No six-cell neighborhood cellular automaton solves the parity problem

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Abstract

The parity problem is one of the best-known classification problems studied to examine the computational abilities of cellular automata. In this inverse problem, one is looking for a cellular automaton that can classify each initial configuration into one of two classes according to its parity. In the case of deterministic one-dimensional cellular automata, there exists a local rule that effectively solves the parity problem, but it is unknown whether it is the simplest possible rule. Specifically, it is known that a cellular automaton with a nine-cell neighborhood can solve the parity problem, whereas no cellular automaton with a five-cell neighborhood is capable of doing so. These findings have remained unimproved for the past 10 years. In this paper, we present novel tools that allow to narrow down the existing gap. With the help of these tools, we are able to demonstrate that there is no cellular automaton with a six-cell neighborhood capable of solving the parity problem.

Keywords: Cellular Automata; classification problems; parity problem

1. Introduction

Cellular automata (CAs) constitute a powerful tool for simulating complex systems. Through the application of simple local rules, they can generate a wide range of behaviors and patterns that can be used to model a variety of physical and biological phenomena. Since CAs can be regarded as models of distributed computation without common global memory, various decision problems have been defined and studied (see, for example, [1, 2, 3, 4]) in order to lay bare the computational capabilities of CAs. One type of computational decision problems presented to CAs, especially in the binary case and for finite grids with periodic boundary conditions, are classification problems in which initial configurations should be properly classified according to one of their properties. In particular, if the initial configuration satisfies a given property, then the global state of the grid should converge to the fixed point of all 1s, and to all 0s otherwise.

One of the most frequently studied decision problems is the parity problem, being almost as popular as the famous Density Classification Problem (DCP). In its classical formulation, this problem consists of finding a CA that can properly classify each initial configuration into two classes according to its parity: if the initial configuration contains an odd number of 1s, then the global state of the grid should converge to the fixed point of all 1s; otherwise, *i.e.*, if the initial configuration contains an even number of 1s, then the global state of the grid should converge to the fixed point of all 0s. However, according to the formulation of the problem, it is immediately clear that it makes no sense for even-sized grids (see, for example, [5]). For this reason, the solution to this problem is considered to be a CA (actually, a local rule f) that correctly classifies all configurations of odd length. This classification problem appears to be much harder than the DCP because the output could be altered simply by a flip in any of the input bits. It was therefore a big surprise when it turned out that, unlike DCP, the parity problem had a solution.

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Betel *et al.* [5] described the local rule with radius 4 (*i.e.*, with a nine-cell neighborhood), called BFO rule, that solves the parity problem[2]. In general, the idea underlying the BFO rule design is to reduce the number of blocks of 0s and 1s present in the initial configuration. This is accomplished by propagating blocks of 1s to the right side, two cells per iteration, combined with the simultaneous propagation of 0s to the left. Actually, the BFO rule is much more complicated, as it must include the appropriate stopping conditions for these two major trends to coexist. Truly, the authors of [5] have shown remarkable ingenuity both in terms of the construction of the BFO rule and the proof that it actually solves the parity problem.

In the same paper, it is shown that there is no binary CA with radius at most two (*i.e.*, for a neighborhood size not greater than 5) that would solve the parity problem for all odd-sized grids. Betel *et al.* [5] also leave open the question of whether there is such a rule of radius 3 (*i.e.*, with a seven-cell neighborhood). They tried, admittedly, to find such a rule using a sophisticated evolutionary algorithm. However, after several failed attempts, they made the following conjecture:

Conjecture 1. *There is no local rule with radius three that can solve the parity problem.*

Although the results presented in [5] are magnificent, there is fairly large gap left, because nothing is known for local rules with neighborhoods of size 6, 7, or 8. Today, more than 10 years after publication, the above results have still not been improved and the gap has not been narrowed. Of course, many attempts have been made to tighten the gap (see, e.g., [6]), but no conclusive results have ever been obtained. This is largely because we cannot just search the entire local rule space and see whether there is one that solves the parity problem or not. Indeed, let d be the size of the neighborhood, then the local rule space has as many as 2^{2^d} elements, and already for $d = 6$ this yields a huge number of 18 446 744 073 709 551 616 local rules to check.

It is worth recalling that the first hopes for a solution to the parity problem by CAs were related to the simplest ones, namely Elementary Cellular Automata (ECAs), *i.e*, one-dimensional binary CAs with radius one. Of course, the natural candidate for a solution among these CAs was ECA150, whose local rule simply computes neighborhood parity (see, for example, [7] for Wolfram's notation). Unfortunately, Sipper [4] demonstrated that no ECA can solve the parity problem. However, he proposed a non-uniform cellular automaton based on two ECAs that could solve a modified version of the parity problem. We can say that the non-uniformity that was used in [4] was spatial because each cell in the grid had a specific local rule assigned to it, which it used at all time steps. Another type of non-uniformity was used by Lee *et al.* in [8], namely temporal non-uniformity. They constructed a sequence of ECAs solving the parity problem: the cells of the grid used the same ECA at each given time step, but the considered ECA was replaced by another one after a certain number of time steps. This approach was later simplified by Martins *et al.* [9] in terms of time complexity. Ninagawa [10] showed that ECA60 alone can solve the parity problem for configurations with length equal to a power of 2. Furthermore, Ruivo *et al.* [11] have designed a solution to the parity problem by using ECA150 with a specific deterministic asynchronous update scheme, in which the cell positions are indexed and the updates alternate between the ones with odd and those with even indexes. Another solution to the parity problem using ECA150 alone is the solution proposed in [12]. This result was obtained by changing the way of connecting the cells in a grid: the authors use a family of directed, non-circulant, regular graphs. Yet another idea for solving the parity problem was proposed by Fatès [13], using non-deterministic CAs and some kind of fully asynchronous updating for this purpose.

In this paper, we return to the classical formulation of the parity problem. Using an innovative method, we are able to get around the difficulties associated with the incredibly high number of 2^{2^6} local rules with a six-cell neighborhood and show that no such local rule solves this problem in the classical sense. In addition, the full proof of this fact does not require the use of a computer, which gives hope that the method

[2]There is a slight inaccuracy in the original formulation in [5], which is easily corrected. Namely, the sentence "*Local shift* a (101) block is transformed into (110) if there are a 0 on its left and at least two 0s on its right (combination of transitions T_7 and T_8)." should be "*Local shift* a (101) block is transformed into (011) if there are at least two 0s on its left and a 0 on its right (combination of transitions T_7 and T_8)." and consequently the transitions T_7 and T_8 should be properly redefined.

presented, in conjunction with powerful computers, will also solve the case of a seven-cell neighborhood (*i.e.*, local rules with radius 3), thus unambiguously resolving Conjecture 1.

The remainder of this paper is organized as follows. The key definitions and facts are presented in Section 2. Section 3 presents some necessary conditions for a six-cell neighborhood local rule to solve the parity problem. The main theorem with its proof is given in Section 4 (some parts of the proof are explained in detail in Appendices A and B). Conclusions are reported in Section 5.

2. Preliminaries

2.1. Notations and problem formulation

In this paper, we consider one-dimensional binary cellular automata on finite grids with periodic boundary conditions. For a positive integer n, the n-cell grid $\mathcal{G}_n$ refers to the grid of cells numbered by $0, 1, \ldots, n-1$, *i.e.*,

$$\mathcal{G}_n = \{0, 1, \ldots, n-1\}\,,$$

and we assume that the last cell $n-1$ is adjacent to the first cell 0 (the cells are arranged along a circle). Any n-tuple $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1}) \in X_n = \{0,1\}^n$ is called *a configuration* (of length n) and represents the states of the cells in $\mathcal{G}_n$: the state of cell $i \in \mathcal{G}_n$ in a configuration $\mathbf{x}$ is denoted by x_i. For the sake of brevity, we will often express configurations in a more compact form and write them without commas or, for instance, using 1^7 as a self-explanatory shorthand for 1111111.

Due to the nature of CAs, it is more convenient not to distinguish configurations that are only shifts of one another. Recall that the shift operator $\sigma : X_n \to X_n$ acts as follows: for $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1}) \in X_n$, we have $\sigma(\mathbf{x}) = (x_{n-1}, x_0, x_1, \ldots, x_{n-2})$ (note that we use the shift to the right).

