Chapter 1
On inverse source problems for space-dependent sources in thermoelasticity

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Abstract The aim of this contribution is to discuss the results in Maes and Van Bockstal (J. Inverse Ill-Posed Probl. (5)4 2022). These uniqueness results deal with inverse source problems of determining a space-dependent load or heat source in thermoelastic systems. The additional measurements are taken to be the final in time or time-average measurement of the displacement and/or time-average measurements of the temperature.

1.1 Introduction and literature overview

We consider an open and bounded domain $\Omega \subset \mathbb{R}^d$, where $d \in \mathbb{N}$, with Lipschitz continuous boundary Γ, which is occupied by an isotropic homogeneous thermoelastic body. We set $Q_T = \Omega \times (0,T)$ and $\Sigma_T = \Gamma \times (0,T)$, where $T > 0$ is a given final time. The elastic an thermal behaviours of the material are described by the functions

$$\boldsymbol{u}\colon Q_T \to \mathbb{R}^d \quad \text{and} \quad \theta\colon Q_T \to \mathbb{R},$$

which denote the displacement and the temperature difference from the reference value $\theta_0 > 0$ respectively, both taken with reference to the unstressed and undeformed state of the material.

The Lamé parameters $\lambda > 0$ and $\mu > 0$ are material-dependent quantities. These parameters can be expressed in terms of the shear modulus $\overline{G} > 0$ and the Poisson's

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ratio $\nu \in (0, \frac{1}{2})$ by the relations $\lambda = \frac{2\nu\overline{G}}{1-2\nu}$ and $\mu = \overline{G}$. We will denote the thermal conductivity, the mass density and the specific heat of the material by the positive constants κ, ρ and C_s respectively. As the material heats up or cools down, it undergoes a deformation in shape. This coupling between the shape and temperature of the material is expressed by the coupling parameter

$$\gamma = \alpha_T(3\lambda + 2\mu) = 2\alpha_T\overline{G}\frac{1+\nu}{1-2\nu} > 0,$$

where $\alpha_T > 0$ denotes the coefficient of linear expansion of the material.

The system of equations – governing the evolution of the displacement $\boldsymbol{u}$ and the temperature θ and their interaction – is given by the following isotropic thermoelastic system (of type-III), see e.g. [9],

$$\begin{cases} \rho\partial_{tt}\boldsymbol{u} - \mu\Delta\boldsymbol{u} - (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u}) + \gamma\nabla\theta = \boldsymbol{p} & \text{in } Q_T \\ \rho C_s\partial_t\theta - \kappa\Delta\theta - (k*\Delta\theta) + \theta_0\gamma\nabla\cdot\partial_t\boldsymbol{u} = h & \text{in } Q_T \\ \boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{0}, \quad \theta(\boldsymbol{x},t) = 0 & (\boldsymbol{x},t) \in \Sigma_T \\ \boldsymbol{u}(\boldsymbol{x},0) = \overline{\boldsymbol{u}}_0(\boldsymbol{x}), \quad \partial_t\boldsymbol{u}(\boldsymbol{x},0) = \overline{\boldsymbol{u}}_1(\boldsymbol{x}), \quad \theta(\boldsymbol{x},0) = \overline{\theta}_0(\boldsymbol{x}) & \boldsymbol{x} \in \Omega. \end{cases} \tag{1.1}$$

The functions $\boldsymbol{p}$ and h on the right hand side in (1.1) stand for the applied load and heat sources respectively. The Laplace convolution $*$ of the kernel k and a function θ is denoted by

$$(k*\theta)(\boldsymbol{x},t) = \int_0^t k(t-s)\theta(\boldsymbol{x},s)\,\mathrm{d}s.$$

The kernel function $k \in \mathrm{C}^2([0,T])$ in (1.1) is assumed to satisfy

$$(-1)^j k^{(j)}(t) \geq 0 \ \text{ and } \ k'(t) \neq 0, \ \text{ for all } t \geq 0, j = 0,1,2, \tag{1.2}$$

and thus is positive definite, see e.g. [10, Corollary 2.2]. A typical example is $k(t) = \alpha\exp(-\beta t)$ with $\alpha, \beta > 0$. The convolution product $k*\Delta\theta$ in (1.1) allows for memory effects and overcomes the infinite speed of propagation of heat in type-I thermoelasticty (where $k = 0$). In case $\kappa = 0$ and $k \neq 0$ the model is referred to as of type-II, whereas $\kappa \neq 0$ and $k \neq 0$ are considered in systems of type-III.

