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Non-isometric translation and modulation invariant Hilbert spaces

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ABSTRACT

Let $\mathcal{H}$ be a Hilbert space, continuously embedded in $\mathscr{S}'(\mathbf{R}^d)$, and which contains at least one non-zero element in $\mathscr{S}'(\mathbf{R}^d)$. If there is a constant $C_0 > 0$ such that

$$\|e^{i\langle\,\cdot\,,\xi\rangle} f(\,\cdot\,-x)\|_{\mathcal{H}} \le C_0\|f\|_{\mathcal{H}}, \qquad f\in\mathcal{H},\ x,\xi\in\mathbf{R}^d,$$

then we prove that $\mathcal{H} = L^2(\mathbf{R}^d)$, with equivalent norms.

0. Introduction

In [18] it is proved that a suitable non-trivial Hilbert space $\mathcal{H} \subseteq \mathscr{S}'(\mathbf{R}^d)$, which is norm preserved under translations

$$f \mapsto f(\,\cdot\,-x)$$

and modulations

$$f \mapsto e^{i\langle\,\cdot\,,\xi\rangle} f$$

is equal to $L^2(\mathbf{R}^d)$. (See [11] or Section 1 for notations.) It is here also proved that the norms between $\mathcal{H}$ and $L^2(\mathbf{R}^d)$ only differ by a multiplicative constant, i.e. for some constant $C > 0$ one has

$$\|f\|_{\mathcal{H}} = C\|f\|_{L^2(\mathbf{R}^d)}, \qquad f\in\mathcal{H} = L^2(\mathbf{R}^d). \tag{0.1}$$

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This property is analogous to Feichtinger's minimization property, which shows that the Feichtinger algebra $S_0(\mathbf{R}^d)$, which is the same as the modulation space $M^{1,1}(\mathbf{R}^d)$, is the smallest non-trivial Banach space of tempered distributions which is norm invariant under translations and modulations. (See e.g. [8] for general facts about modulation spaces.)

The condition on non-triviality in [18] is simply that $\mathcal{H}$ should contain at least one non-trivial element in $M^{1,1}(\mathbf{R}^d)$. Hence the main result in [18] can be formulated as follows.

Theorem 0.1. *Let $\mathcal{H}$ be a Hilbert space which is continuously embedded in $\mathscr{S}'(\mathbf{R}^d)$, and norm preserved under translations and modulations. If $\mathcal{H}$ contains a non-zero element in $M^{1,1}(\mathbf{R}^d)$, then $\mathcal{H} = L^2(\mathbf{R}^d)$, and (0.1) holds for some constant $C > 0$ which is independent of $f \in \mathcal{H} = L^2(\mathbf{R}^d)$.*

An alternative approach, using the Bargmann transform, to reach similar properties is given by Bais, Pinlodi, and Venku Naidu in [2]. In fact, by using their result [2, Theorem 3.1] in combination with some well-known arguments in the distribution theory, one obtains the following improvement of Theorem 0.1.

Theorem 0.2. *Let $\mathcal{H}$ be a Hilbert space which is continuously embedded in $\mathscr{S}'(\mathbf{R}^d)$, and norm preserved under translations and modulations. If $\mathcal{H}$ is non-trivial, then $\mathcal{H} = L^2(\mathbf{R}^d)$, and (0.1) holds for some constant $C > 0$ which is independent of $f \in \mathcal{H} = L^2(\mathbf{R}^d)$.*

We observe that the condition

$$|\langle f, \phi\rangle| \le C\|f\|_{\mathcal{H}}, \qquad f \in \mathcal{H},\ \phi(x) = e^{-\frac{1}{2}|x|^2}, \tag{0.2}$$

for some constant $C > 0$ in the hypothesis in [2, Theorem 3.1] is absent in Theorem 0.2. Hence Theorem 0.2 is slightly more general than [2, Theorem 3.1]. The notation $\langle\,\cdot\,,\,\cdot\,\rangle$ in (0.2) stands for the dual pairing between a distribution and a test function.

Here we remark that we may relax the hypothesis in Theorem 0.2 by assuming that $\mathcal{H}$ is continuously embedded in the space $\mathscr{D}'(\mathbf{R}^d)$ instead of the smaller space $\mathscr{S}'(\mathbf{R}^d)$. In fact, let $\mathcal{B}$ be a translation invariant Banach space which is continuously embedded in $\mathscr{D}'(\mathbf{R}^d)$ and satisfies

$$\|f(\,\cdot\, - x)\|_{\mathcal{B}} \le C\|f\|_{\mathcal{B}},$$

for some constant $C \ge 1$ which is independent of $f \in \mathcal{B}$ and $x \in \mathbf{R}^d$. Then it follows by some standard arguments that $\mathcal{B}$ is continuously embedded in $\mathscr{S}'(\mathbf{R}^d)$ (see e.g. [18, Proposition 1.5]).

In the paper we investigate properties on weaker forms of translation and modulation invariant Hilbert spaces compared to [2,18]. More precisely we show that except for the norm identity (0.1), Theorem 0.2 still holds true after the condition that $\mathcal{H}$ is norm preserved under translations and modulations, is relaxed into the weaker condition

$$\|f(\,\cdot\, - x)e^{i\langle\,\cdot\,,\xi\rangle}\|_{\mathcal{H}} \le C_0\|f\|_{\mathcal{H}}, \qquad f \in \mathcal{H},\ x, \xi \in \mathbf{R}^d. \tag{0.3}$$

In our extension, the identity (0.1) should be replaced by the norm equivalence

$$C_0^{-1}C\|f\|_{L^2(\mathbf{R}^d)} \le \|f\|_{\mathcal{H}} \le C_0C\|f\|_{L^2(\mathbf{R}^d)}, \qquad f \in \mathcal{H} = L^2(\mathbf{R}^d). \tag{0.4}$$

More precisely our extension of Theorem 0.2 is the following.

