Weierstrass transform in discrete Clifford analysis

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Dankwoord

Toen ik in 2016 afstudeerde als master in de wiskunde, wist ik (zoals zovelen op die leeftijd) nog niet zo goed waarheen het leven me zou brengen. Ik wist ook nog niet zo goed naar waar ik wilde dat het leven me zou brengen. Ik startte in de privé-sector, in Brussel, maar dat bleek dan toch zo mijn ding niet. Al vrij snel was ik opnieuw vacatures aan het opzoeken, waarbij ik nu wel zeker wist dat ik me tot regio Gent zou beperken: dat pendelen was niets voor mij. Ze zochten een assistent wiskunde aan de faculteit Ingenieurswetenschappen en Architectuur van de Universiteit Gent: een combinatie van doctoraatsonderzoek en het begeleiden van de studenten in hun wiskundevakken. Ik twijfelde. Lesgeven leek me wel leuk, maar een doctoraat schrijven? Dat is een masterthesis schrijven, maar zes (en uiteindelijk in mijn geval zelfs acht) keer zo lang! Na veel rondvragen en nadenken, greep ik de kans en mocht ik op 7 november, mijn verjaardag nog wel, op gesprek bij professor Hendrik De Bie en professor Marián Slodička. Ik kreeg de job en mocht terugkeren naar mijn welvertrouwde universiteit, weliswaar in een andere functie.

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Ik mocht onderzoek doen in de abstracte en complexe wereld van de wiskunde. Mensen vroegen me soms "Hoe gaat dat dan, doctoreren in de wiskunde?". Men kan zich iets voorstellen bij het ontleden van een orgaan, het onderzoeken van een plant of het bestuderen van een sociaal-economisch fenomeen. Maar onderzoek in de wiskunde? Wel, je vindt dan eigenlijk nieuwe wiskunde uit, door concepten en ideeën uit te werken en te controleren of ze consistent zijn met de wiskunde die al bestaat. Zo bouw je mee aan het immense stratennetwerk dat de wiskunde groot is. Het resultaat daarvan? Dat lees je wat verder[1]. Mijn pad was een hobbelige weg, met omleidingen en doodlopende straten, rode lichten en ellenlange rotondes. En een helse eindsprint, dat ook. Maar wat ben ik blij dat u dit nu aan het lezen bent.

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[1]Er is een Nederlandstalige samenvatting voor de minder wiskundig aangelegde lezer.

wil delen. Hierdoor sta ik ongetwijfeld sterker in mijn schoenen. Ik leerde ook hoe belangrijk de mensen in mijn omgeving voor me zijn en hoe dankbaar ik ben voor hun warme, onvoorwaardelijke steun en liefde.

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Voor Remi

Astrid
Balegem, december 2024

Contents

1
Introduction

In this first introductory chapter, we will clarify the title of the dissertation and put it in the bigger context. We will give some general background about the Weierstrass transform, Clifford analysis and discrete Clifford analysis in particular. Eventually, the overall aim of the thesis is sketched, along with its structure.

1.1 The Weierstrass transform

The Weierstrass transform, [1] (or Gauss transform, Gauss-Weierstrass operator), named after the German mathematician Karl Weierstrass (1815-1897) is a fundamental operation in mathematical analysis and applied mathematics. It is a smoothing operator that transforms a given function into an entire function: a function that is complex differentiable everywhere in the complex plane. It averages the values of a function f by making the convolution with a Gaussian kernel to obtain a 'smoothed' version of f. Evaluating the smoothed function at the point x, points in the original function close to x are given a higher weight, while points that are further away from x get lower weights. Specifically, the most general definition of the Weierstrass transform $\mathcal{W}_t[f]$ of a function $f : \mathbb{R} \to \mathbb{R}$ is given by

$$\mathcal{W}_t[f](u) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} \exp\left(\frac{-|u-x|^2}{2t}\right) f(x)dx.$$

In the classical definition, one sets $t = 1$, which is mostly used for mathematical purposes. For small values of t, $\mathcal{W}_t[f]$ is close to f, but smooth. For larger t, the operator averages out and changes f.

The Weierstrass transform's smoothing properties and its relationship with the Gaussian kernel make it useful in a variety of mathematical and practical applications:

1. The Weierstrass transform is closely related to the solution of the heat equation. Given an initial temperature distribution f of an infinitely long rod with thermal

conductivity equal to 1, the Weierstrass transform $\mathcal{W}_t[f]$ gives the temperature distribution t time units later. This connection makes the transform a valuable tool in studying heat conduction and diffusion processes. See for example [2] and [3].

2. In probability theory, the Weierstrass transform appears in the context of the Central Limit Theorem (CLT). The Gaussian kernel in the transform is the probability density function of a normal distribution, highlighting the transform's role in modeling the distribution of sums of random variables, see for example [4]. Given a function that represents noisy data, one wants to build a function that approximates the important patterns of the data, omitting the noise.

3. The smoothing property of the Weierstrass transform makes it useful in signal processing for noise reduction. By applying the transform to a noisy signal, the high-frequency noise can be attenuated, resulting in a smoother signal. The Weierstrass transform thus acts as a low-pass filter. An example is found in [5].

4. Similar to its application in signal processing, the Weierstrass transform can be used in image processing for tasks such as image smoothing and noise reduction. It helps in removing fine details and noise, producing a cleaner image. In this context, the transform is also known as the Gaussian blur. See for example [6], [7] and [8].

5. The Weierstrass transform is utilized in numerical analysis to approximate functions and integrals. Its ability to convert functions into smoother forms can improve the accuracy and stability of numerical algorithms.

6. In quantum mechanics, the Weierstrass transform is related to the concept of the propagator for a free particle in one dimension. The Gaussian kernel represents the transition probability amplitude, connecting the transform to the evolution of quantum states.

Applications are thus widely around. This dissertation is to be situated in the context of discrete Clifford analysis, where a discrete counterpart of the Weierstrass transform is constructed. In the next section, we will discuss Clifford algebras and Clifford analysis, both in the classical (continuous) setting as in the discrete setting.

1.2 Clifford analysis

Clifford analysis is a quite recent branch of mathematical analysis that extends the techniques of classical real and complex analysis to functions defined on so-called Clifford algebras. It is centred around the notion of a Dirac operator and its null solutions, called monogenic functions. In the most common known case, the two-dimensional algebra of complex numbers, one considers the Cauchy-Riemann operator consisting of a real and a vectorial part. Its null solutions are holomorphic functions. An important property is that, by multiplying this Cauchy-Riemann operator with its complex conjugate, one obtains the two-dimensional Laplacian. This setting may be generalised in a very natural way to higher dimensions by the generalised Cauchy-Riemann operator. This operator contains again a scalar and an m-dimensional vectorial part, the Dirac operator, factorising the $m-$dimensional Laplacian. Clifford analysis can thus be seen as a higher-dimensional theory of holomorphic functions in the complex plane on the one hand. On the other hand, as the Dirac operator factorises the Laplacian, it is also a refinement of classical harmonic analysis. Clifford Analysis bridges the gap between pure mathematical theory and practical applications in science and engineering, providing a rich framework for understanding multidimensional spaces.

Many research has been devoted to function theory in Clifford algebras. In the 19^{th} century, William Kingdon Clifford introduced an algebra with an arbitrary number of anti-commuting basis elements ([9]). This was a generalisation of the already existing function theory of complex analysis, in which big names such as Euler, Gauss, Riemann, Cauchy and Weierstrass contributed. However, it was only until mid 20^{th} century that the setting resulted in the hypercomplex analysis as we know it today because an appropriate differential operator was still missing. It was Paul Dirac who introduced the first "Dirac operator" in 1928 ([10]), followed by a generalisation by Brauer and Weyl in 1935 to any finite dimensional quadratic space with arbitrary signature. The Swiss mathematician Karl Rudolf Fueter was the first to contribute to the study of monogenic functions, the null-solutions of the (generalisation of the) Dirac operator ([11]). In the eighties, the work of Richard Delanghe, Fred Brackx and Franciscus Sommen made this field to really come to existence. A detailed study, constituting the foundations of Clifford Analysis, is the book by Brackx, Delanghe and Sommen, [12]. More standards in the literature are e.g. [13, 14, 15, 16, 17]. We do not claim completeness in this list.

1.3 Discrete Clifford analysis

Discrete Clifford analysis is a more recent branch, which originated from the need for numerical applications. Its development is driven by the desire to study function theory and harmonic analysis on discrete lattices or grids, especially in higher-dimensional settings. The discrete setting introduces new challenges, such as defining a suitable Dirac operator which allows for a discrete counterpart of key results in classical Clifford analysis and function theory. One of those challenges, for example, is the important property of the discrete Dirac operator factorising a discrete Laplacian. Several models for the discrete Clifford algebra have been established, either starting from an application or from a function theoretic point of view.

In the first place, numerical problems related to potential theory and boundary value problems are treated, for example by Gürlebeck and Sprössig in [18] and [15]. They developed continuous strategies to solve boundary value problems based on operator calculus, and gave a basic scheme for the construction of discrete operator calculus. This has led to a suitable numerical approach of boundary value problems. They constructed a function theoretical approach to discrete Clifford analysis for a Dirac operator which only contained forward differences. A major drawback is that the approach of using only one type of difference operator (either forward or backward differences) does not factorise the Star-Laplacian.

The work of Gürlebeck and Sprössig was then extended to a version of a discrete boundary element method by Gürlebeck and Hommel in [19] and [20], with successful application of this theory in [21]. It was based on the concept of discrete fundamental solutions of the discrete Laplace operator. First steps were made in the direction of using a Dirac operator which factorises the Star-Laplacian, as was also done in [21, 22]. Other important contributions regarding the construction of discrete Dirac operators are [23, 24, 25, 26]. In 2006, Faustino and Kähler were one of the first to work on the construction of a discrete Clifford analysis from the theoretical point of view, allowing for the construction of basic polynomial solutions, Fischer decompositions and Taylor series. However, this was also done using the approach of only one type of difference operator, not supporting the factorisation of the Laplace operator. Therefore, both types of difference operators are necessary.

In the so-called discrete Hermitian setting, introduced by Brackx, De Schepper, Sommen and Van de Voorde in [27, 28, 29], the Clifford basis elements $e_j = e_j^+ + e_j^-$ are split in a forward and backward basis element. The discrete Dirac operator was then constructed by combining the forward and backward difference operators with these forward and backward basis elements. While forward and backward difference operators are commuting with each other, their corresponding vector variable operators are not. It was Sommen who came up with the idea of using the so-called skew-Weyl relations to overcome this issue, see [30]. This theory further developed by amongst others De Ridder and colleagues ([31, 32, 33, 34]) and Faustino ([35]). They developed for example discrete analogues of the Cauchy-Kovalevskaya extension, (dual) Taylor series, a Fischer decompostion and discrete Clifford-Hermite and Clifford-Laguerre polynomials. This framework will form the basis for this dissertation.

1.4 Aim and structure of this dissertation

Before we start with the main content of this dissertation, we provide a chapter with preliminaries. Herein, we introduce the basic concepts and definitions of continuous Clifford algebras and analysis. Standard references are, as mentioned above, e.g. [12, 13, 14]. Next, we present its discrete counterpart, the Clifford Hermite framework as it is described in e.g. [27, 28, 29]. We will enlist the necessary definitions and objects to build the theory of this thesis.

The main aim of this dissertation is the definition of a discrete Weierstrass transform, together with the construction of a discrete Weierstrass space. Therefore, we are inspired by the classical definition of the Weierstrass transform in combination with the already defined tools in the discrete Hermitian Clifford analysis. The main idea is to use the discrete Gauss distribution G established in [31], as a 'weight function' and let the composition of a discrete function with this Gaussian act on a well-defined kernel, resulting in a direct analogue to the continuous integral transform. In order to define a discrete Weierstrass space $\mathcal{W}$, we are inspired by the classical L_2-spaces and use the discrete Hermite polynomials as basis elements of this space.

In the third chapter, these ideas will be carried out in dimension $m = 1$. We will define the Weierstrass transform on the discrete radial Hermite polynomials H_n, which will then be naturally extended to possibly infinite linear combinations of these basis functions. We define an inner product and corresponding norm in order to construct the discrete Weierstrass space: discrete functions that are linear combinations of Hermite polynomials, for which their norm is finite. A natural and important question is whether the delta-functions, the building blocks of discrete function theory, are elements of the discrete Weierstrass space. This will be subject of section 3.2.3. Similarly, we investigate if the discrete Hermite functions are contained in this space.

The above will be carried out on a standard grid with mesh width $h = 1$. In section 3.3, we will be concerned about the definitions when $h \neq 1$. In particular, we are interested in the behaviour if h tends to 0 and we will compare this result to the classical setting.

In chapter four, we extend the theory from dimension $m = 1$ to $m > 1$. The situation gets more complex due to the anti-commutativity of the basic Clifford elements and the fact that we now have to consider the *generalised* Hermite polynomials $H_{n,m,r}$ as basis elements for the Weierstrass space. These generalised Hermite polynomials were introduced by Sommen in [36] in the classical case and translated to the discrete setting by De Ridder in [31]. They are formed as the composition of a monogenic polynomial of order r and a Hermite polynomial of degree n. The goal is to obtain recurrence formulae, both in terms of the degree n of the Hermite polynomial and in terms of the degree r of the monogenic, to find the Weierstrass transform of this generalised Hermite polynomial. In order to fix ideas and limit notations, we start in two dimensions. Because of the explicit form of a monogenic polynomial in two dimensions, we will be able to manage the challenges mentioned above. Unfortunately, if $m > 2$, it will turn out that another approach for the definition of the Weierstrass transform is needed, in order to find an explicit expression for the transform of a generalised Hermite polynomial. We give two equivalent definitions, one of which will provide us the formula we are aiming for.

Finally, in chapter 6, we enter a new subject: we discuss the discrete heat equation. In [2], a first sort has been treated: a framework with discrete space and continuous time. We will consider a heat equation in a situation in which both space and time are discrete. Therefore, we introduce a new operator with respect to the time variable. We then discuss the fundamental solution of the discrete heat equation and how to handle an initial value problem. Finally, we obtain the discrete heat polynomials and discuss a family of functions that are orthogonal to and can be interpreted as dual to these heat polynomials.

The content of chapter three, the definitions of the discrete Weierstrass transform and space, has been published in [37], while the generalisations for dimensions $m > 1$ and grid mesh width $h \neq 1$ were subject of a second paper, [38].

2

Preliminaries

This chapter is intended to make the reader familiar with definitions and concepts of Clifford analysis and discrete Clifford analysis in particular. The content is based on a variety of other works, the most prominent ones being [39], [28], [31] and [29].

2.1 Clifford algebras and analysis: continuous setting

2.1.1 Clifford algebras

Consider the m-dimensional real vector space $\mathbb{R}^m$ spanned by the orthonormal basis $\{e_1, \ldots, e_m\}$, endowed with a non-degenerate quadratic form of signature (p, q), $p+q = m$. A non-commutative multiplication then is defined by the rules

$$\begin{aligned} e_j^2 &= 1, & j &= 1, \ldots p, \\ e_j^2 &= -1, & j &= p+1, \ldots, m, \\ e_j e_k + e_k e_j &= 0, & j &\neq k, j, k = 1, \ldots, m, \end{aligned} \tag{2.1}$$

leading to the construction of the real Clifford algebra $\mathbb{R}_{p,q}$. Denote for the set $A = \{j_1, \ldots, j_r\} \subseteq \{1, \ldots, m\} = M$ the element $e_A = e_{j_1} e_{j_2}, \ldots e_{j_r}$, for $1 \leq j_1 < j_2 < \ldots < j_r \leq m$, with $e_\emptyset = 1$ the multiplicative identity element. For every $r \in \{1, \ldots, m\}$, the set $\{e_A \mid A \subseteq M \text{ and } |A| = r\}$ constitutes a basis for the space $\mathbb{R}_{p,q}^r$ of r-vectors. Any element $a \in \mathbb{R}_{p,q}$ can be composed as

$$\sum_{A \subseteq M} a_A e_A, \quad a_A \in \mathbb{R}$$

or equivalently as

$$\sum_{r=0}^{m} [a]_r,$$

where $[.]_r : \mathbb{R}_{p,q} \to \mathbb{R}^r_{p,q}$ is the projection from $\mathbb{R}_{p,q}$ to $\mathbb{R}^r_{p,q}$. For each value of r there are $\binom{m}{r}$ basis elements, so the total dimension of the Clifford algebra is $\sum_{r=0}^{m} \binom{m}{r} = 2^m$.
The space $\mathbb{R}$ of real numbers is identified with the subspace of scalars $\mathbb{R}^0_{p,q}$ and the Euclidean space $\mathbb{R}^m$ is embedded in the Clifford algebra $\mathbb{R}_{p,q}$ by the identification of the point $\underline{x} = (x_1, \dots, x_m) \in \mathbb{R}^m$ with the Clifford vector $\sum_{j=1}^{m} e_j x_j$.

In this work, we will work in the complex Clifford algebra $\mathbb{C}_m$, which is introduced as the composition

$$\mathbb{C}_m \coloneqq \mathbb{C} \otimes \mathbb{R}_{0,m}.$$

It has the same generators $(e_1, \dots, e_m)$ as $\mathbb{R}_{0,m}$, with the same multiplication rules, however allowing for complex constants. As a linear associative algebra over $\mathbb{C}$, it still has dimension 2^m. Any Clifford number $\lambda \in \mathbb{C}_m$ may now be written as

$$\sum_{A \subseteq M} \lambda_A e_A, \quad \lambda_A \in \mathbb{C},$$

or equivalently as $\lambda = a + ib$ with $a, b \in \mathbb{R}_{0,m}$. As in the real Clifford algebra, the space $\mathbb{C}$ can be identified with the subalgebra of scalars $\mathbb{C}^0_m$. The following automorphisms on $\mathbb{C}_m$ are used:

- the main involution or inversion $a \mapsto \tilde{a}$:

$$\begin{aligned} \widetilde{e_j} &= -e_j, & j &= 1, \dots, m, \\ \widetilde{ab} &= \tilde{a}\tilde{b}, & a, b &\in \mathbb{C}_m, \\ \widetilde{(\lambda_A e_A)} &= \lambda_A \widetilde{e_A}, & A &\subset M; \end{aligned}$$

- the Clifford conjugation: $a \mapsto \bar{a}$:

$$\begin{aligned} \overline{e_j} &= -e_j, & j &= 1, \dots, m, \\ \overline{ab} &= \bar{b}\bar{a}, & a, b &\in \mathbb{C}_m, \\ \overline{(\lambda_A e_A)} &= \lambda_A \overline{e_A}, & A &\subset M, \\ \overline{e_A} &= (-1)^{\frac{|A|(|A|+1)}{2}} e_A; \end{aligned}$$

- the Hermitian conjugation: $a \mapsto a^\dagger$:

$$\begin{aligned} {e_j}^\dagger &= -e_j, & j &= 1, \dots, m, \\ (ab)^\dagger &= b^\dagger a^\dagger, & a, b &\in \mathbb{R}_{0,m}, \\ (\lambda_A e_A)^\dagger &= \lambda^c_A {e_A}^\dagger, & A &\subset M, \end{aligned}$$

  with λ^c_A the usual complex conjugate of the complex number λ_A.

It is mainly with the latter involution that we will be concerned. For an arbitrary Clifford number $\lambda \in \mathbb{C}_m$:

$$\lambda^\dagger = \bar{a} - i\bar{b}, \quad a, b \in \mathbb{R}_{0,m}$$

or equivalently

$$\lambda^\dagger = \sum_{A\subseteq M} \lambda_A^c \overline{e_A}, \quad \lambda_A \in \mathbb{C}$$

where $.^c$ is used for complex conjugation.

By means of this Hermitian conjugation $.^\dagger$, one may introduce a sesquilinear inner product and associated norm on $\mathbb{C}_m$:

$$\langle \lambda, \mu \rangle = \left[\lambda^\dagger \mu\right]_0, \tag{2.2}$$

$$|\lambda| = \sqrt{\left[\lambda^\dagger \lambda\right]_0}. \tag{2.3}$$

In this dissertation, we will work with Clifford valued functions: a function $\mathbb{C}^m \to \mathbb{C}_m$ maps a complex variable $\underline{z}$ to a Clifford number and thus can be written as

$$f(\underline{z}) = \sum_{A\subseteq M} e_A f_A(\underline{z}),$$

with $f_A : \mathbb{C}^m \to \mathbb{C}$. We can define the right Clifford-module

$$\mathcal{L}^2\left(\mathbb{C}^m, \mathbb{C}_m, \frac{1}{\pi^m}\exp\left(-|\underline{z}|^2\right)\right), \tag{2.4}$$

as an analogue to the classical weighted $\mathcal{L}_2$ space (the Fock space, see section 3.1). Since the elements of a Clifford algebra do not form a field, this is a Clifford module. The inner product of two functions in this module is then given by

$$\langle f, g \rangle = \frac{1}{\pi^m}\int_{\mathbb{C}^m} f^\dagger(\underline{z}) g(\underline{z}) \exp\left(-|\underline{z}|^2\right) d\underline{x} d\underline{y},$$

with $\underline{z} = \underline{x} + i\underline{y}$. Its associated norm is

$$\|f\|^2 = \left[\langle f, f\rangle\right]_0. \tag{2.5}$$

For a function to be an element of $\mathcal{L}^2\left(\mathbb{C}^m, \mathbb{C}_m, \frac{1}{\pi^m}\exp\left(-|\underline{z}|^2\right)\right)$, its norm (2.5) must be finite.

2.1.2 Clifford analysis

The considered functions are defined on $\mathbb{R}^m$ and take values in the real or complex Clifford algebra $\mathbb{R}_{p,q}$ or $\mathbb{C}_m$. They may thus be written as

$$f = \sum_{A\subset M} f_A(x_1, \ldots, x_m) e_A.$$

The key subjects in Clifford analysis are the Dirac operator and its solutions, monogenic functions. Therefore, consider the generalised Cauchy-Riemann operator (also known as the Fueter-Delanghe operator)

$$D_x = \partial_{x_0} + \sum_{j=1}^m e_j \partial_{x_j} = \partial_{x_0} + \partial_{\underline{x}}.$$

It splits into a scalar part ∂_{x_0} and an $m-$dimensional vectorial part $\partial_{\underline{x}}$, which is called the Dirac operator and is the Fourier dual of the vector variable $\underline{x} = \sum_{j=1}^{m} e_j x_j$. This Dirac operator factorises the $m-$dimensional Laplacian

$$\partial_{\underline{x}}^2 = -\sum_{j=1}^{m} \partial_{x_j}^2 = -\Delta_m,$$

while the Cauchy-Riemann operator factorises the $m+1$-dimensional Laplacian

$$D_x \overline{D_x} = (\partial_{x_0} + \partial_{\underline{x}})(\partial_{x_0} - \partial_{\underline{x}}) = \sum_{j=0}^{m} \partial_{x_j}^2 = \Delta_{m+1}.$$

There is a natural equivalence between solutions of D_x and ∂_x. To this end, one adds an extra generator e_0 to the set of generators of $\mathbb{R}^m$. Consider the corresponding Clifford algebra $\mathbb{R}_{0,m+1}$, then we identify an element of $\mathbb{R}^{m+1}$ with a subspace of $\mathbb{R}_{0,m+1}$ by the correspondence

$$x = \sum_{j=0}^{m} x_j e_j \mapsto x_0 e_0 + \underline{x}.$$

This way, the Dirac operator ∂_x in $m+1$ dimensions is introduced by

$$\partial_x = \sum_{j=0}^{m} \boldsymbol{e}_j \partial_{x_j} = e_0 \partial_{x_0} + \partial_{\underline{x}}.$$

A function f is now called left (resp. right) monogenic if it satisfies $\partial_{\underline{x}} f = 0$ (resp. $f \partial_{\underline{x}} = 0$). Similarly, a function satisfying $\Delta_m = 0$ is called a harmonic function. A function that is monogenic is in particular also harmonic, monogenicity thus being a refinement of harmonicity.

2.2 Clifford algebras and analysis: discrete setting

2.2.1 Discrete Clifford algebras

Consider the discrete grid $\mathbb{Z}_h^m = \{\underline{x} = (n_1 h, n_2 h, \ldots, n_m h) \mid \underline{n} \in \mathbb{Z}^m\}$. Split up the basis elements e_j of $\mathbb{C}_m$ into **forward and backward basic vectors** $\boldsymbol{e}_j^+$ and $\boldsymbol{e}_j^-$, such that

- the forward and the backward basis vector in each particular cartesian direction add up to the traditional basis vector in that direction: $\boldsymbol{e}_j^+ + \boldsymbol{e}_j^- = \boldsymbol{e}_j, j = 1, \ldots m$;
- there are no preferential Cartesian directions, i.e. all Cartesian directions play the same role in the metric (rotational invariance);
- the positive and negative orientations of any cartesian direction play an equivalent role (reflection invariance).

We consider the algebra over the set $\{\boldsymbol{e}_j^+, \boldsymbol{e}_j^- \mid j = 1, .., m\}$, satisfying the following relations:

$$\begin{aligned} \{\boldsymbol{e}_j^-, \boldsymbol{e}_k^-\} &= \boldsymbol{e}_j^- \boldsymbol{e}_k^- + \boldsymbol{e}_k^- \boldsymbol{e}_j^- = 0, \qquad j, k \in \{1, \ldots, m\}, \\ \{\boldsymbol{e}_j^+, \boldsymbol{e}_k^+\} &= \boldsymbol{e}_j^+ \boldsymbol{e}_k^+ + \boldsymbol{e}_k^+ \boldsymbol{e}_j^+ = 0, \qquad j, k \in \{1, \ldots, m\}, \\ \{\boldsymbol{e}_j^+, \boldsymbol{e}_k^-\} &= \boldsymbol{e}_j^+ \boldsymbol{e}_k^- + \boldsymbol{e}_k^- \boldsymbol{e}_j^+ = \delta_{jk}, \qquad j, k \in \{1, \ldots, m\}. \end{aligned} \tag{2.6}$$

Remark that, in contrast to the relations mentioned in (2.1) for the continuous case, the anticommutator of $\boldsymbol{e}_j^+$ and $\boldsymbol{e}_j^-$ equals 1. This implies $e_j^2 = +1$ versus the usual Clifford setting where $e_j^2 = -1$. This is due to historical reasons. The Clifford algebra thus has order $(m, 0)$. Because of this splitting of the basis elements, this setting is sometimes referred to as the 'split' discrete setting. The basis elements $\{\boldsymbol{e}_j^+, \boldsymbol{e}_j^- \mid j = 1, \ldots, m\}$ form a basis of the space of 1-vectors. Let the wedge product (also called exterior product) be defined as follows

$$\boldsymbol{e}_j^\pm \wedge \boldsymbol{e}_k^\pm := \boldsymbol{e}_j^\pm \boldsymbol{e}_k^\pm - \boldsymbol{e}_k^\pm \boldsymbol{e}_j^\pm. \tag{2.7}$$

Then the elements in $\{\boldsymbol{e}_j^\pm \wedge \boldsymbol{e}_k^\pm \mid j, k = 1, \ldots m\}$ form a basis for the space of 2-vectors or bivectors. In general, the set $\{\boldsymbol{e}_{j_1}^\pm \wedge \boldsymbol{e}_{j_2}^\pm \wedge \ldots \wedge \boldsymbol{e}_{j_r}^\pm \mid j_1, \ldots, j_r \in \{1, \ldots, m\}\}$ forms the basis for the r-vectors.

Remark 2.1. We started from the classical Clifford algebra $\mathbb{C}_m$ with dimension 2^m and split up each basis element in a forward and backward vector. Because of that, the dimension raises to $(2^m)^2 = 2^{2m}$. This can also be seen as a lift from $\mathbb{C}_m$ to $\mathbb{C}_{2m}$: identifying $\boldsymbol{e}_j^+ + \boldsymbol{e}_j^-$ with a continuous basis element that squares up to 1 and $\boldsymbol{e}_j^+ - \boldsymbol{e}_j^-$ with one that squares up to -1.

The same involutions as in the continuous case can be introduced. In this work, we will only use the **Hermitian conjugation** 2.1.1 .†, which we now extend to the forward and backward vectors as

$$(\boldsymbol{e}_j^+)^\dagger = \boldsymbol{e}_j^- \text{ and } (\boldsymbol{e}_j^-)^\dagger = \boldsymbol{e}_j^+.$$

Remark 2.2. Hermitian conjugation is the discretisation of hermitian conjugation in $\mathbb{C}_m$ and a generalisation of complex conjugation in $\mathbb{C}$. It holds that $(\boldsymbol{e}_j^+ + \boldsymbol{e}_j^-)^\dagger = \boldsymbol{e}_j^+ + \boldsymbol{e}_j^-$ and $(\boldsymbol{e}_j^+ - \boldsymbol{e}_j^-)^\dagger = \boldsymbol{e}_j^- - \boldsymbol{e}_j^+$.

2.2.2 Discrete Clifford analysis

Classical partial derivatives are replaced by the so-called **forward and backward differences** for $j = 1, \dots, m$, in analogy with the classical left and right limit of differentiation:

$$\Delta_j^+ f(\underline{x}) = \frac{f(\underline{x} + he_j) - f(\underline{x})}{h},$$
$$\Delta_j^- f(\underline{x}) = \frac{f(\underline{x}) - f(\underline{x} - he_j)}{h},$$

with $\underline{x} \in \mathbb{Z}_h^m$ and f defined on $\mathbb{Z}_h^m$.

Definition 2.3. The **star Laplacian**

$$\Delta^*[f](x) = \sum_{j=1}^{m} \frac{f(x+he_j) + f(x-he_j)}{h^2} - 2m\frac{f(x)}{h^2} = \sum_{j=1}^{m} \Delta_j^+ \Delta_j^- [f](x) = \sum_{j=1}^{m} \Delta_j^- \Delta_j^+ [f](x)$$

is the discrete counterpart of the continuous Laplace operator. A function f such that $\Delta^*[f](x) = 0$, is called a (discrete) harmonic function.

Many notions of a discrete Dirac operator are around. As we want discrete monogenicity to be a refinement of discrete harmonicity, we choose the next definition, which makes use of the splitting of the basis vectors e_j.

Definition 2.4. Denote $\partial_j = \boldsymbol{e}_j^+ \Delta_j^+ + \boldsymbol{e}_j^- \Delta_j^-$. The **discrete Dirac operator** then is defined as

$$\partial = \sum_{j=1}^{m} \partial_j = \sum_{j=1}^{m} \boldsymbol{e}_j^+ \Delta_j^+ + \boldsymbol{e}_j^- \Delta_j^-.$$

A discrete function f satisfying $\partial f = 0$, is called (left) **monogenic**.

This Dirac operator factorises the star Laplacian, i.e.

$$\partial^2 = \Delta^*.$$

This is in analogy to classical Clifford analysis where the continuous Dirac operator also factorises the Laplacian Δ_m, monogenicity thus being a refinement of harmonicity.

The Dirac operator and the forward and backward differences are lowering operators: they lower the degree of a polynomial by 1. The corresponding raising operator is the discrete vector variable operator, denoted by

$$\xi = \sum_{j=1}^{m} \xi_j = \sum_{j=1}^{m} \boldsymbol{e}_j^+ X_j^- + \boldsymbol{e}_j^- X_j^+.$$

The interaction of the lowering and rising operators are given by the **skew Weyl-relations** introduced in [30]

$$\begin{aligned} \Delta_j^+ X_j^+ - X_j^- \Delta_j^- &= 1, \\ \Delta_j^- X_j^- - X_j^+ \Delta_j^+ &= 1. \end{aligned} \tag{2.8}$$

This implies for ξ_j and ∂_j

$$\begin{aligned} \partial_j \xi_j - \xi_j \partial_j &= 1, \\ \xi_j \xi_k &= -\xi_k \xi_j, \\ \partial_j \partial_k &= -\partial_k \partial_j, \\ \partial_j \xi_k &= -\xi_k \partial_j, \qquad j \neq k. \end{aligned} \tag{2.9}$$

Definition 2.5. The **discrete Euler operator** is given by

$$\mathbb{E} = \sum_{j=1}^{m} \xi_j \partial_j = \sum_{j=1}^{m} \left(\boldsymbol{e}_j^+ \boldsymbol{e}_j^- X_j^- \Delta_j^- + \boldsymbol{e}_j^- \boldsymbol{e}_j^+ X_j^+ \Delta_j^+ \right). \tag{2.10}$$

The same intertwining relations as in the continuous setting hold:

$$\begin{aligned} \partial \xi + \xi \partial &= 2\mathbb{E} + m, \\ \partial \mathbb{E} &= \mathbb{E} \partial + \partial, \\ \mathbb{E} \xi &= \xi \mathbb{E} + \xi. \end{aligned} \tag{2.11}$$

From the relations (2.8) and (2.11), we find how the raising operators $X_j^{\pm}$ and hence ξ_j act on polynomials. In particular, $\xi_j^n[1]$, which are the **basic discrete polynomials** of degree n were obtained in [33]:

$$\begin{aligned} \xi_j[1](x_j) &= x_j(\boldsymbol{e}_j^+ + \boldsymbol{e}_j^-), \\ \xi_j^{2k+1}[1](x_j) &= x_j \prod_{s=1}^{k} \left(x_j^2 - s^2 h^2 \right) \left(\boldsymbol{e}_j^+ + \boldsymbol{e}_j^- \right), \\ \xi_j^{2k}[1](x_j) &= \left(x_j^2 + k\, h\, x_j \left(\boldsymbol{e}_j^+ \boldsymbol{e}_j^- - \boldsymbol{e}_j^- \boldsymbol{e}_j^+ \right) \right) \prod_{s=1}^{k-1} \left(x_j^2 - s^2 h^2 \right), \end{aligned} \tag{2.12}$$

for $k \geq 1$. These polynomials are, for $k > 2$ unique solutions of the system

$$\begin{cases} \partial_j \xi_j^k[1] = k \xi_j^{k-1}[1], \\ \xi_j^k[1](0) = 0, \\ \xi_j^k[1](h) = 0. \end{cases} \tag{2.13}$$

The roots of $\xi_j^{2k+1}[1]$ and $\xi_j^{2k+2}[1]$ is the set $\{-kh, \ldots, kh\}$. Remark that $\left(\xi_j^n[1]\right)^\dagger = \xi_j^n[1]$.

A polynomial P_k is called **discrete homogeneous** of degree k if it is an eigenfunction of the discrete Euler operator with eigenvalue k:

$$\mathbb{E} P_k = k P_k.$$

The basic discrete polynomials $\xi_j^k[1]$ are discrete homogeneous of degree k. Moreover, they span the subvectorspace of discrete homogeneous polynomials of degree k. It is important to keep in mind that the discrete notion of homogenity does not coincide with the continuous definition. A restriction of a continuous homogeneous polynomial to the grid does not give a discrete homogeneous polynomial in the discrete sense and vice versa: $\xi_j^k[1]$ is not homogeneous in the classical definition.

The natural powers of the Hermitian conjugation $\dagger$ of ξ are analogous: we were able to prove that they only differ by a different sign of the bivector part:

Lemma 2.6. The natural powers of the Hermitian conjugation † of ξ are given by

$$\begin{aligned} \left(\xi_j^\dagger\right)^{2k+1}[1] = \xi_j^{2k+1}[1] = x_j \prod_{s=1}^{k}\left(x_j^2 - s^2h^2\right)\left(\boldsymbol{e}_j^+ + \boldsymbol{e}_j^-\right), \\ \left(\xi_j^\dagger\right)^{2k}[1] = \left(x_j^2 - k\,h\,x_j\left(\boldsymbol{e}_j^+\boldsymbol{e}_j^- - \boldsymbol{e}_j^-\boldsymbol{e}_j^+\right)\right)\prod_{s=1}^{k-1}\left(x_j^2 - s^2h^2\right). \end{aligned} \tag{2.14}$$

Proof. This proof is by induction. Take $h = 1$ and omit the subindex j to not overload notations. The trivial cases with $k = 0$ are

- $\left(\xi^\dagger\right)^0 = 1$
- $\left(\xi^\dagger\right)^1 = x(\boldsymbol{e}^+ + \boldsymbol{e}^-)$

Now let $k \geq 1$. We use lemma 2.9.4 and corollary 2.9.1 from [31]. First, we prove for the odd case that $\left(\xi^\dagger\right)^2\left(\xi^{2k-1}[1](x)\right) = \xi^{2k+1}[1](x)$.

$$\begin{aligned} \xi^\dagger\left(\xi^{2k-1}[1](x)\right) &= (X^+\boldsymbol{e}^- + X^+\boldsymbol{e}^+)\left(\xi^{2k-1}[1](x)\right) \\ &= (x+k)\boldsymbol{e}^-\xi^{2k-1}[1](x) + (x-k)\boldsymbol{e}^+\xi^{2k-1}[1](x) \\ &= x(\boldsymbol{e}^+ + \boldsymbol{e}^-)\xi^{2k-1}[1](x) + k(\boldsymbol{e}^- - \boldsymbol{e}^+)\xi^{2k-1}[1](x). \end{aligned}$$

$$\begin{aligned} \left(\xi^\dagger\right)^2\left(\xi^{2k-1}[1](x)\right) &= \xi^\dagger\left[x(\boldsymbol{e}^+ + \boldsymbol{e}^-)\xi^{2k-1}[1](x)\right] + \xi^\dagger\left[k(\boldsymbol{e}^- - \boldsymbol{e}^+)\xi^{2k-1}[1](x)\right] \\ &= x(\boldsymbol{e}^- + \boldsymbol{e}^+)\xi^\dagger\left(\xi^{2k-1}[1](x)\right) + k(\boldsymbol{e}^- - \boldsymbol{e}^+)\xi^\dagger\left(\xi^{2k-1}[1](x)\right) \\ &= x(\boldsymbol{e}^- + \boldsymbol{e}^+)\left(x(\boldsymbol{e}^+ + \boldsymbol{e}^-)\xi^{2k-1}[1](x) + k(\boldsymbol{e}^- - \boldsymbol{e}^+)\xi^{2k-1}[1](x)\right) \\ &\quad + k(\boldsymbol{e}^- - \boldsymbol{e}^+)\left(x(\boldsymbol{e}^+ + \boldsymbol{e}^-)\xi^{2k-1}[1](x) + k(\boldsymbol{e}^- - \boldsymbol{e}^+)\xi^{2k-1}[1](x)\right) \\ &= x^2\underbrace{(\boldsymbol{e}^- + \boldsymbol{e}^+)^2}_{=1}\xi^{2k-1}[1](x) + xk(\boldsymbol{e}^- + \boldsymbol{e}^+)(\boldsymbol{e}^- - \boldsymbol{e}^+)\xi^{2k-1}[1](x) \\ &\quad + xk(\boldsymbol{e}^- - \boldsymbol{e}^+)(\boldsymbol{e}^- + \boldsymbol{e}^+)\xi^{2k-1}[1](x) + k^2\underbrace{(\boldsymbol{e}^- - \boldsymbol{e}^+)^2}_{=-1}\xi^{2k-1}[1](x) \\ &= (x^2 - k^2)\xi^{2k-1}[1](x). \end{aligned}$$

For the even case, we rely on the property for the odd case.

$$\begin{aligned} \left(\xi^\dagger\right)^{2k+2}[1](x) &= \xi^\dagger\left[\left(\xi^\dagger\right)^{2k+1}[1](x)\right] \\ &= \xi^\dagger\left[\xi^{2k+1}[1](x)\right] \\ &= X^+\boldsymbol{e}^+\left[\xi^{2k+1}[1](x)\right] + X^-\boldsymbol{e}^-\left[\xi^{2k+1}[1](x)\right] \\ &= \left(x - (k+1)\right)\boldsymbol{e}^+\left[\xi^{2k+1}[1](x)\right] + \left(x + (k+1)\right)\boldsymbol{e}^-\left[\xi^{2k+1}[1](x)\right] \\ &= x(\boldsymbol{e}^+ + \boldsymbol{e}^-)\xi^{2k+1}[1](x) + (k+1)(\boldsymbol{e}^- - \boldsymbol{e}^+)\xi^{2k+1}[1](x) \end{aligned}$$

$$= x^2 \prod_{i=1}^{k}(x^2 - i^2)\underbrace{(\boldsymbol{e}^+ + \boldsymbol{e}^-)^2}_{=1} + (k+1)(\boldsymbol{e}^- - \boldsymbol{e}^+)x \prod_{i=1}^{k}(x^2 - i^2)(\boldsymbol{e}^+ + \boldsymbol{e}^-)$$

$$= \left(x^2 + (k+1)x\left(\boldsymbol{e}^-\boldsymbol{e}^+ - \boldsymbol{e}^+\boldsymbol{e}^-\right)\right)\prod_{i=1}^{k}\left(x^2 - i^2\right)$$

$$= \left(x^2 - (k+1)x\left(\boldsymbol{e}^+ \wedge \boldsymbol{e}^-\right)\right)\prod_{i=1}^{k}\left(x^2 - i^2\right).$$

□

Until now, we gave expressions for the discrete vector variable ξ acting from the left on the base state [1]. As the Clifford algebra we are working in is not commutative, the action from the right is, in general, not equal to the action on the left. In [31], it is proven that

$$\left(\xi_1^{\alpha_1} \dots \xi_j^{\alpha_j} \dots \xi_m^{\alpha_m}[1]\right)\partial_j^\dagger = (-1)^{\alpha_{j+1}+\dots+\alpha_m}\xi_1^{\alpha_1}\dots\left(\alpha_j \xi_j^{\alpha_j - 1}\right)\dots\xi_m^{\alpha_m}[1].$$

In particular, if $m = 1$, this reduces to

$$\left(\xi^k[1]\right)\partial^\dagger = \partial\left(\xi^k[1]\right), \tag{2.15}$$

$$\left(\xi^k[1]\right)\xi^\dagger = \xi^{k+1}[1]. \tag{2.16}$$

Due to the anti-commutativity of the basic elements $e_1, \dots, e_m$, also the co-ordinate difference operators and co-ordinate vector variables mutually anti-commute. To overcome this lack of commutativity of ξ and ∂, let us introduce the operators

$$R_j = \boldsymbol{e}_j^+ R_j^+ + \boldsymbol{e}_j^- R_j^-, \qquad j = 1, \dots m, \tag{2.17}$$

which were defined in [40].

The operators interact with $X_j^\pm$ and $\Delta_j^\pm$ in the following way:

$$\begin{aligned}
&R_j^\pm[1] = \boldsymbol{e}_j^\pm, \\
&R_j^+ X_j^+ = X_j^- R_j^-, \quad R_j^- X_j^- = X_j^+ R_j^+, \\
&R_j^+ \Delta_j^- = \Delta_j^+ R_j^-, \quad R_j^- \Delta_j^+ = \Delta_j^- R_j^+
\end{aligned}$$

It follows that, on co-ordinate level, they satisfy the following (anti-)commuting relations:

$$\begin{aligned}
R_j\xi_j - \xi_j R_j &= 0, \\
R_j\partial_j - \partial_j R_j &= 0, \\
R_j\xi_k + \xi_k R_j &= 0, \qquad j \neq k, \\
R_j\partial_k + \partial_k R_j &= 0, \qquad j \neq k, \\
R_j R_k + R_k R_j &= 0, \qquad j \neq k.
\end{aligned} \tag{2.18}$$

In particular, in combination with the operators ξ_j and ∂_j, we obtain mutually commuting operators $\xi_j R_j$ and $\partial_j R_j$, i.e.

$$(\xi_j R_j)(\xi_k R_k) = (\xi_k R_k)(\xi_j R_j) \text{ and } (\partial_j R_j)(\partial_k R_k) = (\partial_k R_k)(\partial_j R_j).$$

These operators R_j will be of interest when we consider structures and definitions in dimensions $m > 1$ in chapter 4.

The **discrete delta functions** are the building blocks of discrete function theory.

$$\delta_{\underline{n}h}(\underline{x}) = \begin{cases} \frac{1}{h^m}, & \text{if } \underline{x} = \underline{n}h, \\ 0, & \text{else.} \end{cases}$$

A discrete function f can be decomposed into discrete delta functions as

$$f(\underline{x}) = \sum_{\underline{n}\in\mathbb{Z}^m} f(\underline{n}h)\, h\, \delta_{\underline{n}h}(\underline{x}), \quad x \in \mathbb{Z}^m.$$

The same function f can also be expressed as an infinite series of powers of the basis vector variables, its **Taylor series**:

$$f(\underline{x}) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\ell_1=1}^{m} \cdots \sum_{\ell_k=1}^{m} \xi_{\ell_1} \ldots \xi_{\ell_k}[1](x)\partial_{\ell_k} \ldots \partial_{\ell_1} f(0). \tag{2.19}$$

In one dimension, this gives

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \xi^k[1](x)[\partial^k f(u)]_{u=0}. \tag{2.20}$$

The Taylor series expansion

$$f(\xi) = \sum_{k\in\mathbb{N}} \frac{\xi_h^k c_k}{k!} \text{ with } c_k = [\partial^k f(u)]_{u=0}$$

is the corresponding operator to this function. Letting this operator act on the identity function 1, one obtains a function in the discrete variable x. By identifying a discrete function with its corresponding operator, the set of discrete functions is a right Clifford module.

In particular, the discrete Taylor series of the delta functions in one dimension are found in [32]:

$$\delta_0(x) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(\ell!)^2\, h^{2\ell+1}} \xi^{2\ell}[1](x) + \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+1}}{(\ell+1)!\,\ell!\, h^{2\ell+2}} \xi^{2\ell+1}[1](x)(\boldsymbol{e}^+ - \boldsymbol{e}^-),$$

For $j = nh$ positive:

$$\delta_{nh}(x) = \sum_{\ell=n}^{\infty} \frac{(-1)^{\ell-n}}{(\ell-n)!(\ell+n)!\, h^{2\ell+1}} \xi^{2\ell}[1](x) +$$

$$\sum_{\ell=n-1}^{\infty} \frac{(-1)^{\ell-n+1}}{(\ell-n+1)!(\ell+n)!} \xi^{2\ell+1}[1](x)\boldsymbol{e}^{+} + \sum_{\ell=n}^{\infty} \frac{(-1)^{\ell-n}}{(\ell-n)!(\ell+n+1)!\, h^{2\ell+2}} \xi^{2\ell+1}[1](x)\boldsymbol{e}^{-},$$

and for $j = nh$ negative:

$$\delta_j(nh) = \sum_{l=|n|}^{\infty} \frac{(-1)^{\ell-n}}{(\ell-n)!(\ell+n)!\, h^{2\ell+1}\, h^{2\ell+1}} \xi^{2\ell}[1](x) +$$

$$\sum_{l=|n|}^{\infty} \frac{(-1)^{\ell-n+1}}{(\ell-n+1)!(\ell+n)!} \xi^{2\ell+1}[1](x)\boldsymbol{e}^{+} + \sum_{l=|n|-1}^{\infty} \frac{(-1)^{\ell-n}}{(l-n)!(l+n+1)!\, h^{2\ell+2}} \xi^{2\ell+1}[1](x)\boldsymbol{e}^{-}.$$

2.2.2.1 Discrete distributions

Similar to classical analysis, distributions are a class of linear functionals acting on a particular space of functions.

A **discrete distribution** is a linear functional defined on the set of discrete polynomials, with values in the Clifford algebra. They were introduced in the discrete Clifford setting in [32]. As in the classical setting, a regular distribution F is one that is associated with a density function f, such that

$$\langle F, g \rangle = \int_{\mathbb{R}} g(x) f(x) dx.$$

The translation to the discrete setting is immediate:

$$\langle F, g \rangle = \sum_{x \in \mathbb{Z}_h} g(x) f(x) h.$$

The inverse is also true: with every discrete function with compact support, a distribution is associated: let $f(x)$ be a function with compact support, then its associated distribution F is defined as

$$\langle F, g \rangle = \sum_{x \in \mathbb{Z}} f(x)\, g(x).$$

Regular distributions are unique, in the sense that there is a one-on-one correspondence between the distribution and the discrete function with compact support.