Definition 1. *Two configurations $\mathbf{x}, \mathbf{y} \in X_n$ are said to be shift equivalent if there exists $k \in \{0, \ldots, n-1\}$ such that $\sigma^k(\mathbf{x}) = \mathbf{y}$. The fact that $\mathbf{x}$ and $\mathbf{y}$ are shift equivalent is denoted by $\mathbf{x} \simeq \mathbf{y}$.*

Note that the relation $\simeq$ is an equivalence relation. The equivalence class of $\mathbf{x} \in X_n$ is denoted by $[\mathbf{x}]$, *i.e.*, $[\mathbf{x}] = \{\mathbf{y} \in X_n \mid \mathbf{x} \simeq \mathbf{y}\}$. The elements of the quotient set $\mathbb{X}_n = X_n/_{\simeq}$ are called *significantly different configurations* (of length n). Whereas $|X_n| = 2^n$, it holds that $|\mathbb{X}_n| = a_n = \dfrac{1}{n}\displaystyle\sum_{d|n} \phi(d) \cdot 2^{\frac{n}{d}}$, where ϕ is Euler's totient function, with $\phi(d)$ expressing the number of positive integers up to d that are relatively prime to d (for the latter see, for example, [14]). If n is a prime number, then the expression for a_n simplifies to $a_n = \frac{1}{n}\big(2^n + 2(n-1)\big)$. Note that the complete sequence $(a_n)_{n=1}^{\infty}$ is registered in The On-Line Encyclopedia of Integer Sequences (OEIS) [15] with number A000031. For example, $|X_3| = 8$, $|X_5| = 32$, and $|X_7| = 128$, while $|\mathbb{X}_3| = 4$, $|\mathbb{X}_5| = 8$, and $|\mathbb{X}_7| = 20$.

In this work, we will not limit ourselves to a single grid, but we will consider an entire family of grids at the same time. For this purpose, we consider two auxiliary sets of configurations:

$$\mathbb{X}_{all} = \bigcup_{n=1}^{\infty} \mathbb{X}_n \quad \text{and} \quad \mathbb{X}_{odd} = \bigcup_{k=1}^{\infty} \mathbb{X}_{2k-1}\,,$$

i.e., the set of all significantly different configurations of all possible lengths and the set of all significantly different configurations of all possible odd lengths, respectively.

The *parity* of a configuration $\mathbf{x} \in X_n$ is denoted by $\mathrm{Par}(\mathbf{x})$, *i.e.*,

$$\mathrm{Par}(\mathbf{x}) = x_0 \oplus x_1 \oplus x_2 \oplus \ldots x_{n-1}\,.$$

In other words, $\mathrm{Par}(\mathbf{x})$ equals 0 or 1, depending on whether there is an even or an odd number of 1s in the configuration $\mathbf{x}$. Note that according to the definition of $\simeq$, the value $\mathrm{Par}([\mathbf{x}]) := \mathrm{Par}(\mathbf{x})$ is well defined.

Generally, it is assumed that in each subsequent time step, each cell $i \in \mathcal{G}_n$ updates its state based on its current state and the states of its l left and r right neighbors, where $l, r \in \mathbb{N} = \{0, 1, 2, \ldots\}$. Formally speaking, there is a local rule $f : \{0,1\}^{l+1+r} \to \{0,1\}$, which generates the global rule $F_n : X_n \to X_n$ in the following way:

$$F_n(\mathbf{x})_i = f(x_{i-l}, x_{i-l+1}, \ldots, x_{i+r-1}, x_{i+r})\,.$$

Furthermore, the global rule F_n induces the global transition function $\mathbb{F} : \mathbb{X}_{all} \to \mathbb{X}_{all}$ as follows: $\mathbb{F}([\mathbf{x}]) = [F_n(\mathbf{x})]$ for any $\mathbf{x} \in X_n$. Note that if a local rule f is given, then for any $k \in \mathbb{Z}$, we can define another global rule by

$$\tilde{F}_n(\mathbf{x})_i = f(x_{i-k}, x_{i-k+1}, \ldots, x_{i-k+(d-1)})\,,$$

where $d = l+1+r$, but we still obtain the same global transition function $\mathbb{F}$. In other words, $\mathbb{F}$ depends on f and on the neighborhood size d, but not on the particular values of l and r. For this reason, without loss of generality, we can set the values of l and r arbitrarily (yet keeping the value $d = l + 1 + r$ unchanged). Usually, it is assumed that l and r are as close as possible, thus, depending on the parity of d, it holds that $l = r$ (then it is said that f has radius r) or $l = r - 1$ (then it is said that f has radius $r - \frac{1}{2}$).

The first property of local rules that we are interested in (in this paper) is *parity conservation* (on selected or all grids).

Definition 2. *Let f be a local rule and $\mathbb{F}$ be the corresponding global transition function. We say that the local rule f conserves parity on the grid $\mathcal{G}_n$ if for each $[\mathbf{x}] \in \mathbb{X}_n$, it holds that* $\mathrm{Par}(\mathbb{F}([\mathbf{x}])) = \mathrm{Par}([\mathbf{x}])$. *If* $\mathrm{Par}(\mathbb{F}([\mathbf{x}])) = \mathrm{Par}([\mathbf{x}])$, *for all $[\mathbf{x}] \in \mathbb{X}_{all}$, then we simply say that f conserves parity.*

The second property of local rules that we are interested in is *correct parity determination* for a given initial configuration.

Definition 3. *Let f be a local rule and $\mathbb{F}$ be the corresponding global transition function. Let $[\mathbf{x}] \in \mathbb{X}_{odd}$ and* $\mathrm{Par}([\mathbf{x}]) = p$. *If for some $k_0 \in \mathbb{N}$, it holds that $\mathbb{F}^k([\mathbf{x}]) = [p^n]$ for all $k \geq k_0$, then we say that f correctly determines the parity of* $[\mathbf{x}]$.

Note that the condition in Definition 3 simply means that the orbit of $\mathbb{F}$ for $[\mathbf{x}]$ is eventually fixed and almost all of its elements (up to a finite number) are equal to $[0^n]$ (if $\mathbf{x}$ contains an even number of 1s) or to $[1^n]$ (if $\mathbf{x}$ contains an odd number of 1s).

As we mentioned in the introduction, the idea of finding a local rule f correctly determining the parity of every $\mathbf{x} \in \mathbb{X}_{all}$ is doomed to failure in advance. Indeed, this would mean that for any even n, the equalities $\mathbb{F}([1^n]) = [0^n]$ and $\mathbb{F}([1^n]) = [1^n]$ would have to hold simultaneously, which is impossible (see [5] for a more detailed explanation). However, the restriction to grids of odd length removes this obstacle.

Definition 4. *A local rule f is said to solve the parity problem if it correctly determines the parity of each* $[\mathbf{x}] \in \mathbb{X}_{odd}$.

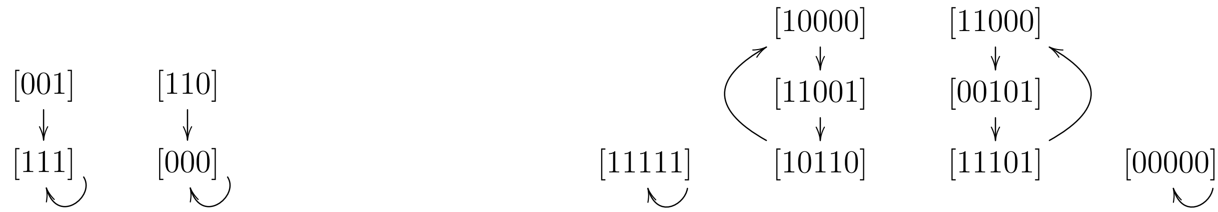
As also mentioned in the introduction, it is known that there exists a local rule that solves the parity problem, but only one such local rule is known: the BFO rule described in [5]. Expressed in terms of the neighborhood size d, the results of [5] can be summarized as follows: $d = 5$ is too small to define a local rule that can solve the parity problem, while $d = 9$ is large enough. This raises the question what is known for $d \in \{6, 7, 8\}$. In this paper, we consider the case $d = 6$ and prove that such d is still not large enough to define a local rule that can solve the parity problem.

2.2. Graphs related to global rules

In this subsection, we introduce some graphs related to local rules, since some ideas and results are easier to express in the language of these graphs.

