda_s \left(\exp(\eta_s(\psi)\zeta_s + \zeta_s^2\psi_s) - 1\right)^2 \right\} ds.$$

Then, thanks to Lemma 4.2-(a), for any $\psi \in \Psi$, there exists $C_\psi \in (0,\infty)$ such that $\langle \overline{M}^{\psi}, \overline{M}^{\psi}\rangle \leq C_\psi T$, and hence $\langle \overline{M}^{\psi}, \overline{M}^{\psi}\rangle$ is bounded. This yields $\overline{D}^{\psi} \in \mathcal{M}$ for any $\psi \in \Psi$, and assertion (a) is proved. Assertion (b) follows from combining Girsanov's theorem and (4.3). Thus, the rest of this proof focuses on proving assertion (c). To this end, we start by considering $(\psi, \psi') \in \Psi \times \Psi$, applying Yor's formula to $1/\overline{D}^{\psi'}$ and after that to the product $\overline{D}^{\psi}/\overline{D}^{\psi'}$, and then to conclude that (4.7) follows immediately. To prove (4.8), we use Lemma 4.2 and deduce that there exists $C'_n \in (0,\infty)$ such that

$$\sup_{\psi\in\Psi_n} |\frac{\eta(\psi)}{\widetilde{\gamma}}| \leq \sup_{\psi\in\Psi_n} |\frac{\eta(\psi)-\eta(0)}{\widetilde{\gamma}}| + |\frac{\eta(0)}{\widetilde{\gamma}}| \leq \frac{\eta(-n)-\eta(n)}{\widetilde{\gamma}} + |\frac{\eta(0)}{\widetilde{\gamma}}| \leq C'_n. \quad \text{(C.1)}$$

Thanks to [15, Proposition 3.5] and (4.7), we calculate the Hellinger process of order p for $D^{(\psi,\psi')}$ under $R_{\psi'}$ as follows:

$$\begin{aligned}
&\frac{dh_t^{(p)}(D^{(\psi,\psi')}, R_{\psi'})}{dt}\\
&= \frac{1}{2}(\eta_t(\psi) - \eta_t(\psi'))^2\sigma_t^2 + \lambda \exp(\eta(\psi')\zeta + \zeta^2\psi')\frac{e^{p\Delta(\psi,\psi')} - 1 - p(e^{\Delta(\psi,\psi')} - 1)}{p(p-1)},\\
&\Delta(\psi,\psi') := (\eta(\psi) - \eta(\psi'))\zeta + \zeta_t^2(\psi - \psi').
\end{aligned}$$

Thus, by combining this with (C.1), we deduce the existence of $C_n \in (0,\infty)$ such that

$$|\eta(\psi)| + |\eta(\psi')| + h_t^{(p)}(D^{(\psi,\psi')}, R_{\psi'}) \leq C_n, \quad \text{for any} \quad \psi \in \Psi_n, \quad P \otimes dt\text{-a.e.}.$$

Hence, thanks to [15, Theorem 3.9], this latter property implies (4.8). This ends the proof assertion (c), and the proof of the lemma is complete. □

Proof of Lemma 4.10. 1) Herein we prove that for any $n \geq 1$, there exists a positive constant C_n such that

$$\| \sup_{0\leq t\leq T} |Y_t^{\psi}| \|_{L^p(R_0)} \leq C_n \|\xi\|_{L^p(R_0)}, \quad \text{for any} \quad \psi \in \Psi_n, \quad \text{(C.2)}$$

and we prove assertion (a) afterwards. To this end, we consider $n \geq 1$, and use Lemma 4.3-(c), more precisely (4.8) with $\psi' \equiv 0$, and deduce that the R_0-martingale $D^{\psi} = D^{(\psi,0)} = \overline{D}^{\psi}/\overline{D}^{0}$ fulfills the reverse-Hölder inequality of order $q := p/(p-1)$ (usually this is denoted by $D^{\psi} \in \mathcal{R}_q(R_0)$). In other words, there exists a positive constant $C_\psi \in (0,\infty)$ such that

$$E_{R_0}\left[\left(\frac{D_T^{\psi}}{D_t^{\psi}}\right)^q \big| \mathcal{F}_t \right] \leq C_\psi, \quad P\text{-a.s.} \quad t \in [0,T].$$

Thus, by combining this fact with [14, Theorem 2.10], $Y_T^{\psi} = \xi$, and the fact that $\exp(-\int_0^{\cdot} r_s ds)Y^{\psi}$ is a martingale under R_ψ, inequality (C.2) follows immediately.