Let F denote a (not necessary regular) discrete distribution, V a discrete polynomial and a a Clifford number. It was proven in [32] that

$$\begin{aligned} \left\langle \partial_j F, V \right\rangle &= -\left\langle F, V \partial_j^{\dagger} \right\rangle, \\ \left\langle \xi_j F, V \right\rangle &= \left\langle F, V \xi_j^{\dagger} \right\rangle, \\ \langle F a, g \rangle &= \langle F, V \rangle\, a, \\ \langle F, aV \rangle &= a \langle F, V \rangle\,. \end{aligned} \tag{2.21}$$

The building blocks of discrete distributions are **the discrete delta distributions $\boldsymbol{\delta}_{\underline{n}h}$**, associated with the discrete delta functions $\delta_{\underline{n}h}$:

$$\langle \boldsymbol{\delta}_{nh}, f \rangle = \sum_{\underline{m} \in \mathbb{Z}^m} f(\underline{m}h)\, h\, \delta_{\underline{n}h}(\underline{m}h) = f(\underline{n}h).$$

Consider the derivatives of the discrete delta distribution, acting on a discrete function f as

$$\left\langle \partial_j^k \boldsymbol{\delta}_{\underline{n}h}, f \right\rangle = (-1)^k \left\langle \boldsymbol{\delta}_{\underline{n}h}, \partial_j^k f \right\rangle = \partial_j^k f(\underline{n}h).$$

In particular, if $f = \xi_j^\ell[1](x)$, then

$$\left\langle \partial_j^k \boldsymbol{\delta}_{\underline{n}h}, \xi_j^\ell[1] \right\rangle = \begin{cases} (-1)^k \dfrac{\ell!}{(\ell-k)!} \xi_j^{\ell-k}[1](\underline{n}h), & k \leq \ell, \\ 0, & k > l. \end{cases} \tag{2.22}$$

The **dual Taylor series** are for distributions what the Taylor series are for functions. Every discrete distribution F can be written in terms of derivatives of the delta distribution as follows

$$F = \sum_{k1,\ldots k_m=0}^{\infty} \frac{(-1)^{|k|}}{k_1!\ldots k_m!} \partial_m^{k_m} \ldots \partial_1^{k_1} \boldsymbol{\delta}_0 \left\langle F, \xi_1^{k_1} \xi_2^{k_2} \ldots \xi_m^{k_m}[1] \right\rangle. \tag{2.23}$$

The **discrete Gauss distribution** is another important distribution in our theory. It is uniquely defined via its action on the discrete homogeneous polynomials:

$$\left\langle G, \xi_1^{k_1} \xi_2^{k_2} \ldots \xi_m^{k_m}[1] \right\rangle = \begin{cases} (2\pi)^{\frac{m}{2}} \prod_{i=1}^m (k_i - 1)!!, & \text{if all } k_i \text{ even}, \\ 0, & \text{else}, \end{cases} \tag{2.24}$$

with the double factorial

$$(k_i - 1)!! = \frac{(2k_i)!}{2^{k_i}\, k_i!}.$$

The dual Taylor series representation, immediately derived from (2.23), is given by:

$$\begin{aligned} G &= \sum_{k_1,\ldots,k_m=0}^{\infty} \frac{(-1)^{|k|}}{k_1!k_2!\ldots k_m!} \partial_1^{k_1} \partial_2^{k_2} \ldots \partial_m^{k_m} \boldsymbol{\delta}_0 \left\langle G, \xi_1^{k_1} \xi_2^{k_2} \ldots \xi_m^{k_m}[1] \right\rangle \\ &= \sum_{k=0}^{\infty} (2\pi)^{\frac{m}{2}} \frac{1}{2^{k_1+\ldots+k_m} k_1!\ldots k_m!} \partial_1^{k_1} \partial_2^{k_2} \ldots \partial_m^{k_m} \boldsymbol{\delta}_0 \\ &= (2\pi)^{\frac{m}{2}} \exp\left(\frac{\partial^2}{2}\right) \boldsymbol{\delta}_0. \end{aligned} \tag{2.25}$$

The following result from [31] for the moments of G is useful for calculations:

$$\begin{cases} \left\langle G, \xi^{2k}[1] \right\rangle &= \sqrt{2\pi}^m 2^k \frac{\Gamma(\frac{m}{2}+k)}{\Gamma(\frac{m}{2})}, \\ \left\langle G, \xi^{2k+1}[1] \right\rangle &= 0. \end{cases} \tag{2.26}$$

Let $P(\xi)$ be a discrete polynomial in the discrete vector variable ξ. The distribution $P(\xi)G$ is well defined and can be calculated using the calculation rule

$$\partial_j G = -\xi_j G, \tag{2.27}$$

and hence also

$$\partial G = -\xi G. \tag{2.28}$$

It enables us to write $P(\xi)G$ as a sum of derivatives of the $\boldsymbol{\delta}_0$-distribution. For a given polynomial P, defined on the grid, there exists a unique operator $P(\xi)$ such that

$$P(\xi)[1](\underline{n}h) = P(\underline{n}h), \qquad \forall \underline{n} \in \mathbb{Z}^m,$$

hence there exists a unique distribution of the form $P(\xi)G$.

The discrete (radial) **Hermite polynomials** are defined using the Gauss distribution. They are polynomials in ξ, defined by the recurrence relation $H_{k+1}G = (-1)^{k+1}\partial H_k G$. Using the relation $\partial G = -\xi G$, they satisfy Rodriguez' formula

$$\begin{aligned} H_{2k}G &= (-1)^k \partial^{2k} G \\ H_{2k+1}G &= (-1)^{k+1}\partial^{2k+1}G. \end{aligned} \tag{2.29}$$

An explicit form for the Hermite polynomials is given by

$$H_{2k,m} = \sum_{j=0}^{k} a_{2j}^{2k}\xi^{2j} \qquad H_{2k+1,m} = \sum_{j=0}^{k} a_{2j+1}^{2k+1}\xi^{2j+1} \tag{2.30}$$

with

$$a_{2j}^{2k} = (-1)^j 2^{k-j}\binom{k}{j}\frac{\Gamma(k+\frac{m}{2})}{\Gamma(j+\frac{m}{2})} \tag{2.31}$$

$$a_{2j+1}^{2k+1} = (-1)^j 2^{k-j}\binom{k}{j}\frac{\Gamma(k+\frac{m}{2}+1)}{\Gamma(j+\frac{m}{2}+1)}. \tag{2.32}$$

or in particular in dimension $m = 1$:

$$H_n(\xi) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^{\lfloor \frac{n}{2} \rfloor - j}\, 2^{-j}\, n!}{j!\,(n-2j)!}\xi^{n-2j} \tag{2.33}$$

The absolute values of the coefficients $a_{2j}^{2\ell}$ and $a_{2j+1}^{2\ell+1}$ are equal to those of the coefficients of the continuous Hermite polynomials H_n: the difference is a factor $(-1)^{\frac{\lfloor n \rfloor}{2}}$.

Even Hermite polynomials and operators have a scalar and a bivectorial part as they only contain even power of ξ, while odd Hermite polynomials only have a vectorial part as they only have odd powers of ξ.

Example 2.7. The first Hermite polynomial operators in dimension $m = 1$ are given by

$$H_1 = \xi$$

$$
\begin{aligned}
H_2 &= -\xi^2 + 1\\
H_3 &= -\xi^3 + 3\xi\\
H_4 &= \xi^4 - 6\xi^2 + 3\\
H_5 &= \xi^5 - 10\xi^3 + 15\xi\\
H_6 &= -\xi^6 + 15\xi^4 - 45\xi^2 + 15\\
H_7 &= -\xi^7 + 21\xi^5 - 105\xi^3 + 105\xi\\
H_8 &= \xi^8 - 28\xi^6 + 210\xi^4 - 420\xi^2 + 105\\
H_9 &= \xi^9 - 36\xi^7 + 378\xi^5 - 1260\xi^3 + 945\xi\\
H_{10} &= -\xi^{10} + 45\xi^8 - 630\xi^6 + 3150\xi^4 - 4725\xi^2 + 945
\end{aligned}
$$

In the classical case, one can use the continuous Hermite polynomials as a basis for a certain kind of L^2-functions: those defined on the real line. However, in [36], Sommen and his colleagues developped a more complex set of polynomials, called the *generalised* Hermite polynomials. They are defined using a monogenic function P_k of degree k and creating the monogenic extension of $\exp\left(-\frac{x^2}{2}\right)P_k(x)$:

$$H_{n,m,k}P_k(x) = \exp\left(-\frac{x^2}{2}\right)(-1)^n\partial_{\underline{x}}^n\left[\exp\left(-\frac{x^2}{2}\right)P_k(x)\right].$$

These can be used to construct a basis for of $\mathcal{L}^2\left(\mathbb{R}^m, \mathbb{R}_{0,m}\right)$, the space of measurable Clifford algebra-valued functions on $\mathbb{R}^m$. We refer to [36] and [41] for a more exhaustive study in the classical setting. Similarly, De Ridder introduced the discrete generalised Hermite polynomials, defined by the same corresponding Rodriguez' formulae as (2.29):

$$
\begin{aligned}
H_{2k,m,r}P_rG &= (-1)^k\partial^{2k}P_rG\\
H_{2k+1,m,r}P_rG &= (-1)^{k+1}\partial^{2k+1}P_rG.
\end{aligned}
\tag{2.34}
$$

Herein, P_r is a discrete monogenic polynomial of degree r. The explicit form for the generalised Hermite polynomials is similar to the expression of the radial Hermite polynomials:

$$H_{2k,m,r} = \sum_{j=0}^{k} a_{2j}^{2k}\xi^{2j} \qquad H_{2k+1,m,r} = \sum_{j=0}^{k} a_{2j+1}^{2k+1}\xi^{2j+1} \tag{2.35}$$

with

$$a_{2j}^{2k} = (-1)^j 2^{k-j}\binom{k}{j}\frac{\Gamma(k+\frac{m}{2}+r)}{\Gamma(j+\frac{m}{2}+r)} \tag{2.36}$$

$$a_{2j+1}^{2k+1} = (-1)^j 2^{k-j}\binom{k}{j}\frac{\Gamma(k+\frac{m}{2}+r+1)}{\Gamma(j+\frac{m}{2}+r+1)}. \tag{2.37}$$

For $r = 0$, we reobtain the radial Hermite polynomials.

3

Weierstrass transform in one dimension

We start with a note on the continuous (classical) Weierstrass transform. The elementary principles of the continuous setting will lead to the definition of a discrete Weierstrass space $\mathcal{W}$ on which a discrete counterpart of the Weierstrass transform will make sense. The set of discrete polynomials should be a dense subset of $\mathcal{W}$ and a basis will be formed by the Hermite polynomials. The inner product will be defined using the discrete Gauss distribution. We will show that the translation to the discrete setting is well-defined and investigate for some basic discrete functions if they are an element of our newly defined space. In particular, we will show in section 3.2.3 that the building blocks of discrete functions, the δ-functions are elements of the discrete Weierstrass space. In section 3.3, we generalise the definitions to a grid with mesh width $h \neq 1$. The fact that, for $h \to 0$, we obtain the classical results, would confirm the accuracy of our definitions.

3.1 Continuous Weierstrass transform

Definition 3.1. The **Weierstrass transform** [1] of a function f on $\mathbb{R}$ is defined as the convolution of f with the Gaussian kernel:

$$\mathcal{W}[f](u) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(\frac{-|u-x|^2}{2}\right) f(x)dx.$$

When considered as an integral transform on $\mathcal{L}^p(\mathbb{R})$, the space of real $p-$th power integrable functions on $\mathbb{R}$, we have that $\mathcal{W}[f] \in \mathcal{L}^p(\mathbb{R})$ and $\|\mathcal{W}[f]\| \leq \|f\|$, hence the Weierstrass transform is a bounded operator on $\mathcal{L}^p(\mathbb{R})$. As is well-known, the Hermite polynomials form a basis for the weighted $\mathcal{L}_2$-space:

$$\mathcal{L}_2\left(\mathbb{R}, \exp\left(-|x|^2/2\right)\right) = \{f : \mathbb{R} \to \mathbb{C} : \int_{\mathbb{R}} \left|f(x)\right|^2 \exp\left(-|x|^2/2\right) dx < \infty\},$$

where we use the probabilistic definition of Hermite polynomials

$$H_n(x) = (-1)^n \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n} \exp\left(-\frac{x^2}{2}\right) = n! \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^j}{j!(n-2j)!} \frac{x^{n-2j}}{2^j},$$

with generating function

$$\exp\left(xz - \frac{1}{2}z^2\right) = \sum_{n=0}^{\infty} H_n(x) \frac{z^n}{n!}. \tag{3.1}$$

Consider the Fock space ([42]) of holomorphic functions which are square integrable with respect to the Gaussian function

$$\frac{1}{\pi} \int_{\mathbb{C}} \exp\left(-|z|^2\right) \left|f(z)\right|^2 dxdy < \infty$$

and equipped with the inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{\mathbb{C}} \exp\left(-|z|^2\right) \overline{f(z)} g(z) dxdy, \qquad z = x + iy.$$

The basis $\{z^k \mid k \in \mathbb{N}\}$ is orthogonal:

$$\langle z^\ell, z^k \rangle = 0, \quad k \neq \ell,$$

and

$$\langle z^k, z^k \rangle = \frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \exp\left(-r^2\right) r^{2k} r dr d\theta = k!, \quad z = r\exp(i\theta).$$

As an informative example, let us calculate the Weierstrass transform of the classical Hermite polynomials in one dimension:

$$\begin{aligned}
\mathcal{W}[H_n](z) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(-(z-x)^2/2\right) (-1)^n \exp\left(x^2/2\right) \frac{d^n}{dx^n} \left[\exp\left(-x^2/2\right)\right] dx \\
&= (-1)^n \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(-z^2/2 + xz\right) \frac{d^n}{dx^n} \left[\exp\left(-x^2/2\right)\right] dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{d^n}{dx^n} \left[\exp\left(-z^2/2 + xz\right)\right] \exp\left(-x^2/2\right) dx \\
&= \frac{z^n}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(-(z-x)^2/2\right) dx = z^n.
\end{aligned} \tag{3.2}$$

This above computation shows that the Weierstrass transform of H_n equals z^n. It proves that we deal with an isometry between $\mathcal{L}_2\left(\mathbb{R}, \exp\left(-|x|^2/2\right)\right)$ and the Fock space. This discussion can be naturally extended from one to several complex variables, see [42], but for us the ideas and construction described above inspire us for the next chapter.

3.2 Discrete Weierstrass transform in one dimension

We want to establish a discrete version of the Weierstrass transform, sending a discrete function (i.e. defined on the grid) to an analytic function in a Clifford-valued Fock space. Therefore, we try to mimic the ideas of the classical setting. Consider again the calculations of $\mathcal{W}[H_n](z) = z^n$ (see (3.2)), we see

$$\mathcal{W}[H_n](z) = (-1)^n \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(-z^2/2 + xz\right) \frac{d^n}{dx^n} \left[\exp\left(-x^2/2\right)\right] dx.$$

It is the product of the $n-$th derivative of the Gaussian function with $\exp\left(-z^2/2 + xz\right)$. We recognise this derivative as the $n-$th degree Hermite polynomial. Both the Hermite polynomials (see (2.29)) and Gaussian function (see (2.24) are introduced in the discrete setting in the previous section. These elements will allow us to translate the Weierstrass transform to the discrete setting:

Definition 3.2. The discrete Weierstrass transformation of the discrete Hermite polynomials in one dimension is defined as

$$\mathcal{W}[H_n](z) := \frac{1}{\sqrt{2\pi}} \langle H_n(\xi)G, \exp\left(-z^2/2 + \xi\, z\right)[1]\rangle. \tag{3.3}$$

We then directly obtain the following result:

Proposition 3.3. $\mathcal{W}[H_n](z) = (-1)^{\lfloor \frac{n}{2} \rfloor} z^n, \forall n \in \mathbb{N}.$

Proof. This is a straightforward calculation.

$$\begin{aligned}\langle H_n(\xi)G, \exp\left(-z^2/2 + \xi\, z\right)[1]\rangle &= \langle (-1)^{\lceil \frac{n}{2} \rceil} \partial^n G, \exp\left(-z^2/2 + \xi\, z\right)[1]\rangle \\ &= (-1)^{\lceil \frac{n}{2} \rceil} \exp\left(-z^2/2\right) \langle \partial^n G, \sum_{i=0}^{\infty} \frac{\xi^i z^i [1]}{i!}\rangle.\end{aligned}$$

We now use (2.21) and (2.15) to see that this equals

$$= (-1)^{\lfloor \frac{n}{2} \rfloor} \exp\left(-z^2/2\right) \sum_{i=0}^{\infty} \frac{z^i}{i!} \langle G, \xi^i [1] (\partial^\dagger)^n \rangle.$$

Acting on the constant function 1, ∂ acts as a formal derivative of ξ (2.13), hence

$$\begin{aligned}&= (-1)^{\lfloor \frac{n}{2} \rfloor} \exp\left(-z^2/2\right) \sum_{i=0}^{\infty} \frac{z^i}{i!} \langle G, \partial^n \xi^i [1]\rangle \\ &= (-1)^{\lfloor \frac{n}{2} \rfloor} \exp\left(-z^2/2\right) \sum_{i=n}^{\infty} \frac{z^i}{i!} \frac{i!}{(i-n)!} \langle G, \xi^{i-n} [1]\rangle \\ &= (-1)^{\lfloor \frac{n}{2} \rfloor} \exp\left(-z^2/2\right) \sum_{j=0}^{\infty} \frac{z^{j+n}}{j!} \langle G, \xi^j [1]\rangle.\end{aligned}$$

The action of the Gauss distribution is only non-zero when acting on even powers of ξ and was given in (2.24)

$$
\begin{aligned}
&= (-1)^{\lfloor \frac{n}{2} \rfloor} \exp\left(-z^2/2\right) z^n \sum_{j=0}^{\infty} \frac{z^{2j}}{(2j)!} \sqrt{2\pi} \frac{(2j)!}{2^j j!} \\
&= (-1)^{\lfloor \frac{n}{2} \rfloor} \sqrt{2\pi}\, z^n.
\end{aligned}
$$

□

These transformations correspond to the continuous case, taking into account that the discrete Hermite polynomials are formally equal to their continuous counterparts, up to signs in the coefficients.

This definition can be naturally extended to any (in)finite right linear combination of Hermite polynomials:

$$
f = \sum_{n=0}^{\infty} H_n\, c_n \Rightarrow \mathcal{W}[f](z) = \sum_{n=0}^{\infty} \mathcal{W}[H_n](z)\, c_n, \qquad c_n \in \mathbb{C}_m.
$$

We now want to extend this definition to general discrete functions and find a condition for a discrete function to possess a Weierstrass transform. Therefore, we aim for an appropriate space of functions similar to the classical weighted L_2-space

$$
\mathcal{L}_2\left(\mathbb{R}^m, \exp\left(-|x|^2/2\right)\right) = \{f : \mathbb{R}^m \to \mathbb{C} : \int_{\mathbb{R}^m} |f(x)|^2 \exp\left(-|x|^2/2\right) dx < \infty\}.
$$

Herein, the set of Hermite polynomials form a basis and the Gaussian is the kernel, exactly the tools we have at hand.

Remark that the set of discrete functions forms a right Clifford module in which the right submodule of polynomials is dense, by writing a discrete function f in its Taylor series $f(\xi)[1]$, with

$$
f(\xi) = \sum_{k=0}^{\infty} \xi^k c_k
$$

the corresponding operator. Moreover, the set of discrete Hermite polynomials in ξ of degree less or equal to n, as its classical counterpart, is a set of $n+1$ linearly independent polynomials. Hence, the space of discrete polynomials of degree $\leq n$ is spanned by these Hermite polynomials, i.e. every discrete polynomial in ξ can be written as a finite linear combination of Hermite polynomials.

Now define a sesquilinear form as follows:

Definition 3.4. Let f and g be two discrete functions.

$$
(f, g) = \big(f(\xi)[1], g(\xi)[1]\big) := \left\langle f(\xi)G, [1](g(\xi))^\dagger \right\rangle. \tag{3.4}
$$

The distribution $f(\xi)G$ is calculated by writing f in its Taylor series expansion, hence a (possibly infinite) series of polynomials in the operator ξ.

Remark that

$$\big(f(\xi)[1]\big)^\dagger = [1]\sum_{k=0}^{\infty} c_k^\dagger (\xi^\dagger)^k = \sum_{k=0}^{\infty} c_k^\dagger \xi^k [1].$$

In particular, if f is a real function (i.e. every $c_k \in \mathbb{R}$) then $[1]\big(f(\xi)\big)^\dagger = f(\xi)[1]$.

Lemma 3.5. The discrete Hermite polynomials are orthogonal with respect to the inner product (3.4) and for $n, m \in \mathbb{N}$:

$$(H_n, H_m) = \sqrt{2\pi}\, n!\, \delta_{n,m} =: \eta_n.$$

Proof. We calculate (H_n, H_m). If $n \neq m$, it follows from [31], p. 9-12, that $(H_n, H_m) = 0$. Now suppose $m = n = 2k$ is even.

$$\begin{aligned}
(H_{2k}, H_{2k}) &= \left\langle \sum_{j=0}^{k} a_{2j}^{2k}\xi^{2j} G, [1]\left(\sum_{i=0}^{k} a_{2i}^{2k}\xi^{2i}\right)^\dagger \right\rangle = \left\langle \sum_{j=0}^{k} a_{2j}^{2k}\xi^{2j} G, \sum_{i=0}^{k} a_{2i}^{2k}[1](\xi^{2i})^\dagger \right\rangle \\
&= \sum_{i,j=0}^{k} a_{2j}^{2k} a_{2i}^{2k} \left\langle G, \xi^{2i+2j}[1]\right\rangle = \sum_{i,j=0}^{k} a_{2j}^{2k} a_{2i}^{2k} \sqrt{2\pi}\, \frac{(2i+2j)!}{2^{i+j}(i+j)!} \\
&= \sqrt{2\pi}\,(2k)! = \eta_{2k}.
\end{aligned}$$

For $m = n = 2k+1$ odd, we similarly have that

$$\begin{aligned}
(H_{2k+1}, H_{2k+1}) &= \sum_{i,j\in\mathbb{N}} a_{2j+1}^{2k+1} a_{2i+1}^{2k+1} \sqrt{2\pi}\, \frac{(2i+2j+2)!}{2^{i+j+1}(i+j+1)!} \\
&= \sqrt{2\pi}\,(2k+1)! = \eta_{2k+1}.
\end{aligned}$$

The last step was calculated by substituting the explicit values of the coefficients a_j^k. □

The results coincide with those for the inner products of Hermite polynomials in the continuous setting.

Lemma 3.6. The bilinear form (3.4) is conjugate symmetric with respect to †.

Proof. Let f, g be two discrete Clifford-valued polynomials and write them in their Hermite polynomial expansion, i.e. $f = \sum_{k=0}^{\infty} H_k a_k$ and $g = \sum_{\ell=0}^{\infty} H_\ell b_\ell$. We then have

$$\begin{aligned}
(f, g) &= \left\langle f(\xi)G, [1](g(\xi))^\dagger \right\rangle \\
&= \sum_{k,\ell=0}^{\infty} \left\langle H_k(\xi) a_k G, [1]\langle H_\ell(\xi) b_\ell\rangle^\dagger \right\rangle \\
&= \sum_{k,\ell=0}^{\infty} b_\ell^\dagger \langle H_k(\xi) G, [1] H_\ell(\xi^\dagger)\rangle a_k
\end{aligned}$$

$$= \sum_{n=0}^{\infty} b_n^\dagger \eta_n a_n,$$

Since G is scalar, the second step $a_k G = G a_k$ is allowed. Also, as $\eta_n \in \mathbb{R}$,

$$(g, f)^\dagger = \sum_{n=0}^{\infty} \left(a_n^\dagger \eta_n b_n \right)^\dagger = \sum_{n=0}^{\infty} b_n^\dagger \eta_n a_n.$$

□

Remark 3.7. It is possible to take the scalar part of the right hand side of definition (3.4) in order to obtain a positive definite inner product. In that case however, one obtains a *complex* Hilbert space in which the discrete Hermite polynomials are no longer basis elements: a function f spanned by discrete Hermite polynomials can be written as

$$\sum_{n=0}^{\infty} (f, H_n) H_n$$

and thus could never be Clifford-valued if both (f, H_n) and H_n are scalar-valued.

Let f be a Clifford-valued polynomial, written as a linear combination of Hermite polynomials: $f = \sum_{k=0}^{\infty} H_k c_k$. Then

$$(f, f) = \sum_{k,\ell=0}^{\infty} c_\ell^\dagger \left(H_k, H_\ell\right) c_k = \sum_{k=0}^{\infty} \eta_k c_k^\dagger c_k. \tag{3.5}$$

Let us now calculate $r^\dagger r$, where r is an arbitrary Clifford element of the form $r = a + b\boldsymbol{e}^+ + c\boldsymbol{e}^- + d(\boldsymbol{e}^+ \wedge \boldsymbol{e}^-)$, where a, b, c, d are complex numbers. We obtain

$$\begin{aligned}
r^\dagger r &= (a + b\boldsymbol{e}^+ + c\boldsymbol{e}^- + d(\boldsymbol{e}^+ \wedge \boldsymbol{e}^-))^\dagger (a + b\boldsymbol{e}^+ + c\boldsymbol{e}^- + d(\boldsymbol{e}^+ \wedge \boldsymbol{e}^-)) \\
&= (\bar{a} + \bar{b}\boldsymbol{e}^- + \bar{c}\boldsymbol{e}^+ + \bar{d}(\boldsymbol{e}^+ \wedge \boldsymbol{e}^-))(a + b\boldsymbol{e}^+ + c\boldsymbol{e}^- + d(\boldsymbol{e}^+ \wedge \boldsymbol{e}^-)) \\
&= |a|^2 + \bar{a}b\boldsymbol{e}^+ + \bar{a}c\boldsymbol{e}^- + \bar{a}d(\boldsymbol{e}^+ \wedge \boldsymbol{e}^-) + a\bar{b}\boldsymbol{e}^- + |b|^2\boldsymbol{e}^-\boldsymbol{e}^+ + \bar{b}d\boldsymbol{e}^-(\boldsymbol{e}^+ \wedge \boldsymbol{e}^-) + a\bar{c}\boldsymbol{e}^+ \\
&\qquad + |c|^2\boldsymbol{e}^+\boldsymbol{e}^- + \bar{c}d\boldsymbol{e}^+(\boldsymbol{e}^+ \wedge \boldsymbol{e}^-) + a\bar{d}(\boldsymbol{e}^+ \wedge \boldsymbol{e}^-) + b\bar{d}(\boldsymbol{e}^+ \wedge \boldsymbol{e}^-)\boldsymbol{e}^+ \\
&\qquad\qquad + c\bar{d}(\boldsymbol{e}^+ \wedge \boldsymbol{e}^-)\boldsymbol{e}^- + |d|^2(\boldsymbol{e}^+ \wedge \boldsymbol{e}^-)^2 \\
&= |a|^2 + |d|^2 + \frac{|b|^2 + |c|^2}{2} + (\bar{a}b + a\bar{c} - \bar{c}d + b\bar{d})\boldsymbol{e}^+ + (\bar{a}c + a\bar{b} + \bar{b}d - c\bar{d})\boldsymbol{e}^- \\
&\qquad + \frac{2\bar{a}d + 2a\bar{d} + |c|^2 - |b|^2}{2}(\boldsymbol{e}^+ \wedge \boldsymbol{e}^-),
\end{aligned} \tag{3.6}$$

where we invoked the relations (2.6), (2.7) and

$$\begin{aligned}
\boldsymbol{e}^\pm(\boldsymbol{e}^+ \wedge \boldsymbol{e}^-) &= \mp\boldsymbol{e}^\pm, \\
(\boldsymbol{e}^+ \wedge \boldsymbol{e}^-)\,\boldsymbol{e}^\pm &= \pm\boldsymbol{e}^\pm,
\end{aligned}$$

$$(\boldsymbol{e}^+ \wedge \boldsymbol{e}^-)^2 = 1,$$
$$(\boldsymbol{e}^+ \wedge \boldsymbol{e}^-)^\dagger = \boldsymbol{e}^+ \wedge \boldsymbol{e}^-.$$

It follows that if $(f, f) = 0$, f must be 0 because of (3.6). Moreover, its scalar part is always positive, thus leading to the definition of the norm of a Clifford number and the norm of a discrete function.

Definition 3.8. The **norm** of a Clifford number a is defined as:

$$\|a\| := \left[a^\dagger a\right]_0.$$

The scalar part of (f, f) is defined as the **norm** of the discrete function f:

$$\|f\| := \left[(f, f)\right]_0. \tag{3.7}$$

Based on the findings in this section, we define the discrete Weierstrass space as follows:

Definition 3.9. The **discrete Weierstrass space** $\mathcal{W}$ is the completion of the right Clifford module of Hermite polynomials in ξ in the norm (3.7):

$$f \in \mathcal{W} \Leftrightarrow f = \sum_{n=0}^{\infty} H_n c_n \text{ with } \|f\| < \infty.$$

With this newly introduced Weierstrass space, we can expand the definition of the discrete Weierstrass transform to all elements in $\mathcal{W}$, as they are a convergent (for the inner product (3.4)) series of Hermite polynomials.

Definition 3.10. For a discrete function $f \in \mathcal{W}$, $f = \sum_{n=0}^{\infty} H_n a_n$, its **Weierstrass transform** is defined as

$$\mathcal{W}[f](z) = \frac{1}{\sqrt{2\pi}} \langle f(\xi) G, \exp\left(-z^2/2 + \xi\, z\right)[1]\rangle = \sum_{n\in\mathbb{N}} \mathcal{W}[H_n](z) a_n = \sum_{n\in\mathbb{N}} (-1)^{\frac{\lfloor n \rfloor}{2}} a_n z^n.$$

The Weierstrass transform of a discrete function $f \in \mathcal{W}$ is a continuous complex Clifford-valued function. In (2.4), we introduced the space $\mathcal{L}^2\left(\mathbb{C}^m, \mathbb{C}_m, \frac{1}{\pi}\exp\left(-\frac{|z|^2}{2}\right)\right)$. We will now show that $\mathcal{W}[f]$ is an element of this module, for $m = 1$. Moreover, this discrete Weierstrass transform of Clifford algebra-valued functions is unitary up to a scaling constant.

Proposition 3.11. If $f \in \mathcal{W}$, then

$$\left\langle \mathcal{W}[f], \mathcal{W}[f] \right\rangle_{\mathcal{L}^2\left(\mathbb{C}, \mathbb{C}, \frac{1}{\pi}\exp\left(-|z|^2\right)\right)} = (f, f)_{\mathcal{W}}.$$

Proof. If $f = \sum_{n=0}^{\infty} H_n a_n \in \mathcal{W}$, then $(f, f) = \sum_{n=0}^{\infty} \eta_n a_n^\dagger a_n$ and $\mathcal{W}[f](z) = \sum_{n=0}^{\infty} z^n a_n$. It then follows that

$$\left\langle \mathcal{W}[f], \mathcal{W}[f] \right\rangle_{\mathcal{L}^2\left(\mathbb{C}^m, \mathbb{C}_m, \frac{1}{\pi^m}\exp\left(-|z|^2\right)\right)} = \left\langle \sum_{n\in\mathbb{N}} z^n a_n, \sum_{m\in\mathbb{N}} z^m a_m \right\rangle_{\mathcal{L}^2\left(\mathbb{C}^m, \mathbb{C}_m, \frac{1}{\pi^m}\exp\left(-|z|^2\right)\right)}$$

$$= \sum_{m,n\in\mathbb{N}} a_m^\dagger a_n \langle z^n, z^m \rangle_{\mathcal{L}^2\left(\mathbb{C}^m, \mathbb{C}_m, \frac{1}{\pi^m}\exp\left(-|z|^2\right)\right)}$$
$$= \sum_{n\in\mathbb{N}} a_n^\dagger a_n n!.$$

Because f is an element of $\mathcal{W}$, $\sum_{n=0}^{\infty} \left[\eta_n a_n^\dagger a_n\right]_0$ is finite, hence $\mathcal{W}[f]$ is an element of $\mathcal{L}^2\left(\mathbb{C}^m, \mathbb{C}_m, \frac{1}{\pi^m}\exp\left(-\frac{|z|^2}{2}\right)\right)$. □

In the next sections, we look at some examples of elements that are in the Weierstrass space their corresponding Weierstrass transforms.

3.2.1 Examples of elements in the Weierstrass space

3.2.1.1 Linear combination of Hermite polynomials

As was already mentioned, any discrete polynomial can be written as a finite linear combination of Hermite polynomials. For any (finite or infinite) linearly combination of Hermite polynomials, one clearly has

$$\sum_{k,\ell=0}^{\infty} (H_k c_k, H_\ell b_\ell) = \sum_{k=0}^{\infty} b_k^\dagger (H_k, H_k)\, c_k = \sum_{k=0}^{\infty} b_k^\dagger \eta_k c_k.$$

3.2.1.2 Basic vectors

The inner product of two basic vectors is:

$$\left(\boldsymbol{e}^\pm, \boldsymbol{e}^\pm\right) = \left\langle \boldsymbol{e}^\pm G, [1]\boldsymbol{e}^\mp \right\rangle = \left\langle G, [1]\boldsymbol{e}^\mp \boldsymbol{e}^\pm \right\rangle = \sqrt{2\pi}\, \boldsymbol{e}^\mp \boldsymbol{e}^\pm = \frac{\sqrt{2\pi}}{2}\left(1 \mp \boldsymbol{e}_j^+ \wedge \boldsymbol{e}_j^-\right),$$
$$\left(\boldsymbol{e}^+, \boldsymbol{e}^-\right) = \left\langle \boldsymbol{e}^+ G, [1]\boldsymbol{e}^+ \right\rangle = \left\langle G, \boldsymbol{e}^+\boldsymbol{e}^+[1] \right\rangle = 0.$$

We can interpret these results as orthogonality relations of the basic elements $\boldsymbol{e}^+$ and $\boldsymbol{e}^-$. As a result, $\left(\boldsymbol{e}^+ \pm \boldsymbol{e}^-, \boldsymbol{e}^+ \pm \boldsymbol{e}^-\right) = \sqrt{2\pi}$

3.2.1.3 Basic discrete polynomials

The inner product of two basic discrete polynomials immediately follows from the definition of the Gaussian distribution G.

$$\left(\xi^k[1], \xi^\ell[1]\right) = \left\langle \xi^k G, [1](\xi^\dagger)^\ell \right\rangle = \left\langle G, [1](\xi^\dagger)^\ell (\xi^\dagger)^k \right\rangle$$
$$= \left\langle G, \xi^{k+\ell}[1] \right\rangle = \begin{cases} \sqrt{2\pi}, & k+\ell = 0, \\ \sqrt{2\pi}\,(k+\ell-1)!!, & k+\ell \text{ even}, \\ 0, & k+\ell \text{ odd}. \end{cases}$$

3.2.1.4 Exponential functions

We calculate $\big(\exp(a\xi), \exp(a\xi)\big), a \in \mathbb{C}$.

$$\begin{aligned}\big(\exp(a\xi), \exp(a\xi)\big) &= \langle \exp(a\xi)\, G, [1] \exp\left(a\xi^{\dagger}\right)\rangle \\ &= \langle G, \exp(a\xi)\, [1] \exp\left(a\xi^{\dagger}\right)[1]\rangle = \langle G, \exp(2a\xi)\,[1]\rangle \\ &= \sum_{s=0}^{\infty} \frac{(2a)^s}{s!}\langle G, \xi^s[1]\rangle = \sum_{\ell=0}^{\infty}\frac{(2a)^{2\ell}}{(2\ell)!}\langle G, \xi^{2\ell}[1]\rangle \\ &= \sum_{\ell=0}^{\infty}\frac{(2a)^{2\ell}}{(2\ell)!}\sqrt{2\pi}\frac{(2\ell)!}{2^{\ell}\ell!} = \sum_{\ell=0}^{\infty}\sqrt{2\pi}\frac{a^{2\ell}2^{\ell}}{\ell!} \\ &= \sqrt{2\pi}\exp\left(2a^2\right).\end{aligned}$$

Hence for every $a \in \mathbb{C}$, $\exp(a\xi)\,[1]$ is an element of $\mathcal{W}$.

On the other side:

$$\begin{aligned}\left(\exp\left(a\xi^2\right)[1], \exp\left(a\xi^2\right)[1]\right) &= \langle G, \exp\left(2a\xi^2\right)[1]\rangle = \sum_{s=0}^{\infty}\frac{(2a)^s}{s!}\langle G, \xi^{2s}[1]\rangle \\ &= \sum_{s=0}^{\infty}\frac{(2a)^s}{s!}\sqrt{2\pi}\,\frac{(2s)!}{2^s s!} = \sum_{s=0}^{\infty}\sqrt{2\pi}\,a^s\binom{2s}{s}.\end{aligned}$$

The convergence of this series depends on the parameter a. Denoting the general term by A_s, this series is convergent by d'Alemberts criterium, if

$$\lim_{s\to\infty}\left|\frac{A_{s+1}}{A_s}\right| < 1 \Leftrightarrow |a| < \frac{1}{4}. \tag{3.8}$$

The cases $a = \pm\frac{1}{4}$ at the boundary of the interval have to be considered separately. For $a = -\frac{1}{4}$, the series (3.8) is convergent with value $\sqrt{\pi}$, while for $a = \frac{1}{4}$, the series is divergent.

For completeness, other examples include

$$\begin{aligned}\left(\exp\left(-\frac{\xi^2}{8}\right), \exp\left(-\frac{\xi^2}{8}\right)\right) &= \frac{2\sqrt{3}}{3}\sqrt{\pi}, \\ \left(\exp\left(-\frac{\xi^2}{6}\right), \exp\left(-\frac{\xi^2}{6}\right)\right) &= \frac{\sqrt{30}}{5}\sqrt{\pi}, \\ \left(\exp\left(\frac{\xi^2}{8}\right), \exp\left(\frac{\xi^2}{8}\right)\right) &= 2\sqrt{\pi}, \\ \left(\exp\left(\frac{\xi^2}{6}\right), \exp\left(\frac{\xi^2}{6}\right)\right) &= \sqrt{6\pi}.\end{aligned}$$

3.2.1.5 Extension of $\mathcal{W}$ to other elements

Suppose that the function $f = \sum_{n=0}^{\infty} \frac{H_n}{\sqrt{\eta_n}} c_n$, $c_n \in \mathbb{R}$ is an element of $\mathcal{W}$. Then it has a Taylor series expansion, for which evaluation of f in zero gives us the constant term in this series. In particular,

$$f(0) = f(\xi)\big|_{\xi=0} = \sum_{n\in\mathbb{N}} \frac{H_n(0)}{\sqrt{\eta_n}} c_n.$$

As f is an element of $\mathcal{W}$, it must hold that $\langle f, f \rangle = \sum_{n\in\mathbb{N}} c_n^2 < \infty$, hence the coefficients $(c_n)_n \in \ell_2(\mathbb{R})$.

Let us first consider the behaviour of $\sum_{k=0}^{\infty} \frac{H_{2k}(0)}{\sqrt{\eta_{2k}}}$ (remark that $H_{2k+1}(0) = 0, \forall k \in \mathbb{N}$, as constant coefficients only occur in even Hermite polynomials). Using Stirlings asymptotic behaviour for factorials, we have on the one hand

$$\begin{aligned} H_{2k}(0) = a_0^{2k} &= \frac{2^k \Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{2^k (2k)!}{4^k k!} \\ &\sim \frac{1}{2^k} \frac{\sqrt{2k}(2k)^{2k} \exp(k)}{\sqrt{k} \exp(2k) k^k} \\ &\sim \frac{2^{k+\frac{1}{2}} k^k}{\exp(k)} \\ &\sim \sqrt{2} \left(\frac{2k}{\exp(1)}\right)^k, \end{aligned}$$

and on the other hand

$$\begin{aligned} \sqrt{\eta_{2k}} &= \sqrt{(2k)! \sqrt{2\pi}} \\ &\sim (2\pi)^{\frac{1}{4}} \left[\sqrt{2\pi}\sqrt{2k} \left(\frac{2k}{\exp(1)}\right)^{2k}\right]^{\frac{1}{2}} \\ &= (2\pi)^{\frac{1}{2}} (2k)^{\frac{1}{4}} \left(\frac{2k}{\exp(1)}\right)^k. \end{aligned}$$

which gives us as asymptotic behaviour for the general term

$$\frac{H_{2k}(0)}{\sqrt{\eta_{2k}}} \sim k^{-\frac{1}{4}}.$$

If we take

$$(c_k)_k = \left(\frac{1}{(2k)^s}\right)_k,$$

with $s > \frac{1}{2}$ so that $(c_k)_k \in \ell_2(\mathbb{R})$, then

$$f(0) = \sum_{k=0}^{\infty} \frac{1}{2^s\, k^{s+\frac{1}{4}}},$$

which diverges if $s < \frac{3}{4}$. This shows that not every converging combination of Hermite polynomials with coefficients in ℓ_2 results in pointwise convergence. However, a discrete function must have finite values in every point of the grid, by definition. Hence it is necessary to add the condition that f is a function in the definition of the Weierstrass space. On the other hand, it would also be possible to extend this definition to general discrete 'elements'. We will not dig deeper into this subject, but this can be a topic for further research.

Let us now calculate the Weierstrass transform of the previous examples.

3.2.2 Examples of Weierstrass transforms

3.2.2.1 Basic discrete polynomials

Consider the basic discrete homogeneous polynomials of even degree.

$$\begin{aligned}
\mathcal{W}\left[\xi^{2k}[1]\right](z) &= \frac{1}{\sqrt{2\pi}}\langle \xi^{2k}G, \exp\left(-z^2/2+\xi\, z\right)[1]\rangle \\
&= \frac{1}{\sqrt{2\pi}}\exp\left(-z^2/2\right)\langle \xi^{2k}G, \sum_{l=0}^{\infty}\frac{\xi^l z^l}{l!}[1]\rangle \\
&= \frac{1}{\sqrt{2\pi}}\exp\left(-z^2/2\right)\sum_{l=0}^{\infty}\frac{z^l}{l!}\langle G, \xi^{l+2k}[1]\rangle \\
&= \frac{1}{\sqrt{2\pi}}\exp\left(-z^2/2\right)\sum_{l=0}^{\infty}\frac{z^{2l}}{(2l)!}\langle G, \xi^{2l+2k}[1]\rangle \\
&= \exp\left(-z^2/2\right)\sum_{l=0}^{\infty}\frac{z^{2l}}{(2l)!}\frac{(2l+2k)!}{2^{l+k}(l+k)!} \\
&= \exp\left(-z^2/2\right)\frac{(2k)!}{2^k k!}\,{}_1F_1\left(k+\frac{1}{2};\frac{1}{2};\frac{z^2}{2}\right) \\
&= \frac{(2k)!}{2^k k!}\,{}_1F_1\left(-k;\frac{1}{2};-\frac{z^2}{2}\right) \\
&= (-1)^k H_{2k}(iz),
\end{aligned}$$

where now $H_{2k}(z)$ is the continuous Hermite polynomial of degree $2k$, i is the imaginary unit and ${}_1F_1(a;b;z)$ is Kummer's confluent hypergeometric function, converging for all finite values of z. For odd powers of ξ, we find

$$\begin{aligned}
\mathcal{W}\left[\xi^{2k+1}[1]\right](z) &= \exp\left(-z^2/2\right)\sum_{l=0}^{\infty}\frac{z^{2l+1}}{(2l+1)!}\langle G, \xi^{2l+2k+2}[1]\rangle \\
&= \exp\left(-z^2/2\right)\sum_{l=0}^{\infty}\frac{z^{2l+1}}{(2l+1)!}\frac{(2l+2k+2)!}{2^{l+k+1}(l+k+1)!} \\
&= \exp\left(-z^2/2\right)\, z\,\frac{(2k+1)!}{2^k k!}\,{}_1F_1\left(k+\frac{3}{2};\frac{3}{2};\frac{z^2}{2}\right) \\
&= z\,\frac{(2k+1)!}{2^k\, k!}\,{}_1F_1\left(-k;\frac{3}{2};-\frac{z^2}{2}\right)
\end{aligned}$$

k	$\mathcal{W}\left[\xi^k[1]\right](z)$
1	z
2	z^2+1
3	z^3+3z
4	z^4+6z^2+3
5	z^5+10z^3+15z
6	$z^6+15z^4+45z^2+15$
7	$z^7+21z^5+105z^3+105z$
8	$z^8+28z^6+210z^4+420z^2+105$
9	$z^9+36z^7+378z^5+1260z^3+945z$
10	$z^{10}+45z^8+630z^6+3150z^4+4725z^2+945$
11	$z^{11}+55z^9+990z^7+6930z^5+17325z^3+10395z$
12	$z^{12}+66z^{10}+1485z^8+13860z^6+51975z^4+62370z^2+10395$

Table 3.1: Weierstrass transforms of $\xi^k[1] k \in \mathbb{N}$.

$$= (-1)^{k+1} i H_{2k+1}(iz).$$

For low values of k, we find the results listed in Table 3.1. These are the classical Hermite polynomials, but with all coefficients taken positive. In section 4.2.3.1, we will meet these polynomials again and introduce the notation b_n^j for their coefficients.

Remark 3.12. The classical Hermite polynomials can be expressed in terms of Kummer's confluent hypergeometric function by ([43])

$$2^n H_{2n}\left(\sqrt{2}x\right) = (-1)^n \frac{(2n)!}{n!}\, {}_1F_1\left(-n, \frac{1}{2}; x^2\right),$$
$$2^n H_{2n+1}\left(\sqrt{2}x\right) = (-1)^n \frac{(2n+1)!}{n!} \sqrt{2}x\, {}_1F_1\left(-n, \frac{3}{2}; x^2\right).$$

A natural question is to compare these results with the classical (continuous) case. Indeed: the continuous Weierstrass transform of the (continuous) polynomials x^k are the same as in Table 3.1.

3.2.2.2 Exponential functions

We calculate the Weierstrass transform of an exponential function ($a \in \mathbb{C}$).

$$\begin{aligned}
\mathcal{W}[\exp(a\xi)](z) &= \langle \exp(a\xi)\, G, \exp\left(-z^2/2 + \xi\, z\right)[1]\rangle \\
&= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} a^\ell \langle \xi^\ell G, \exp\left(-z^2/2 + \xi\, z\right)[1]\rangle \\
&= \sum_{\ell=0}^{\infty} \frac{a^\ell}{\ell!} \mathcal{W}[\xi^\ell](z) \\
&= \sum_{\ell=0}^{\infty} \frac{a^{2\ell}}{(2\ell)!} (-1)^\ell H_{2\ell}(iz) + \frac{a^{2\ell+1}}{(2\ell+1)!} (-1)^{\ell+1} i H_{2\ell+1}(iz).
\end{aligned}$$

To calculate this sum of Hermite polynomials, we use the identities [1]

$$
\begin{aligned}
\sum_{k=0}^{\infty} \frac{H_{2k}(\sqrt{2}x)\,(2t)^k}{(2k)!} &= \exp(-t)\cos\left(2x\sqrt{-t}\right), \\
\sum_{k=0}^{\infty} \frac{H_{2k+1}(\sqrt{2}x)\sqrt{-2t}\,(2t)^k}{(2k+1)!} &= \exp(-t)\sin\left(2x\sqrt{-t}\right).
\end{aligned}
\tag{3.9}
$$

Now put $\sqrt{2}x = iz$ and $2t = -a^2$, then

$$
\begin{aligned}
\mathcal{W}[\exp(a\xi)](z) &= \exp\left(\frac{a^2}{2}\right)\big(\cos(iza) + i\sin(iza)\big) \\
&= \exp\left(\frac{a^2}{2}\right)\big(\cosh(az) + \sinh(az)\big) \\
&= \exp\left(\frac{a^2}{2} + az\right).
\end{aligned}
$$

The function $\exp(a\xi)$ is thus an eigenfunction of the Weierstrass transform, with eigenvalue $\exp\left(\frac{a^2}{2}\right)$.

