A dissection of the monotonicity property of binary operations from a dominance point of view

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Abstract

In this paper, we expound weaker forms of increasingness of binary operations on a lattice by reducing the number of variables involved in the classical formulation of the increasingness property as seen from the viewpoint of dominance between binary operations. We investigate the relationships among these weaker forms. Furthermore, we demonstrate the role of these weaker forms in characterizing the meet and join operations of a lattice and a chain in particular. Finally, we provide ample generic examples.

Keywords: Binary operation, increasingness, dominance relation, lattice

1. Introduction

Over the past years, the theory of aggregation, initiated by researchers in the field of fuzzy set theory, has grown into a field of its own, with far-reaching applications in data science, statistics and machine learning [13, 29, 31]. The focal object in this theory is the concept of an aggregation function allowing to turn a list of input values into a single representative output value. The original setting of the real unit interval has long been replaced by more general structures, with a growing interest in partially ordered sets, and lattices [10] in particular.

Increasingness (equivalently, increasingness in each argument) is fundamental to the notion of an aggregation function. However, it has been recognized that this may exclude certain sound

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functions used in practice for fusing data, such as the mode function or the Lehmer mean. To overcome this problem, *weak monotonicity* [3] and *directional monotonicity* [6, 19, 20] (for a discussion of maximal directions of monotonicity, see [8]) were proposed as relaxations of the increasingness property. These notions were further extended to structures more general than the real unit interval, such as Riesz spaces [26], bounded chains [16], lattices and even trellises [32]. Further, in the interval-valued setting, directional monotonicity w.r.t. an admissible order [25] and *conditional monotonicity* [17] were studied. Besides, there exist also other different weaker forms of increasingness that are restricted to the real-valued setting, such as *cone monotonicity* [3] and *curve-based monotonicity* [14].

Given the availability of an order relation, increasingness is also a natural property of aggregation functions on bounded posets and lattices. In the context of lattices, by reducing the number of variables involved in the classical formulation of the increasingness property, Zedam and De Baets [32] explored *cross-increasingsness* and some *restriction-less* forms of increasingness, which can be employed as weaker forms of increasingness when characterizing the meet and/or join operation of a lattice. This study also considered the less-known structure of a trellis (a pseudo-ordered set with meet and join operations [27]), showing that a further generalization of increasingness is not always that trivial.

Along a different line, the concept of dominance as a binary relation on the set of binary operations defined on the same poset was introduced in the field of probabilistic metric spaces [28], where it appears in connection with the construction of Cartesian products of probabilistic metric spaces. Later, the dominance relation between t-norms was investigated in the context of the construction of fuzzy equivalence relations [9, 11, 22] and fuzzy order relations [5]. A long-standing conjecture about the transitivity of the dominance relation between t-norms was disproven at last [18, 24]. With the growing interest in the dominance relation, more general classes of binary operations were considered in [21, 22, 23].

A well-known basic result is that the increasingness of a binary operation on the real unit interval is equivalent to this operation dominating the maximum operation or, equivalently, being dominated by the minimum operation. Inspired by this intimate relationship between the dominance relation and the increasingness property as well as the reduction of the number of variables considered in [32], we develop a systematic way of proposing weaker forms of increasingness of binary operations on lattices. More specifically, since the classical increasingness property is expressed in terms of four variables, we reduce the number of variables to three or two in different ways. The relationships between the resulting weaker forms of increasingness are explored, especially for commutative binary operations. As a continuation of the work in [32], we also show that several of these weaker forms (which are incomparable in general) allow for characterizing the meet and join operations as specific idempotent operations.

This paper is structured as follows. In Section 2, we recall some basic notions on lattices and the dominance relation needed in this paper. In Section 3, we reduce the number of variables by one in the above three equivalent forms of increasingness, to obtain three families of weaker forms of increasingness expressed in terms of three variables. We carry out a similar analysis in Section 4, where weaker forms of increasingness are expressed in terms of two variables. In both sections, we consider the relationships among the different weaker forms. We discuss the characterization of the meet and join operations on lattices and chains as a by-product. In Section 5, we explore the relationships among the weaker forms of increasingness expressed in terms of two and three variables. Next, in Section 6, we present Hasse diagrams summarizing and unraveling the intricate relationships among the various weaker forms of increasingness introduced, with particular attention to commutative operations. Section 7 is devoted to examples. Finally, we present some concluding remarks and future research lines in Section 8.

2. Preliminaries

In this section, we recall some basic concepts and results about lattices, existing weaker forms of increasingness of binary operations on lattices and the dominance relation between binary operations.

2.1. Basic definitions of lattices

In this subsection, we recall some essential notions from lattice theory [4, 7, 12].

A binary relation $\leq$ on a set L is called a partial order (relation) if it is reflexive, antisymmetric and transitive. A set L equipped with a partial order $\leq$ is denoted by $(L, \leq)$ and is called a partially ordered set, or a poset for short. A poset $(L, \leq)$ that has a bottom element 0 and a top element 1 (*i.e.*, $0 \leq x \leq 1$, for any $x \in L$) is called a bounded poset and is denoted by $(L, \leq, 0, 1)$.

A meet semilattice is a poset $(L, \leq)$ in which every two-element subset $\{x, y\}$ has an infimum, denoted by $x \wedge y$; a join semilattice is a poset $(L, \leq)$ in which every two-element subset $\{x, y\}$ has a supremum, denoted by $x \vee y$. A lattice, denoted by $\mathbb{L} = (L, \leq, \wedge, \vee)$, is a poset that is both a meet semilattice and a join semilattice. Note that $x \leq y$ if and only if $x \wedge y = x$ if and only if $x \vee y = y$. A bounded lattice is denoted by $\mathbb{L} = (L, \leq, \wedge, \vee, 0, 1)$.

If $x \leq y$ and $x \neq y$, then we write $x < y$. If neither $x \leq y$ nor $y \leq x$, then we say that x and y are incomparable, and write $x \parallel y$. If any two elements of a poset $(L, \leq, \wedge, \vee)$ are comparable, *i.e.*, $x \leq y$ or $y \leq x$, for any $x, y \in L$, then $(L, \leq, \wedge, \vee)$ is called a totally ordered set or simply a chain.

The dual poset $\mathbb{L}^{\mathrm{d}}$ of a bounded poset $\mathbb{L} = (L, \leq, 0, 1)$ is defined as $\mathbb{L}^{\mathrm{d}} = (L^{\mathrm{d}}, \leq^{\mathrm{d}}, 1, 0)$, where $\leq^{\mathrm{d}}$ is the converse of $\leq$, *i.e.*, $x \leq^{\mathrm{d}} y$ if and only if $y \leq x$. Similarly, the dual lattice $\mathbb{L}^{\mathrm{d}}$ of a bounded lattice $\mathbb{L} = (L, \leq, \wedge, \vee, 0, 1)$ is defined as $\mathbb{L}^{\mathrm{d}} = (L^{\mathrm{d}}, \leq^{\mathrm{d}}, \vee, \wedge, 1, 0)$.

2.2. Properties of binary operations

In this subsection, we first recall the classical notion of increasingness of binary operations and the dominance relation between binary operations. We also introduce some shorthand notations that will become clear further on.

Definition 2.1. [13] Let $\mathbb{L} = (L, \leq, \wedge, \vee)$ be a lattice. A binary operation F on L is called:

(i) $\leq^r$-increasing (referred to as property $\mathcal{I}_3^r$), if for any $x, y, v \in L$, it holds that

$$y \leq v \Rightarrow F(x, y) \leq F(x, v); \tag{2.1}$$

(ii) $\leq^\ell$-increasing (referred to as property $\mathcal{I}_3^\ell$), if for any $x, y, u \in L$, it holds that

$$x \leq u \Rightarrow F(x, y) \leq F(u, y); \tag{2.2}$$

(iii) $\leq$-increasing (referred to as property $\mathcal{I}_4$), if it is $\leq^r$-increasing and $\leq^\ell$-increasing.

Note that for a commutative binary operation, properties $\mathcal{I}_3^r$ and $\mathcal{I}_3^\ell$ coincide. Obviously, $\mathcal{I}_4$ implies both $\mathcal{I}_3^r$ and $\mathcal{I}_3^\ell$.

Next, we recall the dominance relation between two binary operations on a lattice.

Definition 2.2. [1] Let $\mathbb{L} = (L, \leq, \wedge, \vee)$ be a lattice. A binary operation F on L is said to dominate a binary operation G on L (or, also, G is said to be dominated by F), denoted as $F \gg G$, if for any $x, y, u, v \in L$, it holds that

$$F\left(G\left(x, y\right), G\left(u, v\right)\right) \geq G\left(F\left(x, u\right), F\left(y, v\right)\right). \tag{2.3}$$

We recall a well-known result.

Theorem 2.1. [1] Let $\mathbb{L} = (L, \leq, \wedge, \vee)$ be a lattice and F be a binary operation on L. Then the following statements are equivalent:

(i) F is $\leq$-increasing, *i.e.*, for any $x, y, u, v \in L$, it holds that

$$x \leq u, y \leq v \Rightarrow F(x, y) \leq F(u, v); \tag{2.4}$$

(ii) $\wedge \gg F$, *i.e.*, for any $x, y, u, v \in L$, it holds that

$$F(x, y) \wedge F(u, v) \geq F(x \wedge u, y \wedge v); \tag{2.5}$$

(iii) $F \gg \vee$, *i.e.*, for any $x, y, u, v \in L$, it holds that

$$F(x \vee u, y \vee v) \geq F(x, y) \vee F(u, v). \tag{2.6}$$

Consider two properties p and q that an operation on $\mathbb{L} = (L, \leq, \wedge, \vee)$ can satisfy or not. We use the notation $p \Rightarrow q$ to denote that p implies q and $p \Leftrightarrow q$ to denote that p and q are equivalent, $p \sqcap q$ to denote their conjunction and $p \sqcup q$ to denote their disjunction. Let $\mathcal{P}$ represent a set of properties of operations of interest on $\mathbb{L}$, then $(\mathcal{P}, \Rightarrow)$ is a poset. If any operation satisfies a property q on $\mathbb{L}^{\mathrm{d}}$ whenever it satisfies a property p on $\mathbb{L}$, then p is called dual to q.

3. Weaker forms of increasingness expressed in terms of three variables

3.1. Definitions

In this subsection, we introduce three families of weaker forms of increasingness obtained by reducing the number of variables by one in Theorem 2.1. We use the symbols $\mathscr{I}$, $\mathscr{M}$ and $\mathscr{J}$ to distinguish the weaker forms obtained from Eqs. (2.4), (2.5) and (2.6), respectively, complemented with a subscript to indicate the number of variables involved and a superscript allowing to distinguish the different weaker forms in the same family.

In the following table, the leftmost column indicates the two variables identified in Theorem 2.1, whereas the other columns indicate the resulting property.

	$\mathscr{I}_3$	$\mathscr{M}_3$	$\mathscr{J}_3$
$x = y$	$\mathcal{I}_3^1 : x \leq u, x \leq v \Rightarrow F(x,x) \leq F(u,v)$	$\mathcal{M}_3^1 : F(x,x) \wedge F(u,v) \geq F(x \wedge u, x \wedge v)$	$\mathcal{J}_3^1 : F(x \vee u, x \vee v) \geq F(x,x) \vee F(u,v)$
$x = u$	$\mathcal{I}_3^2 : y \leq v \Rightarrow F(x,y) \leq F(x,v)$	$\mathcal{M}_3^2 : F(x,y) \wedge F(x,v) \geq F(x, y \wedge v)$	$\mathcal{J}_3^2 : F(x, y \vee v) \geq F(x,y) \vee F(x,v)$
$x = v$	$\mathcal{I}_3^3 : y \leq x, x \leq u \Rightarrow F(x,y) \leq F(u,x)$	$\mathcal{M}_3^3 : F(x,y) \wedge F(u,x) \geq F(x \wedge u, y \wedge x)$	$\mathcal{J}_3^3 : F(x \vee u, y \vee x) \geq F(x,y) \vee F(u,x)$
$y = u$	$\mathcal{I}_3^4 : x \leq y, y \leq v \Rightarrow F(x,y) \leq F(y,v)$	$\mathcal{M}_3^4 : F(x,y) \wedge F(y,v) \geq F(x \wedge y, y \wedge v)$	$\mathcal{J}_3^4 : F(x \vee y, y \vee v) \geq F(x,y) \vee F(y,v)$
$y = v$	$\mathcal{I}_3^5 : x \leq u \Rightarrow F(x,y) \leq F(u,y)$	$\mathcal{M}_3^5 : F(x,y) \wedge F(u,y) \geq F(x \wedge u, y)$	$\mathcal{J}_3^5 : F(x \vee u, y) \geq F(x,y) \vee F(u,y)$
$u = v$	$\mathcal{I}_3^6 : x \leq u, y \leq u \Rightarrow F(x,y) \leq F(u,u)$	$\mathcal{M}_3^6 : F(x,y) \wedge F(u,u) \geq F(x \wedge u, y \wedge u)$	$\mathcal{J}_3^6 : F(x \vee u, y \vee u) \geq F(x,y) \vee F(u,u)$

Table 1. Properties resulting from the reduction of the number of variables by one in Theorem 2.1.

The following remark can be easily verified from Table 1.

Remark 3.1. The following equivalences hold among the properties in Table 1 (equivalent properties are indicated in the same color).