Definition 5. *Let f be a local rule and $\mathbb{F}$ be the corresponding global transition function. The graph $\Gamma_n(f)$ related to f on $\mathcal{G}_n$ is a directed graph (V, A), where the set of vertices V consists of all elements of $\mathbb{X}_n$, while the set of arcs A is given by:*

$$A = \big\{([x], [y]) \mid [x] \in \mathbb{X}_n \wedge [y] = \mathbb{F}([x])\big\}\,.$$



(a) (b)

Figure 1: Graphs corresponding with the local rule f of ECA150: (a) $\Gamma_3(f)$, (b) $\Gamma_5(f)$.

Example 1. *Let f be the local rule of* ECA150. *The graphs related to f on $\mathcal{G}_3$ and $\mathcal{G}_5$ are shown in Figure 1.*

Note that if a local rule f solves the parity problem, then for any odd n, the graph $\Gamma_n(f)$ is disconnected and has exactly two connected components: $\Gamma_n^{even}(f)$, consisting only of significantly different configurations with parity 0, and $\Gamma_n^{odd}(f)$, consisting only of significantly different configurations with parity 1. Moreover, both graphs $\Gamma_n^{even}(f)$ and $\Gamma_n^{odd}(f)$ are directed rooted in-trees with a loop in the root $[0^n]$ or $[1^n]$, respectively (for example, Figure 1(b) shows that ECA150 does not solve the parity problem). Furthermore, since each $\Gamma_n^{even}(f)$ and $\Gamma_n^{odd}(f)$ have exactly $a_n/2$ vertices, the longest path to the root can have a length of at most $s_n = a_n/2 - 1$, which directly leads to the following statement.

Theorem 1. *Let f be a local rule and $\mathbb{F}$ be the corresponding global transition function. If f solves the parity problem, then for each $[\mathbf{x}] \in \mathbb{X}_{odd}$, it holds that $\mathbb{F}^{s_n}([\mathbf{x}]) = [(\mathrm{Par}(\mathbf{x}))^n]$, where n is the length of $\mathbf{x}$.*

The above condition is a very useful necessary condition. For example, if for some local rule f there is a configuration $\mathbf{x} \in X_7$ with parity $p \in \{0, 1\}$ and $\mathbb{F}^9([\mathbf{x}]) \neq [p^7]$ (note that $a_7 = 20$, so $s_7 = 9$), then f does not solve the parity problem. Hence, we do not need to check values $\mathbb{F}^k([\mathbf{x}])$ for larger time steps k.

Furthermore, note that if a local rule f solves the parity problem, then each of the graphs $\Gamma_3^{even}(f)$ and $\Gamma_3^{odd}(f)$ has only two vertices, hence there is only one possibility for their shape and they have to look as in Figure 1(a). We immediately obtain the following conclusion.

Corollary 1. *Let f be a local rule and $\mathbb{F}$ be the corresponding global transition function. If f solves the parity problem, then $\mathbb{F}([001]) = [111]$, $\mathbb{F}([111]) = [111]$, $\mathbb{F}([110]) = [000]$, and $\mathbb{F}([000]) = [000]$.*

Note that any local rule that solves the parity problem should treat state 0 and state 1 differently. Therefore, there is no reason to expect that for every odd n, the subgraphs $\Gamma_n^{even}(f)$ and $\Gamma_n^{odd}(f)$ would be isomorphic. For example, for the BFO rule f, the graphs $\Gamma_5(f)$ and $\Gamma_7(f)$ are shown in Figure 2.

2.3. Number conservation

Obviously, if a local rule f solves the parity problem, then for every odd n, the corresponding global rule F_n must conserve parity on $\mathcal{G}_n$. This means that for any odd number n, the local rule f is number-conserving in the field $\mathbb{Z}_2$, *i.e.*, for each $\mathbf{x} \in X_n$ the sum of states in $F_n(\mathbf{x})$ is the same as the sum of states in $\mathbf{x}$, where the states of both configurations are considered as elements of $\mathbb{Z}_2$:

$$F_n(\mathbf{x})_0 \oplus F_n(\mathbf{x})_1 \oplus \cdots \oplus F_n(\mathbf{x})_{n-1} = x_0 \oplus x_1 \oplus \cdots \oplus x_{n-1} .$$

In this case, we get the following necessary condition characterizing number conservation (see, for example, [16, 17, 18, 19, 20]).

Theorem 2. *Let f be a local rule with a d-cell neighborhood solving the parity problem. For any $x_1, x_2, \dots, x_d \in \{0,1\}$, it holds that:*

$$f(x_1, x_2, \dots, x_d) = x_1 \oplus \sum_{k=1}^{d-1} \big(f(\underbrace{0, \dots, 0}_{k}, x_2, x_3, \dots, x_{d-k+1}) \oplus f(\underbrace{0, \dots, 0}_{k}, x_1, x_2, \dots, x_{d-k}) \big) , \qquad (1)$$

and also

$$f(x_1, x_2, \dots, x_d) = x_1 \oplus \sum_{k=1}^{d-1} \big(f(\underbrace{1, \dots, 1}_{k}, x_2, x_3, \dots, x_{d-k+1}) \oplus f(\underbrace{1, \dots, 1}_{k}, x_1, x_2, \dots, x_{d-k}) \big) . \qquad (2)$$

We omit the proof, as it is a simple modification of the proof of Theorem 2.1 in [16].

Remark 1. *It is quite easy to see that if $x_1 = 0$, then Eq. (1) reduces to $0 = 0$ and there is no need to consider it. The same holds for Eq. (2) if $x_1 = 1$. Moreover, if f satisfies Eq. (1) (or Eq. (2)), then F_n conserves parity on $\mathcal{G}_n$ for each positive integer n (not just for odd ones).*

Inserting $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, and so on sequentially in Eq. (1), we get the following handy conclusion. Note the use of the generally accepted notation for binary states: $\bar{s} = 1 \oplus s$.

Corollary 2. *Let f be a local rule with a d-cell neighborhood solving the parity problem. If d is even, then $f(1010\dots10) = \overline{f(0101\dots01)}$, while if d is odd, then $f(1010\dots101) = \overline{f(0101\dots010)}$.*

3. The case of a six-cell neighborhood

In this section, we consider the case $d = 6$, which means that a local rule depends on at most two left and three right neighbors. In the next section, we show that such a local rule cannot solve the parity problem. In order to prove this, we first establish some necessary conditions for a local rule with a six-cell neighborhood to be a candidate for solving the parity problem. We start with rewriting the results of the previous section for this particular d.

To define a local rule f with a six-cell neighborhood, we need to declare its $2^6 = 64$ values (namely: $f(000000)$, $f(000001), \dots, f(111111)$). However, if f solves the parity problem, then, according to Corollary 1, eight of these values are fixed.

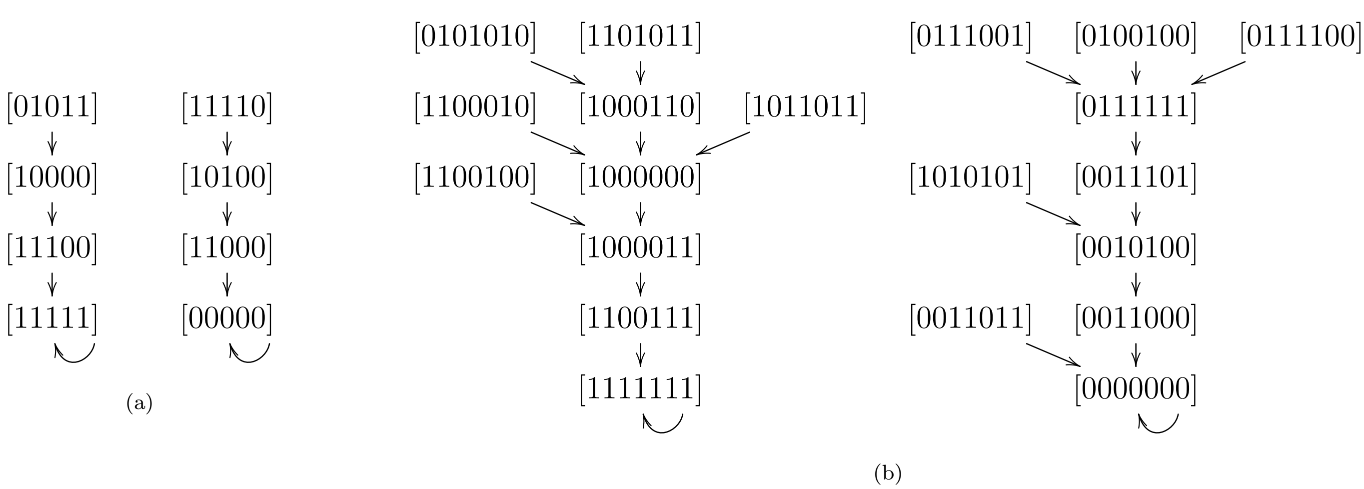


Figure 2: Graphs corresponding with the BFO rule f: (a) $\Gamma_5(f)$, (b) $\Gamma_7(f)$.