Let $|a| < \frac{1}{4}$, such that $\exp\left(a\xi^2\right) \in \mathcal{W}$, then

$$
\begin{aligned}
\mathcal{W}[\exp\left(a\xi^2\right)](z) &= \langle \exp\left(a\xi^2\right) G, \exp\left(-z^2/2 + \xi z\right)[1]\rangle \\
&= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} a^{\ell} \langle \xi^{2\ell} G, \exp\left(-z^2/2 + \xi z\right)[1]\rangle \\
&= \sum_{\ell=0}^{\infty} \frac{a^{\ell}}{\ell!} \mathcal{W}[\xi^{2\ell}](z) \\
&= \sum_{\ell=0}^{\infty} \frac{(-a)^{\ell}}{\ell!} H_{2\ell}(iz).
\end{aligned}
\tag{3.10}
$$

Using next formula

$$
\sum_{n=0}^{\infty} \frac{H_n(z)t^n}{\lfloor \frac{n}{2} \rfloor!} = \left(1 + 2zt + 4t^2\right)\left(1 + 4t^2\right)^{-3/2} e^{\frac{4t^2 z^2}{1+4t^2}}
\tag{3.11}
$$

with $t^2 = -a$,

$$
\mathcal{W}[\exp\left(a\xi^2\right)](z) = \exp\left(\frac{4az^2}{1-4a}\right)(1-4a)^{-\frac{1}{2}}.
$$

This result confirms our previous finding that $|a| < \frac{1}{4}$ in order for $\exp\left(a\xi^2\right)$ to belong to $\mathcal{W}$.

Remark 3.13. These results are again in line with the continuous setting. Moreover, the same boundary $|a| < \frac{1}{4}$ such that $\mathcal{W}[\exp\left(ax^2\right)]$ exists, is also found in the classical case.

[1] https://functions.wolfram.com/Polynomials/HermiteH/23/02/

3.2.2.3 Ladder operators in the Weierstrass space

For the discrete Hermite polynomials, one has the recurrence formulae, see [31]

$$(\partial - \xi)H_{2k-1}[1](x) = H_{2k}[1](x),$$
$$-(\partial - \xi)H_{2k}[1](x) = H_{2k+1}[1](x),$$

whence $\partial - \xi$ can be seen as a raising operator, up to sign, for the discrete Hermite polynomials in the Weierstrass space .

After applying the Weierstrass transform, we then find

$$\mathcal{W}[H_{2k}[1]](z) = (-1)^k z^{2k} = (-1)^k z\, z^{2k-1} = -z\mathcal{W}[H_{2k-1}[1]](z)$$

and

$$\mathcal{W}[H_{2k+1}[1]](z) = (-1)^k z^{2k+1} = (-1)^k z\, z^{2k} = z\mathcal{W}[H_{2k}[1]](z).$$

So the raising operator $-(\partial - \xi)$ in the discrete Weierstrass space corresponds to the raising operator z in the (continuous) Fock space.

$$\begin{array}{ccc} H_{2k} & \xrightarrow{-(\partial-\xi)} & H_{2k+1} \\ \big\downarrow \mathcal{W} & & \big\downarrow \mathcal{W} \\ (-1)^k z^{2k} & \xrightarrow{z} & (-1)^k z^{2k+1} \end{array} \qquad \begin{array}{ccc} H_{2k-1} & \xrightarrow{(\partial-\xi)} & H_{2k} \\ \big\downarrow \mathcal{W} & & \big\downarrow \mathcal{W} \\ (-1)^{k-1} z^{2k-1} & \xrightarrow{-z} & (-1)^k z^{2k} \end{array}$$

On the other hand, as $\partial\xi^{2k}[1] = 2k\xi^{2k-1}[1]$,

$$\begin{aligned} \partial H_{2k}[1] &= \partial\left(\sum_{j=0}^{k} a_{2j}^{2k}\xi^{2j}[1]\right) \\ &= \sum_{j=1}^{k}(2j)a_{2j}^{2k}\xi^{2j-1}[1] \\ &\overset{*}{=} -\sum_{j=1}^{k}(2k)a_{2j-1}^{2k-1}\xi^{2j-1}[1] \\ &= -(2k)\sum_{j=0}^{k-1}a_{2j+1}^{2k-1}\xi^{2j+1}[1] \\ &= -(2k)H_{2k-1}[1]. \end{aligned}$$

The equality $*$ is from [31], lemma 9.1.2. Also, as $\partial\xi^{2k+1}[1] = (2k+1)\xi^{2k}[1]$,

$$\begin{aligned} \partial H_{2k+1}[1] &= \partial\left(\sum_{j=0}^{k} a_{2j+1}^{2k+1}\xi^{2j+1}[1]\right) \\ &= \sum_{j=0}^{k}(2j+1)a_{2j+1}^{2k+1}\xi^{2j}[1] \end{aligned}$$

$$
\begin{aligned}
&\overset{*}{=} \sum_{j=0}^{k} (2k+1) a_{2j}^{2k} \xi^{2j}[1] \\
&= (2k+1) \sum_{j=0}^{k} a_{2j}^{2k} \xi^{2j}[1] \\
&= (2k+1) H_{2k}[1],
\end{aligned}
$$

where we used the same lemma for the equality $*$. Hence the dirac operator ∂ can be seen as the corresponding lowering operator in the Weierstrass space. When we apply the Weierstrass transform:

$$
\begin{aligned}
\mathcal{W}[\partial H_{2k}[1]](z) &= (-2k)\mathcal{W}[H_{2k-1}[1]](z) = (-2k)(-1)^k z^{2k-1} = \frac{d}{dz}\mathcal{W}[H_{2k}[1]](z), \\
\mathcal{W}[\partial H_{2k+1}[1]](z) &= (2k+1)\mathcal{W}[H_{2k}[1]](z) = (2k+1)(-1)^k z^{2k} = \frac{d}{dz}\mathcal{W}[H_{2k+1}[1]](z).
\end{aligned}
$$

The lowering operator ∂ in $\mathcal{W}$ thus corresponds to the lowering operator $\frac{d}{dz}$ in the (continuous) Fock space.

$$
\begin{array}{ccc}
H_{2k+1} & \xrightarrow{\partial} & (2k+1)H_{2k} \\
\downarrow \mathcal{W} & & \downarrow \mathcal{W} \\
(-1)^k z^{2k+1} & \xrightarrow{\frac{d}{dz}} & (-1)^k (2k+1) z^{2k}
\end{array}
\qquad
\begin{array}{ccc}
H_{2k} & \xrightarrow{\partial} & -(2k)H_{2k-1} \\
\downarrow \mathcal{W} & & \downarrow \mathcal{W} \\
(-1)^k z^{2k} & \xrightarrow{\frac{d}{dz}} & (-1)^k (2k) z^{2k-1}
\end{array}
$$

This is confirmed by calculating the commutator relations:

$$
\begin{aligned}
\big[\partial, -(\partial - \xi)\big] &= \partial\xi - \xi\partial = 1, \\
\left[\frac{d}{dz}, z\right] &= \frac{d}{dz} z - z \frac{d}{dz} = 1.
\end{aligned}
$$

3.2.3 Discrete δ-functions

As they are the building blocks of discrete function theory, we investigate whether the discrete $\delta-$functions are elements of the Weierstrass space $\mathcal{W}$.
Let $\delta_0 = \delta_0(\xi)[1]$ be the δ_0-function which takes values 0 everywhere except in the origin where it is 1. Let

$$
\delta_0(\xi) = \sum_{\ell=0}^{\infty} \xi^{\ell} c_{\ell}
$$

the corresponding Taylor series operator. We search for an expression

$$
\delta_0 = \delta_0(\xi)[1] = \sum_{n=0}^{\infty} H_n(\xi) d_n[1],
$$

writing δ_0 in terms of the Hermite polynomials. It holds that $\delta_0(\xi)[1] = \delta_0 = \delta_0^{\dagger} = (\delta_0(\xi)[1])^{\dagger}$, because δ_0 is real. If the coefficients d_n exist, they must satisfy the relations

$$
(H_n, \delta_0) = \left\langle H_n(\xi)G, [1] \sum_{\ell=0}^{\infty} d_{\ell}^{\dagger} H_{\ell}(\xi)^{\dagger} \right\rangle = \left\langle H_n(\xi)G, \sum_{\ell=0}^{\infty} d_{\ell}^{\dagger} H_{\ell}(\xi)[1] \right\rangle
$$

$$= d_n^\dagger \left\langle H_n(\xi)G, H_n \right\rangle = d_n^\dagger \eta_n,$$

from which it follows that

$$d_n^\dagger = \frac{(H_n, \delta_0)}{\eta_n}.$$

On the one hand, we have

$$(H_n, \delta_0) = \left\langle H_n(\xi)\, G, \delta_0 \right\rangle = \sum_{s \in \mathbb{Z}} \big(H_n(\xi)G\big)\,(s)\,\delta_0(s),$$

because the distribution $H_k(\xi)\,G$ corresponds to a function on $\mathbb{Z}$, where the action is pointwise. In this way, we can use δ_0 as a density function on $\mathbb{Z}$. So $(H_n, \delta_0) = H_n(\xi)G(0)$. On the other hand, one can also consider the Taylor series $\delta_0(\xi) = \sum_{\ell=0}^{\infty} \xi^\ell c_\ell$ and substitute it into the inner product (H_n, δ_0):

$$(H_n, \delta_0) = \left\langle H_n(\xi)G, [1] \sum_{\ell=0}^{\infty} c_\ell^\dagger \xi^\ell \right\rangle = \sum_{\ell=0}^{\infty} c_\ell^\dagger \left\langle H_n(\xi)G, [1]\xi^\ell \right\rangle,$$

which should equal $d_n^\dagger \eta_n$. This gives us two ways to calculate the coefficients d_n, as we will show in the next sections.

3.2.3.1 First method: Taylor series

Expanding δ_0 by its Taylor series, we have (see [44])

$$\delta_0(x) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(\ell!)^2} \xi^{2\ell}[1](x) + \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+1}}{(\ell+1)!\ell!} \xi^{2\ell+1}[1](x)(\boldsymbol{e}^+ - \boldsymbol{e}^-).$$

We will now let the discrete Gaussian distribution G act on this expression.

Calculation of (H_{2k}, δ_0)

As $\left\langle G, \xi^s[1] \right\rangle = 0$ for odd s, only the first part of the Taylor series is of importance in order to calculate (H_{2k}, δ_0). First, use Rodriguez' formula (2.29), then the rules (2.15):

$$\begin{aligned}
\left\langle H_{2k}(\xi)\, G, [1]\delta_0^\dagger \right\rangle &= \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(\ell)^2} \left\langle (-1)^k\, \partial^{2k}\, G, \xi^{2\ell}[1] \right\rangle \\
&= \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+k}}{(\ell)^2} \left\langle G,\, \partial^{2k}\, \xi^{2\ell}[1] \right\rangle \\
&= \sum_{\ell=k}^{\infty} \frac{(-1)^{\ell+k}}{(\ell)^2} \frac{(2\ell)!}{(2\ell-2k)!} \left\langle G, \xi^{2\ell-2k}\,[1] \right\rangle.
\end{aligned}$$

Now use the expression for the moments of the Gaussian distribution (2.24) and simplify:

$$= \sum_{\ell=k}^{\infty} \frac{(-1)^{\ell+k}}{(\ell)^2} \frac{(2\ell)!}{(2\ell-2k)!} \sqrt{2\pi}\, \frac{(2\ell-2k)!}{2^{\ell-k}(\ell-k)!}$$

$$= \sum_{\ell=k}^{\infty} (-1)^{\ell+k} \frac{\sqrt{2\pi}}{(\ell)^2} \frac{(2\ell)!}{2^{\ell-k}(\ell-k)!}$$
$$= \sqrt{2\pi} \binom{2k}{k} {}_1F_1\left(k+\frac{1}{2}, k+1, -2\right).$$

Calculation of (H_{2k+1}, δ_0)

We calculate (H_{2k+1}, δ_0). Similarly as in the case above, now only the second part of the Taylor series is of importance.

$$\begin{aligned}
\left\langle H_{2k+1}(\xi)\, G, [1]\delta_0^\dagger \right\rangle &= \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+1}}{(\ell+1)!\ell!}(\boldsymbol{e}^- - \boldsymbol{e}^+) \left\langle H_{2k+1}(\xi)\, G, [1]\, (\xi^{2\ell+1})^\dagger \right\rangle \\
&= \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+1}}{(\ell+1)!\ell!}(\boldsymbol{e}^- - \boldsymbol{e}^+) \left\langle (-1)^{k+1}\partial^{2k+1} G, \xi^{2\ell+1}\, [1] \right\rangle \\
&= \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+k+1}}{(\ell+1)!\ell!}(\boldsymbol{e}^- - \boldsymbol{e}^+) \left\langle G, \xi^{2\ell+1}\, [1]\, (\partial^{2k+1})^\dagger \right\rangle \\
&= \sum_{l=k}^{\infty} \frac{(-1)^{\ell+k+1}}{(\ell+1)!\ell!} \frac{(2\ell+1)!}{(2\ell-2k)!}(\boldsymbol{e}^- - \boldsymbol{e}^+) \left\langle G, \xi^{2\ell-2k}\, [1] \right\rangle.
\end{aligned}$$

Let G act on $\xi^{2\ell-2k}$ and simplify again

$$\begin{aligned}
&= \sum_{\ell=k}^{\infty} \frac{(-1)^{\ell+k+1}}{(\ell+1)!\ell!} \frac{(2\ell+1)!}{(2\ell-2k)!} \sqrt{2\pi} \frac{(2\ell-2k)!}{2^{\ell-k}(\ell-k)!} (\boldsymbol{e}^- - \boldsymbol{e}^+) \\
&= \sum_{\ell=k}^{\infty} (-1)^{\ell+k+1} \frac{\sqrt{2\pi}}{(\ell+1)!\ell!} \frac{(2\ell+1)!}{2^{\ell-k}(\ell-k)!}(\boldsymbol{e}^- - \boldsymbol{e}^+) \\
&= \sqrt{2\pi} \binom{2k+1}{k} {}_1F_1\left(k+\frac{3}{2}; k+2; -2\right)(\boldsymbol{e}^+ - \boldsymbol{e}^-).
\end{aligned}$$

After normalising, i.e. dividing by η_{2k}, respectively η_{2k+1} and applying the conjugation $\dagger$, we obtain the following result:

Proposition 3.14. The discrete function δ_0 can be written as a linear combination of discrete Hermite polynomials, i.e. $\delta_0 = \sum_{n\in\mathbb{Z}} H_n(\xi) d_n^0[1]$, with coefficients d_n^0

$$d_{2k}^0 = \frac{1}{(k!)^2}\, {}_1F_1\left(k+\frac{1}{2}; k+1; -2\right), \tag{3.12}$$

$$d_{2k+1}^0 = \frac{1}{k!(k+1)!}\, {}_1F_1\left(k+\frac{3}{2}; k+2; -2\right)(\boldsymbol{e}^+ - \boldsymbol{e}^-). \tag{3.13}$$

This generalized hypergeometric series ${}_1F_1$ is also known as (Kummer's) confluent hypergeometric function of the first kind (see e.g. [45]). In general, the hypergeometric function ${}_pF_q\,(a; b; z)$ with $p \leq q$ converges for all finite values of z and defines an entire function, [46, Sec. 16.2].

The same question can be asked for any other $\delta_j, j \in \mathbb{Z}$ and the reasoning will be completely similar. The Taylor series are given by

$$\delta_j(x) = \sum_{\ell=j}^{\infty} \frac{(-1)^{\ell-j}}{(\ell-j)!(\ell+j)!}\xi^{2\ell}[1](x)+$$
$$\sum_{\ell=j-1}^{\infty} \frac{(-1)^{\ell-j+1}}{(\ell-j+1)!(\ell+j)!}\xi^{2\ell+1}[1](x)\boldsymbol{e}^{+} + \sum_{\ell=j}^{\infty} \frac{(-1)^{\ell-j}}{(\ell-j)!(\ell+j+1)!}\xi^{2\ell+1}[1](x)\boldsymbol{e}^{-} \quad (3.14)$$

for positive j and

$$\delta_j(x) = \sum_{l=|j|}^{\infty} \frac{(-1)^{\ell-j}}{(\ell-j)!(\ell+j)!}\xi^{2\ell}[1](x)+$$
$$\sum_{l=|j|}^{\infty} \frac{(-1)^{\ell-j+1}}{(\ell-j+1)!(\ell+j)!}\xi^{2\ell+1}[1](x)\boldsymbol{e}^{+} + \sum_{l=|j|-1}^{\infty} \frac{(-1)^{\ell-j}}{(l-j)!(l+j+1)!}\xi^{2\ell+1}[1](x)\boldsymbol{e}^{-} \quad (3.15)$$

for negative j.

Calculation of (H_{2k}, δ_j)

Let us calculate the inner product for $j > 0$:

$$\begin{aligned}
\left\langle H_{2k}(\xi)\, G, [1]\delta_j^{\dagger} \right\rangle &= \sum_{\ell=j}^{\infty} \frac{(-1)^{\ell-j}}{(\ell-j)!(\ell+j)!} \left\langle H_{2k}(\xi)\, G, [1]\, (\xi^{2\ell})^{\dagger} \right\rangle \\
&= \sum_{\ell=j}^{\infty} \frac{(-1)^{\ell-j}}{(\ell-j)!(\ell+j)!} \left\langle (-1)^k \partial^{2k}\, G, \xi^{2\ell}\, [1] \right\rangle \\
&= \sum_{\ell=j}^{\infty} \frac{(-1)^{\ell+k-j}}{(\ell-j)!(\ell+j)!} \left\langle G, \xi^{2\ell}[1]\, (\partial^{2k})^{\dagger} \right\rangle
\end{aligned}$$

Denote $\mu := \max(k, j)$, then

$$\begin{aligned}
&= \sum_{\ell=\mu}^{\infty} \frac{(-1)^{\ell+k-j}}{(\ell-j)!(\ell+j)!} \frac{(2\ell)!}{(2\ell-2k)!} \left\langle G, \xi^{2\ell-2k}[1] \right\rangle \\
&= \sum_{\ell=\mu}^{\infty} \frac{(-1)^{\ell+k-j}}{(\ell-j)!(\ell+j)!} \frac{(2\ell)!}{(2\ell-2k)!} \sqrt{2\pi}\, \frac{(2\ell-2k)!}{2^{\ell-k}(\ell-k)!} \\
&= \frac{(-1)^{\mu-j+k}\, \sqrt{2\pi}\, 2^{k-\mu}\, (2\mu)!}{(\mu-j)!\,(\mu+j)!\,(\mu-k)!} \\
&\quad \times {}_3F_3\left(1, \mu+\frac{1}{2}, \mu+1; \mu-j+1, \mu+j+1, \mu-k+1; -2\right).
\end{aligned}$$

This generalized hypergeometric function ${}_3F_3$ will simplify to ${}_2F_2$ in both cases $\mu = k$ and $\mu = j$, because

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = {}_{p-1}F_{q-1}(a_1, \ldots, a_{p-1}; b_1, \ldots, b_{q-1}; z),$$

whenever, without loss of generality, $a_p = b_q$. We then obtain one of the next results:

- $k > j$, hence $\mu = k$: $\dfrac{(-1)^j \sqrt{2\pi}\,(2k)!}{(k-j)!\,(k+j)!}\,{}_2F_2\left(k+\dfrac{1}{2}, k+1; k-j+1, k+j+1; -2\right)$,
- $k < j$, hence $\mu = j$: $\dfrac{(-1)^k \sqrt{2\pi}\,2^{k-j}}{(j-k)!}\,{}_2F_2\left(j+\dfrac{1}{2}, j+1; 2j+1, j-k+1; -2\right)$,
- $k = j$, hence $\mu = j = k$: $(-1)^k \sqrt{2\pi}\,{}_2F_2\left(k+\dfrac{1}{2}, k+1; 2k+1, 1; -2\right)$.

Calculation of (H_{2k+1}, δ_j)

For the odd Hermite polynomials, we have that

$$\begin{aligned}
\left\langle H_{2k+1}(\xi)\,G, [1]\delta_j^\dagger \right\rangle &= \left\langle (-1)^{k+1}\partial^{2k+1}\,G, [1]\sum_{\ell=j-1}^{\infty}\frac{(-1)^{\ell-j+1}}{(\ell-j+1)!\,(\ell+j)!}(\xi^{2\ell+1})^\dagger \right\rangle \boldsymbol{e}^- \\
&\quad + \left\langle (-1)^{k+1}\partial^{2k+1}\,G, [1]\sum_{\ell=j}^{\infty}\frac{(-1)^{\ell-j}}{(\ell-j)!\,(\ell+j+1)!}(\xi^{2\ell+1})^\dagger \right\rangle \boldsymbol{e}^+ \\
&= \sum_{\ell=j-1}^{\infty}\frac{(-1)^{\ell-j+k+1}}{((\ell-j+1)!\,(\ell+j)!}\left\langle G, \partial^{2k+1}\,\xi^{2\ell+1}[1]\right\rangle \boldsymbol{e}^- \\
&\quad + \sum_{\ell=j}^{\infty}\frac{(-1)^{\ell-j+k}}{((\ell-j)!\,(\ell+j+1)!}\left\langle G, \partial^{2k+1}\,\xi^{2\ell+1}[1]\right\rangle \boldsymbol{e}^+
\end{aligned}$$

Let again $\mu = \max(k, j)$ and $\mu' = \max(k, j-1)$

$$\begin{aligned}
&= \sum_{\ell=\mu'}^{\infty}\frac{(-1)^{\ell-j+k+1}\,(2\ell+1)!}{(\ell-j+1)!\,(\ell+j)!(2\ell-2k)!}\left\langle G, \xi^{2\ell-2k}[1]\right\rangle \boldsymbol{e}^- \\
&\quad + \sum_{\ell=\mu}^{\infty}\frac{(-1)^{\ell-j+k}\,(2\ell+1)!}{(\ell-j)!\,(\ell+j+1)!(2\ell-2k)!}\left\langle G, \xi^{2\ell-2k}[1]\right\rangle \boldsymbol{e}^+ \\
&= \sum_{\ell=\mu'}^{\infty}\frac{(-1)^{\ell-j+k+1}\,(2\ell+1)!}{(\ell-j+1)!\,(\ell+j)!\,(2\ell-2k)!}\,\frac{\sqrt{2\pi}\,(2\ell-2k)!}{2^{\ell-k}\,(\ell-k)!}\boldsymbol{e}^- + \\
&\quad + \sum_{\ell=\mu}^{\infty}\frac{(-1)^{\ell-j+k}\,(2\ell+1)!}{(\ell-j)!\,(\ell+j+1)!\,(2\ell-2k)!}\,\frac{\sqrt{2\pi}\,(2\ell-2k)!}{2^{\ell-k}\,(\ell-k)!}\boldsymbol{e}^+ \\
&= \frac{\sqrt{2\pi}\,(-1)^{1-j+k+\mu'}\,2^{k-\mu'}\,(1+2\mu')!}{(1-j+\mu')!\,(j+\mu')!\,(\mu'-k)!} \\
&\qquad \times\ {}_3F_3(1, 1+\mu', \frac{3}{2}+\mu'; 2-j+\mu', 1+j+\mu', 1-k+\mu'; -2)\boldsymbol{e}^- \\
&\quad + \frac{\sqrt{2\pi}\,(-1)^{-j+k+\mu}\,2^{k-\mu}\,(1+2\mu)!}{(1+j+\mu)!\,(\mu-j)!\,(\mu-k)!} \\
&\qquad \times {}_3F_3(1, 1+\mu, \frac{3}{2}+\mu; 1-j+\mu, 2+j+\mu, 1-k+\mu; -2)\boldsymbol{e}^+.
\end{aligned}$$

We obtain one of next possibilities:

- $k \geq j,$ hence $\mu = \mu' = k:$

$$\frac{\sqrt{2\pi}\,(-1)^{1-j}\,(1+2k)!}{(1-j+k)!\,(j+k)!}\,{}_2F_2(1+k,\frac{3}{2}+k;2-j+k,1+j+k;-2)\boldsymbol{e}^- \\ +\frac{\sqrt{2\pi}\,(-1)^{-j}\,(1+2k)!}{(1+j+k)!\,(k-j)!}\,{}_2F_2(1+k,\frac{3}{2}+k;1-j+k,2+j+k;-2)\boldsymbol{e}^+,$$

- $k \leq j-1,$ hence $\mu = j$ and $\mu' = j-1:$

$$\frac{\sqrt{2\pi}\,(-1)^{k}\,2^{k-j+1}}{(j-1-k)!}\,{}_2F_2(j,\frac{3}{2}+j-1;2j,j-k;-2)\boldsymbol{e}^- \\ +\frac{\sqrt{2\pi}\,(-1)^{k}\,2^{k-j}}{(j-k)!}\,{}_2F_2(1+j,\frac{3}{2}+j;2+2j,1-k+j;-2)\boldsymbol{e}^+.$$

The results for $j < 0$ are completely analoguous, and it is immediately clear from the Taylor series (see (3.14) and (3.15) that coefficients of $\boldsymbol{e}^+$ and $\boldsymbol{e}^-$ will be interchanged. For both inner products above, we end by normalizing H_{2k} and H_{2k+1} to obtain the following result:

Proposition 3.15. The discrete function $\delta_j, j > 0$, can be written as a linear combination of discrete Hermite polynomials, i.e. $\delta_j = \sum_{k\in\mathbb{Z}} H_k(\xi) d_k^j[1]$, with coefficients d_k^j given by

$$d_{2k}^j = \frac{(-1)^{\mu-j+k}\,2^{k-\mu}\,(2\mu)!}{(2k)!\,(\mu-j)!\,(\mu+j)!\,(\mu-k)!} \\ \times\ {}_3F_3\left(1,\mu+\frac{1}{2},\mu+1;\mu-j+1,\mu+j+1,\mu-k+1;-2\right),$$

$$d_{2k+1}^j = \frac{(-1)^{1-j+k+\mu'}\,2^{k-\mu'}\,(1+2\mu')!}{(1-j+\mu')!\,(j+\mu')!\,(\mu'-k)!\,(2k+1)!} \\ \times\ {}_3F_3(1,1+\mu',\frac{3}{2}+\mu';2-j+\mu',1+j+\mu',1-k+\mu';-2)\boldsymbol{e}^+ \\ +\frac{(-1)^{-j+k+\mu}\,2^{k-\mu}\,(1+2\mu)!}{(1+j+\mu)!\,(\mu-j)!\,(\mu-k)!\,(2k+1)!} \\ \times\ {}_3F_3(1,1+\mu,\frac{3}{2}+\mu;1-j+\mu,2+j+\mu,1-k+\mu;-2)\boldsymbol{e}^-.$$

If $j < 0$,

$$d_{2k}^j = \frac{(-1)^{\mu-j+k}\,2^{k-\mu}\,(2\mu)!}{(2k)!\,(\mu-j)!\,(\mu+j)!\,(\mu-k)!} \\ \times\ {}_3F_3\left(1,\mu+\frac{1}{2},\mu+1;\mu-j+1,\mu+j+1,\mu-k+1;-2\right),$$

$$d_{2k+1}^j = \frac{(-1)^{1-j+k+\mu}\,2^{k-\mu}\,(1+2\mu)!}{(1-j+\mu)!\,(j+\mu)!\,(\mu-k)!\,(2k+1)!} \\ \times\ {}_3F_3(1,1+\mu,\frac{3}{2}+\mu;2-j+\mu,1+j+\mu,1-k+\mu;-2)\boldsymbol{e}^+$$

$$+\frac{(-1)^{-j+k+\mu}\,2^{k-\mu}\,(1+2\mu)!}{(1+j+\mu)!\,(\mu-j)!\,(\mu-k)!\,(2k+1)!}$$
$$\times\ {}_3F_3(1,1+\mu,\frac{3}{2}+\mu;1-j+\mu,2+j+\mu,1-k+\mu;-2)\boldsymbol{e}^-.$$

We used the notation $\mu=\max(k,|j|)$ and $\mu'=\max(k,|j|-1)$.

For the special case $j=0$, we reobtain the result from proposition 3.14. To avoid confusion, we will always write j in superscript of the coefficients d_n^j, also for δ_0.

3.2.3.2 Second method: pointwise evaluation

In [32], for the derivatives of the delta functions δ_j the following results were established:

$$\partial^{2k}\delta_j=\sum_{i=0}^{2k}(-1)^i\binom{2k}{i}\delta_{j-(k-i)}, \tag{3.16}$$
$$\partial^{2k+1}\delta_j=\sum_{i=0}^{2k+1}(-1)^i\binom{2k+1}{i}\left(\delta_{j-(k+1-i)}\boldsymbol{e}^++\delta_{j-(k-i)}\boldsymbol{e}^-\right). \tag{3.17}$$

We will now use these expressions to evaluate the action of the Hermite operators on the Gauss distribution G in $x=0$.

Calculation of $(H_{2k},\delta_0)=H_{2k}(\xi)G(0)$

For the action of H_{2k}, we have

$$\begin{aligned}H_{2k}G&=(-1)^k\,\partial^{2k}\sqrt{2\pi}\,\exp\left(\frac{\partial^2}{2}\right)\delta_0=\sum_{\ell=0}^{\infty}(-1)^k\,\partial^{2k}\sqrt{2\pi}\,\frac{\partial^{2\ell}}{2^\ell\ell!}\delta_0\\&=\sum_{\ell=0}^{\infty}\sqrt{2\pi}\,\frac{(-1)^k}{2^\ell\ell!}\partial^{2(\ell+k)}\delta_0\\&=\sum_{\ell=0}^{\infty}\sqrt{2\pi}\,\frac{(-1)^k}{2^\ell\ell!}\left(\sum_{i=0}^{2(\ell+k)}(-1)^i\binom{2\ell+2k}{i}\delta_{-(\ell+k)+i}\right).\end{aligned} \tag{3.18}$$

To obtain the coefficients in δ_0, we have to look at the case $-(\ell+k)+i=0$, or $i=k+\ell$:

$$\langle H_{2k}G,\delta_0\rangle=\sum_{\ell=0}^{\infty}\sqrt{2\pi}\,\frac{(-1)^\ell}{2^\ell\ell!}\binom{2\ell+2k}{k+\ell}=\sqrt{2\pi}\binom{2k}{k}\,{}_1F_1\left(k+\frac{1}{2};k+1;-2\right).$$

Calculation of $(H_{2k+1}, \delta_0) = H_{2k+1}(\xi)G(0)$

For the odd case, we have

$$\begin{aligned}
H_{2k+1}G &= (-1)^{k+1}\partial^{2k+1}\sqrt{2\pi}\,\exp\left(\partial^2/2\right)\delta_0 = \sum_{\ell=0}^{\infty}(-1)^{k+1}\partial^{2k+1}\sqrt{2\pi}\,\frac{\partial^{2\ell}}{2^\ell \ell!}\\
&= \sum_{\ell=0}^{\infty}\sqrt{2\pi}\,\frac{(-1)^{k+1}}{2^\ell \ell!}\partial^{2(\ell+k)+1}\delta_0\\
&= \sum_{\ell=0}^{\infty}\sqrt{2\pi}\,\frac{(-1)^{k+1}}{2^\ell \ell!}\left(\sum_{i=0}^{2(\ell+k)+1}(-1)^i\binom{2\ell+2k+1}{i}\delta_{-(\ell+k+1-i)}\boldsymbol{e}^+\right)\\
&\qquad + \sum_{\ell=0}^{\infty}\sqrt{2\pi}\,\frac{(-1)^{k+1}}{2^\ell \ell!}\left(\sum_{i=0}^{2(\ell+k)+1}(-1)^i\binom{2\ell+2k+1}{i}\delta_{-(l+k-i)}\boldsymbol{e}^-\right).
\end{aligned} \tag{3.19}$$

We look for the coefficients in δ_0: for $\boldsymbol{e}^+$, we need to consider $i = k+1+\ell$, while for $\boldsymbol{e}^-$, we look at $i = k+\ell$.

$$\begin{aligned}
H_{2k+1}G(0) &= \sum_{\ell=0}^{\infty}\sqrt{2\pi}\,\frac{(-1)^{k+1}}{2^\ell \ell!}\times\\
&\qquad \left((-1)^{k+\ell+1}\binom{2\ell+2k+1}{k+\ell+1}\boldsymbol{e}^+ + (-1)^{\ell+k}\binom{2\ell+2k+1}{\ell+k}\boldsymbol{e}^-\right)\\
&= \sum_{\ell=0}^{\infty}\sqrt{2\pi}\,\frac{(-1)^{\ell}}{2^\ell \ell!}\left(\binom{2\ell+2k+1}{k+\ell+1}\boldsymbol{e}^+ - \binom{2\ell+2k+1}{\ell+k}\boldsymbol{e}^-\right)\\
&= \sum_{\ell=0}^{\infty}\sqrt{2\pi}\,\frac{(-1)^{\ell}}{2^\ell \ell!}\binom{2\ell+2k+1}{k+\ell}(\boldsymbol{e}^+ - \boldsymbol{e}^-)\\
&= \sqrt{2\pi}\binom{2k+1}{k}\,{}_1F_1\left(k+\frac{3}{2}; k+2; -2\right)(\boldsymbol{e}^+ - \boldsymbol{e}^-).
\end{aligned}$$

These are indeed the same results as in proposition 3.14.

Calculation of $(H_{2k}, \delta_j) = H_{2k}(\xi)G(j)$

For δ_j, we can do similar calculations. For the even Hermite functions, starting from (3.18), we are now interested in the coefficients for $i = j+\ell+k$:

$$\sum_{\ell=|j|-k}^{\infty}\sqrt{2\pi}\,\frac{(-1)^{\ell+j}}{2^\ell \ell!}\binom{2\ell+2k}{j+k+\ell}\delta_j. \tag{3.20}$$

Since $0 \le i \le 2\ell+2k$, also $0 \le j+k+\ell \le 2\ell+2k$, or $|j| \le \ell+k$. This value is

$$\sqrt{2\pi}\,(-1)^k 2^{k-j}\,{}_2F_2\left(j+1, j+\frac{1}{2}; 1+2j, j-k+1; -2\right)$$

in the point $j \in \mathbb{Z}$.

Calculation of $(H_{2k+1}, \delta_j) = H_{2k+1}(\xi)G(j)$

For the odd case, we have to look at $i = j + \ell + k + 1$ for $\boldsymbol{e}^+$ and at $i = j + k + \ell$ for $\boldsymbol{e}^-$ in (3.19). This implies that $-k - \ell - 1 \leq j \leq \ell + k$ and $-k - \ell \leq j \leq k + \ell + 1$. If we notate $\nu := \max(j - k, 0)$ and $\nu' := \max(j - k - 1, 0)$, the coefficient in δ_j is

$$\sum_{\ell=\nu}^{\infty} \sqrt{2\pi}\, \frac{(-1)^{k+1}}{2^\ell \ell!} (-1)^{j+\ell+k+1} \binom{2k + 2\ell + 1}{j + k + \ell + 1} \boldsymbol{e}^+ - \sum_{\ell=\nu'}^{\infty} \sqrt{2\pi}\, \frac{(-1)^{k+1}}{2^\ell \ell!} (-1)^{j+\ell+k} \binom{2\ell + 2k + 1}{j + k + \ell} \boldsymbol{e}^-$$

$$= \sum_{\ell=\nu}^{\infty} \sqrt{2\pi}\, \frac{(-1)^{\ell+j}}{2^\ell \ell!} \binom{2k + 2\ell + 1}{j + k + \ell + 1} \boldsymbol{e}^+ - \sum_{\ell=\nu'}^{\infty} \sqrt{2\pi}\, \frac{(-1)^{\ell+j+1}}{2^\ell \ell!} \binom{2\ell + 2k + 1}{j + k + \ell} \boldsymbol{e}^-.$$

This result equals

$$= \sqrt{2\pi}\, \frac{(-1)^{j+\nu}}{2^\nu\, \nu!} \binom{2k + 2\nu + 1}{j + k + \nu + 1} \times {}_3F_3\left(1, k + \nu + 1, k + \nu + \frac{3}{2}; \nu + 1, -j + k + \nu + 1, j + k + \nu + 2; -2\right) \boldsymbol{e}^+$$
$$+ \sqrt{2\pi}\, \frac{(-1)^{j+\nu'+1}}{2^{\nu'}\, \nu'!} \binom{2k + 2\nu' + 1}{j + k + \nu'} \times {}_3F_3\left(1, k + \nu' + 1, k + \nu' + \frac{3}{2}; \nu' + 1, -j + k + \nu' + 2, j + k + \nu' + 1; -2\right) \boldsymbol{e}^-.$$

Again, this generalised hypergeometric function ${}_3F_3$ is a ${}_2F_2$ when we substitute the value of $\nu = \max(j - k, 0)$ or $\nu' := \max(j - k - 1, 0)$:

- $j > k$, hence $\nu = j - k$ and $\nu' = j - k - 1$:
$$\sqrt{2\pi}\, \frac{(-1)^k}{2^{j-k}\,(j-k)!}\, {}_2F_2\left(j + 1, j + \frac{3}{2}; j - k + 1, 2j + 2; -2\right) \boldsymbol{e}^+ + \sqrt{2\pi}\, \frac{(-1)^k}{2^{j-k-1}\,(j-k-1)!}\, {}_2F_2\left(j, j - 1 + \frac{3}{2}; j - k, 2j; -2\right) \boldsymbol{e}^-.$$

- $j \leq k$, hence $\nu = \nu' = 0$:
$$= \sqrt{2\pi}(-1)^j \binom{2k+1}{j + k + 1}\, {}_2F_2\left(k + 1, k + \frac{3}{2}; -j + k + 1, j + k + 2; -2\right) \boldsymbol{e}^+ + \sqrt{2\pi}\,(-1)^{j+1} \binom{2k+1}{j+k}\, {}_2F_2\left(k + 1, k + \frac{3}{2}; -j + k + 2, j + k + 1; -2\right) \boldsymbol{e}^-.$$

We now have another expression for the coefficients d_n^j, after normalizing by η_{2k}, respectively η_{2k+1}:

Proposition 3.16. The discrete function $\delta_j, j > 0$, can be written as a linear combination of discrete Hermite polynomials, i.e. $\delta_j = \sum_{k \in \mathbb{Z}} H_k(\xi) d_k^j[1]$, with coefficients d_k^j

$$d_{2k}^j = \frac{(-1)^k 2^{k-j}}{(2k)!}\, {}_2F_2\left(j + 1, j + \frac{1}{2}; 1 + 2j, j - k + 1; -2\right),$$

$$d^j_{2k+1} = \frac{(-1)^{j+\nu}}{2^\nu\,\nu!\,(2k+1)!}\binom{2k+2\nu+1}{j+k+\nu+1}$$
$$ {}_3F_3\left(1, k+\nu+1, k+\nu+\frac{3}{2}; \nu+1, -j+k+\nu+1, j+k+\nu+2; -2\right)\boldsymbol{e}^+$$
$$+\frac{(-1)^{j+\nu'+1}}{2^{\nu'}\,\nu'!\,(2k+1)!}\binom{2k+2\nu'+1}{j+k+\nu'}$$
$${}_3F_3\left(1, k+\nu'+1, k+\nu'+\frac{3}{2}; \nu'+1, -j+k+\nu'+2, j+k+\nu'+1; -2\right)\boldsymbol{e}^-.$$

We used the notation $\nu = \max(j-k, 0)$ and $\nu' = \max(j-k-1, 0)$.

It is easily checked that the expressions in propositions 3.15 and 3.16 are equal for concrete values of j and k.

3.2.3.3 Conclusion

In the previous section, we found explicit expressions for the coefficients d^j_n, such that δ_j can be written as a linear combination of Hermite polynomials. First of all, let us check if if this expression is well-defined, i.e. if this infinite series of Hermite polynomials defines a discrete function. Therefore, its value for every $x \in \mathbb{Z}$ must be finite. Let us rewrite the infinite series of Hermite polynomials again as an infinite series of powers of the discrete vector variable ξ. If all coefficients are finite, we can conclude pointwise convergence: for fixed $x \in \mathbb{Z}$, $\xi^k[1](x) = 0, \forall k > 2|x|$. Let us calculate this for δ_0:

$$\begin{aligned}
\delta_0 &= \sum_{n\in\mathbb{N}} H_n(\xi)d^0_n[1] = \sum_{k\in\mathbb{N}} H_{2k}(\xi)d^0_{2k}[1] + H_{2k+1}(\xi)d^0_{2k+1}[1]\\
&= \sum_{k\in\mathbb{N}}\left(\sum_{j=0}^{k} a^{2k}_{2j}\xi^{2j}\right)d^0_{2k}[1] + \left(\sum_{j=0}^{k} a^{2k+1}_{2j+1}\xi^{2j+1}\right)d^0_{2k+1}[1]\\
&= \sum_{k\in\mathbb{N}}\left(\sum_{j=k}^{\infty} a^{2j}_{2k}d^0_{2j}\right)\xi^{2k}[1] + \left(\sum_{j=k}^{\infty} a^{2j+1}_{2k+1}d^0_{2j+1}\right)\xi^{2k+1}[1].
\end{aligned}$$

To check the convergence of the coefficients in ξ^{2k}, we use d'Alembert's ratio test:

$$\begin{aligned}
\lim_{j\to\infty}\left|\frac{a^{2j+2}_{2k}d^0_{2j+2}}{a^{2j}_{2k}d^0_{2j}}\right| &= \frac{2^{j-k+1}\binom{j+1}{k}\Gamma\left(j+\frac{3}{2}\right)}{2^{j-k}\binom{j}{k}\Gamma\left(j+\frac{1}{2}\right)}\frac{j!^2\,{}_1F_1\left(j+\frac{3}{2}; j+2; -2\right)}{(j+1)!^2\,{}_1F_1\left(j+\frac{1}{2}; j+1; -2\right)}\\
&= \lim_{j\to\infty}\frac{(2j+1)}{(j+1-k)(j+1)}\frac{{}_1F_1\left(j+\frac{3}{2}; j+2; -2\right)}{{}_1F_1\left(j+\frac{1}{2}; j+1; -2\right)}.
\end{aligned}$$

In view of the properties of the confluent hypergeometric function, it holds that[2]

$$\lim_{k\to\infty}{}_1F_1\left(k+\frac{1}{2}; k+1; -2\right) = \lim_{k\to\infty}{}_1F_1\left(k+\frac{3}{2}; k+2; -2\right) = \exp(-2). \qquad (3.21)$$

[2]See formulae 13.8.17 and 5.11.3 from [46]

D'Alembert's ratio test hence results in

$$\lim_{j\to\infty}\left|\frac{a_{2k}^{2j+2}d_{2j+2}^0}{a_{2k}^{2j}d_{2j}^0}\right| = 0.$$

For the convergence of the coefficients in ξ^{2k+1}, as well as for the delta functions δ_j, the method and result is completely analogous, hence this infinite series of Hermite polynomials indeed defines a discrete function.

Still, it does not imply that δ_0 is an element of the discrete Weierstrass space $\mathcal{W}$. In order to be an element of the discrete Weierstrass space, we need that

$$\sum_{k=0}^{\infty}\eta_{2k}(d_{2k}^0)^\dagger d_{2k}^0 + \eta_{2k+1}(d_{2k+1}^0)^\dagger d_{2k+1}^0 < \infty.$$

In order to proof the convergence of this series, we want to use d'Alembert's ratio test again and find the limits

$$\lim_{k\mapsto\infty}\left|\frac{\eta_{2k+2}(d_{2k+2}^0)^\dagger d_{2k+2}^0}{\eta_{2k}(d_{2k}^0)^\dagger d_{2k}^0}\right| \text{ and } \lim_{k\mapsto\infty}\left|\frac{\eta_{2k+3}(d_{2k+3}^0)^\dagger d_{2k+3}^0}{\eta_{2k+1}(d_{2k+1}^0)^\dagger d_{2k+1}^0}\right|.$$

Using the same limit (3.21), it then follows immediately that

$$\lim_{k\mapsto\infty}\left|\frac{\eta_{2k+2}(d_{2k+2}^0)^\dagger d_{2k+2}^0}{\eta_{2k}(d_{2k}^0)^\dagger d_{2k}^0}\right| = \lim_{k\mapsto\infty}\left|\frac{\eta_{2k+3}(d_{2k+3}^0)^\dagger d_{2k+3}^0}{\eta_{2k+1}(d_{2k+1}^0)^\dagger d_{2k+1}^0}\right| = 0,$$

from which we conclude that $\delta_0 \in \mathcal{W}$.

