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A Mathematical Classification of Kinds of Quantities and Quantity Equations

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Philippe A.J.G. Chevalier

Nederlandse samenvatting

De beroemdste natuurkundige vergelijking is Einstein's vergelijking $E = mc_0^2$. Het heeft de ternaire relatievorm $Z = XY$ waarbij, X, Y, en Z soorten grootheden zijn. De door wetenschappers gebruikte vergelijkingen hebben algemene vormen die grootheidsvergelijkingen worden genoemd en die allemaal aan het homogeniteitsbeginsel moeten voldoen. De termen in de grootheidsvergelijkingen zijn de soorten grootheden die voorkomen in de experimenten en theorieën.

Soorten grootheden worden gebruikt in het dagelijkse werk van miljoenen wetenschappers wereldwijd. De meeste wetenschappers zijn zich niet bewust van de wiskundige eigenschappen van deze soorten grootheden en wat er a priori uit deze wiskundige eigenschappen afgeleid kan worden voor het ontwerpen van experimenten.

Het Bureau international des poids et mesures (BIPM) heeft geen onderzoeksactiviteit naar de wiskundige classificatie van de fysische grootheden. Soorten grootheden komen in de wetenschappelijke literatuur voor in de vorm van tensoren (scalairen, vectoren, matrices) en pseudo-tensoren. Elke soort grootheid wordt voorgesteld door een symbool. Alle soorten grootheden worden bepaald door hun respectieve *componenten*, en daarom hebben we onze studie beperkt tot de componenten van soorten grootheden. We zullen ons focussen op twee onopgeloste problemen van de dimensieanalyse.

Onze eerste onderzoeksvraag is: Is het mogelijk om soorten grootheden wiskundig te classificeren? Deze onderzoeksvraag is onopgelost sinds Maxwell. Soorten grootheden worden nog steeds geclassificeerd op basis van hun technische discipline.

Onze tweede onderzoeksvraag is: Is het mogelijk om wiskundig te selecteren welke grootheidsvergelijkingen 'natuurkundige wetten' zijn? Feynman beschreef het proces voor het ontdekken van een nieuwe 'natuurkundige wet' als een proces dat begint met een gok. Er is geen methode bekend om 'natuurkundige wetten' te vinden. De enige manier volgens Feynman is 'raden' en het raden verifiëren door een experiment.

Dit proefschrift geeft een uitgewerkt 'ja' antwoord op beide onderzoeksvragen. In dit proefschrift worden grootheidsvergelijkingen bestudeerd binnen het raamwerk van dimensieanalyse. Dimensieanalyse beschouwt soorten grootheden en zoekt naar producten van soorten grootheden die resulteren in een dimensieloze grootheid.

De eerste onderzoeksvraag heeft een zekere mate van overeenkomst met het werk van Dmitri Mendelejev in 1869, waarin hij het periodiek systeem creëerde en de chemische elementen classificeerde. In 1871 vestigde Maxwell de aandacht in zijn presentatie 'Opmerkingen over de wiskundige classificatie van fysische grootheden' op het belang van de wiskundige classificatie van fysische grootheden. Helaas, stierf Maxwell in 1879, slechts 48 jaar oud, zonder de wiskundige classificatie van de fysische grootheden te hebben voltooid.

Dit proefschrift presenteert de wiskundige classificaties van de soorten grootheden die resulteren in de 'Tabellen van de natuurkunde'. Een 'natuurkundige tabel' wordt op unieke wijze gegenereerd zodra het alfabet van de natuurkunde is gekozen. De woorden die met dat alfabet gevormd worden, vertegenwoordigen soorten grootheden. Het aantal letters in het alfabet komt overeen met het aantal basisgrootheden dat per conventie is gekozen om de natuurkunde te beschrijven.

Het Internationale Stelsel van Eenheden (SI), dat wereldwijd wordt gebruikt door de wetenschappelijke gemeenschap, heeft zeven basisgrootheden: tijd, lengte, massa, elektrische stroom, thermodynamische temperatuur, hoeveelheid stof en lichtsterkte. Elke basisgrootheid vertegenwoordigt een unieke letter die wordt gekoppeld aan een N-tupel van gehele getallen waarbij N, dimensie genaamd, overeenkomt met het aantal letters van het alfabet. De letters in de woorden die kunnen gemaakt worden, zijn gerangschikt in de volgende volgorde gedefinieerd door de SI: T, L, M, I, Θ, N, J .

De soort grootheid energie in SI wordt weergegeven door het woord T^{-1} T^{-1} L L M en heeft de unieke zeven tupel $(-2, 2, 1, 0, 0, 0, 0) \in \mathbb{Z}^7$. De gehele getallen van het N-tupel kunnen gepermuteerd worden, en die "signed permutations"leiden tot verzamelingen N-tupels die *banen* genoemd worden. De kardinaliteiten van de banen van het rooster worden bepaald door de dimensie N . Bij iedere N bestaat er een unieke verzameling van kardinaliteiten. Voor $N = 7$ bestaat de verzameling van kardinaliteiten uit 30 gehele getallen. We kunnen aan elk van de getallen een unieke kleur associëren. Bij iedere baan hoort één bepaald getal uit die verzameling. Om de kardinaliteit van de baan te visualiseren, kunnen we de kleur kiezen die toehoort aan de kardinaliteit en elk roosterpunt van de baan inkleuren.

De Euclidische afstand van een N-tupel tot de oorsprong $(0, 0 \ldots, 0)$ wordt aangeduid als de 2-norm en is een speciaal geval van de p-norm. Het is bekend dat het gehele rooster $\mathbb{Z}^N$ gepartitioneerd kan worden met behulp van de supremum norm. Er ontstaat een structuur van hyperoppervlakken van genestelde hyperkubussen.

De creatie van de banen en de creatie van hypervlakken van hyperkubussen zijn equivalentierelaties, en dus verzamelingen. De doorsnede van die verzamelingen resulteert in de wiskundige classificatie van de soorten grootheden. We verkrijgen een 'natuurkundige tabel' voor de gekozen dimensie N . De rijen in de tabel verwijzen naar de supremum norm en de kolommen naar de kardinaliteit van de baan. Het aantal banen met elementen met dezelfde supremum norm en met dezelfde kardinaliteit wordt geschreven in een cel van de 'natuurkundige tabel'.

Het is duidelijk uit de dimensieanalyse dat er een oneindig aantal grootheidsvergelijkingen bestaan, maar dat slechts een zeer klein aantal grootheidsvergelijkingen 'natuurkundige wetten' zijn. Dit vraagt om een verklaring en resulteert in het tweede onderzoeksprobleem.

We zullen enkele belangrijke meetkundige eigenschappen bespreken van grootheidsvergelijkingen. We bewijzen dat ternaire relaties $Z = XY$ tussen soorten grootheden voorgesteld kunnen worden door parallellogrammen $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$ in het rooster $\{0,1\} \times \mathbb{Z}^N$.

We creëeren een 3D-faseruimte met als coördinaten: de $\|\boldsymbol{x}\|_1$, de $\|\boldsymbol{y}\|_1$, en de oppervlakte van het parallellogram in het kwadraat A^2 . De multipliciteit van elke parallellogramtoestand is als attribuut verbonden aan de drie coördinaten van de faseruimte.

We besteden speciale aandacht aan parallellogrammen met multipliciteit één. Bijvoorbeeld, de formule $E = mc_0^2$ komt overeen met een parallellogram, dat tevens een rechthoek is, met een omtrek dat uniek is in het histogram van de omtrekken van parallellogrammen die een energie relatie voorstellen van het type $E = XY$. De normalisatie van het histogram leidt tot waarschijnlijkheden. De unieke omtrekken komen overeen met de laagste waarschijnlijkheid. Gebruik makend van de Shannon informatietheorie, kan met de laagste waarschijnlijkheid de grootste informatie-inhoud geassocieerd worden.

De ternaire relatie $Z = X_1 X_2$ kan worden uitgebreid tot een $(m+1)$ relatie $Z = X_1 X_2 \ldots X_m$. Deze relatie tussen SI soorten grootheden vertegenwoordigt een constellatie van roosterpunten. Alle $(m+1)$ relaties kunnen uiteindelijk gereduceerd worden op basis van de associativiteit eigenschap tot ternaire relaties voorgesteld door parallellogrammen. Daarom spitsen we het onderzoek toe op ternaire relaties, maar we zullen $(m+1)$ relaties tegenkomen in dit proefschrift.

We bestuderen ook de afhankelijkheid van de multipliciteit van parallellogrammen als functie van de dimensie N van het rooster. Resultaten van computersimulaties hebben het bestaan van een minimale dimensie N_{min} voor een gegeven grootheid Z aangetoond, waarbij het aantal unieke ternaire vergelijkingen constant is wanneer de dimensie N van het gehele rooster toeneemt. Er moeten voldoende letters in het alfabet zijn om de uniciteit van parallellogram te waarborgen. Bijvoorbeeld, de verzameling unieke omtrekken van parallellogrammen voor de soort grootheid energiedichtheid heeft een grotere kardinaliteit wanneer deze wordt berekend in het gehele rooster $\mathbb{Z}^3$ dan in $\mathbb{Z}^7$.

Niet alle relaties tussen fysische grootheden worden beschouwd als 'natuurwetten'. Wat zijn de eigenschappen van relaties tussen fysische grootheden die een specifieke relatie tot een 'natuurwet' maakt? Einstein postuleerde in de speciale relativiteitstheorie het principe van relativiteit en de constantheid van de lichtsnelheid in vacuum. Kip Thorne stelt dat Einstein, door het relativiteitsprincipe te postuleren een wet formuleerde die de 'wetten van de natuurkunde' beheerst. We zullen een wet over de 'wetten van de natuurkunde' een metawet noemen. De relativiteit metawet van Einstein stelt dat elke 'natuurkundewet' hetzelfde moet zijn in coordinatenstelsels die zich eenparig rechtlijning voortbewegen. Bestaan er nog andere metawetten? Zo ja, wat zijn deze metawetten? Zouden we deze metawetten kunnen ontdekken?

In dit proefschrift formuleren we een vermoeden dat elke 'natuurwet' een maximale informatie-inhoud moet hebben. Alle door ons gekende natuurwetten voldoen aan deze nieuwe metawet. Dit vermoeden verklaart waarom er weinig natuurwetten zijn doordat natuurwetten slechts weergegeven worden door unieke parallellogrammen. Dit vermoeden vermindert het 'raden',vermeld door Feynman, en zou de weg kunnen openen naar het waarnemen van nieuwe 'natuurkundige wetten'. Een software code om unieke parallellogrammen te zoeken werd geschreven. Deze metawet, die evenwel nog een vermoeden is, beantwoordt de tweede onderzoeksvraag met een bevestigend 'ja'.

English summary

The most famous equation of physics is Einstein's equation $E = mc_0^2$. It has the ternary relation form $Z = XY$, in which X, Y, Z are kinds of quantities. The equations used by the scientists have generic forms that are denoted quantity equations and all those equations have to comply with the homogeneity principle. The terms in the quantity equations are the kinds of quantities which occur in experiments and theories.

Kinds of quantities are used in the daily work of millions of scientists worldwide. Most scientists are not aware of the mathematical properties of those kinds of quantities and what a priori can be derived from these mathematical properties to design experiments.

The International Bureau of Weights and Measures (BIPM) shows no research activity toward the mathematical classification of the physical quantities. Kinds of quantities occur in the scientific literature in the form of tensors (scalars, vectors, matrices) and pseudo-tensors. Each kind of quantity is represented by a symbol. All the kinds of quantities are eventually defined through their respective *components*, and thus we have restricted our study to the components of kinds of quantities. We will focus on two unsolved problems of dimensional analysis.

The first research question is: Is it possible to mathematically classify kinds of quantities? This research question has been unsolved since Maxwell. Kinds of quantities are still today classified according to their engineering discipline.

The second research question is: Is it possible to mathematically select which quantity equations are 'laws of physics'? Feynman described the process for discovering a new 'law of physics' as a process that starts with a guess. No method is known to find 'laws of physics'. The only way according to Feynman is 'guessing' and verify the guess by experiment.

This dissertation provides an elaborated answer 'yes' to both research questions. In this work, *quantity equations* are studied within the framework of dimensional analysis. Dimensional analysis considers kinds of quantities and aims to find products of kinds of quantities resulting in a dimensionless quantity.

The first question has some degree of similarity with the work done by Dmitri Mendeleev in 1869, in which he created the periodic table, classifying the chemical elements. In 1871, Maxwell drew attention in his presentation 'Remarks on the mathematical classification of physical quantities' to the importance of the mathematical classification of physical quantities. Unfortunately, Maxwell died in 1879, being 48 years old, without having finished the mathematical classification of the physical quantities.

This dissertation presents the mathematical classifications of the kinds of

quantities resulting in the ‘Table(s) of Physics’. A ‘Table of physics’ is uniquely generated once the *alphabet* of physics is chosen. The words formed with that alphabet represent kinds of quantities. The number of letters in the alphabet corresponds to the number of base quantities that are selected by convention to describe physics.

The International System of Units (SI), adopted worldwide by the scientific community, has seven base quantities: time, length, mass, electric current, thermodynamic temperature, amount of substance, and luminous intensity. Each base quantity represents a unique letter that is mapped to an N-tuple of integers in which N, called dimension, corresponds to the number of letters of the alphabet. The letters, in Roman font, in the words that can be created are ordered to the following sequence defined by the SI: T, L, M, I, Θ, N, J .

The kind of quantity energy in SI is represented by the word $\mathsf{T}^{-1}\ \mathsf{T}^{-1}\ \mathsf{L}\ \mathsf{L}\ \mathsf{M}$ and has the unique seven tuple $(-2, 2, 1, 0, 0, 0, 0) \in \mathbb{Z}^7$. The integers of the N-tuple can be permuted, and those signed permutations lead to sets of N-tuples that are called *orbits*. The cardinalities of the orbits of the integer lattice are determined by the dimension N . To each N corresponds a unique set of cardinalities. For $N = 7$, we find 30 integers. We can associate a unique color to each of the integers. Each orbit is associated with an integer from the set of cardinalities. The orbit’s cardinality can be visualized by coloring each lattice point that is an element of the orbit.

The Euclidean distance from an N-tuple to the origin $(0, 0, \dots, 0)$ is denoted 2-norm, being a special case of the p-norm. It is known that the integer lattice $\mathbb{Z}^N$ can be partitioned using the infinity norm. A structure of hypersurfaces of nested hypercubes emerges.

The creation of orbits and the creation of hypersurfaces of the hypercubes are equivalence relations, and thus specific sets. The intersection of these sets results in the mathematical classification of the kinds of quantities. We obtain a ‘Table of physics’ for the selected N that is the dimension of the integer lattice. The rows in the table refer to the infinity norm and the columns to the orbit’s cardinality. The number of orbits having elements with the same infinity norm and with the same cardinality is written in a cell of the ‘Table of physics’.

It is evident from dimensional analysis that an infinity of quantity equations exist but only a very small number of quantity equations are ‘laws of physics’. This needs an explanation and results in the second research problem.

We will discover some important geometric properties of quantity equations. We prove that ternary relations $Z = XY$ between kinds of quantities represent parallelograms $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$ in the integer lattice $\{0, 1\} \times \mathbb{Z}^N$.

We create a 3D-phase space of parallelograms with coordinates: the $\|\boldsymbol{x}\|_1$, the $\|\boldsymbol{y}\|_1$, and the area of the parallelogram squared A^2 . The multiplicity of each *parallelogram state* is attached as an attribute to the three phase space coordinates.

We pay special attention to parallelograms having multiplicity one. For example, the equation $E = mc_0^2$ corresponds to a parallelogram, that is also a rectangle, with a unique perimeter in the histogram of perimeters of parallelograms which represent an energy relation of the type $E = XY$. The normalization of the histogram yields probabilities. Those unique parallelograms have the lowest probability. Using Shannon's information theory, we associate with the lowest probability the largest information content.

The ternary relation $Z = X_1 X_2$ can be expanded to a $(m+1)$-ary relation $Z = X_1 X_2 \ldots X_m$. This relation between SI kinds of quantities represents a constellation of lattice points. All the $(m+1)$-ary relations can be reduced using the associativity property to ternary relations represented by parallelograms. We therefore focus the research on the ternary relations, however we will encounter $(m+1)$-ary relations in this dissertation.

We also study the dependence of parallelogram state's multiplicity as function of the dimension N of the integer lattice. Results of computer simulations have revealed the existence of a minimum dimension N_{min} for a given kind of quantity Z in which the number of unique ternary equations is constant when the dimension N of the integer lattice increases. There should be a minimum of letters in the alphabet to guarantee the unicity of the parallelogram. For example, the set of unique perimeters of parallelograms for the kind of quantity energy density has a larger cardinality when it is calculated in the integer lattice $\mathbb{Z}^3$ than in $\mathbb{Z}^7$.

Not all relations between physical quantities are considered 'laws of physics'. What are the properties of relations between physical quantities that promotes a specific relation to a 'law of physics'? In the special relativity theory Einstein postulated the principle of relativity and the principle of the constancy of the velocity of light in vacuum. Kip Thorne states that Einstein, by postulating the principle of relativity, formulated a law that governs the 'laws of physics'. We will denote a law about the 'laws of physics' as a meta-law. Einstein's meta-law of relativity states that each 'law of physics' must be the same for systems of coordinates in uniform translatory motion. Do other meta-laws exist? If so, what are these meta-laws? Could we discover these meta-laws?

In this dissertation, we conjecture that each 'law of physics' must have maximum information content. All the 'laws of physics´, known to us, satisfy this new meta-law. This conjecture explains why there are so few 'laws of physics' because 'laws of physics' are represented by unique parallelo-grams. This result reduces the 'guessing', mentioned by Feynman, and should open the path to observing new 'laws of physics'. A software code has been created to find the unique parallelograms. This meta-law, which is still a conjecture, answers the second research question with an affirmative 'yes'.