For the direct problem associated to (1.1), the following lemma obtained in [13, Theorem 4.1] is crucial. It addresses the existence and uniqueness questions for the direct problem (1.1).

Lemma 1 (Well-posedness)

Let $k \in \mathrm{C}^2([0,T])$ fulfill the condition (1.2)*.*

(i) Assume that $\boldsymbol{p} \in \mathrm{L}^2\left((0,T),\mathbf{L}^2(\Omega)\right)$ and $h \in \mathrm{L}^2\left((0,T),\mathrm{L}^2(\Omega)\right)$ and that the initial data satisfy $\overline{\boldsymbol{u}}_0 \in \mathbf{H}_0^1(\Omega)$, $\overline{\boldsymbol{u}}_1 \in \mathbf{L}^2(\Omega)$ and $\overline{\theta}_0 \in \mathrm{L}^2(\Omega)$. Then, the problem (1.1) *has a unique weak solution $(\boldsymbol{u},\theta)$ satisfying*

$$\begin{aligned}
&\boldsymbol{u} \in \mathrm{C}\left([0,T],\mathbf{L}^2(\Omega)\right) \cap \mathrm{L}^2\left((0,T),\mathbf{H}_0^1(\Omega)\right), \quad \partial_t \boldsymbol{u} \in \mathrm{L}^2\left((0,T),\mathbf{L}^2(\Omega)\right), \\
&\partial_{tt} \boldsymbol{u} \in \mathrm{L}^2\left((0,T),\mathbf{H}_0^1(\Omega)^*\right), \\
&\theta \in \mathrm{C}\left([0,T],\mathrm{L}^2(\Omega)\right) \cap \mathrm{L}^2\left((0,T),\mathrm{H}_0^1(\Omega)\right), \quad \partial_t \theta \in \mathrm{L}^2\left((0,T),\mathrm{H}_0^1(\Omega)^*\right).
\end{aligned}$$

(ii) Assume that $\boldsymbol{p} \in \mathbf{H}\left((0,T),\mathbf{L}^2(\Omega)\right)$ *and* $h \in \mathrm{H}^1\left((0,T),\mathrm{L}^2(\Omega)\right)$ *and that the initial data satisfy* $\overline{\boldsymbol{u}}_0 \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$, $\overline{\boldsymbol{u}}_1 \in \mathbf{H}_0^1(\Omega)$ *and* $\overline{\theta}_0 \in \mathrm{H}_0^1(\Omega)$. *Let the PDEs in* (1.1) *be fulfilled for* $t = 0$. *Then, the problem* (1.1) *has a unique weak solution* $(\boldsymbol{u}, \theta)$ *satisfying*

$$\begin{aligned}
&\boldsymbol{u} \in \mathrm{C}\left([0,T],\mathbf{H}_0^1(\Omega)\right), \quad \partial_t \boldsymbol{u} \in \mathrm{C}\left([0,T],\mathbf{L}^2(\Omega)\right) \cap \mathrm{L}^2\left((0,T),\mathbf{H}_0^1(\Omega)\right), \\
&\partial_{tt} \boldsymbol{u} \in \mathrm{L}^2\left((0,T),\mathrm{L}^2(\Omega)\right), \\
&\theta \in \mathrm{C}\left([0,T],\mathrm{H}_0^1(\Omega)\right), \quad \partial_t \theta \in \mathrm{L}^2\left((0,T),\mathbf{H}_0^1(\Omega)\right).
\end{aligned}$$