For coefficients d_n^j, the reasoning is very similar. We want the convergence of the series

$$\sum_{k=0}^{\infty}\eta_{2k}(d_{2k}^j)^\dagger d_{2k}^j + \eta_{2k+1}(d_{2k+1}^j)^\dagger d_{2k+1}^j < \infty$$

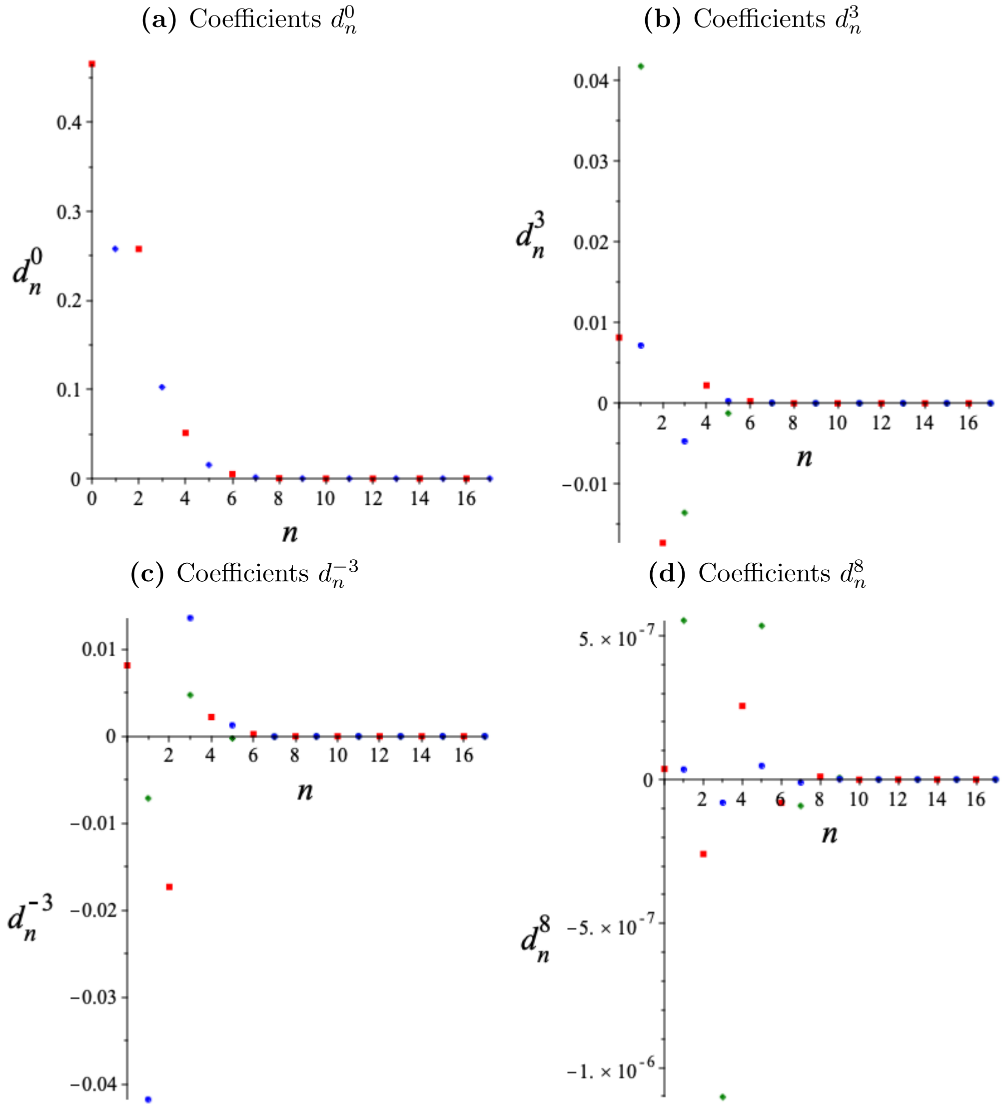
and need to investigate the corresponding limit of the generalized hypergeometric function ${}_2F_2$. As $k\to\infty$, $\mu = k$ and for fixed j

$$\begin{aligned}
&\lim_{k\to\infty} {}_2F_2\left(k+\frac{1}{2}, k+1; k-j+2, k+j+1; -2\right)\\
&= \lim_{k\to\infty} {}_2F_2\left(k+\frac{3}{2}, k+1; k-j+1, k+j+1; -2\right)\\
&= \lim_{k\to\infty} {}_2F_2\left(k+\frac{3}{2}, k+1; k-j+1, k+j+2; -2\right)\\
&= \exp(-2).
\end{aligned}$$

We thus can conclude that $\delta_j \in \mathcal{W}$ for every $j \in \mathbb{Z}$. Furthermore, it immediately follows that

$$\lim_{n\to\infty} d_n^j = 0, \forall j \in \mathbb{Z}. \tag{3.22}$$

Figure 3.1: Plots of δ-coefficients



To give a visual idea of the results, the coefficients d_n^0, d_n^3, d_n^{-3}, d_n^8, d_n^{30} and d_n^{35} are plotted in Figure 3.1. On the x-axis, the n-th Hermite polynomial is represented, while on the y-axis, the respective coefficients are depicted. For n even, the values of the delta-coefficients are always scalar and plotted in red. For n odd, the values are vectorial: in green, one finds the coefficents for $\boldsymbol{e}^+$, in blue, we find the coefficients for $\boldsymbol{e}^-$.

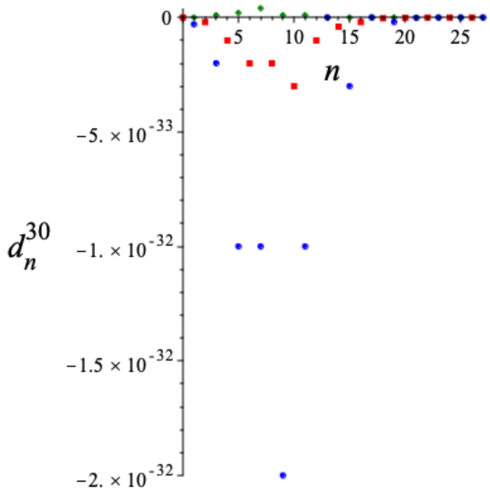
In particular, for δ_0, the odd coefficients are opposite for $\boldsymbol{e}^+$ and $\boldsymbol{e}^-$, i.e. $d_{2k+1}^0 = r(\boldsymbol{e}^- - \boldsymbol{e}^+)$, with r scalar. This r is depicted by the blue dots.

In the plots of d_n^3 and d_n^{-3}, it is clear that the coefficients of $\boldsymbol{e}^\pm$ reflected with respect to the $x-$axis give the coefficients of $\boldsymbol{e}^\mp$ of the delta function with opposite index.

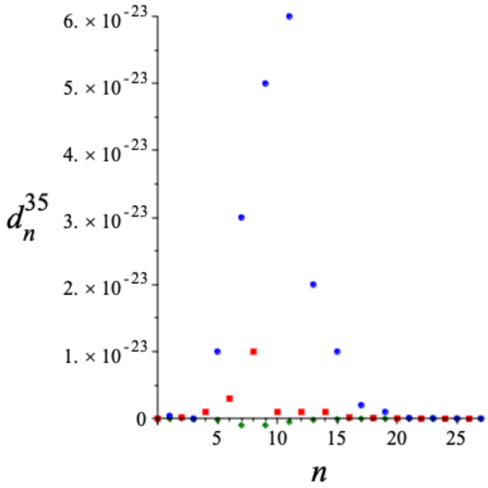
The absolute values of all coefficients d_n^j converge to 0 for fixed j if n enlarges, which confirms (3.22).

Having found the coefficients of the δ_j functions with respect to the discrete Hermite

(e) Coefficients d_n^{30}



(f) Coefficients d_n^{35}



polynomials in the discrete Weierstrass space, we can now calculate the Weierstrass transform of these delta functions. Considering the most simple case $\delta_0 = \sum_{n\in\mathbb{N}} H_n(\xi) d_n^0$, we obtain:

$$\begin{aligned}\mathcal{W}\left[\delta_0(\xi)[1]\right](z) &= \sum_{n\in\mathbb{N}} \mathcal{W}[H_n(\xi)](z) d_n^0 = \sum_{n\in\mathbb{N}} d_n^0 \, (-1)^{\lfloor \frac{n}{2} \rfloor} \, z^n \\ &= \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{(k!)^2} \, {}_1F_1\left(k+\frac{1}{2}; k+1; -2\right) z^{2k} \right. \\ &\qquad \left. + \frac{1}{k!(k+1)!} \, {}_1F_1\left(k+\frac{3}{2}; k+2; -2\right) (\boldsymbol{e}^- - \boldsymbol{e}^+) \, z^{2k+1} \right).\end{aligned}$$

For every $z \in \mathbb{C}$, this number is finite. However, $\lim_{|z|\to\infty} \mathcal{W}\left[\delta_0(\xi)[1]\right](z) = +\infty$, indicating that this function is unbounded and will certainly not converge to the Gaussian kernel, as is the case in the continuous case. In analogy, it was already found out in [33] that also the CK-extension of δ_0 'explodes' at infinity. An expected result would be that the Weierstrass transform of the discrete delta distribution $\boldsymbol{\delta}_0$ is the Gaussian function. Of course, this raises the question to define a Weierstrass transform for discrete distributions, and hence a condition for a distribution to *have* a Weierstrass transform. One idea could be to take the Fourier transform of the distribution (which is a function), to check if this function is an element of the discrete Weierstrass space $\mathcal{W}$ and calculate its transform. However, this needs to be accurately defined and investigated and could be subject to further research.

Conclusion Let us now come back to the question how we can describe the elements in the discrete Weierstrass space $\mathcal{W}$. Take an element $f \in \mathcal{W}$ and consider fG. On the one hand, fG acts on functions in $\mathcal{W}$, in particular polynomials, by means of the inner product (3.4). It means fG is a distribution. On the other hand, as the delta functions $\delta_j \in \mathcal{W}$, fG acts on δ_j, which is pointwise evaluation of fG. This means fG can be interpreted as a function on $\mathbb{Z}$. With every distribution fG, there is associated a density function on $\mathbb{Z}$, which reflects the dual aspect in this theory. In this way, the set $\mathcal{W}G = \{fG \mid f \in G\}$ can be seen to contain both functions and distributions and thus is a subspace of $\mathcal{D} \cap \mathcal{F}$, where $\mathcal{D}$ and $\mathcal{F}$ are the spaces of distributions and functions, respectively.

An element $f \in \mathcal{W}$ however is not a distribution: it does not act on polynomials and has no compact support. In particular polynomials, which are in $\mathcal{W}$, are not distributions.

3.2.4 Discrete Hermite functions

Obviously, Hermite polynomials are elements of the Weierstrass space $\mathcal{W}$. In paragraph 3.2.1.4, we have seen that also $\exp\left(\frac{-\xi^2}{4}\right)$ is contained in $\mathcal{W}$. Discrete Hermite functions are defined as the product of a Hermite polynomial operator and $\exp\left(\frac{-\xi^2}{4}\right)$, in analogy to the definition from [42].

Definition 3.17. The discrete Hermite functions are defined as

$$\psi_n(\xi)[1] = H_n(\xi) \exp\left(\frac{-\xi^2}{4}\right)[1],$$

where H_n is the n-th degree Hermite polynomial.

It is important to remark that this is the composition of the operators $H_n(\xi)$ and $\exp\left(\frac{-\xi^2}{4}\right)$ and not the multiplication of the corresponding functions in ξ.

In this section, we investigate if these Hermite functions are elements of $\mathcal{W}$. This is done in the same way as we did for the delta functions δ_j: by means of its Taylor series. Writing the Hermite functions as an (infinite) linear combination of Hermite polynomials, then

$$\psi_m(\xi) = \sum_{n\in\mathbb{N}} H_n(\xi) s_n^m[1].$$

We can obtain the coefficients s_n^m as follows:

$$\begin{aligned}(H_n, \psi_m) &= \left\langle H_n(\xi)G, [1]\sum_{\ell\in\mathbb{N}} \left(s_\ell^m\right)^\dagger H_\ell(\xi)^\dagger \right\rangle = \left\langle H_n(\xi)G, \sum_{\ell\in\mathbb{N}} \left(s_\ell^m\right)^\dagger H_\ell(\xi)[1] \right\rangle \\ &= (s_n^m)^\dagger \left\langle H_n(\xi)G, H_n \right\rangle = (s_n^m)^\dagger \eta_n,\end{aligned}$$

From which it follows that

$$(s_n^m)^\dagger = \frac{(H_n, \psi_m)}{\eta_n}.$$

We will make a distinction between even and odd Hermite functions to make the calculations. Remark that the inner product of an even Hermite function and an odd Hermite polynomial, and vice versa, will be zero because the action of the Gaussian on odd powers of ξ is zero, hence we only need to consider the remaining combinations.

First, consider an **even Hermite function** ψ_{2m} and combine it with an even Hermite polynomial H_{2k}.

$$\left(s_{2k}^{2m}\right)^{\dagger}\eta_{2k} = \left\langle H_{2k}G, [1]\psi_{2m}^{\dagger}\right\rangle$$
$$= \sum_{j=0}^{m}\sum_{i=0}^{\infty}\frac{(-1)^i}{4^i i!}a_{2j}^{2m}(-1)^k\left\langle \partial^{2k}G, \xi^{2i+2j}[1]\right\rangle .$$

Denote $\rho = \max(0, k-j)$ and use equation (2.26) then

$$= \sum_{j=0}^{m}\sum_{i=\rho}^{\infty}\frac{(-1)^{i+k+j}}{4^i i!}2^{m-j}\frac{m!}{j!(m-j)!}\frac{\Gamma(m+\frac{1}{2})}{\Gamma(j+\frac{1}{2})}\frac{(2i+2j)!}{(2i+2j-2k)!}$$
$$\times \frac{\Gamma(\frac{1}{2}+i+j-k)}{\Gamma(\frac{1}{2})}2^{i+j-k}\sqrt{2\pi}.$$

Now use the rules

$$\Gamma\left(n+\frac{1}{2}\right) = \sqrt{\pi}\,\frac{(2n)!}{4^n n!} \qquad \text{and} \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

and simplify the derived expression

$$\left(s_{2k}^{2m}\right)^{\dagger}\eta_{2k} = \sqrt{\pi}\sum_{j=0}^{m}\sum_{i=\rho}^{\infty}\frac{(-1)^{i+j+k}}{4^i i!}2^{m+i-k+\frac{1}{2}}\frac{m!}{j!(m-j)!}\frac{(2m)!}{4^m m!}\frac{4^j j!}{(2j)!}$$
$$\times \frac{(2i+2j)!}{(2i+2j-2k)!}\frac{(2i+2j-2k)!}{4^{i+j-k}(i+j-k)!}$$
$$= \sqrt{\pi}\sum_{j=0}^{m}\sum_{i=\rho}^{\infty}\frac{(-1)^{i+j+k}\,2^{-3i+k-m+\frac{1}{2}}\,(2m)!\,(2i+2j)!}{i!(m-j)!\,(2j)!\,(i+j-k)!}$$
$$= \sqrt{\pi}\sum_{j=0}^{m}\frac{(2m)!(-1)^{\rho+j+k}2^{-3\rho+k-m+\frac{1}{2}}(2\rho+2j)!}{(2j)!\rho!(m-j)!(\rho+j-k)!}$$
$$\times\ {}_3F_2\left(1, 1+\rho+j, j+\rho+\frac{1}{2}; \rho+1, \rho+1+j-k; -\frac{1}{2}\right).$$

Clearly, the coefficients $\left(s_{2k}^{m}\right)^{\dagger}$ are scalar, hence $\left(s_{2k}^{m}\right)^{\dagger} = s_{2k}^{m}$. Depending on ρ, this expression can be simplified:

- $k > j$: $\rho = k - j$:

$$s_{2k}^{2m}\eta_{2k} = \sqrt{\pi}\sum_{j=0}^{m}\frac{(2m)!2^{-2k+3j-m+\frac{1}{2}}(2k)!}{(2j)!(k-j)!(m-j)!}\,{}_2F_1\left(1+k, k+\frac{1}{2}; k-j+1; -\frac{1}{2}\right), \tag{3.23}$$

- $k < j$: $\rho = 0$:

$$s_{2k}^{2m}\eta_{2k} = \sqrt{\pi}\sum_{j=0}^{m}\frac{(2m)!(-1)^{j+k}2^{k-m+\frac{1}{2}}}{(m-j)!(j-k)!}\,{}_2F_1\left(1+j, j+\frac{1}{2}; 1+j-k; -\frac{1}{2}\right),$$

- $k = j$: $\rho = 0$:

$$s_{2k}^{2m}\eta_{2k} = \sqrt{\pi}\sum_{j=0}^{m}\frac{(2m)!2^{k-m+\frac{1}{2}}}{(m-k)!}\,{}_2F_1\left(1+k,k+\frac{1}{2};1;-\frac{1}{2}\right).$$

Now consider an **odd Hermite function** ψ_{2m+1} combined with an odd Hermite polynomial H_{2k+1}. Similarly,

$$\begin{aligned}\left(s_{2k+1}^{2m+1}\right)^{\dagger}\eta_{2k+1} &= \left\langle H_{2k+1}G,[1]\psi_{2m+1}^{\dagger}\right\rangle \\ &= \sum_{j=0}^{m}\sum_{i=0}^{\infty}\frac{(-1)^i}{4^i i!}a_{2j+1}^{2m+1}(-1)^{k+1}\left\langle\partial^{2k+1}G,\xi^{2i+2j+1}[1]\right\rangle\end{aligned}$$

Again we use the notation $\rho = \max(0,k-j)$

$$\begin{aligned}&= \sum_{j=0}^{m}\sum_{i=\rho}^{\infty}\frac{(-1)^{i+k+j}}{4^i i!}2^{m-j}\frac{m!}{j!(m-j)!}\frac{\Gamma(m+\frac{3}{2})}{\Gamma(j+\frac{3}{2})}\frac{(2i+2j+1)!}{(2i+2j-2k)!} \\ &\quad\times\frac{\Gamma(\frac{1}{2}+i+j-k)}{\Gamma(\frac{1}{2})}2^{i+j-k}\sqrt{2\pi}\end{aligned}$$

Use the formulae for the Gamma function of a half-integer and simplify this expression

$$\begin{aligned}&= \sqrt{\pi}\sum_{j=0}^{m}\sum_{i=\rho}^{\infty}\frac{(-1)^{i+j+k}}{4^i i!}2^{m+i-k+\frac{1}{2}}\frac{m!}{j!(m-j)!}\frac{(2m+2)!}{4^{m+1}(m+1)!} \\ &\quad\times\frac{4^{j+1}(j+1)!}{(2j+2)!}\frac{(2i+2j+1)!}{(2i+2j-2k)!}\frac{(2i+2j-2k)!}{4^{i+j-k}(i+j-k)!} \\ &= \sqrt{\pi}\sum_{j=0}^{m}\sum_{i=\rho}^{\infty}\frac{(-1)^{i+j+k}\,2^{-3i+k-m+\frac{1}{2}}\,(2m+1)!\,(2i+2j+1)!}{i!(m-j)!\,(2j+1)!\,(i+j-k)!} \\ &= \sqrt{\pi}\sum_{j=0}^{m}\frac{(2m+1)!(-1)^{\rho+j+k}2^{-3\rho+k-m+\frac{1}{2}}(2\rho+2j+1)!}{(2j+1)!\rho!(m-j)!(\rho+j-k)}! \\ &\quad\times{}_3F_2\left(1,1+\rho+j,j+\rho+\frac{3}{2};\rho+1,\rho+1+j-k;-\frac{1}{2}\right).\end{aligned}$$

The generalised hypergeometric function ${}_3F_2$ again can be simplified to the hypergeometric ${}_2F_1$, depending on ρ:

- $k > j$: $\rho = k - j$:

$$\begin{aligned}s_{2k+1}^{2m+1}\eta_{2k+1} = \sqrt{\pi}\sum_{j=0}^{m}&\frac{(2m+1)!2^{-2k+3j-m+\frac{1}{2}}(2k+1)!}{(2j+1)!(k-j)!(m-j)!} \\ &\times{}_2F_1\left(1+k,k+\frac{3}{2};k-j+1;-\frac{1}{2}\right),\end{aligned}$$

- $k < j$: $\rho = 0$:

$$s_{2k+1}^{2m+1}\eta_{2k+1} = \sqrt{\pi}\sum_{j=0}^{m}\frac{(2m+1)!(-1)^{j+k}2^{k-m+\frac{1}{2}}}{(m-j)!(j-k)!}\,{}_2F_1\left(1+j, j+\frac{3}{2}; 1+j-k; -\frac{1}{2}\right),$$

- $k = j$: $\rho = 0$:

$$s_{2k+1}^{2m+1}\eta_{2k+1} = \sqrt{\pi}\sum_{j=0}^{m}\frac{(2m+1)!2^{k-m+\frac{1}{2}}}{(m-k)!}\,{}_2F_1\left(1+k, k+\frac{3}{2}; 1; -\frac{1}{2}\right).$$

These scalar coefficients are very similar to the ones for even Hermite functions above. We will further discuss the calculations and results explicitly for the even Hermite functions as the conclusions will be identical.

Although we did not found a closed form for the coefficients, we can calculate its value for low values of n and m and make a plot to visualise its behaviour.

From the examples in figures 3.2a to 3.2k, it is clear that for a fixed Hermite function ψ_m, the coefficients s_n^m tend to zero as n goes to infinity. In order to proof this analytically, let us first find an upper bound for this hypergeometric series in the next lemma.

Lemma 3.18.

$$\left|{}_2F_1\left(1+k, k+\frac{1}{2}; k-j+1; -\frac{1}{2}\right)\right| \leq (j+1)(2j)!\left(\frac{2}{3}\right)^{k+\frac{1}{2}},$$

$$\left|{}_2F_1\left(1+k, k+\frac{3}{2}; k-j+1; -\frac{1}{2}\right)\right| \leq (j+1)(2j+1)!\left(\frac{2}{3}\right)^{k+\frac{3}{2}}.$$

Proof. Both inequalities are proven in the same way. Let us do the calculations for the first expression. From [46], formula 15.8.1, we have that

$${}_2F_1(a, b; c; z) = (1-z)^{c-b-a}\,{}_2F_1(c-a, c-b; c; z).$$

Applied to our parameters, this becomes

$${}_2F_1\left(k+1, k+\frac{1}{2}; k-j+1; -\frac{1}{2}\right) = \left(\frac{2}{3}\right)^{k+j+\frac{1}{2}}\,{}_2F_1\left(-j, -j+\frac{1}{2}; k-j+1; -\frac{1}{2}\right).$$

Because the negative integer $-j$ appears as the first parameter in this hypergeometric function, this infinite sum reduces to a finite one. We use the definitions of ${}_2F_1(a, b; c; z)$ in this case [46](15.2.4) and the Pochhammer symbol:

$$\begin{aligned}
&\left(\frac{2}{3}\right)^{k+j+\frac{1}{2}}\,{}_2F_1\left(-j, -j+\frac{1}{2}; k-j+1; -\frac{1}{2}\right)\\
&\quad= \left(\frac{2}{3}\right)^{k+j+\frac{1}{2}}\sum_{i=0}^{j}\left(\frac{1}{2}\right)^{i}\frac{j!}{i!(j-i)!}\frac{(-j+\frac{1}{2})_i}{(k+1-j)_i}\\
&\quad= \left(\frac{2}{3}\right)^{k+j+\frac{1}{2}}\sum_{i=0}^{j}\left(\frac{1}{2}\right)^{i}\frac{j!}{i!(j-i)!}\frac{(k-j)!}{(k-j+i)!}\frac{\Gamma(-j+i+\frac{1}{2})}{\Gamma(-j+\frac{1}{2})}
\end{aligned}$$

$$= \left(\frac{2}{3}\right)^{k+j+\frac{1}{2}} \sum_{i=0}^{j} \left(\frac{1}{2}\right)^{i} \frac{j!}{i!(j-i)!} \frac{(k-j)!}{(k-j+i)!} \frac{(-4)^{j-i}(j-i)!}{(2j-2i)!} \frac{(2j)!}{(-4)^{j}j!}$$
$$= \left(\frac{2}{3}\right)^{k+j+\frac{1}{2}} \sum_{i=0}^{j} (-1)^{i} \frac{(k-j)!(2j)!}{i!(k-j+i)!(2j-2i)!8^{i}}.$$

Now take the absolute value of this sum and maximise it.

$$\left| {}_2F_1\left(1+k, k+\frac{1}{2}; k-j+1; -\frac{1}{2}\right)\right| = \left| \left(\frac{2}{3}\right)^{k+j+\frac{1}{2}} \sum_{i=0}^{j} (-1)^{i} \frac{(k-j)!(2j)!}{i!(k-j+i)!(2j-2i)!8^{i}} \right|$$
$$\leq \left(\frac{2}{3}\right)^{k+\frac{1}{2}} (j+1) \frac{(k-j)!(2j)!}{0!(k-j+0)!(2j-2j)!8^{0}}$$
$$= \left(\frac{2}{3}\right)^{k+\frac{1}{2}} (j+1)(2j)!.$$

□

To calculate the limit as $n = 2k$ tends to infinity, we need the expressions for which $\rho = k - j$. It follows that

$$\lim_{k\to\infty} \left|s_{2k}^{2m}\right| = \left| \sum_{j=0}^{m} \frac{(2m)!2^{-2k+3j-m}}{(2j)!(k-j)!(m-j)!} {}_2F_1\left(1+k, k+\frac{1}{2}; k-j+1; -\frac{1}{2}\right) \right|$$
$$\leq \sum_{j=0}^{m} \frac{(2m)!2^{-2k+3j-m}}{(2j)!(k-j)!(m-j)!} (j+1)(2j)! \left(\frac{2}{3}\right)^{k+\frac{1}{2}}$$
$$\leq (m+1) \frac{2^{-2k+2m}}{(2.0)!(k-m)!(m-m)!} (m+1)(2m)! \left(\frac{2}{3}\right)^{k+\frac{1}{2}}$$
$$= 0.$$

Before we check if the Hermite functions are elements of the Weierstrass space, we verify if the infinite series of Hermite polynomials is well-defined and has pointwise convergence in every point of the grid: the coefficient in every power $\xi^n[1]$ must be finite.

$$\psi_{2m}(\xi)[1] = \sum_{k\in\mathbb{N}} H_{2k}(\xi) s_{2k}^{2m}$$
$$= \sum_{k\in\mathbb{N}} \sum_{j=0}^{k} a_{2j}^{2k} \xi^{2j} s_{2k}^{2m}$$
$$= \sum_{k\in\mathbb{N}} \sum_{j=k}^{\infty} a_{2k}^{2j} s_{2j}^{2m} \xi^{2k}.$$

The infinite sum $\sum_{j=k}^{\infty} a_{2k}^{2j} s_{2j}^{2m}$ must be convergent. We use the expression for a_{2k}^{2j} from (2.30) and the formula

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{4^{n}\, n!}.$$

The aim is to find a convergent majorant series. As we will let j tend to infinity, take the first expression for s_{2j}^{2m}, (3.23).

$$\left|a_{2k}^{2j} s_{2j}^{2m}\right| = \left|\frac{2^{j-k} j!}{k!(j-k)!} \frac{(2j)!}{4^j j!} \frac{4^k k!}{(2k)!} \sum_{i=0}^{m} \frac{(2m)! 2^{-2j+3i-m}}{(2i)!(j-i)!(m-i)!} {}_2F_1\left(1+j, j+\frac{1}{2}; j-i+1; -\frac{1}{2}\right)\right|.$$

Use lemma 3.18:

$$\begin{aligned} &\leq \frac{2^{j-k}(2j)!}{4^{j-k}(j-k)!(2k)!} \sum_{i=0}^{m} \frac{(2m)! 2^{-2j+3i-m}}{(2i)!(j-i)!(m-i)!}(i+1)(2i)! \left(\frac{2}{3}\right)^{j+\frac{1}{2}} \\ &\leq \frac{2^{k-j}(2j)!}{(j-k)!(2k)!}(m+1) \frac{(2m)! 2^{-2j+2m}}{(2.0)!(j-m)!(m-m)!}(m+1)(2m)! \left(\frac{2}{3}\right)^{j+\frac{1}{2}}. \end{aligned}$$

Now use d'Alemberts ratio test to prove that this majorant series converges:

$$\begin{aligned} \lim_{j\to\infty} \left|\frac{a_{2k}^{2j+2} s_{2j+2}^{2m}}{a_{2k}^{2j} s_{2j}^{2m}}\right| &= \lim_{j\to\infty} \left|\frac{\frac{2^{-j-1}(2j+2)!}{(j+1-k)!} \frac{2^{-2j-2}}{(j+1-m)!} \left(\frac{2}{3}\right)^{j+\frac{3}{2}}}{\frac{2^{-j}(2j)!}{(j-k)!} \frac{2^{-2j}}{(j-m)!} \left(\frac{2}{3}\right)^{j+\frac{1}{2}}}\right| \\ &= \lim_{j\to\infty} \frac{2^{-1}(2j+2)(2j+1)}{(j+1-k)} \frac{2^{-2}}{(j+1-m)} \frac{2}{3} \\ &= \frac{1}{3}. \end{aligned}$$

We conclude that the series defines a discrete function.

If the Hermite function ψ_m is an element of the Weierstrass space, its coefficients must satisfy

$$\sum_{n=0}^{\infty} \eta_n (s_n^m)^\dagger s_n^m = \sum_{k=0}^{\infty} \eta_{2k} \left(s_{2k}^m\right)^2 + \eta_{2k+1} \left(s_{2k+1}^m\right)^2 < \infty. \tag{3.24}$$

Depending on the parity of m, only one term will be non-zero. Let us again discuss the case for **even Hermite functions**, hence we want to prove that

$$\sum_{k=0}^{\infty} \eta_{2k} \left(s_{2k}^{2m}\right)^2 < \infty. \tag{3.25}$$

There are only a finite number of terms for which $k \leq m$, so the question of convergence reduces to the series

$$\sum_{k=m+1}^{\infty} \eta_{2k} \left(s_{2k}^{2m}\right)^2 < \infty. \tag{3.26}$$

In this case, $\rho = k - j$ and we are led to the examination of the series

$$\sum_{k=m}^{\infty} \eta_{2k} \left(s_{2k}^{2m}\right)^2$$

$$= \sum_{k=m}^{\infty} \frac{(2k)!(2m)!^2}{\sqrt{2\pi}} \left[\sum_{j=0}^{m} \frac{2^{-2k+3j-m}}{(2j)!(k-j)!(m-j)!} \, {}_2F_1 \left(1+k, k+\frac{1}{2}; k-j+1; -\frac{1}{2} \right) \right]^2 . \tag{3.27}$$

We again want to find a convergent majorant series.

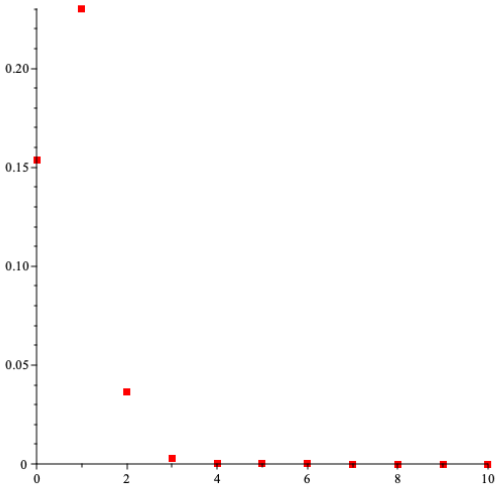
$$\begin{aligned}
&\frac{(2k)!(2m)!^2}{\sqrt{2\pi}} \left[\sum_{j=0}^{m} \frac{2^{-2k+3j-m}}{(2j)!(k-j)!(m-j)!} \, {}_2F_1 \left(1+k, k+\frac{1}{2}; k-j+1; -\frac{1}{2} \right) \right]^2 \\
&\quad \le (2k)!(2m)!^2 \left[(m+1) \frac{2^{-2k+3m-m}}{(2.0)!(k-m)!(m-m)!} \, {}_2F_1 \left(1+k, k+\frac{1}{2}; k-j+1; -\frac{1}{2} \right) \right]^2 \\
&\quad \le (2k)!(2m)!^2 (m+1)^2 \frac{4^{-2k+2m}}{(k-m)!^2} (j+1)^2 (2j)!^2 \left(\frac{2}{3} \right)^{2k+1} \\
&\quad \le (2k)!(2m)!^4 (m+1)^4 \frac{4^{-2k+2m}}{(k-m)!^2} \left(\frac{2}{3} \right)^{2k+1} \quad := S_k.
\end{aligned}$$

Now use d'Alembert's ratio test once more:

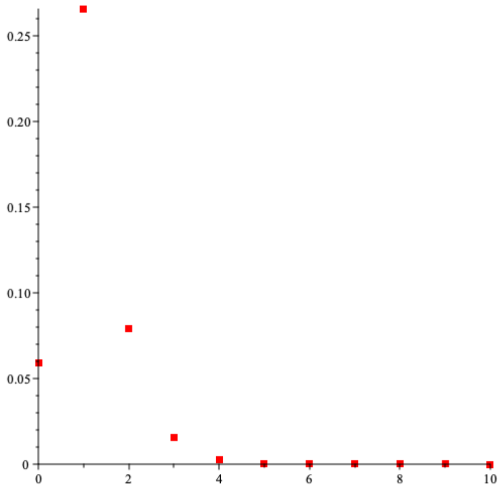
$$\lim_{k\to\infty} \left| \frac{S_{k+1}}{S_k} \right| = \lim_{k\to\infty} \frac{(2k+2)(2k+1)4^{-2}}{(k+1-m)^2} \left(\frac{2}{3} \right)^2 = \frac{1}{4}\frac{4}{9} = \frac{1}{9} < 1.$$

We conclude that ψ_{2m} is indeed a function in the discrete Weierstrass space. As mentioned above, the calculations and conclusions for ψ_{2m+1} are identical.

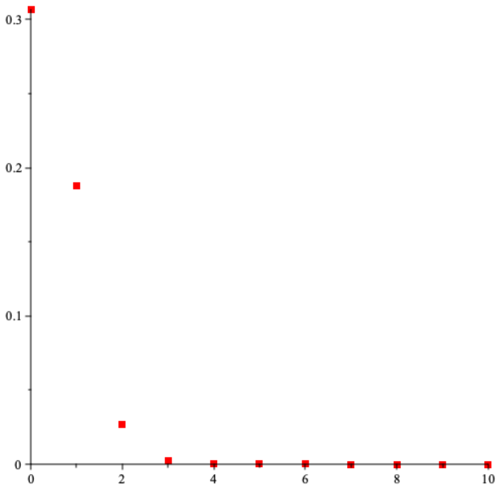
(a) Coefficients s_{2k}^2



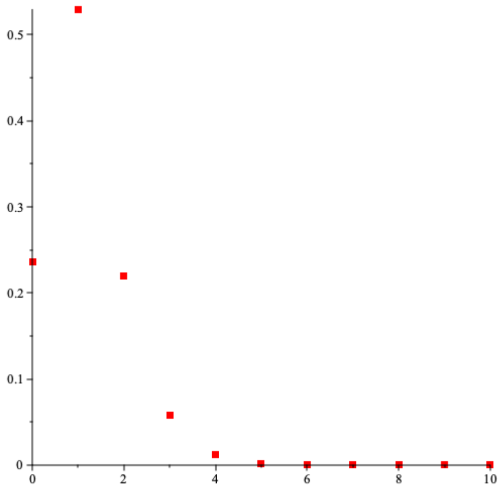
(b) Coefficients $\eta_{2k}\left(s_{2k}^2\right)^2$



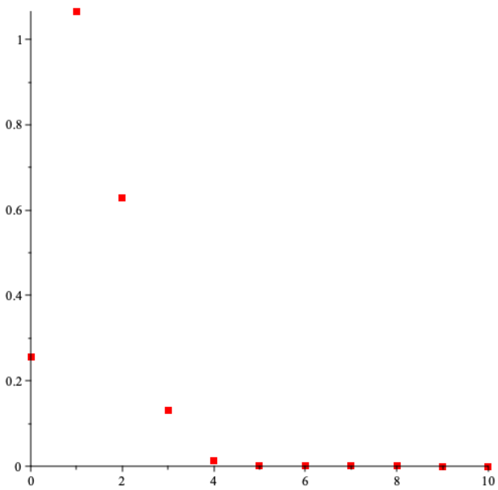
(c) Coefficients s_{2k+1}^3



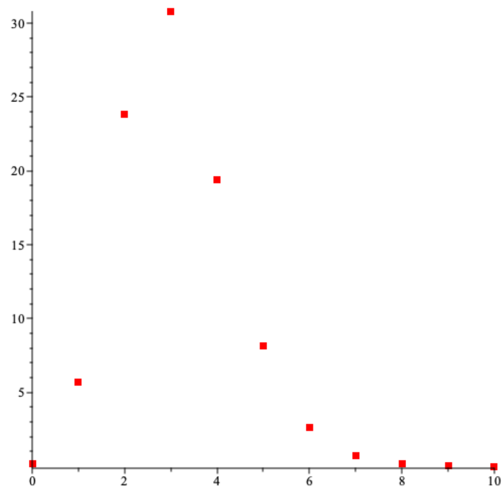
(d) Coefficients $\eta_{2k+1}\left(s_{2k+1}^3\right)^2$



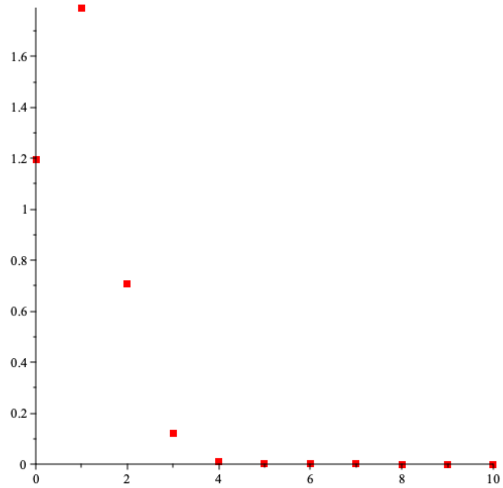
(e) Coefficients s_{2k}^6



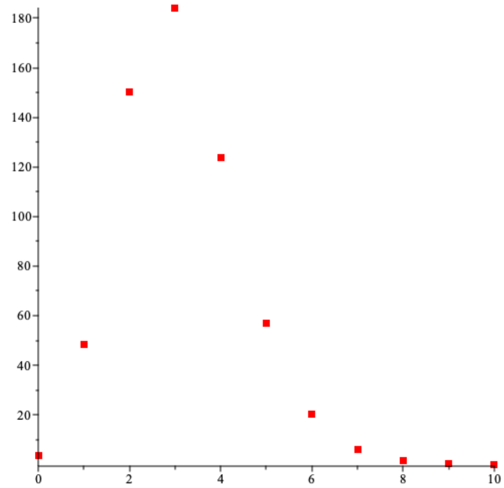
(f) Coefficients $\eta_{2k}\left(s_{2k}^6\right)^2$



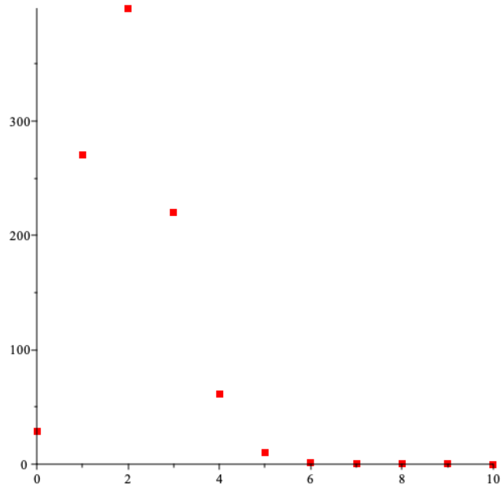
(g) Coefficients s^{7}_{2k+1}



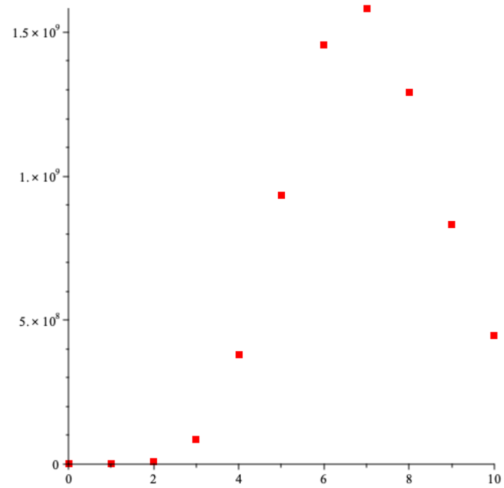
(h) Coefficients $\eta_{2k+1}\left(s^{7}_{2k+1}\right)^{2}$



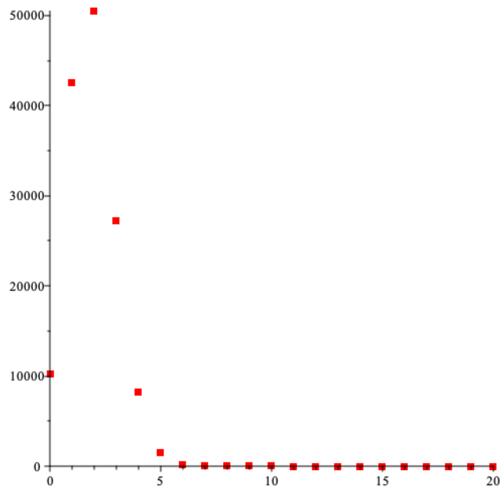
(i) Coefficients s^{14}_{2k}



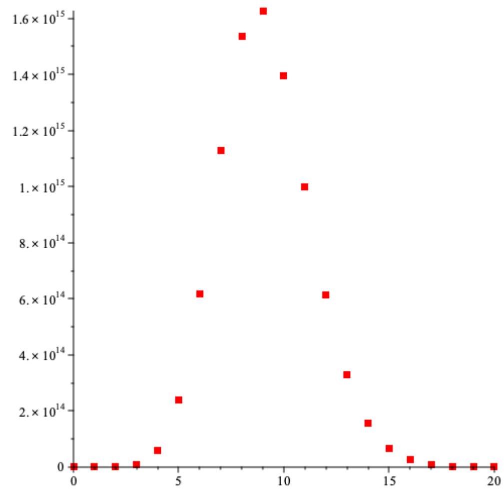
(j) Coefficients $\eta_{2k}\left(s^{14}_{2k}\right)^{2}$



(k) Coefficients s^{19}_{2k+1}



(l) Coefficients $\eta_{2k+1}\left(s^{19}_{2k+1}\right)^{2}$



3.3 Mesh width $h \neq 1$

In the next section, we investigate how the value h influences the definition and results of the Weierstrass transform and in particular the behaviour for h tending to 0.
Therefore, we return to the basic definitions we have used to define the Weierstrass transform and space.

Let us first consider the Gaussian distribution. As seen in the preliminaries, the mesh width h does not appear explicitly in the definition of G. It does however appear in its density function.

To calculate its density function, we will need an expression for the even derivatives of $\boldsymbol{\delta}_0$:

$$\begin{aligned}\partial^{2k}\boldsymbol{\delta}_0 &= \sum_{i=0}^{2k} \frac{(-1)^i}{h^{2k}} \binom{2k}{i} \boldsymbol{\delta}_{-(k-i)h} \\ &= \sum_{i=-k}^{k} \frac{(-1)^{k+i}}{h^{2k}} \binom{2k}{i+k} \boldsymbol{\delta}_{ih} \\ &= \sum_{i=-k}^{k} \frac{(-1)^{k+i}}{h^{2k}} \binom{2k}{i+k} \boldsymbol{\delta}_{ih}.\end{aligned}$$

Then

$$\begin{aligned}G &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \partial_h^k \boldsymbol{\delta}_0 \left\langle G, \xi_h^k[1] \right\rangle \\ &= \sum_{k=0}^{\infty} \frac{(-1)^{2k}\sqrt{2\pi}(2k)!}{(2k)!2^k k!} \partial_h^{2k}\boldsymbol{\delta}_0 \\ &= \sum_{k=0}^{\infty} \frac{\sqrt{2\pi}}{2^k k!} \left[\sum_{i=-k}^{k} \frac{(-1)^{k+i}}{h^{2k}} \binom{2k}{i+k} \boldsymbol{\delta}_{ih} \right] \\ &= \sqrt{2\pi} \sum_{n\in\mathbb{Z}} \left[\sum_{\ell=|n|}^{\infty} \frac{(-1)^{n+l}}{2^\ell \ell! h^{2\ell}} \binom{2\ell}{n+\ell} \right] \boldsymbol{\delta}_{nh} \\ &= \sqrt{2\pi} \sum_{n\in\mathbb{Z}} \mathcal{I}_n\left(\frac{1}{h^2}\right) \exp\left(-\frac{1}{h^2}\right) \boldsymbol{\delta}_{nh},\end{aligned}$$

with $n = \frac{x}{h}$. Here, $\mathcal{I}_n(z)$ is the modified Bessel function of the first kind. Therefore, in a point $x = nh \in \mathbb{Z}_h$, the value of G is

$$\frac{\sqrt{2\pi}}{h} \exp\left(-\frac{1}{h^2}\right) \mathcal{I}_{\frac{x}{h}}\left(\frac{1}{h^2}\right) \tag{3.28}$$

and this is thus the density function of the Gaussian distribution, denoted by $g(x)$.

Remark 3.19. For general dimension m, this density function is

$$g(\underline{x}) = \frac{(\sqrt{2\pi})^m}{h^m} \exp\left(\frac{m}{h^2}\right) \prod_{j=1}^{m} \mathcal{I}_{\frac{x_j}{h}}\left(\frac{1}{h^2}\right) \text{ with } \frac{x_j}{h} \in \mathbb{Z}.$$

To analytically investigate the asymptotic behaviour of the Gauss distribution as $h \to 0$, we use formula 9.7.7 from [47]:

$$\mathcal{I}_\nu(\nu z) \approx \frac{1}{\sqrt{2\pi\nu}} \frac{\exp(\nu\eta)}{(1+z^2)^{1/4}} \left(1 + \sum_{k=1}^{\infty} \frac{U_k(p)}{\nu^k}\right), \qquad \eta = \sqrt{1+z^2} + \ln\left(\frac{z}{1+\sqrt{1+z^2}}\right), \tag{3.29}$$

which describes the uniform asymptotic expansion for large orders $\nu \to +\infty$ and is valid for z in the sector

$$\left|\arg(z)\right| < \frac{\pi}{2}.$$

The terms $U_k(p)$ are polynomials in $p = (1+z^2)^{-\frac{1}{2}}$ of degree $3k$, recursively given by

$$\begin{aligned} U_0(p) &= 1, \\ U_{k+1}(p) &= \frac{1}{2}p^2(1-p^2)U_k'(p) + \frac{1}{8}\int_0^p (1-5t^2)U_k(t)dt. \end{aligned}$$

As $\left(\frac{1}{h^2}\right) > 0$, we may apply it for the density function g. As seen in figure 3.3, cases for $h = 1, h = \frac{1}{10}$ and $h = \frac{1}{100}$ approach the continuous Gaussian. This is clearly confirmed by substituting formula (3.29) into the definition of g with $\nu = \frac{x}{h}$ and $z = \frac{1}{xh}$. Taking the limit for $h \to 0$ gives $\exp\left(-\frac{x^2}{2}\right)$, the continuous Gaussian distribution. This can be checked, for example with Maple: see the appendix 7.3.

Another way to visualise the effect in G of h approaching 0 is given in figure 3.4, where we let h approach 0 in (3.28). Be aware that $x = nh$, hence the absolute value of x increases with the same factor as h decreases, resulting in a rescaling of the plots and x-axis. As $h \to 0$, all points of the grid collapse to the origin, hence g tends to 1.

The discrete Weierstrass transform was defined based on the transform of the Hermite polynomials. Because there is no explicit appearance of the mesh width h in formulas (4.6), the calculation in (3.3) can be replicated and the outcome of the Weierstrass transform will not change, as h tends to 0.

$$\begin{aligned} \langle H_n(\xi_h)G, \exp\left(-z^2/2 + \xi_h z\right)[1]\rangle &= \langle (-1)^{\lceil \frac{n}{2} \rceil} \partial^n G, \exp\left(-z^2/2 + \xi_h z\right)[1]\rangle \\ &= (-1)^{\lceil \frac{n}{2} \rceil} \exp\left(-z^2/2\right) \langle \partial^n G, \sum_{i=0}^{\infty} \frac{\xi_h^i z^i [1]}{i!}\rangle \\ &= (-1)^{\lfloor \frac{n}{2} \rfloor} \exp\left(-z^2/2\right) \sum_{i=0}^{\infty} \frac{z^i}{i!} \langle G, \xi_h^i[1](\partial^\dagger)^n\rangle \\ &= (-1)^{\lfloor \frac{n}{2} \rfloor} \exp\left(-z^2/2\right) \sum_{i=0}^{\infty} \frac{z^i}{i!} \langle G, \partial^n \xi_h^i[1]\rangle \\ &= (-1)^{\lfloor \frac{n}{2} \rfloor} \exp\left(-z^2/2\right) \sum_{i=n}^{\infty} \frac{z^i}{i!} \frac{i!}{(i-n)!} \langle G, \xi_h^{i-n}[1]\rangle. \end{aligned}$$

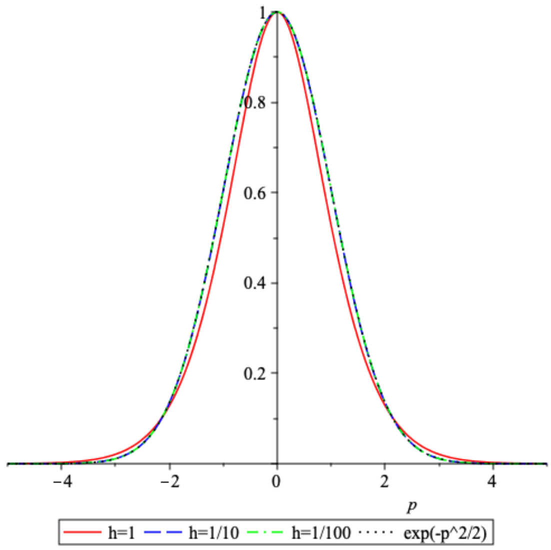


Figure 3.3: Asymptotic behaviour of discrete Gauss distribution for $h = 1, h = \frac{1}{10}, h = \frac{1}{100}$, compared to the continuous Gauss distribution.