CHAPTER 0

Preface

This PhD dissertation is a summary of 46 years of reflection on a topic that was put forward for the first time by J. C. Maxwell (1871a) and solved in this PhD dissertation by presenting the 'Tables of Physics', classifying mathematically kinds of quantities in $\{0,1\} \times \mathbb{Z}^N$ in which $1 \leq N \leq 7$.

While searching for the 'Tables of Physics' we invented a method and system for encoding/decoding variables in engineering problems. The encoding/decoding method obtains a more efficient mathematical model than the models obtained using the Buckingham π theorem of dimensional analysis. This new method gives answers on how to define a dimensional measurement model in the design of experiments.

The first research question is: *Is it possible to mathematically classify kinds of quantities?*

The second research question is: *Is it possible to mathematically select which quantity equations are 'laws of physics'?*

The conceptualization, methodology, software, writing, visualization, conjectures, and results in this dissertation are the sole work of the author and they are, in the light of the present state-of-the-art of mathematical modeling, innovative to the best of the author's knowledge. The results of this dissertation have presently no restrictions in their applicability.

Before diving into the N-dimensional mathematics, we present a $N = 3$ view of the answers to the two research questions. We have compiled in Appendix B the relevant mathematical properties for an integer lattice of dimension N, where $2 \leq N \leq 7$.

0.1 Is it possible to mathematically classify kinds of quantities?

We call the $N = 3$ dimensional case the 'Length-Mass-Time(LMT)'-physics view. We start from the definition in three-dimensions of an integer lattice that we visualize as an cubic array of lattice points and shown in Figure 1.

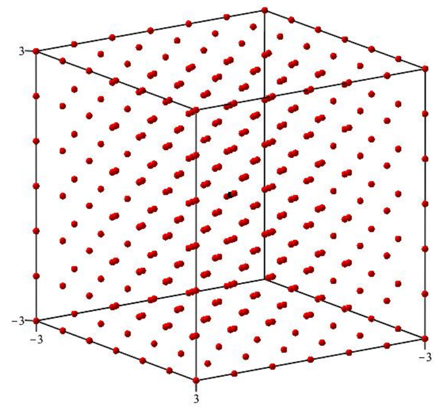

Figure 1: Cubic lattice in $\mathbb{Z}^3$ of size three.

Such an integer lattice can be partitioned in centrally symmetric cubes. The partioning is based on the infinity norm s. Figure 2 shows the partioning in three centrally symmetric cubes for $s \leq 3$.

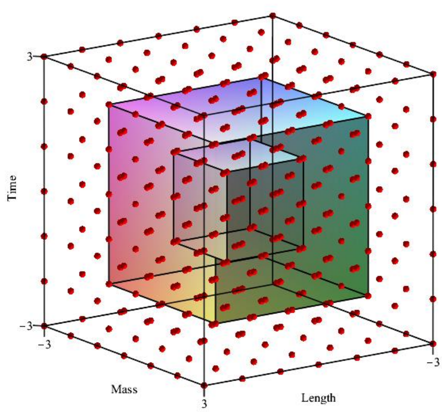

Figure 2: Cubic lattice in $\mathbb{Z}^3$ for $s \leq 3$.

We will show that a lattice can be partitioned in sets of lattice point. We call such a set an orbit. The number of elements in such a set is called the cardinality of the set. The values that the cardinalities can take are in a finite set of natural numbers. The cardinalities of the orbits $\#([w])$ of $\mathbb{Z}^3$ are 1, 6, 8, 12, 24, and 48. The number of cardinalities in an integer lattice changes as function of the dimension N of the integer lattice.

All the cardinalities of the orbits are observed when the cube in $\mathbb{Z}^3$ has its infinity norm $s \leq 3$. We will show step-by-step the filling-up of the outer surface of a cube with $s = 3$ by adding the orbits one-by-one. We visualize the orbits of lattice points with an infinity norm $s = 3$ in $\mathbb{Z}^3$. The color codes are:

- Aquamarine: $\#([w]) = 6$;
- Black: $\#([w]) = 8$;
- Blue: $\#([w]) = 12$;
- Brown: $\#([w]) = 24$;
- Coral: $\#([w]) = 24$;
- Cyan: $\#([w]) = 24$;
- Gold: $\#([w]) = 24$;
- Grey: $\#([w]) = 24$;
- Green: $\#([w]) = 24$; and,
- Maroon: $\#([w]) = 48$.

We add to an empty cube the orbit $[(0 \mid 3, 0, 0)]$, with color code aquamarine, having cardinality six and $\|z\|_2^2 = 9$ resulting in Figure 3.

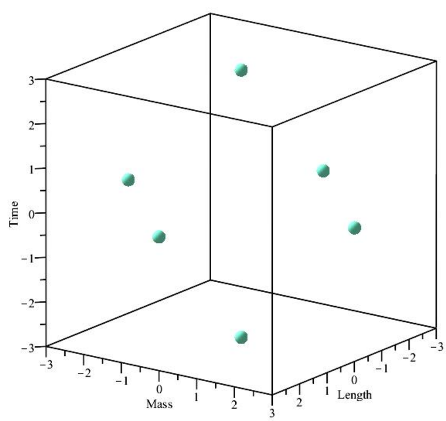

Figure 3: Addition of an orbit with cardinality 6 in $\mathbb{Z}^3$ for $s = 3$.

We add an orbit $[(0 \mid 3,3,3)]$, with color code black, having cardinality eight and $\|z\|_2^2 = 27$ resulting in Figure 4.

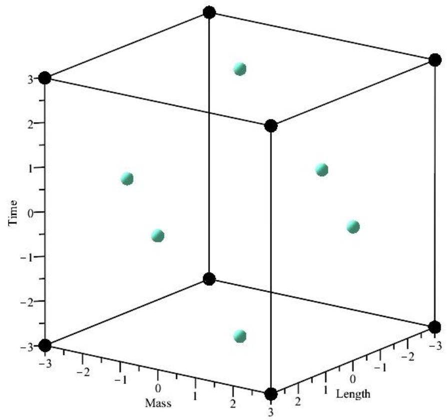

Figure 4: Addition of an orbit with cardinality 8 in $\mathbb{Z}^3$ for $s = 3$.

We add an orbit $[(0 \mid 3,3,0)]$, with color code blue, having cardinality twelve and $\|z\|_2^2 = 18$ resulting in Figure 5.

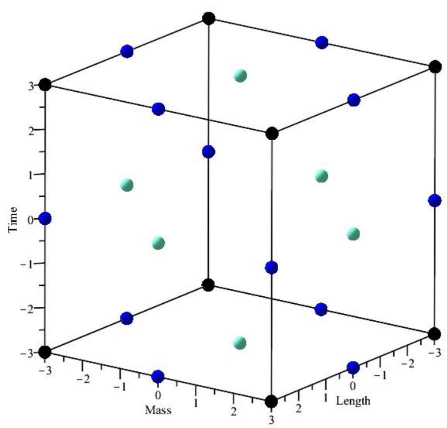

Figure 5: Addition of an orbit with cardinality 12 in $\mathbb{Z}^3$ for $s = 3$.

We add an orbit $[(0 \mid 3,2,0)]$, with color code brown, having cardinality 24, and $\|z\|_2^2 = 13$ resulting in Figure 6.

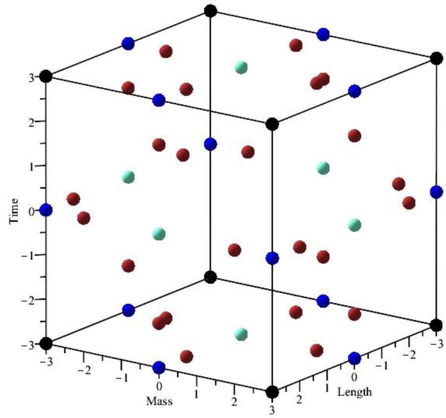

Figure 6: Addition of an orbit with cardinality 24 in $\mathbb{Z}^3$ for $s = 3$.

We add another orbit $[(0 \mid 3,1,0)]$, with color code coral, having cardinality 24, and $\|z\|_2^2 = 10$ resulting in Figure 7.

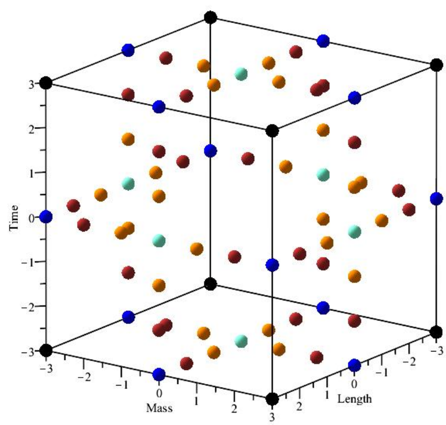

Figure 7: Addition of an orbit with cardinality 24 in $\mathbb{Z}^3$ for $s = 3$.

We add another orbit $[(0 \mid 3,1,1)]$, with color code cyan, having cardinality 24, and $\|z\|_2^2 = 11$ resulting in Figure 8.

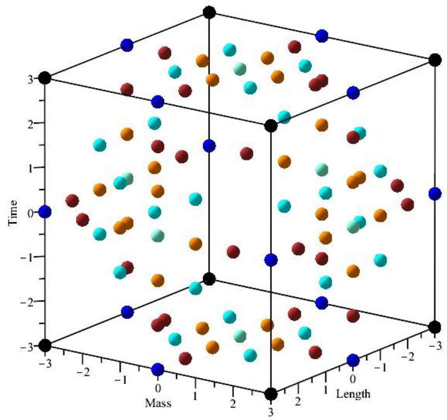

Figure 8: Addition of an orbit with cardinality 24 in $\mathbb{Z}^3$ for $s = 3$.

We add another orbit $[(0 \mid 3,2,2)]$, with color code gold, having cardinality 24, and $\|z\|_2^2 = 17$ resulting in Figure 9.

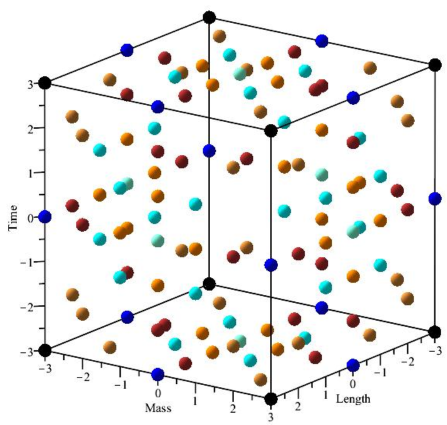

Figure 9: Addition of an orbit with cardinality 24 in $\mathbb{Z}^3$ for $s = 3$.

We add another orbit $[(0 \mid 3,3,1)]$, with color code grey, having cardinality 24, and $\|z\|_2^2 = 19$ resulting in Figure 10.

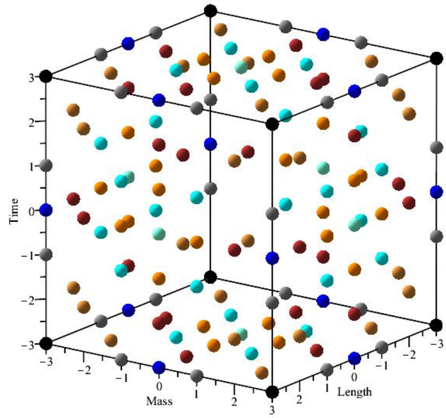

Figure 10: Addition of an orbit with cardinality 24 in $\mathbb{Z}^3$ for $s = 3$.

We add another orbit $[(0 \mid 3,3,2)]$, with color code green, having cardinality 24, and $\|z\|_2^2 = 22$ resulting in Figure 11.

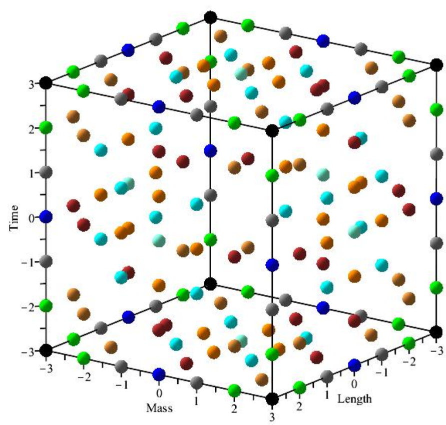

Figure 11: Addition of an orbit with cardinality 24 in $\mathbb{Z}^3$ for $s = 3$.

We add an orbit $[(0 \mid 3,2,1)]$, with color code maroon, having cardinality 48, and $\|z\|_2^2 = 14$ resulting in Figure 12.

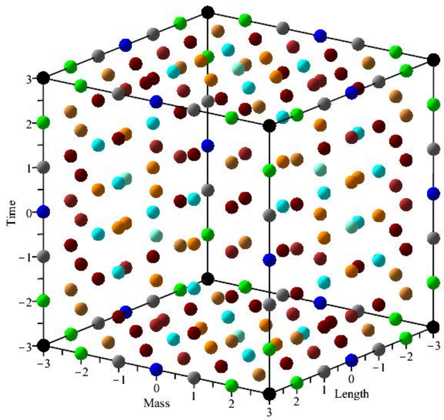

Figure 12: Addition of an orbit with cardinality 48 in $\mathbb{Z}^3$ for $s = 3$.

Figure 12 fills completely the array of lattice points at the surface of the cube with $s = 3$.

We can create a 'Table of LMT physics' in which the rows are representing the infinity norm $\|z\|_\infty = s$ and the columns are the cardinalities $\#([w])$ of the orbits of $\mathbb{Z}^3$. We write in each cell of the Table 1 the number of orbits that have the cell coordinates $(s, \#([w]))$ and call this the mass m of the element. We call this cell an element of physics in the $\mathbb{Z}^3$ framework.

Hence, each element of physics is a set of orbits of $\mathbb{Z}^3$ and each orbit is a set of integer lattice points of $\mathbb{Z}^3$. Thus an element of physics is a set of sets of integer lattice points of $\mathbb{Z}^3$. These masses are given in each cell of the Table 1.

We have limited the number of rows in the Table 1 to infinity norms $s \leq 10$. The row where $\|z\|_\infty = 3$ summarizes the information visualized in Figure 12. It is the union of ten orbits and this is expressed in the column called *RowSum*. Each of the orbits has a distinct Euclidean norm squared $\|z\|_2^2$.

Table 1: Elements of physics in $\mathbb{Z}^3$.

$\|z\|_\infty$	1	6	8	12	24	48	RowSum
0	1	0	0	0	0	0	1
1	0	1	1	1	0	0	3
2	0	1	1	1	3	0	6
3	0	**1**	**1**	**1**	**6**	**1**	10
4	0	1	1	1	9	3	15
5	0	1	1	1	12	6	21
6	0	1	1	1	15	10	28
7	0	1	1	1	18	15	36
8	0	1	1	1	21	21	45
9	0	1	1	1	24	28	55
10	0	1	1	1	27	36	66
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Table 1 contains elements with mass $m = 1$ indicating the existence of unique orbits in $\mathbb{Z}^3$.

Table 1 is the *answer* to the research question 1 for the case $N = 3$: *Is it possible to mathematically classify kinds of quantities?* A solution for the case $N = 7$, that corresponds to the SI framework, will be discussed in the Table of SI physics.

0.2 Is it possible to mathematically select which quantity equations are 'laws of physics'?

We will prove in theorem1 that a ternary equation $Z = XY$ is equivalent to a parallelogram. We apply the theorem1 on the quantity equation representing work $\mathrm{d}E \propto F \cdot \mathrm{d}s$ and obtain Figure 13:

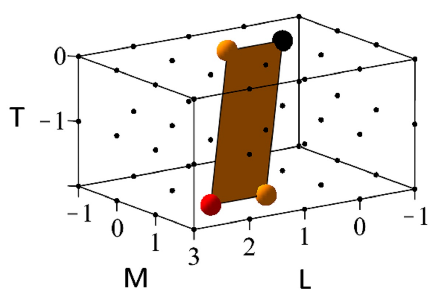

Figure 13: Addition of quantity equation $\Delta E \propto F \cdot \Delta s$ in $\mathbb{Z}^3$ for $s \leq 3$.

We apply the theorem1 on the quantity equation representing thermodynamic work $\Delta E \propto P\Delta V$ and add it to Figure 13 to obtain Figure 14:

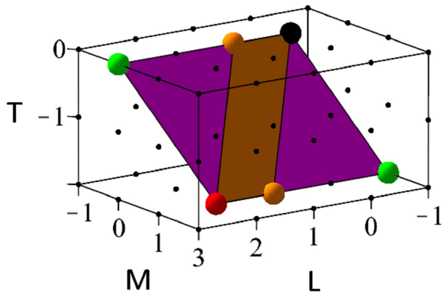

Figure 14: Addition of quantity equation $\Delta E \propto P\mathrm{d}V$ in $\mathbb{Z}^3$ for $s \leq 3$.

We apply the theorem1 on Planck's quantity equation $\Delta E \propto h\nu$ and add it to Figure 14 to obtain Figure 15:

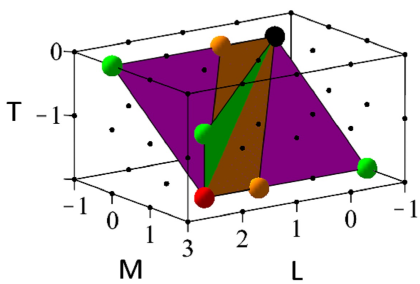

Figure 15: Addition of quantity equation $\Delta E \propto h\nu$ in $\mathbb{Z}^3$ for $s \leq 3$.

We apply the theorem1 on Einsteins quantity equation $\Delta E \propto mv^2$ and add it to Figure 15 to obtain Figure 16:

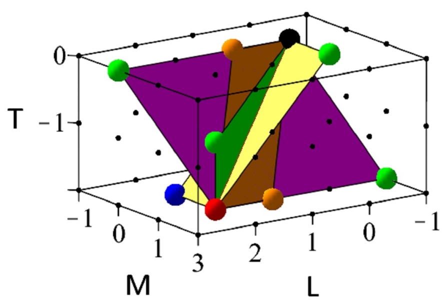

Figure 16: Addition of quantity equation $\Delta E \propto mv^2$ in $\mathbb{Z}^3$ for $s \leq 3$.

