UNIVERSITEIT GENT

Laplace transform method for hypercomplex Fourier transforms

Laplacetransformatiemethode voor hypercomplexe fouriertransformaties

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Contents

Chapter 1

Introduction

1.1 Overview

Clifford analysis [4, 33] is a theory that offers a natural generalization of complex analysis to higher dimensions. It is a well established mathematical discipline closely related but complementary to harmonic analysis. It focuses on the null solutions of various special partial differential operators arising naturally within the Clifford algebra language. Lots of papers, conference proceedings and books have studied this theory and shown its ability for applications.

The Fourier transform (see [2, 47, 65]) is by far the most important integral transform. It was first extensively studied in the setting of Clifford analysis by Sommen in [66, 67]. Subsequently, generalized Fourier transforms have received a lot of attention over the past decades, from various points of view as explained in several review papers [7, 19, 21]. Among these various generalizations, the pair of so-called Clifford-Fourier transforms $\mathcal{F}_{\pm}$, initially introduced in [5] by Brackx, De Schepper and Sommen, have in particular received a lot of attention. It took a long time before explicit expressions were made available for the kernels $K_{\pm}$ of the Clifford-Fourier transforms. Initially the two dimensional case was treated in [6], and using the Clifford-Bessel transform introduced in [8] an iterative procedure was given for the even dimensional kernels [9]. Finally in [24] explicit expressions were obtained for the even dimensional case, the odd case still being an open problem. Alternative approaches were considered in [15, 18]. In [27] it was shown that there is in fact an entire family of solutions of a system of PDEs. All these solutions can be used as kernels of integral transforms that behave in ways comparable to the Fourier transform. Further extensions were also investigated from a representation theory point of view in [22]. In later work, the fractional version of the Clifford-Fourier transform was introduced in [18, 26]. The fractional Clifford-Fourier transform $\mathcal{F}_{\alpha,\beta}$ with two numerical parameters α and β was defined in [26]. A series expansion of its kernel was derived and analytic properties were studied in details. These two parameters can be chosen equal to each other as in [25], in which the kernel of $\mathcal{F}_{\pm,\alpha}$ in dimension 2 was obtained using Clifford algebra techniques.

The Clifford-Helmholtz system is a system of partial differential equations in a Clifford algebra that refines the classical Helmholtz equation. The latter has been used in the study of electromagnetic radiation, seismology, acoustics and other physical problems. In earlier work, Oste studied an entire class of independent solutions of the Clifford-Helmholtz system in [58], each solution consisting of a finite sum of Bessel functions. Recursion relations between solutions led to series expansions in terms of Bessel functions and Gegenbauer polynomials. The author also considered the Fourier-type integral transforms $\mathcal{F}_m^n$, $0 \leq n \leq m-2$ having a kernel function $K_m^n(x, y)$ that is solution of the Clifford-Helmholtz system. Further properties and the unitary Fourier transform $\mathcal{F}_m^\lambda$, $\lambda = \frac{m}{2} - 1$ were introduced subsequently.

This thesis aims at two types of contributions: properties of the Fourier transforms associated with the Clifford-Helmholtz system and further development of Laplace transform methods.

The first main goal of this thesis is to prove uncertainty inequalities and real Paley-Wiener theorems for the new Fourier transforms $\mathcal{F}_m^n$ associated with the Clifford-Helmholtz system when $0 \leq n \leq m-2$, defined in [58]. Various uncertainty relations have been established for the existing hypercomplex Fourier transforms in the past few years, see e.g. [14, 49, 50, 52, 55]. By comparison with the results for the classic Fourier transform, we will establish the local, global uncertainty principles and Pitt's inequality for these novel Fourier transforms. Due to Bang and Tuan [11, 72], there also has been a great interest in the real Paley-Wiener theorem, which independently characterized functions with compact support of the Fourier transform only through some norms of their derivatives on $\mathbb{R}^m$ at infinity. In the hypercomplex setting, several versions of the real Paley-Wiener theorems have been established, see for instance [41, 53, 54]. We will give two characterization of the Fourier transforms of functions vanishing outside and inside a ball in this thesis, which should be considered as some kind of uncertainty relations.

More recently, Laplace transform methods have started to play an important role in the study of the kernels of hypercomplex Fourier transforms. In [15], Constales, De Bie and Lian developed a new Laplace transform method to compute the Clifford-Fourier kernel $K_m(x, y)$ by introducing an auxiliary variable t. Then the expressions of the Laplace transform was given in terms of the Cauchy kernel for the Dirac operator. This leads to integral expressions and generating functions of the Clifford-Fourier kernel for all even dimensions. Later, another interesting method was developed to obtain explicit expressions for the kernel of the (κ, a)-generalized Fourier transform for $\kappa = 0$ in [16]. This was achieved by introducing a variable t in the Bessel functions of the kernel and subsequently taking the Laplace transform. By means of the Poisson kernel, the Laplace domain expression of the kernel can be simplified. Finally, using the generalized Mittag-Leffler function, the Laplace inversion can be used to obtain the integral expressions. This will be our second goal: to derive explicit expressions for the kernels of hypercomplex Fourier transforms. In this thesis four Fourier transforms will be considered: the Fourier transforms associated with the Clifford-Helmholtz system [58], the radially deformed Fourier transform [28, 30, 31], the Clifford-Fourier transform [24] and the fractional Clifford-Fourier transforms [25, 26], by modifying these Laplace transform methods. Let us first focus on the kernels of the Fourier transform $\mathcal{F}_m^n$, $0 \leq n \leq m-2$ associated with the Clifford-Helmholtz system. In [23], we adapt the parameter $a \in \mathbb{R}_{>0}$ to $t \in \mathbb{R}_{>0}$ and Laplace transform the kernels with respect to t. This leads to hypergeometric functions expressions and recursion relations for the kernels in the Laplace domain. Exponential generating functions for two cases are obtained using Laplace inversions. The second Laplace transform method from [16] is modified to find the kernel of the radially deformed Fourier transform $\mathcal{F}_D$ for the cases of $m = 2$ and $m \geq 3$. Introducing the variable t for the Bessel functions in the kernel, the series expansions of the Laplace transforms with respect to t will be given. The Poisson kernels and the generating function of the Gegenbauer polynomials lead to a simpler form of the Laplace domain expressions for $m > 2$. The kernel in dimension 2 will be considered separately by taking the limit $\lambda = 0$ from [28]. This extends to a similar decomposition for the expressions in the general dimensions when m is even. New integral expressions of the kernel are discovered in terms of the Mittag-Leffler functions by transforming back and setting $t = 1$. Much of the results obtained for the kernel of the deformed Fourier transform can be translated to similar results for the Clifford-Fourier transform $\mathcal{F}_{-}$ and its fractional versions $\mathcal{F}_{\alpha,\beta}$. These expressions are also presented by means of the Bessel functions and the Prabhakar functions. The procedure of the two kinds of modified Laplace transform methods is shown in Figure 1.2. Note that:

- The notation $\propto$ means that the equations hold up to constant factors.
- The expressions of the kernels in the Laplace domain are indicated by the dashed box.

1.2 Outline

The organization of this thesis is as follows. In Chapter 2 we recall some basic facts on special functions, Clifford algebras and Fourier transforms in hypercomplex analysis necessary for the following chapters.

For the Clifford-Helmholtz system, a class of solutions in terms of Bessel functions were found in [58]. These solutions were used to construct a new class of Fourier transforms. In Chapter 3, we consider several versions of uncertainty principles and real Paley-Wiener theorems for this novel class of Fourier transforms. Since the majority of these integral transforms is not unitary, the results and proofs deviate from the classical ones.

In Chapter 4 we still focus on the family of solutions of the Clifford-Helmholtz system. In [58] one shows that all these solutions can be used as integral kernels of generalized Fourier transforms in hypercomplex analysis. We show that in the Laplace domain they have interesting expressions in terms of terminating hypergeometric functions. This allows us to compute recursion relations between the different kernels. Moreover, we show that it is now possible to compute generating functions of the kernels in different dimensions as

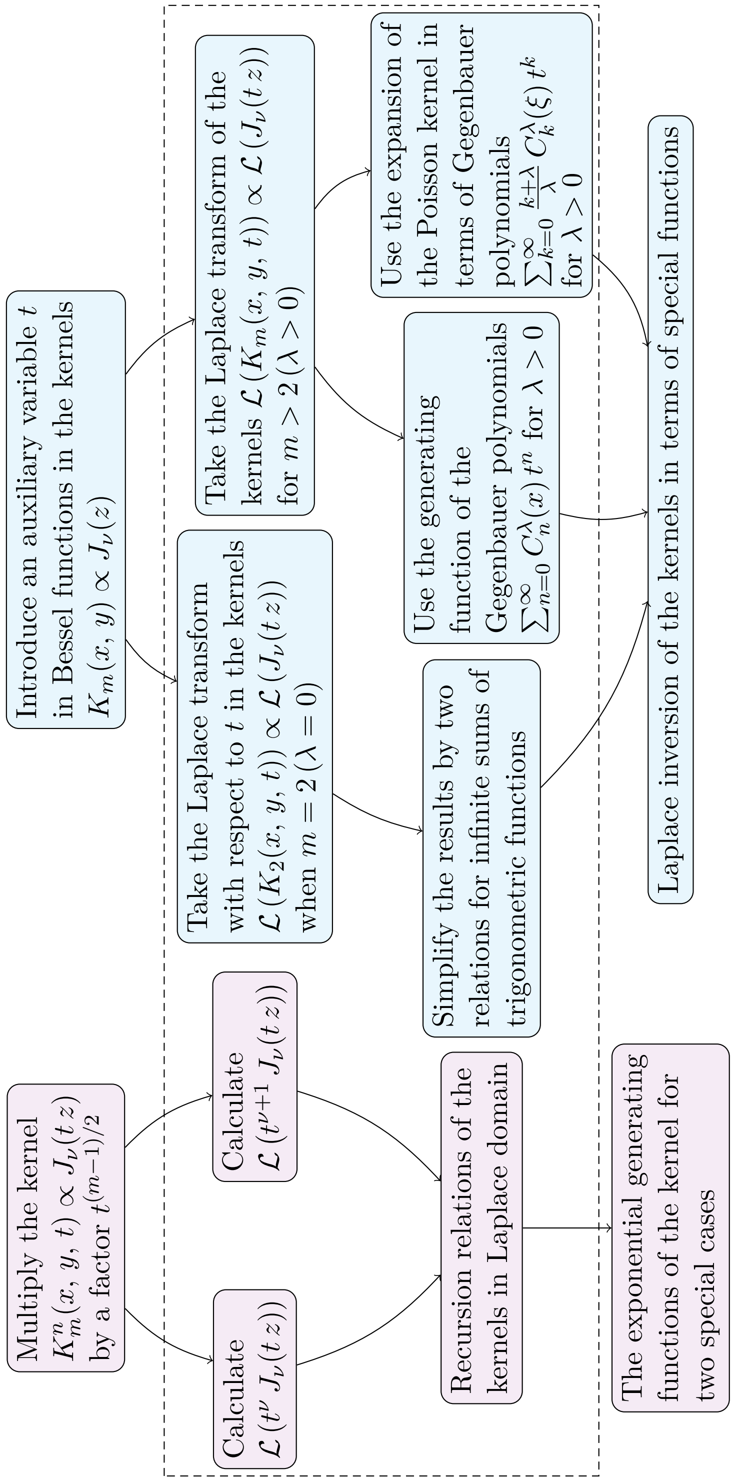

Figure 1.1: The structure of the Laplace transform method

well. As a consequence we obtain the generating function of the Clifford-Bessel transform (see [8]).

In Chapter 5 we turn our attention to the kernel of the radially deformed Fourier transform (see [30, 31]) in all dimensions when $1 + c = \frac{1}{n}, n \in \mathbb{N}_0 \backslash \{1\}$ with n odd introduced in [28]. By adapting the Laplace transform method from [16], we can obtain the Laplace domain expressions of the kernels. Once the expressions are simplified using the Poisson kernel and the generating function of the Gegenbauer polynomial, the inverse formulas can be used subsequently to get the integral expressions of the kernels in terms of Mittag-Leffler functions.

Chapter 6 is an application of the modified Laplace transform method from Chapter 5. In this chapter we study the kernels of the Clifford-Fourier transform and their fractional versions (see [24–26]). Combining the Laplace transform method in the previous chapter with some new Laplace formulas, new explicit expressions of the kernels will be given by means of the Bessel functions and the Prabhakar functions.

A final short Chapter 7 discusses some interesting avenues for further researches.

The text and notations have been adapted to make the thesis more readable. We remark that the Laplace transform method considered in Chapter 4 is shown in the purple boxes in Figure 1.2 and the method developed in Chapter 5 and Chapter 6 is shown in the blue boxes.

Chapter 2

Preliminary notions

2.1 Special functions

2.1.1 Gamma function

The Gamma function $\Gamma(z)$ (see e.g. [62]) is defined by the integral

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, \mathrm{d}t,$$

which converges in the right half of the complex plane $\mathrm{Re}\,(z) > 0$. One of the basic properties of the gamma function is that it satisfies the following functional equation:

$$\Gamma(z+1) = z\,\Gamma(z), \tag{2.1}$$

which can be easily proved by integrating by parts. Obviously, $\Gamma(1) = 1$, and using (2.1) we obtain for $n \in \mathbb{N}$:

$$\Gamma(n+1) = n!.$$

The Beta function is defined as a certain combination of values of the Gamma function

$$B(z,\, w) = \int_0^1 \tau^{z-1}\,(1-\tau)^{w-1}\,\mathrm{d}\tau, \quad (\mathrm{Re}\,(z) > 0,\, \mathrm{Re}\,(w) > 0).$$

In the sequel we need the following explicit expression

$$B(z,\, w) = \frac{\Gamma(z)\,\Gamma(w)}{\Gamma(z+w)}.$$

The rising factorial

$$(x)_n = x(x+1)(x+2)\cdots(x+n-1)$$

or the so-called Pochhammer symbol can be given in terms of the Gamma function as

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}, \quad x \in \mathbb{R}_+,$$

where $(x)_0 = 1$. The rising factorials can be used to express a binomial coefficient

$$\frac{(x)_n}{n!} = \binom{x+n-1}{n},$$

where the binomial coefficient is defined as

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

2.1.2 Hypergeometric function

The Gauss hypergeometric function (see e.g. [57]) is a function represented by

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n}\,\frac{z^n}{n!},$$

which converges if c is not a negative integer, for all of $|z| < 1$, and on the unit circle $|z| = 1$ if $\mathrm{Re}(c-a-b) > 0$. Note that when a or b is a non-positive integer, the hypergeometric series reduces to a polynomial,

$${}_2F_1\left(\begin{matrix} -m, b \\ c \end{matrix}; z\right) = \sum_{n=0}^{m} (-1)^n \binom{m}{n} \frac{(b)_n}{(c)_n}\,z^n. \tag{2.2}$$

In the sequel, we will need Euler's transformation

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = (1-z)^{c-a-b}\,{}_2F_1\left(\begin{matrix} c-a, c-b \\ c \end{matrix}; z\right), \tag{2.3}$$

and the formulas (see `https://dlmf.nist.gov/15.4`)

$$\begin{aligned} &{}_2F_1\left(\begin{matrix} a, 1-a \\ \frac{1}{2} \end{matrix}; -z^2\right) \\ &= \frac{1}{2\sqrt{1+z^2}}\left(\left(\sqrt{1+z^2}+z\right)^{2a-1} + \left(\sqrt{1+z^2}-z\right)^{2a-1}\right); \end{aligned} \tag{2.4}$$

$$
\begin{aligned}
&{}_2F_1\left(\begin{matrix} a, 1-a \\ \frac{3}{2} \end{matrix}; -z^2\right) \\
&= \frac{1}{(2-4a)z}\left(\left(\sqrt{1+z^2}+z\right)^{1-2a} - \left(\sqrt{1+z^2}-z\right)^{1-2a}\right).
\end{aligned} \tag{2.5}
$$

2.1.3 Bessel function

Bessel's differential equation is given by

$$
x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x \frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 - \nu^2)\, y = 0 \tag{2.6}
$$

for an arbitrary complex number ν , which represents the order of the Bessel function. The Bessel function of the first kind is given by the infinite power series

$$
J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, \Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+\nu}. \tag{2.7}
$$

This power series is a holomorphic function of x on the complex plane cut along the negative real axis. In particular, we have [1, p. 202]

$$
J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x \qquad \text{and} \qquad J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x
$$

and the recurrence formula [73, p. 45]

$$
\frac{2\,\nu}{x} J_\nu(x) = J_{\nu-1}(x) + J_{\nu+1}(x). \tag{2.8}
$$

When solving the Helmholtz equation in spherical coordinates by separation of variables, the radial equation has the form

$$
x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2x \frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 - n(n+1))\, y = 0
$$

The two linearly independent solutions to this equation are called the spherical Bessel functions j_n and y_n, and are related to the ordinary Bessel function J_n via:

$$
\begin{aligned}
j_n(x) &= \sqrt{\frac{\pi}{2x}}\, J_{n+\frac{1}{2}}(x); \\
y_n(x) &= (-1)^{n+1} \sqrt{\frac{\pi}{2x}}\, J_{-n-\frac{1}{2}}(x).
\end{aligned}
$$

2.1.4 Gegenbauer polynomial

For $n \in \mathbb{N}$ and $\lambda > -1/2$, the Gegenbauer polynomials are defined as

$$C_n^\lambda(w) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \frac{\Gamma(n-j-\lambda)}{\Gamma(\lambda)\, j!\, (n-2j)!} (2w)^{n-2j}.$$

In this thesis, we need the well-known relation from [70], p. 80 (4.7.8):

$$\lim_{\lambda \to 0} \lambda^{-1} C_k^\lambda(w) = (2/k) \cos(k\,\theta), \quad w = \cos\theta,\ k \geq 1; \tag{2.9}$$

and the relation (see [70], p. 83 (4.7.25)

$$\lim_{\lambda \to 0} C_k^\lambda(w) = \begin{matrix} 1 & \text{if } k = 0, \\ 0 & \text{if } k \geq 1; \end{matrix} \tag{2.10}$$

and the following properties of Gegenbauer polynomials (see [70])

$$\frac{\lambda + n}{\lambda} C_n^\lambda(w) = C_n^{\lambda+1}(w) - C_{n-2}^{\lambda+1}(w) \tag{2.11}$$

and

$$w\, C_{n-1}^{\lambda+1}(w) = \frac{n}{2(n+\lambda)} C_n^{\lambda+1}(w) + \frac{n+2\lambda}{2(n+\lambda)} C_{n-2}^{\lambda+1}(w). \tag{2.12}$$

We also need the following expansion of the Poisson kernel for the unit ball in terms of Gegenbauer polynomials.

Theorem 2.1.1. *[36] For $x, y \in \mathbb{R}^m$ and $|y| \leq |x| = 1$, the Poisson kernel for the unit ball is*

$$\begin{aligned} P(x,y) &= \frac{1-|y|^2}{|x-y|^m} \\ &= \frac{1-|y|^2}{(1-2\xi|y|+|y|^2)^{m/2}} \\ &= \sum_{j=0}^{\infty} \frac{j+m/2-1}{m/2-1} C_j^{m/2-1}(\xi)\,|y|^j, \quad \xi = \langle x, \frac{y}{|y|} \rangle. \end{aligned}$$

This result can be extended for $\lambda > 0$, we have

$$\frac{1-|y|^2}{(1-2\xi|y|+|y|^2)^{\lambda+1}} = \sum_{j=0}^{\infty} \frac{j+\lambda}{\lambda} C_j^\lambda(\xi)\,|y|^j. \tag{2.13}$$

It is still valid for $z \in \mathbb{C}$, $|z| < 1$ *and* $|\xi| < 1$, *(see [57])*

$$\frac{1 - z^2}{(1 - 2\xi z + z^2)^{\lambda+1}} = \sum_{j=0}^{\infty} \frac{j+\lambda}{\lambda} C_j^{\lambda}(\xi)\, z^j. \tag{2.14}$$

Remark 2.1.2. *The validity of the analytic continuation of* (2.13) *to* (2.14) *for the whole unit disk has been proved in [16].*

The generating function of the Gegenbauer polynomials (see e.g. [38], p.177 (29)) is given by

$$\frac{1}{(1 - 2xt + t^2)^{\alpha}} = \sum_{n=0}^{\infty} C_n^{(\alpha)}(x)\, t^n \tag{2.15}$$

for $0 \le |x| < 1$, $|t| \le 1$, $\alpha > 0$.

2.1.5 Mittag-Leffler functions

Next we give definitions of some Mittag-Leffler functions in [45, 56].

Definition 2.1.3. *The two-parametric Mittag-Leffler function is defined by*

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \mathrm{Re}\,\alpha > 0, \mathrm{Re}\,\beta > 0. \tag{2.16}$$

Definition 2.1.4. *The Prabhakar generalized Mittag-Leffler function is*

$$E^{\delta}_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{(\delta)_n\, z^n}{\Gamma(\alpha n + \beta)\, n!}, \tag{2.17}$$

where $\alpha, \beta, \delta \in \mathbb{C}$ *with* $\mathrm{Re}\,\alpha > 0$.

Remark 2.1.5. *When* $\delta = 1$, *we have* $E^1_{\alpha,\beta}(z) = E_{\alpha,\beta}(z)$ *(see e.g. [56], (2.3.3)).*

In [62], the contour integral representation of the reciprocal Gamma function $1/\Gamma(z)$ *is given by*

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{\gamma(\epsilon,\,\mu)} e^{\tau}\, \tau^{-z}\, \mathrm{d}\tau, \tag{2.18}$$

where $\epsilon > 0$, $\frac{\pi}{2} < \mu < \pi$. *When* $\epsilon > 0$ *and* $0 < \mu < \pi$, $\gamma(\epsilon,\,\mu)$ *is the contour shown in Figure 2.1.*

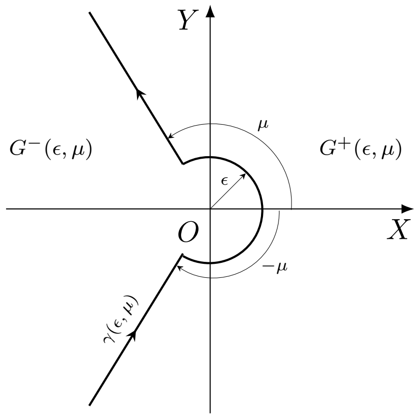

Figure 2.1: The contour $\gamma(\epsilon,\mu)$ has three parts: i) $\arg\tau = -\mu, \quad |\tau| \geq \epsilon$; ii) $-\mu \leq \arg\tau \leq \mu, \quad |\tau| = \epsilon$; iii) $\arg\tau = \mu, \quad |\tau| \geq \epsilon$. It divides the complex plane τ into two domains: $G^-(\epsilon,\mu)$ on the left side of the contour and $G^+(\epsilon,\mu)$ on the right.

If we substitute $\tau = \zeta^{1/\alpha}\,(\alpha < 2)$ *in* (2.18)*, it yields the following integral representation*

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi\alpha i}\int_{\gamma(\epsilon,\mu)} \exp\left(\zeta^{1/\alpha}\,\zeta^{(1-z-\alpha)/\alpha}\right)\mathrm{d}\zeta, \tag{2.19}$$

for $\alpha < 2$ *and* $\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}$ *([62], (1.52)).*

2.2 The Clifford analysis toolkit

Clifford analysis is a higher-dimensional function theory of functions taking values in a Clifford algebra, offering a natural generalization of complex analysis to higher dimensions [4, 33]. The real Clifford algebra $\mathcal{C}\ell_{0,m}$ over the m-dimensional Euclidean space $\mathbb{R}^m$ with a negative-definite quadratic form is generated by the canonical basis e_i, $i = 1, \ldots, m$. These generators e_i satisfy the multiplication rules

$$e_ie_j + e_je_i = -2\delta_{ij}, \quad 1 \leq i,\, j \leq m,$$

where δ_{ij} is the Kronecker delta.
Denote $e_{i_1\cdots i_k} = e_{i_1}\cdots e_{i_k}$, a basis for $\mathcal{C}\ell_{0,m}$ is given by

$$\{e_A : A \subset \{1, \ldots, m\} \text{ with } e_\emptyset = 1\}.$$

The Clifford algebra $\mathcal{C}\ell_{0,m}$ can be decomposed as $\mathcal{C}\ell_{0,m} = \oplus_{k=0}^{m}\mathcal{C}\ell_{0,m}^{k}$ with the space of k-vectors

$$\mathcal{C}\ell_{0,m}^{k} = \operatorname{span}\{e_A : A \subset \{1,\ldots,m\} \text{ and } |A| = k\}.$$

We identify the point $(x_1,\ldots,x_m)$ in $\mathbb{R}^m$ with the vector variable $\underline{x}$ given by

$$\underline{x} := \sum_{i=1}^{m} e_i x_i.$$

It is seen that $\underline{x}^2 = -\langle x,x\rangle = -|x|^2$. For any two vectors $\underline{x}$ and $\underline{y}$, their Clifford product splits into a scalar part and a bivector part:

$$\underline{x}\,\underline{y} = -\langle x,y\rangle + \underline{x}\wedge\underline{y},$$

where

$$\langle x,y\rangle := \sum_{j=1}^{m} x_j y_j = -\frac{1}{2}\left(\underline{x}\underline{y} + \underline{y}\underline{x}\right)$$

and

$$\underline{x}\wedge\underline{y} := \sum_{j<k} e_j e_k \left(x_j y_k - x_k y_j\right) = \frac{1}{2}\left(\underline{x}\underline{y} - \underline{y}\underline{x}\right).$$

For the sequel we need the square of $\underline{x}\wedge\underline{y}$ (see [24])

$$(\underline{x}\wedge\underline{y})^2 = -|x|^2|y|^2 + \langle x,y\rangle^2 = -\sum_{j<k}(x_j y_k - x_k y_j)^2.$$

We see that $(\underline{x}\wedge\underline{y})^2$ is real-valued and furthermore we have $|\underline{x}\wedge\underline{y}| = \sqrt{|x|^2|y|^2 - \langle x,y\rangle^2}$.
The Dirac operator is the first order vector differential operator (see [33])

$$\partial_{\underline{x}} = \sum_{j=1}^{m} \partial_{x_j} e_j.$$

Its square equals the Laplace operator up to a minus sign: $\partial_{\underline{x}}^2 = -\Delta$.
The complex Clifford algebra, denoted by $\mathbb{C}_m$, is the complexification of $\mathcal{C}\ell_{0,m}$, i.e.

$$\mathbb{C}_m = \mathbb{C}\otimes\mathcal{C}\ell_{0,m}.$$

The conjugation on $\mathcal{C}\ell_{0,m}$ is the main anti-involution, defined on the generators as follows

$$\overline{e_j} = -e_j$$

and extended to the whole Clifford algebra by

$$\overline{xy} = \overline{y}\,\overline{x},$$

for $x, y \in \mathcal{C}\ell_{0,m}$.

Each function f over $\mathbb{R}^m$ taking values in $\mathcal{C}\ell_{0,m}$ or $\mathbb{C}_m$ can be explicitly expressed by the reduced products as

$$f = f_0 + \sum_{i=1}^{m} e_i f_i + \sum_{i<j} e_i e_j f_{ij} + \ldots + e_1 \ldots e_m f_{1\ldots m}, \tag{2.20}$$

where $f_0, f_i, f_{ij}, \ldots, f_{1\ldots m}$ are real- or complex-valued functions over $\mathbb{R}^m$. These components will be denoted by f_A as well. The scalar part of the product $f\,\overline{g}$ for $\mathbb{C}_m$-valued functions f and g is

$$[f\,\overline{g}]_0 := \sum_A f_A\,\overline{g}_A, \tag{2.21}$$

where $[\cdot]_0$ is the projection operator which maps an element $a \in \mathcal{C}\ell_{0,m}$ to its scalar valued part, i.e. $[a]_0 = a_\emptyset$.
When $f = g$ in (2.21), we have the modulus $|f|$ defined as

$$|f| := \sqrt{\left[f\,\overline{f}\right]_0} = \sqrt{\sum_A |f_A|^2}. \tag{2.22}$$

It is seen that $|f| = \left|\overline{f}\right|$.

Furthermore, the $L^p(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m}$ space is defined as the module of all Clifford-valued functions $f : \mathbb{R}^m \to \mathbb{C}_m$ with finite norm, i.e.

$$\|f\|_p = \begin{cases} \left(\int_{\mathbb{R}^m} |f(\underline{x})|^p \,\mathrm{d}x\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \operatorname{ess\,sup}_{x\in\mathbb{R}^m} |f(\underline{x})|, & p = \infty, \end{cases}$$

where $dx = dx_1 dx_2 \cdots dx_m$ is the usual Lebesgue measure in $\mathbb{R}^m$. In particular, the space $L^2(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m}$ carries the following inner product,

$$\langle f, g\rangle := \int_{\mathbb{R}^m} \left[f(x)\,\overline{g(x)}\right]_0 \mathrm{d}x, \qquad f, g \in L^2(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m}. \tag{2.23}$$

Definition 2.2.1. *A function f defined in some open domain $\Omega \subset \mathbb{R}^m$ taking values in the Clifford algebra $\mathcal{C}\ell_{0,m}$ is called monogenic if it is in the kernel of the Dirac operator, i.e. $\partial_{\underline{x}} f = 0$.*

Obviously, each component f_α in (2.20) of a monogenic function f is harmonic in the ordinary sense, i.e. $\Delta f = 0$.
The Gamma operator or the spherical Dirac operator is given by

$$\Gamma_{\underline{x}} := -\sum_{j<k} e_j e_k (x_j \partial_{x_k} - x_k \partial_{x_j}) = -\underline{x}\partial_{\underline{x}} - \mathbb{E}_{\underline{x}} = -\underline{x} \wedge \partial_{\underline{x}}, \quad (2.24)$$

where $\mathbb{E}_{\underline{x}} = \sum_{i=1}^m x_i \partial_{x_i}$ is the Euler operator. Note that $\Gamma_{\underline{x}}$ commutes with radial functions, i.e.

$$[\Gamma_{\underline{x}}\,,\, f(|\underline{x}|)] = 0,$$

where f is a real valued function.

Denote by $\mathcal{P}$ the space of polynomials taking values in $\mathcal{C}\ell_{0,m}$, i.e.

$$\mathcal{P} := \mathbb{R}[x_1, \ldots, x_m] \otimes \mathcal{C}\ell_{0,m}$$

and $\mathcal{P}_k$ the subspace of homogeneous polynomials of degree k. The space $\mathcal{M}_k := \ker \partial_{\underline{x}} \cap \mathcal{P}_k$ is called the space of inner homogeneous monogenic polynomials of degree k. An arbitrary element in $\mathcal{M}_k$ is called an inner spherical monogenic of degree k. Similarly, the space $\mathcal{H}_k := \ker \Delta \cap \mathcal{P}_k$ is the space of Clifford-valued spherical harmonics of degree k. It follows that $\mathcal{M}_k \subset \mathcal{H}_k$ by definition. More precisely, the following Fischer decomposition

$$\mathcal{H}_k = \mathcal{M}_k \oplus \underline{x}\mathcal{M}_{k-1}$$

holds, see [33].

The reproducing kernel of $\mathcal{H}_k$ is given by $\frac{\lambda+k}{\lambda} C_k^\lambda(\langle \zeta, \tau \rangle)$ with $\lambda = (m-2)/2$. This means that for $\zeta, \tau \in \mathbb{S}^{m-1}$

$$\frac{\lambda+k}{\lambda} \int_{\mathbb{S}^{m-1}} C_k^\lambda(\langle \zeta, \tau \rangle)\, H_\ell(\zeta)\, \mathrm{d}\sigma(\zeta) = \sigma_m\, \delta_{k\ell}\, H_\ell(\tau), \qquad H_\ell \in \mathcal{H}_k \quad (2.25)$$

with $\sigma_m = 2\pi^{m/2}/\Gamma(m/2)$, see e.g. [1, 36].

For α real, the generalized Laguerre polynomials are given by

$$L_p^\alpha(t) = \sum_{n=0}^{p} \frac{\Gamma(p+\alpha+1)}{n!\,(p-n)!\,\Gamma(n+\alpha+1)} (-t)^n. \quad (2.26)$$

We further introduce a basis $\{\psi_{p,k,\ell}\}$ for the space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m} \subset L^2(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m}$, where $\mathcal{S}(\mathbb{R}^m)$

denotes the Schwartz space. This basis is given by the generalized Clifford-Hermite functions (see [68])

$$\begin{aligned} \psi_{2p,k,\ell}(x) &:= L_p^{\frac{m}{2}+k-1}(|x|^2)\ M_k^{(\ell)}(x)\ e^{-|x|^2/2}, \\ \psi_{2p+1,k,\ell}(x) &:= L_p^{\frac{m}{2}+k}(|x|^2)\ \underline{x}\ M_k^{(\ell)}(x)\ e^{-|x|^2/2}, \end{aligned} \tag{2.27}$$

where $p, k, \ell \in \mathbb{Z}_{\geq 0}$, $\{M_k^\ell \in \mathcal{M} \mid \ell = 1, \ldots, \dim(\mathcal{M}_k)\}$ is a basis for $\mathcal{M}_k$ and L_p^α is the Laguerre polynomials (see (2.26)). Note that the action of the dual pair $(\mathrm{Spin}(m), \mathfrak{osp}(1|2))$ realizes the complete decomposition of $\mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m}$ into irreducible subspaces, see [33].

2.3 Fourier transforms in Clifford analysis

The classical Fourier transform FT in $\mathbb{R}^m$ can be defined in many ways and each formulation has its specific advantages and uses (see [25, 27]). The first and most basic formulation is given by the integral transform

$$\boldsymbol{F1} \quad \mathcal{F}[f](y) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{-i\langle x,\, y\rangle}\, f(x)\, \mathrm{d}x;$$

with i the complex unit, $\langle x,\, y\rangle$ the standard inner product and dx the Lebesgue measure on $\mathbb{R}^m$. ***F1*** immediately yields a bound of the kernel and is hence ideal to study the transform on L_1 spaces or more general function spaces. Alternatively, one can rewrite the transform

$$\boldsymbol{F2} \quad \mathcal{F}[f](y) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} K(x,\, y)\, f(x)\, \mathrm{d}x;$$

where $K(x,\, y)$ is, up to a multiplicative constant, the unique solution of the system of PDEs

$$\partial_{y_j} K(x,\, y) = -i\, x_j\, K(x,\, y), \quad j = 1, \ldots, m.$$

Formulation ***F2*** gives the calculus properties of the transform. Yet another formulation is given by

$$\boldsymbol{F3} \quad \mathcal{F} = e^{\frac{i\pi m}{4}}\, e^{\frac{i\pi}{4}(\Delta - |x|^2)},$$

with Δ the Laplacian in $\mathbb{R}^m$. This expression connects the Fourier transform with the Lie algebra $\mathfrak{sl}_2$ generated by Δ and $|\underline{x}^2|$ and with the theory of the quantum harmonic oscillator. ***F*3** emphasizes the structural (Lie algebraic) properties of the Fourier transform and allows to compute its eigenfunctions and spectrum. Finally, the kernel can also be expressed as an infinite series in terms of special functions as (see [73], Section 11.5)

$$\textbf{\textit{F}4} \quad K(x,\, y) = 2^{\lambda}\, \Gamma(\lambda) \sum_{k=0}^{\infty} (k+\lambda)\, (-i)^k (|x||y|)^{-\lambda}\, J_{k+\lambda}(|x||y|) C_k^{\lambda}(\langle \zeta,\, \tau \rangle),$$

where $\zeta = \underline{x}/|x|$, $\tau = \underline{y}/|y|$ and $\lambda = (m-2)/2$. Here, J_ν is the Bessel function and C_k^λ the Gegenbauer polynomial. ***F*4** connects the Fourier transform with the theory of special functions and is the ideal formulation to obtain, for example, the Bochner identities.

These 4 definitions serve as guidance in defining hypercomplex Fourier transforms, as each definition gives access to a crucial piece of information about the transform. A suitable hypercomplex transform should hence be expressible in these 4 different ways.

2.3.1 The Clifford-Fourier transform

The Clifford-Fourier transform (CFT) is a generalization of the Fourier transform in the framework of Clifford analysis (see [5–7, 24]). ***F*3** was adapted to a pair of Clifford algebra-valued operator exponentials

$$\mathcal{F}_{\pm} = e^{\frac{i\pi m}{4}}\, e^{\mp \frac{i\pi}{2}\Gamma_{\underline{x}}}\, e^{\frac{i\pi}{4}(\Delta - |x|^2)}. \tag{2.28}$$

The motivation behind this definition was to find a couple of transforms $\mathcal{F}_\pm$ such that

$$\mathcal{F}_+\, \mathcal{F}_- = \mathcal{F}_-\, \mathcal{F}_+ = \mathcal{F}^2.$$

They can equivalently be written as integral transforms

$$\mathcal{F}_\pm[f(x)](y) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} K_\pm(x, y)\; f(x)\, \mathrm{d}x,$$

with integral kernels given by

$$K_\pm(x, y) = e^{\mp i \frac{\pi}{2} \Gamma_{\underline{y}}} \left(e^{-i\langle x, y\rangle}\right). \tag{2.29}$$

In [6], the Clifford-Fourier kernel (2.29) was determined explicitly in the case $m = 2$. For higher even dimensions, a complicated iterative procedure for constructing the kernel can be found in [9].

A breakthrough was obtained in [24], where a formulation similar to $\boldsymbol{F4}$ was found. This form of kernels from [24] and some new integral expressions are given in Section 6.2 in Chapter 6. The authors also showed that the even dimensional kernels can be expressed in terms of a finite sum of Bessel functions. Using the notations $\xi = \langle x, y \rangle$ and $\eta = |\underline{x} \wedge \underline{y}| = \sqrt{|x|^2|y|^2 - \langle x, y \rangle^2}$, one has

$$K_-(x,y) = (-1)^{m/2} \sqrt{\frac{\pi}{2}} \Big(A(\xi,\eta) + B(\xi,\eta) + (\underline{x} \wedge \underline{y})\; C(\xi,\eta) \Big) \tag{2.30}$$

with

$$\begin{aligned}
A(\xi,\eta) &= \sum_{\ell=0}^{\lfloor \frac{m-3}{4} \rfloor} \xi^{m/2-2-2\ell} \frac{1}{2^\ell \ell!} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2} - 2\ell - 1\right)} \frac{J_{(m-2\ell-3)/2}(\eta)}{\eta^{(m-2\ell-3)/2}} \\
B(\xi,\eta) &= -\sum_{\ell=0}^{\lfloor \frac{m-2}{4} \rfloor} \xi^{m/2-1-2\ell} \frac{1}{2^\ell \ell!} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2} - 2\ell\right)} \frac{J_{(m-2\ell-3)/2}(\eta)}{\eta^{(m-2\ell-3)/2}} \\
C(\xi,\eta) &= -\sum_{\ell=0}^{\lfloor \frac{m-2}{4} \rfloor} \xi^{m/2-1-2\ell} \frac{1}{2^\ell \ell!} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2} - 2\ell\right)} \frac{J_{(m-2\ell-1)/2}(\eta)}{\eta^{(m-2\ell-1)/2}}.
\end{aligned}$$

Here $\lfloor \ell \rfloor$ denotes the largest $n \in \mathbb{N}$ which satisfies $n \leq \ell$. The other Clifford-Fourier kernel follows from the relation

$$K_+(x,y) = \big(K_-(x,-y)\big)^c$$

where c denotes complex conjugation. In [27], the formulation $\boldsymbol{F2}$ for the CFT was studied, which determines the Clifford-Fourier kernel as solutions of the system of Clifford-algebra valued PDEs

$$\begin{aligned}
\partial_{\underline{y}} \left[K_+(x,\, y)\right] &= (-i)^m \, K_-(x,\, y)\, \underline{x} \\
\left[K_+(x,\, y)\right] \partial_{\underline{x}} &= (-i)^m \, \underline{y}\, K_-(x,\, y).
\end{aligned} \tag{2.31}$$

In the second equation, the Dirac operator is acting from the right on K_+. However, contrary to the classic Fourier transform, this system has no unique solution, but instead $m-1$ linearly independent solutions $K^i_{+,m}$ (satisfying additional constraints). For m even, each of the solutions give rise to an associated integral transform

$$\mathcal{F}^i_{+,m}[f(x)](y) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} K^i_{+,m}(x,\, y)\, f(x)\, \mathrm{d}x,$$

and its further properties were obtained subsequently, similar to the results for the Clifford-Helmholtz system (2.32) in [58]. Both the formulation $\boldsymbol{F4}$ and $\boldsymbol{F1}$ were considered for all relevant solutions of (2.31) in [27]. Later, in [15], a new Laplace transform method was developed to compute the integral kernel K_m of $\mathcal{F}_-$ (in $\boldsymbol{F3}$) by introducing an auxiliary variable t and taking Laplace transform for $\mathcal{L}\left(t^{m/2-1}\, e^{-i\,t\,\langle x,\, y\rangle}\right)$. Then Laplace inversion yields the new integral representations for the Clifford-Fourier kernel in all dimensions $m \geq 3$.

2.3.2 The fractional Clifford-Fourier transform

In [26], the fractional version of the Clifford-Fourier transform was initially introduced as a generalization of the fractional Fourier transform (see [59])

$$\mathcal{F}_{\alpha,\,\beta} = e^{\frac{i\alpha m}{2}}\, e^{i\beta\Gamma}\, e^{\frac{i\alpha}{2}(\Delta - |x|^2)}$$

where $\alpha,\, \beta \in [-\pi,\, \pi]$. Note that when $\alpha = \frac{\pi}{2}$ and $\beta = \pm\frac{\pi}{2}$, $\mathcal{F}_{\frac{\pi}{2},\pm\frac{\pi}{2}} = \mathcal{F}_{\pm}$ (see (2.28)). It is an adaptation of the definition $\boldsymbol{F3}$. Notice that we immediately have

$$\mathcal{F}_{\alpha,\,\beta} \circ \mathcal{F}_{\alpha,\,-\beta} = \mathcal{F}_{\alpha,\,-\beta} \circ \mathcal{F}_{\alpha,\,\beta} = \mathcal{F}_{\alpha}^2$$

with $\mathcal{F}_\alpha = e^{\frac{i\alpha m}{2}}\, e^{\frac{i\alpha}{2}(\Delta - |x|^2)}$ the fractional version of the ordinary Fourier transform (see [59]), given explicitly by

$$\mathcal{F}_\alpha[f](y) = (\pi(1-e^{-2i\alpha}))^{-m/2} \int_{\mathbb{R}^m} e^{-i\langle x,y\rangle/\sin\alpha} e^{\frac{i}{2}(\cot\alpha)(|x|^2+|y|^2)} f(x)\mathrm{d}x.$$

One may compute formally

$$\mathcal{F}_{\alpha,\beta} = (\pi(1-e^{-2i\alpha}))^{-m/2} e^{i\beta\Gamma_{\underline{y}}} \int_{\mathbb{R}^m} e^{-i\langle x,y\rangle/\sin\alpha} e^{\frac{i}{2}(\cot\alpha)(|x|^2+|y|^2)} (.)\mathrm{d}x.$$

The fractional Clifford-Fourier kernel is therefore given by

$$\begin{aligned} K_{\alpha,\,\beta}\,(x,\, y) &= e^{i\beta\Gamma_{\underline{y}}} \left(e^{-i\langle x,y\rangle/\sin\alpha}\, e^{\frac{i}{2}(\cot\alpha)(|x|^2+|y|^2)}\right) \\ &= e^{\frac{i}{2}(\cot\alpha)(|x|^2+|y|^2)}\, e^{i\beta\Gamma_{\underline{y}}} \left(e^{-i\langle x,y\rangle/\sin\alpha}\right) \end{aligned}$$

where the last line follows because $\Gamma_{\underline{y}}$ commutes with $|y|$. In [18], a new construction of the fractional Clifford-Fourier kernels was given by solving wave-type problems. Later, the kernel of the fractional CFT is computed as

$$K_{\frac{\pi}{2},\beta}(x,\, y) = e^{i\,\beta\,\Gamma_y}\, e^{-i\,\langle x,\, y\rangle}$$

to give the plane wave decomposition in [15].
For the case when $\alpha = \beta$, the series representation for the kernel of the fractional CFT

$$\mathcal{F}_{\pm,\,\alpha} = e^{\frac{i\alpha m}{2}} e^{\frac{i\alpha}{2}(\Delta - |x|^2 \mp 2\Gamma)}$$

where $\alpha \in [-\pi,\, \pi]$ and $\alpha \neq 0$ and further properties are given in [25].
Notice that we have

$$\mathcal{F}_{+,\,\alpha} \circ \mathcal{F}_{-,\,\alpha} = \mathcal{F}_{\alpha}^{2}.$$

The fractional CFT was computed formally

$$\mathcal{F}_{-,\,\alpha} = (\pi(1 - e^{-2i\alpha}))^{-m/2} e^{i\alpha\Gamma_{\underline{y}}} \int_{\mathbb{R}^m} e^{-i\langle x,y\rangle/\sin\alpha} e^{\frac{i}{2}(\cot\alpha)(|x|^2+|y|^2)}(.)\mathrm{d}x.$$

with the kernel given by

$$\begin{aligned} K_{-,\,\alpha}\,(x,\,y) &= e^{i\alpha\Gamma_{\underline{y}}}\left(e^{-i\langle x,\,y\rangle/\sin\alpha}\, e^{\frac{i}{2}(\cot\alpha)(|x|^2+|y|^2)}\right) \\ &= e^{\frac{i}{2}(\cot\alpha)(|x|^2+|y|^2)}\, e^{i\alpha\Gamma_{\underline{y}}}\left(e^{-i\langle x,\,y\rangle/\sin\alpha}\right) \end{aligned}$$

More generalizations of formulation ***F4*** from [25, 26] and new results on the kernel of the fractional Clifford-Fourier transform (in ***F4***) will be introduced in Section 6.3 in Chapter 6.

