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Explicit Solutions for Coupled Parallel Queues

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Abstract: We consider a system of two coupled parallel queues with infinite waiting rooms. The time setting is *discrete*. In either queue, the service of a customer requires exactly one discrete time slot. Arrivals of new customers occur independently from slot to slot, but the numbers of arrivals into both queues within a slot may be mutually dependent. Their joint probability generating function (*pgf*) is indicated as $A(z_1, z_2)$ and characterizes the whole model. In general, determining the steady-state joint probability mass function (*pmf*) $u(m, n), m, n \geq 0$ or the corresponding joint pgf $U(z_1, z_2)$ of the numbers of customers present in both queues is a formidable task. Only for very specific choices of the arrival pgf $A(z_1, z_2)$ are explicit results known. In this paper, we identify a multi-parameter, *generic class* of arrival pgfs $A(z_1, z_2)$, for which we can explicitly determine the system-content pgf $U(z_1, z_2)$. We find that, for arrival pgfs of this class, $U(z_1, z_2)$ has a denominator that is a product, say $r_1(z_1)r_2(z_2)$, of two univariate functions. This property allows a straightforward inversion of $U(z_1, z_2)$, resulting in a pmf $u(m, n)$ which can be expressed as a finite linear combination of bivariate geometric terms. We observe that our generic model encompasses most of the previously known results as special cases.

Keywords: parallel queues; discrete time; joint system-content distribution; explicit solutions

MSC: 60K25; 90B22; 68M20; 32A08; 32A10; 32A20

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1. Introduction

This paper fits into a greater scientific effort that aims to find *explicit analytic solutions* for the joint stationary probability distribution (or probability generating function) of the numbers of customers in a system of two *coupled* discrete-time single-server queues. Various instances of such systems have been studied before, both differing in the *cause of the coupling* between the two queues or in the *scientific perspective* taken in the study.

With no claim on completeness, we mention a number of *possible causes* for the presence of coupling between queues. A first cause may be that the arrival streams into the queues are *mutually interdependent* or *state-dependent*—that is to say, dependent on the *system contents*, i.e., the numbers of customers present in the queues. Mutual dependence between arrivals occurs, for instance, in the context of communications networks, where the nodes of the network contain switching systems that have to forward digital packets from many different origins to many different destinations. In such switches, each destination has (at least, conceptually) its own dedicated buffer to temporarily store arriving packets, and, since packets destined for one destination do not enter the output buffer associated with another destination, the arrivals within such output buffers are mutually correlated. Buffered slotted switches have been studied, e.g., in [1–6]. Specifically, in [1–3], the authors analyze a symmetric 2×2 switch with Bernoulli arrivals on the two input lines of the switch; [4] extends this analysis to asymmetric switches; [5] gives an overview of a large number of analysis techniques for coupled queues (including the example of the 2×2 switch with Bernoulli arrivals); [6] focuses on the asymptotic behavior of two discrete-time coupled parallel queues, for general arrival processes (including the 2×2 switch model), under the condition that one of the two queues is highly loaded. *State dependence* of arrivals occurs, for instance, in *join-the-shortest-queue* systems, where arriving customers adapt their behavior at the entrance of the two-queue system to the system contents upon arrival; see, e.g., ref. [7] for the case of two asymmetric regular queues, ref. [8] for the specific case of orbit queues in the context of systems with retrials and [9] in the setting of polling models. More conceptual studies of queues with interdependent arrivals include [6,10–12]: ref. [10] discusses a continuous-time two-queue model with coupled Poisson inputs; ref. [11] provides some initial results for discrete-time coupled parallel queues with various specific arrival processes; ref. [6] treats more general arrival streams but focuses on the asymptotic behavior, whereas the focus of [12] is to examine in what circumstances the (exact) joint system-content pmf in a system of two coupled queues can be expressed as a finite linear combination of bivariate geometric product-form terms, and it includes (among others) parallel-queue systems with interdependent arrivals.

Another (major) cause of coupling may be that the queues of the system have to *share the same service facilities*. This situation occurs, for instance, in *polling systems*, where one server periodically visits multiple queues to serve a number of customers and then goes to the next queue; various variants of polling systems have been studied quite intensively in the past. Good general overviews of different kinds of polling models and their applications can be found, for instance, in [13–17]; random polling systems are considered in [18]; time-limited polling systems are studied in [19,20]; whereas [21] discusses a specific polling model with an autonomous server.

Sharing of servers also occurs in so-called *alternating service* systems, where one server is allocated for alternating random durations of time to either of two queues, regardless of the states of these queues. Continuous-time models with Poisson arrivals and general service-time distributions were considered, for instance, in [22,23]; in [24], the concept of switchover times is added to this concept. Discrete-time models were treated in [12,25–27]. An exact analysis of the case of independent Bernoulli arrivals is reported in [25] by means of a transform-based technique involving analytic continuation and also in [27] by means of a probability-based compensation technique, while the case of global geometric arrivals is treated in depth in [26,27], again using transform-based and probability-based methods, respectively. In [12], exact solutions are also derived for more general arrival distributions.

Priority queuing models, where one common service facility gives preferential service to one class of customers over other class(es) of customers, also introduce coupling between the class-dedicated queues; a large body of research results, both in a continuous-time setting and a discrete-time setting, is available on this topic. For pioneering work on continuous-time models, we refer to [28–30]. In a discrete-time setting, which is most relevant for the present paper, various models have also been examined, differing as follows: in the nature of the service-time distributions (deterministic [31], general); in the specifics of the priority scheduling rule (non-pre-emptive [32], pre-emptive resume [33], pre-emptive repeat identical [34], pre-emptive repeat with resampling [35], priority jumps [36], accumulating priority [37]); in the nature of the performance metrics studied (system contents, customer delays [32]); and/or in the nature of the arrival processes (independent arrivals, correlated arrivals [38]). A detailed discussion of all these models is beyond the scope of this paper.

Similar ideas are also implemented in so-called *(generalized) processor sharing (GPS)* systems, whereby the service facility is randomly allocated to multiple queues according to preset weights, as opposed to *alternating service* systems. However, *GPS systems* usually allow the server to deliver service to customers of a queue to which it is not allocated when the queue to which it is allocated is empty, thus making the system work-conserving. In fact, *GPS systems* can also be viewed as systems with *alternating priorities*; see, e.g., ref. [39]. Some papers dealing with *GPS systems* are [40–44]: ref. [40] presents a basic continuous-time model, and [41] adds server interruptions; refs. [42–44] deal with discrete-time models—specifically, ref. [42] provides an approximate analysis based on the power series approach, ref. [43] discusses an approximative approach for the case of Bernoulli arrivals, and [44] compares *GPS* with *alternating service*, based on a heavy-traffic approximation.

In the context of server sharing, we should also mention *serve-the-longest-queue* systems, where, upon a service completion, a server can autonomously decide to give preference to the queue that contains the largest number of customers; see, e.g, refs. [45,46]. Recently, some authors have examined the combined *join-the-shortest-queue* and *serve-the-longest-queue* scenario; see [47,48].

A third important cause of coupling in two-queue systems can be that (part of) the output stream of one queue constitutes (part of) the input stream into the other queue, such as in the context of *tandem queues* (see, e.g., refs. [49–51]), or, more generally, in a network environment.

As far as the *scientific perspective* taken by various authors in the literature is concerned, we see a significant difference between considering the involved (two-queue) queuing system as the basic concept of the study—where the determination of the joint (or total) system-content distribution of both queues, the overflow probabilities, the customer delays, etc., is the main objective—versus a more fundamental, mathematically-oriented point of view, whereby the underlying *random walks* that model the system contents of both queues are the basic concepts of the study. We refer to [52] for a thorough discussion of random walks in the quarter-plane, ref. [53] for a pioneering paper on the subject, ref. [54] for the definition and analysis of an interesting subclass of random walks, ref. [55] for the detailed analysis of a specific random walk by means of techniques from complex analysis and [56] for an analysis based on the compensation approach. In [27], the aim is to shed more light on the structural properties of random walks required to admit elegant solutions. Very often, in these more theoretically oriented studies, the involved random walks are of nearest-neighbor type, which is rather restrictive in a queuing context, and the structure of their transition probabilities may be quite arbitrary and may not necessarily reflect the behavior of a queuing system.

The present paper does not take the mathematical study of the random walk that models the two-queue system explicitly as a major point of interest, but rather concentrates on the explicit determination of the joint pgf of the two system contents in the two queues of the system. Specifically, we consider a conceptually very simple system of two coupled *parallel* discrete-time queues. The queues are named queue 1 and queue 2; both have their own dedicated server and infinite storage capacity. Customers arriving to queue 1 and to queue 2 are referred to as type-1 and type-2 customers, respectively. The service times of the customers are deterministically equal to one time slot, regardless of the customer type. New customer arrivals of both types occur independently from slot to slot, but are possibly *type-interdependent* within a slot. This is the only source of coupling in this model. Earlier studies of various instances of this type of two-queue system include the aforementioned papers [1–6,10–12].

In general, determining the steady-state joint pgf $U(z_1, z_2)$ of the system contents in a system of two coupled queues is a formidable task, because it requires the solution of a possibly complicated, nonlinear kernel-type *functional equation* for $U(z_1, z_2)$, which contains the unknown boundary functions $U(z_1, 0)$ and/or $U(0, z_2)$. A well-established generic technique to solve such equations is the so-called *boundary-value approach*, which is described in great detail in the classic texts [52,57]. Although this approach can deal with various kinds of kernel-type functional equations, it has the disadvantage that it involves singular integrals and conformal mappings, which may be very complicated, and also requires considerable additional numerical work. Another useful method is the *compensation approach*, a rather versatile technique for the analysis of two-dimensional Markov chains satisfying certain conditions without transforms (pgfs) [56,58]. Basically, in this method the desired unknown joint distribution is expressed as a sum of bivariate geometric product forms satisfying the inner balance equations of the Markov chain, and the coefficients of the individual terms in the sum are determined in a clever way. Other researchers have successfully applied the so-called *uniformization method*, a complex function-based technique, for various models; see, e.g., refs. [1,59] for the analysis of a clocked buffered switch. This method introduces a parameter representation for the algebraic curve that represents the set of zero-tuples of the kernel. The unknown boundary functions in the functional equation are also expressed in terms of this parameter, referred to as the "uniformization variable", and equations are derived and subsequently solved for these boundary functions. Another technique that is frequently used involves *analytic (or meromorphic) continuation* of complex (boundary) functions; see, for instance, refs. [25,55]. We also mention the so-called *power series technique* [42,60,61] as a useful *approximative* method. This technique consists of expanding the desired distribution (or the associated pgf) as a power series, in terms of some system parameter (e.g., the load in some queue, the probability that the server is allocated to some queue), and then finding the coefficients of the consecutive powers in that power series as the solutions of "easier" equations than the original functional equation. An excellent overview of methods that have been used to find the joint system-content distribution in coupled-queue systems can be found in [5].

In this paper, we use a different, purely *algebraic*, transform-based technique, which aims at a direct derivation of the joint system-content pgf $U(z_1, z_2)$ without the need of first determining the boundary functions $U(z_1, 0)$ and $U(0, z_2)$. Our technique can be best described as a two-step process: first, we make an *educated guess* at the solution for $U(z_1, z_2)$ that corresponds to a given arrival pgf $A(z_1, z_2)$; next, we prove that the proposed expression of $U(z_1, z_2)$ indeed satisfies the functional equation. Of course, in this approach, the choice of a suitable arrival pgf $A(z_1, z_2)$ and making an *educated guess* at the corresponding system-content pgf $U(z_1, z_2)$ is crucial. In fact, this step is essentially a process of trial and error, based on the intuition gained from the preliminary study of a large number of simple special cases and a generalization and adaptation of these, until a class of arrival pgfs $A(z_1, z_2)$ is found for which $U(z_1, z_2)$ can be conjectured. The second step is then just a matter of rather simple algebra, and—unlike the aforementioned methods—does not involve complicated mathematics such as conformal mappings, singular integrals, uniformization, analytic continuation, power series expansion, etc.

For the specific coupled-queues system considered in this paper, explicit results have been obtained thus far only in a number of isolated cases, for very specific choices of the arrival pgf $A(z_1, z_2)$ (see, e.g., ref. [11]). Furthermore, these special cases are of a rather simple nature: the arrivals of both types should be *mutually independent* or the two queues should receive *identical numbers of arrivals* in each slot or one of both queues should receive *no more than one single arrival per slot*, implying that in this queue no accumulation of customers occurs. Some initial indications to extend the class of "solvable" arrival pgfs $A(z_1, z_2)$ are also given in [11], but the extensions are limited.

In this paper, we identify a multi-parameter, *generic class* of arrival pgfs $A(z_1, z_2)$, for which we go on to explicitly determine the joint pgf $U(z_1, z_2)$, using the *algebraic* approach described above. By making specific choices for the many parameters of the model, we also define three interesting *subclasses* of arrival pgfs that lead to even more explicit solutions. We find that for arrival pgfs of the classes considered in this paper, the bivariate joint system-content pgf $U(z_1, z_2)$ has a denominator that is a product (say, $r_1(z_1)r_2(z_2)$) of two univariate functions. This property allows a straightforward inversion of the pgf $U(z_1, z_2)$ by means of an inversion technique we developed in a previous paper [12], resulting in a pmf $u(m, n)$ that can be expressed as a *finite linear combination* of bivariate geometric terms. We observe that, in addition to providing explicit solutions for a great variety of arrival pgfs, for which no solution was known until now, our generic model also encompasses most of the previously known results as special cases. In fact, it was by studying these special cases that we developed the intuition needed to be able to identify the class of arrival pgfs introduced in this paper.