Literature overview. The inverse source problems for the systems described in (1.1) are listed in the next parts below. The listed theorems extend and complement several previously reported results. An overview of the available literature is exposed in great detail in [12]. Let us mention here the works [3, 13, 14, 15, 16, 17]. The approaches in [3, 17] consider the case $\Omega \subset \mathbb{R}^3$ and involved the clever use of Carleman estimates. In [3], the measurements $\boldsymbol{u}|_{\omega\times(0,T)}$ and $\theta|_{t=t_0}$ were used to determine the spatial component $f(\boldsymbol{x})$ in the heat source for type-I thermoelasticity of the form $h(\boldsymbol{x},t) = f(\boldsymbol{x})k(\boldsymbol{x},t)$. Here $t_0 \in (0,T)$ is an appropriate given time and $\omega \subset \Omega$ is a given subdomain whose boundary includes the boundary of Ω, i.e. $\partial\omega \supset \Gamma$. The recovery of a spatially varying load source $\boldsymbol{p}(\boldsymbol{x},t) = \sigma(\boldsymbol{x},t)\boldsymbol{f}(\boldsymbol{x})$ in type-II thermoelasticity, where σ is a known scalar field was discussed in [17] with measurement $\boldsymbol{u}|_{\omega\times(0,T)}$ under the assumption that $\boldsymbol{f}$ is known in a neighborhood of $\partial\Omega$. The final time measurement $\boldsymbol{u}(\boldsymbol{x},T)$ was used in [15] to recover $\boldsymbol{f}(\boldsymbol{x})$ in a load source of the form $\boldsymbol{p}(\boldsymbol{x},t) = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{r}(\boldsymbol{x},t)$ for all types of thermoelasticity, however including an extra damping term $\widetilde{\boldsymbol{g}}(\partial_t \boldsymbol{u})$ in the hyperbolic equation. An extension of these results to the anisotropic case for type-III thermoelasticity was investigated in [13]. The work [16] considered the recovery of the unknown time-dependent function $g(t)$ in the heat source $h(\boldsymbol{x},t) = g(t)f(\boldsymbol{x}) + r(\boldsymbol{x},t)$ in the one-dimensional case, using a space-average measurement of the temperature. There, numerical schemes are presented and the existence and uniqueness of a weak solution is proven based on Rothe's method. Finally, in [14] the time dependent-function $g(t)$ in $h(\boldsymbol{x},t) = g(t)f(\boldsymbol{x})$ is theoretically and numerically recovered from the integral measurement $\int_\Omega w(\boldsymbol{x})\theta(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}$, $t \in [0,T]$, where $w(t)$ is a certain weight function. Their technique was to reformulate the inverse problem into a direct problem. Numerical schemes are presented and the convergence of them is shown and illustrated with numerical examples.

In our work, as opposed to [3, 17], no Carleman estimates are used. Compared to [15], we focus on solving the inverse problems without damping term in the hyperbolic PDE. The results presented in this contribution are valid for all types of thermoelasticity.

Applications for the inverse source problems discussed in [7] range (among others) from the fitting and influence of a prosthetic socket on a residual limb after amputation, to the thermoforming/molding processes of polymer sheets and the solidification of metals on a mold. We refer to [1, 5, 8, 11, 18] for a detailed discussion.

1.2 Inverse problems of determining a space-dependent load source

Consider the decomposition of the load source $\boldsymbol{p}$ as

$$\boldsymbol{p}(\boldsymbol{x},t) = g(t)\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{s}(\boldsymbol{x},t),$$

where the functions $g\colon [0,T] \to \mathbb{R}$ and $\boldsymbol{s}\colon Q_T \to \mathbb{R}^d$ are given. Find the function $\boldsymbol{f}(\boldsymbol{x})$ from either the final-time measurement

$$\boldsymbol{u}(\boldsymbol{x},T) = \boldsymbol{\xi}_T(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{1.3}$$

or the time-average measurement

$$\int_0^T \boldsymbol{u}(\boldsymbol{x},t)\,\mathrm{d}t = \boldsymbol{\chi}_T(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega. \tag{1.4}$$

We first introduce several spaces of functions, allowing to conveniently state our results. For a function $g\colon [0,T] \to \mathbb{R}$, we define

$$G(t) = \frac{1}{g(t)} \quad \text{and} \quad H(t) = \int_0^t g(s)\,\mathrm{d}s.$$