Then, due to assumption (4.2), the inequality implies that Y^{ψ} is a special semimartingale under R_0, and its Doob-Meyer's decomposition is given by

$$Y^{\psi} = Y_0^{\psi} + M^{\psi} + A^{\psi}, \quad M^{\psi} \in \mathcal{M}_{loc}(R_0), \quad A^{\psi} \in \mathcal{A}_{loc}(R_0) \text{ and is predictable.} \tag{C.3}$$

Thus, Lemma 4.3-(c) again guarantees that $M^{\psi} \in \mathrm{bmo}_q(R_0)$, and hence, thanks to [14, Theorem 4.5], we conclude the existence of $\overline{C}_n \in (0,\infty)$ such that

$$\|Y_0^{\psi}\|_{L^p(R_0)} + \|[M^{\psi}, M^{\psi}]_T^{1/2}\|_{L^p(R_0)} + \|\mathrm{Var}_T(A^{\psi})\|_{L^p(R_0)} \leq \overline{C}_n \| \sup_{0\leq t\leq T} |Y_t^{\psi}| \|_{L^p(R_0)}. \tag{C.4}$$

By combining (C.2) and (C.4), under assumption (4.2), we get $M^{\psi} \in \mathcal{M}^p(R_0)$, and due to the martingale representation theorem, we obtain a unique pair (Z^{ψ}, U^{ψ}) that belongs to $\mathbb{L}^p(W, R_0) \times \mathbb{L}^p(N, R_0)$ and

$$Y^{\psi} = Y_0^{\psi} + Z^{\psi} \bullet W^0 + U^{\psi} \bullet \widetilde{N}^0 + A^{\psi}. \tag{C.5}$$

Then, as $\exp(-\int_0^{\cdot} r_s ds) Y^{\psi} \in \mathcal{M}_{loc}(R_{\psi})$, this equality and Girsanov's theorem yield

$$A^{\psi} = -\int_0^{\cdot} \left((\eta_s(\psi) - \eta_s(0)) \frac{\sigma_s}{\widetilde{\gamma}_s} (\widetilde{\gamma}_s Z_s^{\psi} - \sigma U_s) - r_s Y_s^{\psi} \right) ds. \tag{C.6}$$

On the one hand, by combining this latter claim with (C.4), (C.3), and (C.2), inequality (4.19) follows immediately. On the other hand, by combining (C.6), (C.5), and $Y_T^{\psi} = \xi$, the second equality in (4.17) follows immediately, while the first equality is a direct consequence of the second one and Girsanov's theorem. This ends the proof of assertion (a).

2) Here we prove assertion (b). Then, we remark that for $\psi \in \Psi$, the processes $Z^{\psi} \bullet W^0$ and $U^{\psi} \bullet \widetilde{N}^0$ are R_0-martingales (due to assertion (a)), where $(Y^{\psi}, Z^{\psi}, U^{\psi})$ denotes the solution to (4.17). Thus, by taking the conditional expectation under R_0 on both sides of (4.17), equality (4.21) follows immediately.