Corollary 3. *If a local rule f with a six-cell neighborhood solves the parity problem, then*

$$f(111111) = 1,\ f(001001) = 1,\ f(010010) = 1,\ f(100100) = 1,$$

$$f(000000) = 0,\ f(110110) = 0,\ f(101101) = 0,\ f(011011) = 0.$$

A reformulation of Theorem 2 (taking into account Remark 1) and Corollary 2 yields the following necessary condition.

Theorem 3. *If a local rule f with a six-cell neighborhood solves the parity problem, then for any $x_2, x_3, x_4, x_5, x_6 \in \{0, 1\}$ it holds that*

$$\begin{aligned} f(1x_2x_3x_4x_5x_6) = 1 &\oplus f(00000x_2) \oplus f(000001) \oplus f(0000x_2x_3) \oplus f(00001x_2) \oplus f(000x_2x_3x_4) \\ &\oplus f(0001x_2x_3) \oplus f(00x_2x_3x_4x_5) \oplus f(001x_2x_3x_4) \oplus f(0x_2x_3x_4x_5x_6) \oplus f(01x_2x_3x_4x_5) \end{aligned} \tag{3}$$

and

$$\begin{aligned} f(0x_2x_3x_4x_5x_6) =& f(11111x_2) \oplus f(111110) \oplus f(1111x_2x_3) \oplus f(11110x_2) \oplus f(111x_2x_3x_4) \oplus f(1110x_2x_3) \\ &\oplus f(11x_2x_3x_4x_5) \oplus f(110x_2x_3x_4) \oplus f(1x_2x_3x_4x_5x_6) \oplus f(10x_2x_3x_4x_5)\,. \end{aligned} \tag{4}$$

In particular, if $f(010101) = v$, then $f(101010) = \bar{v}$.

For the sake of convenience, we introduce the following notations:

$$\begin{aligned} f(000001) = \mathrm{A}, f(000010) = \mathrm{B}, f(000100) = \mathrm{C}, f(001000) = \mathrm{D}, f(010000) = \mathrm{E}. \\ f(111110) = \mathrm{a}, f(111101) = \mathrm{b}, f(111011) = \mathrm{c}, f(110111) = \mathrm{d}, f(101111) = \mathrm{e}. \end{aligned} \tag{5}$$

The purpose of the remainder of this section is to determine a total of 24 values of the local rule f that would be a candidate for solving the parity problem.

Lemma 1. *Let f be a local rule with a six-cell neighborhood and $\mathbb{F}$ be the corresponding global transition function. If f solves the parity problem, then $\mathbb{F}([11100]) = [1^5]$ and $\mathbb{F}([1111000]) = [0^7]$ cannot occur simultaneously.*

Proof. Let us suppose that both $\mathbb{F}([11100]) = [1^5]$ and $\mathbb{F}([1111000]) = [0^7]$. Then, in particular, $f(111100) = f(001111) = f(011110) = 0$, and $f(001110) = f(011100) = 1$. However, this contradicts Eq. (3) for $x_2 = x_3 = x_4 = 1$ and $x_5 = x_6 = 0$, which states that:

$$f(111100) = 1 \oplus f(001110) \oplus f(001111) \oplus f(011100) \oplus f(011110)\,.$$

□

Using Eq. (4) for $x_2 = x_3 = 1$ and $x_4 = x_5 = x_6 = 0$, in a similar way one can prove the analogous fact.

Lemma 2. *Let f be a local rule with a six-cell neighborhood and $\mathbb{F}$ be the corresponding global transition function. If f solves the parity problem, then $\mathbb{F}([11000]) = [0^5]$ and $\mathbb{F}([1110000]) = [1^7]$ cannot occur simultaneously.*

For a given global transition function, a *pre-image* of a significantly different configuration is any significantly different configuration that is mapped to it by that function. If a local rule f solves the parity problem, then for every odd n, there should be at least one non-trivial pre-image of $[0^n]$ and at least one non-trivial image of $[1^n]$ (*i.e.*, different from $[0^n]$ and $[1^n]$, respectively). Following [5], we will often use this simple observation in the proofs.

Theorem 4. *Let f be a local rule with a six-cell neighborhood and $\mathbb{F}$ be the corresponding global transition function. If f solves the parity problem, then $\mathbb{F}([10000]) = [11111]$. Moreover, $f(100001) = 1$ and $f(100000) = \bar{\mathrm{A}}$.*

Proof. If f solves the parity problem, then the graph $\Gamma_5^{odd}(f)$ has four vertices: [10000], [11100], [11010], [11111], which are arranged as a rooted in-tree with a loop in the root [11111]. According to Corollary 1 and our notations, we have

$$\begin{array}{cccc} [10000] & [11100] & [11010] & [11111] \\ \downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} \\ [\mathrm{DE}\phi\mathrm{BC}] & [\cdot\cdot\cdot\cdot\cdot] & [0\cdot\cdot\cdot\cdot] & [11111] \end{array}$$

where $\phi = f(100001)$.

Let us assume that $\mathbb{F}([10000]) \neq [11111]$. Then $\mathbb{F}([10000]) = [11100]$ or $\mathbb{F}([10000]) = [11010]$, so there are two 0s among the values $\mathrm{D}, \mathrm{E}, \phi, \mathrm{B}, \mathrm{C}$. Moreover, the only non-trivial pre-image of [11111] is [11100], which gives

$$f(111001) = f(110011) = f(100111) = f(001110) = f(011100) = 1\,. \tag{6}$$

If so, then taking into account the above values and Corollary 1, we have the following values of $\mathbb{F}$ on the vertices of $\Gamma_7^{even}(f)$:

$$\begin{array}{ccccc} [0111111] & [0001111] & [0010111] & [0100111] & [0011011] \\ \downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} \\ [\cdot\cdot\cdot1\cdot\cdot\cdot] & [\cdot\cdot\cdot\cdot\cdot\cdot\cdot] & [\cdot\cdot\cdot\cdot\cdot11] & [\cdot\cdot\cdot11\cdot\cdot] & [1\cdot\cdot00\cdot\cdot] \\ \\ [0101011] & [0000011] & [0000101] & [0001001] & [0000000] \\ \downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} \\ [0\bar{v}v\cdot\cdot0\cdot] & [\cdot\cdot\mathrm{A}\cdot\cdot\cdot\cdot] & [\mathrm{E}\phi\mathrm{B}\cdot\cdot\cdot\cdot] & [\cdot\cdot01110] & [0000000] \end{array}$$

Additionally, according to Lemma 1, [0001111] cannot be a pre-image of $[0^7]$. Moreover, $\mathrm{E}\phi\mathrm{B} \neq 000$ (since among the values $\mathrm{D}, \mathrm{E}, \phi, \mathrm{B}, \mathrm{C}$ there are two 0s). Thus [0000011] is the only candidate for a non-trivial pre-image of $[0^7]$, which gives, in particular, that

$$f(000110) = f(001100) = f(011000) = 0\,.$$

But then, using Eq. (3) for $x_2 = x_3 = 1$ and $x_4 = x_5 = x_6 = 0$, we get

$$f(111000) = 1 \oplus f(000110) \oplus f(000111) \oplus f(001100) \oplus f(001110) \oplus f(011000) \oplus f(011100)\,,$$

which gives (according to Eq. (6)), that $f(111000) = \overline{f(000111)} = z$. However, then $\mathbb{F}([1110000]) = [11z000\bar{z}] = [1110000]$, for both possibilities of z. And this is a contradiction, since we get an illegal loop in $\Gamma_7^{odd}(f)$. Thus $\mathbb{F}([10000]) = [11111]$.