We will consider the spaces

$$\begin{aligned}
\mathscr{C}_0 &:= \{g \in \mathrm{C}([0,T]) \mid 0 \notin g([0,T])\}; \\
\mathscr{C}_1 &:= \left\{g \in \mathrm{C}^1([0,T]) \mid \partial_t(g(t)^2) > 0, \forall t \in [0,T]\right\}; \\
\mathscr{C}_{1,*} &:= \left\{g \in \mathrm{C}^1([0,T]) \cap \mathscr{C}_0 \mid \partial_t(g(t)^2) \geq 0, \forall t \in [0,T]\right\}.
\end{aligned}$$

The class $\mathscr{C}_0$ consists of all continuous functions g on $[0,T]$ satisfying $g(t) > 0$ or $g(t) < 0$ for all $t \in [0,T]$. If $g \in \mathscr{C}_1$, then g is continuously differentiable, g and its derivative g' have the same sign and they both never attain the value 0, that is, $\mathscr{C}_1$ includes the functions g with either $g, \partial_t g > 0$ or $g, \partial_t g < 0$. For functions g in the class $\mathscr{C}_{1,*}$ we allow, besides the same sign behavior of g and g', that g' can vanish.

Theorem 1 *Let the conditions of Lemma 1(ii) be fulfilled. Assume that $g \in \mathscr{C}_1$. Then, given $\boldsymbol{\xi}_T \in \mathbf{L}^2(\Omega)$, there exists at most one triple*

$$(\boldsymbol{u}, \theta, \boldsymbol{f}) \in \mathrm{C}\left([0,T], \mathbf{H}_0^1(\Omega)\right) \times \mathrm{C}\left([0,T], \mathrm{H}_0^1(\Omega)\right) \times \mathbf{L}^2(\Omega),$$

that solves the problem (1.1) *and for which* (1.3) *holds.*

Theorem 2 *Let the conditions of Lemma 1(i) be fulfilled. Assume that $g \in \mathscr{C}_0$. Then, given $\boldsymbol{\chi}_T \in \mathbf{L}^2(\Omega)$, there exists at most one triple*

$$(\boldsymbol{u}, \theta, \boldsymbol{f}) \in \mathrm{C}\left([0,T], \mathbf{H}_0^1(\Omega)\right) \times \mathrm{C}\left([0,T], \mathrm{H}_0^1(\Omega)\right) \times \mathbf{L}^2(\Omega),$$

that solves the problem (1.1) *and for which* (1.4) *holds.*

From Lemma 1(i), it is clear that the measurements $\boldsymbol{\xi}_T$ and $\boldsymbol{\chi}_T$ are well-defined elements of $\mathbf{L}^2(\Omega)$. The stronger conditions in Lemma 1(ii), allow to test the equation with (time-dependent multiples) of $\partial_t \boldsymbol{u}(t)$. More precisely, we set $\boldsymbol{\varphi} = G(t)\partial_t \boldsymbol{u}(t) \in \mathbf{H}_0^1(\Omega)$ and $\psi = \theta_0^{-1} G(t)\theta(t) \in \mathrm{H}_0^1(\Omega)$ in the weak formulation associated to (1.1), and integrate the resulting equations in time, adding them together afterwards. If we consider this procedure for the difference $(\widetilde{\boldsymbol{u}}, \widetilde{\theta})$ of two solutions to (1.1), then the crucial observations for Theorem 1 are the cancellation of the coupled terms and that

$$\int_0^T (\boldsymbol{f}, \partial_t \widetilde{\boldsymbol{u}}(t))\, \mathrm{d}t = (\boldsymbol{f}, \widetilde{\boldsymbol{u}}(T) - \widetilde{\boldsymbol{u}}(0))\,.$$

For Theorem 2, an integration of the equations in the weak formulation in time is employed in the first step of the proof. Thereafter, $\boldsymbol{\varphi} = H(t)^{-1}\boldsymbol{u}(t) \in \mathbf{H}_0^1(\Omega)$ and $\psi = (\theta_0 H(t))^{-1} \int_0^t \theta(s)\mathrm{d}s \in \mathrm{H}_0^1(\Omega)$ are chosen in the weak formulation. The crucial observation here is that

$$\int_0^T (\boldsymbol{f}, \widetilde{\boldsymbol{u}}(t))\, \mathrm{d}t = \left(\boldsymbol{f}, \int_0^T \widetilde{\boldsymbol{u}}(t)\, \mathrm{d}t\right).$$

We refer to [7, Theorem 2.2] and [7, Theorem 2.7] respectively, for the complete details and calculations.

