A Combined Field Integral Equation for Scattering by Open Surfaces

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***Abstract*—It is well-established that the magnetic field integral equation (MFIE) cannot be used to model scattering by open surfaces. In the context of multi-trace spaces it becomes apparent that the reason for this is that the MFIE does not uniquely determine the radiating component of the induced current. However, when the MFIE is combined with the electric field integral equation, a combined field integral equation results that can be uniquely solved for the currents on both sides of the open surface. This removes the necessity to compute these two currents in post-processing. It also renders the treatment of closed and open surfaces more consistent in codes for application to general geometries.**

I. Introduction

The electric field integral equation (EFIE) is a versatile tool in the modelling of scattering of time-harmonic electromagnetic fields by perfect electrical conductors (PECs). It is often preferred over the magnetic field integral equation because it can be applied to closed and open surfaces, and because it is fairly straightforward to account for the effects of non-perfect conductivity through the inclusion of a surface impedance condition. The quadratic increase of the condition number of linear systems resulting from the discretisation of the EFIE can be mitigated by Calderón preconditioners.

The magnetic field integral equation (MFIE) is used in the construction of combined field integral equations (CFIEs) for the modelling of scattering by closed surfaces that are injective at all frequencies. The MFIE states that the total tangential magnetic field vanishes at points in the interior of the scatterer near its surface. From this it is clear why the MFIE does not make sense for open surfaces, which do not partition the space into an exterior and an interior region.

The advent of multi-trace techniques changes this state-of-affairs: the open surface is interpreted as an inflated double-sided screen of infinitesimal thickness. For both sides of this double-sides screen, it is now clear in which direction the interior and exterior region can be found. In [1], it is shown that the representation theorem usually applied to closed surfaces holds here as well: the total field can be represented as the sum of the incident field and the fields radiated by the currents on the front and back of the open surface.

Here we will investigate what the magnetic field integral equation looks like on double-sided open surfaces and what information this provides about the induced current. We will combine this with the electric field integral equation to arrive at a combined field integral equation that can be uniquely solved for the currents on the front and back of the inflated surface. Correctness of the equation is verified by comparing the results with those of the classic EFIE, and by computing the tangential traces of the magnetic field.

A. Equations and Discretisation

A rectangular planar PEC illuminated by an electromagnetic field (e^{inc}, h^{inc}) with time variation $e^{-\iota\omega t}$ is modelled as an double-sided screen with top and bottom (Γ_1, Γ_2) and *outward* pointing normals (n_1, n_2). The medium surrounding Γ is characterised by permittivity ϵ and permeability μ, or equivalently by wavenumber $\kappa = \omega\sqrt{\epsilon\mu}$ and impedance $\eta = \sqrt{\mu/\epsilon}$. The EFIE and MFIE operators are

$$\begin{aligned} T_{ij}(u_j)(x) =& -\iota\kappa n_i \times \int_{\Gamma_j} \frac{e^{-\iota\kappa|x-y|}}{4\pi|x-y|} u_j(y) dy \\ &+ \frac{1}{\iota\kappa} n_i \times \mathrm{grad} \int_{\Gamma_j} \frac{e^{-\iota\kappa|x-y|}}{4\pi|x-y|} \mathrm{div}_{\Gamma_j} u_j(y) dy, \\ K_{ij}(u_j)(x) =& n_i \times \int_{\Gamma_j} \mathrm{grad}_x \frac{e^{-\iota\kappa|x-y|}}{4\pi|x-y|} \times u_j(y) dy \end{aligned}$$

If Γ_i and Γ_j are co-planar, $K_{ij} = 0$. This significantly simplifies the MFIE. What remains are the *geometric* identities, with signs that depend on the relative positioning and orientation of the top and bottom surface (see [2]):

$$\begin{pmatrix} -\frac{1}{2} + K_{11} & \frac{1}{2} + K_{12} \\ \frac{1}{2} + K_{21} & -\frac{1}{2} + K_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = - \begin{pmatrix} n_1 \times h^{inc} \\ n_2 \times h^{inc} \end{pmatrix}. \tag{1}$$

Its solutions are all pairs of top-bottom currents for which

$$u_1 - u_2 = 2n_1 \times h^{inc}. \tag{2}$$

The electric field integral equation for this geometry reads

$$\eta \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = - \begin{pmatrix} n_1 \times e^{inc} \\ n_2 \times e^{inc} \end{pmatrix}. \tag{3}$$

Because top and bottom surface coincide and are oppositely oriented, this is consistent with the single layer EFIE, which is solved for the total induced current:

$$\eta T_{11}(u_1 + u_2) = -n_1 \times e^{inc} \tag{4}$$

Neither equation gives access to the currents induced in the top and bottom layers, but provide complementary information! This leads us to propose the following multi-trace CFIE:

$$\begin{aligned} &\left[\begin{pmatrix} \frac{1}{2} + \tilde{K}_{11} & \\ & \frac{1}{2} + \tilde{K}_{22} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} + K_{11} & \frac{1}{2} + K_{12} \\ \frac{1}{2} + K_{21} & -\frac{1}{2} + K_{22} \end{pmatrix} \right. \\ &\left. \quad + \begin{pmatrix} T_{11} & \\ & T_{22} \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \right] \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ &= - \begin{pmatrix} \left(\frac{1}{2} + \tilde{K}_{11}\right)(n_1 \times h^{inc}) \\ \left(\frac{1}{2} + \tilde{K}_{22}\right)(n_1 \times h^{inc}) \end{pmatrix} - \begin{pmatrix} \frac{1}{\eta} T_{11}(n_1 \times e^{inc}) \\ \frac{1}{\eta} T_{22}(n_2 \times e^{inc}) \end{pmatrix}, \end{aligned} \tag{5}$$

where $\tilde{K}$ is the double layer boundary operator at imaginary wavenumber $\iota\kappa$. This CFIE is designed to be free from dense grid breakdown [3], coercive for closed surfaces [4], and is cast in a form that makes it amenable to low-frequency stabilisation using quasi-Helmholtz projectors, even in the presence of holes and handles [5].