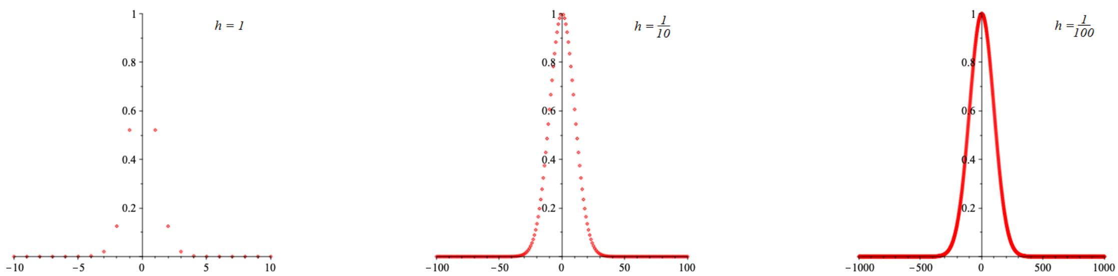


Figure 3.4: Density function of discrete Gauss distribution for $h = 1, h = \frac{1}{10}, h = \frac{1}{100}$ respectively.

Now change the summation index i to j to make it start from 0, then recall that G is only non-trivial if acting on even powers of ξ.

$$\begin{aligned}
&= (-1)^{\lfloor \frac{n}{2} \rfloor} \exp\left(-z^2/2\right) \sum_{j=0}^{\infty} \frac{z^{j+n}}{j!} \langle G, \xi_h^j[1] \rangle \\
&= (-1)^{\lfloor \frac{n}{2} \rfloor} \exp\left(-z^2/2\right) z^n \sum_{j=0}^{\infty} \frac{z^{2j}}{(2j)!} \sqrt{2\pi} \frac{(2j)!}{2^j j!} \\
&= (-1)^{\lfloor \frac{n}{2} \rfloor} \sqrt{2\pi}\, z^n.
\end{aligned}$$

3.4 Conclusion

In this chapter, we laid the foundation for the Weierstrass transform in the discrete Hermitian Clifford setting. We defined the discrete Weierstrass transform of a discrete function f as

$$\mathcal{W}[f](z) = \frac{1}{\sqrt{2\pi}} \langle f(\xi) G, \exp\left(-z^2/2 + \xi\, z\right)[1] \rangle.$$

In order to put a condition on the functions for which this Weierstrass transform is makes sense, we also defined a Weierstrass space: it is the completion of the (right) Clifford module of the discrete Hermite polynomials $H_n(\xi)$ for the norm

$$\left[\sum_{n=0}^{\infty} \eta_n c_n^\dagger c_n \right]_0,$$

where $f = \sum_{n=0}^{\infty} H_n c_n$ and $\eta_n = \sqrt{2\pi} n!$. We gave some examples of elementary functions that are contained in the Weierstrass space and gave their corresponding Weierstrass transforms. The discrete delta functions, which are the building blocks of discrete function theory, are elements of the Weierstrass space, and so are the Hermite functions. Eventually, we showed that this definition is consistent with the definition in the classical (continuous) setting. First, we generalised the definitions on a grid with general mesh width h, then let this mesh width approach zero, where we found the same properties and results as in the classical case.

4

Discrete Weierstrass transform in dimension $m > 1$

Having defined and investigated the discrete Weierstrass transform and its functions in one dimension, we will now further explore how we need to adapt these definitions in the higher-dimensional context. There will be two obstacles to overcome.

First, we must take into account the non-commutativity of the basis Clifford elements $e_1, \ldots, e_m$. In particular, it holds that $e_j e_k = -e_k e_j$ if $j \neq k$. To handle this, we will use the **discrete rotation invariant operators** R_j**,** introduced in [40]:

$$R_j = \boldsymbol{e}_j^+ R_j^+ + \boldsymbol{e}_j^- R_j^-,$$

where $R_j^{\pm}$ are scalar operators.

For a detailed study of these operators, we refer to [40]. For now, we will summarise the properties that we will need in this thesis. The interaction of R_j with other fundamental operators is as follows:

$$R_j[c] = e_j c, \quad c \in \mathbb{C}_m, \tag{4.1}$$

$$[R_j, \xi_j] = [R_j, \partial_j] = 0, \tag{4.2}$$

$$\{R_j, \xi_k\} = \{R_j, \partial_k\} = 0, \quad j \neq k. \tag{4.3}$$

As a result:

$$(\xi_j R_j)(\xi_k R_k) = (\xi_k R_k)(\xi_j R_j) \text{ and } (\partial_j R_j)(\partial_k R_k) = (\partial_k R_k)(\partial_j R_j),$$

which we can interpret as commutativity for the operators $\xi_j R_j$ and $\xi_k R_k (j \neq k)$ and similar for ∂_j. We will implement the operator R_j in the kernel of the Weierstrass transform: $\xi_j z_j$ becomes $\xi_j R_j z_j$, which will allow for commutativity between the different indices $j = 1, \ldots, m$. As a result of the extra operator R_j in the definition of the Weierstrass transform, due to the rule $R_j[1] = e_j$, a basis element e_j will emerge in the

Clifford-Fock space: every occurrence of the complex variable z_j will be accompanied by the basis element e_j, resulting in a complex Clifford variable.

Besides the anti-commutativity of the basis elements, there is the need to review the discrete Hermite polynomials in order to form a basis in the Weierstrass space in higher dimensions. In one dimension, it suffices to consider the *radial* Hermite polynomials, as defined in the preliminaries (see (2.29)) and as used in the previous chapter. In this form, the radial Hermite polynomials form a basis for the space of functions of the form $f(\xi)$, where f is a function on the discrete 'line'. In higher dimensions, $m > 1$, they no longer constitute a basis, as they are polynomials in $\xi = \xi_1 + \xi_2 + \ldots + \xi_m$ and do not allow for other forms. Therefore, we use the **generalised discrete Hermite polynomials**, introduced by Sommen in [36] in the classical setting, and established in the discrete setting by De Ridder in [31]. These generalised discrete Hermite polynomials $H_{n,m,r}$ are defined based on the composition of a discrete spherical monogenic operator P_r, i.e.

$$\partial P_r = 0 \text{ and } \mathbb{E} P_r = r P_r,$$

of degree r with the (discrete) Hermite polynomial $H_{n,m,r}$ of degree n. Its recurrence relation (with respect to the Hermite degree n) is

$$H_{n,m,r} P_r G = (-1)^n \partial H_{n-1,m,r} P_r G. \tag{4.4}$$

The defining Rodriguez' formula is

$$\begin{aligned} H_{2k,m,r} P_r G &= (-1)^k \partial^{2k} P_r G, \\ H_{2k+1,m,r} P_r G &= (-1)^{k+1} \partial^{2k+1} P_r G. \end{aligned} \tag{4.5}$$

Note the dependency of $H_{n,m,r}$ on the degree r of the monogenic P_r. The explicit form is given by

$$H_{2k,m,r} = \sum_{j=0}^{k} a_{2j}^{2k} \xi^{2j}, \qquad H_{2k+1,m,r} = \sum_{j=0}^{k} a_{2j+1}^{2k+1} \xi^{2j+1} \tag{4.6}$$

with

$$a_{2j}^{2k} = (-1)^j 2^{k-j} \binom{k}{j} \frac{\Gamma(k + \frac{m}{2} + r)}{\Gamma(j + \frac{m}{2} + r)}, \tag{4.7}$$

$$a_{2j+1}^{2k+1} = (-1)^j 2^{k-j} \binom{k}{j} \frac{\Gamma(k + \frac{m}{2} + r + 1)}{\Gamma(j + \frac{m}{2} + r + 1)}. \tag{4.8}$$

The combination of the new kernel with the product $\xi_j R_j z_j$ and the generalised Hermite polynomials will lead to definition (4.3).

But first, we recall the definition of the discrete Weierstrass space and its inner product, which are directly extendable to dimension $m > 1$.

Definition 4.1. Let f and g be two discrete functions.

$$(f, g) = \big(f(\xi)[1], g(\xi)[1]\big) \coloneqq \left\langle f(\xi) G, [1](g(\xi))^\dagger \right\rangle. \tag{4.9}$$

The scalar part of (f, f) is defined as the **norm** of the discrete function f:

$$\|f\| \coloneqq \big[(f, f)\big]_0. \tag{4.10}$$

A discrete function f is an element of the discrete Weierstrass space if it is a finite or infinite linear combination of generalised Hermite polynomials, for which its norm is finite.

Definition 4.2. The **discrete Weierstrass space** $\mathcal{W}$ is the completion of the right Clifford module of generalised Hermite polynomials in ξ in the norm (4.10):

$$f \in \mathcal{W} \Leftrightarrow f = \sum_{n\in\mathbb{N}}^{\infty} H_{n,m,r}P_r c_n \text{ with } \|f\| < \infty, c_n \in \mathbb{C}_m.$$

We are now led to the definition of the discrete Weierstrass transform.

Definition 4.3. The **discrete Weierstrass transform** of a discrete function $f \in \mathcal{W}$ is defined by the transforms of the $n-$th degree generalised Hermite polynomial in dimension m

$$\mathcal{W}[H_{n,m,r}P_r](\underline{z}) := \sqrt{2\pi}^{-m} \left\langle H_{n,m,r}P_r G, \exp\left(\frac{-|\underline{z}|^2}{2} + \xi R\underline{z}\right)[1]\right\rangle,$$

where $\xi R\underline{z} = \sum_{j=1}^{m} \xi_j R_j z_j$ and $\underline{z} = \sum_{j=1}^{m} z_j e_j$ is a continuous complex Clifford variable.

Our aim is to generate an explicit formula for this transform. As there are two positive integer parameters, n and r, the natural way is to try to obtain a recurrence relation.

4.1 Recurrence relation in terms of the degree n of the Hermite polynomial

Our first goal is to establish an expression for $\mathcal{W}[H_{n,m,r}P_r](\underline{z})$ in terms of $\mathcal{W}[H_{n-1,m,r}P_r](\underline{z})$. This is mainly based on the recurrence relation of the generalised Hermite polynomials (4.4), complemented by some additional technical lemmas. In order to fix ideas and limit notations, let us do the calculations and check some examples in dimension $m = 2$. Afterwards, we address for $m > 2$.

We subsequently obtain

$$\mathcal{W}[H_{n,2,r}P_r](\underline{z}) = \frac{1}{2\pi}\left\langle H_{n,2,r}P_rG, \exp\left(\frac{-|\underline{z}|^2}{2} + \xi R\underline{z}\right)[1]\right\rangle$$
$$\overset{(4.4)}{=} \frac{1}{2\pi}\exp\left(\frac{-|\underline{z}|^2}{2}\right)\left\langle (-1)^n\partial H_{n-1,2,r}P_rG, \sum_{\ell=0}^{\infty}\frac{(\xi R\underline{z})^\ell}{\ell!}[1]\right\rangle$$
$$= \frac{(-1)^{n+1}}{2\pi}\exp\left(\frac{-|\underline{z}|^2}{2}\right)$$
$$\times\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\left\langle H_{n-1,2,r}P_rG, (\xi_1R_1z_1+\xi_2R_2z_2)^\ell[1](\partial_1+\partial_2)^\dagger\right\rangle.$$

Here is were the role of the operators R_j becomes clear: we can expand the sum $(\xi_1R_1z_1+\xi_2R_2z_2)^\ell$

$$= \frac{(-1)^{n+1}}{2\pi}\exp\left(\frac{-|\underline{z}|^2}{2}\right)\sum_{\ell=0}^{\infty}\sum_{j=0}^{\ell}\binom{\ell}{j}\frac{1}{\ell!}\times$$
$$\left[\left\langle H_{n-1,2,r}P_rG, \xi_1^jR_1^jz_1^j\xi_2^{\ell-j}R_2^{\ell-j}z_2^{\ell-j}[1]\partial_1^\dagger\right\rangle\right.$$
$$\left.+\left\langle H_{n-1,2,r}P_rG, \xi_1^jR_1^jz_1^j\xi_2^{\ell-j}R_2^{\ell-j}z_2^{\ell-j}[1]\partial_2^\dagger\right\rangle\right]. \tag{4.11}$$

In order to move $\partial_j^\dagger$ to work from the left, we now need the following lemma.

Lemma 4.4. For a fixed i and $\forall j,k,\ell\in\mathbb{N}$: If j and k have equal parity, then

$$\xi_i^jR_i^k[1]\left(\partial_i^\dagger\right)^\ell = e_i^k\partial_i^\ell\xi_i^j[1].$$

Proof. If j and k are even, then $R_i^k[1]=1$, hence apply calculation rule (2.15) to find

$$\xi_i^jR_i^k[1]\left(\partial_i^\dagger\right)^\ell = \xi_i^j[1]\left(\partial_i^\dagger\right)^\ell = \partial_i^\ell\xi_i^j[1] = e_i^k\partial_i^\ell\xi_i^j[1].$$

If j and k are odd, then $\xi_i^j[1] = \left(\xi_i^\dagger\right)^j[1]$ and $R_i^k[1]=R_i[1]=e_i[1]$. There are two options for the dirac operator ∂_i, keeping in mind that $\partial_i^2 = \left(\partial_i^\dagger\right)^2$ is scalar. If ℓ is even, say 2ℓ, then scalar

$$\xi_i^jR_i[1]\left(\partial_i^\dagger\right)^{2\ell} = \xi_i^je_i[1]\left(\partial_i^\dagger\right)^{2\ell} = \left(\partial_i^\dagger\right)^{2\ell}\left(\xi_i^\dagger\right)^je_i[1] = e_i\partial_i^{2\ell}\xi_i^j[1].$$

If ℓ is odd, say $2\ell+1$, then

$$\xi_i^jR[1]\left(\partial_i^\dagger\right)^{2\ell+1} = \partial_i^{2\ell}\left(\xi_i^je[1]\partial_i^\dagger\right) = \partial_i^{2\ell}\left(\partial_i^\dagger\left(\xi_i^\dagger\right)^je_i\right)[1] = \partial_i^{2\ell}\left(e_i\partial_i\xi^j[1]\right) = e_i\partial_i^{2\ell+1}\xi_i^j[1].$$

□

Remark 4.5. In our context, we will often see combinations as, for example,

$$\xi_1^{j_1} R_1^{k_1} \xi_2^{j_2} R_2^{k_2}[1] \left(\partial_2^\dagger\right)^\ell,$$

or with even more $\xi_i R_i$ depending on the dimension we are working in. The interaction of the different indices does not affect the outcome of this lemma, as ∂_k and $\xi_j R_j$ $(j \neq k)$ are commutative:

$$\partial_k(\xi_j R_j) = -\xi_j \partial_k R_j = \xi_j R_j \partial_k.$$

We continue from (4.11), invoking the above lemma, to obtain

$$\begin{aligned}
\mathcal{W}[H_{n,2,r}P_r](\underline{z}) &= \frac{(-1)^{n+1}}{2\pi} \exp\left(\frac{-|\underline{z}|^2}{2}\right) \sum_{\ell=0}^{\infty} \sum_{j=0}^{\ell} \binom{\ell}{j} \frac{1}{\ell!} \times \\
&\quad \left[\left\langle H_{n-1,2,r}P_r G, e_1^j \underbrace{\partial_1 \xi_1^j}_{j\xi_1^{j-1}} z_1^j \xi_2^{\ell-j} R_2^{\ell-j} z_2^{\ell-j}[1] \right\rangle \right. \\
&\quad \left. + \left\langle H_{n-1,2,r}P_r G, \xi_1^j R_1^j z_1^j e_2^{\ell-j} \underbrace{\partial_2 \xi_2^{\ell-j}}_{(\ell-j)\xi_2^{\ell-j-1}} z_2^{\ell-j}[1] \right\rangle \right] \\
&= \frac{(-1)^{n+1}}{2\pi} \exp\left(\frac{-|\underline{z}|^2}{2}\right) \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \times \\
&\quad \left[\sum_{j=1}^{l} \binom{\ell}{j} j \left\langle H_{n-1,2,r}P_r G, e_1^j \xi_1^{j-1} z_1^j \xi_2^{\ell-j} R_2^{\ell-j} z_2^{\ell-j}[1] \right\rangle \right. \\
&\quad \left. + \sum_{j=0}^{\ell-1} \binom{\ell}{j} (\ell-j) \left\langle H_{n-1,2,r}P_r G, \xi_1^j R_1^j z_1^j e_2^{\ell-j} \xi_2^{\ell-j-1} z_2^{\ell-j}[1] \right\rangle \right].
\end{aligned} \tag{4.12}$$

We want to re-introduce the operators R_1 and R_2 in the first, resp. second term, in order to go back to the definition of the Weierstrass transform.

Lemma 4.6. For any index $i = 1 \dots m$ and any power $j \in \mathbb{N}$,

$$e_i^j \xi_i^{j-1}[1] = e_i \xi_i^{j-1} R_i^{j-1}[1].$$

Proof. The aim is to bring the factor e_i^{j-1} in the left hand side through the basic discrete polynomial ξ_i^{j-1}, in order to re-write it as the operator R_i^{j-1}, acting on [1]. However, $e_i \xi_i = \xi_i^\dagger e_i$. If j is even, $j-1$ is odd, which means $\left(\xi_i^\dagger\right)^{j-1}[1] = \xi_i^{j-1}[1]$. If j is odd, $j-1$ is even and $e_i^{j-1} = 1$. In both cases, the lemma is proven. □

Let us proceed with (4.12)

$$\begin{aligned}
\mathcal{W}[H_{n,2,r}P_r](\underline{z}) &= \frac{(-1)^{n+1}}{2\pi} \exp\left(\frac{-|\underline{z}|^2}{2}\right) \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \\
&\quad \times \left[\sum_{j=1}^{\ell} \binom{\ell}{j} j e_1 z_1 \left\langle H_{n-1,2,r}P_r G, \xi_1^{j-1} R_1^{j-1} z_1^{j-1} \xi_2^{\ell-j} R_2^{\ell-j} z_2^{\ell-j}[1] \right\rangle \right. \\
&\quad \left. + \sum_{j=0}^{\ell-1} \binom{\ell}{j} (\ell-j) e_2 z_2 \left\langle H_{n-1,2,r}P_r G, \xi_1^{j} R_1^{j} z_1^{j} \xi_2^{\ell-j-1} R_2^{\ell-j-1} z_2^{\ell-j-1}[1] \right\rangle \right] \\
&= \frac{(-1)^{n+1}}{2\pi} \exp\left(\frac{-|\underline{z}|^2}{2}\right) \sum_{\ell=0}^{\infty} \sum_{j=0}^{\ell-1} \frac{1}{\ell!} \left(\binom{\ell}{j+1} (j+1) z_1 e_1 + \binom{\ell}{j} (\ell-j) z_2 e_2 \right) \\
&\quad \times \left\langle H_{n-1,2,r}P_r G, \xi_1^{j} R_1^{j} z_1^{j} \xi_2^{\ell-j-1} R_2^{\ell-j-1} z_2^{\ell-j-1}[1] \right\rangle \\
&= \frac{(-1)^{n+1}}{2\pi} \exp\left(\frac{-|\underline{z}|^2}{2}\right) \sum_{\ell=1}^{\infty} \sum_{j=0}^{\ell-1} \frac{1}{(\ell-1)!} \binom{\ell-1}{j} (z_1 e_1 + z_2 e_2) \\
&\quad \times \left\langle H_{n-1,2,r}P_r G, \xi_1^{j} R_1^{j} z_1^{j} \xi_2^{\ell-j-1} R_2^{\ell-j-1} z_2^{\ell-j-1}[1] \right\rangle \\
&= \frac{(-1)^{n+1}}{2\pi} \exp\left(\frac{-|\underline{z}|^2}{2}\right) \sum_{\ell=1}^{\infty} \frac{1}{(\ell-1)!} (z_1 e_1 + z_2 e_2) \\
&\quad \times \left\langle H_{n-1,2,r}P_r G, (\xi_1 R_1 z_1 + \xi_2 R_2 z_2)^{\ell-1}[1] \right\rangle \\
&= (-1)^{n+1} (z_1 e_1 + z_2 e_2)\, \mathcal{W}[H_{n-1,2,r}P_r](\underline{z}).
\end{aligned} \tag{4.13}$$

In this way, we have proven the following recurrence relation.

Theorem 4.7. For the discrete Weierstrass transform of the discrete generalised Hermite polynomials in dimension $m = 2$, it holds that

$$\mathcal{W}[H_{n,2,r}P_r](\underline{z}) = (-1)^{n+1}\,(z_1 e_1 + z_2 e_2) \mathcal{W}[H_{n-1,2,r}P_r](\underline{z}).$$

Let us illustrate this theorem by looking at some examples for low values of n. For $r = 0$, the results of the general definition 4.3 for the discrete Weierstrass transform must coincide with the former definition of chapter 3.2, i.e. the transform of the $n-$th degree Hermite polynomial should be the $n-$th power of $\underline{z}$. As $P_0 = 1$, we will omit it in the notation.

Example 4.8. Let us check this for $n = 0$.

$$\begin{aligned}
\mathcal{W}[H_{0,2,0}](\underline{z}) &= \frac{1}{2\pi} \exp\left(\frac{-|\underline{z}|^2}{2}\right) \left\langle H_{0,2,0} G, \exp\left(\xi R \underline{z}\right)[1] \right\rangle \\
&= \frac{1}{2\pi} \exp\left(\frac{-|\underline{z}|^2}{2}\right) \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left\langle G, (\xi_1 R_1 z_1 + \xi_2 R_2 z_2)^{\ell}[1] \right\rangle
\end{aligned}$$

$$= \frac{1}{2\pi} \exp\left(\frac{-|\underline{z}|^2}{2}\right) \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{j=0}^{\ell} \binom{\ell}{j} \left\langle G, \xi_1^j R_1^j z_1^j \xi_2^{\ell-j} R_2^{\ell-j} z_2^{\ell-j}[1] \right\rangle$$

$$= \frac{1}{2\pi} \exp\left(\frac{-|\underline{z}|^2}{2}\right) \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{j=0}^{\ell} \binom{\ell}{j} z_1^j z_2^{\ell-j} \left\langle G, e_1^j \xi_1^j e_2^{\ell-j} \xi_2^{\ell-j}[1] \right\rangle.$$

The Gaussian distribution vanishes when acting on odd powers of $\xi[1]$, see (2.24). Hence, the only remaining terms are those where j and ℓ are both even, whence

$$\begin{aligned}
\mathcal{W}[H_{0,2,0}](\underline{z}) &= \frac{1}{2\pi} \exp\left(\frac{-|\underline{z}|^2}{2}\right) \sum_{\ell=0}^{\infty} \frac{1}{(2\ell)!} \sum_{j=0}^{\ell} \binom{2\ell}{2j} z_1^{2j} z_2^{2\ell-2j} \left\langle G, e_1^{2j} \xi_1^{2j} e_2^{2\ell-2j} \xi_2^{2\ell-2j}[1] \right\rangle \\
&= \exp\left(\frac{-|\underline{z}|^2}{2}\right) \sum_{\ell=0}^{\infty} \frac{1}{(2\ell)!} \sum_{j=0}^{\ell} \frac{(2\ell)!}{(2j)!(2\ell-2j)!} z_1^{2j} z_2^{2\ell-2j} \frac{(2j)!}{2^j j!} \frac{(2\ell-2j)!}{2^{\ell-j}(\ell-j)!} \\
&= \exp\left(\frac{-|\underline{z}|^2}{2}\right) \sum_{\ell=0}^{\infty} z_1^{2j} z_2^{2\ell-2j} \frac{1}{j!(\ell-j)!2^\ell} \\
&= \exp\left(\frac{-|\underline{z}|^2}{2}\right) \frac{(z_1^2+z_2^2)^\ell}{2^\ell \ell!} = 1.
\end{aligned}$$

Example 4.9. The next example for $n = 1$ uses the same calculations as seen in the general proof and again the fact that the Gaussian vanishes when acting on odd powers of $\xi[1]$.

$$\begin{aligned}
\mathcal{W}[H_{1,2,0}](\underline{z}) &= \frac{1}{2\pi} \exp\left(\frac{-|\underline{z}|^2}{2}\right) \left\langle H_{1,2,0} G, \exp\left(\xi R \underline{z}\right)[1] \right\rangle \\
&= \frac{1}{2\pi} \exp\left(\frac{-|\underline{z}|^2}{2}\right) \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left\langle -\left(\partial_1+\partial_2\right) G, \left(\xi_1 R_1 z_1 + \xi_2 R_2 z_2\right)^\ell[1] \right\rangle \\
&= \frac{1}{2\pi} \exp\left(\frac{-|\underline{z}|^2}{2}\right) \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{j=0}^{\ell} \binom{\ell}{j} \\
&\quad \times \left[\left\langle G, \xi_1^j R_1^j z_1^j \xi_2^{\ell-j} R_2^{\ell-j} z_2^{\ell-j}[1]\partial_1^\dagger \right\rangle + \left\langle G, \xi_1^j R_1^j z_1^j \xi_2^{\ell-j} R_2^{\ell-j} z_2^{\ell-j}[1]\partial_2^\dagger \right\rangle \right].
\end{aligned}$$

Use lemma 4.4 to bring $\partial_j^\dagger$ forward

$$\begin{aligned}
&= \frac{1}{2\pi} \exp\left(\frac{-|\underline{z}|^2}{2}\right) \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left[\sum_{j=1}^{\ell} \binom{\ell}{j} z_1^j z_2^{\ell-j} j \left\langle G, e_1^j \xi_1^{j-1} \xi_2^{\ell-j} e_2^{\ell-j}[1] \right\rangle \right. \\
&\quad \left. + \sum_{j=0}^{\ell-1} \binom{\ell}{j} z_1^j z_2^{\ell-j} (\ell-j) \left\langle G, \xi_1^j e_1^j \xi_2^{\ell-j-1} e_2^{\ell-j}[1] \right\rangle \right] \\
&= \frac{1}{2\pi} \exp\left(\frac{-|\underline{z}|^2}{2}\right) \sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)!}
\end{aligned}$$

$$
\begin{aligned}
&\times \left[\sum_{j=0}^{\ell} \binom{2\ell+1}{2j+1} z_1^{2j+1} z_2^{2\ell-2j} e_1 (2j+1) \left\langle G, \xi_1^{2j} \xi_2^{2\ell-2j}[1] \right\rangle \right. \\
&\qquad \left. + \sum_{j=0}^{\ell} \binom{2\ell+1}{2j} z_1^{2j} z_2^{2\ell+1-2j} e_2 (2\ell+1-2j) \left\langle G, \xi_1^{2j} \xi_2^{2\ell-2j}[1] \right\rangle \right] \\
&= \exp\left(\frac{-|\underline{z}|^2}{2}\right) \sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)!} \\
&\times \left[\sum_{j=0}^{\ell} \binom{2\ell+1}{2j+1} z_1^{2j+1} z_2^{2\ell-2j} e_1 (2j+1) \frac{(2j)!}{2^j j!} \frac{(2\ell-2j)!}{2^{\ell-j}(\ell-j)!} \right. \\
&\qquad \left. + \sum_{j=0}^{\ell} \binom{2\ell+1}{2j} z_1^{2j} z_2^{2\ell+1-2j} e_2 (2\ell+1-2j) \frac{(2j)!}{2^j j!} \frac{(2\ell-2j)!}{2^{\ell-j}(\ell-j)!} \right] \\
&= \exp\left(\frac{-|\underline{z}|^2}{2}\right) \sum_{\ell=0}^{\infty} \sum_{j=0}^{\ell} \frac{1}{2^\ell j! (\ell-j)!} \left[z_1^{2j+1} z_2^{2\ell-2j} e_1 + z_1^{2j} z_2^{2\ell+1-2j} e_2 \right] \\
&= \exp\left(\frac{-|\underline{z}|^2}{2}\right) \sum_{\ell=0}^{\infty} \sum_{j=0}^{\ell} \frac{(z_1^{2j} z_2^{2\ell-2j})}{2^\ell j! (\ell-j)!} (e_1 z_1 + e_2 z_2) \\
&= \exp\left(\frac{-|\underline{z}|^2}{2}\right) \sum_{\ell=0}^{\infty} \frac{(z_1^2 + z_2^2)^\ell}{2^\ell \ell!} (e_1 z_1 + e_2 z_2) \\
&= \exp\left(\frac{-|\underline{z}|^2}{2}\right) \exp\left(\frac{|\underline{z}|^2}{2}\right) (e_1 z_1 + e_2 z_2) \\
&= e_1 z_1 + e_2 z_2.
\end{aligned}
$$

Based on the calculations above in examples 4.8 and 4.9, together with the results of theorem 4.7, we obtain an explicit expression for the discrete Hermite polynomial of $n-th$ in dimension 2 with $r = 0$.

$$
\begin{aligned}
\mathcal{W}[H_{2k,2,0}](\underline{z}) &= (-1)^k (z_1 e_1 + z_2 e_2)^{2k}, \\
\mathcal{W}[H_{2k+1,2,0}](\underline{z}) &= (-1)^k (z_1 e_1 + z_2 e_2)^{2k+1}.
\end{aligned}
\tag{4.14}
$$

The calculations and results from section 4.1 can be directly extended to $m > 2$. However, due to the overload in notations, we limit ourselves to the results.

Theorem 4.10. The discrete Weierstrass transform of the discrete generalised Hermite polynomials in m dimensions is recursively given by

$$
\mathcal{W}[H_{n,m,r} P_r](\underline{z}) = \left(\sum_{j=1}^{m} z_j e_j \right) (-1)^{n+1} \mathcal{W}[H_{n-1,m,r} P_r](\underline{z}). \tag{4.15}
$$

To start the recursive definition, it can be easily calculated that

$$\begin{aligned} \mathcal{W}[H_{0,m,0}](\underline{z}) &= 1, \\ \mathcal{W}[H_{1,m,0}](\underline{z}) &= \sum_{j=1}^{m} z_j e_j. \end{aligned} \tag{4.16}$$

Having found a recurrence relation for the degree n of the Hermite polynomial, we seek for an analogous formula, expressing the Weierstrass transform of $H_{n,m,r}P_r$ in terms of $H_{n,m,r-1P_{r-1}}$.

4.2 Recurrence relation in terms of the degree r of the monogenic

A basis for the space $\mathcal{M}_r^{(m)}$ of spherical discrete monogenics of degree r in m variables is given by the so-called Fueter polynomials: they are the Cauchy-Kovalevskaya (CK in short) extension of the discrete homogeneous polynomials of degree r in $m-1$ variables. Let us therefore rephrase this important theorem and accompanying notations.

Theorem 4.11 (Cauchy-Kovalevskaya extension for discrete monogenic functions, [31])**.** Let f be a discrete function in the variables $x_2, \ldots, x_m$, defined on the grid $\mathbb{Z}^{m-1}$ and taking values in the algebra over $\{\boldsymbol{e}_2^+, \boldsymbol{e}_2^-, \ldots, \boldsymbol{e}_m^+, \boldsymbol{e}_m^-\}$. Then there exists a unique discrete monogenic function F in the variables $x_1, \ldots, x_m$, defined on the grid $\mathbb{Z}^m$ and taking values in the algebra over $\{\boldsymbol{e}_1^+, \boldsymbol{e}_1^-, \ldots, \boldsymbol{e}_m^+, \boldsymbol{e}_m^-\}$, such that the restriction of F to the hyperplane $x_1 = 0$ equals f. This function F is given by

$$\mathrm{CK}[f](x_1, \ldots, x_m) = \sum_{k=0}^{\infty} \frac{\xi_1^k[1](x_1)}{k!} f_k(x_2, \ldots, x_m),$$

where $f_0 = f$ and $f_{k+1} = (-1)^{k+1} \sum_{j=2}^{m} \partial_j f_k$.

Theorem 4.12. The set

$$\{\mathrm{CK}[\xi^{\underline{\alpha}}] \mid \underline{\alpha} = (\alpha_2, \ldots, \alpha_m), \alpha_2 + \ldots + \alpha_m = r\}$$

constitutes a basis for the set of discrete spherical monogenics of degree r in dimension m.

This discrete CK-extension was introduced and investigated by De Ridder in [33].

Let us introduce some notations.

Notation 4.13. We use the notations

$$\eta_i = \xi_i - \xi_1 \text{ and } \hat{\eta_i} = \xi_i + \xi_1.$$

For $\underline{\alpha} = (\alpha_2, \ldots, \alpha_m) \in \mathbb{N}^{m-1}$, let $\xi^{\underline{\alpha}} = \xi_2^{\alpha_2} \ldots \xi_m^{\alpha_m}$. The degree of the operator $\xi^{\underline{\alpha}}$ is $k = \alpha_2 + \ldots + \alpha_m$. With every $\underline{\alpha}$, we associate the $k-$tupple $(\ell_1, \ldots, \ell_k)$, with every $\ell_j \in \{2, \ldots, m\}$, $\ell_i \leq \ell_j$ if $i \leq j$ and the number of times that j appears in $(\ell_1, \ldots, \ell_k)$ is α_j.

Notation 4.14. On a k-tuple $(\eta_{\ell_1}, \dots, \eta_{\ell_k})$ we define three operators:

- $(\eta_{\ell_1}, \dots, \eta_{\ell_k})^{E_j}$ means that every even occurrence (i.e. second, fourth, sixth,…) of η_j is replaced by $\hat{\eta_j}$ and vice versa. The composition of $E_{r_1}, \dots, E_{r_k}$ is denoted in short by $E_{r_1,\dots,r_k}$.
- Analogously, $(\eta_{\ell_1}, \dots, \eta_{\ell_k})^{O_j}$ means that every odd occurrence (i.e. first, third, fifth,…) of η_j is replaced by $\hat{\eta_j}$ and vice versa. The composition of $O_{r_1}, \dots, O_{r_k}$ is denoted in short by $O_{r_1,\dots,r_k}$.
- $(\eta_{\ell_1}, \dots, \eta_{\ell_k})^{*_j}$ denotes that every η_j is replaced by $\hat{\eta_j}$ and vice versa, every $\hat{\eta_j}$ is replaced by η_j.

From [34], we know that

$$\mathrm{CK}[\xi^{\underline{\alpha}}] = \frac{\alpha_2! \dots \alpha_m!}{k!} \sum_{\pi(\ell_1,\dots,\ell_k)} \mathrm{sgn}(\pi) (\eta_{\pi(\ell_1)} \dots \eta_{\pi(\ell_k)})^{E_{2,\dots,m}},$$

where the sum runs over all distinguishable permutations π of $(\ell_1, \dots, \ell_k)$ and $\mathrm{sgn}(\pi)$ is $+1$ or -1, according to the signature of the permutation π.

As in the first section, let us start in dimension $m = 2$ in order to fix ideas and limit notations.

4.2.1 Recurrence relation in terms of the degree r in dimension $m = 2$

In what follows, results will be proven for the basis monogenic polynomials, and hence we will use the notations

$$P_r = \underbrace{(\xi_2 - \xi_1)(\xi_2 + \xi_1) \dots (\xi_2 \pm \xi_1)}_{r \text{ times}} = \eta_2 \hat{\eta}_2 \dots$$
$$\tilde{P}_r := \underbrace{(\xi_2 + \xi_1)(\xi_2 - \xi_1) \dots (\xi_2 \mp \xi_1)}_{r \text{ times}} = \hat{\eta}_2 \eta_2 \dots$$

as well as

$$\begin{aligned} \partial &= \partial_2 + \partial_1, \\ \tilde{\partial} &:= \partial_2 - \partial_1. \end{aligned} \tag{4.17}$$

In combination with the discrete Gauss distribution, we know that $\partial P_r G = -\xi P_r G$, which we now can write as

$$\partial P_r G = -\xi P_r G = -\tilde{P}_{r+1} G.$$

We now try to exploit this relationship to find the recurrence relation we are looking for. Recall that $H_{0,m,r'} = 1, \forall m, \forall r'$, hence

$$W[\tilde{P}_r](\underline{z}) = \mathcal{W}[H_{0,2,r} \tilde{P}_r](\underline{z}).$$

The calculation uses the same method as in (4.13):

$$\begin{aligned}
\mathcal{W}[H_{0,2,r}\tilde{P}_r](\underline{z}) &= \frac{1}{2\pi}\exp\left(-\frac{|\underline{z}|^2}{2}\right)\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\sum_{j=0}^{\ell}\binom{\ell}{j} \\
&\quad \times \left\langle -(\partial_1+\partial_2)P_{r-1}G, \xi_1^j R_1^j z_1^j \xi_2^{\ell-j} R_2^{\ell-j} z_2^{\ell-j}[1]\right\rangle \\
&= \frac{1}{2\pi}\exp\left(-\frac{|\underline{z}|^2}{2}\right)\sum_{\ell=0}^{\infty}\frac{1}{\ell!} \\
&\quad \times \sum_{j=0}^{\ell}\binom{\ell}{j}\left\langle P_{r-1}G, \xi_1^j R_1^j z_1^j \xi_2^{\ell-j} R_2^{\ell-j} z_2^{\ell-j}[1](\partial_1^\dagger+\partial_2^\dagger)\right\rangle.
\end{aligned}$$

Split up $\partial_1^\dagger+\partial_2^\dagger$ in two separate terms, for each of them we apply lemma 4.4.

$$\begin{aligned}
&= \frac{1}{2\pi}\exp\left(-\frac{|\underline{z}|^2}{2}\right)\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\left[\sum_{j=1}^{\ell}\binom{\ell}{j} j\left\langle P_{r-1}G, e_1^j\xi_1^{j-1}z_1^j\xi_2^{\ell-j}R_2^{\ell-j}z_2^{\ell-j}[1]\right\rangle\right. \\
&\quad \left. + \sum_{j=0}^{\ell-1}\binom{\ell}{j}(\ell-j)\left\langle P_{r-1}G, \xi_1^j R_1^j z_1^j e_2^{\ell-j}\xi_2^{\ell-j-1}z_2^{\ell-j}[1]\right\rangle\right] \\
&= \frac{1}{2\pi}\exp\left(-\frac{|\underline{z}|^2}{2}\right)\sum_{l=1}^{\infty}\frac{1}{\ell!} \\
&\quad \times\left[\sum_{j=0}^{\ell-1}\binom{\ell}{j+1}(j+1)z_1e_1\left\langle P_{r-1}G, \xi_1^j R_1^j z_1^j\xi_2^{\ell-j-1}R_2^{\ell-j-1}z_2^{\ell-j-1}[1]\right\rangle\right. \\
&\quad \left. +\sum_{j=0}^{\ell-1}\binom{\ell}{j}(\ell-j)z_2e_2\left\langle P_{r-1}G, \xi_1^j R_1^j z_1^j\xi_2^{\ell-j-1}R_2^{\ell-j-1}z_2^{\ell-j-1}[1]\right\rangle\right] \\
&= \frac{1}{2\pi}\exp\left(-\frac{|\underline{z}|^2}{2}\right)\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\sum_{j=0}^{\ell}\binom{\ell}{j}(z_1e_1+z_2e_2) \\
&\quad \times\left\langle P_{r-1}G, (\xi_1R_1z_1+\xi_2R_2z_2)^l[1]\right\rangle \\
&= (z_1e_1+z_2e_2)\,\mathcal{W}[H_{0,2,r-1}P_{r-1}](\underline{z}).
\end{aligned}$$

Similarly, it is easily checked that, with the new notation, $\tilde{\partial}\tilde{P}_r = 0$. It then holds that

$$\tilde{\partial}\tilde{P}_r G = -(\xi_2-\xi_1)\tilde{P}_r G = -P_{r+1}G.$$

The proof of the first equality is completely analogous as the proof for the equality $\partial P_r G = -\xi P_r G$, which can be found in [31]. We obtain

$$\mathcal{W}[H_{0,2,r}P_r](\underline{z}) = \frac{1}{2\pi}\exp\left(-\frac{|\underline{z}|^2}{2}\right)\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\sum_{j=0}^{\ell}\binom{\ell}{j}\left\langle -(\partial_2-\partial_1)\tilde{P}_{r-1}G, \xi_1^j R_1^j z_1^j \xi_2^{\ell-j} R_2^{\ell-j} z_2^{\ell-j}[1]\right\rangle$$
$$= \frac{1}{2\pi}\exp\left(-\frac{|\underline{z}|^2}{2}\right)\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\sum_{j=0}^{\ell}\binom{\ell}{j}\left\langle \tilde{P}_{r-1}G, \xi_1^j R_1^j z_1^j \xi_2^{\ell-j} R_2^{\ell-j} z_2^{\ell-j}[1](\partial_1^\dagger-\partial_2^\dagger)\right\rangle.$$

Split up and use lemma 4.4 once more:

$$= \frac{1}{2\pi}\exp\left(-\frac{|\underline{z}|^2}{2}\right)\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\left[\sum_{j=1}^{\ell}\binom{\ell}{j} j\left\langle \tilde{P}_{r-1}G, e_1^j \xi_1^{j-1} z_1^j \xi_2^{\ell-j} R_2^{\ell-j} z_2^{\ell-j}[1]\right\rangle\right.$$
$$\left. - \sum_{j=0}^{\ell-1}\binom{\ell}{j}(\ell-j)\left\langle \tilde{P}_{r-1}G, \xi_1^j R_1^j z_1^j e_2^{\ell-j} \xi_2^{\ell-j-1} z_2^{\ell-j}[1]\right\rangle\right]$$
$$= \frac{1}{2\pi}\exp\left(-\frac{|\underline{z}|^2}{2}\right)\sum_{l=1}^{\infty}\frac{1}{\ell!}$$
$$\left[\sum_{j=1}^{\ell}\binom{\ell}{j} j z_1 e_1\left\langle \tilde{P}_{r-1}G, \xi_1^j R_1^j z_1^j \xi_2^{\ell-j-1} R_2^{\ell-j-1} z_2^{\ell-j-1}[1]\right\rangle\right.$$
$$\left. - \sum_{j=0}^{\ell-1}\binom{\ell}{j}(\ell-j) z_2 e_2\left\langle \tilde{P}_{r-1}G, \xi_1^j R_1^j z_1^j \xi_2^{\ell-j-1} R_2^{\ell-j-1} z_2^{\ell-j-1}[1]\right\rangle\right]$$
$$= \frac{1}{2\pi}\exp\left(-\frac{|\underline{z}|^2}{2}\right)\sum_{l=1}^{\infty}\frac{1}{\ell!}\sum_{j=0}^{\ell-1}\binom{\ell}{j}(z_1e_1 - z_2e_2)$$
$$\times\left\langle \tilde{P}_{r-1}G, \xi_1^j R_1^j z_1^j \xi_2^{\ell-j-1} R_2^{\ell-j-1} z_2^{\ell-j-1}[1]\right\rangle$$
$$= (z_1e_1 - z_2e_2)\,\mathcal{W}[H_{0,2,r-1}\tilde{P}_{r-1}](\underline{z}). \tag{4.18}$$

We conclude the results of the previous calculations in the next theorem:

Theorem 4.15. For the discrete Weierstrass transform in dimension 2, it holds that

$$\mathcal{W}[H_{0,2,r}\tilde{P}_r](\underline{z}) = (z_1e_1 + z_2e_2)\ \mathcal{W}[H_{0,2,r-1}P_{r-1}(\underline{z})],$$
$$\mathcal{W}[H_{0,2,r}P_r](\underline{z}) = (z_1e_1 - z_2e_2)\ \mathcal{W}[H_{0,2,r-1}\tilde{P}_{r-1}](\underline{z}).$$

Combining the recurrence relations in theorems 4.7, 4.15 and the trivial example in (4.14), we can calculate the Weierstrass transform of every generalised Hermite polynomial in two dimensions. We made an overview for low values of r and n in Tables 7.1, complemented by Table 7.2.

The previous method in section 4.2.1 is unfortunately not directly extendable to higher dimensions, because it relies on the specific representation of the basic monogenics P_r for

$m = 2$. However, it gives us a good idea of the results when $m > 2$: the Weierstrass transform of a generalised Hermite polynomial $H_{n,m,r}P_r$ is the product of n factors $\left(\sum_{j=1}^{m} z_j e_j\right)$ and r factors $\left(\sum_{\ell=1}^{m} \pm z_\ell e_\ell\right)$, with $\pm$ depending on the form of the monogenic. To prove this result in higher dimensions, we need another approach. Before we do so, let us further illustrate the structure of the Weierstrass transform for some concrete examples of monogenics for $m > 2$.

