Geometric properties of ternary fuzzy relations

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Abstract

Ternary fuzzy relations, and fuzzy betweenness relations in particular, are witnessing increasing attention in recent years. A key reason is that *axiomatic properties* of ternary fuzzy relations seem to be ideally suited to capture *geometric characteristics* of the abstract notion of betweenness. In this paper, we introduce several new properties of ternary fuzzy relations, including the Peano property, the Pasch property and the sand-glass property, that can be qualified as *geometric properties*. We investigate their interrelationships as well as their connections with various types of fuzzy betweenness relations. Additionally, in the context of our study of the Pasch property and the sand-glass property, we introduce the convexity property of ternary fuzzy relations by taking inspiration from the solid theoretical basis of the theory of fuzzy convex structures.

Keywords: Ternary fuzzy relation, Fuzzy betweenness relation, Fuzzy convex structure, Peano property, Pasch property, Sand-glass property

1. Introduction

Ternary relations are seldom studied in a generic manner, a rare exception being a recent study of the composition of ternary relations [5]. Indeed, it is common practice to limit a study to ternary relations satisfying some set of additional conditions, such as those leading to various types of betweenness relations. The origin of betweenness relations can be traced back to the letter by Gauss to Bolyai in 1832, pointing out the absence of betweenness postulates in the Euclidean treatment. Only fifty years later, Pasch [21] first formalized the notion of a betweenness relation and used it to describe when an element lies in between two other ones. Later, Huntington [12] and Huntington and Kline [13] further studied different postulates for betweenness relations. Subsequently, many scholars investigated betweenness relations that could be derived from various mathematical structures. For instance, Blumenthal and Ellis [4], Chvátal [6], and Smiley and Transue [35] characterized betweenness relations induced by metrics. Düvelmeyer and Wenzel [8], Fishburn [9],

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Rautenbach and Schäfer [25], and Sholander [33] investigated betweenness relations induced by binary relations such as order relations, weak order relations and semiorder relations. Cibulskis [7], Hedlíková and Katrinák [11], Ploščica [24], Padmanabhan [17] and Transue [36] studied betweenness relations induced by lattices. In particular, Smiley [34] investigated how betweenness relations induced by a real vector space, a metric or a lattice on the same given set relate. Recently, Pérez-Fernández and De Baets [23] introduced a type of betweenness relation for defining the concept of a monometric, which is closely related to that of a metric but substitutes the symmetry and the triangle inequality by the compatibility with a betweenness relation. Zhang et al. [42] adopted the latter betweenness relation to induce topologies. It is important to stress again that all the works above consider different axiomatizations, resulting in different types of betweenness relations.

With the development of fuzzy set theory, also ternary relations have been generalized to the fuzzy setting. In different contexts, ternary fuzzy relations are required to satisfy additional conditions. In particular, this leads to different types of fuzzy betweenness relations. Jacas and Recasens [14, 15] first introduced a notion of a fuzzy betweenness relation, taking values in the real unit interval equipped with a strict Archimedean t-norm T, and showed that there is a one-to-one correspondence between these fuzzy betweenness relations and separating T-equivalences. Zhang et al. [43] proposed fuzzy (pre)betweenness relations with respect to a fuzzy equivalence relation, taking values in a bounded lattice equipped with a t-norm, and discussed the connections with fuzzy orders. Additionally, given a t-norm on the real unit interval, Zhang et al. [44] presented three methods for constructing a fuzzy betweenness relation from a given metric by making use of the pseudo-inverse of either a continuous additive generator or a continuous multiplicative generator of the t-norm, or by making use of the residual implication of the t-norm. Shi [30] studied different postulates for fuzzy betweenness relations on a fuzzy metric space or on a fuzzy normed space. In the context of fuzzy convex structures, Shi et al. [31, 32] proposed lattice-valued betweenness relations and investigated their relationships with lattice-valued interval operators, lattice-valued orders and lattice-valued betweenness fields.

As in the classical setting, fuzzy betweenness relations can be regarded as ternary fuzzy relations that satisfy additional *betweenness axioms* describing some classical properties such as reflexivity, symmetry and transitivity. However, as the main aim is to describe the geometric characteristics of a point lying in between two other points, additional properties with a truly geometric flavor should come into play. Inspiration can be taken from the study of fuzzy convex structures [26, 28, 29]. In that context, Shi et al. [31, 32] laid bare the relationship between fuzzy betweenness relations and fuzzy interval operators [19, 20, 38, 41]. It is well known that (fuzzy) interval operators are basic tools to characterize geometric properties of (fuzzy) convex structures, such as the Peano property, the Pasch property and the sand-glass property. That is to say, (fuzzy) interval operators provide a framework for studying various geometric properties [27, 39, 40]. So the relationship between fuzzy betweenness relations and fuzzy interval operators motivates us to study geometric properties of ternary fuzzy relations. Consequently, in the present paper, we treat the axioms for defining a fuzzy betweenness relation as geometric properties of a ternary fuzzy relation. From this point of view, we propose new properties of ternary fuzzy relations, including the Peano property, the Pasch property and the sand-glass property. Moreover, we introduce the convexity property, which originates from the theory of convex structures, and study the connections

with the fuzzy betweenness axioms.

The remainder of this paper is organized as follows. We recall some basic notions and results related to residuated lattices, fuzzy sets, fuzzy convex structures and fuzzy betweenness relations in Section 2. The Peano property of ternary fuzzy relations and its connections with fuzzy betweenness relations are studied in Section 3. Taking (fuzzy) convex structures as point of departure, the (fuzzy) convexity axiom is introduced for ternary (fuzzy) relations in Section 4. The Pasch property of ternary fuzzy relations and its connections with fuzzy betweenness relations are studied in Section 5. Finally, in Section 6, the sand-glass property of ternary fuzzy relations is proposed and its connections with fuzzy betweenness relations as well as the Peano property and the Pasch property are investigated.

2. Preliminaries

Throughout this paper, X always denotes a nonempty set. In this section, we recall some basic notions and results related to residuated lattices, fuzzy sets, convex structures and ternary (fuzzy) relations.

2.1. Residuated lattices and fuzzy sets

Definition 2.1. A complete residuated lattice is a triple $(L, \leq, *)$, where $(L, \leq)$ is a complete lattice with the top element $\top$ and the bottom element $\bot$, and $*$ is a commutative, associative binary operation such that

(1) $\top$ is the unit element of $*$, *i.e.*, $\top * a = a$, for any $a \in L$;

(2) for any $a \in L$ and $\{b_j \mid j \in J\} \subseteq L$,

$$a * \bigvee_{j\in J} b_j = \bigvee_{j\in J} (a * b_j)\,.$$

The set of all subsets of X is denoted by $P(X)$ and the characteristic function of $A \in P(X)$ is denoted by χ_A. A subset $\mathcal{A}$ of $P(X)$ is called *directed* if for any $A, B \in \mathcal{A}$, there exists $C \in \mathcal{A}$ such that $A \subseteq C$ and $B \subseteq C$.