To prove the second part of the statement, let us note that $\mathbb{F}([10000]) = [11111]$ means that $f(100001) = \phi = 1$ and $\mathrm{B} = \mathrm{C} = \mathrm{D} = \mathrm{E} = 1$. The latter with Eq. (3) for $x_2 = x_3 = x_4 = x_5 = x_6 = 0$ gives $f(100000) = \bar{\mathrm{A}}$. □

Analogously, using $\Gamma_7^{odd}(f)$ and Eq. (4), we get the following result.

Theorem 5. *Let f be a local rule with a six-cell neighborhood and $\mathbb{F}$ be the corresponding global transition function. If f solves the parity problem, then $\mathbb{F}([01111]) = [00000]$. Moreover, $f(011110) = 0$ and $f(011111) = \bar{\mathrm{a}}$.*

Below we summarize the results of this section.

Corollary 4. *If a local rule f with a six-cell neighborhood solves the parity problem, then it satisfies the following necessary conditions:*

$$\begin{array}{llll}
f(111111)=1 & f(001001)=1 & f(010010)=1 & f(100100)=1 \\
f(000000)=0 & f(110110)=0 & f(101101)=0 & f(011011)=0 \\
f(000010)=1 & f(000100)=1 & f(001000)=1 & f(010000)=1 \\
f(111101)=0 & f(111011)=0 & f(110111)=0 & f(101111)=0 \\
f(000001)=\mathrm{A} & f(100000)=\bar{\mathrm{A}} & f(111110)=\mathrm{a} & f(011111)=\bar{\mathrm{a}} \\
f(011110)=0 & f(100001)=1 & f(010101)=v & f(101010)=\bar{v}
\end{array} \tag{7}$$

for some $\mathrm{A}, \mathrm{a}, v \in \{0,1\}$.

4. The main theorem

In this section, we prove the main theorem stating that a six-cell neighborhood is still not enough for defining a local rule that can solve the parity problem.

Theorem 6. *A local rule f with a six-cell neighborhood cannot solve the parity problem.*

Proof. Suppose on the contrary that f solves the parity problem and let $\mathbb{F}$ be the corresponding global transition function. Taking into account Corollary 4, we have the following values of $\mathbb{F}$ on the vertices of $\Gamma_7^{even}(f)$:

$$\begin{array}{ccccc}
[0111111] & [0001111] & [0010111] & [0100111] & [0011011] \\
\downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} \\
[00\bar{\mathrm{a}}1\mathrm{a}00] & [\cdot\cdot\cdot\cdot0\cdot\cdot] & [\cdot\cdot\cdot\cdot\cdot\cdot\cdot] & [\cdot\cdot\cdot\cdot\cdot\cdot\cdot] & [\cdot\cdot\cdot00\cdot\cdot] \\
\\
[0101011] & [0000011] & [0000101] & [0001001] & [0000000] \\
\downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} \\
[\cdot\bar{v}v\cdot\cdot0\cdot] & [\cdot\bar{\mathrm{A}}\mathrm{A}\cdot\cdot\cdot\cdot] & [111\cdot\cdot\cdot\cdot] & [\cdot\cdot11111] & [0000000]
\end{array}$$

Since $\Gamma_7^{even}(f)$ should be a directed in-tree with the loop in the root $[0^7]$, at least one of the vertices [0001111], [0010111], [0100111], [0011011] must be a non-trivial pre-image of $[0^7]$. In the following lemmas, however, we will show that none of these four vertices can satisfy this condition, which completes the proof of our statement. $\square$

Lemma 3. *If a local rule f with a six-cell neighborhood solves the parity problem, then* [0001111] *cannot be a pre-image of* $[0^7]$.

Proof. Supposing that [0001111] is a pre-image of $[0^7]$, we obtain six new values for f, namely $f(000111) = f(001111) = f(111100) = f(111000) = f(110001) = f(100011) = 0$, which, combined with Corollary 4, give the following values of $\mathbb{F}$ on the vertices of $\Gamma_7^{odd}(f)$:

$$\begin{array}{ccccc}
[1000000] & [1110000] & [1101000] & [1011000] & [1100100] \\
\downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} \\
[11\bar{\mathrm{A}}\,0\mathrm{A}11] & [\cdot\cdot0\cdot1\cdot0] & [\cdot\cdot\cdot\cdot\cdot0\cdot] & [\cdot\cdot\cdot\cdot0\cdot\cdot] & [\cdot\cdot\cdot11\cdot\cdot] \\
\\
[1010100] & [1111100] & [1111010] & [1110110] & [1111111] \\
\downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} \\
[\cdot v\bar{v}\cdot\cdot1\cdot] & [0\bar{\mathrm{a}}\mathrm{a}0\cdot\cdot\cdot] & [000\cdot\cdot\cdot\cdot] & [\cdot\cdot00000] & [1111111]
\end{array}$$

Thus [1100100] must be a pre-image of $[1^7]$, which implies $f(001100) = 1$. But then $\mathbb{F}([00011]) = [00\cdot1\cdot]$, which means (since $\mathbb{F}$ conserves parity) that $\mathbb{F}([00011]) = [00011]$, leading to a contradiction. $\square$

Lemma 4. *If a local rule f with a six-cell neighborhood solves the parity problem, then* $[0011011]$ *cannot be a pre-image of* $[0^7]$.

Proof. Similarly as in the proof of Lemma 3, the assumption that $[0011011]$ is a pre-image of $[0^7]$ implies:

$$
\begin{array}{ccccc}
[1000000] & [1110000] & [1101000] & [1011000] & [1100100] \\
\downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} \\
[11\bar{\mathrm{A}}\,0\mathrm{A}11] & [\cdot\cdot\cdot\cdot 1\cdot\cdot] & [0\cdot\cdot\cdot\cdot\cdot\cdot] & [\cdot\cdot 0\cdot\cdot\cdot\cdot] & [\cdot 0\cdot 11\cdot 0] \\
\\
[1010100] & [1111100] & [1111010] & [1110110] & [1111111] \\
\downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} \\
[\cdot v\bar{v}\cdot\cdot 1\cdot] & [\cdot\bar{\mathrm{a}}\mathrm{a}\cdot\cdot 0\cdot] & [000\cdot\cdot\cdot\cdot] & [\cdot\cdot 00000] & [1111111]
\end{array}
$$

which gives that $[1110000]$ must be a pre-image of $[1^7]$. In particular, $f(001110) = 1$ and $f(011100) = 1$. But then $\mathbb{F}([11100]) = [11\cdot 0\cdot]$, thus $\mathbb{F}([11100]) = [11100]$, again a contradiction. □

Lemma 5. *If a local rule f with a six-cell neighborhood solves the parity problem, then* $[0010111]$ *cannot be a pre-image of* $[0^7]$.

Proof. The reasoning in this case is a little bit more complicated. If we assume that $\mathbb{F}([0010111]) = [0^7]$, then

$$
\begin{array}{ccccc}
[1000000] & [1110000] & [1101000] & [1011000] & [1100100] \\
\downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} \\
[11\bar{\mathrm{A}}\,0\mathrm{A}11] & [\cdot 0\cdot\cdot 1\cdot\cdot] & [\cdot\cdot\cdot\cdot\cdot\cdot\cdot] & [0\cdot\cdot\cdot\cdot\cdot\cdot] & [\cdot\cdot 011\cdot\cdot] \\
\\
[1010100] & [1111100] & [1111010] & [1110110] & [1111111] \\
\downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} & \downarrow\mathbb{F} \\
[\cdot v\bar{v}\cdot\cdot 10] & [\cdot\bar{\mathrm{a}}\mathrm{a}\cdot 0\cdot\cdot] & [000\cdot\cdot\cdot 0] & [0\cdot 00000] & [1111111]
\end{array}
$$

and $[1101000]$ must be a pre-image of $[1^7]$. Note that $\mathbb{F}([0010111]) = [0^7]$ and $\mathbb{F}([1101000]) = [1^7]$ determine the values of f on 14 additional arguments not present in Corollary 4:

$$
\begin{array}{llll}
f(001011) = 0 & f(010111) = 0 & f(101110) = 0 & f(011100) = 0 \\
f(111001) = 0 & f(110010) = 0 & f(100101) = 0 & \\
f(110100) = 1 & f(101000) = 1 & f(010001) = 1 & f(100011) = 1 \\
f(000110) = 1 & f(001101) = 1 & f(011010) = 1 &
\end{array} \tag{8}
$$