3) This part proves assertions (c) and (d). To this end, we remark that assertion (d) is a direct consequence of assertion (c). In fact, for any $\psi_i \in \Psi$, $i = 1, 2$, we set $\Gamma := \{g_{\psi_1} > g_{\psi_2}\}$, and then $\psi_3 := \psi_1 I_{\Gamma} + \psi_2 I_{\Gamma^c}$ satisfies (4.23). As a direct consequence of these inequalities, we deduce that all four sets of assertion (d) are upward directed. This proves assertion (d), and the rest of this proof focuses on proving assertion (c). Therefore, on the one hand, we consider $\psi_i \in \Psi$, $i = 1, 2$, Γ a predictable set, and define the quadruplet $(\psi_3, \overline{Y}, \overline{Z}, \overline{U})$ of processes by

$$\begin{aligned} &\psi_3 := \psi_1 I_{\Gamma} + \psi_2 I_{\Gamma^c},\ d\overline{Y} := I_{\Gamma} d(e^{-B} Y^{\psi_1}) + I_{\Gamma^c} d(e^{-B} Y^{\psi_2}),\ \overline{Y}_T := e^{-B_T}\xi, \\ &\overline{Z} := e^{-B} Z^{\psi_1} I_{\Gamma} + e^{-B} Z^{\psi_2} I_{\Gamma^c}, \quad \overline{U} := e^{-B} U^{\psi_1} I_{\Gamma} + e^{-B} U^{\psi_2} I_{\Gamma^c}. \end{aligned} \tag{C.7}$$

On the other hand, for any $\psi \in \Psi$, the triplet $(e^{-B} Y^{\psi}, e^{-B} Z^{\psi}, e^{-B} U^{\psi})$ is the unique solution to the following BSDE:

$$Y_t = e^{-B_T}\xi + \int_t^T (\eta_s(\psi) - \eta_s(0)) \frac{\sigma_s}{\widetilde{\gamma}_s} \left(\widetilde{\gamma}_s Z_s - \sigma_s U_s\right) ds - \int_t^T Z_s dW_s^0 - \int_t^T U_s d\widetilde{N}_s^0. \tag{C.8}$$

Thus, by virtue of the notation in (C.7) and using (C.8) for each $e^{-B} Y^{\psi_i}$, $i = 1, 2$, we easily derive

$$\overline{Y}_t := e^{-B_T}\xi + \int_t^T (\eta_s(\psi_3) - \eta_s(0)) \frac{\sigma_s}{\widetilde{\gamma}_s} \left(\widetilde{\gamma}_s \overline{Z}_s - \sigma_s \overline{U}_s\right) ds - \int_t^T \overline{Z}_s dW_s^0 - \int_t^T \overline{U}_s d\widetilde{N}_s^0.$$

This proves that $(\overline{Y},\overline{Z},\overline{U})$ is a solution to (C.8) when $\psi=\psi_3$. Then, due to the uniqueness of the solution to this BSDE, we deduce that $(\overline{Y},\overline{Z},\overline{U})$ coincides with $(e^{-B}Y^{\psi_3},e^{-B}Z^{\psi_3},e^{-B}U^{\psi_3})$, and hence (4.22) follows immediately. This proves assertion (c) and completes the proof of the lemma. $\square$

Proof of Lemma 4.11 . It is clear that assertion (a) follows immediately from Lemma 4.10 (see assertion (b), (c), and (d)). To prove assertion (b), we remark that there exists a sequence $(\psi_k)_k\subset\Psi_n$ such that $|g_{\psi_k}|$ converges $P\otimes dt$-almost everywhere to $\operatorname*{ess\,sup}_{\psi\in\Psi_n}|g_\psi|$. Then, thanks to Fatou, we derive

$$
\begin{aligned}
E_{R_0}\left[\int_0^T\widetilde{g}_n(t)dt\right] &\leq E_{R_0}\left[\int_0^T\operatorname*{ess\,sup}_{\psi\in\Psi_n}|g_\psi(t)|dt\right]=E_{R_0}\left[\int_0^T\lim_{k\longrightarrow\infty}|g_{\psi_k}(t)|dt\right]\\
&\leq\lim_{k\longrightarrow\infty}E_{R_0}\left[\int_0^T|g_{\psi_k}(t)|dt\right]\leq\sup_{\psi\in\Psi_n}E_{R_0}\left[\int_0^T|g_\psi(t)|dt\right]=\widetilde{C}_n.
\end{aligned}
$$

Thus, by combining this inequality with Lemma 4.10-(a) (see (4.19) precisely), assertion (b) follows immediately.