An L-set on X is a map from X to L. The family of all L-sets on X is called the *L-powerset* of X and is denoted by L^X. Equipping L^X with the pointwise order $\leq$, *i.e.*, $U \leq V$ if $U(x) \leq V(x)$, for any $x \in X$, we obtain a complete lattice $(L^X, \leq)$. The top element and the bottom element of L^X are denoted by $\underline{\top}$ and $\underline{\bot}$, respectively. The definition of a *directed* subset of L^X is identical to the one above, using the pointwise order $\leq$ instead of inclusion. All algebraic operations on L can be extended to the L-powerset L^X in a pointwise manner, *i.e.*, for any $U, V \in L^X$, $\alpha \in L$ and $x \in X$:

(i) $(U \vee V)(x) = U(x) \vee V(x)$;

(ii) $(U \wedge V)(x) = U(x) \wedge V(x)$;

(iii) $(U * V)(x) = U(x) * V(x)$;

(iv) $(\alpha * U)(x) = \alpha * U(x)$.

2.2. Convex structures

By making abstraction of the common characteristics of convex sets in different mathematical settings, van de Vel [27] introduced the concept of a convex structure.

Definition 2.2 ([27])**.** A convex structure on X is a subset C of $P(X)$ that satisfies:

(CS1) $\emptyset, X \in \mathrm{C}$;

(CS2) $\{C_k\}_{k\in K} \subseteq \mathrm{C}$ implies $\bigcap_{k\in K} C_k \in \mathrm{C}$;

(CS3) if $\{C_j\}_{j\in J} \subseteq \mathrm{C}$ is directed, then $\bigcup_{j\in J} C_j \in \mathrm{C}$.

For a convex structure C on X, the pair (X, C) is called a convex space.

The notion of a convex structure has been generalized to the fuzzy setting in different ways. Here, we recall the definition of an L-convex structure.

Definition 2.3 ([16, 26])**.** An L-convex structure on X is a subset $\mathscr{C}$ of L^X that satisfies:

(LCS1) $\underline{\bot}, \underline{\top} \in \mathscr{C}$;

(LCS2) $\{U_k\}_{k\in K} \subseteq \mathscr{C}$ implies $\bigwedge_{k\in K} U_k \in \mathscr{C}$;

(LCS3) if $\{U_j\}_{j\in J} \subseteq \mathscr{C}$ is directed, then $\bigvee_{j\in J} U_j \in \mathscr{C}$.

For an L-convex structure $\mathscr{C}$ on X, the pair $(X, \mathscr{C})$ is called an L-convex space.

Next, we present the notion of a stratified L-convex structure with L a complete residuated lattice.

Definition 2.4. An L-convex structure $\mathscr{C}$ on X is called *stratified* if it satisfies:

(SLCS) $U \in \mathscr{C}$ and $\alpha \in L$ imply $\alpha * U \in \mathscr{C}$.

For a stratified L-convex structure $\mathscr{C}$ on X, the pair $(X, \mathscr{C})$ is called a stratified L-convex space.

Note that the notion of a stratified L-convex structure coincides with that in the sense of Pang and Shi [18] when L is a completely distributive lattice, and that in the sense of Wang and Shi [37] when L is the real unit interval equipped with a left continuous t-norm.

2.3. Ternary (fuzzy) relations

A ternary relation on X is a subset T of X^3. The following properties of a ternary relation are of importance in this paper (see, e.g., [10, 23, 42]).

(T1) *Symmetry*: for any $a, b, c \in X$,

$$(a, c, b) \in \mathrm{T} \Leftrightarrow (b, c, a) \in \mathrm{T}\,.$$

(T2) *Reflexivity*: for any $a, b \in X$,

$$(a, b, b) \in \mathrm{T}\,.$$

(T3) *Antisymmetry*: for any $a, b, c \in X$,

$$((a, b, c) \in \mathrm{T} \wedge (a, c, b) \in \mathrm{T}) \Rightarrow b = c\,.$$

(T4) *Type-1 transitivity*: for any $a, b, c, o \in X$,

$$((o, a, b) \in \mathrm{T} \wedge (o, b, c) \in \mathrm{T}) \Rightarrow (o, a, c) \in \mathrm{T}\,.$$

(T5) *Type-2 transitivity*: for any $a, b, c, o \in X$,

$$((o, a, b) \in \mathrm{T} \wedge (o, b, c) \in \mathrm{T}) \Rightarrow (a, b, c) \in \mathrm{T}\,.$$

By selecting different subsets of the above (and more) properties, different types of betweenness relations have been proposed [10, 22, 23]. Recently, the one in the sense of Pérez-Fernéndez and De Baets [23] received some attention: it is a basic betweenness relation in the sense of [3] that additionally satisfies axiom (R7) defined in the same paper. Note that (R7) in [3] is exactly (T3) herein.

Definition 2.5 ([23])**.** A ternary relation T on X is called a betweenness relation if it satisfies (T1), (T4) and

(T2&3) *Closure*: for any $a, b, c \in X$,

$$((a, b, c) \in \mathrm{T} \wedge (a, c, b) \in \mathrm{T}) \Leftrightarrow b = c\,.$$

It is obvious that (T2&3) is the conjunction of (T2) and (T3).

A ternary L-relation is a map from X^3 to L. Axioms (T1)–(T5) have been generalized to the fuzzy setting, see, e.g., [32, 43]:

(LT1) *Symmetry*: for any $a, b, c \in X$,

$$T(a, c, b) = T(b, c, a)\,.$$

(LT2) *Reflexivity*: for any $a, b \in X$,

$$T(a, b, b) = \top .$$

(LT3) $*$*-antisymmetry*: for any $a, b, c \in X$,

$$T(a, b, c) * T(a, c, b) \neq \bot \Rightarrow b = c .$$

(LT4) *Type-1 transitivity*: for any $a, b, c, o \in X$,

$$T(o, a, b) * T(o, b, c) \leq T(o, a, c) .$$

(LT5) *Type-2 transitivity*: for any $a, b, c, o \in X$,

$$T(o, a, b) * T(o, b, c) \leq T(a, b, c) .$$

It is easy to see that (LT3) is a fuzzy counterpart of (T3). There exists another weaker version (LT3') that is called crisp antisymmetry:

(LT3') *Crisp antisymmetry*: for any $a, b, c \in X$,

$$T(a, b, c) = T(a, c, b) = \top \Longrightarrow b = c .$$

In case of the real unit interval $L = [0, 1]$ equipped with a t-norm, Zhang et al. [43] proposed several types of fuzzy betweenness relations. Here, we generalize these notions to the L-valued case, but for some of them we will use different axioms than those in [43].

Definition 2.6. A ternary L-relation on X is called an

(1) L-betweenness relation if it satisfies (LT1)–(LT4);

(2) L-strobetweenness relation if it satisfies (LT1)–(LT5);

(3) L-prebetweenness relation if it satisfies (LT1), (LT2) and (LT4);

(4) L-semibetweenness relation if it satisfies (LT1)–(LT3);

(5) L-tolerance relation if it satisfies (LT1) and (LT2).

Remark 2.7. As the terminology in Definition 2.6 slightly differs from existing ones, we provide some clarification:

(1) (LT1)–(LT4) are used to define a strong $*$-betweenness relation in the sense of Zhang et al. [43] in the case of a bounded lattice with a t-norm.

(2) (LT1)–(LT5) are used to define a fuzzy betweenness relation in the sense of Shi et al. [32], which is closely related to fuzzy geometric interval operators in the theory of fuzzy convex structures.

(3) (LT1), (LT2) and (LT4) are used to define a $*$-prebetweenness relation in the sense of Zhang et al. [44] in the case of the real unit interval equipped with a t-norm.