We apply the theorem1 on Maxwell's kinetic energy quantity equation

$\Delta E \propto \boldsymbol{p} \cdot \boldsymbol{v}$ and add it to Figure 16 to obtain Figure 17:

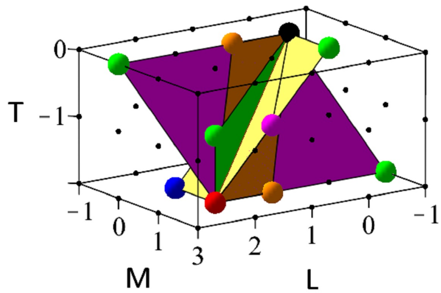

Figure 17: Addition of quantity equation $\Delta E \propto \boldsymbol{p} \cdot \boldsymbol{v}$ in $\mathbb{Z}^3$ for $s \leq 3$.

We apply the theorem1 on the kinetic energy of the Hamiltonian quantity equation $\Delta E \propto \dfrac{p^2}{m}$ and add it to Figure 17 to obtain Figure 18:

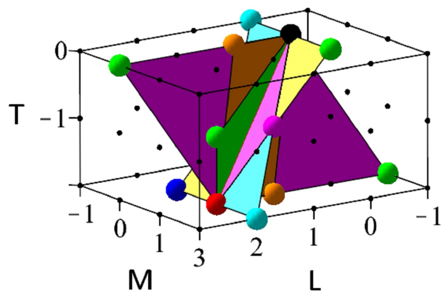

Figure 18: Addition of quantity equation $\Delta E \propto \dfrac{p^2}{m}$ in $\mathbb{Z}^3$ for $s \leq 3$.

We apply the theorem1 on the harmonic oscillator energy equation $\Delta E \propto mx^2\omega^2$ and add it to Figure 18 to obtain Figure 19:

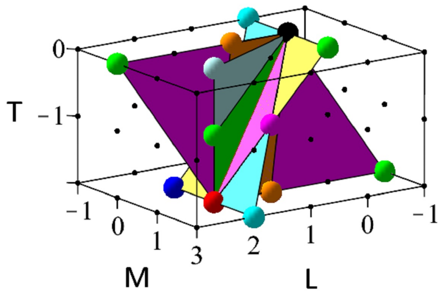

Figure 19: Addition of quantity equation $\Delta E \propto mx^2\omega^2$ in $\mathbb{Z}^3$ for $s \leq 3$.

Some of these parallelograms have a *unique* semi-perimeter. Figure 20 shows the unique parallelograms for the kind of quantity energy.

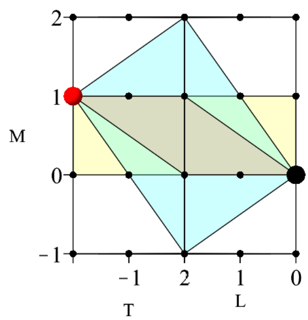

Figure 20: Unique parallelograms for the kind of quantity energy in $\mathbb{Z}^3$ for $s \leq 3$.

These unique semi-perimeters of parallelograms are corresponding to the following quantity equations of the kind of quantity energy:

$$E \propto \boldsymbol{p} \cdot \boldsymbol{v}\,,$$
$$E \propto mv^2\,,$$
$$E \propto \frac{p^2}{m}\,,$$

We recognize the product of velocity and linear impulse, kinetic energy and total energy, and the kinetic energy of the Hamiltonian.

We observed that the quantity equation containing Einstein's equation $E = mc^2$ is a special parallelogram: it is a *rectangle* with a *unique* semi-perimeter. Figure 21 shows the rectangle:

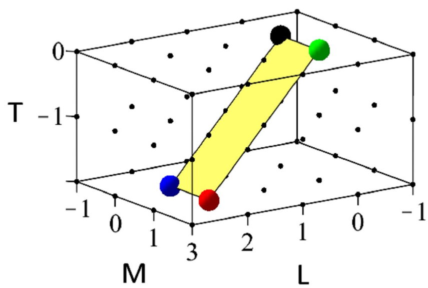

Figure 21: Unique rectangle for the kind of quantity energy in $\mathbb{Z}^3$ for $s \leq 3$.

We will formulate in Chapter 5 a conjecture that 'laws of physics' are represented by parallelograms having *unique* semi-perimeters in $\{0, 1\} \times \mathbb{Z}^N$, with the dimension N of the integer lattice properly chosen.

This conjecture is the *answer* to the research question 2: *Is it possible to mathematically select which quantity equations are 'laws of physics'?*

0.3 List of Publications

A list of peer reviewed publications related to the research topic is given subsequently.

0.3.1 Patent application

A patent application with identifier US 2023/0153487 A1 and title: MACHINE-IMPLEMENTABLE METHOD AND SYSTEM FOR ENCODING/DECODING VARIABLES IN ENGINEERING PROBLEMS and published at the USPTO on the 18th of May 2023.

0.3.2 Publication in an international journal

An A1 publication of part of the research of this dissertation was done in the Journal of the Franklin Institute with title: Method for encoding and decoding variables in engineering problems `https://www.sciencedirect.com/science/article/pii/S0016003222006792`.

0.3.3 Publication in the Online Encyclopedia of Integer Sequences (OEIS)

Comments have been given on the sequences: A000579, A001477, A002412, A008586, A045943, A102860, A115067, A128891, A128892, and A154286. New sequences have been added to the OEIS: A240934, A247557, A266387, A266395, A266396, A266397, A266398, and A270950.

CHAPTER 1

Introduction

The goal of this chapter is to elaborate on the terminology that will be used in this dissertation to create a common ground of understanding of the concepts. Our guideline is the International Vocabulary of Metrology VIM3 (BIPM et al., 2012).

We will indicate where discussions exist among scientists about the internationally accepted vocabulary. A new VIM is under preparation but has not yet been accepted by the international community and thus we use the VIM3 as baseline. We also recall in this chapter the axioms of quantity calculus and recall the state-of-the-art dimensional analysis methods: Classical Dimensional Analysis (CDA) and Modern Dimensional Analysis (MDA).

The definition of **quantity** according to the VIM3 is, and we quote (BIPM et al., 2012):

> **quantity**
> property of a phenomenon, body, or substance, where the property
> has a magnitude that can be expressed as a number and a reference.

Example 1.0.1 (Length)**.** *The generic concept length can be considered as a set of individual quantities. A non-extensive list of these individual quantities is: height, breadth, width, thickness, distance, radius, diameter, path length, persistence length, length of arc, Planck length, wavelength, Compton wavelength, relaxation length, luminous distance, proper length ...*

Example 1.0.2 (Energy)**.** *The generic concept energy can be considered as a set of individual quantities. A non-extensive list of these individual quantities is: kinetic energy, potential energy, Lagrange function, Hamilton function, elastic energy, electrostatic energy, internal energy, electrical energy, total energy, radiant energy, work, Gibbs free energy, Fermi energy, Hartree energy, Planck energy, dissociation energy, chemical potential, ionization energy ...*

In the examples about the concepts length and energy, we show that a list of individual quantities can be associated to the concept. The VIM3 expresses this in the terminology of kind of quantity, and we quote:

> **kind of quantity**
> kind
> aspect common to mutually comparable quantities

This concept of a *kind of quantity* is also valid in general relativity, given in Appendix S, and in quantum mechanics, given in Appendix T. The next step in the study is to understand the system of quantities, and we quote (BIPM et al., 2012):

> **system of quantities**
> set of quantities together with a set of non-contradictory equations relating those quantities

Example 1.0.3 (System of quantities)**.** *A set of quantities and a set of non-contradictory equations in the framework of the special relativity theory are the set of quantities consisting of total energy E, the mass m, the velocity vector $\boldsymbol{v}$, the speed of light in vacuum c_0 and the linear momentum vector $\boldsymbol{p}$ of a free body and the set of non-contradictory equations:*

$$1 = \frac{m^2 c_0^4}{E^2} + \frac{c_0^2(\boldsymbol{p}\cdot\boldsymbol{p})}{E^2}, \tag{1.1}$$

$$\boldsymbol{p} = \frac{E}{c_0^2}\boldsymbol{v}\,. \tag{1.2}$$

We adopt the convention that the concept quantity is divided in base quantities and derived quantities.

A subset of the system of quantities is called set of base quantities, and we quote (BIPM et al., 2012):

> **base quantity**
> **quantity** in a conventionally chosen subset of a given **system of quantities**, where no subset quantity can be expressed in terms of the others

We denote the set of base quantities $\mathcal{B}$. The other subset contains the derived quantities, and we quote (BIPM et al., 2012):

> **derived quantity**
> **quantity**, in a **system of quantities**, defined in terms of the **base quantities** of that system

A common system of quantities that is adopted by convention is the International System of Quantities (ISQ) (BIPM et al., 2012), and we quote:

> **International System of Quantities**
> **ISQ**
> **system of quantities** based on the seven **base quantities**: length, mass, time, electric current, thermodynamic temperature, amount of substance, and luminous intensity

The International System of Quantities is isomorphic to a seven-dimensional vector space in which each base quantity is mapped to a basis vector of the vector space.

An important concept is what is called quantity dimension, and we quote (BIPM et al., 2012):

> **quantity dimension**
> dimension of a quantity
> dimension
> expression of the dependence of a **quantity** on the **base quantities** of a **system of quantities** as a product of powers of factors corresponding to the base quantities, omitting any numerical factor

The factors are the symbols of the base quantities of the chosen system of quantities. The symbols representing the dimensions of the base quantities in the ISQ are found in Table 1.1: The definitions of the SI units are given in Appendix Q.

Table 1.1: Symbols representing the dimensions of the base quantities in the ISQ. Adapted from (JCGM, 2012).

Symbols for SI dimensions and SI units			
Base quantity Name	ISQ Symbol	Unit Name	Unit Symbol
time	T	second	s
length	L	metre	m
mass	M	kilogram	kg
electric current	I	ampere	A
thermodynamic temperature	Θ	kelvin	K
amount of substance	N	mole	mol
luminous intensity	J	candela	cd

Some products of SI units have received special names. A list of such special names is given in Appendix R. The seven ISQ symbols form an alphabet. Observe that the dimensions of the base quantities have a unique symbol that is written in Roman font. Any quantity can be considered as a word formed by a combination of the seven ISQ symbols and by inference we find the dimension of a quantity Q. In the ISQ we denote the dimension of a quantity Q by the monomial:

$$\dim Q = \mathsf{T}^{\alpha}\mathsf{L}^{\beta}\mathsf{M}^{\gamma}\mathsf{I}^{\delta}\Theta^{\epsilon}\mathsf{N}^{\zeta}\mathsf{J}^{\eta} \,. \tag{1.3}$$

We made a compilation of physical quantities that can be found in scientific literature (IUPAP, 1978), (Cohen, Giacomo, et al., 1987), (Cohen et al., 2007), (Szirtes, 2007, p.57-59), (Feynman, 1977a; R. P. Feynman, Leighton, & Sands, 1971, 1966),(Butcher & Cotter, 1998, p.26-27) and, (List of Physical quantities, 2024) . The compilation is given in Appendix A. The compilation resulted in 139 kinds of quantities which take SI dimensional exponent values in the following sets of integers:

- $\alpha \in \{-12, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 6, 7, 10\}$;
- $\beta \in \{-4, -3, -2, -1, 0, 1, 2, 3, 4, 6\}$;
- $\gamma \in \{-3, -2, -1, 0, 1, 2, 3, 4\}$;
- $\delta \in \{-4, -2, -1, 0, 1, 2, 3, 4\}$;
- $\epsilon \in \{-4, -1, 0, 1\}$;
- $\zeta \in \{-1, 0, 1\}$;
- $\eta \in \{0, 1\}$.

Observe that the SI dimensional exponents are small integers.

The dimension of a quantity Q is denoted by $\dim Q$.

Example 1.0.4 (dimension)**.** *The quantity dimension of energy E is denoted by* $\dim E = \mathsf{T}^{-2}\mathsf{L}^2\mathsf{M}$. *The quantity dimension of moment of force M is denoted by* $\dim M = \mathsf{T}^{-2}\mathsf{L}^2\mathsf{M}$.

Observe that energy and moment of force have the same quantity dimension but it should be obvious that these physical quantities are not of the same kind. The quantity dimension is defined independently of the type of quantity (e.g. scalar, vector, tensor,. . .). The VIM3 definition of kind of quantity refers to comparable quantities without defining comparable quantities. Hence, quantities having the same quantity dimension are not necessarily of the same kind and this results in classification problems of kinds of quantities. A new VIM version is under review and we hope it will address the above issues.

We redefine *kind of quantity* as:

Definition 1.0.1. *kind of quantity is a set of physical quantities having the same quantity dimension expressed using the base quantities in the ISQ and the type of the quantity, called tensorness.*

Definition 1.0.2. *comparable quantities are subsets of a kind of quantity having identical mathematical properties in the isomorphic structure of the ISQ.*

We define a dimensionless quantity, and we quote (BIPM et al., 2012):

> **quantity of dimension one**
> dimensionless quantity
> **quantity** for which all the exponents of the factors corresponding
> to the **base quantities** in its **quantity dimension** are zero.

We adopt in this dissertation the term *dimensionless quantity* instead of quantity of dimension one. These dimensionless quantities occur in the celebrated Buckingham π theorem (Buckingham, 1914) also known as the π theorem. Dimensionless kinds of quantities have the property of being scale-invariant.

Some dimensionless kinds of quantities are derived from fundamental physical constants as those given in Appendix K. A well-known dimensionless quantity based on fundamental physical constants is the fine-structure constant α expressed in the International System of Units (SI):

$$\alpha = \frac{\mu_0 c e^2}{2h},$$

where e is the elementary charge, h is Planck's constant, μ_0 is the permeability of vacuum, and c is the speed of light in vacuum.

We recall the trialogue between Michael J. Duff, Lev B. Okun and Gabriele Veneziano and quote their common agreement in Duff, Okun, and Veneziano (2002) about dimensionless quantities:
Lev B. Okun states, and we quote:

> Theoretical equations describing the physical world deal with dimensionless quantities and their solutions depend on dimensionless fundamental parameters.

Gabriele Veneziano states, and we quote:

> Physics is always dealing, in the end, with dimensionless quantities, typically representing ratios of quantities having the same dimensions, e.g.
>
> $$\alpha = \frac{e^2}{\hbar c}, \qquad \frac{m_e}{m_p}, \ldots$$

where m_e is the rest mass of an electron and m_p is the rest mass of a proton.
Michael J. Duff states, and we quote:

> Weinberg [1] defines constants to be fundamental if we cannot calculate their values in terms of more fundamental constants, not just because the calculation is too hard, but because we do not know of anything more fundamental. This definition is fine, but does not resolve the dispute between Gabriel, Lev and me. It is the purpose of this section to propose one that does. I will conclude that, according to this definition, the dimensionless parameters, such as the fine structure constant, are fundamental, whereas all dimensionful constants, including $\hbar$, c and G, are not.

The VIM3 defines a system of units, and we quote (BIPM et al., 2012):

> **system of units**
> set of **base units** and **derived units**, together with their multiples and submultiples, defined in accordance with given rules, for a given **system of quantities**.

The history of the system of units is given in the review of (de Boer, 1995).

In high-energy physics the physicists choose a system of units such that the unit of energy is MeV. They choose the time unit $t_u = 6.582449 \times 10^{-22}\,\mathrm{s}$ and the length unit $l_u = 1.97327 \times 10^{-13}\,\mathrm{m}$ such that the symbols c_0 and $\hbar$ disappear from their equations, resulting in more symmetric mathematical equations. In this way, light will travel 1 length unit in 1 time unit and the reduced Planck constant is equal to 1 MeV times the time unit (Veltman, 2003). Typical spacetime (s, t) graphs using those units of length and time have straight lines inclined at 45 degree showing the propagation of light in vacuum.

Another approach is to use the system of atomic units in quantum mechanical calculations (Teo & Li, 2011). In that system we have $m_e(au) = e(au) = \hbar(au) = a_0(au) = E_h(au) = 1\,au$, in which E_h denotes the Hartree energy and a_0 denotes the Bohr radius of the hydrogen atom. The atomic unit system can be considered as a system of 1 unit that is the au. All the physical quantities are derived, in that atomic unit system, from four fundamental constants, namely m_e the electron rest mass, e the charge of the electron, $\hbar$ the reduced Planck constant and ϵ_0 the vacuum permittivity. The set $\{m_e, e, \hbar, \epsilon_0\}$ could be replaced by the set $\{m_e, e, \hbar, a_0\}$ or the set $\{m_e, e, \hbar, E_h\}$ or the set $\{m_e, E_h, \hbar, a_0\}$ or another independent set. The atomic unit system has a time unit $t_u = \dfrac{\hbar}{E_h} = 2.41888 \times 10^{-17}\,\mathrm{s}$. The atomic unit length scale is the Bohr radius of the hydrogen atom $a_0 = 5.29177 \times 10^{-11}\,\mathrm{s}$ and thus we find as unit of velocity in the atomic unit system $v(au) = \dfrac{a_0(au)}{t(au)} = 2.18769 \times 10^{6}\,\mathrm{ms}^{-1}$. Observe that the atomic time unit is different from the time unit used in high-energy physics in which $\hbar = 1$ and $c_0 = 1$. Here, in the atomic unit system the speed of light has a value $c_0 = 137.036$.

Since the publication of ‘Remarks on the Mathematical Classification of Physical Quantities’ by Maxwell (J. C. Maxwell, 1871b), many discussions on the units and dimensions of quantities have taken place. We refer to the discussions in Hering (1910, 1911), Page (1975), Siano (1985a, 1985b), de Boer (1995), Duff et al. (2002), Aragon (2004), Wilczek (2005, 2006b, 2006a), Uzan, Clarkson, and Ellis (2008),Uzan (2015), Foster (2010), and Isaev, Kononogov, and Khruschov (2013).

The VIM3 defines the International System of Units (SI), and we quote (BIPM et al., 2012):

> **International System of Units**
> **SI**
> **system of units**, based on the **International System of Quantities**, their names and symbols, including a series of prefixes and their names and symbols, together with rules for their use, adopted by the General Conference on Weights and Measures (CGPM).