2.4 The Clifford-Helmholtz system

The Clifford-Helmholtz system

$$\partial_y K(x,y) = a\, i\, K(x,y)\, \underline{x} \tag{2.32}$$

with $a \in \mathbb{R}_{>0}$ is a system of partial differential equations in a Clifford algebra that forms a refinement of the classical Helmholtz equation

$$\Delta_y K + a^2\, |x|^2\, K = 0.$$

In [58], the author found suitable kernel functions

$$K(x,y) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{C}_m$$

that satisfy the system (2.32). A class of solutions can be given in terms of a finite sum of Bessel functions. Using $\xi = \langle x,\, y\rangle$ and $\eta = |\underline{x} \wedge \underline{y}| = \sqrt{|x|^2|y|^2 - \langle x,\, y\rangle^2}$, one has

Theorem 2.4.1. *[58] Let m be even. A class of solutions of the Clifford-Helmholtz system (2.32) is given by*

$$K_m^n(x,y) = \widetilde{f}_m^n(\xi,\eta) + \widehat{f}_m^n(\xi,\eta) + (\underline{x} \wedge \underline{y})\, g_m^n(\xi,\eta), \tag{2.33}$$

where $n = 0, 1, 2, \ldots, m-2$, $\tilde{f}$, $\hat{f}$ and g are real valued functions explicitly given by

$$\widetilde{f}_m^n = -\gamma_m \sum_{\ell=0}^{\lfloor \frac{n-1}{2} \rfloor} \xi^{n-1-2\ell}\, \frac{1}{a^{\ell+1}} \frac{1}{2^\ell \ell!} \frac{\Gamma(n+1)}{\Gamma(n-2\ell)} \frac{J_{(m-2\ell-3)/2}(a\eta)}{\eta^{(m-2\ell-3)/2}}, \quad n \geq 1$$

$$\widehat{f}_m^n = -i\,\gamma_m \sum_{\ell=0}^{\lfloor \frac{n}{2} \rfloor} \xi^{n-2\ell}\, \frac{1}{a^{\ell}} \frac{1}{2^\ell \ell!} \frac{\Gamma(n+1)}{\Gamma(n+1-2\ell)} \frac{J_{(m-2\ell-3)/2}(a\eta)}{\eta^{(m-2\ell-3)/2}}, \quad n \geq 0$$

$$g_m^n = \gamma_m \sum_{\ell=0}^{\lfloor \frac{n}{2} \rfloor} \xi^{n-2\ell}\, \frac{1}{a^{\ell}} \frac{1}{2^\ell \ell!} \frac{\Gamma(n+1)}{\Gamma(n+1-2\ell)} \frac{J_{(m-2\ell-1)/2}(a\eta)}{\eta^{(m-2\ell-1)/2}}, \quad n \geq 0,$$

in which

$$\gamma_m = \begin{cases} \sqrt{\frac{\pi}{2}} & \text{if } m \text{ even}, \\ 1 & \text{if } m \text{ odd}, \end{cases}$$

Note that each $K_m^n(x,\, y)$ is a solution of (2.32). These solutions were used to define a novel class of Fourier transforms in Chapter 4 in [58]:

Definition 2.4.2. *Let m be even. For each $n = 0, 1, 2, \ldots, m-2$, the Fourier transforms associated with the Clifford-Helmholtz system are defined by*

$$\mathcal{F}_m^n[f(x)](y) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} K_m^n(x,\, y)\, f(x)\, \mathrm{d}x, \tag{2.34}$$

where the kernel $K_m^n(x,\, y)$ is given in (2.33).

Remark 2.4.3. *[58] Note that for $a = 1$ and $n = 0$, the kernel $K_m^0(x,y)$ resembles the Fourier-Bessel kernel (see [8]), as it differs only by a factor $-i$ in front of the scalar part. We have*

$$K_m^0(x,\, y) = \gamma_m \left(-i\ \widetilde{J}_{(m-3)/2}(t) + (\underline{x} \wedge \underline{y})\, \widetilde{J}_{(m-1)/2}(t) \right),$$

while

$$K_m^{\text{Bessel}}(x,\, y) = \gamma_m \left(\widetilde{J}_{(m-3)/2}(t) + (\underline{x} \wedge \underline{y})\, \widetilde{J}_{(m-1)/2}(t) \right)$$

with $\widetilde{J}_\nu(t) = t^{-\nu} J_\nu(t)$.

It is easy to verify that for the new class of Fourier transforms the following differentiation rule holds (see [58], Proposition 4.13).

Theorem 2.4.4. *(differential property) For $f \in \mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$, it holds that*

$$\mathcal{F}^n_m[\partial_{\underline{x}} f(x)](y) = -a\, i\, \underline{y}\, \mathcal{F}^n_m[f(x)](y) \tag{2.35}$$

and as a consequence applying (2.35) twice yields

$$\mathcal{F}^n_m[\Delta_{\underline{x}} f(x)](y) = -a^2 |\underline{y}|^2 \mathcal{F}^n_m[f(x)](y). \tag{2.36}$$

Similar to the Clifford-Bessel transform [8] and the Clifford-Fourier transform [24], an analog of formulation ***F4*** was obtained in terms of the Bessel functions and Gegenbauer polynomials.

Theorem 2.4.5. *[58] For m even, the following series expansions hold:*

Case 1: n even $(n = 0, 2, 4, \ldots)$

$$\begin{aligned}
\widetilde{f}^n_m(w,z) &= -\, n\, a^{-n-1/2}\, \Gamma\Big(\frac{m}{2}-1\Big)\, 2^{m/2-1} \sum_{j=0}^{\infty} \Big(2j+\frac{m}{2}\Big) \\
&\quad \times \frac{(2j+n-1)!!}{(2j+m-n-1)!!}\, z^{-m/2+1}\, J_{2j+m/2}(az)\, C^{m/2-1}_{2j+1}(w) \\
\widehat{f}^n_m(w,z) &= -\, i\, a^{-n-1/2}\, \Gamma\Big(\frac{m}{2}-1\Big)\, 2^{m/2-1} \sum_{j=0}^{\infty} \Big(2j+\frac{m}{2}-1\Big) \\
&\quad \times \frac{(2j+n-1)!!}{(2j+m-n-3)!!}\, z^{-m/2+1}\, J_{2j+m/2-1}(az)\, C^{m/2-1}_{2j}(w) \\
g^n_m(w,z) &= a^{-n-1/2}\, \Gamma\Big(\frac{m}{2}\Big)\, 2^{m/2} \sum_{j=0}^{\infty} \Big(2j+\frac{m}{2}\Big) \\
&\quad \times \frac{(2j+n-1)!!}{(2j+m-n-1)!!}\, z^{-m/2}\, J_{2j+m/2}(az)\, C^{m/2}_{2j}(w)
\end{aligned}$$

Case 2: n odd $(n = 1, 3, 5, \ldots)$

$$\begin{aligned}
\widetilde{f}^n_m(w,z) &= -\, n\, a^{-n-1/2}\, \Gamma\Big(\frac{m}{2}-1\Big)\, 2^{m/2-1} \sum_{j=0}^{\infty} \Big(2j+\frac{m}{2}-1\Big) \\
&\quad \times \frac{(2j+n-2)!!}{(2j+m-n-2)!!}\, z^{-m/2+1}\, J_{2j+m/2-1}(az)\, C^{m/2-1}_{2j}(w)
\end{aligned}$$

$$\begin{aligned}\hat{f}^n_m(w,z) =& - i\, a^{-n-1/2}\, \Gamma\Big(\frac{m}{2}-1\Big)\, 2^{m/2-1} \sum_{j=0}^{\infty} \Big(2j+\frac{m}{2}\Big) \\ & \times \frac{(2j+n)!!}{(2j+m-n-2)!!}\, z^{-m/2+1}\, J_{2j+m/2}(az)\, C^{m/2-1}_{2j+1}(w) \\ g^n_m(w,z) =& a^{-n-1/2}\, \Gamma\Big(\frac{m}{2}\Big)\, 2^{m/2} \sum_{j=1}^{\infty} \Big(2j+\frac{m}{2}-1\Big) \\ & \times \frac{(2j+n-2)!!}{(2j+m-n-2)!!}\, z^{-m/2}\, J_{2j+m/2-1}(az)\, C^{m/2}_{2j-1}(w).\end{aligned}$$

Here the double factorial is defined as $u!! = u(u-2)(u-4)\cdots 5\cdot 3\cdot 1$ for u odd, and $u!! = u(u-2)\cdots 6\cdot 4\cdot 2$ for u even.
For general $a \in \mathbb{R}_{>0}$, we need the following definition for new basis functions for $\mathcal{S}(\mathbb{R}^m)\otimes \mathcal{C}l_{0,m}$.

Definition 2.4.6. *Taking $\{\psi_{p,k,\ell}\}$ as given in (2.27), we define a new basis $\{\phi_{p,k,\ell}\}$ for $\mathcal{S}(\mathbb{R}^m)\otimes \mathcal{C}l_{0,m}$ as*

$$\begin{aligned}\phi_{2p,k,\ell}(x) &:= \psi_{2p,k,\ell}(\sqrt{a}x) = L_p^{k+\lambda}(a|x|^2)\, \big(\sqrt{a}\big)^k\, M_k^{(\ell)}(x)\, e^{-a|x|^2/2} \\ \phi_{2p+1,k,\ell}(x) &:= \psi_{2p+1,k,\ell}(\sqrt{a}x) = L_p^{k+\lambda+1}(a|x|^2)\, \big(\sqrt{a}\big)^{k+1}\, \underline{x} M_k^{(\ell)}(x) e^{-a|x|^2/2}.\end{aligned}$$

When $a=1$, these new basis functions reduce to $\{\psi_{p,k,\ell}\}$.

Furthermore, the generalized Clifford-Hermite functions are shown to be the eigenfunctions of $\mathcal{F}^n_m$ and the corresponding eigenvalues are determined in [58].

Theorem 2.4.7. *The integral transforms $\mathcal{F}^n_m$ act as follows on the basis $\{\phi_{p,k,\ell}\}$ of $\mathcal{S}(\mathbb{R}^m)\otimes \mathcal{C}l_{0,m}$:*

Case 1: n even $(n = 0, 2, 4, \dots)$
a) *k even*

$$\begin{aligned}\mathcal{F}^n_m[\phi_{2p,k,\ell}]\,(y) &= i\, a^{-n-3/2} \frac{(k+n-1)!!}{(k+m-n-3)!!}\, (-1)^{p+1}\, \phi_{2p,k,\ell}\,(y) \\ \mathcal{F}^n_m[\phi_{2p+1,k,\ell}]\,(y) &= a^{-n-3/2}\, \frac{(k+n-1)!!}{(k+m-n-3)!!}\, (-1)^{p}\, \phi_{2p+1,k,\ell}\,(y)\end{aligned}$$

b) *k odd*

$$\mathcal{F}^n_m[\phi_{2p,k,\ell}]\,(y) = a^{-n-3/2}\, \frac{(k+n)!!}{(k+m-n-2)!!}\, (-1)^{p+1}\, \phi_{2p,k,\ell}\,(y)$$

$$\mathcal{F}_m^n[\phi_{2p+1,k,\ell}]\,(y) = i\,a^{-n-3/2}\,\frac{(k+n)!!}{(k+m-n-2)!!}\,(-1)^{p+1}\,\phi_{2p+1,k,\ell}\,(y)$$

Case 2: n odd $(n = 1, 3, 5, \dots)$

a) *k even*

$$\mathcal{F}_m^n[\phi_{2p,k,\ell}]\,(y) = a^{-n-3/2}\,\frac{(k+n)!!}{(k+m-n-2)!!}\,(-1)^{p+1}\,\phi_{2p,k,\ell}\,(y)$$

$$\mathcal{F}_m^n[\phi_{2p+1,k,\ell}]\,(y) = i\,a^{-n-3/2}\,\frac{(k+n)!!}{(k+m-n-2)!!}\,(-1)^{p+1}\,\phi_{2p+1,k,\ell}\,(y)$$

b) *k odd*

$$\mathcal{F}_m^n[\phi_{2p,k,\ell}]\,(y) = i\,a^{-n-3/2}\,\frac{(k+n-1)!!}{(k+m-n-3)!!}\,(-1)^{p+1}\,\phi_{2p,k,\ell}\,(y)$$

$$\mathcal{F}_m^n[\phi_{2p+1,k,\ell}]\,(y) = a^{-n-3/2}\,\frac{(k+n-1)!!}{(k+m-n-3)!!}\,(-1)^{p}\,\phi_{2p+1,k,\ell}\,(y).$$

Restricting the transform $\mathcal{F}_m^n$ to the space $\mathcal{S}(\mathbb{R}^m)\otimes \mathcal{C}\ell_{0,m}$ of Schwartz class functions, we have the following inversion theorem.

Theorem 2.4.8. *[58] Let m be even. The following equality, involving the integral transform $\mathcal{F}_m^n$ and $\mathcal{F}_m^{m-2-n}$, holds on $\mathcal{S}(\mathbb{R}^m)\otimes \mathcal{C}\ell_{0,m}$.*

$$(-1)^{n+1}a^{m+1}\mathcal{F}_m^n\mathcal{F}_m^{m-2-n} = (-1)^{n+1}a^{m+1}\mathcal{F}_m^{m-2-n}\mathcal{F}_m^n = \mathcal{P},$$

where $\mathcal{P}$ is the parity operator, of which the action on a function f is defined by

$$\mathcal{P}[f](y) := f(-y).$$

Therefore, the inverse of the transform $\mathcal{F}_m^n$ is given by

$$(\mathcal{F}_m^n)^{-1} = (-1)^{n+1}a^{m+1}\mathcal{P}\mathcal{F}_m^{m-2-n},$$

In particular, for $\lambda = (m-2)/2$, we have

$$(\mathcal{F}_m^\lambda)^{-1} = (-1)^{\lambda+1}a^{m+1}\mathcal{P}\mathcal{F}_m^\lambda.$$

In the following we give the extension of the transforms $\mathcal{F}_m^n$ to $L^2(\mathbb{R}^m)\otimes \mathcal{C}l_{0,m}$.

Theorem 2.4.9. *[58] For $a = 1$, the transform $\mathcal{F}_m^n$ extends from $\mathcal{S}(\mathbb{R}^m)\otimes \mathcal{C}l_{0,m}$ to a continuous map on $L^2(\mathbb{R}^m)\otimes \mathcal{C}l_{0,m}$ for all $n \leq \lambda$, but not for $n > \lambda$.*

In particular, only when m is even and $n = \lambda$, the transform $\mathcal{F}_m^\lambda$ is unitary, i.e.

$$\|\mathcal{F}_m^\lambda(f)\| = \|f\|$$

for all $f \in L^2(\mathbb{R}^m)\otimes \mathcal{C}l_{0,m}$.

2.5 The Laplace transform method

Suppose that f is a real or complex valued function of the variable $t > 0$ and s is a complex parameter. The (one-sided) Laplace transform of f is defined as (see e.g. [64])

$$F(s) = \mathcal{L}\left(f(t)\right) = \int_0^{\infty} e^{-st} f(t)\, \mathrm{d}t. \tag{2.37}$$

By Lerch's theorem, if we restrict to continuous functions on $[0, \infty)$, the inverse transform is uniquely determined $\mathcal{L}^{-1}\left(F(s)\right) = f(t)$.

In the paper, we will need the following Laplace transform formulas from [39]:

$$\mathcal{L}(e^{-\alpha t}) = \frac{1}{s+\alpha}, \quad \mathrm{Re}\, s > \mathrm{Re}\, \alpha; \tag{2.38}$$

$$\mathcal{L}\left(\cos at\right) = \frac{s}{s^2+a^2}, \quad \mathrm{Re}\, s > |\mathrm{Im}\, a|; \tag{2.39}$$

$$\mathcal{L}\left(\sin at\right) = \frac{a}{s^2+a^2}, \quad \mathrm{Re}\, s > |\mathrm{Im}\, a|; \tag{2.40}$$

$$\mathcal{L}\left(\frac{t^{k-1}\, e^{-\alpha\, t}}{\Gamma(k)}\right) = \frac{1}{(s+\alpha)^k}, \quad k > 0. \tag{2.41}$$

For $\mathrm{Re}\,(\nu) > -1/2$, we have

$$\mathcal{L}\left(t^{\nu}\, J_{\nu}(a\, t)\right) = 2^{\nu}\, \pi^{-1/2}\, \Gamma\left(\nu + \tfrac{1}{2}\right) a^{\nu}\, r^{-2\nu-1}; \tag{2.42}$$

and for $\mathrm{Re}\,(\nu) > -1$,

$$\mathcal{L}\left(t^{\nu+1}\, J_{\nu}(a\, t)\right) = 2^{\nu+1}\, \pi^{-1/2}\, \Gamma\left(\nu + \tfrac{3}{2}\right) a^{\nu}\, r^{-2\nu-3}\, s \tag{2.43}$$

where $r = \left(s^2 + a^2\right)^{1/2}$ and $\mathrm{Re}\,(s) > |\mathrm{Im}\,(a)\,|$.
For $\mathrm{Re}\, s > |\mathrm{Im}\, b|$, it holds

$$\mathcal{L}\left(J_{\nu}(b\, t)\right) = \frac{1}{\sqrt{s^2+b^2}} \left(\frac{b}{s+\sqrt{s^2+b^2}}\right)^{\nu}, \quad \mathrm{Re}\, \nu > -1. \tag{2.44}$$

The Laplace transform of the Prabhakar generalized Mittag-Leffler function (2.17) is

$$\mathcal{L}\left(t^{\beta-1} E^{\delta}_{\alpha,\beta}(b\, t^{\alpha})\right) = \frac{1}{s^{\beta}} \frac{1}{(1 - b\, s^{-\alpha})^{\delta}} \tag{2.45}$$

where $\operatorname{Re}\alpha > 0, \operatorname{Re}\beta > 0, \operatorname{Re}s > 0$ and $s > |b|^{1/(\operatorname{Re}\alpha)}$, see [56].
We also list some inverse transforms from [63]. For $\operatorname{Re}\nu > -1$, $\operatorname{Re}s > |\operatorname{Im}a|$, we have

$$\begin{aligned} &\mathcal{L}^{-1}\left(\frac{\left(\sqrt{s^2+a^2}-s\right)^{\nu}}{\sqrt{s^2+a^2}} F\left(\sqrt{s^2+a^2}-s\right)\right) \\ &= (a^2\, t)^{\nu/2}\int_0^t (t+2\tau)^{-\nu/2} J_\nu(a\sqrt{t^2+2\tau t})\, f(\tau)\mathrm{d}\tau. \end{aligned} \tag{2.46}$$

where $\mathcal{L}(f(t)) = F(s)$. For $\operatorname{Re}\nu > -1$, $\operatorname{Re}s > |\operatorname{Im}a|$, we have

$$\begin{aligned} &\mathcal{L}^{-1}\left(\frac{\left(\sqrt{s^2+a^2}-s\right)^{\nu}}{\sqrt{s^2+a^2}}\, F\left(\sqrt{s^2+a^2}\right)\right) \\ &= a^{\nu}\int_0^t \left(\frac{t-u}{t+u}\right)^{\nu/2} J_\nu[a\,(t^2-u^2)^{1/2}]\, f(u)\,\mathrm{d}u; \end{aligned} \tag{2.47}$$

and for $\operatorname{Re}s > |\operatorname{Im}a|$,

$$\mathcal{L}^{-1}\left(\frac{F(\sqrt{s^2+a^2})}{\sqrt{s^2+a^2}}\right) = \int_0^t J_0[a\,(t^2-u^2)^{1/2}]\, f(u)\,\mathrm{d}u. \tag{2.48}$$

In [39], we also have:

$$\begin{aligned} &\mathcal{L}^{-1}\left(F\left(\sqrt{s^2+a^2}\right)\right) \\ &= f(t) - a\int_0^t f[(t^2-u^2)^{1/2}] J_1(a\,u)\,\mathrm{d}u; \end{aligned} \tag{2.49}$$

$$\begin{aligned} &\mathcal{L}^{-1}\left(\frac{s\,F\left(\sqrt{s^2+a^2}\right)}{\sqrt{s^2+a^2}}\right) \\ &= f(t) - a\,t\int_0^t (t^2-u^2)^{-1/2}\, J_1(a(t^2-u^2)^{1/2})\, f(u)\,\mathrm{d}u. \end{aligned} \tag{2.50}$$

Inverse Laplace transform formulas for the logarithmic functions are given by

$$\mathcal{L}^{-1}\left(\ln\frac{s^2+\beta^2}{s^2+\alpha^2}\right) = \frac{2}{t}\,\left(\cos\left(\alpha\,t\right) - \cos\left(\beta\,t\right)\right); \tag{2.51}$$

$$\begin{aligned} \mathcal{L}^{-1}\left(s\,\ln\frac{s^2+\beta^2}{s^2+\alpha^2}\right) = \frac{2}{t^2}\,&(\cos\left(\beta\,t\right) + \beta\,t\sin\left(\beta\,t\right) - \cos\left(\alpha\,t\right) \\ &- \alpha\,t\sin\left(\alpha\,t\right)). \end{aligned} \tag{2.52}$$

We also need the following formula from [39]

$$\mathcal{L}^{-1}\left(\frac{\lambda\, s+\mu}{(s+\alpha)\,(s+\beta)}\right)=\frac{\alpha\,\lambda-\mu}{\alpha-\beta}\,e^{-\alpha\, t}+\frac{\beta\,\lambda-\mu}{\beta-\alpha}\,e^{-\beta\, t}, \tag{2.53}$$

and the convolution formula of the Laplace transform. Denoting $\mathcal{L}(g(t))=G(s)$ and $\mathcal{L}(f(t))=F(s)$, we have

$$\mathcal{L}^{-1}(F(s)\,G(s))=\int_0^t f(\tau)\,g(t-\tau)\,\mathrm{d}\tau. \tag{2.54}$$

Chapter 3

Uncertainty principles and real Paley-Wiener theorems

3.1 Introduction

Uncertainty principles (see e.g. [14, 49, 50, 52, 55]) are one of the key research topics in harmonic analysis and quantum mechanics. They also play an important role in signal processing. Another crucial topic in the study of generalized Fourier transforms is the Paley-Wiener theorem (see e.g. [41, 53, 54]), which relates decay properties of a function at infinity with the analyticity of its Fourier transform. In this chapter we consider the local and global uncertainty inequalities, Pitt's inequality, and real Paley-Wiener theorems for the novel Fourier transforms associated with the Clifford-Helmholtz system (2.32) introduced in [58].

The main difference with the current literature is that the majority of the Fourier transforms in Section 2.4 are not unitary. This prevents us from using Parseval's identity in the proof and thus makes it not so easy to recognize the correct representation for the Paley-Wiener theorems and uncertainty inequalities. On the other hand, our non-unitary cases provide a deeper understanding as to why the uncertainty inequalities and Paley-Wiener theorems assume their current forms. Moreover, the relations between different transforms in this class are considered, see Theorem 3.3.16 and 3.3.18 as well as Remark 3.3.20.

The structure of this chapter is as follows. In Section 3.2, we give some necessary facts of the Fourier transform $\mathcal{F}_m^n$. In Section 3.3, we introduce some uncertainty inequalities for the classic Fourier transform. Then based on those results we obtain three versions of the uncertainty inequalities for the Fourier transforms associated with the Clifford-Helmholtz system. Two real Paley-Wiener theorems for these transforms are discussed in Section 3.4, which characterize functions such that their Fourier transform vanishes outside or inside a ball.

3.2 Basic facts of the Fourier transform $\mathcal{F}_m^n$

Let m be even and $n = 0, 1, 2, \ldots, m-2$. We first give some necessary facts on the Fourier transforms (2.34) associated with the Clifford-Helmholtz system

$$\mathcal{F}_m^n[f(x)](y) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} K_m^n(x,\, y)\, f(x)\, \mathrm{d}x$$

with the kernel $K_m^n(x,\, y)$ given in (2.33).

Based on Definition 2.4.2 and Remark 2.4.3 in [58], one shows that there exists a constant c such that

$$|K_m^0(x,\, y)| \leq c.$$

Thus, the Hausdorff-Young inequality holds for the operator $\mathcal{F}_m^0$ and the Fourier-Bessel transform $\mathcal{F}_{\text{Bessel}}$. More precisely, there exists a constant α such that for any $f \in L^p(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,\,m}$, $1 \leq p \leq 2$,

$$\|\mathcal{F}_m^0\, f\|_{L^q(\mathbb{R}^m)\otimes \mathcal{C}\ell_{0,m}} \leq \alpha\, \|f\|_{L^p(\mathbb{R}^m)\otimes \mathcal{C}\ell_{0,\,m}},$$

where $1/p + 1/q = 1$.

We next introduce the following two important properties from [54], which were deduced for the Clifford-Fourier transform (see [24])

$$\mathcal{F}_-[f](y) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} K_-(x, y)\; f(x)\, \mathrm{d}x.$$

by using the inverse formula of $\mathcal{F}_-$ and the relation $K_-(y, x) = \overline{K_-(x, y)}$.

Proposition 3.2.1 (Plancherel Theorem)**.** *[54] If $f,\, g \in \mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m}$, then*

$$\int_{\mathbb{R}^m} f(y)\, g(y)\, \mathrm{d}y = \int_{\mathbb{R}^m} \overline{\mathcal{F}_-(\overline{f})(x)}\, \mathcal{F}_-(g)(x)\, \mathrm{d}x.$$

Proposition 3.2.2 (Parseval's Identity)**.** *[54] If $g \in \mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m}$, then*

$$\|g\|_2 = \|\mathcal{F}_-(g)\|_2.$$

Since most of the Fourier transforms (2.34) are not unitary, we can not obtain Parseval's Identity for $\mathcal{F}_m^n$, $0 \le n \le m-2$, unless $n = \lambda = \frac{m-2}{2}$. However, according to Theorem 2.4.8, the boundedness of $\mathcal{F}_m^n$ on $L^2(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m}$ is easy to obtain by comparing the eigenvalues when $a = 1$ in Theorem 2.4.7.

Theorem 3.2.3. *For $a = 1$, the Fourier transform $\mathcal{F}_m^n$ extends from $\mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m}$ to a continuous map on $L^2(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m}$ for all $n \le \lambda$, but not for $n > \lambda$. In particular, we have for any $f \in L^2(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m}$,*

$$\begin{aligned} \|\mathcal{F}_m^n f\|_2 &\le \|f\|_2, \qquad n \le \lambda; \\ \|\mathcal{F}_m^n f\|_2 &\ge \|f\|_2, \qquad n \ge \lambda. \end{aligned} \tag{3.1}$$

Only when $n = \lambda$, the transform $\mathcal{F}_m^\lambda$ is unitary.

Note that the Fourier transforms in Definition 2.4.2 are well-defined on $\mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m}$ when m is even. We introduce the following function space for $1 \le p < \infty$:

$$B^p(\mathbb{R}^m; \mathcal{C}\ell_{0,m}) =: \{f \in L^p(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m} : \\ \|f\|_{w,p} =: \left(\int_{\mathbb{R}^m} |f(y)|^p \, w_m(y) \, \mathrm{d}y\right)^{1/p} < \infty\},$$

where $w_m(y) = (1 + |y|)^{(m-2)/2}$. Then the Fourier transforms $\mathcal{F}_m^n$ are well-defined on $B^1(\mathbb{R}^m; \mathcal{C}\ell_{0,m})$.

When m is odd, a novel class of Fourier transforms associated with the Clifford-Helmholtz system (2.32) is defined in [23] and [58]. The uncertainty inequalities and Paley-Wiener theorem can be studied in a similar way. In the following, we will only consider the even dimensional cases.

3.3 Uncertainty principles

According to [42], "the uncertainty principle is partly a description of a characteristic feature of quantum mechanical systems, partly a statement about the limitations of one's ability to perform measurements on a system without disturbing it, and partly a meta-theorem in harmonic analysis that can be summed up as follows.

$$\text{\textit{A nonzero function and its Fourier transform cannot both be sharply localized."}} \tag{3.2}$$

In Heisenberg's seminal paper [48], the physical ideas of (3.2) were first cast in a mathematical form. The properties of a function f and its Fourier transform $\mathcal{F}(f)$ were further investigated in various uncertainty principles thereafter.

In this section, we study some uncertainty principles for the Fourier transforms $\mathcal{F}_m^n$, $0 \leq n \leq m-2$, given in (2.34) in Definition 2.4.2.

3.3.1 The classical Fourier transform

In this subsection, we first introduce some uncertainty principles for the classical Fourier transform from [42].

We first define the Fourier transform on $L^1(\mathbb{R}^n)$ by

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \langle \xi, x \rangle} f(x)\, \mathrm{d}x.$$

Then the inversion theorem and Parseval's formula take the form $\mathcal{F}^{-1}f(y) = \mathcal{F}f(-y)$ and $\|\mathcal{F}\|_2 = \|f\|_2$. In particular, $\mathcal{F}$ is a unitary operator on $L^2(\mathbb{R})$.

Recall the definition of unitary operator on a Hilbert space.

Definition 3.3.1. *A unitary operator is a bounded linear operator $U : H \to H$ on a Hilbert space H for which the following holds:*

- *U is surjective, and*
- *U preserves the inner product of the Hilbert space H. In other words, for all vectors x and y in H we have:*

$$\langle Ux, Uy \rangle_H = \langle x, y \rangle_H.$$

Heisenberg's Inequality for the classical Fourier transform was studied by Kennard [51] and Weyl ([74], Appendix 1).

Theorem 3.3.2 (Heisenberg's Inequality)**.** *For any $f \in L^2(\mathbb{R})$ and any $u, v \in \mathbb{R}$,*

$$\left(\int_{\mathbb{R}} (x-u)^2 \, |f(x)|^2 \, \mathrm{d}x\right) \left(\int_{\mathbb{R}} (\xi-v)^2 \, |\mathcal{F}(\xi)|^2 \, \mathrm{d}\xi\right) \geq \frac{\|f\|_2^4}{16\pi^2}. \qquad (3.3)$$

Equality holds in (3.3) if and only if $f(x) = Ce^{2\pi i v x} \, e^{-\gamma(x-u)^2}$ for some $C \in \mathbb{C}$ and $\gamma > 0$.

More generally, one can consider some modifications and generalizations of Theorem 3.3.2. In this chapter we observe inequalities of the following form

$$\||x|^a \, f\|_p^{\gamma} \, \||\xi|^b \, \mathcal{F}\|_q^{1-\gamma} \geq K\|f\|_2, \quad (f \in L^2(\mathbb{R})). \qquad (3.4)$$

where $a, b \in (0, \infty)$, $p, q \in [1, \infty]$ and $\gamma \in (0, 1)$. Based on the invariance under dilations, (3.4) is equivalent to an additive inequality by the restriction on parameters a, b, p, q, γ. We now give the following lemma from [42].

Lemma 3.3.3. *A necessary condition for the validity of (3.4) is that*

$$\gamma\left(a + \frac{1}{p} - \frac{1}{2}\right) = (1-\gamma)\left(b + \frac{1}{q} - \frac{1}{2}\right). \qquad (3.5)$$

Moreover, if (3.5) is satisfied, (3.4) is equivalent to

$$\gamma\||x|^a \, f\|_p + (1-\gamma)\||\xi|^b \, \mathcal{F}\|_q \geq K\|f\|_2, \quad (f \in L^2(\mathbb{R})). \qquad (3.6)$$

Inequality (3.4) implies (3.6) because of the elementary inequality

$$s^{\gamma} \, t^{1-\gamma} \leq \gamma \, s + (1-\gamma) \, t,$$

where $s, t \geq 0, \gamma \in (0,\, 1)$.

Another necessary uncertainty principle is the Local uncertainty inequality.

Theorem 3.3.4 (Local Uncertainty Inequalities)**.** *Let $f \in L^2(\mathbb{R}^n)$ and $E \subset \mathbb{R}^n$ measurable. Denote by $|E|$ the measure of E.*
(I)*. If $0 < \alpha < n/2$, there exists a constant K_α such that*

$$\int_E |\mathcal{F}(f)(x)|^2 \, \mathrm{d}x \leq K_\alpha |E|^{2\alpha/n} \||x|^{\alpha} f\|_2^2;$$

(II)*. If $\alpha > n/2$, there exists a constant C_α such that*

$$\int_E |\mathcal{F}(f)(x)|^2 \, \mathrm{d}x \leq C_\alpha |E| \|f\|_2^{2-(n/\alpha)} \||x|^{\alpha} f\|_2^{n/\alpha}.$$

These two inequalities for the classical Fourier transform were considered and generalized in [40, 60, 61]. For more versions of uncertainty principles on the classical Fourier transform, we refer readers to [42].

3.3.2 Specific case: the unitary transform $\mathcal{F}_m^\lambda$

For this subsection we put $a = 1$. In order to establish general uncertainty inequalities for the new class of Fourier transforms, we first consider the specific unitary transform $\mathcal{F}_m^\lambda$ in Theorem 2.4.9, where $\lambda = (m-2)/2$. Direct verification of eigenfunctions and the corresponding eigenvalues shows that $\mathcal{F}_m^\lambda$ admits an elegant operator exponential representation.

Theorem 3.3.5. *[58] Let m be even. The action of $\mathcal{F}_m^\lambda$ on the basis $\{\psi_{p,k,l}\}$ of $\mathcal{S}(\mathbb{R}^m) \otimes C\ell_{0,m}$ coincides with the following operator exponential:*
Case 1*: λ odd or $m \equiv 0 \,(\mathrm{mod}\, 4)$,*

$$e^{i\pi} e^{i\frac{\pi}{2}\left(\mathcal{H}+\Gamma+\Gamma^2\right)},$$

Case 2*: λ even or $m \equiv 2 \,(\mathrm{mod}\, 4)$,*

$$e^{i\frac{3\pi}{2}} e^{i\frac{\pi}{2}\left(\mathcal{H}+\Gamma-\Gamma^2\right)},$$

where $\mathcal{H} = \frac{1}{2}\left(-\Delta_x + |x|^2 - m\right)$ and Γ is the spherical Dirac operator defined in (2.24).

The operator exponential definition immediately implies the following important uncertainty inequalities for the Fourier transform $\mathcal{F}_m^\lambda$.

Theorem 3.3.6. *Let $\lambda = (m-2)/2$. For any $\alpha, \beta > 0$ and $f \in \mathcal{S}(\mathbb{R}^m) \otimes C\ell_{0,m}$, there exist two positive constants c_1 and c_2 such that*

$$\| |x|^\alpha f \|_2^{\frac{\beta}{\alpha+\beta}} \cdot \| |y|^\beta \mathcal{F}_m^\lambda(f) \|_2^{\frac{\alpha}{\alpha+\beta}} \geq c_1 \|f\|_2,$$
$$\int_{\mathbb{R}^m} |f(x)|^2 \ln|x| \, dx + \int_{\mathbb{R}^m} |\mathcal{F}_m^\lambda(f)(y)|^2 \ln|y| \, dy \geq c_2 \|f\|_2^2, \qquad (3.7)$$

where the sharp constants c_1, c_2 are the same as the ones for the ordinary Fourier transform over $\mathbb{R}^m$.

Proof. It is seen from Theorem 3.3.5 that the operator is a composition of the operator $e^{i\frac{\pi}{2}(\Gamma+\Gamma^2)}$ or $e^{i\frac{\pi}{2}(\Gamma-\Gamma^2)}$ with the classical Fourier transform $\mathcal{F}$.

On the other hand, since both the operators $e^{i\frac{\pi}{2}(\Gamma+\Gamma^2)}$ and $e^{i\frac{\pi}{2}(\Gamma-\Gamma^2)}$ are unitary and commute with radial functions, it follows that

$$\begin{aligned} \||y|^\alpha \mathcal{F}_m^\lambda(f)\|_2 &= \||y|^\alpha \mathcal{F}(f)\|_2, \\ \int_{\mathbb{R}^m} |\mathcal{F}_m^\lambda(f)(y)|^2 \ln|y|\,\mathrm{d}y &= \int_{\mathbb{R}^m} |\mathcal{F}(f)(y)|^2 \ln|y|\,\mathrm{d}y. \end{aligned}$$

Now, the two uncertainty inequalities claimed follow from the corresponding inequalities of the classical Fourier transform. □

Remark 3.3.7. *The sharpness of c_1 can be obtained from the factors γ and $(1-\gamma)$ in (3.4). For more definitive results we refer readers to [17]. The constant c_2 can be directly derived from the following inequality in [3]. For all $f \in L^2(\mathbb{R}^n)$, it holds*

$$\begin{aligned} &\int_{\mathbb{R}^n} |f(x)|^2 \log|x-a|\,\mathrm{d}x + \int_{\mathbb{R}^n} |\mathcal{F}(f)(\xi)|^2 \log|\xi - b|\,\mathrm{d}\xi \\ \geq &\left[\psi\left(\frac{1}{4}n\right) - \log\pi\right] \int_{\mathbb{R}^n} |f(x)|^2\,\mathrm{d}x, \end{aligned}$$

where ψ is the logarithmic derivative of the Gamma function.

Remark 3.3.8. *Heisenberg's uncertainty inequality is included as a special case. More precisely, we have for $\alpha = \beta = 1$*

$$\||x|f\|_2 \cdot \||y|\mathcal{F}_m^\lambda(f)\|_2 \geq \frac{m}{2}\|f\|_2,$$

for $f \in \mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m}$ and the constant $m/2$ here is sharp.

Remark 3.3.9. *Recently in [52], the authors proved the logarithmic uncertainty inequality (3.7) for the unitary Clifford-Fourier transform*

$$\mathcal{F}_- := e^{i\pi} e^{i\frac{\pi}{2}(\mathcal{H}+\Gamma)} \tag{3.8}$$

with a completely different method.

3.3.3 Local and global uncertainty inequalities

According to the original Heisenberg's inequality, if f is highly localized, then its Fourier transform could potentially still be concentrated in a small neighborhood of two or more widely separated points. However, the local uncertainty principles shows in a precise way that this also cannot happen, see [40]. In this section, we consider the local and global uncertainty inequalities for the transforms $\mathcal{F}_m^n$, $n \leq \lambda$ by putting $a = 1$. The inequalities for the other cases, i.e. $n > \lambda$ may be obtained immediately by the relations in Theorem 2.4.8. Note that the majority of the Fourier transforms $\mathcal{F}_m^n$ are not unitary and therefore they do not admit an equivalent unitary exponential operator representation. The above mentioned two kinds of uncertainty inequalities will be established by comparing $\mathcal{F}_m^n$, $0 \leq n \leq m-2$ with the classical Fourier transform.

Theorem 3.3.10. *(Local uncertainty inequalities) Let $n \leq \lambda$.*
(I). *If $0 < \alpha < m/2$, there exists a constant K_α such that for all $f \in L^2(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m}$ and all balls $B(0, \varsigma)$ with radius ς centered at the origin,*

$$\int_{B(0,\varsigma)} |\mathcal{F}_m^n f|^2 \,\mathrm{d}x \leq K_\alpha \, |B(0,\varsigma)|^{2\alpha/m} \, \| |x|^\alpha f\|_2^2;$$

(II). *If $\alpha > m/2$, there exists a constant C_α such that for all $f \in L^2(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m}$ and all balls $B(0, \varsigma)$,*

$$\int_{B(0,\varsigma)} |\mathcal{F}_m^n f|^2 \,\mathrm{d}x \leq C_\alpha \, |B(0,\varsigma)| \, \|f\|_2^{2-(m/\alpha)} \, \| |x|^\alpha f\|_2^{m/\alpha}.$$

Proof. We first show that the following inequality holds

$$\int_{B(0,\varsigma)} |\mathcal{F}_m^n(f)(y)|^2 \,\mathrm{d}y \leq \int_{B(0,\varsigma)} |\mathcal{F}(f)(y)|^2 \,\mathrm{d}y$$

for any $B(0, \varsigma)$.

Let $\{Y_k^j\}_{j=1}^{l_k}$ be an orthonormal basis of $\mathcal{H}_k \otimes \mathcal{C}\ell_{0,m}$ on $\mathbb{S}^{m-1}$. Using the spherical coordinate $\underline{x} = r\underline{x}'$ with $\underline{x}' \in \mathbb{S}^{m-1}$ and $r = |x|$, each $f \in \mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m}$ can then be decomposed orthogonally as

$$f(r\,x') = \sum_{k=0}^{\infty} \sum_{j=1}^{l_k} f_{kj}(r) \, Y_k^j(x'), \tag{3.9}$$

where

$$f_{kj}(r) = \int_{\mathbb{S}^{m-1}} \left[f(r\,x') \, \overline{Y_k^j(x')} \right]_0 \mathrm{d}x'.$$

By (3.9), it is seen that the integral of f over the sphere $\mathbb{S}^{m-1}$ is given by

$$\int_{\mathbb{S}^{m-1}} |f(r\,x')|^2\,\mathrm{d}x' = \sum_{k=0}^{\infty}\sum_{j=1}^{l_k} |f_{kj}(r)|^2.$$

Now, by the series expansion of the integral kernel in Theorem 2.4.5 as well as Theorem 2.4.7, there exist constants $C(k,m,j)$ satisfying $|C(k,m,j)| \le 1$ such that

$$\begin{aligned}
&\mathcal{F}_m^n(f)(y)\\
&= \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} K_m^n(x,\,y) f(x)\,\mathrm{d}x\\
&= \frac{1}{(2\pi)^{m/2}} \sum_{k=0}^{\infty}\sum_{j=1}^{l_k} \int_0^{\infty}\int_{\mathbb{S}^{m-1}} K_m^n(x,\,y) Y_k^j(x')\,\mathrm{d}x' f_{kj}(r) r^{m-1}\,\mathrm{d}r\\
&= \frac{1}{(2\pi)^{m/2}} \sum_{k=0}^{\infty}\sum_{j=1}^{l_k} C(k,m,j) Y_k^j(y')\\
&\quad\times \int_0^{\infty} f_{kj}(r) J_{k+m/2-1}(\rho r)(\rho r)^{1-m/2} r^{m+n-1}\,\mathrm{d}r,
\end{aligned} \tag{3.10}$$

where $J_k(x)$ is the Bessel function of the first kind. Therefore, we have

$$\begin{aligned}
&\int_{B(0,\varsigma)} |\mathcal{F}_m^n(f)(y)|^2\,\mathrm{d}y\\
&= \frac{1}{(2\pi)^m} \sum_{k=0}^{\infty}\sum_{j=1}^{l_k} C(k,m,j)^2\\
&\quad\times \int_0^{\varsigma} \left| \int_0^{\infty} f_{kj}(r)\, J_{k+m/2-1}(\rho r)\, (\rho r)^{1-m/2}\, r^{m+k-1}\,\mathrm{d}r \right|^2 \rho^{m-1}\,\mathrm{d}\rho\\
&\le \int_{B(0,\varsigma)} |\mathcal{F}(f)(y)|^2\,\mathrm{d}y,
\end{aligned}$$

where the last step is because the constants $C(k,m,j)^2$ are all smaller than 1. Now the desired inequality follows from the local uncertainty inequality for the classical Fourier transform $\mathcal{F}$. □

Remark 3.3.11. *Instead of comparing the norm of the Fourier transform $\mathcal{F}_m^n$ and the classical Fourier transform of a given function, the result in Theorem 3.3.10 could equally be proved starting from (3.10)* *and then using the local uncertainty principle for the Hankel transform. After carefully estimating the coefficients, the sharp constant in the uncertainty inequality can be obtained for a concrete Fourier transform* $\mathcal{F}_m^n$.

Remark 3.3.12. *The constants* $C(k,m,j)$ *are indeed the eigenvalues given in Theorem 2.4.7. When* $n=\lambda$*, they can also be explicitly given by the method in Theorem 6.4 in [24], using relations* (2.11), (2.12) *and* (2.25).

Using the local uncertainty principle, the global uncertainty principle, which greatly extends Heisenberg's inequality, can be obtained without difficulties following the classical approach (see for instance [44]). Because we do not have Parseval's identity, the right-hand side is $\|\mathcal{F}_m^n f\|_2$ instead of $\|f\|_2$. Here we give a different proof for its dual result.

Theorem 3.3.13. *(Global uncertainty inequality) Let* $n \le \lambda$*. For* $\alpha, \beta > 0$*, there exists a positive constant* $c(\alpha,\beta)$ *such that for all* $f \in L^2(\mathbb{R}^m)\otimes \mathcal{C}\ell_{0,m}$*,*

$$\||x|^\alpha f\|_2^{\frac{\beta}{\alpha+\beta}} \cdot \||y|^\beta \mathcal{F}_m^n f\|_2^{\frac{\alpha}{\alpha+\beta}} \ge c(\alpha,\beta)\|\mathcal{F}_m^n f\|_2. \tag{3.11}$$

Remark 3.3.14. *By Lemma 3.3.3, the product uncertainty inequality* (3.11) *is equivalent with the following sum uncertainty inequality*

$$\frac{\beta}{\alpha+\beta}\||x|^\alpha f\|_2 + \frac{\alpha}{\alpha+\beta}\left\||y|^\beta \mathcal{F}_m^n f\right\|_2 \ge c(\alpha,\beta)\,\|\mathcal{F}_m^n f\|_2$$

with $n \le \lambda$.

For the Fourier transform $\mathcal{F}_m^n$ with $n \ge \lambda$, replacing f by its Fourier transform $\mathcal{F}_m^n f$, then we have by Theorem 2.4.8,

Theorem 3.3.15. *(Global uncertainty inequality) Let* $n \ge \lambda$*. For* $\alpha, \beta > 0$*, there exists a positive constant* $c(\alpha,\beta)$ *such that for all* $\mathcal{F}_m^n f \in L^2(\mathbb{R}^m)\otimes \mathcal{C}\ell_{0,m}$*,*

$$\||x|^\alpha f\|_2^{\frac{\beta}{\alpha+\beta}} \cdot \left\||y|^\beta \mathcal{F}_m^n f\right\|_2^{\frac{\alpha}{\alpha+\beta}} \ge c(\alpha,\beta)\|f\|_2. \tag{3.12}$$

Proof. We have

$$\left\||y|^\beta \mathcal{F}_m^n f\right\|_2 = \left(\int_{\mathbb{R}^m} \left||y|^\beta \mathcal{F}_m^n f(y)\right|^2 \mathrm{d}y\right)^{\frac{1}{2}}$$

$$
\begin{aligned}
&=\Bigg(\sum_{k=0}^{\infty}\sum_{j=1}^{l_k}|C(k,m,j)|^2\int_0^{\infty}\rho^{2\beta+m-1}\\
&\quad\times\left|\int_0^{\infty}f_{kj}(r)J_{k+m/2-1}(\rho r)(\rho r)^{1-m/2}r^{m+k-1}\,\mathrm{d}r\right|^2\mathrm{d}\rho\Bigg)^{\frac{1}{2}}\\
&\geq\left(\int_{\mathbb{R}^m}\left||y|^{\beta}\,\mathcal{F}f(y)\right|^2\mathrm{d}y\right)^{\frac{1}{2}}\\
&=\left\||y|^{\beta}\,\mathcal{F}f\right\|_2, \qquad (3.13)
\end{aligned}
$$

where the inequality is because when $n \geq \lambda$, $|C(k,m,j)|^2 \geq 1$. Now the uncertainty inequality (3.12) follows from (3.13) and the global uncertainty inequality of the ordinary Fourier transform. □

3.3.4 Heisenberg's inequality for general a

We now consider the cases for general a, the parameter appearing in the Clifford-Helmholtz system (2.32). When $a = 1$, comparing the eigenvalues determined in Theorem 2.4.7, it is seen that the L^2-norms of the Fourier transforms of a given function satisfy the following relations.

Theorem 3.3.16. *Let m be even and $a = 1$. For any $f \in \mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m}$, it holds that,*

$$
\begin{aligned}
\|\mathcal{F}_m^0 f\|_2 \leq \|\mathcal{F}_m^2 f\|_2 \leq \cdots \leq \|\mathcal{F}_m^{m-2} f\|_2;\\
\|\mathcal{F}_m^1 f\|_2 \leq \|\mathcal{F}_m^3 f\|_2 \leq \cdots \leq \|\mathcal{F}_m^{m-3} f\|_2.
\end{aligned}
$$

For general a we now have the following theorem.