Inserting the above 14 values and the 24 values from Corollary 4 into Eq. (3), we obtain a system of linear equations, from which we can calculate the following eight values:

$$
\begin{array}{llll}
f(000011) = 0 & f(000101) = 0 & f(010110) = 0 & f(100010) = 0 \\
f(101001) = 1 & f(111010) = 1 & f(111100) = 1 & f(011101) = 1
\end{array} \tag{9}
$$

and the remaining 18 values of f can be expressed in terms of five parameters:

$$
\begin{array}{lllll}
f(010011) = \alpha & f(000111) = \beta & f(001100) = \gamma & f(001010) = \delta & f(011000) = \eta \\
f(001110) = \beta & f(001111) = \beta & f(010100) = \delta & f(011001) = \gamma & f(100110) = \bar{\alpha} \\
f(100111) = \alpha \oplus \beta & f(101011) = \delta \oplus \bar{v} & f(101100) = \bar{\gamma} & f(110000) = \gamma \oplus \eta & f(110001) = \gamma \oplus \eta \\
f(110011) = \bar{\alpha} & f(110101) = \delta \oplus \bar{v} & f(111000) = \gamma \oplus \eta & &
\end{array} \tag{10}
$$

Since $\alpha, \beta, \gamma, \delta, \eta$, as well as $\mathrm{A}, \mathrm{a}, v$ belong to $\{0, 1\}$, we are left with $2^8 = 256$ local rules that could possibly be a solution to the parity problem. We checked each of them by computer on a significantly different configuration of length 9, namely $[000010000]$, and it appeared that none of these rules correctly determines this configuration (it is sufficient to use Theorem 1 to confirm this). Thus, the proof of this lemma is completed. However, using a computer is not necessary and we present a detailed proof (which is not particularly complicated at all) in Appendix A as an alternative. □

Lemma 6. *If a local rule f with a six-cell neighborhood solves the parity problem, then* $[0100111]$ *cannot be a pre-image of* $[0^7]$.

Proof. Similarly as in the proof of Lemma 5, let us assume that $\mathbb{F}([0100111]) = [0^7]$, then

$$
\begin{array}{ccccc}
[1000000] & [1110000] & [1101000] & [1011000] & [1100100] \\
\downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} \\
[11\bar{\mathrm{A}}\,0\mathrm{A}11] & [0\cdots1\cdot\cdot] & [\cdot\cdot0\cdots\cdot] & [\cdots\cdots\cdot] & [\cdots110\cdot] \\
\\
[1010100] & [1111100] & [1111010] & [1110110] & [1111111] \\
\downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} \\
[\cdot v\bar{v}\cdot01\cdot] & [\cdot\bar{\mathrm{a}}\mathrm{a}\cdots0] & [0000\cdot\cdot\cdot] & [\cdot000000] & [1111111]
\end{array}
$$

and $[1011000]$ must be a pre-image of $[1^7]$. From $\mathbb{F}([0100111]) = [0^7]$ and $\mathbb{F}([1011000]) = [1^7]$, we obtain

$$
\begin{array}{llll}
f(010011)=0 & f(100111)=0 & f(001110)=0 & f(011101)=0 \\
f(111010)=0 & f(110100)=0 & f(101001)=0 & \\
f(101100)=1 & f(011000)=1 & f(110001)=1 & f(100010)=1 \\
f(000101)=1 & f(001011)=1 & f(010110)=1 &
\end{array}
\tag{11}
$$

This time, inserting the above 14 values and the 24 values from Corollary 4 into Eq. (3) gives:

$$
\begin{array}{llll}
f(010001)=0 & f(110000)=0 & f(011010)=0 & f(101000)=0 \\
f(100101)=1 & f(001111)=1 & f(101110)=1 & f(010111)=1
\end{array}
\tag{12}
$$

and the remaining 18 values of f can again be expressed in terms of five parameters:

$$
\begin{array}{lllll}
f(000011)=\alpha & f(000111)=\alpha & f(100011)=\alpha & f(000110)=\beta & f(100110)=\alpha\oplus\beta \\
f(001100)=\alpha\oplus\beta & f(011001)=\delta & f(110010)=\bar{\delta} & f(110011)=\delta & f(001101)=\bar{\alpha}\oplus\beta \\
f(001010)=\gamma & f(010100)=\gamma & f(101011)=\gamma\oplus v & f(011100)=\eta & f(111000)=\eta \\
f(111001)=\bar{\delta}\oplus\eta & f(111100)=\eta & f(110101)=\gamma\oplus v & &
\end{array}
\tag{13}
$$

Thus, again, we are left with only $2^8 = 256$ local rules that could possibly be a solution to the parity problem. Similarly as before, we checked each of them by computer on $[000010000]$, and it appeared that none of these rules correctly classifies this configuration. A detailed proof without the use of a computer is presented in Appendix B. □

5. Conclusions

In this paper, we have continued the investigation initiated by Betel *et al.* in [5]. Using an innovative approach, we were able to improve the lower bound on the neighborhood size of local rules that solve the parity problem. Moreover, the complete proof of this theorem does not use a computer, which gives hope that our methodology, combined with high-performance computers, will also solve the case of seven-cell neighborhoods, thus resolving Conjecture 1.

The approach presented in this work also seems to be suitable for exploring a kind of "halfway solutions", *i.e.*, CAs that correctly classify all even configurations or all odd configurations.

Definition 6 (Perfect solution to the even or odd parity problems)**.** *Let f be a local rule and F be the corresponding global rule. We say that f* perfectly classifies even configurations *if for any odd integer n and any initial configuration $\mathbf{x} \in X_n$, it holds that* $\mathrm{Par}(F(\mathbf{x})) = \mathrm{Par}(\mathbf{x})$, *and that, moreover, there exists $k_0 > 0$ such that for all $k \geq k_0$:*

$$
\mathrm{Par}(\mathbf{x}) = 0 \quad \Rightarrow \quad F^k(\mathbf{x}) = (0, 0, \ldots, 0)\,.
$$

Similarly, we say that f perfectly classifies odd configurations *if for any odd integer n and any initial configuration $\mathbf{x} \in X_n$, it holds that* $\mathrm{Par}(F(\mathbf{x})) = \mathrm{Par}(\mathbf{x})$*, and that, moreover, there exists $k_0 > 0$ such that for all $k \geq k_0$:*

$$\mathrm{Par}(\mathbf{x}) = 1 \quad \Rightarrow \quad F^k(\mathbf{x}) = (1, 1, \ldots, 1)\,.$$

If by d_P we denote the size of the smallest neighborhood for a CA solving the parity problem, while by d_0 (resp. d_1) we denote the size of the smallest neighborhood for a CA perfectly classifying even (resp. odd) configurations, then it is obvious that $d_0 \leq d_P$. Moreover, the use of the conjugation operator gives at once that $d_0 = d_1$. Finding the value of d_0 is just as striking as finding the value of d_P, because an intriguing question arises regarding d_0 compared to d_P. Are they the same, or is the latter smaller than the former? We hope that the tools presented in this study will also allow to answer this question.

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References

[1] S. C. Benjamin, N. F. Johnson, A possible nanometer-scale computing device based on an adding cellular automaton, Applied Physics Letters 70 (17) (1997) 2321–2323. `doi:10.1063/1.118851`.

[2] M. Mitchell, P. T. Hraber, J. P. Crutchfield, Revisiting the edge of chaos: Evolving cellular automata to perform computations, Complex Systems 7 (1993) 89–130. `doi:10.1016/0167-2789(94)90293-3`.

[3] M. Montalva-Medel, P. P. B. de Oliveira, E. Goles, A portfolio of classification problems by one-dimensional cellular automata, over cyclic binary configurations and parallel update, Natural Computing 17 (2018) 663 – 671. `doi:10.1007/s11047-017-9650-1`.

[4] M. Sipper, Computing with cellular automata: Three cases for nonuniformity, Physical Review E 57 (3) (1998) 3589–3592. `doi:10.1103/PhysRevE.57.3589`.