The definitions of the SI2019 base units are given in Appendix Q.

The subject of this dissertation is the mathematical classification of kinds of quantities and quantity equations.

A quantity equation is, and we quote (BIPM et al., 2012):

> **quantity equation**
> mathematical relation between quantities in a given system of quantities, independent of measurement units.

Observe that quantity equations are independent of the choice of measurement units.

Example 1.0.5 (quantity equation)**.** $z = f(\pi)xy$ *in which* x, y, z *denote different kinds of quantities and* $f(\pi)$ *is a function of a dimensionless quantity* π*.*

The VIM3 gives the following examples of quantity equations, and we quote (BIPM et al., 2012):

> EXAMPLE 1 $Q_1 = \zeta Q_2 Q_3$ where Q_1, Q_2 and Q_3 denote different quantities, and where ζ is a numerical factor.
> EXAMPLE 2 $T = (1/2)mv^2$ where T is the kinetic energy and v the speed of a specified particle of mass m.
> EXAMPLE 3 $n = It/F$ where n is the amount of substance of a univalent component, I is the electric current and t the duration of the electrolysis, and where F is the Faraday constant.

The VIM3 defines quantity calculus, and we quote (BIPM et al., 2012):

> **quantity calculus**
> set of mathematical rules and operations applied to **quantities** other than **ordinal quantities**.

Dimensional analysis is the study of the relationships between kinds of quantities. These kinds of quantities are expressed as being derived from base quantities. Different coherent systems of base quantities can be created. The international convention for engineering and science resulted in the SI (BIPM, 2019) that uses seven base quantities. The base quantities are time, length, mass, electric current, thermodynamic temperature, amount of substance and luminous intensity.

Observe that the *order of base quantities was changed* in 2019. We denote it as SI2019. The previous order was: length, mass, time, electric current, thermodynamic temperature, amount of substance and luminous intensity. The *dimension concept* and the construction of equations of physics has been reviewed by Tatjana Ehrenfest-Afanassjewa (Ehrenfest-Afanassjewa, 1916). Whitney (1968a) formalized mathematically dimensional analysis. Kasprzak, Lysik, and Rybaczuk (1990) consider dimensional analysis as a problem of identification of mathematical models (Kasprzak et al., 1990, p.76). Tao (2012) considers dimensional analysis as being the representation theory of groups.

Kinds of quantities are used in the daily work of millions of scientists all over the world. Most scientists are not aware of the mathematical properties of those kinds of quantities and what a priori can be derived from these mathematical properties to guide the physicist or engineer in the design of experiments (DoE). Unfortunately, the information from the BIPM (BIPM, 2006) shows no research activity toward the mathematical classification of kinds of quantities. Kinds of quantities occur in the scientific literature in the form of tensors (scalars, vectors, matrices. . .) and pseudo-tensors. All the kinds of quantities are eventually defined through their respective *components* and thus we restrict our study to the components of kinds of quantities.

J. C. Maxwell (1871b) referred to the kind of quantity energy E through the dimension $\frac{\mathsf{ML}^2}{\mathsf{T}^2}$ in a LMT language of physics with base quantities length l, mass m, and time t. Maxwell in collaboration with Jenkin described in 1863 electrical and magnetic phenomena from a dimensional point of view in his article 'On the Elementary Relations of Electrical Quantities' in which he introduced the dimensional symbols $\mathsf{L}, \mathsf{M}, \mathsf{T}$ (Peruzzi, 2009; Rivadulla, 2017, p.89). Maxwell called vectors which refer to unit of length 'Forces' and those which refer to unit of area 'Fluxes'. It is clear from his article that he thought of physics in terms of the dimensions: length L, mass M, and time T. His comments on this topic have been forgotten by the physics and mathematics communities.

Kinds of quantities are still today classified according to their engineering discipline. For example, Sykora published in 2006 such a lexicon (Sykora, 2006). But as early as 1910 Hering organized the kinds of quantities in increasing order of the exponent of the base quantities (Hering, 1910). His goals were the relations of quantities, the search for new quantities and the search for new fundamental systems.

Our research has some similarities with that of Hering's but differs in the classification methods and results. Hering arranged the quantity equations in groups. All the quantities containing length formed the first group. Within this group, the quantities were arranged starting with the smallest exponent of L. He used the following base quantities: length l, mass m, time t, magnetic permeability μ, electric inductive capacity κ, temperature Θ and plane angle a in which μ, and κ are suppressed factors with unknown dimensions. He considered μ, κ, Θ, and a as *auxiliary fundamental quantities*. Hence, he solved the absurdity of 'moment of force = energy' by allocating to energy the dimensions $\mathsf{L}^2\mathsf{M}^1\mathsf{T}^{-2}$ and to moment of force the dimensions $\mathsf{L}^2\mathsf{M}^1\mathsf{T}^{-2}a^{-1}$.

Other mathematical classification models have been created in Tonti (2003, 2013) and in Whitney (1968a, 1968b), but they all differ from our approach. Unlike Hering, we solve the problem of 'moment of force = energy' by considering energy as a component of a *tensor* and considering moment of force as a component of a *pseudo-tensor*.

Unanswered questions of dimensional analysis, according to G. I. Barenblatt are, and we quote (Roche, 1998, p.213):

> Which cluster of related quantities fixed the value of the quantity being investigated, and did not under or over-determine it? Which variables, material constants and universal constants should appear multiplied together in a trial law? Which quantities should be excluded? This knowledge had to be derived from familiarity, inspectional analysis, experiment, deductive exploration from first principles or from a combination of all of these.

Dimensional analysis assumes the following:

- identification of the dependent variable;
- identification of the set of independent variables; and,
- completeness of the set of variables.

To the best of our knowledge there is no conclusive method to verify the correctness of the assumptions in dimensional analysis. In many cases the success of dimensional analysis relies on the skills and know-how of the scientist that creates the model of the system under investigation. The dimensional independent variables can be found using matrix algebra by calculating the rank of the dimensional matrix.

Robust dimensional analysis methods using N-optimal designs are presented in Albrecht, Albrecht, Nachtsheim, and Cook (2013) but they also conclude that informed decisions are needed based on intuition and physical understanding of the system.

Pexton (2014) claims that dimensional analysis has explanatory capabilities and refers for that claim to the use of dimensional analysis in astrophysics. Dimensional analysis applied to astrophysics was exemplified in (Kurth, 1972).

1.1 Axioms of quantity calculus

In this section we consider the set of mathematical rules and operations applied to kinds of quantities other than ordinal quantities. The foundations of quantity calculus were given by Wallot (Wallot, 1922b, 1922a, 1957). According to Wallot, the kinds of quantities are the primary concepts (de Boer, 1995).

An algebraic structure for quantity calculus was proposed for the first time in 1951 by Fleischmann (1951) as reported by de Boer (1995). Fleischmann also introduced the idea of 'Verknüpfungsgleichung'(Fleischmann, 1951, 1954) that we translate as 'quantity equation'. Further developments of quantity calculus were discussed by others (Alten & Quade, 1971; Landolt, 1943). We posit from the 9th edition of the SI (BIPM, 2019) a set of axioms derived from promoting some of the SI conventions to mathematical axioms in a similar way as was proposed by de Boer (1995).

Axiom 1.1.1. *The base quantities are time, length, mass, electric current, thermodynamic temperature, amount of substance and luminous intensity.*

Axiom 1.1.2. *The base quantities are independent.*

Axiom 1.1.3. *The quantities are organized according to a system of dimensions.*

Axiom 1.1.4. *For each base quantity of the SI, there exists one and only one dimension.*

Axiom 1.1.5. *The product of two quantities is the product of their numerical values and units.*

Axiom 1.1.6. *The quotient of two quantities is the quotient of their numerical values and units.*

1.2 Classical Dimensional Analysis (CDA)

Some research problems have unknown sets of equations, and in those problems, one relies on the Classical Dimensional Analysis (CDA) method, with a large set of examples in diverse fields of engineering and physics, as well as in other sciences.

Classical Dimensional Analysis (CDA) transforms the dimensional matrix by a row reduction process to obtain Buckingham or B-numbers, products of kinds of quantities that result in a dimensionless quantity, with some further optional optimization schemes (Langhaar, 1946; Decius, 1948; Langhaar, 1951; W. W. Happ, 1967; Sloan & Happ, 1969; Deb & Deb, 1986). Staicu (1971) introduces the concept of progressive homogeneity in dimensional analysis. Huntley (1958) introduces directed dimensions by considering distinct dimensions of length in the x, y and z directions to express vector properties of a physical quantity. Araneda (1996) introduces quaternions to tackle orientational data in physical quantities.

J.J. Roche gives a critical history of dimensional analysis and the quantity calculus (Roche, 1998, pp.188-207). Other historico-critical reviews of dimensional analysis exist (Ravetz, 1961; Macagno, 1971; Martins, 1981; Rivadulla, 2017; Clark, 2017).

The cornerstone of dimensional analysis is the Buckingham (1914) π theorem, in which dimensionless products of quantities are formed from the set of variables that the researcher has identified.

Theorem 1.2.1 (Buckingham π theorem)**.** *Any physically meaningful relation* $\Phi(R_1, \ldots, R_n) = 0$*, with* $R_j \neq 0$*, is equivalent to a relation of the form* $\Psi(\pi_1, \ldots, \pi_{n-r}) = 0$ *involving a maximal set of independent dimensionless combinations.*

Application of the theorem results in a reduction of the number M of dimensional variables Q^m, in which m is a superscript index, in N dimensional units to a set of $(M - r)$ dimensionless products in which r is the rank of the $N \times M$ dimensional matrix. Hence, the researcher starts with a formal description of the measurement model (JCGM, 2012) $F(Q^1, \ldots, Q^M) = 0$ and obtains a formal description of lower dimension and thus reducing the computational load and the cost of the experimental setup, through the dimensionless measurement model $f(\pi_1, \ldots, \pi_{(M-r)}) = 0$ in which π_i are dimensionless products with $i \in \{1, \ldots, (M - r)\}$.

Many famous physicists have used the technique of dimensional analysis as tool to find 'their' equation. Among these physicists we find names as Fourier, Maxwell, Rayleigh, Reynolds, Planck, Einstein and Bohr. Rayleigh (1899) applied the dimensional analysis on the problem of the effect of temperature on the viscosity of a gas. Einstein (1911) applied dimensional analysis on the problem of molecular vibrations in solid state physics.

Several researchers have studied quantity calculus, the mathematics of physical quantities (Bridgman, 1922; Whitney, 1968b, 1968a; Barr, 1971; Moran, 1971; Sharp, 1975; Szekeres, 1978; Siano, 1985a; Deb & Deb, 1986; Araneda, 1996; Sonin, 2004; Raposo, 2018, 2019; Meinsma, 2019).

Quantity calculus exhibits group properties (Fleischmann, 1954; Fleury & de Boer, 1962; kai Chen, 1971; Hainzl, 1971; de Boer, 1995; Emerson, 2008).

Graph theoretical approaches to dimensional analysis have also been explored (W. Happ, 1971).

For real world problems engineers use the Système International SI (BIPM, 2019) unit system in which the number of units is given by $N = 7$. Most theoretical physicists use a time, length, and mass unit system in which $N = 3$ but there is not really a consensus (Duff et al., 2002). Other sciences (Coyle & Ballico-Lay, 1984; Fröhlich, 2010) (e.g. econometrics…) use different unit systems.

In all problem descriptions the researchers start with a finite set of base quantities. Those researchers agree by convention to associate to each base quantity a base unit (Mari, Ehrlich, & Pendrill, 2018).

The convention of the Système International (SI) (BIPM, 2019) recommends researchers to use seven base quantities: time, length, mass, electric current, thermodynamic temperature, amount of substance and luminous intensity.

1.2.1 Rational or integer exponents?

We found that some researchers are defining physical variables with rational exponents e.g. noise-equivalent power having dimension $\mathrm{W}/\sqrt{\mathrm{Hz}}$. We discourage the usage of rational exponents for physical variables based on the axioms of quantity calculus (Fleischmann, 1951, 1954; de Boer, 1995) that is related to the existence of a unimodular matrix $\mathbf{U}$ with $\det \mathbf{U} = \pm 1$ connecting congruent lattices such that the two generator matrices $\mathbf{M}_1$ and $\mathbf{M}_2$, in which

$\mathbf{M_2} = \mathbf{M_1} \cdot \mathbf{U}$, generate the same lattice (J. Conway, Sloane, & Bannai, 1999, p.10), (Micciancio & Goldwasser, 2002, p.4).

Consider the base quantities: $length$, $mass$, and $time$ with generator matrix $\mathbf{M_1}$:

$$\mathbf{M_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{1.4}$$

The determinant of the lattice L_1 is the determinant of the Gram matrix $\mathbf{A_1} = \mathbf{M_1} \cdot \mathbf{M_1}^{tr}$

$$\mathbf{A_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \tag{1.5}$$

and thus $\det L_1 = \det \mathbf{A_1} = 1$.

Consider a new basis: $length$, $mass$, and Noise Equivalent Power (NEP), expressed in watt per square root hertz, having dimension $(2, 1, -5/2)$ with generator matrix $\mathbf{M_2}$:

$$\mathbf{M_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & -5/2 \end{bmatrix} . \tag{1.6}$$

The determinant of the lattice L_2 is the determinant of the Gram matrix $\mathbf{A_2} = \mathbf{M_2} \cdot \mathbf{M_2}^{tr}$

$$\mathbf{A_2} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & -45/4 \end{bmatrix} , \tag{1.7}$$

with $\det L_2 = \det \mathbf{A_2} = 25/4$. Two generator matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ define equivalent lattices if and only if they are related by (J. Conway et al., 1999, p.10):

$$\mathbf{M_2} = c\mathbf{U}\mathbf{M}_1\mathbf{B} , \tag{1.8}$$

in which c is a nonzero constant, $\mathbf{U}$ is a matrix with integer entries and determinant ± 1, and $\mathbf{B}$ is a real orthogonal matrix. The corresponding Gram matrices are related by (J. Conway et al., 1999, p.10):

$$\mathbf{A_2} = c^2 \mathbf{U}\mathbf{A}_1\mathbf{U}^{tr} . \tag{1.9}$$

If $\mathbf{U}$ has determinant ± 1 and $c = 1$ then $\mathbf{M_1}$ and $\mathbf{M_2}$ are *congruent* lattices. They are *directly congruent* if $\det \mathbf{U} = +1$ (J. Conway et al., 1999, p.10).

The lattices L_1 and L_2 are not equivalent and not congruent. The quantity NEP is an element of the set $\mathbb{Q}^3$ but the quantity NEP^2 is an element of $\mathbb{Z}^3$. If we consider the basis: $length$, $mass$, NEP^2 then the lattice L_3 is an equivalent lattice but not a congruent lattice because the scale factor is $c = -5$ being the determinant of the L_3 lattice. Hence, rational exponents for the dimensions generate interpretation problems because we should be able to write for the generator matrices $\mathbf{M_2} = \mathbf{M_1} \cdot \mathbf{U}$ and $\mathbf{M_2} \cdot \mathbf{U}^{-1} = \mathbf{M_1} \cdot \mathbf{U} \cdot \mathbf{U}^{-1} = \mathbf{M_1}$ in which $\mathbf{U}$ is a uni-modular matrix. It should be obvious from the comparison of the generator matrices $\mathbf{M_1}$ and $\mathbf{M_2}$ that the reciprocal expression of physical concepts is not possible in the case of rational exponents.

The reciprocity of expressions of physical concepts should not be broken and thus we recommend not to use rational exponents. The dual lattice or reciprocal lattice of a cubic lattice is a cubic lattice.

We encourage these researchers to exponentiate variables with rational exponents such that a kind of quantity Q is formed that has a dimension $\dim Q$ having integer exponents and to consider only physical quantities having integer exponents in the mathematical modeling of phenomena.

The restriction that the exponents of physical quantities should be integers is not restricting the modeling of physical phenomena. The coordinates in lattices are most of the time real numbers. Those lattices are subjected to rescaling to obtain general integer coordinates (J. Conway et al., 1999).

1.2.2 CDA example applied to $F(G, T, c_0, t) = 0$

We exemplify classical dimensional analysis by considering a hypothetical phenomenon that is described by $F(G, T, c_0, t) = 0$ in which G is the universal gravitational constant, T is an energy density, c_0 is the speed of light in vacuum and t is a characteristic time. We apply classical dimensional analysis (CDA) to reduce the number of variables. We create Table 1.2:

Table 1.2: Dimensional matrix for G, T, c_0, t.

	t	T	G	c_0
s	1	-2	-2	-1
m	0	-1	3	1
kg	0	1	-1	0

The 3×4 dimensional matrix has rank equal to three and thus there is one dimensionless quantity π that can be formed.
We convert the function $F(G, T, c_0, t) = 0$ to the function:

$$f(\pi) = f(\frac{t\sqrt{G \cdot T}}{c_0}) = 0\,. \tag{1.10}$$

We assume that the equation 1.10 has a solution and denote the solution as U that satisfies $f(U) = 0$. We remove the square root and postulate the equation:

$$U^2 = \frac{t^2 \cdot G \cdot T}{c_0^2}\,, \tag{1.11}$$

in which U^2 is a dimensionless constant. If we plot the range of solutions we obtain the 3D-plot given in Figure 1.1.

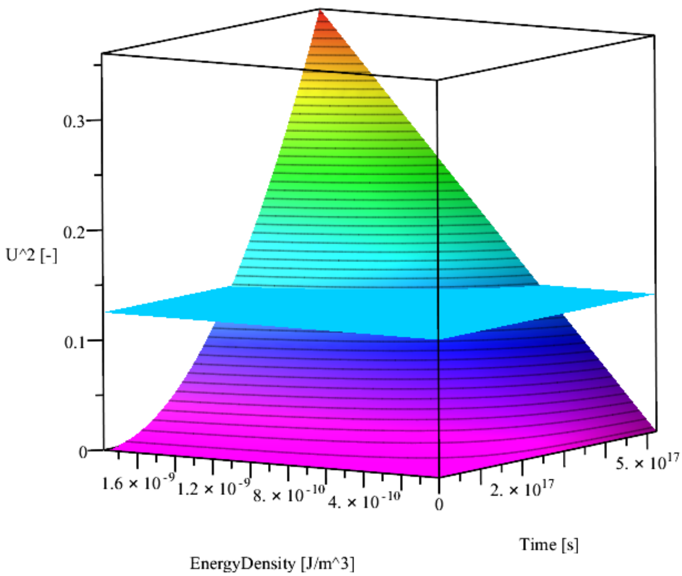

Figure 1.1: Range of values for the universal constant U^2.