Theorem 0.3. *Let $\mathcal{H}$ be a Hilbert space which is continuously embedded in $\mathscr{S}'(\mathbf{R}^d)$, and such that (0.3) holds true for some constant $C_0 \ge 1$ which is independent of $f \in \mathcal{H}$ and $x, \xi \in \mathbf{R}^d$. If $\mathcal{H}$ is non-trivial, then $\mathcal{H} = L^2(\mathbf{R}^d)$, and (0.4) holds for some constant $C > 0$ which is independent of $f \in \mathcal{H} = L^2(\mathbf{R}^d)$.*

The key step for the proof of Theorem 0.3 is to find an equivalent Hilbert norm to $\| \cdot \|_{\mathcal{H}}$, which is norm preserved under translations and modulations. The result then follows from Theorem 0.2. As a first idea one may try to use the equivalent norm

$$\|f\|_{\mathcal{B}} \equiv \sup_{x,\xi\in\mathbf{R}^d} \left(\|f(\,\cdot\, - x)e^{i\langle\,\cdot\,,\xi\rangle}\|_{\mathcal{H}} \right).$$

A straight-forward control shows that this norm is invariant under translations and modulations. On the other hand, it seems $\|f\|_{\mathcal{B}}$ might fail to be a Hilbert norm, and thereby not being suitable for applying Theorem 0.2.

In our approach to find the sought Hilbert norm, we use some ideas in the construction of $\| \cdot \|_{\mathcal{B}}$ above, but replace the supremum with mean-values of the form

$$\|f\|_{[R]}^2 \equiv (2R)^{-2d} \iint\limits_{[-R,R]^{2d}} \|f(\,\cdot\, - x)e^{i\langle\,\cdot\,,\xi\rangle}\|_{\mathcal{H}}^2 \, dx d\xi,$$

when $R > 0$. It follows that each $\| \cdot \|_{[R]}$ is a Hilbert norm which is uniformly equivalent to $\| \cdot \|_{\mathcal{H}}$. On the other hand, none of $\| \cdot \|_{[R]}$ need to be translation nor modulation invariant. However, by increasing R, it follows that $\| \cdot \|_{[R]}$ becomes, in some sense, closer to being translation and modulation invariant. From suitable limit process, letting R tending to infinity, we are able to extract a sought equivalent Hilbert norm which is norm preserved under translations and modulations. (See Lemma 1.3.)

In Section 3 we also present some improvements of this result, where it is assumed that $\mathcal{H}$ in Theorem 0.3 is embedded in suitable larger (ultra-)distribution spaces which contain $\mathscr{S}'(\mathbf{R}^d)$ (see Theorem 3.1). These investigations are based on some general properties of translation and modulation invariant Banach spaces (see Proposition 3.3).

The proofs of Theorem 0.3 and its extensions, are, among others, based on the fact that the involved Hilbert spaces are separable. In Section 2 we verify such facts, by using the Bargmann transform to transfer the questions to Hilbert spaces of *entire functions*. In the end we conclude that any Hilbert space which is continuously embedded in $\mathscr{S}'(\mathbf{R}^d)$ (or in some even larger distribution spaces), must be separable. (See Propositions 1.2 and 2.2.)

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1. Proof of the main result

In this section we prove our main result Theorem 0.3. First we show how the condition (0.2) in [2, Theorem 3.1] can be removed, which leads to Theorem 0.2. Then we prove a lemma, which in combination with Theorem 0.2, essentially leads to Theorem 0.3.

First we have the following lemma.

Lemma 1.1. *Suppose that $\mathcal{H}$ is continuously embedded in $\mathscr{S}'(\mathbf{R}^d)$. Then there is a constant $C > 0$ such that* (0.2) *holds.*

Lemma 1.1 follows by a straight-forward continuity argument. (See e.g. the arguments in [14, p. 35].) In order to assist the reader we present a proof in Section 2 for a more general result (see Lemma 2.1 and its proof).

Proof of Theorem 0.2. The result follows by combining [2, Theorem 3.1] with Lemma 1.1. The details are left for the reader. □

The proof of Theorem 0.3 is based on the fact that the involved Hilbert spaces are separable. This fact is guaranteed by the following proposition.

Proposition 1.2. *Suppose that the Hilbert space $\mathcal{H}$ is continuously embedded in $\mathscr{S}'(\mathbf{R}^d)$. Then $\mathcal{H}$ is separable.*

We observe that neither translation nor modulation invariant hypotheses are imposed on the Hilbert spaces in Proposition 1.2.

It is expected that Proposition 1.2 is available in the literature. For completeness we give a proof of a generalized result in Section 2 (see Proposition 2.2 and its proof).

Lemma 1.3. *Suppose that $\mathcal{H}$ satisfies the hypothesis in Theorem 0.3. Then there is a norm $\|\cdot\|$ on $\mathcal{H}$ with the following properties:*

(1) $\|\cdot\|$ *is equivalent to* $\|\cdot\|_{\mathcal{H}}$;
(2) $\|\cdot\|$ *is a Hilbert norm;*
(3) $\|e^{i\langle\,\cdot\,,\xi\rangle}f(\,\cdot\,-x)\| = \|f\|$ *for every* $f\in\mathcal{H}$ *and* $x,\xi\in\mathbf{R}^d$.

