Newton-Raphson Nonlinear Solver for Electric Field Integral Equation and Resistive Boundary Condition

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***Abstract*—This contribution deals with the transient scattering of electromagnetic fields by conductors whose surface dynamics are captured by a strongly non-linear boundary condition. The scattering process is modeled by coupling the electric field integral equation with the non-linear current-field characteristic. The marching-on-in-time method is combined with a Newton-Raphson non-linear solver to produce an approximate solution. The method is demonstrated on an example exhibiting a negative differential resistance.**

I. Introduction

In many cases, the interaction of systems with the electromagnetic field is accurately described by including appropriate scalar values for the permittivity, permeability, and conductivity, resulting in a linear problem. With the advent of two-dimensional materials and their applications in communications this is often no longer the case [1] [2]. Coupling the linear integral equations of electromagnetics to non-linear characteristics for the field traces leads to a non-linear model that needs to be solved in the time-domain and requires non-linear solution methods such as Newton-Raphson.

In [3], Newton's method is applied to the time-domain finite element method incorporating non-linear material to model dielectric breakdown and achieve second-order convergence. The differential equation solver, however, suffers from numerical dispersion errors and the Courant-Friedrich-Lewy (CFL) limitation on the time step size. To overcome these limitations, we seek the solution to Maxwell's equation via an integral equation method.

In this contribution, we extend the work in [1] to build a solver robust to the inclusion of divergence and curl conforming basis functions for the induced current and electric field respectively. We formulate and solve the time domain Electric Field Integral Equation (EFIE) and the auxiliary equation mathematically described by Resistive Boundary Condition (RBC) for thin surfaces. This is achieved by applying the Newton-Raphson non-linear solver for the RBC. The paper is organized as follows. Section II details the formulation of the equations and the Newton-Raphson solver, followed by the discretization scheme and the solution procedure. Section III presents a numerical example and a discussion on the outcome of the simulation results. Section IV summarises the conclusions and some directions for future work.

II. Formulation

The electromagnetic scattering problem for an imperfect conductor can be solved by equating the trace of the total electric field to the sum of the traces of incident and scattered electric field as follows,

$$\dot{\mathscr{T}}\mathbf{j} - \hat{\mathbf{n}} \times \frac{\partial \mathbf{e}}{\partial t} = -\hat{\mathbf{n}} \times \frac{\partial \mathbf{e}^{inc}}{\partial t} \tag{1}$$

where $\dot{\mathscr{T}}$, the time derivative of the time-domain single layer operator is given by,

$$\dot{\mathscr{T}}\mathbf{j} = \hat{\mathbf{n}} \times \int_S \Big[\frac{\partial^2}{\partial t^2}\frac{\mathbf{j}(\mathbf{r}', t-\frac{R}{c})}{4\pi c R} - c\nabla\frac{\nabla' \cdot \mathbf{j}(\mathbf{r}', t-\frac{R}{c})}{4\pi R}\Big] dS'. \tag{2}$$

The auxiliary equation in our case is the RBC. It describes the inhomogeneous, non-linear relationship between the current density and the electric field and is given by,

$$\mathbf{j} = -\sigma \cdot \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{e}). \tag{3}$$

In this work, we model an inhomogeneous but isotropic material resulting in the following equation to be solved:

$$\mathbf{j} = f(\mathbf{e}) = \sigma(|\mathbf{e}|)\mathbf{e}. \tag{4}$$

Here, $\mathbf{e}$ is the tangential component of the total electric field. We aim to solve (4) using Newton-Rapshon solver. To do this, we need the Taylor expansion of (4) up to the first-order terms:

$$\mathbf{j}_n = \nabla_{\mathbf{e}} f(\mathbf{e}_{n-1})(\mathbf{e}_n - \mathbf{e}_{n-1}) \tag{5}$$

where

$$\nabla_{\mathbf{e}} f(\mathbf{e}_{n-1}) = \sigma(|\mathbf{e}_{n-1}|)\frac{\mathbf{e}_{n-1} \otimes \mathbf{e}_{n-1}}{|\mathbf{e}_{n-1}|} + \sigma'(|\mathbf{e}_{n-1}|)\mathbf{I}, \tag{6}$$

and n is the iteration index of the solver.

A. Discretization

Equations (1) and (5) must be discretized before they can be solved numerically. This is achieved by first expanding the unknown variables $\mathbf{j}$ and $\mathbf{e}$ into spatial and temporal basis functions. The current density $\mathbf{j}$ is required to be discretized using divergence conforming basis functions [4]. The electric field is a 1-form and so $\mathbf{e}$ is expanded using curl-conforming basis functions [5].

Thus, the current density $\mathbf{j}$ and the electric field $\mathbf{e}$ are expanded as follows:

$$\mathbf{j}(\mathbf{r}, t) = \sum_{m=1}^{N_s} \sum_{l=0}^{N_t-1} J_m^{(l)} \mathbf{f}_m(\mathbf{r}) T^{(l)}(t), \tag{7}$$

and

$$\mathbf{e}(\mathbf{r}, t) = \sum_{m=1}^{N_s} \sum_{l=0}^{N_t-1} E_m^{(l)} \mathbf{g}_m(\mathbf{r}) T^{(l)}(t), \tag{8}$$

where $\mathbf{f}_m$s are divergence-conforming Rao-Wilton-Glisson (RWG) functions and $\mathbf{g}_m$s are curl-conforming order 1 Nédélec functions of the first-kind. T^ls are shifted Lagrange bases.