(i) $\mathcal{M}_3^2 \Leftrightarrow \mathcal{J}_3^2 \Leftrightarrow \mathcal{I}_3^2 \Leftrightarrow \mathcal{I}_3^r$ and $\mathcal{M}_3^5 \Leftrightarrow \mathcal{J}_3^5 \Leftrightarrow \mathcal{I}_3^5 \Leftrightarrow \mathcal{I}_3^\ell$.

(ii) $\mathcal{M}_3^1 \Leftrightarrow \mathcal{M}_3^6$, $\mathcal{J}_3^1 \Leftrightarrow \mathcal{J}_3^6$, $\mathcal{M}_3^3 \Leftrightarrow \mathcal{M}_3^4$ and $\mathcal{J}_3^3 \Leftrightarrow \mathcal{J}_3^4$, which can be obtained directly by relabelling the variables.

Therefore, the eighteen weaker forms of increasingness expressed in terms of three variables collapse into ten different weaker forms, namely $\mathscr{W}_3 = \mathscr{I}_3 \cup \mathscr{M}_3 \cup \mathscr{J}_3 = \left\{\mathcal{I}_3^1, \mathcal{I}_3^2, \mathcal{I}_3^3, \mathcal{I}_3^4, \mathcal{I}_3^5, \mathcal{I}_3^6, \mathcal{M}_3^1, \mathcal{M}_3^3, \mathcal{J}_3^1, \mathcal{J}_3^3\right\}$.

Moreover, for commutative binary operations, some additional equivalences hold.

Remark 3.2. Let $\mathbb{L} = (L, \leq, \wedge, \vee)$ be a lattice and F be a binary operation on L. If F is commutative, then the following equivalences hold:

(i) F satisfies $\mathcal{I}_3^3$ if and only if F satisfies $\mathcal{I}_3^4$;

(ii) F satisfies $\mathcal{I}_3^2$ if and only if F satisfies $\mathcal{I}_3^5$ if and only if F satisfies $\mathcal{I}_4$.

Therefore, for commutative binary operations, there are only seven different weaker forms of increasingness expressed in terms of three variables. Moreover, for commutative binary operations, $\mathcal{M}_3^3$ and $\mathcal{J}_3^3$ can be further simplified.

Proposition 3.1. *Let $\mathbb{L} = (L, \leq, \wedge, \vee)$ be a lattice and F be a binary operation on L. If F is commutative, then the following equivalences hold:*

(i) *F satisfies $\mathcal{M}_3^3$ if and only if for any $x, y, u \in L$, it holds that*

$$F(x, y) \geq F(x \wedge u, y \wedge x);$$

(ii) *F satisfies $\mathcal{J}_3^3$ if and only if for any $x, y, u \in L$, it holds that*

$$F(x, y) \leq F(x \vee u, y \vee x)\,.$$

Proof. We only prove (i), since the proof of (ii) is similar. Suppose that $F(x, y) \geq F(x \wedge u, y \wedge x)$, for any $x, y, u \in L$. Then we have $F(u, x) = F(x, u) \geq F(x \wedge y, u \wedge x) = F(x \wedge u, y \wedge x)$. Hence, $F(x, y) \wedge F(u, x) \geq F(x \wedge u, y \wedge x)$. Hence, F satisfies $\mathcal{M}_3^3$. The converse implication is obvious. □

The following remark expresses the duality among the weaker forms of increasingness.

Remark 3.3.

(i) $\mathcal{M}_3^1$ is dual to $\mathcal{J}_3^1$; $\mathcal{M}_3^3$ is dual to $\mathcal{J}_3^3$;

(ii) $\mathcal{I}_3^1$ is dual to $\mathcal{I}_3^6$; $\mathcal{I}_3^3$ is dual to $\mathcal{I}_3^4$;

(iii) $\mathcal{I}_3^2$ and $\mathcal{I}_3^5$ are self-dual.

3.2. Weaker forms of increasingness on lattices

In this subsection, we study the relationships among the different weaker forms expressed in terms of three variables of lattice-valued binary operations. In this subsection, we always assume that the underlying structure is a lattice $\mathbb{L} = (L, \leq, \wedge, \vee)$.

The following proposition directly follows from the definition of increasingness.

Proposition 3.2. $\mathcal{I}_3^2 \sqcap \mathcal{I}_3^5 \Leftrightarrow \mathcal{I}_4$.

Proposition 3.3.

(i) $\mathcal{M}_3^1 \sqcup \mathcal{J}_3^1 \Rightarrow \mathcal{I}_3^1 \sqcap \mathcal{I}_3^6$*;*

(ii) $\mathcal{M}_3^3 \sqcup \mathcal{J}_3^3 \Rightarrow \mathcal{I}_3^3 \sqcap \mathcal{I}_3^4$*.*

Proof. We only prove (i), since the proof of (ii) is similar. We prove that $\mathcal{M}_3^1 \Rightarrow \mathcal{I}_3^1 \sqcap \mathcal{I}_3^6$. Suppose that F satisfies $\mathcal{M}_3^1$. For any $x, u, v \in L$ such that $x \leq u, x \leq v$, it holds that $F(x, x) =$

$F(x \wedge u, x \wedge v) \leq F(u, v)$; for any $x, y, u \in L$ such that $x \leq u, y \leq u$, it holds that $F(x, y) = F(u \wedge x, u \wedge y) \leq F(u, u)$. Hence, F satisfies $\mathcal{I}_3^1 \sqcap \mathcal{I}_3^6$. The proof in the case that $\mathcal{J}_3^1 \Rightarrow \mathcal{I}_3^1 \sqcap \mathcal{I}_3^6$ is similar. $\square$

Next, we give some examples to illustrate that the converse implications do not hold in general.

Example 3.1.

(i) Consider the bounded lattice $\mathbb{L}_1 = (L, \leq, \wedge, \vee)$ shown in the Hasse diagram in Figure 1 and the binary operation F on L in Table 2 [1]. The operation F satisfies $\mathcal{I}_3^1$ and $\mathcal{I}_3^6$, but not $\mathcal{M}_3^1$, since $F(b \wedge a, b \wedge c) = F(0, b) = b > 0 = b \wedge a = F(b, b) \wedge F(a, c)$.

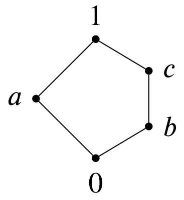

Figure 1. Hasse diagram of $\mathbb{L}_1$.

F	0	a	b	c	1
0	0	0	b	b	c
a	0	0	a	a	0
b	b	a	b	b	c
c	b	c	b	c	1
1	a	c	b	c	1

Table 2. The binary operation F on L.

(ii) Consider the bounded lattice $\mathbb{L}_2 = (L, \leq, \wedge, \vee)$ shown in the Hasse diagram in Figure 2 and the binary operation G on L in Table 3. The operation G satisfies $\mathcal{I}_3^3$ and $\mathcal{I}_3^4$, but not $\mathcal{M}_3^3$, since $G(a \wedge 1, 1 \wedge b) = G(a, b) = a > 0 = a \wedge b = G(a, 1) \wedge G(1, b)$.

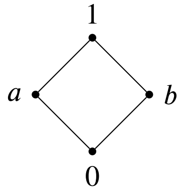

Figure 2. Hasse diagram of $\mathbb{L}_2$.

G	0	a	b	1
0	0	0	b	0
a	a	a	a	a
b	b	b	b	b
1	0	a	b	1

Table 3. The binary operation G on L.

To characterize the properties in more detail, we will separate $\mathcal{I}_3^2$ and $\mathcal{I}_3^5$ into two parts relative to the diagonal section.

Definition 3.1. Let $\mathbb{L} = (L, \leq, \wedge, \vee)$ be a lattice. A binary operation F on L is said to satisfy property $\mathcal{I}_3^{r,a}$, $\mathcal{I}_3^{r,b}$, $\mathcal{I}_3^{\ell,a}$ or $\mathcal{I}_3^{\ell,b}$, if it satisfies the corresponding implication listed in Table 4.

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[1] Note that in all the tables, the first column represents the values of the first variable, whereas the first row represents the values of the second variable.

	above the diagonal	below the diagonal
$\mathcal{I}_3^2$	$\mathcal{I}_3^{r,a}$: $x \leq y \leq v \Rightarrow F(x,y) \leq F(x,v)$	$\mathcal{I}_3^{r,b}$: $y \leq v \leq x \Rightarrow F(x,y) \leq F(x,v)$
$\mathcal{I}_3^5$	$\mathcal{I}_3^{\ell,a}$: $x \leq u \leq y \Rightarrow F(x,y) \leq F(u,y)$	$\mathcal{I}_3^{\ell,b}$: $y \leq x \leq u \Rightarrow F(x,y) \leq F(u,y)$

Table 4. Properties obtained from $\mathcal{I}_3^2$ and $\mathcal{I}_3^5$.

Remark 3.4. Let $\mathbb{L} = (L, \leq, \wedge, \vee)$ be a lattice and F be a binary operation on L. If F is commutative, then the following equivalences hold:

(i) F satisfies $\mathcal{I}_3^{r,a}$ if and only if F satisfies $\mathcal{I}_3^{\ell,b}$;

(ii) F satisfies $\mathcal{I}_3^{r,b}$ if and only if F satisfies $\mathcal{I}_3^{\ell,a}$.

The following remark expresses the duality among the weaker forms of increasingness.

Remark 3.5.

(i) $\mathcal{I}_3^{r,a}$ is dual to $\mathcal{I}_3^{r,b}$;

(ii) $\mathcal{I}_3^{\ell,a}$ is dual to $\mathcal{I}_3^{\ell,b}$.

The following propositions are immediate.

Proposition 3.4.

(i) $\mathcal{I}_3^2 \Rightarrow \mathcal{I}_3^{r,b} \sqcap \mathcal{I}_3^{r,a}$;

(ii) $\mathcal{I}_3^5 \Rightarrow \mathcal{I}_3^{\ell,b} \sqcap \mathcal{I}_3^{\ell,a}$.

Proposition 3.5.

(i) $\mathcal{M}_3^1 \sqcup \mathcal{M}_3^3 \Rightarrow \mathcal{I}_3^{r,a} \sqcap \mathcal{I}_3^{\ell,b}$;

(ii) $\mathcal{J}_3^1 \sqcup \mathcal{J}_3^3 \Rightarrow \mathcal{I}_3^{r,b} \sqcap \mathcal{I}_3^{\ell,a}$.

Combining Propositions 3.3 and 3.5 leads to the following result.

Corollary 3.1.

(i) $\mathcal{M}_3^1 \Rightarrow \mathcal{I}_3^1 \sqcap \mathcal{I}_3^6 \sqcap \mathcal{I}_3^{r,a} \sqcap \mathcal{I}_3^{\ell,b}$;

(ii) $\mathcal{M}_3^3 \Rightarrow \mathcal{I}_3^3 \sqcap \mathcal{I}_3^4 \sqcap \mathcal{I}_3^{r,a} \sqcap \mathcal{I}_3^{\ell,b}$;

(iii) $\mathcal{J}_3^1 \Rightarrow \mathcal{I}_3^1 \sqcap \mathcal{I}_3^6 \sqcap \mathcal{I}_3^{r,b} \sqcap \mathcal{I}_3^{\ell,a}$;

(iv) $\mathcal{J}_3^3 \Rightarrow \mathcal{I}_3^3 \sqcap \mathcal{I}_3^4 \sqcap \mathcal{I}_3^{r,b} \sqcap \mathcal{I}_3^{\ell,a}$.

3.3. Weaker forms of increasingness on chains

In this subsection, we study the relationships among the different weaker forms expressed in terms of three variables of chain-valued binary operations. In this subsection, we always assume that the underlying structure is a chain $\mathbb{C} = (C, \leq, \wedge, \vee)$.

Proposition 3.6.

(i) $\mathcal{I}_3^2 \Leftrightarrow \mathcal{I}_3^{r,b} \sqcap \mathcal{I}_3^{r,a}$;

(ii) $\mathcal{I}_3^5 \Leftrightarrow \mathcal{I}_3^{\ell,b} \sqcap \mathcal{I}_3^{\ell,a}$.

The following proposition is related to Corollary 3.1.

Proposition 3.7.

(i) $\mathcal{M}_3^1 \Leftrightarrow \mathcal{I}_3^1 \sqcap \mathcal{I}_3^6 \sqcap \mathcal{I}_3^{r,a} \sqcap \mathcal{I}_3^{\ell,b}$;

(ii) $\mathcal{M}_3^3 \Leftrightarrow \mathcal{I}_3^3 \sqcap \mathcal{I}_3^4 \sqcap \mathcal{I}_3^{r,a} \sqcap \mathcal{I}_3^{\ell,b}$;

(iii) $\mathcal{J}_3^1 \Leftrightarrow \mathcal{I}_3^1 \sqcap \mathcal{I}_3^6 \sqcap \mathcal{I}_3^{r,b} \sqcap \mathcal{I}_3^{\ell,a}$;

(iv) $\mathcal{J}_3^3 \Leftrightarrow \mathcal{I}_3^3 \sqcap \mathcal{I}_3^4 \sqcap \mathcal{I}_3^{r,b} \sqcap \mathcal{I}_3^{\ell,a}$.