Theorem 3.3.17. *Let $m \geq 4$ be even, $n \geq \lambda$ and $a \in \mathbb{R}_{>0}$. Then for any $f \in \mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m}$,*

$$\|\mathcal{F}_m^n f\|_2 \geq \|f\|_2 \tag{3.14}$$

if and only if

$$
a \leq \begin{cases} \left(\frac{(n-1)!!}{(m-n-3)!!}\right)^{n+3/2}, & \text{when } n \text{ is even;}\\ \left(\frac{(n)!!}{(m-n-2)!!}\right)^{n+3/2}, & \text{when } n \text{ is odd.}\end{cases} \tag{3.15}
$$

Proof. For any $f \in \mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m}$, the relation

$$\|\mathcal{F}_m^n f\|_2 \geq \|f\|_2$$

holds if and only if all the eigenvalues in Theorem 2.4.7 are bigger than 1. More precisely, when n is even, this is equivalent for every k even with

$$a^{-n-3/2} \, \frac{(k+n-1)!!}{(k+m-n-3)!!} \geq 1, \tag{3.16}$$

and for every k odd with

$$a^{-n-3/2} \, \frac{(k+n)!!}{(k+m-n-2)!!} \geq 1. \tag{3.17}$$

It is seen that the smallest value of the left-hand sides of (3.16) and (3.17) is achieved at $k = 0$. Thus, (3.16) and (3.17) are equivalent to

$$a^{-n-3/2} \, \frac{(n-1)!!}{(m-n-3)!!} \geq 1.$$

Hence we have

$$a \leq \left(\frac{(n-1)!!}{(m-n-3)!!} \right)^{n+3/2}.$$

Similarly, when n is odd, we obtain

$$a \leq \left(\frac{(n)!!}{(m-n-2)!!} \right)^{n+3/2}.$$

$\square$

For general a in (3.15), we have the following relations for the norm of the Fourier transform $\mathcal{F}_m^n$, $0 \leq n \leq m-2$ when $m \geq 4$ is even and and $n \geq \lambda$.

Theorem 3.3.18. *Let $m \geq 4$ be even and and $n \geq \lambda$. For any given a satisfying (3.15), and any $f \in \mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m}$, the following relations hold,*

$$\left\| \mathcal{F}_m^{2\lfloor \frac{m}{4} \rfloor} f \right\|_2 \leq \left\| \mathcal{F}_m^{2\lfloor \frac{m}{4} \rfloor + 2} f \right\|_2 \leq \cdots \leq \|\mathcal{F}_m^{m-2} f\|_2;$$
$$\left\| \mathcal{F}_m^{2\lfloor \frac{m-2}{4} \rfloor + 1} f \right\|_2 \leq \left\| \mathcal{F}_m^{2\lfloor \frac{m-2}{4} \rfloor + 3} f \right\|_2 \leq \cdots \leq \|\mathcal{F}_m^{m-3} f\|_2.$$

Now, we are in position to reconsider Heisenberg's inequality for $\mathcal{F}_m^n$ with a satisfying (3.15).

Theorem 3.3.19. *Let m be even, $n \geq \lambda$ and $a > 0$ satisfying* (3.15). *Then for any $f \in \mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m}$, we have*

$$\||x|f\|_2 \cdot \||y|\mathcal{F}_m^n(f)\|_2 \geq \frac{m}{2a}\|f\|_2,$$

Proof. By the differential property (2.35) and the relation (3.14), we have

$$\begin{aligned}\||y|\,\mathcal{F}_m^n(f)\|_2^2 &= a^{-2}\|\mathcal{F}_m^n\,[\partial_{\underline{x}}\,f(x)\,](\underline{y})\|_2^2\\ &\geq a^{-2}\|\partial_{\underline{x}} f(x)\|_2^2.\end{aligned}$$

Then we have

$$\begin{aligned}\||\underline{x}|f\|_2^2 \cdot \||\underline{y}|\mathcal{F}_m^n(f)\|_2^2 &\geq a^{-2}\||\underline{x}|f\|_2^2 \cdot \|\partial_{\underline{x}} f(\underline{x})\|_2^2\\ &\geq \frac{m^2}{4a^2}\|f\|_2,\end{aligned}$$

where the last inequality is by the Heisenberg's inequality of the ordinary Fourier transform. □

Remark 3.3.20. *By Theorem 3.3.18, for given $a > 0$ satisfying* (3.15)*, the following relation holds*

$$\||x|f\|_2 \cdot \||y|\mathcal{F}_m^n(f)\|_2 \geq \||x|f\|_2 \cdot \left\||y|\mathcal{F}_m^{n-2}(f)\right\|_2 \geq \frac{m}{2a}\|f\|_2,$$

where $n > 2\left\lfloor \frac{m}{4} \right\rfloor$ and even.

3.3.5 Pitt's inequality

In this section, we consider Pitt's inequality for $\mathcal{F}_m^n$, $n \leq \lambda$. We give a proof based on a comparison with the classical Fourier transform. Note that we have set $a = 1$ in this section. The cases for general a can be considered similarly as in Section 3.3.4.

Theorem 3.3.21. *(Pitt's inequality) For $f \in \mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m}$ and $0 \leq \alpha < m$, there exists a constant c_α such that*

$$\int_{\mathbb{R}^m} |y|^{-\alpha}\,|\mathcal{F}_m^n f(y)|^2\,\mathrm{d}y \leq c_\alpha \int_{\mathbb{R}^m} |x|^{\alpha}|\,f(x)|^2\,\mathrm{d}x,$$

where $n \leq \lambda = (m-2)/2$.

Proof. We show that

$$\int_{\mathbb{R}^m} |y|^{-\alpha} |\mathcal{F}_m^n f(y)|^2 \, \mathrm{d}y \leq \int_{\mathbb{R}^m} |y|^{-\alpha} |\mathcal{F} f(y)|^2 \, \mathrm{d}y.$$

Using the notations of the proof of Theorem 3.3.10, we have

$$\begin{aligned} & \int_{\mathbb{R}^m} |y|^{-\alpha} \, |\mathcal{F}_m^n f(y)|^2 \, \mathrm{d}y \\ = \; & \sum_{k=0}^{\infty} \sum_{j=1}^{l_k} |C(k,m,j)|^2 \int_0^{\infty} \left| \int_0^{\infty} f_{kj}(r) J_{k+m/2-1}(\rho \, r) \right. \\ & \left. \times \, (\rho \, r)^{1-m/2} \, r^{m+k-1} \, \mathrm{d}r \right|^2 \rho^{-\alpha+m-1} \, \mathrm{d}\rho \qquad (3.18) \\ \leq \; & \int_{\mathbb{R}^m} |y|^{-\alpha} \, |\mathcal{F} f(y)|^2 \, \mathrm{d}y. \end{aligned}$$

Then the result follows from Pitt's inequality for the classical Fourier transform. □

Remark 3.3.22. *For a concrete Fourier transform $\mathcal{F}_m^n$, the sharp constants in Pitt's inequality can be determined using Pitt's inequality for the Hankel transform and (3.18), see e.g. [52].*

Remark 3.3.23. *Note that as most of the Fourier transforms $\mathcal{F}_m^n$, $0 \leq n \leq m-2$ are not unitary, it is not possible to obtain the logarithmic uncertainty inequality for Pitt's inequality in the above theorem by the classical differentiation approach.*

3.4 Real Paley-Wiener theorems

Paley-Wiener theorems describe Fourier transforms of L^2-functions in various set ups. Rather than on $\mathbb{C}$ as in the classical Paley-Wiener theorem, Bang and Tuan in [11, 72] consider that the support of the Fourier transform comes from growth rates associated to functions on $\mathbb{R}$. In the hypercomplex setting, several versions of the real Paley-Wiener theorems have been established, see for instance [41, 53, 54].

We now give two characterizations of the Fourier transforms (2.34) of functions vanishing outside and inside a ball. In this entire section, we set $a = 1$.

3.4.1 Fourier transforms of functions with compact support

Our first real Paley-Wiener theorem for the Fourier transform $\mathcal{F}_m^n$, $0 \leq n \leq m-2$ is given as follows.

Theorem 3.4.1. *Let $f \in L^2(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m}$ with $\partial_{\underline{x}}^s f \in L^2(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m}$ for any $s \in \mathbb{N}_0$.*
(I) *For $n \leq \lambda$, the Fourier transform $\mathcal{F}_m^n f$ of f vanishes outside a ball $B(0,\sigma)$, if and only if*

$$\sigma \leq \liminf_{s\to\infty} \|\partial_{\underline{x}}^s f\|_2^{1/s} < \infty, \tag{3.19}$$

where $\sigma := \sup\{|y| : y \in \operatorname{supp} \mathcal{F}_m^n f(y)\}$.
(II). *For $n \geq \lambda$, suppose the Fourier transform $\mathcal{F}_m^n f$ of f vanishes outside a ball $B(0,\sigma)$, then $\partial_{\underline{x}}^s f$ satisfies*

$$\limsup_{s\to\infty} \|\partial_{\underline{x}}^s f\|_2^{1/s} \leq \sigma,$$

where $\sigma := \sup\{|y| : y \in \operatorname{supp}\mathcal{F}_m^n f(y)\}$.

Proof. (I). "$\Longrightarrow$": Put $n \leq \lambda$. We suppose that $f \neq 0$ otherwise the result is trivial. Assume $\operatorname{supp} \mathcal{F}_m^n f \subset B(0,\sigma)$. By (3.1), we have

$$\begin{aligned} \|\partial_{\underline{x}}^s f\|_2^2 &\geq \|\mathcal{F}_m^n(\partial_{\underline{x}}^s f)\|_2^2 \qquad (3.20)\\ &= \|-i^s \underline{y}^s \mathcal{F}_m^n(f)\|_2^2 \\ &= \int_{\mathbb{R}^m} |y^s \mathcal{F}_m^n f(\underline{y})|^2 \,\mathrm{d}y \\ &= \int_{\operatorname{supp} \mathcal{F}_m^n f} |y^s \mathcal{F}_m^n f(y)|^2 \,\mathrm{d}y \\ &= \|\mathcal{F}_m^n f\|_2^2 \left(\frac{1}{\|\mathcal{F}_m^n f\|_2^2} \int_{\operatorname{supp} \mathcal{F}_m^n f} |y|^{2s} |\mathcal{F}_m^n f(y)|^2 \,\mathrm{d}y \right). \end{aligned}$$

Alternatively, it is known that if μ is a Lebesgue measure on a set Ω with $\mu(\Omega) = 1$, then

$$\lim_{p\to\infty} \|\phi\|_{L^p(\Omega;\,\mathrm{d}\mu)} = \|\phi\|_{L^\infty(\Omega;\,\mathrm{d}\mu)}. \tag{3.21}$$

Here we set $\Omega = \operatorname{supp} \mathcal{F}_m^n f$, $p = 2s$, $d\mu = \|\mathcal{F}_m^n f\|_2^{-2} |\mathcal{F}_m^n f|^2 \,\mathrm{d}y$ and $\phi = |y|$. Then we have

$$\mu(\operatorname{supp} \mathcal{F}_m^n f) = \|\mathcal{F}_m^n f\|_2^{-2} \int_{\operatorname{supp} \mathcal{F}_m^n f} |\mathcal{F}_m^n f(y)|^2 \,\mathrm{d}y$$

$$= \quad \|\mathcal{F}_m^n f\|_2^{-2} \|\mathcal{F}_m^n f\|_2^2 = 1.$$

Therefore, (3.21) implies

$$\lim_{s\to\infty} \left(\frac{1}{\|\mathcal{F}_m^n f\|_2^2} \int_{\mathrm{supp}\mathcal{F}_m^n f} |y|^{2s} |\mathcal{F}_m^n f(y)|^2 \, \mathrm{d}y \right)^{\frac{1}{2s}} = \sup_{y \,\in\, \mathrm{supp}\mathcal{F}_m^n f} |y| = \sigma.$$

Taking the $\frac{1}{2s}$ power on both sides of (3.20) and then letting $s \to \infty$, we obtain

$$\liminf_{s\to\infty} \left\| \partial_{\underline{x}}^s f \right\|_2^{\frac{1}{s}} \quad \geq \quad \sigma. \tag{3.22}$$

"$\Longleftarrow$": we suppose $\partial_{\underline{x}}^s f \in L^2(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m}$ for any $s \in \mathbb{N}_0$ and

$$\liminf_{s\to\infty} \left\| \partial_{\underline{x}}^s f \right\|_2^{\frac{1}{s}} = d < \infty. \tag{3.23}$$

We need to show $d \geq \sigma$, where $\sigma := \sup\{|y| : \underline{y} \in \mathrm{supp}\, \mathcal{F}_m^n f(y)\}$. We first prove $\mathcal{F}_m^n f$ is compactly supported. For any positive constant M, we let $U_M := \{y : |y| \geq M\}$. Then there holds that

$$\begin{aligned}
\left\| \partial_{\underline{x}}^s f \right\|_2^2 &\geq \left\| \mathcal{F}_m^n \left(\partial_{\underline{x}}^s f \right) \right\|_2^2 \\
&\geq \int_{U_M} |y|^{2s} |\mathcal{F}_m^n f(y)|^2 \, \mathrm{d}y \\
&\geq M^{2s} \int_{U_M} |\mathcal{F}_m^n f(y)|^2 \, \mathrm{d}y \\
&= C M^{2s}
\end{aligned}$$

where C is a positive constant. This leads to

$$\liminf_{s\to\infty} \left\| \partial_{\underline{x}}^s f \right\|_2^{\frac{1}{s}} = \infty.$$

This is a contradiction to (3.23), which implies that $\mathcal{F}_m^n f$ is compactly supported. The same discussion as in (3.20) yields $d = \sigma$.

(II). When $n \geq \lambda$, using Theorem 3.2.3, we have

$$\begin{aligned}
\left\| \partial_{\underline{x}}^s f \right\|_2^2 &\leq \left\| \mathcal{F}_m^n (\partial_{\underline{x}}^s f) \right\|_2^2 \\
&= \int_{B(0,\sigma)} |y|^{2s} \left| \mathcal{F}_m^n f(y) \right|^2 \, \mathrm{d}\underline{y} \\
&\leq \sigma^{2s} \left\| \mathcal{F}_m^n f(y) \right\|_2^2 .
\end{aligned}$$

Thus we have

$$\limsup_{s\to\infty} \|\partial_{\underline{x}}^s f\|_2^{\frac{1}{s}} \leq \sigma.$$

$\square$

3.4.2 Fourier transforms of functions vanishing on a ball

In this subsection, we consider the real Paley-Wiener theorem for functions vanishing on a ball for the Fourier transform $\mathcal{F}_m^n$ in (2.34) in Definition 2.4.2. We again set $a = 1$.

Theorem 3.4.2. *Let $f \in \mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m}$. When $n \geq \lambda$, the following relation holds:*

$$\limsup_{s\to\infty} \left\| \sum_{k=0}^{\infty} \frac{s^k \Delta^k f}{k!} \right\|_p^{\frac{1}{s}} \leq \exp\left(- \inf_{y \in \operatorname{supp} \mathcal{F}_m^n f} |\underline{y}|^2 \right), \quad 1 \leq p \leq 2; \tag{3.24}$$

Proof. We assume $f \neq 0$ in $\mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}\ell_{0,m}$ and let $q = \frac{p}{p-1}$ if $1 < p < \infty$. By the differential property (2.36) of the Fourier transform $\mathcal{F}_m^n$, it follows that

$$\mathcal{F}_m^n \left(\sum_{k=0}^{\infty} \frac{s^k \Delta^k f}{k!} \right) = \exp(-s|y|^2) \mathcal{F}_m^n f(y). \tag{3.25}$$

Let $n \geq \lambda$. For $1 \leq p < 2$, by Hölder's inequality and (3.1), we have

$$\begin{aligned}
\|f\|_p^p &= \int_{\mathbb{R}^m} (1+|x|^2)^{-mp} |(1+|x|^2)^m f(x)|^p \, dx \\
&\leq \left\| (1+|\cdot|^2)^{-mp} \right\|_{\frac{2}{2-p}} \cdot \left\| (1+|\cdot|^2)^m f \right\|_2^p \\
&= C \left\| (1+|\cdot|^2)^m f \right\|_2^p \\
&\leq C \left\| (1-\Delta)^m \mathcal{F}_m^n f \right\|_2^p .
\end{aligned} \tag{3.26}$$

Therefore,

$$\left\| \sum_{k=0}^{\infty} \frac{s^k \Delta^k f}{k!} \right\|_p \leq C \left\| (1-\Delta)^m \left(\exp(-s|\cdot|^2) \mathcal{F}_m^n f \right) \right\|_2 . \tag{3.27}$$

Next, we expand the function $(1-\Delta)^m \left(\exp(-s|\cdot|^2)\mathcal{F}_m^n f\right)$ by introducing a new function ϕ_n, i.e.

$$(1-\Delta)^m \left(\exp\left(-s|\cdot|^2\right) \mathcal{F}_m^n f(y) \right) =: \exp\left(-s|\cdot|^2\right) \phi_n(y), \tag{3.28}$$

where $\operatorname{supp}\phi_n \subset \operatorname{supp}\mathcal{F}_m^n f$ and $\|\phi_n\|_2 < Cs^{2m}$. From (3.27) and (3.28), we have

$$\begin{aligned}\left\|\sum_{k=0}^{\infty}\frac{s^k\Delta^k f}{k!}\right\|_p &\le C\left\|\exp(-t|y|^2)\phi_n(\cdot)\right\|_2\\ &\le C\sup_{y\,\in\,\operatorname{supp}\mathcal{F}_m^n f}\exp\left(-s|y|^2\right)\|\phi_n\|_2\\ &\le Cs^{2m}\exp\left(-s\inf_{y\,\in\,\operatorname{supp}\mathcal{F}_m^n f}|y|^2\right),\end{aligned}$$

which implies that

$$\limsup_{s\to\infty}\left\|\sum_{k=0}^{\infty}\frac{s^k\Delta^k f}{k!}\right\|_p^{\frac{1}{s}} \le \exp\left(-\inf_{y\,\in\,\operatorname{supp}\mathcal{F}_m^n f}|y|^2\right). \tag{3.29}$$

For $p=2$, we have by (3.1),

$$\begin{aligned}&\left\|\sum_{k=0}^{\infty}\frac{s^k\Delta^k f}{k!}\right\|_2\\ &\le\left\|\mathcal{F}_m^n\left(\sum_{k=0}^{\infty}\frac{s^k\Delta^k f}{k!}\right)\right\|_2\\ &=\left\|\exp(-s|y|^2)\,\mathcal{F}_m^n(y)\right\|_2\\ &=\left(\int_{\operatorname{supp}\mathcal{F}_m^n f}\exp(-2s|y|^2)\,|\mathcal{F}_m^n f(y)|^2\,\mathrm{d}y\right)^{\frac{1}{2}}\\ &=\|\mathcal{F}_m^n f\|_2\left(\frac{1}{\|\mathcal{F}_m^n f\|_2^2}\int_{\operatorname{supp}\mathcal{F}_m^n f}\exp(-2s|y|^2)|\mathcal{F}_m^n f(y)|^2\,\mathrm{d}y\right)^{\frac{1}{2}},\end{aligned} \tag{3.30}$$

where the second step is by (3.25). Similar to the proof of Theorem 3.4.1, we take $\Omega = \operatorname{supp}\mathcal{F}_m^n f$, $\phi = \exp(-|y|^2)$, $p = 2s$ and $d\mu = \|\mathcal{F}_m^n f\|_2^{-2}|\mathcal{F}_m^n f|^2 dy$, which yields

$$\limsup_{s\to\infty}\left\|\sum_{k=0}^{\infty}\frac{s^k\Delta^k f}{k!}\right\|_2^{\frac{1}{s}} \le \exp\left(-\inf_{y\,\in\,\operatorname{supp}\mathcal{F}_m^n f}|y|^2\right).$$

□

Remark 3.4.3. *Since there is no Plancherel Theorem for the Fourier transform $\mathcal{F}_m^n$, $0 \le n \le m-2$, we can not derive the corresponding estimates when $n \le \lambda$.*

Chapter 4

Fourier kernels in the Laplace domain

The content of this chapter is based on the paper [23]:
H. De Bie, R. Oste and Z. Yang. *Fourier kernels associated with the Clifford-Helmholtz system. Complex Anal. Oper. Theory.* (2024).
Here we give some more details in the computations.

4.1 Introduction

The Clifford-Helmholtz system (2.32) is a factorization (and hence a refinement) of the standard Helmholtz equation, making it a much more natural starting point for constructing generalized Fourier transforms than (2.31). In this chapter, we turn our attention to the solutions of the Clifford-Helmholtz system (2.32) in the Laplace domain. This system will be considered after a parameter adaptation from a to t according to the notational conventions for the Laplace transform (2.37). The Clifford-Helmholtz system then is

$$\partial_{\underline{y}} K(x, y, t) = t\, i\, K(x, y, t)\, \underline{x} \tag{4.1}$$

with $t \in \mathbb{R}_{>0}$. We start with a finite family of solutions of this Clifford-Helmholtz equation, of parabivector type, given in Theorem 3.3 in [23]. The Laplace transform of solutions takes on a particularly simple form as terminating hypergeometric functions. The presence of the parameter t in the Clifford-Helmholtz system precisely enables this, and was motivated by the fact that in the Laplace domain many generalized Fourier transforms in harmonic or Clifford analysis take on a particularly interesting form (see e.g. [15]). The Laplace domain expression turns out to be indeed beneficial, as it allows to obtain recurrence relations between the different kernels in an easier way than the techniques developed in [27] for (2.31). Furthermore, it allows us to compute formal generating functions of the obtained kernels, which we do explicitly in two cases. In particular, this allows us to find the generating function of the Clifford-Bessel transform introduced in [8].

The chapter is organized as follows. In Section 4.2 we make the translation to the Laplace domain, which allows us to simplify the kernels and to obtain recursion relations. In Section 4.3 we determine the generating function in some special cases. Finally some figures explaining the recursion relations between the kernels are given.

4.2 The new kernels in the Laplace domain

4.2.1 Laplace transform of solutions

In order to simplify the computations, we have multiplied the solutions of the Clifford-Helmholtz system in Theorem 2.4.1 by a factor $t^{(m-1)/2}$ and we keep the factor $\sqrt{\frac{\pi}{2}}$ for all dimensions in the sequel. Therefore the Fourier kernels associated with the Clifford-Helmholtz system are concluded in the following theorem (see [23], Section 3).

Theorem 4.2.1. *[23] For $0 \leq n \leq m-2$, a class of solutions of the Clifford-Helmholtz system (4.1) is given by*

$$K_m^n(x,y) = \widetilde{f}_m^n + \widehat{f}_m^n + (\underline{x} \wedge \underline{y})\; g_m^n, \tag{4.2}$$

for a non-negative integer n, where $\widetilde{f}_m^0 = 0$ and

$$\widetilde{f}_m^n = -\sqrt{\frac{\pi}{2}} \sum_{\ell=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{\xi^{n-1-2\ell}}{2^\ell \ell!} \frac{\Gamma(n+1)}{\Gamma(n-2\ell)} \frac{J_{(m-2\ell-3)/2}(t\,\eta)}{\eta^{(m-2\ell-3)/2}} t^{(m-2\ell-3)/2},$$

$$\widehat{f}_m^n = -i\sqrt{\frac{\pi}{2}} \sum_{\ell=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\xi^{n-2\ell}}{2^\ell \ell!} \frac{\Gamma(n+1)}{\Gamma(n+1-2\ell)} \frac{J_{(m-2\ell-3)/2}(t\,\eta)}{\eta^{(m-2\ell-3)/2}} t^{(m-2\ell-1)/2},$$

$$g_m^n = \sqrt{\frac{\pi}{2}} \sum_{\ell=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\xi^{n-2\ell}}{2^\ell \ell!} \frac{\Gamma(n+1)}{\Gamma(n+1-2\ell)} \frac{J_{(m-2\ell-1)/2}(t\eta)}{\eta^{(m-2\ell-1)/2}} t^{(m-2\ell-1)/2}.$$

with $\xi = \langle x, y \rangle$ and $\eta = |\underline{x} \wedge \underline{y}|$.

Remark 4.2.2. *Note that the solutions $K(x,y,t)$, $\tilde{f}(\xi,\eta,t)$, $\hat{f}(\xi,\eta,t)$ and $g(\xi,\eta,t)$ depend on the parameter t. In order to not overload notations, we suppress the dependence on t and use $K(x,y)$, $\tilde{f}^n_m$, $\hat{f}^n_m$ and g^n_m in the sequel.*

We are interested in finding the Laplace transform of the kernels in (4.2). For this, we will use the Laplace transform formulas (2.42) and (2.43). Note that the restrictions on the order of the Bessel functions in these formulas are satisfied as long as $0 \leq n \leq m-2$.

Using formula (2.42) with $\nu = (m-2\ell-1)/2$, it follows that

$$\begin{aligned}\mathcal{L}\left(g^n_m\right) &= \sum_{\ell=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\sqrt{\frac{\pi}{2}}\,\xi^{n-2\ell}}{\eta^{(m-2\ell-1)/2}} \frac{1}{2^\ell \ell!} \frac{\Gamma(n+1)}{\Gamma(n+1-2\ell)} \\ &\quad \times \mathcal{L}\left(t^{(m-2\ell-1)/2}\, J_{(m-2\ell-1)/2}(t\eta)\right) \\ &= \frac{2^{m/2-1}\,\xi^n\,\Gamma(n+1)}{(s^2+\eta^2)^{m/2}} \sum_{\ell=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{\ell!} \frac{\Gamma\left(\frac{m}{2}-\ell\right)}{\Gamma(n+1-2\ell)} \left(\frac{s^2+\eta^2}{4\xi^2}\right)^\ell\end{aligned} \tag{4.3}$$

and in a similar way:

$$\begin{aligned}\mathcal{L}(\widetilde{f}^n_m) &= -2^{m/2-2}\,\xi^{n-1}(s^2+\eta^2)^{-(m-2)/2}\Gamma(n+1) \\ &\quad \times \sum_{\ell=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{\ell!} \frac{\Gamma\left(\frac{m}{2}-\ell-1\right)}{\Gamma(n-2\ell)} \left(\frac{s^2+\eta^2}{4\xi^2}\right)^\ell.\end{aligned}$$

Furthermore, using formula (2.43) with $\nu = (m-2\ell-3)/2$, we have

$$\begin{aligned}\mathcal{L}(t^{(m-2\ell-1)/2}\, J_{(m-2\ell-3)/2}(t\eta)) &= 2^{(m-2\ell-1)/2}\,\pi^{-1/2}\,\Gamma(m/2-\ell) \\ &\quad \times \eta^{(m-2\ell-3)/2}(s^2+\eta^2)^{-(m-2\ell)/2}\,s,\end{aligned}$$

it follows that

$$\begin{aligned}\mathcal{L}(\widehat{f}^n_m) &= -i\,2^{m/2-1}\,\xi^n(s^2+\eta^2)^{-m/2}\,s\,\Gamma(n+1) \\ &\quad \times \sum_{\ell=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{\ell!} \frac{\Gamma\left(\frac{m}{2}-\ell\right)}{\Gamma(n+1-2\ell)} \left(\frac{s^2+\eta^2}{4\xi^2}\right)^\ell.\end{aligned} \tag{4.4}$$

Remark 4.2.3. *By comparing formulas (4.3) and (4.4), we see that:*

$$\mathcal{L}(\widehat{f}^n_m) = -i\,s\,\mathcal{L}\left(g^n_m\right), \quad 0 \leq n \leq m-2. \tag{4.5}$$

In the following we will therefore focus on $\mathcal{L}(\widetilde{f}^n_m)$ and $\mathcal{L}(g^n_m)$, as the results for $\mathcal{L}(\widehat{f}^n_m)$ can be obtained immediately by (4.5).

4.2.2 Simplified representation

We now determine the series expansion in terms of hypergeometric functions of the kernels. We note that there is a distinction between m and n being odd or even respectively.

Theorem 4.2.4. *Put $0 \leq n \leq m-2$. The following series expansions hold:*
Case 1: m even $(m = 2q)$
a) *n even $(n = 2p)$*

$$\begin{aligned}
\mathcal{L}(g_{2q}^{2p}) &= 2^{q-1-2p}\, (s^2+\eta^2)^{-q+p}\, \frac{\Gamma(2p+1)\Gamma(q-p)}{p!} \\
&\quad \times {}_2F_1\left(\begin{matrix} -p, q-p \\ \frac{1}{2} \end{matrix}; -\frac{\xi^2}{s^2+\eta^2}\right), \\
\mathcal{L}(\widetilde{f}_{2q}^{2p}) &= -2^{q-2p}\, \xi\, (s^2+\eta^2)^{-q+p}\, \frac{\Gamma(2p+1)\Gamma(q-p)}{(p-1)!} \\
&\quad \times {}_2F_1\left(\begin{matrix} 1-p, q-p \\ \frac{3}{2} \end{matrix}; -\frac{\xi^2}{s^2+\eta^2}\right);
\end{aligned}$$

b) *n odd $(n = 2p+1)$*

$$\begin{aligned}
\mathcal{L}(g_{2q}^{2p+1}) &= 2^{q-1-2p}\xi\, (s^2+\eta^2)^{-q+p}\, \frac{\Gamma(2p+2)\Gamma(q-p)}{p!} \\
&\quad \times {}_2F_1\left(\begin{matrix} -p, q-p \\ \frac{3}{2} \end{matrix}; -\frac{\xi^2}{s^2+\eta^2}\right), \\
\mathcal{L}(\widetilde{f}_{2q}^{2p+1}) &= -2^{q-2-2p}\, (s^2+\eta^2)^{-q+1+p}\, \frac{\Gamma(2p+2)\Gamma(q-p-1)}{p!} \\
&\quad \times {}_2F_1\left(\begin{matrix} -p, q-p-1 \\ \frac{1}{2} \end{matrix}; -\frac{\xi^2}{s^2+\eta^2}\right);
\end{aligned}$$

Case 2: m odd $(m = 2q+1)$
a) *n even $(n = 2p)$*

$$\begin{aligned}
\mathcal{L}(g_{2q+1}^{2p}) &= 2^{q-1/2-2p}\, (s^2+\eta^2)^{-q-1/2+p}\, \frac{\Gamma(2p+1)\Gamma(q-p+\frac{1}{2})}{p!} \\
&\quad \times {}_2F_1\left(\begin{matrix} -p, q-p+\frac{1}{2} \\ \frac{1}{2} \end{matrix}; -\frac{\xi^2}{s^2+\eta^2}\right),
\end{aligned}$$

$$\mathcal{L}(\widetilde{f}^{2p}_{2q+1}) = -2^{q+1/2-2p}\,\xi\,(s^2+\eta^2)^{-q-1/2+p}\,\frac{\Gamma(2p+1)\Gamma(q-p+\frac{1}{2})}{(p-1)!}$$
$$\times\,{}_2F_1\left(\begin{matrix}1-p, q-p+\frac{1}{2}\\ \frac{3}{2}\end{matrix}; -\frac{\xi^2}{s^2+\eta^2}\right);$$

b) *n odd* $(n = 2p+1)$

$$\mathcal{L}(g^{2p+1}_{2q+1}) = 2^{q-1/2-2p}\,\xi\,(s^2+\eta^2)^{-q-1/2+p}\,\frac{\Gamma(2p+2)\Gamma(q-p+\frac{1}{2})}{p!}$$
$$\times\,{}_2F_1\left(\begin{matrix}-p, q-p+\frac{1}{2}\\ \frac{3}{2}\end{matrix}; -\frac{\xi^2}{s^2+\eta^2}\right),$$
$$\mathcal{L}(\widetilde{f}^{2p+1}_{2q+1}) = -2^{q-3/2-2p}\,(s^2+\eta^2)^{-q+1/2+p}\,\frac{\Gamma(2p+2)\Gamma(q-p-\frac{1}{2})}{p!}$$
$$\times\,{}_2F_1\left(\begin{matrix}-p, q-p-\frac{1}{2}\\ \frac{1}{2}\end{matrix}; -\frac{\xi^2}{s^2+\eta^2}\right).$$

Proof. We only prove the case of m and n even. We first start with calculating the finite sum in the Laplace transform of $g^n_m(\xi,\eta,a)$. Applying the Legendre duplication formula

$$\Gamma(z)\,\Gamma\left(z+\tfrac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z). \tag{4.6}$$

to the second equality, it follows that

$$\sum_{\ell=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{\ell!}\frac{\Gamma\left(\frac{m}{2}-\ell\right)}{\Gamma(n+1-2\ell)}\left(\frac{s^2+\eta^2}{4\xi^2}\right)^{\ell}$$
$$=\sum_{\ell=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{\ell!}\frac{\Gamma\left(\frac{m}{2}-\ell\right)}{\Gamma\left(2\left(\frac{n}{2}+\frac{1}{2}-\ell\right)\right)}\left(\frac{s^2+\eta^2}{4\xi^2}\right)^{\ell}$$
$$=\sum_{\ell=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{\ell!}\frac{\Gamma\left(\frac{m}{2}-\ell\right)\sqrt{\pi}}{2^{n-2\ell}\,\Gamma\left(\frac{n}{2}+\frac{1}{2}-\ell\right)\,\Gamma\left(\frac{n}{2}+1-\ell\right)}\left(\frac{s^2+\eta^2}{4\xi^2}\right)^{\ell}$$
$$=2^{-n}\sqrt{\pi}\sum_{\ell=0}^{\lfloor\frac{n}{2}\rfloor}\frac{1}{\ell!}\frac{\Gamma\left(\frac{m}{2}-\ell\right)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}-\ell\right)\,\Gamma\left(\frac{n}{2}+1-\ell\right)}\left(\frac{s^2+\eta^2}{\xi^2}\right)^{\ell},$$

Substituting m for $m = 2q$ and n for $n = 2p$ yields

$$2^{-2p}\sqrt{\pi}\sum_{\ell=0}^{p}\frac{1}{\ell!}\frac{\Gamma(q-\ell)}{\Gamma\left(p+\frac{1}{2}-\ell\right)\Gamma\left(p+1-\ell\right)}\left(\frac{s^2+\eta^2}{\xi^2}\right)^{\ell}.$$

Next, executing the substitution $\ell \mapsto p-\ell$ and applying the formula (2.2) yields the desired result:

$$\begin{aligned}
&\sum_{\ell=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{\ell!} \frac{\Gamma\left(\frac{m}{2}-\ell\right)}{\Gamma(n+1-2\ell)} \left(\frac{s^2+\eta^2}{4\xi^2}\right)^{\ell} \\
&= 2^{-2p}\sqrt{\pi} \sum_{\ell=0}^{p} \frac{1}{(p-\ell)!} \frac{\Gamma\left(q-p+\ell\right)}{\Gamma\left(\ell+\frac{1}{2}\right)\Gamma\left(\ell+1\right)} \left(\frac{s^2+\eta^2}{\xi^2}\right)^{p-\ell} \\
&= 2^{-2p}\sqrt{\pi}\, \frac{1}{p!} \sum_{\ell=0}^{p} \binom{p}{\ell} \frac{\Gamma(q-p+\ell)}{\Gamma\left(\ell+\frac{1}{2}\right)} \left(\frac{s^2+\eta^2}{\xi^2}\right)^{p-\ell} \\
&= 2^{-2p}\, \frac{\Gamma(q-p)}{p!} \left(\frac{s^2+\eta^2}{\xi^2}\right)^{p} {}_2F_1\left(\begin{matrix}-p, q-p \\ \frac{1}{2}\end{matrix}; -\frac{\xi^2}{s^2+\eta^2}\right).
\end{aligned}$$

We now have the new form of the finite sum in terms of a hypergeometric function. Combining with the formula (4.3), this leads to

$$\begin{aligned}
\mathcal{L}(g_{2q}^{2p}) = 2^{q-1-2p}\,(s^2+\eta^2)^{-q+p}\, &\frac{\Gamma(2p+1)\,\Gamma(q-p)}{p!} \\
&\times {}_2F_1\left(\begin{matrix}-p, q-p \\ \frac{1}{2}\end{matrix}; -\frac{\xi^2}{s^2+\eta^2}\right).
\end{aligned}$$

Similarly, we need to simplify the infinite sum in $\mathcal{L}(\tilde{f}_{2q}^{2p})$

$$\begin{aligned}
&\sum_{\ell=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{\ell!} \frac{\Gamma\left(\frac{m}{2}-\ell-1\right)}{\Gamma(n-2\ell)} \left(\frac{s^2+\eta^2}{4\xi^2}\right)^{\ell} \\
&= \sum_{\ell=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{\ell!} \frac{\Gamma\left(\frac{m}{2}-\ell-1\right)\sqrt{\pi}}{2^{n-2\ell-1}\,\Gamma\left(\frac{n}{2}-\ell\right)\Gamma\left(\frac{n}{2}+\frac{1}{2}-\ell\right)} \left(\frac{s^2+\eta^2}{4\xi^2}\right)^{\ell} \\
&= 2^{-n+1}\sqrt{\pi} \sum_{\ell=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{\ell!} \frac{\Gamma\left(\frac{m}{2}-\ell-1\right)}{\Gamma\left(\frac{n}{2}-\ell\right)\Gamma\left(\frac{n}{2}+\frac{1}{2}-\ell\right)} \left(\frac{s^2+\eta^2}{\xi^2}\right)^{\ell} \\
&= 2^{-2p+1}\sqrt{\pi} \sum_{\ell=0}^{p-1} \frac{1}{\ell!} \frac{\Gamma\left(q-\ell-1\right)}{\Gamma\left(p-\ell\right)\Gamma\left(p+\frac{1}{2}-\ell\right)} \left(\frac{s^2+\eta^2}{\xi^2}\right)^{\ell}
\end{aligned}$$

by substituting m for $m=2q$ and n for $n=2p$. Next, executing the substitution $\ell \mapsto p-1-\ell$

$$
\begin{aligned}
&\sum_{\ell=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{\ell!} \frac{\Gamma\left(\frac{m}{2}-\ell-1\right)}{\Gamma(n-2\ell)} \left(\frac{s^2+\eta^2}{4\xi^2}\right)^{\ell} \\
&= 2^{-2p+1}\sqrt{\pi} \sum_{\ell=0}^{p-1} \frac{1}{(p-1-\ell)!} \frac{\Gamma(q-p+\ell)}{\Gamma(\ell+1)\,\Gamma\left(\ell+\frac{3}{2}\right)} \left(\frac{s^2+\eta^2}{\xi^2}\right)^{p-1-\ell} \\
&= 2^{-2p+1}\sqrt{\pi} \frac{1}{(p-1)!} \sum_{\ell=0}^{p-1} \binom{p-1}{\ell} \frac{\Gamma(q-p+\ell)}{\Gamma\left(\ell+\frac{3}{2}\right)} \left(\frac{s^2+\eta^2}{\xi^2}\right)^{p-1-\ell} \\
&= 2^{-2p+1}\sqrt{\pi} \frac{1}{(p-1)!} \left(\frac{s^2+\eta^2}{\xi^2}\right)^{p-1} \sum_{\ell=0}^{p-1} (-1)^{\ell} \binom{p-1}{\ell} \\
&\quad \times \frac{(q-p)_{\ell}\,\Gamma(q-p)}{\left(\frac{3}{2}\right)_{\ell}\Gamma\left(\frac{3}{2}\right)} \left(-\frac{\xi^2}{s^2+\eta^2}\right)^{\ell} \\
&= 2^{-2p+2} \frac{\Gamma(q-p)}{(p-1)!} \left(\frac{s^2+\eta^2}{\xi^2}\right)^{p-1} {}_2F_1\left(\begin{matrix} 1-p, q-p \\ \frac{3}{2} \end{matrix}; -\frac{\xi^2}{s^2+\eta^2}\right),
\end{aligned}
$$

we obtain the desired result

$$
\begin{aligned}
\mathcal{L}(\widetilde{f}_{2q}^{2p}) = -2^{q-2p}\,\xi\,(s^2+\eta^2)^{-q+p} &\frac{\Gamma(2p+1)\Gamma(q-p)}{(p-1)!} \\
\times {}_2F_1&\left(\begin{matrix} 1-p, q-p \\ \frac{3}{2} \end{matrix}; -\frac{\xi^2}{s^2+\eta^2}\right).
\end{aligned}
$$

Using the same techniques as in the case m and n even, the proof for the other cases can be done. □

To summarize our results, we use floor functions to substitute $q = \lfloor \frac{m}{2} \rfloor$ and $p = \lfloor \frac{n}{2} \rfloor$ in Theorem 4.2.4. Then the Laplace transform of solutions of the Clifford-Helmholtz system is as follows:

$$
\begin{aligned}
\mathcal{L}(\widehat{f}_m^n) &= -i\,s\,\mathcal{L}(g_m^n), \\
\mathcal{L}(g_m^n) &= \frac{2^{\frac{m}{2}-2\lfloor \frac{n}{2} \rfloor -1}\,\xi^{n-2\lfloor \frac{n}{2} \rfloor}}{(s^2+\eta^2)^{\frac{m}{2}-\lfloor \frac{n}{2} \rfloor}} \frac{\Gamma(n+1)\,\Gamma\left(\frac{m}{2}-\lfloor \frac{n}{2} \rfloor\right)}{\lfloor \frac{n}{2} \rfloor !} \\
&\quad \times {}_2F_1\left(\begin{matrix} -\lfloor \frac{n}{2} \rfloor, \frac{m}{2}-\lfloor \frac{n}{2} \rfloor \\ n+\frac{1}{2}-2\lfloor \frac{n}{2} \rfloor \end{matrix}; -\frac{\xi^2}{s^2+\eta^2}\right),
\end{aligned}
$$

$$\mathcal{L}(\widetilde{f}_m^n) = -\frac{2^{\frac{m}{2}+2\lfloor\frac{n}{2}\rfloor-2n}\,\xi^{-n+1+2\lfloor\frac{n}{2}\rfloor}}{(s^2+\eta^2)^{\frac{m}{2}-n+\lfloor\frac{n}{2}\rfloor}}\frac{\Gamma(n+1)\,\Gamma\left(\frac{m}{2}-n+\lfloor\frac{n}{2}\rfloor\right)}{(n-\lfloor\frac{n}{2}\rfloor-1)!}$$
$$\times\,{}_2F_1\left(\begin{matrix}-n+\lfloor\frac{n}{2}\rfloor+1,\frac{m}{2}-n+\lfloor\frac{n}{2}\rfloor\\ -n+\frac{3}{2}+2\lfloor\frac{n}{2}\rfloor\end{matrix};-\frac{\xi^2}{s^2+\eta^2}\right).$$

The series expressions for $\mathcal{L}(\widetilde{f}_m^n)$ and $\mathcal{L}(g_m^n)$ obtained in Theorem 4.2.4 allow us to study the recursion relations between the kernels $K_m^n(x,y)$. This is the subject of the following subsection.

4.2.3 Recursion relations of the kernels

We can now determine recursion relations between the components of $\mathcal{L}(K_m^n(x,y))$ for different values of m. These relations will be based upon the operator

$$\mathbf{d} := \partial_\xi - \frac{\xi}{\eta}\partial_\eta. \tag{4.7}$$

Remark 4.2.5. *In [27, 58] the notations $z=|x||y|$ and $w=\langle\zeta,\tau\rangle$ are used with $\underline{\zeta}=\underline{x}/|x|$, $\underline{\tau}=\underline{y}/|y|$. These variables are related to ξ and η by the transformation formulas $\xi=zw$ and $\eta=z\sqrt{1-w^2}$. Applying the chain rule, the differential operator $z^{-1}\partial_w$ can be written as*

$$z^{-1}\partial_w = z^{-1}\partial_w[\xi]\,\partial_\xi + z^{-1}\partial_w[\eta]\,\partial_\eta = \partial_\xi - \frac{w}{\sqrt{1-w^2}}\,\partial_\eta = \partial_\xi - \frac{\xi}{\eta}\,\partial_\eta = \boldsymbol{d}.$$

Proposition 4.2.6. *For $1\le n\le m-2$ we have the following recursion relations:*

$$\mathcal{L}(g_{m+2}^{n+1}) = \boldsymbol{d}[\mathcal{L}(g_m^n)] \tag{4.8}$$
$$\mathcal{L}(\widetilde{f}_{m+2}^{n+1}) = \frac{n+1}{n}\,\boldsymbol{d}\left[\mathcal{L}(\widetilde{f}_m^n)\right] \tag{4.9}$$

with starting values given by

$$\begin{aligned}\mathcal{L}(g_{m+2}^1) &= \boldsymbol{d}\left[\mathcal{L}(g_m^0)\right]\\ \mathcal{L}(g_m^1) &= -\boldsymbol{d}\left[\mathcal{L}(\widetilde{f}_m^1)\right].\end{aligned} \tag{4.10}$$

For $1\le n\le m-2$ we have

$$\begin{aligned}\mathcal{L}(g_m^n) &= -\frac{1}{i\,s}\mathcal{L}(\widehat{f}_m^n)\\ &= -\frac{1}{n}\,\boldsymbol{d}\left[\mathcal{L}(\widetilde{f}_m^n)\right].\end{aligned} \tag{4.11}$$

Proof. By the series expansion of $\mathcal{L}(g_m^0)$ and $\mathcal{L}(\widetilde{f}_m^1)$, we can first prove the two starting relations (4.10):

$$\mathbf{d}\left[\mathcal{L}(g_m^0)\right] = \mathbf{d}\left[2^{m/2-1}\,(s^2+\eta^2)^{-m/2}\,\Gamma\left(\frac{m}{2}\right)\right] = \mathcal{L}(g_{m+2}^1)$$

and

$$\begin{aligned}
\mathbf{d}\left[\mathcal{L}(\widetilde{f}_m^1)\right] &= -\frac{\xi}{\eta}\,\partial_\eta\left[\mathcal{L}(\widetilde{f}_m^1)\right]\\
&= -2^{m/2-1}\,\xi\,(s^2+\eta^2)^{-m/2}\,\Gamma\left(\frac{m}{2}\right)\\
&= -\mathcal{L}(g_m^1).
\end{aligned}$$

For the next part of the proof, we make a distinction between $n = 2j$ and $n = 2j+1$. We now check (4.8) for the case $n = 2j$.

$$\begin{aligned}
\mathbf{d}\left[\mathcal{L}(g_m^{2j})\right] &= \left(\partial_\xi - \frac{\xi}{\eta}\,\partial_\eta\right)\left[\mathcal{L}(g_m^{2j})\right]\\
&= (2j-2\ell)\,2^{m/2-1}\,\xi^{2j-1}(s^2+\eta^2)^{-m/2}\,\Gamma(2j+1)\\
&\quad\times\sum_{\ell=0}^{j}\frac{1}{\ell!}\frac{\Gamma\left(\frac{m}{2}-\ell\right)}{\Gamma(2j+1-2\ell)}\left(\frac{s^2+\eta^2}{4\xi^2}\right)^{\ell}\\
&\quad+\left(\frac{m}{2}-\ell\right)\,2^{m/2}\,\xi^{2j+1}(s^2+\eta^2)^{-m/2-1}\,\Gamma(2j+1)\\
&\quad\times\sum_{\ell=0}^{j}\frac{1}{\ell!}\frac{\Gamma\left(\frac{m}{2}-\ell\right)}{\Gamma(2j+1-2\ell)}\left(\frac{s^2+\eta^2}{4\xi^2}\right)^{\ell}.
\end{aligned}$$

In the second equation we let $\ell \longmapsto \ell-1$ for the first term, then it turns into

$$\begin{aligned}
\mathbf{d}\left[\mathcal{L}(g_m^{2j})\right] &= 2^{m/2-1}\,\xi^{2j-1}(s^2+\eta^2)^{-m/2}\,\Gamma(2j+1)\\
&\quad\times\sum_{\ell=1}^{j+1}\frac{1}{(\ell-1)!}\frac{\Gamma\left(\frac{m}{2}-\ell+1\right)}{\Gamma(2j-2\ell+2)}\left(\frac{s^2+\eta^2}{4\xi^2}\right)^{\ell-1}\\
&\quad+2^{m/2}\,\xi^{2j+1}(s^2+\eta^2)^{-m/2-1}\,\Gamma(2j+1)\\
&\quad\times\sum_{\ell=0}^{j}\frac{1}{\ell!}\frac{\Gamma\left(\frac{m}{2}-\ell+1\right)}{\Gamma(2j+1-2\ell)}\left(\frac{s^2+\eta^2}{4\xi^2}\right)^{\ell}\\
&= (2j+1)\,2^{m/2}\,\xi^{2j+1}(s^2+\eta^2)^{-m/2-1}\,\Gamma(2j+1)
\end{aligned}$$

$$
\begin{aligned}
&\times \sum_{\ell=1}^{j} \frac{1}{\ell!} \frac{\Gamma\left(\frac{m}{2}-\ell+1\right)}{\Gamma(2j+2-2\ell)} \left(\frac{s^2+\eta^2}{4\ell^2}\right)^{\ell} \\
&+ (2j+1)\, 2^{m/2}\, \xi^{2j+1} (s^2+\eta^2)^{-m/2-1}\, \Gamma(2j+1) \frac{\Gamma\left(\frac{m}{2}+1\right)}{\Gamma(2j+2)} \\
= {}& 2^{m/2}\, \xi^{2j+1} (s^2+\eta^2)^{-m/2-1}\, \Gamma(2j+1)(2j+1) \\
&\times \sum_{\ell=0}^{j} \frac{1}{\ell!} \frac{\Gamma\left(\frac{m}{2}-\ell+1\right)}{\Gamma(2j+2-2\ell)} \left(\frac{s^2+\eta^2}{4\xi^2}\right)^{\ell} \\
= {}& \mathcal{L}(g_{m+2}^{2j+1}).
\end{aligned}
$$

The case $n = 2j+1$ is treated similarly.