(4) The adjective "semi" refers to the fact that a ternary relation satisfying (T1)–(T3) can be constructed from a semimetric in a natural way. Explicitly, let $\rho : X \times X \to [0, \infty[$ be a semimetric, *i.e.*, $\rho(a, b) = \rho(b, a)$ and $\rho(a, b) = 0 \Leftrightarrow a = b$, for any $a, b \in X$. Adopting the standard way of constructing a betweenness relation from a metric, we obtain that

$$\mathrm{T}_\rho = \{(a, b, c) \in X^3 \mid \rho(a, c) = \rho(a, b) + \rho(b, c)\}$$

satisfies (T1)–(T3).

(5) The term "tolerance" refers to the fact that a reflexive and symmetric binary relation is usually called a tolerance relation (see, e.g., [1, 2]).

In the sequel, let $\chi_=$ denote the characteristic mapping of the equality relation "=".

Proposition 2.8. *Consider a ternary L-relation T on X.*

(1) *If T is reflexive and symmetric, then*

$$T(a, a, b) = T(a, b, b) = \top, \text{ for any } a, b \in X\,.$$

(2) *If T is an L-semibetweenness relation, then*

$$T(a, b, a) = \chi_=(a, b), \text{ for any } a, b \in X\,.$$

Proof. (1) is straightforward. We only verify (2). If $a = b$, then it follows from the reflexivity of T that

$$T(a, b, a) = \top = \chi_=(a, b)\,.$$

If $a \neq b$, then it follows from (1) and (LT3) that

$$T(a, b, a) = T(a, b, a) * T(a, a, b) = \bot = \chi_=(a, b)\,.$$

□

For the development of the subsequent sections, we propose the following definition.

Definition 2.9. A ternary L-relation T on X is called idempotent if it satisfies:

(LT6) *Idempotence*: for any $a, b \in X$,

$$T(a, b, a) = \chi_=(a, b)\,.$$

In the classical setting, the idempotence can be described as follows: "if the two endpoints are the same, then there is exactly one point, *i.e.*, the same endpoint, between them."

3. Peano ternary (L-)relations

A ternary relation can be used to describe the positional relationships among three points. By imposing additional conditions on a ternary relation, particular types of ternary relations have been introduced (see, e.g., Definition 2.6). Following this approach, we introduce the notion of a Peano relation by imposing the Peano property on a ternary relation.

Definition 3.1. A ternary relation T on X is called a *Peano* relation on X if it satisfies the following property (see Figure 1):

Peano Property: for any $a, b, c, x, y \in X$,

$$((a, x, b) \in \mathrm{T} \wedge (c, y, x) \in \mathrm{T}) \Rightarrow (\exists z \in X)\, ((a, z, c) \in \mathrm{T} \wedge (b, y, z) \in \mathrm{T})\,.$$

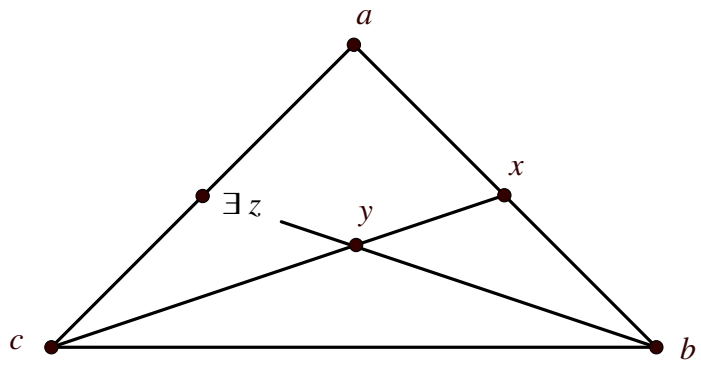

Figure 1: Peano property

Remark 3.2. The term *Peano property* originates from the theory of convex structures [27], which can also describe the positional relationships among different points. Here, we also introduce this term in the context of ternary relations.

Example 3.3. Let $(X, +, \cdot)$ be a real vector space. Define $\mathrm{T} \subseteq X^3$ as

$$\mathrm{T} = \{(a, c, b) \mid (\exists t \in [0, 1])(c = t \cdot a + (1 - t) \cdot b)\}\,.$$

Then T satisfies the Peano property.

Now, let us generalize the Peano property to the fuzzy setting.

Definition 3.4. A ternary L-relation T on X is called a *Peano* L-relation on X if it satisfies the following property:

Peano Property: for any $a, b, c, x, y \in X$,

$$T(a, x, b) * T(c, y, x) \leq \bigvee_{z \in X} T(a, z, c) * T(b, y, z)\,.$$

It can be easily observed that the Peano property in the fuzzy setting reduces to the classical one when $L = \{0, 1\}$.

Example 3.5. Let $(X, +, \cdot)$ be a real vector space and $\alpha \in L \setminus \{\bot, \top\}$. Define $T : X^3 \to L$ by, for any $(a, c, b) \in X^3$:

$$T(a,c,b) = \begin{cases} \top & , (c = a) \vee (c = b), \\ \alpha & , (\exists t \in]0,1[)(c = t \cdot a + (1-t) \cdot b) \wedge (a \neq b), \\ \bot & , \text{elsewhere.} \end{cases}$$

It is routine to verify that T satisfies the Peano property.

Given a ternary relation T on X, for any $a, b \in X$, a subset of X can be determined by

$$\mathrm{T}(a, \cdot, b) = \{x \in X \mid (a, x, b) \in \mathrm{T}\}.$$

For convenience, we denote $\mathrm{T}(a, \cdot, b)$ by $a \bullet b$. Similarly, if T is a ternary L-relation on X, an L-set $T(a, \cdot, b) : X \to L$ can be determined by, for any $x \in X$:

$$T(a, \cdot, b)(x) = T(a, x, b).$$

We will also use $a \bullet b$ to denote $T(a, \cdot, b)$ in the fuzzy case since no confusion can arise. The above process essentially defines a map $\bullet : X \times X \to L^X$. Further, we can extend the domain of this map from $X \times X$ to $P(X) \times P(X)$ and $L^X \times L^X$, respectively. Specifically, for any $A, B \in P(X)$ and $U, V \in L^X$,

$$(A \bullet B)(x) = \bigvee_{a \in A} \bigvee_{b \in B} T(a, x, b), \text{ for any } x \in X,$$

and

$$(U \bullet V)(x) = \bigvee_{a \in X} \bigvee_{b \in X} T(a, x, b) * U(a) * V(b), \text{ for any } x \in X.$$

In particular,

$$(U \bullet \chi_B)(x) = \bigvee_{a \in X} \bigvee_{b \in B} T(a, x, b) * U(a), \text{ for any } x \in X,$$

and

$$(U \bullet \chi_{\{b\}})(x) = \bigvee_{a \in X} T(a, x, b) * U(a), \text{ for any } x \in X.$$

In short, $U \bullet \chi_B$ and $U \bullet \chi_{\{b\}}$ will be denoted by $U \bullet B$ and $U \bullet b$, respectively.

By means of the map $\bullet$, we can provide some characterizations of a Peano L-relation. To this end, the following lemma is essential.

Lemma 3.6. *Let T be a ternary L-relation on X. Then the following statements are equivalent:*

(1) *T is symmetric;*

(2) *$a \bullet b = b \bullet a$, for any $a, b \in X$;*

(3) *$A \bullet B = B \bullet A$, for any $A, B \in P(X)$;*

(4) *$U \bullet b = b \bullet U$, for any $b \in X$ and $U \in L^X$;*

(5) $U \bullet B = B \bullet U$, *for any* $U \in L^X$ *and* $B \in P(X)$;

(6) $U \bullet V = V \bullet U$, *for any* $U, V \in L^X$.