Proof. We only prove (i), since the proofs of (ii)–(iv) are similar. Due to Corollary 3.1, we only need to prove that $\mathcal{I}_3^1 \sqcap \mathcal{I}_3^6 \sqcap \mathcal{I}_3^{r,a} \sqcap \mathcal{I}_3^{\ell,b} \Rightarrow \mathcal{M}_3^1$. Suppose that F satisfies $\mathcal{I}_3^1 \sqcap \mathcal{I}_3^6 \sqcap \mathcal{I}_3^{r,a} \sqcap \mathcal{I}_3^{\ell,b}$ and consider $x, u, v \in C$. We distinguish the following cases:

(1) $x \leq u, x \leq v$: it follows directly from property $\mathcal{I}_3^1$;

(2) $u \leq x, v \leq x$: it follows directly from property $\mathcal{I}_3^6$;

(3) $u \leq x \leq v$: then $F(x \wedge u, x \wedge v) = F(u, x)$. From property $\mathcal{I}_3^6$, we obtain $F(x, x) \geq F(u, x)$, whereas from property $\mathcal{I}_3^{r,a}$, we obtain $F(u, v) \geq F(u, x)$. Hence, $F(x, x) \wedge F(u, v) \geq F(x \wedge u, x \wedge v)$.

(4) $v \leq x \leq u$: then $F(x \wedge u, x \wedge v) = F(x, v)$. From property $\mathcal{I}_3^1$, we obtain $F(x, x) \geq F(x, v)$, whereas from property $\mathcal{I}_3^{\ell,b}$, we obtain $F(u, v) \geq F(x, v)$. Hence, $F(x, x) \wedge F(u, v) \geq F(x \wedge u, x \wedge v)$.

Therefore, we conclude that F satisfies $\mathcal{M}_3^1$. □

A counterexample is given here to illustrate that this equivalence does not hold in general on a lattice.

Example 3.2.

(i) Consider the bounded lattice $\mathbb{L}_1 = (L, \leq, \wedge, \vee)$ shown in the Hasse diagram in Figure 1 and the binary operation F on L in Table 5. The operation F satisfies $\mathcal{I}_3^1 \sqcap \mathcal{I}_3^6 \sqcap \mathcal{I}_3^{r,a} \sqcap \mathcal{I}_3^{\ell,b}$, but not $\mathcal{M}_3^1$, since $F(a \wedge b, c \wedge b) = F(0, b) = b > 0 = a \wedge b = F(a, c) \wedge F(b, b)$.

F	0	a	b	c	1
0	0	0	b	c	c
a	0	a	a	a	1
b	b	a	b	c	1
c	c	a	c	c	1
1	c	1	1	1	1

Table 5. The binary operation F on L.

(ii) Consider the bounded lattice $\mathbb{L}_2 = (L, \leq, \wedge, \vee)$ shown in the Hasse diagram in Figure 2 and the binary operation G on L in Table 6. The operation G satisfies $\mathcal{I}_3^3 \sqcap \mathcal{I}_3^4 \sqcap \mathcal{I}_3^{r,a} \sqcap \mathcal{I}_3^{\ell,b}$, but not $\mathcal{M}_3^3$, since $G(a \wedge a, b \wedge a) = G(a, 0) = a > 0 = b \wedge a = G(a, b) \wedge G(a, a)$.

G	0	a	b	1
0	0	a	b	1
a	a	a	b	a
b	b	b	b	1
1	1	a	1	1

Table 6. The binary operation G on L.

Proposition 3.8. $\mathcal{I}_3^1 \sqcap \mathcal{I}_3^6 \Leftrightarrow \mathcal{I}_3^3 \sqcap \mathcal{I}_3^4$.

Proof. First, we prove that $\mathcal{I}_3^1 \sqcap \mathcal{I}_3^6 \Rightarrow \mathcal{I}_3^3 \sqcap \mathcal{I}_3^4$. Suppose that F satisfies $\mathcal{I}_3^1 \sqcap \mathcal{I}_3^6$. For any $x, y, u \in C$ such that $y \leq x \leq u$, it holds that $F(x, y) \leq F(x, x) \leq F(u, x)$. Hence, F satisfies $\mathcal{I}_3^3$. For any $x, y, v \in C$ such that $x \leq y \leq v$, it holds that $F(x, y) \leq F(y, y) \leq F(y, v)$. Hence, F satisfies $\mathcal{I}_3^4$.

On the other hand, suppose that F satisfies $\mathcal{I}_3^3 \sqcap \mathcal{I}_3^4$. We first prove that F satisfies $\mathcal{I}_3^1$. For any $x, u, v \in C$ such that $x \leq u \leq v$, it holds that $F(x, x) \leq F(x, u) \leq F(u, v)$. The proof in the case that $x \leq v \leq u$ is similar. Next, we prove that F satisfies $\mathcal{I}_3^6$. For any $x, y, u \in C$ such that $x \leq y \leq u$, it holds that $F(x, y) \leq F(y, u) \leq F(u, u)$. The proof in the case that $y \leq x \leq u$ is similar.

Therefore, we conclude that $\mathcal{I}_3^1 \sqcap \mathcal{I}_3^6 \Leftrightarrow \mathcal{I}_3^3 \sqcap \mathcal{I}_3^4$. □

Combining Propositions 3.7 and 3.8 leads to the following corollary.

Corollary 3.2.

(i) $\mathcal{M}_3^1 \Leftrightarrow \mathcal{M}_3^3$;

(ii) $\mathcal{J}_3^1 \Leftrightarrow \mathcal{J}_3^3$.

The following example shows that $\mathcal{J}_3^1 \not\Leftrightarrow \mathcal{M}_3^1$ and $\mathcal{J}_3^3 \not\Leftrightarrow \mathcal{M}_3^3$ on a chain.

Example 3.3. Consider the bounded chain $\mathbb{C}_1 = (L, \leq, \wedge, \vee)$ shown in the Hasse diagram in Figure 3.

(i) The binary operation F on L in Table 7 satisfies $\mathcal{M}_3^1$, but not $\mathcal{J}_3^1$, since $F(b \vee a, b \vee 1) = F(b, 1) = a < b = a \vee b = F(b, b) \vee F(a, 1)$.

(ii) The binary operation G on L in Table 8 satisfies $\mathcal{M}_3^3$, but not $\mathcal{J}_3^3$, since $G(a \vee 1, b \vee a) = G(1, b) = a < b = a \vee b = G(a, b) \vee G(1, a)$.

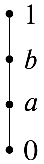

Figure 3. Hasse diagram of $\mathbb{C}_1$.

F	0	a	b	1
0	0	a	b	b
a	a	0	a	b
b	b	a	a	a
1	b	b	a	1

Table 7. The binary operation F on L.

G	0	a	b	1
0	0	0	0	a
a	0	a	a	b
b	a	a	a	b
1	a	b	a	1

Table 8. The binary operation G on L

The following proposition expresses increasingness on chains in equivalent manners.

Proposition 3.9. $\mathcal{M}_3^1 \sqcap \mathcal{J}_3^1 \Leftrightarrow \mathcal{I}_3^2 \sqcap \mathcal{I}_3^5 \Leftrightarrow \mathcal{M}_3^3 \sqcap \mathcal{J}_3^3 \Leftrightarrow \mathcal{I}_4$.

Proof. Due to Corollary 3.2, we only need to prove that $\mathcal{M}_3^3 \sqcap \mathcal{J}_3^3 \Leftrightarrow \mathcal{I}_4$. It is obvious that $\mathcal{I}_4 \Rightarrow \mathcal{M}_3^3 \sqcap \mathcal{J}_3^3$, therefore, we only need to prove the converse implication. Suppose that F satisfies $\mathcal{M}_3^3 \sqcap \mathcal{J}_3^3$. Consider $x \leq u$ and $y \leq v$ and suppose w.l.o.g. that $x \leq y$. The proof can be split into the following cases:

(1) $x \leq y \leq u \leq v$: it follows from $\mathcal{J}_3^3$ and $\mathcal{M}_3^3$ that

$$F(x, y) \leq F(x \vee y, x \vee u) = F(y, u) \leq F(y \wedge u, u \wedge v) = F(u, v);$$

(2) $x \leq u \leq y \leq v$: it follows from $\mathcal{J}_3^3$ and $\mathcal{M}_3^3$ that

$$F(x, y) \leq F(x \vee u, x \vee y) = F(u, y) = F(u \wedge v, v \wedge y) \leq F(u, v);$$

(3) $x \leq y \leq v \leq u$: it follows from $\mathcal{J}_3^3$ that

$$F(x, y) \leq F(x \vee y, y \vee v) = F(y, v) \leq F(y \vee u, y \vee v) = F(u, v)\,.$$

Hence, F satisfies $\mathcal{I}_4$ and $\mathcal{M}_3^3 \sqcap \mathcal{J}_3^3 \Leftrightarrow \mathcal{I}_4$. □

Therefore, the ten weaker forms of increasingness expressed in terms of three variables on chains collapse into eight different weaker forms, namely $\mathscr{W}_3 = \{\mathcal{I}_3^1, \mathcal{I}_3^2, \mathcal{I}_3^3, \mathcal{I}_3^4, \mathcal{I}_3^5, \mathcal{I}_3^6, \mathcal{M}_3^1, \mathcal{M}_3^3\}$.

A counter-example is given here to illustrate that $\mathcal{M}_3^1 \sqcap \mathcal{J}_3^1 \sqcap \mathcal{M}_3^3 \sqcap \mathcal{J}_3^3$ does not suffice to characterize property $\mathcal{I}_4$, *i.e.*, classical increasingness, on a lattice.

Example 3.4. Consider the bounded lattice $\mathbb{L}_3 = (L, \leq, \wedge, \vee)$ shown in the Hasse diagram in Figure 4 and the binary operation F on L in Table 9. The operation F satisfies $\mathcal{M}_3^1 \sqcap \mathcal{J}_3^1 \sqcap \mathcal{M}_3^3 \sqcap \mathcal{J}_3^3$, but $\mathcal{I}_4$, since $F(a, c) = a \parallel d = F(b, d)$.

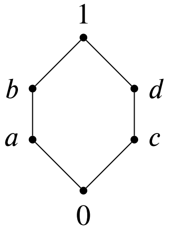

Figure 4. Hasse diagram of $\mathbb{L}_3$.

F	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	b	a	c	1
b	0	b	b	c	d	1
c	0	a	c	c	c	d
d	0	c	d	c	d	d
1	0	1	1	d	d	1

Table 9. The binary operation F on L.

3.4. Weaker forms for idempotent and averaging binary operations

If the binary operation considered satisfies some additional conditions, then some interesting observations can be made. Here, some propositions are presented that foreshadow the later characterizations of the meet and join operations.

Recall that a binary operation F on a lattice $\mathbb{L} = (L, \leq, \wedge, \vee)$ is called averaging if $x \wedge y \leq F(x, y) \leq x \vee y$, for any $x, y \in L$; and idempotent if $F(x, x) = x$, for any $x \in L$ [13, 15]. An averaging operation is obviously idempotent.

Proposition 3.10. *Let $\mathbb{L} = (L, \leq, \wedge, \vee)$ be a lattice and F be a binary operation on L. If F is idempotent, then the following equivalences hold:*

(i) *F satisfies $\mathcal{I}_3^1$ if and only if $F(x, y) \geq x \wedge y$, for any $x, y \in L$;*

(ii) *F satisfies $\mathcal{I}_3^6$ if and only if $F(x, y) \leq x \vee y$, for any $x, y \in L$.*

Proof. We only prove (i), since the proof of (ii) is similar. Suppose that F satisfies $\mathcal{I}_3^1$. For any $x, y \in L$, it holds that $F(x, y) \geq F(x \wedge y, x \wedge y) = x \wedge y$. On the other hand, for any $x, u, v \in L$ such that $x \leq u, x \leq v$, it holds that $F(x, x) = x \leq u \wedge v \leq F(u, v)$. Hence, F satisfies $\mathcal{I}_3^1$. $\square$

Proposition 3.10 leads to the following corollary.

Corollary 3.3. *Let $\mathbb{L} = (L, \leq, \wedge, \vee)$ be a lattice and F be a binary operation on L. Then F is averaging if and only if F is idempotent and satisfies $\mathcal{I}_3^1 \sqcap \mathcal{I}_3^6$.*

Proposition 3.11. *Let $\mathbb{L} = (L, \leq, \wedge, \vee)$ be a lattice and F be a binary operation on L. If F is averaging, then it satisfies $\mathcal{I}_3^3 \sqcap \mathcal{I}_3^4$.*

Proof. Suppose that F is averaging. For any $x, y, u \in L$ such that $y \leq x \leq u$, it holds that $F(x, y) \leq x \vee y = x = x \wedge u \leq F(x, u)$. Hence, F satisfies $\mathcal{I}_3^3$. Similarly, for any $x, y, v \in L$ such that $x \leq y \leq v$, it holds that $F(x, y) \leq x \vee y = y = y \wedge v \leq F(y, v)$. Hence, F satisfies $\mathcal{I}_3^4$. $\square$

Combining Proposition 3.8 and Corollary 3.3 leads to the following result.

Corollary 3.4. *Let $\mathbb{C} = (C, \leq, \wedge, \vee)$ be a chain and F be a binary operation on C. Then F is averaging if and only if F is idempotent and satisfies $\mathcal{I}_3^3 \sqcap \mathcal{I}_3^4$.*

We give an example here to show that Corollary 3.4 does not hold in general on a lattice.

Example 3.5. Consider the bounded lattice $\mathbb{L}_4 = (L, \leq, \wedge, \vee)$ shown in the Hasse diagram in Figure 5 and the binary operation F on L in Table 10. Then F satisfies $\mathcal{I}_3^3 \sqcap \mathcal{I}_3^4$, but it is not averaging, since $F(b, c) = 0 \notin [a, 1]$.