The rest of this paper is organized as follows. In Section 2 we introduce the detailed mathematical model of the system under study and establish a *functional equation* for the joint pgf $U(z_1, z_2)$. The solution of this functional equation is, in fact, the main purpose of the paper. Section 3 defines the generic class of arrival pgfs $A(z_1, z_2)$ that will be studied in this paper. In Section 4, we present and prove the main result of the paper, in the form of Theorem 1, which gives an explicit expression for the joint system-content pgf $U(z_1, z_2)$ associated with the joint arrival pgf $A(z_1, z_2)$ defined in Section 3. Section 5 defines three interesting subclasses, named A, B and C, of the generic class of arrival pgfs $A(z_1, z_2)$ defined in Section 3 and establishes even more explicit formulas for the associated system-content pgfs $U(z_1, z_2)$ in these cases, in the form of three corollaries of Theorem 1, also named A, B and C. In Section 6, we consider several instances of subclasses A, B and C, whereby specific choices are made for the various parameters and functions appearing in the formulations of corollaries A, B and C. Section 7 discusses a fundamental method to invert the system-content pgf $U(z_1, z_2)$, i.e., to determine the pmf $u(m, n)$ from the pgf $U(z_1, z_2)$, and it illustrates this technique by means of specific examples within subclasses A, B and C. Finally, we state some concluding remarks in Section 8.

2. Mathematical Model and Queuing Analysis

We define the random variables $a_{1,k}$ and $a_{2,k}$ as the numbers of type-1 and type-2 arrivals, respectively, during slot k. Their joint probability mass function (pmf) $a(i, j)$ and probability generating function (pgf) $A(z_1, z_2)$ are indicated as

$$a(i,j) \triangleq \text{Prob}[a_{1,k} = i \text{ and } a_{2,k} = j] \ , \quad i, j \geq 0 \ ,$$

$$A(z_1, z_2) \triangleq E\Big[z_1^{a_{1,k}} z_2^{a_{2,k}}\Big] \triangleq \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a(i,j)\, z_1^i z_2^j \ , \tag{1}$$

which are independent of k. The (marginal) pgfs of $a_{1,k}$ and $a_{2,k}$ are given by

$$A_1(z_1) \triangleq E\Big[z_1^{a_{1,k}}\Big] = A(z_1, 1) \quad , \quad A_2(z_2) \triangleq E\Big[z_2^{a_{2,k}}\Big] = A(1, z_2) \ , \tag{2}$$

respectively. The mean arrival rates, i.e., the mean numbers of arrivals per slot of types 1 and 2 are denoted as

$$\lambda_1 \triangleq A_1'(1) \ , \ \lambda_2 \triangleq A_2'(1) \ . \tag{3}$$

A graphical representation of the system under study is shown in Figure 1:

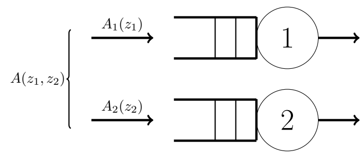

Figure 1. System of two coupled parallel queues.

Let $u_{1,k}$ and $u_{2,k}$ indicate the *system contents*, i.e., the *total* numbers of customers present in queue 1 and queue 2, respectively, *including* the customer(s) in service, if any, at the beginning of slot k. We indicate their joint pgf as

$$U_k(z_1, z_2) \triangleq E\Big[z_1^{u_{1,k}} z_2^{u_{2,k}}\Big] \ . \tag{4}$$

Furthermore, let $q_{1,k}$ and $q_{2,k}$ indicate the *queue contents*, i.e., the numbers of *waiting* customers in queue 1 and queue 2, respectively, *excluding* the customer(s) in service, if any, at the beginning of slot k. We indicate their joint pgf as

$$Q_k(z_1, z_2) \triangleq E\Big[z_1^{q_{1,k}} z_2^{q_{2,k}}\Big] \ . \tag{5}$$

It is not difficult to see that the following relationships exist between the *system contents* and the *queue contents*:

$$q_{1,k} = (u_{1,k} - 1)^+ \quad , \quad q_{2,k} = (u_{2,k} - 1)^+ \ , \tag{6}$$

where we have introduced the notation $(x)^+$ to indicate the quantity $\max(0, x)$.

The main purpose of the paper is to analyze the steady-state behavior of the queuing system under study, i.e., we are interested in determining the steady-state joint pgfs of the two *system contents* and/or *queue contents*, provided that such a steady state exists. Specifically, we wish to study the following limit functions:

$$U(z_1, z_2) \triangleq \lim_{k\to\infty} U_k(z_1, z_2) \quad , \quad Q(z_1, z_2) \triangleq \lim_{k\to\infty} Q_k(z_1, z_2) \tag{7}$$

if they exist. A steady state exists if and only if both queues are stable—that is to say, they receive, on average, less customers per slot than they can serve, i.e., if and only if the following stability conditions are fulfilled:

$$\lambda_1 < 1 \ , \ \lambda_2 < 1 \ , \tag{8}$$

where λ_1 and λ_2 denote the mean arrival rates, as defined in (3).

As mentioned in, e.g., refs. [6,11], the evolution of the *system contents* is described by the following *system equations*:

$$u_{1,k+1} = a_{1,k} + (u_{1,k} - 1)^+ \quad , \quad u_{2,k+1} = a_{2,k} + (u_{2,k} - 1)^+ \ . \tag{9}$$

Using standard z-transform techniques, the equations (9) can be translated into one corresponding transform equation between the joint pgfs $U_k(z_1, z_2)$ and $U_{k+1}(z_1, z_2)$ by using definition (4). Assuming the system reaches a steady state, i.e., assuming the stability conditions (8) are met, letting the time parameter k go to infinity, and using definitions (4) and (7), the latter transform equation translates into the following *functional equation* for the steady-state *system-content* pgf $U(z_1, z_2)$:

$$K(z_1, z_2)U(z_1, z_2) = A(z_1, z_2)L(z_1, z_2) \ , \tag{10}$$

where the unknown function $L(z_1, z_2)$ is defined as

$$L(z_1, z_2) \triangleq (z_2 - 1)U(z_1, 0) + (z_1 - 1)U(0, z_2) + (z_1 - 1)(z_2 - 1)U(0, 0) \tag{11}$$

and the *kernel* $K(z_1, z_2)$ is given by

$$K(z_1, z_2) \triangleq z_1 z_2 - A(z_1, z_2) \ . \tag{12}$$

Although, in general, the functional Equation (10) is hard to solve for $U(z_1, z_2)$ it is fairly easy to derive explicit expressions for the marginal pgfs $U(z_1, 1)$ and $U(1, z_2)$ of the individual system contents in queues 1 and 2 by choosing either $z_2 = 1$ or $z_1 = 1$ in (10), because such choices greatly simplify the L-function. As a result, we then obtain

$$U(z_1, 1) = \frac{U(0, 1)(z_1 - 1)A_1(z_1)}{z_1 - A_1(z_1)} \ , \ U(1, z_2) = \frac{U(1, 0)(z_2 - 1)A_2(z_2)}{z_2 - A_2(z_2)} \ . \tag{13}$$

Invoking the normalization condition $U(1, 1) = 1$ yields $U(0, 1) = 1 - \lambda_1$ and $U(1, 0) = 1 - \lambda_2$. The expressions in (13) are very well-known in the context of discrete-time queuing theory; see, for instance [62]; they will be very useful later in this paper.

We now turn our attention to the *queue contents*. Using (6) in (9), we readily obtain

$$u_{1,k+1} = a_{1,k} + q_{1,k} \quad , \quad u_{2,k+1} = a_{2,k} + q_{2,k} \ .$$

Transforming these relationships to pgfs, we obtain, on account of the definitions (4) and (5),

$$U_{k+1}(z_1, z_2) = A(z_1, z_2)Q_k(z_1, z_2) \ .$$

In view of (7), this implies that

$$U(z_1,z_2) = A(z_1,z_2)Q(z_1,z_2) \quad , \quad Q(z_1,z_2) = \frac{U(z_1,z_2)}{A(z_1,z_2)} \ . \tag{14}$$

Equation (14) makes clear that $Q(z_1,z_2)$ is known as soon as $U(z_1,z_2)$ is known, and vice versa. In the remainder of this paper, we mainly concentrate on the determination of $U(z_1,z_2)$.

3. Defining a Class of Arrival Pgfs $A(z_1,z_2)$

Let $f_1(z_1)$, $f_2(z_2)$, $g_1(z_1)$, $g_2(z_2)$, $h_1(z_1)$, $h_2(z_2)$ denote one-dimensional probability generating functions. We note that this implies

$$f_1(1) = f_2(1) = g_1(1) = g_2(1) = h_1(1) = h_2(1) = 1 \ . \tag{15}$$

Furthermore, let n_{11}, n_{12}, n_{21}, n_{22} denote a set of normalized probabilities, i.e.,

$$n_{11}, n_{12}, n_{21}, n_{22} \geq 0 \quad , \quad n_{11} + n_{12} + n_{21} + n_{22} = 1 \ , \tag{16}$$

and d_1 and d_2 two non-negative real parameters. We then use all the above quantities to define a whole class of bivariate functions $A(z_1,z_2)$, as follows:

$$A(z_1,z_2) = \frac{n_{11}z_1z_2 + n_{12}z_1f_2(z_2) + n_{21}z_2f_1(z_1) + n_{22}h_1(z_1)h_2(z_2)}{1 + d_1 + d_2 - d_1g_1(z_1) - d_2g_2(z_2)} \ . \tag{17}$$

We now show that the above function is a genuine joint pgf, i.e., it can be developed as a two-dimensional power series in z_1 and z_2 with non-negative coefficients that add up to 1.

Let us denote the numerator and the denominator of (17) as $n(z_1,z_2)$ and $d(z_1,z_2)$, respectively, i.e.,

$$A(z_1,z_2) = \frac{n(z_1,z_2)}{d(z_1,z_2)} = n(z_1,z_2)\left(\frac{1}{d(z_1,z_2)}\right) \ , \tag{18}$$

with

$$\begin{aligned} n(z_1,z_2) &\triangleq n_{11}z_1z_2 + n_{12}z_1f_2(z_2) + n_{21}z_2f_1(z_1) + n_{22}h_1(z_1)h_2(z_2) \ , \\ d(z_1,z_2) &\triangleq 1 + d_1 + d_2 - d_1g_1(z_1) - d_2g_2(z_2) \ . \end{aligned} \tag{19}$$

In view of (15) and (16), it is clear that both $n(z_1,z_2)$ and $d(z_1,z_2)$ are normalized, i.e., $n(1,1) = 1$ and $d(1,1) = 1$, which implies that $A(z_1,z_2)$ is also normalized, i.e., $A(1,1) = 1$. The numerator $n(z_1,z_2)$ is a probabilistic mixture of four valid pgfs and, therefore, is a valid pgf too. The function $1/d(z_1,z_2)$ is also a genuine pgf, since it can be developed as a two-dimensional power series in z_1 and z_2 with non-negative coefficients as follows:

$$\begin{aligned} \frac{1}{d(z_1,z_2)} &= \frac{1}{(1 + d_1 + d_2) - d_1g_1(z_1) - d_2g_2(z_2)} \\ &= \frac{1}{(1 + d_1 + d_2)(1 - \pi_1g_1(z_1) - \pi_2g_2(z_2))} \\ &= \left(\frac{1}{1 + d_1 + d_2}\right)\sum_{i=0}^{\infty}\left(\pi_1g_1(z_1) + \pi_2g_2(z_2)\right)^i \ , \end{aligned}$$

where the probabilities π_1 and π_2 have been defined as

$$\pi_1 = \frac{d_1}{1 + d_1 + d_2} \ , \ \pi_2 = \frac{d_2}{1 + d_1 + d_2} \ .$$

Equation (18) thus shows that $A(z_1, z_2)$ can be expressed as the product of two valid joint pgfs and, therefore, is a valid joint pgf too.

In this paper, we will examine a *parallel-queues* system with joint arrival pgf $A(z_1, z_2)$, as defined in (17) and (18). The corresponding marginal arrival pgfs $A_1(z_1)$ and $A_2(z_2)$ are

$$\begin{aligned} A_1(z_1) &= A(z_1, 1) = \frac{n(z_1, 1)}{d(z_1, 1)} = \frac{(n_{11} + n_{12})z_1 + n_{21} f_1(z_1) + n_{22} h_1(z_1)}{1 + d_1 - d_1 g_1(z_1)} \ , \\ A_2(z_2) &= A(1, z_2) = \frac{n(1, z_2)}{d(1, z_2)} = \frac{(n_{11} + n_{21})z_2 + n_{12} f_2(z_2) + n_{22} h_2(z_2)}{1 + d_2 - d_2 g_2(z_2)} \ . \end{aligned} \tag{20}$$

The mean arrival rates λ_1 and λ_2 are

$$\begin{aligned} \lambda_1 &\triangleq A_1'(1) = n_{11} + n_{12} + n_{21} f_1'(1) + n_{22} h_1'(1) + d_1 g_1'(1) \\ \lambda_2 &\triangleq A_2'(1) = n_{11} + n_{21} + n_{12} f_2'(1) + n_{22} h_2'(1) + d_2 g_2'(1) \ . \end{aligned} \tag{21}$$

We assume that $\lambda_1 < 1, \lambda_2 < 1$.

4. The Main Result

According to (13), the marginal system-content pgfs are given by

$$\begin{aligned} U(z_1, 1) &= \frac{U(0,1)(z_1 - 1)A_1(z_1)}{z_1 - A_1(z_1)} = \frac{U(0,1)(z_1 - 1)n(z_1, 1)}{k_1(z_1)} \ , \\ U(1, z_2) &= \frac{U(1,0)(z_2 - 1)A_2(z_2)}{z_2 - A_2(z_2)} = \frac{U(1,0)(z_2 - 1)n(1, z_2)}{k_2(z_2)} \ , \end{aligned} \tag{22}$$

where

$$k_1(z_1) \triangleq z_1 d(z_1, 1) - n(z_1, 1) \quad , \quad k_2(z_2) \triangleq z_2 d(1, z_2) - n(1, z_2) \ . \tag{23}$$

We are now ready to formulate the main result of this paper.