4.2.2 Examples of the Weierstrass transform of spherical monogenics

In the next section, we will give some examples of the Weierstrass transform of $\mathrm{CK}[\xi^{\underline{\alpha}}]$, for low values of $\|\underline{\alpha}\| = r$. This can be interpreted as the Weierstrass transform $\mathcal{W}[H_{0,m,r}\mathrm{CK}[\xi^{\underline{\alpha}}]]$, as any Hermite polynomial of degree 0 is 1. This will give us some inspiration concerning the form of the results and how to calculate them. In order to aid to clarify the structure, let us introduce another notation:

Notation 4.16. Denote

$$\begin{aligned} y_i &= z_i - z_1 \\ \hat{y}_i &= z_i + z_1. \end{aligned}$$

4.2.2.1 $\|\underline{\alpha}\| = 1$

Here is $\mathrm{CK}[\xi^{\underline{\alpha}}] = \eta_j$, when $\underline{\ell} = (j)$. Hence $P_1 = \xi_j - \xi_1$. Having in mind that $\xi G = -\partial G$, this leads us to the calculation of example 4.9:

$$\begin{aligned} \mathcal{W}[\eta_j](\underline{z}) &= \mathcal{W}[\xi_j - \xi_1](\underline{z}) \\ &= \sqrt{2\pi}^{-\frac{m}{2}} \exp\left(\frac{-|\underline{z}|^2}{2}\right) \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left\langle -\left(\partial_j - \partial_1\right) G, \left(\xi R\underline{z}\right)^{\ell} [1] \right\rangle \\ &= e_j z_j - e_1 z_1 = y_j. \end{aligned}$$

4.2.2.2 $\|\underline{\alpha}\| = 2$

Two combinations are possible $(i, j \neq 1, i \neq j)$:

- $\underline{\ell} = (i, j)$: In this case,

$$\mathrm{CK}[\xi^{\underline{\alpha}}] = \frac{1}{2}\left(\eta_i\eta_j - \eta_j\eta_i\right) = \xi_i\xi_j - \xi_1\xi_j + \xi_1\xi_i.$$

  This will result in the Weierstrass transform $\frac{1}{2}\left(y_i y_j - y_j y_i\right)$. For example, take $m = 3$ and $\ell = (2, 3)$.

$$\begin{aligned} &\mathcal{W}[\xi_2\xi_3 - \xi_1\xi_3 + \xi_1\xi_2] \\ &\quad= \sqrt{2\pi}^{-\frac{3}{2}} \exp\left(\frac{-|\underline{z}|^2}{2}\right) \Big[\left\langle \xi_2\xi_3 G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle - \left\langle \xi_1\xi_3 G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle \\ &\qquad + \left\langle \xi_1\xi_2 G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle\Big]. \end{aligned}$$

The three terms in the right hand side are completely similar, so we will only work out the first term.

$$\left\langle \xi_2\xi_3 G, \exp\left(\xi R \underline{z}\right)[1]\right\rangle$$

$$= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{j_1+j_2+j_3=\ell} \binom{\ell}{j_1 j_2 j_3} \left\langle \xi_2\xi_3 G, \xi_1^{j_1} R_1^{j_1} z_1^{j_1} \xi_2^{j_2} R_2^{j_2} z_2^{j_2} \xi_3^{j_3} R_3^{j_3} z_3^{j_3}[1]\right\rangle$$

$$= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{j_1+j_2+j_3=\ell} \binom{\ell}{j_1 j_2 j_3} \left\langle \partial_2\partial_3 G, \xi_1^{j_1} R_1^{j_1} z_1^{j_1} \xi_2^{j_2} R_2^{j_2} z_2^{j_2} \xi_3^{j_3} R_3^{j_3} z_3^{j_3}[1]\right\rangle$$

$$= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{j_1+j_2+j_3=\ell} \binom{\ell}{j_1 j_2 j_3} \left\langle G, \xi_1^{j_1} R_1^{j_1} z_1^{j_1} \xi_2^{j_2} R_2^{j_2} z_2^{j_2} \xi_3^{j_3} R_3^{j_3} z_3^{j_3}[1]\partial_2^{\dagger}\partial_3^{\dagger}\right\rangle.$$

Use lemma 4.4 to bring $\partial_2^{\dagger}\partial_3^{\dagger}$ forward.

$$= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{j_1+j_2+j_3=\ell} \binom{\ell}{j_1 j_2 j_3} \left\langle G, \xi_1^{j_1} R_1^{j_1} z_1^{j_1} e_2^{j_2} \partial_2 \xi_2^{j_2} z_2^{j_2} e_3^{j_3} \partial_3 \xi_3^{j_3} z_3^{j_3}[1]\right\rangle$$

$$= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{j_1+j_2+j_3=\ell} \binom{\ell}{j_1 j_2 j_3} j_2 j_3 \left\langle G, \xi_1^{j_1} R_1^{j_1} z_1^{j_1} e_2^{j_2} \xi_2^{j_2-1} z_2^{j_2} e_3^{j_3} \xi_3^{j_3-1} z_3^{j_3}[1]\right\rangle$$

$$= \sum_{\ell=0}^{\infty} \frac{1}{(2\ell+2)!} \sum_{j_1+j_2+j_3=\ell} \frac{(2\ell+2)!}{(2j_1)!(2j_2+1)!(2j_3+1)!} (2j_2+1)(2j_3+1)$$
$$\times \left\langle G, \xi_1^{2j_1} R_1^{2j_1} z_1^{2j_1} e_2^{2j_2+1} \xi_2^{2j_2} z_2^{2j_2+1} e_3^{2j_3+1} \xi_3^{2j_3} z_3^{2j_3+1}[1]\right\rangle.$$

Now use the fact that $R_j^2 = e_j^2 = 1$.

$$= \sum_{\ell=0}^{\infty} \frac{1}{(2\ell+2)!} \sum_{j_1+j_2+j_3=\ell} \frac{(2\ell+2)!}{(2j_1)!(2j_2)!(2j_3)!}$$
$$\times z_1^{2j_1} z_2^{2j_2+1} z_3^{2j_3+1} e_2 e_3 \left\langle G, \xi_1^{2j_1} \xi_2^{2j_2} \xi_3^{2j_3}[1]\right\rangle$$

$$= \sum_{\ell=0}^{\infty} \sum_{j_1+j_2+j_3=\ell} \frac{1}{(2j_1)!(2j_2)!(2j_3)!} z_1^{2j_1} z_2^{2j_2+1} z_3^{2j_3+1} e_2 e_3 \frac{(2j_1)!}{2^{j_1} j_1!} \frac{(2j_2)!}{2^{j_2} j_2!} \frac{(2j_3)!}{2^{j_3} j_3!}$$

$$= \sum_{\ell=0}^{\infty} \sum_{j_1+j_2+j_3=\ell} \left(\frac{z_1^2}{2}\right)_1^j \frac{1}{j_1!} \left(\frac{z_2^2}{2}\right)_2^j \frac{1}{j_2!} \left(\frac{z_3^2}{2}\right)_3^j \frac{1}{j_3!} z_2 e_2 z_3 e_3$$

$$= \exp\left(\frac{|\underline{z}|^2}{2}\right) z_2 e_2 z_3 e_3.$$

The result of the three terms together will then be

$$\mathcal{W}\left[\frac{1}{2}\left(\eta_2\eta_3 - \eta_3\eta_2\right)\right] = z_2 e_2 z_3 e_3 - z_1 e_1 z_3 e_3 + z_1 e_1 z_2 e_2 = \frac{1}{2}\left(y_2 y_3 - y_3 y_2\right).$$

- $\underline{\ell} = (j, j)$: Here,

$$\mathrm{CK}[\xi^{\underline{\alpha}}] = \eta_j \hat{\eta}_j = \xi_j^2 - 2\xi_1\xi_j - \xi_1^2,$$

which will give $y_j\hat{y}_j$ as a result. Let us verify this for $m = 3$ and $\ell = (2,2)$.

$$\begin{aligned}
&\mathcal{W}[\xi_2^2 - 2\xi_1\xi_2 - \xi_1^2] \\
&\quad = \sqrt{2\pi}^{-\frac{3}{2}} \exp\left(\frac{-|\underline{z}|^2}{2}\right) \\
&\qquad \times \left[\left\langle \xi_2^2 G, \exp(\xi R\underline{z})[1]\right\rangle - 2\left\langle \xi_1\xi_2 G, \exp(\xi R\underline{z})[1]\right\rangle - \left\langle \xi_1^2 G, \exp(\xi R\underline{z})[1]\right\rangle\right].
\end{aligned}$$

Using the result of the previous calculation, we only need to know $\left\langle \xi_2^2 G, \exp(\xi R\underline{z})[1]\right\rangle$. Remark that

$$\xi_j^2 G = -\xi_j\partial_j G = -(\partial_j\xi_j - 1)G = (\partial_j^2 + 1)G,$$

because $\xi_j G = -\partial_j G$ and $\xi_j\partial_j = \partial_j\xi_j - 1$.

$$\begin{aligned}
&\left\langle \xi_2^2 G, \exp(\xi R\underline{z})[1]\right\rangle \\
&\quad = \sum_{\ell=0}^{\infty}\frac{1}{\ell!}\sum_{j_1+j_2+j_3=\ell}\binom{\ell}{j_1 j_2 j_3}\left\langle \xi_2^2 G, \xi_1^{j_1}R_1^{j_1}z_1^{j_1}\xi_2^{j_2}R_2^{j_2}z_2^{j_2}\xi_3^{j_3}R_3^{j_3}z_3^{j_3}[1]\right\rangle \\
&\quad = \sum_{\ell=0}^{\infty}\frac{1}{\ell!}\sum_{j_1+j_2+j_3=\ell}\binom{\ell}{j_1 j_2 j_3}\left\langle (\partial_2^2+1)G, \xi_1^{j_1}R_1^{j_1}z_1^{j_1}\xi_2^{j_2}R_2^{j_2}z_2^{j_2}\xi_3^{j_3}R_3^{j_3}z_3^{j_3}[1]\right\rangle \\
&\quad = \sum_{\ell=0}^{\infty}\frac{1}{\ell!}\sum_{j_1+j_2+j_3=\ell}\binom{\ell}{j_1 j_2 j_3}\left\langle G, \xi_1^{j_1}R_1^{j_1}z_1^{j_1}\xi_2^{j_2}R_2^{j_2}z_2^{j_2}\xi_3^{j_3}R_3^{j_3}z_3^{j_3}[1](\partial_2^{\dagger 2}+1)\right\rangle.
\end{aligned}$$

∂_j^2 is scalar, hence commutative with any other operator

$$\begin{aligned}
&= \sum_{\ell=0}^{\infty}\frac{1}{\ell!}\sum_{j_1+j_2+j_3=\ell}\binom{\ell}{j_1 j_2 j_3}\left\langle G, \xi_1^{j_1}R_1^{j_1}z_1^{j_1}(\partial_2^2+1)\xi_2^{j_2}R_2^{j_2}z_2^{j_2}\xi_3^{j_3}R_3^{j_3}z_3^{j_3}[1]\right\rangle \\
&= \sum_{\ell=0}^{\infty}\frac{1}{\ell!}\sum_{j_1+j_2+j_3=\ell}\binom{\ell}{j_1 j_2 j_3}\Big(j_2(j_2-1) \\
&\qquad \times \left\langle G, \xi_1^{j_1}R_1^{j_1}z_1^{j_1}\xi_2^{j_2-2}R_2^{j_2}z_2^{j_2}\xi_3^{j_3}R_3^{j_3}z_3^{j_3}[1]\right\rangle \\
&\qquad\quad + \left\langle G, \xi_1^{j_1}R_1^{j_1}z_1^{j_1}\xi_2^{j_2}R_2^{j_2}z_2^{j_2}\xi_3^{j_3}R_3^{j_3}z_3^{j_3}[1]\right\rangle\Big).
\end{aligned}$$

The action of G is only non-zero when all of the powers of ξ_j are even, thus

$$\begin{aligned}
&= \sum_{\ell=0}^{\infty}\frac{1}{(2\ell)!}\sum_{j_1+j_2+j_3=\ell}\frac{(2\ell)!}{(2j_1)!(2j_2)!(2j_3)!}\Big(2j_2(2j_2-1) \\
&\qquad \times \left\langle G, \xi_1^{2j_1}R_1^{2j_1}z_1^{2j_1}\xi_2^{2j_2-2}R_2^{2j_2}z_2^{2j_2}\xi_3^{2j_3}R_3^{2j_3}z_3^{2j_3}[1]\right\rangle \\
&\qquad\quad + \left\langle G, \xi_1^{2j_1}R_1^{2j_1}z_1^{2j_1}\xi_2^{2j_2}R_2^{2j_2}z_2^{2j_2}\xi_3^{2j_3}R_3^{2j_3}z_3^{2j_3}[1]\right\rangle\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\ell=0}^{\infty} \sum_{j_1+j_2+j_3=\ell} \frac{1}{(2j_1)!(2j_2-2)!(2j_3)!} \\
&\quad \times \left(\left\langle G, \xi_1^{2j_1} R_1^{2j_1} z_1^{2j_1} \xi_2^{2j_2-2} R_2^{2j_2} z_2^{2j_2} \xi_3^{2j_3} R_3^{2j_3} z_3^{2j_3}[1] \right\rangle \right. \\
&\qquad \left. + \left\langle G, \xi_1^{2j_1} R_1^{2j_1} z_1^{2j_1} \xi_2^{2j_2} R_2^{2j_2} z_2^{2j_2} \xi_3^{2j_3} R_3^{2j_3} z_3^{2j_3}[1] \right\rangle \right).
\end{aligned}
$$

Now re-arrange so we can write it in closed form again.

$$
\begin{aligned}
&= \sum_{\ell=0}^{\infty} \sum_{j_1+j_2+j_3=\ell} \frac{1}{(2j_1)!(2j_2-2)!(2j_3)!} \\
&\quad \times \left(z_2^2 \left\langle G, \xi_1^{2j_1} z_1^{2j_1} \xi_2^{2j_2-2} z_2^{2j_2-2} \xi_3^{2j_3} z_3^{2j_3}[1] \right\rangle \right. \\
&\qquad \left. + \left\langle G, \xi_1^{2j_1} z_1^{2j_1} \xi_2^{2j_2} z_2^{2j_2} \xi_3^{2j_3} z_3^{2j_3}[1] \right\rangle \right) \\
&= z_2^2 + 1 = (z_2 e_2)^2 + 1.
\end{aligned}
$$

As a result,

$$
\begin{aligned}
\mathcal{W}[\eta_2 \hat{\eta}_2] &= \mathcal{W}[\xi_2^2 - 2\xi_1\xi_2 - \xi_1^2] \\
&= z_2^2 + 1 - 2z_1 e_1 z_2 e_2 - z_1^2 - 1 \\
&= z_2^2 - 2z_1 e_1 z_2 e_3 - z_1^2 \\
&= y_2 \hat{y}_2.
\end{aligned}
$$

4.2.2.3 $\|\underline{\alpha}\| = 3$

Three combinations are possible $(i, j \neq 1, i \neq j)$

- $\ell = (i, j, k)$: here we have that

  $$\mathrm{CK}[\xi^{\underline{\alpha}}] = \frac{1}{3!} \left(\eta_i\eta_j\eta_k - \eta_j\eta_i\eta_k + \eta_j\eta_k\eta_i - \eta_k\eta_j\eta_i - \eta_i\eta_k\eta_j + \eta_k\eta_i\eta_j \right).$$

  Using the same reasoning as in the previous examples, this is transformed into

  $$\frac{1}{3!} \left(y_i y_j y_k - y_j y_i y_k + y_j y_k y_i - y_k y_j y_i - y_i y_k y_j + y_k y_i y_j \right).$$

- $\ell = (i, j, j)$ or $\ell = (i, i, j)$: The second case splits up in two sub cases, however both are identically calculated as a combination of the two options when $\|\alpha\| = 2$.

  $$\mathcal{W}\left[\mathrm{CK}[\xi^{\underline{\alpha}}]\right] = \frac{1}{3} \mathcal{W}\left[\eta_i\hat{\eta}_i\eta_j \;\; - \eta_i\eta_j\hat{\eta}_i + \eta_j\eta_i\hat{\eta}_i\right] = y_i\hat{y}_i y_j \;\; - y_i y_j \hat{y}_i + y_j y_i \hat{y}_i.$$

  Therefore, we need a combination of previous calculations.

- $\ell = (j, j, j)$: Finally,

  $$\mathrm{CK}[\xi_j^3] = \eta_j\hat{\eta}_j\eta_j = \xi_j^3 + \xi_1^3 - 3\xi_1\xi_j^2 - 3\xi_1^2\xi_j$$

becomes

$$\begin{aligned}
&z_j^3 e_j + 3z_3 e_3 + z_1^3 e_1 + 3z_1 e_1 - 3(z_1 e_1 (z_j^2 + 1)) - 3((z_1^2 + 1) z_j e_j) \\
&\qquad = z_j^3 e_j + z_1^3 e_1 - 3z_1 e_1 z_j^2 - 3z_1^2 z_j e_j \\
&\qquad = \frac{1}{3}\left(y_j \hat{y}_j y_j\right).
\end{aligned}$$

On the basis of these examples, the structure of the Weierstrass transform of the basic monogenic polynomials $\mathrm{CK}[\xi^{\underline{\alpha}}]$ is clear: every factor η_j or $\hat{\eta}_j$ translates into y_j or $\hat{y}_j$, in the same order:

$$\mathcal{W}\left[\sum_{\pi(\ell_1,\ldots\ell_k)} \mathrm{sgn}(\pi)(\eta_{\pi(\ell_1)}\ldots\eta_{\pi(\ell_k)})^{E_{2,\ldots,m}}\right](\underline{z}) = \sum_{\pi(\ell_1,\ldots\ell_k)} \mathrm{sgn}(\pi)(y_{\pi(\ell_1)}\ldots y_{\pi(\ell_k)})^{E_{2,\ldots,m}}.$$

Indeed, anticipating on proposition 4.17, every power of ξ_j, acting on G, corresponds to a polynomial of the same degree in ∂_j, acting on G. This polynomial will result in the same polynomial in $z_j e_j$.

In the next section, we will try to prove this structure for $\mathcal{W}[\mathrm{CK}[\xi^{\underline{\alpha}}]]$. Therefore, we accessed three different approaches.

4.2.3 Explicit expression for the Weierstrass transform of spherical monogenics

Having a clear idea about the expected result, we now still need to prove it. Several options were investigated, with the idea to obtain a recurrence formula for the degree of the monogenic. In the next subsections, we will discuss three different strategies.

4.2.3.1 First attempt

The discrete Hermite polynomial of degree n is a polynomial in ξ, defined by as the $n-$th derivative of the Gaussian polynomial. One can of course also reverse this relationship: the action of the $n-$th power of ξ on G is a polynomial in ∂, acting on G. We claim that the coefficients of this polynomial in ∂ are the same as the coefficients for the Hermite polynomials given in (4.6), up to signs.

Proposition 4.17. The action of natural powers of ξ on the discrete Gauss distribution is as follows:

$$\xi^{2l}G = \sum_{j=0}^{\ell} b_{2j}^{2\ell} \partial^{2j} G,$$

$$\xi^{2l+1}G = -\sum_{j=0}^{\ell} b_{2j+1}^{2\ell+1} \partial^{2j+1} G,$$

with

$$b_{2j}^{2\ell} = 2^{\ell-j} \binom{\ell}{j} \frac{\Gamma(\ell + \frac{m}{2})}{\Gamma(j + \frac{m}{2})},$$

$$b_{2j+1}^{2\ell+1} = 2^{\ell-j} \binom{\ell}{j} \frac{\Gamma(\ell + \frac{m}{2} + 1)}{\Gamma(j + \frac{m}{2} + 1)}.$$

To prove this proposition, we first look for some recurrence relations on the coefficients b_j^n in the following lemmata.

Lemma 4.18.

$$b_{2j+1}^{2\ell+1} = (2j+2) b_{2j+2}^{2\ell} + b_{2j}^{2\ell}, \tag{4.19}$$

$$b_{2j}^{2\ell} = (2j+m) b_{2j+1}^{2\ell-1} + b_{2j-1}^{2\ell-1}. \tag{4.20}$$

Proof. For the odd coefficients,

$$\begin{aligned}\xi^{2l+1}G &= \xi \sum_{j=0}^{\ell} b_{2j}^{2\ell} \partial^{2j} G \\ &= \sum_{j=0}^{\ell} b_{2j}^{2\ell} \left(-2j\partial^{2j-1} - \partial^{2j+1}\right) G,\end{aligned}$$

so

$$-\sum_{j=0}^{\ell} b_{2j+1}^{2\ell+1} \partial^{2j+1} G = \sum_{j=0}^{\ell} b_{2j}^{2\ell} \left((-2j\partial^{2j-1} - \partial^{2j+1}\right) G,$$

hence

$$b^{2\ell+1}_{2j+1} = b^{2\ell}_{2j} + 2(j+1) b^{2\ell}_{2j+2}.$$

For the even coefficients,

$$\begin{aligned} \xi^{2l} G &= -\xi \sum_{j=0}^{\ell-1} b^{2\ell-1}_{2j+1} \partial^{2j+1} G \\ &= -\sum_{j=0}^{\ell-1} b^{2\ell-1}_{2j+1} \left((-2j+m)\partial^{2j} + 2\partial^{2j}\mathbb{E} - \partial^{2j+1}\xi \right) G \\ &= -\sum_{j=0}^{\ell-1} b^{2\ell-1}_{2j+1} \left((-2j+m)\partial^{2j} - 2\partial^{2j}(\partial^2+m) + \partial^{2j+2} \right) G. \end{aligned}$$

So

$$\sum_{j=0}^{\ell} b^{2\ell}_{2j} \partial^{2j} G = \sum_{j=0}^{\ell-1} b^{2\ell-1}_{2j+1} \left((2j+m)\partial^{2j} + \partial^{2j+2} \right) G.$$

□

We now use the previous relations (4.19) and (4.20) in order to proof a first order recurrence relation for $b^{2\ell+1}_{2j+1}$ and $b^{2\ell}_{2j}$.

Lemma 4.19.

$$(2j+m)\, b^{2\ell+1}_{2j+1} = (2\ell+m)\, b^{2\ell}_{2j}, \tag{4.21}$$

$$2j\, b^{2\ell}_{2j} = 2\ell\, b^{2\ell-1}_{2j-1}. \tag{4.22}$$

Proof. The proof is again by induction on ℓ. The statements are true for $\ell = 0$, as $b^j_j = 1, \forall j$ and $\xi G = -\partial G$. For $\ell = 1$, we rely on the second order Hermite polynomial $H_{2,m}(\xi) = -\xi^2 + m$, for which we know that $H_{2,m} G = -\partial^2 G$, hence $\xi^2 G = (\partial^2 + m) G$. Further,

$$\begin{aligned} \xi^3 G &= \xi(\partial^2+m)G \\ &= (2\mathbb{E} + m - \partial\xi)\partial G - m\partial G \\ &= 2(\partial\mathbb{E} - \partial)G + m\partial G - \partial(2\partial + m - \partial\xi)G - m\partial G \\ &= -2\partial G + \partial^2\xi G - m\partial G \\ &= -(m+2)\partial G - \partial^3 G, \end{aligned}$$

both are in accordance to $b^2_0 = m$ and $b^3_1 = m+2$. For larger ℓ:

$$\begin{aligned} (2j+m) b^{2\ell+1}_{2j+1} &\overset{(4.19)}{=} (2j+m)\left[(2j+2) b^{2\ell}_{2j+2} + b^{2\ell}_{2j} \right] \\ &\overset{(4.22)}{=} (2j+m)\left[2\ell b^{2\ell-1}_{2j+1} + b^{2\ell}_{2j} \right] \\ &\overset{(4.20)}{=} 2\ell \left[b^{2\ell}_{2j} - b^{2\ell-1}_{2j-1} \right] + (2j+m) b^{2\ell}_{2j} \\ &= 2j\, b^{2\ell}_{2j} - 2\ell\, b^{2\ell-1}_{2j-1} + (2\ell+m) b^{2\ell}_{2j} \end{aligned}$$

$$\overset{(4.22)}{=} 0 + (2\ell + m) b_{2j}^{2\ell} = (2\ell + m) b_{2j}^{2\ell}.$$

and

$$\begin{aligned}
2j\, b_{2j}^{2\ell} &\overset{(4.20)}{=} 2j \left[(2j+m) b_{2j+1}^{2\ell-1} + b_{2j-1}^{2\ell-1} \right] \\
&\overset{(4.21)}{=} (2\ell - 2 + m) 2j\, b_{2j}^{2\ell-2} + 2j\, b_{2j-1}^{2\ell-1} \\
&\overset{(4.19)}{=} (2\ell - 2 + m) \left[-b_{2j-2}^{2\ell-2} + b_{2j-1}^{2\ell-1} \right] + 2j\, b_{2j-1}^{2\ell-1} \\
&= -(2\ell - 2 + m) b_{2j-2}^{2\ell-2} + 2\ell\, b_{2j-1}^{2\ell-1} + (2j - 2 + m) b_{2j-1}^{2\ell-1} \\
&\overset{(4.21)}{=} (2\ell - 2 + m) b_{2j-2}^{2\ell-2} + 2\ell\, b_{2j-1}^{2\ell-1} + (2\ell - 2 + m) b_{2j-2}^{2\ell-2} \\
&= 2\ell\, b_{2j-1}^{2\ell-1}.
\end{aligned}$$

□

We now use the above recurrence relations to prove Proposition 4.17.

Proof.

$$\begin{aligned}
b_{2j}^{2\ell} &\overset{(4.22)}{=} \frac{l}{j} b_{2j-1}^{2\ell-1} \\
&\overset{(4.21)}{=} \frac{l}{j} \frac{\big(2(\ell-1)+m\big)}{\big(2(j-1)+m\big)} b_{2j-2}^{2\ell-2} \\
&= \frac{\ell(\ell-1)}{j(j-1)} \frac{\big(2(\ell-1)+m\big)\big(2(\ell-2)+m\big)}{\big(2(j-1)+m\big)\big(2(j-2)+m\big)} b_{2j-4}^{2\ell-4}
\end{aligned}$$

which eventually results in

$$b_{2j}^{2\ell} = \binom{\ell}{j} \frac{\Gamma(\ell + \frac{m}{2})}{\Gamma(\ell - j + \frac{m}{2})} \frac{\Gamma(\frac{m}{2})}{\Gamma(j + \frac{m}{2})} b_0^{2\ell-2j}. \tag{4.23}$$

Now consider b_0^{2k} separately and obtain the following result:

$$\begin{aligned}
b_0^{2k} &\overset{(4.20)}{=} m b_1^{2k-1} \\
&\overset{(4.21)}{=} m \frac{\big(2(k-1)+m\big)}{m} b_0^{2k-2} = 2\left(k - 1 + \frac{m}{2}\right) b_0^{2k-2}
\end{aligned}$$

Repeatedly applying the same relations gives us

$$\begin{aligned}
b_0^{2k} &= 2^k \left(k - 1 + \frac{m}{2}\right) \cdots \left(k + \frac{m}{2}\right) b_0^0 \\
&= 2^k \frac{\Gamma(k + \frac{m}{2})}{\Gamma(\frac{m}{2})}.
\end{aligned} \tag{4.24}$$

Combining (4.23) and (4.24), we arrive at the expression for the even coefficient

$$b_{2j}^{2\ell} = 2^{\ell-j} \binom{\ell}{j} \frac{\Gamma(\ell + \frac{m}{2})}{\Gamma(j + \frac{m}{2})}.$$

The expression for the odd coefficient follows from (4.21):

$$b_{2j+1}^{2\ell+1} = \frac{(2\ell+m)}{(2j+m)}\, b_{2j}^{2\ell} = 2^{\ell-j}\binom{\ell}{j}\frac{\Gamma(\ell+\frac{m}{2}+1)}{\Gamma(j+\frac{m}{2}+1)}.$$

□

Recall the Weierstrass transform in one dimension of the discrete basic polynomials $\xi^n[1]$, from section 3.2.2.1:

$$\mathcal{W}\left[\xi^{2k}[1]\right](z) = (-1)^k H_{2k}(iz),$$
$$\mathcal{W}\left[\xi^{2k+1}[1]\right](z) = (-1)^{k+1} i H_{2k+1}(iz).$$

For low values of k, the results were given in table 3.1. The coefficients in the polynomials of $\mathcal{W}\left[\xi^n[1]\right](z)$ are exactly the values b_j^ℓ, so we can rewrite:

$$\mathcal{W}\left[\xi^{2k}[1]\right](z) = \sum_{j=0}^{n} b_{2j}^{2n} z^{2j},$$
$$\mathcal{W}\left[\xi^{2k+1}[1]\right](z) = \sum_{j=0}^{n} b_{2j+1}^{2n+1} z^{2j+1} e.$$

Using the proposition, we can easily calculate the Weierstrass transform of any polynomial of the form $\xi^{\underline{\alpha}} = \xi_1^{\alpha_1}\ldots\xi_m^{\alpha_m}$: the result holds for any of the $\xi_j^{\alpha_j}$ and using property (2.21) and lemma 4.4, we can transfer any ∂_j acting on G - in the left side of the inner product - to a derivative of ξ_j in the right side of the inner product. The final result of the Weierstrass transform will be the a product of factors of the form

$$\sum_{i=0}^{\ell} b_{2i}^{2\ell} z_j^{2i} e_j^{2i} = \sum_{i=0}^{\ell} b_{2i}^{2\ell} z_j^{2i}$$

and/or

$$-\sum_{i=0}^{\ell} b_{2i+1}^{2\ell+1} z_j^{2i+1} e_j^{2i+1} = -\sum_{i=0}^{\ell} b_{2i+1}^{2\ell+1} z_j^{2i+1} e_j,$$

according to an even or odd power of ξ_j, in the same order as they appeared in the original product. The examples in section 4.2.2 are already an illustration of this method. For completeness, let us calculate the Weierstrass transform of the product $\xi_1^{2k}\xi_2^{2n+1}$, a combination of an even and an odd power of ξ_j in dimension 3. This will be directly generalisable to other (more complex) combinations of different ξ_j.

$$\mathcal{W}[\xi_1^{2k}\xi_2^{2n+1}] = \frac{1}{\sqrt{2\pi}^{\frac{m}{2}}}\exp\left(\frac{-|\underline{z}|^2}{2}\right)\left[\left\langle \xi_1^{2k}\xi_2^{2n+1}G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle\right]$$

For shortness in notation, write $C = \sqrt{2\pi}^{-\frac{m}{2}}\exp\left(\frac{-|\underline{z}|^2}{2}\right)$

$$= C\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\sum_{j_1+j_2+j_3=\ell}\binom{\ell}{j_1 j_2 j_3}\left\langle \xi_1^{2k}\xi_2^{2n+1}G, \xi_1^{j_1}R_1^{j_1}z_1^{j_1}\xi_2^{j_2}R_2^{j_2}z_2^{j_2}\xi_3^{j_3}R_3^{j_3}z_3^{j_3}[1]\right\rangle$$

$$= C \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{j_1+j_2+j_3=\ell} \binom{\ell}{j_1 j_2 j_3} \sum_{i=0}^{k} b_{2i}^{2k} \sum_{r=0}^{n} b_{2r+1}^{2n+1}$$
$$\times \left\langle \partial_1^{2i} \partial_2^{2r+1} G, \xi_1^{j_1} R_1^{j_1} z_1^{j_1} \xi_2^{j_2} R_2^{j_2} z_2^{j_2} \xi_3^{j_3} R_3^{j_3} z_3^{j_3} [1] \right\rangle$$
$$= C \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{j_1+j_2+j_3=\ell} \binom{\ell}{j_1 j_2 j_3} \sum_{i=0}^{k} b_{2i}^{2k} \sum_{r=0}^{n} b_{2r+1}^{2n+1}$$
$$\times \left\langle G, \xi_1^{j_1} R_1^{j_1} z_1^{j_1} \xi_2^{j_2} R_2^{j_2} z_2^{j_2} \xi_3^{j_3} R_3^{j_3} z_3^{j_3} [1] (\partial_1^{\dagger})^{2i} (\partial_2^{\dagger})^{2r+1} \right\rangle$$
$$= C \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{j_1+j_2+j_3=\ell} \binom{\ell}{j_1 j_2 j_3} \sum_{i=0}^{k} b_{2i}^{2k} \sum_{r=0}^{n} b_{2r+1}^{2n+1}$$
$$\times \left\langle G, e_1^{j_1} \partial_1^{2i} \xi_1^{j_1} z_1^{j_1} \xi_2^{j_2} R_2^{j_2} z_2^{j_2} \xi_3^{j_3} R_3^{j_3} z_3^{j_3} [1] (\partial_2^{\dagger})^{2r+1} \right\rangle$$
$$= C \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{j_1+j_2+j_3=\ell} \binom{\ell}{j_1 j_2 j_3} \sum_{i=0}^{k} b_{2i}^{2k} \sum_{r=0}^{n} b_{2r+1}^{2n+1}$$
$$\times \left\langle G, e_1^{j_1} \partial_1^{2i} \xi_1^{j_1} z_1^{j_1} e_2^{j_2} \partial_2^{2r+1} \xi_2^{j_2} z_2^{j_2} \xi_3^{j_3} R_3^{j_3} z_3^{j_3} [1] \right\rangle .$$

Let the Dirac operators act on the corresponding vector variables, then recall again that G vanishes on odd powers of these vector variables:

$$= C \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{j_1+j_2+j_3=\ell} \binom{\ell}{j_1 j_2 j_3} \sum_{i=0}^{k} b_{2i}^{2k} \sum_{r=0}^{n} b_{2r+1}^{2n+1} \frac{j_1!}{(j_1-2i)!} \frac{j_2!}{(j_2-2r-1)!}$$
$$\times \left\langle G, e_1^{j_1} \xi_1^{j_1-2i} z_1^{j_1} e_2^{j_2} \xi_2^{j_2-2r-1} z_2^{j_2} \xi_3^{j_3} R_3^{j_3} z_3^{j_3} [1] \right\rangle$$
$$= C \sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)!} \sum_{j_1+j_2+j_3=\ell} \frac{(2\ell+1)!}{(2j_1)!(2j_2+1)!(2j_3)!}$$
$$\times \sum_{i=0}^{k} b_{2i}^{2k} \sum_{r=0}^{n} b_{2r+1}^{2n+1} \frac{(2j_1)!}{(2j_1-2i)!} \frac{(2j_2+1)!}{(2j_2-2r)!}$$
$$\times \left\langle G, e_1^{2j_1} \xi_1^{2j_1-2i} z_1^{2j_1} e_2^{2j_2+1} \xi_2^{2j_2-2r} z_2^{2j_2+1} \xi_3^{2j_3} R_3^{2j_3} z_3^{2j_3} [1] \right\rangle .$$

The Gaussian distribution acts on the vector variables, then simplify the expression and combine until the final result.

$$= C \sum_{\ell=0}^{\infty} \sum_{j_1+j_2+j_3=\ell} \sum_{i=0}^{k} b_{2i}^{2k} \sum_{r=0}^{n} b_{2r+1}^{2n+1} \frac{1}{(2j_1-2i)!} \frac{1}{(2j_2-2r)!} \frac{1}{(2j_3)!}$$
$$\times z_1^{2j_1} z_2^{2j_2+1} z_3^{2j_3} e_2 \left\langle G, \xi_1^{2j_1-2i} \xi_2^{2j_2-2r} \xi_3^{2j_3} [1] \right\rangle$$
$$= C \sum_{\ell=0}^{\infty} \sum_{j_1+j_2+j_3=\ell} \sum_{i=0}^{k} b_{2i}^{2k} \sum_{r=0}^{n} b_{2r+1}^{2n+1} \frac{1}{(2j_1-2i)!} \frac{1}{(2j_2-2r)!} \frac{1}{(2j_3)!}$$
$$\times z_1^{2j_1} z_2^{2j_2+1} z_3^{2j_3} e_2 \frac{(2j_1-2i)!}{2^{j_1-i}(j_1-i)!} \frac{(2j_2-2r)!}{2^{j_2-r}(j_2-r)!} \frac{(2j_3)!}{2^{j_3}(j_3)!}$$
$$= \frac{1}{\sqrt{2\pi}^{\frac{m}{2}}} \exp\left(\frac{-|\underline{z}|^2}{2}\right) \sum_{\ell=0}^{\infty} \sum_{j_1+j_2+j_3=\ell} \sum_{i=0}^{k} b_{2i}^{2k} \sum_{r=0}^{n} b_{2r+1}^{2n+1}$$

$$\times\, z_1^{2i} \frac{1}{(j_1-i)!}\left(\frac{z_1^2}{2}\right)^{j_1-i} z_2^{2r+1} \frac{1}{(j_2-r)!}\left(\frac{z_2^2}{2}\right)^{j_2-r} \frac{1}{j_3!}\left(\frac{z_3^2}{2}\right)^{j_3} e_2$$

$$= \frac{1}{\sqrt{2\pi}^{\frac{m}{2}}} \sum_{i=0}^{k} b_{2i}^{2k} z_1^{2i} \sum_{r=0}^{n} b_{2r+1}^{2n+1} z_2^{2r+1} e_2.$$

Recall that every spherical monogenic P_r is a linear combination of the set $\mathrm{CK}[\xi^{\underline{\alpha}}]$, each of them being of the form

$$\mathrm{CK}[\xi^{\underline{\alpha}}] = \frac{\alpha_2!\ldots\alpha_m!}{k!} \sum_{\pi(\ell_1,\ldots\ell_k)} \mathrm{sgn}(\pi)(\eta_{\pi(\ell_1)}\ldots\eta_{\pi(\ell_k)})^{E_{2,\ldots,m}},$$

i.e. basically a sum of products of η_j and $\hat{\eta}_j$. Due to the distributive property, this is a sum of elements of the form $\xi_1^{\alpha_1}\ldots\xi_m^{\alpha_m}$, for which we can now calculate its Weierstrass transform. The extra (lower order) terms in $z_j e_j$ arriving in the transform of ξ_j^n, evidently cancel out in the bigger picture of the spherical monogenic: this is what we see in the examples of section 4.2.2. Unfortunately, so far, we could not find an explanation for this observation, so with this method, we have also hit a dead end.

4.2.3.2 Second attempt

An important lemma on which we based our initial ideas comes from [34]:

Lemma 4.20. Denote

$$V_{\ell_1,\ldots,\ell_n} = \frac{1}{k!} \sum_{\pi(l_1,\ldots l_k)} \mathrm{sgn}(\pi)(\eta_{\pi(l_1)}\ldots\eta_{\pi(l_k)})^{E_{2,\ldots,m}}.$$

Let P_k be a discrete spherical monogenic of degree k. For every $1 \le n \le k$

$$P_k = \frac{n!(k-n)!}{k!} \sum_{(\ell_1,\ldots,\ell_n)} V_{\ell_1,\ldots,\ell_n} \partial_{\ell_n}\ldots\partial_{\ell_1} P_k,$$

with $\ell_j \in \{2,\ldots,m\}$ and where every subset $\{\ell_1,\ldots,\ell_n\}$ only appears once in the sum.

Remark 4.21. With the notation in the lemma, $\mathrm{CK}[\xi^{\underline{\alpha}}] = \alpha_2!\ldots\alpha_m! V_{\ell_1,\ldots,\ell_n}$.

By taking $n = 1$ and $P_k = \mathrm{CK}[\xi^{\underline{\alpha}}]$ with k the degree of $\underline{\alpha}$, this reduces to

$$P_k = \frac{1}{k}\sum_{j=2}^{m} \eta_j \partial_j P_k.$$

It should allow us to gradually take one η_j out of P_k and lower its degree in order to construct a recurrence relation. However unfortunately, CK and ∂_j are not commutative. More precisely, it was proven in [34]

Lemma 4.22. Let $\underline{\alpha} = (\alpha_2,\ldots,\alpha_m)$ and $\xi^{\underline{\alpha}} = \xi_2^{\alpha_2}\ldots\xi_m^{\alpha_m}$. For $j = 2,\ldots,m$, it holds that

$$\begin{aligned}
\mathrm{CK}[\partial_j^2 \xi^{\underline{\alpha}}] &= \partial_j^2 \mathrm{CK}[\xi^{\underline{\alpha}}],\\
\partial_j \mathrm{CK}[\xi^{\underline{\alpha}}] &= \left(\mathrm{CK}[\partial_j \xi^{\underline{\alpha}}]\right)^{*_j},\\
\mathrm{CK}[\partial_j \xi^{\underline{\alpha}}] &= \left(\partial_j \mathrm{CK}[\xi^{\underline{\alpha}}]\right)^{*_j}.
\end{aligned}$$

When calculating $\mathcal{W}\left[\mathrm{CK}[\xi^{\underline{\alpha}}]\right]$, we thus get

$$\begin{aligned}\exp\left(\frac{-|\underline{z}|^2}{2}\right)&\left\langle \mathrm{CK}[\xi^{\underline{\alpha}}]G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle \\ &= \exp\left(\frac{-|\underline{z}|^2}{2}\right)\frac{1}{k}\sum_{j=2}^{m}\left\langle \eta_j\partial_j \mathrm{CK}[\xi^{\underline{\alpha}}]G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle \\ &= \exp\left(\frac{-|\underline{z}|^2}{2}\right)\frac{1}{k}\sum_{j=2}^{m}\left\langle \eta_j \mathrm{CK}[\partial_j\xi^{\underline{\alpha}}]^{*_j}G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle .\end{aligned}$$

Hence the question arises how to deal with the $*_j$. Can its action be transferred to the right so that G acts on it? For some inspiration, let us have a look at the next two examples.

Example 4.23. Take $P_2 = V_{2,3} = \frac{1}{2}\left(\eta_2\eta_3 - \eta_3\eta_2\right)$. On the one side, we know that

$$\begin{aligned}2\left\langle P_2G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle &= \left\langle \eta_2\eta_3G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle - \left\langle \eta_3\eta_2G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle \\ &= \left\langle \eta_3G, \exp\left(\xi R\underline{z}\right)[1]\eta_2^\dagger\right\rangle - \left\langle \eta_2G, \exp\left(\xi R\underline{z}\right)[1]\eta_3^\dagger\right\rangle . \end{aligned} \tag{4.25}$$

On the other side, applying the lemmata above, we obtain

$$\begin{aligned}&2\left\langle P_2G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle \\ &\overset{4.20}{=} \left\langle \eta_2\partial_2\mathrm{CK}[\xi_2\xi_3]G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle + \left\langle \eta_3\partial_3\mathrm{CK}[\xi_2\xi_3]G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle \\ &\overset{4.22}{=} \left\langle \eta_2\left(\mathrm{CK}[\xi_3]\right)^{*_2}G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle + \left\langle \eta_3\left(\mathrm{CK}[-\xi_2]\right)^{*_3}G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle \\ &= \left\langle \left(\mathrm{CK}[\xi_3]\right)^{*_2}G, \exp\left(\xi R\underline{z}\right)[1]\eta_2^\dagger\right\rangle + \left\langle \left(\mathrm{CK}[-\xi_2]\right)^{*_3}G, \exp\left(\xi R\underline{z}\right)[1]\eta_3^\dagger\right\rangle\end{aligned}$$

Now $\mathrm{CK}[\xi_3] = \eta_3$, so the operation $*_2$ has no effect on this factor. Similarly, $\mathrm{CK}[\xi_2] = \eta_2$ which is not affected by $*_3$. Thus this equals:

$$= \left\langle \eta_3G, \exp\left(\xi R\underline{z}\right)[1]\eta_2^\dagger\right\rangle - \left\langle \eta_2G, \exp\left(\xi R\underline{z}\right)[1]\eta_3^\dagger\right\rangle . \tag{4.26}$$

Both calculations (4.25) and (4.26) are equal, without any transfer of $*_j$ to the right hand side of the brackets. It means that, in this example, $*_j$ should not be transferred.

Example 4.24. Take $P_2 = V_{2,2} = \eta_2\hat{\eta}_2 = \xi_2^2 - 2\xi_1\xi_2 - \xi_1^2$. On the one side, we know that

$$\begin{aligned}\left\langle P_2G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle &\overset{4.20}{=} \left\langle \eta_2\hat{\eta}_2G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle \\ &\overset{4.22}{=} \left\langle \hat{\eta}_2G, \exp\left(\xi R\underline{z}\right)[1]\eta_2^\dagger\right\rangle \\ &= \left\langle G, \exp\left(\xi R\underline{z}\right)[1]\eta_2^\dagger\hat{\eta}_2^\dagger\right\rangle .\end{aligned} \tag{4.27}$$

On the other side,

$$\begin{aligned}\left\langle P_2G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle &= \frac{1}{2}\left\langle \eta_2\partial_2\mathrm{CK}[\xi_2^2]G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle \\ &= \frac{1}{2}\left\langle \eta_2\left(\mathrm{CK}[2\xi_2]\right)^{*_2}G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left\langle \left(\mathrm{CK}[2\xi_2]\right)^{*2} G, \exp\left(\xi R\underline{z}\right)[1]\eta_2^{\dagger}\right\rangle \\
&= \left\langle \left(\eta_2\right)^{*2} G, \exp\left(\xi R\underline{z}\right)[1]\eta_2^{\dagger}\right\rangle .
\end{aligned}
$$

We transfer the $*_2$ from the left side of the brackets (acting on $\eta_2^{\dagger}$) to the right side (acting on $\hat{\eta}_2$), let us denote it by $?_2$:

$$
\begin{aligned}
&= \left\langle \eta_2 G, \exp\left(\xi R\underline{z}\right)[1]\left(\eta_2^{\dagger}\right)^{?_2}\right\rangle \\
&= \left\langle G, \exp\left(\xi R\underline{z}\right)[1]\left(\eta_2^{\dagger}\right)^{?_2}\eta_2^{\dagger}\right\rangle . \qquad (4.28)
\end{aligned}
$$

(4.27) and (4.28) should be equal. However, this is impossible due to $\hat{\eta}_2^{\dagger}$ versus $\eta_2^{\dagger}$ as the last factor in the respective equations.

Both examples imply that the appearance of $*_j$ is not transferable in a 'nice' way to the right. Although we do not claim that is impossible, the possible result will not be easy to work with. Let us try another argument.

4.2.3.3 Third attempt

In order to find any recurrence relation in the Weierstrass transform of the spherical monogenic polynomials, we tried to write the transform of a product of r elements η_i or $\hat{\eta}_i$ of different combinations of indices in terms of the transform of an analogue product of $r-1$ of these elements, by 'peeling off' the first η_i. We hope to find a structure that will allow us to look for a general rule. Here are some examples:

$$
\begin{aligned}
\mathcal{W}[\eta_i] &= y_i, \\
\mathcal{W}[\hat{\eta}_i] &= \hat{y}_i, \\
\mathcal{W}[\eta_i\hat{\eta}_i] &= y_i\hat{y}_i = y_i\mathcal{W}[\hat{\eta}_i], \\
\mathcal{W}[\hat{\eta}_i\eta_i] &= \hat{y}_i y_i = \hat{y}_i\mathcal{W}[y_i], \\
\mathcal{W}[\eta_i\eta_j] &= y_i y_j + 1 = y_i\mathcal{W}[\eta_j] + 1, \\
\mathcal{W}[\eta_i\hat{\eta}_j] &= y_i\hat{y}_j - 1 = y_i\mathcal{W}[\hat{\eta}_j] - 1, \\
\mathcal{W}[\hat{\eta}_i\eta_j] &= \hat{y}_i y_j - 1 = \hat{y}_i\mathcal{W}[\eta_j] - 1, \\
\mathcal{W}[\hat{\eta}_i\hat{\eta}_j] &= \hat{y}_i\hat{y}_j + 1 = \hat{y}_i\mathcal{W}[\hat{\eta}_j] + 1
\end{aligned}
$$

So far, it looks good: the Weierstrass transform of a product of two η_i (same index) is a product of the corresponding y_i and a lower degree transform. If the indices are different with both $\hat{}$ or both no $\hat{}$, we add 1 to the product with the lower degree transform. If the indices are different and one of both have $\hat{}$, we add -1. Unfortunately, when we look at third degree polynomials, this behaviour does not sustain.

$$
\begin{aligned}
\mathcal{W}[\eta_i\hat{\eta}_i\eta_i] &= y_i\hat{y}_i y_i = y_i\mathcal{W}[\hat{y}_i y_i], \\
\mathcal{W}[\hat{\eta}_i\eta_i\hat{\eta}_i] &= \hat{y}_i y_i\hat{\eta}_i = y_i\mathcal{W}[\eta_i\hat{\eta}_i], \\
\mathcal{W}[\eta_i\eta_j\hat{\eta}_j] &= y_i y_j\hat{y}_j + 2\hat{y}_j = y_i\mathcal{W}[\eta_j\hat{\eta}_j] + 2\hat{y}_j, \\
\mathcal{W}[\eta_i\hat{\eta}_j\eta_j] &= y_i\hat{y}_j y_j - 2y_j = y_i\mathcal{W}[\hat{\eta}_j\eta_j] - 2y_j, \\
\mathcal{W}[\hat{\eta}_i\eta_j\hat{\eta}_j] &= \hat{y}_i y_j\hat{y}_j - 2\hat{y}_j = \hat{y}_i\mathcal{W}[\eta_j\hat{\eta}_j] - 2\hat{y}_j,
\end{aligned}
$$

$$\begin{aligned}
\mathcal{W}[\eta_i\hat{\eta}_i\eta_j] &= y_i\hat{y}_iy_j - 2y_i = y_i\mathcal{W}[\hat{\eta}_i\eta_j] - y_i,\\
\mathcal{W}[\hat{\eta}_i\eta_i\eta_j] &= \hat{y}_iy_iy_j + 2\hat{y}_i = \hat{y}_i\mathcal{W}[\eta_i\eta_j] + \hat{y}_i,\\
\mathcal{W}[\hat{\eta}_i\eta_i\hat{\eta}_j] &= \hat{y}_iy_i\hat{y}_j - 2\hat{y}_i = \hat{y}_i\mathcal{W}[\eta_i\hat{\eta}_j] - \hat{y}_i,\\
\mathcal{W}[\eta_i\eta_j\hat{\eta}_i] &= y_iy_j\hat{y}_i + 4z_1e_1 = y_i\mathcal{W}[\eta_j\hat{\eta}_i] + y_i + 4z_1e_1,\\
\mathcal{W}[\hat{\eta}_i\hat{\eta}_j\eta_i] &= \hat{y}_i\hat{y}_jy_i - 4z_1e_1 = \hat{y}_i\mathcal{W}[\hat{y}_jy_i] + \hat{y}_i - 4z_1e_1.
\end{aligned}$$

While any combination of indices $'iii'$ or $'ijj'$ or $'iij'$ shows a nice behaviour regarding the lower order transform, this is not the case for the combination $'iji'$: the extra terms $\pm 4z_1e_1$ appear, which is not in line with the other findings. From the above, it unfortunately follows that also the third attempt fails.