To prove assertion (c) we notice that due to Lemma 4.10-(c)-(d), we always have $g_n=g_{\psi_n}\leq\widetilde{g}_n$, and the rest of the proof focuses on the converse inequality. Let $(\psi_k)_k\subset\Psi_n$ such that g_{ψ_k} converges to $\widetilde{g}_n$, $P\otimes dt$-a.e.. Hence, by virtue of assertion (b) (which implies that $\operatorname*{ess\,sup}_{\psi\in\Psi_n}|g_\psi|\in L^1(R_0\otimes dt)$) and the dominated convergence theorem, this convergence holds in $L^1(P\otimes dt)$. Thus, we derive

$$
\begin{aligned}
&E\left[e^{B_t-B_T}\xi+\int_t^T e^{B_t-B_u}\widetilde{g}_n(u)du\big|\mathcal{F}_t\right]\\
&=\lim_{k\longrightarrow\infty}E\left[e^{B_t-B_T}\xi+\int_t^T e^{B_t-B_u}g_{\psi_k}(u)du\big|\mathcal{F}_t\right]\\
&=\lim_{k\longrightarrow\infty}Y_t^{\psi_k}\leq Y_t^{(n)}=E\left[e^{B_t-B_T}\xi+\int_t^T e^{B_t-B_u}g_n(u)du\big|\mathcal{F}_t\right].
\end{aligned}
$$

Then, by setting $t=0$ and taking expectation under R_0 afterwards, we get

$$
E\left[\int_0^T e^{-B_u}\widetilde{g}_n(u)du\right]\leq E\left[\int_0^T e^{-B_u}g_n(u)du\right].
$$

Thus, by combining this with $g_n\leq\widetilde{g}_n$ $P\otimes dt$-a.e., we obtain $g_n=\widetilde{g}_n$, $P\otimes dt$-a.e.. This ends the proof of the lemma. $\square$

Proof of Lemma 4.12. Let τ be a stopping time. Thanks to Lemma 4.11-(a), for any $n\geq 1$, we derive

$$
\begin{aligned}
E_{R_0}\left[\int_\tau^T g_n(s)ds\big|\mathcal{F}_\tau\right]&=E_{R_0}\left[-\xi+Y_\tau^{(n)}\big|\mathcal{F}_\tau\right]=E_{R_0}\left[-\xi+Y_\tau^{\psi_n}\big|\mathcal{F}_\tau\right]\\
&\leq E_{R_0}\left[\xi^-+\operatorname*{ess\,sup}_{\psi\in\Psi}\sup_{\tau\in\mathcal{T}}(Y_\tau^\psi)^+\big|\mathcal{F}_\tau\right].
\end{aligned}
$$

Then, by combining this inequality with (4.9), assertion (a) follows immediately.

To prove assertion (b), we apply Garsia's lemma to the predictable process $\int_0^\cdot \widetilde{g}(u)du$, see [18, Theorem 99], and use assertion (a) to derive

$$
\begin{aligned}
E_{R_0}\left[\left(\int_0^T \widetilde{g}(u)du\right)^p\right] &\leq C_p E_{R_0}\left[(\xi^-)^p\right] + C_p E_{R_0}\left[\left(\operatorname*{ess\,sup}_{\psi\in\Psi}\sup_{\tau\in\mathcal{T}}(Y_\tau^\psi)^+\right)^p\right] \\
&= C_p E_{R_0}\left[(\xi^-)^p\right] + C_p \sup_{\psi\in\Psi} E_{R_0}\left[\left(\sup_{\tau\in\mathcal{T}}(Y_\tau^\psi)^+\right)^p\right].
\end{aligned}
\tag{C.9}
$$

The latter equality follows from combining the convergence monotone theorem and Lemma 4.10-(d), which states that the family $\{Y^\psi:\ \psi\in\Psi\}$ is upward direct.