Theorem 1. *In the stable parallel-queues system with joint arrival pgf*

$$A(z_1, z_2) = \frac{n(z_1, z_2)}{d(z_1, z_2)} = \frac{n_{11} z_1 z_2 + n_{12} z_1 f_2(z_2) + n_{21} z_2 f_1(z_1) + n_{22} h_1(z_1) h_2(z_2)}{1 + d_1 + d_2 - d_1 g_1(z_1) - d_2 g_2(z_2)} \ , \tag{24}$$

where $f_1(z_1)$, $f_2(z_2)$, $g_1(z_1)$ and $g_2(z_2)$ are arbitrary one-dimensional pgfs and

$$n_{11}, n_{12}, n_{21}, n_{22} \geq 0 \quad , \quad n_{11} + n_{12} + n_{21} + n_{22} = 1 \ . \tag{25}$$

The steady-state joint system-content pgf $U(z_1, z_2)$ is given by

$$U(z_1, z_2) = \frac{n(z_1, z_2) U(z_1, 1) U(1, z_2)}{n(z_1, 1) n(1, z2)} = \frac{U(0,1) U(1,0)(z_1 - 1)(z_2 - 1) n(z_1, z_2)}{k_1(z_1) k_2(z_2)} \ , \tag{26}$$

provided that the functions $h_1(z_1)$ and $h_2(z_2)$ are genuine one-dimensional pgfs, given by

$$\begin{aligned} h_1(z_1) &= \alpha_1 z_1 + \beta_1 f_1(z_1) + \gamma_1 z_1 g_1(z_1) \ , \\ h_2(z_2) &= \alpha_2 z_2 + \beta_2 f_2(z_2) + \gamma_2 z_2 g_2(z_2) \ , \end{aligned} \tag{27}$$

where $\{\alpha_1, \beta_1, \gamma_1\}$ and $\{\alpha_2, \beta_2, \gamma_2\}$ are two sets of "normalized constants", i.e.,

$$\alpha_1 + \beta_1 + \gamma_1 = 1 \ , \ \alpha_2 + \beta_2 + \gamma_2 = 1 \ , \tag{28}$$

satisfying the restrictions

$$\beta_1 > 0 \ , \ ,\beta_2 > 0 \quad , \quad \gamma_1 = \frac{d_1\beta_1}{n_{21}} \geq 0 \ , \ \gamma_2 = \frac{d_2\beta_2}{n_{12}} \geq 0 \ . \tag{29}$$

4.1. Some Remarks on the Terms of Theorem 1

According to Equation (27), Theorem 1 requires that $h_1(z_1)$ be a linear combination of z_1, $f_1(z_1)$ and $z_1g_1(z_1)$ and, similarly, that $h_2(z_2)$ be a linear combination of z_2, $f_2(z_2)$ and $z_2g_2(z_2)$, with coefficients that add up to 1. It is easily seen that, as required, this implies that $h_1(z_1)$ and $h_2(z_2)$ are normalized, i.e., $h_1(1) = h_2(1) = 1$. However, for arbitrary choices of the parameters β_1, β_2, d_1, d_2, n_{12} and n_{21}, the functions $h_1(z_1)$ and $h_2(z_2)$, as given in (27), could, in general, contain linear terms in z_1 or z_2 with a negative coefficient, which would prevent them from being genuine pgfs. Indeed, whereas the coefficients β_1, γ_1, β_2 and γ_2 are certainly non-negative, this not necessarily the case for the coefficients of the linear terms in (27). It is clear that *sufficient* conditions to guarantee that $h_1(z_1)$ and $h_2(z_2)$ are valid pgfs are

$$\alpha_1 \geq 0 \ , \ \alpha_2 \geq 0 \ , \tag{30}$$

but these conditions are not *necessary*. In order to determine the linear terms in (27) completely, it is useful to decompose the functions $f_1(z_1)$, $f_2(z_2)$, $g_1(z_1)$ and $g_2(z_2)$, as follows:

$$f_1(z_1) = f_{10} + f_{11}z_1 + z_1^2v_1(z_1) \ , \ f_2(z_2) = f_{20} + f_{21}z_2 + z_2^2v_2(z_2) \tag{31}$$

and

$$g_1(z_1) = g_{10} + z_1w_1(z_1) \ , \ g_2(z_2) = g_{20} + z_2w_2(z_2) \ . \tag{32}$$

Substitution of (31) and (32) in (27) yields

$$\begin{aligned} h_1(z_1) &= \beta_1f_{10} + \big(\alpha_1 + \beta_1f_{11} + \gamma_1g_{10}\big)z_1 + \big(\beta_1v_1(z_1) + \gamma_1w_1(z_1)\big)z_1^2 \\ h_2(z_2) &= \beta_2f_{20} + \big(\alpha_2 + \beta_2f_{21} + \gamma_2g_{20}\big)z_2 + \big(\beta_2v_2(z_2) + \gamma_2w_2(z_2)\big)z_2^2 \ . \end{aligned} \tag{33}$$

The above expressions make clear that $h_1(z_1)$ and $h_2(z_2)$ are genuine pgs *if and only if*

$$\alpha_1 + \beta_1f_{11} + \gamma_1g_{10} \geq 0 \ , \ \alpha_2 + \beta_2f_{21} + \gamma_2g_{20} \geq 0 \ , \tag{34}$$

which are milder conditions than (30).

4.2. Proving Theorem 1

In order to prove Theorem 1, we first establish a technical lemma. Let us define the bivariate function $e(z_1,z_2)$ as

$$\begin{aligned} e(z_1,z_2) \triangleq\ & n(0,1)n(1,0)n(z_1,z_2) + n(0,1)n(z_1,0)k_2(z_2) + n(1,0)n(0,z_2)k_1(z_1) \\ & + n(0,0)k_1(z_1)k_2(z_2) \ . \end{aligned} \tag{35}$$

Lemma 1. *The function $e(z_1,z_2)$ can be expressed as*

$$e(z_1,z_2) = n(0,1)n(1,0)z_1z_2d(z_1,z_2) \ .$$

Proof. Combining (27), (28) and (29), and due to the fact that β_1 and β_2 are assumed to be strictly positive, we can compute the functions $n_{21}f_1(z_1)$ and $n_{12}f_2(z_2)$ in terms of $g_1(z_1)$, $g_2(z_2)$, $h_1(z_1)$ and $h_2(z_2)$, as follows:

$$
\begin{aligned}
n_{21} f_1(z_1) &= \frac{\beta_1 z_1 \big(n_{21} + d_1(1 - g_1(z_1))\big) + n_{21}\big(z_1 - h_1(z_1)\big)}{\beta_1} \ , \\
n_{12} f_2(z_2) &= \frac{\beta_2 z_2 \big(n_{12} + d_2(1 - g_2(z_2))\big) + n_{12}\big(z_2 - h_2(z_2)\big)}{\beta_2} \ .
\end{aligned} \tag{36}
$$

Inserting (36) into (19), we then obtain

$$
n(z_1, z_2) = n_{22}\big(h_1(z_1)h_2(z_2) - z_1 z_2\big) + z_1 z_2 d(z_1, z_2) - \frac{n_{21} z_2 \big(z_1 - h_1(z_1)\big)}{\beta_1} - \frac{n_{12} z_1 \big(z_2 - h_2(z_2)\big)}{\beta_2} \ , \tag{37}
$$

where we have also used the definition in (19) of $d(z_1, z_2)$.

Choosing either $z_1 = 0$ or $z_2 = 0$ in (37) yields

$$
n(z_1, 0) = f_2(0)\big(n_{12} z_1 + \beta_2 n_{22} h_1(z_1)\big) \ , \ n(0, z_2) = f_1(0)\big(n_{21} z_2 + \beta_1 n_{22} h_2(z_2)\big) \ , \tag{38}
$$

and, from this,

$$
n(0,0) = f_1(0) f_2(0) n_{22} \beta_1 \beta_2 \ . \tag{39}
$$

On the other hand, choosing either $z_1 = 1$ or $z_2 = 1$ in (37) leads to

$$
\begin{aligned}
n(z_1, 1) &= n_{22} h_1(z_1) + z_1\big(d(z_1, 1) - n_{22}\big) - \frac{n_{21}\big(z_1 - h_1(z_1)\big)}{\beta_1} \ , \\
n(1, z_2) &= n_{22} h_2(z_2) + z_2\big(d(1, z_2) - n_{22}\big) - \frac{n_{12}\big(z_2 - h_2(z_2)\big)}{\beta_2} \ .
\end{aligned} \tag{40}
$$

Equations (27) imply that $h_1(0) = \beta_1 f_1(0)$ and $h_2(0) = \beta_2 f_2(0)$, and, hence, choosing $z_1 = 0$ and $z_2 = 0$ in (40) yields

$$
n(0,1) = f_1(0)\big(\beta_1 n_{22} + n_{21}\big) \ , \ n(0,1) = f_2(0)\big(\beta_2 n_{22} + n_{12}\big) \ . \tag{41}
$$

Using (40) and (41) in (23), we can express $k_1(z_1)$ and $k_2(z_2)$ as

$$
\begin{aligned}
k_1(z_1) &\triangleq z_1 d(z_1, 1) - n(z_1, 1) = \frac{\beta_1 n_{22} + n_{21}}{\beta_1}\big(z_1 - h_1(z_1)\big) = \frac{n(0,1)}{\beta_1 f_1(0)}\big(z_1 - h_1(z_1)\big) \ , \\
k_2(z_2) &\triangleq z_2 d(1, z_2) - n(1, z_2) = \frac{\beta_2 n_{22} + n_{12}}{\beta_2}\big(z_2 - h_2(z_2)\big) = \frac{n(1,0)}{\beta_2 f_2(0)}\big(z_2 - h_2(z_2)\big) \ .
\end{aligned} \tag{42}
$$

Substitution of (37), (38), (39) and (42) in (35) then leads to

$$
\begin{aligned}
e(z_1, z_2) = {} & n(0,1) n(1,0) \bigg(n_{22}\big(h_1(z_1)h_2(z_2) - z_1 z_2\big) + z_1 z_2 d(z_1, z_2) - \frac{n_{21} z_2 \big(z_1 - h_1(z_1)\big)}{\beta_1} \\
& - \frac{n_{12} z_1 \big(z_2 - h_2(z_2)\big)}{\beta_2} \bigg) + n(0,1) f_2(0)\big(n_{12} z_1 + \beta_2 n_{22} h_1(z_1)\big) \frac{n(1,0)}{\beta_2 f_2(0)}\big(z_2 - h_2(z_2)\big) \\
& + n(1,0) f_1(0)\big(n_{21} z_2 + \beta_1 n_{22} h_2(z_2)\big) \frac{n(0,1)}{\beta_1 f_1(0)}\big(z_1 - h_1(z_1)\big) \\
& + f_1(0) f_2(0) n_{22} \beta_1 \beta_2 \frac{n(0,1)}{\beta_1 f_1(0)}\big(z_1 - h_1(z_1)\big) \frac{n(1,0)}{\beta_2 f_2(0)}\big(z_2 - h_2(z_2)\big) \ .
\end{aligned} \tag{43}
$$

It is then a matter of straightforward algebra to prove that all the terms in the above equation containing $h_1(z_1)$ and/or $h_2(z_2)$ compensate each other. The final result is

$$e(z_1,z_2) = n(0,1)n(1,0)z_1z_2d(z_1,z_2) \ .$$

This concludes the proof of Lemma 1. □

Proof of Theorem 1. The proof of Theorem 1 consists of two steps. In the first step, we show that the function $U(z_1,z_2)$, defined in (26), is a genuine joint pgf. In the second step, we prove that, under the conditions of Theorem 1, $U(z_1,z_2)$ satisfies the functional Equation (10) of the system.

Step 1: $U(z_1,z_2)$ is a genuine joint pgf *Proof.* In view of the relationship (14) between $U(z_1,z_2)$ and $Q(z_1,z_2)$ (and the corresponding marginal pgfs), Equation (26) can be rewritten as

$$U(z_1,z_2) = \frac{n(z_1,z_2)U(z_1,1)U(1,z_2)}{n(z_1,1)n(1,z_2)} = \frac{n(z_1,z_2)A(z_1,1)Q(z_1,1)A(1,z_2)Q(1,z_2)}{n(z_1,1)n(1,z_2)} \ .$$

Using (20), we then easily obtain

$$U(z_1,z_2) = \frac{n(z_1,z_2)Q(z_1,1)Q(1,z_2)}{d(z_1,1)d(1,z_2)} = n(z_1,z_2)Q(z_1,1)Q(1,z_2)\left(\frac{1}{d(z_1,1)}\right)\left(\frac{1}{d(1,z_2)}\right) \ ,$$

where the right-hand side is a product of five valid pgfs. Hence, $U(z_1,z_2)$ is a valid pgf as well.

Step 2: $U(z_1,z_2)$ satisfies the functional equation *Proof.* Combining (10) and (12), we can express the functional equation as

$$\big(z_1z_2 - A(z_1,z_2)\big)U(z_1,z_2) = A(z_1,z_2)L(z_1,z_2) \ ,$$

and, from this,

$$A(z_1,z_2)\big(L(z_1,z_2) + U(z_1,z_2)\big) = z_1z_2U(z_1,z_2) \ .$$

Inserting the expression (18) for $A(z_1,z_2)$ then leads to

$$L(z_1,z_2) + U(z_1,z_2) = \frac{z_1z_2d(z_1,z_2)U(z_1,z_2)}{n(z_1,z_2)} \ . \tag{44}$$

The function $L(z_1,z_2)$ can be computed from (11) as

$$L(z_1,z_2) \triangleq (z_2-1)U(z_1,0) + (z_1-1)U(0,z_2) + (z_1-1)(z_2-1)U(0,0) \ , \tag{45}$$

where $U(z_1,0)$ and $U(0,z_2)$ can be derived from (26) as

$$U(z_1,0) = \frac{U(0,1)U(1,0)(z_1-1)n(z_1,0)}{n(1,0)\big(k_1(z_1)\big)} \ , \ U(0,z_2) = \frac{U(0,1)U(1,0)(z_2-1)n(0,z_2)}{n(0,1)\big(k_2(z_2)\big)} \ , \tag{46}$$

which also implies

$$U(0,0) = \frac{U(0,1)U(1,0)n(0,0)}{n(0,1)n(1,0)} \ . \tag{47}$$

Substitution of (46) and (47) in (48) then leads to

$$L(z_1,z_2) = \ell(z_1,z_2)b(z_1,z_2) \ , \tag{48}$$

where we have defined $\ell(z_1,z_2)$ and $b(z_1,z_2)$ as short-hand notations for

$$\ell(z_1,z_2) \triangleq n(0,1)n(z_1,0)k_2(z_2) + n(1,0)n(0,z_2)k_1(z_1) + n(0,0)k_1(z_1)k_2(z_2) \ ,$$

$$b(z_1,z_2) \triangleq \frac{U(0,1)U(1,0)(z_1-1)(z_2-1)}{n(0,1)n(1,0)k_1(z_1)k_2(z_2)} \ .$$

On the other hand, in view of (26), $U(z_1,z_2)$ can be expressed as

$$U(z_1,z_2) = n(0,1)n(1,0)n(z_1,z_2)b(z_1,z_2) \ . \quad (49)$$

Inserting (48) and (49) into (44), we then obtain

$$\big(\ell(z_1,z_2) + n(0,1)n(1,0)n(z_1,z_2)\big)b(z_1,z_2) = \big(n(0,1)n(1,0)z_1z_2d(z_1,z_2)\big)b(z_1,z_2) \ . \quad (50)$$

It thus suffices to show that the expressions between the large parentheses on the left-hand side and on the right-hand side of (50) are equal. The expression on the left-hand side is exactly the function $e(z_1,z_2)$, defined in (35). Lemma 1 thus proves that (50) is fulfilled. This concludes the proof of Theorem 1. □

5. Subclasses A, B and C

Theorem 1 provides an explicit solution for the steady-state joint system-contents pgf $U(z_1,z_2)$ for any joint arrival pgf $A(z_1,z_2)$ that satisfies the shape specified in Equations (24), (25), (27), (28) and (29), *on condition that* $h_1(z_1)$ and $h_2(z_2)$ are valid pgfs. We now define three interesting subclasses of the generic class of arrival pgfs considered in Theorem 1, for which this crucial condition is fulfilled.