Proof. Let $Q_R=[-R,R]^{2d}$ be the cube in $\mathbf{R}^{2d}$ with center at the origin and side length $2R$, $R\ge 1$. Then the volume of Q_R is given by $|Q_R|=(2R)^{2d}$. We define the norm $\|\cdot\|_{[R]}$ by the formula

$$\|f\|_{[R]}^2 = \frac{1}{|Q_R|}\iint_{Q_R}\|e^{i\langle\,\cdot\,,\xi\rangle}f(\,\cdot\,-x)\|_{\mathcal{H}}^2\,dxd\xi,\qquad f\in\mathcal{H}.$$

By the definition it follows that $\|\cdot\|_{[R]}$ is equivalent to $\|\cdot\|_{\mathcal{H}}$, and (0.3) gives

$$C_0^{-1}\|f\|_{\mathcal{H}}\le\|f\|_{[R]}\le C_0\|f\|_{\mathcal{H}},\qquad f\in\mathcal{H}. \tag{1.1}$$

Furthermore, $\|\cdot\|_{[R]}$ is a norm which arises from the scalar product

$$(f,g)_{[R]} = \frac{1}{|Q_R|}\iint_{Q_R}(e^{i\langle\,\cdot\,,\xi\rangle}f(\,\cdot\,-x),e^{i\langle\,\cdot\,,\xi\rangle}g(\,\cdot\,-x))_{\mathcal{H}}\,dxd\xi,\quad f,g\in\mathcal{H}.$$

Now let $\mho_0$ be a dense countable subset of $\mathcal{H}$ containing 0, and let $\mho$ be the smallest set which contains $\mho_0$, and is closed under multiplications by $\pm i$, additions and subtractions. Then $\mho$ is countable. Note that $\mho_0$ and $\mho$ exist, because $\mathcal{H}$ is separable, due to Proposition 1.2. Let $\mho=\{f_j\}_{j=1}^\infty$ be a counting of $\mho$. Since

$$C_0^{-1}\|f_1\|_{\mathcal{H}}\le\|f_1\|_{[m]}\le C_0\|f_1\|_{\mathcal{H}},$$

for every $m\in\mathbf{N}$, there is a subsequence $\{m_k\}_{k=1}^\infty$ of $\mathbf{N}$ such that

$$\|f_1\|\equiv\lim_{k\to\infty}\|f_1\|_{[m_k]}$$

exists. By Cantor's diagonalization principle, there is a subsequence $\{n_k\}_{k=1}^{\infty}$ of $\{m_k\}_{k=1}^{\infty}$ such that

$$\|f_j\| \equiv \lim_{k\to\infty} \|f_j\|_{[n_k]}$$

exists for every $f_j \in \mho$, and by (1.1) we get

$$C_0^{-1}\|f\|_{\mathcal{H}} \le \|f\| \le C_0\|f\|_{\mathcal{H}}, \tag{1.2}$$

when $f \in \mho$.

Next suppose that $f \in \mathcal{H}$ is arbitrary, and let $\{f_{0,j}\}_{j=1}^{\infty} \subseteq \mho_0$, be chosen such that

$$\lim_{j\to\infty} \|f - f_{0,j}\|_{\mathcal{H}} = 0.$$

Then

$$\|f_{0,j} - f_{0,k}\| \le C_0\|f_{0,j} - f_{0,k}\|_{\mathcal{H}} \to 0, \quad \text{as} \quad j, k \to \infty.$$

Since $|\|f_{0,j}\| - \|f_{0,k}\|| \le \|f_{0,j} - f_{0,k}\|$, it follows that $\{\|f_{0,j}\|\}_{j=1}^{\infty}$ is a Cauchy sequence in $\mathbf{R}$. Hence

$$\|f\| \equiv \lim_{j\to\infty} \|f_{0,j}\|$$

exists and is independent of the chosen particular sequence $\{f_{0,j}\}_{j=1}^{\infty}$. Since (1.2) holds for any $f_{0,j} \in \mho$, it follows from the recent estimates and limit properties that (1.2) extends to any $f \in \mathcal{H}$. This gives (1).

For $f, g \in \mathcal{H}$, their scalar product $(f, g)_{[n_k]}$ can be evaluated by

$$(f,g)_{[n_k]} = \frac{1}{4}\left(\|f+g\|_{[n_k]}^2 - \|f-g\|_{[n_k]}^2 + i\|f+ig\|_{[n_k]}^2 - i\|f-ig\|_{[n_k]}^2.\right)$$

By letting k tends to ∞, it follows that $\| \cdot \|$ is a Hilbert norm with scalar product

$$(f,g) \equiv \frac{1}{4}\left(\|f+g\|^2 - \|f-g\|^2 + i\|f+ig\|^2 - i\|f-ig\|^2\right),$$

giving that $\| \cdot \|$ fulfills (2).

It remains to prove that (3) holds. By repetition it suffices to prove

$$\|e^{i\langle\,\cdot\,,\xi\rangle} f(\,\cdot\, - x)\| = \|f\| \tag{1.3}$$

when $x = 0$ and $\xi = \xi_j e_j$ or when $\xi = 0$ and $x = x_j e_j$ for some $x_j, \xi_j \in \mathbf{R}$. Here e_j denotes the vector of order j in the standard basis of $\mathbf{R}^d$. Then we only prove (1.3) for $\xi = 0$ and $x = x_j e_j$. The other cases follow by similar arguments and are left for the reader.

Let R be chosen such that $R > |x_j|$. Then Q_R and $-x_j e_j + Q_R$ intersect. We have

$$\begin{aligned}
\|f(\,\cdot\, - x_j e_j)\|_{[R]}^2 &= \frac{1}{|Q_R|}\iint_{Q_R} \|e^{i\langle\,\cdot\,,\eta\rangle} f(\,\cdot\, - (x_j e_j + y))\|_{\mathcal{H}}^2 \, dy d\eta \\
&= \frac{1}{|Q_R|}\iint_{-x_j e_j + Q_R} \|e^{i\langle\,\cdot\,,\eta\rangle} f(\,\cdot\, - y)\|_{\mathcal{H}}^2 \, dy d\eta \\
&= \frac{1}{|Q_R|}\iint_{Q_R} \|e^{i\langle\,\cdot\,,\eta\rangle} f(\,\cdot\, - y)\|_{\mathcal{H}}^2 \, dy d\eta + E_R(f),
\end{aligned}$$

where

$$E_R(f) = \frac{1}{|Q_R|} \iint\limits_{-x_j e_j + Q_R} \|e^{i\langle\,\cdot\,,\eta\rangle} f(\,\cdot\, - y)\|_{\mathcal{H}}^2 \, dy d\eta - \frac{1}{|Q_R|} \iint\limits_{Q_R} \|e^{i\langle\,\cdot\,,\eta\rangle} f(\,\cdot\, - y)\|_{\mathcal{H}}^2 \, dy d\eta.$$