Now note that $\sup_{\tau\in\mathcal{T}}(Y_\tau^\psi)^+ \leq \sup_{0\leq t\leq T} E_{R_0}[D_T^\psi\xi^+(D_t^\psi)^{-1}\Big|\mathcal{F}_t]$, and consider

$$\tau_a := \inf\{t\geq 0:\ (Y_t^\psi)^+ > a\},\quad a\in(0,\infty),\quad \inf(\emptyset):=\infty.$$

Then, we have

$$
\begin{aligned}
&pa^{p-1}R_0\left(\sup_{0\leq t\leq T}(Y_t^\psi)^+ > a\right)\\
&\leq pa^{p-1}R_0\left(\tau_a\leq T\right)\\
&\leq pa^{p-2}E_{R_0}[(Y_{\tau_a}^\psi)^+ I_{\{\tau_a\leq T\}}]\\
&\leq pa^{p-2}E_{R_0}\left[\frac{D_T^\psi}{D_{T\wedge\tau_a}^\psi}\xi^+ I_{\{\sup\limits_{0\leq t\leq T\wedge\tau_a}(Y_t^\psi)^+\geq a\}}\right].
\end{aligned}
\tag{C.10}
$$

These are due to $(\tau_a\leq T)\subset((Y_{T\wedge\tau_a}^\psi)^+\geq a)\subset(\sup_{0\leq t\leq T\wedge\tau_a}(Y_t^\psi)^+\geq a)$. Then, by integrating with respect to da on both sides of (C.10) and setting $q:=p/(p-1)$, we get

$$
\begin{aligned}
&\|\sup_{0\leq t\leq T}(Y_t^\psi)^+\|_{L^p(R_0)}^p\\
&\leq qE_{R_0}\left[\frac{D_T^\psi}{D_{T\wedge\tau_a}^\psi}\xi^+\left(\sup_{0\leq t\leq T\wedge\tau_a}(Y_t^\psi)^+\right)^{p-1}\right]\\
&= qE_{R_\psi}\left[\frac{\xi^+}{D_{T\wedge\tau_a}^\psi}\left(\sup_{0\leq t\leq T\wedge\tau_a}(Y_t^\psi)^+\right)^{p-1}\right]\\
&\leq q\|\xi^+(D_{T\wedge\tau_a}^\psi)^{-1/p}\|_{L^p(R_\psi)}\|(D_{T\wedge\tau_a}^\psi)^{-1/q}\left(\sup_{0\leq t\leq T\wedge\tau_a}(Y_t^\psi)^+\right)^{p-1}\|_{L^q(R_\psi)}\\
&= q\left(E_{R_0}[D_T^\psi(D_{T\wedge\tau_a}^\psi)^{-1}(\xi^+)^p]\right)^{1/p}\|\sup_{0\leq t\leq T\wedge\tau_a}(Y_t^\psi)^+\|_{L^p(R_0)}^{p/q}\\
&\leq q\left(E_{R_0}[D_T^\psi(D_{T\wedge\tau_a}^\psi)^{-1}(\xi^+)^p]\right)^{1/p}\|\sup_{0\leq t\leq T}(Y_t^\psi)^+\|_{L^p(R_0)}^{p/q}.
\end{aligned}
$$

By combining this latter inequality with $\|\sup_{0\leq t\leq T}(Y_t^\psi)^+\|_{L^p(R_0)}<\infty$ (which is due to Lemma 4.10-(a)), we get

$$\|\sup_{0\leq t\leq T}(Y_t^\psi)^+\|_{L^p(R_0)}\leq q\sup_{\tau\in\mathcal{T},\psi\in\Psi}\left(E_{R_0}[D_T^\psi(D_{T\wedge\tau_a}^\psi)^{-1}(\xi^+)^p]\right)^{1/p}. \tag{C.11}$$

Thus, assertion (b) follows from combining (C.9) and (C.11). This ends the proof of the lemma. $\square$

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REFERENCES

[1] A. Aksamit, T. Choulli, J. Deng and M. Jeanblanc, No-arbitrage up to random horizon for quasi-left-continuous models, *Finance and Stochastics*, **21** (2017), 1103-1139.