5.1. Subclass A: $d_1=0, d_2=0, \beta_1 \le 1, \beta_2 \le 1$

If $d_1=0$ and $d_2=0$ then the denominator $d(z_1,z_2)$ is equal to 1 and the arrival pgf $A(z_1,z_2)$, defined in (17), simplifies to

$$A(z_1,z_2) = n(z_1,z_2) = n_{11}z_1z_2 + n_{12}z_1f_2(z_2) + n_{21}z_2f_1(z_1) + n_{22}h_1(z_1)h_2(z_2) \ . \quad (51)$$

We note that, in this case, the pgfs $g_1(z_1)$ and $g_2(z_2)$ play no role anymore in $A(z_1,z_2)$. We also observe that, since $A(z_1,z_2)=n(z_1,z_2)$, the joint *queue-content* pgf $Q(z_1,z_2)$, given in (14), reduces to

$$Q(z_1,z_2) = \frac{U(z_1,z_2)}{n(z_1,z_2)} \ ,$$

and, hence, our main result (26) in Theorem 1 is equivalent to

$$Q(z_1,z_2) = Q(z_1,1)Q(1,z_2) \ , \quad (52)$$

i.e., the *queue contents* of both queues are mutually independent.

The parameters γ_1 and γ_2, defined in (29), are both equal to zero, and, hence, the functions $h_1(z_1)$ and $h_2(z_2)$, defined in (27), are given by

$$h_1(z_1) = (1-\beta_1)z_1 + \beta_1f_1(z_1) \ , \ \ h_2(z_2) = (1-\beta_2)z_2 + \beta_2f_2(z_2) \ , \quad (53)$$

and are certainly genuine pgfs if we assume

$$\beta_1 \le 1 \ , \ \ \beta_2 \le 1 \ . \quad (54)$$

In the sequel, we assume condition (54) fulfilled, and we formally define *subclass A* of the generic class of arrival pgfs considered in this paper by the conditions $d_1=0, d_2=0$, $\beta_1 \le 1, \beta_2 \le 1$.

Substitution of (53) in (51) then leads to

$$A(z_1,z_2) = n(z_1,z_2) = n_{11}z_1z_2 + n_{12}z_1f_2(z_2) + n_{21}z_2f_1(z_1) + n_{22}\big((1-\beta_1)z_1 + \beta_1 f_1(z_1)\big)\big((1-\beta_2)z_2 + \beta_2 f_2(z_2)\big) \ , \tag{55}$$

which can be rewritten as

$$A(z_1,z_2) = n(z_1,z_2) = m_{11}z_1z_2 + m_{12}z_1f_2(z_2) + m_{21}z_2f_1(z_1) + m_{22}f_1(z_1)f_2(z_2)$$

if we define the new parameters m_{11}, m_{12}, m_{21} and m_{22} as

$$m_{11} \triangleq (1-\beta_1)(1-\beta_2)n_{22} \ , \ m_{12} \triangleq n_{12} + (1-\beta_1)\beta_2 n_{22} \ ,$$

$$m_{21} \triangleq n_{21} + \beta_1(1-\beta_2)n_{22} \ , \ m_{22} \triangleq \beta_1\beta_2 n_{22} \ .$$

It is easily seen that, since $\beta_1 \leq 1, \beta_2 \leq 1$ and $\alpha_1 = 1-\beta_1, \alpha_2 = 1-\beta_2$, the sufficient conditions (30) are fulfilled and the parameters m_{11}, m_{12}, m_{21} and m_{22} also represent a normalized set of probabilities, just as the original parameters n_{11}, n_{12}, n_{21} and n_{22}, i.e.,

$$m_{11}, m_{12}, m_{21}, m_{22} \geq 0 \quad , \quad m_{11} + m_{12} + m_{21} + m_{22} = 1 \ . \tag{56}$$

The marginal arrival pgfs $A_1(z_1)$ and $A_2(z_2)$, given in (20), reduce to

$$A_1(z_1) = (m_{11} + m_{12})z_1 + (m_{21} + m_{22})f_1(z_1) \ , \ A_2(z_2) = (m_{11} + m_{21})z_2 + (m_{12} + m_{22})f_2(z_2) \ , \tag{57}$$

whereas the marginal mean arrival rates, given in (21), simplify to

$$\lambda_1 = 1 - (m_{21} + m_{22})(1 - f_1'(1)] \ , \ \lambda_2 = 1 - (m_{12} + m_{22})(1 - f_2'(1)] \ . \tag{58}$$

Hence, the stability conditions $\lambda_1 < 1, \lambda_2 < 1$ are equivalent to

$$f_1'(1) < 1 \ , \ f_2'(1) < 1 \ .$$

The functions $k_1(z_1)$ and $k_2(z_2)$, defined in (23), reduce to

$$k_1(z_1) = (m_{21} + m_{22})\big(z_1 - f_1(z_1)\big) \ , \ k_2(z_2) = (m_{12} + m_{22})\big(z_2 - f_2(z_2)\big) \ ,$$

and, hence, the joint pgf $U(z_1,z_2)$ can be derived from (26) as

$$U(z_1,z_2) = \frac{M(z_1-1)(z_2-1)n(z_1,z_2)}{\big(z_1 - f_1(z_1)\big)\big(z_2 - f_2(z_2)\big)} \ , \tag{59}$$

where the constant M is defined as

$$M \triangleq \frac{U(0,1)U(1,0)}{(m_{21} + m_{22})(m_{12} + m_{22})} \ .$$

The only remaining unknown M in Equation (59) can be computed from the normalization condition $U(1,1) = 1$, which results in

$$M = \big(1 - f_1'(1)\big)\big(1 - f_2'(1)\big) \ . \tag{60}$$

A fully explicit expression for $U(z_1,z_2)$ then follows from (59) and (60).

In summary, we have, thus, proven the following corollary of Theorem 1:

Corollary 1. *In the stable parallel-queues system with joint arrival pgf*

$$A(z_1,z_2) = m_{11}z_1z_2 + m_{12}z_1f_2(z_2) + m_{21}z_2f_1(z_1) + m_{22}f_1(z_1)f_2(z_2) \ , \tag{61}$$

where $f_1(z_1)$ and $f_2(z_2)$ are arbitrary one-dimensional pgfs, and

$$m_{11}, m_{12}, m_{21}, m_{22} \geq 0 \quad , \quad m_{11} + m_{12} + m_{21} + m_{22} = 1 \ , \tag{62}$$

the steady-state queue contents of both queues are mutually independent, and the steady-state joint system-content pgf $U(z_1, z_2)$ is given by

$$U(z_1, z_2) = \frac{(1 - f_1'(1))(1 - f_2'(1))(z_1 - 1)(z_2 - 1)A(z_1, z_2)}{(z_1 - f_1(z_1))(z_2 - f_2(z_2))} \ . \tag{63}$$

Remark 1. *If we define the* discriminant *D of $A(z_1, z_2)$ as $D \triangleq m_{12}m_{21} - m_{11}m_{22}$ then it is easily seen that if $D = 0$ then the pgf $A(z_1, z_2)$ has a product form, i.e.,*

$$A(z_1, z_2) = \frac{(m_{11}z_1 + m_{21}f_1(z_1))(m_{11}z_2 + m_{12}f_2(z_2))}{m_{11}} \ ,$$

and the arrivals of both customer types are mutually independent. It is clear from (63) that, in this case, the pgf $U(z_1, z_2)$ reduces to a product form too, i.e., the two steady-state system contents *are also mutually independent.*

5.2. Subclass B: $\alpha_1 = 0, \alpha_2 = 0$

The requirements (30) represent sufficient conditions to guarantee that the functions $h_1(z_1)$ and $h_2(z_2)$ are valid pgfs. A trivial way to satisfy (30) is to choose

$$\alpha_1 = 0 \ , \ \alpha_2 = 0 \ . \tag{64}$$

We formally define the *subclass B* of the generic class of arrival pgfs considered in this paper by condition (64). From (28) and (29), it follows that (64) is equivalent to

$$\gamma_1 = \frac{d_1}{d_1 + n_{21}} \ , \ \gamma_2 = \frac{d_2}{d_2 + n_{12}} \ . \tag{65}$$

Substitution of (64) in (27) then yields

$$h_1(z_1) = (1 - \gamma_1)f_1(z_1) + \gamma_1 z_1 g_1(z_1) \ , \ h_2(z_2) = (1 - \gamma_2)f_2(z_2) + \gamma_2 z_2 g_2(z_2) \ . \tag{66}$$

Equations (66) and (15) imply that

$$h_1'(1) = (1 - \gamma_1)f_1'(1) + \gamma_1[1 + g_1'(1)] \ , \ h_2'(1) = (1 - \gamma_2)f_2'(1) + \gamma_2[1 + g_2'(1)] \ ,$$

so that the mean arrival rates, given in (21), can be expressed as

$$\begin{aligned} \lambda_1 &= (n_{11} + n_{12} + \gamma_1 n_{22}) + (n_{21} + (1 - \gamma_1)n_{22})f_1'(1) + (d_1 + \gamma_1 n_{22})g_1'(1) \ , \\ \lambda_2 &= (n_{11} + n_{21} + \gamma_2 n_{22}) + (n_{12} + (1 - \gamma_2)n_{22})f_2'(1) + (d_2 + \gamma_2 n_{22})g_2'(1) \ . \end{aligned} \tag{67}$$

The functions $k_1(z_1)$ and $k_2(z_2)$, defined in (23), are given by

$$\begin{aligned} k_1(z_1) &= \frac{n_{21} + (1 - \gamma_1)n_{22}}{1 - \gamma_1}([1 - \gamma_1 g_1(z_1)]z_1 - (1 - \gamma_1)f_1(z_1)) \ , \\ k_2(z_2) &= \frac{n_{12} + (1 - \gamma_2)n_{22}}{1 - \gamma_2}([1 - \gamma_2 g_2(z_2)]z_2 - (1 - \gamma_2)f_2(z_2)) \ . \end{aligned} \tag{68}$$

The joint pgf $U(z_1, z_2)$ can be derived from (26) and (68) as

$$U(z_1, z_2) = \frac{M(z_1 - 1)(z_2 - 1)n(z_1, z_2)}{([1 - \gamma_1 g_1(z_1)]z_1 - (1 - \gamma_1)f_1(z_1))([1 - \gamma_2 g_2(z_2)]z_2 - (1 - \gamma_2)f_2(z_2))} \ , \tag{69}$$

where the constant M has been defined as

$$M \triangleq \frac{U(0,1)U(1,0)(1-\gamma_1)(1-\gamma_2)}{\big(n_{21}+(1-\gamma_1)n_{22}\big)\big(n_{12}+(1-\gamma_2)n_{22}\big)} \ .$$

The only remaining unknown M in Equation (93) can be computed from the normalization condition $U(1,1)=1$, which results in

$$M = \big((1-\gamma_1)[1-f_1'(1)]-\gamma_1 g_1'(1)\big)\big((1-\gamma_2)[1-f_2'(1)]-\gamma_2 g_2'(1)\big) \ . \tag{70}$$

A fully explicit expression for $U(z_1,z_2)$ then follows from (74) and (75).