The equation 1.11, defining the universal number U^2 is unknown to the author. Due to its universal character, we *speculate* that the equation can be applied to the universe. The age and energy density of the universe are respectively denoted by t_0 and T_0. Based on the NASA WMAP and the ESA PLANCK data, we have $t_0 = 4.351653 \times 10^{17}\,\mathrm{s}$ and $T_0 = 8.897676 \times 10^{-10}\,\mathrm{J} \cdot \mathrm{m}^{-3}$. By substitution of these values in the equation 1.11 we find the value for the universal constant $U^2 = 0.125127$ and $U = 0.353732$. The closest rational to $U^2 = \dfrac{14\,163}{113\,189}$.

1.3 Modern Dimensional Analysis (MDA)

Classical Dimensional Analysis (CDA) has evolved since Buckingham (1914) towards Modern Dimensional Analysis (MDA). The state-of-the-art method is the matrix method proposed by Szirtes (2007) which is called Modern Dimensional Analysis (MDA) by some researchers (Gálfi, Száva, Șova, & Vlase, 2021; Száva et al., 2022). The MDA method is exemplified in Appendix O by analyzing the modeling of the kind of quantity second order derivative of energy density with respect to time and modeling the case of a simple pendulum. Szirtes (2007, p.164) defines an array called *dimensional set* $\mathbf{DS}$ based on the sub-matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ in the following arrangement:

$$\mathbf{DS} = \begin{bmatrix} \mathbf{B} & \mathbf{A} \\ \mathbf{D} & \mathbf{C} \end{bmatrix} . \tag{1.12}$$

The research of Szirtes boils down to the matrix equation called *Fundamental Formula* (Szirtes, 2007, p.177):

$$\mathbf{C} = -\mathbf{D} \cdot (\mathbf{A}^{-1} \cdot \mathbf{B})^{\mathrm{T}} . \tag{1.13}$$

The rows in the dimensional matrix have a $[\mathsf{L}], [\mathsf{M}], [\mathsf{T}]$ order (Szirtes, 2007).

For dimensionless quantities the *Fundamental Formula* reduces to (Szirtes, 2007, p.176):

$$\mathbf{C} = -(\mathbf{A}^{-1} \cdot \mathbf{B})^{\mathrm{T}} , \tag{1.14}$$

in which $\mathbf{A}$ is a $r \times r$ matrix, $\mathbf{B}$ is an $r \times (M - r)$ matrix, $\mathbf{D} = \mathbf{I}$ is an $(M - r) \times (M - r)$ identity matrix, and $\mathbf{C}$ is an $(M - r) \times r$ matrix.

1.4 Conclusion

From the literature study of the prior art and the state-of-the-art in dimensional analysis, we conclude that there is no answer to the first research question: *Is it possible to mathematically classify kinds of quantities?* This question is unanswered since 1871. Maxwell considered this an important issue, and we quote (J. C. Maxwell, 1871b):

> The first part of the growth of a physical science consists in the discovery of a system of quantities on which its phenomena may be conceived to depend. The next stage is the discovery of the mathematical form of the relations between these quantities. After this, the science may be treated as a mathematical science, and the verification of the laws is effected by a theoretical investigation of the conditions under which certain quantities can be most accurately measured, followed by an experimental realisation of these conditions, and actual measurement of the quantities.

This dissertation answers the question of Maxwell by constructing the 'Table of SI physics' from an equivalence relation, as will be shown in Chapter 3. We published the 'Table of SI physics' in 2022 (P. Chevalier & Constales, 2022). The creation of the 'Table of SI physics' was done in Academic year 2014-2015.

Moreover, we don't find any answer in the literature to the second research question: *Is it possible to mathematically select which quantity equations are* *'laws of physics'?* Feynman described the process for discovering a new 'law of physics' as a process that starts with a guess. No method is presently known to find 'laws of physics'. The only way according to Feynman is 'guessing' and verify the guess by experiment, and we quote (R. Feynman, 1967, p.156):

> Now I am going to discuss how we would look for a new law. In general we look for a new law by the following process. First we guess it. Then we compute the consequences of the guess to see what would be implied if this laws that we guessed is right. Then we compare the result of the computation to nature, with experiment or experience, compare it directly with observation, to see if it works. If it disagrees with experiment it is wrong. In that simple statement is the key to science. It does not make any difference how beautiful your guess is. It does not make any difference how smart you are, who made the guess, or what his name is - if it disagrees with experiment it is wrong. That is all there is to it.

An article published by Dijkgraaf (2020), director and Leon Levy Professor at the Institute of Advanced Study in Princeton, caught our attention. He made the following statement, and we quote:

> Confronted with the endless number of physical systems we could fabricate out of the currently known fundamental pieces of the universe, I begin to imagine an upside-down view of physics. Instead of studying a natural phenomenon, and subsequently discovering a law of nature, one could first design a new law and then reverse engineer a system that actually displays the phenomena described by the law.

Dijkgraaf doesn't explain how to design a new law, but this dissertation gives methods and tools to find unknown 'laws of nature' or to find unknown relations between existing laws, and certainly find the known 'laws of nature' as will be shown in the next chapters. A patent application about the method has been submitted to the United States Patent and Trademark Office (USPTO) by Ghent University in 2021.

CHAPTER 2

Isomorphism between the quotient set Q/$\sim$ and the set $\{0, 1\} \times \mathbb{Z}^N$

In this chapter we formalize the mapping of a physical quantity to an integer lattice point of $\{0, 1\} \times \mathbb{Z}^N$. The complete mapping consists of the composition of two maps.

The first map groups the physical quantities in equivalence classes, denoted q_R. A projection map relates each quantity q to its equivalence class q_R. Such an equivalence class is called a *kind of quantity*. Each kind of quantity is an element of the set of kind of quantities. These kinds of quantities are the major subject of this dissertation.

To discover the mathematical relation between these kinds of quantities, we use a second map that relates one-to-one the kinds of quantities to the lattice points of an integer lattice. The integer lattice can be at first visualized as a three-dimensional grid of lattice points as shown in Figure 2.1. Each lattice point $\boldsymbol{q}$ of the integer lattice is a kind of quantity having integer coordinates $\boldsymbol{q} = (i, j, k)$ where $i, j, k \in \mathbb{Z}$.

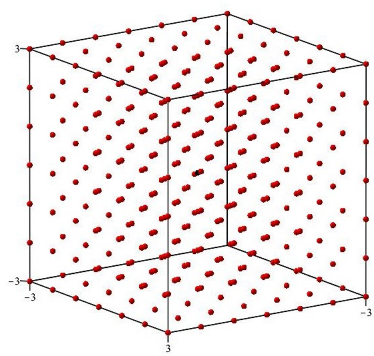

Figure 2.1: Integer lattice in $\mathbb{Z}^3$ for a truncated lattice.

Instead of working with three coordinates we have to work with seven coordinates to be compliant with the SI convention because it uses seven base quantities. Unfortunately, our brain cannot visualize this seven-dimensional integer lattice. To comply with the SI convention, we have to execute these steps in a seven-dimensional integer lattice. We will find that 30 cardinalities are needed to classify the kinds of quantities in a seven-dimensional lattice and that the number of possible transformations in $\mathbb{Z}^7$ is $2^N N! = 2^7 7! = 645\,120$. This chapter creates the mathematical framework for the process illustrated for the three-dimensional case but generalized for dimensions $1 \leq N \leq 7$.

2.1 Generalized dimensional exponent of a physical quantity

Let the set of all physical quantities be a universe denoted by Q. Physical quantities are described by tensors and pseudo-tensors and we can without loss of generality consider a component of a tensor or pseudo-tensor and denote it as q. The VIM3 (BIPM et al., 2012) states that the quantity *dimension* is defined independently of the type of quantity (e.g. scalar, vector, tensor,...).

Here we extend the system of quantities (ISQ) with a new base quantity B_0, called tensorness, related to the tensor transformation properties of the physical quantity q, having the symbol $d_0 = (-1)$. We define a Boolean variable α_0 that takes the value 0 when the physical quantity q is a tensor and the value 1 when the physical quantity is a pseudo-tensor.

The surjective map $q \to q_R$ is the canonical projection map which maps each quantity q to its equivalence class q_R in which the given equivalence relation qRx is defined by a generalized dimension, unlike the SI dimension of the VIM3, where $\mathrm{gdim}\,(q) = \mathrm{gdim}\,(x)$ and in which

$$\mathrm{gdim}\,(q) = \mathrm{d_0}^{\alpha_0} \mathrm{d_1}^{\alpha_1} \mathrm{d_2}^{\alpha_2} \ldots \mathrm{d_N}^{\alpha_\mathrm{N}} ,$$

with $\mathrm{d_n}$ the symbol of the n-th extended base quantity B_n and x a representative physical quantity of the equivalence class.

Example 2.1.1 (equivalence relation)**.** *An example of this equivalence relation is that kinetic energy and potential energy are represented by the representative element x, that we denote by $x = E_R$, of the quotient set Q/R. However, we will see that the distinction between kinetic energy and potential energy remains preserved based on their walk, which is a cycle, in the integer lattice.*

Observe that by the equivalence relation R a difference can be made between the kind of quantity energy and the kind of quantity moment of force. This is not possible when the SI dimension dim defined in the VIM3 is used. This justifies the definition of a generalized dimension function $\mathrm{gdim}\,()$. The distinction is made through the value α_0 of the kind of tensor. The physical quantity energy is mapped to the equivalence class kind of quantity energy E_R in which the value of $\alpha_0 = 0$ and the physical quantity moment of force is mapped to the equivalence class kind of quantity moment of force M_R in which the value of $\alpha_0 = 1$ because the component of the moment of force belongs to a pseudo-tensor.

The equivalence class of dimensionless quantities is denoted $\mathit{1}_R$. We consider a multiplicative binary operator $\{\cdot\}$ between the equivalence classes of $\mathsf{Q}/\!\sim$. The algebraic properties of the composition of the equivalence classes result in a multiplicative commutative group $\mathsf{Q}/\!\sim, \{\cdot\}$.

For the sequel of the dissertation we will do an abuse of notation on the kind of quantity q_R and denote it as q when no confusion is possible.

We define a mapping $\operatorname{dex}()$ called ‘dimensional exponent of a kind of quantity’ that maps a kind of quantity to an integer lattice point. A clear distinction exists between the dimension of a physical quantity according to the VIM3, a generalized dimension of a physical quantity, and the mapping $\operatorname{dex}()$ of a kind of quantity q_R.

The mapping $\operatorname{dex}()$ is defined from $\mathsf{Q}/\!\sim$ into $\{0,1\} \times \mathbb{Z}^N$, and formally as:

Definition 2.1.1.

$$\begin{aligned}&\operatorname{dex}() : \mathsf{Q}/\sim \rightarrow \{\mathit{0}, \mathit{1}\} \times \mathbb{Z}^N \ :\\ &\operatorname{dex}(q_R) \doteq \boldsymbol{z} = (\alpha_{\mathit{0}} \mid \alpha_{\mathit{1}}, \ldots, \alpha_N)\,,\end{aligned}$$

in which $\alpha_n \in \mathbb{Z}$ for $n \neq \mathit{0}$, and $n \in \{1, \ldots, N\}$; $\alpha_0 \in \{0,1\}$, and $\operatorname{gdim}(q) = d_{\mathit{0}}^{\alpha_{\mathit{0}}} d_{\mathit{1}}^{\alpha_{\mathit{1}}} \ldots d_N^{\alpha_N}$, with $d_0 = (-1)$, and qRx .

Observe that $\alpha_n = \log_{\alpha_n} d_n^{\alpha_n}$ in which d_n is the symbol of an extended base quantity B_n . The α_n’s are the components of an integer lattice point $\boldsymbol{z} = (\alpha_0 \mid \alpha_1, \ldots, \alpha_N)$ of $\{0,1\} \times \mathbb{Z}^N$.

We map the unit element $\mathit{1}_R$ of $\mathsf{Q}/\sim, \{\cdot\}$ on the unit element $\boldsymbol{o} = (0 \mid 0, \ldots, 0)$ of $\{0,1\} \times \mathbb{Z}^N, \{+\}$. We obtain $\operatorname{dex}(\mathit{1}_R) \doteq \boldsymbol{o} = (0 \mid 0, \ldots, 0)$.

We claim, without giving proofs, the following $\operatorname{dex}()$ identities:

$$\forall\, a, b \in \mathsf{Q}/\sim \mid \operatorname{dex}(ab) = \operatorname{dex}(a) + \operatorname{dex}(b)\ , \tag{2.1}$$

$$\forall\, a, b \in \mathsf{Q}/\sim \mid \operatorname{dex}\left(\frac{a}{b}\right) = \operatorname{dex}(a) - \operatorname{dex}(b)\ , \tag{2.2}$$

$$\forall\, a, b, c \in \mathsf{Q}/\sim \mid \operatorname{dex}(abc) = \operatorname{dex}(a(bc)) = \operatorname{dex}((ab)c)\ , \tag{2.3}$$

$$\forall\, a \in \mathsf{Q}/\sim, \forall p \in \mathbb{Z} \mid \operatorname{dex}(a^p) = p \operatorname{dex}(a)\ . \tag{2.4}$$

The inverse of the $\operatorname{dex}()$ mapping is a mapping of $\mathbb{Z}^N$ into $\mathsf{Q}/\!\sim$, and formally as:

Definition 2.1.2. $\operatorname{dex}^{-1}() : \forall \boldsymbol{a} \in \{0,1\} \times \mathbb{Z}^{\mathsf{N}}, \exists\, \mathsf{a} \in \mathsf{Q}/\sim :$
$\operatorname{dex}^{-1}(\boldsymbol{a}) \doteq a$.

We claim, without giving proofs, the following $\mathrm{dex}^{-1}()$ identities:

$$\forall \boldsymbol{a},\boldsymbol{b} \in \{0,1\} \times \mathbb{Z}^N \mid ab = \mathrm{dex}^{-1}(\boldsymbol{a}+\boldsymbol{b}) \ , \tag{2.5}$$

$$\forall \boldsymbol{a},\boldsymbol{b} \in \{0,1\} \times \mathbb{Z}^N \mid \frac{a}{b} = \mathrm{dex}^{-1}(\boldsymbol{a}-\boldsymbol{b}) \ , \tag{2.6}$$

$$\forall \boldsymbol{a},\boldsymbol{b},\boldsymbol{c} \in \{0,1\} \times \mathbb{Z}^N \mid \ abc = \tag{2.7}$$

$$= \mathrm{dex}^{-1}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}) \tag{2.8}$$

$$= \mathrm{dex}^{-1}(\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})) \tag{2.9}$$

$$= \mathrm{dex}^{-1}((\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}) \ , \tag{2.10}$$

$$\forall \boldsymbol{a} \in \{0,1\} \times \mathbb{Z}^N , \forall p \in \mathbb{Z} \mid a^p = \mathrm{dex}^{-1}(p\boldsymbol{a}) \ . \tag{2.11}$$

A homomorphism $\mathrm{f} : \mathsf{Q}/\sim \to \{0,1\} \times \mathbb{Z}^{\mathrm{N}}$ is an isomorphism if there exists a homomorphism $\mathrm{g} : \{0,1\} \times \mathbb{Z}^{\mathrm{N}} \to \mathsf{Q}/\sim$ such that $\mathrm{f} \circ \mathrm{g}$ and $\mathrm{g} \circ \mathrm{f}$ are the identity mappings of $\{0,1\} \times \mathbb{Z}^N$ and $\mathsf{Q}/\sim$ respectively (Lang, 2005). We identify $\mathrm{f} = \mathrm{dex}()$ and $\mathrm{g} = \mathrm{dex}^{-1}()$ and infer that a group isomorphism exists between $\mathsf{Q}/\sim$ and $\{0,1\} \times \mathbb{Z}^N$, that we denote $\{0,1\} \times \mathbb{Z}^N \approx \mathsf{Q}/\sim$ (Lang, 2005).

The set $\mathbb{Z}^N$ is known as the N-dimensional integer lattice (J. Conway et al., 1999, p.106) that is a discrete subgroup of the real vector space $\mathbb{R}^N$ with Euclidean norm. The properties of the integer lattice $\mathbb{Z}^N$ are found in several publications (Birkhoff, 1967; J. Conway et al., 1999).

2.2 Lattice constellations

A set of lattice points is called a lattice constellation (Forney & Ungerboeck, 1998). An arbitrary set of kinds of quantities is represented by a constellation of lattice points in $\{0,1\} \times \mathbb{Z}^N$. We are interested in the properties of these constellations of integer lattice points.

2.3 Geometric representation of quantity equations

A relationship between p quantities, represented by individual symbols, which may be used to describe a phenomenon, without exception, is a p-ary operation between quantities. The symbols are called terms.

A finite sequence of terms is called a formula. Formulas can be combined with the relational operator '=' to generate *equations*. First we investigate the non-trivial p-ary operation in which $p = 3$. The ternary relation under study, results in quantity equations between x, y, z of the type $f(\pi) = \frac{z}{xy}$ and the binary operator is the multiplication operator.

A physics equation containing tensors and or pseudo-tensors is meaningful when expressed between the components of the tensor and/or pseudo-tensor. We call these relations between components of tensors and/or pseudo-tensors *physically valid*. Each of the physical quantities in these physically valid relations is mapped to its respective kind of quantity resulting in physically valid quantity equations.