Proof. It is easy and is omitted. □

Theorem 3.7. *Let T be a symmetric ternary L-relation on X. Then the following statements are equivalent:*

(1) *T satisfies the Peano property;*

(2) $(a \bullet b) \bullet c = a \bullet (b \bullet c)$, *for any* $a, b, c \in X$.

Proof. (1)⇒(2): Consider any $a, b, c, x \in X$. Then

$$((a \bullet b) \bullet c)(x) = \bigvee_{y \in X} T(y, x, c) * (a \bullet b)(y) = \bigvee_{y \in X} T(y, x, c) * T(a, y, b)$$

and

$$(a \bullet (b \bullet c))(x) = \bigvee_{z \in X} T(a, x, z) * (b \bullet c)(z) = \bigvee_{z \in X} T(a, x, z) * T(b, z, c)\,.$$

By (LT1) and the L-Peano property, we have

$$\begin{aligned} T(y, x, c) * T(a, y, b) &= T(b, y, a) * T(c, x, y) \\ &\leq \bigvee_{z \in X} T(b, z, c) * T(a, x, z) \\ &= (a \bullet (b \bullet c))(x)\,. \end{aligned}$$

This implies that $(a \bullet b) \bullet c \leq a \bullet (b \bullet c)$. Similarly, it can be verified that $a \bullet (b \bullet c) \leq (a \bullet b) \bullet c$.

(2)⇒(1): Consider any $a, b, c, x, y \in X$. Then

$$\begin{aligned} T(a, x, b) * T(c, y, x) &\leq \bigvee_{z \in X} T(a, z, b) * T(c, y, z) \\ &= \bigvee_{z \in X} T(a, z, b) * T(z, y, c) \\ &= ((a \bullet b) \bullet c)(y) = (c \bullet (a \bullet b))(y) \\ &= ((c \bullet a) \bullet b)(y) = ((a \bullet c) \bullet b)(y) \\ &= \bigvee_{z \in X} T(a, z, c) * T(z, y, b) \\ &= \bigvee_{z \in X} T(a, z, c) * T(b, y, z)\,. \end{aligned}$$

This shows that T satisfies the Peano property. □

Lemma 3.8. *Let T be a ternary L-relation on X. Then the following properties hold:*

(1) $(A \bullet B) \bullet C = \bigvee_{a\in A}\bigvee_{b\in B}\bigvee_{c\in C}(a \bullet b) \bullet c$, *for any* $A, B, C \in P(X)$.

(2) $A \bullet (B \bullet C) = \bigvee_{a\in A}\bigvee_{b\in B}\bigvee_{c\in C} a \bullet (b \bullet c)$, *for any* $A, B, C \in P(X)$.

Proof. (1) Consider any $x \in X$. Then

$$\begin{aligned}((A \bullet B) \bullet C)(x) &= \bigvee_{y\in X}\bigvee_{c\in C} T(y, x, c) * (A \bullet B)(y)\\ &= \bigvee_{y\in X}\bigvee_{c\in C} T(y, x, c) * \bigvee_{a\in A}\bigvee_{b\in B} T(a, y, b)\\ &= \bigvee_{a\in A}\bigvee_{b\in B}\bigvee_{c\in C}\bigvee_{y\in X} T(y, x, c) * T(a, y, b)\\ &= \bigvee_{a\in A}\bigvee_{b\in B}\bigvee_{c\in C}((a \bullet b) \bullet c)(x)\,.\end{aligned}$$

(2) It can be verified in a similar way. □

Theorem 3.9. *Let T be a symmetric ternary L-relation on X. Then the following statements are equivalent:*

(1) *T satisfies the Peano property;*

(2) $(A \bullet B) \bullet C = A \bullet (B \bullet C)$, *for any* $A, B, C \in P(X)$.

Proof. (1)⇒(2): By Theorem 3.7 and Lemma 3.8, we have

$$\begin{aligned}(A \bullet B) \bullet C &= \bigvee_{a\in A}\bigvee_{b\in B}\bigvee_{c\in C}(a \bullet b) \bullet c\\ &= \bigvee_{a\in A}\bigvee_{b\in B}\bigvee_{c\in C} a \bullet (b \bullet c)\\ &= A \bullet (B \bullet C)\,.\end{aligned}$$

(2)⇒(1): Consider any $a, b, c \in X$ and let $A = \{a\}$, $B = \{b\}$ and $C = \{c\}$. Then it follows that

$$(a \bullet b) \bullet c = (A \bullet B) \bullet C = A \bullet (B \bullet C) = a \bullet (b \bullet c)\,.$$

By Theorem 3.7, we know that T satisfies the Peano property. □

Lemma 3.10. *Let T be a ternary L-relation on X. Then the following statements hold:*

(1) $(U \bullet V) \bullet W = \bigvee_{a\in X}\bigvee_{b\in X}\bigvee_{c\in X} U(a) * V(b) * W(c) * ((a \bullet b) \bullet c)$, *for any* $U, V, W \in L^X$.

(2) $U \bullet (V \bullet W) = \bigvee_{a\in X}\bigvee_{b\in X}\bigvee_{c\in X} U(a) * V(b) * W(c) * (a \bullet (b \bullet c))$, *for any* $U, V, W \in L^X$.

Proof. (1) Consider any $x \in X$. Then

$$\begin{aligned}((U \bullet V) \bullet W)(x) &= \bigvee_{y\in X}\bigvee_{c\in X} T(y,x,c) * (U \bullet V)(y) * W(c)\\ &= \bigvee_{y\in X}\bigvee_{c\in X} T(y,x,c) * \bigvee_{a\in X}\bigvee_{b\in X}(T(a,y,b) * U(a) * V(b)) * W(c)\\ &= \bigvee_{a\in X}\bigvee_{b\in X}\bigvee_{c\in X} U(a) * V(b) * W(c) * \bigvee_{y\in X} T(a,y,b) * T(y,x,c)\\ &= \bigvee_{a\in X}\bigvee_{b\in X}\bigvee_{c\in X} U(a) * V(b) * W(c) * ((a \bullet b) \bullet c)(x)\,.\end{aligned}$$

(2) It can be verified in a similar way. □

Theorem 3.11. *Let T be a symmetric ternary L-relation on X. Then the following statements are equivalent:*

(1) *T satisfies the Peano property;*

(2) *$(U \bullet V) \bullet W = U \bullet (V \bullet W)$, for any $U, V, W \in L^X$.*

Proof. (1)⇒ (2): By Theorem 3.7 and Lemma 3.10, we have

$$\begin{aligned}(U \bullet V) \bullet W &= \bigvee_{a\in X}\bigvee_{b\in X}\bigvee_{c\in X} U(a) * V(b) * W(c) * ((a \bullet b) \bullet c)\\ &= \bigvee_{a\in X}\bigvee_{b\in X}\bigvee_{c\in X} U(a) * V(b) * W(c) * (a \bullet (b \bullet c))\\ &= U \bullet (V \bullet W)\,.\end{aligned}$$

(2)⇒ (1): For any $A, B, C \in P(X)$, let $U = \chi_A$, $V = \chi_B$ and $W = \chi_C$. By Lemma 3.10, we have

$$(A \bullet B) \bullet C = (U \bullet V) \bullet W = U \bullet (V \bullet W) = A \bullet (B \bullet C)\,.$$

By Theorem 3.9, we know that T satisfies the Peano property. □

Corollary 3.12. *Let T be a symmetric ternary L-relation on X. Then the following statements are equivalent:*

(1) *T satisfies the Peano property;*

(2) *$(a \bullet b) \bullet c = a \bullet (b \bullet c)$, for any $a, b, c \in X$;*

(3) *$(A \bullet B) \bullet C = A \bullet (B \bullet C)$, for any $A, B, C \in P(X)$;*

(4) *$(U \bullet V) \bullet W = U \bullet (V \bullet W)$, for any $U, V, W \in L^X$.*

There exists a close relationship between L-betweenness relations and Peano L-relations. To unveil this relationship, we first give the following proposition.