Summarizing again, we have, thus, proven the following corollary of Theorem 1:

Corollary 2. *In the stable parallel-queues system with joint arrival pgf*

$$A(z_1,z_2) = \frac{n(z_1,z_2)}{1+d_1+d_2-d_1g_1(z_1)-d_2g_2(z_2)} \ , \tag{71}$$

where $n(z_1,z_2)$ is defined as

$$\begin{aligned} n(z_1,z_2) \triangleq\ & n_{11}z_1z_2 + n_{12}z_1f_2(z_2) + n_{21}z_2f_1(z_1) \\ & + n_{22}\big((1-\gamma_1)f_1(z_1)+\gamma_1z_1g_1(z_1)\big)\big((1-\gamma_2)f_2(z_2)+\gamma_2z_2g_2(z_2)\big) \ , \end{aligned} \tag{72}$$

with $f_1(z_1)$, $f_2(z_2)$, $g_1(z_1)$, $g_2(z_2)$ arbitrary one-dimensional pgfs, and

$$n_{11}, n_{12}, n_{21}, n_{22} \geq 0 \ , \ \ n_{11}+n_{12}+n_{21}+n_{22}=1$$

and

$$\gamma_1 \triangleq \frac{d_1}{d_1+n_{21}} \ , \ \ \gamma_2 \triangleq \frac{d_2}{d_2+n_{12}} \ , \tag{73}$$

the steady-state joint system-content pgf $U(z_1,z_2)$ is given by

$$U(z_1,z_2) = \frac{M(z_1-1)(z_2-1)n(z_1,z_2)}{\big([1-\gamma_1g_1(z_1)]z_1-(1-\gamma_1)f_1(z_1)\big)\big([1-\gamma_2g_2(z_2)]z_2-(1-\gamma_2)f_2(z_2)\big)} \ , \tag{74}$$

with

$$M \triangleq \big((1-\gamma_1)[1-f_1'(1)]-\gamma_1 g_1'(1)\big)\big((1-\gamma_2)[1-f_2'(1)]-\gamma_2 g_2'(1)\big) \ . \tag{75}$$

5.3. Subclass C: No Linear Terms in $h_1(z_1)$ and $h_2(z_2)$

The requirements of (34) are *necessary and sufficient* conditions in order for $h_1(z_1)$ and $h_2(z_2)$ to represent genuine pgfs. In this subsection, we examine the extreme case whereby the inequalities in (34) are replaced by equalities, i.e., where

$$\alpha_1+\beta_1f_{11}+\gamma_1g_{10}=0 \ , \ \ \alpha_2+\beta_2f_{21}+\gamma_2g_{20}=0 \ . \tag{76}$$

In view of (28) and (29), (76) can be rewritten as

$$(1-\alpha_1)(n_{21}f_{11}+d_1g_{10})+\alpha_1(n_{21}+d_1)=0 \ , \ \ (1-\alpha_2)(n_{12}f_{21}+d_2g_{20})+\alpha_2(n_{12}+d_2)=0 \ . \tag{77}$$

Solving (77) for α_1 and α_2, we find

$$\alpha_1 = -\frac{n_{21}f_{11}+d_1g_{10}}{n_{21}(1-f_{11})+d_1(1-g_{10})} \ , \ \ \alpha_2 = -\frac{n_{12}f_{21}+d_2g_{20}}{n_{12}(1-f_{21})+d_2(1-g_{20})} \ . \tag{78}$$

In these circumstances, due to (33), the functions $h_1(z_1)$ and $h_2(z_2)$ can be expressed as

$$h_1(z_1) = \frac{n_{21}f_{10} + z_1^2\big(n_{21}v_1(z_1) + d_1w_1(z_1)\big)}{n_{21}(1-f_{11}) + d_1(1-g_{10})} \ ,$$
$$h_2(z_2) = \frac{n_{12}f_{20} + z_2^2\big(n_{12}v_2(z_2) + d_2w_2(z_2)\big)}{n_{12}(1-f_{21}) + d_2(1-g_{20})} \ , \tag{79}$$

and, hence, contain no linear terms in z_1 and z_2, respectively.

In order to further simplify the expressions, let us consider the (further) special case where

$$v_1(z_1) = w_1(z_1) \ , \ v_2(z_2) = w_2(z_2) \ . \tag{80}$$

Of course, we then also have $v_1(1) = w_1(1), v_2(1) = w_2(1)$. From (31) and (32), we readily obtain

$$v_1(1) = 1 - \big(f_{10} + f_{11}\big) \ , \ v_2(1) = 1 - \big(f_{20} + f_{21}\big) \ , \ w_1(1) = 1 - g_{10} \ , \ w_2(1) = 1 - g_{20} \ ,$$

and, hence, $v_1(1) = w_1(1), v_2(1) = w_2(1)$ implies

$$f_{11} = g_{10} - f_{10} \ , \ f_{21} = g_{20} - f_{20} \ .$$

We now choose to additionally simplify the model by assuming

$$g_{10} = f_{10} \triangleq 1 - \omega_1 \ , \ g_{20} = f_{20} \triangleq 1 - \omega_2 \quad \Leftrightarrow \quad f_{11} = 0 \ , f_{20} = 0 \ , \tag{81}$$

where we have introduced the new parameters ω_1 and ω_2, which are valid probabilities.

By definition, we refer to arrival pgfs $A(z_1, z_2)$ (of the form considered in Theorem 1) as pgfs of *subclass C* if and only if they comply with the conditions (76), (80) and (81).

Using (80) and (81), we obtain the following expressions for $h_1(z_1)$ and $h_2(z_2)$ from (79):

$$h_1(z_1) = \frac{n_{21}(1-\omega_1) + (n_{21}+d_1)z_1^2v_1(z_1)}{n_{21} + d_1\omega_1} \ , \ h_2(z_2) = \frac{n_{12}(1-\omega_2) + (n_{12}+d_2)z_2^2v_2(z_2)}{n_{12} + d_2\omega_2} \ , \tag{82}$$

From their definitions in (31) and (32), it follows that the functions $v_1(z_1)$ and $v_2(z_2)$ only contain powers of z_1 and z_2 with non-negative coefficients, but are not necessarily normalized. It is useful to replace them by new functions, say $c_1(z_1)$ and $c_2(z_2)$, that do satisfy a normalization condition, and, hence, are genuine pgfs, as follows:

$$c_1(z_1) \triangleq \frac{v_1(z_1)}{v_1(1)} = \frac{v_1(z_1)}{\omega_1} \ , \ c_2(z_2) \triangleq \frac{v_2(z_2)}{v_2(1)} = \frac{v_2(z_2)}{\omega_2} \ . \tag{83}$$

All the defining functions of our model can then be expressed in terms of the pgfs $c_1(z_1)$ and $c_2(z_2)$, as follows:

$$f_1(z_1) = 1 - \omega_1 + \omega_1 z_1^2 c_1(z_1) \ , \ f_2(z_2) = 1 - \omega_2 + \omega_2 z_2^2 c_2(z_2) \ , \tag{84}$$

$$g_1(z_1) = 1 - \omega_1 + \omega_1 z_1 c_1(z_1) \ , \ g_2(z_2) = 1 - \omega_2 + \omega_2 z_2 c_2(z_2) \ , \tag{85}$$

$$h_1(z_1) = 1 - \theta_1 + \theta_1 z_1^2 c_1(z_1) \ , \ h_2(z_2) = 1 - \theta_2 + \theta_2 z_2^2 c_2(z_2) \ , \tag{86}$$

where we have defined the probabilities θ_1 and θ_2 as

$$\theta_1 \triangleq \frac{(n_{21}+d_1)\omega_1}{n_{21} + d_1\omega_1} \ , \ \theta_2 \triangleq \frac{(n_{12}+d_2)\omega_2}{n_{12} + d_2\omega_2} \ . \tag{87}$$

The arrival pgf $A(z_1,z_2)$ can be determined by the substitution of (84), (85) and (86) in (24):

$$A(z_1,z_2) = \frac{n(z_1,z_2)}{1 + d_1\omega_1(1 - z_1c_1(z_1)) + d_2\omega_2(1 - z_2c_2(z_2))} \ , \tag{88}$$

where

$$\begin{aligned} n(z_1,z_2) \triangleq\ & n_{11}z_1z_2 + n_{12}z_1(1 - \omega_2 + \omega_2 z_2^2 c_2(z_2)) + n_{21}z_2(1 - \omega_1 + \omega_1 z_1^2 c_1(z_1)) \\ & + n_{22}(1 - \theta_1 + \theta_1 z_1^2 c_1(z_1))(1 - \theta_2 + \theta_2 z_2^2 c_2(z_2)) \ . \end{aligned} \tag{89}$$

The marginal mean arrival rates can be computed from (21), which results in

$$\begin{aligned} \lambda_1 &= 1 - \frac{n_{21} + n_{22} + \omega_1 d_1}{n_{21} + \omega_1 d_1}\Big(n_{21} + \omega_1 d_1 - \omega_1(n_{21} + d_1)(2 + c_1'(1))\Big) \\ \lambda_2 &= 1 - \frac{n_{12} + n_{22} + \omega_2 d_2}{n_{12} + \omega_2 d_2}\Big(n_{12} + \omega_2 d_2 - \omega_2(n_{12} + d_2)(2 + c_2'(1))\Big) \ . \end{aligned} \tag{90}$$

Consequently, the stability conditions $\lambda_1 < 1, \lambda_2 < 1$ are equivalent to

$$c_1'(1) < \frac{n_{21} - \omega_1(2n_{21} + d_1)}{\omega_1(n_{21} + d_1)} \ , \ c_2'(1) < \frac{n_{12} - \omega_2(2n_{12} + d_2)}{\omega_2(n_{12} + d_2)} \ .$$

The functions $k_1(z_1)$ and $k_2(z_2)$ that constitute the denominator of $U(z_1,z_2)$ can be derived from (42):

$$\begin{aligned} k_1(z_1) &= \frac{n_{21} + n_{22} + \omega_1 d_1}{n_{21} + \omega_1 d_1}(n_{21}(z_1 - 1) + \omega_1(n_{21} + d_1z_1) - \omega_1(n_{21} + d_1)z_1^2c_1(z_1)) \ , \\ k_2(z_2) &= \frac{n_{12} + n_{22} + \omega_2 d_2}{n_{12} + \omega_2 d_2}(n_{12}(z_2 - 1) + \omega_2(n_{12} + d_2z_2) - \omega_2(n_{12} + d_2)z_2^2c_2(z_2)) \ , \end{aligned} \tag{91}$$

where we have also used (87). Introducing the notations $V_1(z_1)$ and $V_2(z_2)$ as

$$\begin{aligned} V_1(z_1) &\triangleq \frac{z_1 - 1}{n_{21}(z_1 - 1) + \omega_1(n_{21} + d_1z_1) - \omega_1(n_{21} + d_1)z_1^2c_1(z_1)} \ , \\ V_2(z_2) &\triangleq \frac{z_2 - 1}{n_{12}(z_2 - 1) + \omega_2(n_{12} + d_2z_2) - \omega_2(n_{12} + d_2)z_2^2c_2(z_2)} \ , \end{aligned} \tag{92}$$

we can compute the joint pgf $U(z_1,z_2)$ from (26) and (91) as

$$U(z_1,z_2) = MV_1(z_1)V_2(z_2)n(z_1,z_2) \ , \tag{93}$$

where the constant M has been defined as

$$M \triangleq \frac{U(0,1)U(1,0)(n_{21} + \omega_1 d_1)(n_{12} + \omega_2 d_2)}{(n_{21} + n_{22} + \omega_1 d_1)(n_{12} + n_{22} + \omega_2 d_2)} \ .$$

As before, the remaining unknown M can be determined by invoking the normalization condition $U(1,1) = 1$, which results in

$$M = \Big(n_{21} + \omega_1 d_1 - \omega_1(n_{21} + d_1)(2 + c_1'(1))\Big)\Big(n_{12} + \omega_2 d_2 - \omega_2(n_{12} + d_2)(2 + c_2'(1))\Big) \ . \tag{94}$$

A fully explicit expression for $U(z_1,z_2)$ then follows from (93), (104) and (94).

Summarizing again, we have thus proven the following corollary of Theorem 1:

Corollary 3. *In the stable parallel-queues system with joint arrival pgf*

$$A(z_1,z_2) = \frac{n(z_1,z_2)}{1+d_1\omega_1\big(1-z_1c_1(z_1)\big)+d_2\omega_2\big(1-z_2c_2(z_2)\big)} \ , \tag{95}$$

where

$$\begin{aligned} n(z_1,z_2) \triangleq\ & n_{11}z_1z_2 + n_{12}z_1\big(1-\omega_2+\omega_2z_2^2c_2(z_2)\big) + n_{21}z_2\big(1-\omega_1+\omega_1z_1^2c_1(z_1)\big) \\ & + n_{22}\big(1-\theta_1+\theta_1z_1^2c_1(z_1)\big)\big(1-\theta_2+\theta_2z_2^2c_2(z_2)\big) \ , \end{aligned} \tag{96}$$

with $c_1(z_1)$ and $c_2(z_2)$ arbitrary one-dimensional pgfs, and

$$n_{11}, n_{12}, n_{21}, n_{22} \geq 0 \quad , \quad n_{11}+n_{12}+n_{21}+n_{22} = 1$$

and

$$0 \leq \omega_1,\omega_2 \leq 1 \quad , \quad \theta_1 \triangleq \frac{(n_{21}+d_1)\omega_1}{n_{21}+d_1\omega_1} \ , \ \theta_2 \triangleq \frac{(n_{12}+d_2)\omega_2}{n_{12}+d_2\omega_2} \ ,$$

the steady-state joint system-content pgf $U(z_1,z_2)$ is given by

$$U(z_1,z_2) = MV_1(z_1)V_2(z_2)n(z_1,z_2) \ , \tag{97}$$

where

$$\begin{aligned} V_1(z_1) &\triangleq \frac{z_1-1}{n_{21}(z_1-1)+\omega_1(n_{21}+d_1z_1)-\omega_1(n_{21}+d_1)z_1^2c_1(z_1)} \ , \\ V_2(z_2) &\triangleq \frac{z_2-1}{n_{12}(z_2-1)+\omega_2(n_{12}+d_2z_2)-\omega_2(n_{12}+d_2)z_2^2c_2(z_2)} \ , \end{aligned} \tag{98}$$

and

$$M \triangleq \Big(n_{21}+\omega_1d_1-\omega_1(n_{21}+d_1)\big(2+c_1'(1)\big)\Big)\Big(n_{12}+\omega_2d_2-\omega_2(n_{12}+d_2)\big(2+c_2'(1)\big)\Big) \ . \tag{99}$$

6. Special Cases within Subclasses A, B and C

In this section, we consider several instances of subclasses A, B and C, whereby specific choices are made for the various parameters and functions appearing in the formulations of corollaries 1, 2 and 3.

6.1. Special Cases within Subclass A

6.1.1. At Most, One Arrival per Slot in Queue 1

Here, we choose

$$f_1(z_1) = 1-\sigma_1+\sigma_1z_1 \ , \ f_1'(1) = \sigma_1 \ , \tag{100}$$

which implies that the pgf $A(z_1,z_2)$, given in (61), reduces to

$$A(z_1,z_2) = \big((m_{21}\sigma_1+m_{11})z_2+(m_{22}\sigma_1+m_{12})f_2(z_2)\big)z_1+(1-\sigma_1)\big(m_{21}z_2+m_{22}f_2(z_2)\big) \ ,$$

which is clearly linear in z_1, meaning that queue 1 receives, at most, one arrival per slot. The marginal arrival pgf $A_2(z_2)$ and the mean arrival rate λ_2 follow from (57) and (58) as

$$A_2(z_2) = (m_{11}+m_{21})z_2+(m_{12}+m_{22})f_2(z_2) \ , \ \lambda_2 = 1-(m_{12}+m_{22})[1-f_2'(1)] \ ,$$

from which we can deduce that

$$f_2(z_2) = \frac{A_2(z_2)-(m_{11}+m_{21})z_2}{m_{12}+m_{22}} \ , \ 1-f_2'(1) = \frac{1-\lambda_2}{m_{12}+m_{22}} \ . \tag{101}$$

According to corollary A, the pgf $U(z_1, z_2)$ can be obtained from (63) by substitution of (100), i.e.,

$$U(z_1, z_2) = \frac{(1-\sigma_1)\big(1-f_2'(1)\big)(z_1-1)(z_2-1)A(z_1,z_2)}{\big(z_1-(1-\sigma_1+\sigma_1 z_1)\big)\big(z_2-f_2(z_2)\big)} = \frac{\big(1-f_2'(1)\big)(z_2-1)A(z_1,z_2)}{z_2-f_2(z_2)} \ .$$

Owing to (101) and (56), this can be rewritten as

$$U(z_1, z_2) = \frac{(1-\lambda_2)(z_2-1)A(z_1,z_2)}{z_2-A_2(z_2)} \ .$$

This particular result is well-known. We first established it through an alternative, more direct, approach in our earlier short paper [11]. It is interesting that we retrieve it here as a very simple special case of our more general results.