Next we check formula (4.9) for the case $n = 2j$. Applying the differential operator $\mathbf{d}$ yields

$$
\begin{aligned}
\mathbf{d}\left[\mathcal{L}(\widetilde{f}_m^{2j})\right] = {}& -(2j-1-2\ell)\, 2^{m/2-2}\, \xi^{2j-2} (s^2+\eta^2)^{-m/2+1}\, \Gamma(2j+1) \\
&\times \sum_{\ell=0}^{j-1} \frac{1}{\ell!} \frac{\Gamma\left(\frac{m}{2}-\ell-1\right)}{\Gamma(2j-2\ell)} \left(\frac{s^2+\eta^2}{4\xi^2}\right)^{\ell} \\
&- \left(\frac{m}{2}-1-\ell\right) 2^{m/2-1}\, \xi^{2j} (s^2+\eta^2)^{-m/2}\, \Gamma(2j+1) \\
&\times \sum_{\ell=0}^{j-1} \frac{1}{\ell!} \frac{\Gamma\left(\frac{m}{2}-\ell-1\right)}{\Gamma(2j-2\ell)} \left(\frac{s^2+\eta^2}{4\xi^2}\right)^{\ell}.
\end{aligned}
$$

In the first term we execute the substitution $\ell \longmapsto \ell - 1$, which leads to

$$
\begin{aligned}
&\mathbf{d}\left[\mathcal{L}(\widetilde{f}_m^{2j})\right] \\
= {}& -2^{m/2-2}\, \xi^{2j-2} (s^2+\eta^2)^{-m/2+1}\, \Gamma(2j+1) \\
&\times \sum_{\ell=1}^{j} \frac{1}{(\ell-1)!} \frac{\Gamma\left(\frac{m}{2}-\ell\right)}{\Gamma(2j-2\ell+1)} \left(\frac{s^2+\eta^2}{4\xi^2}\right)^{\ell-1} \\
&- \frac{2^{m/2-1}\, \xi^{2j}}{(s^2+\eta^2)^{m/2}}\, \Gamma(2j+1) \sum_{\ell=0}^{j-1} \frac{1}{\ell!} \frac{\Gamma\left(\frac{m}{2}-\ell\right)}{\Gamma(2j-2\ell)} \left(\frac{s^2+\eta^2}{4\xi^2}\right)^{\ell} \\
= {}& -(2j)\, \frac{2^{m/2-1}\, \xi^{2j}}{(s^2+\eta^2)^{m/2}}\, \Gamma(2j+1) \sum_{\ell=1}^{j-1} \frac{1}{\ell!} \frac{\Gamma\left(\frac{m}{2}-\ell\right)}{\Gamma(2j-2\ell+1)} \left(\frac{s^2+\eta^2}{4\xi^2}\right)^{\ell} \\
&- \frac{2^{m/2-1}\, \xi^{2j}}{(s^2+\eta^2)^{m/2}}\, \Gamma(2j+1) \frac{2}{(j-1)!} \Gamma\left(\frac{m}{2}-j\right) \left(\frac{s^2+\eta^2}{4\xi^2}\right)^{j}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{2^{m/2-1}\,\xi^{2j}}{(s^2+\eta^2)^{m/2}}\,\Gamma(2j+1)\frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma(2j)} \\
&= -\frac{2j}{2j+1}\mathcal{L}(\widetilde{f}^{2j+1}_{m+2}).
\end{aligned}
$$

The proof of formula (4.11) runs along the same lines. We have

$$
\begin{aligned}
&\mathbf{d}\left[\mathcal{L}(\widetilde{f}^{2j}_m)\right] \\
&= -(2j)\,\frac{2^{m/2-1}\,\xi^{2j}}{(s^2+\eta^2)^{m/2}}\,\Gamma(2j+1)\sum_{\ell=0}^{j}\frac{1}{\ell!}\frac{\Gamma\left(\frac{m}{2}-\ell\right)}{\Gamma(2j-2\ell+1)}\left(\frac{s^2+\eta^2}{4\xi^2}\right)^{\ell} \\
&= -2j\,\mathcal{L}(g^{2j}_m) \\
&= \frac{2j}{i\,s}\mathcal{L}(\widehat{f}^{2j}_m).
\end{aligned}
$$

This completes the proof. □

Lemma 4.2.7. *For all $m \geq 3$, the kernels K^0_m satisfy the relations*

$$
\begin{aligned}
\mathcal{L}(g^0_m) &= \frac{1}{\xi}\,\mathcal{L}(g^1_m), \\
\mathcal{L}(g^0_m) &= -\mathcal{L}(\widetilde{f}^1_{m+2}), \\
\mathcal{L}(\widehat{f}^0_m) &= \frac{1}{\xi}\,\mathcal{L}(\widehat{f}^1_m).
\end{aligned}
\tag{4.12}
$$

Proof. We have, by using the series expansions of the terms $\mathcal{L}(g^0_m)$, $\mathcal{L}(g^1_m)$ and $\mathcal{L}(\widetilde{f}^1_{m+2})$,

$$
\begin{aligned}
\mathcal{L}(g^0_m) &= 2^{m/2-1}\,(s^2+\eta^2)^{-m/2}\,\Gamma\left(\frac{m}{2}\right), \\
\mathcal{L}(g^1_m) &= 2^{m/2-1}\,\xi\,(s^2+\eta^2)^{-m/2}\,\Gamma\left(\frac{m}{2}\right), \\
\mathcal{L}(\widetilde{f}^1_{m+2}) &= -2^{m/2-1}\,(s^2+\eta^2)^{-m/2}\,\Gamma\left(\frac{m}{2}\right),
\end{aligned}
$$

yielding the first two relations of (4.12). Then the third can be immediately derived from the first relation because of (4.5). □

Lemma 4.2.8. *For the kernels K^{m-2}_m, $m \geq 4$, we have*

$$
\mathcal{L}(g^{m-2}_m) = \xi\,\mathcal{L}(g^{m-3}_m) + (m-3)\,\mathcal{L}(g^{m-4}_{m-2}). \tag{4.13}
$$

and

$$
\mathcal{L}(\widetilde{f}^{m-2}_m) = \frac{m-2}{m-3}\,\xi\,\mathcal{L}(\widetilde{f}^{m-3}_m) + (m-2)\,\mathcal{L}(\widetilde{f}^{m-4}_{m-2}) \tag{4.14}
$$

Proof. The proof can be carried out by induction on the dimension m. We first consider the relation (4.13).

$$\begin{aligned}
&\xi\, \mathcal{L}(g_m^{m-3}) + (m-3)\, \mathcal{L}(g_{m-2}^{m-4}) \\
&= 2^{m/2-1}\, \xi^{m-2}\, (s^2+\eta^2)^{-m/2}\, \Gamma(m-2) \\
&\quad \times \sum_{\ell=0}^{\frac{m}{2}-2} \frac{1}{\ell!} \frac{\Gamma(m/2-\ell)}{\Gamma(m-2-2\ell)} \left(\frac{s^2+\eta^2}{4\xi^2}\right)^{\ell} \\
&\quad + 2^{m/2-2}\, \xi^{m-4}\, (s^2+\eta^2)^{-m/2+1}\, \Gamma(m-2) \\
&\quad \times \sum_{\ell=0}^{\frac{m}{2}-2} \frac{1}{\ell!} \frac{\Gamma(m/2-1-\ell)}{\Gamma(m-3-2\ell)} \left(\frac{s^2+\eta^2}{4\xi^2}\right)^{\ell}
\end{aligned}$$

Now substituting $\ell \longmapsto \ell - 1$ in the second term, it yields

$$\begin{aligned}
&\xi\, \mathcal{L}(g_m^{m-3}) + (m-3)\, \mathcal{L}(g_{m-2}^{m-4}) \\
&= 2^{m/2-1}\, \xi^{m-2}\, (s^2+\eta^2)^{-m/2}\, \Gamma(m-2) \\
&\quad \times \sum_{\ell=0}^{\frac{m}{2}-2} \frac{1}{\ell!} \frac{\Gamma(m/2-\ell)}{\Gamma(m-2-2\ell)} \left(\frac{s^2+\eta^2}{4\xi^2}\right)^{\ell} \\
&\quad + 2^{m/2-2}\, \xi^{m-4}\, (s^2+\eta^2)^{-m/2+1}\, \Gamma(m-2) \\
&\quad \times \sum_{\ell=1}^{\frac{m}{2}-1} \frac{1}{(\ell-1)!} \frac{\Gamma(m/2-\ell)}{\Gamma(m-1-2\ell)} \left(\frac{s^2+\eta^2}{4\xi^2}\right)^{\ell-1} \\
&= (m-2)\, 2^{m/2-1}\, \xi^{m-2}\, (s^2+\eta^2)^{-m/2}\, \Gamma(m-2) \\
&\quad \times \sum_{\ell=1}^{\frac{m}{2}-2} \frac{1}{\ell!} \frac{\Gamma(m/2-\ell)}{\Gamma(m-1-2\ell)} \left(\frac{s^2+\eta^2}{4\xi^2}\right)^{\ell} \\
&\quad + (m-2)\, 2^{m/2-1}\, \xi^{m-2}\, (s^2+\eta^2)^{-m/2}\, \Gamma(m-2)\, \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma(m-1)} \\
&\quad + (m-2)\, 2^{m/2-1}\, \xi^{m-2}\, (s^2+\eta^2)^{-m/2}\, \Gamma(m-2)\, \frac{1}{(\frac{m}{2}-1)!} \\
&\quad \times \left(\frac{s^2+\eta^2}{4\xi^2}\right)^{\frac{m}{2}-1} \\
&= (m-2)\, 2^{m/2-1}\, \xi^{m-2}\, (s^2+\eta^2)^{-m/2}\, \Gamma(m-2) \\
&\quad \times \sum_{\ell=0}^{\frac{m}{2}-1} \frac{1}{\ell!} \frac{\Gamma(m/2-\ell)}{\Gamma(m-1-2\ell)} \left(\frac{s^2+\eta^2}{4\xi^2}\right)^{\ell}
\end{aligned}$$

$$= \mathcal{L}(g_m^{m-2}).$$

The formula (4.14) is treated similarly. By executing the substitution $\ell \longmapsto \ell - 1$ in $\mathcal{L}(\widetilde{f}_{m-2}^{m-4})$, we arrive at

$$
\begin{aligned}
&\frac{m-2}{m-3}\,\xi\,\mathcal{L}(\widetilde{f}_m^{m-3}) + (m-2)\,\mathcal{L}(\widetilde{f}_{m-2}^{m-4}) \\
&= -(m-2)\,2^{m/2-2}\,\xi^{m-3}(s^2+\eta^2)^{-m/2+1}\,\Gamma(m-3) \\
&\quad\times \sum_{\ell=0}^{\frac{m}{2}-2} \frac{1}{\ell!}\frac{\Gamma\left(\frac{m}{2}-\ell-1\right)}{\Gamma(m-3-2\ell)}\left(\frac{s^2+\eta^2}{4\xi^2}\right)^{\ell} \\
&\quad - (m-2)\,2^{m/2-3}\,\xi^{m-5}(s^2+\eta^2)^{-m/2+2}\,\Gamma(m-3) \\
&\quad\times \sum_{\ell=1}^{\frac{m}{2}-2} \frac{1}{(\ell-1)!}\frac{\Gamma\left(\frac{m}{2}-\ell-1\right)}{\Gamma(m-2-2\ell)}\left(\frac{s^2+\eta^2}{4\xi^2}\right)^{\ell-1} \\
&= -(m-3)(m-2)\,2^{m/2-2}\,\xi^{m-3}(s^2+\eta^2)^{-m/2+1}\,\Gamma(m-3) \\
&\quad\times \sum_{\ell=1}^{\frac{m}{2}-2} \frac{1}{\ell!}\frac{\Gamma\left(\frac{m}{2}-\ell-1\right)}{\Gamma(m-2-2\ell)}\left(\frac{s^2+\eta^2}{4\xi^2}\right)^{\ell} \\
&\quad - (m-3)(m-2)\,2^{m/2-2}\,\xi^{m-3}(s^2+\eta^2)^{-m/2+1}\Gamma(m-3)\frac{\Gamma\left(\frac{m}{2}-1\right)}{\Gamma(m-2)} \\
&= -2^{m/2-2}\,\xi^{m-3}(s^2+\eta^2)^{-m/2+1}\,\Gamma(m-1) \\
&\quad\times \sum_{\ell=0}^{\frac{m}{2}-2} \frac{1}{\ell!}\frac{\Gamma\left(\frac{m}{2}-\ell-1\right)}{\Gamma(m-2-2\ell)}\left(\frac{s^2+\eta^2}{4\xi^2}\right)^{\ell} \\
&= \mathcal{L}(\widetilde{f}_m^{m-2}).
\end{aligned}
$$

We conclude the proof of the lemma. □

Based on the recursion relations of kernels obtained in this section, we can summarize all the kernels as in Figure 4.1.

4.3 Generating functions

The exponential generating function of a sequence $(a_n)_{n=0}^{\infty}$ is given by

$$\mathrm{EG}\left((a_n)_{n=0}^{\infty}; X\right) = \sum_{n=0}^{\infty} a_n \frac{X^n}{n!}.$$

In this section we want to determine formal exponential generating functions for the kernels $K_m^n(x,y)$. We will do this for specific values of n, in even dimension m, that is, for sequences of the form $K^n = (K_{2q+2}^n(x,y))_{q=0}^{\infty}$:

$$\mathrm{EG}\,(K^n;X) = \sum_{q=0}^{\infty} K_{2q+2}^n(x,y)\,\frac{X^q}{q!} = \sum_{m=2,4,\ldots} K_m^n(x,y)\,\frac{X^{m/2-1}}{\Gamma\left(\frac{m}{2}\right)}$$

by obtaining the exponential generating functions of the Laplace transforms and then calculating the inverse Laplace transform.

Remark 4.3.1. *In the formal generating function, we are adding kernels in different even dimensions. Although the interpretation of the variables x and y of course changes with the dimension, the components f and g of the kernel only depend on the inner product and the norm of the wedge product (but not on the individual vectors x and y). If we consider the inner product and norm of the wedge product as 'new' variables, it therefore becomes possible to compute the generating function. The wedge product, connecting the scalar and bivector parts, has to be interpreted as a formal object in our final result, which only acquires geometrical meaning after selecting a specific term in the expansion in X of the formal generating function.*

We introduce the following shorthand notation for some sequences:

$$\mathcal{L}g^0 = \left(\mathcal{L}(g_{2q+2}^0)\right)_{q=0}^{\infty} \quad \mathcal{L}\widetilde{f}^0 = \left(\mathcal{L}(\widetilde{f}_{2q+2}^0)\right)_{q=0}^{\infty} \quad \mathcal{L}\widehat{f}^0 = \left(\mathcal{L}(\widehat{f}_{2q+2}^0)\right)_{q=0}^{\infty}$$
$$\mathcal{L}g^\lambda = \left(\mathcal{L}(g_{2q+2}^q)\right)_{q=0}^{\infty} \quad \mathcal{L}\widetilde{f}^\lambda = \left(\mathcal{L}(\widetilde{f}_{2q+2}^q)\right)_{q=0}^{\infty} \quad \mathcal{L}\widehat{f}^\lambda = \left(\mathcal{L}(\widehat{f}_{2q+2}^q)\right)_{q=0}^{\infty}.$$

4.3.1 The case $n = 0$

In the diagram of Figure 4.2, the kernels for $n = 0$ are in the first diagonal row. When $n = 0$, the kernel $K_m^0(x,y)$ reduces to

$$\begin{aligned} K_m^0(x,y) &= \widehat{f}_m^0 + (\underline{x}\wedge\underline{y})\,g_m^0 \\ &= -i\sqrt{\frac{\pi}{2}}\,J_{(m-3)/2}(t\eta)\,\eta^{-(m-3)/2}\,t^{(m-1)/2} \\ &\quad + (\underline{x}\wedge\underline{y})\sqrt{\frac{\pi}{2}}J_{(m-1)/2}(t\eta)\,\eta^{-(m-1)/2}\,t^{(m-1)/2}. \end{aligned} \tag{4.15}$$

It is easy to obtain the Laplace transform:

$$\mathcal{L}\left(K_m^0(x,y)\right) = 2^{m/2-1}\,(s^2+\eta^2)^{-m/2}\,\Gamma(m/2)\left(-i\,s + (\underline{x}\wedge\underline{y})\right).$$

Next, we determine the exponential generating functions of $\mathcal{L}g^0$,

$$\begin{aligned}\mathrm{EG}\left(\mathcal{L}g^0; X\right) &= \sum_{q=1}^{\infty} (2X)^{q-1} (s^2+\eta^2)^{-q} \\ &= \frac{1}{s^2+\eta^2} \frac{1}{1-\frac{2X}{s^2+\eta^2}} \\ &= \frac{1}{s^2+\eta^2-2X},\end{aligned}$$

and $EG(\mathcal{L}\widehat{f}^0_m; X)$ is calculated in the same way. We then have

$$EG\left((\mathcal{L}(K^0_{2q+2}(x,y)))_{q=0}^{\infty}; X\right) = \frac{-i\,s + \underline{x}\wedge\underline{y}}{s^2+\eta^2-2X}.$$

By formulas (2.39) and (2.40), we arrive at the following result.

Theorem 4.3.2 (m even). *The exponential generating function of the even dimensional kernels $K^0 = (K^0_{2q+2}(x,y))_{q=0}^{\infty}$ associated with the Clifford-Helmholtz system is given by*

$$\begin{aligned}EG\left(K^0; X\right) = &-i\,\cos\left(t\sqrt{\eta^2-2X}\right) \\ &+ \frac{\underline{x}\wedge\underline{y}}{\sqrt{\eta^2-2X}} \sin\left(t\sqrt{\eta^2-2X}\right).\end{aligned}$$

Remark 4.3.3. *Note that for the case $n=0$ and $t=1$ the formula (4.15) resembles the Fourier-Bessel kernel (see [8], Definition 1)*

$$K^{\mathrm{Bessel}}(x,y) = \sqrt{\frac{\pi}{2}} \left(\frac{J_{(m-3)/2}(\eta)}{\eta^{(m-3)/2}} + (\underline{x}\wedge\underline{y})\frac{J_{(m-1)/2}(\eta)}{\eta^{(m-1)/2}}\right),$$

as it differs only by a factor $-i$ in front of the scalar part, so this yields:

$$\begin{aligned}EG\left((K^{\mathrm{Bessel}}(x,y))_{m=2}^{\infty}; X\right) = &\cos\left(t\sqrt{\eta^2-2X}\right) \\ &+ \frac{\underline{x}\wedge\underline{y}}{\sqrt{\eta^2-2X}} \sin\left(t\sqrt{\eta^2-2X}\right).\end{aligned}$$

Remark 4.3.4. *Recall the spherical Bessel functions of the first and second kinds $j_n(z) = \sqrt{\frac{\pi}{2z}} J_{n+\frac{1}{2}}(z)$ and $y_n(z) = (-1)^{n+1}\sqrt{\frac{\pi}{2z}} J_{-n-\frac{1}{2}}(z)$. The generating functions of $j_n(z)$ and $y_n(z)$ take the following forms*

$$\frac{\cos\sqrt{z^2-2zt}}{z} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{j}_{n-1}(z);$$

$$\frac{\sin\sqrt{z^2+2zt}}{z} = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \mathsf{y}_{n-1}(z)$$

and can be compared with Theorem 4.3.2.

Proposition 4.3.5. *For m even and $n=1$, the exponential generating function of the kernels $K^1 = (K^1_{2q+2}(x,y))_{q=0}^{\infty}$ associated with the Clifford-Helmholtz system takes the form*

$$\begin{aligned} EG\left(K^1; X\right) &= \frac{1}{t}\cos(\eta\, t) - \left(i\,\xi + \frac{1}{t}\right)\cos\left(t\sqrt{\eta^2-2X}\right) \\ &\quad + (\underline{x}\wedge\underline{y})\,\frac{\xi}{\sqrt{\eta^2-2X}}\sin\left(t\sqrt{\eta^2-2X}\right). \end{aligned}$$

Proof. When $n=1$, the Laplace transform of the kernel $K^1_m(x,y)$ in the second diagonal row of Figure 2 is as follows

$$\begin{aligned} \mathcal{L}\left(K^1_m(x,y)\right) &= \mathcal{L}(\tilde{f}^1_m) + \mathcal{L}(\hat{f}^1_m) + (\underline{x}\wedge\underline{y})\,\mathcal{L}(g^1_m) \\ &= -2^{m/2-2}\,(s^2+\eta^2)^{-m/2+1}\,\Gamma\left(\frac{m}{2}-1\right) \\ &\quad - i\,2^{m/2-1}\,\xi\,(s^2+\eta^2)^{-m/2}\,s\,\Gamma\left(\frac{m}{2}\right) \\ &\quad + (\underline{x}\wedge\underline{y})\,2^{m/2-1}\,\xi\,(s^2+\eta^2)^{-m/2}\,\Gamma\left(\frac{m}{2}\right). \end{aligned}$$

In the same way as for $n=0$, we determine immediately the exponential generating functions for $\hat{f}^1_m$ and g^1_m

$$\mathrm{EG}\left(\hat{f}^1; X\right) = -i\,\xi\,\cos\left(t\sqrt{\eta^2-2X}\right)$$

and

$$\mathrm{EG}\left(g^1; X\right) = \frac{\xi}{\sqrt{\eta^2-2X}}\sin\left(t\sqrt{\eta^2-2X}\right).$$

For $\tilde{f}^1_m$, we first obtain the exponential generating functions of the Laplace transform of the kernel $\mathcal{L}\tilde{f}^1$ by the substitution $q \longmapsto q+1$ in the second step

$$\begin{aligned} \mathrm{EG}\left(\mathcal{L}\tilde{f}^1; X\right) &= -\frac{1}{2}\sum_{q=1}^{\infty} \frac{1}{q-1}\left(\frac{2X}{s^2+\eta^2}\right)^{q-1} \\ &= -\frac{1}{2}\sum_{q=0}^{\infty} \frac{1}{q}\left(\frac{2X}{s^2+\eta^2}\right)^{q} \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\left(-\ln\left(1-\frac{2X}{s^2+\eta^2}\right)\right)\\
&= \frac{1}{2}\ln\left(\frac{s^2+\eta^2-2X}{s^2+\eta^2}\right).
\end{aligned}
$$

where in the third step we have used the Taylor expansion of $\ln(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n}$. By means of (2.51), the exponential generating function $\tilde{f}^1_m$ arrives at

$$
\begin{aligned}
\mathrm{EG}\left(\tilde{f}^1;X\right) &= \mathcal{L}^{-1}\left(\frac{1}{2}\ln\left(\frac{s^2+\eta^2-2X}{s^2+\eta^2}\right)\right)\\
&= \frac{1}{t}\left(\cos(\eta\, t) - \cos\left(t\sqrt{\eta^2-2X}\right)\right).
\end{aligned}
$$

These yield the results. □

Proposition 4.3.6. *For m even and $n=2$, the exponential generating function of the kernels $K^2 = (K^2_{2q+2}(x,y))_{q=0}^{\infty}$ associated with the Clifford-Helmholtz system is given by*

$$
\begin{aligned}
EG\left(K^2;X\right) = &\left(-\frac{2\xi}{t}+\frac{\xi^2}{t}\left(\underline{x}\wedge\underline{y}\right) - i\,\xi^2 + \frac{i\xi^2}{t^2}\right)\cos\left(t\sqrt{\eta^2-2X}\right)\\
&+\left(\frac{\xi^2\left(\underline{x}\wedge\underline{y}\right)}{\sqrt{\eta^2-2X}} + \frac{i\,\xi^2\sqrt{\eta^2-2X}}{t}\right)\sin\left(t\sqrt{\eta^2-2X}\right)\\
&+\left(\frac{2\xi}{t} - \frac{\xi^2\left(\underline{x}\wedge\underline{y}\right)}{t} - \frac{i\xi^2}{t^2}\right)\cos(\eta\, t) - \frac{i\,\eta\,\xi^2}{t}\sin(\eta\, t).
\end{aligned}
$$

Proof. For the kernels in the third diagonal row of Figure 2, the relations can be derived from the series expansions of the Laplace transforms of kernels in section 4.2.1:

$$
\begin{aligned}
\mathcal{L}(\tilde{f}^2_m) &= 2\xi\,\mathcal{L}(\tilde{f}^1_m);\\
\mathcal{L}(\hat{f}^2_m) &= \xi^2\left(\mathcal{L}(\hat{f}^0_m) + i\,s\,\mathcal{L}(\tilde{f}^1_m)\right);\\
\mathcal{L}(g^2_m) &= \xi^2\left(\mathcal{L}(g^0_m) - \mathcal{L}(\tilde{f}^1_m)\right).
\end{aligned}
$$

Next, the generating functions of kernels $\tilde{f}^2_m$ and g^2_m follow from the same techniques as in the case $n=0$ and $n=1$:

$$
\mathrm{EG}\left(\tilde{f}^2;X\right) = 2\xi\,\mathrm{EG}\left(\tilde{f}^1;X\right)
$$

$$= \frac{2\xi}{t} \cos(\eta\, t) - \frac{2\xi}{t} \cos\left(t \sqrt{\eta^2 - 2X}\right)$$

and

$$\begin{aligned}
\mathrm{EG}\left(g^2; X\right) &= \xi^2 \left(\mathrm{EG}\left(g^0; X\right) - \mathrm{EG}\left(\tilde{f}^1; X\right)\right) \\
&= \frac{\xi^2}{\sqrt{\eta^2 - 2X}} \sin\left(t \sqrt{\eta^2 - 2X}\right) \\
&\quad - \frac{\xi^2}{t} \cos(\eta\, t) + \frac{\xi^2}{t} \cos\left(t \sqrt{\eta^2 - 2X}\right).
\end{aligned}$$

For $\hat{f}^2_m$ we observe

$$\mathrm{EG}\left(\mathcal{L}\hat{f}^2; X\right) = \xi^2 \left(\mathrm{EG}\left(\mathcal{L}\hat{f}^0; X\right) + i\, s\, \mathrm{EG}\left(\mathcal{L}\tilde{f}^1; X\right)\right),$$

where

$$\begin{aligned}
i\, s\, \mathrm{EG}\left(\mathcal{L}\hat{f}^1; X\right) &= -\frac{i\, s}{2} \sum_{q=1}^{\infty} \frac{1}{q-1} \left(\frac{2X}{s^2 + \eta^2}\right)^{q-1} \\
&= -\frac{i\, s}{2} \sum_{q=0}^{\infty} \frac{1}{q} \left(\frac{2X}{s^2 + \eta^2}\right)^{q} \\
&= \frac{i}{2} s \ln\left(\frac{s^2 + \eta^2 - 2X}{s^2 + \eta^2}\right).
\end{aligned}$$

Using the formula (2.52) yields the desired results for $\mathrm{EG}\left(\hat{f}^2; X\right)$

$$\begin{aligned}
\mathrm{EG}\left(\hat{f}^2; X\right) = &-i\, \xi^2 \cos\left(t \sqrt{\eta^2 - 2X}\right) + \frac{i\xi^2}{t^2} \cos\left(t \sqrt{\eta^2 - 2X}\right) \\
&- \frac{i\xi^2}{t^2} \cos(\eta\, t) - \frac{i\, \eta\, \xi^2}{t} \sin(\eta\, t) \\
&+ \frac{i\, \xi^2}{t} \sqrt{\eta^2 - 2X} \sin\left(t \sqrt{\eta^2 - 2X}\right).
\end{aligned}$$

□

4.3.2 The case $n = \frac{m}{2} - 1$

Denote $\lambda := \frac{m}{2} - 1$. In the diagram of Figure 4.2, the kernels for $n = \lambda$ when m is even are in the horizontal row in the middle. We now calculate the generating function for these kernels. From Theorem

4.2.4, it is clear that we have to make a distinction between n even and n odd: i) $m = 4p+2$ and $\lambda = 2p$; ii) $m = 4p$ and $\lambda = 2p-1$.

We first compute the exponential generating function of $\mathcal{L}(g_m^\lambda)$

$$\begin{aligned}
&\mathrm{EG}\left(\mathcal{L}g^\lambda; X\right) \\
&= \sum_{m=2,4,\ldots} \mathcal{L}(g_m^\lambda) \frac{X^{m/2-1}}{\Gamma\left(\frac{m}{2}\right)} \\
&= \sum_{\substack{m=4p \\ p=1}}^{\infty} \mathcal{L}(g_{4p}^\lambda) \frac{X^{2p-1}}{\Gamma(2p)} + \sum_{\substack{m=4p+2 \\ p=1}}^{\infty} \mathcal{L}(g_{4p+2}^\lambda) \frac{X^{2p}}{\Gamma(2p+1)}.
\end{aligned} \tag{4.16}$$

Applying the formulas in case 1 of Theorem 4.2.4 for $m = 4p+2$, $\lambda = 2p$, we have

$$\mathcal{L}(g_{4p+2}^\lambda) = (s^2+\eta^2)^{-p-1}\,\Gamma(2p+1)\,{}_2F_1\left(\begin{matrix}-p, p+1 \\ \frac{1}{2}\end{matrix}; -\frac{\xi^2}{s^2+\eta^2}\right).$$

Then using formula (2.4) with $a = -p$ and $z^2 = \frac{\xi^2}{s^2+\eta^2}$ yields

$$\begin{aligned}
&{}_2F_1\left(\begin{matrix}-p, p+1 \\ \frac{1}{2}\end{matrix}; -z^2\right) \\
&= \frac{1}{2\sqrt{1+z^2}}\left(\left(\sqrt{1+z^2}+z\right)^{-2p-1} + \left(\sqrt{1+z^2}-z\right)^{-2p-1}\right)
\end{aligned}$$

from which we can obtain

$$\begin{aligned}
\mathcal{L}(g_{4p+2}^\lambda) &= \frac{(s^2+\eta^2)^{-p-1/2}}{2\sqrt{s^2+\eta^2+\xi^2}}\,\Gamma(2p+1)\left(\left(\sqrt{1+z^2}+z\right)^{-2p-1}\right. \\
&\quad \left. + \left(\sqrt{1+z^2}-z\right)^{-2p-1}\right) \\
&= \frac{\Gamma(2p+1)}{2\sqrt{s^2+\eta^2+\xi^2}}\left(\left(\sqrt{s^2+\eta^2+\xi^2}+\xi\right)^{-2p-1}\right. \\
&\quad \left. + \left(\sqrt{s^2+\eta^2+\xi^2}-\xi\right)^{-2p-1}\right).
\end{aligned} \tag{4.17}$$

Subsequently, for $m = 4p > 0$, $\lambda = 2p-1$ we have

$$\mathcal{L}(g_{4p}^\lambda) = 2\xi\,(s^2+\eta^2)^{-p-1}\,p\,\Gamma(2p)\,{}_2F_1\left(\begin{matrix}-p+1, p+1 \\ \frac{3}{2}\end{matrix}; -\frac{\xi^2}{s^2+\eta^2}\right).$$

By means of formulas (2.3) and (2.5) by substituting $a = p + \frac{1}{2}$ and $z^2 = \frac{\xi^2}{s^2+\eta^2}$, we have

$$\begin{aligned}
&{}_2F_1\left(\begin{matrix}-p+1, p+1\\ \frac{3}{2}\end{matrix}; -z^2\right)\\
&= (1-(-z^2))^{-1/2}\, {}_2F_1\left(\begin{matrix}p+\frac{1}{2}, -p+\frac{1}{2}\\ \frac{3}{2}\end{matrix}; -z^2\right)\\
&= (1+z^2)^{-1/2}\frac{1}{-4pz}\left(\left(\sqrt{1+z^2}+z\right)^{-2p} - \left(\sqrt{1+z^2}-z\right)^{-2p}\right),
\end{aligned}$$

leading to

$$\begin{aligned}
\mathcal{L}(g_{4p}^{\lambda}) = -\frac{\Gamma(2p)}{2\sqrt{s^2+\eta^2+\xi^2}}\Bigg(&\left(\sqrt{s^2+\eta^2+\xi^2}+\xi\right)^{-2p}\\
&-\left(\sqrt{s^2+\eta^2+\xi^2}-\xi\right)^{-2p}\Bigg).
\end{aligned} \tag{4.18}$$

Now for $q > 0$, combining this with (4.17) leads to

$$\begin{aligned}
\mathcal{L}(g_{2q}^{\lambda}) = \frac{\Gamma(q)}{2\sqrt{s^2+\eta^2+\xi^2}}\Bigg(&\left(\sqrt{s^2+\eta^2+\xi^2}-\xi\right)^{-q}\\
&-(-1)^q\left(\sqrt{s^2+\eta^2+\xi^2}+\xi\right)^{-q}\Bigg).
\end{aligned}$$

For the exponential generating function of $\mathcal{L}(g_m^{\lambda})$ we then find

$$\begin{aligned}
\mathrm{EG}\left(\mathcal{L}g^{\lambda}; X\right) &= \sum_{q=1}^{\infty}\frac{X^{q-1}}{2\sqrt{s^2+\eta^2+\xi^2}}\Bigg(\left(\frac{1}{\sqrt{s^2+\eta^2+\xi^2}-\xi}\right)^q\\
&\qquad -\left(-\frac{1}{\sqrt{s^2+\eta^2+\xi^2}+\xi}\right)^q\Bigg)\\
&= \frac{1}{2\sqrt{s^2+\eta^2+\xi^2}}\Bigg(\frac{1}{\sqrt{s^2+\eta^2+\xi^2}-\xi-X}\\
&\qquad +\frac{1}{\sqrt{s^2+\eta^2+\xi^2}+\xi+X}\Bigg)\\
&= \frac{1}{s^2+\eta^2-X^2-2\xi X}.
\end{aligned}$$

By means of the above result, $\mathrm{EG}\left(\mathcal{L}\widehat{f}^{\lambda}; X\right)$ is obtained immediately by applying the relation (4.5)

$$\begin{aligned}\mathrm{EG}\left(\mathcal{L}\widehat{f}^{\lambda};X\right) &= -\frac{i\,s}{s^2+\eta^2-X^2-2\xi X}\\ &= -i\,s\,\mathrm{EG}(\mathcal{L}g_m^{\lambda};X).\end{aligned} \tag{4.19}$$

In a similar way as for $\mathcal{L}(g_m^{\lambda})$, we obtain the exponential generating function of the kernel $\mathcal{L}(\widetilde{f}_m^{\lambda})$, thus we restrict ourselves to giving some important steps.
For the case i) $m = 4p+2$ and $\lambda = 2p$, we have

$$\begin{aligned}&\mathcal{L}(\widetilde{f}_{4p+2}^{\lambda})\\ &= \frac{-2\xi}{(s^2+\eta^2)^{p+1}}\frac{\Gamma(2p+1)\,\Gamma(p+1)}{(p-1)!}\,{}_2F_1\left(\begin{matrix}-p+1,p+1\\ \frac{3}{2}\end{matrix};-\frac{\xi^2}{s^2+\eta^2}\right)\\ &= -\frac{\Gamma(2p+1)}{2\sqrt{s^2+\eta^2+\xi^2}}\\ &\quad\times\left(\left(\sqrt{s^2+\eta^2+\xi^2}-\xi\right)^{-2p}-\left(\sqrt{s^2+\eta^2+\xi^2}+\xi\right)^{-2p}\right),\end{aligned}$$

then for the other case ii) $m = 4p$ and $\lambda = 2p-1$, it follows that

$$\begin{aligned}\mathcal{L}(\widetilde{f}_{4p}^{\lambda}) &= -(s^2+\eta^2)^{-p}\,\Gamma(2p)\,{}_2F_1\left(\begin{matrix}-p+1,p\\ \frac{1}{2}\end{matrix};-\frac{\xi^2}{s^2+\eta^2}\right)\\ &= -\frac{\Gamma(2p)}{2\sqrt{s^2+\eta^2+\xi^2}}\Bigg(\left(\sqrt{s^2+\eta^2+\xi^2}+\xi\right)^{-2p+1}\\ &\quad+\left(\sqrt{s^2+\eta^2+\xi^2}-\xi\right)^{-2p+1}\Bigg).\end{aligned}$$

We can also arrive at

$$\begin{aligned}&\mathrm{EG}\left(\mathcal{L}\widetilde{f}^{\lambda};X\right)\\ &= -\frac{1}{2\sqrt{s^2+\eta^2+\xi^2}}\left(\frac{1}{1-\frac{X}{\sqrt{s^2+\eta^2+\xi^2}-\xi}}-\frac{1}{1+\frac{X}{\sqrt{s^2+\eta^2+\xi^2}+\xi}}\right)\\ &= -\frac{X}{s^2+\eta^2-X^2-2\xi X}.\end{aligned}$$

where we have used the formulas (2.3–2.5). Now we find the exponential generating function of $\mathcal{L}\left(K_m^{\lambda}(x,y)\right)$

$$\mathrm{EG}\left((\mathcal{L}(K_{2q+2}^{q}(x,y))_{q=0}^{\infty};X\right) = \frac{\underline{x}\wedge\underline{y}-X-i\,s}{s^2+\eta^2-X^2-2\xi X}.$$

Transforming back by formulas (2.39) and (2.40) yields

$$\begin{aligned}
&\mathrm{EG}\left(K^{\lambda};X\right)\\
&=\mathcal{L}^{-1}\left(\frac{\underline{x}\wedge\underline{y}-X-i\,s}{s^2+\eta^2-X^2-2\xi X}\right)\\
&=-i\,\mathcal{L}^{-1}\left(\frac{s}{s^2+\eta^2-X^2-2\xi X}\right)\\
&\quad+\frac{\underline{x}\wedge\underline{y}-X}{\sqrt{\eta^2-X^2-2\xi X}}\mathcal{L}^{-1}\left(\frac{\sqrt{\eta^2-X^2-2\xi X}}{s^2+\eta^2-X^2-2\xi X}\right)\\
&=-i\,\cos\left(t\sqrt{\eta^2-X^2-2\xi X}\right)\\
&\quad+\frac{\underline{x}\wedge\underline{y}-X}{\sqrt{\eta^2-X^2-2\xi X}}\,\sin\left(t\sqrt{\eta^2-X^2-2\xi X}\right).
\end{aligned}$$

We arrive at the following theorem:

Theorem 4.3.7. *The exponential generating function for the even dimensional Fourier transform kernels $K^{\lambda}=(K^q_{2q+2}(x,y))_{q=0}^{\infty}$ is given by*

$$\begin{aligned}
\mathrm{EG}\left(K^{\lambda};X\right)=-i\,\cos&\left(t\sqrt{\eta^2-X^2-2\xi X}\right)\\
&+\frac{\underline{x}\wedge\underline{y}-X}{\sqrt{\eta^2-X^2-2\xi X}}\,\sin\left(t\sqrt{\eta^2-X^2-2\xi X}\right).
\end{aligned}$$

Remark 4.3.8. *In a similar way, one can obtain an exponential generating function for the even dimensional kernels when $n=\lambda+1=\frac{m}{2}$.*

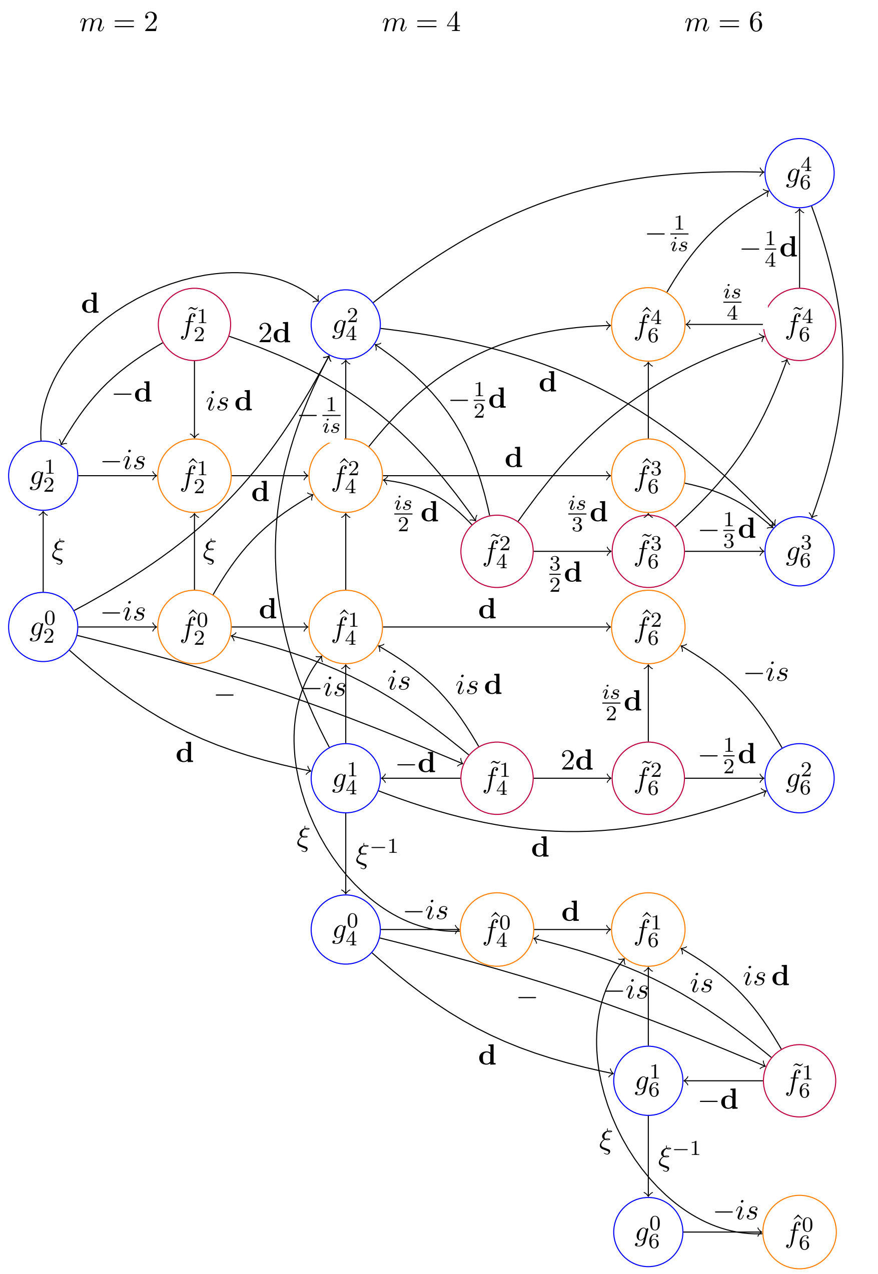

Figure 4.1: Explicit relations for even dimensions. Different colours are used to mark the individual components of each kernel. The components with the same indices and different colours stand for a kernel. This clearly shows that all the components of each kernel can be obtained from K_2^0 by the recursion relations we have found in Section 4.2.3.

$$
\begin{tikzcd}
 & m=2 & m=4 & m=6 & m=8 \\
 & & & & K_8^6 \\
 & & & K_6^4 \arrow[r, "\mathbf{d}"] \arrow[ur, bend left] & K_8^5 \arrow[u] \\
 & & K_4^2 \arrow[r, "\mathbf{d}"] \arrow[ur, bend left] & K_6^3 \arrow[r, "\mathbf{d}"] \arrow[u] & K_8^4 \\
n = \frac{m}{2} - 1 & K_2^0 \arrow[r, "\mathbf{d}"] \arrow[ur, bend left] & K_4^1 \arrow[r, "\mathbf{d}"] \arrow[u] \arrow[d] & K_6^2 \arrow[r, "\mathbf{d}"] & K_8^3 \\
 & & K_4^0 \arrow[r, "\mathbf{d}"] & K_6^1 \arrow[r, "\mathbf{d}"] \arrow[d] & K_8^2 \\
 & & & K_6^0 \arrow[r, "\mathbf{d}"] & K_8^1 \arrow[d] \\
 & & & & K_8^0
\end{tikzcd}
$$

Figure 4.2: Kernel recursion relations for even dimensions. The kernels for $n = \frac{m}{2} - 1$ are in the horizontal row in the middle.

$$
\begin{tikzcd}
m=3 & m=5 & m=7 & m=9 \\
 & & & K_9^7 \\
 & & K_7^5 \arrow[r, "\mathbf{d}"] \arrow[ur, bend left] & K_9^6 \arrow[u] \\
 & K_5^3 \arrow[r, "\mathbf{d}"] \arrow[ur, bend left] & K_7^4 \arrow[r, "\mathbf{d}"] \arrow[u] & K_9^5 \\
K_3^1 \arrow[r, "\mathbf{d}"] \arrow[ur, bend left] & K_5^2 \arrow[r, "\mathbf{d}"] \arrow[u] & K_7^3 \arrow[r, "\mathbf{d}"] & K_9^4 \\
K_3^0 \arrow[r, "\mathbf{d}"] & K_5^1 \arrow[r, "\mathbf{d}"] \arrow[d] & K_7^2 \arrow[r, "\mathbf{d}"] & K_9^3 \\
 & K_5^0 \arrow[r, "\mathbf{d}"] & K_7^1 \arrow[r, "\mathbf{d}"] \arrow[d] & K_9^2 \\
 & & K_7^0 \arrow[r, "\mathbf{d}"] & K_9^1 \arrow[d] \\
 & & & K_9^0
\end{tikzcd}
$$

Figure 4.3: Kernel recursion relations for odd dimensions.