4.3 Conclusion

In this chapter, we generalised the definition from chapter one for the Weierstrass transform to dimension $m > 1$. Firstly, we needed to handle the anti-commutativity of the basis elements. This was done by implementing the operators $R_j (j = 1, \dots m)$. Secondly, the basis for the discrete Weierstrass space now consists of the generalised Hermite polynomials $H_{n,m,r}$, the composition of the n-th degree Hermite polynomial and a monogenic polynomial of degree r. Both considerations lead to

$$\mathcal{W}[H_{n,m,r}P_r](\underline{z}) = \sqrt{2\pi}^{-m}\left\langle H_{n,m,r}P_rG, \exp\left(\frac{-|\underline{z}|^2}{2} + \xi R\underline{z}\right)[1]\right\rangle.$$

In order to calculate this, we found a recurrence relation for the Weierstrass transform of these generalised Hermite polynomials in terms of n. In dimension $m = 2$, we also found a recurrence relation in terms of r. This was done based on the explicit expression for spherical monogenics. Unfortunately, finding a recurrence relation or an explicit expression for the Weierstrass transform of a spherical monogenic, based on this definition, in dimension $m > 2$ did not work. Although we have a clear idea of the outcome, inspired by examples and the cases for $m = 2$, we were not able to prove it. Therefore, let us go back to the continuous (classical) Weierstrass transform and take a different tack. This is the subject of the next chapter.

5

Two alternative definitions

In the previous chapters, we defined a discrete version of the Weierstrass transform, using the discrete Hermite polynomials, which form a basis for the elements in the discrete Weierstrass space, and we provided several examples. However, we were unable to prove an explicit formula for the Weierstrass transform of the generalised Hermite polynomials in dimension $m > 1$. In this chapter, we provide two alternative ways to define the Weierstrass transform, for which we revisited the classical setting in order to obtain inspiration. Using the second alternative form, we finally will be able to prove the desired expression for the Weierstrass transform of the generalised Hermite polynomials.

5.1 Discrete translations

Consider once again the classical definition of the Weierstrass transform

$$\mathcal{W}[f](u) = \frac{1}{\sqrt{2\pi}^m} \int_{\mathbb{R}^m} \exp\left(-\frac{|u-x|^2}{2}\right) f(x)\, dx,$$

defined as the convolution of the function f with the Gaussian function $\sqrt{2\pi}^{-m} \exp\left(-|x|^2/2\right)$. Exploiting the basic property of convolutions, this can be rewritten as

$$\mathcal{W}[f](u) = \frac{1}{\sqrt{2\pi}^m} \int_{\mathbb{R}^m} \exp\left(\frac{-|x|^2}{2}\right) f(x-u)\, dx,$$

showing the product of the Gaussian function with a translation of the function f in the integrand.

In [40], De Ridder defined a discrete translation T_j ($j = 1, \ldots, m$), using the operator R_j, see definition 2.17. It allows to introduce discrete infinitesimal translations $R_j\partial_j$ in such a way that they satisfy the same properties as in the continuous case: they are symmetries of the Dirac operator. In the following definition, we use them to generate discrete translations. For more in depth information, we refer to [40].

Definition 5.1. The translation T_j of a polynomial V in the $j-$direction is given by the action on V of the operator

$$T_j := \exp\left(z_j R_j \partial_j\right), \qquad z_j \in \mathbb{C}.$$

We investigate if the discrete Weierstrass transform, as defined in definition 4.3,

$$\mathcal{W}[f(\xi)](z) = \left\langle f(\xi)G, \exp\left(\frac{-|\underline{z}|^2}{2} + \xi R z\right)[1]\right\rangle,$$

can be written in terms of this discrete translation. Based on the continuous analogue, it can be expected that:

$$\mathcal{W}[f(\xi)] = \sqrt{2\pi}^{-m}\left\langle G, T[f(\xi)]\right\rangle, \tag{5.1}$$

where $T := T_1 \dots T_m$. Since the operators $R_j\partial_j$ mutually commute, so do the translations T_j. We now want to check if this alternative form has the same properties as the original definition, based on its action on the basis elements.

As the Hermite polynomials form the basis of the discrete Weierstrass space, we first calculate

$$T[H_{n,1,0}P_0(\xi)] = T[H_{n,1,0}(\xi)],$$

where we will make a distinction between even and odd Hermite polynomials. First, we know from [40], that

$$T_j\left[\xi_j^n[1]\right] = \sum_{i=0}^{n}\binom{n}{i}\xi_j^i[1](z_je_j)^{n-i} = \sum_{i=0}^{n}\binom{n}{i}\xi_j^{n-i}[1](z_je_j)^{i}.$$

We use this in the calculation of $T[H_{2n,1,0}(\xi)]$ (omiting the subindex 1).

$$\begin{aligned}
T\left[H_{2n}(\xi)[1]\right] &= T\left[\sum_{j=0}^{n} a_{2j}^{2n}\xi^{2j}[1]\right] \\
&= \sum_{j=0}^{n}\sum_{s=0}^{2j} a_{2j}^{2n}\binom{2j}{s}\xi^s[1]e^{2j-s}z^{2j-s}.
\end{aligned}$$

As the action of the Gaussian G is zero for odd powers of ξ, only even powers of ξ are of interest:

$$\begin{aligned}
&= \sum_{j=0}^{n}\sum_{s=0}^{j} a_{2j}^{2n}\binom{2j}{2s}\xi^{2s}[1]e^{2j-2s}z^{2j-2s} \\
&= \sum_{j=0}^{n}\sum_{s=0}^{j} \frac{(-1)^j 2^{n-j} n!}{j!(n-j)!}\frac{\Gamma\left(n+\frac{m}{2}\right)}{\Gamma\left(j+\frac{m}{2}\right)}\frac{(2j)!}{(2s)!(2j-2s)!}z^{2j-2s}\xi^{2s}[1].
\end{aligned}$$

Let now G act on this expression:

$$\left\langle G, T[H_{2n}(\xi)[1]]\right\rangle = \sqrt{2\pi}^{-m}\sum_{j=0}^{n}\sum_{s=0}^{j}\frac{(-1)^j 2^{n-j} n!}{j!(n-j)!}\frac{\Gamma\left(n+\frac{m}{2}\right)}{\Gamma\left(j+\frac{m}{2}\right)}\frac{(2j)!}{(2j-2s)!}z^{2j-2s}\frac{1}{2^s s!}.$$

Likewise, we can calculate

$$\begin{aligned}
\left\langle G, T[H_{2n+1}(\xi)[1]]\right\rangle &= \sqrt{2\pi}^{-m}\left\langle G, T\left[\sum_{j=0}^{n} a_{2j+1}^{2n+1}\xi^{2j+1}[1]\right]\right\rangle \\
&= \sqrt{2\pi}^{-m}\left\langle G, \sum_{j=0}^{n}\sum_{s=0}^{2j+1} a_{2j+1}^{2n+1}\binom{2j+1}{s}\xi^{s}[1]e^{2j+1-s}z^{2j+1-s}\right\rangle \\
&= \sqrt{2\pi}^{-m}\left\langle G, \sum_{j=0}^{n}\sum_{s=0}^{j} a_{2j+1}^{2n+1}\binom{2j+1}{2s}\xi^{2s}[1]e^{2j+1-2s}z^{2j+1-2s}\right\rangle \\
&= \sqrt{2\pi}^{-m}\left\langle G, \sum_{j=0}^{n}\sum_{s=0}^{j} \frac{(-1)^j 2^{n-j} n!}{j!(n-j)!}\frac{\Gamma\left(n+\frac{m}{2}+1\right)}{\Gamma\left(j+\frac{m}{2}+1\right)}\right. \\
&\qquad \left.\times\frac{(2j+1)!}{(2s)!(2j-2s+1)!}\xi^{2s}[1]ez^{2j+1-2s}\right\rangle \\
&= \sqrt{2\pi}^{-m}\sum_{j=0}^{n}\sum_{s=0}^{j} \frac{(-1)^j 2^{n-j} n!}{j!(n-j)!}\frac{\Gamma\left(n+\frac{m}{2}+1\right)}{\Gamma\left(j+\frac{m}{2}+1\right)} \\
&\qquad \times\frac{(2j+1)!}{(2j-2s+1)!}\xi^{2s}[1]ez^{2j+1-2s}\frac{1}{2^s s!}.
\end{aligned}$$

Simplification of the above expression (using Maple) yields

$$\sqrt{2\pi}^{m}\left\langle G, T[H_{2n}(\xi)[1](z)]\right\rangle = (-1)^n z^{2n}, \tag{5.2}$$

$$\sqrt{2\pi}^{m}\left\langle G, T[H_{2n+1}(\xi)[1](z)]\right\rangle = (-1)^n z^{2n+1}e. \tag{5.3}$$

This is exactly the result of the Weierstrass transform as defined via the original definition 4.3. To confirm this alternative definition, let us check some other examples and compare them to the results in section 3.2.2.

5.1.1 Example 1: basic discrete polynomials ($m = 1$)

Let us calculate $\left\langle G, T[\xi^n]\right\rangle$. Again we make a distinction between even and odd powers. First we obtain, for the even powers

$$\begin{aligned}
\left\langle G, T[\xi^{2n}]\right\rangle &= \left\langle G, \sum_{i=0}^{2n}\binom{2n}{i}\xi^{i}[1](ze)^{2n-i}\right\rangle \\
&= \left\langle G, \sum_{i=0}^{n}\binom{2n}{2i}\xi^{2i}[1](ze)^{2n-2i}\right\rangle \\
&= \sum_{i=0}^{n}\frac{(2n)!}{(2i)!(2n-2i)!}(ze)^{2n-2i}\frac{(2i)!}{2^i\, i!} \\
&= z^{2n}\,{}_2F_0\left[-n, -n+\frac{1}{2};\,;\frac{2}{z^2}\right] \\
&\overset{*}{=} (-1)^n H_{2n}(iz).
\end{aligned}$$

and next, for the odd powers

$$\begin{aligned}\left\langle G, T[\xi^{2n+1}]\right\rangle &= \left\langle G, \sum_{i=0}^{2n+1}\binom{2n+1}{i}\xi^i[1](ze)^{2n+1-i}\right\rangle \\ &= \left\langle G, \sum_{i=0}^{n}\binom{2n+1}{2i}\xi^{2i}[1](ze)^{2n+1-2i}\right\rangle \\ &= \sum_{i=0}^{n}\frac{(2n+1)!}{(2i)!(2n+1-2i)!}(ze)^{2n+1-2i}\frac{(2i)!}{2^i\, i!} \\ &= z^{2n+1}\,{}_2F_0\left[-n,-n-\frac{1}{2};\,;\frac{2}{z^2}\right] \\ &\overset{*}{=} (-1)^{n+1} i H_{2n+1}(iz).\end{aligned}$$

For the transitions $*$, we made use of the formula[1]

$${}_2F_0\left[-\frac{n}{2},\frac{1-n}{2};\,;z\right] = \left(-\frac{z}{2}\right)^{\frac{n}{2}} H_n\left(\sqrt{-\frac{2}{z}}\right).$$

In terms of the coefficients $b_{2j}^{2\ell}$ and $b_{2j+1}^{2\ell+1}$, introduced in section 4.2.3.1, this means

$$\left\langle G, T[\xi^{2n}]\right\rangle = \sum_{j=0}^{n} b_{2j}^{2n} z^{2j}, \tag{5.4}$$

$$\left\langle G, T[\xi^{2n+1}]\right\rangle = \sum_{j=0}^{n} b_{2j+1}^{2n+1} z^{2j+1} e. \tag{5.5}$$

These results coincide with the calculations in section 3.2.2.

Now move to dimension $m > 1$ and recall that $T = T_1 \dots T_m$ with $T_j\xi_k = \xi_k T_j$ because of the commutator relations (2.18). Hence we can calculate the translation T of $\xi^{\underline{\alpha}} = \xi_1^{\alpha_1}\dots\xi_m^{\alpha_m}$ by letting each individual T_j act on its corresponding $\xi_j^{\alpha_j}$. Therefore,

$$\left\langle G, T[\xi_1^{2k}\xi_2^{2n+1}]\right\rangle = \sum_{j=0}^{k} b_{2j}^{2k} z^{2j} \sum_{i=0}^{n} b_{2j+1}^{2n+1} z^{2j+1} e,$$

and this result is immediately applicable for general $\xi^{\underline{\alpha}}$.

5.1.2 Example 2: Exponential functions ($m = 1$)

First, consider $T\left[\exp(a\xi)\right], a \in \mathbb{C}$. We subsequently obtain

$$T\left[\exp(a\xi)\right] = T\left[\sum_{j=0}^{\infty}\frac{a^j\xi^j}{j!}\right]$$

[1] http://functions.wolfram.com/07.31.03.0083.01

$$
\begin{aligned}
&= \sum_{j=0}^{\infty} \frac{a^j}{j!} T\left[\xi^j\right] \\
&= \sum_{j=0}^{\infty} \frac{a^j}{j!} \sum_{i=0}^{j} \frac{j!}{i!(j-i)!} (ze)^{j-i} \xi^i \\
&= \sum_{j=0}^{\infty} \sum_{i=0}^{j} \frac{a^j}{i!(j-i)!} (ze)^{j-i} \\
&= \exp\left(a(\xi+z)\right),
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
\left\langle G, \exp\left(a(\xi+z)\right)\right\rangle &= \exp(az) \left\langle G, \exp(\xi a)\right\rangle \\
&= \exp(az) \left\langle G, \sum_{j=0}^{\infty} \frac{a^j \xi^j}{j!} \right\rangle \\
&= \exp(az) \sum_{j=0}^{\infty} \frac{a^{2j}}{(2j)!} \left\langle G, \xi^{2j} \right\rangle \\
&= \exp(az) \sum_{j=0}^{\infty} \frac{a^{2j}}{(2j)!} \frac{(2j)!}{2^j j!} \\
&= \exp\left(az + \frac{a^2}{2}\right).
\end{aligned}
$$

Next, we consider

$$
T\left[\exp\left(a\xi^2\right)\right] = \sum_{j=0}^{\infty} \sum_{i=0}^{2j} \frac{a^j (2j)!}{j! i! (2j-i)!} \xi^i (ze)^{2j-i}
$$

yielding

$$
\begin{aligned}
\left\langle G, T\left[\exp\left(a\xi^2\right)\right]\right\rangle &= \sum_{j=0}^{\infty} \sum_{i=0}^{j} \frac{a^j (2j)!}{j!(2i)!(2j-2i)!} \frac{(2i)!}{2^i i!} (ze)^{2j-2i} \\
&= \sum_{j=0}^{\infty} \frac{(az^2)^j}{j!} \, {}_2F_0\left[-j, -j+\frac{1}{2};\,;\frac{2}{z^2}\right] \\
&= \sum_{j=0}^{\infty} \frac{(-a)^j}{j!} H_{2j}(iz) \\
&= \exp\left(\frac{4az^2}{1-4a}\right) (1-4a)^{-\frac{1}{2}}.
\end{aligned}
$$

Again, both results coincide with the original definition 4.3.

5.1.3 Weierstrass transform of monogenics using discrete translations

Now the same question as in the previous chapter arises: we want to establish an explicit expression, or a recurrence relation for the discrete Weierstrass transform of a monogenic polynomial P_r of degree r. If successful, then we also are able to find an explicit form for the transform of a generalised Hermite polynomial, i.e. of the basis elements of the discrete Weierstrass space.

The translations T_j used in definition 5.1, are invariant under the discrete Dirac operator. This means that the translation of a monogenic is again monogenic of the same degree. The only conclusion that we can make thus far is:

$$\left\langle G, T[P_r] \right\rangle = \left\langle G, P'_{r'} \right\rangle .$$

It is necessary to further investigate the properties of the translation on monogenic functions in order to use this definition to establish an explicit expression for the Weierstrass transform of these monogenic functions. This can be an interesting topic for further research.

5.2 Another formal expression

Although clear from the examples and despite various attempts to find a recurrence relation on the degree r of the monogenic, we were not able to prove the general formula for the discrete Weierstrass transform of a generalized Hermite polynomial in dimension $m > 2$. Let us again look at the classical setting: there, the Weierstrass transform has a an alternative formal expression. However more informal, it leads to certain advantages such as the idea of the inverse of the Weierstrass transform, see [48].

Definition 5.2. The continuous Weierstrass transform can be written as

$$W[f] = \exp\left(\frac{1}{2}\partial_x^2\right) f(x) = \sum_{j=0}^{\infty} \frac{1}{j!\,2^j} \partial_x^{2j} f(x).$$

This definition plays with convergence of the series. There are functions that are Weierstrass transformable, but for which this series does not converge. For details, also see [48]. Nonetheless, it inspires us to look for an alternative definition for the discrete Weierstrass transform. Therefore, let us calculate $\exp\left(-\frac{\partial^2}{2}\right) \xi^n P_r(\xi)[1]$, with P_r a monogenic homogeneous polynomial of degree r (as is $CK[\xi^{\underline{\alpha}}]$).

We will use the following lemma:

Lemma 5.3.

$$\begin{aligned}
\partial^{2j}\xi^{2\ell}P_r[1] &= 4^j \frac{\ell!}{(\ell-j)!} \frac{(\ell-1+r+\frac{m}{2})!}{(\ell-1+r+\frac{m}{2}-j)!} \xi^{2\ell-2j} P_r[1] \\
&= \frac{4^j\,\Gamma(\ell+r+\frac{m}{2})\,\Gamma(\ell+1)}{\Gamma(\ell+r+\frac{m}{2}-j)\,\Gamma(\ell+1-j)} \xi^{2\ell-2j} P_r[1], \\
\partial^{2j}\xi^{2\ell+1}P_r[1] &= 4^j \frac{\ell!}{(\ell-j)!} \frac{(\ell+r+\frac{m}{2})!}{(\ell+r+\frac{m}{2}-j)!} \xi^{2\ell+1-2j} P_r[1] \\
&= \frac{4^j\,\Gamma(\ell+r+1+\frac{m}{2})\,\Gamma(\ell+1)}{\Gamma(\ell+r+1+\frac{m}{2}-j)\,\Gamma(\ell+1-j)} \xi^{2\ell+1-2j} P_r[1].
\end{aligned}$$

Proof. This is a consequence of the relations

$$\partial\xi^{2j+1}P_r[1] = (2j+2r+m)\xi^{2j}P_r, \text{ and } \partial\xi^{2j}P_r[1] = 2j\xi^{2j-1}P_r,$$

which in turn are a consequence of the intertwining relations $\partial\xi + \xi\partial = 2\mathbb{E} + m$ and $\partial\mathbb{E} = \mathbb{E}\partial + \partial$. □

Let us now calculate $\exp\left(-\frac{\partial^2}{2}\right) \xi^n P_r(\xi)[1]$, making the distinction between even and odd powers of ξ.

For the even case, i.e. $n = 2\ell$, we obtain

$$\begin{aligned}
\exp\left(-\frac{\partial^2}{2}\right)\xi^{2\ell}P_r(\xi)[1] &= \sum_{j=0}^{\ell}\frac{(-1)^j}{j!\,2^j}\partial^{2j}\xi^{2\ell}P_r(\xi)[1] \\
&= \sum_{j=0}^{\ell}\frac{(-1)^j}{j!\,2^j}\frac{4^j\,\Gamma(\ell+r+\frac{m}{2})\,\Gamma(\ell+1)}{\Gamma(\ell+r+\frac{m}{2}-j)\,\Gamma(\ell+1-j)}\xi^{2\ell-2j}P_r[1](\xi) \\
&= \sum_{i=0}^{\ell}\frac{(-1)^{\ell-i}\,2^{\ell-i}}{(\ell-i)!}\frac{\Gamma(\ell+r+\frac{m}{2})\,\Gamma(\ell+1)}{\Gamma(i+r+\frac{m}{2})\,\Gamma(i+1)}\xi^{2i}P_r[1](\xi) \\
&= (-1)^{\ell}\sum_{i=0}^{\ell}(-1)^i\,2^{\ell-i}\binom{\ell}{i}\frac{\Gamma(\ell+\frac{m}{2}+r)}{\Gamma(i+\frac{m}{2}+r)}\xi^{2i}P_r[1](\xi) \\
&= (-1)^{\ell}\,H_{2\ell,m,r}(\xi)P_r[1](\xi).
\end{aligned}$$

For the odd case, i.e. $n = 2\ell+1$, we obtain

$$\begin{aligned}
\exp\left(-\frac{\partial^2}{2}\right)\xi^{2\ell+1}P_r(\xi)[1] &= \sum_{j=0}^{\ell}\frac{(-1)^j}{j!\,2^j}\partial^{2j}\xi^{2\ell+1}P_r[1](\xi) \\
&= \sum_{j=0}^{\ell}\frac{(-1)^j}{j!\,2^j}\frac{4^j\,\Gamma(\ell+r+\frac{m}{2}+1)\,\Gamma(\ell+1)}{\Gamma(\ell+r+\frac{m}{2}-j+1)\,\Gamma(\ell+1-j)}\xi^{2\ell+1-2j}P_r[1](\xi) \\
&= \sum_{i=0}^{\ell}\frac{(-1)^{\ell-i}\,2^{\ell-i}}{(\ell-i)!}\frac{\Gamma(\ell+r+\frac{m}{2}+1)\,\Gamma(\ell+1)}{\Gamma(i+r+\frac{m}{2}+1)\,\Gamma(i+1)}\xi^{2i+1}P_r[1](\xi) \\
&= (-1)^{\ell}\sum_{i=0}^{\ell}(-1)^i\,2^{\ell-i}\binom{\ell}{i}\frac{\Gamma(\ell+\frac{m}{2}+r+1)}{\Gamma(i+\frac{m}{2}+r+1)}\xi^{2i+1}P_r[1](\xi) \\
&= (-1)^{\ell}\,H_{2\ell+1,m,r}(\xi)P_r[1](\xi).
\end{aligned}$$

Together, we see that

$$\exp\left(-\frac{\partial^2}{2}\right)\xi^n P_r(\xi)[1] = (-1)^{\lfloor\frac{n}{2}\rfloor}H_{n,m,r}(\xi)P_r(\xi)[1],$$

or

$$\exp\left(\frac{\partial^2}{2}\right)H_{n,m,r}(\xi)P_r(\xi)[1] = (-1)^{\lfloor\frac{n}{2}\rfloor}\xi^n P_r(\xi)[1].$$

Remark 5.4. These results also come back in the differential operator representation of the continuous Hermite polynomials, i.e. in the continuous case it holds that

$$H_n(x) = \exp\left(-\frac{\partial^2}{2}\right)x^n.$$

After formally replacing the discrete variable ξ into the continuous variable $\underline{z} = \sum_{j=1}^{m} z_j e_j$, we see that

1. For $r = 0$, this is exactly the result obtained by calculation with the original definition.
2. For $n = 0$, this is exactly the result obtained by direct calculation of the examples in paragraph 4.2.2.

As the generalised Hermite polynomials form a basis for the space of functions which are Weierstrass transformable and as the result obtained by the original definition 4.3 is identical as the one obtained above, we can state that this alternative definition makes sense and is valid to work with.

Definition 5.5. The discrete Weierstrass transform of a discrete function f can be written as

$$\mathcal{W}[f](z) = \exp\left(\frac{\partial^2}{2}\right) f(\xi)\bigg|_{\xi=\underline{z}}.$$

With this definition, we can finally deduce a formula for the discrete Weierstrass transform of the generalised Hermite polynomials.

Conjecture 5.6. The discrete Weierstrass transform of the generalised Hermite polynomials is given by

$$\mathcal{W}\left[H_{n,m,r}P_r(\xi)\right](\underline{z}) = (-1)^{\lfloor \frac{n}{2} \rfloor} \underline{z}^n P_r(\underline{z}) \tag{5.6}$$

where $\underline{z} = \sum_{j=1}^{m} z_j e_j$.

As the generalised Hermite polynomials form a basis for the discrete Weierstrass space, we now have an expression for the discrete Weierstrass transform of any discrete function contained in this space.

5.3 Conclusion

Inspired by the classical Weierstrass transform, we defined a discrete analogue, based on the discrete Gaussian distribution G and the discrete (generalised) Hermite polynomials $H_{n,m,r}$. The latter form a basis for a function space, which we called the discrete Weierstrass space. We defined an inner product and a corresponding norm on it, in order to put a condition on such that a discrete function contained in this space, would have a meaningful Weierstrass transform. Our goal was to find an explicit expression for the Weierstrass transform of a discrete function contained in this Weierstrass space. As the (generalised) Hermite polynomials form a basis and the transform is linear, it is sufficient to find an expression for the transform of those Hermite polynomials. Our initial idea was to find a recurrence relation, both in terms of the degree n of the Hermite polynomial and in terms of r of the monogenic polynomial occurring in $H_{n,m,r}P_r$. Based on Rodriguez' formula, we successfully found a recurrence relation in terms of n. However, for the recurrence relation in terms of r, we did not succeed. Instead, we used an alternative definition for the discrete Weierstrass transform. Although more informal, it led us to an explicit expression for the Weierstrass transform of a generalised Hermite polynomial.

6

Discrete heat equation

The heat equation is a second order partial differential equation originally developed and solved to describe the heat flow over time through a solid medium, as it flows from spots with higher density to spots with lower density. It is however valid for the distribution of any quantity. Physics, engineering, financial mathematics and probability theory (e.g. in [49], [50], [51], [52]) are only a few domains in which this well-known differential equation has its applications. For a function $u(x,t), x \in \mathbb{R}^m, t \in \mathbb{R}^+$, the heat equation is given by

$$\left(\frac{\partial}{\partial t} - \alpha \Delta_x\right) u(x,t) = 0.$$

The coefficient $\alpha > 0$, called the thermal diffusivity, affects the speed and spatial scale of the process. For mathematical purposes, it suffices to set $\alpha = 1$, which makes the heat equation a prototypical parabolic partial differential equation. In this chapter, we will focus on the one-dimensional heat equation, i.e. $x \in \mathbb{R}$.

Much research has been devoted to the heat equation. In 2014, first steps were taken by Baaske, Bernstein and De Ridder to discretize the heat equation in a discrete Clifford analysis setting, see [2]. They considered discrete space and continuous time and constructed a fundamental solution together with related heat polynomials. This chapter will be devoted to finding solutions to the discrete heat equation, in which both space and time are discrete.

6.1 Continuous heat equation

6.1.1 Classical setting

The classical, continuous heat equation with initial condition u_0 is given by

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta_x\right) u(x,t) = 0, & t \in \mathbb{R}^+, x \in \mathbb{R} \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

The initial temperature function u_0 is an element of the Schwartz space of rapidly decreasing infinitely differentiable functions on $\mathbb{R}$. The solution $u(x,t)$ can be found, for example, using Fourier series and is given by

$$u(x,t) = \frac{1}{(4\pi t)^{m/2}} \int_{\mathbb{R}} \exp\left(\frac{-|x-y|^2}{4t}\right) u_0(y) dy.$$

This is the convolution of the Gaussian function $G(x,t)$ with u_0, which is the Weierstrass transform.

Remark 6.1. Note that the Gaussian function appearing here is a rescaling of the one we used in the first part of this thesis. This equivalent definition however is more commonly used in physics, hence we proceed with this rescaling in the current section.

The above result is intimately related to the fact that the convolution kernel

$$G(x,t) = \frac{1}{(4\pi t)^{m/2}} \exp\left(-\frac{|x|^2}{4t}\right)$$

is a fundamental solution of the heat equation, i.e. a solution to

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta_x\right) u(x,t) = 0, & t \in \mathbb{R}^+, x \in \mathbb{R} \\ u(x,0) = \delta(x), & x \in \mathbb{R}. \end{cases}$$

with $\delta(x)$ the Dirac delta function.

The heat polynomials $p_n(x,t)$ are defined as the polynomial solutions to the heat equation with initial condition $u_0(x) = x^n$. They are found as the coefficient of $\frac{z^n}{n!}$ in the power series expansion of $\exp\left(zx + z^2 t\right)$:

$$\exp\left(zx + z^2 t\right) = \sum_{n=0}^{\infty} p_n(x,t) \frac{z^n}{n!}. \tag{6.1}$$

We refer to [53] for a proof. The heat polynomials are suitable to determine the general solution of the heat equation with a given initial condition. They are used to construct an approximate solution of a given problem in a form of a linear combination of polynomials ([54]).

An explicit formula for the heat polynomials is given by

$$p_n(x,t) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{x^{n-2k} t^k}{(n-2k)! k!}. \tag{6.2}$$

In their paper from 1959, [53], Rosenbloom and Widder introduced the associated functions in one dimension, which are obtained by the Appell transformation ([55], [56]) of the heat polynomials:

$$q_n(x,t) = G(x,t)\, p_n\left(\frac{x}{t}, -\frac{1}{t}\right), \quad t > 0. \tag{6.3}$$

Inherent to the Appell transform is that it maps one solution of the heat equation to another. Alternatively, $q_n(x,t)$ may be defined by use of the fundamental solution $G(x,t)$ as a generating function:

$$G(x-2z,t) = \frac{1}{(4\pi t)^{m/2}} \exp\left(-\frac{(x-2z)^2}{4t}\right) = \sum_{n=0}^{\infty} q_n(x,t)\frac{z^n}{n!}.$$

It then follows that the associated functions are also given by the formula

$$q_n(x,t) = \frac{\partial^n}{\partial z^n} G(x-2z,t)\big|_{z=0} = (-2)^n \frac{\partial^n}{\partial z^n} G(x,t). \tag{6.4}$$

Remarkable is the orthogonality relation between the heat polynomials and their associated functions:

$$\int_{\mathbb{R}} p_n(x,-t)\, q_m(x,t)\, dx = \delta_{m,n}. \tag{6.5}$$

6.1.2 Discrete heat equation: continuous time, discrete space

A first step to discretise the heat equation is to use a discrete space variable and a continuous time variable. We will refer to this as the 'mixed' heat equation, to distinguish between the fully continuous heat equation and the fully discrete one. In [2], the authors considered this situation, found a fundamental solution and studied solutions of the initial value problem by taking the discrete convolution with the fundamental solution.

In a discrete space setting, the heat equation is given by

$$\left(\partial_t - \Delta^*\right) u(x,t) = 0, \qquad x \in \mathbb{Z}, t \in \mathbb{R}^+. \tag{6.6}$$

Here Δ^* is the discrete star Laplacian and $\boldsymbol{\delta}_0$ the discrete delta distribution in the space variable x. To determine solutions of the heat equation with a given initial temperature is, as in the continuous case, based on a fundamental solution of the heat equation, satisfying in distributional sense

$$\left(\partial_t - \Delta^*\right) u(x,t) = \delta(t)\boldsymbol{\delta}_0, \qquad x \in \mathbb{Z}, t \in \mathbb{R}^+. \tag{6.7}$$

As every discrete distribution can be written in its dual Taylor series, the fundamental solution to (6.7) is found by use of those. Based on the equivalence with the continuous heat equation, where the fundamental solution is basically a Gaussian function, the following form for the fundamental solution is proposed:

$$G(t) = \sum_{\ell=0}^{\infty} c_\ell(t) \partial^{2\ell} \boldsymbol{\delta}_0.$$

The coefficients $c_\ell(t)$ are to be determined in order for this distribution $G(t)$ to fulfill (6.6). Substituting the proposed form of $G(t)$ in (6.7), gives us

$$G(t) = H(t) \sum_{\ell=0}^{\infty} \frac{t^\ell}{\ell!} \partial^{2\ell} \boldsymbol{\delta}_0 = H(t) \exp\left(t\partial^2\right) \boldsymbol{\delta}_0, \tag{6.8}$$

where $H(t)$ is the continuous Heaviside function. Observe that $G(t)$ consists of continuous distributions in t combined with discrete distributions in x. We refer to [2] for the explicit calculations.

The density function of the distribution $G(t)$ is the function $g(n,t)(n \in \mathbb{Z})$ with

$$g(n,t) = (-1)^n \frac{\mathcal{I}(|n|, 2t)}{\exp(2t)} H(t), \tag{6.9}$$

where again $\mathcal{I}$ denotes the modified Bessel function of the first kind. For $t = 0$, this discrete density function reduces to the discrete delta function δ_0.

In order to find a solution to the initial value problem

$$\begin{cases} (\partial_t - \Delta^*)\ u(x,t) = 0, & x \in \mathbb{Z}, t \in \mathbb{R}^+, \\ u(x,0) = u_0(x), & x \in \mathbb{Z}, \end{cases} \tag{6.10}$$

where $u_0(x)$ is a given discrete function, we need a discrete analogue for the convolution.

6.1.3 Discrete convolution theory

In this section, we will briefly introduce some concepts of discrete convolution theory. Again, we revert to a summary of the results obtained in [2] without proofs.

Definition 6.2. Consider two discrete functions f and g and define their convolution $f * g$ as the discrete function

$$(f * g)(n) := \sum_{x \in \mathbb{Z}} f(x) g(n-x) = \sum_{x \in \mathbb{Z}} f(n-x)\, g(x), \tag{6.11}$$

in any point n where convergence is ensured.

Note that, whenever one of the discrete functions f or g have compact support, the convolution is defined. The interaction with Clifford constants is as follows:

Lemma 6.3. Given two discrete functions f and g and a Clifford constant $a \in \mathbb{C}_m$. It holds that

$$(af) * g = a(f * g), \qquad f * (ag) = (fa) * g, \qquad f * (ga) = (f * g)a.$$

Useful for calculations is to know how ∂ interacts with convolutions, see [2].

Lemma 6.4. For the discrete convolution, it holds that $(f\partial) * g = f * (\partial g)$.

Example 6.5. The convolution of the discrete functions $\partial^k \delta$ and f are given by

$$\left(\partial^k \delta * f\right)(n) = \left(\partial^k f\right)(n),$$
$$\left(f * \partial^k \delta\right)(n) = \left(f \partial^k\right)(n).$$

The discrete convolution is also defined for regular distributions:

Definition 6.6. Let T_f and T_g be two regular distributions with respective density functions f and g. The convolution of T_f and T_g is the distribution $T_f * T_g$, which is defined by its action on polynomials as follows:

$$\left\langle T_f * T_g \right\rangle = \left\langle T_g(y), \left\langle T_f(x), V(x+y) \right\rangle \right\rangle = \sum_{y \in \mathbb{Z}} \left(\sum_{x \in \mathbb{Z}} V(x+y) f(x) \right) g(y),$$

whenever this double series converges.

It is proven that the convolution of two distributions T_f and T_g equals the regular distribution with density function T_{f*g}. Similar to the convolution of the functions $\partial^k\delta$ and f, it holds that

$$\partial^j \boldsymbol{\delta}_0 * G = \partial^j G,$$

for a general distribution G.

6.1.4 Solutions of the mixed heat equation

Using convolutions, we are now able to describe a solution to the problem

$$\left(\partial_t - \Delta^*\right) u(x,t) = f(x,t),$$

given $f(x,t)$. Therefore, consider the convolution $u(x,t) := g(x,t) * f(x,t)$, where $g(x,t)$ denotes the density function of the fundamental solution G, see (6.9). If $f(x,t)$ is the density function of a regular distribution T_f, then $u(x,t)$ is the density function of the distribution $G * T_f$ and will satisfy the heat equation (6.10) in distributional sense. Note that $g(x,t) * f(x,t)$ is a combination of a discrete (in the space variable x) and a continuous (in the time variable t) convolution.

In particular, we can look for discrete polynomial solutions $p_n(x,t)$ of the mixed heat equation (6.10), with initial condition $p_n(x,0) = \xi^n[1](x)$. Using the Fourier transform, the authors of [2] were able to construct a discrete analogue of the heat polynomials:

$$h_n(x,t) = n! \sum_{\ell=0}^{\lfloor \frac{n}{2} \rfloor} \frac{t^\ell}{\ell!} \frac{1}{(n-2\ell)!} \xi^{n-2\ell}[1](x).$$

They formally resemble the continuous heat polynomials (see (6.2)). In order to find a solution to the (mixed) heat equation with a given general initial condition (6.10) one proceeds as follows: first develop the function $u_0(x)$ in its discrete Taylor series (2.20). Then substitute every discrete basis vector variable $\xi^n[1]$ by its corresponding heat polynomial $h_n(x,t)$ This solution will satisfy the heat equation with initial condition $u(x,0) = u_0(x)$.

6.2 Discrete time variable

As our aim is to formulate and solve a heat equation in which both time and space are discrete, we still need some definitions and notations for the discrete time variable.

6.2.1 Discrete time variable

In order to formulate and investigate the fully discrete heat equation, we need to introduce the discrete time variable t together with its corresponding differential operators. For completeness, we again mention the difference operators related to the discrete space variable.

Definition 6.7 (Difference operators)**.** Let f be a function defined on the discrete variables $x \in Z$ and $t \in \mathbb{N}$.

$$\partial_t^{\pm} f(x,t) \coloneqq \pm \left(f(x, t \pm 1) - f(x,t) \right), \tag{6.12}$$
$$\Delta_j^{\pm} f(x,t) \coloneqq \pm \left(f(x \pm 1, t) - f(x,t) \right). \tag{6.13}$$

Related to the space variable x, there is the discrete (raising) vector variable operator $\xi = \boldsymbol{e}^- X^+ + \boldsymbol{e}^+ X^-$, where $X^{\pm}$ are (scalar) operators acting on discrete functions. Their action is defined by the skew Weyl relations (2.8) with initial value $X^{\pm}[1] = x$.

Similarly, we now also define the raising operator θ with respect to the variable t. To this aim, we define the **Weyl relation**

$$\partial_t^+ \theta - \theta \partial_t^+ = 1, \tag{6.14}$$

together with the initial value $\theta[1] = t$.

Note that, as opposed to the discrete vector variable ξ, we only have one Weyl relation for the positive time differentiation. As ξ and θ act on different variables, their action is commutative, i.e. $\theta\xi = \xi\theta$.

Let us calculate how θ acts on powers of t, i.e. $\theta[t^k]$.

Proposition 6.8. For the discrete time variable θ, it holds that

$$\theta[t^k] = t\,(t-1)^k, \tag{6.15}$$
$$\theta^k[1](t) = \frac{t!}{(t-k)!}. \tag{6.16}$$

Proof. For $k = 0$, formula (6.15) is true by definition. For $k = 1$, the Weyl relation (6.14) implies

$$\partial_t^+ \theta[t] - \theta \partial_t^+[t] = t \Leftrightarrow \partial_t^+ \theta[t] = t + \theta[t + 1 - t] = t + \theta[1] = 2t.$$

Let $\theta[t] = at^2 + bt + c$, then also

$$\partial_t^+ \theta[t] = 2at + a + b,$$

Hence $a = 1$ and $b = -1$. By determining the action of the raising operator θ on t^k, for every $k \in \mathbb{N}$, by means of the Weyl relation, the resulting constant will always remain undetermined. At this point, we will always make the choice of fixing the constant term to be zero, which completely determines the operator. It then follows that $\theta[t] = t^2 - t = t\,(t-1)$. For a general k, we proceed by induction. Suppose the formula holds for all natural numbers $j < k \in \mathbb{N}$. First, notice that

$$\partial_t^+[t^k] = (t+1)^k - t^k = \sum_{j=0}^{k} \binom{k}{j} t^j - t^k = \sum_{j=0}^{k-1} \binom{k}{j} t^j.$$

We calculate $\theta[t^k]$ by induction, starting from

$$\partial_t^+ \theta[t^k] - \theta \partial_t^+[t^k] = t^k$$

which yields

$$\partial_t^+ \theta[t^k] = t^k + \sum_{j=0}^{k-1} \binom{k}{j} \theta[t^j]$$

where we used the definition of ∂_t^+. By the induction hypothesis, we then obtain

$$\partial_t^+ \theta[t^k] = t^k + \sum_{j=0}^{k-1} \binom{k}{j} t(t-1)^j$$

or still

$$\partial_t^+ \theta[t^k] = t^k + t\left(t^k - (t-1)^k\right) = (t+1)t^k - t\,(t-1)^k.$$

On the other side, It is directly obtained that

$$\partial_t^+[t(t-1)^k] = (t+1)t^k - t\,(t-1)^k.$$

It is easily seen for discrete functions f and g in t that

$$\partial_t^+[f] = \partial_t^+[g] \Rightarrow f = g + c,$$

where c is a constant. As we chose to take constants zero, we can conclude that

$$\theta[t^k] = t\,(t-1)^k.$$

This proves the first part of the proposition.

For $k = 0$ or $k = 1$, formula (6.16) is true by definition. For general k, we calculate $\theta^{k+1}[1]$ by induction. The induction hypothesis yields

$$\begin{aligned}\theta^{k+1}[1] &= \theta\left[\theta^k[1]\right] = \theta\left[\frac{t!}{(t-k)!}\right] \\ &= \theta\,[\sum_{j=0}^{k} s(k,j)\, t^j] = \sum_{j=0}^{k} s(k,j)\, \theta\left[t^j\right].\end{aligned}$$

Using the first part of the proposition, we then obtain

$$
\begin{aligned}
&= t \sum_{j=0}^{k} s(k,j)\,(t-1)^j \\
&= t \frac{(t-1)!}{(t-1-k)!} \\
&= \frac{t!}{(t-(k+1))!}.
\end{aligned}
$$

This proves the second part of the proposition. □

Using this discrete time variabele θ, we are able to express discrete functions by means of a Taylor series expansion. There is a natural isomorphism between the space of discrete operators onto the space of discrete functions, by

$$F(\xi,\theta) \mapsto f(x,t) = F(\xi,\theta)[1], \qquad \text{on } \mathbb{Z} \times \mathbb{N}.$$

Definition 6.9 (Taylor Series in $(x,t) \in \mathbb{Z} \times \mathbb{N}$)**.** The formal Taylor series of a discrete operator $F(\xi,\theta)$ is defined as

$$F(\xi,\theta) = \sum_{k,\ell=0}^{\infty} \xi^k \theta^\ell a_{k,\ell}. \tag{6.17}$$

The coefficients $a_{k,\ell}$ are calculated as

$$a_{k,\ell} = \frac{\partial^k}{k!} \frac{(\partial_t^+)^\ell}{\ell!} F(\xi,\theta)[1]\big|_{t=x=0}.$$

The values $\frac{\partial^k}{k!} \frac{\partial_t^{+\ell}}{\ell!} f(x,t)\big|_{t=x=0}$ are determined by the values of $f(x,t)$ on the half plane $\mathbb{Z} \times \mathbb{N}$.

6.2.2 Discrete distributions for the time variable

Consider the dual space of distributions, which we can now expand to be defined on the space of discrete polynomials in x and t. Similar to the discrete delta distributions with respect to the space variable, we can now define discrete delta distributions with respect to the (fixed) tuple $(X,T) \in \mathbb{Z} \times \mathbb{N}$. The discrete distribution

$$\underline{\delta}_{X,T} : f \to \underline{\delta}_{X,T}[f] = \sum_{(x,t)\in\mathbb{Z}^2} \delta(x-X)\delta(t-T) f(x,t) = f(X,T)$$

sends a discrete function f to its value at the tuple $(X,T) \in \mathbb{Z}^2$. Any finite linear combination

$$F = \sum_{i,j} \underline{\delta}_{X_i,T_j} c_{i,j} : f \to \sum_{(x,t)\in\mathbb{Z}} f(x,t)F(x,t) = \sum_{(x,t)\in\mathbb{Z}} f(X_i,T_i) c_{i,j}$$

is a distribution with compact support.

The following definition complements and extends (2.21).

Definition 6.10. For a discrete function g and a discrete distribution F, we put

$$\begin{aligned}\langle \partial_x F, g\rangle &= -\left\langle F, [g]\partial_x^\dagger \right\rangle,\\ \langle \xi F, g\rangle &= \left\langle F, [g]\xi^\dagger \right\rangle,\\ \left\langle \partial_t^- F, g\right\rangle &= -\left\langle F, \partial_t^+ g\right\rangle,\\ \langle \theta F, g\rangle &= \langle F, \theta g\rangle .\end{aligned}$$

With these definitions, the following **Weyl relations** for distributions are valid:

$$\begin{aligned}\partial_x \xi - \xi\partial_x &= 1,\\ \partial_t^- \theta - \theta\partial_t^- &= 1.\end{aligned}$$

Remark that we use the negative time differentiation for distributions, while we used the positive differentiation for functions.

The dual aspect of distributions versus functions is now clear: ∂_t^+ acting on functions is equivalent as ∂_t^- acting on distributions. For example, let ∂_t^- act on the discrete δ distribution in X and T:

$$\partial_t^- \delta_{X,T} = \delta(x-X)\left(\delta(t-T) - \delta(t-1-T)\right).$$

Now let $\partial_t^- \delta_{X,T}$ act on the discrete function f and use definition 6.10:

$$\begin{aligned}\left\langle \partial_t^- \delta_{X,T}, f\right\rangle &= -\left\langle \delta_{XT}, \partial_t^+ f(x,t)\right\rangle\\ &= -\left\langle \delta_{X,T}, f(x,t+1) - f(x,t)\right\rangle\\ &= -\left(f(X,T+1) - f(X,T)\right)\\ &= f(X,T) - f(X,T+1)\\ &= \partial_t^+ f(X,T)\end{aligned}$$

Because of the skew Weyl relations, $\xi\partial_x F = \partial_x \xi F - F$. Now let $F = \partial_x^{\ell-1}\boldsymbol{\delta}_{0,0}$. It then follows that

$$\xi\partial_x^\ell \boldsymbol{\delta}_{0,0} = \partial_x \xi \partial_x^{\ell-1}\boldsymbol{\delta}_{0,0} - \partial_x^{\ell-1}\boldsymbol{\delta}_{0,0} = \ldots = -\ell\partial_x^{\ell-1}\boldsymbol{\delta}_{0,0}. \tag{6.18}$$

This is done by consecutive application of the Weyl relations and the fact that $\xi\boldsymbol{\delta}_{0,0} = 0$. In general, as an immediate consequence, we can state next proposition, which, in its turn, complements and extends (2.22).

Proposition 6.11.

$$\left\langle \partial_x^k \boldsymbol{\delta}_{0,0}, \xi^\ell[1]\right\rangle = \delta_{\ell k}(-1)^k k!, \tag{6.19}$$

$$\left\langle \partial_t^{-k} \boldsymbol{\delta}_{0,0}, \theta^\ell[1]\right\rangle = \delta_{\ell k}(-1)^k k!. \tag{6.20}$$

This again leads to the dual Taylor series expansion of a distribution F. The dual Taylor series of a discrete distribution F is given by

$$F = \sum_{k,\ell} \frac{(-\partial_x)^k}{k!}\frac{\left(-\partial_t^-\right)^\ell}{\ell!}\boldsymbol{\delta}_{0,0}\left\langle F, \xi^k\theta^\ell[1]\right\rangle. \tag{6.21}$$

6.3 Discrete heat equation: discrete time and discrete space

The heat equation in a setting with discrete space variable x and discrete time variable $t > 0$ is given by

$$\left(\partial_t^+ - \alpha\Delta^*\right) f(x,t) = 0. \tag{6.22}$$

Let us furthermore define the anti-heat equation by

$$\left(\partial_t^- + \alpha\Delta^*\right) f(x,t) = 0. \tag{6.23}$$

Note that there is no physical process linked to the anti-heat equation, as we cannot (yet) travel back in time. We will, as in the continuous and mixed cases, again set $\alpha = 1$.