This gives

$$\|f(\,\cdot\, - x_j e_j)\|_{[R]}^2 = \|f\|_{[R]}^2 + E_R(f). \tag{1.4}$$

If

$$\Delta_R = \big((-x_j e_j + Q_R) \setminus Q_R\big) \bigcup \big(Q_R \setminus (-x_j e_j + Q_R)\big),$$

then it follows that $|\Delta_R| = 2|x_j|(2R)^{2d-1}$ and that

$$|E_R(f)| \le \frac{1}{|Q_R|} \iint\limits_{\Delta_R} \|e^{i\langle\,\cdot\,,\eta\rangle} f(\,\cdot\, - y)\|_{\mathcal{H}}^2 \, dy d\eta.$$

This gives

$$\begin{aligned} |E_R(f)| &\le \frac{1}{|Q_R|} \iint\limits_{\Delta_R} \|e^{i\langle\,\cdot\,,\eta\rangle} f(\,\cdot\, - y)\|_{\mathcal{H}}^2 \, dy d\eta \\ &\le \frac{C_0}{|Q_R|} \iint\limits_{\Delta_R} \|f\|_{\mathcal{H}}^2 \, dy d\eta \\ &= \frac{C_0 |\Delta_R| \|f\|_{\mathcal{H}}^2}{|Q_R|} = \frac{C_0 |x_j| \|f\|_{\mathcal{H}}^2}{R}, \end{aligned}$$

which tends to zero as R turns to infinity.

By letting $R = n_k$ in the previous analysis, (1.4) gives

$$\begin{aligned} \|f(\,\cdot\, - x_j e_j)\| &= \lim_{k\to\infty} \|f(\,\cdot\, - x_j e_j)\|_{[n_k]}^2 \\ &= \lim_{k\to\infty} \left(\frac{1}{|Q_{n_k}|} \iint\limits_{Q_{n_k}} \|e^{i\langle\,\cdot\,,\eta\rangle} f(\,\cdot\, - y)\|_{\mathcal{H}}^2 \, dy d\eta + E_{n_k}(f) \right) \\ &= \|f\| + 0, \end{aligned}$$

which gives (3) and thereby the result. □

Proof of Theorem 0.3. By Lemma 1.3, we may replace the norm for $\mathcal{H}$ by an equivalent norm which is invariant under translations and modulations. The result now follows from Theorem 0.2. □

2. Separability of Hilbert spaces embedded in distribution spaces

In this section we prove that Hilbert spaces which are continuously embedded in suitable distribution spaces are separable. We remark that these distribution spaces can be significantly larger than the set of tempered distributions, $\mathscr{S}'(\mathbf{R}^d)$. Especially we deduce a generalization of Proposition 1.2 (see Proposition 2.2 below).

First we introduce some test function spaces and their distribution spaces, which are under consideration. We recall that the Pilipović space $\mathcal{H}_\flat(\mathbf{R}^d)$ consists of all Hermite function expansions

$$f(x) = \sum_{\alpha \in \mathbf{N}^d} c(f,\alpha) h_\alpha(x), \tag{2.1}$$

where the Hermite coefficients $c(f,\alpha)$ should satisfy

$$|c(f,\alpha)| \lesssim h^{|\alpha|}\alpha!^{-\frac{1}{2}},$$

for some $h > 0$. As in [16], we equip $\mathcal{H}_\flat(\mathbf{R}^d)$ with the inductive limit topology of $\mathcal{H}_{\flat,h}(\mathbf{R}^d)$ with respect to $h > 0$. Here $\mathcal{H}_{\flat,h}(\mathbf{R}^d)$ is the Banach space of all smooth functions f on $\mathbf{R}^d$ such that

$$\|f\|_{\mathcal{H}_{\flat,h}} \equiv \sup_{\alpha \in \mathbf{N}^d} \left(\frac{|c(f,\alpha)|\alpha!^{\frac{1}{2}}}{h^{|\alpha|}} \right)$$

is finite. In particular, $\mathcal{H}_\flat(\mathbf{R}^d)$ is the union of all $\mathcal{H}_{\flat,h}(\mathbf{R}^d)$, $h > 0$.

The distribution space $\mathcal{H}_\flat'(\mathbf{R}^d)$ can be identified with the set of all formal expansions in (2.1) such that

$$|c(f,\alpha)| \lesssim h^{|\alpha|}\alpha!^{\frac{1}{2}},$$

for every $h > 0$. The topology of $\mathcal{H}_\flat'(\mathbf{R}^d)$ is the projective limit topology of $\mathcal{H}_{\flat,h}'(\mathbf{R}^d)$ with respect to $h > 0$. Here $\mathcal{H}_{\flat,h}'(\mathbf{R}^d)$ is the Banach space of all formal expansions f in (2.1) such that

$$\|f\|_{\mathcal{H}_{\flat,h}'} \equiv \sup_{\alpha \in \mathbf{N}^d} \left(\frac{|c(f,\alpha)|\alpha!^{-\frac{1}{2}}}{h^{|\alpha|}} \right)$$

is finite. In particular, $\mathcal{H}_\flat'(\mathbf{R}^d)$ is the intersection of all $\mathcal{H}_{\flat,h}'(\mathbf{R}^d)$, $h > 0$. It follows that $\mathcal{H}_\flat(\mathbf{R}^d)$ is a complete (LB)-space (by e.g. [6, Proposition 15 and Theorem 4]) and that $\mathcal{H}_\flat'(\mathbf{R}^d)$ is a Fréchet space. These spaces are reflexive.