Chapter 5

Kernel of the radially deformed Fourier transform

The content of this chapter is written into the manuscript [32]:

Hendrik De Bie and Ze Yang. *On the radially deformed Fourier transform*. (2024). arXiv: 2404.06839.

5.1 Introduction

The radially deformed Fourier transform was initially introduced in [30] as the Fourier transform associated with the so-called radially deformed Dirac operator **D**. In [31], a series representation for the kernel of the integral representation of the deformed Fourier transform was determined explicitly, and the recursion relations for the kernel were shown subsequently. Explicit formulas for the kernel of the Fourier transform were obtained when the dimension is even and $1 + c = \frac{1}{n}, n \in \mathbb{N}_0 \backslash \{1\}$ with n odd in [28], where c is a deformation parameter of the deformed Fourier transform.

In this chapter, we turn our attention to the kernel of the deformed Fourier transform in both even and odd dimensions by adapting the Laplace transform method in [16]. By introducing an auxiliary variable t in the kernel, we can take the Laplace transform in t. For the case when $m = 2$, we begin by considering the kernel shown in Section 5.3 in [28]. The Laplace transform of the kernel can be simplified using two relations for infinite sums of trigonometric functions. For the case when $m > 2$, the Laplace domain expression of kernel can be obtained by means of the Poisson kernel and the generating function of the Gegenbauer polynomial. Furthermore, these results allow us to derive the explicit expressions for the kernel in terms of the Mittag-Leffler functions for $\lambda = 0$ and $\lambda > 0$ respectively.

This chapter is organized as follows. In Section 5.2 we introduce basic facts on the radially deformed Fourier transform and some results necessary for the sequel. In Section 5.3.1 we obtain the kernel of the Fourier transform in dimension 2 in the Laplace domain. In Section 5.3.2 we derive the Laplace transform of the kernel when $m > 2$. Finally, the integral expressions of the kernel in terms of the Mittag-Leffler functions are given in Section 5.4.

5.2 The Radially Deformed Fourier transform

In [30], the radially deformed Dirac operator

$$\mathbf{D} := \partial_{\underline{x}} + c|x|^{-2}\underline{x}\,\mathbb{E}, \quad c > -1$$

and the associated Fourier transform were studied. The investigation of this radially deformed Fourier transform

$$\mathcal{F}_D = e^{i\frac{\pi}{2}\left(\frac{1}{2}+\frac{m-1}{2(1+c)}\right)} e^{\frac{-i\pi}{4(1+c)^2}\left(\mathbf{D}^2-(1+c)^2\underline{x}^2\right)}$$

where $L = \mathbf{D}^2 - (1+c)^2\underline{x}^2$ is the generalized Hamiltonian, was continued in [31] by a group theoretical approach. The integral representation is given by

$$\mathcal{F}_D\,(f)\,(y) = \frac{\Gamma(\frac{m}{2})}{2\pi^{m/2}} \int_{\mathbb{R}^m} K_m^c(x,y)\, f(x)\, h(r)\, \mathrm{d}x,$$

with $h(r) = r^{1-\frac{1+mc}{1+c}}$. The kernel K_m^c of the deformed Fourier transform $\mathcal{F}_D$ can be written as (see [31], Section 8)

$$K_m^c = \frac{1}{2\lambda} z^{-\frac{\mu-2}{2}} A_\lambda + \frac{1}{2} z^{-\frac{\mu-2}{2}} B_\lambda - z^{-\frac{\mu}{2}} (\underline{x} \wedge \underline{y})\, C_\lambda \tag{5.1}$$

with

$$
\begin{aligned}
A_\lambda &= \sum_{k=0}^{+\infty}(k+\lambda)\left(\alpha_k\, J_{\frac{\gamma_k}{2}-1}(z) - i\,\alpha_{k-1}\, J_{\frac{\gamma_{k-1}}{2}}(z)\right) C_k^\lambda(w),\\
B_\lambda &= \sum_{k=0}^{+\infty}\left(\alpha_k\, J_{\frac{\gamma_k}{2}-1}(z) + i\,\alpha_{k-1}\, J_{\frac{\gamma_{k-1}}{2}}(z)\right) C_k^\lambda(w), \qquad (5.2)\\
C_\lambda &= \sum_{k=1}^{+\infty}\left(\alpha_k\, J_{\frac{\gamma_k}{2}-1}(z) + i\,\alpha_{k-1}\, J_{\frac{\gamma_{k-1}}{2}}(z)\right) C_{k-1}^{\lambda+1}(w),
\end{aligned}
$$

where $\underline{x}\wedge\underline{y} := \sum_{j<k} e_j e_k\,(x_j\, y_k - x_k\, y_j)$ is the wedge product of the vectors $\underline{x}$ and $\underline{y}$, $z = |x||y|$, $w = \frac{\langle x, y\rangle}{z}$. J_ν denotes the Bessel function and C_k^λ the Gegenbauer polynomial. We list the remaining notations in Table 5.1 for readability.

Table 5.1: Notations

Symbol	Definition
λ	$\frac{m-2}{2}$
μ	$1+\frac{m-1}{1+c}$
α_k	$e^{-\frac{i\pi k}{2(1+c)}}$
α_{-1}	0
γ_k	$\frac{2}{1+c}\left(k+\frac{m-2}{2}\right)+\frac{c+2}{1+c}$
c	bigger than -1

In [28], the authors obtained explicit formulas for the above kernel when the dimension is even and $1+c=\frac{1}{n}, n\in\mathbb{N}_0\backslash\{1\}$ with n odd by adapting the method developed in [20].

In case of $m=2$, the kernel was given explicitly in the following theorem:

Theorem 5.2.1. *[28] If $m=2\,(\lambda=0)$ and $1+\frac{1}{c}=\frac{1}{n}, n\in\mathbb{N}_0\backslash\{1\}$ with n odd, then the kernel of the deformed Fourier transform $\mathcal{F}_D = e^{i\frac{\pi}{2}\left(\frac{1}{2}+\frac{m-1}{2(1+c)}\right)} e^{\frac{-i\pi}{4(1+c)^2}\left(\mathbf{D}^2-(1+c)^2\underline{x}^2\right)}$ takes the form*

$$
K_2^c = \frac{1}{n} i^{\frac{n-1}{2}} \left\{ z^{\frac{1-n}{2}} \sum_{j=0}^{n-1} \cos\left(\left(\frac{n-1}{2}\right)\left(\frac{\theta+2\pi j}{n}\right)\right) e^{-iz\cos\left(\frac{\theta+2\pi j}{n}\right)}\right.
$$

$$+ \frac{1}{\sin\theta} z^{-\frac{n+1}{2}} (\underline{x} \wedge \underline{y}) \\ \times \sum_{j=0}^{n-1} \sin\left(\left(\frac{n-1}{2}\right)\left(\frac{\theta+2\pi j}{n}\right)\right) e^{-iz\cos\left(\frac{\theta+2\pi j}{n}\right)} \Bigg\}$$

with $\theta = \arccos w$, $z = |x||y|$, $w = \frac{\langle x,y\rangle}{z}$.

Next, the kernel in even dimensions was derived by the recursion relations obtained in [31].

Theorem 5.2.2. *[28] When the dimension* $m = 2k$, $k \in \mathbb{N}_0\backslash\{1\}$, *is even and* $1 + c = \frac{1}{n}, n \in \mathbb{N}_0\backslash\{1\}$ *with* n *odd, then the kernel of the deformed Fourier transform*

$$\mathcal{F}_D = e^{i\frac{\pi}{2}\left(\frac{1}{2}+\frac{m-1}{2(1+c)}\right)} e^{\frac{-i\pi}{4(1+c)^2}\left(\mathbf{D}^2-(1+c)^2\underline{x}^2\right)}$$

is given by

$$K_{2k}^c = \frac{1}{2k-2} z^{-\frac{\mu-2}{2}} A_{k-1} + \frac{1}{2} z^{-\frac{\mu-2}{2}} B_{k-1} - z^{-\frac{\mu}{2}} (\underline{x} \wedge \underline{y})\, C_{k-1}$$

with

$$A_{k-1} = \frac{i^{(k-1)n}\, i^{\frac{n-1}{2}}}{2^{k-2}\,(k-2)!} \\ \times \partial_\omega^{k-1}\left[\frac{1}{n}\sum_{j=0}^{n-1} \cos\left(\left(\frac{n-1}{2}\right)\left(\frac{\theta+2\pi j}{n}\right)\right) e^{-iz\cos\left(\frac{\theta+2\pi j}{n}\right)}\right]$$

$$B_{k-1} = -\frac{i^{(k-1)n}\, i^{\frac{n-1}{2}}}{2^{k-2}\,(k-2)!} \\ \times \partial_\omega^{k-2}\left[\frac{1}{\sin\theta}\frac{1}{n}\sum_{j=0}^{n-1} \sin\left(\left(\frac{n-1}{2}\right)\left(\frac{\theta+2\pi j}{n}\right)\right) e^{-iz\cos\left(\frac{\theta+2\pi j}{n}\right)}\right]$$

$$C_{k-1} = -\frac{i^{(k-1)n}\, i^{\frac{n-1}{2}}}{2^{k-1}\,(k-1)!} \\ \times \partial_\omega^{k-1}\left[\frac{1}{\sin\theta}\frac{1}{n}\sum_{j=0}^{n-1} \sin\left(\left(\frac{n-1}{2}\right)\left(\frac{\theta+2\pi j}{n}\right)\right) e^{-iz\cos\left(\frac{\theta+2\pi j}{n}\right)}\right]$$

and $\theta = \arccos w$, $z = |x||y|$, $w = \frac{\langle x,y\rangle}{z}$, $\mu = 1 + (2k-1)n$.

The iterated derivatives in the above theorem can in principle be computed recursively.

5.3 The kernel in the Laplace domain

5.3.1 The dimension $m = 2$

In [28], when $m = 2$, i.e. $\lambda = 0$, the kernel of the deformed Fourier transform in case of $1 + c = \frac{1}{n}, n \in \mathbb{N}_0\backslash\{1\}$ with n odd and $n = 2\tilde{n} + 1, \tilde{n} \in \mathbb{N}_0$ takes the form

$$K_2^c = \frac{1}{2} z^{\frac{c}{2(1+c)}} \lim_{\lambda\to 0} \frac{1}{\lambda} A_\lambda + \frac{1}{2} z^{\frac{c}{2(1+c)}} \lim_{\lambda\to 0} B_\lambda - z^{-\frac{2+c}{2(1+c)}} (\underline{x} \wedge \underline{y}) \lim_{\lambda\to 0} C_\lambda$$

with

$$\begin{aligned}
\lim_{\lambda\to 0} \frac{1}{\lambda} A_\lambda &= -J_{\tilde{n}}(z) + 2 \sum_{k=0}^{+\infty} (-i)^{kn} J_{kn+\tilde{n}}(z) \cos(k\theta) \\
&\quad + 2(-1)^{\tilde{n}} \sum_{k=1}^{+\infty} (-i)^{kn} J_{kn-\tilde{n}}(z) \cos(k\theta), \quad w = \cos\theta; \\
\lim_{\lambda\to 0} B_\lambda &= J_{\tilde{n}}(z); \\
\lim_{\lambda\to 0} C_\lambda &= \frac{1}{\sin\theta} \sum_{k=1}^{+\infty} (-i)^{kn} J_{kn+\tilde{n}}(z) \sin(k\theta) \\
&\quad - (-1)^{\tilde{n}} \frac{1}{\sin\theta} \sum_{k=1}^{+\infty} (-i)^{kn} J_{kn-\tilde{n}}(z) \sin(k\theta), \quad w = \cos\theta.
\end{aligned}$$

Inspired by the Laplace transform method in [16], we introduce an auxiliary variable t in the Bessel functions of the kernel. For the scalar part of K_2^c, we have

$$\begin{aligned}
K_{2,scal}^c &= \frac{1}{2} z^{\frac{c}{2(1+c)}} \left(\lim_{\lambda\to 0} \frac{1}{\lambda} A_\lambda + \lim_{\lambda\to 0} B_\lambda \right) \\
&= z^{-\tilde{n}} \sum_{k=0}^{+\infty} (-i)^{kn} J_{kn+\tilde{n}}(z\,t) \cos(k\theta) \\
&\quad + (-1)^{\tilde{n}} z^{-\tilde{n}} \sum_{k=1}^{+\infty} (-i)^{kn} J_{kn-\tilde{n}}(z\,t) \cos(k\theta),
\end{aligned} \tag{5.3}$$

and for the bivector part

$$\begin{aligned} K^c_{2,biv} &= -z^{-\frac{2+c}{2(1+c)}} \lim_{\lambda\to 0} C_\lambda \\ &= -\frac{1}{\sin\theta} z^{-\tilde{n}-1} \sum_{k=1}^{+\infty} (-i)^{kn} J_{kn+\tilde{n}}(z\,t) \sin(k\theta) \\ &\quad + (-1)^{\tilde{n}} \frac{1}{\sin\theta} z^{-\tilde{n}-1} \sum_{k=1}^{+\infty} (-i)^{kn} J_{kn-\tilde{n}}(z\,t) \sin(k\theta). \end{aligned} \tag{5.4}$$

Next, let us first consider the scalar part of the kernel. For $\operatorname{Re} s$ big enough, we take the Laplace transform of the scalar part with respect to t by formula (2.44), which yields

$$\begin{aligned} &\mathcal{L}\left(K^c_{2,scal}\right) \\ &= \frac{1}{r}\left(\frac{1}{s+r}\right)^{\tilde{n}} \sum_{k=0}^{+\infty} (-i)^{kn} \left(\frac{z}{s+r}\right)^{kn} \cos(k\theta) \\ &\quad + (-1)^{\tilde{n}} z^{-2\tilde{n}} \frac{1}{r}\left(\frac{1}{s+r}\right)^{-\tilde{n}} \sum_{k=1}^{+\infty} (-i)^{kn} \left(\frac{z}{s+r}\right)^{kn} \cos(k\theta) \\ &= \frac{1}{r}\left(\frac{1}{s+r}\right)^{\tilde{n}} \sum_{k=0}^{+\infty} (-i)^{kn} \left(\frac{z}{s+r}\right)^{kn} \cos(k\theta) \\ &\quad + (-1)^{\tilde{n}} z^{-2\tilde{n}} \frac{1}{r}\left(\frac{1}{s+r}\right)^{-\tilde{n}} \sum_{k=0}^{+\infty} (-i)^{kn} \left(\frac{z}{s+r}\right)^{kn} \cos(k\theta) \\ &\quad - (-1)^{\tilde{n}} z^{-2\tilde{n}} \frac{1}{r}\left(\frac{1}{s+r}\right)^{-\tilde{n}}. \end{aligned}$$

with $r = \sqrt{s^2+z^2}$, $\underline{z} = |x||y|$, $w = \frac{\langle x,y\rangle}{z}$. The validity of transforming term by term in (5.3) is guaranteed by the following theorem.

Theorem 5.3.1. *[34] Let the function $F(s)$ be represented by a series of $\mathcal{L}$-transforms*

$$F(s) = \sum_{v=0}^{\infty} F_v(s), \quad F_v(s) = \mathcal{L}(f_v(t)),$$

where all integrals

$$\mathcal{L}(f_v) = \int_0^\infty e^{-st} f_v(t)dt = F_v(s), \quad (v = 0, 1, \cdots)$$

converge in a common half-plane $\operatorname{Re} s \geq x_0$*. Moreover, we require that the integrals*

$$\mathcal{L}(|f_v|) = \int_0^\infty e^{-st}|f_v(t)|dt = G_v, \quad (v = 0, 1, \cdots)$$

and the series

$$\sum_{v=0}^{\infty} G_v(x_0)$$

converge which implies that $\sum_{v=0}^{\infty} F_v(s)$ *converges absolutely and uniformly in* $\operatorname{Re} s \geq x_0$*. Then* $\sum_{v=0}^{\infty} f_v(t)$ *converges, absolutely, towards a function* $f(t)$ *for all* $t \geq 0$*; this* $f(t)$ *is the original function of* $F(s)$*;*

$$\mathcal{L}\left(\sum_{v=0}^{\infty} f_v(t)\right) = \sum_{v=0}^{\infty} F_v(s).$$

Using the well-known relation (see for e.g. [69], p. 136 (21.41)):

$$\sum_{k=0}^{\infty} u^k \cos kx = \frac{1 - u\cos x}{1 - 2u\cos x + u^2}, \quad |u| < 1, \tag{5.5}$$

we obtain the following expression for the Laplace transform of the scalar part of K_2^c

$$\begin{aligned}
\mathcal{L}\left(K_{2,scal}^c\right) &= \frac{1}{r}\left(\frac{1}{s+r}\right)^{\tilde{n}} \frac{1 - wu_R}{1 - 2wu_R + u_R^2} \\
&\quad + (-1)^{\tilde{n}} z^{-2\tilde{n}} \frac{1}{r}\left(\frac{1}{s+r}\right)^{-\tilde{n}} \frac{1 - wu_R}{1 - 2wu_R + u_R^2} \\
&\quad - (-1)^{\tilde{n}} z^{-2\tilde{n}} \frac{1}{r}\left(\frac{1}{s+r}\right)^{-\tilde{n}} \\
&= \frac{1}{r}\left(\frac{1}{s+r}\right)^{\tilde{n}} \frac{1}{1 - 2wu_R + u_R^2} \\
&\quad - \frac{1}{r}\left(\frac{1}{s+r}\right)^{\tilde{n}} \frac{wu_R}{1 - 2wu_R + u_R^2} \\
&\quad + (-1)^{\tilde{n}} z^{-2\tilde{n}} \frac{1}{r}\left(\frac{1}{s+r}\right)^{-\tilde{n}} \frac{wu_R}{1 - 2wu_R + u_R^2} \\
&\quad - (-1)^{\tilde{n}} z^{-2\tilde{n}} \frac{1}{r}\left(\frac{1}{s+r}\right)^{-\tilde{n}} \frac{u_R^2}{1 - 2wu_R + u_R^2}
\end{aligned} \tag{5.6}$$

where $u_R = \left(\frac{-iz}{s+r}\right)^n$. Simplifying the results yields

$$\begin{aligned}\mathcal{L}\left(K^c_{2,scal}\right) = & \frac{1}{r}\frac{(s+r)^{n-\tilde{n}} + (-1)^{\tilde{n}}(r-s)^{n-\tilde{n}}}{(s+r)^n - 2w(-iz)^n + (-1)^n(r-s)^n} \\ & - w(-i)^n\frac{z}{r}\frac{(r-s)^{\tilde{n}} - (-1)^n(s+r)^{\tilde{n}}}{(s+r)^n - 2w(-iz)^n + (-1)^n(r-s)^n}.\end{aligned}$$

Similarly, we can compute the Laplace transform of the bivector part

$$\begin{aligned}& \mathcal{L}\left(K^c_{2,biv}\right) \\ = & -\frac{1}{\sin\theta}z^{-1}\frac{1}{r}\left(\frac{1}{s+r}\right)^{\tilde{n}}\sum_{k=1}^{+\infty}(-i)^{kn}\left(\frac{z}{s+r}\right)^{kn}\sin(k\theta) \\ & + (-1)^{\tilde{n}}\frac{1}{\sin\theta}z^{-2\tilde{n}-1}\frac{1}{r}\left(\frac{1}{s+r}\right)^{-\tilde{n}}\sum_{k=0}^{+\infty}(-i)^{kn}\left(\frac{z}{s+r}\right)^{kn}\sin(k\theta).\end{aligned}$$

By the relation ([69], p. 136 (21.42))

$$\sum_{k=1}^{\infty} u^k \sin kx = \frac{u\sin x}{1-2u\cos x + u^2}, \qquad |u| < 1, \tag{5.7}$$

it follows that

$$\begin{aligned}& \mathcal{L}\left(K^c_{2,biv}\right) \\ = & -\frac{1}{zr}\left(\frac{1}{s+r}\right)^{\tilde{n}}\frac{u_R}{1-2wu_R+u_R^2} \\ & + (-1)^{\tilde{n}}z^{-2\tilde{n}}\frac{1}{zr}\left(\frac{1}{s+r}\right)^{-\tilde{n}}\frac{u_R}{1-2wu_R+u_R^2}\end{aligned} \tag{5.8}$$

where $u_R = \left(\frac{-iz}{s+r}\right)^n$.
By direct computation, we obtain

$$\begin{aligned}\mathcal{L}\left(K^c_{2,biv}\right) & = -(-i)^n\frac{1}{r}\frac{(r-s)^{\tilde{n}} - (-1)^n(s+r)^{\tilde{n}}}{(s+r)^n - 2w(-iz)^n + (-1)^n(r-s)^n} \\ & = \frac{i}{r}\frac{(-1)^{\tilde{n}}(r-s)^{\tilde{n}} - (s+r)^{\tilde{n}}}{(s+r)^n - 2w(-iz)^n + (-1)^n(r-s)^n},\end{aligned}$$

since

$$\begin{aligned}(s+r)^{\tilde{n}}(r-s)^{\tilde{n}} & = (r^2-s^2)^{\tilde{n}} \\ & = \left(\left(\sqrt{s^2+z^2}\right)^2 - s^2\right)^{\tilde{n}} \\ & = z^{2\tilde{n}}.\end{aligned} \tag{5.9}$$

Hence we can summarize the results in the following theorem.

Theorem 5.3.2. *If $m = 2$ ($\lambda = 0$), $1+c = \frac{1}{n}$, $n \in \mathbb{N}_0 \setminus \{1\}$ with n odd, i.e. $n = 2\tilde{n}+1$, $\tilde{n} \in \mathbb{N}_0$, then the kernel of the deformed Fourier transform $\mathcal{F}_D = e^{i\frac{\pi}{2}\left(\frac{1}{2}+\frac{m-1}{2(1+c)}\right)} e^{\frac{-i\pi}{4(1+c)^2}\left(\mathbf{D}^2-(1+c)^2\underline{x}^2\right)}$ in the Laplace domain is given by*

$$\begin{aligned}
\mathcal{L}\left(K_2^c\right) &= \frac{1}{r}\frac{(s+r)^{\tilde{n}+1}+(-1)^{\tilde{n}}(r-s)^{\tilde{n}+1}}{(s+r)^n-2w(-iz)^n+(-1)^n(r-s)^n} \\
&+ wz\frac{i}{r}\frac{(-1)^{\tilde{n}}(r-s)^{\tilde{n}}-(s+r)^{\tilde{n}}}{(s+r)^n-2w(-iz)^n+(-1)^n(r-s)^n} \\
&+ (\underline{x}\wedge\underline{y})\,\frac{i}{r}\frac{(-1)^{\tilde{n}}(r-s)^{\tilde{n}}-(s+r)^{\tilde{n}}}{(s+r)^n-2w(-iz)^n+(-1)^n(r-s)^n}
\end{aligned}$$

where $r = \sqrt{s^2+z^2}$, $z = |x||y|$, $w = \frac{\langle x,y\rangle}{z}$.

By adapting the factorization for the polynomials utilized in Lemma 1 in [16], we rewrite Theorem 5.3.2 in the following theorem.

Theorem 5.3.3. *If the dimension $m = 2\,(\lambda = 0)$ and $1+c = \frac{1}{n}$, $n \in \mathbb{N}_0 \setminus \{1\}$ odd, $n = 2\tilde{n}+1$, $\tilde{n} \in \mathbb{N}_0$, the following expansions hold:*
Case 1: $\tilde{n}$ odd $\quad(\tilde{n} = 1,3,5,\dots)$

$$\begin{aligned}
\mathcal{L}\left(K_2^c\right) &= \frac{1}{2^{\tilde{n}}}\frac{\prod_{l=0,\,l\neq\frac{\tilde{n}+1}{2}}^{\tilde{n}}\left(s-iz\sin\left(\frac{l\pi}{\tilde{n}+1}\right)\right)}{\prod_{l=0}^{n-1}\left(s+iz\cos\left(\frac{\theta+2\pi l}{n}\right)\right)} \\
&- \frac{i\,\omega\,z}{2^{\tilde{n}+1}}\frac{\prod_{l=1}^{\tilde{n}-1}\left(s-iz\cos\left(\frac{l\pi}{\tilde{n}}\right)\right)}{\prod_{l=0}^{n-1}\left(s+iz\cos\left(\frac{\theta+2\pi l}{n}\right)\right)} \\
&- \frac{i}{2^{\tilde{n}+1}}(\underline{x}\wedge\underline{y})\,\frac{\prod_{l=1}^{\tilde{n}-1}\left(s-iz\cos\left(\frac{l\pi}{\tilde{n}}\right)\right)}{\prod_{l=0}^{n-1}\left(s+iz\cos\left(\frac{\theta+2\pi l}{n}\right)\right)};
\end{aligned}$$

Case 2: $\tilde{n}$ even $\quad(\tilde{n} = 0,2,4,\dots)$

$$\begin{aligned}
\mathcal{L}\left(K_2^c\right) &= \frac{1}{2^{\tilde{n}}}\frac{\prod_{l=1}^{\tilde{n}}\left(s-iz\cos\left(\frac{l\pi}{\tilde{n}+1}\right)\right)}{\prod_{l=0}^{n-1}\left(s+iz\cos\left(\frac{\theta+2\pi l}{n}\right)\right)} \\
&- \frac{i\,\omega\,z}{2^{\tilde{n}+1}}\frac{\prod_{l=0,\,l\neq\frac{\tilde{n}}{2}}^{\tilde{n}-1}\left(s-iz\sin\left(\frac{l\pi}{\tilde{n}}\right)\right)}{\prod_{l=0}^{n-1}\left(s+iz\cos\left(\frac{\theta+2\pi l}{n}\right)\right)}
\end{aligned}$$

$$-\frac{i}{2^{\tilde{n}+1}}(\underline{x}\wedge\underline{y})\frac{\prod_{l=0,\,l\neq\frac{\tilde{n}}{2}}^{\tilde{n}-1}\left(s-iz\sin\left(\frac{l\pi}{\tilde{n}}\right)\right)}{\prod_{l=0}^{n-1}\left(s+iz\cos\left(\frac{\theta+2\pi l}{n}\right)\right)}$$

where $\theta=\arccos w$, $w=\frac{\langle x,y\rangle}{z}$, $z=|x||y|$.

Proof. We only prove the results for the case $\tilde{n}$ odd. Let us define the denominator and numerator polynomials of the kernel in Theorem 5.3.2 as

$$\begin{aligned}
P_n(s) &:= (s+r)^n-2\omega(-iz)^n+(-1)^n(r-s)^n,\\
Q_{\tilde{n}}(s) &:= \frac{(s+r)^{\tilde{n}+1}+(-1)^{\tilde{n}}(r-s)^{\tilde{n}+1}}{r},\\
O_{\tilde{n}-1}(s) &:= \frac{(-1)^{\tilde{n}}(r-s)^{\tilde{n}}-(s+r)^{\tilde{n}}}{r}.
\end{aligned}$$

1. We first consider the denominator $P_n(s)$,

$$\begin{aligned}
P_n(s) &= (s+r)^n-2\omega(-iz)^n+(-1)^n\,(r-s)^n\\
&=\sum_{k=0}^{n}\binom{n}{k}s^{n-k}r^k-2\omega(-iz)^n\\
&\quad+(-1)^n\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\,s^{n-k}r^k\\
&=\left(\sum_{k=0}^{n}\binom{n}{k}s^{n-k}r^k\,(1+(-1)^k)\right)-2\omega(-iz)^n\\
&=2\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}s^{n-2k}\,(s^2+z^2)^k-2\omega(-iz)^n.
\end{aligned}$$

This means that $P_n(s)$ is a polynomial of degree n in s. The coefficient of s^n is $2\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}=2^n$.
Next, we verify $P_n(s_l)=0$ with $s_l=-iz\cos\left(\frac{\theta+2\pi l}{n}\right)$, $l=0,\cdots,n-1$. Denote $w=\cos(\theta)=\frac{e^{i\theta}+e^{-i\theta}}{2}$. When $\sin\left(\frac{\theta+2\pi l}{n}\right)\geq 0$, we have

$$\begin{aligned}
P_n(s_l) &= (s_l+r_l)^n-2\omega(-iz)^n+(-1)^n\,(r_l-s_l)^n\\
&=(-iz)^n\Bigg[\left(\cos\left(\frac{\theta+2\pi l}{n}\right)+i\sin\left(\frac{\theta+2\pi l}{n}\right)\right)^n
\end{aligned}$$

$$
\begin{aligned}
&+\left(\cos\left(\frac{\theta+2\pi l}{n}\right)-i\sin\left(\frac{\theta+2\pi l}{n}\right)\right)^{n}-2\omega\Bigg]\\
&=(-iz)^n\left(e^{i\theta}-2\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)+e^{-i\theta}\right)\\
&=0.
\end{aligned}
$$

where in the second step we have used Euler's formula. When $\sin\left(\frac{\theta+2\pi l}{n}\right)<0$ we have $P_n(s_j)=0$ by a similar calculation. Therefore, $s_l, l=0,\dots,n-1$ are all roots of P_n and we get the factorization

$$
\begin{aligned}
P_n(s)&=2\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}\prod_{l=0}^{n-1}\left(s+iz\cos\left(\frac{\theta+2\pi l}{n}\right)\right)\\
&=2^n\prod_{l=0}^{n-1}\left(s+iz\cos\left(\frac{\theta+2\pi l}{n}\right)\right).
\end{aligned}
$$

2. For $Q_{\tilde{n}}(s)$, we first claim that $Q_{\tilde{n}}(s)$ is a polynomial of degree $\tilde{n}$ in s. Indeed, we have

$$
\begin{aligned}
Q_{\tilde{n}}(s)&=\frac{(s+r)^{\tilde{n}+1}+(-1)^{\tilde{n}}(r-s)^{\tilde{n}+1}}{r}\\
&=\frac{1}{r}\Bigg[\sum_{k=0}^{\tilde{n}+1}\binom{\tilde{n}+1}{k}s^{\tilde{n}+1-k}\,r^k\\
&\quad+(-1)^{\tilde{n}}\sum_{k=0}^{\tilde{n}+1}\binom{\tilde{n}+1}{k}(-1)^{\tilde{n}+1-k}\,s^{\tilde{n}+1-k}\,r^k\Bigg]\\
&=\frac{1}{r}\sum_{k=0}^{\tilde{n}+1}\binom{\tilde{n}+1}{k}\,s^{\tilde{n}+1-k}\,r^k\,(1-(-1)^k)\\
&=2\sum_{k=0}^{\lfloor\frac{\tilde{n}+1}{2}\rfloor}\binom{\tilde{n}+1}{2k+1}\,s^{\tilde{n}-2k}\,(s^2+z^2)^k.
\end{aligned}
$$

The coefficient of $s^{\tilde{n}}$ is $2\sum_{k=0}^{\lfloor\frac{\tilde{n}+1}{2}\rfloor}\binom{\tilde{n}+1}{2k+1}=2^{\tilde{n}+1}$. When $\tilde{n}$ is odd, we verify $r_l\,Q_{\tilde{n}}(s_l)=0$ with $s_l=iz\sin\left(\frac{l\pi}{\tilde{n}+1}\right)$, $l=0,\dots,\tilde{n}$. For $l\le\frac{\tilde{n}+1}{2}$, we have $r_l=\sqrt{z^2+s_l^2}=z\cos\left(\frac{l\pi}{\tilde{n}+1}\right)$ and

$$
r_l\,Q_{\tilde{n}}(s_l)=(s_l+r_l)^{\tilde{n}+1}-(r_l-s_l)^{\tilde{n}+1}
$$

$$
\begin{aligned}
&= z^{\tilde{n}+1} \left[\left(\cos\left(\frac{l\pi}{\tilde{n}+1}\right) + i \sin\left(\frac{l\pi}{\tilde{n}+1}\right) \right)^{\tilde{n}+1} \right. \\
&\quad \left. - \left(\cos\left(\frac{l\pi}{\tilde{n}+1}\right) - i \sin\left(\frac{l\pi}{\tilde{n}+1}\right) \right)^{\tilde{n}+1} \right] \\
&= z^{\tilde{n}+1} \left(e^{il\pi} - e^{-il\pi} \right) \\
&= 2\, i z^{\tilde{n}+1} \sin(l\pi) \\
&= 0
\end{aligned}
$$

For $l > \frac{\tilde{n}+1}{2}$, $r_l\, Q_{\tilde{n}}(s_l) = 0$ can be proved in the same way. Moreover, we have $r_l = 0$ if and only if $l = \frac{\tilde{n}+1}{2}$. Hence, s_l, $l \neq \frac{\tilde{n}+1}{2}$ are the $\tilde{n}$ roots of the polynomial $Q_{\tilde{n}}(s)$. We get the factorization

$$
Q_{\tilde{n}}(s) = 2^{\tilde{n}+1} \prod_{\substack{l=0,\\ l\neq \frac{\tilde{n}+1}{2}}}^{\tilde{n}} \left(s - iz \sin\left(\frac{l\pi}{\tilde{n}+1}\right) \right).
$$

3. For the numerator $O_{\tilde{n}-1}(s)$, since

$$
\begin{aligned}
O_{\tilde{n}-1}(s) &= \frac{(-1)^{\tilde{n}}(r-s)^{\tilde{n}} - (s+r)^{\tilde{n}}}{r} \\
&= -\frac{1}{r} \sum_{k=0}^{\tilde{n}} \binom{\tilde{n}}{k} s^{\tilde{n}-k}\, r^k \left(1 - (-1)^k\right) \\
&= -2 \sum_{k=0}^{\lfloor \frac{\tilde{n}}{2} \rfloor} \binom{\tilde{n}}{2k+1} s^{\tilde{n}-2k-1} \left(s^2 + z^2\right)^k,
\end{aligned}
$$

we have that $O_{\tilde{n}-1}(s)$ is a polynomial of degree $\tilde{n}-1$ in s. The coefficient of $s^{\tilde{n}-1}$ is $-2\sum_{k=0}^{\lfloor \frac{\tilde{n}}{2} \rfloor} \binom{\tilde{n}}{2k+1} = -2^{\tilde{n}}$.
Next, we verify that $s_l = iz\cos\left(\frac{l\pi}{\tilde{n}}\right) = iz \sin\left(\frac{\pi}{2} + \frac{l\pi}{\tilde{n}}\right)$, $l = 0, \dots, \tilde{n}-1$ are $\tilde{n}$ roots of $O_{\tilde{n}-1}(s_l) = 0$ when $\tilde{n}$ is odd. We compute

$$
\begin{aligned}
r_l\, O_{\tilde{n}-1}(s_l) &= -\left((s_l + r_l)^{\tilde{n}} + (r_l - s_l)^{\tilde{n}} \right) \\
&= -z^{\tilde{n}} \left[\left(\cos\left(\frac{\pi}{2} + \frac{l\pi}{\tilde{n}}\right) + i \sin\left(\frac{\pi}{2} + \frac{l\pi}{\tilde{n}}\right) \right)^{\tilde{n}} \right. \\
&\quad \left. + \left(\cos\left(\frac{\pi}{2} + \frac{l\pi}{\tilde{n}}\right) - i \sin\left(\frac{\pi}{2} + \frac{l\pi}{\tilde{n}}\right) \right)^{\tilde{n}} \right]
\end{aligned}
$$

$$
\begin{aligned}
&= -z^{\tilde{n}} \left(e^{i\left(\frac{\pi}{2}\tilde{n}+l\pi\right)} + e^{-i\left(\frac{\pi}{2}\tilde{n}+l\pi\right)} \right) \\
&= -2z^{\tilde{n}} \cos\left(\frac{\pi}{2}\tilde{n}+l\pi\right) \\
&= 0.
\end{aligned}
$$

Note that $r_l = 0$ if and only if $l = 0$. So s_l, $l = 1, \dots, \tilde{n}-1$ are the $n-1$ roots of the polynomial $O_{\tilde{n}-1}(s)$. Hence, we have

$$
O_{\tilde{n}-1}(s) = -2^{\tilde{n}} \prod_{l=1}^{\tilde{n}-1} \left(s - iz\sin\left(\frac{\pi}{2} + \frac{l\pi}{\tilde{n}}\right)\right).
$$

The case $\tilde{n}$ even is treated similarly. □

Remark 5.3.4. *It is seen that all the rational functions in the kernel are proper (the degree of the numerator polynomial is smaller than the degree of the denominator polynomial). Therefore the function in the bivector part of $\mathcal{L}(K_2^c)$ for case $\tilde{n}$ odd*

$$
\frac{\prod_{l=1}^{\tilde{n}-1}\left(s - iz\cos\left(\frac{l\pi}{\tilde{n}}\right)\right)}{\prod_{l=0}^{n-1}\left(s + iz\cos\left(\frac{\theta+2\pi l}{n}\right)\right)} := X(s)
$$

has the partial fraction expansion

$$
\frac{c_0}{s+iz\cos\left(\frac{\theta}{n}\right)} + \frac{c_1}{s+iz\cos\left(\frac{\theta+2\pi}{n}\right)} + \cdots + \frac{c_{n-1}}{s+iz\cos\left(\frac{\theta+2\pi(n-1)}{n}\right)}
$$

where the coefficients $c_0, \dots, c_{n-1}$ (called the residues of $X(s)$) are determined via the formula

$$
c_l = \left[\left(s + iz\cos\left(\frac{\theta+2\pi l}{n}\right)\right) X(s)\right]_{s=-iz\cos\left(\frac{\theta+2\pi l}{n}\right)}
$$

for $l = 0, \dots, n-1$. Using linearity and formula (2.38) for each term, we have

$$
\begin{aligned}
K^c_{2,biv} &= \frac{i}{2^{\tilde{n}+1}} \mathcal{L}^{-1}\left(\sum_{l=0}^{n-1} \frac{c_l}{s+iz\cos\left(\frac{\theta+2\pi l}{n}\right)} \right) \\
&= \frac{i}{2^{\tilde{n}+1}} \sum_{l=0}^{n-1} c_l\, e^{-izt\cos\left(\frac{\theta+2\pi l}{n}\right)}, \quad l = 0, \dots, n-1.
\end{aligned}
$$

The scalar part of the kernel can be considered by analogy with the bivector case. This form of kernel can be compared with the results of [28], given in Theorem 5.2.1 by setting $t = 1$.

5.3.2 The dimension $m > 2$

For dimensions $m > 2$, we rewrite the kernel in (5.1) by introducing an auxiliary variable t in the Bessel functions. For $1+c = \frac{1}{n}$, $n \in \mathbb{N}_0 \setminus \{1\}$ odd and $n = 2\tilde{n} + 1$, $\tilde{n} \in \mathbb{N}_0$, the kernel of the deformed Fourier transform is given by

$$
\begin{aligned}
A_\lambda &= \sum_{k=0}^{+\infty} (k+\lambda)\,(-i)^{kn} J_{nk+n\lambda+\tilde{n}}(z\,t)\, C_k^\lambda(w) \\
&\quad + i^{n-1} \sum_{k=0}^{+\infty} (k+\lambda)\,(-i)^{kn} J_{nk+n\lambda-\tilde{n}}(z\,t)\, C_k^\lambda(w); \\
B_\lambda &= \sum_{k=0}^{+\infty} (-i)^{kn} J_{nk+n\lambda+\tilde{n}}(z\,t)\, C_k^\lambda(w) \\
&\quad - i^{n-1} \sum_{k=0}^{+\infty} (-i)^{kn} J_{nk+n\lambda-\tilde{n}}(z\,t)\, C_k^\lambda(w); \\
C_\lambda &= (-i)^n \sum_{k=0}^{+\infty} (-i)^{kn} J_{nk+n\lambda+3\tilde{n}+1}(z\,t)\, C_k^{\lambda+1}(w) \\
&\quad + i \sum_{k=0}^{+\infty} (-i)^{kn} J_{nk+n\lambda+\tilde{n}+1}(z\,t)\, C_k^{\lambda+1}(w)
\end{aligned}
\tag{5.10}
$$

where we substituted $k \mapsto k+1$ in C_λ.
To consider the Laplace transform of A_λ, we need the expansion of the Poisson kernel in terms of Gegenbauer polynomials (2.1.1). By Theorem 5.3.1, Theorem 2.1.1 and formula (2.44), we take the Laplace transform with respect to t in (5.10) for $\operatorname{Re} s$ big enough. With $r = \sqrt{s^2+z^2}$, $u_R = \left(\frac{-iz}{s+r}\right)^n$, $\lambda = \frac{m-2}{2} > 0$, $\underline{z} = |x||y|$, $w = \frac{\langle x, y\rangle}{z}$, we obtain

$$
\begin{aligned}
&\mathcal{L}\left(\frac{1}{2\lambda} z^{-\frac{\mu-2}{2}} A_\lambda\right) \\
&= \frac{1}{2r}\left(\frac{1}{s+r}\right)^{n\lambda+\tilde{n}} \frac{1-u_R^2}{(1-2wu_R+u_R^2)^{\lambda+1}} \\
&\quad + \left(\frac{i}{z}\right)^{n-1} \frac{1}{2r}\left(\frac{1}{s+r}\right)^{n\lambda-\tilde{n}} \frac{1-u_R^2}{(1-2wu_R+u_R^2)^{\lambda+1}}.
\end{aligned}
\tag{5.11}
$$

For B_λ and C_λ, using the formula (2.15), we have, for $\lambda > 0$,

$$\begin{aligned}
&\mathcal{L}\left(\frac{1}{2}z^{-\frac{\mu-2}{2}}B_\lambda\right)\\
&=\frac{1}{2r}\left(\frac{1}{s+r}\right)^{n\lambda+\tilde{n}}\frac{1}{(1-2wu_R+u_R^2)^\lambda}\\
&\quad-\left(\frac{i}{z}\right)^{n-1}\frac{1}{2r}\left(\frac{1}{s+r}\right)^{n\lambda-\tilde{n}}\frac{1}{(1-2wu_R+u_R^2)^\lambda}
\end{aligned}\tag{5.12}$$

and

$$\begin{aligned}
&\mathcal{L}\left(z^{-\frac{\mu}{2}}C_\lambda\right)\\
&=(-i)^n z^{n-1}\frac{1}{r}\left(\frac{1}{s+r}\right)^{n\lambda+3\tilde{n}+1}\frac{1}{(1-2wu_R+u_R^2)^{\lambda+1}}\\
&\quad+\frac{i}{r}\left(\frac{1}{s+r}\right)^{n\lambda+\tilde{n}+1}\frac{1}{(1-2wu_R+u_R^2)^{\lambda+1}}.
\end{aligned}\tag{5.13}$$

By relation (5.9), the kernels can be further simplified as follows

$$\begin{aligned}
&\mathcal{L}\left(\frac{1}{2\lambda}z^{-\frac{\mu-2}{2}}A_\lambda\right)\\
&=\frac{1}{2r}\left(\frac{1}{s+r}\right)^{\tilde{n}}\frac{(s+r)^n-\frac{(-iz)^{2n}}{(s+r)^n}}{((s+r)^n-2w\,(-iz)^n+\frac{(-iz)^{2n}}{(s+r)^n})^{\lambda+1}}\\
&\quad+\left(\frac{i}{z}\right)^{n-1}\frac{1}{2r}\left(\frac{1}{s+r}\right)^{-\tilde{n}}\frac{(s+r)^n-\frac{(-iz)^{2n}}{(s+r)^n}}{((s+r)^n-2w\,(-iz)^n+\frac{(-iz)^{2n}}{(s+r)^n})^{\lambda+1}}\\
&=\frac{1}{2r}\frac{(s+r)^{-\tilde{n}}\left((s+r)^n-(-1)^n\,(r-s)^n\right)}{((s+r)^n-2w\,(-iz)^n+(-1)^n\,(r-s)^n)^{\lambda+1}}\\
&\quad+\left(\frac{i}{z}\right)^{n-1}\frac{1}{2r}\frac{(s+r)^{\tilde{n}}\left((s+r)^n-(-1)^n\,(r-s)^n\right)}{((s+r)^n-2w\,(-iz)^n+(-1)^n\,(r-s)^n)^{\lambda+1}}\\
&=\frac{1}{2r}\frac{\left((s+r)^{-\tilde{n}}+(-1)^{\tilde{n}}(r-s)^{-\tilde{n}}\right)\left((s+r)^n-(-1)^n(r-s)^n\right)}{\left((s+r)^n-2w(-iz)^n+(-1)^n(r-s)^n\right)^{\lambda+1}}.
\end{aligned}$$

The corresponding results for B_λ and C_λ can be derived immediately by the following expressions

$$\mathcal{L}\left(\frac{1}{2}z^{-\frac{\mu-2}{2}}B_\lambda\right)$$

$$= \frac{1}{2r}\left(\frac{1}{s+r}\right)^{\tilde{n}} \frac{1}{((s+r)^n - 2w\,(-iz)^n + (-1)^n(r-s)^n)^\lambda}$$
$$- \left(\frac{i}{z}\right)^{n-1} \frac{1}{2r}\left(\frac{1}{s+r}\right)^{-\tilde{n}} \frac{1}{((s+r)^n - 2w\,(-iz)^n + (-1)^n(r-s)^n)^\lambda}$$

and

$$\mathcal{L}\left(z^{-\frac{\mu}{2}} C_\lambda\right)$$
$$= (-i)^n z^{n-1}\frac{1}{r}\left(\frac{1}{s+r}\right)^{\tilde{n}} \frac{1}{((s+r)^n - 2w\,(-iz)^n + (-1)^n(r-s)^n)^{\lambda+1}}$$
$$+ \frac{i}{r}\left(\frac{1}{s+r}\right)^{-\tilde{n}} \frac{1}{((s+r)^n - 2w\,(-iz)^n + (-1)^n(r-s)^n)^{\lambda+1}}.$$

Hence, we conclude the following theorem.