After substituting (7) and (8) in (1) and (5), (1) is tested using $\hat{\mathbf{n}} \times \mathbf{f}_m$s, time delta functions and (5) is tested using $\mathbf{g}_m$s, time delta functions to obtain

$$\mathbf{Z}^{(0)}\mathbf{J}^{(i)} + \dot{\mathbf{G}}^{(0)}\mathbf{E}^{(i)} = \dot{\mathbf{E}}^{inc(i)} - \mathbf{E}^{res(i)} \tag{9}$$

where

$$\mathbf{E}^{res(i)} = \sum_{j=0}^{i-1} \mathbf{Z}^{(i-j)}\mathbf{J}^{(j)} + \sum_{j=0}^{i-1} \dot{\mathbf{G}}^{(i-j)}\mathbf{E}^{(j)},$$

and

$$\mathbf{G}\mathbf{J}_n^{(i)} - \mathbf{Q}_{(n-1)}^{(i)}\mathbf{E}_n^{(i)} = -\mathbf{Q}_{(n-1)}^{(i)}\mathbf{E}_{n-1}^{(i)} \tag{10}$$

where

$$\left[\mathbf{Q}_{(n-1)}^{(i)}\right]_{MN} = \langle \mathbf{g}_M(\mathbf{r}), \nabla_{\mathbf{e}} f(\mathbf{e}_{n-1})\mathbf{g}_N(\mathbf{r})\rangle. \tag{11}$$

Here, n and i are the n^{th} Newton-Raphson iteration and the time step index respectively.

Here, $\mathbf{e}_{n-1}$ is the electric field obtained from the expansion coefficients $\mathbf{E}_n^{i-1}$ i.e. in the previous time index if it is the first iteration of the non-linear solver for the current time index i, else from the coefficients $\mathbf{E}_{n-1}^{i}$ obtained in the previous iteration of the non-linear solver. The solution to (9) and (10) is obtained as follows:

for $i \leq i_{max}$ **do**
  while *norm* $(\mathbf{E}_n^{(i)} - \mathbf{E}_{n-1}^{(i)}) <$ *tol* **do**
   assemble equation (11)
   and solve

$$\begin{bmatrix} \mathbf{J}_n^{(i)} \\ \mathbf{E}_n^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{Z}^{(0)} & \dot{\mathbf{G}}^{(0)} \\ \mathbf{G} & -\mathbf{Q}_{(n-1)}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} \dot{\mathbf{E}}^{inc(i)} - \mathbf{E}^{res(i)} \\ -\mathbf{Q}_{(n-1)}^{(i)}\mathbf{E}_{n-1}^{(i)} \end{bmatrix}$$

   $n \leftarrow n + 1$
  end while
  $i \leftarrow i + 1$
end for

III. Numerical Example

Scattering of a Gaussian pulse of width $cT = 2.95$ light meter by a sphere of radius 1.0 m with $\sigma(\mathbf{e}) = (1.2|\mathbf{e}|^2 - 1.65|\mathbf{e}| + 0.681)c\ \Omega^{-1}\cdot \mathrm{m}^{-1}$ was solved. Here, c is the speed of light.

Experiments with various mesh size confirms that the non-linearity can be modeled in an error controlled manner by choosing a fine enough mesh as shown in Fig. 1. In this work, the spatial variation of induced current is described by a 3rd-degree polynomial due to the choice of σ and an approximate linear variation of the incident electric field. The error can be attributed to the fact that the spatial non-linear relation is only imposed in a tested sense, and pointwise discrepancies remain. This error can be mitigated by reduction of the mesh size.

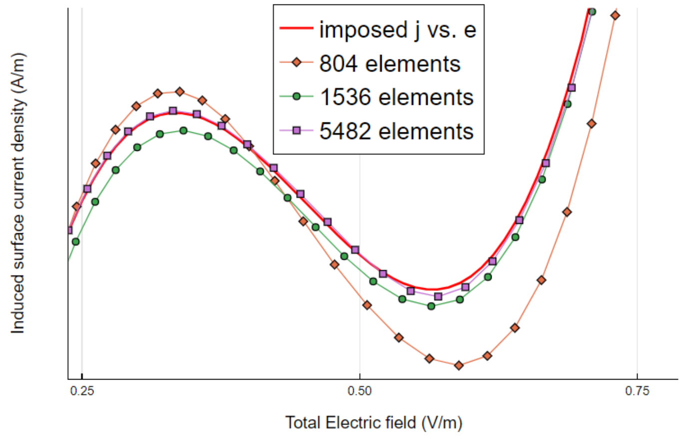

Fig. 1. The solved RBC at a fixed point on the sphere as the incident Gaussian pulse sweeps over it. The solution using 804 mesh elements shows greater deviation than with 5482 mesh elements owing to the non-linear spatial variation of induced current supported by piecewise linear basis functions.

IV. Conclusion

We have demonstrated that the time domain electric field integral equation can be coupled to a strongly non-linear impedance boundary condition and that the resulting system can be solved by combining the marching-on-in-time method with a Newton-Raphson solver. The currents and fields approximately obey the non-linear relation with an error that can be controlled by an appropriate choice of mesh size. Future work is devoted towards an implementation of this method towards solving scattering problem for thin open sheets and usage of ports as the excitation source.

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