Proposition 3.13. *Let T be a symmetric and idempotent ternary L-relation on X. If T satisfies the Peano property, then T satisfies* (LT4).

Proof. Consider any $a, b, c, o \in X$. Then it follows that

$$\begin{aligned} T(o,a,b) * T(o,b,c) &= T(o,b,c) * T(o,a,b) \\ &\leq \bigvee_{x\in X} T(o,x,o) * T(c,a,x) \quad \text{(Peano Property)} \\ &= \bigvee_{x\in X} \chi_{\{o\}}(x) * T(c,a,x) \quad \text{((LT6))} \\ &= T(c,a,o) \\ &= T(o,a,c)\,. \end{aligned}$$

This shows that (LT4) holds. □

Theorem 3.14. *Let T be an L-semibetweenness relation on X with the Peano property. Then T is an L-betweenness relation on X.*

Proof. Since T is an L-semibetweenness relation on X, it suffices to show that T satisfies (LT4). By Proposition 2.8, we know that T is idempotent. Then it follows from Proposition 3.13 that T satisfies (LT4). This confirms that T is an L-betweenness relation on X. □

4. Convex ternary (L-)relations

Convexity is omnipresent in many mathematical frameworks, such as vector spaces, metric spaces and preordered sets. For a ternary relation T on X, a convex subset of X can be defined in a natural way.

Definition 4.1. Consider a ternary relation T on X. A subset C of X is called *convex* with respect to T if it satisfies

$$\text{(CS)}\ \ ((a,b \in C) \wedge (a,c,b) \in \mathrm{T}) \Rightarrow c \in C, \text{ for any } a,b,c \in X\,.$$

By collecting all convex subsets, a convex structure C on X can be obtained.

Theorem 4.2. *Let* T *be a ternary relation on X and define* $\mathrm{C} \subseteq P(X)$ *as*

$$\mathrm{C} = \{C \in P(X) \mid\ C \textit{ satisfies } \text{(CS)}\}\,.$$

Then C *is a convex structure on X.*

Proof. It suffices to verify that C satisfies (CS1)–(CS3). (CS1) and (CS2) are obvious.

(CS3) Consider any directed subfamily $\{C_j\}_{j\in J} \subseteq \mathrm{C}$. For any $a, b, c \in X$, if $a, b \in \bigcup_{j\in J} C_j$ and $(a,c,b) \in \mathrm{T}$, then there exist $i, k \in J$ such that $a \in C_i$ and $b \in C_k$. By the directedness of $\{C_j\}_{j\in J}$, there exists $l \in J$ such that $C_i \subseteq C_l$ and $C_k \subseteq C_l$. This implies that $a, b \in C_l$. Since $C_l \in \mathrm{C}$ and $(a,c,b) \in \mathrm{T}$, we have $c \in C_l \subseteq \bigcup_{j\in J} C_j$. This shows that $\bigcup_{j\in J} C_j \in \mathrm{C}$. □

Given a convex structure C on X, each $C \in \mathrm{C}$ is usually called convex. From this point of view, Theorem 4.2 exactly explains the reason why a subset C satisfying (CS) is called convex with respect to a ternary relation T.

The main purpose of a betweenness relation is to describe whether or not a point is between two other points. This can be expressed via the operation $\bullet$ that is defined in Section 3. Explicitly, given three points $a, b, c \in X$, if $c \in \mathrm{T}(a, \cdot, b)$, then we say c is between a and b. It seems that $\mathrm{T}(a, \cdot, b)$ represents some kind of "interval" with two endpoints a and b. Intuitively, $\mathrm{T}(a, \cdot, b)$ should be a convex subset of X (see, e.g., Examples 4.4 and 4.5). However, in general, $\mathrm{T}(a, \cdot, b)$ may be not convex since it is not really an interval (see, e.g., Example 4.6). Due to this fact, we introduce the following property for a ternary relation.

Definition 4.3. A ternary relation T on X is called *convex* if $\mathrm{T}(a, \cdot, b)$ is convex with respect to T for any $a, b \in X$. That is, for any $a, b, x, y, z \in X$,

$$\text{(CT)} \quad ((x, y \in \mathrm{T}(a, \cdot, b)) \wedge z \in \mathrm{T}(x, \cdot, y)) \Rightarrow z \in \mathrm{T}(a, \cdot, b)\,.$$

Example 4.4. Let $(X, +, \cdot)$ be a real vector space and define $\mathrm{T} \subseteq X^3$ as

$$\mathrm{T} = \{(a, c, b) \in X^3 \mid (\exists t \in [0, 1])(c = t \cdot a + (1 - t) \cdot b)\}\,.$$

Then it is easy to verify that T is a convex ternary relation on X.

Example 4.5. Let R be a preorder on X and define $\mathrm{T} \subseteq X^3$ as

$$\mathrm{T} = \{(a, c, b) \in X^3 \mid aRc \wedge cRb\}\,.$$

Then it is easy to verify that T is a convex ternary relation on X.

Example 4.6. Let $X = \{a, b, c, d, e\}$ and define $\rho : X \times X \to [0, \infty[$ as

$\rho(\cdot, \cdot)$	a	b	c	d	e
a	0	2	1	1	1.5
b	2	0	1	1	1.5
c	1	1	0	1	0.5
d	1	1	1	0	0.5
e	1.5	1.5	0.5	0.5	0

It is easy to verify that ρ is a metric on X. The corresponding ternary relation T_ρ defined as

$$\mathrm{T}_\rho = \{(x, y, z) \in X^3 \mid \rho(x, z) = \rho(x, y) + \rho(y, z)\}$$

is not convex since $c, d \in \mathrm{T}_\rho(a, \cdot, b)$ and $e \in \mathrm{T}_\rho(c, \cdot, d)$, but $e \notin \mathrm{T}_\rho(a, \cdot, b)$.

Next, we will generalize the above-mentioned concepts to the L-valued setting. In order to present the definition of a convex ternary L-relation, we first define convex L-sets with respect to a ternary L-relation.

Definition 4.7. Consider a ternary L-relation T on X. An L-set U on X is called *convex* with respect to T if it satisfies

$$\text{(LCS)}\quad U(a) * U(b) * T(a,c,b) \leq U(c), \text{ for any } a,b,c \in X\,.$$

The reason why an L-set U satisfying (LCS) is called convex is that all such L-sets constitute an L-convex structure on X. Furthemore, this L-convex structure is stratified.

Theorem 4.8. *Let T be a ternary L-relation on X and define $\mathscr{C} \subseteq L^X$ as*

$$\mathscr{C} = \{U \in L^X \mid U \text{ satisfies (LCS)}\}\,.$$

Then $\mathscr{C}$ is a stratified L-convex structure on X.

Proof. It suffices to verify that $\mathscr{C}$ satisfies (LCS1)–(LCS3) and (SLCS).

(LCS1) It is obvious.