Theorem 2.3.1. *The quantity equation* $f(\pi) = \dfrac{z}{xy}$ *is physically valid, with* $f(\pi), x, y, z$ *distinct kinds of quantities obeying the properties:*

$$\operatorname{dex}^{-1}(\operatorname{dex}(z)) = z,\ \operatorname{dex}^{-1}(\operatorname{dex}(f(\pi))) = f(\pi)\,,$$
$$\operatorname{dex}^{-1}(\operatorname{dex}(x)) = x,\ \operatorname{dex}^{-1}(\operatorname{dex}(y)) = y\,,$$

if and only if, the 4-cycle ***oyzxo*** *is a parallelogram in the integer lattice* $\{0,1\} \times \mathbb{Z}^N$ *and* $\operatorname{dex}(x) = \boldsymbol{x}$*,* $\operatorname{dex}(y) = \boldsymbol{y}$*,* $\operatorname{dex}(z) = \boldsymbol{z}$*,* $\operatorname{dex}(f(\pi)) = \boldsymbol{o}$ *are distinct integer lattice points of* $\{0,1\} \times \mathbb{Z}^N$ *with* ***o*** *being the origin of the integer lattice* $\{0,1\} \times \mathbb{Z}^N$*. The proof is of the 'if and only if'–type in which it is split in a necessary and sufficient condition.*

Proof. We aim to prove that a ternary relation between kinds of quantities is equivalent with a 4-cycle, being a parallelogram and vice-versa. [Necessary] Let $f(\pi), x, y, z \in \mathsf{Q}/\sim$ be distinct kinds of quantities and π be a dimensionless quantity.
Suppose that the quantity equation $f(\pi) = \dfrac{z}{xy}$ is physically valid. By the $\operatorname{dex}()$ identity (2.1) we obtain $\operatorname{dex}(f(\pi)) = \operatorname{dex}(z) - \operatorname{dex}(xy) = \operatorname{dex}(z) - \operatorname{dex}(x) - \operatorname{dex}(y)$. By the definition of $\operatorname{dex}()$, see 2.1.1, one writes

$$\boldsymbol{o} = \boldsymbol{z} - \boldsymbol{x} - \boldsymbol{y}\,, \tag{2.12}$$

in which the addition is performed component-wise.
The coordinates $(x_0 \mid x_1, \ldots, x_N)$ of $\boldsymbol{x}$, $(y_0 \mid y_1, \ldots, y_N)$ of $\boldsymbol{y}$ and $(0 \mid 0, \ldots, 0)$ of the origin $\boldsymbol{o}$ determine uniquely the coordinates of an integer lattice point $\boldsymbol{z}$ according to the above equation (2.12). As no degree of freedom is left over for the coordinates of $\boldsymbol{z}$, we infer that a parallelogram (Fig. 2.2) represented by the 4-cycle ***oyzxo*** has been constructed in $\{0,1\} \times \mathbb{Z}^N$.

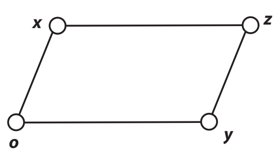

Figure 2.2: Parallelogram ***oyzxo*** representing the quantity equation $f(\pi) = \dfrac{z}{xy}$ in $\{0,1\} \times \mathbb{Z}^N$.

[Sufficient] Let the 4-cycle ***oyzxo*** be a parallelogram (Fig. 2.2) with as diagonals the lines ***oz*** and ***xy***. Let ***o*** be, without loss of generality, the origin of the integer lattice $\{0,1\} \times \mathbb{Z}^N$. By the definition of a 4-cycle one writes $\boldsymbol{o} = \boldsymbol{z} - \boldsymbol{x} - \boldsymbol{y}$. This equation is rewritten as

$$\boldsymbol{z} = \boldsymbol{o} + \boldsymbol{x} + \boldsymbol{y}\,, \tag{2.13}$$

in which the addition is performed component-wise.
We apply on both sides of the equation (2.13) the mapping $\mathrm{dex}^{-1}()$, and obtain the equation $\mathrm{dex}^{-1}(\boldsymbol{z}) = \mathrm{dex}^{-1}(\boldsymbol{o}+\boldsymbol{x}+\boldsymbol{y})$. By the definition, of $\mathrm{dex}^{-1}()$ identity (2.5) we obtain:

$$\mathrm{dex}^{-1}(\mathrm{dex}(z)) = \mathrm{dex}^{-1}(\mathrm{dex}(f(\pi))) \cdot \mathrm{dex}^{-1}(\mathrm{dex}(x)) \cdot \mathrm{dex}^{-1}(\mathrm{dex}(y))\,. \tag{2.14}$$

As the product mapping $(\mathrm{dex}^{-1}() \circ \mathrm{dex}())$ results in the identity mapping we claim that there exists a set $\{f(\pi), x, y, z\} \subset \mathsf{Q}/\sim$ for which

$$\mathrm{dex}^{-1}(\mathrm{dex}(z)) = z,\; \mathrm{dex}^{-1}(\mathrm{dex}(f(\pi))) = f(\pi)\,,$$
$$\mathrm{dex}^{-1}(\mathrm{dex}(x)) = x,\; \mathrm{dex}^{-1}(\mathrm{dex}(y)) = y\,.$$

Hence, we rewrite the equation (2.14) to obtain the quantity equation $f(\pi) = \dfrac{z}{xy}$ that is physically valid. □

Dimensionless physics is obtained in the integer lattice $\{0,1\} \times \mathbb{Z}^N$ by considering dimensionless products of kinds of quantities as k-cycles containing the origin of the integer lattice $\{0,1\} \times \mathbb{Z}^N$ and having parallelogram properties between its vertices. It is known that the parallelogram law $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{z}$ in which $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \{0,1\} \times \mathbb{Z}^N$ is valid.

We find that parallelograms are the fundamental geometric form of all kinds of quantity equations because each non-trivial quantity equation can be reduced, based on the associativity of the multiplication of kinds of quantities X, Y, Z, to the algebraic form $Z = f(\boldsymbol{\pi})XY$.

Moreover, as corollary of Theorem 2.3.1, we conclude that any dimensionless quantity equation $f(\pi) = \dfrac{Z}{X_1 X_2 \ldots X_M}$, where $X_1, X_2, \ldots, X_M$ are kinds of quantities, represents a *closed path*, defined by the vector equation in the integer lattice $\{0,1\} \times \mathbb{Z}^N$:

$$\boldsymbol{o} = \boldsymbol{z} - \sum_{m=1}^{M} \boldsymbol{x}_m\,, \tag{2.15}$$

where $\mathrm{dex}(X)_m = \boldsymbol{x}_m$, $\mathrm{dex}(f(\pi)) = \boldsymbol{o}$, and $\mathrm{dex}(Z) = \boldsymbol{z}$.

2.4 Semi-perimeter of a parallelogram

Let SP be the semi-perimeter of a parallelogram, representing a ternary quantity equation, formed by the 4-cycle $\boldsymbol{oyzxo}$ where $\boldsymbol{o}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \{0,1\} \times \mathbb{Z}^N$. The value of the semi-perimeter SP is obtained by the equation:

$$SP = \sqrt{u} + \sqrt{v}\,,$$

with $u, v \in \mathbb{N}$ and expressed through the following equations:

$$u = \sum_{n=1}^{N} x_n^2 \qquad\qquad v = \sum_{n=1}^{N} y_n^2 = \sum_{n=1}^{N} (x_n - z_n)^2\,,$$

in which the integer lattice points have the coordinates $(x_0 \mid x_1, x_2, \ldots, x_N)$. The first coordinate x_0 can take only the values 0 or 1 .

The property of the semi-perimeter of a parallelogram will be used in Chapter 5 for exploring the integer lattice $\{0,1\} \times \mathbb{Z}^N$. Histograms of the semi-perimeter SP of parallelograms will be used in this exploration to find 'laws of physics'. A software code to find the relevant semi-perimeters is given in Appendix V. In this search for 'laws of physics' we will study the uniqueness of the semi-perimeter of a parallelogram. Unique semi-perimeters of parallelograms of the lattice point $(0 \mid -4, -1, 1, 0^4)$, representing the kind of quantity of the partial second derivative of the energy density with respect to time, are analyzed in Appendix I. The maximum number of distinct parallelogram semi-perimeters as function of the dimension N of the integer lattice and the infinity norm s are given in Appendix J.

2.5 Fundamental $(N+1)$-dimensional confocal ellipsoid

We know that the integer lattice is infinite and we know that the dimensional exponents are small integers. Hence, we need to find an appropriate way to limit the search for parallelograms. We will investigate the distributions of semi-perimeters of parallelograms as function of the dimension N of the integer lattice $\mathbb{Z}^N$ and the infinity norm s .

Let the kind of quantity be represented by the visible lattice point $\boldsymbol{z}$. We consider in increasing order of parallelogram semi-perimeter SP the degenerated parallelograms given by the equations $\boldsymbol{o} = \boldsymbol{z} - \boldsymbol{z}$, with semi-perimeter $SP_0 = \|\boldsymbol{z}\|_2$ and $\boldsymbol{z} = 2\boldsymbol{z} - \boldsymbol{z}$ with $SP_1 = 3\|\boldsymbol{z}\|_2$. The next degenerated parallelogram in increasing order has the equation $\boldsymbol{z} = 3\boldsymbol{z} - 2\boldsymbol{z}$ with $SP_2 = 5\|\boldsymbol{z}\|_2$. It is obvious that the semi-perimeter of the n-th degenerated parallelogram is equal to $SP_n = (2n+1)\|\boldsymbol{z}\|_2$. Hence, the semi-perimeter of a degenerated parallelogram has a periodicity of $P = 2\|\boldsymbol{z}\|_2$.

To truncate the histogram of semi-perimeters of parallelograms we define a fundamental $(N+1)$-dimensional confocal ellipsoid where $\|\boldsymbol{x} - \frac{\boldsymbol{z}}{2}\|_2 \leq \frac{3}{2}\|\boldsymbol{z}\|_2$ and $\|\boldsymbol{y} - \frac{\boldsymbol{z}}{2}\|_2 \leq \frac{3}{2}\|\boldsymbol{z}\|_2$. Histograms of semi-perimeters of parallelograms of kind of quantities will be limited to their fundamental $(N+1)$-dimensional confocal ellipsoid by the condition:

$$SP = \sqrt{u} + \sqrt{v} \leq \frac{3}{2}\|\boldsymbol{z}\|_2 \,. \tag{2.16}$$

2.6 Rectangles

The decomposition of a lattice point $\boldsymbol{z}$ of $\{0,1\} \times \mathbb{Z}^N$ in two pairwise orthogonal lattice points $\boldsymbol{x}$ and $\boldsymbol{y}$ assumes the existence of a system of Diophantine equations given by equation (2.17a) and equation (2.17b) :

$$\text{parallelogram law: } \boldsymbol{x} + \boldsymbol{y} - \boldsymbol{z} = \boldsymbol{o} \,, \tag{2.17a}$$

$$\text{inner product: } \boldsymbol{x} \cdot \boldsymbol{y} = 0 \,, \tag{2.17b}$$

in which $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \{0,1\} \times \mathbb{Z}^N$. We eliminate $\boldsymbol{y}$ from the equation (2.17a) and find:

$$\boldsymbol{x} \cdot \boldsymbol{x} - \boldsymbol{x} \cdot \boldsymbol{z} = 0 \,. \tag{2.18}$$

We apply the method of completing the square and write equation (2.18) as:

$$\left(\boldsymbol{x} - \frac{\boldsymbol{z}}{2}\right)^2 = \left(\frac{\boldsymbol{z}}{2}\right)^2 , \tag{2.19}$$

that represents a $(N+1)$-dimensional hypersphere in $\{0,1\} \times \mathbb{R}^N$ with center at $\frac{\boldsymbol{z}}{2}$ and radius $\|\frac{\boldsymbol{z}}{2}\|_2$.

Lohmann, Mendlovic, Zalevsky, and Shabtay (1997) and Foadi and Evans (2008) relate the $(N+1)$-dimensional hypersphere to a $(N+1)$-dimensional Helmholtz wave equation. The solutions of the differential equation are given by the lattice points of the integer lattice that are incident on the $(N+1)$-dimensional hypersphere.

The center of the $(N+1)$-dimensional hypersphere is a lattice point only if the coordinates of $\boldsymbol{z}$ are even. If the coordinates of $\boldsymbol{z}$ are even then $\boldsymbol{z}$ is not a visible point of the integer lattice. If the coordinates of $\boldsymbol{z}$ are odd, then this has no impact in finding rectangles.

Observe from equation (2.19) that there exists a unique $(N+1)$-dimensional hypersphere (2.19), with radius squared $\left(\frac{\boldsymbol{z}}{2}\right)^2$ and center equal to $\frac{\boldsymbol{z}}{2}$, for each lattice point $\boldsymbol{z}$. This unique $(N+1)$-dimensional hypersphere determines the *finite* set of pairwise orthogonal vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ resulting in the kinds of quantities x and y that satisfy the dimensionless quantity equation $f(\pi) = \frac{z}{xy}$. The four lattice points $\boldsymbol{o}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ form a rectangle in $\{0,1\} \times \mathbb{Z}^N$.

A closed form for the solution set of equation (2.19) is unknown to the author and thus we use a brute force method, using the supercomputer of

Ghent University, and list the vertices of the parallelograms ***oxzy*** embedded in $\{0,1\} \times \mathbb{Z}^N$ representing quantity equations $f(\pi) = \dfrac{z}{xy}$.

This situation is similar to what can be observed in X-ray diffraction at cubic crystals. A diffracted beam from the crystal is detected when the reciprocal lattice of the crystal crosses the Ewald sphere during the rotation of the crystal in the X-ray diffraction equipment. The detection of a scattered X-ray photon is similar to the observation of a 'law of nature' for the kind of quantity ***z*** under consideration. We are motivated to make this conjecture by the observation that the law $E = mc_0^2$ is representing a unique rectangle in $\{0,1\} \times \mathbb{Z}^N$.

The situation is visualized in Figure 2.3 showing an Ewald sphere in a cubic lattice:

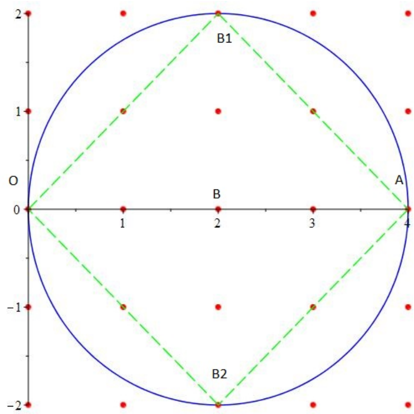

Figure 2.3: Ewald sphere of a cubic lattice.

Figure 2.3 shows that the point A is similar to the lattice point ***z***, the point B, being the center of the circle is similar to $\frac{\boldsymbol{z}}{2}$, the point O is similar to the origin ***o*** of the integer lattice. Rectangles are formed by forming the path OB_1AB_2O, which in Figure 2.3 is a special rectangle being a square.

We enumerate for $N = 7$ the number of rectangles formed by the orbit representatives of $\{0,1\}\times\mathbb{Z}^7$ as a discrete function of the infinity norm given in the On-line Encyclopedia of Integer Sequences (OEIS), by identifier A240934 and those with a unique perimeter by the OEIS integer sequence A247557.

For $\|\boldsymbol{z}\|_\infty \leq 10$, we find in $\mathbb{Z}_+^7$ a total of 7 747 *unique rectangles* out of 6 510 466 998 rectangles, obtained using the supercomputer of Ghent University. These unique rectangles are enumerated in an *atlas*, that have been generated with the pseudocode algorithm given in Appendix M, of canonical ternary quantity equations of SI physics. This atlas of rectangles can be made available after approval by UGent TechTransfer in accordance with valorization project: P2021/066 – Mathematical Classification. The data are presently in the author's personal repository. A small part of the atlas of rectangles is given by enumerating: 60 rectangles in Appendix F for the kind of quantity energy represented by the lattice point $(0 \mid -2, 2, 1, 0^4)$ and 341 rectangles in Appendix H for the kind of quantity second order partial derivative of the energy density with respect to time represented by the lattice point $(0 \mid -4, -1, 1, 0^4)$. Unique rectangles for the kind of quantity power represented by the lattice point $(0 \mid -3, 2, 1, 0^4)$, are discussed in Appendix G. The creation of an application giving public access to the research data is part of the future work.

2.7 Coloring of the integer lattice

Coloring maps used for studying further properties of $\{0,1\} \times \mathbb{Z}^N$ are found in Manturov (2016).

Definition 2.7.1. *Let the surjective mapping* soc*, represent the sum of coordinates of a lattice point of* $\{0,1\} \times \mathbb{Z}^N$ *and define:*

$$\mathrm{soc} : \{0,1\} \times \mathbb{Z}^{\mathrm{N}} \to \mathbb{Z} \mid \mathrm{soc}\,(\boldsymbol{x}) = \sum_{n=1}^{N} x_n \,,\; x_n \in \mathbb{Z}\,.$$

The sum of coordinates define different hyperplanes that contain kinds of quantities. The hyperplane equations are given by $x_1 + \ldots + x_7 = \mathrm{soc}\,(\boldsymbol{x})$ in which $\boldsymbol{x} \in \{0,1\} \times \mathbb{Z}^7$. We could give to each integer lattice point a color based on the value of $\mathrm{soc}\,(\boldsymbol{x})$ as a way of classifying the physical quantities. This classification occurs also when using the singular value decomposition (SVD) or the principal component analysis (PCA) on the set of physical quantities of the lexicon A. We used the PCA method using the software VisuMap.

Definition 2.7.2. *Let the surjective mapping 'psoc', represent the parity of the sum of coordinates of a lattice point of* $\mathbb{Z}^N$ *and define:*

$$\mathrm{psoc} : \{0,1\} \times \mathbb{Z}^{\mathrm{N}} \to \{0,1\} \mid \mathrm{psoc}\,(\boldsymbol{x}) = \sum_{n=1}^{N} x_n \pmod{2}\,,\; x_n \in \mathbb{Z}\,.$$

The 'psoc' mapping is bi-coloring. We have an even sum lattice point when $\mathrm{psoc}\,(\boldsymbol{x}) = 0$ and an odd sum lattice point when $\mathrm{psoc}\,(\boldsymbol{x}) = 1$ in which $\boldsymbol{x} \in \mathbb{Z}^N$.