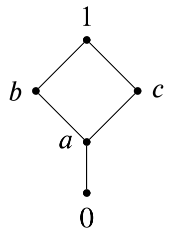

Figure 5. Hasse diagram of $\mathbb{L}_4$.

F	0	a	b	c	1
0	0	0	0	a	a
a	0	a	a	a	b
b	0	a	b	0	b
c	a	a	0	c	c
1	a	b	b	c	1

Table 10. The binary operation F on L.

Combining Proposition 3.3 (i) and Corollary 3.3 leads to the following result.

Corollary 3.5. *Let $\mathbb{L} = (L, \leq, \wedge, \vee)$ be a lattice and F be a binary operation on L. If F is idempotent and satisfies $\mathcal{M}_3^1 \sqcup \mathcal{J}_3^1$, then it is averaging.*

Proposition 3.12. *Let $\mathbb{L} = (L, \leq, \wedge, \vee)$ be a lattice and F be a binary operation on L. If F is idempotent and satisfies $\mathcal{M}_3^3 \sqcup \mathcal{J}_3^3$, then it is averaging.*

Proof. Suppose that F is idempotent and satisfies $\mathcal{M}_3^3$. For any $x, y \in L$, it holds that $F(x, y) \geq F(x \wedge y, y \wedge x) = x \wedge y$; and

$$\begin{aligned} F(x, y) &= F(x \wedge (x \vee y), (x \vee y) \wedge y) \\ &\leq F(x, x \vee y) \\ &= F(x \wedge (x \vee y), (x \vee y) \wedge (x \vee y)) \\ &\leq F(x \vee y, x \vee y) \\ &= x \vee y\,. \end{aligned}$$

Therefore, we conclude that F is averaging. The proof in the case that F satisfies $\mathcal{J}_3^3$ is similar. □

Proposition 3.13. *Let $\mathbb{C} = (C, \leq, \wedge, \vee)$ be a chain and F be a binary operation on C. If F is idempotent, then the following implications hold:*

(i) *if F satisfies $\mathcal{I}_3^{r,a} \sqcap \mathcal{I}_3^{\ell,b}$, then $F(x, y) \geq x \wedge y$, for any $x, y \in C$;*

(ii) *if F satisfies $\mathcal{I}_3^{r,b} \sqcap \mathcal{I}_3^{\ell,a}$, then $F(x, y) \leq x \vee y$, for any $x, y \in C$.*

Proof. We only prove (i), since the proof of (ii) is similar. For any $x, y \in C$, if $x \leq y$, then it holds that $x \wedge y = x = F(x, x) \leq F(x, y)$, since F satisfies $\mathcal{I}_3^{r,a}$; if $x \geq y$, then it holds that $x \wedge y = y = F(y, y) \leq F(x, y)$, since F satisfies $\mathcal{I}_3^{\ell,b}$. □

3.5. *Characterization of the meet and join operations of a lattice and a chain*

Based on the above work, the results in this subsection can be obtained straightforwardly. As the meet and join operations of a lattice naturally satisfy all the above-mentioned weaker forms of increasingness, next we only give the weakest or incomparable ones to characterize them.

Recall that a binary operation F on a lattice $\mathbb{L} = (L, \leq, \wedge, \vee)$ is called conjunctive (resp. disjunctive) if $F(x, y) \leq x \wedge y$ (resp. $F(x, y) \geq x \vee y$), for any $x, y \in L$ [13, 15].

Proposition 3.10 leads to the following corollary.

Corollary 3.6. *Let $\mathbb{L} = (L, \leq, \wedge, \vee)$ be a lattice and F be a binary operation on L. Then the following equivalences hold:*

(i) *F is idempotent, conjunctive and satisfies $\mathcal{I}_3^1$ if and only if F is the meet operation;*

(ii) *F is idempotent, disjunctive and satisfies $\mathcal{I}_3^6$ if and only if F is the join operation.*

Proposition 3.12 leads to the following corollary.

Corollary 3.7. *Let $\mathbb{L} = (L, \leq, \wedge, \vee)$ be a lattice and F be a binary operation on L. Then the following equivalences hold:*

(i) *F is idempotent, conjunctive and satisfies $\mathcal{M}_3^3 \sqcup \mathcal{J}_3^3$ if and only if F is the meet operation;*

(ii) *F is idempotent, disjunctive and satisfies $\mathcal{M}_3^3 \sqcup \mathcal{J}_3^3$ if and only if F is the join operation.*

Proposition 3.14. *Let $\mathbb{L} = (L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and F be a binary operation on L. Then the following implications hold:*

(i) *if F has 1 as the neutral element and satisfies $\mathcal{M}_3^3$, then F is conjunctive;*

(ii) *if F has 0 as the neutral element and satisfies $\mathcal{J}_3^3$, then F is disjunctive.*

Proof. We only prove (i), since the proof of (ii) is similar. Suppose that F has 1 as the neutral element and satisfies $\mathcal{M}_3^3$. It holds for any $x, y \in L$ that

$$F(x, y) = F(x \wedge 1, 1 \wedge y) \leq F(x, 1) \wedge F(1, y) = x \wedge y\,.$$

Therefore, F is conjunctive. □

Combining Corollary 3.7 and Proposition 3.14 leads to the following corollary.

Corollary 3.8. *Let $\mathbb{L} = (L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and F be a binary operation on L. Then the following equivalences hold:*

(i) *F is idempotent, has 1 as the neutral element and satisfies $\mathcal{M}_3^3$ if and only if F is the meet operation;*

(ii) *F is idempotent, has 0 as the neutral element and satisfies $\mathcal{J}_3^3$ if and only if F is the join operation.*

Corollary 3.4 leads to the following corollary.

Corollary 3.9. *Let $\mathbb{C} = (C, \leq, \wedge, \vee)$ be a chain and F be a binary operation on C. Then the following equivalences hold:*

(i) *F is idempotent, conjunctive and satisfies $\mathcal{I}_3^3 \sqcap \mathcal{I}_3^4$ if and only if F is the meet operation;*

(ii) *F is idempotent, disjunctive and satisfies $\mathcal{I}_3^3 \sqcap \mathcal{I}_3^4$ if and only if F is the join operation.*

The following example illustrates that the conjunctivity condition cannot be replaced with the the neutral element condition in the previous corollary.

Example 3.6.

(i) Let $\mathbb{C}_2 = (C, \leq, \wedge, \vee)$ be a finite chain with at least three elements, where $C = \{x_0, x_1, \dots, x_n, x_{n+1}\}$ such that $x_0 < x_1 < \cdots < x_n < x_{n+1}$. Define the binary operation F on C as follows:

$$F(x_i, x_j) = \begin{cases} x_{i \wedge j} & \text{, if } i = j \text{ or } i \vee j = n+1, \\ x_{i \wedge j+1} & \text{, otherwise.} \end{cases}$$

The operation F is idempotent, has 1 as the neutral element and satisfies $\mathcal{I}_3^3 \sqcap \mathcal{I}_3^4$. However, F is not the meet operation.

(ii) Consider the bounded chain $\mathbb{C}_1 = (L, \leq, \wedge, \vee)$ shown in the Hasse diagram in Figure 3 and the binary operation G on $\mathbb{C}_1$ in Table 11. The operation G is idempotent, conjunctive and satisfies $\mathcal{I}_3^4$. However, G is not the meet operation.

G	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	a	1

Table 11. The binary operation G on L.

Proposition 3.13 leads to the following corollary.

Corollary 3.10. *Let $\mathbb{C} = (C, \leq, \wedge, \vee)$ be a chain and F be a binary operation on C. Then the following equivalences hold:*

(i) *F is idempotent, conjunctive and satisfies $\mathcal{I}_3^{r,a} \sqcap \mathcal{I}_3^{\ell,b}$ if and only if F is the meet operation;*

(ii) *F is idempotent, disjunctive and satisfies $\mathcal{I}_3^{\ell,a} \sqcap \mathcal{I}_3^{r,b}$ if and only if F is the join operation.*

Proposition 3.15. *Let $\mathbb{C} = (C, \leq, \wedge, \vee, 0, 1)$ be a bounded chain and F be a binary operation on C. Then the following implications hold:*

(i) *if F has 1 as the neutral element and satisfies $\mathcal{I}_3^{r,a} \sqcap \mathcal{I}_3^{\ell,b}$, then F is conjunctive;*

(ii) *if F has 0 as the neutral element and satisfies $\mathcal{I}_3^{\ell,a} \sqcap \mathcal{I}_3^{r,b}$, then F is disjunctive.*

Proof. We only prove (i), since the proof of (ii) is similar. Suppose that F has 1 as the neutral element and satisfies $\mathcal{I}_3^{r,a} \sqcap \mathcal{I}_3^{\ell,b}$. For any $x, y \in C$, if $x \leq y$, then it holds that $F(x, y) \leq F(x, 1) = x$; if $y \leq x$, then it holds that $F(x, y) \leq F(1, y) = y$. Therefore, F is conjunctive. $\square$

Combining Corollary 3.10 and Proposition 3.15 leads to the following corollary.

Corollary 3.11. *Let $\mathbb{C} = (C, \leq, \wedge, \vee, 0, 1)$ be a bounded chain and F be a binary operation on C. Then the following equivalences hold:*

(i) *F is idempotent, has 1 as the neutral element and satisfies $\mathcal{I}_3^{r,a} \sqcap \mathcal{I}_3^{\ell,b}$ if and only if F is the meet operation;*

(ii) *F is idempotent, has 0 as the neutral element and satisfies $\mathcal{I}_3^{\ell,a} \sqcap \mathcal{I}_3^{r,b}$ if and only if F is the join operation.*

Among all the weaker forms introduced in this section, properties $\mathcal{I}_3^2$ and $\mathcal{I}_3^5$ cannot be used together with idempotence to characterize the meet and join operations on a lattice or a chain.

4. Weaker forms of increasingness expressed in terms of two variables

In this section, we introduce three families of weaker forms of increasingness obtained by reducing the number of variables by two in Theorem 2.1.

First, we present several weaker forms of increasngness that have already been put forward.

Definition 4.1. [32] Let $\mathbb{L} = (L, \leq, \wedge, \vee)$ be a lattice. A binary operation F on L is called:

(i) $\times^r$-increasing (referred to as property $\mathcal{C}_3^r$), if for any $x, y, v \in L$, it holds that

$$y \leq x \leq v \Rightarrow F(x, y) \leq F(x, v); \tag{4.1}$$

(ii) $\times^\ell$-increasing (referred to as property $\mathcal{C}_3^\ell$), if for any $x, y, u \in L$, it holds that

$$x \leq y \leq u \Rightarrow F(x, y) \leq F(u, y)\,. \tag{4.2}$$

Definition 4.2. [32] Let $\mathbb{L} = (L, \leq, \wedge, \vee)$ be a lattice. A binary operation F on L is called:

(i) $\wedge$-increasing (referred to as property $\mathcal{O}_2^m$), if for any $x, y \in L$, it holds that

$$F(x \wedge y, x \wedge y) \leq F(x, y); \tag{4.3}$$

(ii) $\vee$-increasing (referred to as property $\mathcal{O}_2^j$), if for any $x, y \in L$, it holds that

$$F(x, y) \leq F(x \vee y, x \vee y)\,. \tag{4.4}$$

Definition 4.3. [13] Let $\mathbb{L} = (L, \leq, \wedge, \vee)$ be a lattice. A binary operation F on L is called $\angle$-increasing (referred to as property $\mathcal{D}_2$) if the diagonal section of F is increasing, *i.e.*, for any $x, y \in L$, it holds that

$$x \leq y \Rightarrow F(x, x) \leq F(y, y)\,. \tag{4.5}$$

4.1. Definitions

In the following table, we give definitions of the weaker forms of increasingness obtained by reducing the number of variables by two in Eqs. (2.4), (2.5) and (2.6), respectively.

	$\mathscr{I}_2$	$\mathscr{M}_2$	$\mathscr{J}_2$
$x=y$ $u=v$	$\mathcal{I}_2^1: x \leq u \Rightarrow F(x,x) \leq F(u,u)$	$\mathcal{M}_2^1: F(x,x) \wedge F(u,u) \geq F(x \wedge u, x \wedge u)$	$\mathcal{J}_2^1: F(x \vee u, x \vee u) \geq F(x,x) \vee F(u,u)$
$x=v$ $y=u$	$\mathcal{I}_2^2$: $x \leq y, y \leq x \Rightarrow F(x,y) \leq F(y,x)$	$\mathcal{M}_2^2: F(x,y) \wedge F(y,x) \geq F(x \wedge y, y \wedge x)$	$\mathcal{J}_2^2: F(x \vee y, y \vee x) \geq F(x,y) \vee F(y,x)$
$x=y$ $x=u$	$\mathcal{I}_2^3: x \leq v \Rightarrow F(x,x) \leq F(x,v)$	$\mathcal{M}_2^3: F(x,x) \wedge F(x,v) \geq F(x, x \wedge v)$	$\mathcal{J}_2^3: F(x, x \vee v) \geq F(x,x) \vee F(x,v)$
$x=y$ $x=v$	$\mathcal{I}_2^4: x \leq u \Rightarrow F(x,x) \leq F(u,x)$	$\mathcal{M}_2^4: F(x,x) \wedge F(u,x) \geq F(x \wedge u, x)$	$\mathcal{J}_2^4: F(x \vee u, x) \geq F(x,x) \vee F(u,x)$
$x=u$ $u=v$	$\mathcal{I}_2^5: y \leq x \Rightarrow F(x,y) \leq F(x,x)$	$\mathcal{M}_2^5: F(x,y) \wedge F(x,x) \geq F(x, y \wedge x)$	$\mathcal{J}_2^5: F(x, y \vee x) \geq F(x,y) \vee F(x,x)$
$y=u$ $u=v$	$\mathcal{I}_2^6: x \leq y \Rightarrow F(x,y) \leq F(y,y)$	$\mathcal{M}_2^6: F(x,y) \wedge F(y,y) \geq F(x \wedge y, y)$	$\mathcal{J}_2^6: F(x \vee y, y) \geq F(x,y) \vee F(y,y)$

Table 12. Properties resulting from the reduction of the number of variables by two in Theorem 2.1.