(LCS2) Consider any $\{U_k\}_{k\in K} \subseteq \mathscr{C}$. Then for any $a,b,c \in X$,

$$\begin{aligned}
&\Big(\bigwedge_{k\in K} U_k\Big)(a) * \Big(\bigwedge_{k\in K} U_k\Big)(b) * T(a,c,b)\\
\leq &\bigwedge_{k\in K} U_k(a) * U_k(b) * T(a,c,b)\\
\leq &\Big(\bigwedge_{k\in K} U_k\Big)(c)\,.
\end{aligned}$$

This shows that $\bigwedge_{k\in K} U_k \in \mathscr{C}$.

(LCS3) Consider any directed family $\{U_j\}_{j\in J} \subseteq \mathscr{C}$. Then for any $a,b,c \in X$,

$$\begin{aligned}
&\Big(\bigvee_{j\in J} U_j\Big)(a) * \Big(\bigvee_{j\in J} U_j\Big)(b) * T(a,c,b)\\
= &\bigvee_{j\in J}\bigvee_{i\in J} U_j(a) * U_i(b) * T(a,c,b)\\
\leq &\bigvee_{j\in J} U_j(a) * U_j(b) * T(a,c,b) \text{ (due to the directedness of } \{U_j\}_{j\in J})\\
\leq &\Big(\bigvee_{j\in J} U_j\Big)(c)\,.
\end{aligned}$$

This shows that $\bigvee_{j\in J} U_j \in \mathscr{C}$.

(SCS) Consider any $\alpha \in L$ and $U \in \mathscr{C}$. Then for any $a,b,c \in X$,

$$(\alpha * U)(a) * (\alpha * U)(b) * T(a,c,b)$$

$$
\begin{aligned}
&= \alpha * U(a) * \alpha * U(b) * T(a,c,b) \\
&\leq \alpha * U(a) * U(b) * T(a,c,b) \\
&\leq \alpha * U(c)\,.
\end{aligned}
$$

This shows that $\alpha * U \in \mathscr{C}$. □

Given two points $a, b \in X$, the convexity of the L-set $T(a,\cdot,b)$ with respect to a ternary L-relation can be described by (LCS). This determines the convexity of a ternary L-relation T.

Definition 4.9. A ternary L-relation T on X is called *convex* if it satisfies

$$\text{(LCT)}\quad T(a,x,b) * T(a,y,b) * T(x,z,y) \leq T(a,z,b), \text{ for any } a,b,x,y,z \in X\,.$$

Example 4.10. Let $(X,+,\cdot)$ be a real vector space and $U \in L^X$. Define $T : X^3 \to L$ by, for any $(a,c,b) \in X^3$:

$$T(a,c,b) = \begin{cases} U(c) & , (\exists t \in [0,1])(c = t\cdot a + (1-t)\cdot b), \\ \bot & , \text{elsewhere}, \end{cases}$$

Then T is a convex ternary L-relation on X.

Example 4.11. Let E be an L-preorder on X. Define $T : X^3 \to L$ by, for any $(a,c,b) \in X^3$:

$$T(a,c,b) = E(a,c) * E(c,b)\,.$$

Then T is a convex ternary L-relation on X.

Proposition 4.12. *Let T be an L-tolerance relation on X. If T is convex, then T satisfies* (LT4).

Proof. Consider any $a,b,c,o \in X$. Then it follows that

$$T(o,a,b) * T(o,b,c) = T(o,o,c) * T(o,b,c) * T(o,a,b) \leq T(o,a,c)\,.$$

This shows that (LT4) holds. □

Note that the convexity of a ternary L-relation will play an important role in the following sections.

5. Pasch ternary (L-)relations

In this section, we present another type of ternary L-relations by adding the Pasch property, with the term *Pasch property* also originating from the theory of convex structures.

Definition 5.1. A ternary relation T on X is called a *Pasch* relation on X if it satisfies the following property (see Figure 2):

Pasch Property: for any $a,b,c,x,y \in X$,

$$((a,x,b) \in \mathrm{T} \wedge (a,y,c) \in \mathrm{T}) \Rightarrow (\exists z \in X)\,((b,z,y) \in \mathrm{T} \wedge (c,z,x) \in \mathrm{T})\,.$$

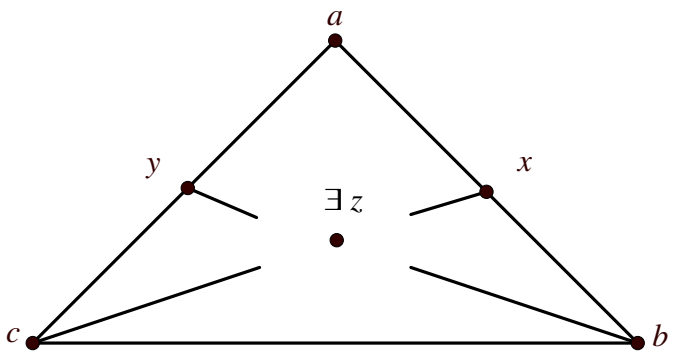

Figure 2: Pasch property

Now, we provide the fuzzy counterpart of the Pasch property for a ternary L-relation T on X.

Definition 5.2. A ternary L-relation T on X is called a *Pasch* L-relation on X if it satisfies the following property:

Pasch Property: for any $a, b, c, x, y \in X$,

$$T(a, x, b) * T(a, y, c) \leq \bigvee_{z \in X} T(b, z, y) * T(c, z, x)\,.$$

In case L is the two-point chain $L = \{\bot, \top\}$, the Pasch property of a ternary L-relation reduces to that of a classical ternary relation.

Example 5.3. By routine computations, it can be verified that

(1) T defined in Example 3.3 satisfies the Pasch property.

(2) T defined in Example 3.5 satisfies the Pasch property.

In order to present characterizations of the Pasch property of a ternary L-relation T on X, for any $a, b \in X$, we define $a/b : X \to L$ by

$$(a/b)(x) = T(b, a, x), \text{ for any } x \in X\,.$$

This essentially determines a map $(\cdot)/(\cdot) : X \times X \to L^X$. Next, we generalize this map by extending its domain from $X \times X$ to $P(X) \times P(X)$ and $L^X \times L^X$. Explicitly, for any $A, B \in P(X)$ and $U, V \in L^X$,

$$(A/B)(x) = \bigvee_{a \in A} \bigvee_{b \in B} (a/b)(x) = \bigvee_{a \in A} \bigvee_{b \in B} T(b, a, x), \text{ for any } x \in X\,,$$

$$(U/V)(x) = \bigvee_{a \in X} \bigvee_{b \in X} U(a) * V(b) * (a/b)(x) = \bigvee_{a \in X} \bigvee_{b \in X} U(a) * V(b) * T(b, a, x), \text{ for any } x \in X\,.$$

In particular, for any $b \in X$ and $U \in L^X$,

$$(U/b)(x) = (U/\chi_{\{b\}})(x) = \bigvee_{a \in X} U(a) * T(b, a, x), \text{ for any } x \in X\,.$$

Lemma 5.4. *Let T be a ternary L-relation on X. Then it holds that*

$$((a \bullet b)/c)(x) = \bigvee_{d \in X} T(a,d,b) * T(c,d,x), \text{ for any } a,b,c,x \in X\,.$$

Proof. It is easy and is omitted. □

Theorem 5.5. *Let T be a convex L-tolerance relation on X. Then the following statements are equivalent:*