6.1.2. The Case $m_{12} = m_{21} = 0$

Again, in our earlier paper [11] we stated (without proof) the following theorem. Later, we also provided a formal proof in [12].

Theorem 2. *If $U_E(z_1, z_2)$ denotes the joint system-content pgf in a parallel-queues system with joint arrival pgf $E(z_1, z_2)$, and a new arrival pgf $A(z_1, z_2)$ is defined as*

$$A(z_1, z_2) \triangleq (1-\nu)z_1 z_2 + \nu E(z_1, z_2) \ , \quad \textit{where} \quad 0 < \nu \leq 1 \ , \tag{102}$$

then the joint system-content pgf $U_A(z_1, z_2)$ corresponding to arrival pgf $A(z_1, z_2)$ is given by

$$U_A(z_1, z_2) = \frac{U_E(z_1,z_2)A(z_1,z_2)}{E(z_1,z_2)} \ . \tag{103}$$

Specifically, if the arrivals of both types are mutually independent in the original system, i.e., if $E(z_1, z_2)$ has a product form, $E(z_1, z_2) = E_1(z_1)E_2(z_2)$, then $U_E(z_1, z_2)$ has a product form too, i.e., $U_E(z_1, z_2) = U_E(z_1, 1)U_E(1, z_2)$, with, similar to (13),

$$U_E(z_1, 1) = \frac{\big(1-E_1'(1)\big)(z_1-1)E_1(z_1)}{z_1-E_1(z_1)} \ , \ U_E(1, z_2) = \frac{\big(1-E_2'(1)\big)(z_2-1)E_2(z_2)}{z_2-E_2(z_2)} \ , \tag{104}$$

and (103) reduces to

$$U_A(z_1, z_2) = \frac{U_E(z_1,1)U_E(1,z_2)A(z_1,z_2)}{E_1(z_1)E_2(z_2)} = \frac{\big(1-E_1'(1)\big)\big(1-E_2'(1)\big)(z_1-1)(z_2-1)A(z_1,z_2)}{\big(z_1-E_1(z_1)\big)\big(z_2-E_2(z_2)\big)} \ . \tag{105}$$

It is remarkable that we can easily retrieve this property as a simple instance of our subclass-A results if we choose

$$m_{11} = 1-\nu, m_{12} = m_{21} = 0, m_{22} = \nu, f_1(z_1) = E_1(z_1), f_2(z_2) = E_2(z_2) \ .$$

Indeed, Equations (61) and (63) from the formulation of corollary A are then equivalent to Equations (102) and (103) from the formulation of Theorem 2. We do emphasize that Theorem 2 was proven to be valid also if $E(z_1, z_2)$ does not have a product form.

6.1.3. Geometric f-Distributions

Here, we choose *geometric* distributions with respective mean values σ_1 and σ_2 for the pgfs $f_1(z_1)$ and $f_2(z_2)$:

$$f_1(z_1) = \frac{1}{1+\sigma_1-\sigma_1 z_1} \ , \ f_2(z_2) = \frac{1}{1+\sigma_2-\sigma_2 z_2} \ , \ f_1'(1) = \sigma_1 \ , \ f_2'(1) = \sigma_2 \ . \tag{106}$$

The arrival pgf $A(z_1, z_2)$ then follows from (61) as

$$A(z_1, z_2) = \frac{F(z_1, z_2)}{(1 + \sigma_1 - \sigma_1 z_1)(1 + \sigma_2 - \sigma_2 z_2)} \ ,$$

where $F(z_1, z_2)$ is a quadratic polynomial in both z_1 and z_2, defined as

$$F(z_1, z_2) \triangleq m_{22} + m_{12} z_1 (1 + \sigma_1 - \sigma_1 z_1) + m_{21} z_2 (1 + \sigma_2 - \sigma_2 z_2) \\ + m_{11} z_1 z_2 (1 + \sigma_1 - \sigma_1 z_1)(1 + \sigma_2 - \sigma_2 z_2) \ .$$

The marginal mean arrival rates follow from (58):

$$\lambda_1 = 1 - (m_{21} + m_{22})(1 - \sigma_1) \ , \ \lambda_2 = 1 - (m_{12} + m_{22})(1 - \sigma_2) \ . \tag{107}$$

The stability conditions $\lambda_1 < 1, \lambda_2 < 1$ are, therefore, equivalent to $\sigma_1 < 1, \sigma_2 < 1$.

The system-content pgf $U(z_1, z_2)$ can be obtained by using (106) in (63):

$$U(z_1, z_2) = \frac{(1 - \sigma_1)(1 - \sigma_2) F(z_1, z_2)}{(1 - \sigma_1 z_1)(1 - \sigma_2 z_2)} \ , \tag{108}$$

a remarkably simple expression. The zeroes of the denominator are

$$z_1 = \frac{1}{\sigma_1} > 1 \ , \ z_2 = \frac{1}{\sigma_2} > 1 \ .$$

We return to this special case further in the paper.

6.1.4. Binomial f-Distributions

Here, we choose *binomial* distributions of order 2, again with respective mean values σ_1 and σ_2, for the pgfs $f_1(z_1)$ and $f_2(z_2)$:

$$f_1(z_1) = (1 - \frac{\sigma_1}{2} - \frac{\sigma_1}{2} z_1)^2 \ , \ f_2(z_2) = (1 - \frac{\sigma_2}{2} - \frac{\sigma_2}{2} z_2)^2 \ , \ f_1'(1) = \sigma_1 \ , \ f_2'(1) = \sigma_2 \ . \tag{109}$$

The arrival pgf $A(z_1, z_2)$ follows from (61) as

$$A(z_1, z_2) = m_{11} z_1 z_2 + m_{12} z_1 (1 - \frac{\sigma_2}{2} - \frac{\sigma_2}{2} z_2)^2 + m_{21} z_2 (1 - \frac{\sigma_1}{2} - \frac{\sigma_1}{2} z_1)^2 \\ + m_{22} (1 - \frac{\sigma_1}{2} - \frac{\sigma_1}{2} z_1)^2 (1 - \frac{\sigma_2}{2} - \frac{\sigma_2}{2} z_2)^2 \ ,$$

and is a quadratic polynomial in both z_1 and z_2. Again, the stability conditions $\lambda_1 < 1$, $\lambda_2 < 1$ are equivalent to $\sigma_1 < 1, \sigma_2 < 1$.

The system-content pgf $U(z_1, z_2)$ can be obtained by using (109) in (63):

$$U(z_1, z_2) = \frac{16(1 - \sigma_1)(1 - \sigma_2) A(z_1, z_2)}{\left((2 - \sigma_1)^2 - \sigma_1^2 z_1\right)\left((2 - \sigma_2)^2 - \sigma_2^2 z_2\right)} \ ;$$

again, a rather simple expression. The zeroes of the denominator are

$$z_1 = \left(\frac{2 - \sigma_1}{\sigma_1}\right)^2 > 1 \ , \ z_2 = \left(\frac{2 - \sigma_2}{\sigma_2}\right)^2 > 1 \ .$$

6.1.5. Batch-2-Geometric f-Distributions

Here, we choose *batch-2-geometric* distributions with respective mean values σ_1 and σ_2 for the pgfs $f_1(z_1)$ and $f_2(z_2)$:

$$f_1(z_1) = \frac{2}{2+\sigma_1-\sigma_1 z_1^2} \;,\; f_2(z_2) = \frac{2}{2+\sigma_2-\sigma_2 z_2^2} \;,\; f_1'(1) = \sigma_1 \;,\; f_2'(1) = \sigma_2 \;. \tag{110}$$

The terminology *batch-2-geometric* reflects the fact that a random variable with this distribution can only take values equal to geometrically distributed multiples of batch-size 2. The arrival pgf $A(z_1,z_2)$ follows from (61) as

$$A(z_1,z_2) = \frac{F(z_1,z_2)}{\left(2+\sigma_1-\sigma_1 z_1^2\right)\left(2+\sigma_2-\sigma_2 z_2^2\right)} \;,$$

where $F(z_1,z_2)$ is a cubic polynomial in both z_1 and z_2, defined as

$$F(z_1,z_2) \triangleq 4m_{22} + 2m_{12}z_1(2+\sigma_1-\sigma_1 z_1^2) + 2m_{21}z_2(2+\sigma_2-\sigma_2 z_2^2)$$
$$+ m_{11}z_1z_2(2+\sigma_1-\sigma_1 z_1^2)(2+\sigma_2-\sigma_2 z_2^2) \;.$$

Once again, the stability conditions $\lambda_1 < 1, \lambda_2 < 1$ are equivalent to $\sigma_1 < 1, \sigma_2 < 1$.

The system-content pgf $U(z_1,z_2)$ can be obtained by using (110) in (63):

$$U(z_1,z_2) = \frac{(1-\sigma_1)(1-\sigma_2)F(z_1,z_2)}{\left(2-\sigma_1 z_1(1+z_1)\right)\left(2-\sigma_2 z_2(1+z_2)\right)} \;,$$

a remarkably simple expression. The zeroes of the denominator can be explicitly determined as the solutions of quadratic equations; we omit the results here, for the sake of brevity.

6.2. Special Cases within Subclass B

In subclass A, the bivariate arrival pgf $A(z_1,z_2)$ is completely determined by the univariate pgfs $f_1(z_1)$ and $f_2(z_2)$, and the pgfs $g_1(z_1)$ and $g_2(z_2)$ play no role. In subclass B, however, all the defining one-dimensional pgfs contribute to $A(z_1,z_2)$. In order to specifically examine the effect of $g_1(z_1)$ and $g_2(z_2)$, we consider two examples where $f_1(z_1) = f_2(z_2) = 1$, combined with different choices for $g_1(z_1)$ and $g_2(z_2)$.

6.2.1. The Case $f_1(z_1) = f_2(z_2) = 1, g_1(z_1) = z_1, g_2(z_2) = z_2$

Here, we choose

$$f_1(z_1) = f_2(z_2) = 1, g_1(z_1) = z_1, g_2(z_2) = z_2 \;,\; f_1'(1) = f_2'(1) = 0, g_1'(1) = 1, g_2'(1) = 1 \;. \tag{111}$$

The arrival pgf $A(z_1,z_2)$ follows from (71), (72) and (73) as

$$A(z_1,z_2) = \frac{n_{11}z_1z_2 + n_{12}z_1 + n_{21}z_2 + n_{22}(1-\gamma_1+\gamma_1 z_1^2)(1-\gamma_2+\gamma_2 z_2^2)}{1 + \frac{\gamma_1 n_{21}}{1-\gamma_1}(1-z_1) + \frac{\gamma_2 n_{12}}{1-\gamma_2}(1-z_2)} \;. \tag{112}$$

In view of (67), the marginal mean arrival rates are

$$\lambda_1 = 1 - \frac{1-2\gamma_1}{1-\gamma_1}\left(n_{21} + (1-\gamma_1)n_{22}\right) \;,\; \lambda_2 = 1 - \frac{1-2\gamma_2}{1-\gamma_2}\left(n_{12} + (1-\gamma_2)n_{22}\right) \;,$$

and, hence, the stability conditions $\lambda_1 < 1, \lambda_2 < 1$ are equivalent to $\gamma_1 < 1/2, \gamma_2 < 1/2$.