Remark 4.1. The following equivalences hold among the properties in Table 12 (equivalent properties are indicated in the same color).

(i) $\mathcal{M}_2^1 \Leftrightarrow \mathcal{J}_2^1 \Leftrightarrow \mathcal{I}_2^1 \Leftrightarrow \mathcal{D}_2$.

(ii) $\mathcal{M}_2^3 \Leftrightarrow \mathcal{M}_2^5$, $\mathcal{J}_2^3 \Leftrightarrow \mathcal{J}_2^5$, $\mathcal{M}_2^4 \Leftrightarrow \mathcal{M}_2^6$ and $\mathcal{J}_2^4 \Leftrightarrow \mathcal{J}_2^6$.

Note that $\mathcal{I}_2^2$ is trivially satisfied and will not be considered further. Therefore, the eighteen weaker forms of increasingness expressed in terms of two variables collapse into eleven weaker forms, namely $\mathscr{W}_2 = \mathscr{I}_2 \cup \mathscr{M}_2 \cup \mathscr{J}_2 = \{\mathcal{I}_2^1, \mathcal{I}_2^3, \mathcal{I}_2^4, \mathcal{I}_2^5, \mathcal{I}_2^6, \mathcal{M}_2^2, \mathcal{M}_2^3, \mathcal{M}_2^4, \mathcal{J}_2^2, \mathcal{J}_2^3, \mathcal{J}_2^4\}$.

Remark 4.2. Some weaker forms of increasingness on lattices mentioned in [32] are included.

(i) $\mathcal{C}_3^r \Leftrightarrow \mathcal{I}_2^3 \sqcap \mathcal{I}_2^5$ and $\mathcal{C}_3^\ell \Leftrightarrow \mathcal{I}_2^4 \sqcap \mathcal{I}_2^6$;

(ii) $\mathscr{O}_2^m \Leftrightarrow \mathcal{M}_2^2$ and $\mathscr{O}_2^j \Leftrightarrow \mathcal{J}_2^2$.

Remark 4.3. Let $\mathbb{L} = (L, \leq, \wedge, \vee)$ be a lattice and F be a binary operation on L. If F is commutative, then the following equivalences hold:

(i) F satisfies $\mathcal{M}_2^3$ if and only if F satisfies $\mathcal{M}_2^4$;

(ii) F satisfies $\mathcal{J}_2^3$ if and only if F satisfies $\mathcal{J}_2^4$;

(iii) F satisfies $\mathcal{I}_2^3$ if and only if F satisfies $\mathcal{I}_2^4$;

(iv) F satisfies $\mathcal{I}_2^5$ if and only if F satisfies $\mathcal{I}_2^6$.

Therefore, for commutative binary operations, there are only seven different weaker forms of increasingness expressed in terms of two variables.

The following remark expresses the duality among the weaker forms of increasingness.

Remark 4.4.

(i) $\mathcal{M}_2^i$ is dual to $\mathcal{J}_2^i$, for $i = 2, 3, 4$;

(ii) $\mathcal{I}_2^3$ is dual to $\mathcal{I}_2^5$,

(iii) $\mathcal{I}_2^4$ is dual to $\mathcal{I}_2^6$;

(iv) $\mathcal{I}_2^1$ is self-dual.

4.2. Weaker forms of increasingness on lattices

In this subsection, we study the relationships among the different weaker forms expressed in terms of two variables of lattice-valued binary operations. In this subsection, we always assume that the underlying structure is a lattice $\mathbb{L} = (L, \leq, \wedge, \vee)$.

Proposition 4.1.

(i) $\mathcal{M}_2^3 \sqcup \mathcal{J}_2^3 \Rightarrow \mathcal{I}_2^3 \sqcap \mathcal{I}_2^5$*;*

(ii) $\mathcal{M}_2^4 \sqcup \mathcal{J}_2^4 \Rightarrow \mathcal{I}_2^4 \sqcap \mathcal{I}_2^6$.

Proof. We only prove (i), since the proof of (ii) is similar. Suppose that F satisfies $\mathcal{M}_2^3 \cup \mathcal{J}_2^3$. For any $x, v \in L$ such that $x \leq v$, it holds that $F(x, x) = F(x, x \wedge v) \leq F(x, v)$ from property $\mathcal{M}_2^3$; $F(x, x) \leq F(x, x \vee v) = F(x, v)$ from property $\mathcal{J}_2^3$. Hence, $\mathcal{M}_2^3 \sqcup \mathcal{J}_2^3 \Rightarrow \mathcal{I}_2^3$.

Suppose that F satisfies $\mathcal{M}_2^3 \cup \mathcal{J}_2^3$. For any $x, y \in L$ such that $y \leq x$, it holds that $F(x, y) = F(x, x \wedge y) \leq F(x, x)$ from property $\mathcal{M}_2^3$; $F(x, y) \leq F(x, x \vee y) = F(x, x)$ from property $\mathcal{J}_2^3$. Hence, $\mathcal{M}_2^3 \sqcup \mathcal{J}_2^3 \Rightarrow \mathcal{I}_2^5$. Thus, $\mathcal{M}_2^3 \sqcup \mathcal{J}_2^3 \Rightarrow \mathcal{I}_2^3 \sqcap \mathcal{I}_2^5$. ☐

The following example illustrates that the converse implications in the previous proposition do not hold in general.

Example 4.1. Consider the bounded lattice $\mathbb{L}_2 = (L, \leq, \wedge, \vee)$ shown in the Hasse diagram in Figure 2 and the binary operation F on L in Table 13. The operation F satisfies $\mathcal{I}_2^3 \sqcap \mathcal{I}_2^5$, but not $\mathcal{M}_2^3$, since $F(a, a \wedge b) = F(a, 0) = a > 0 = a \wedge b = F(a, a) \wedge F(a, b)$.

F	0	a	b	1
0	0	a	a	b
a	a	a	b	1
b	0	a	b	b
1	a	a	1	1

Table 13. The binary operation F on L.

Next, we consider the relationships with $\mathcal{M}_2^2$ and $\mathcal{J}_2^2$.

Proposition 4.2.

(i) $\mathcal{M}_2^2 \Rightarrow \mathcal{I}_2^3 \sqcap \mathcal{I}_2^4$;

(ii) $\mathcal{J}_2^2 \Rightarrow \mathcal{I}_2^5 \sqcap \mathcal{I}_2^6$.

Proof. We only prove (i), since the proof of (ii) is similar. Suppose that F satisfies $\mathcal{M}_2^2$. For any $x, v \in L$ such that $x \leq v$, it holds that $F(x,x) = F(x \wedge v, x \wedge v) \leq F(x,v)$ and $F(x,x) = F(v \wedge x, v \wedge x) \leq F(v,x)$. Hence, F satisfies $\mathcal{I}_2^3 \sqcap \mathcal{I}_2^4$. □

Proposition 4.3.

(i) $\mathcal{M}_2^3 \sqcap \mathcal{M}_2^4 \Rightarrow \mathcal{M}_2^2$;

(ii) $\mathcal{J}_2^3 \sqcap \mathcal{J}_2^4 \Rightarrow \mathcal{J}_2^2$.

Proof. We only prove (i), since the proof of (ii) is similar. Suppose that F satisfies $\mathcal{M}_2^3 \sqcap \mathcal{M}_2^4$. For any $x, y \in L$, it holds that $F(x,y) \geq F(x \wedge y, y) \geq F(x \wedge y, y \wedge (x \wedge y)) = F(x \wedge y, x \wedge y)$. Hence, F satisfies $\mathcal{M}_2^2$. □

The following example illustrates that the converse implications do not hold in general.

Example 4.2. Consider the bounded lattice $\mathbb{L}_2 = (L, \leq, \wedge, \vee)$ shown in the Hasse diagram in Figure 2 and the binary operation F on L in Table 14. The operation F satisfies $\mathcal{M}_2^2 \sqcap \mathcal{J}_2^2$, but not $\mathcal{M}_2^3$ or $\mathcal{M}_2^4$, since F is commutative and $F(a, a \wedge b) = F(a, 0) = a > 0 = a \wedge b = F(a,a) \wedge F(a,b)$.

F	0	a	b	1
0	0	a	0	a
a	a	a	b	a
b	0	b	b	1
1	a	a	1	1

Table 14. The binary operation F on L.

The following proposition is about conjunctions that imply $\mathcal{I}_2^1$.

Proposition 4.4.

(i) $\mathcal{M}_2^3 \sqcap \mathcal{M}_2^4 \Rightarrow \mathcal{I}_2^1$;

(ii) $\mathcal{J}_2^3 \sqcap \mathcal{J}_2^4 \Rightarrow \mathcal{I}_2^1$;

(iii) $\mathcal{I}_2^3 \sqcap \mathcal{I}_2^6 \Rightarrow \mathcal{I}_2^1$;

(iv) $\mathcal{I}_2^4 \sqcap \mathcal{I}_2^5 \Rightarrow \mathcal{I}_2^1$.

4.3. Weaker forms of increasingness on chains

In this subsection, we study the relationships among the different weaker forms expressed in terms of two variables of chain-valued binary operations. In this subsection, we always assume that the underlying structure is a chain $\mathbb{C} = (C, \leq, \wedge, \vee)$.

Proposition 4.5.

(i) $\mathcal{M}_2^3 \Leftrightarrow \mathcal{J}_2^3$;

(ii) $\mathcal{M}_2^4 \Leftrightarrow \mathcal{J}_2^4$.

Proof. We only prove (i), since the proof of (ii) is similar. Suppose that F satisfies $\mathcal{J}_2^3$. For any $x, v \in C$, it holds that $F(x,x) = F(x, x \vee (x \wedge v)) \geq F(x, x \wedge v)$. If $x \geq v$, then $F(x,v) = F(x \wedge v)$; if $x \leq v$, then $F(x,v) = F(x, x \vee v) \geq F(x,x) = F(x, x \wedge v)$. Therefore, we conclude that $F(x,x) \wedge F(x,v) \geq F(x, x \wedge v)$ and F satisfies $\mathcal{M}_2^3$.

On the other hand, suppose that F satisfies $\mathcal{M}_2^3$. For any $x, v \in C$, it holds that $F(x, x \vee v) \geq F(x, (x \vee v) \wedge x) = F(x,x)$. If $x \geq v$, then $F(x, x \vee v) = F(x,x) \geq F(x, x \wedge v) = F(x,v)$; if $x \leq v$, then $F(x \vee v) = F(x,v)$. Therefore, we conclude that $F(x, x \vee v) \geq F(x,x) \vee F(x,v)$ and F satisfies $\mathcal{J}_2^3$. Hence, $\mathcal{M}_2^3 \Leftrightarrow \mathcal{J}_2^3$. ☐

Therefore, the eleven weaker forms of increasingness expressed in terms of two variables on chains collapse into nine forms, namely $\mathscr{W}_2 = \mathscr{I}_2 \cup \mathscr{M}_2 \cup \mathscr{J}_2 = \left\{\mathcal{I}_2^1, \mathcal{I}_2^3, \mathcal{I}_2^4, \mathcal{I}_2^5, \mathcal{I}_2^6, \mathcal{M}_2^2, \mathcal{M}_2^3, \mathcal{M}_2^4, \mathcal{J}_2^2\right\}$.

The following example illustrates that Proposition 4.5 does not hold in general on a lattice.

Example 4.3. Consider the bounded lattice $\mathbb{L}_2 = (L, \leq, \wedge, \vee)$ shown in the Hasse diagram in Figure 2 and the binary operations F in Table 15 and G in Table 16. Then F satisfies $\mathcal{M}_2^3$ and G satisfies $\mathcal{J}_2^3$. However, F does not satisfy $\mathcal{J}_2^3$, whereas G does not satisfy $\mathcal{M}_2^3$, since $F(a,a) \vee F(a,b) = a \vee b = 1 > a = F(a,1) = F(a, a \vee b)$ and $G(a, a \wedge b) = G(a, 0) = a > 0 = a \wedge b = G(a,a) \wedge G(a,b)$.

F	0	a	b	1
0	0	a	b	b
a	0	a	b	a
b	0	b	b	b
1	0	a	b	1

Table 15. The binary operation F on L.

G	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	a	b	1	b
1	1	1	b	1

Table 16. The binary operation G on L.

Proposition 4.6.

(i) $\mathcal{M}_2^2 \Leftrightarrow \mathcal{I}_2^3 \sqcap \mathcal{I}_2^4$;

(ii) $\mathcal{J}_2^2 \Leftrightarrow \mathcal{I}_2^5 \sqcap \mathcal{I}_2^6$.