The equation is discretised by (i) building a triangular meshes $(\Gamma_{1,h}, \Gamma_{2,h})$ that agree on both sides of the open surface, (ii) restricting the search for (u_1, u_2) to the space of Rao-Wilton-Glisson functions $\mathrm{RWG}(\Gamma_{1,h}) \times \mathrm{RWG}(\Gamma_{2,h})$, (iii) testing the equation by functions in the Buffa-Christiansen space $\mathrm{BC}(\Gamma_{1,h}) \times \mathrm{BC}(\Gamma_{2,h})$. Operator products are dealt with by introducing the appropriate inverse Gram matrices observing conformity and stability [5].

II. Numerical Results

An incident wave with signature $e^{inc}(x) = (1,0,0)^T \exp(-\iota\kappa x_3)$, with $\kappa = 4.0m^{-1}$ illuminated a planar square in the xy-plane with side $1.0m^{-1}$, The surface is modelled by a double-sided mesh $(\Gamma_{1,h}, \Gamma_{2,h})$ with $h = 0.05m^{-1}$. The current is computed by the classic, single-trace EFIE and the multi-trace CFIE introduced here. Fig. 1 shows agreement for the total current. The tangential magnetic field along a vertical line is computed in post-processing. Single-trace EFIE and multi-trace CFIE completely agree (Fig. 2). At the point of intersection of this vertical line and Γ, the norm of the induced current is computed directly from the solution of the multi-trace CFIE: $\eta u_1 \approx 0.48$ and $\eta u_2 \approx 2.47$. These values agree with the limits of the tangential magnetic field, but only the multi-trace CFIE provides them without post-processing!

ACKNOWLEDGEMENT

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (Grant agreement No. 101001847).

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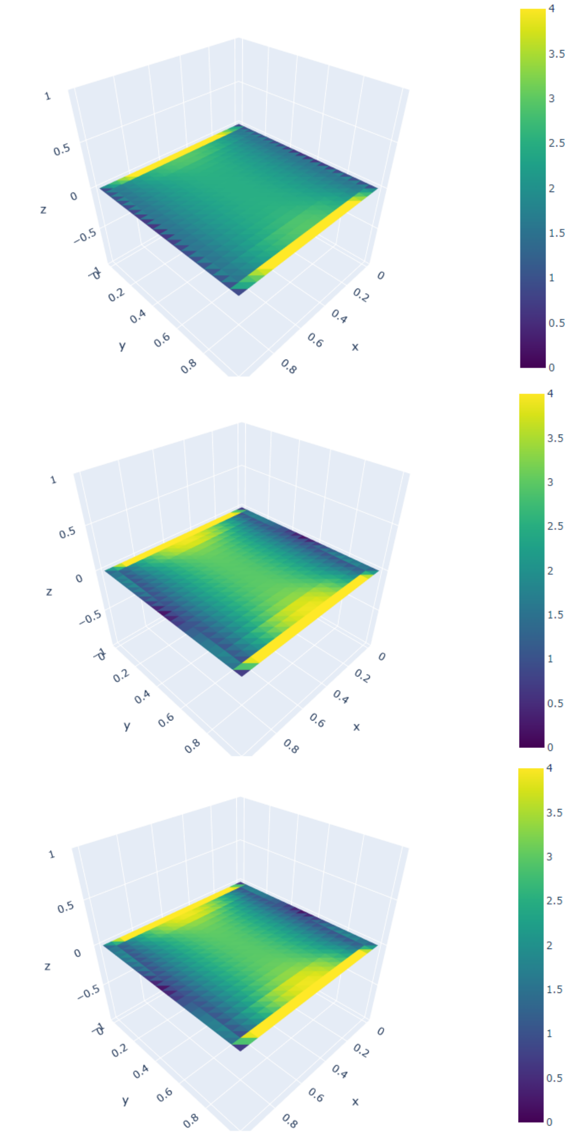

Fig. 1. Norm of the induced currents times η in the top layer computed by the multi-trace CFIE (top), in both layers computed by the multi-trace CFIE (center), and in both layers computed by the classic single-trace EFIE (bottom), showing complete agreement.

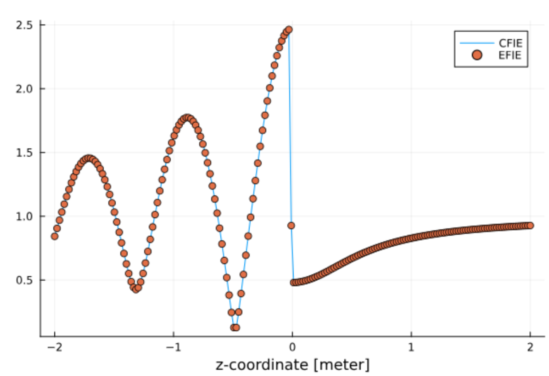

Fig. 2. Norm of the tangential magnetic field along a vertical line through the center of Γ. Even though EFIE and CFIE solutions differ, their radiated fields and their traces agree.