The distribution action between $\phi \in \mathcal{H}_\flat(\mathbf{R}^d)$ and $f \in \mathcal{H}_\flat'(\mathbf{R}^d)$ is then given by

$$\langle f, \phi \rangle = \sum_{\alpha \in \mathbf{N}^d} c(f,\alpha) c(\phi,\alpha). \tag{2.2}$$

Next we recall some facts concerning the Gelfand-Shilov space $\Sigma_1(\mathbf{R}^d)$ and its distribution space $\Sigma_1'(\mathbf{R}^d)$, originally introduced by Silva in [15] (see also [12,16]). The space $\Sigma_1'(\mathbf{R}^d)$ is also known as the space of Fourier ultrahyperfunctions [7]. We recall that $\Sigma_1(\mathbf{R}^d)$ consists of all $f \in C^\infty(\mathbf{R}^d)$ such that for every $h > 0$, there is a constant $C_h > 0$ such that

$$|x^\alpha \partial^\beta f(x)| \le C_h h^{|\alpha+\beta|}\alpha!\beta!\,.$$

The smallest choice of C_h defines a semi-norm on $\Sigma_1(\mathbf{R}^d)$, and by defining a (project limit) topology from these semi-norms, it follows that $\Sigma_1(\mathbf{R}^d)$ is a Fréchet space.

There are several characterizations of $\Sigma_1(\mathbf{R}^d)$ and its (strong) dual or distribution space $\Sigma_1'(\mathbf{R}^d)$. For example, $\Sigma_1(\mathbf{R}^d)$ is the set of all expansions in (2.1) such that

$$|c(f,\alpha)| \le C_r e^{-r|\alpha|^{\frac{1}{2}}}$$

for every $r > 0$. The smallest C_r defines a semi-norm for $\Sigma_1(\mathbf{R}^d)$, and the (project limit) topology in this setting agrees with the topology for $\Sigma_1(\mathbf{R}^d)$. We may then identify $\Sigma_1'(\mathbf{R}^d)$ with all expansions in (2.1) such that

$$|c(f,\alpha)| \le Ce^{r_0|\alpha|^{\frac{1}{2}}},$$

for some constants $C > 0$ and $r_0 > 0$, with distribution action given by (2.2) when $f \in \Sigma_1'(\mathbf{R}^d)$ and $\phi \in \Sigma_1(\mathbf{R}^d)$.

In the same way we may identify $\mathscr{S}(\mathbf{R}^d)$ and $\mathscr{S}'(\mathbf{R}^d)$ as the sets of all expansions in (2.1) such that

$$|c(f,\alpha)| \le C_r(1+|\alpha|)^{-r}$$

for every $r > 0$, and

$$|c(f,\alpha)| \le C(1+|\alpha|)^{r_0}$$

for some $C > 0$ and $r > 0$, respectively (see e.g. [13]). The distribution action is given by (2.2) when $f \in \mathscr{S}'(\mathbf{R}^d)$ and $\phi \in \mathscr{S}(\mathbf{R}^d)$.

From these identifications it is evident that

$$\mathcal{H}_\flat(\mathbf{R}^d) \subseteq \Sigma_1(\mathbf{R}^d) \subseteq \mathscr{S}(\mathbf{R}^d) \subseteq \mathscr{S}'(\mathbf{R}^d) \subseteq \Sigma_1'(\mathbf{R}^d) \subseteq \mathcal{H}_\flat'(\mathbf{R}^d), \tag{2.3}$$

with dense and continuous embeddings.

We are now prepared to state the generalizations of Lemma 1.1 and Proposition 1.2 in Section 1.

Lemma 2.1. *Suppose that $\mathcal{H}$ is continuously embedded in $\mathcal{H}_\flat'(\mathbf{R}^d)$. Then there is a constant $C > 0$ such that (0.2) holds.*

Proof. Note that $p_\phi(f) = |\langle f,\phi\rangle|$, $f \in \mathcal{H}_\flat'(\mathbf{R}^d)$, is a continuous seminorm on $\mathcal{H}_\flat'(\mathbf{R}^d)$. The continuity of the inclusion mapping $\mathcal{H} \to \mathcal{H}_\flat'(\mathbf{R}^d)$ then yields $p_\phi(f) \le C_\phi \|f\|_{\mathcal{H}}$ for some $C_\phi > 0$ and all $f \in \mathcal{H}$, which is the same as (0.2), completing the proof. □

We observe that the Hilbert structure of $\mathcal{H}$ actually plays no role in the previous proof and Lemma 2.1 therefore holds if we just assume that $\mathcal{H}$ is a Banach space that is continuously embedded in $\mathcal{H}_\flat'(\mathbf{R}^d)$.

Our generalization of Proposition 1.2 is the following.

Proposition 2.2. *Suppose that the Hilbert space $\mathcal{H}$ is continuously embedded in $\mathcal{H}_\flat'(\mathbf{R}^d)$. Then $\mathcal{H}$ is separable.*

Evidently, by (2.3) it follows that Proposition 2.2 is true with Σ_1' in place of $\mathcal{H}_\flat'$.

We need some preparations for the proof of Proposition 2.2. Especially we shall make use of the Bargmann transform, $\mathfrak{V}_d$, defined by

$$(\mathfrak{V}_d f)(z) = \pi^{-\frac{d}{4}} \int_{\mathbf{R}^d} \exp\Big(-\frac{1}{2}(\langle z,z\rangle + |y|^2) + 2^{1/2}\langle z,y\rangle\Big) f(y)\,dy, \quad z \in \mathbf{C}^d.$$

We have

$$(\mathfrak{V}_d f)(z) = \int_{\mathbf{R}^d} \mathfrak{A}_d(z,y) f(y)\,dy, \quad z \in \mathbf{C}^d,$$

or

$$(\mathfrak{V}_d f)(z) = \langle f, \mathfrak{A}_d(z, \cdot\,)\rangle, \quad z \in \mathbf{C}^d, \tag{2.4}$$

where the Bargmann kernel $\mathfrak{A}_d$ is given by

$$\mathfrak{A}_d(z,y) = \pi^{-\frac{d}{4}} \exp\Big(-\frac{1}{2}(\langle z,z\rangle + |y|^2) + 2^{1/2}\langle z,y\rangle\Big), \quad z \in \mathbf{C}^d, y \in \mathbf{R}^d.$$

(Cf. [3,4].) Here

$$\langle z,w\rangle = \sum_{j=1}^{d} z_j w_j \quad \text{and} \quad (z,w) = \langle z, \overline{w}\rangle$$

when

$$z = (z_1, \dots, z_d) \in \mathbf{C}^d \quad \text{and} \quad w = (w_1, \dots, w_d) \in \mathbf{C}^d.$$

Otherwise $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the duality between test function spaces and their corresponding duals, as above, which is clear from the context.