Theorem 5.3.5. *If the dimension $m > 2\,(\lambda > 0)$ and $1 + c = \frac{1}{n}$, $n \in \mathbb{N}_0 \setminus \{1\}$ odd, $n = 2\tilde{n} + 1$, $\tilde{n} \in \mathbb{N}_0$, then the kernel of the radially deformed Fourier transform $\mathcal{F}_D = e^{i\frac{\pi}{2}\left(\frac{1}{2}+\frac{m-1}{2(1+c)}\right)} e^{\frac{-i\pi}{4(1+c)^2}\left(\mathbf{D}^2-(1+c)^2\underline{x}^2\right)}$ in the Laplace domain takes the form*

$$\mathcal{L}\left(K_m^c\right) = \mathcal{L}\left(\frac{1}{2\lambda} z^{-\frac{\mu-2}{2}} A_\lambda\right) + \mathcal{L}\left(\frac{1}{2} z^{-\frac{\mu-2}{2}} B_\lambda\right) - (\underline{x} \wedge \underline{y})\,\mathcal{L}\left(z^{-\frac{\mu}{2}}\, C_\lambda\right)$$

with

$$\mathcal{L}\left(\frac{1}{2\lambda} z^{-\frac{\mu-2}{2}} A_\lambda\right) = \frac{1}{2r}\frac{((s+r)^n - (-1)^n(r-s)^n)}{((s+r)^n - 2w(-iz)^n + (-1)^n(r-s)^n)^{\lambda+1}}$$
$$\times \left((s+r)^{-\tilde{n}} + (-1)^{\tilde{n}}(r-s)^{-\tilde{n}}\right);$$
$$\mathcal{L}\left(\frac{1}{2} z^{-\frac{\mu-2}{2}} B_\lambda\right) = \frac{1}{2r}\frac{(s+r)^{-\tilde{n}} - (-1)^{\tilde{n}}(r-s)^{-\tilde{n}}}{((s+r)^n - 2w(-iz)^n + (-1)^n(r-s)^n)^{\lambda}};$$
$$\mathcal{L}\left(z^{-\frac{\mu}{2}} C_\lambda\right) = \frac{i}{r}\frac{(s+r)^{\tilde{n}} - (-1)^{\tilde{n}}(r-s)^{\tilde{n}}}{((s+r)^n - 2w(-iz)^n + (-1)^n(r-s)^n)^{\lambda+1}}$$

where $r = \sqrt{s^2+z^2}$, $z = |x||y|$, $w = \frac{\langle x,y\rangle}{z}$.

Remark 5.3.6. *When $\lambda > 0$, $\mathcal{L}\left(\frac{1}{2\lambda} z^{-\frac{\mu-2}{2}} A_\lambda\right)$ can also be written as*

$$\mathcal{L}\left(\frac{1}{2\lambda} z^{-\frac{\mu-2}{2}} A_\lambda\right) = -\frac{1}{2\lambda n}\left((s+r)^{-\tilde{n}} + (-1)^n(r-s)^{-\tilde{n}}\right)$$
$$\times \frac{d}{ds}\left(\frac{1}{((s+r)^n - 2w(-iz)^n + (-1)^n(r-s)^n)^{\lambda}}\right)$$

where $r = \sqrt{s^2+z^2}$.

Remark 5.3.7. *When $\lambda > 0$, we can rewrite the expressions into the following forms*

$$\begin{aligned}
\mathcal{L}\left(\frac{1}{2\lambda} z^{-\frac{\mu-2}{2}} A_\lambda\right) &= \frac{1}{2rz^{2\tilde{n}}} \frac{(-1)^{\tilde{n}}(s+r)^{3\tilde{n}+1} + (r-s)^{3\tilde{n}+1}}{((s+r)^n - 2w(-iz)^n + (-1)^n(r-s)^n)^{\lambda+1}} \\
&\quad + \frac{1}{2r} \frac{(s+r)^{\tilde{n}+1} + (-1)^{\tilde{n}}(r-s)^{\tilde{n}+1}}{((s+r)^n - 2w(-iz)^n + (-1)^n(r-s)^n)^{\lambda+1}}; \\
\mathcal{L}\left(\frac{1}{2} z^{-\frac{\mu-2}{2}} B_\lambda\right) &= \frac{1}{2rz^{2\tilde{n}}} \frac{(r-s)^{\tilde{n}} - (-1)^{\tilde{n}}(s+r)^{\tilde{n}}}{((s+r)^n - 2w(-iz)^n + (s-r)^n)^{\lambda}}; \\
\mathcal{L}\left(z^{-\frac{\mu}{2}} C_\lambda\right) &= \frac{i}{r} \frac{(s+r)^{\tilde{n}} - (-1)^{\tilde{n}}(r-s)^{\tilde{n}}}{((s+r)^n - 2w(-iz)^n + (s-r)^n)^{\lambda+1}},
\end{aligned}$$

by the relation

$$\begin{aligned}
(s+r)^{-\tilde{n}} - (-1)^{\tilde{n}}(r-s)^{-\tilde{n}} &= \frac{1}{(s+r)^{\tilde{n}}} - \frac{(-1)^{\tilde{n}}}{(r-s)^{\tilde{n}}} \\
&= \frac{(r-s)^{\tilde{n}} - (-1)^{\tilde{n}}(s+r)^{\tilde{n}}}{(s+r)^{\tilde{n}}\,(r-s)^{\tilde{n}}} \\
&= \frac{((r-s)^{\tilde{n}} - (-1)^{\tilde{n}}(s+r)^{\tilde{n}})}{z^{2\tilde{n}}}.
\end{aligned}$$

Similar to Theorem 5.3.3, the Laplace transforms of K_m^c of the deformed Fourier transform obtained in Remark 5.3.7 for dimension $m > 2$ can be given in more detail as follows.

Theorem 5.3.8. *If the dimension $m > 2\,(\lambda > 0)$ and $1 + c = \frac{1}{n}$ with $n \in \mathbb{N}_0 \setminus \{1\}$ odd, $n = 2\tilde{n}+1$, $\tilde{n} \in \mathbb{N}_0$, the following expansions hold:*
<u>Case 1</u>: $\tilde{n}$ odd $\quad (\tilde{n} = 1, 3, 5, \dots)$

$$\begin{aligned}
\mathcal{L}\left(\frac{1}{2\lambda} z^{-\frac{\mu-2}{2}} A_\lambda\right) &= -\frac{2^{\tilde{n}-n\lambda-1}}{z^{2\tilde{n}}} \frac{\prod_{l=0,\, l\neq\frac{3\tilde{n}+1}{2}}^{3\tilde{n}} \left(s - iz\sin\left(\frac{l\pi}{3\tilde{n}+1}\right)\right)}{\prod_{l=0}^{n-1}\left(s + iz\cos\left(\frac{\theta+2\pi l}{n}\right)\right)^{\lambda+1}} \\
&\quad + 2^{\tilde{n}-n\lambda-n} \frac{\prod_{l=0,\, l\neq\frac{\tilde{n}+1}{2}}^{\tilde{n}} \left(s - iz\sin\left(\frac{l\pi}{\tilde{n}+1}\right)\right)}{\prod_{l=0}^{n-1}\left(s + iz\cos\left(\frac{\theta+2\pi l}{n}\right)\right)^{\lambda+1}}; \\
\mathcal{L}\left(\frac{1}{2} z^{-\frac{\mu-2}{2}} B_\lambda\right) &= -\frac{2^{\tilde{n}-n\lambda-1}}{z^{2\tilde{n}}} \frac{\prod_{l=1}^{\tilde{n}-1} \left(s - iz\cos\left(\frac{l\pi}{\tilde{n}}\right)\right)}{\prod_{l=0}^{n-1}\left(s + iz\cos\left(\frac{\theta+2\pi l}{n}\right)\right)^{\lambda}};
\end{aligned}$$

$$\mathcal{L}\left(z^{-\frac{\mu}{2}} C_\lambda\right) = i\, 2^{\tilde{n}-n\lambda-n} \frac{\prod_{l=1}^{\tilde{n}-1}\left(s - iz\cos\left(\frac{l\pi}{\tilde{n}}\right)\right)}{\prod_{l=0}^{n-1}\left(s + iz\cos\left(\frac{\theta+2\pi l}{n}\right)\right)^{\lambda+1}}.$$

Case 2: $\tilde{n}$ even $\quad(\tilde{n} = 2, 4, 6, \dots)$

$$\begin{aligned}
\mathcal{L}\left(\frac{1}{2\lambda} z^{-\frac{\mu-2}{2}} A_\lambda\right) &= \frac{2^{\tilde{n}-n\lambda-1}}{z^{2\tilde{n}}} \frac{\prod_{l=1}^{3\tilde{n}}\left(s - iz\cos\left(\frac{l\pi}{3\tilde{n}+1}\right)\right)}{\prod_{l=0}^{n-1}\left(s + iz\cos\left(\frac{\theta+2\pi l}{n}\right)\right)^{\lambda+1}} \\
&\quad + 2^{\tilde{n}-n\lambda-n} \frac{\prod_{l=1}^{\tilde{n}}\left(s - iz\cos\left(\frac{l\pi}{\tilde{n}+1}\right)\right)}{\prod_{l=0}^{n-1}\left(s + iz\cos\left(\frac{\theta+2\pi l}{n}\right)\right)^{\lambda+1}}; \\
\mathcal{L}\left(\frac{1}{2} z^{-\frac{\mu-2}{2}} B_\lambda\right) &= -\frac{2^{\tilde{n}-n\lambda-1}}{z^{2\tilde{n}}} \frac{\prod_{l=0,\, l\neq\frac{\tilde{n}}{2}}^{\tilde{n}-1}\left(s - iz\sin\left(\frac{l\pi}{\tilde{n}}\right)\right)}{\prod_{l=0}^{n-1}\left(s + iz\cos\left(\frac{\theta+2\pi l}{n}\right)\right)^{\lambda}}; \\
\mathcal{L}\left(z^{-\frac{\mu}{2}} C_\lambda\right) &= i\, 2^{\tilde{n}-n\lambda-n} \frac{\prod_{l=0,\, l\neq\frac{\tilde{n}}{2}}^{\tilde{n}-1}\left(s - iz\sin\left(\frac{l\pi}{\tilde{n}}\right)\right)}{\prod_{l=0}^{n-1}\left(s + iz\cos\left(\frac{\theta+2\pi l}{n}\right)\right)^{\lambda+1}}
\end{aligned}$$

where $\theta = \arccos(\omega)$, $\omega = \frac{\langle x,y\rangle}{z}$, $z = |x||y|$.

Proof. The factorization for the polynomial $P_n(s) = (s+r)^n - 2\omega(-iz)^n + (-1)^n(r-s)^n$ can be found in the proof of Theorem 5.3.3. Hence, we have immediately

$$P_n(s)^{\lambda+1} = 2^{n(\lambda+1)} \prod_{l=0}^{n-1}\left(s + iz\cos\left(\frac{\theta+2\pi l}{n}\right)\right)^{\lambda+1}.$$

As we have already calculated other numerator polynomials for $\tilde{n}$ odd, here we only prove the numerator of the first term of $\mathcal{L}\left(\frac{1}{2\lambda} z^{-\frac{\mu-2}{2}} A_\lambda\right)$ (see Remark 5.3.7) for the cases $\tilde{n}$ odd and even. Denote

$$U_{3\tilde{n}}(s) := \frac{(-1)^{\tilde{n}}(s+r)^{3\tilde{n}+1} + (r-s)^{3\tilde{n}+1}}{2rz^{2\tilde{n}}}.$$

We first show the $U_{3\tilde{n}}(s)$ is a polynomial of degree $3\tilde{n}$ in s.

$$\begin{aligned}
2z^{2\tilde{n}} U_{3\tilde{n}}(s) &= \frac{(-1)^{\tilde{n}}(s+r)^{3\tilde{n}+1} + (r-s)^{3\tilde{n}+1}}{r} \\
&= \frac{1}{r}\left[(-1)^{\tilde{n}} \sum_{k=0}^{3\tilde{n}+1} \binom{3\tilde{n}+1}{k} s^{3\tilde{n}+1-k}\, r^k\right.
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{k=0}^{3\tilde{n}+1} \binom{3\tilde{n}+1}{k} (-1)^{3\tilde{n}+1-k}\, s^{3\tilde{n}+1-k}\, r^k \Big] \\
&= (-1)^{\tilde{n}} \frac{1}{r} \sum_{k=0}^{3\tilde{n}+1} \binom{3\tilde{n}+1}{k} s^{3\tilde{n}+1-k}\, r^k \,(1-(-1)^k) \\
&= 2(-1)^{\tilde{n}} \sum_{k=0}^{\lfloor \frac{3\tilde{n}+1}{2} \rfloor} \binom{3\tilde{n}+1}{2k+1} s^{3\tilde{n}-2k}\,(s^2+z^2)^k.
\end{aligned}
$$

The coefficient of $s^{3\tilde{n}}$ is $2(-1)^{\tilde{n}} \sum_{k=0}^{\lfloor (3\tilde{n}+1)/2 \rfloor} \binom{3\tilde{n}+1}{2k+1} = (-1)^{\tilde{n}}\, 2^{3\tilde{n}+1}$. When $\tilde{n}$ is odd, we have $s_l = iz \sin\left(\frac{l\pi}{3\tilde{n}+1}\right)$, $l = 0, \ldots, 3\tilde{n}$. For $l \leq \frac{3\tilde{n}+1}{2}$, we find

$$
\begin{aligned}
2 r_l z^{2\tilde{n}} U_{3\tilde{n}}(s_l) &= -(s_l + r_l)^{3\tilde{n}+1} + (r_l - s_l)^{3\tilde{n}+1} \\
&= -z^{3\tilde{n}+1}\left(e^{il\pi} - e^{-il\pi}\right) \\
&= 0.
\end{aligned}
$$

For $l \geq \frac{3\tilde{n}+1}{2}$, we have $2 r_l z^{2\tilde{n}} U_{3\tilde{n}}(s_l) = 0$. Moreover, we have $r_l = 0$ if and only if $l = \frac{3\tilde{n}+1}{2}$. Hence, s_l, $l \neq \frac{3\tilde{n}+1}{2}$ are the $3\tilde{n}$ roots of the polynomial $U_{3\tilde{n}}(s)$. Therefore, we have the factorization

$$
U_{3\tilde{n}}(s) = (-1)^{\tilde{n}}\, 2^{3\tilde{n}+1} \prod_{\substack{l=0,\\ l \neq \frac{3\tilde{n}+1}{2}}}^{3\tilde{n}} \left(s - iz \sin\left(\frac{l\pi}{3\tilde{n}+1}\right)\right).
$$

When $\tilde{n}$ is even, $s_l = iz \cos\left(\frac{l\pi}{3\tilde{n}+1}\right) = iz \sin\left(\frac{\pi}{2} + \frac{l\pi}{3\tilde{n}+1}\right)$, $l = 0, \ldots, 3\tilde{n}$ are $3\tilde{n}$ roots of $2 r_l z^{2\tilde{n}} U_{3\tilde{n}}(s_l) = 0$. Indeed, we have $r_l = \sqrt{s_l^2 + z^2} = z \cos\left(\frac{\pi}{2} + \frac{l\pi}{3\tilde{n}+1}\right)$ and

$$
\begin{aligned}
2 r_l z^{2\tilde{n}} U_{3\tilde{n}}(s_l) &= (s_l + r_l)^{3\tilde{n}+1} + (r_l - s_l)^{3\tilde{n}+1} \\
&= z^{3\tilde{n}+1}\left(e^{i\frac{\pi}{2}(3\tilde{n}+1) + il\pi} + e^{-i\frac{\pi}{2}(3\tilde{n}+1) - il\pi}\right) \\
&= 0.
\end{aligned}
$$

Note that $r_l = 0$ if and only if when $l = 0$. Hence, s_l, $l = 1, \ldots, 3\tilde{n}+1$ are the $3\tilde{n}$ roots of the polynomial $U_{3\tilde{n}}(s)$. We get

$$
U_{3\tilde{n}}(s) = (-1)^{\tilde{n}}\, 2^{3\tilde{n}+1} \prod_{l=1}^{3\tilde{n}} \left(s - iz \cos\left(\frac{l\pi}{3\tilde{n}+1}\right)\right).
$$

This completes the proof. □

If we consider the Laplace transform of B_λ in Theorem 5.3.5 in the following forms

$$\begin{aligned}\mathcal{L}\left(\frac{1}{2}z^{-\frac{\mu-2}{2}}B_\lambda\right) = &\frac{1}{2r}\frac{(s+r)^{\tilde{n}+1}+(-1)^{\tilde{n}}(r-s)^{\tilde{n}+1}}{((s+r)^n-2w(-iz)^n+(-1)^n(r-s)^n)^{\lambda+1}}\\ &+\frac{i\,wz}{r}\frac{(-1)^{\tilde{n}}(r-s)^{\tilde{n}}-(s+r)^{\tilde{n}}}{((s+r)^n-2w(-iz)^n+(-1)^n(r-s)^n)^{\lambda+1}}\\ &-\frac{z^{-2\tilde{n}}}{2r}\frac{(-1)^{\tilde{n}}(s+r)^{3\tilde{n}+1}+(r-s)^{3\tilde{n}+1}}{((s+r)^n-2w(-iz)^n+(-1)^n(r-s)^n)^{\lambda+1}},\end{aligned}$$

then summing the results of A_λ in Remark 5.3.7 together as the scalar part of the kernel, we arrive at the following result:

Theorem 5.3.9. *If the dimension $m > 2\,(\lambda > 0)$ and $1+c = \frac{1}{n}$, $n \in \mathbb{N}_0\backslash\ \{1\}$ odd, $n = 2\tilde{n}+1$, $\tilde{n} \in \mathbb{N}_0$, the kernel of the radially deformed Fourier transform $\mathcal{F}_D = e^{i\frac{\pi}{2}\left(\frac{1}{2}+\frac{m-1}{2(1+c)}\right)}e^{\frac{-i\pi}{4(1+c)^2}\left(\mathbf{D}^2-(1+c)^2\underline{x}^2\right)}$ is given by*

$$K_m^c = K_{m,scal}^c + (\underline{x}\wedge\underline{y})\,K_{m,biv}^c$$

with

$$\begin{aligned}K_{m,scal}^c = &\frac{1}{r}\frac{(s+r)^{\tilde{n}+1}+(-1)^{\tilde{n}}(r-s)^{\tilde{n}+1}}{((s+r)^n-2w(-iz)^n+(-1)^n(r-s)^n)^{\lambda+1}}\\ &+\frac{wz\,i}{r}\frac{(-1)^{\tilde{n}}(r-s)^{\tilde{n}}-(s+r)^{\tilde{n}}}{((s+r)^n-2w(-iz)^n+(-1)^n(r-s)^n)^{\lambda+1}}\end{aligned}$$

and

$$K_{m,biv}^c = \frac{i}{r}\frac{(s+r)^{\tilde{n}}-(-1)^{\tilde{n}}(r-s)^{\tilde{n}}}{((s+r)^n-2w(-iz)^n+(-1)^n(r-s)^n)^{\lambda+1}}$$

where $r = \sqrt{s^2+z^2}$, $z = |x||y|$, $w = \frac{\langle x,y\rangle}{z}$.

Remark 5.3.10. *If we take the limit $\lambda = 0$, we can recover the same expressions for dimension 2 in Theorem 5.3.2.*

Remark 5.3.11. *The factorization for the polynomials in the above theorem can be derived immediately from Theorem 5.3.3.*

5.4 Explicit expressions of the kernel

In this section, we give the integral expressions for the kernel in all dimensions. These results will be given in terms of the Mittag-Leffler functions in Definition 2.16 and Definition 2.17 by adapting the method introduced in [16].

5.4.1 The case of $m = 2$

We consider the kernel of the deformed Fourier transform in the forms of (5.6) and (5.8) in dimension 2, i.e. $\lambda = 0$, for special values of the deformation parameter c.

Let us first calculate the 4 different terms separately given in (5.6)

$$\mathcal{L}\left(K^c_{2,scal}\right) := I_1 + I_2 + I_3 + I_4, \tag{5.14}$$

with

$$\begin{aligned}
I_1 &= \frac{1}{r}\left(\frac{1}{s+r}\right)^{\tilde{n}} \frac{1}{1-2w\,u_R+u_R^2},\\
I_2 &= -\frac{1}{r}\left(\frac{1}{s+r}\right)^{\tilde{n}} \frac{w\,u_R}{1-2w\,u_R+u_R^2},\\
I_3 &= (-1)^{\tilde{n}} z^{-2\tilde{n}}\frac{1}{r}\left(\frac{1}{s+r}\right)^{-\tilde{n}} \frac{w\,u_R}{1-2w\,u_R+u_R^2},\\
I_4 &= -(-1)^{\tilde{n}} z^{-2\tilde{n}}\frac{1}{r}\left(\frac{1}{s+r}\right)^{-\tilde{n}} \frac{u_R^2}{1-2w\,u_R+u_R^2}
\end{aligned}$$

where $u_R = \left(\frac{-iz}{s+r}\right)^n$, $r = \sqrt{s^2+z^2}$ and $w = \cos\theta$. Denote $b_\pm = i^n\, e^{\pm i\theta} z^n$. Then the common fraction can be equivalently written as

$$\begin{aligned}
\frac{1}{1-2w\,u_R+u_R^2} &= \frac{1}{u_R-e^{i\theta}}\frac{1}{u_R-e^{-i\theta}}\\
&= \frac{i^n\,z^n}{(r-s)^n-b_+}\frac{i^n\,z^n}{(r-s)^n-b_-}\\
&= (-1)^n\,z^{2n}\frac{1}{(r-s)^n-b_+}\frac{1}{(r-s)^n-b_-}.
\end{aligned} \tag{5.15}$$

where we used (5.9) to rewrite:

$$u_R = \left(\frac{-iz}{s+r}\right)^n = (-iz)^n\,\frac{(r-s)^n}{z^{2n}} = (-i)^n\,\frac{(r-s)^n}{z^n}.$$

Take into account (2.46), we rewrite I_1 up to I_4 as

$$
\begin{aligned}
I_1 &= -\frac{1}{r}\, z^{n+1}\, (r-s)^{\tilde{n}}\, \frac{1}{(r-s)^n - b_+}\, \frac{1}{(r-s)^n - b_-};\\
I_2 &= -\frac{1}{r}\, \omega\, i^n\, z\, (r-s)^{n+\tilde{n}}\, \frac{1}{(r-s)^n - b_+}\, \frac{1}{(r-s)^n - b_-};\\
I_3 &= \frac{i}{r}\, \omega\, z^n\, (r-s)^{\tilde{n}+1}\, \frac{1}{(r-s)^n - b_+}\, \frac{1}{(r-s)^n - b_-};\\
I_4 &= -(-1)^{\tilde{n}} \frac{1}{r}\, (r-s)^{3\tilde{n}+2}\, \frac{1}{(r-s)^n - b_+}\, \frac{1}{(r-s)^n - b_-}
\end{aligned}
$$

in the form $F(r-s) = F\left(\sqrt{s^2+z^2} - s\right)$ by using relation (5.9) again. Next, we consider the inverse Laplace transform of $\frac{1}{s^n - b_+}\frac{1}{s^n - b_-}$. By Remark 2.1.5 and Formula (2.45), we derive the Laplace transform of the Mittag-Leffler function (2.16)

$$
\mathcal{L}\left(t^{\beta-1} E_{\alpha,\beta}(b\, t^{\alpha})\right) = \frac{1}{s^{\beta}}\frac{1}{1 - b s^{-\alpha}}
$$

where $\operatorname{Re}\alpha > 0, \operatorname{Re}\beta > 0, \operatorname{Re}s > 0$ and $s > |b|^{1/(\operatorname{Re}\alpha)}$. Then we obtain the inverse transform

$$
\mathcal{L}^{-1}\left(\frac{1}{s^n - b}\right) = t^{n-1}\, E_{n,n}\left(b\, t^n\right).
$$

Setting $F(s) = \frac{1}{s^n - b_+}$, $G(s) = \frac{1}{s^n - b_-}$ and using formula (2.54), the inverse Laplace transform is given by

$$
\begin{aligned}
h_1(t) &:= \mathcal{L}^{-1}\left(\frac{1}{s^n - b_+}\frac{1}{s^n - b_-}\right)\\
&= \int_0^t \zeta^{n-1} E_{n,n}\left(b_+\, \zeta^n\right) (t-\zeta)^{n-1} E_{n,n}(b_-(t-\zeta)^n)\, \mathrm{d}\zeta.
\end{aligned}
$$

Therefore, adding these 4 terms and using (2.46), we arrive at the following result.

$$
\begin{aligned}
&K^c_{2,scal}\\
&= -z^{3\tilde{n}+2} \int_0^1 (1+2\tau)^{-\tilde{n}/2}\, J_{\tilde{n}}\left(z\sqrt{1+2\tau}\right)\, h_1(\tau)\, \mathrm{d}\tau\\
&\quad - iw\, (-1)^{\tilde{n}}\, z^{3\tilde{n}+2} \int_0^1 (1+2\tau)^{-(3\tilde{n}+1)/2}\, J_{3\tilde{n}+1}\left(z\sqrt{1+2\tau}\right)\, h_1(\tau)\, \mathrm{d}\tau
\end{aligned}
$$

$$+ iw\, z^{3\tilde{n}+2} \int_0^1 (1+2\tau)^{-(\tilde{n}+1)/2} J_{\tilde{n}+1}\left(z\sqrt{1+2\tau}\right) h_1(\tau)\, \mathrm{d}\tau$$
$$- (-1)^{\tilde{n}} z^{3\tilde{n}+2} \int_0^1 (1+2\tau)^{-(3\tilde{n}+2)/2} J_{3\tilde{n}+2}\left(z\sqrt{1+2\tau}\right) h_1(\tau)\, \mathrm{d}\tau.$$

Similarly, we deduce the Laplace transform of the bivector part (5.8)

$$\begin{aligned}\mathcal{L}\left(K^c_{2,biv}\right) &= -i^n \frac{1}{r}(r-s)^{n+\tilde{n}} \frac{1}{(r-s)^n - b_+} \frac{1}{(r-s)^n - b_-}\\ &\quad + iz^{2\tilde{n}} \frac{1}{r}(r-s)^{n-\tilde{n}} \frac{1}{(r-s)^n - b_+} \frac{1}{(r-s)^n - b_-}.\end{aligned}$$

It then follows that

$$\begin{aligned}K^c_{2,biv} &= -i^n z^{3\tilde{n}+1} \int_0^1 (1+2\tau)^{-(3\tilde{n}+1)/2} J_{3\tilde{n}+1}\left(z\sqrt{1+2\tau}\right) h_1(\tau)\, \mathrm{d}\tau\\ &\quad + iz^{3\tilde{n}+1} \int_0^1 (1+2\tau)^{-(\tilde{n}+1)/2} J_{\tilde{n}+1}\left(z\sqrt{1+2\tau}\right) h_1(\tau)\, \mathrm{d}\tau.\end{aligned}$$

These results immediately lead to the following theorem.

Theorem 5.4.1. *Let* $b_\pm := i^n e^{\pm i\theta} z^n$ *and*

$$h_1(t) := \int_0^t \zeta^{n-1} E_{n,n}\left(b_+ \zeta^n\right) (t-\zeta)^{n-1} E_{n,n}(b_-(t-\zeta)^n)\, \mathrm{d}\zeta.$$

Then for $m = 2\,(\lambda = 0)$, $1 + c = \frac{1}{n}$, $n \in \mathbb{N}_0 \backslash \{1\}$ *with* n *odd and* $n = 2\tilde{n}+1$, $\tilde{n} \in \mathbb{N}_0$, *the kernel of the deformed Fourier transform* $\mathcal{F}_D = e^{i\frac{\pi}{2}\left(\frac{1}{2}+\frac{m-1}{2(1+c)}\right)} e^{\frac{-i\pi}{4(1+c)^2}\left(\mathbf{D}^2-(1+c)^2\underline{x}^2\right)}$ *takes the form*

$$K^c_2 = K^c_{2,scal} + (\underline{x} \wedge \underline{y})\, K^c_{2,biv}$$

with

$$\begin{aligned}&K^c_{2,scal}\\ &= -z^{3\tilde{n}+2} \int_0^1 (1+2\tau)^{-\tilde{n}/2} J_{\tilde{n}}\left(z\sqrt{1+2\tau}\right) h_1(\tau)\, \mathrm{d}\tau\\ &\quad - iw\,(-1)^{\tilde{n}} z^{3\tilde{n}+2} \int_0^1 (1+2\tau)^{-(3\tilde{n}+1)/2} J_{3\tilde{n}+1}\left(z\sqrt{1+2\tau}\right) h_1(\tau)\, \mathrm{d}\tau\\ &\quad + iw\, z^{3\tilde{n}+2} \int_0^1 (1+2\tau)^{-(\tilde{n}+1)/2} J_{\tilde{n}+1}\left(z\sqrt{1+2\tau}\right) h_1(\tau)\, \mathrm{d}\tau\\ &\quad - (-1)^{\tilde{n}} z^{3\tilde{n}+2} \int_0^1 (1+2\tau)^{-(3\tilde{n}+2)/2} J_{3\tilde{n}+2}\left(z\sqrt{1+2\tau}\right) h_1(\tau)\, \mathrm{d}\tau.\end{aligned}$$

and

$$K^c_{2,biv} = -i^n\, z^{3\tilde{n}+1} \int_0^1 (1+2\tau)^{-(3\tilde{n}+1)/2}\, J_{3\tilde{n}+1}\left(z\sqrt{1+2\tau}\right)\, h_1(\tau)\,\mathrm{d}\tau$$
$$+\, i z^{3\tilde{n}+1} \int_0^1 (1+2\tau)^{-(\tilde{n}+1)/2}\, J_{\tilde{n}+1}\left(z\sqrt{1+2\tau}\right)\, h_1(\tau)\,\mathrm{d}\tau$$

where $w = \cos\theta = \frac{\langle x,y\rangle}{z}$, $z = |x||y|$.

5.4.2 The case of $m > 2$

In this subsection, we begin by considering the Laplace transform of the kernel given in (5.11-5.13) in Section 5.3.2. Subsequently, we give the explicit expressions of kernel in terms of the Prabhakar generalized Mittag-Leffler functions (2.17) in dimension $m > 2$.

We start with the expression of $\mathcal{L}\left(\frac{1}{2\lambda} z^{-\frac{\mu-2}{2}} A_\lambda\right)$ shown in (5.11). It can be spilt in 4 different equations that we calculate separately:

$$\begin{aligned}
\mathcal{L}\left(\frac{1}{2\lambda} z^{-\frac{\mu-2}{2}} A_\lambda\right) &= \frac{1}{2r}\left(\frac{r-s}{z^2}\right)^{n\lambda+\tilde{n}} \frac{1}{(1-2w\,u_R+u_R^2)^{\lambda+1}} \\
&\quad - \frac{1}{2r}\left(\frac{r-s}{z^2}\right)^{n\lambda+\tilde{n}} \frac{u_R^2}{(1-2w\,u_R+u_R^2)^{\lambda+1}} \\
&\quad + \left(\frac{i}{z}\right)^{n-1} \frac{1}{2r}\left(\frac{r-s}{z^2}\right)^{n\lambda-\tilde{n}} \frac{1}{(1-2w\,u_R+u_R^2)^{\lambda+1}} \\
&\quad - \left(\frac{i}{z}\right)^{n-1} \frac{1}{2r}\left(\frac{r-s}{z^2}\right)^{n\lambda-\tilde{n}} \frac{u_R^2}{(1-2w\,u_R+u_R^2)^{\lambda+1}} \\
&:= I_1' + I_2' + I_3' + I_4'.
\end{aligned}$$

As we have established in the previous subsection,

$$\frac{1}{(1-2w\,u_R+u_R^2)^{\lambda+1}} = \frac{(-1)^{n(\lambda+1)}\, z^{2n(\lambda+1)}}{((r-s)^n - b_+)^{\lambda+1}} \frac{1}{((r-s)^n - b_-)^{\lambda+1}}$$

where $b_\pm = e^{\pm i\theta} i^n z^n$, we subsequently obtain

$$I_1' = \frac{1}{2r} z^{n+1} (-1)^{n(\lambda+1)} \frac{(r-s)^{n\lambda+\tilde{n}}}{((r-s)^n - b_+)^{\lambda+1}} \frac{1}{((r-s)^n - b_-)^{\lambda+1}}$$

using (5.9). Similarly for I_2'

$$I_2' = \frac{1}{2r} z^{1-n} (-1)^{n\lambda+1} \frac{(r-s)^{n\lambda+2n+\tilde{n}}}{((r-s)^n - b_+)^{\lambda+1}} \frac{1}{((r-s)^n - b_-)^{\lambda+1}}$$

and also

$$I_3' = i^{n-1}\frac{1}{2r}z^{2n}(-1)^{n(\lambda+1)}\frac{(r-s)^{n\lambda-\tilde{n}}}{((r-s)^n-b_+)^{\lambda+1}}\frac{1}{((r-s)^n-b_-)^{\lambda+1}};$$
$$I_4' = i^{n-1}\frac{1}{2r}(-1)^{n\lambda+1}\frac{(r-s)^{n\lambda+2n-\tilde{n}}}{((r-s)^n-b_+)^{\lambda+1}}\frac{1}{((r-s)^n-b_-)^{\lambda+1}}.$$

Next we deduce the inverse Laplace transform by means of formulas (2.45) and (2.46). Using the convolution formula (2.54) again, we have

$$\begin{aligned} h_{\lambda+1}(t) &:= \mathcal{L}^{-1}\left(\frac{1}{(s^n-b_+)^{\lambda+1}}\frac{1}{(s^n-b_-)^{\lambda+1}}\right) \\ &= \int_0^t \zeta^{n(\lambda+1)-1}E^{\lambda+1}_{n,n(\lambda+1)}\left(b_+\zeta^n\right) \\ &\quad \times (t-\zeta)^{n(\lambda+1)-1}E^{\lambda+1}_{n,n(\lambda+1)}\left(b_-(t-\zeta)^n\right)\mathrm{d}\tau \end{aligned}$$

Now summing the inverse Laplace transform of $I_1'+I_2'+I_3'+I_4'$, we conclude the results as below:

$$\begin{aligned} &\mathcal{L}^{-1}\left(\frac{1}{2\lambda}z^{-\frac{\mu-2}{2}}A_\lambda\right) \\ &= c_1\, z^{n\lambda+3\tilde{n}+2}\left(\int_0^1(1+2\tau)^{-(n\lambda+\tilde{n})/2}J_{n\lambda+\tilde{n}}\left(z\sqrt{1+2\tau}\right)h_{\lambda+1}(\tau)\,\mathrm{d}\tau\right. \\ &\quad + \int_0^1(1+2\tau)^{-(n\lambda+2n+\tilde{n})/2}J_{n\lambda+2n+\tilde{n}}\left(z\sqrt{1+2\tau}\right)h_{\lambda+1}(\tau)\,\mathrm{d}\tau \\ &\quad + i^{n-1}\int_0^1(1+2\tau)^{-(n\lambda-\tilde{n})/2}J_{n\lambda-\tilde{n}}\left(z\sqrt{1+2\tau}\right)h_{\lambda+1}(\tau)\,\mathrm{d}\tau \\ &\quad \left. + i^{n-1}\int_0^1(1+2\tau)^{-(n\lambda+2n-\tilde{n})/2}J_{n\lambda+2n-\tilde{n}}\left(z\sqrt{1+2\tau}\right)h_{\lambda+1}(\tau)\mathrm{d}\tau\right), \end{aligned}$$

with $c_1 = \frac{1}{2}(-1)^{n(\lambda+1)}$.
The expression for C_λ in (5.13) yields

$$\begin{aligned} &\mathcal{L}\left(z^{-\frac{\mu}{2}}C_\lambda\right) \\ &= (-i)^n(-1)^{n(\lambda+1)}\frac{1}{r}\frac{(r-s)^{n\lambda+3\tilde{n}+1}}{((r-s)^n-b_+)^{\lambda+1}}\frac{1}{((r-s)^n-b_-)^{\lambda+1}} \\ &\quad + i(-1)^{n(\lambda+1)}z^{2\tilde{n}}\frac{1}{r}\frac{(r-s)^{n\lambda+\tilde{n}+1}}{((r-s)^n-b_+)^{\lambda+1}}\frac{1}{((r-s)^n-b_-)^{\lambda+1}}. \end{aligned}$$

It follows readily that

$$\begin{aligned}\mathcal{L}^{-1}\left(z^{-\frac{\mu}{2}}C_\lambda\right) = c_3\, z^{n\lambda+3\tilde{n}+1}\Big(& i\int_0^1 (1+2\tau)^{-(n\lambda+\tilde{n}+1)/2} \\ &\times J_{n\lambda+\tilde{n}+1}\left(z\sqrt{1+2\tau}\right) h_{\lambda+1}(\tau)\,\mathrm{d}\tau \\ &+ (-i)^n \int_0^1 (1+2\tau)^{-(n\lambda+3\tilde{n}+1)/2} \\ &\times J_{n\lambda+3\tilde{n}+1}\left(z\sqrt{1+2\tau}\right) h_{\lambda+1}(\tau)\,\mathrm{d}\tau\Big)\end{aligned}$$

where $c_3 = (-1)^{n(\lambda+1)}$. Similarly, we have by (2.54),

$$\begin{aligned}h_\lambda(t) &:= \mathcal{L}^{-1}\left(\frac{1}{(s^n-b_+)^\lambda}\frac{1}{(s^n-b_-)^\lambda}\right) \\ &= \int_0^t \zeta^{n\lambda-1}E^\lambda_{n,n\lambda}\left(b_+\zeta^n\right)(t-\zeta)^{n\lambda-1}E^\lambda_{n,n\lambda}(b_-(t-\zeta)^n)\,\mathrm{d}\zeta.\end{aligned}$$

This allows us to compute the inverse Laplace transform of B_λ obtained in (5.12)

$$\begin{aligned}&\mathcal{L}^{-1}\left(\frac{1}{2}z^{-\frac{\mu-2}{2}}B_\lambda\right) \\ &= \frac{1}{2}(-1)^{n\lambda}z^{-2\tilde{n}}\,\mathcal{L}^{-1}\left(\frac{1}{r}\frac{(r-s)^{n\lambda+\tilde{n}}}{((r-s)^n-b_+)^\lambda}\frac{1}{((r-s)^n-b_-)^\lambda}\right) \\ &\quad -\frac{1}{2}i^{n-1}(-1)^{n\lambda}\,\mathcal{L}^{-1}\left(\frac{1}{r}\frac{(r-s)^{n\lambda-\tilde{n}}}{((r-s)^n-b_+)^\lambda}\frac{1}{((r-s)^n-b_-)^\lambda}\right) \\ &= c_2\, z^{n\lambda-\tilde{n}}\int_0^1 (1+2\tau)^{-(n\lambda+\tilde{n})/2}J_{n\lambda+\tilde{n}}\left(z\sqrt{1+2\tau}\right)h_\lambda(\tau)\,\mathrm{d}\tau \\ &\quad - i^{n-1}c_2\, z^{n\lambda-\tilde{n}}\int_0^1 (1+2\tau)^{-(n\lambda-\tilde{n})/2}J_{n\lambda-\tilde{n}}\left(z\sqrt{1+2\tau}\right)h_\lambda(\tau)\,\mathrm{d}\tau,\end{aligned}$$

with $c_2 = \frac{1}{2}(-1)^{n\lambda}$.
Collecting all results then gives the desired theorem.

Theorem 5.4.2. *Let* $b_\pm := e^{\pm i\theta}i^n z^n$ *and*

$$h_\lambda(t) := \int_0^t \zeta^{n\lambda-1}E^\lambda_{n,n\lambda}\left(b_+\zeta^n\right)(t-\zeta)^{n\lambda-1}E^\lambda_{n,n\lambda}(b_-(t-\zeta)^n)\,\mathrm{d}\zeta.$$

Then for all the dimensions $m > 2\,(\lambda > 0)$ and $1+c = \frac{1}{n}$, $n \in \mathbb{N}_0 \setminus \{1\}$ odd, $n = 2\tilde{n}+1$, $\tilde{n} \in \mathbb{N}_0$, the kernel of the radially deformed Fourier transform $\mathcal{F}_D = e^{i\frac{\pi}{2}\left(\frac{1}{2}+\frac{m-1}{2(1+c)}\right)} e^{\frac{-i\pi}{4(1+c)^2}\left(\mathbf{D}^2-(1+c)^2\underline{x}^2\right)}$ is given by

$$K_m^c = \frac{1}{2\lambda} z^{-\frac{\mu-2}{2}} A_\lambda + \frac{1}{2} z^{-\frac{\mu-2}{2}} B_\lambda - z^{-\frac{\mu}{2}} \left(\underline{x} \wedge \underline{y}\right)\ C_\lambda$$

with

$$\begin{aligned}
&\frac{1}{2\lambda} z^{-\frac{\mu-2}{2}} A_\lambda \\
&= c_1\, z^{n\lambda+3\tilde{n}+2} \Bigg(\int_0^1 (1+2\tau)^{-(n\lambda+\tilde{n})/2} J_{n\lambda+\tilde{n}}\left(z\sqrt{1+2\tau}\right) h_{\lambda+1}(\tau)\,\mathrm{d}\tau \\
&\quad + \int_0^1 (1+2\tau)^{-(n\lambda+2n+\tilde{n})/2} J_{n\lambda+2n+\tilde{n}}\left(z\sqrt{1+2\tau}\right) h_{\lambda+1}(\tau)\,\mathrm{d}\tau \\
&\quad + (-1)^{\tilde{n}} \Bigg(\int_0^1 (1+2\tau)^{-(n\lambda-\tilde{n})/2} J_{n\lambda-\tilde{n}}\left(z\sqrt{1+2\tau}\right) h_{\lambda+1}(\tau)\,\mathrm{d}\tau \\
&\quad + \int_0^1 (1+2\tau)^{-(n\lambda+2n-\tilde{n})/2} J_{n\lambda+2n-\tilde{n}}\left(z\sqrt{1+2\tau}\right) h_{\lambda+1}(\tau)\,\mathrm{d}\tau \Bigg); \\
&\frac{1}{2} z^{-\frac{\mu-2}{2}} B_\lambda \\
&= c_2\, z^{n\lambda-\tilde{n}} \Bigg(\int_0^1 (1+2\tau)^{-(n\lambda+\tilde{n})/2} J_{n\lambda+\tilde{n}}\left(z\sqrt{1+2\tau}\right) h_\lambda(\tau)\,\mathrm{d}\tau \\
&\quad - (-1)^{\tilde{n}} \int_0^1 (1+2\tau)^{-(n\lambda-\tilde{n})/2} J_{n\lambda-\tilde{n}}\left(z\sqrt{1+2\tau}\right) h_\lambda(\tau)\,\mathrm{d}\tau \Bigg); \\
&z^{-\frac{\mu}{2}} C_\lambda \\
&= c_3 z^{n\lambda+3\tilde{n}+1} \Bigg(i \int_0^1 (1+2\tau)^{-(n\lambda+\tilde{n}+1)/2} J_{n\lambda+\tilde{n}+1}\left(z\sqrt{1+2\tau}\right) h_{\lambda+1}(\tau)\,\mathrm{d}\tau \\
&\quad + (-i)^n \int_0^1 (1+2\tau)^{-(n\lambda+3\tilde{n}+1)/2} J_{n\lambda+3\tilde{n}+1}\left(z\sqrt{1+2\tau}\right) h_{\lambda+1}(\tau)\,\mathrm{d}\tau \Bigg)
\end{aligned}$$

where $w = \cos\theta = \frac{\langle x,y\rangle}{z}$, $z = |x||y|$, $c_1 = \frac{1}{2}(-1)^{n(\lambda+1)}$, $c_2 = \frac{1}{2}(-1)^{n\lambda}$ and $c_3 = (-1)^{n(\lambda+1)}$.

It is now possible to present this result in a more compact form. Indeed, combining the equations in (5.11) and (5.12), one can see that the scalar part of the kernel $K^c_{m,scal}$, $m > 2$ reduces to

$$\mathcal{L}\left(K^c_{m,scal}\right)$$

$$
\begin{aligned}
&= \mathcal{L}\left(\frac{1}{2\lambda} z^{-\frac{\mu-2}{2}} A_\lambda\right) + \mathcal{L}\left(\frac{1}{2} z^{-\frac{\mu-2}{2}} B_\lambda\right)\\
&= (-1)^{n(\lambda+1)}\, z^{2\tilde{n}+2}\, \frac{1}{r} \frac{(r-s)^{n\lambda+\tilde{n}}}{((r-s)^n - b_+)^{\lambda+1}} \frac{1}{((r-s)^n - b_-)^{\lambda+1}}\\
&\quad + i\, w\, (-1)^{n(\lambda+1)}\, (-1)^{\tilde{n}}\, z\, \frac{1}{r} \frac{(r-s)^{n\lambda+n+\tilde{n}}}{((r-s)^n - b_+)^{\lambda+1}} \frac{1}{((r-s)^n - b_-)^{\lambda+1}}\\
&\quad - i\, w\, (-1)^{n(\lambda+1)}\, z^n\, \frac{1}{r} \frac{(r-s)^{n\lambda+n-\tilde{n}}}{((r-s)^n - b_+)^{\lambda+1}} \frac{1}{((r-s)^n - b_-)^{\lambda+1}}\\
&\quad + i^{n-1}\, (-1)^{n(\lambda+1)}\, \frac{1}{r} \frac{(r-s)^{n\lambda+2n-\tilde{n}}}{((r-s)^n - b_+)^{\lambda+1}} \frac{1}{((r-s)^n - b_-)^{\lambda+1}}.
\end{aligned}
$$

Adding the bivector part $K^c_{m,biv} = -z^{-\frac{\mu}{2}} C_\lambda$, $m > 2$ in the above theorem and using (2.46), we thus conclude the results:

Theorem 5.4.3. *Let $b_\pm := e^{\pm i\theta} i^n z^n$ and*

$$
\begin{aligned}
h_{\lambda+1}(t) := \int_0^t &\zeta^{n(\lambda+1)-1} E^{\lambda+1}_{n,n(\lambda+1)}\left(b_+ \zeta^n\right)\\
&\times (t-\zeta)^{n(\lambda+1)-1} E^{\lambda+1}_{n,n(\lambda+1)}(b_-(t-\zeta)^n) d\zeta.
\end{aligned}
$$

Then for the dimension $m > 2\,(\lambda > 0)$ and $1 + c = \frac{1}{n}$, $n \in \mathbb{N}_0 \setminus \{1\}$ odd, $n = 2\tilde{n} + 1$, $\tilde{n} \in \mathbb{N}_0$, the kernel of the radially deformed Fourier transform $\mathcal{F}_D = e^{i\frac{\pi}{2}\left(\frac{1}{2} + \frac{m-1}{2(1+c)}\right)} e^{\frac{-i\pi}{4(1+c)^2}\left(\mathbf{D}^2 - (1+c)^2 \underline{x}^2\right)}$ takes the form

$$
K^c_m = K^c_{m,scal} + (\underline{x} \wedge \underline{y})\, K^c_{m,biv}
$$

with

$$
\begin{aligned}
&K^c_{m,scal}\\
&= c_3\, z^{n\lambda+3\tilde{n}+2} \Bigg(\int_0^1 (1+2\tau)^{-(n\lambda+\tilde{n})/2} J_{n\lambda+\tilde{n}}\left(z\sqrt{1+2\tau}\right) h_{\lambda+1}(\tau)\, \mathrm{d}\tau\\
&\quad - i\, w \int_0^1 (1+2\tau)^{-(n\lambda+\tilde{n}+1)/2} J_{n\lambda+\tilde{n}+1}\left(z\sqrt{1+2\tau}\right) h_{\lambda+1}(\tau)\, \mathrm{d}\tau\\
&\quad + (-1)^{\tilde{n}} \int_0^1 (1+2\tau)^{-(n\lambda+3\tilde{n}+2)/2} J_{n\lambda+3\tilde{n}+2}\left(z\sqrt{1+2\tau}\right) h_{\lambda+1}(\tau) \mathrm{d}\tau \Bigg)\\
&\quad + (-1)^{\tilde{n}} \int_0^1 (1+2\tau)^{-(n\lambda+n+\tilde{n})/2} J_{n\lambda+n+\tilde{n}}\left(z\sqrt{1+2\tau}\right) h_{\lambda+1}(\tau)\, \mathrm{d}\tau \Bigg).
\end{aligned}
$$

and

$$
K^c_{m,biv}
$$

$$= -i\, c_3\, z^{n\lambda+3\tilde{n}+1} \Bigg(\int_0^1 (1+2\tau)^{-(n\lambda+\tilde{n}+1)/2} J_{n\lambda+\tilde{n}+1}\left(z\sqrt{1+2\tau}\right) h_{\lambda+1}(\tau)\, \mathrm{d}\tau$$

$$- (-1)^{\tilde{n}} \int_0^1 (1+2\tau)^{-(n\lambda+3\tilde{n}+1)/2} J_{n\lambda+3\tilde{n}+1}\left(z\sqrt{1+2\tau}\right) h_{\lambda+1}(\tau)\, \mathrm{d}\tau \Bigg)$$

where $w = \cos\theta = \frac{\langle x, y\rangle}{z}$, $z = |x||y|$ *and* $c_3 = (-1)^{n(\lambda+1)}$.