[5] H. Betel, P. P. Oliveira, P. Flocchini, Solving the parity problem in one-dimensional cellular automata, Natural Computing: An International Journal 12 (3) (2013) 323–337. `doi:10.1007/s11047-013-9374-9`.

[6] D. Wolz, P. P. B. de Oliveira, Very effective evolutionary techniques for searching cellular automata rule spaces, Journal of Cellular Automata 3 (2008) 289–312.

[7] S. Wolfram, Statistical mechanics of cellular automata, Reviews of Modern Physics 55 (1983) 601–644. `doi:10.1103/RevModPhys.55.601`.

[8] K. M. Lee, H. Xu, H. F. Chau, Parity problem with a cellular automaton solution, Physical Review E 64 (2001) 026702. `doi:10.1103/PhysRevE.64.026702`.

[9] C. L. Martins, P. P. de Oliveira, Improvement of a result on sequencing elementary cellular automata rules for solving the parity problem, Electronic Notes in Theoretical Computer Science 252 (2009) 103–119, 15th International Workshop on Cellular Automata and Discrete Complex Systems. `doi:10.1016/j.entcs.2009.09.017`.

[10] S. Ninagawa, Solving the parity problem with rule 60 in array size of the power of two, Journal of Cellular Automata 8 (3-4) (2013) 189–203.

[11] E. L. Ruivo, P. P. de Oliveira, A perfect solution to the parity problem with elementary cellular automaton 150 under asynchronous update, Information Sciences 493 (2019) 138–151. doi:10.1016/j.ins.2019.04.045.

[12] P. P. Balbi, E. Ruivo, F. Faria, Synchronous solution of the parity problem on cyclic configurations, with elementary cellular automaton rule 150, over a family of directed, non-circulant, regular graphs, Information Sciences 615 (2022) 578–603. doi:10.1016/j.ins.2022.10.045.

[13] N. A. Fatès, An asynchronous cellular system that solves the parity problem, inria Nancy Grand-Est. hal-03894581 (2022).

[14] J. Riordan, An Introduction to Combinatorial Analysis, Princeton Legacy Library, Princeton University Press, 2014.

[15] N. J. A. Sloane, The on-line encyclopedia of integer sequences, Published electronically at http://oeis.org.

[16] N. Boccara, H. Fukś, Number-conserving cellular automaton rules, Fundamenta Informaticae 52 (1-3) (2002) 1–13.

[17] B. Durand, E. Formenti, Z. Róka, Number-conserving cellular automata i: decidability, Theoretical Computer Science 299 (1) (2003) 523–535. doi:10.1016/S0304-3975(02)00534-0.

[18] E. Formenti, A. Grange, Number conserving cellular automata ii: dynamics, Theoretical Computer Science 304 (1) (2003) 269–290. doi:10.1016/S0304-3975(03)00134-8.

[19] A. Moreira, Universality and decidability of number-conserving cellular automata, Theoretical Computer Science 292 (3) (2003) 711–721, algorithms in Quantum Information Prcoessing. doi:10.1016/S0304-3975(02)00065-8.

[20] M. Pivato, Conservation laws in cellular automata, Nonlinearity 15 (6) (2002) 1781 – 1793. doi:10.1088/0951-7715/15/6/305.

Appendix A

Theorem 7. *Let a local rule f with a six-cell neighborhood satisfy Eqs. (7), (8), (9) and (10). Then f cannot solve the parity problem.*

Proof. We start by showing that if η equals 0, then f cannot solve the parity problem. Indeed, if $\eta = f(011000) = 0$, then $\mathbb{F}([00011]) = [\gamma 11\gamma 0]$, so $\gamma = 1$, otherwise $\mathbb{F}([00011]) = [00011]$. Thus

$$
\begin{array}{ccccc}
[0111111] & [0001111] & [0010111] & [0100111] & [0011011] \\
\big\downarrow \mathbb{F} & \big\downarrow \mathbb{F} & \big\downarrow \mathbb{F} & \big\downarrow \mathbb{F} & \big\downarrow \mathbb{F} \\
[00\bar{\mathrm{a}}1\mathrm{a}00] & [11\cdot\cdot 011] & [0000000] & [11\cdots 11] & [\cdot\cdot\ 10001] \\
\\
[0101011] & [0000011] & [0000101] & [0001001] & [0000000] \\
\big\downarrow \mathbb{F} & \big\downarrow \mathbb{F} & \big\downarrow \mathbb{F} & \big\downarrow \mathbb{F} & \big\downarrow \mathbb{F} \\
[p\bar{v}vp001] & [1\bar{\mathrm{A}}\mathrm{A}0110] & [1110\cdot\cdot 1] & [1011111] & [0000000]
\end{array}
$$

where $p = \bar{v} \oplus \delta$. If so, then [0010111] is the only non-trivial pre-image of $[0^7]$. However, for both values of p the pre-image of [0010111] is empty, which means that the graph $\Gamma_7^{even}(f)$ is not connected, which yields a contradiction. In the following part of the proof, we assume that $\eta = f(011000) = 1$.

Now, we show that if $\alpha \neq \beta$, then f cannot solve the parity problem. Indeed, if $f(100111) = \alpha \oplus \beta = 1$, then $\alpha = 0$ and $\beta = 1$ or $\alpha = 1$ and $\beta = 0$. However, in the first case $\mathbb{F}([11100]) = [10011]$, so we get an illegal loop. In the second case, *i.e.*, when $\alpha = 1$ and $\beta = 0$, we have for the vertices of $\Gamma_7^{odd}(f)$:

$$\begin{array}{ccccc}
[1000000] & [1110000] & [1101000] & [1011000] & [1100100] \\
\downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} \\
[11\bar{\mathrm{A}}\,0\mathrm{A}11] & [00\cdot\cdot100] & [1111111] & [00\cdot1\cdot00] & [\cdot\cdot01110] \\[2ex]
[1010100] & [1111100] & [1111010] & [1110110] & [1111111] \\
\downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} \\
[\delta v\bar{v}\delta110] & [0\bar{\mathrm{a}}\mathrm{a}1001] & [0001\cdot\cdot0] & [0100000] & [1111111]
\end{array}$$

Thus [1101000] is the only non-trivial pre-image of $[1^7]$. However, for both values of δ the pre-image of [1101000] is empty, which means that the graph $\Gamma_7^{odd}(f)$ is not connected, again a contradiction.

Using the fact that $\eta = f(011000) = 1$ and $\alpha = \beta$, we obtain the following values of $\mathbb{F}$ on the vertices of $\Gamma_7^{odd}(f)$:

$$\begin{array}{ccccc}
[1000000] & [1110000] & [1101000] & [1011000] & [1100100] \\
\downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} \\
[11\bar{\mathrm{A}}\,0\mathrm{A}11] & [\alpha0\bar{\gamma}\bar{\gamma}10\alpha] & [1111111] & [00\bar{\gamma}1\bar{\gamma}00] & [\gamma\gamma011\alpha\bar{\alpha}] \\[2ex]
[1010100] & [1111100] & [1111010] & [1110110] & [1111111] \\
\downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} \\
[\delta v\bar{v}\delta110] & [\alpha\bar{\mathrm{a}}\mathrm{a}10\bar{\alpha}0] & [0001pp0] & [0100000] & [1111111]
\end{array}$$

Note that $\alpha = \gamma = 0$ is impossible, since then $\mathbb{F}([1110000]) = [1110000]$. Thus [1101000] is the only non-trivial pre-image of $[1^7]$, whereas the only candidate for a pre-image of [1101000] is [1111100], which implies $\alpha = \bar{\mathrm{a}}$.

On the other hand, using the fact that $\eta = f(011000) = 1$, $\alpha = \beta = \bar{\mathrm{a}}$ (and we know that $\alpha = \gamma = 0$ cannot hold) we obtain the following values of $\mathbb{F}$ on the vertices of $\Gamma_7^{even}(f)$:

$$\begin{array}{ccccc}
[0111111] & [0001111] & [0010111] & [0100111] & [0011011] \\
\downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} \\
[00\alpha1\bar{\alpha}00] & [\bar{\gamma}1\alpha\alpha01\bar{\gamma}] & [0000000] & [11\alpha0\alpha11] & [\bar{\alpha}\bar{\alpha}100\bar{\gamma}\gamma] \\[2ex]
[0101011] & [0000011] & [0000101] & [0001001] & [0000000] \\
\downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} \\
[p\bar{v}vp001] & [\bar{\gamma}\bar{\mathrm{A}}\mathrm{A}01\gamma1] & [1110\delta\delta1] & [1011111] & [0000000]
\end{array}$$

Thus [0010111] is the only non-trivial pre-image of $[0^7]$, whereas the only candidate for a pre-image of [0010111] is [0000011] (since it cannot hold that $\alpha = \gamma = 0$), which implies $\gamma = \mathrm{A}$.