[2] A. Badescu, R. J. Elliott and T. K. Siu, Esscher transforms and consumption-based models, *Insurance: Mathematics and Economics*, **45** (2009), 337-347.

[3] F. E. Benth and C. Sgarra, The risk premium and the Esscher transform in power markets, *Stochastic Analysis and Applications*, **30** (2012), 20-43.

[4] M. Bernhart, H. Pham, P. Tankov and X. Warin, Swing options valuation: A BSDE with constrained jumps approach, In: *Numerical Methods in Finance*, Springer, 2012, 379-400.

[5] A. Bondi, D. Radojičić and T. Rheinländer, Comparing two different option pricing methods, *Risks*, **8** (2020), 108.

[6] W. Boughamoura and F. Trabelsi, On two-parametric Esscher transform for geometric CGMY Lévy processes, *International Journal of Operational Research*, **19** (2014), 280-301.

[7] H. Bühlmann, An economic premium principle, *ASTIN Bulletin: The Journal of the IAA*, **11** (1980), 52-60.

[8] H. Bühlmann, F. Delbaen, P. Embrechts and A. N. Shiryaev, No-arbitrage, change of measure and conditional Esscher transforms, *CWI quarterly*, **9** (1996), 291-317.

[9] H. Bühlmann, F. Delbaen, P. Embrechts and A. N. Shiryaev, On Esscher transforms in discrete finance models, *ASTIN Bulletin: The Journal of the IAA*, **28** (1998), 171-186.

[10] T. Choulli, J. Deng and J. Ma, How non-arbitrage, viability and numéraire portfolio are related, *Finance and Stochastics*, **19** (2015), 719-741.

[11] T. Choulli, E. Elazkany and M. Vanmaele, Applications of the second-order Esscher pricing in risk management, Work in progress. A version is available at arXiv:2407.03960, (2024)

[12] T. Choulli and Ch. Stricker, More on minimal entropy-Hellinger martingale measure, *Mathematical Finance*, **16** (2006), 1-19.

[13] T. Choulli and Ch. Stricker, Minimal entropy-Hellinger martingale measure in incomplete markets, *Mathematical Finance*, **15** (2005), 465-490.

[14] T. Choulli, Ch. Stricker and L. Krawczyk, On Fefferman and Burkholder-Davis-Gundy inequalities for $\mathcal{E}$-martingales, *Probability Theory and Related Fields*, **113** (1999), 571-597.

[15] T. Choulli, Ch. Stricker and J. Li, Minimal Hellinger martingale measures of order q, *Finance and Stochastics*, **11** (2007), 399-427.

[16] J. Cvitanic, I. Karatzas and H. M. Soner, Backward stochastic differential equations with constraints on the gains-process, *The Annals of Probability*, **26** (1998), 1522-1551.

[17] F. Delbaen and J. Haezendonck, A martingale approach to premium calculation principles in an arbitrage free market, *Insurance: Mathematics and Economics*, **8** (1989), 269-277.

[18] C. Dellacherie and P.-A. Meyer, *Probabilités et Potentiel*, Volume 2, chapitres V-VIII: Théorie des Martingales, Hermann, Paris, 1980.

[19] M. Dellacherie, B. Maisonneuve and P.-A. Meyer, *Probabilités et Potentiel*, Volume 5, chapitres XVII-XXIV: Processus de Markov, compléments aux Calculs Stochastiques. Hermann, Paris, 1992.

[20] R. Elie and I. Kharroubi, Adding constraints to BSDEs with jumps: An alternative to multidimensional reflections, *ESAIM: Probability and Statistics*, **18** (2014), 233-250.