Remark 5.4.4. *If we take the limit $\lambda = 0$, the results for dimension 2 in Theorem 5.4.1 can be obtained again.*

By Theorem 5.4.1, Theorem 5.4.3 and Remark 5.4.4, the explicit expressions of the kernel of the radially deformed Fourier transform $\mathcal{F}_D$ can be considered together as K_m^c, $m \geq 2$.

Chapter 6

Applications on the Clifford-Fourier kernels

6.1 Introduction

The Clifford-Fourier transform (CFT) (see [5–7]) is a generalization of the Fourier transform in the framework of Clifford analysis. The explicit formulas of the Clifford-Fourier kernel on $\mathbb{R}^m$ were considered when m is even and odd in [24]. So far the explicit descriptions of the kernel are found in even dimensions. For the case $m = 3$, the kernel can be expressed as a single integral of a combination of Bessel functions. For higher odd dimensions, a iterative procedure for constructing the kernels can be used to deduce the kernels. In [15], the kernel for the m-dimensional Clifford-Fourier transform was explicitly given for both even and odd dimension when $m \geq 3$. More recently, a new fractional version of the Clifford-Fourier transform was developed. In [25], a series expansion of kernel in all dimensions was initially determined. When restricting to the case of dimension 2, an explicit expression for the kernel was obtained and further properties of the fractional CFT were derived subsequently. A generalized version of the fractional CFT with two numerical parameters α and β was introduced in [26]. We refer the readers to [26] for details on the series expansion of its kernel in even dimension.

In this chapter we apply the modified Laplace transform method introduced in Chapter 5 to the kernels of the Clifford-Fourier transform and its fractional versions. By this method, we re-obtain the explicit expressions of the kernels in dimension 2, shown in Sections 6.2.1 and

6.3.1 respectively. In dimension $m \geq 3$, some new integral expressions for the kernels in terms of the Bessel function and the Prabhakar function are derived. The expressions for the Clifford-Fourier kernel can be found in Section 6.2.2 and the corresponding results for the kernel of the fractional CFT are shown in Section 6.3.2.

6.2 The Clifford-Fourier kernel

In this section, we derive explicit expressions for the kernel of the Clifford-Fourier transform introduced in Section 2.3.1. We first give a necessary decomposition of the kernel in terms of Bessel functions and Gegenbauer polynomials.

The following kernel of the Clifford-Fourier transform (see [24], Section 3) is given by

$$K_{-}(x,\, y) = A_\lambda + B_\lambda + (\underline{x} \wedge \underline{y})\, C_\lambda \tag{6.1}$$

with

$$A_\lambda(w,\, z) = 2^{\lambda-1}\Gamma\,(\lambda+1)\sum_{k=0}^{\infty}(i^{2\lambda+2} + (-1)^k)\, z^{-\lambda} J_{k+\lambda}(z)\, C_k^\lambda(w),$$
$$B_\lambda(w,\, z) = -2^{\lambda-1}\Gamma\,(\lambda)\sum_{k=0}^{\infty}(k+\lambda)(i^{2\lambda+2} - (-1)^k)\, z^{-\lambda} J_{k+\lambda}(z)\, C_k^\lambda(w),$$
$$C_\lambda(w,\, z) = -2^{\lambda}\Gamma\,(\lambda+1)\sum_{k=1}^{\infty}(i^{2\lambda+2} + (-1)^k)\, z^{-\lambda-1} J_{k+\lambda}(z)\, C_{k-1}^{\lambda+1}(w),$$

where $z = |x||y|$, $w = \langle \zeta,\, \tau\rangle = \langle \frac{x}{|x|}, \frac{y}{|y|}\rangle = \cos\theta$ and $\lambda = \frac{m-2}{2}$.

The following explicit representation of the Clifford-Fourier kernel was obtained when $m = 2$.

Theorem 6.2.1. *[6, 24] The kernel of the Clifford-Fourier transform is given by*

$$K_{-}(x,\, y) = e^{i\frac{\pi}{2}\Gamma_{\underline{y}}} e^{-i\langle \underline{x},\, \underline{y}\rangle} = \cos\eta + (\underline{x} \wedge \underline{y})\,\frac{\sin\eta}{\eta}$$

with $\eta = |\underline{x} \wedge \underline{y}| = \sqrt{|x|^2\,|y|^2 - \langle x,\, y\rangle^2}$.

6.2.1 Clifford-Fourier kernel in dimension 2

When $m = 2\,(\lambda = 0)$, the Clifford-Fourier kernel takes the form:

$$\lim_{\lambda\to 0} K_{-}(x,\, y) = \lim_{\lambda\to 0} B_\lambda + (\underline{x} \wedge \underline{y})\, \lim_{\lambda\to 0} C_\lambda \tag{6.2}$$

with

$$\begin{aligned}
\lim_{\lambda\to 0} B_\lambda &= J_0(z) + \sum_{k=1}^{\infty} J_k(z) \cos{(k\,\theta)} \\
&\quad + \sum_{k=1}^{\infty} (-1)^k\, J_k(z) \cos{(k\,\theta)}; \\
\lim_{\lambda\to 0} C_\lambda &= \frac{1}{\sin\theta}\, z^{-1} \sum_{k=1}^{\infty} J_k(z) \cos{(k\,\theta)} \\
&\quad - \frac{1}{\sin\theta}\, z^{-1} \sum_{k=1}^{\infty} (-1)^k\, J_k(z) \cos{(k\,\theta)},
\end{aligned}$$

where we have used the relations (2.9) and (2.10).
Similar with the constructions on the kernel of the radially deformed Fourier transform in Chapter 5, we introduce an auxiliary variable t in the Bessel functions. For $\mathrm{Re}\, s$ big enough, we take the Laplace transform with respect to t term by term using formula (2.44) and Theorem 5.3.1,

$$\begin{aligned}
\mathcal{L}\left(\lim_{\lambda\to 0} B_\lambda\right) &= \frac{1}{r} \sum_{k=0}^{\infty} \left(\frac{z}{s+r}\right)^k \cos{(k\,\theta)} \\
&\quad + \frac{1}{r} \sum_{k=1}^{\infty} (-1)^k \left(\frac{z}{s+r}\right)^k \cos{(k\,\theta)}; \\
\mathcal{L}\left(\lim_{\lambda\to 0} C_\lambda\right) &= \frac{1}{\sin\theta}\, z^{-1}\, \frac{1}{r} \sum_{k=1}^{\infty} \left(\frac{z}{s+r}\right)^k \sin{(k\,\theta)} \\
&\quad - \frac{1}{\sin\theta}\, z^{-1}\, \frac{1}{r} \sum_{k=1}^{\infty} (-1)^k \left(\frac{z}{s+r}\right)^k \sin{(k\,\theta)}.
\end{aligned}$$

where $r = \sqrt{s^2 + z^2}$ and $z = |x||y|$. Using the well-known relation (5.5)

$$\sum_{k=0}^{\infty} u^k \cos kx = \frac{1 - u\cos x}{1 - 2u\cos x + u^2}, \quad |u| < 1,$$

the Laplace transform of B_λ in dimension 2 becomes

$$\begin{aligned}
\mathcal{L}\left(\lim_{\lambda\to 0} B_\lambda\right) &= \frac{1}{r}\,\frac{1-w\left(\frac{z}{s+r}\right)}{1-2w\left(\frac{z}{s+r}\right)+\left(\frac{z}{s+r}\right)^2} - \frac{1}{r} \\
&\quad + \frac{1}{r}\,\frac{1+w\left(\frac{z}{s+r}\right)}{1+2w\left(\frac{z}{s+r}\right)+\left(\frac{z}{s+r}\right)^2} \\
&= \frac{1}{r}\,\frac{s+r-w\,z}{2r-2w\,z} - \frac{1}{r} + \frac{1}{r}\,\frac{s+r+w\,z}{2r+2w\,z} \\
&= \frac{s}{s^2+z^2\,\sin^2\theta}
\end{aligned}$$

by the relation $(s+r)(r-s) = r^2 - s^2 = z^2$.
Transforming back by using formula (2.39), we arrive at

$$\mathcal{L}^{-1}\left(\lim_{\lambda\to 0} B_\lambda\right) = \cos\left(t\,z\,\sin\theta\right). \tag{6.3}$$

By the relation (5.7)

$$\sum_{k=1}^{\infty} u^k \sin kx = \frac{u\sin x}{1-2u\cos x+u^2}, \quad |u|<1,$$

the bivector part of the kernel when $m=2$ can be simplified as

$$\begin{aligned}
\mathcal{L}\left(\lim_{\lambda\to 0} C_\lambda\right) &= \frac{1}{\sin\theta}\,z^{-1}\,\frac{1}{r}\,\frac{\sin\theta\left(\frac{z}{s+r}\right)}{1-2w\left(\frac{z}{s+r}\right)+\left(\frac{z}{s+r}\right)^2} \\
&\quad - \frac{1}{\sin\theta}\,z^{-1}\,\frac{1}{r}\,\frac{\sin\theta\left(-\frac{z}{s+r}\right)}{1+2w\left(\frac{z}{s+r}\right)+\left(\frac{z}{s+r}\right)^2} \\
&= \frac{1}{r}\,\frac{1}{s+r}\,\frac{1}{1-2w\left(\frac{z}{s+r}\right)+\left(\frac{z}{s+r}\right)^2} \\
&\quad + \frac{1}{r}\,\frac{1}{s+r}\,\frac{1}{1+2w\left(\frac{z}{s+r}\right)+\left(\frac{z}{s+r}\right)^2} \\
&= \frac{1}{s^2+z^2\,\sin^2\theta}.
\end{aligned}$$

Using the inverse formula of (2.40), we obtain

$$\mathcal{L}^{-1}\left(\lim_{\lambda\to 0} C_\lambda\right) = \frac{1}{z\,\sin\theta}\,\sin\left(t\,z\,\sin\theta\right). \tag{6.4}$$

Summing (6.3) and (6.4) and setting $t=1$, the Clifford-Fourier kernel in dimension 2 is given by

$$\begin{aligned} K_{-}(x,\,y) &= \cos\left(z\,\sin\theta\right) + \left(\underline{x}\wedge\underline{y}\right)\frac{\sin\left(z\,\sin\theta\right)}{z\,\sin\theta} \\ &= \cos\eta + \left(\underline{x}\wedge\underline{y}\right)\frac{\sin\eta}{\eta} \end{aligned}$$

where $\eta = |\underline{x}\wedge\underline{y}| = \sqrt{|x|^2\,|y|^2 - \langle x,\,y\rangle^2}$. This form of the kernel is consistent with the result in Theorem 6.2.1 (See [24], Theorem 4.1).

6.2.2 The kernel in $m \geq 3$

In this subsection, we give two new integral expressions of the Clifford-Fourier kernel in terms of the Prabhakar function and the Bessel function in dimension $m \geq 3$.
Introducing a variable t in Bessel functions in the kernel (6.1) and executing the substitution $k \mapsto k+1$ in C_λ, we obtain

$$\begin{aligned} A_\lambda &= 2^{\lambda-1}\Gamma\left(\lambda+1\right)\sum_{k=0}^{\infty}(i^{2\lambda+2}+(-1)^k)\,z^{-\lambda}J_{k+\lambda}(t\,z)\,C_k^\lambda(w), \\ B_\lambda &= -2^{\lambda-1}\Gamma\left(\lambda+1\right)\sum_{k=0}^{\infty}\frac{k+\lambda}{\lambda}(i^{2\lambda+2}-(-1)^k)\,z^{-\lambda}J_{k+\lambda}(t\,z)\,C_k^\lambda(w), \\ C_\lambda &= -2^{\lambda}\,\Gamma\left(\lambda+1\right)\sum_{k=0}^{\infty}(i^{2\lambda+2}+(-1)^{k+1})\,z^{-\lambda-1}J_{k+1+\lambda}(t\,z)\,C_k^{\lambda+1}(w). \end{aligned}$$

Using (2.44) and Theorem 5.3.1 again, we get the Laplace transform for B_λ, $\lambda > 0$,

$$\begin{aligned} &\mathcal{L}\left(B_\lambda\right) \\ &= -2^{\lambda-1}\,i^{2\lambda+2}\,\Gamma(\lambda+1)\,\frac{1}{r}\left(\frac{1}{s+r}\right)^{\lambda}\sum_{k=0}^{\infty}\frac{k+\lambda}{\lambda}\left(\frac{z}{s+r}\right)^{k}C_k^\lambda(w) \\ &\quad + 2^{\lambda-1}\,\Gamma(\lambda+1)\,\frac{1}{r}\left(\frac{1}{s+r}\right)^{\lambda}\sum_{k=0}^{\infty}\frac{k+\lambda}{\lambda}\,(-1)^k\left(\frac{z}{s+r}\right)^{k}C_k^\lambda(w) \end{aligned}$$

where $r=\sqrt{s^2+z^2}$ and $z=|x||y|$.
To simplify the Laplace transform of B_λ, we need the expansion of the Poisson kernel in terms of Gegenbauer polynomials. When $\lambda > 0$,

we have, by Theorem 2.1.1 and Remark 2.1.2

$$\begin{aligned}
&\mathcal{L}\left(B_\lambda\right)\\
&= -2^{\lambda-1}\, i^{2\lambda+2}\, \Gamma(\lambda+1)\frac{1}{r}\left(\frac{1}{s+r}\right)^{\lambda} \frac{1-\left(\frac{z}{s+r}\right)^2}{\left(1-2w\left(\frac{z}{s+r}\right)+\left(\frac{z}{s+r}\right)^2\right)^{\lambda+1}}\\
&\quad + 2^{\lambda-1}\, \Gamma(\lambda+1)\frac{1}{r}\left(\frac{1}{s+r}\right)^{\lambda} \frac{1-\left(\frac{z}{s+r}\right)^2}{\left(1+2w\left(\frac{z}{s+r}\right)+\left(\frac{z}{s+r}\right)^2\right)^{\lambda+1}}.
\end{aligned} \tag{6.5}$$

For A_λ and C_λ, we utilize the generating function (2.15) of the Gegenbauer polynomials. When $\lambda > 0$, we have that the Laplace transform of A_λ is given by

$$\begin{aligned}
&\mathcal{L}\left(A_\lambda\right)\\
&= 2^{\lambda-1}\, i^{2\lambda+2}\, \Gamma(\lambda+1)\frac{1}{r}\left(\frac{1}{s+r}\right)^{\lambda} \frac{1}{\left(1-2w\left(\frac{z}{s+r}\right)+\left(\frac{z}{s+r}\right)^2\right)^{\lambda}}\\
&\quad + 2^{\lambda-1}\, \Gamma(\lambda+1)\frac{1}{r}\left(\frac{1}{s+r}\right)^{\lambda} \frac{1}{\left(1+2w\left(\frac{z}{s+r}\right)+\left(\frac{z}{s+r}\right)^2\right)^{\lambda}}.
\end{aligned} \tag{6.6}$$

The bivector part follows as

$$\begin{aligned}
&\mathcal{L}\left(C_\lambda\right)\\
&= -2^{\lambda}\, i^{2\lambda+2}\, \Gamma(\lambda+1)\frac{1}{r}\left(\frac{1}{s+r}\right)^{\lambda+1} \frac{1}{\left(1-2w\left(\frac{z}{s+r}\right)+\left(\frac{z}{s+r}\right)^2\right)^{\lambda+1}}\\
&\quad + 2^{\lambda}\, \Gamma(\lambda+1)\frac{1}{r}\left(\frac{1}{s+r}\right)^{\lambda+1} \frac{1}{\left(1+2w\left(\frac{z}{s+r}\right)+\left(\frac{z}{s+r}\right)^2\right)^{\lambda+1}}.
\end{aligned}$$

Next, we consider A_λ and B_λ together as the scalar part of the kernel: $K^-_{scal}(x,y) := A_\lambda + B_\lambda$ in the sequel. Adding (6.5) and (6.6) yields

$$\begin{aligned}
&\mathcal{L}\left(K^-_{scal}(x,y)\right)\\
&= -2^\lambda\, i^{2\lambda+2}\, w\, z\, \Gamma(\lambda+1)\frac{1}{r}\left(\frac{1}{s+r}\right)^{\lambda+1}\frac{1}{\left(1-2w\left(\frac{z}{s+r}\right)+\left(\frac{z}{s+r}\right)^2\right)^{\lambda+1}}\\
&\quad + 2^\lambda\, i^{2\lambda+2}\, z^2\, \Gamma(\lambda+1)\frac{1}{r}\left(\frac{1}{s+r}\right)^{\lambda+2}\frac{1}{\left(1-2w\left(\frac{z}{s+r}\right)+\left(\frac{z}{s+r}\right)^2\right)^{\lambda+1}}\\
&\quad + 2^\lambda\, \Gamma(\lambda+1)\frac{1}{r}\left(\frac{1}{s+r}\right)^{\lambda}\frac{1}{\left(1+2w\left(\frac{z}{s+r}\right)+\left(\frac{z}{s+r}\right)^2\right)^{\lambda+1}}\\
&\quad + 2^\lambda\, w\, z\, \Gamma(\lambda+1)\frac{1}{r}\left(\frac{1}{s+r}\right)^{\lambda+1}\frac{1}{\left(1+2w\left(\frac{z}{s+r}\right)+\left(\frac{z}{s+r}\right)^2\right)^{\lambda+1}},
\end{aligned} \tag{6.7}$$

and the bivector part satisfies

$$\mathcal{L}\left(K^-_{biv}(x,\, y)\right) = \mathcal{L}\left(C_\lambda\right). \tag{6.8}$$

6.2.2.1 Inversion using Bessel functions

We now simplify the Laplace transform of the kernel (6.7) and (6.8) as follows

$$\begin{aligned}
\mathcal{L}\left(K^-_{scal}(x,y)\right) &= -2^{-1}\, i^{2\lambda+2}\, w\, z\, \Gamma(\lambda+1)\,\frac{1}{r}\frac{1}{(r-wz)^{\lambda+1}}\\
&\quad + 2^{-1}\, i^{2\lambda+2}\, z^2\, \Gamma(\lambda+1)\,\frac{1}{r}\,\frac{1}{s+r}\frac{1}{(r-wz)^{\lambda+1}}\\
&\quad + 2^{-1}\,\Gamma(\lambda+1)\,\frac{s+r}{r}\frac{1}{(r+wz)^{\lambda+1}}\\
&\quad + 2^{-1}\, w\, z\, \Gamma(\lambda+1)\,\frac{1}{r}\,\frac{1}{(r+wz)^{\lambda+1}}, \quad \lambda>0;\\
\mathcal{L}\left(K^-_{biv}(x,\, y)\right) &= -2^{-1}\, i^{2\lambda+2}\, \Gamma(\lambda+1)\,\frac{1}{r}\,\frac{1}{(r-wz)^{\lambda+1}}\\
&\quad + 2^{-1}\,\Gamma(\lambda+1)\frac{1}{r}\,\frac{1}{(r+wz)^{\lambda+1}}, \quad \lambda>0.
\end{aligned}$$

The results are concluded in the following theorem.

Theorem 6.2.2. *For $\underline{x}, \underline{y} \in \mathbb{R}^m$, the kernel of the Clifford-Fourier transform for dimension $m \geq 3$ in the Laplace domain takes the form*

$$\begin{aligned}
\mathcal{L}\left(K_-(x,\, y)\right) = & -i^{2\lambda+2}\, w\, z\, \Gamma(\lambda+1)\, \frac{1}{2r} \frac{1}{(r - wz)^{\lambda+1}} \\
& + i^{2\lambda+2}\, z^2\, \Gamma(\lambda+1)\, \frac{1}{2r}\, \frac{1}{s+r}\, \frac{1}{(r - wz)^{\lambda+1}} \\
& + \Gamma(\lambda+1)\, s\, \frac{1}{2r} \frac{1}{(r + wz)^{\lambda+1}} + \Gamma(\lambda+1)\, \frac{1}{2} \frac{1}{(r + wz)^{\lambda+1}} \\
& + w\, z\, \Gamma(\lambda+1)\, \frac{1}{2r}\, \frac{1}{(r + wz)^{\lambda}} \\
& - (\underline{x} \wedge \underline{y})\, i^{2\lambda+2}\ \Gamma(\lambda+1)\, \frac{1}{2r} \frac{1}{(r - wz)^{\lambda+1}} \\
& + (\underline{x} \wedge \underline{y})\, \Gamma(\lambda+1) \frac{1}{2r}\, \frac{1}{(r + wz)^{\lambda+1}}
\end{aligned}$$

where $z = |x||y|$, $w = \langle \xi,\, \eta \rangle = \langle \frac{x}{|x|}, \frac{y}{|y|} \rangle = \cos\theta$, $\lambda = \frac{m-2}{2}$ *and* $r = \sqrt{s^2 + z^2}$.

The inverse Laplace transform of kernel $K_-(x,\, y)$, $m > 2$ can be split in 7 different integrals that we calculate separately.
We first derive the inverse transform

$$\mathcal{L}^{-1}\left(\frac{1}{(s \pm \omega\, z)^{\lambda+1}}\right) = \frac{t^{\lambda}}{\Gamma(\lambda+1)}\, e^{\mp \omega\, z\, t}.$$

by formula (2.41). Then applying (2.48), the first term arrives at

$$\begin{aligned}
I_1'' &= -2^{-1}\, i^{2\lambda+2}\, w\, z\, \Gamma(\lambda+1)\, \mathcal{L}^{-1}\left(\frac{1}{r} \frac{1}{(r - wz)^{\lambda+1}}\right) \\
&= -2^{-1}\, i^{2\lambda+2}\, w\, z \int_0^t J_0\left(z\sqrt{t^2 - u^2}\right)\, u^{\lambda}\, e^{\omega\, z\, u}\, \mathrm{d}u;
\end{aligned}$$

as well as the fifth term

$$I_5'' = 2^{-1}\, w\, z \int_0^t J_0\left(z\sqrt{t^2 - u^2}\right) u^{\lambda} e^{-\omega\, z\, u}\, \mathrm{d}u.$$

Similarly, we find by (2.47)

$$I_2'' = 2^{-1}\, i^{2\lambda+2}\, z \int_0^t \left(\frac{t-u}{t+u}\right)^{1/2}\ J_1\left(z\sqrt{t^2 - u^2}\right)\, u^{\lambda}\, e^{\omega\, z\, u}\, \mathrm{d}u.$$

Using (2.50), we subsequently have

$$I_3'' = 2^{-1}\left(t^{\lambda}\, e^{-w\,z\,t} - z\,t\int_0^t (t^2-u^2)^{-1/2}\, J_1\left(z\sqrt{t^2-u^2}\right)\, u^{\lambda}\, e^{-\omega\,z\,u}\,\mathrm{d}u\right).$$

By (2.49), we get

$$I_4'' = 2^{-1}\left(t^{\lambda}\, e^{-w\,z\,t} - z\int_0^t (t^2-u^2)^{\lambda/2}\, e^{-w\,z\,(t^2-u^2)^{1/2}}\, J_1(z\,u)\,\mathrm{d}u\right).$$

The bivector part follows

$$\begin{aligned} I_6'' &= -2^{-1}\, i^{2\lambda+2}\int_0^t J_0\left(z\sqrt{t^2-u^2}\right)\, u^{\lambda}\, e^{\omega\,z\,u}\,\mathrm{d}u; \\ I_7'' &= 2^{-1}\int_0^t J_0\left(z\sqrt{t^2-u^2}\right)\, u^{\lambda}\, e^{-\omega\,z\,u}\,\mathrm{d}u. \end{aligned}$$

Now summing $I_1''+I_2''+I_3''+I_4''+I_5''+I_6''+I_7''$ and setting $t=1$ leads to the following results.

Theorem 6.2.3. *For $x,\, y\in\mathbb{R}^m, m\geq 3$, the kernel of the Clifford-Fourier transform is given by*

$$\begin{aligned} K_-(x,\,y) = &-\frac{1}{2}\, i^{2\lambda+2}\, w\, z\int_0^1 J_0\left(z\sqrt{1-u^2}\right) u^{\lambda}\, e^{w\,z\,u}\,\mathrm{d}u \\ &+\frac{1}{2} i^{2\lambda+2}\, z\int_0^1\left(\frac{1-u}{1+u}\right)^{1/2} J_1\left(z\sqrt{1-u^2}\right) u^{\lambda}\, e^{\omega\,z\,u}\,\mathrm{d}u \\ &+ e^{-w\,z} - \frac{1}{2}\, z\int_0^1 (1-u^2)^{-1/2}\, J_1\left(z\sqrt{1-u^2}\right) u^{\lambda}\, e^{-\omega\,z\,u}\,\mathrm{d}u \\ &-\frac{1}{2}\, z\int_0^1 (1-u^2)^{\lambda/2}\, e^{-w\,z\,(1-u^2)^{1/2}}\, J_1(z\,u)\, u^{\lambda}\,\mathrm{d}u \\ &+\frac{1}{2}\, w\, z\int_0^1 J_0\left(z\sqrt{1-u^2}\right) u^{\lambda}\, e^{-w\,z\,u}\,\mathrm{d}u \\ &-(\underline{x}\wedge\underline{y})\,\frac{1}{2}\, i^{2\lambda+2}\int_0^1 J_0\left(z\sqrt{1-u^2}\right) u^{\lambda}\, e^{\omega\,z\,u}\,\mathrm{d}u \\ &+(\underline{x}\wedge\underline{y})\,\frac{1}{2}\int_0^1 J_0\left(z\sqrt{1-u^2}\right) u^{\lambda}\, e^{-\omega\,z\,u}\,\mathrm{d}u. \end{aligned}$$

where $\lambda=\frac{m-2}{2}$, $z=|x||y|$ and $w=\langle\xi,\,\eta\rangle=\langle\frac{x}{|x|},\,\frac{y}{|y|}\rangle=\cos\theta$.

Remark 6.2.4. *The expressions of the Clifford-Fourier kernel in the above theorem can be compared with the results in Theorem 6 in [15].*

Remark 6.2.5. *When $m = 3$, the explicit expression of the Clifford-Fourier kernel can also be obtained by means of the Legendre polynomial of degree k $P_k(\omega) = C_k^{1/2}(\omega)$ and its generating function (see e.g. [69], (28.19))*

$$\left(1 - 2x\, t + t^2\right)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

where $|x| \leq 1$ and $|t| < 1$.

6.2.2.2 Inversion using Prabhakar functions

We next give the explicit expression of the Clifford-Fourier kernel in terms of the Prabhakar function (2.17).
In (6.7) and (6.8), the common fraction can be equivalently written as

$$\begin{aligned}\frac{1}{\left(1 \pm \frac{2w\,z}{s+r} + \left(\frac{z}{s+r}\right)^2\right)^{\lambda+1}} &= \frac{1}{\left(\frac{z}{s+r} \pm e^{i\theta}\right)^{\lambda+1}} \frac{1}{\left(\frac{z}{s+r} \pm e^{-i\theta}\right)^{\lambda+1}} \\ &= \frac{z^{2(\lambda+1)}}{\left((r-s) \pm e^{i\theta} z\right)^{\lambda+1}} \frac{1}{\left((r-s) \pm e^{-i\theta} z\right)^{\lambda+1}},\end{aligned}$$

where $\lambda = \frac{m-2}{2}$ and $w = \cos\theta$. In the second step we have used the relation $(s+r)(r-s) = (r^2 - s^2) = \left(\sqrt{s^2+z^2}\right)^2 - s^2 = z^2$. Take into account formula (2.46), we rewrite the kernel in the form $F(r-s) = F\left(\sqrt{s^2+z^2} - s\right)$. The result is

$$\begin{aligned}&\mathcal{L}\left(K^{-}_{scal}(x,y)\right) \\ &= -2^{\lambda}\, i^{2\lambda+2}\, \Gamma(\lambda+1) \frac{(r-s)^{\lambda+1}}{r} \frac{w\,z}{\left((r-s) - e^{i\theta} z\right)^{\lambda+1}} \frac{1}{\left((r-s) - e^{-i\theta} z\right)^{\lambda+1}} \\ &\quad + 2^{\lambda}\, i^{2\lambda+2}\, \Gamma(\lambda+1) \frac{(r-s)^{\lambda+2}}{r} \frac{1}{\left((r-s) - e^{i\theta} z\right)^{\lambda+1}} \frac{1}{\left((r-s) - e^{-i\theta} z\right)^{\lambda+1}} \\ &\quad + 2^{\lambda}\, z^2\, \Gamma(\lambda+1) \frac{(r-s)^{\lambda}}{r} \frac{1}{\left((r-s) + e^{i\theta} z\right)^{\lambda+1}} \frac{1}{\left((r-s) + e^{-i\theta} z\right)^{\lambda+1}} \\ &\quad + 2^{\lambda}\, w\, z\, \Gamma(\lambda+1) \frac{(r-s)^{\lambda+1}}{r} \frac{1}{\left((r-s) + e^{i\theta} z\right)^{\lambda+1}} \frac{1}{\left((r-s) + e^{-i\theta} z\right)^{\lambda+1}}\end{aligned} \tag{6.9}$$

and

$$\begin{aligned}
&\mathcal{L}\left(K_{biv}^{-}(x,\,y)\right)\\
&= -\Gamma(\lambda+1)\frac{(r-s)^{\lambda+1}}{r}\frac{2^{\lambda}\,i^{2\lambda+2}}{\left((r-s)-e^{i\theta}z\right)^{\lambda+1}}\frac{1}{\left((r-s)-e^{-i\theta}z\right)^{\lambda+1}}\\
&\quad + 2^{\lambda}\,\Gamma(\lambda+1)\frac{(r-s)^{\lambda+1}}{r}\frac{1}{\left((r-s)+e^{i\theta}z\right)^{\lambda+1}}\frac{1}{\left((r-s)+e^{-i\theta}z\right)^{\lambda+1}}.
\end{aligned} \tag{6.10}$$

Furthermore, we consider the inverse Laplace transform of the term $\frac{1}{(s\pm e^{i\theta}z)^{\lambda+1}}\frac{1}{(s\pm e^{-i\theta}z)^{\lambda+1}}$ by the convolution formula of the Laplace transform (2.54). Setting $F(s)=\frac{1}{(s\pm e^{i\theta}z)^{\lambda+1}}$, $G(s)=\frac{1}{(s\pm e^{-i\theta}z)^{\lambda+1}}$, then by (2.45) and (2.54), the inverse transform follows

$$\begin{aligned}
h_{\pm,\,\lambda+1}(t) &:= \mathcal{L}^{-1}\left(\frac{1}{(s\pm e^{i\theta}z)^{\lambda+1}}\frac{1}{(s\pm e^{-i\theta}z)^{\lambda+1}}\right)\\
&= \int_0^t \zeta^{\lambda}E_{1,\lambda+1}^{\lambda+1}\left(\mp e^{i\theta}z\,\zeta\right)(t-\zeta)^{\lambda}E_{1,\lambda+1}^{\lambda+1}\left(\mp e^{-i\theta}z\,(t-\zeta)\right)\mathrm{d}\zeta.
\end{aligned}$$

Taking the inverse Laplace transform of (6.9) and (6.10) by formula (2.46) and putting $t=1$, we conclude the following theorem.

Theorem 6.2.6. *Let*

$$h_{\pm,\,\lambda+1}(t) = \int_0^t \zeta^{\lambda}E_{1,\lambda+1}^{\lambda+1}\left(\mp e^{i\theta}z\,\zeta\right)(t-\zeta)^{\lambda}E_{1,\lambda+1}^{\lambda+1}\left(\mp e^{-i\theta}z\,(t-\zeta)\right)\mathrm{d}\zeta.$$

Then for the dimension $m\geq 3\,(\lambda>0)$, the kernel of the Clifford-Fourier transform takes the form

$$K_{-}(x,\,y) = K_{scal}^{-}(x,\,y) + (\underline{x}\wedge\underline{y})\,K_{biv}^{-}(x,\,y)$$

with

$$\begin{aligned}
&K_{scal}^{-}(x,\,y)\\
&= -i^{2\lambda+2}\,c_{\lambda}\,w\,z^{\lambda+2}\int_0^1(1+2\tau)^{-(\lambda+1)/2}J_{\lambda+1}\left(z\sqrt{1+2\tau}\right)h_{-,\,\lambda+1}(\tau)\,\mathrm{d}\tau\\
&\quad + i^{2\lambda+2}\,c_{\lambda}\,z^{\lambda+2}\int_0^1(1+2\tau)^{-(\lambda+2)/2}J_{\lambda+2}\left(z\sqrt{1+2\tau}\right)h_{-,\,\lambda+1}(\tau)\,\mathrm{d}\tau\\
&\quad + c_{\lambda}\,z^{\lambda+2}\int_0^1(1+2\tau)^{-\lambda/2}J_{\lambda}\left(z\sqrt{1+2\tau}\right)h_{+,\,\lambda+1}(\tau)\,\mathrm{d}\tau
\end{aligned}$$

$$+\, c_\lambda\, w\, z^{\lambda+2} \int_0^1 (1+2\tau)^{-(\lambda+1)/2} J_{\lambda+1}\left(z\sqrt{1+2\tau}\right) h_{+,\,\lambda}(\tau)\,\mathrm{d}\tau.$$

and

$$\begin{aligned}
&K^-_{biv}(x,\, y)\\
&= -i^{2\lambda+2}\, c_\lambda\, z^{\lambda+1} \int_0^1 (1+2\tau)^{-(\lambda+1)/2} J_{\lambda+1}\left(z\sqrt{1+2\tau}\right) h_{-,\,\lambda+1}(\tau)\,\mathrm{d}\tau\\
&\quad + c_\lambda\, z^{\lambda+1} \int_0^1 (1+2\tau)^{-(\lambda+1)/2} J_{\lambda+1}\left(z\sqrt{1+2\tau}\right) h_{+,\,\lambda+1}(\tau)\,\mathrm{d}\tau
\end{aligned}$$

where $\lambda = \frac{m-2}{2}$, $w = \cos\theta$, $z = |x||y|$ *and* $c_{\lambda+1} = 2^\lambda\,\Gamma(\lambda+1)$.

6.3 The fractional Clifford-Fourier kernel

In this section, we determine the explicit expressions for kernels of the fractional CFT with two numerical parameters α and β and $\alpha,\, \beta \in [-\pi,\, \pi]$.
In [26], the series representation for the kernel of the fractional CFT $\mathcal{F}_{\alpha,\,\beta} = e^{\frac{i\alpha m}{2}} e^{i\beta\Gamma} e^{\frac{i\alpha}{2}(\Delta - |x|^2)}$ is given in the following theorem.

Theorem 6.3.1. *The fractional Clifford-Fourier transform* $\mathcal{F}_{\alpha,\,\beta} = e^{\frac{i\alpha m}{2}} e^{i\beta\Gamma} e^{\frac{i\alpha}{2}(\Delta - |\underline{x}|^2)}$ *is given explicitly by the integral transform*

$$\left(\pi\left(1-e^{-2i\alpha}\right)\right)^{-m/2} \int_{\mathbb{R}^m} K_{\alpha,\,\beta}(x,\, y)\, f(\underline{x})\,\mathrm{d}x$$

with kernel

$$K_{\alpha,\,\beta}(x,\, y) = \left(A^{\alpha,\,\beta}_\lambda + B^{\alpha,\,\beta}_\lambda + \left(\underline{x}\wedge\underline{y}\right) C^{\alpha,\,\beta}_\lambda\right) e^{\frac{i}{2}(\cot\alpha)(|x|^2+|y|^2)}$$

with

$$\begin{aligned}
A^{\alpha,\,\beta}_\lambda(\omega,\, \tilde{z}) &= -2^{\lambda-1}\,\Gamma(\lambda+1) \sum_{k=0}^\infty i^{-k} e^{i\beta(k+2\lambda)}\, \tilde{z}^{-\lambda}\, J_{k+\lambda}(\tilde{z})\, C^\lambda_k(\omega)\\
&\quad + 2^{\lambda-1}\,\Gamma(\lambda+1) \sum_{k=0}^\infty i^{-k} e^{-i\beta k}\, \tilde{z}^{-\lambda}\, J_{k+\lambda}(\tilde{z})\, C^\lambda_k(\omega);\\
B^{\alpha,\,\beta}_\lambda(\omega,\, \tilde{z}) &= 2^{\lambda-1}\,\Gamma(\lambda) \sum_{k=0}^\infty (k+\lambda)\, i^{-k} e^{i\beta(k+2\lambda)}\, \tilde{z}^{-\lambda}\, J_{k+\lambda}(\tilde{z})\, C^\lambda_k(\omega)\\
&\quad + 2^{\lambda-1}\,\Gamma(\lambda) \sum_{k=0}^\infty (k+\lambda)\, i^{-k} e^{-i\beta k}\, \tilde{z}^{-\lambda}\, J_{k+\lambda}(\tilde{z})\, C^\lambda_k(\omega);
\end{aligned}$$

$$C_\lambda^{\alpha,\beta}(\omega,\,\tilde{z}) = \frac{2^\lambda\,\Gamma(\lambda+1)}{\sin\alpha}\sum_{k=1}^{\infty} i^{-k} e^{i\beta(k+2\lambda)} \tilde{z}^{-\lambda-1}\, J_{k+\lambda}(\tilde{z})\, C_{k-1}^{\lambda+1}(\omega)$$
$$- \frac{2^\lambda\,\Gamma(\lambda+1)}{\sin\alpha}\sum_{k=1}^{\infty} i^{-k} e^{-i\beta k} \tilde{z}^{-\lambda-1}\, J_{k+\lambda}(\tilde{z})\, C_{k-1}^{\lambda+1}(\omega).$$

where $\tilde{z} = (|x||y|)/\sin\alpha$, $\omega = \langle \zeta,\,\tau\rangle$ *and* $\lambda = (m-2)/2$.

The authors determined the explicit formulas of $K_{\alpha,\beta}\,(x,\,y)$ on $\mathbb{R}^m$ in even dimensions and obtained the expressions of the kernel in dimension 2 by using a series expansion.

Theorem 6.3.2. *[26] The kernel of the fractional Clifford-Fourier transform in dimension $m=2$ is given by*

$$K_{\alpha,\beta}\,(x,\,y) = e^{\frac{i}{2}(\cot\alpha)(|x|^2+|y|^2)} e^{i\beta\Gamma_{\underline{y}}}\left(e^{-i\langle x,\,y\rangle/\sin\alpha}\right)$$
$$= \left(\cos\left(\eta\,\frac{\sin\beta}{\sin\alpha}\right) + \left(\underline{x}\wedge\underline{y}\right)\frac{\sin\left(\eta\,\frac{\sin\beta}{\sin\alpha}\right)}{\eta}\right)$$
$$\times\, e^{-i\,\langle x,\,y\rangle\frac{\cos\beta}{\sin\alpha}} e^{\frac{i}{2}(\cot\alpha)(|x|^2+|y|^2)}$$

with $\eta = |\underline{x}\wedge\underline{y}| = \sqrt{|x|^2|y|^2 - \langle x,\,y\rangle^2}$.

The kernel for even dimensions larger than 2 was derived subsequently using recursion relations.

6.3.1 The kernel in dimension 2

In this subsection, we determine the kernels of the fractional CFT by the Laplace transform method. The following results were derived by relation (2.9), given in Section 4 in [26].
When $m=2$, the kernel of the fractional Clifford-Fourier transform can be written as the following form (see [26]):

$$K_{\alpha,\beta}(x,\,y) = \left(B_0^{\alpha,\beta}(\omega,\,\tilde{z}) + (\underline{x}\wedge\underline{y})\,C_0^{\alpha,\beta}(\omega,\,\tilde{z})\right) e^{\frac{i}{2}(\cot\alpha)(|x|^2+|y|^2)} \tag{6.11}$$

with

$$B_0^{\alpha,\beta}(\omega,\,\tilde{z}) = J_0(\tilde{z}) + \sum_{k=1}^{\infty} i^{-k}\, J_k(\tilde{z})\cos\left(k(\beta-\theta)\right)$$
$$+ \sum_{k=1}^{\infty} i^{-k}\, J_k(\tilde{z})\cos\left(k(\beta+\theta)\right);$$

$$
\begin{aligned}
C_0^{\alpha,\beta}(\omega,\,\tilde{z}) = & \frac{i}{z\,\sin\theta}\sum_{k=1}^{\infty} i^{-k}\,J_k(\tilde{z})\cos\left(k(\beta-\theta)\right) \\
& -\frac{i}{z\,\sin\theta}\sum_{k=1}^{\infty} i^{-k}\,J_k(\tilde{z})\cos\left(k(\beta+\theta)\right).
\end{aligned}
$$

We introduce a variable t in the Bessel functions in the kernels and take the Laplace transform with respect to t using (2.44) and Theorem 5.3.1. We obtain

$$
\begin{aligned}
& \mathcal{L}\left(B_0^{\alpha,\beta}(\omega,\,\tilde{z},\,t)\right) \\
&= \frac{1}{r}\sum_{k=0}^{\infty} i^{-k}\left(\frac{\tilde{z}}{s+r}\right)^k \cos\left(k(\beta-\theta)\right) - \frac{1}{r} \\
&\quad + \frac{1}{r}\sum_{k=0}^{\infty} i^{-k}\left(\frac{\tilde{z}}{s+r}\right)^k \cos\left(k(\beta+\theta)\right) \\
&= \frac{1}{r}\frac{s+r+\cos\left(\beta-\theta\right)i\,\tilde{z}}{s+r}\,\frac{1}{1-2\left(\frac{i^{-1}\,\tilde{z}}{s+r}\right)\cos\left(\beta-\theta\right)+\left(\frac{i^{-1}\,\tilde{z}}{s+r}\right)^2} - \frac{1}{r} \\
&\quad + \frac{1}{r}\frac{s+r+\cos\left(\beta+\theta\right)i\,\tilde{z}}{s+r}\,\frac{1}{1-2\left(\frac{i^{-1}\,\tilde{z}}{s+r}\right)\cos\left(\beta+\theta\right)+\left(\frac{i^{-1}\,\tilde{z}}{s+r}\right)^2} \\
&= \frac{1}{2r}\frac{\left(s+r+\cos\left(\beta-\theta\right)i\,\tilde{z}\right)\left(s+\cos\left(\beta+\theta\right)i\,\tilde{z}\right)}{\left(s+\cos\left(\beta+\theta\right)i\,\tilde{z}\right)\left(s+\cos\left(\beta-\theta\right)i\,\tilde{z}\right)} - \frac{1}{r} \\
&\quad + \frac{1}{2r}\frac{\left(s+r+\cos\left(\beta+\theta\right)i\,\tilde{z}\right)\left(s+\cos\left(\beta-\theta\right)i\,\tilde{z}\right)}{\left(s+\cos\left(\beta+\theta\right)i\,\tilde{z}\right)\left(s+\cos\left(\beta-\theta\right)i\,\tilde{z}\right)} \\
&= \frac{s+\cos\beta\,\cos\theta\,i\,\tilde{z}}{\left(s+\cos\left(\beta+\theta\right)i\,\tilde{z}\right)\left(s+\cos\left(\beta-\theta\right)i\,\tilde{z}\right)},
\end{aligned}
$$

where in the second equation we have used the relation (5.5).
By means of formula (2.53), we obtain

$$
\begin{aligned}
\mathcal{L}^{-1}\left(B_0^{\alpha,\beta}(\omega,\,\tilde{z},\,t)\right) &= \frac{\cos\left(\beta-\theta\right)-\cos\beta\,\cos\theta}{\cos\left(\beta-\theta\right)-\cos\left(\beta+\theta\right)}\,e^{-i\,\tilde{z}\,\cos\left(\beta-\theta\right)t} \\
&\quad + \frac{\cos\left(\beta+\theta\right)-\cos\beta\,\cos\theta}{\cos\left(\beta+\theta\right)-\cos\left(\beta-\theta\right)}\,e^{-i\,\tilde{z}\cos\left(\beta+\theta\right)t} \\
&= e^{-i\,\tilde{z}\,\cos\theta\frac{\cos\beta}{\sin\alpha}}\left(\frac{e^{-i\,z\,\frac{\sin\beta}{\sin\alpha}\sin\theta\,t}+e^{i\,z\,\frac{\sin\beta}{\sin\alpha}\sin\theta\,t}}{2}\right) \\
&= e^{-i\,z\,\cos\theta\,\frac{\cos\beta}{\sin\alpha}}\cos\left(z\,\frac{\sin\beta}{\sin\alpha}\sin\theta\,t\right).
\end{aligned}
$$

For $C_0^{\alpha,\beta}$, we have

$$
\begin{aligned}
&\mathcal{L}\left(\lim_{\lambda\to 0} C_0^{\alpha,\beta}(\omega,\,\tilde{z},\,t)\right)\\
&=\frac{i}{\sin\theta\, z}\frac{1}{r}\sum_{k=0}^{\infty} i^{-k}\left(\frac{\tilde{z}}{s+r}\right)^k \cos\left(k(\beta-\theta)\right)-\frac{i}{\sin\theta\, z}\frac{1}{r}\\
&\quad-\frac{i}{\sin\theta\, z}\frac{1}{r}\sum_{k=0}^{\infty} i^{-k}\left(\frac{\tilde{z}}{s+r}\right)^k \cos\left(k(\beta+\theta)\right)+\frac{i}{\sin\theta\, z}\frac{1}{r}\\
&=\frac{i}{\sin\theta\, z}\frac{1}{2r}\frac{s+r+\cos(\beta-\theta)\, i\,\tilde{z}}{s+\cos(\beta-\theta)\, i\,\tilde{z}}-\frac{i}{\sin\theta\, z}\frac{1}{2r}\frac{s+r+\cos(\beta+\theta)\, i\,\tilde{z}}{s+\cos(\beta+\theta)\, i\,\tilde{z}}\\
&=\frac{\sin\beta}{\sin\alpha}\frac{1}{(s+\cos(\beta-\theta)\, i\,\tilde{z})\ (s+\cos(\beta+\theta)\, i\,\tilde{z})}.
\end{aligned}
$$

Take into account the inverse formula of the Laplace transform and formula (2.38), we have by putting $t=1$

$$
\begin{aligned}
\mathcal{L}^{-1}\left(\lim_{\lambda\to 0} C_\lambda(\omega,\,\tilde{z},\,t)\right)&=\frac{\sin\beta}{\sin\alpha}\int_0^1 e^{-i\,\tilde{z}\,\cos(\beta-\theta)\,(1-\tau)}e^{-i\,\tilde{z}\,\cos(\beta+\theta)\,\tau}\,\mathrm{d}\tau\\
&=\frac{\sin\beta}{\sin\alpha}e^{-i\,\tilde{z}\,\cos(\beta-\theta)}\int_0^1 e^{-i\,\tilde{z}\,2\sin\beta\,\sin\theta\,\tau}\,\mathrm{d}\tau\\
&=\frac{1}{\sin\theta\, z}\,e^{-i\,\langle x,\,y\rangle\,\frac{\cos\beta}{\sin\alpha}}\,\sin\left(z\,\frac{\sin\beta}{\sin\alpha}\,\sin\theta\right).
\end{aligned}
$$

Collecting the results and setting $t=1$, the kernel of the fractional Clifford-Fourier transform in dimension 2 becomes

$$
K_{\alpha,\beta}(x,y,t)=\left(B_0^{\alpha,\beta}(\omega,\tilde{z},t)+(\underline{x}\wedge\underline{y})\,C_0^{\alpha,\beta}(\omega,\tilde{z},t)\right)e^{\frac{i}{2}(\cot\alpha)(|x|^2+|y|^2)}
$$

with

$$
\begin{aligned}
B_0^{\alpha,\beta}(\omega,\,\tilde{z},\,t)&=e^{-i\,\langle x,\,y\rangle\,\frac{\cos\beta}{\sin\alpha}}\,\cos\left(\eta\,\frac{\sin\beta}{\sin\alpha}\right);\\
C_0^{\alpha,\beta}(\omega,\,\tilde{z},\,t)&=\frac{\sin\left(\eta\,\frac{\sin\beta}{\sin\alpha}\right)}{\eta}\,e^{-i\,\langle x,\,y\rangle\,\frac{\cos\beta}{\sin\alpha}}
\end{aligned}
$$

where $\eta=|\underline{x}\wedge\underline{y}|=\sqrt{|x|^2\,|y|^2-\langle x,\,y\rangle^2}$. By this Laplace transform method, we obtain the same result with Theorem 6.3.2 (see [26]), although the proof given there is completely different.