(1) *T satisfies the Pasch property;*

(2) *$((a \bullet b)/c) \bullet ((a \bullet b)/c) \leq (a \bullet b)/c$, for any $a,b,c \in X$.*

Proof. (1)⇒ (2): By Proposition 4.12, we know that T satisfies (LT4). Consider any $x \in X$. Then

$$\begin{aligned}
&(((a \bullet b)/c) \bullet ((a \bullet b)/c))(x)\\
&= \bigvee_{y \in X} \bigvee_{z \in X} ((a \bullet b)/c)(y) * ((a \bullet b)/c)(z) * T(y,x,z)\\
&= \bigvee_{y,z \in X} \Big(\bigvee_{d \in X} T(a,d,b) * T(c,d,y) \Big) * \Big(\bigvee_{e \in X} T(a,e,b) * T(c,e,z) \Big) * T(y,x,z)\\
&= \bigvee_{y,z \in X} \bigvee_{d,e \in X} (T(y,x,z) * T(y,d,c)) * T(z,e,c) * T(a,e,b) * T(a,d,b) \quad \text{(Symmetry)}\\
&= \bigvee_{d,e,z \in X} \bigvee_{g \in X} T(c,g,x) * T(z,g,d) * T(z,e,c) * T(a,e,b) * T(a,d,b) \quad \text{(Pasch property)}\\
&= \bigvee_{d,e,g \in X} \bigvee_{z \in X} (T(z,g,d) * T(z,e,c)) * T(c,g,x) * T(a,e,b) * T(a,d,b)\\
&\leq \bigvee_{d,e,g \in X} \bigvee_{h \in X} T(c,h,g) * T(d,h,e) * T(c,g,x) * T(a,e,b) * T(a,d,b) \quad \text{(Pasch property)}\\
&= \bigvee_{d,e \in X} \bigvee_{h,g \in X} (T(c,h,g) * T(c,g,x)) * (T(a,d,b) * T(a,e,b) * T(d,h,e))\\
&\leq \bigvee_{h \in X} T(c,h,x) * T(a,h,b) \quad \text{((LT4) and (LCT))}\\
&= ((a \bullet b)/c)(x).
\end{aligned}$$

(2)⇒ (1): Consider $a,b,c,x,y \in X$ and let $U = (b \bullet y)/c$. By Proposition 2.8, we have

$$U(b) = \bigvee_{z \in X} T(b,z,y) * T(c,z,b) \geq T(b,b,y) * T(c,b,b) = \top$$

and

$$U(a) = \bigvee_{z \in X} T(b,z,y) * T(c,z,a) \geq T(b,y,y) * T(c,y,a) = T(a,y,c)\,.$$

Then it follows that

$$T(a,x,b) * T(a,y,c) \leq T(a,x,b) * U(b) * U(a)$$

$$
\begin{aligned}
&\leq \bigvee_{d\in X}\bigvee_{e\in X}(T(d,x,e)*U(d)*U(e))\\
&= (U\bullet U)(x)\\
&\leq U(x)\\
&= \bigvee_{z\in X} T(b,z,y)*T(c,z,x)\,.
\end{aligned}
$$

This shows that T satisfies the Pasch property. □

Proposition 5.6. *Let T be a symmetric and idempotent ternary L-relation on X. If T satisfies the Pasch property, then T satisfies* (LT5).

Proof. Consider $a, b, c, o \in X$. Then it follows that

$$
\begin{aligned}
T(o,a,b)*T(o,b,c) &\leq \bigvee_{x\in X} T(c,x,a)*T(b,x,b)\\
&= \bigvee_{x\in X} T(c,x,a)*\chi_{\{b\}}(x)\\
&= T(c,b,a)\\
&= T(a,b,c)\,.
\end{aligned}
$$

This shows that (LT5) holds. □

Theorem 5.7. *Let T be an L-semibetweenness relation on X with the Peano property and the Pasch property. Then T is a strong L-betweenness relation on X.*

Proof. Since T is an L-semibetweenness relation on X, it suffices to show that T satisfies (LT4) and (LT5). By Proposition 2.8, we know T is idempotent. Then it follows from Propositions 3.13 and 5.6 that T satisfies (LT4) and (LT5). This confirms that T is a strong L-betweenness relation on X. □

6. Sand-glass ternary (L-)relations

In this section, we will introduce another property for a ternary L-relation, called the sand-glass property.

Definition 6.1. A ternary relation T on X is called a *sand-glass* relation if it satisfies the following property (see Figure 3):

Sand-glass Property: for any $a, b, c, d, o, x \in X$,

$$
((a,o,c)\in \mathrm{T}\wedge(b,o,d)\in \mathrm{T}\wedge(a,x,b)\in \mathrm{T})\Rightarrow(\exists y\in X)\,((c,y,d)\in \mathrm{T}\wedge(x,o,y)\in \mathrm{T})\,.
$$

Note that the term *sand-glass* also originates from the theory of convex structures [27]. Next, let us generalize the sand-glass property to the fuzzy setting with respect to a ternary L-relation T on X.

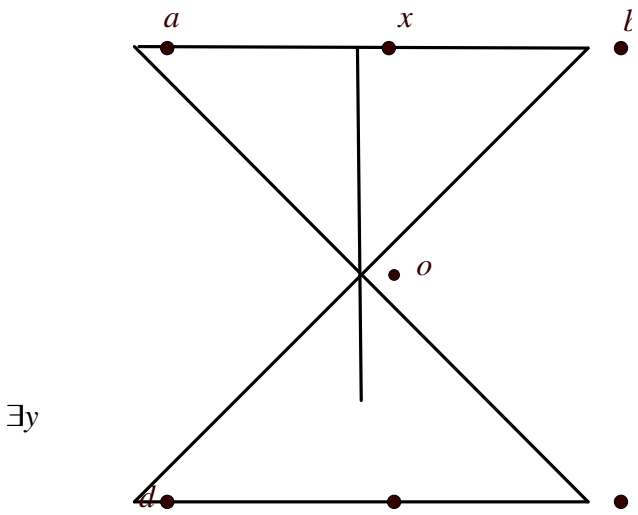

Figure 3: Sand-glass property

Definition 6.2. A ternary L-relation T on X is called a *sand-glass* L-relation on X if it satisfies the following property:

Sand-glass Property: for any $a, b, c, d, o, x \in X$,

$$T(a, o, c) * T(b, o, d) * T(a, x, b) \leq \bigvee_{y \in X} T(c, y, d) * T(x, o, y)\,.$$

When $L = \{\bot, \top\}$, it is easy to see that the sand-glass property of a ternary L-relation reduces to that of a classical ternary relation.

In order to present characterizations of the sand-glass property of a ternary L-relation T on X, the following lemma is essential. We first recall the definition of U/V for any $U, V \in L^X$, i.e., for any $x \in X$:

$$(U/V)(x) = \bigvee_{a \in X} \bigvee_{b \in X} U(a) * V(b) * T(b, a, x)\,.$$

In particular, when $U = \chi_{\{c\}}$ and $V = a \bullet b$ for some $a, b, c \in X$, the following lemma holds.