1.3 Inverse problem of determining a space-dependent heat source

Consider the decomposition of the heat source h as

$$h(\boldsymbol{x}, t) = g(t) f(\boldsymbol{x}) + r(\boldsymbol{x}, t),$$

where the functions $g\colon [0,T] \to \mathbb{R}$ and $r\colon Q_T \to \mathbb{R}$ are given. Find the function $f(\boldsymbol{x})$ from the time-average measurement

$$\int_0^T \theta(\boldsymbol{x}, t)\, \mathrm{d}t = \psi_T(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega. \tag{1.5}$$

Theorem 3 *Let the conditions of Lemma 1(ii) be fulfilled. Assume that $g \in \mathscr{C}_{1,*}$. Then, given $\psi_T \in \mathrm{L}^2(\Omega)$, there exists at most one triple*

$$(\boldsymbol{u},\theta,f)\in \mathrm{C}\left([0,T],\mathbf{H}_0^1(\Omega)\right)\times \mathrm{C}\left([0,T],\mathrm{H}_0^1(\Omega)\right)\times \mathrm{L}^2(\Omega),$$

that solves the problem (1.1) *and for which* (1.5) *holds.*

The crucial ingredient here, is that for the difference $(\widetilde{\boldsymbol{u}},\widetilde{\theta})$ of two solutions with the same measurements, initial and boundary data, the following estimate holds

$$\int_0^T (-G'(t))\left\|\widetilde{\theta}(t)\right\|_{\mathrm{L}^2(\Omega)}^2\,\mathrm{d}t+\int_0^T G(t)\left\|\nabla\widetilde{\theta}(t)\right\|_{\mathrm{L}^2(\Omega)}^2\leq C\left(f,\int_0^T\widetilde{\theta}(t)\,\mathrm{d}t\right),$$

for some constant $C>0$, from which the conclusion can be drawn. This estimate is achieved by taking $\boldsymbol{\varphi}=G(t)\partial_t\boldsymbol{u}(t)\in\mathbf{H}_0^1(\Omega)$ and $\psi=(\theta_0)^{-1}G(t)\theta(t)\in\mathrm{H}_0^1(\Omega)$ in the weak formulation. The full details can be found in [7, Theorem 3.1].

1.4 Combined problems

Consider the following decomposition of the load and heat sources,

$$\begin{aligned}\boldsymbol{p}(\boldsymbol{x},t)&=g_{\boldsymbol{u}}(t)\boldsymbol{f}_{\boldsymbol{u}}(\boldsymbol{x})+\boldsymbol{s}(\boldsymbol{x},t),\\ h(\boldsymbol{x},t)&=g_\theta(t)f_\theta(\boldsymbol{x})+r(\boldsymbol{x},t),\end{aligned}$$

where the functions $g_{\boldsymbol{u}},g_\theta\colon[0,T]\to\mathbb{R}$, $\boldsymbol{s}\colon Q_T\to\mathbb{R}^d$ and $r\colon Q_T\to\mathbb{R}$ are given. Find the functions $(\boldsymbol{f}_{\boldsymbol{u}},f_\theta)$ simultaneously from either the measurements

$$\boldsymbol{u}(\boldsymbol{x},t)=\boldsymbol{\xi}_T(\boldsymbol{x})\quad\text{ and }\quad\int_0^T\theta(\boldsymbol{x},t)\,\mathrm{d}t=\psi_T(\boldsymbol{x}),\quad\boldsymbol{x}\in\Omega,\tag{1.6}$$

or the measurements

$$\int_0^T\boldsymbol{u}(\boldsymbol{x},t)\,\mathrm{d}t=\boldsymbol{\chi}_T(\boldsymbol{x}),\quad\int_0^T\theta(\boldsymbol{x},t)\,\mathrm{d}t=\psi_T(\boldsymbol{x})$$
$$\text{and}\quad\int_0^T t\theta(\boldsymbol{x},t)\,\mathrm{d}t=\phi_T(\boldsymbol{x}),\quad\boldsymbol{x}\in\Omega.\tag{1.7}$$

By combining the techniques used for the separate inverse problems, one natural deduces [7, Theorem 4.1] and [7, Theorem 4.2]:

Theorem 4 *Let the conditions of Lemma 1(ii) be fulfilled. Assume that $g_{\boldsymbol{u}},g_\theta\in\mathscr{C}_1$ satisfy $g_{\boldsymbol{u}}=cg_\theta$ for some $c>0$. Then, given $\boldsymbol{\xi}_T\in\mathbf{L}^2(\Omega)$ and $\psi_T\in\mathrm{L}^2(\Omega)$, there exists at most one solution*

$$(\boldsymbol{u},\theta,\boldsymbol{f}_{\boldsymbol{u}},f_\theta)\in\mathrm{C}\left([0,T],\mathbf{H}_0^1(\Omega)\right)\times\mathrm{C}\left([0,T],\mathrm{H}_0^1(\Omega)\right)\times\mathbf{L}^2(\Omega)\times\mathrm{L}^2(\Omega),$$

that solves the problem (1.1) *and for which* (1.6) *holds.*

Theorem 5 *Let the conditions of Lemma 1(i) be fulfilled. Assume that $g_{\boldsymbol{u}},g_\theta\in\mathscr{C}_0$ satsify $g_{\boldsymbol{u}}=cg_\theta$ for some $c>0$. Then, given $\boldsymbol{\chi}_T\in\mathbf{L}^2(\Omega)$ and $\psi_T,\phi_T\in\mathrm{L}^2(\Omega)$,*

there exists at most one solution

$$(\boldsymbol{u},\theta,\boldsymbol{f_u},f_\theta)\in \mathrm{C}\left([0,T],\mathbf{H}_0^1(\Omega)\right)\times \mathrm{C}\left([0,T],\mathrm{H}_0^1(\Omega)\right)\times \mathbf{L}^2(\Omega)\times \mathrm{L}^2(\Omega),$$

that solves the problem (1.1) *and for which* (1.7) *holds.*

The condition $g_{\boldsymbol{u}} = cg_\theta$ for some $c>0$ impose that both intensities over time of the load and the heat sources are proportional to each other. On the application side, this means that the intensity of the pressure and heating of the material follow the same profile in time.

1.5 Discussion conditions and outlook

The necessity of the sign conditions on g and on its derivative can be shown. In fact, the results in Theorem 1, 2 and 3 remain valid when replacing the homogeneous Dirichlet condition for θ to a Neumann boundary condition. Several counterexamples to the uniqueness of a solution are reported in [7, Example 2.6, Remark 2.8, Remark 3.2]. We here present the one related to Theorem 3, in which the sign condition on g is violated and at least two solutions to the problem exist.

Let $T=\pi$ and $\Omega=(0,2\pi)$. Consider the problem one-dimensional problem

$$\begin{cases} \partial_{tt}u(x,t)-\partial_{xx}u(x,t)+\partial_x\theta(x,t)=0 & (x,t)\in Q_T \\ \partial_t\theta(x,t)-\partial_{xx}\theta(x,t)-k*\partial_{xx}\theta(x,t)+\partial_{xt}u(x,t)=g(t)f(x) & (x,t)\in Q_T \\ u(x,t)=0,\quad \partial_x\theta(x,t)=0 & (x,t)\in \Sigma_T \\ u(x,0)=0,\quad \partial_t u(x,0)=0,\quad \theta(x,0)=0 & x\in\Omega, \end{cases} \tag{1.8}$$

where we take $k(t)=\exp(-2t)$ and consider vanishing measurement $\int_0^T \theta(\cdot,t)\,\mathrm{d}t = \psi_T = 0$ in Ω. The function

$$g(t)=\left(\frac{7}{5}+\frac{t}{2}\right)\cos(t)-\frac{3\sin(t)}{10}-\frac{2\exp(-2t)}{5},$$

has changing sign behavior. Therefore, besides the trivial solution $(u,\theta,f)=(0,0,0)$, the set of functions

$$u(x,t)=\frac{t}{2}\sin(t)\sin(x),\quad \theta(x,t)=\cos(t)\cos(x),\quad f(x)=\cos(x)$$

form a non-trivial solution to (1.8). Hence, there is no uniqueness in this case.