In [16], the images of the spaces (2.3) under the Bargmann transform are presented. Let $A(\mathbf{C}^d)$ be the set of entire functions on $\mathbf{C}^d$, and let

$$\mathcal{A}'_{0,\infty}(\mathbf{C}^d) \equiv \bigcup_{r\ge 0}\{\, F \in A(\mathbf{C}^d)\,;\, |F(z)| \le Ce^{\frac{1}{2}|z|^2}(1+|z|)^r \text{ for some } C, r > 0\,\}$$

$$\mathcal{A}'_{0,1}(\mathbf{C}^d) \equiv \bigcup_{r\ge 0}\{\, F \in A(\mathbf{C}^d)\,;\, |F(z)| \le Ce^{\frac{1}{2}|z|^2+r|z|} \text{ for some } C, r > 0\,\}$$

and

$$\mathcal{A}'_\flat(\mathbf{C}^d) \equiv A(\mathbf{C}^d).$$

We equip $\mathcal{A}'_{0,\infty}(\mathbf{C}^d)$ and $\mathcal{A}'_{0,1}(\mathbf{C}^d)$ with inductive limit topologies through the semi-norms

$$\|F\|_{\mathcal{A}'_{0,\infty},r} \equiv \|F\cdot e^{-\frac{1}{2}|\,\cdot\,|^2}(1+|\,\cdot\,|)^{-r}\|_{L^\infty}, \quad r > 0,$$

and

$$\|F\|_{\mathcal{A}'_{0,1},r} \equiv \|F\cdot e^{-(\frac{1}{2}|z|^2+r|\,\cdot\,|)}\|_{L^\infty}, \qquad r > 0,$$

respectively. We also equip $\mathcal{A}'_\flat(\mathbf{C}^d) = A(\mathbf{C}^d)$ with its canonical projective limit topology, given by the semi-norms

$$\|F\|_{\mathcal{A}'_\flat,r} \equiv \|F\|_{B_r(0)}, \qquad r > 0.$$

Here $B_r(z)$ denotes the open ball in $\mathbf{C}^d$ with center at $z \in \mathbf{C}^d$ and radius $r > 0$.

In [16] it is proved that the Bargmann mappings

$$\mathfrak{V}_d : \mathscr{S}'(\mathbf{R}^d) \to \mathcal{A}'_{0,\infty}(\mathbf{C}^d), \tag{2.5}$$

$$\mathfrak{V}_d : \Sigma'_1(\mathbf{R}^d) \to \mathcal{A}'_{0,1}(\mathbf{C}^d) \tag{2.6}$$

and

$$\mathfrak{V}_d : \mathcal{H}'_\flat(\mathbf{R}^d) \to \mathcal{A}'_\flat(\mathbf{C}^d) \tag{2.7}$$

are linear homeomorphisms.

Proof of Proposition 2.2. We consider the Hilbert space of entire functions $\tilde{\mathcal{H}} = \mathfrak{V}_d(\mathcal{H})$, provided with the scalar product $(\mathfrak{V}_d f, \mathfrak{V}_d g)_{\tilde{\mathcal{H}}} = (f, g)_{\mathcal{H}}$. It suffices to show that $\tilde{\mathcal{H}}$ is separable. Let $z \in \mathbf{C}^d$. Reasoning as in the proof of Lemma 1.1 with the test function $\mathfrak{A}_d(z, \cdot\,)$ instead of ϕ, we obtain the existence of a constant $C_z > 0$ such that

$$|\mathfrak{V}_d f(z)| = |\,\langle f, \mathfrak{A}_d(z, \cdot\,)\rangle\,| \le C_z \|f\|_{\mathcal{H}} = C_z \|\mathfrak{V}_d f\|_{\tilde{\mathcal{H}}}, \qquad f \in \mathcal{H},$$

which shows that $\tilde{\mathcal{H}}$ is a reproducing kernel Hilbert space. (See e.g. [1].)

Let $K_z(w) = K(z, w)$ be its reproducing kernel and fix a countable and dense subset D of $\mathbf{C}^d$. We claim that the linear span of $\{\, K_z\,;\, z \in D\,\}$ is a dense subset of $\tilde{\mathcal{H}}$.

Indeed, if ℓ is a continuous linear functional on $\tilde{\mathcal{H}}$ which vanishes on $\{\, K_z\,;\, z \in D\,\}$, then by Riesz representation theorem, there is a unique $G \in \tilde{\mathcal{H}}$ such that $\ell(F) = (F, G)_{\tilde{\mathcal{H}}}$, for every $F \in \tilde{\mathcal{H}}$. Since ℓ vanish on $\{\, K_z\,;\, z \in D\,\}$, we have

$$G(z) = (G, K_z)_{\tilde{\mathcal{H}}} = \overline{\ell(K_z)} = 0 \quad \text{for each } z \in D,$$

giving that G, and thereby ℓ, are identically zero, in view of the density of D and the continuity of G. The result now follows by the Hahn-Banach theorem. □

Remark 2.3. Let $d\mu(z) = e^{-|z|^2}\, d\lambda(z)$, $z \in \mathbf{C}^d$, where $d\lambda(z)$ is the Lebesgue measure on $\mathbf{C}^d$. Then the Bargmann-Fock space, $A^2(\mathbf{C}^d)$, consists of all $F \in A(\mathbf{C}^d)$ such that

$$\|F\|_{A^2} \equiv \left(\int_{\mathbf{C}^d} |F(z)|^2\, d\mu(z) \right)^{\frac{1}{2}}$$

is finite. We recall that $A^2(\mathbf{C}^d)$ is a Hilbert space with scalar product

$$(F, G)_{A^2} = \int_{\mathbf{C}^d} F(z)\overline{G(z)}\, d\mu(z), \qquad F, G \in A^2(\mathbf{C}^d).$$

In [3] it is proved that if $\mathcal{H}$ and its norm, are equal to $L^2(\mathbf{R}^d)$ and its norm, then $\tilde{\mathcal{H}}$ and its norm, are equal to $A^2(\mathbf{C}^d)$ and its norm.