2.8 Bi-coloring of a 4-cycle

The ternary quantity equations obey a rule that constrains the bi-coloring of 4-cycles in $\mathbb{Z}^N$ (Quattrocchi, 2001; L. Gionfriddo, Gionfriddo, & Ragusa, 2010; M. Gionfriddo & Ragusa, 2010). We will use the method *proof by exhaustion* for proving Theorem 2.8.1.

Theorem 2.8.1. *Any ternary quantity equation* $f(\pi) = \dfrac{z}{xy}$ *between distinct quantities* $f(\pi), x, y, z$ *represents a distinct coloring pattern* $(\mathrm{psoc}(\boldsymbol{o}), \mathrm{psoc}(\boldsymbol{x}), \mathrm{psoc}(\boldsymbol{y}), \mathrm{psoc}(\boldsymbol{z}))$*, that is an element of the set of coloring patterns* $\{(0,0,0,0),(0,0,1,1),(0,1,0,1),(0,1,1,0)\}$.

Proof. Let the four distinct integer lattice points $\boldsymbol{o}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{y}$ be the vertices of a parallelogram, represented by the 4-cycle ***oyzxo***. The parallelogram is the representation of the ternary quantity equation $f(\pi) = \dfrac{z}{xy}$ in the integer lattice $\mathbb{Z}^N$. Let the coloring pattern be defined by the 4-tuple $(\mathrm{psoc}(\boldsymbol{o}), \mathrm{psoc}(\boldsymbol{x}), \mathrm{psoc}(\boldsymbol{y}), \mathrm{psoc}(\boldsymbol{z}))$ in which $\boldsymbol{o}$ is the origin of $\mathbb{Z}^N$. By convention $\mathrm{psoc}(\boldsymbol{o})$ is placed as the first element and $\mathrm{psoc}(\boldsymbol{z})$ as the last element in the coloring patterns. By the definition of the 'psoc' mapping we obtain $\mathrm{psoc}(\boldsymbol{o}) = 0$. A 4-tuple having two characters {0,1} has in total $2^4 = 16$ combinations of 4-tuples. Hence, we will review the 16 cases. The condition that the first element of the 4-tuple has to be 0 reduces the number of combinations to $2^3 = 8$ being the set of coloring patterns {(0,0,0,0), (0,0,0,1), (0,0,1,0), (0,0,1,1), (0,1,0,0), (0,1,0,1),(0,1,1,0), (0,1,1,1)}. The four distinct integer lattice points $\boldsymbol{o}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ of the parallelogram have the property $\boldsymbol{x}+\boldsymbol{y} = \boldsymbol{z}$, see Figure 2.2. From elementary number theory, it is known that (Epp, 1990):

- even $\pm$ even = even
- odd $\pm$ odd = even
- even $\pm$ odd = odd

The mapping 'psoc' is binary-valued on $\mathbb{Z}^N$ satisfying $\mathrm{psoc}(\boldsymbol{x}+\boldsymbol{y}) = \mathrm{psoc}(\boldsymbol{x}) + \mathrm{psoc}(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}^N$.
Thus the 4-tuples have the form $(0, \mathrm{psoc}(\boldsymbol{x}), \mathrm{psoc}(\boldsymbol{y}), \mathrm{psoc}(\boldsymbol{x}) + \mathrm{psoc}(\boldsymbol{y}))$ resulting in the following cases:

Case 2.8.1. *(0,0,0,0)*
If $\mathrm{psoc}(\boldsymbol{x}) = \mathrm{psoc}(\boldsymbol{y}) = 0$ *then by number theory* $\mathrm{psoc}(\boldsymbol{x}) + \mathrm{psoc}(\boldsymbol{y}) = 0$*. The coloring pattern* ***(0,0,0,0)*** *satisfies the above property and is a* valid *coloring pattern. This coloring pattern is called monochromatic.*

Case 2.8.2. *(0,0,0,1)*
If $\mathrm{psoc}(\boldsymbol{x}) = \mathrm{psoc}(\boldsymbol{y}) = 0$ *then by number theory* $\mathrm{psoc}(\boldsymbol{x}) + \mathrm{psoc}(\boldsymbol{y}) = 0$*. The coloring pattern (0,0,0,1) violates the above property and is a* forbidden *coloring pattern.*

Case 2.8.3. *(0,0,1,0)*
If $\mathrm{psoc}(\boldsymbol{x}) = 0$ *and* $\mathrm{psoc}(\boldsymbol{y}) = 1$ *then by number theory* $\mathrm{psoc}(\boldsymbol{x}) + \mathrm{psoc}(\boldsymbol{y}) = 1$*. The coloring pattern (0,0,1,0) violates the above property and is a* forbidden *coloring pattern.*

Case 2.8.4. *(0,0,1,1)*
If $\mathrm{psoc}(\boldsymbol{x}) = 0$ *and* $\mathrm{psoc}(\boldsymbol{y}) = 1$ *then by number theory* $\mathrm{psoc}(\boldsymbol{x}) + \mathrm{psoc}(\boldsymbol{y}) =$

*1. The coloring pattern **(0,0,1,1)** satisfies the above property and is a* valid *coloring pattern. This coloring pattern is called two-coloured of pattern 2+2.*

Case 2.8.5. *(0,1,0,0)*
If $\mathrm{psoc}\,(\boldsymbol{x}) = 1$ *and* $\mathrm{psoc}\,(\boldsymbol{y}) = 0$ *then by number theory* $\mathrm{psoc}\,(\boldsymbol{x}) + \mathrm{psoc}\,(\boldsymbol{y}) = 1$*. The coloring pattern (0,1,0,0) violates the above property and is a* forbidden *coloring pattern.*

Case 2.8.6. *(0,1,0,1)*
If $\mathrm{psoc}\,(\boldsymbol{x}) = 1$ *and* $\mathrm{psoc}\,(\boldsymbol{y}) = 0$ *then by number theory* $\mathrm{psoc}\,(\boldsymbol{x}) + \mathrm{psoc}\,(\boldsymbol{y}) = 1$*. The coloring pattern **(0,1,0,1)** satisfies the above property and is a* valid *coloring pattern. This coloring pattern is called mixed two-coloured.*

Case 2.8.7. *(0,1,1,0)*
If $\mathrm{psoc}\,(\boldsymbol{x}) = 1$ *and* $\mathrm{psoc}\,(\boldsymbol{y}) = 1$ *then by number theory* $\mathrm{psoc}\,(\boldsymbol{x}) + \mathrm{psoc}\,(\boldsymbol{y}) = 0$*. The coloring pattern **(0,1,1,0)** satisfies the above property and is a* valid *coloring pattern. This coloring pattern is called two-coloured of pattern 1+2+1.*

Case 2.8.8. *(0,1,1,1)*
If $\mathrm{psoc}\,(\boldsymbol{x}) = 1$ *and* $\mathrm{psoc}\,(\boldsymbol{y}) = 1$ *then by number theory* $\mathrm{psoc}\,(\boldsymbol{x}) + \mathrm{psoc}\,(\boldsymbol{y}) = 0$*. The coloring pattern (0,1,1,1) violates the above property and is a* forbidden *coloring pattern.*

We obtain as valid coloring patterns:

$$(0,0,0,0), (0,0,1,1), (0,1,0,1), (0,1,1,0)\,.$$

□

Corollary 2.8.1. *If* $\mathrm{psoc}\,(\boldsymbol{z}) = 0$ *then* $\mathrm{psoc}\,(\boldsymbol{x}) = \mathrm{psoc}\,(\boldsymbol{y})$.
If $\mathrm{psoc}\,(\boldsymbol{z}) = 1$ *then* $\mathrm{psoc}\,(\boldsymbol{x})$ *is the opposite of* $\mathrm{psoc}\,(\boldsymbol{y})$.

2.9 Linear independence and orthogonality

Ternary quantity equations $f(\pi) = \dfrac{z}{xy}$ are only one of the six following cases:

1. $\boldsymbol{x} \cdot \boldsymbol{y} > 0$ and $2 \times N$ matrix rank $= 2$ (not orthogonal with positive inner product, linearly independent)
2. $\boldsymbol{x} \cdot \boldsymbol{y} = 0$ and $2 \times N$ matrix rank $= 2$ (orthogonal, linearly independent)
3. $\boldsymbol{x} \cdot \boldsymbol{y} < 0$ and $2 \times N$ matrix rank $= 2$ (not orthogonal with negative inner product, linearly independent)
4. $\boldsymbol{x} \cdot \boldsymbol{y} > 0$ and $2 \times N$ matrix rank < 2 (not orthogonal with positive inner product, linearly dependent)
5. $\boldsymbol{x} \cdot \boldsymbol{y} = 0$ and $2 \times N$ matrix rank < 2 (orthogonal, linearly dependent)
6. $\boldsymbol{x} \cdot \boldsymbol{y} < 0$ and $2 \times N$ matrix rank < 2 (not orthogonal with negative inner product, linearly dependent)

2.10 Cardinality of an orbit

Berstel and Perrin (2007) define a *word*, and we quote:

> A *word* is a finite, infinite from left to right or two-sided infinite sequence of symbols taken in a finite set called *alphabet*.

We choose an alphabet Σ that consists of the first m natural numbers and zero and denote $\Sigma = \{0, 1, 2, \ldots, m\}$. The alphabet Σ is a totally ordered alphabet. We consider only finite words. The set of finite words over an alphabet Σ is denoted by Σ^* . The representative of an orbit is a word w constructed from the alphabet Σ^* . The words w have a length $(N+1)$ where $(N+1)$ corresponds to the dimension of the integer lattice $\{0,1\} \times \mathbb{Z}^N$. A factor of a word w is a block of consecutive symbols of w (Berstel & Perrin, 2007). A word is considered square-free if it does not contain two adjacent identical factors (Berstel & Perrin, 2007).

Let q_i be the number of letters i of the alphabet Σ^* . Suppose that the letters of the word w are subjected to a signed permutation, then the cardinality of the orbit is given by Equation (2.20):

$$\#\left(w\right) = 2^{N-q_0} \frac{N!}{q_0!q_1!q_2!\ldots q_m!} \,. \tag{2.20}$$

Observe that $q_0 + q_1 + q_2 \ldots + q_m = N$ and thus $[q_0, q_1, \ldots, q_m]$ is representing an additive partition of a natural number N .

The only method to find the number of distinct cardinalities of an integer lattice $\{0,1\} \times \mathbb{Z}^N$ was through a brute-force approach. We developed a new method, given in Appendix U to find the number of distinct cardinalties of orbits of $\{0,1\} \times \mathbb{Z}^N$ and to list the values of the cardinalities. The results have been published in the OEIS A270950 integer sequence.

Example 2.10.1 (Cardinality of an orbit)**.** *Consider the orbit* $w = (0 \mid 1, 1, 0)$ *and let us exhaustively list all the signed permutations of it.*
We find twelve possibilities:
$(0 \mid 1, 1, 0)$, $(0 \mid 1, 0, 1)$, $(0 \mid 0, 1, 1)$, $(0 \mid -1, 1, 0)$,
$(0 \mid 1, -1, 0)$, $(0 \mid 0, -1, 1)$, $(0 \mid 0, 1, -1)$, $(0 \mid -1, 0, 1)$,
$(0 \mid 1, 0, -1)$, $(0 \mid -1, -1, 0)$, $(0 \mid -1, 0, -1)$, $(0 \mid 0, -1, -1)$.
Applying the Equation(2.20) yields:

$$\#\left((0 \mid 1, 1, 0)\right) = 2^{3-1} \frac{3!}{1!2!} = 12 \,. \tag{2.21}$$

We denote an orbit of $\{0,1\} \times \mathbb{Z}^N$ as $w = (b \mid w_1, \ldots, w_N)$, in which $(w_1, \ldots, w_N)$ are the coordinates of the representative lattice point ordered in graded reverse lexicographical order (Cox, Little, & O'Shea, 1997) and thus $w_1 \geq w_2 \geq w_3 \ldots \geq w_N$ and $w_i \in \mathbb{Z}_+$ and $b \in \{0, 1\}$.

Each orbit forms a set of integer lattice points that are centrally symmetric about the origin $\boldsymbol{o}$ (Coxeter, 1973). The union of all orbits is called in information theory a codebook (Vasilache & Tăbuş, 2003).

2.11 Number of orbits as function of s and N

The number of orbits O_N^s that can be formed in a N-dimensional lattice when the infinity norm $\|\boldsymbol{z}\|_\infty = s$ and $s \in \mathbb{N}$ is given by:

$$\#\left(O_N^s\right) = \binom{s+(N-1)}{(N-1)}.$$

For $N = 7$ and $s = 7$ we find $\#\left(O_7^7\right) = 1716$.

2.12 Gödel walk

We encode each integer lattice point of $\{0,1\} \times (\mathbb{Z}^7)$ by calculating the orbit representative $\mathrm{Orb}\,(\boldsymbol{q})$, by taking the absolute value of the coordinates of the integer lattice point $\boldsymbol{q} = (q_0 \mid q_1, \ldots, q_N)$, sorting them in decreasing order, and renaming the coordinates such that $\mathrm{Orb}\,(\boldsymbol{q}) = (z_0 \mid z_1, \ldots, z_N)$ in which $z_1 \geq z_2 \geq \ldots \geq z_N$ by using a similar scheme to the Gödel encoding applied to 7 non-negative integer variables (Gödel et al., 2001).

We map the orbit representative $\mathrm{Orb}\,(\boldsymbol{q})$ to the integer $G(\mathrm{Orb}\,(\boldsymbol{q}))$ of signed Gödel numbers SG, where $SG \subset 2\mathbb{Z}$ according to the injective mapping:

$$G\colon \{0,1\} \times (\mathbb{Z}^7) \to SG \subset 2\mathbb{Z},$$
$$\mathrm{Orb}\,(\boldsymbol{q}) \mapsto G(\mathrm{Orb}\,(\boldsymbol{q})) := (-1)^{z_0}\, p_1^{z_1} \ldots p_7^{z_7},$$

of the orbit representative $\mathrm{Orb}\,(\boldsymbol{q})$ in the lattice $\{0,1\} \times \mathbb{Z}^7$, in which z_n is the n-th power of the n-th prime number with $n \in \{1, \ldots, 7\}$.

All the signed Gödel numbers of orbit representatives are even integers because the coordinate z_1 of each orbit representative fulfills the condition $z_1 \geq 1$ and $z_1 \geq z_2 \geq z_3 \geq z_4 \geq z_5 \geq z_6 \geq z_7$ and yields a power of two resulting in only even integers.

Consider the orthants of $\{0,1\} \times \mathbb{Z}^7$. There are $2^7 = 128$ orthants in the sub-lattice $\{0\} \times \mathbb{Z}^7$ and the same number in the sub-lattice $\{1\} \times \mathbb{Z}^7$. The signs of the orthants are given in Appendix N. Orthant 1 of $\{0\} \times \mathbb{Z}^7$ and orthant 1 of $\{1\} \times \mathbb{Z}^7$ have sign $(+,+,+,+,+,+,+)$ and contain *all* the orbit representatives. It is also known as the non-negative orthant.

The set of SI base quantities is $\mathcal{B} = \{\mathsf{T}, \mathsf{L}, \mathsf{M}, \mathsf{I}, \Theta, \mathsf{N}, \mathsf{J}\}$. Observe that each of the SI base quantities of the set $\mathcal{B}$ are assigned to a prime number. We have the following assignments: $\mathsf{T} \mapsto 2, \mathsf{L} \mapsto 3, \mathsf{M} \mapsto 5, \mathsf{I} \mapsto 7, \Theta \mapsto 11, \mathsf{N} \mapsto 13, \mathsf{J} \mapsto 17$. The base quantities play the same role as the prime numbers, being the atoms in number theory (Apostol, 1998).

Example 2.12.1 (Electric displacement)**.** *Consider the quantity electric displacement* $\boldsymbol{D}$ *represented by the lattice point* $(0 \mid -2, 2, 0, 1, 0^3)$. *The corresponding orbit representative for the quantity electric displacement is the* *lattice point with coordinates* $(0 \mid 2,2,1,0^4)$ *that is obtained by a signed permutation of the original coordinates. We calculate for this orbit its Gödel number.*

$$G((0 \mid 2,2,1,0^4)) = (-1)^0 2^2 3^2 5^1 7^0 11^0 13^0 17^0 = 180\,.$$

Observe that the factorization of $G((0 \mid 2,2,1,0^4)) = 180$ based on the divisibility relation $n|m$ between the natural numbers n and m induces a lattice structure (See Figure 2.4). The number of divisors of 180 is $\tau(180) = 18$. The divisor set is $\{1,2,3,4,5,6,9,10,12,15,18,20,30,36,45,60,90,180\}$.

The construction of the lattice is as follows: the prime numbers 2, 3, and 5 are forming the edges $[1,2]$, $[1,3]$, and $[1,5]$. The lattice point 6 is connected by forming an edge $[2,6]$ and the edge $[3,6]$ because $6 = 2 \times 3$. The lattice point 10 is connected by forming the edge $[2,10]$ and the edge $[5,10]$ because $10 = 2 \times 5$. Continuing this process results in the factorization of 180 shown in Figure 2.4. The graph has an infimum and a supremum and is a lattice.

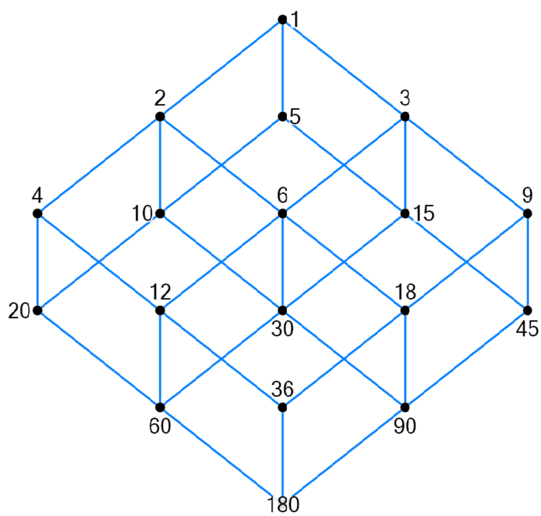

Figure 2.4: Factorization of orbit $[(0 \mid 2^2,1,0^4)]$ based on the relation $m = n \mod 0$ between the natural numbers n and m.