The joint system-content pgf $U(z_1,z_2)$ can be obtained by using (111) in (72), (74) and (75), which results in

$$U(z_1,z_2) = \frac{(1-2\gamma_1)(1-2\gamma_2)\left(n_{11}z_1z_2 + n_{12}z_1 + n_{21}z_2 + n_{22}[1-\gamma_1+\gamma_1 z_1^2][1-\gamma_2+\gamma_2 z_2^2]\right)}{(1-\gamma_1-\gamma_1 z_1)(1-\gamma_2-\gamma_2 z_2)} \;. \tag{113}$$

The zeroes of the denominator are given by

$$z_1 = \frac{1-\gamma_1}{\gamma_1} > 1 \ , \ z_2 = \frac{1-\gamma_2}{\gamma_2} > 1 \ .$$

Remark 2. *It is worth mentioning that a special instance of this case was treated in our recent paper [6]. There, we considered a parallel-queues system, whereby the total number of arrivals per slot (of both customer types together) has a shifted geometric distribution with pgf* $C(z)$ *and mean value* $q \geq 1$*, i.e.,* $C(z) = z/[q-(q-1)z]$*, and new arrivals are routed independently and probabilistically to queue* 1 *or* 2 *with probabilities* p *and* $1-p$*, respectively, implying that the joint arrival pgf* $A(z_1, z_2)$ *is given by*

$$A(z_1,z_2) = C(pz_1 + (1-p)z_2) = \frac{pz_1 + (1-p)z_2}{q-(q-1)pz_1 - (q-1)(1-p)z_2} \ . \tag{114}$$

In Appendix A of [6], we formally proved that the joint system-content pgf $U(z_1, z_2)$ *for this system is*

$$U(z_1,z_2) = \frac{(\kappa_1 - 1)(\tau_1 - 1)\big(pz_1 + (1-p)z_2\big)}{(\kappa_1 - z_1)(\tau_1 - z_2)} \ , \ \text{with} \ \kappa_1 \triangleq \frac{1-p}{p(q-1)} \ , \tau_1 \triangleq \frac{p}{(1-p)(q-1)} \ . \tag{115}$$

The proof in [6] was a (rather complicated) constructive *proof, whereby we explicitly solved the functional Equation (10),* $K(z_1,z_2)U(z_1,z_2) = A(z_1,z_2)L(z_1,z_2)$*, by expressing that the unknown function* $L(z_1,z_2)$ *should vanish for all* (z_1,z_2) *in the area of convergence of* $U(z_1,z_2)$ *for which the kernel* $K(z_1,z_2)$ *vanishes. This allowed us to determine the boundary functions* $U(z_1,0)$ *and* $U(0,z_2)$*, and, from this, the function* $L(z_1,z_2)$*, and, eventually, the pgf* $U(z_1,z_2)$*, as given in (115).*

The function $A(z_1,z_2)$ *in (114) is clearly of the form (112) considered in the current subsection, provided we choose*

$$n_{11} = 0 \ , \ n_{12} = p \ , \ n_{21} = 1-p \ , \ n_{22} = 0 \ , \ d_1 = p(q-1) \ , \ d_2 = (1-p)(q-1) \ . \tag{116}$$

We now show that the solution (115) can be retrieved from the results in this subsection. Indeed, using (116) in the definitions (65) of our current parameters γ_1 *and* γ_2 *leads to*

$$\gamma_1 \triangleq \frac{d_1}{d_1 + n_{21}} = \frac{p(q-1)}{p(q-1)+1-p} \ , \ \gamma_2 \triangleq \frac{d_2}{d_2 + n_{12}} = \frac{(1-p)(q-1)}{(1-p)(q-1)+p} \ ,$$

and, from this,

$$\frac{1-\gamma_1}{\gamma_1} = \frac{1-p}{p(q-1)} = \kappa_1 \ , \ \frac{1-\gamma_2}{\gamma_2} = \frac{p}{(1-p)(q-1)} = \tau_1 \ , \tag{117}$$

and

$$\frac{1-2\gamma_1}{\gamma_1} = \frac{1-p}{p(q-1)} = \kappa_1 - 1 \ , \ \frac{1-2\gamma_2}{\gamma_2} = \frac{p}{(1-p)(q-1)} = \tau_1 - 1 \ . \tag{118}$$

Inserting (116) in (113) yields

$$U(z_1,z_2) = \frac{(1-2\gamma_1)(1-2\gamma_2)\big(pz_1 + (1-p)z_2\big)}{(1-\gamma_1-\gamma_1 z_1)(1-\gamma_2-\gamma_2 z_2)} \ . \tag{119}$$

Division of both the numerator and the denominator of the above expression by $\gamma_1\gamma_2$ *and substitution of (117) and (118) then clearly shows that (119) is identical to (115).*

We have, thus, been able, once again, to recover a specific existing result as a particular case of the results of the current paper.

6.2.2. The Case $f_1(z_1) = f_2(z_2) = 1, g_1(z_1) = z_1^2, g_2(z_2) = z_2^2$

Here, we choose

$$f_1(z_1) = f_2(z_2) = 1, g_1(z_1) = z_1^2, g_2(z_2) = z_2^2 \ , \ f_1'(1) = f_2'(1) = 0, g_1'(1) = 2, g_2'(1) = 2 \ . \tag{120}$$

The arrival pgf $A(z_1, z_2)$ follows from (71), (72) and (73) as

$$A(z_1, z_2) = \frac{n_{11} z_1 z_2 + n_{12} z_1 + n_{21} z_2 + n_{22}(1 - \gamma_1 + \gamma_1 z_1^3)(1 - \gamma_2 + \gamma_2 z_2^3)}{1 + \frac{\gamma_1 n_{21}}{1-\gamma_1}(1 - z_1) + \frac{\gamma_2 n_{12}}{1-\gamma_2}(1 - z_2)} \ .$$

From (67), we find the marginal mean arrival rates as

$$\lambda_1 = 1 - \frac{1 - 3\gamma_1}{1 - \gamma_1}\big(n_{21} + (1 - \gamma_1) n_{22}\big) \ , \ \lambda_2 = 1 - \frac{1 - 3\gamma_2}{1 - \gamma_2}\big(n_{12} + (1 - \gamma_2) n_{22}\big) \ ,$$

and, hence, the stability conditions $\lambda_1 < 1, \lambda_2 < 1$ are equivalent to $\gamma_1 < 1/3, \gamma_2 < 1/3$. The joint system-content pgf $U(z_1, z_2)$ is obtained by using (120) in (72), (74) and (75):

$$U(z_1, z_2) = \frac{(1 - 3\gamma_1)(1 - 3\gamma_2)\big(n_{11} z_1 z_2 + n_{12} z_1 + n_{21} z_2 + n_{22}[1 - \gamma_1 + \gamma_1 z_1^3][1 - \gamma_2 + \gamma_2 z_2^3]\big)}{\big(1 - \gamma_1 - \gamma_1 z_1(1 + z_1)\big)\big(1 - \gamma_2 - \gamma_2 z_2(1 + z_2)\big)} \ . \tag{121}$$

Again, the zeroes of the denominator can be explicitly computed. We return to this special case in more detail further in the paper.

6.3. Special Cases within Subclass C

In order to simplify the expressions in this subsection, we first make the following assumptions:

$$n_{11} = 0 \ , \ n_{12} = n_{21} = d_1 = d_2 = d \ , \ n_{22} = 1 - 2d \ , \ \omega_1 = \omega_2 = \frac{1}{4} \ . \tag{122}$$

According to (95) and (96), the arrival pgf is given by

$$A(z_1, z_2) = \frac{n(z_1, z_2)}{1 + \frac{d}{4}\big(2 - z_1 c_1(z_1) - z_2 c_2(z_2)\big)} \ ,$$

where

$$n(z_1, z_2) = \frac{d}{4}\Big(z_1\big(3 + z_2^2 c_2(z_2)\big) + z_2\big(3 + z_1^2 c_1(z_1)\big)\Big) + \frac{1 - 2d}{25}\big(3 + z_2^2 c_2(z_2)\big)\big(3 + z_1^2 c_1(z_1)\big) \ . \tag{123}$$

The marginal mean arrival rates are

$$\lambda_1 = 1 - \frac{4 - 3d}{20}\big(1 - 2c_1'(1)\big) \ , \ \lambda_2 = 1 - \frac{4 - 3d}{20}\big(1 - 2c_2'(1)\big) \ ,$$

which implies that the stability conditions $\lambda_1 < 1, \lambda_2 < 1$ are equivalent to $c_1'(1) < 1/2, c_2'(1) < 1/2$. From (98) and (99), we obtain

$$V_1(z_1) = \frac{4(z_1 - 1)}{d\big(5z_1 - 2z_1^2 c_1(z_1) - 3\big)} \ , \ V_2(z_2) = \frac{4(z_2 - 1)}{d\big(5z_2 - 2z_2^2 c_2(z_2) - 3\big)} \ ,$$

and

$$M = \frac{d^2}{16}\big(1 - 2c_1'(1)\big)\big(1 - 2c_2'(1)\big) \ .$$

It then follows from (97) that the system-content pgf is given by

$$U(z_1,z_2) = \frac{\big(1-2c_1'(1)\big)\big(1-2c_2'(1)\big)(z_1-1)(z_2-1)n(z_1,z_2)}{\big(3-5z_1+2z_1^2c_1(z_1)\big)\big(3-5z_2+2z_2^2c_2(z_2)\big)} \ , \tag{124}$$

We now make different choices for $c_1(z_1)$ and $c_2(z_2)$ that are interesting from a computational point-of-view because they lead to easily computable zeroes of the denominator of (124).

6.3.1. Bernoulli c-distributions

Here, we choose Bernoulli distributions with parameters σ_1 and σ_2 for $c_1(z_1)$ and $c_2(z_2)$:

$$c_1(z_1) = 1-\sigma_1+\sigma_1 z_1 \ , \ c_2(z_2) = 1-\sigma_2+\sigma_2 z_2 \ , \ c_1'(1) = \sigma_1 \ , \ c_2'(1) = \sigma_2 \ .$$

The pgf $U(z_1,z_2)$ then reduces to

$$U(z_1,z_2) = \frac{(1-2\sigma_1)(1-2\sigma_2)n(z_1,z_2)}{(3-2z_1-2\sigma_1 z_1^2)(3-2z_2-2\sigma_2 z_2^2)} \ . \tag{125}$$

We come back to this special case later in this paper.

6.3.2. Geometric c-Distributions

We now consider geometric distributions with mean values σ_1 and σ_2 for $c_1(z_1)$ and $c_2(z_2)$:

$$c_1(z_1) = \frac{1}{1+\sigma_1-\sigma_1 z_1} \ , \ c_2(z_2) = \frac{1}{1+\sigma_2-\sigma_2 z_2} \ , \ c_1'(1) = \sigma_1 \ , \ c_2'(1) = \sigma_2 \ . \tag{126}$$

The pgf $U(z_1,z_2)$ then reduces to

$$U(z_1,z_2) = \frac{(1-2\sigma_1)(1-2\sigma_2)(1+\sigma_1-\sigma_1 z_1)(1+\sigma_2-\sigma_2 z_2)n(z_1,z_2)}{\big(3(1+\sigma_1)-(2+5\sigma_1)z_1\big)\big(3(1+\sigma_2)-(2+5\sigma_2)z_2\big)} \ .$$

The zeroes of the denominator are given by

$$z_1 = \frac{3(1+\sigma_1)}{2+5\sigma_1} > 1 \ , \ z_2 = \frac{3(1+\sigma_2)}{2+5\sigma_2} > 1 \ .$$

6.3.3. Negative Binomial c-Distributions

Here, we choose negative binomial distributions of order two for the pgfs $c_1(z_1)$ and $c_2(z_2)$:

$$c_1(z_1) = \frac{4}{(2+\sigma_1-\sigma_1 z_1)^2} \ , \ c_2(z_2) = \frac{4}{(2+\sigma_2-\sigma_2 z_2)^2} \ , \ c_1'(1) = \sigma_1 \ , \ c_2'(1) = \sigma_2 \ . \tag{127}$$

In this case, the pgf $U(z_1,z_2)$ is given by

$$U(z_1,z_2) = W_1(z_1)W_2(z_2)n(z_1,z_2) \ ,$$

where

$$W_1(z_1) \triangleq \frac{(1-2\sigma_1)(2+\sigma_1-\sigma_1 z_1)^2}{\big(3(2+\sigma_1)^2-4(2+\sigma_1)(1+2\sigma_1)z_1+5\sigma_1^2 z_1^2\big)} \ ,$$

$$W_2(z_2) \triangleq \frac{(1-2\sigma_2)(2+\sigma_2-\sigma_2 z_2)^2}{\big(3(2+\sigma_2)^2-4(2+\sigma_2)(1+2\sigma_2)z_2+5\sigma_2^2 z_2^2\big)} \ .$$

Also in this case, the zeroes of the denominator are the solutions of quadratic equations and can be computed explicitly; it is not difficult to show that they lie outside the unit disks on the z_1-plane and the z_2-plane.

7. Inverting the Joint pgf $U(z_1, z_2)$

In this section, we focus on the derivation of the steady-state joint *probability mass function* (pmf) $u(m,n)$ of the system contents in queues 1 and 2, which is defined as

$$u(m,n) \triangleq \lim_{k\to\infty} \text{Prob}[u_{1,k} = m, u_{2,k} = n] \ ,$$

and is related to the joint pgf $U(z_1, z_2)$ by the equation

$$U(z_1, z_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u(m,n) z_1^m z_2^n \ .$$

In an earlier paper [12], we proved (with slightly different notations) the following useful theorem to determine $u(m,n)$ from $U(z_1, z_2)$ for "interior states" (m,n) in the state space.

Theorem 3. *If the joint pgf $U(z_1, z_2)$ is a* rational *function of z_1 and z_2 of the form*

$$U(z_1, z_2) = \frac{B(z_1, z_2)}{r_1(z_1) r_2(z_2)} = \frac{B(z_1, z_2)}{\prod_{i=1}^{L_1}(z_1 - \kappa_i) \prod_{j=1}^{L_2}(z_2 - \tau_j)} \ , \tag{128}$$

where the numerator $B(z_1, z_2)$ is a bivariate polynomial of degree K_1 in z_1 and K_2 in z_2 and the denominator is a product of two univariate functions $r_1(z_1)$ and $r_2(z_2)$, having only zeroes of multiplicity 1*, and the numerator and the denominator are mutually prime, then threshold values m_0 and n_0 can be defined as $m_0 \triangleq max(0, K_1 - L_1 + 1), n_0 \triangleq max(0, K_2 - L_2 + 1)$, such that for $m \geq m_0, n \geq n_0$ the pmf $u(m,n)$ is given by a finite linear combination of bivariate geometric terms, i.e.,*

$$u(m,n) = \sum_{i=1}^{L_1} \sum_{j=1}^{L_2} \mu_{i,j} \Big(\frac{1}{\kappa_i}\Big)^m \Big(\frac{1}{\tau_j}\Big)^n \ , \ m \geq m_0, n \geq n_0 \ , \tag{129}$$

where

$$\mu_{i,j} \triangleq \frac{B(\kappa_i, \tau_j)}{\kappa_i \tau_j r_1'(\kappa_i) r_2'(\tau_j)} \ . \tag{130}$$

7.1. Some Comments

In all the examples that we have considered in this paper, we have chosen rational functions for constituting one-dimensional pgfs $f_1(z_1)$, $f_2(z_2)$, $g_1(z_1)$, $g_2(z_2)$, $h_1(z_1)$, $h_2(z_2)$ of the joint arrival pgf $A(z_1, z_2)$, defined in (24). This implies that the pgf $U(z_1, z_2)$, given in (26) by

$$U(z_1, z_2) = \frac{U(0,1)U(1,0)(z_1 - 1)(z_2 - 1)n(z_1, z_2)}{k_1(z_1) k_2(z_2)} \ , \tag{131}$$

is a rational bivariate function whose denominator is a product of two univariate functions, and that it can, therefore, be expressed in the form required to apply Theorem 3.