Let $\mathcal{A}_\flat(\mathbf{C}^d)$ be the space of entire functions of exponential type, given by

$$\mathcal{A}_\flat(\mathbf{C}^d) \equiv \bigcup_{r \ge 0} \{\, F \in A(\mathbf{C}^d)\,;\, |F(z)| \le Ce^{r|z|} \text{ for some } C > 0\,\},$$

equipped with the inductive limit topology through the semi-norms

$$\|F\|_{\mathcal{A}_\flat, r} \equiv \|F \cdot e^{-r|\,\cdot\,|}\|_{L^\infty}.$$

Evidently, $\mathcal{A}_\flat(\mathbf{C}^d)$ is continuously embedded in $A^2(\mathbf{C}^d)$. In [16] it is proved that

(1) the Bargmann transform is a homeomorphism from $\mathcal{H}_\flat(\mathbf{R}^d)$ to $\mathcal{A}_\flat(\mathbf{C}^d)$;
(2) $\mathcal{A}_\flat(\mathbf{C}^d) \subseteq A^2(\mathbf{C}^d) \subseteq \mathcal{A}_\flat'(\mathbf{C}^d)$ with dense embeddings;
(3) the map $(F, G) \mapsto (F, G)_{A^2}$ from $\mathcal{A}_\flat(\mathbf{C}^d) \times \mathcal{A}_\flat(\mathbf{C}^d)$ to $\mathbf{C}$ extends uniquely to a continuous map from $\mathcal{A}_\flat'(\mathbf{C}^d) \times \mathcal{A}_\flat(\mathbf{C}^d)$ or from $\mathcal{A}_\flat(\mathbf{C}^d) \times \mathcal{A}_\flat'(\mathbf{C}^d)$ to $\mathbf{C}$;
(4) the (strong) dual of $\mathcal{A}_\flat(\mathbf{C}^d)$ can be identified with $\mathcal{A}_\flat'(\mathbf{C}^d)$, through the (extension of the) form $(\,\cdot\,,\,\cdot\,)_{A^2}$ as

$$(\mathfrak{V}_d f, \mathfrak{V}_d \phi)_{A^2} = \langle f, \overline{\phi} \rangle, \qquad f \in \mathcal{H}_\flat'(\mathbf{R}^d),\ \phi \in \mathcal{H}_\flat(\mathbf{R}^d).$$

The reproducing kernel of $A^2(\mathbf{C}^d)$ is given by $K_z(w) = e^{(w,z)}$ (see e.g. [3]). Hence

$$F(z) = (F, e^{(\,\cdot\,,z)})_{A^2}, \tag{2.8}$$

when $F \in A^2(\mathbf{C}^d)$. We observe that $K_z \in \mathcal{A}_\flat(\mathbf{C}^d)$, for every $z \in \mathbf{C}^d$. Hence, the right-hand side of (2.8) makes sense for any $F \in \mathcal{A}_\flat'(\mathbf{C}^d)$, and because $A^2(\mathbf{C}^d)$ is dense in $F \in \mathcal{A}_\flat'(\mathbf{C}^d)$, it follows that the identity (2.8) still holds for any $F \in \mathcal{A}_\flat'(\mathbf{C}^d)$.

We observe that the reproducing kernel in the proof of Proposition 2.2 is chosen with respect to the scalar product $(\,\cdot\,,\,\cdot\,)_{\tilde{\mathcal{H}}}$, while the reproducing kernel in (2.8) is defined with respect to $(\,\cdot\,,\,\cdot\,)_{A^2}$. It follows that these kernels are not the same, when $\mathcal{H}$ differs from $L^2(\mathbf{R}^d)$.

3. Extensions and variations

In this section we slightly improve Theorem 0.3, and show that the result is still true when $\mathcal{H}$ is continuously embedded in larger distribution spaces. Our extension of Theorem 0.3 is the following.

Theorem 3.1. *Let $\mathcal{H}$ be a Hilbert space which is continuously embedded in $\mathscr{D}'(\mathbf{R}^d)$ or in $\mathcal{H}_\flat'(\mathbf{R}^d)$, and such that (0.3) holds true for some constant $C_0 \ge 1$ which is independent of $f \in \mathcal{H}$ and $x, \xi \in \mathbf{R}^d$. If $\mathcal{H}$ is non-trivial, then $\mathcal{H} = L^2(\mathbf{R}^d)$, and (0.4) holds for some constant $C > 0$ which is independent of $f \in \mathcal{H} = L^2(\mathbf{R}^d)$.*

The proof of Theorem 3.1 follows by a combination of [18, Proposition 1.5], Theorem 0.3, and Proposition 3.3 below.

We need some preparations for the proof of Theorem 3.1. Since translation and modulation invariant Hilbert spaces are in focus, we here notice that all the spaces in Theorem 3.1 are invariant under such actions, which is explained in the next result.

Proposition 3.2. *Let $x, \xi \in \mathbf{R}^d$. Then the map $f \mapsto f(\,\cdot - x)e^{i\langle\,\cdot\,,\xi\rangle}$ is a homeomorphism on each one of the spaces in (2.3) and in $\mathscr{D}'(\mathbf{R}^d)$.*

Proof. The result is evidently true for the spaces

$$\mathscr{S}(\mathbf{R}^d), \quad \Sigma_1(\mathbf{R}^d), \quad \mathscr{S}'(\mathbf{R}^d), \quad \Sigma_1'(\mathbf{R}^d) \quad \text{and} \quad \mathscr{D}'(\mathbf{R}^d).$$

We need to prove the result for $\mathcal{H}_\flat(\mathbf{R}^d)$ and $\mathcal{H}_\flat'(\mathbf{R}^d)$.