Lemma 6.3. *Let T be a ternary L-relation on X. Then it holds that*

$$(c/(a \bullet b))(x) = \bigvee_{d \in X} T(d, c, x) * T(a, d, b), \text{ for any } a, b, c, x \in X\,.$$

Proof. It is easy and is omitted. □

Theorem 6.4. *Let T be a convex L-tolerance relation on X. Then the following statements are equivalent:*

(1) *T satisfies the sand-glass property;*

(2) *$(c/(a \bullet b)) \bullet (c/(a \bullet b)) \leq c/(a \bullet b)$, for any $a, b, c \in X$.*

Proof. (1)⇒(2): Consider $x \in X$. Then

$$\begin{aligned}
&((c/(a \bullet b)) \bullet (c/(a \bullet b)))(x)\\
&= \bigvee_{y\in X} \bigvee_{z\in X} (c/(a \bullet b))(y) * (c/(a \bullet b))(z) * T(y,x,z)\\
&= \bigvee_{y,z\in X} \Big(\bigvee_{d\in X} T(d,c,y) * T(a,d,b) \Big) * \Big(\bigvee_{e\in X} T(e,c,z) * T(a,e,b) \Big) * T(y,x,z)\\
&= \bigvee_{y,z\in X} \bigvee_{d,e\in X} T(d,c,y) * T(a,d,b) * T(e,c,z) * T(a,e,b) * T(y,x,z)\\
&= \bigvee_{d,e\in X} \bigvee_{y,z\in X} (T(y,c,d) * T(z,c,e) * T(y,x,z)) * T(a,d,b) * T(a,e,b) \quad \text{((LT1))}\\
&\leq \bigvee_{d,e\in X} \bigvee_{p\in X} T(d,p,e) * T(x,c,p) * T(a,d,b) * T(a,e,b) \quad \text{(Sand-glass property)}\\
&= \bigvee_{d,e\in X} \bigvee_{p\in X} T(p,c,x) * (T(a,d,b) * T(a,e,b) * T(d,p,e))\\
&\leq \bigvee_{p\in X} T(p,c,x) * T(a,p,b) \qquad \text{((LCT))}\\
&= (c/(a \bullet b))(x).
\end{aligned}$$

This shows that $(c/(a \bullet b)) \bullet (c/(a \bullet b)) \leq c/(a \bullet b)$.

(2)⇒(1): Consider any $a, b, c, d, o \in X$. By (LT1) and (LT2), we have

$$T(a,o,c) \leq \bigvee_{x\in X} T(x,o,a) * T(c,x,d) = (o/(c \bullet d))(a)$$

and

$$T(b,o,d) \leq \bigvee_{x\in X} T(x,o,b) * T(c,x,d) = (o/(c \bullet d))(b)\,.$$

Then

$$\begin{aligned}
&T(a,o,c) * T(b,o,d) * T(a,x,b)\\
&\leq (o/(c \bullet d))(a) * (o/(c \bullet d))(b) * T(a,x,b)\\
&\leq \bigvee_{y\in X} \bigvee_{z\in X} (o/(c \bullet d))(y) * (o/(c \bullet d))(z) * T(y,x,z)\\
&= ((o/(c \bullet d)) \bullet (o/(c \bullet d)))(x)\\
&\leq (o/(c \bullet d))(x)\\
&= \bigvee_{y\in X} T(y,o,x) * T(c,y,d)\\
&= \bigvee_{y\in X} T(c,y,d) * T(x,o,y)\,.
\end{aligned}$$

This shows that T satisfies the sand-glass property. □

Proposition 6.5. *Let T be a convex L-tolerance relation on X. If T satisfies the sand-glass property, then T satisfies* (LT5).

Proof. Consider $a, b, c, o \in X$. Then

$$\begin{aligned}
(b/(a \bullet c))(o) &= \bigvee_{x \in X} T(x, b, o) * T(a, x, c) \geq T(c, b, o) * T(a, c, c) = T(o, b, c), \\
(b/(a \bullet c))(b) &= \bigvee_{x \in X} T(x, b, b) * T(a, x, c) \geq T(a, b, b) * T(a, a, c) = \top, \\
(b/(a \bullet c))(a) &= \bigvee_{x \in X} T(x, b, a) * T(a, x, c) \\
&= \bigvee_{x \in X} T(a, x, c) * T(a, a, c) * T(x, b, a) \quad \text{(Proposition 2.8)} \\
&\leq T(a, b, c)\,. \quad \text{((LCT))}
\end{aligned}$$

This implies that

$$\begin{aligned}
T(o, a, b) * T(o, b, c) &\leq (b/(a \bullet c))(b) * (b/(a \bullet c))(o) * T(o, a, b) \\
&\leq \bigvee_{x \in X} \bigvee_{y \in X} (b/(a \bullet c))(x) * (b/(a \bullet c))(y) * T(x, a, y) \\
&= ((b/(a \bullet c)) \bullet (b/(a \bullet c)))(a) \\
&\leq (b/(a \bullet c))(a) \\
&\leq T(a, b, c)\,.
\end{aligned}$$

This shows that T satisfies (LT5). □

Theorem 6.6. *Let T be a convex L-semibetweenness relation on X. If T satisfies the sand-glass property, then T is a strong L-betweenness relation on X.*

Proof. By Propositions 4.12 and 6.5, we know that T satisfies (LT4) and (LT5). Thus, T is a strong L-betweenness relation on X. □

To conclude, we explore the relationship among the Peano property, the Pasch property and the sand-glass property.

Theorem 6.7. *Let T be a symmetric and idempotent ternary L-relation on X. If T satisfies the Peano property and the Pasch property, then T satisfies the sand-glass property.*

Proof. By Proposition 5.6, we know that T satisfies (LT5). Consider any $a, b, c, d, o, x \in X$. Then

$$\begin{aligned}
& T(a, o, c) * T(b, o, d) * T(a, x, b) \\
= \; & T(b, o, d) * (T(a, o, c) * T(a, x, b)) \\
\leq \; & T(b, o, d) * \bigvee_{z \in X} (T(b, z, o) * T(c, z, x)) \quad \text{(Pasch property)} \\
= \; & \bigvee_{z \in X} (T(b, o, d) * T(b, z, o)) * T(c, z, x)
\end{aligned}$$

$$
\begin{aligned}
&\leq \bigvee_{z\in X} (T(z,o,d) * T(c,z,x) \qquad ((\mathrm{LT5}))\\
&= \bigvee_{z\in X} T(c,z,x) * T(d,o,z) \qquad ((\mathrm{LT1}))\\
&\leq \bigvee_{y\in X} T(c,y,d) * T(x,o,y)\,. \quad (\text{Peano property})
\end{aligned}
$$

This shows that T satisfies the sand-glass property. □

Using Theorem 6.7, we provide the following example.

Example 6.8. It is easily observed that T in Example 3.3 is symmetric and idempotent in the classical setting and T in Example 3.5 is symmetric and idempotent in the L-valued setting. By Theorem 6.7, we obtain that

(1) T defined in Example 3.3 satisfies the sand-glass property.

(2) T defined in Example 3.5 satisfies the sand-glass property.

7. Conclusions and future work

In this paper, we have introduced the Peano property, the Pasch property and the sand-glass property of a ternary L-relation. Moreover, we have discussed their characterizations and their connections with L-betweenness relations. Based on the theory of fuzzy convex structures, we have introduced the notion of a convex ternary L-relation allowing to investigate the Pasch property and the sand-glass property of a ternary L-relation. Since we have included L-betweenness axioms into the definition of geometric properties of ternary L-relations, we have thus established a basic framework for geometric properties of ternary L-relations.

It is easily observed that ternary L-relations provide an excellent framework for investigating geometric properties. Due to the close relationship between fuzzy convex structures and ternary fuzzy relations, we will use ternary L-relations, or L-betweenness relations (in either sense), as basic tools to study geometric properties of fuzzy convex structures in the near future.

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