Let F be a regular distribution with density function f. Then whenever f fulfills the heat equation, F will fulfill the anti-heat equation, because of the relations in definition 6.10. The anti-heat equation is thus dual to the heat equation, as functions are dual to distributions.

6.3.1 Fundamental Solution

Denote by χ_A the characteristic function on the set $A \subset \mathbb{R}$ and consider

$$E(x,t) = \chi_{t>0}(t)\left(1+\Delta^*\right)^{t-1}\delta_0(x),$$

which is both a function and a distribution with compact support with respect to x: the density function of the distribution $\mathbf{E}(x,t)$ is the function $E(x,t)$. For $t < 0$, $E(x,t+1) = 0$, while for $t \geq 0$:

$$E(x,t+1) = \left(1+\Delta^*\right)^t \delta_0(x) = \begin{cases} \delta_0(x) = \delta_0(x) + E(x,0), & \text{if } t = 0, \\ (1+\Delta^*)\, E(x,t), & \text{if } t > 0. \end{cases}$$

By definition,

$$\partial_t^+ E(x,t) = E(x,t+1) - E(x,t) = \Delta^* E(x,t).$$

Hence it follows that

$$\left(\partial_t^+ - \Delta^*\right) E(x,t) = \delta_0(x)\delta_0(t), \tag{6.24}$$

hence $E(x,t)$ is a fundamental solution to the heat equation. Remark that $E(x,-t)$ fulfills the anti-heat equation.

6.3.2 The Cauchy-Kovalevskaya extension

We now aim for a general solution of the heat equation, given an initial condition, i.e. a function f that represents the initial temperature in the medium. Therefore, we use once more the Cauchy-Kowalevskaya extension, which we already mentioned in theorem 4.11. It is not only a powerful tool to construct monogenic functions, it is also useful to solve differential equations, such as the heat equation.

Theorem 6.12. Cauchy-Kovalevskaya extension Let $f(x)$ be a function defined on the discrete grid $\mathbb{Z}$. Then there is a unique function $f(x,t)$ on $\mathbb{Z} \times \mathbb{N}$ such that

(i) $f(x,0) = f(x)$,

(ii) $\left(\partial_t^+ - \Delta^*\right) f(x,t) = 0$ on $\mathbb{Z} \times \mathbb{N}$.

Based on the analogy with the continuous settings, we propose the next form of this CK-extension of f:

$$\mathrm{CK}[f(x)] = f(x,t) := E(x,t+1) * f(x) = \sum_{u\in\mathbb{Z}} E(u-x,t+1)f(u) = \left(1+\Delta^*\right)^t f(x), \tag{6.25}$$

where E is the fundamental solution to the heat equation and $*$ denotes the discrete convolution.

Now consider the general statement of an initial value problem

$$\begin{cases} \left(\partial_t^+ - \Delta^*\right) u(x,t) = 0, \\ u(x,0) = f(x). \end{cases} \tag{6.26}$$

with $f(x)$ a given discrete function. To find the solution $u(x,t)$, we check if the CK-extension of f is a possible solution. It is immediate that it satisfies the initial condition in (6.26) and

$$\begin{aligned} \left(\partial_t^+ - \Delta^*\right)\left[E(x,t+1) * f(x)\right] &= \partial_t^+ \left(1+\Delta^*\right)^t f(x) - \Delta^* \left(1+\Delta^*\right)^t f(x) \\ &= \left(1+\Delta^*\right)^{t+1} f(x) - \left(1+\Delta^*\right)^t f(x) - \Delta^* \left(1+\Delta^*\right)^t f(x) \\ &= \left(1+\Delta^*\right)^{t+1} f(x) - \left(1+\Delta^*\right)\left(1+\Delta^*\right)^t f(x) \\ &= 0. \end{aligned} \tag{6.27}$$

The CK-extension of f hence is a solution to the heat equation with initial profile f.

This form of the CK−extension is unique, as proven in the following theorem.

Theorem 6.13. Let F be a discrete function defined on $\mathbb{Z}\times\mathbb{N}$, fulfilling the heat equation, with $F|_{t=0} = 0$. Then F is the null function.

Proof. As F fulfills the heat equation, we can write

$$\left(\partial_t^+ - \Delta^*\right) F(x,t) = 0,$$

or equivalently

$$F(x,t+1) - F(x,t) = \big(F(x+1,t) + F(x-1,t)\big) - 2F(x,t).$$

For $t = 0$, the above expression reduces to

$$F(x,1) = 0,$$

because $F(x,0) = 0$, for every $x \in \mathbb{Z}$. Next, consider again

$$F(x,t+1) - F(x,t) = \big(F(x+1,t) + F(x-1,t)\big) - 2F(x,t),$$

where we now let $t = 1$. This yields

$$F(x,2) = 0, \forall x \in \mathbb{Z}.$$

Repeating this procedure, we end up with

$$F(x,t) = 0, \qquad \forall (x,t) \in \mathbb{Z} \times \mathbb{N}.$$

□

Example 6.14. For $f(x) = \delta_0(x)$, we find that

$$\mathrm{CK}[\delta_0(x)](x,t) = \sum_{u\in\mathbb{Z}} E(u-x,t+1)\delta_0(u) = E(-x,t+1) = E(x,t+1) = \left(1+\Delta^*\right)^t \delta_0(x),$$

where the third equality follows from the even behaviour of $E(x,t)$ as function of x. It means the CK−extension of the discrete delta function is a shift of the fundamental solution. More general, we see that for $a \in \mathbb{Z}$

$$\mathrm{CK}[\delta_a(x)](x,t) = E(a-x,t+1) = \left(1+\Delta^*\right)^t \delta_0(a-x).$$

Proposition 6.15. In operator form, the CK-extension of f is given by

$$\mathrm{CK}[f(x)] = f(x,t) = \exp\left(\theta\Delta^*\right) F(\xi)[1](x,t), \tag{6.28}$$

where $f(x) = F(\xi)[1]$ is the Taylor series expansion of the discrete function f.

Proof. One has

$$\begin{aligned}\partial_t^+ \exp\left(\theta\Delta^*\right) &= \partial_t^+ \sum_{k=0}^{\infty} \frac{\theta^k}{k!} \left(\Delta^*\right)^k \\ &= \sum_{k=0}^{\infty} \frac{\partial_t^+\theta^k}{k!} \left(\Delta^*\right)^k\end{aligned}$$

Repeatedly applying the Weyl relation (6.14)

$$\begin{aligned}&= \sum_{k=1}^{\infty} \frac{k\theta^{k-1}}{k!} \left(\Delta^*\right)^k + \frac{\theta^k}{k!} \left(\Delta^*\right)^k \partial_t^+ \\ &= \Delta^* \exp\left(\theta\Delta^*\right) + \exp\left(\theta\Delta^*\right)\partial_t^+.\end{aligned} \tag{6.29}$$

Letting the left hand side of (6.29) act on $f(x) = F(\xi)[1]$, the second term of (6.29) will vanish as $f(x)$ is only a function of the space variable x. Hence (6.28) fulfills the heat equation. □

Remark 6.16. The operator form of the CK-extension matches the fundamental solution of the mixed heat equation (6.8), where now the variable t has been replaced by the operator θ.

Remark 6.17. In the continuous setting, the correspondence is obvious: the fundamental solution to the heat equation is the Gaussian function. Hence a solution to an initial value problem is found by the convolution of the initial function with the Gaussian kernel, which is the Weierstrass transform. The same obvious relationship is not immediately found in the discrete setting, mainly due to the discretisation of the time variable. The correspondence between the heat equation and the Weierstrass transform is only straightforward when looking at the operational form: if we formally substitute $\theta = \frac{1}{2}$, we find the alternative definition 5.5 of the Weierstrass transform. Because of the particularities of the discrete setting, the correspondence is much less clear when considering the points on the grid.

Example 6.18. Let (6.29) act on $2^t f(x)$ and take in account that $\partial_t^+ \left(2^t f(x)\right) = 2^t f(x)$. Then

$$\begin{aligned} \partial_t^+ \exp\left(\theta\Delta^*\right)[2^t f(x)] &= \Delta^* \exp\left(\theta\Delta^*\right) 2^t f(x) + \exp\left(\theta\Delta^*\right) 2^t f(x) \\ &= \left(1+\Delta^*\right) \exp\left(\theta\Delta^*\right)[2^t f(x)]. \end{aligned} \tag{6.30}$$

In other words: the CK-extension of $2^t f(x)$ is an eigenfunction of the heat equation.

6.3.3 Discrete heat polynomials

The definition of the Cauchy-Kovalevskaya extension in the previous section enables us to define the discrete heat polynomials as solutions to the heat equation with initial condition $p(x,0) = \xi^n[1](x)$.

Definition 6.19. The discrete heat polynomials $h_n(x,t)$ are the solutions of the system

$$\begin{cases} \left(\partial_t^+ - \Delta^*\right) u(x,t) = 0, \\ u(x,0) = \xi^n[1](x), \end{cases} \tag{6.31}$$

i.e.

$$h_n(x,t) = \mathrm{CK}\left[\xi^n[1]\right](x,t). \tag{6.32}$$

To obtain an explicit formula, we calculate:

$$\begin{aligned} h_n(x,t) &= \mathrm{CK}\left[\xi^n[1]\right](x,t) \\ &= \sum_{u\in\mathbb{Z}} E(u-x,t+1)\,\xi^n[1](u) \\ &= \left(1+\Delta^*\right)^t \xi^n[1](x) \\ &= \sum_{j=0}^{t} \binom{t}{j} (\Delta^*)^j \xi^n[1](x). \end{aligned}$$

The star Laplacian Δ^* is factorised by the Dirac operator, hence

$$h_n(x,t) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{t!}{(t-j)!\,j!} \frac{n!}{(n-2j)!} \xi^{n-2j}[1](x). \tag{6.33}$$

Proposition 6.20. The generating function of the discrete heat polynomials is

$$\mathrm{CK}\left[\exp(z\,\xi)[1]\right](x,t) = \exp\left(\theta z^2 + \xi z\right)[1](x,t).$$

Proof. This follows from the explicit calculation:

$$\begin{aligned}\mathrm{CK}\left[\exp(z\,\xi)[1]\right](x,t) &= \left(1+\Delta^*\right)^t \exp(z\,\xi)[1](x) \\ &= \sum_{j=0}^{t}\binom{t}{j}(\Delta^*)^j \sum_{k=0}^{\infty}\frac{z^k \xi^k[1](x)}{k!}.\end{aligned}$$

The star Laplacian acts as the second order derivative

$$= \sum_{k=0}^{\infty}\sum_{j=0}^{t}\binom{t}{j} z^k \frac{\xi^{k-2j}}{(k-2j)!}.$$

Herein, we recognise the explicit form of the heat polynomials

$$\begin{aligned}&= \sum_{k=0}^{\infty}\sum_{j=0}^{t}\frac{\theta^j[1](t)}{j!}\frac{\xi^{k-2j}}{(k-2j)!}\,z^k \\ &= \sum_{k=0}^{\infty} h_k(x,t)\frac{z^k}{k!}.\end{aligned}$$

This clearly yields the same result as

$$\begin{aligned}\exp\left(\theta z^2 + \xi z\right)[1](x,t) &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\theta z^2 + \xi z\right)^k[1](x,t) \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{1}{k!}\frac{k!}{j!\,(k-j)!}\theta^j[1](t)z^{2j}\xi^{k-j}[1](x)z^{k-j} \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{1}{j!\,(k-j)!}\theta^j[1](t)\xi^{k-j}[1](x)z^{j+k}.\end{aligned}$$

Implement a change of summation index an find

$$\begin{aligned}\exp\left(\theta z^2 + \xi z\right)[1](x,t) &= \sum_{n=0}^{\infty}\sum_{j=0}^{\lfloor n/2\rfloor}\frac{1}{j!(n-2j)!}\theta^j[1](t)\xi^{n-2j}[1](x)\,z^n \\ &= \sum_{n=0}^{\infty} h_n(x,t)\,\frac{z^n}{n!}.\end{aligned}$$

□

Example 6.21. Explicitly, we find the following polynomials:

$h_0(x,t) = 1$
$h_1(x,t) = \xi[1](x)$

$$
\begin{aligned}
h_2(x,t) &= \xi^2[1](x) + 2t \\
h_3(x,t) &= \xi^3[1](x) + 6\,t\,\xi[1](x) \\
h_4(x,t) &= \xi^4[1](x) + 12\,t\,\xi^2[1](x) + 12\,t^2 - 12t \\
h_5(x,t) &= \xi^5[1](x) + 20\,t\,\xi^3[1](x) + 60\,t\,(t-1)\xi[1](x) \\
h_6(x,t) &= \xi^6[1](x) + 30\,t\,\xi^4[1](x) + (180\,t^2 - 180\,t)\,\xi^2[1](x) + 120\,t\,(t-1)\,(t-2) \\
h_7(x,t) &= \xi^7[1](x) + 42\,t\,\xi^5[1](x) + 420\,t\,(t-1)\xi^3[1](x) + 840\,t\,(t-1)\,(t-2)\xi[1](x) \\
h_8(x,t) &= \xi^8[1](x) + 56\,t\,\xi^6[1](x) + 840\,t\,(t-1)\xi^4[1](x) + 3360\,t\,(t-1)\,(t-2)\xi^2[1](x) \\
&\quad + 1680\,t\,(t-1)\,(t-2)\,(t-3)
\end{aligned}
$$

Define the operator $\Psi_n(\xi,\theta)$ as

$$\Psi_n(\xi,\theta) = \exp\left(\theta\Delta^*\right)\xi^n,$$

i.e. the operator form of the CK-extension of ξ^n. Based on (6.33), we find

$$\Psi_n(\xi,\theta) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\theta^j}{j!}\frac{n!}{(n-2j)!}\xi^{n-2j}. \tag{6.34}$$

Consider the functions $\Psi_n(\xi,\frac{1}{2})$: these are the polynomials enlisted in table 3.1. Hence, the heat polynomials are connected to the Weierstrass transform of the discrete homogeneous polynomials $\xi^n[1]$. This is of course an immediate consequence of the fact that the operator form of the CK extension, with $\theta = \frac{1}{2}$ is the Weierstrass transform (see remark 6.17).

Comparing the explicit formula of the discrete radial Hermite polynomials (2.33) with the above expression for the discrete heat polynomial operators, the next relation is obtained:

$$(-1)^{\lfloor \frac{n}{2} \rfloor}\Psi_n\left(\xi,-\frac{1}{2}\right) = H_n(\xi).$$

Moreover, if we formally let $\theta = -\frac{1}{2}$ and $\xi[1] = x$ in (6.34), we find the continuous Hermite polynomials. This can also be easily seen by comparing the corresponding generating functions.

It is important to remark that $h_k(x,-t) \neq \Psi_k(\xi,-\theta)[1](x,t)$: it is not sufficient to map t to $-t$ and θ to $-\theta$ to obtain again the same polynomials in x and t. For example:

$$h_5(x,-t) = \xi^5[1](x) - 20\,t\,\xi^3[1](x) - 60\,t\,(-t-1)\xi[1](x)$$

and

$$\Psi_5(\xi,-\theta) = \xi^5 - 20\,\theta\xi^3 + 60\,\theta^2\xi,$$

hence

$$
\begin{aligned}
\Psi_5(\xi,-\theta)[1](x,t) &= \xi^5[1](x) - 20\theta[1](t)\xi^3[1](x) + 60\theta^2[1](t)\xi[1](x) \\
&= \xi^5[1](x) - 20t\xi^3[1](x) + 60\,t\,(t-1)\xi[1](x) \\
&\neq h_5(x,-t).
\end{aligned}
$$

Definition 6.22. We define the heat-Euler operator as

$$\mathbb{E}_{\xi,\theta} := 2\theta\partial_t^+ + \xi\partial_x. \tag{6.35}$$

Lemma 6.23. The discrete heat polynomials are eigenfunctions of the discrete heat-Euler operator: $\mathbb{E}_{\xi,\theta}h_k(x,t) = kh_k(x,t)$.

Proof. Consider $2\theta\partial_t^+ h_k(x,t)$:

$$2\theta\partial_t^+ h_k(x,t) = \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{2\theta\partial_t^+\theta^j[1]}{j!}\frac{k!}{(k-2j)!}\xi^{k-2j}[1].$$

First, for the numerator $2\theta\partial_t^+\theta^j[1]$, we use the Weyl relation $\partial_t^+\theta - \theta\partial_t^+ = 1$ to obtain

$$\begin{aligned} 2\theta\partial_t^+\theta^j[1] &= 2\theta\left(1+\theta\partial_t^+\right)\theta^{j-1}[1] \\ &= 2\theta^j + 2\theta^2\left(1+\theta\partial_t^+\right)\theta^{j-2}[1]. \end{aligned}$$

Repeating the same formula eventually yields

$$2\theta\partial_t^+\theta^j[1] = 2j\theta^j + 2\theta^{j+1}\partial_t^+[1] = 2j\theta^j. \tag{6.36}$$

Secondly, consider

$$\xi\partial_x h_k(x,t) = \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{\theta^j[1](t)}{j!}\frac{k!}{(k-2j)!}\xi\partial_x\xi^{k-2j}[1](x).$$

The last part can be simplified, also using the corresponding Weyl relations:

$$\begin{aligned} \xi\partial_x\xi^{k-2j}[1](x) &= \xi\left(1+\xi\partial_x\right)\xi^{k-2j-1}[1](x) \\ &= \xi^{k-2j} + \xi^2\left(1+\xi\partial_x\right)\xi^{k-2j-2}[1](x). \end{aligned}$$

Repeatedly applying the same reasoning yields

$$\xi\partial_x\xi^{k-2j}[1](x) = (k-2j)\xi^{k-2j} + \xi^{k-2j+1}\partial_x[1](x) = (k-2j)\xi^{k-2j}[1](x). \tag{6.37}$$

Combining (6.36) and (6.37), we arrive at

$$\begin{aligned} \mathbb{E}_{\xi,\theta}h_k(x,t) &= 2\theta\partial_t^+ h_k(x,t) + \xi\partial_x h_k(x,t) \\ &= \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{2j\theta^j[1]}{j!}\frac{k!}{(k-2j)!}\xi^{k-2j}[1](x) + \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{\theta^j[1](t)}{j!}\frac{k!}{(k-2j)!}(k-2j)\xi^{k-2j}[1](x) \\ &= k\sum_{j=0}^{\lfloor k/2 \rfloor} \frac{\theta^j[1](t)}{j!}\frac{k!}{(k-2j)!}\xi^{k-2j}[1](x) = k\,h_k(x,t). \end{aligned}$$

This concludes the proof. □

6.3.4 Associated functions

In the continuous setting, the functions $q_n(x,t)$, (6.3) are called *associated* to the heat polynomials: they are dual, in the sense that there is an orthogonality relation with the heat polynomials, see (6.5). We investigate whether a similar result exists in this discrete Clifford setting, being inspired by the fact that - in the continuous setting - the associated functions are derivatives of the fundamental solution $G(x,t)$. Define

$$q_n(x,t) := \frac{1}{n!}\partial^n E(x,-t), \tag{6.38}$$

i.e. the n-th derivative of the fundamental solution. This function $q_n(x,t)$ satisfies the anti-heat equation. We can interpret it as a distribution because $E(x,t)$ is the density function of the distribution $\mathbb{E}(x,t)$. The interpretation as a distribution will stress the duality argument between $h_n(x,t)$ and $q_n(x,t)$. To proceed, we will need some discrete analogues to continuous integral formulae, with respect to the differential operators $\partial_t^{\pm}$ and the star Laplacian Δ^*. The first lemma is a Green's formula, an analogue of integration by parts.

Lemma 6.24 (Green's formula)**.** Let f and g be two discrete functions, depending on time and space. If at least one of both functions has compact support with respect to the time variable t, then

$$\sum_{t\in\mathbb{Z}} f(x,t)\partial_t^+ g(x,t) = -\sum_{t\in\mathbb{Z}} \partial_t^- f(x,t) g(x,t). \tag{6.39}$$

If at least one of both functions has compact support with respect to the space variable x, then

$$\sum_{x\in\mathbb{Z}} \Delta^* f(x,t)g(x,t) - f(x,t)\Delta^* g(x,t) = 0. \tag{6.40}$$

Proof. Let f and g be as stated in the proposition, with at least one of both functions having compact support with respect to t. It holds that

$$\begin{aligned}\sum_{t\in\mathbb{Z}} f(x,t)\partial_t^+ g(x,t) &= \sum_{t\in\mathbb{Z}} f(x,t)\left(g(x,t+1) - g(x,t)\right)\\ &= \sum_{t\in\mathbb{Z}} \left(f(x,t-1) - f(x,t)\right) g(x,t)\\ &= -\sum_{t\in\mathbb{Z}} \partial_t^- f(x,t) g(x,t).\end{aligned} \tag{6.41}$$

Analogously, we have that

$$\sum_{x\in\mathbb{Z}} f(x,t)\left(\Delta^* g(x,t)\right) = \sum_{x\in\mathbb{Z}} f(x,t)\left(\Delta_x^+\Delta_x^- g(x,t)\right) = -\sum_{x\in\mathbb{Z}}\left(\Delta_x^- f(x,t)\right)\left(\Delta_x^- g(x,t)\right).$$

and

$$\sum_{x\in\mathbb{Z}} \left(\Delta^* f(x,t)\right) g(x,t) = \sum_{x\in\mathbb{Z}} \left(\Delta_x^+\Delta_x^- f(x,t)\right) g(x,t) = -\sum_{x\in\mathbb{Z}}\left(\Delta_x^- f(x,t)\right)\left(\Delta_x^- g(x,t)\right).$$

□

Remark 6.25. The condition of one of both functions having compact support is actually too strong: it might be possible for the sums in (6.39) and (6.40) to converge, even if f nor g has compact support.

The next lemma shows that we can shift solutions in time: solutions to the (anti-)heat equation are invariant for translations in time.

Lemma 6.26. Fix a and $b \in \mathbb{N}$, let $f(x,t)$ be a solution to the anti-heat equation in $[a,b[$ and $g(x,t)$ a solution to the heat equation, with one of both functions having compact support with respect to x. Then

$$\sum_{x\in\mathbb{Z}} f(x,b-1)g(x,b) = \sum_{x\in\mathbb{Z}} f(x,a-1)g(x,a). \tag{6.42}$$

Proof. Let f and g be as stated in the proposition, i.e.

$$\begin{aligned} \partial_t^- f(x,t) &= -\Delta^* f(x,t), \qquad a \le t < b, \\ \left(\partial_t^+ - \Delta^*\right) g(x,t) &= 0. \end{aligned}$$

Denote by χ the characteristic function on the interval $[a,b[$:

$$\chi(t) = \begin{cases} 1, & \text{if } a \le t < b; \\ 0, & \text{else.} \end{cases}$$

Then it holds that

$$\partial_t^+ \chi(t) = \chi(t+1) - \chi(t) = \delta_{a-1} - \delta_{b-1}$$

and $g(x,t)\chi(t)$ now is a solution to the heat equation on $[a,b[$.

It holds that

$$\begin{aligned} \partial_t^+ \left[\chi(t)g(x,t)\right] &= \chi(t+1)g(x,t+1) - \chi(t)g(x,t) \\ &= \big(\chi(t+1) - \chi(t)\big)\, g(x,t+1) + \chi(t)\, \big(g(x,t+1) - g(x,t)\big) \\ &= \left(\delta_{a-1} - \delta_{b-1}\right) g(x,t+1) + \chi(t)\partial_t^+ g(x,t). \end{aligned} \tag{6.43}$$

As g fulfills the heat equation, this equals

$$= \left(\delta_{a-1} + \delta_{b-1}\right) g(x,t+1) + \chi(t)\Delta^* g(x,t). \tag{6.44}$$

Hence,

$$\begin{aligned} f(x,t)\partial_t^+ \left[\chi(t)g(x,t)\right] &= f(x,t)\left[\left(\delta_{a-1} - \delta_{b-1}\right) g(x,t+1)\right] + f(x,t)\chi(t)\Delta^* g(x,t) \\ &= -f(x,t-1)g(x,t)\Big|_a^b + f(x,t)\chi(t)\Delta^* g(x,t). \end{aligned}$$

We now use (6.39), which is allowed since χg has compact support with respect to t.

$$\sum_{(x,t)\in\mathbb{Z}\times\mathbb{N}} \left(\partial_t^- f\right)\chi g + f\left(\partial_t^+(\chi g)\right) = 0$$

which yields

$$\sum_{(x,t)\in\mathbb{Z}\times\mathbb{N}} -\Delta^* f(x,t)\chi(t)g(x,t) - f(x,t-1)g(x,t)\Big|_a^b + f(x,t)\Delta^* g(x,t) = 0$$

and is equivalent to

$$\sum_{x\in\mathbb{Z}} f(x,b-1)g(x,b) = \sum_{x\in\mathbb{Z}} f(x,a-1)g(x,a).$$

This sum is well-defined, because at least one of both functions has compact support with respect to x. □

Let us now come back to the associated functions and apply lemma 6.26 to the functions $q_n(x,t)$ and $h_m(x,t)$, on the interval $[-T,0[, T>0$:

$$\sum_{x\in\mathbb{Z}} h_m(x,0)q_n(x,-1) = \sum_{x\in\mathbb{Z}} h_m(x,-T)q_n(x,-T-1). \tag{6.45}$$

On the one side, because $\delta_0(x)$ is the density function of the distribution $\boldsymbol{\delta}_0(x)$, the left hand side of (6.45) is equal to

$$\begin{aligned}\sum_{x\in\mathbb{Z}} h_m(x,0)\frac{\partial^n}{n!}\delta_0(x) &= \sum_{x\in\mathbb{Z}} \xi^m[1](x)\,\frac{1}{n!}\partial^n\delta_0(x)\\ &= \sum_{x\in\mathbb{Z}} \frac{1}{n!}\left\langle \partial^n\boldsymbol{\delta}_0, \xi^m[1](x)\right\rangle\\ &= \sum_{x\in\mathbb{Z}} \frac{(-1)^n}{n!}\left\langle \boldsymbol{\delta}_0, \partial^n\xi^m[1](x)\right\rangle\\ &= \frac{(-1)^n}{n!}\partial^n\xi^m[1](0)\\ &= \frac{(-1)^n}{n!}\frac{m!}{(m-n)!}\xi^{m-n}[1](0)\\ &= (-1)^m\delta_{m,n}.\end{aligned}$$

The last equality follows from the fact that $\xi^k[1](0) = 0, \forall k\in\mathbb{N}$. On the other side, the right hand side of (6.45) is

$$\sum_{x\in\mathbb{Z}} h_m(x,-T))\frac{1}{n!}\partial^n E(x,T+1).$$

All together, we obtain

$$\sum_{x\in\mathbb{Z}} h_m(x,-T)\left(\frac{(-\partial)^n}{n!}\left(1+\Delta^*\right)^T\delta_0(x)\right) = \delta_{m,n}. \tag{6.46}$$

In other words, the $m-$th order heat polynomial $h_m(x,t)$ is orthogonal to the $n-$th derivative of the fundamental solution $E(x,t)$. This is exact the result that was established in the continuous case, see (6.5), by Rosenbloom and Widder in [53].

6.4 Conclusion

In this chapter, we formulated a heat equation, for a setting in which both time and space are discrete:

$$\left(\partial_t^+ - \Delta^*\right) f(x,t) = 0. \tag{6.47}$$

Here, ∂_t^+ is the forward difference operator with respect to the time variable t. Similar to the operator calculus with respect to x, we defined a raising operator θ, acting on polynomials as

$$\theta[t^k] = t\,(t-1)^k,$$
$$\theta^k[1](t) = \frac{t!}{(t-k)!.}$$

The interaction of θ and ∂_t^+ is given by the Weyl relation

$$\partial_t^+\theta - \theta\partial_t^+ = 1.$$

We expanded the dual space of distributions, now to be defined on the space of discrete polynomials in x and t. The dual aspect becomes clear when we consider the Weyl relation for distributions

$$\partial_t^-\theta - \theta\partial_t^-.$$

A fundamental solution for the discrete heat equation is given by

$$E(x,t) = \chi_{t>0}(t)\left(1+\Delta^*\right)^{t-1}\delta_0(x).$$

We furthermore defined the anti-heat equation as

$$\left(\partial_t^- + \Delta^*\right) f(x,t) = 0. \tag{6.48}$$

A fundamental solution for the anti-heat equation is then given by $E(x,-t)$. In order to find a solution of the initial value problem

(i) $f(x,0) = f(x)$,

(ii) $\left(\partial_t^+ - \Delta^*\right) f(x,t) = 0$ on $\mathbb{Z}\times\mathbb{N}$,

we constructed the Cauchy Kovalevskaya extension:

$$\mathrm{CK}[f(x)] = E(x,t+1) * f(x) = \left(1+\Delta^*\right)^t f(x). \tag{6.49}$$

The CK-extension of f fulfills this initial value problem. In particular, the heat polynomials $h_n(x,t)$ are the solutions to the heat equation with initial condition $f(x,0) = \xi^n[1](x)$. Finally, we constructed a family of functions $q_n(x,t)$, orthogonal to the heat polynomials:

$$q_n(x,t) := \frac{1}{n!}\partial_x^n E(x,-t).$$

These functions satisfy the anti-heat equation, again reflecting the dual aspect between the heat and anti-heat equation.

7

Appendix

7.1 English summary

In this summary, we give a short overview on the most important concepts and results in this thesis. We give a general overview of the content by chapter.

The Weierstrass transform, named after the German mathematician Karl Weierstrass, is a fundamental operator in mathematical analysis en applied mathematics. It averages the values of a function f by making the convolution with a Gaussian kernel to obtain a 'smoothed' version of f. The Weierstrass transform's smoothing properties and its relationship with the Gaussian kernel make it useful in a variety of mathematical and practical applications, such as solving the heat equation, signal processing, image processing, quantum mechanics and numerical analysis.

This thesis is to be situated in the context of discrete Clifford analysis. Euclidean Clifford analysis is a recent branch of mathematics. It studies functions defined in Euclidean space, with values in a real or complex Clifford algebra. Clifford analysis can be seen as a higher-dimensional theory of complex analysis as well as a refinement of harmonic analysis. Similar to complex analysis, the central objects are a differential operator, called the Dirac operator and its solutions, called monogenic functions. They are the subject of a variety of important results and have a range of uses, amongst others solving boundary value problems and harmonic analysis.

Discrete Clifford analysis originated from the need for numerical applications. Its development is driven by the desire to study function theory and harmonic analysis on discrete lattices or grids, especially in higher-dimensional settings. The discrete setting introduces new challenges, such as defining a suitable Dirac operator which allows for a discrete counterpart of key results in classical Clifford analysis. Several models for the discrete Clifford algebra have been established, either starting from an application or from a function theoretic point of view. The discrete Hermitian setting was introduced by Brackx, De Schepper, Sommen and Van de Voorde ([27, 28, 29]) and will form the basis for this dissertation.

The main aim of this dissertation is the definition of a discrete Weierstrass transform, together with a function space for which the transform is well-defined, in the discrete Hermitian Clifford setting. Therefore, we are inspired by the classical definition of the Weierstrass transform in combination with the already defined tools in the discrete Hermitian Clifford analysis, by (amongst others) De Ridder, De Schepper, Sommen en Van de Voorde, [30, 31, 44, 33, 34, 35]. The main idea is to use the discrete Gauss distribution G as a weight function and the discrete Hermite polynomials will form the basis of the space of functions which admit a Weierstrass transform.

Chapter 2 comprises some basic notions and preliminaries on the classical and discrete Clifford analysis that are being used in this thesis.

In the third chapter, we get started with the definition of the Weierstrass transform in one dimension. As already mentioned, the discrete Gauss distribution G has a crucial role: we let the composition of a discrete function with this Gaussian act on a well-defined kernel, resulting in a direct analogue to the continuous integral transform. We define an inner product and corresponding norm in order to construct the discrete Weierstrass space: discrete functions that are linear combinations of Hermite polynomials, for which their norm is finite. A natural and important question is whether the delta-functions, the building blocks of discrete function theory, are elements of the discrete Weierstrass space. Similarly, we investigate if the discrete Hermite functions are contained in this space. Indeed, for both families of functions, we are able to prove that they can be written as (infinite) linear combination of discrete Hermite polynomials and that they fullfill the condition to be elements of the discrete Weierstrass space.

The above is carried out on a standard grid with mesh width $h = 1$. As a last part of chapter three, we will generalise this to a grid with general mesh width h. In particular, we are interested in the asymptotic behaviour if h tends to 0, as this is the continuous (classical) situation. It turns out that our definitions are indeed consistent with the classical case.

In chapter four, we extend the theory from dimension $m = 1$ to $m > 1$. this brings some complications. Firstly, we need to handle the anti-commutativity of the basic Clifford elements: $e_i e_j = -e_j e_i$. This problem will be solved by using a rotation invariant operator. Secondly, the radial Hermite polynomials have to be replaced by the generalised Hermite polynomials, in order to form a basis for the discrete Weierstrass space. Those generalised Hermite polynomials are formed as the composition of a monogenic polynomial of order r and a Hermite polynomial of degree n. The goal is to obtain recurrence formulae, both in terms of the degree n of the Hermite polynomial and in terms of the degree r of the monogenic, for the Weierstrass transform of this generalised Hermite polynomial. In order to fix ideas and limit notations, we start in dimension $m = 2$, this gives us a good idea of the results when $m \geq 2$. However, the reasonings and proofs in dimension two relies on the explicit form of the monogenic polynomials. This is not extendable to higher dimensions. We need another approach to find an expression for the Weierstrass transform of the generalised Hermite polynomials. We give two alternative definitions: one is based on discrete translations, the other is based on the classical formal expression of the Weierstrass transform. The latter will lead us to the formula we are looking for.

In the last chapter, we focus on a different subject: the discrete heat equation. As a logical progression of a first discretisation in [2], where space is discrete but time is continuous, we now formulate a heat equation where both space and time are discrete. Therefore, we introduce a new operator with respect to the time variable. We discuss the fundamental solutions of the discrete heat equation and how we solve an initial value problem. Finally, we obtain the discrete heat polynomials and discuss a family of functions that are orthogonal to and can be interpreted as dual to these heat polynomials.

The content of chapter three, the definitions of the discrete Weierstrass transform and space, has been published in [37], while the generalisations for dimensions $m > 1$ and grid mesh width $h \neq 1$ were subject of a second paper, [38].

7.2 Nederlandstalige samenvatting

In deze Nederlandstalige samenvatting zal ik kort overlopen waar dit proefschrift zich situeert en wat het doel is. Vervolgens geef ik een algemeen overzicht van de inhoud, geordend per hoofdstuk.

De Weierstrass transformatie, genoemd naar de Duitse wiskundige Karl Weierstrass, is een fundamentele operatie in de wiskundige analyse en toegepaste wiskunde. Het is een operator die een gegeven functie omzet in een 'gladdere' functie: een functie met mooiere en interessantere eigenschappen. Het gaat om een integraaltransformatie, waarbij de convolutie met een Gaussische kern wordt gemaakt. De Weierstrass transformatie heeft tal van toepassingen, in onder meer het oplossen van de warmtevergelijking, signaalprocessen, beeldverwerking, kwantummechanica en numerieke analyse.

Deze thesis situeert zich in de context van discrete Clifford analyse. Euclidische Clifford analyse is een vrij recente tak van de wiskunde die, eenvoudig gezegd, functies bestudeert die gedefinieerd zijn in de Euclidische ruimte en waarden hebben in een reële of complexe Clifford algebra. Clifford analyse kan zowel gezien worden als een hogerdimensionale uitbreiding op complexe analyse alsook als een verfijning van de klassieke harmonische analyse. Net zoals in de complexe analyse, is het centrale object een differentiaaloperator: de zogenaamde Dirac operator. Oplossingen hiervan worden monogene functies genoemd. Een andere belangrijke operator, de Laplaciaan, wordt gefactoriseerd door de Dirac operator, waardoor oplossingen van de Dirac operator ook oplossingen van de Laplaciaan zijn. De Dirac operator en zijn oplossingen zijn het onderwerp van tal van belangrijke resultaten en kennen meerdere toepassingen, in onder andere het oplossen van randwaardeproblemen en harmonische analyse.

Discrete Clifford analyse is ontstaan uit de nood aan numerieke toepassingen. Men wilde de resultaten uit de klassieke Clifford analyse kunnen toepassen op roosters, in het bijzonder in hogere dimensies. Dit bracht nieuwe uitdagingen met zich mee, zoals het definiëren van een discrete Dirac operator, die ook de discrete Laplaciaan factoriseert. Verschillende methoden en invalshoeken werden voorgesteld en uitgewerkt, sommigen waren eerder theoretische van aard, anderen meer toepassingsgericht. De discrete Hermitische setting werd geïntroduceerd door Brackx, De Schepper, Sommen en Van de Voorde ([27, 28, 29]) en vormt de basis voor dit proefschrift.

Het belangrijkste doel van deze thesis is om een discrete versie van de Weierstrass transformatie te definiëren in de discrete Hermitische Clifford setting. We baseren ons daarvoor op de bestaande definities in het continue geval en op de reeds geïntroduceerde concepten in de discrete Clifford analyse, door onder andere De Ridder, De Schepper, Sommen en Van de Voorde ([30, 31, 44, 33, 34, 35]). De discrete Gauss distributie zal dienen als de gewichtsfunctie en de discrete Hermite polynomen zullen de basis vormen van de ruimte van functies waarop we de Weierstrass transformatie willen definiëren.

We starten in hoofdstuk twee met de herhaling van de definities, concepten en resultaten uit de klassieke en discrete Clifford analyse, dewelke nodig zijn in dit werk.

In hoofdstuk drie gaan we aan de slag met de definitie van de Weierstrass transformatie in één dimensie. Zoals reeds vermeld, speelt de discrete Gauss distributie G hier een cruciale rol: we laten de samenstelling van de te transformeren functie f en G inwerken op een slim gekozen kern. Op deze manier vinden we een directe analogie met de continue integraaltransformatie. We construeren ook een Weierstrassruimte $\mathcal{W}$, een ruimte van functies analoog aan de continue $\mathcal{L}_2-$ ruimte, met bijhorend inproduct en norm. De discrete Hermite veeltermen vormen nu een basis van $\mathcal{W}$. We onderzoeken wat de voorwaarden zijn opdat een discrete functie tot de Weierstrassruimte behoort. We geven enkele voorbeelden van eenvoudige discrete functies met hun Weierstrass transformatie. Een belangrijke vraag is of de discrete deltafuncties ook bevat zijn in $\mathcal{W}$, vermits dit de bouwblokken zijn van de discrete functietheorie. Hetzelfde onderzoeken we voor de discrete Hermitefuncties. Voor beide verzamelingen van functies vinden we inderdaad dat ze kunnen geschreven worden als (oneindige) lineaire combinatie van de discrete Hermiteveeltermen én dat de voorwaarde opdat ze bevat zijn in de discrete Weierstrass ruimte, vervuld is.

Bovenstaande wordt allemaal uitgevoerd op een standaardrooster met stapgrootte $h = 1$. Als laatste onderdeel van hoofdstuk drie, veralgemenen we dit naar een rooster met algemene stapgrootte h. We zijn in het bijzonder geïnteresseerd in het asymptotisch gedrag als h naar 0 nadert. Dat is namelijk de situatie van de continue (klassieke) Clifford analyse. Het blijkt dat onze definities inderdaad in overeenstemming zijn met het klassieke geval, als we h tot 0 laten naderen.

In hoofdstuk vier veralgemenen we de theory van dimensie $m = 1$ naar dimensie $m > 1$. Dit brengt enkele complicaties met zich mee. Enerzijds stuiten we nu op het feit dat de basisvectoren van de Clifford algebra anticommutatief zijn: $e_ie_j = -e_je_i$. De invoering van een rotatie-invariante operator zal hier een oplossing bieden. Anderzijds volstaan de radiale Hermite veeltermen niet meer als basisfuncties voor de Weierstrassruimte. De *veralgemeende* Hermite veeltermen, gevormd door de samenstelling van een Hermite veelterm van graad n en een monogene veelterm van graad r brengen soelaas. We proberen recurrente betrekkingen te vinden, zowel in functie van de graad n als in functie van de graad r, om de Weierstrass transformatie van deze veralgemeende Hermite veeltermen te beschrijven. In eerste instantie doen we dit in dimensie $m = 2$, wat ons een goed beeld geeft van de resultaten als $m > 2$. Echter, het bewijs in dimensie twee steunt op de expliciete vorm van de monogene veeltermen, iets wat niet veralgemeenbaar is in hogere dimensies. Een andere aanpak is nodig om een uitdrukking te vinden voor de Weierstrass transformatie van de veralgemeende Hermite veeltermen. We geven twee alternatieve definities: de eerste is gebaseerd op discrete translaties, de tweede op een formele vorm van de Weierstrass transformatie in het klassieke geval. Deze laatste levert ons de gezochte formule op.

In het laatste hoofdstuk focussen we op een ander onderwerp: de discrete warmtevergelijking. In navolging van een eerste discretisering van de warmtevergelijking in [2], waarin de ruimte discreet is, maar de tijd continu, formuleren we nu een warmtevergelijking waarbij zowel de tijd als de ruimte discreet zijn. Daartoe introduceren we een nieuwe operator met betrekking tot de tijdsvariabele. We bediscussiëren de fundamentele oplossingen van de discrete warmetevergelijking en hoe we een beginvoorwaardeprobleem moeten oplossen. Tot slot verkrijgen we de discrete warmteveeltermen en bespreken we een set veeltermen die we kunnen interpreteren als duaal aan de warmteveeltermen.

In een eerste paper, [37], werden de definities van de discrete Weierstrasstransformatie en -ruimte gepubliceerd. De veralgemeningen hiervan naar stapgrootte $h \neq 1$ en naar hogere dimensies $m > 1$ waren het onderwerp van een tweede paper [38].

7.3 Maple calculations for section 3.3

Asymptotic behaviour density function g for h->0

$$B := p \to \frac{\sqrt{2}\sqrt{\pi}\,\mathrm{BesselI}\left(\frac{p}{h}, \frac{1}{h^2}\right)}{e^{\frac{1}{h^2}}} \cdot \frac{1}{\mathrm{abs}(h)}$$

$$B := p \mapsto \frac{\sqrt{2}\cdot\sqrt{\pi}\cdot\mathrm{BesselI}\left(\frac{p}{h}, \frac{1}{h^2}\right)}{e^{\frac{1}{h^2}}\cdot|h|} \quad (1.1)$$

$$e := z \to \mathrm{sqrt}(1+z^2) + \ln\left(\frac{z}{1+\mathrm{sqrt}(1+z^2)}\right)$$

$$e := z \mapsto \sqrt{1+z^2} + \ln\left(\frac{z}{1+\sqrt{1+z^2}}\right) \quad (1.2)$$

$$Lim := (\mathrm{nu}, z) \to \frac{1}{\mathrm{sqrt}(2\cdot\mathrm{Pi}\cdot\mathrm{nu})}\cdot\frac{\exp(\mathrm{nu}\cdot e(z))}{(1+z^2)^{\frac{1}{4}}}$$

$$Lim := (\nu, z) \mapsto \frac{e^{\nu\cdot e(z)}}{\sqrt{2\cdot\pi\cdot\nu}\cdot(1+z^2)^{1/4}} \quad (1.3)$$

$$Blim := p \to \frac{\mathrm{sqrt}(2\cdot\mathrm{Pi})\cdot Lim\left(\frac{p}{h}, \frac{1}{p\cdot h}\right)}{\exp\left(\frac{1}{h^2}\right)}\cdot\frac{1}{\mathrm{abs}(h)}$$

$$Blim := p \mapsto \frac{\sqrt{2\cdot\pi}\cdot Lim\left(\frac{p}{h}, \frac{1}{p\cdot h}\right)}{e^{\frac{1}{h^2}}\cdot|h|} \quad (1.4)$$

$$Blimeven := p \to piecewise(p > 0, Blim(p), p < 0, Blim(-p))$$

$$Blimeven := p \mapsto \begin{cases} Blim(p) & 0 < p \\ Blim(-p) & p < 0 \end{cases} \quad (1.5)$$

$$simplify(Blim(p))$$

$$\frac{\left(\frac{1}{p\,h\left(1+\sqrt{\frac{p^2 h^2+1}{p^2 h^2}}\right)}\right)^{\frac{p}{h}} e^{\frac{p\,h\sqrt{\frac{p^2h^2+1}{p^2h^2}}-1}{h^2}}}{|h|\sqrt{\frac{p}{h}}\left(\frac{p^2 h^2+1}{p^2 h^2}\right)^{1/4}} \quad (1.6)$$

7.4 Tables 7.1 and 7.2

H_n \ P_r	0	1	2	3
0	1	$(z_1e_1 - z_2e_2)$	$(z_1e_1 - z_2e_2)(z_1e_1 + z_2e_2)$	$(z_1e_1 - z_2e_2)(z_1e_1 + z_2e_2)(z_1e_1 - z_2e_2)$
1	$(z_1e_1 + z_2e_2)$	$(z_1e_1 + z_2e_2)(z_1e_1 - z_2e_2)$	$(z_1e_1 + z_2e_2)(z_1e_1 - z_2e_2)(z_1e_1 + z_2e_2)$	$(z_1e_1 + z_2e_2)(z_1e_1 - z_2e_2)(z_1e_1 + z_2e_2)(z_1e_1 - z_2e_2)$
2	$-(z_1e_1 + z_2e_2)^2$	$-(z_1e_1 + z_2e_2)^2(z_1e_1 - z_2e_2)$	$-(z_1e_1 + z_2e_2)^2(z_1e_1 - z_2e_2)(z_1e_1 + z_2e_2)$	$-(z_1e_1 + z_2e_2)^2(z_1e_1 - z_2e_2)(z_1e_1 + z_2e_2)(z_1e_1 - z_2e_2)$
3	$-(z_1e_1 + z_2e_2)^3$	$-(z_1e_1 + z_2e_2)^3(z_1e_1 - z_2e_2)$	$-(z_1e_1 + z_2e_2)^3(z_1e_1 - z_2e_2)(z_1e_1 + z_2e_2)$	$-(z_1e_1 + z_2e_2)^3(z_1e_1 - z_2e_2)(z_1e_1 + z_2e_2)(z_1e_1 - z_2e_2)$
4	$(z_1e_1 + z_2e_2)^4$	$(z_1e_1 + z_2e_2)^4(z_1e_1 - z_2e_2)$	$(z_1e_1 + z_2e_2)^4(z_1e_1 - z_2e_2)(z_1e_1 + z_2e_2)$	$(z_1e_1 + z_2e_2)^4(z_1e_1 - z_2e_2)(z_1e_1 + z_2e_2)(z_1e_1 - z_2e_2)$
5	$(z_1e_1 + z_2e_2)^5$	$(z_1e_1 + z_2e_2)^5(z_1e_1 - z_2e_2)$	$(z_1e_1 + z_2e_2)^5(z_1e_1 - z_2e_2)(z_1e_1 + z_2e_2)$	$(z_1e_1 + z_2e_2)^5(z_1e_1 - z_2e_2)(z_1e_1 + z_2e_2)(z_1e_1 - z_2e_2)$

Table 7.1: Weierstrass transform of generalised Hermite polynomials $H_{n,2,r}P_r$ in two dimensions.

H_n \ $\tilde{P}_r$	0	1	2	3
0	1	$(z_1e_1 + z_2e_2)$	$(z_1e_1 + z_2e_2)(z_1e_1 - z_2e_2)$	$(z_1e_1 + z_2e_2)(z_1e_1 - z_2e_2)(z_1e_1 + z_2e_2)$

Table 7.2: Weierstrass transform of $H_{0,2,r}\tilde{P}_r = \tilde{P}_r$ in two dimensions.

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