Note that we can recover the kernel of the fractional Clifford-Fourier transform $\mathcal{F}_{-,\alpha}=e^{\frac{i\alpha m}{2}}\,e^{\frac{i\alpha}{2}(\Delta-|x|^2+2\Gamma)}$ for the case when $\alpha=\beta$ in Theorem 4 in [25].

6.3.2 The kernels in $m \geq 3$

Since we have already conducted the Laplace transform method in Chapter 5 and in the previous section, in this case we restrict ourselves to give some important steps only. We consider the kernels of the fractional Clifford-Fourier transform in the following forms:

$$\begin{aligned}
A_\lambda^{\alpha,\beta} &= -2^{\lambda-1}\Gamma(\lambda+1)\,\tilde{z}^{-\lambda}\sum_{k=0}^{\infty} i^{-k}e^{i\beta(k+2\lambda)}\,J_{k+\lambda}(t\,\tilde{z})\,C_k^\lambda(\omega)\\
&\quad + 2^{\lambda-1}\Gamma(\lambda+1)\,\tilde{z}^{-\lambda}\sum_{k=0}^{\infty} i^{-k}e^{-i\beta k}\,J_{k+\lambda}(t\,\tilde{z})\,C_k^\lambda(\omega);\\
B_\lambda^{\alpha,\beta} &= 2^{\lambda-1}\Gamma(\lambda+1)\tilde{z}^{-\lambda}\sum_{k=0}^{\infty}\frac{k+\lambda}{\lambda} i^{-k}e^{i\beta(k+2\lambda)}\,J_{k+\lambda}(t\,\tilde{z})\,C_k^\lambda(\omega)\\
&\quad + 2^{\lambda-1}\Gamma(\lambda+1)\,\tilde{z}^{-\lambda}\sum_{k=0}^{\infty}\frac{k+\lambda}{\lambda}\, i^{-k}e^{-i\beta k}\,J_{k+\lambda}(t\,\tilde{z})\,C_k^\lambda(\omega);\\
C_\lambda^{\alpha,\beta} &= \frac{2^\lambda\Gamma(\lambda+1)}{\sin\alpha\,\tilde{z}^{\lambda+1}}\sum_{k=0}^{\infty} i^{-k-1}e^{i\beta(k+1+2\lambda)}\,J_{k+1+\lambda}(t\,\tilde{z})\,C_k^{\lambda+1}(\omega)\\
&\quad - \frac{2^\lambda\Gamma(\lambda+1)}{\sin\alpha\,\tilde{z}^{\lambda+1}}\sum_{k=0}^{\infty} i^{-k-1}e^{-i\beta k-i\beta}\,J_{k+1+\lambda}(t\,\tilde{z})\,C_k^{\lambda+1}(\omega).
\end{aligned}$$

In a similar way, the Laplace transforms of $A_\lambda^{\alpha,\beta}$ and $B_\lambda^{\alpha,\beta}$ when $m \geq 3$ are given by

$$\begin{aligned}
&\mathcal{L}\left(A_\lambda^{\alpha,\beta}\right)\\
&= -2^{\lambda-1}\Gamma(\lambda+1)\,\frac{1}{r}\left(\frac{1}{s+r}\right)^\lambda \frac{e^{i\beta 2\lambda}}{\left(1-2w\left(\frac{e^{i\beta}i^{-1}\tilde{z}}{s+r}\right)+\left(\frac{e^{i\beta}i^{-1}\tilde{z}}{s+r}\right)^2\right)^\lambda}\\
&\quad + 2^{\lambda-1}\Gamma(\lambda+1)\,\frac{1}{r}\left(\frac{1}{s+r}\right)^\lambda \frac{1}{\left(1-2w\left(\frac{e^{-i\beta}i^{-1}\tilde{z}}{s+r}\right)+\left(\frac{e^{-i\beta}i^{-1}\tilde{z}}{s+r}\right)^2\right)^\lambda};\\
&\mathcal{L}\left(B_\lambda^{\alpha,\beta}\right)\\
&= 2^{\lambda-1}\Gamma(\lambda+1)\,\frac{1}{r}\left(\frac{1}{s+r}\right)^\lambda \frac{e^{i\beta 2\lambda}}{\left(1-2w\left(\frac{e^{i\beta}i^{-1}\tilde{z}}{s+r}\right)+\left(\frac{e^{i\beta}i^{-1}\tilde{z}}{s+r}\right)^2\right)^{\lambda+1}}
\end{aligned}$$

$$+\,2^{\lambda-1}\Gamma(\lambda+1)\,\frac{1}{r}\left(\frac{1}{s+r}\right)^{\lambda+2}\frac{\tilde{z}^2\,e^{i\beta 2\lambda+2i\beta}}{\left(1-2w\left(\frac{e^{i\beta}i^{-1}\tilde{z}}{s+r}\right)+\left(\frac{e^{i\beta}i^{-1}\tilde{z}}{s+r}\right)^2\right)^{\lambda+1}}$$

$$+\,2^{\lambda-1}\Gamma(\lambda+1)\,\frac{1}{r}\left(\frac{1}{s+r}\right)^{\lambda}\frac{1}{\left(1-2w\left(\frac{e^{-i\beta}i^{-1}\tilde{z}}{s+r}\right)+\left(\frac{e^{-i\beta}i^{-1}\tilde{z}}{s+r}\right)^2\right)^{\lambda+1}}$$

$$+\,2^{\lambda-1}\Gamma(\lambda+1)\frac{1}{r}\left(\frac{1}{s+r}\right)^{\lambda+2}\frac{\tilde{z}^2\,e^{-2i\beta}}{\left(1-2w\left(\frac{e^{-i\beta}i^{-1}\tilde{z}}{s+r}\right)+\left(\frac{e^{-i\beta}i^{-1}\tilde{z}}{s+r}\right)^2\right)^{\lambda+1}},$$

where $\omega=\cos\theta$, $\tilde{z}=(|x||y|)/\sin\alpha$, $r=\sqrt{s^2+\tilde{z}^2}$ and $\lambda=(m-2)/2$. Summing the above results yields the scalar part of $K_{\alpha,\beta}(x,\,y)$

$$\mathcal{L}\left(K^{\alpha,\beta}_{scal}\right)$$

$$=-2^{\lambda}\,w\,\Gamma(\lambda+1)\,\frac{1}{r}\left(\frac{1}{s+r}\right)^{\lambda+1}\frac{i\,\tilde{z}\,e^{i\beta 2\lambda+i\beta}}{\left(1-2w\left(\frac{e^{i\beta}i^{-1}\tilde{z}}{s+r}\right)+\left(\frac{e^{i\beta}i^{-1}\tilde{z}}{s+r}\right)^2\right)^{\lambda+1}}$$

$$+\,2^{\lambda}\,\Gamma(\lambda+1)\,\frac{1}{r}\left(\frac{1}{s+r}\right)^{\lambda+2}\frac{\tilde{z}^2\,e^{i\beta 2\lambda+2i\beta}}{\left(1-2w\left(\frac{e^{i\beta}i^{-1}\tilde{z}}{s+r}\right)+\left(\frac{e^{i\beta}i^{-1}\tilde{z}}{s+r}\right)^2\right)^{\lambda+1}}$$

$$+\,2^{\lambda}\,\Gamma(\lambda+1)\,\frac{1}{r}\left(\frac{1}{s+r}\right)^{\lambda}\frac{1}{\left(1-2w\left(\frac{e^{-i\beta}i^{-1}\tilde{z}}{s+r}\right)+\left(\frac{e^{-i\beta}i^{-1}\tilde{z}}{s+r}\right)^2\right)^{\lambda+1}}$$

$$+\,2^{\lambda}\,w\,\Gamma(\lambda+1)\,\frac{1}{r}\left(\frac{1}{s+r}\right)^{\lambda+1}\frac{i\,\tilde{z}\,e^{-i\beta}}{\left(1-2w\left(\frac{e^{-i\beta}i^{-1}\tilde{z}}{s+r}\right)+\left(\frac{e^{-i\beta}i^{-1}\tilde{z}}{s+r}\right)^2\right)^{\lambda+1}}.$$

Next we calculate $\mathcal{L}\left(C^{\alpha,\beta}_{\lambda}\right)$ as the bivector part

$$\mathcal{L}\left(K^{\alpha,\beta}_{biv}\right)$$

$$=\frac{2^{\lambda}\Gamma(\lambda+1)}{i\sin\alpha}\,\frac{1}{r}\left(\frac{1}{s+r}\right)^{\lambda+1}\frac{e^{i\beta(2\lambda+1)}}{\left(1-2w\left(\frac{e^{i\beta}i^{-1}\tilde{z}}{s+r}\right)+\left(\frac{e^{i\beta}i^{-1}\tilde{z}}{s+r}\right)^2\right)^{\lambda+1}}$$

$$-\,\frac{2^{\lambda}\Gamma(\lambda+1)}{i\sin\alpha}\,\frac{1}{r}\left(\frac{1}{s+r}\right)^{\lambda+1}\frac{e^{-i\beta}}{\left(1-2w\left(\frac{e^{-i\beta}i^{-1}\tilde{z}}{s+r}\right)+\left(\frac{e^{-i\beta}i^{-1}\tilde{z}}{s+r}\right)^2\right)^{\lambda+1}}.$$

The common fraction can be written as

$$
\begin{aligned}
&\frac{1}{\left(1-2w\left(\frac{e^{\pm i\beta}i^{-1}\tilde{z}}{s+r}\right)+\left(\frac{e^{\pm i\beta}i^{-1}\tilde{z}}{s+r}\right)^2\right)^{\lambda+1}}\\
&=\frac{1}{\left(e^{\pm i\beta}\frac{i^{-1}\tilde{z}}{s+r}-e^{i\theta}\right)^{\lambda+1}}\frac{1}{\left(e^{\pm i\beta}\frac{i^{-1}\tilde{z}}{s+r}-e^{-i\theta}\right)^{\lambda+1}}\\
&=\frac{(-1)^{\lambda+1}\,\tilde{z}^{2(\lambda+1)}\,e^{\mp 2\,i\beta\,(\lambda+1)}}{((r-s)-e^{i\theta}e^{\mp i\beta}i\,\tilde{z})^{\lambda+1}}\frac{1}{((r-s)-e^{-i\,\theta}e^{\mp i\beta}i\,\tilde{z})^{\lambda+1}}.
\end{aligned}
$$

We find the inverse Laplace transform

$$
\begin{aligned}
&\mathcal{L}\left(\frac{1}{(s-e^{i\theta}e^{\mp i\beta}i\tilde{z})^{\lambda+1}}\frac{1}{(s-e^{-i\theta}e^{\mp i\beta}i\tilde{z})^{\lambda+1}}\right)\\
&=\int_0^t\zeta^\lambda E^{\lambda+1}_{1,\lambda+1}\left(e^{i\theta}\,e^{\mp i\beta}\,i\,\tilde{z}\,\zeta\right)(t-\zeta)^\lambda E^{\lambda+1}_{1,\lambda+1}\left(e^{-i\theta}\,e^{\mp i\beta}\,i\,\tilde{z}\,(t-\zeta)\right)\,\mathrm{d}\zeta\\
&:=h^{\alpha,\,\beta}_{\mp,\,\lambda+1}(t)
\end{aligned}
$$

using (2.45) and (2.54).
As a consequence, it suffices to explicitly give the integral expressions for the kernel by means of the inversion formula (2.46).

Theorem 6.3.3. *Denote*

$$
\begin{aligned}
h^{\alpha,\,\beta}_{\mp,\,\lambda+1}(t):=&\int_0^t\zeta^\lambda E^{\lambda+1}_{1,\lambda+1}\left(e^{i\theta}\,e^{\mp i\beta}\,i\,\tilde{z}\,\zeta\right)\\
&\times(t-\zeta)^\lambda E^{\lambda+1}_{1,\lambda+1}\left(e^{-i\theta}\,e^{\mp i\beta}\,i\,\tilde{z}\,(t-\zeta)\right)\,\mathrm{d}\zeta.
\end{aligned}
$$

Then for the dimension $m\geq 3$, the kernel of the fractional Clifford-Fourier transform $\mathcal{F}_{\alpha,\,\beta}=e^{\frac{i\alpha m}{2}}e^{i\beta\Gamma}\,e^{\frac{i\alpha}{2}(\Delta-|x|^2)}$ takes the form

$$
K_{\alpha,\,\beta}(x,\,y)=\left(K^{\alpha,\,\beta}_{scal}(x,\,y)+(\underline{x}\wedge\underline{y})\,K^{\alpha,\,\beta}_{biv}(x,\,y)\right)e^{\frac{i}{2}(\cot\alpha)(|x|^2+|y|^2)}
$$

with

$$
\begin{aligned}
&K^{\alpha,\,\beta}_{scal}(x,\,y)\\
&=-c^{\alpha,\,\beta}_\lambda\,i\,w\,\tilde{z}^{\lambda+2}\,e^{-i\beta}\int_0^1(1+2\tau)^{-(\lambda+1)/2}J_{\lambda+1}\left(\tilde{z}\sqrt{1+2\tau}\right)h^{\alpha,\,\beta}_{-,\,\lambda+1}(\tau)\,\mathrm{d}\tau\\
&\quad+c^{\alpha,\,\beta}_\lambda\,\tilde{z}^{\lambda+2}\int_0^1(1+2\tau)^{-(\lambda+2)/2}J_{\lambda+2}\left(\tilde{z}\sqrt{1+2\tau}\right)h^{\alpha,\,\beta}_{-,\,\lambda+1}(\tau)\,\mathrm{d}\tau
\end{aligned}
$$

$$+ c_\lambda^{\alpha,\beta}\,\tilde{z}^{\lambda+2}\,e^{2i\beta(\lambda+1)}\int_0^1 (1+2\tau)^{-\lambda/2} J_\lambda\left(\tilde{z}\sqrt{1+2\tau}\right) h_{+,\lambda+1}^{\alpha,\beta}(\tau)\,\mathrm{d}\tau$$
$$+ c_\lambda^{\alpha,\beta} iw\tilde{z}^{\lambda+2}e^{2i\beta\lambda+i\beta}\int_0^1 (1+2\tau)^{-(\lambda+1)/2} J_{\lambda+1}\left(\tilde{z}\sqrt{1+2\tau}\right) h_{+,\lambda+1}^{\alpha,\beta}(\tau)\mathrm{d}\tau$$

and

$$K_{biv}^{\alpha,\beta}(x,y)$$
$$= \frac{2\,i\,c_\lambda^{\alpha,\beta}\,\tilde{z}^{\lambda+1}}{\sin\alpha\, e^{i\beta}}\Bigg(\int_0^1 (1+2\tau)^{-(\lambda+1)/2} J_{\lambda+1}\left(\tilde{z}\sqrt{1+2\tau}\right) h_{-,\lambda+1}^{\alpha,\beta}(\tau)\,\mathrm{d}\tau$$
$$- e^{2i\beta(\lambda+1)}\int_0^1 (1+2\tau)^{-(\lambda+1)/2} J_{\lambda+1}\left(\tilde{z}\sqrt{1+2\tau}\right) h_{+,\lambda+1}^{\alpha,\beta}(\tau)\,\mathrm{d}\tau\Bigg)$$

where $w = \cos\theta$, $\tilde{z} = (|x||y|)/\sin\alpha$, $c_\lambda^{\alpha,\beta} = (-1)^{\lambda+1}\,2^\lambda\,\Gamma(\lambda+1)$ *and* $\lambda = (m-2)/2$.

Remark 6.3.4. *When $m \geq 3$, we can obtain the integral expressions for the kernel of the fractional Clifford-Fourier transform for the case introduced in [25] by letting $\alpha = \beta$.*

Remark 6.3.5. *If we take $\alpha = \beta = \frac{\pi}{2}$, the results in Theorem 6.3.3 reduce to Theorem 6.2.6.*

Chapter 7

Further research

In Chapter 3 we established three versions of the uncertainty inequalities in the L^2-setting and two real Paley-Wiener theorems for the novel class of Fourier transforms associated with the Clifford-Helmholtz system introduced in [23] and [58]. The difficulties to study further inequalities are mainly in the following two aspects: one is that most of these transforms are not unitary; the other is that the Fourier kernels associated with the Clifford-Helmholtz system are polynomially bounded. Therefore the Hausdorff-Young inequality is not available for that case. It might be worthwhile to consider a suitable Hausdorff-Young inequality and then derive the uncertainty inequality and Paley-Wiener theorem for L^p functions.
For many generalized integral transforms it is possible to define a generalized translation operator, that plays the same role as the standard translation for the standard Fourier transform, see e.g. [71]. It might be possible to establish such a translation operator in our setting, and investigate its most important properties. In particular, the translation of radial functions should be computed by using techniques from special functions.
In the same line of research, we can then introduce a convolution that interacts nicely with the Fourier transform associated with the Clifford-Helmholtz system using the generalized translation operator. Subsequently, an inversion formula could be established by the convolution. This will make use of the translation of a Gaussian function. The latter is radial, therefore its translation will have been determined in the previous step. From this one can investigate whether a Young inequality can be established.

Another possible direction of further research is to establish estimates for the integral expressions of the kernel of the radially deformed Fourier transform studied in Chapter 5. In [29], the authors point out that the Prabhakar function (2.17) has integral expressions in terms of one of the integral representations of the reciprocal gamma function (2.19) when $0 < \alpha < 2$.

Theorem 7.0.1. *[29] Let $\delta > 0$, $0 < \alpha < 2$ and $\beta \in \mathbb{C}$ and let $\epsilon > 0$ and μ be given such that*

$$\frac{\pi\alpha}{2} < \mu < \min\{\pi, \alpha\pi\}.$$

Then for $z \in G^-(\epsilon,\, \mu)$ we have the formula

$$E^\delta_{\alpha,\,\beta}(z) = \frac{1}{2\pi\alpha i}\int_{\gamma(\epsilon,\,\mu)} \frac{\exp\left(\zeta^{1/\alpha}\right)\zeta^{\frac{1-\beta}{\alpha}+\delta-1}}{(\zeta - z)^\delta}$$

and for $z \in G^+(\epsilon,\, \mu)$ in case δ is a positive integer

$$E^\delta_{\alpha,\,\beta}(z) = \frac{1}{2\pi\alpha i}\int_{\gamma(\epsilon,\,\mu)} \frac{\exp\left(\zeta^{1/\alpha}\right)\zeta^{\frac{1-\beta}{\alpha}+\delta-1}}{(\zeta - z)^\delta} + \frac{1}{\alpha\,\Gamma(\delta)}\frac{\mathrm{d}^{\delta-1}}{\mathrm{d}z^{\delta-1}} g(z),$$

where $g(z) = \exp\left(z^{1/\alpha}\right) z^{(1-\beta)/\alpha+\delta-1}$.

Based on the above theorem, one shows that the Prabhakar function satisfies the following extimates.

Theorem 7.0.2. *[29] Let $\delta > 0$, $0 < \alpha < 2$ and $\beta > 0$. Assume that $\alpha\pi/2 < \mu < \min\{\pi, \alpha\pi\}$ and $\mu \le |\arg z| \le \pi$. Then there exists a constant $C > 0$ only depending on μ, α and β such that*

$$|E^\delta_{\alpha,\beta}(z)| \le \frac{C}{1 + |z|^\delta}. \tag{7.1}$$

The integral expressions for the kernel of the deformed Fourier transform in Theorem 5.4.1 and Theorem 5.4.3 can be concluded together for K^c_m, $m \ge 2$. Inspired by the estimates for the Mittag-Leffler function $E_{\alpha,\,\beta}$ in [62], we found the convolution formula for $|h_{\lambda+1}(t)|$ in Theorem 5.4.3 can be estimated as follows.

Lemma 7.0.3. *Let $m \geq 2$, $\alpha > 0$ and $\beta \in \mathbb{C}$. Assume that $\alpha\pi/2 < \mu < \min\{\pi, \alpha\pi\}$ and $\mu \leq |\arg e^{\pm i\theta}| \leq \pi$. Then there exists a constant $C > 0$ only depending on μ, α and β such that*

$$\begin{aligned}
&\left| \int_0^t \zeta^{\beta-1} E^{\delta}_{\alpha,\,\beta}\,(b_+ \zeta^{\alpha})\,(t-\zeta)^{\beta-1} E^{\delta}_{\alpha,\,\beta}(b_-(t-\zeta)^{\alpha})\,\mathrm{d}\zeta \right| \\
&\leq t^{2\beta-1}\,\left| E^{2\delta}_{\alpha,\,2\beta}(b_- t^{\alpha}) \right|, \quad t \in [0,\,1]
\end{aligned} \tag{7.2}$$

where $b_{\pm} = e^{\pm i\theta} i^n z^n$.

Proof. Using the definition of the Prabhakar function (2.17) we have

$$\begin{aligned}
&\int_0^t \zeta^{\beta-1} E^{\delta}_{\alpha,\,\beta}\,(b_+ \zeta^{\alpha})\,(t-\zeta)^{\beta-1} E^{\delta}_{\alpha,\,\beta}(b_-(t-\zeta)^{\alpha})\,\mathrm{d}\zeta \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\delta)_n\,(\delta)_m\,b_+^n\,b_-^m}{n!\,m!\,\Gamma(\alpha n+\beta)\,\Gamma(\alpha m+\beta)} \int_0^t \zeta^{\alpha n+\beta-1}(t-\zeta)^{\alpha m+\beta-1}\,\mathrm{d}\zeta \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\delta)_n\,(\delta)_m\,b_+^n\,b_-^m\,t^{\alpha(n+m)+2\beta-1}}{n!\,m!\,\Gamma(\alpha n+\beta)\,\Gamma(\alpha m+\beta)}\,B\,(\alpha n+\beta,\,\alpha m+\beta) \\
&= \sum_{n=0}^{\infty}\sum_{k=n}^{\infty} \frac{(\delta)_n\,(\delta)_{k-n}\,b_+^n\,b_-^{k-n}}{n!\,(k-n)!\,\Gamma\,(\alpha k+2\beta)}\,t^{\alpha k+2\beta-1} \\
&= \sum_{n=0}^{\infty}\sum_{k=n}^{\infty} \binom{\delta+n-1}{n}\binom{\delta+k-n-1}{k-n} \frac{\left(\frac{b_+}{b_-}\right)^n b_-^k}{\Gamma\,(\alpha k+2\beta)}\,t^{\alpha k+2\beta-1}.
\end{aligned}$$

Recall the generalized identity of the Vandermonde's identity (see [46])

$$\sum_{k=0}^{r} \binom{\alpha+k}{k}\binom{\gamma+r-k}{r-k} = \binom{\alpha+\gamma+r+1}{r},$$

for any nonnegative integers r, m and n, it follows that

$$\begin{aligned}
&\left| \sum_{n=0}^{\infty}\sum_{k=n}^{\infty} \binom{\delta+n-1}{n}\binom{\delta+k-n-1}{k-n} \frac{\left(\frac{b_+}{b_-}\right)^n b_-^k}{\Gamma\,(\alpha k+2\beta)}\,t^{\alpha k+2\beta-1} \right| \\
&= \left| \sum_{k=0}^{\infty}\sum_{n=0}^{k} \binom{\delta+n-1}{n}\binom{\delta+k-n-1}{k-n} \frac{\left(\frac{b_+}{b_-}\right)^n b_-^k}{\Gamma\,(\alpha k+2\beta)}\,t^{\alpha k+2\beta-1} \right| \\
&\leq t^{2\beta-1} \sum_{k=0}^{\infty}\sum_{n=0}^{k} \left| \frac{t^{\alpha k}\,b_-^k}{\Gamma\,(\alpha k+2\beta)} \right| \left| \binom{\delta+n-1}{n}\binom{\delta+k-n-1}{k-n} \right| \left| \left(\frac{b_+}{b_-}\right)^n \right|
\end{aligned}$$

$$
\begin{aligned}
&= t^{2\beta-1}\sum_{k=0}^{\infty}\left|\frac{t^{\alpha k}\,b_-^k}{\Gamma\left(\alpha k+2\beta\right)}\right|\left(\sum_{n=0}^{k}\binom{\delta+n-1}{n}\binom{\delta+k-n-1}{k-n}\right)\\
&= t^{2\beta-1}\sum_{k=0}^{\infty}\binom{2\delta+k-1}{k}\left|\frac{t^{\alpha k}\,b_-^k}{\Gamma\left(\alpha k+2\beta\right)}\right|\\
&= t^{2\beta-1}\left|E_{\alpha,\,2\beta}^{2\delta}(b_-t^{\alpha})\right|.
\end{aligned}
$$

This completes the proof. □

It would be interesting to obtain the bounds for the Prabhakar function $E_{\alpha,\,\beta}^{\delta}$ in terms of another integral representation (2.18) of $1/\Gamma(z)$ for the case $\alpha > 2$. The right way to proceed might be found in Theorem 1.7 in [62].

Appendix A

Nederlandstalige samenvatting

Het Clifford-Helmholtz systeem is een stelsel van partiële differentiaalvergelijkingen binnen een bepaalde Clifford algebra. Het is mogelijk om hiervoor een integraaltransformatie te definiëren die gelijkenissen vertoont met de fouriertransformatie en waarvan de kern overeenkomt met een oplossing van dit systeem. Belangrijke eigenschappen, zoals de afleidingseigenschap, eigenfuncties en bijhorende eigenwaarden worden voor deze nieuwe soort fouriertransformaties bestudeerd. Op basis van eerder onderzoek kunnen onder meer onzekerheidsongelijkheden en stellingen van Paley-Wiener bewezen worden. Recentelijk zijn nieuwe en elegante methoden op basis van de laplacetransformatie ontwikkeld om expliciete voorstellingen van de kern horende bij hypercomplexe fouriertransformaties te berekenen. Deze tools bieden inzicht en inspiratie om de kernen van de fouriertransformaties in deze thesis te bestuderen.

Deze scriptie onderzoekt eigenschappen van de fouriertransformaties gelinkt aan het Clifford-Helmholtz systeem en presenteert laplacetransformatiemethoden om uitdrukkingen te vinden voor de fourierkernen in de context van de Clifford analyse.

In Hoofdstuk 3 werden lokale en globale onzekerheidsongelijkheden en de ongelijkheid van Heisenberg voor deze nieuwe klasse van fouriertransformaties verkregen door $\mathcal{F}_m^\lambda$ te vergelijken met de klassieke fouriertransformatie. De ongelijkheid van Pitt, die een belangrijke rol zal spelen bij het opstellen van de logaritmische onzekerheidsongelijkheid, werd bewezen. In hetzelfde hoofdstuk werden twee reële Paley-Wiener stellingen voor de fouriertransformaties $\mathcal{F}_m^n, 0 \leq n \leq m-2$ van functies die nul worden buiten en binnen een zekere bal opgesteld, in het geval $a = 1$.

In Hoofdstuk 4 construeerden we een familie van oplossingen K_m^n van het Clifford-Helmholtz systeem in het laplacedomein, door de oplossingen met de factor $t^{(m-1)/2}$ te vermenigvuldigen. Door de laplacetransformatie uit te voeren ten opzichte van t, vonden we eindige hypergeometrische uitdrukkingen voor deze oplossingen. Er werd aangetoond hoe de recursieve relaties berekend kunnen worden door middel van reeksontwikkeling van de verkregen laplacebeelden. We waren in staat om interessante genererende functies voor de kernen neer te schrijven in de gevallen $n = 0$ en $n = \frac{m}{2} - 1$.

De radiaal gedeformeerde fouriertransformatie is de fouriertransformatie geassocieerd aan de radiaal gedeformeerde Dirac-operator in de setting van de Clifford analyse. Voorheen werden reeds verschillende functionaalanalytische aspecten van deze operator bestudeerd. In Hoofdstuk 5 ontwikkelden we een nieuwe methode op basis van de laplacetransformatie om de kern van de gedeformeerde fouriertransformatie in het geval dat $1 + c = \frac{1}{n}, n \in \mathbb{N}_0 \setminus \{1\}$ met n oneven te berekenen. Door het toevoegen van de hulpvariabele t in de reeksontwikkeling van de kern, werd het mogelijk om het laplacebeeld ervan te verkrijgen. Deze resultaten werden verder vereenvoudigd door gebruik te maken van de Poisson kern en de genererende functie van de Gegenbauerveeltermen. Vervolgens werden de rationale functies die in de kern optreden, ontbonden in partiëelbreuken, afzonderlijk voor het geval de dimensie m even is. Bovendien werden expliciete integraalvoorstellingen van de kern voor alle dimensies gevonden door laplaceinversie die leidde tot Mittag-Lefflerfuncties.

Door de aangepaste laplacetransformatiemethode uit Hoofdstuk 5 om te zetten naar de Clifford-Fourier kern en de fractionele Clifford-Fourier kern, werden nieuwe uitdrukkingen voor deze kernen gevonden en gepresenteerd in Hoofdstuk 6. Met behulp van deze methode legden we het verband tussen de gekende integraalvoorstellingen en onze nieuwe resultaten. De kernen van de Clifford-Fouriertransformatie en de fractionele variant kunnen worden gegeven in termen van de Bessel- en Prabhakarfuncties.

Appendix B

English summary

The Clifford-Helmholtz system is a system of partial differential equations in a Clifford algebra. It is possible to define Fourier-type integral transforms with as kernels the solutions of this system. Some important properties such as the differential property, the eigenfunctions and corresponding eigenvalues were studied for these new Fourier-type integral transforms. Based on those previous works, properties like uncertainty inequalities and Paley-Wiener theorems can be established. In recent years, some new and elegant Laplace transform methods were developed to compute the explicit representations of the kernels of hypercomplex Fourier transforms, which offer some inspirations in understanding the kernels of the Fourier transforms in our thesis.

This thesis considers properties of Fourier transforms associated with the Clifford-Helmholtz system and presents Laplace transform methods to obtain some expressions of Fourier kernels in the Clifford analysis setting.

In Chapter 3, local and global uncertainty inequalities and Heisenberg's inequality for this novel class of Fourier transforms were obtained by comparing $\mathcal{F}_m^\lambda$ with classical Fourier transform. Pitt's inequality, which will play an important role in establishing the logarithmic uncertainty inequality, was proven. In the same chapter two real Paley-Wiener theorems of the Fourier transforms $\mathcal{F}_m^n$, $0 \leq n \leq m-2$ of functions vanishing outside and inside a ball were established, by setting $a = 1$.

In Chapter 4, we constructed the family of solutions K_m^n of the Clifford-Helmholtz system in the Laplace domain by multiplying the solutions by the factor $t^{(m-1)/2}$. Taking Laplace transform with repect to t, we were able to obtain terminating hypergeometric function expressions for the solutions. It is shown how the recursion relations can be computed by the series expansion of the Laplace transforms. We also were able to exhibit interesting generating functions for the kernels in the cases $n = 0$ and $n = \frac{m}{2} - 1$.

The radially deformed Fourier transform is the Fourier transform associated with the radially deformed Dirac operator in the Clifford analysis setting. It was previously studied as one of several function theoretical aspects of the operator. In Chapter 5 we developed a new Laplace transform method for the kernel of the deformed Fourier transform for the case of $1 + c = \frac{1}{n}, n \in \mathbb{N}_0 \backslash \{1\}$ with n odd. By introducing an auxiliary variable t in the series expansion of the kernel, the Laplace transform of the kernel was obtained. We further simplified these results by making use of the Poisson kernel and the generating function of the Gegenbauer polynomials. Then the rational functions in the kernels were given in partial fraction decomposition when dimension m is even. Moreover, explicit and integral expressions of the kernel for all dimensions were given using the Laplace inversion of the Mittag-Leffler functions.

Transforming the modified Laplace transform method in Chapter 5 to the Clifford-Fourier kernel and the fractional Clifford-Fourier kernel, new explicit expressions of the kernels were given in Chapter 6. By this method, the connection is made between the known integral representations and our new results. The kernels of the Clifford-Fourier transform and its fractional versions can be given in terms of Bessel functions and Prabhakar functions.

Bibliography

[1] G.E. Andrews, R. Askey, R. Roy. Special functions. Vol. 71 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press 1, Cambridge, 1999.

[2] R.N. Bracewell. The Fourier Transform and Its Applications. New York: McGraw-Hill, 1978.

[3] W. Beckner. Pitt's inequality and the uncertainty principle. *Proc. Amer Math. Soc.*. **123** (1995), 1897-1905.

[4] F. Brackx, R. Delanghe and F. Sommen. Clifford Analysis. Vol. 76 of *Research Notes in Mathematics*. Pitman (Advanced Publishing program), Boston, MA, 1982.

[5] F. Brackx, N. De Schepper and F. Sommen. The Clifford-Fourier transforms. *J. Fourier Anal. Appl.*. **11** (2005), 669-681.

[6] F. Brackx, N. De Schepper and F. Sommen. The two-dimensional Clifford-Fourier transform. *J. Math. Imaging Vision* **26** (2006), 5–18.

[7] F. Brackx, N. De Schepper and F. Sommen. The Fourier transforms in Clifford analysis. *Advances in Imaging and Electron Physics*. **156** (2008), 55-203.

[8] F. Brackx, N. De Schepper and F. Sommen. The Fourier-Bessel Transform. *In 18th International Conference on the Application of Computer Science and Mathematics in Architecture and Civil Engineering*. K. Cürlebeck and C. Könke(eds), Weimar, Germany, 07-09 July 2009.

[9] F. Brackx, N. De Schepper and F. Sommen. The Clifford-Fourier integral kernel in even dimensional Euclidean space. *J. Math. Anal. Appl.* **365** (2010), 718-728.

[10] F. Brackx, E. Hitzer, S. Sangwine. History of quaternion and Clifford-fourier transforms and wavelets. Trends in Mathematics 27. (2013).

[11] H.H. Bang. A property of infinitely differentiable functions. *Proc. Amer. Math. Soc.* **108**(1990), 73-76.

[12] E. Bayro-Corrochano, N. Trujillo, M. Naranjo. Quaternion Fourier descriptors for the preprocessing and recognition of spoken words using images of spatiotemporal representations. *J. Math. Imaging Vision.* **28**(2007), 179-190 .

[13] S. Breuils, K. Tachibana, E. Hitzer. New applications of Clifford's geometric algebra. *Adv. Appl. Clifford Algebr.* **32** (2022), 1-39.

[14] L. Chen, K. Kou, M. Liu. Pitt's inequality and the uncertainty principle associated with the quaternion Fourier transform. *J. Math. Anal. Appl.* **423** (2015), 681-700.

[15] D. Constales, H. De Bie, P. Lian. A new construction of the Clifford-Fourier kernel. *J. Fourier Anal. Appl.* **23** (2017) 462-483.

[16] D. Constales, H. De Bie, P. Lian. Explicit formulas for the Dunkl dihedral kernel and the (κ, a)-generalized Fourier kernel. *J. Math. Anal. Appl.* **460**(2018), 900-926.

[17] M.G. Cowling, F. Price. Bandwidth versus time concentration: the Heisenberg-Pauli-Weyl inequality. *SIAM J. Math. Anal.*. **15**(1984), 151-165.

[18] M.J. Craddock, J.A. Hogan. The Fractional Clifford-Fourier Kernel. *J. Fourier Anal. Appl.* **19** (2013), 683-711.

[19] H. De Bie. Clifford algebras, Fourier transforms and quantum mechanics. *Math. Methods Appl. Sci.* **35** (2012), 2198-2228.

[20] H. De Bie. The kernel of the radially deformed Fourier transform. *Integral Transforms Spec. Funct.* **24** (2013), 1000-1008.

[21] H. De Bie. Fourier transforms in Clifford analysis in Colombo, Sabadini, Shapiro, eds. Reference Work Operator Theory, pp 1651-1672, Springer Basel, 2015.

[22] H. De Bie, R. Oste and J. Van der Jeugt. Generalized Fourier transforms arising from the enveloping algebras of $sl(2)$ and $osp(1|2)$. *Int. Math. Res. Not. IMRN.* **15** (2016), 4649-4705.

[23] H. De Bie, R. Oste and Z. Yang. Fourier Kernels Associated with the Clifford–Helmholtz System. *Complex Anal. Oper. Theory.* https://doi.org/10.1007/s11785-023-01463-4. **18** (2024).

[24] H. De Bie, Y. Xu. On the Clifford-Fourier transform. *Int. Math. Res. Not. IMRN.* **22** (2011), 5123-5163.

[25] H. De Bie, N. De Schepper. Fractional Fourier transforms of hypercomplex signals. *Signal Image Video Process.* **6**(2012), 381-388.

[26] H. De Bie, N. De Schepper. The Fractional Clifford-Fourier Transform. *Complex Anal. Oper. Theory.* **6**(2012), 1047-1067.

[27] H. De Bie, N. De Schepper and F. Sommen. The class of Clifford-Fourier transforms. *J. Fourier Anal. Appl.*. **17** (2011), 1198-1231.

[28] H. De Bie, N. De Schepper, D. Eelbode. New Results on the Radially Deformed Dirac Operator. *Comples Anal. Oper. Theory.* **11** (2017), 1283-1307.

[29] H. De Bie, P. Lian, F. Maes. Bounds for the kernel of the (κ, a)-generalized Fourier transform. https://doi.org/10.48550/arXiv.2310.14229. 2023.

[30] H. De Bie, B. Ørsted, P. Somberg, V. Souček. Dunkl operators and a family of realizations of $\mathfrak{osp}(1|2)$. *Trans. Am. Math. Soc.* **364** (2012), 3875-3902.

[31] H. De Bie, B. Ørsted, P. Somberg, V. Souček. The Clifford deformation of the Hermite semigroup. SIGMA. **9** (2013), 010.

[32] H. De Bie, Z. Yang. On the radially deformed Fourier transform. arXiv: 2404.06839. (2024).

[33] R. Delanghe, F. Sommen and V. Soucek. Clifford algebra and spinor-valued functions. Vol. 53 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1992.

[34] G. Doetsch. Introduction to the theory and application of the Laplace transformation. Springer Science and Business Media, 2012.

[35] L. Dorst, D. Fontijne, S. Mann. Geometric Algebra for Computer Science. Morgan Kaufmann: Burlington, MA 01803, USA, 2007.

[36] C.F. Dunkl, Y. Xu. Orthogonal Polynomials of Several Variables. Cambridge University Press, 2014.

[37] T.A. Ell, N. Le Bihan, S.J. Sangwine. Quaternion Fourier transforms for signal and image processing. John Wiley Sons, (2014).

[38] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomt. Higher transcendental functions, Vol. II, McGraw-Hill, New York, 1953.

[39] A. Erdélyi (Ed.). Tables of Integral Transforms, Vol. I. McGraw-Hill, New York, 1954.

[40] W.G. Faris. Inequalities and uncertainty principles. *J. Math. Phys.* **19** (1978), 461-466.

[41] M. Fei, Y. Xu , J. Yan. Real Paley-Wiener theorem for the quaternion Fourier transform. *Complex Var. Elliptic Equ.* **62** (2017), 1072-1080.

[42] G.B. Folland, A. Sitaram. The Uncertainty Principle: A Mathematical Survey. *J. Fourier Anal. Appl.*. **3** (1997), 207-238.

[43] R. Garra, R. Garrappa. The Prabhakar or three parameter Mittag-Leffler function: Theory and application. *Commun. Nonlinear Sci. Numer. Dimul.* **56**(2018), 314-329.

[44] S. Ghobber, P. Jaming. Uncertainty principles for integral operators. *Studia Math.* **220** (2014), 197-220.

[45] R. Gorenflo, A.A. Kilbas, F. Mainardi, S.V. Rogosin. Mittag-Leffler functions, related topics and applications. Springer Berlin, 2014.

[46] H.W. Gould. Some Generalizations of Vandermonde's Convolution. The American Mathematical Monthly, Vol. 63. 84-91 (1956).

[47] E. Hansen. Fourier Transforms: Principles and Applications. Wiley, 2014.

[48] W. Heisenberg. Über den anschaulichen Inhalt der quantentheoretischen $\mathcal{K}$inematric und Mechanik. *Zeit. Physik.* **43** (1927), 172-198.

[49] E. Hitzer. Directional uncertainty principle for quaternion Fourier transform. *Adv. Appl. Clifford Algebr.* **20** (2010), 271-284.

[50] Y.E. Haoui, S. Fahlaoui. The uncertainty principle for the two-sided quaternion Fourier transform. *Mediterr. J. Math.* **14** (2017), 1-8.

[51] E.H. Kennard. Zur Quantenmechanik einfacher Bewegungstypen. *Zeit. Physik.* **44** (1927), 326-352.

[52] S. Li, M. Fei. Pitt's inequality and logarithmic uncertainty principle for the Clifford-Fourier transform. *Adv. Appl. Clifford Algebr.* **33**(2023), 1-12.

[53] S. Li, J. Leng, M. Fei. Spectrums of functions associated to the fractional Clifford-Fourier transform. *Adv. Appl. Clifford Algebr.* **30** (2020), 1-15.

[54] S. Li, J. Leng, M. Fei. Paley-Wiener-Type theorems for the Clifford-Fourier transform. *Math. Meth. Appl. Sci.* **42** (2019), 6101-6113.

[55] P. Lian. Uncertainty principle for the quaternion Fourier transform. *J. Math. Anal. Appl.* **467** (2018), 1258-1269.

[56] A.M. Mathai, H.J. Haubold. Special Functions for Applied Scientists. Springer, New York, 2008.

[57] F.W.J. Olver, D.W. Lozier, R.F. Boisvert, C.W. Clark. *NIST* handbook of mathematical functions. Cambridge: Cambridge University Press, 2010.

[58] R. Oste. The Clifford-Helmholtz system and related Fourier transforms. Master thesis, Ghent University, 2013.

[59] H. Ozaktas, Z. Zalevsky and M. Kutay. The fractional Fourier transform. Wiley, Chichester, 2001.

[60] J.F. Price. Inequalities and local uncertainty principles. *J. Math. Phys.* **24** (1983), 1711-1714.

[61] J.F. Price. Sharp local uncertainty inequalities. *Studia Math..* **85** (1987), 37-45.

[62] I. Podlubny. Fractional differential equations. Academic press, San Diego, 1999.

[63] A.P. Prudnikov, Y.A. Brychkov, O.I. Marichev. Integrals and Series, Vol. V: Inverse Laplace Transforms. Gordon, Breach Science Publishers, Philadelphia, PA, 1992.

[64] J. L. Schiff. The Laplace transform: theory and application. Springer, 1999.

[65] V. Serov. Fourier series, Fourier transform and their applications to Mathematical Physics. Applied Mathematical Sciences, 2017.

[66] F. Sommen. Hypercomplex Fourier and Laplace transforms I. *Illinois J. Math.* **26**(2) (1982), 332–352.

[67] F. Sommen. Hypercomplex Fourier and Laplace transforms II. *Complex Variables: Theory and Application.* **1** (1983), 209–238.

[68] F. Sommen. Special functions in Clifford analysis and axial symmetry. *J. Math. Anal. Appl.* **130** (1988), 110–133.

[69] M.R. Spiegel, S. Lipschutz and J. Liu. Schaum's Outline of Mathematical Handbook of Formulas and Tables (Third edition). McGraw-Hill, 2009.

[70] G. Szegö. Orthogonal polynomials. American Mathematical Society Colloquium Publications, Vol. 23, 4th edn. American Mathematical Society, Providence, 1975.

[71] S. Thangavelu, Y. Xu. Convolution operator and maximal function for the Dunkl transform. *J. Anal. Math.* **97** (2005), 25-55.

[72] V.K. Tuan. On the Paley-Wiener theorem. Theory of Functions and Applications. Collection of Works dedicated to the Memory of Mkhitar M. Djrbashian. Yerevan: Louys Publishing House; pp: 193-196, 1995.

[73] G.N. Watson. A treatise on the theory of Bessel functions. Cambridge University Press, Cambridge, England; The Macmillan Company, New York, 1944.

[74] H. Weyl. Gruppentheorie und Quantenmechanik. S. Hirzel, Leipzig. Revised English edition: The Theory of Groups and Quantum Mechanics Methuen, London, 1931; reprinted by Dover, New York, 1950.

[75] Z. Yang. Further properties of Fourier transforms associated to the Clifford-Helmholtz system. Preprint.