Finally, it suffices to consider the cases when $\eta = f(011000) = 1$, $\gamma = \mathrm{A}$ and $\alpha = \beta = \bar{\mathrm{a}}$. However, if we check few steps of $\mathbb{F}$ on [000010000], then it appears that we always get an illegal loop, as shown below:

$$
\begin{tikzcd}[column sep=small]
 & & {[000010000]} \arrow[d] & & \\
 & & {[001111100]} \arrow[d] & & \\
 & & {[0\bar{a}\bar{a}\bar{a}a1\bar{A}\bar{A}1]} \arrow[dll, "\text{a=1,A=0}"'] \arrow[dl, "\text{a=1,A=1}"] \arrow[dr, "\text{a=0,A=0}"'] \arrow[drr, "\text{a=0,A=1}"] & & \\
{[000011111]} & {[000011001]} \arrow[d] & & {[011101111]} \arrow[d] & {[011101001]} \arrow[d] \\
 & {[110111011]} \arrow[d] & & {[001000000]} & {[001111100]} \\
 & {[000010000]} & & &
\end{tikzcd}
$$

The proof is completed. $\square$

Appendix B

Theorem 8. *Let a local rule f with a six-cell neighborhood satisfy Eqs. (7), (11), (12) and (13). Then f cannot solve the parity problem.*

Proof. Similarly as in the proof of Theorem 7, we show that if $\beta = f(000110) = 0$, then f cannot solve the parity problem. Indeed, we have $\mathbb{F}([00011]) = [1\alpha 0\alpha 1]$, so $\alpha = 1$, otherwise $\mathbb{F}([00011]) = [00011]$. Thus, in particular,

$$
\begin{array}{ccccc}
[0111111] & [0001111] & [0010111] & [0100111] & [0011011] \\
\downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} \\
[00\bar{\mathrm{a}}1\mathrm{a}00] & [11110\eta\eta] & [\cdot 1111\cdot\cdot] & [0000000] & [\cdot 10001\cdot] \\
 & & & & \\
[0101011] & [0000011] & [0000101] & [0001001] & [0000000] \\
\downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} \\
[p\bar{v}vp100] & [0\bar{\mathrm{A}}\mathrm{A}1011] & [1111\cdot\cdot 0] & [0111111] & [0000000]
\end{array}
$$

where $p = v \oplus \gamma$. If so, then [0100111] is the only one non-trivial pre-image of $[0^7]$. However, for both values of p the pre-image of [0100111] is empty, which means that the graph $\Gamma_7^{even}(f)$ is not connected, which yields a contradiction. In the next part of the proof, we assume that $\beta = f(000110) = 1$.

Now, we show that if $f(111001) = \bar{\delta} \oplus \eta = 1$, then f can also not solve the parity problem. Indeed, then $\delta = 0$ and $\eta = 0$, or $\delta = 1$ and $\eta = 1$. However, in the latter case, it holds that $\mathbb{F}([11100]) = [0\eta 1\delta 0] = [01110]$, so again we get an illegal loop. In the first case, *i.e.*, when $\delta = 0$ and $\eta = 0$, we have

$$
\begin{array}{ccccc}
[1000000] & [1110000] & [1101000] & [1011000] & [1100100] \\
\downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} \\
[11\bar{\mathrm{A}}\,0\mathrm{A}11] & [00001\alpha\alpha] & [\alpha 0000\alpha 1] & [1111111] & [\bar{\alpha}01110\bar{\alpha}] \\
 & & & & \\
[1010100] & [1111100] & [1111010] & [1110110] & [1111111] \\
\downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} \\
[\gamma v\bar{v}\gamma 011] & [1\bar{\mathrm{a}}\mathrm{a}0100] & [0000pp1] & [1000000] & [1111111]
\end{array}
$$

Thus [1011000] is the only non-trivial pre-image of $[1^7]$. However, for both values of γ the pre-image of [1011000] is empty, which means that the graph $\Gamma_7^{odd}(f)$ is not connected, again a contradiction. Therefore, $f(111001) = \bar{\delta} \oplus \eta \neq 1$, *i.e.*, $\delta \neq \eta$. Using the fact that $\beta = f(000110) = 1$ and $\delta \neq \eta$, we obtain the following values of $\mathbb{F}$ on the vertices of $\Gamma_7^{odd}(f)$:

$$\begin{array}{ccccc}
[1000000] & [1110000] & [1101000] & [1011000] & [1100100] \\
\downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} \\
[11\bar{\mathrm{A}}\,0\mathrm{A}11] & [0\eta\eta01\alpha\alpha] & [\alpha0000\alpha1] & [1111111] & [\bar{\alpha}\bar{\eta}\eta110\bar{\alpha}] \\
 & & & & \\
[1010100] & [1111100] & [1111010] & [1110110] & [1111111] \\
\downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} \\
[\gamma v\bar{v}\gamma011] & [1\bar{\mathrm{a}}\mathrm{a}\eta0\bar{\eta}0] & [0000pp1] & [1000000] & [1111111]
\end{array}$$

Note that it cannot hold that $\alpha = 1$ and $\eta = 0$ since then $\mathbb{F}([1110000]) = [1110000]$. Thus the only non-trivial pre-image of $[1^7]$ is[1011000], whereas the only candidate for a pre-image of [1011000] is [1111100], which implies $\eta = \mathrm{a}$.

On the other hand, again using the fact that $\beta = f(000110) = 1$ and $\delta \neq \eta$, we obtain the following values of $\mathbb{F}$ on the vertices of $\Gamma_7^{even}(f)$:

$$\begin{array}{ccccc}
[0111111] & [0001111] & [0010111] & [0100111] & [0011011] \\
\downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} \\
[00\bar{\mathrm{a}}1\mathrm{a}00] & [1\alpha\alpha10\eta\eta] & [\bar{\delta}1111\eta0] & [0000000] & [\bar{\eta}\bar{\alpha}\alpha001\bar{\eta}] \\
 & & & & \\
[0101011] & [0000011] & [0000101] & [0001001] & [0000000] \\
\downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} & \downarrow{\scriptstyle\mathbb{F}} \\
[p\bar{v}vp100] & [0\bar{\mathrm{A}}\mathrm{A}\alpha1\bar{\alpha}1] & [1111\gamma\gamma0] & [0111111] & [0000000]
\end{array}$$

Thus [0100111] is the only non-trivial pre-image of $[0^7]$, whereas the only candidate for a pre-image of [0100111] is [0000011] (since it cannot hold that $\alpha = 1$ and $\eta = 0$), which implies $\alpha = \mathrm{A}$.

Finally, it suffices to consider the case when $\beta = f(000110) = 1$, $\delta \neq \eta$, $\alpha = \bar{\mathrm{A}}$ and $\eta = \mathrm{a}$. However, if we check a few steps of $\mathbb{F}$ on [000010000], then it appears that we always get an illegal loop, as shown below:

$$\begin{tikzcd}
 & & {[000010000]} \arrow[d] & & \\
 & & {[001111100]} \arrow[d] & & \\
 & & {[\alpha\alpha1\bar{\mathrm{a}}\mathrm{aaa}01]} \arrow[dll, "{\mathrm{a}=1,\alpha=0}"'] \arrow[dl, "{\mathrm{a}=0,\alpha=0}"] \arrow[dr, "{\mathrm{a}=1,\alpha=1}"'] \arrow[drr, "{\mathrm{a}=0,\alpha=1}"] & & \\
{[001011101]} \arrow[d] & {[001100001]} \arrow[d] & & {[111011101]} \arrow[d] & {[111100001]} \\
{[111110000]} & {[011101111]} \arrow[d] & & {[000010000]} & \\
 & {[010000000]} & & &
\end{tikzcd}$$

The proof is completed. $\square$