Our argument relies on the Weyl maps, which are defined as the linear operators

$$(W_w F)(z) \equiv e^{-\frac{1}{2}|w|^2 + (z,w)} F(z - w), \quad F \in A(\mathbf{C}^d),\ z, w \in \mathbf{C}^d. \tag{3.1}$$

The relevance of these Weyl maps for us is that the Bargmann transform transfers time-frequency shifts into these Weyl maps. In fact, as observed in [2, Lemma 2.3], we have

$$
\begin{aligned}
(\mathfrak{V}_d(e^{i\langle\,\cdot\,,\xi\rangle}f(\,\cdot\,-x)))(z) &= e^{\frac{i}{2}\langle x,\xi\rangle}(W_{\overline{w}/\sqrt{2}}(\mathfrak{V}_d f))(z),\\
& z\in\mathbf{C}^d,\ w=x+i\xi\in\mathbf{C}^d.
\end{aligned}
\tag{3.2}
$$

Therefore, the transfer formula (3.2) yields that a distribution or function space is continuously invariant under the action of translations and modulations, if and only if its image space under the Bargmann transform is continuously invariant under the Weyl maps. Since it is evident that each Weyl map is a homeomorphism on the space of all entire functions, which by definition is $\mathcal{A}'_\flat(\mathbf{C}^d)$, the assertion for $\mathcal{H}'_\flat(\mathbf{R}^d)$ is a consequence of the homeomorphism (3.2) and the fact that (2.7) is a topological isomorphism of Fréchet spaces. By duality it now follows that translations and modulations are homeomorphisms on $\mathcal{H}_\flat(\mathbf{R}^d)$ as well, and the result follows. □

The next proposition gives useful inclusion relations for Banach subspaces of $\mathcal{H}'_\flat(\mathbf{R}^d)$ that are invariant under translations and modulations.

Proposition 3.3. *Let $\mathcal{B}$ be a Banach space which is continuously embedded in $\mathcal{H}'_\flat(\mathbf{R}^d)$, and let*

$$
v(x,\xi)=\sup_{\|f\|_{\mathcal{B}}\le 1}\|e^{i\langle\,\cdot\,,\xi\rangle}f(\,\cdot\,-x)\|_{\mathcal{B}},\quad x,\xi\in\mathbf{R}^d. \tag{3.3}
$$

Then the following is true:

(1) *if $v\in L^\infty_{loc}(\mathbf{R}^{2d})$, then $\mathcal{B}$ is continuously embedded in $\Sigma'_1(\mathbf{R}^d)$;*
(2) *if in addition $v(x,\xi)\le C_0(1+|x|+|\xi|)^N$, for some constants $C_0>0$ and $N>0$, then $\mathcal{B}$ is continuously embedded in $\mathscr{S}'(\mathbf{R}^d)$.*

Proof. Suppose that $v\in L^\infty_{loc}(\mathbf{R}^{2d})$. Note that v is submultiplicative, i.e., $v(x,\xi)>0$ and

$$
v(x+y,\xi+\eta)\le v(x,\xi)v(y,\eta),\qquad x,y,\xi,\eta\in\mathbf{R}^d. \tag{3.4}
$$

Since v is locally bounded, a classical result due to Beurling [5] (see also [9]) then yields

$$
v(x)\le Ce^{r(|x|+|\xi|)},\quad x,\xi\in\mathbf{R}^d, \tag{3.5}
$$

for some constants $C>0$ and $r>0$.

Suppose that $f\in\mathcal{B}$ and let $F=\mathfrak{V}_d f$. By the assumptions, (3.2) and that (2.7) is a homeomorphism, it follows that for some $C>0$ we have

$$
|F(z-w)e^{-\frac{1}{2}|w|^2+\operatorname{Re}(z,w)}|v(\sqrt{2}\overline{w})^{-1}\le C,\qquad z\in B_r(0),\ w\in\mathbf{C}^d,
$$

which implies

$$
\sup_{z\in\mathbf{C}^d}\Big(|F(z)|e^{-\frac{1}{2}|z|^2}v(-\sqrt{2}\overline{z})^{-1}\Big)<\infty. \tag{3.6}
$$

Combining the latter estimate with (3.5) shows that $F\in\mathcal{A}'_{0,1}(\mathbf{C}^d)$. The asserted embedding in (1) now follows from the homeomorphic property of (2.6).

The assertion (2) can be found in e.g. [10]. Here we give alternative arguments. Therefore, suppose that v satisfies the estimates in (2). Then (3.6) shows that $F\in\mathcal{A}'_{0,\infty}(\mathbf{C}^d)$. The asserted embedding in (2) now follows from the homeomorphic property of (2.5), giving the result. □

Remark 3.4. Proposition 3.3 still holds under the weaker assumption that $\mathcal{B}$ is a quasi-Banach space [16,17].

Remark 3.5. By the definition of modulation spaces and their mapping properties under the Bargmann transform, it follows from the estimate (3.6) that $\mathcal{B}$ in Proposition 3.3 (1) is continuously embedded in the modulation space $M^\infty_{(\omega)}(\mathbf{R}^d)$, with $\omega(x,\xi) = v(-x - i\xi)^{-1}$, $x, \xi \in \mathbf{R}^d$. (See e.g. [16].)

Proof of Theorem 3.1. If $\mathcal{H}$ is continuously embedded in $\mathscr{D}'(\mathbf{R}^d)$ or in $\mathcal{H}'_\flat(\mathbf{R}^d)$, then [18, Proposition 1.5] and Proposition 3.3 (2) show that $\mathcal{H}$ is continuously embedded in $\mathscr{S}'(\mathbf{R}^d)$. The result now follows from Theorem 0.3. $\square$

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