Proof. We only prove (i), since the proof of (ii) is similar. Due to Proposition 4.2, we only need to prove that $\mathcal{I}_2^3 \sqcap \mathcal{I}_2^4 \Rightarrow \mathcal{M}_2^2$. Suppose that F satisfies $\mathcal{I}_2^3 \sqcap \mathcal{I}_2^4$. For any $x, y \in C$, if $x \leq y$, then it holds that $F(x \wedge y, x \wedge y) = F(x, x) \leq F(x, y)$; if $x \geq y$, then it holds that $F(x \wedge y, x \wedge y) = F(y, y) \leq F(x, y)$. Hence, F satisfies $\mathcal{M}_2^2$. □

Proposition 4.7.

(i) $\mathcal{M}_2^3 \Leftrightarrow \mathcal{I}_2^3 \sqcap \mathcal{I}_2^5$;

(ii) $\mathcal{M}_2^4 \Leftrightarrow \mathcal{I}_2^4 \sqcap \mathcal{I}_2^6$.

Proof. We only prove (i), since the proof of (ii) is similar. Due to Proposition 4.1, we only need to prove that $\mathcal{I}_2^3 \sqcap \mathcal{I}_2^5 \Rightarrow \mathcal{M}_2^3$. Suppose that F satisfies $\mathcal{I}_2^3 \sqcap \mathcal{I}_2^5$, then for any $x, v \in C$, it follows from $\mathcal{I}_2^5$ that $F(x, x) \geq F(x, x \wedge v)$. If $x \geq v$, then $F(x, v) = F(x, x \wedge v)$; if $x \leq v$, then it follows from $\mathcal{I}_2^3$ that $F(x, v) \geq F(x, x \wedge v)$. Therefore, $F(x, x) \wedge F(x, v) \geq F(x, x \wedge v)$. Hence, F satisfies $\mathcal{M}_2^3$. □

Combining Propositions 4.6 and 4.7 leads to the following corollary.

Corollary 4.1. $\mathcal{M}_2^3 \sqcap \mathcal{M}_2^4 \Leftrightarrow \mathcal{M}_2^2 \sqcap \mathcal{J}_2^2 \Leftrightarrow \mathcal{I}_2^3 \sqcap \mathcal{I}_2^4 \sqcap \mathcal{I}_2^5 \sqcap \mathcal{I}_2^6$.

4.4. Weaker forms for idempotent and averaging binary operations

Note that any idempotent binary operation satisfies $\mathcal{I}_2^1$.

Proposition 4.8. *Let* $\mathbb{L} = (L, \leq, \wedge, \vee)$ *be a lattice and* F *be a binary operation on* L*. Then* F *is averaging if and only if* F *is idempotent and satisfies* $\mathcal{M}_2^2 \sqcap \mathcal{J}_2^2$*.*

Proof. Since any averaging operation is idempotent, it holds for any $x, y \in L$ that $F(x \wedge y, x \wedge y) = x \wedge y \leq F(x, y) \leq x \vee y = F(x \vee y, x \vee y)$. Hence, F satisfies $\mathcal{M}_2^2 \sqcap \mathcal{J}_2^2$.

On the other hand, suppose that F is idempotent and satisfies $\mathcal{M}_2^2 \sqcap \mathcal{J}_2^2$. Then for any $x, y \in L$, it holds that $x \wedge y = F(x \wedge y, x \wedge y) \leq F(x, y) \leq F(x \vee y, x \vee y) = x \vee y$. Hence, F is averaging. □

Combining Propositions 4.3 and 4.8 leads to the following proposition.

Proposition 4.9. *Let* $\mathbb{L} = (L, \leq, \wedge, \vee)$ *be a lattice and* F *be a binary operation on* L*. If* F *is idempotent, then the following implications hold:*

(i) *if* F *satisfies* $\mathcal{M}_2^3 \sqcap \mathcal{M}_2^4$*, then* $F(x, y) \geq x \wedge y$ *for any* $x, y \in L$*;*

(ii) *if* F *satisfies* $\mathcal{J}_2^3 \sqcap \mathcal{J}_2^4$*, then* $F(x, y) \leq x \vee y$ *for any* $x, y \in L$*.*

The following example illustrates that the converse implications in Proposition 4.9 do not hold in general.

Example 4.4. Consider the bounded lattice $\mathbb{L}_2 = (L, \leq, \wedge, \vee)$ shown in the Hasse diagram in Figure 2 and the binary operation F on L in Table 17. The operation F is averaging, but it does not satisfy $\mathcal{M}_2^3$, since $F(a, a \wedge b) = F(a, 0) = a > 0 = a \wedge 0 = F(a, a) \wedge F(a, b)$.

F	0	a	b	1
0	0	a	0	b
a	a	a	0	a
b	0	0	b	1
1	b	a	1	1

Table 17. The binary operation F on L.

Combining Corollary 4.1 and Proposition 4.8 leads to the following result.

Corollary 4.2. *Let $\mathbb{C} = (C, \leq, \wedge, \vee)$ be a chain and F be a binary operation on C. Then F is averaging if and only if F is idempotent and satisfies $\mathcal{M}_2^3 \sqcap \mathcal{M}_2^4$.*

Proposition 4.10. *Let $\mathbb{C} = (C, \leq, \wedge, \vee)$ be a chain and F be a binary operation on C. If F is idempotent, then the following equivalences hold:*

- (i) *F satisfies $\mathcal{I}_2^3 \sqcap \mathcal{I}_2^4$ if and only if $F(x, y) \geq x \wedge y$, for any $x, y \in C$;*
- (ii) *F satisfies $\mathcal{I}_2^5 \sqcap \mathcal{I}_2^6$ if and only if $F(x, y) \leq x \vee y$, for any $x, y \in C$.*

4.5. Characterization of the meet and join operations of a lattice and a chain

Based on the above work, the results in this subsection can be obtained straightforwardly.

Proposition 4.11. [32]. Let $\mathbb{L} = (L, \leq, \wedge, \vee)$ be a lattice and F be a binary operation on L. Then the following equivalences hold:

- (i) F is idempotent, conjunctive and satisfies $\mathcal{M}_2^2$ if and only if F is the meet operation;
- (ii) F is idempotent, disjunctive and satisfies $\mathcal{J}_2^2$ if and only if F is the join operation.

Proposition 4.9 leads to the following corollary.

Corollary 4.3. *Let $\mathbb{L} = (L, \leq, \wedge, \vee)$ be a lattice and F be a binary operation on L. Then the following equivalences hold:*

- (i) *F is idempotent, conjunctive and satisfies $\mathcal{M}_2^3 \sqcap \mathcal{M}_2^4$ if and only if F is the meet operation;*
- (ii) *F is idempotent, disjunctive and satisfies $\mathcal{J}_2^3 \sqcap \mathcal{J}_2^4$ if and only if F is the join operation.*

Proposition 4.10 leads to the following corollary.

Corollary 4.4. *Let $\mathbb{C} = (C, \leq, \wedge, \vee)$ be a chain and F be a binary operation on C. Then the following equivalences hold:*

(i) *F is idempotent, conjunctive and satisfies $\mathcal{I}_2^3 \sqcap \mathcal{I}_2^4$ if and only if F is the meet operation;*

(ii) *F is idempotent, disjunctive and satisfies $\mathcal{I}_2^5 \sqcap \mathcal{I}_2^6$ if and only if F is the join operation.*

Among all the weaker forms introduced in this section, property $\mathcal{I}_2^1$ cannot be used together with idempotence to characterize the meet and join operations on a lattice or a chain.

5. Relationships among the weaker forms of increasingness expressed in terms of two and three variables

The weaker forms of increasingness expressed in terms of three variables are obviously stronger than the corresponding forms expressed in terms of two variables. The following table is straightforward to obtain.

	$\mathscr{I}_3$ and $\mathscr{I}_2$	$\mathscr{M}_3$ and $\mathscr{M}_2$	$\mathscr{J}_3$ and $\mathscr{J}_2$
$x = y$	$\mathcal{I}_3^1 \Rightarrow \mathcal{I}_2^1 \sqcap \mathcal{I}_2^3 \sqcap \mathcal{I}_2^4$	$\mathcal{M}_3^1 \Rightarrow \mathcal{M}_2^1 \sqcap \mathcal{M}_2^3 \sqcap \mathcal{M}_2^4$	$\mathcal{J}_3^1 \Rightarrow \mathcal{J}_2^1 \sqcap \mathcal{J}_2^3 \sqcap \mathcal{J}_2^4$
$x = u$	$\mathcal{M}_3^2 \Leftrightarrow \mathcal{J}_3^2 \Leftrightarrow \mathcal{I}_3^2 \Rightarrow \mathcal{M}_2^3 \sqcap \mathcal{J}_2^3 \sqcap \mathcal{I}_2^3 \sqcap \mathcal{I}_2^5$		
$x = v$	$\mathcal{I}_3^3 \Leftrightarrow \mathcal{I}_2^4 \sqcap \mathcal{I}_2^5$	$\mathcal{M}_3^3 \Rightarrow \mathcal{M}_2^2 \sqcap \mathcal{M}_2^3 \sqcap \mathcal{M}_2^4$	$\mathcal{J}_3^3 \Rightarrow \mathcal{J}_2^2 \sqcap \mathcal{J}_2^3 \sqcap \mathcal{J}_2^4$
$y = u$	$\mathcal{I}_3^4 \Leftrightarrow \mathcal{I}_2^3 \sqcap \mathcal{I}_2^6$	$\mathcal{M}_3^3 \Leftrightarrow \mathcal{M}_3^4 \Rightarrow \mathcal{M}_2^2 \sqcap \mathcal{M}_2^3 \sqcap \mathcal{M}_2^4$	$\mathcal{J}_3^3 \Leftrightarrow \mathcal{J}_3^4 \Rightarrow \mathcal{J}_2^2 \sqcap \mathcal{J}_2^3 \sqcap \mathcal{J}_2^4$
$y = v$	$\mathcal{M}_3^5 \Leftrightarrow \mathcal{J}_3^5 \Leftrightarrow \mathcal{I}_3^5 \Rightarrow \mathcal{M}_2^4 \sqcap \mathcal{J}_2^4 \sqcap \mathcal{I}_2^4 \sqcap \mathcal{I}_2^6$		
$u = v$	$\mathcal{I}_3^6 \Rightarrow \mathcal{I}_2^1 \sqcap \mathcal{I}_2^5 \sqcap \mathcal{I}_2^6$	$\mathcal{M}_3^1 \Leftrightarrow \mathcal{M}_3^6 \Rightarrow \mathcal{M}_2^1 \sqcap \mathcal{M}_2^3 \sqcap \mathcal{M}_2^4$	$\mathcal{J}_3^1 \Leftrightarrow \mathcal{J}_3^6 \Rightarrow \mathcal{J}_2^1 \sqcap \mathcal{J}_2^3 \sqcap \mathcal{J}_2^4$

Table 18. Relationships among the weaker forms of increasingness in Tables 1 and 12.

The following remark follows immediately from the increasingness of $\wedge$ and $\vee$.

Remark 5.1.

(i) $\mathcal{I}_3^1 \Rightarrow \mathcal{M}_2^2$ and $\mathcal{I}_3^6 \Rightarrow \mathcal{J}_2^2$;

(ii) $\mathcal{I}_3^2 \Rightarrow \mathcal{M}_2^3 \sqcap \mathcal{J}_2^3$ and $\mathcal{I}_3^5 \Rightarrow \mathcal{M}_2^4 \sqcap \mathcal{J}_2^4$;

(iii) $\mathcal{I}_3^{r,a} \Rightarrow \mathcal{I}_2^3$, $\mathcal{I}_3^{\ell,b} \Rightarrow \mathcal{I}_2^4$, $\mathcal{I}_3^{r,b} \Rightarrow \mathcal{I}_2^5$ and $\mathcal{I}_3^{\ell,a} \Rightarrow \mathcal{I}_2^6$.

Proposition 5.1.

(i) $\mathcal{M}_2^3 \sqcap \mathcal{M}_2^4 \Rightarrow \mathcal{I}_3^1$;

(ii) $\mathcal{J}_2^3 \sqcap \mathcal{J}_2^4 \Rightarrow \mathcal{I}_3^6$.

Proof. We only prove (i), since the proof of (ii) is similar. Suppose that F satisfies $\mathcal{M}_2^3 \sqcap \mathcal{M}_2^4$. For any $x, u, v \in L$ such that $x \leq u, x \leq v$, it holds that $x \leq u \wedge v$. Therefore, we have that

$$\begin{aligned} F(x,x) = F(u \wedge v \wedge x, u \wedge v \wedge x) &\leq F(u \wedge v, u \wedge v \wedge x) \\ &\leq F(u \wedge v, u \wedge v) \leq F(u, u \wedge v) \leq F(u, v)\,. \end{aligned}$$

Hence, F satisfies $\mathcal{I}_3^1$. □

The following propositions hold on a chain.

Proposition 5.2. *Let $\mathbb{C} = (C, \leq, \wedge, \vee)$ be a chain. Then the following equivalences hold:*

(i) $\mathcal{I}_3^1 \Leftrightarrow \mathcal{I}_2^1 \sqcap \mathcal{I}_2^3 \sqcap \mathcal{I}_2^4$;

(ii) $\mathcal{I}_3^6 \Leftrightarrow \mathcal{I}_2^1 \sqcap \mathcal{I}_2^5 \sqcap \mathcal{I}_2^6$.