By definition, the quantities κ_i and τ_j occurring in (129) are the zeroes of $r_1(z_1)$ (or $k_1(z_1)$) and $r_2(z_2)$ (or $k_2(z_2)$). According to (129), the geometric decay rates of the system-content distribution are the inverse values of these zeroes, i.e., the ith decay rate for queue 1 is equal to $1/\kappa_i$ and the jth decay rate for queue 2 is given by $1/\tau_j$. Each bivariate geometric term in $u(m,n)$ thus corresponds to a couple (κ_i, τ_j) of zeroes of $r_1(z_1)$ and $r_2(z_2)$, but the opposite is not necessarily true, since, for some i and j, it may happen that the coefficient $\mu_{i,j}$ in Equation (129) is zero. According to (130), this situation occurs if $B(\kappa_i, \tau_j) = 0$. If this is the case for one or more couples (κ_i, τ_j), the number of nonzero bivariate geometric terms in $u(m,n)$ is lower than the product $L1 \times L2$.

7.2. Specific Examples

In this subsection, we apply Theorem 3 in some examples of arrival pgfs $A(z_1, z_2)$ belonging to subclasses A, B and C, as defined before.

7.2.1. An Example within Subclass A

In this example, we revisit the model with geometric f-distributions of Section 6.1.3. The system-content pgf $U(z_1, z_2)$ is given in (108). The parameters and functions appearing in the formulation of Theorem 3 are $K_1 = K_2 = 2, L_1 = L_2 = 1$ and

$$B(z_1, z_2) = (1 - \sigma_1)(1 - \sigma_2)F(z_1, z_2) \ , \ r_1(z_1) = 1 - \sigma_1 z_1 \ , \ r_2(z_2) = 1 - \sigma_2 z_2 \ ,$$

with

$$F(z_1, z_2) \triangleq m_{22} + m_{12}z_1(1 + \sigma_1 - \sigma_1 z_1) + m_{21}z_2(1 + \sigma_2 - \sigma_2 z_2) \\ + m_{11}z_1z_2(1 + \sigma_1 - \sigma_1 z_1)(1 + \sigma_2 - \sigma_2 z_2) \ .$$

The zeroes of $r_1(z_1)$ and $r_2(z_2)$ are $\kappa_1 = \frac{1}{\sigma_1} > 1, \tau_1 = \frac{1}{\sigma_2} > 1$. The coefficient $\mu_{1,1}$ can be computed from (130) as $\mu_{1,1} = (1 - \sigma_1)(1 - \sigma_2)$. Finally, the pmf $u(m, n)$ for interior states (m, n) follows from (129) as

$$u(m, n) = (1 - \sigma_1)(1 - \sigma_2)\left(\frac{1}{\sigma_1}\right)^m\left(\frac{1}{\sigma_2}\right)^n \ , \ m \geq 2, n \geq 2 \ .$$

7.2.2. An example within subclass B

Here, we consider a symmetric instance of the model with quadratic g-functions of Section 6.2.2, with the following specific parameter choices: $n_{12} = n_{21} = n_0, d_1 = d_2 = d$, $\gamma_1 = \gamma_2 = \gamma$. The joint system-content pgf $U(z_1, z_2)$ can be obtained from (121):

$$U(z_1, z_2) = \frac{(1 - 3\gamma)^2\big(n_{11}z_1z_2 + n_0z_1 + n_0z_2 + n_{22}[1 - \gamma + \gamma z_1^3][1 - \gamma + \gamma z_2^3]\big)}{\big(1 - \gamma - \gamma z_1(1 + z_1)\big)\big(1 - \gamma - \gamma z_2(1 + z_2)\big)} \ .$$

We can apply Theorem 3 with $K_1 = K_2 = 3, L_1 = L_2 = 2$,

$$B(z_1, z_2) \triangleq (1 - 3\gamma)^2\big(n_{11}z_1z_2 + n_0z_1 + n_0z_2 + n_{22}[1 - \gamma + \gamma z_1^3][1 - \gamma + \gamma z_2^3]\big) \ ,$$

and

$$r_1(z_1) \triangleq 1 - \gamma - \gamma z_1(1 + z_1) \ , \ r_2(z_2) \triangleq 1 - \gamma - \gamma z_2(1 + z_2) \ .$$

The zeroes of $r_1(z_1)$ and $r_2(z_2)$ lie outside the unit disks $\{z_1 : |z_1| \leq 1\}$ and $\{z_2 : |z_2| \leq 1\}$ on the complex z_1-plane and z_2-plane, respectively, and are given by

$$\kappa_1 = \tau_1 = \frac{\sqrt{\gamma(4 - 3\gamma)} - \gamma}{2\gamma} > 1 \ , \ \kappa_2 = \tau_2 = -\frac{\sqrt{\gamma(4 - 3\gamma)} + \gamma}{2\gamma} < -1 \ .$$

The coefficients $\mu_{i,j}$ can be computed from (130):

$$\mu_{1,1} = \mu\frac{1 - \gamma + (1 - 4n_0)(1 + \sqrt{\gamma(4 - 3\gamma)})}{2 - \gamma - \sqrt{\gamma(4 - 3\gamma)}} \ , \ \mu_{2,2} = \mu\frac{1 - \gamma + (1 - 4n_0)(1 - \sqrt{\gamma(4 - 3\gamma)})}{2 - \gamma + \sqrt{\gamma(4 - 3\gamma)}} \\ \mu_{1,2} = \mu_{2,1} = \mu\frac{1 - \gamma - n_0(2 - 3\gamma)}{1 - \gamma} \ , \ \text{where} \ \ \mu \triangleq \frac{(1 - 3\gamma)^2}{\gamma(4 - 3\gamma)} \ . \tag{132}$$

In general, all these coefficients are nonzero, and the linear combination in (129) contains four terms:

$$u(m,n) = \mu_{1,1}(\frac{1}{\kappa_1})^{m+n} + \mu_{12}\Big((\frac{1}{\kappa_1})^m(\frac{1}{\kappa_2})^n + (\frac{1}{\kappa_2})^m(\frac{1}{\kappa_1})^n\Big) + \mu_{2,2}(\frac{1}{\kappa_2})^{m+n} \ , \ m \geq 2, n \geq 2 \ .$$

Careful study shows that it is impossible to choose the parameter n_0, appearing in (132), in such a way that the coefficients $\mu_{1,1}$, $\mu_{1,2}$ or $\mu_{2,1}$ are zero, but there does exist a value of n_0, such that $\mu_{2,2}$ vanishes; the required n_0-value is

$$n_0 = \frac{2-\gamma+\sqrt{\gamma(4-3\gamma)}}{4(1+\sqrt{\gamma(4-3\gamma)})} > 0 \ . \tag{133}$$

This is an acceptable value, since it implies that

$$n_{11} + n_{22} = 1 - n_{12} - n_{21} = 1 - 2n_0 = \frac{\gamma+\sqrt{\gamma(4-3\gamma)}}{2(1+\sqrt{\gamma(4-3\gamma)})} > 0$$

So, in case n_0 is chosen in accordance with (133), the linear combination in (129) contains only three bivariate geometric terms:

$$u(m,n) = \mu_{1,1}(\frac{1}{\kappa_1})^{m+n} + \mu_{12}\Big((\frac{1}{\kappa_1})^m(\frac{1}{\kappa_2})^n + (\frac{1}{\kappa_2})^m(\frac{1}{\kappa_1})^n\Big) \ , \ m \geq 2, n \geq 2 \ ,$$

and does not contain a term with two negative decay rates.

7.2.3. An example within subclass *C*

We now go back to the model with Bernoulli c-distributions in Section 6.3.1. The pgf $U(z_1,z_2)$ is given by (125). We can apply Theorem 3, provided we choose $K_1 = K_2 = 3$, $L_1 = L_2 = 2$:

$$B(z_1,z_2) = (1-2\sigma_1)(1-2\sigma_2)n(z_1,z_2) \ , \ r_1(z_1) = (3-2z_1-2\sigma_1 z_1^2) \ , \ r_2(z_2) = (3-2z_2-2\sigma_1 z_2^2) \ ,$$

where, owing to (123), $n(z_1,z_2)$ is given by

$$\begin{aligned} n(z_1,z_2) = {} & \frac{d}{4}\Big(z_1(3+z_2^2(1-\sigma_2+\sigma_2 z_2)) + z_2(3+z_1^2(1-\sigma_1+\sigma_1 z_1))\Big) \\ & + \frac{1-2d}{25}(3+z_1^2(1-\sigma_1+\sigma_1 z_1))(3+z_2^2(1-\sigma_2+\sigma_2 z_2)) \ . \end{aligned} \tag{134}$$

The zeroes of $r_1(z_1)$ and $r_2(z_2)$ lie outside the unit disks $\{z_1 : |z_1| \leq 1\}$ and $\{z_2 : |z_2| \leq 1\}$ on the complex z_1-plane and z_2-plane, respectively, and are given by

$$\kappa_1 = \frac{\sqrt{1+6\sigma_1}-1}{2\sigma_1} > 1 \ , \ \kappa_2 = -\frac{\sqrt{1+6\sigma_1}+1}{2\sigma_1} < -1$$

$$\tau_1 = \frac{\sqrt{1+6\sigma_2}-1}{2\sigma_2} > 1 \ , \ \tau_2 = -\frac{\sqrt{1+6\sigma_2}+1}{2\sigma_2} < -1 \ .$$

The coefficients $\mu_{i,j}$ can be computed from (130):

$$\mu_{1,1} = \mu\frac{4+d(\sigma_1\kappa_1+\sigma_2\tau_1-1)}{(1+2\sigma_1\kappa_1)(1+2\sigma_2\tau_1)} \ , \ \mu_{1,2} = \mu\frac{4+d(\sigma_1\kappa_1+\sigma_2\tau_2-1)}{(1+2\sigma_1\kappa_1)(1+2\sigma_2\tau_2)} \ ,$$

$$\mu_{2,1} = \mu\frac{4+d(\sigma_1\kappa_2+\sigma_2\tau_1-1)}{(1+2\sigma_1\kappa_2)(1+2\sigma_2\tau_1)} \ , \ \mu_{2,2} = \mu\frac{4+d(\sigma_1\kappa_2+\sigma_2\tau_2-1)}{(1+2\sigma_1\kappa_2)(1+2\sigma_2\tau_2)} \ ,$$

where we have defined μ as

$$\mu \triangleq \frac{(1-2\sigma_1)(1-2\sigma_2)}{16} \ .$$

Again, in general, all these coefficients are nonzero, and the linear combination in (129) contains four terms:

$$u(m,n) = \mu_{1,1}(\frac{1}{\kappa_1})^m(\frac{1}{\tau_1})^n + \mu_{12}(\frac{1}{\kappa_1})^m(\frac{1}{\tau_2})^n + (\frac{1}{\kappa_2})^m(\frac{1}{\tau_1})^n + \mu_{2,2}(\frac{1}{\kappa_2})^m(\frac{1}{\tau_2})^n \ , \ m \geq 2, n \geq 2 \ . \tag{135}$$

Let d_{ij} denote the value of d that makes $\mu_{i,j}$ zero; then, we can easily compute the following values:

$$d_{11} = \frac{8}{4-\sqrt{1+6\sigma_1}-\sqrt{1+6\sigma_2}} \ , \ d_{12} = \frac{8}{4-\sqrt{1+6\sigma_1}+\sqrt{1+6\sigma_2}} \ ,$$

$$d_{21} = \frac{8}{4+\sqrt{1+6\sigma_1}-\sqrt{1+6\sigma_2}} \ , \ d_{22} = \frac{8}{4+\sqrt{1+6\sigma_1}+\sqrt{1+6\sigma_2}} \ .$$

Taking into account the stability conditions $\sigma_1 < 1/2, \sigma_2 < 1/2$, as we have shown in Section 6.3.1, it is readily seen that all these d-values are positive, as required, but none of them are lower than $1/2$, which is also necessary, because in this model, according to (122), $n_{22} = 1-2d$ and needs to be positive. We conclude that, in this particular case, the pmf $u(m,n)$ always contains exactly four bivariate geometric terms, as shown in (135).

8. Concluding Remarks

This paper has considered the steady-state queuing analysis of a system of two *parallel* discrete-time single-server queues with mutually interdependent arrivals, characterized by the joint arrival pgf $A(z_1,z_2)$. We have identified a very broad, multi-parameter, generic class of arrival pgfs $A(z_1,z_2)$, for which we were able to determine explicit analytic solutions for the joint system-content pgf $U(z_1,z_2)$. We think this is the main virtue of the paper. It is also interesting to observe that our results encompass most of the previously known results for this kind of system, which is known to be hard to analyze.

Although the class of arrival pgfs $A(z_1,z_2)$ examined in this paper is very broad, it still has its limitations, which are mainly due to the shape of the arrival pgf, i.e., Equation (17):

$$A(z_1,z_2) = \frac{n_{11}z_1z_2 + n_{12}z_1f_2(z_2) + n_{21}z_2f_1(z_1) + n_{22}h_1(z_1)h_2(z_2)}{1+d_1+d_2-d_1g_1(z_1)-d_2g_2(z_2)} \ ,$$

and the requirement that the pgfs appearing in the above expression should be related as stated in Equations (27) or (36), which can be rewritten as

$$\beta_1 n_{21}(z_1 - f_1(z_1)) + \beta_1 d_1 z_1(1-g_1(z_1)) + n_{21}(z_1 - h_1(z_1)) = 0 \ ,$$

$$\beta_2 n_{12}(z_2 - f_2(z_2)) + \beta_2 d_2 z_2(1-g_2(z_2)) + n_{12}(z_2 - h_2(z_2)) = 0 \ .$$

Since the parameters β_1 and β_2 need to be strictly positive – we need this in the proof of Lemma 1—we can, thus, not have a constant numerator for $A(z_1,z_2)$ without the requirement that the denominator be also constant. Hence, a seemingly simple arrival pgf like

$$A(z_1,z_2) = \frac{1}{1+d_1+d_2-d_1z_1-d_2z_2} \tag{136}$$

is not a special case of our model. So far, we have not seen a solution for the *"global geometric"* arrival pgf in (136), and the current paper also does not provide one.

Future work could go in several directions. We may try to further extend the class of arrival pgfs that lead to explicit solutions for the *parallel-queues* system, dealt with in this paper, but we may also consider other types of coupled queues, such as the (other) ones mentioned in the introduction section of this paper. An even more challenging task could be to extend the analysis from two queues to more than two queues. We expect such an extension to be far from obvious, because the number and the nature of the boundary functions in the functional equation make the problem much more complex than in the two-queue case.

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