Outlook. Further research could be undertaken to investigate the monotonicity conditions on the function g arising in the proofs, as it might be possible to obtain uniqueness of the considered inverse problems for a broader class of functions. Additionally, at the moment, the uniqueness for the inverse problem discussed in Section 1.3 under the measurement $\theta(\cdot,T)$ in Ω is still unanswered. The coupling between the parabolic and hyperbolic equations in (1.1) complicates the analysis.

Furthermore, the numerical reconstruction of the spatial dependent sources in the inverse problems, gives a third direction of future research. Lastly, the studied problems could be investigated for other systems of thermoealasticity, e.g. in the dual-phase-lag setting [4, 6].

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References

1. Alphonse, A., Rautenberg, C. N., Rodrigues, J. F.: Analysis of a quasi-variational contact problem arising in thermoelasticity. Nonlinear Anal. **217**, 112728 (2022)
2. Biot, M. A.: Thermoelasticity and irreversible thermodynamics. J. Appl. Phys. **27**(3), 240–253 (1956)
3. Bellassoued, M., Yamamoto, M.: Carleman estimates and an inverse heat source problem for the thermoelasticity system. Inverse Problems **27**(1), ID 015006 (2011)
4. Chandrasekharaiah, D. S.: Thermoelasticity with second sound: A review. Appl. Mech. Rev. **39**(3), 355–376 (1986)
5. Demir, M. H., Yigit, F.: Thermoelastic stability analysis of solidification of pure metals on a coated planar mold of finite thickness. Metallurgical Mat. Trans. B **48**(2), 966–982 (2017)
6. Maes, F., Van Bockstal, K.: Thermoelastic problem in the setting of dual-phase-lag heat conduction: Existence and uniqueness of a weak solution.: J. Math. Anal. Appl. **503**(1) 125304 (2021)
7. Maes, F., Van Bockstal, K.: Uniqueness for inverse source problems of determining a space-dependent source in thermeoleastic systems. J. Inverse Ill-Posed Probl. **5**(4), 1–12 (2022)
8. Mak, A. F., Zhang, M., Boone, D. A.: State-of-the-art research in lower-limb prosthetic biomechanics-socket interface: A review. J. Rehabil. Res. Dev. **38**(2), 161–174 (2001)
9. Muñoz Rivera, J. E., Qin, Y.: Global Existence and Exponential Stability in One-dimensional Nonlinear Thermoelasticity with Thermal Memory. Nonlinear Anal. **51**(1), 11–32 (2002)
10. Nohel, J. A., Shea, D. F.: Frequency domain methods for Volterra equations. Adv. Math. **22**(3), 278–304 (1976)
11. Tönük, E., Silver-Thorn, M. B.: Nonlinear elastic material property estimation of lower extremity residual limb tissues. IEEE Trans. Neural Syst. Rehabil. Eng **11**(1), 43–53 (2003)
12. Van Bockstal, K.: Inverse source problems in thermoelasticity. In: Ruzhanksy, M., Torebek, B. (eds.) Extended Abstracts MWCAPDE 2023, Birkhäuser Cham (2023) (to appear)
13. Van Bockstal, K., Marin, L.: Recovery of a space-dependent vector source in anisotropic thermoelastic systems. Comput. Methods Appl. Mech. Engrg. **321**, 269–293 (2017)
14. Van Bockstal, K., Marin, L.: Finite element method for the reconstruction of a time-dependent heat source in isotropic thermoelasticity systems of type-III. Z. Angew. Math. Phys. **73**(3), 27 (2022)
15. Van Bockstal, K., Slodička, M.: Recovery of a space-dependent vector source in thermoelastic systems. Inverse Probl. Sci. Eng. **23**(6), 956–968 (2015)
16. Van Bockstal, K., Slodička, M.: Recovery of a time-dependent heat source in one-dimensional thermoelasticity of type-III. Inverse Probl. Sci. Eng. **25**(5), 749–770 (2017)
17. Wu, B., Liu, J.: Determination of an unknown source for a thermoelastic system with a memory effect. Inverse Problems **28**(9), ID 095012 (2012)
18. Warby, M. K., Whiteman, J. R., Jiang, W. -G., Warwick, P., Wright, T.: Finite element simulation of thermoforming processes for polymer sheets. Math. Comput. Simulation **61**(3–6), 209–218 (2003)