Proposition 5.3. *Let $\mathbb{C} = (C, \leq, \wedge, \vee)$ be a chain. Then $\mathcal{M}_2^3 \sqcap \mathcal{M}_2^4 \Leftrightarrow \mathcal{I}_3^3 \sqcap \mathcal{I}_3^4$.*

Proof. Suppose that F satisfies $\mathcal{I}_3^3 \sqcap \mathcal{I}_3^4$. For any $x, v \in C$, if $x \leq v$, then it holds that $F(x, x \wedge v) = F(x,x) \leq F(x,v)$; if $v \leq x$, then it holds that $F(x, x \wedge v) = F(x,v) \leq F(x,x)$. Hence, F satisfies $\mathcal{M}_2^3$. The proof in the case that F satisfies $\mathcal{M}_2^4$ is similar.

On the other hand, suppose that F satisfies $\mathcal{M}_2^3 \sqcap \mathcal{M}_2^4$. For any $x, y, u \in C$ such that $y \leq x \leq u$, it holds that $F(x,y) = F(x, x \wedge y) \leq F(x,x) = F(x \wedge u, x) \leq F(u,x)$. Hence, F satisfies $\mathcal{I}_3^3$. For any $x, y, u \in C$ such that $x \leq y \leq u$, it holds that $F(x,y) = F(x \wedge y, y) \leq F(y,y) = F(y, y \wedge u) \leq F(y,u)$. Hence, F satisfies $\mathcal{I}_3^4$. □

6. Hasse diagrams

In this section, we provide summarizing Hasse diagrams that collect the implications among the weaker forms of increasingness studied in this paper. We distinguish between lattices and chains, and discuss the case of commutative binary operations separately.

These implications evidently establish posets, and, in some cases, even meet semilattices.

To ensure an accurate reading of these Hasse diagrams, it is important to take note of the following:

(i) we exclusively focus on the conjunctions of weaker forms; disjunctions therefore do not appear in the diagrams;

(ii) two solid lines converging downward onto a single vertex indicate that the conjunction of the corresponding weaker forms occurs at that point; lines with a dash-dot pattern merely represent the order relationship;

(iii) in case multiple solid lines converge downward onto a single vertex, we indicate the pairwise conjunctions to avoid confusion;

(iv) we use the notation $p \equiv q$ instead of $p \Leftrightarrow q$ for readability.

6.1. Hasse diagrams of $\mathscr{W}_3$

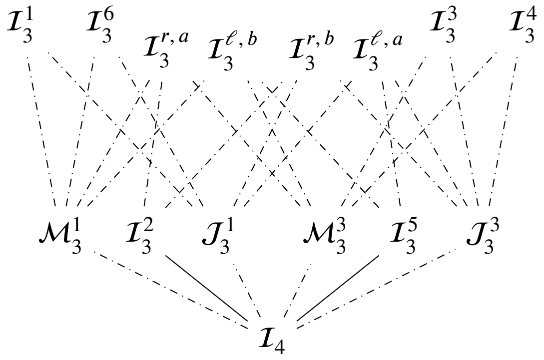

Figure 6. Hasse diagrams of the weaker forms of increasingness expressed in terms of three variables on a lattice.

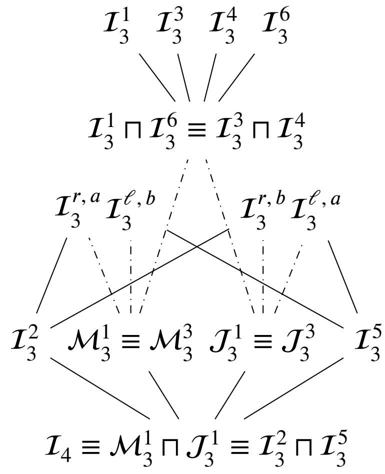

Figure 7. Hasse diagram of the weaker forms of increasingness expressed in terms of three variables on a chain.

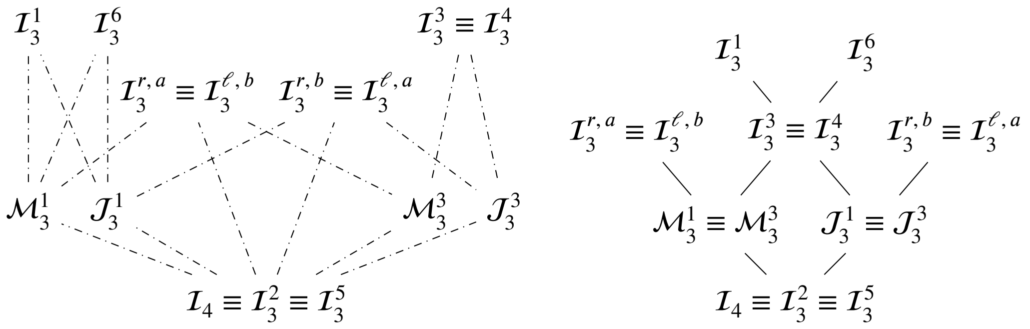

Figure 8. Hasse diagrams of the weaker forms of increasingness for commutative binary operations expressed in terms of three variables on a lattice (left) and on a chain (right).

6.2. *Hasse diagrams of* $\mathscr{W}_2$

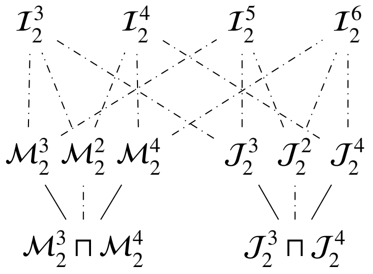

Figure 9. Hasse diagram of the weaker forms of increasingness expressed in terms of two variables on a lattice.

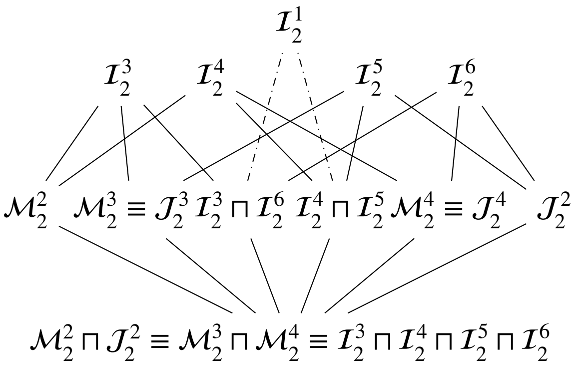

Figure 10. Hasse diagram of the weaker forms of increasingness expressed in terms of two variables on a chain. The dash-dotted lines also hold on a lattice (not shown in the previous Hasse diagram).

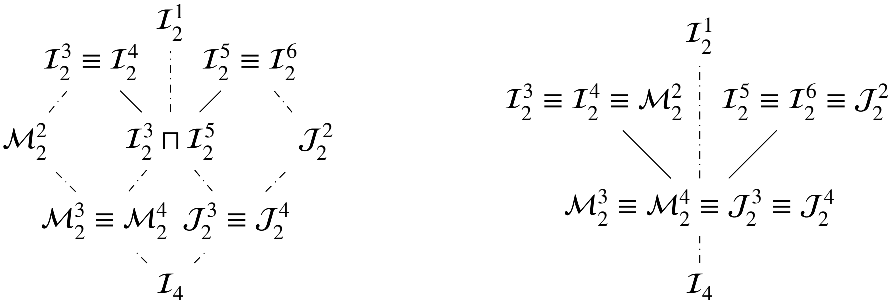

Figure 11. Hasse diagrams of the weaker forms of increasingness for commutative binary operations expressed in terms of two variables on a lattice (left) and on a chain (right).

6.3. *Hasse diagrams of the relationship between $\mathscr{W}_2$ and $\mathscr{W}_3$*

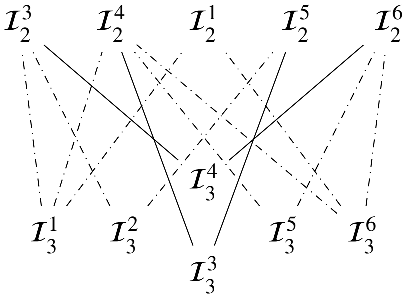

Figure 12. Hasse diagram of the weaker forms of increasingness in $\mathscr{I}$ on a lattice.

Figure 13. Hasse diagrams of the weaker forms of increasingness in $\mathscr{M}$ (left) and $\mathscr{J}$ (right) on a lattice.

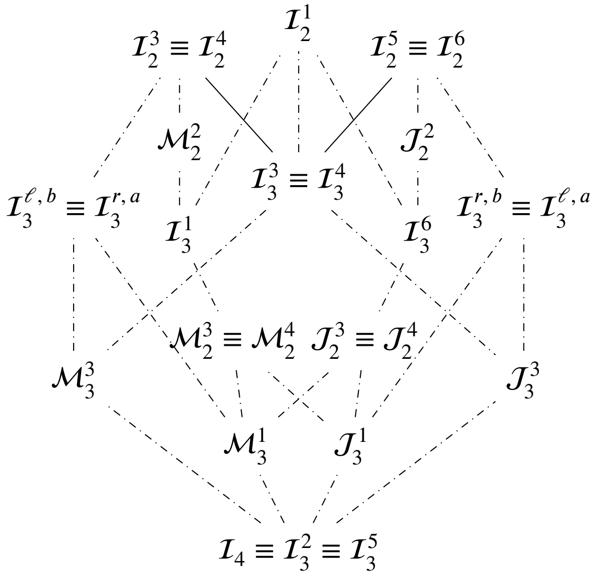

Figure 14. Hasse diagram of the weaker forms of increasingness for commutative binary operations on a lattice.

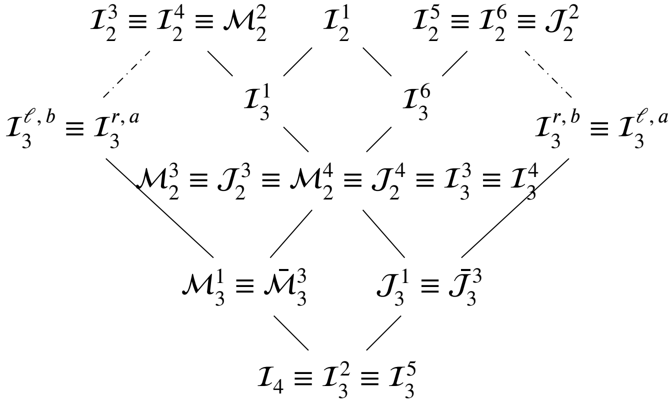

Figure 15. Hasse diagram of the weaker forms of increasingness for commutative binary operations on a chain.

7. Examples

In this section, we give some generic examples on lattices, chains or even the real unit interval.

7.1. *Examples for* $\mathscr{W}_3$

In this subsection, we give examples of binary operations satisfying some weaker forms of increasingness mentioned in Section 3 on lattices, chains or the real unit interval. Some already-known operations are also included.

Example 7.1. Let $\mathbb{L} = (L, \leq, \wedge, \vee)$ be a lattice. Define the binary operations $F_\wedge$ and $F_\vee$ on L as follows:

$$F_\wedge(x, y) = \begin{cases} x & \text{, if } x \leq y, \\ y & \text{, otherwise,} \end{cases} \qquad F_\vee(x, y) = \begin{cases} y & \text{, if } x \leq y, \\ x & \text{, otherwise.} \end{cases}$$

Then $F_\wedge$ and $F_\vee$ satisfy $\mathcal{I}_3^1 \sqcap \mathcal{I}_3^3 \sqcap \mathcal{I}_3^4 \sqcap \mathcal{I}_3^6$. However, they do not satisfy $\mathcal{M}_3^1 \sqcup \mathcal{M}_3^3 \sqcup \mathcal{M}_3^4 \sqcup \mathcal{M}_3^6$. Note that $F_\wedge$ becomes the meet operation and $F_\vee$ becomes the join operation when $\mathbb{L}$ is a chain.

Example 7.2.

(i) For $m \in \mathbb{R}$ the Lehmer mean $L_m : [0, \infty[^2 \to [0, \infty[$, and, for $p, q \in \mathbb{R}$, $p \neq q$, the Gini mean $G^{p,q} : [0, \infty[^2 \to [0, \infty[$ are defined by

$$L_m(x, y) = \frac{x^{m+1} + y^{m+1}}{x^m + y^m} \qquad G^{p,q}(x, y) = \left(\frac{x^p + y^p}{x^q + y^q} \right)^{\frac{1}{p-q}},$$

with the convention $0/0 = 0$. They are averaging and thus satisfy $\mathcal{I}_3^1 \sqcap \mathcal{I}_3^3 \sqcap \mathcal{I}_3^4 \sqcap \mathcal{I}_3^6$.

(ii) Let $\mathbb{I} = ([0, 1], \leq, \wedge, \vee)$ be the real unit interval. Consider the mixture function M_w on $[0, 1]$ defined by

$$M_w(x, y) = \frac{w(x)x + w(y)y}{w(x) + w(y)},$$

with the convention $0/0 = 0$, where $w : [0, 1] \to [0, \infty[$ is a weight function. If w is increasing, then M_w satisfies $\mathcal{I}_3^{r,a}$, whereas if w is decreasing, then M_w satisfies $\mathcal{I}_3^{\ell,b}$.

Example 7.3. Let $\mathbb{I} = ([0, 1], \leq, \wedge, \vee)$ be the real unit interval.