[21] R. J. Elliott and T. K. Siu, A generalized Esscher transform for option valuation with regime switching risk, *Quantitative Finance*, **22** (2022), 691-705.

[22] F. Esscher, On the probability function in collective theory of risk, *Skandinavisk Aktuarietidskrift*, **1932** (1932), 175-195.

[23] H. Föllmer and D. Kramkov, Optional decompositions under constraints, *Probability Theory and Related Fields*, **109** (1997), 1-25.

[24] H. U. Gerber and E. S. Shiu, Martingale approach to pricing perpetual American options, *Astin Bulletin*, **24** (1994), 195-220.

[25] H. U. Gerber and E. S. Shiu, Option pricing by Esscher transform, *Transaction of the Society of Actuaries*, **46** (1994), 99-140.

[26] H. U. Gerber and E. S. Shiu, Actuarial bridges to dynamic hedging and option pricing, *Insurance: Mathematics and Economics*, **18** (1996), 183-218.

[27] S.-w. He, J.-g. Wang and J.-a. Yan, *Semimartingale Theory and Stochastic Calculus*, Routledge, New York, 2019.

[28] F. Hubalek and C. Sgarra, On the Esscher transforms and other equivalent martingale measures for Barndorff-Nielsen and Shephard stochastic volatility models with jumps, *Stochastic Processes and their Applications*, **119** (2009), 2137-2157.

[29] F. Hubalek and C. Sgarra, Esscher transforms and the minimal entropy martingale measure for exponential Lévy models, *Quantitative Finance*, **6** (2006), 125-145.

[30] J. Jacod and A. N. Shiryaev, *Limit Theorems for Stochastic Processes*, vol. 288, Springer Science & Business Media, 2013.

[31] J. Kallsen and A. N. Shiryaev, The cumulant process and Esscher's change of measure, *Finance and Stochastics*, **6** (2002), 397-428.

[32] I. Kharroubi, J. Ma, H. Pham and J. Zhang, Backward SDEs with constrained jumps and quasi-variational inequalities, *The Annals of Probability*, **38** (2010), 794-840.

[33] J. W. Lau, and T. K. Siu, On option pricing under a completely random measure via a generalized Esscher transform, *Insurance: Mathematics and Economics*, **43** (2008), 99-107.

[34] J.-P. Lepeltier and M. Xu, Penalization method for reflected backward stochastic differential equations with one r.c.l.l. barrier, *Statistics & Probability Letters*, **75** (2005), 58-66.

[35] A. Monfort and F. Pegoraro, Asset pricing with second-order Esscher transforms, *Journal of Banking & Finance*, **36** (2012), 1678-1687.

[36] M. Monoyios, The minimal entropy measure and an Esscher transform in an incomplete market model, *Statistics & Probability Letters*, **77** (2007), 1070-1076.

[37] S. Peng, Monotonic limit theorem of BSDE and nonlinear decomposition theorem of Doob–Meyer's type, *Probability Theory and Related Fields*, **113** (1999), 473-499.

[38] S. Peng and M. Xu, Reflected BSDE with a constraint and its applications in an incomplete market, *Bernoulli*, **16** (2010), 614-640.

[39] Th. Rheinländer and J. Sexton, *Hedging Derivatives,* Advanced Series on Statistical Science & Applied Probability, Volume 15. World Scientific Publishing Company, 2011.

[40] W. Schoutens, *Lévy Processes in Finance: Pricing Financial Derivatives*, Wiley Series in Probability and Statistics, John Wiley & Sons Ltd., Chichester, 2003.

[41] C. Stricker, Quelques remarques sur la topologie des semimartingales. Applications aux intégrales stochastiques, In: Azéma, J., Yor, M. (eds), *Séminaire de Probabilités XV 1979/80*, Lecture Notes in Mathematics, volume 850, Springer, Berlin, Heidelberg, 1981, 499-522.

[42] M. Xu, Superhedging problem under ratio constraint: BSDE approaches with Malliavin calculus, *Numerical Algebra, Control and Optimization*, **13** (2023), 664-680.

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