(i) Define the binary operation F on $[0, 1]$ by $F(x, y) = |x - y|$. Then F satisfies $\mathcal{I}_3^1 \sqcap \mathcal{I}_3^{r,a} \sqcap \mathcal{I}_3^{\ell,b}$.

(ii) Define the binary operation G on $[0, 1]$ by $G(x, y) = 1 - |x - y|$. Then G satisfies $\mathcal{I}_3^6 \sqcap \mathcal{I}_3^{r,b} \sqcap \mathcal{I}_3^{\ell,a}$.

More generally, the difference between two increasing operations may still show some weaker forms of increasingness.

Example 7.4. Let $\mathbb{I} = ([0,1], \leq, \wedge, \vee)$ be the real unit interval. Consider two OWA operators [2] OWA_1 and OWA_2 such that $\mathrm{OWA}_1 \geq \mathrm{OWA}_2$, *i.e.*, $w_{11} \geq w_{21}$, and a negation N [3].

(i) Define the binary operation F_{diff} on $[0,1]$ as $F_{\mathrm{diff}}(x,y) = \mathrm{OWA}_1(x,y) - \mathrm{OWA}_2(x,y)$. Then F_{diff} satisfies $\mathcal{I}_3^1 \sqcap \mathcal{I}_3^{r,a} \sqcap \mathcal{I}_3^{\ell,b}$.

(ii) Define the binary operation F^N_{diff} on $[0,1]$ as $F^N_{\mathrm{diff}}(x,y) = N\left(\mathrm{OWA}_1(x,y) - \mathrm{OWA}_2(x,y)\right)$. Then F^N_{diff} satisfies $\mathcal{I}_3^6 \sqcap \mathcal{I}_3^{r,b} \sqcap \mathcal{I}_3^{\ell,a}$.

Next, we give some examples that are exclusive to finite chains.

Example 7.5. Let $\mathbb{C}_2 = (C, \leq, \wedge, \vee)$ be a finite chain with at least three elements, where $C = \{x_0, x_1, \ldots, x_n, x_{n+1}\}$ such that $x_0 < x_1 < \cdots < x_n < x_{n+1}$.

(i) Define the binary operation F_1 on C as follows:

$$F_1(x_i, x_j) = \begin{cases} x_{i \vee j} & \text{, if } |i-j| = 2, \\ x_{i \wedge j} & \text{, if } |i-j| = 1, \\ x_{\frac{i+j}{2}} & \text{, if } |i-j| \neq 2 \text{ and } i-j \equiv 0 \,(\mathrm{mod} 2)\,, \\ x_{\frac{i+j+1}{2}} & \text{, if } |i-j| \neq 1 \text{ and } i-j \equiv 1 \,(\mathrm{mod} 2)\,. \end{cases}$$

Then F_1 satisfies $\mathcal{M}_3^1$.

(ii) Define the binary operation F_2 on C as follows:

$$F_2(x_i, x_j) = \begin{cases} x_{i \wedge j} & \text{, if } |i-j| = 2, \\ x_{i \vee j} & \text{, if } |i-j| = 1, \\ x_{\frac{i+j}{2}} & \text{, if } |i-j| \neq 2 \text{ and } i-j \equiv 0 \,(\mathrm{mod} 2)\,, \\ x_{\frac{i+j-1}{2}} & \text{, if } |i-j| \neq 1 \text{ and } i-j \equiv 1 \,(\mathrm{mod} 2)\,. \end{cases}$$

Then $F_2 \in \mathcal{J}_3^1$.

(iii) Define the binary operation F_3 on C by $F_3(x_i, x_j) = x_{n+1-x \wedge j}$, then F_3 satisfies $\mathcal{I}_3^{r,a} \sqcap \mathcal{I}_3^{\ell,b}$.

(iv) Define the binary operation F_4 on C by $F_4(x_i, x_j) = x_{n+1-x \vee j}$, then F_4 satisfies $\mathcal{I}_3^{\ell,a} \sqcap \mathcal{I}_3^{r,b}$.

(v) Define the binary operation F_5 on C as follows:

$$F_5(x_i, x_j) = \begin{cases} x_{i \vee j} & \text{, if } |i-j| = 1, \\ x_{i \wedge j} & \text{, otherwise.} \end{cases}$$

---

[2] For any weight vector $\mathbf{w} = \left(w_1, w_2\right) \in [0,1]^2$ such that $w_1 + w_2 = 1$, the ordered weighted averaging function $\mathrm{OWA}_{\mathbf{w}} : [0,1]^2 \to [0,1]$ (associated with $\mathbf{w}$) is defined by $\mathrm{OWA}_{\mathbf{w}}(x,y) := w_1 \max(x,y) + w_2 \min(x,y)$ [30].

[3] A decreasing function $N : [0,1] \to [0,1]$ is called a negation if $N(0) = 1$ and $N(1) = 0$ [2].

Then F_5 satisfies $\mathcal{I}_3^3 \sqcap \mathcal{I}_3^4$.

(vi) Define the binary operation F_6 on C as follows:

$$F_6(x_i, x_j) = \begin{cases} x_{|i-j|+|i\wedge j-m|} & \text{, if } i \wedge j \geq m, \\ x_{m+i\wedge j-|i-j|} & \text{, if } i \wedge j < m \text{ and } i \neq j, \\ x_{|i-j|} & \text{, if } i = j < m. \end{cases}$$

Then F_6 satisfies $\mathcal{I}_3^1$.

(vii) Define the binary operation F_7 on C as follows:

$$F_7(x_i, x_j) = \begin{cases} x_{n+1-|i-j|-|i\wedge j-m|} & \text{, if } i \wedge j \geq m \text{ and } i \neq j, \\ x_{n+1} & \text{, if } i = j \geq m, \\ x_{i\vee j} & \text{, if } i \wedge j < m. \end{cases}$$

Then F_7 satisfies $\mathcal{I}_3^6$.

7.2. Examples for $\mathscr{W}_2$

In this subsection, we give examples of binary operations satisfying some weaker forms of increasingness mentioned in Section 4 on lattices, finite chains or the real unit interval.

Example 7.6. Let $\mathbb{I} = ([0,1], \leq, \wedge, \vee)$ be the real unit interval.

(i) Define the binary operation F on $[0,1]$ as follows:

$$F(x, y) = \begin{cases} 1 - x \wedge y & \text{, if } x \neq y \text{ and } x \wedge y \leq \frac{1}{2}, \\ x \vee y & \text{, if } x = y \text{ or } x \wedge y > \frac{1}{2}. \end{cases}$$

Then F satisfies $\mathcal{I}_2^3 \sqcap \mathcal{I}_2^4$.

(ii) Define the binary operation G on $[0,1]$ as follows:

$$G(x, y) = \begin{cases} x \wedge y & \text{, if } x = y \text{ or } x \vee y < \frac{1}{2}, \\ 1 - x \vee y & \text{, if } x \neq y \text{ and } x \vee y \geq \frac{1}{2}. \end{cases}$$

Then G satisfies $\mathcal{I}_3^5 \sqcap \mathcal{I}_3^6$.

Example 7.7. Let $\mathbb{L} = (L, \leq, \wedge, \vee)$ be a lattice and $a \in L$.

(i) Define the binary operation F_ℓ^a on L as follows:

$$F_\ell^a(x, y) = \begin{cases} x & \text{, if } x \geq y, \\ a & \text{, otherwise.} \end{cases}$$

Then F_ℓ^a satisfies $\mathcal{I}_2^4 \sqcap \mathcal{I}_2^5$.

(ii) Define the binary operation F_{r}^{a} on L as follows:

$$F_{\mathrm{r}}^{a}(x,y)=\begin{cases} y & \text{, if } x \geq y, \\ a & \text{, otherwise.} \end{cases}$$

Then F_{r}^{a} satisfies $\mathcal{I}_2^3 \sqcap \mathcal{I}_2^6$.

Example 7.8. Let $\mathbb{L}=(L,\leq,\wedge,\vee)$ be a lattice and $b,c\in L$ such that $b\leq c$.

(i) Define the binary operation $F_{\ell}^{b,c}$ on L as follows:

$$F_{\ell}^{b,c}(x,y)=\begin{cases} c & \text{, if } x\wedge y \parallel b, \\ x & \text{, otherwise.} \end{cases}$$

Then $F_{\ell}^{b,c}$ satisfies $\mathcal{M}_2^2$.

(ii) Define the binary operation $F_{\mathrm{r}}^{b,c}$ on L as follows:

$$F_{\mathrm{r}}^{b,c}(x,y)=\begin{cases} b & \text{, if } x\vee y \parallel c, \\ y & \text{, otherwise.} \end{cases}$$

Then $F_{\mathrm{r}}^{b,c}$ satisfies $\mathcal{J}_2^2$.

Example 7.9. Let $\mathbb{L}=(L,\leq,\wedge,\vee)$ be a lattice and $d\in L$. Define the binary operation F^d on L as follows:

$$F^{d}(x,y)=\begin{cases} x\vee y & \text{, if } x,y\leq d, \\ d & \text{, if } x,y\geq d, \\ x\wedge y\wedge d & \text{, otherwise.} \end{cases}$$

Then F^d satisfies $\mathcal{M}_2^2 \sqcap \mathcal{J}_2^2$.

Example 7.10. Let $\mathbb{L}=(L,\leq,\wedge,\vee)$ be a lattice.

(i) Define the binary operations $F_{\wedge}^{r}$ and $F_{\vee}^{r}$ on L as follows:

$$F_{\wedge}^{r}(x,y)=\begin{cases} y & \text{, if } x\parallel y, \\ x\wedge y & \text{, otherwise,} \end{cases} \qquad F_{\vee}^{r}(x,y)=\begin{cases} y & \text{, if } x\parallel y, \\ x\vee y & \text{, otherwise.} \end{cases}$$

Then $F_{\wedge}^{r}$ satisfies $\mathcal{M}_2^3$, but not $\mathcal{J}_2^3$, whereas $F_{\vee}^{r}$ satisfies $\mathcal{J}_2^3$, but not $\mathcal{M}_2^3$.

(ii) Define the binary operations $F_{\wedge}^{\ell}$ and $F_{\vee}^{\ell}$ on L as follows:

$$F_{\wedge}^{\ell}(x,y)=\begin{cases} x & \text{, if } x\parallel y, \\ x\wedge y & \text{, otherwise,} \end{cases} \qquad F_{\vee}^{\ell}(x,y)=\begin{cases} x & \text{, if } x\parallel y, \\ x\vee y & \text{, otherwise.} \end{cases}$$

Then $F_{\wedge}^{\ell}$ satisfies $\mathcal{M}_2^4$, but not $\mathcal{J}_2^4$, whereas $F_{\vee}^{\ell}$ satisfies $\mathcal{J}_2^4$, but not $\mathcal{M}_2^4$.

Finally, we give some examples that are exclusive to finite chains.

Example 7.11. Let $\mathbb{C}_2 = (C, \leq, \wedge, \vee)$ be a finite chain with at least three elements, where $C = \{x_0, x_1, \ldots, x_n, x_{n+1}\}$ such that $x_0 < x_1 < \cdots < x_n < x_{n+1}$.

(i) Define the binary operation G_1 on C as follows:

$$G_1(x_i, x_j) = \begin{cases} x_{n+1-i\wedge j} & \text{, if } |i-j| \neq 1 \text{ or } i \wedge j = 0, \\ x_{n+2-i\wedge j} & \text{, if } |i-j| = 1 \text{ and } i \wedge j \neq 0. \end{cases}$$

Then G_1 satisfies $\mathcal{M}_2^2$.

(ii) Define the binary operation G_2 on C as follows:

$$G_2(x_i, x_j) = \begin{cases} x_{n+1-i\vee j} & \text{, if } |i-j| \neq 1 \text{ or } i \vee j = n+1, \\ x_{n-i\vee j} & \text{, if } |i-j| = 1 \text{ and } i \vee j \neq n+1. \end{cases}$$

Then G_2 satisfies $\mathcal{J}_2^2$.

(iii) Define the binary operation G_3 on C as follows:

$$G_3(x_i, x_j) = \begin{cases} x_{i\vee j} & \text{, if } |i-j| \leq 1, \\ x_{i\wedge j} & \text{, if } |i-j| = 2, \\ x_{i\wedge j+2} & \text{, otherwise.} \end{cases}$$

Then G_3 satisfies $\mathcal{M}_2^3$.

8. Concluding remarks

In this paper, we have introduced and studied weaker forms of increasingness of binary operations on lattices from the perspective of the dominance relation. We have started from a well-known theorem connecting the dominance relation and the classical notion of increasingness. Based on this theorem, we have obtained weaker forms by reducing the number of variables involved in the classical notion of increasingness or one of the two equivalent formulations by one or two.

We have laid bare all the implications among these weaker forms (on lattices and chains) and provided a counter-example when it is not so obvious that the converse of an implication does not hold. The results are summarized by providing Hasse diagrams that collect the implications and equivalences between the weaker forms of increasingness in a visually attractive manner. We have provided several generic examples to illustrate weaker forms of increasingness on lattices, chains or the real unit interval.

We have also paid attention to the characterization of the meet and join operations on lattices and chains. We anticipate that other well-known (families of) binary operations that fail to satisfy the classical increasingness property can be shown to satisfy some of the weaker forms, which could constitute a first step towards their characterization.

Note that the discussion in this paper can easily be extended to higher dimensions, the only difficulty being a combinatorial explosion of the number of weaker forms one can consider and the corresponding cumbersome notations.

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