If we walk through the integer sub-lattice $\mathbb{Z}_+^7$ respecting the ordering created by the Gödel encoding, then we generate a sequence of segments in $\mathbb{Z}_+^7$. We call this walk a Gödel walk through the integer sub-lattice $\mathbb{Z}_+^7$. The segments are known in number theory as the prime gaps $g(p) = n$ of gap length n. All the orbit representatives are located on the Gödel walk. When the Gödel walk is performed in $\mathbb{Z}_+^{25}$ then all the first 100 non-negative integers will be visited (Fig. 2.5).

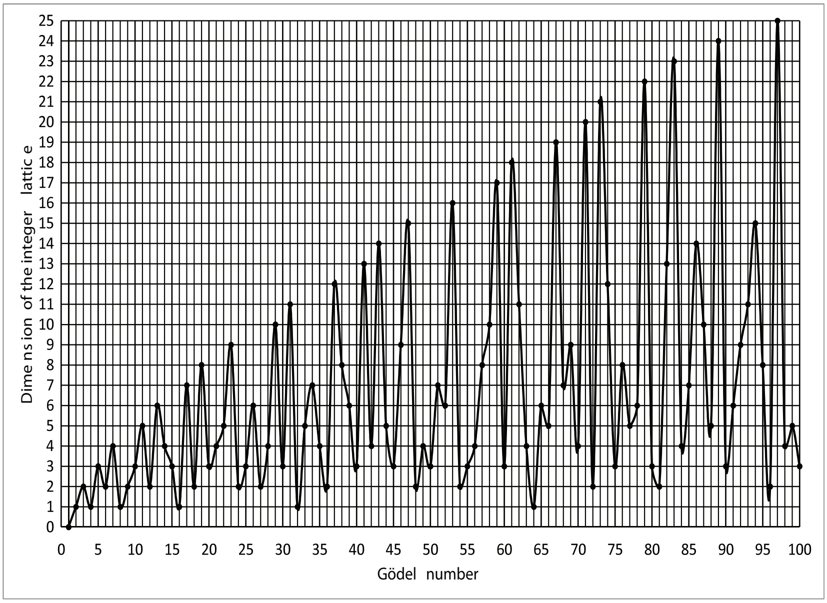

Figure 2.5: Gödel walk in $\mathbb{Z}_+^{25}$.

The large peaks in Fig. 2.5 are represented by the prime counting function $\pi(x)$ that gives the number of prime numbers less or equal to x . We obtain as result that the minimal dimension N_{min} necessary to have a Gödel walk visiting all the natural numbers up to x is $N_{min} = \pi(x)$.

When restricting the dimension from $N = 25$ to $N = 7$ we find 67 non-negative integers that will be visited. This can be found in the data of Fig. 2.5 by tracing a horizontal line at the selected dimension of the integer lattice and counting the Gödel numbers in the prime gaps. The data of Fig. 2.5 for $\mathbb{Z}^7$ has been enumerated in (Table B.26) of Appendix B resulting in the first 67 lattice points and showing also the crossings of the Gödel walk with the hypercube HC_7^s .

The successive lattice points of the Gödel walk are orthogonal when calculated for the first 100 lattice points in the integer lattice $\mathbb{Z}_+^{25}$. The Gödel walk represents a unique walk in $\mathbb{Z}_+^k$, where $k \in \mathbb{Z}_+$ because it requires orthogonality between successive lattice points. There are 23 segments in $\mathbb{Z}_+^7$ and 28 segments in $\mathbb{Z}_+^3$ for the first 100 non-negative integers.

The orthogonality between successive lattice points remains valid within the segments that have more than one lattice point. Physicists visualize correlations between quantities graphically in the form of cubes that contain the respective quantities as the axes of the cube.

The Gödel walk presents a natural way of selecting mutually orthogonal sequential kinds of quantities. Inspection of the list (Table B.26) results in cubes $C(i, j, k)$, where i, j, k are successive Gödel numbers. We find 7 cubes

$C(3,4,5) = \{\mathsf{L}, \mathsf{T}^2, \mathsf{M}\}$, $C(5,6,7) = \{\mathsf{M}, \mathsf{TL}, \mathsf{I}\}$, $C(7,8,9) = \{\mathsf{I}, \mathsf{T}^3, \mathsf{L}^2\}$, $C(9,10,11) = \{\mathsf{L}^2, \mathsf{TM}, \Theta\}$, $C(11,12,13) = \{\Theta, \mathsf{LT}^2, \mathsf{N}\}$, $C(13,14,15) = \{\mathsf{N}, \mathsf{TI}, \mathsf{LM}\}$, $C(15,16,17) = \{\mathsf{LM}, \mathsf{T}^4, \mathsf{J}\}$ where we use the SI symbol for the dimensions (BIPM, 2006). The mutual orthogonality in the 7 cubes is invariant when the integer lattice points representing the cube axes are subject to a signed permutation.

2.13 Conclusion

This chapter contains the mathematical building blocks and tools for solving the first research question: *Is it possible to mathematically classify kinds of quantities?* These mathematical building blocks and tools are:

- the map of a physical quantity to a kind of quantity,
- the map of a kind of quantity to an N-dimensional cubic integer lattice point,
- the partitioning of integer lattice points in orbits of the lattice,
- the partitioning of the integer lattice based on the infinity norm,
- the map of ternary quantity equations to parallelograms,
- the bi-coloring of parallelograms,
- the selection of parallelograms that have unique semi-perimeters,
- the selection of parallelograms that are rectangles,
- the selection of parallelograms that are rectangles having a unique perimeter,
- the identification of orbit representatives of the integer lattice,
- the mapping of orbit representatives to signed Gödel numbers,
- the definition of a Gödel walk.

These mathematical building blocks and tools will be used in the next chapter for the mathematical classification of the SI kind of quantities where $N = 7$.

CHAPTER 3

SI elements of physics in $\{0,1\} \times \mathbb{Z}^7$

The goal in this chapter is to answer the first research question: *Is it possible to mathematically classify kinds of quantities?* The mathematical classification of the kinds of quantities required the exploration of the mathematical properties of the integer lattice points of $\{0,1\} \times \mathbb{Z}^7$.

We search for an equivalence relation that could create the 'Table of SI physics'. The motivation for this search strategy is that an equivalence relation results in the generation of equivalence classes, each being a partition of the integer lattice $\{0,1\} \times \mathbb{Z}^7$. Those partitions are disjoint sets resulting in a mathematical classification of the integer lattice points.

In this search we will focus on the sub-lattice $\{0\} \times \mathbb{Z}^7$. Within this sub-lattice we select another sub-lattice $\mathbb{Z}^7$ that is known in the literature as a N-dimensional cubic lattice where $N = 7$. The properties of $\mathbb{Z}^7$ are directly transferred to $\{0,1\} \times \mathbb{Z}^7$ by forming a Cartesian product with the set $\{0,1\}$.

3.1 Equivalence relation R_1

The 7-dimensional cubic lattice $\mathbb{Z}^7$ can be partitioned based on the norm squared $m = \boldsymbol{z} \cdot \boldsymbol{z}$ of the integer lattice point $\boldsymbol{z}$ (J. Conway et al., 1999, p.106-107). It results in centrally symmetric 7-hyperspheres with the square of the 7-hypersphere radius being a natural number $m \in \mathbb{N}$. Each 7-hypersphere is a finite set of integer lattice points. The number of lattice points on the 7-hypersphere corresponds to the number of ways to write a natural number as the sum of 7 squares and thus $m = \sum_{i=1}^{7} q_i^2$. These values are tabulated in the OEIS integer sequence A008451 (N. J. A. Sloane, 2006).

Those 7-hyperspheres treated as partitions of $\mathbb{Z}^7$ correspond to an equivalence relation R_1 on $\{0,1\} \times \mathbb{Z}^7$. The equivalence relation R_1 expresses that the integer lattice point has the same norm squared. We denote each 7-hypersphere as an equivalence class with the label $[a]_m$. It is obvious that the equivalence relation R_1 is reflexive, symmetric, and transitive and that the equivalence classes $[a]_m$ are disjoint. The number of equivalence classes $[a]_m$ of $\{0,1\} \times \mathbb{Z}^7$ is infinite.

3.2 Equivalence relation R_2

The 7-dimensional cubic lattice $\mathbb{Z}^7$ can be partitioned based on the infinity norm $s = \|z\|_\infty$ of the integer lattice point z. It results in centrally symmetric 7-hypercubes having edges equal to $2s$. The number of lattice points on the 7-hypercube is equal to $(2s+1)^7$. Those 7-hypercubes treated as partitions of $\mathbb{Z}^7$ correspond to an equivalence relation R_2 on $\{0,1\} \times \mathbb{Z}^7$. The equivalence relation R_2 expresses that the integer lattice point has the same infinity norm . We denote each 7-hypercube as an equivalence class with the label $[b]_s$. It is obvious that the equivalence relation R_2 is reflexive, symmetric, and transitive and that the equivalence classes $[b]_s$ are disjoint. The number of equivalence classes $[b]_s$ of $\{0,1\} \times \mathbb{Z}^7$ is infinite.

3.3 Equivalence relation R_3

The 7-dimensional cubic lattice $\mathbb{Z}^7$ can be partitioned based on the orbits of $\{0,1\} \times \mathbb{Z}^7$. We find 30 distinct cardinalities. The cardinalities of the orbits of $\{0,1\} \times \mathbb{Z}^7$ are 1, 14, 84, 128, 168, 280, 448, 560, 672, 840, 896, 1 680, 2 240, 2 688, 3 360, 4 480, 5 376, 6 720, 8 960, 13 440, 17 920, 20 160, 26 880, 40 320, 53 760, 80 640, 107 520, 161 280, 322 560, 645 120. The number of orbits of $\{0,1\} \times \mathbb{Z}^7$ is infinite.

3.4 Equivalence relation $R_1 \cap R_3$

The result of this intersection can be made available after approval by UGent TechTransfer in accordance with valorization project: P2021/066 – Mathematical Classification. The data are presently in the author's personal repository. The creation of an application giving public access to the research data is part of the future work.

3.5 Equivalence relation $R_2 \cap R_3$

The equivalence relation $R_2 \cap R_3$ is given in Table 3.1 in which the rows are representing the infinity norms $\ell_\infty = s$ of the orbit representative lattice points and the columns are the cardinalities $\#([w])$ of the orbits of the integer lattice $\{0,1\} \times \mathbb{Z}^7$. Thus, each orbit $[w]$ of $\{0,1\} \times \mathbb{Z}^7$ is mapped to a cell in Table 3.1. Each *SI element of physics* has a coordinate $(s, \#([w]))$ and to that coordinate we associate a value m that we call the *mass of the SI element*. The value of the mass m is the number of orbits of $\{0,1\} \times \mathbb{Z}^7$ having the properties s and $\#([w])$ in common.

Definition 3.5.1. *An SI element of physics is a set of orbits of* $\{0,1\} \times \mathbb{Z}^7$ *where each orbit has the same infinity norm* s *and the same orbit cardinality*

$\#([w])$ *and where the mass of the element is given by the number of orbits in the selected set of orbits.*

The numbers in font bold and red color are the elements that contain orbit representatives listed in the lexicon Appendix A. We see that a lot of elements contain physical quantities that are not yet explored or that have not been published. It is known that sets having the same cardinality can be mapped through a bijection. When the mass has the value of 1 then a unique bijection exists between these orbits. This can be observed for the cardinalities 14, 84, 128, 280, 448, 560 and 672. Increasing the infinity norm for the selected cardinality is not increasing the value of the mass of the element. For the cardinalities 168, 8 960 we observe an arithmetic progression with a common difference of 1. For the cardinalities 840, 896, 2 240, 4 480 we observe an arithmetic progression with a common difference of 2. For the cardinality 2 688 we observe an arithmetic progression with a common difference of 4.

Table 3.1: Table of SI physics.

s	1	14	84	128	280	448	560	672	168	840	896	1680	2240	2688	3360	4480	5376	6720	8960	13440	17920	20160	26880	40320	53760	80640	107520	161280	322560	645120	RowSum
0	**1**	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
1	0	**1**	**1**	1	**1**	1	**1**	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	7
2	0	**1**	**1**	1	1	1	1	1	**1**	**2**	**2**	0	**2**	4	**3**	2	0	**4**	1	0	0	0	0	0	0	0	0	0	0	0	28
3	0	**1**	1	1	1	1	1	1	**2**	**4**	4	**1**	**4**	8	**6**	4	3	**11**	2	12	3	3	9	1	0	0	0	0	0	0	84
4	0	**1**	1	1	1	1	1	1	**3**	6	6	**3**	6	12	9	6	9	**21**	3	**37**	9	9	31	7	16	10	0	0	0	0	210
5	0	1	1	1	1	1	1	1	**4**	8	8	6	8	16	12	8	18	34	4	76	18	18	70	22	64	41	5	15	0	0	462
6	0	1	1	1	1	1	1	1	**5**	10	10	10	**10**	20	15	10	30	50	5	130	30	30	130	50	160	105	25	75	7	0	924
7	0	1	1	1	1	1	1	1	6	12	12	15	12	24	18	12	45	69	6	**200**	45	45	215	95	320	215	75	225	42	1	1716
8	0	1	1	1	1	1	1	1	7	14	14	21	14	28	21	14	63	91	7	287	63	63	329	161	560	385	175	525	147	7	3003
9	0	1	1	1	1	1	1	1	8	16	16	28	**16**	32	24	16	84	116	8	392	84	84	476	252	896	630	350	1050	392	28	5005
10	0	1	1	1	1	1	1	1	9	18	18	36	18	36	27	18	108	144	9	**516**	108	108	660	372	1344	966	630	1890	882	84	8008
11	0	1	1	1	1	1	1	1	10	20	20	45	20	40	30	20	135	175	10	660	135	135	885	525	1920	1410	1050	3150	1764	210	12376
12	0	1	1	1	1	1	1	1	11	22	22	55	**22**	44	33	22	165	209	11	**825**	165	165	1155	715	2640	1980	1650	4950	3234	462	18564
13	0	1	1	1	1	1	1	1	12	24	24	66	24	48	36	24	198	246	12	1012	198	198	1474	946	3520	2695	2475	7425	5544	924	27132
14	0	1	1	1	1	1	1	1	13	26	26	78	26	52	39	26	234	286	13	1222	234	234	1846	1222	4576	3575	3575	10725	9009	1716	38760
15	0	1	1	1	1	1	1	1	14	28	28	91	28	56	42	28	273	329	14	1456	273	273	2275	1547	5824	4641	5005	15015	14014	3003	54264
16	0	1	1	1	1	1	1	1	15	30	30	105	30	60	45	30	315	375	15	1715	315	315	2765	1925	7280	5915	6825	20475	21021	5005	74613
17	0	1	1	1	1	1	1	1	16	32	32	120	32	64	48	32	360	424	16	2000	360	360	3320	2360	8960	7420	9100	27300	30576	8008	100947
18	0	1	1	1	1	1	1	1	17	34	34	136	34	68	51	34	408	476	17	2312	408	408	3944	2856	10880	9180	11900	35700	43316	12376	134596
19	0	1	1	1	1	1	1	1	18	36	36	153	36	72	54	36	459	531	18	2652	459	459	4641	3417	13056	11220	15300	45900	59976	18564	177100
20	0	1	1	1	1	1	1	1	19	38	38	171	38	76	57	38	513	589	19	3021	513	513	5415	4047	15504	13566	19380	58140	81396	27132	230230
21	0	1	1	1	1	1	1	1	20	40	40	190	40	80	60	40	570	650	20	3420	570	570	6270	4750	18240	16245	24225	72675	108528	38760	296010
22	0	1	1	1	1	1	1	1	21	42	42	210	42	84	63	42	630	714	21	3850	630	630	7210	5530	21280	19285	29925	89775	142443	54264	376740
23	0	1	1	1	1	1	1	1	22	44	44	231	44	88	66	44	693	781	22	4312	693	693	8239	6391	24640	22715	36575	109725	184338	74613	475020
24	0	1	1	1	1	1	1	1	23	46	46	253	46	92	69	46	759	851	23	4807	759	759	9361	7337	28336	26565	44275	132825	235543	100947	593775
25	0	1	1	1	1	1	1	1	24	48	48	276	48	96	72	48	828	924	24	5336	828	828	10580	8372	32384	30866	53130	159390	297528	134596	736281
26	0	1	1	1	1	1	1	1	25	50	50	300	50	100	75	50	900	1000	25	5900	900	900	11900	9500	36800	35650	63250	189750	371910	177100	906192

The columns with cells containing only *one* orbit have been placed to the left side of the table in increasing cardinality value. The remaining columns are placed in increasing cardinality from left to right starting with cardinality 168. The total number of orbits in the hypercube with infinity norm $s \leq 7$ yields $\sum_{s=0}^{7} \mathrm{RowSum(s)} = 3\,432$. The list of orbit representatives can be made available after approval by UGent TechTransfer in accordance with valorization project: P2021/066 – Mathematical Classification. The data are presently in the author's personal repository. The creation of an application giving public access to the research data is part of the future work.

Observe that each term of the OEIS sequence A000579 in which n is substituted by $(s+6)$ is obtained by taking the sum of the masses in each row of the Table 3.1. The OEIS sequence A000579 represents the binomial coefficients $\binom{n}{7-1}$ in N. J. A. Sloane (2015).

The column denoted RowSum in Table 3.1 is obtained using the equation $a(s) = \binom{s+6}{6}$.

Example 3.5.1. *The SI element of physics with coordinates* $(s, \#\,([w])) = (2, 2\,240)$ *has mass* $m = 2$. *It means that the element contains* $m = 2$ *orbits having each the cardinality 2 240.*

The sequences of integers in the columns of Table 3.1 can be referenced to sequences in the On-line Encyclopedia of Integer Sequences (OEIS). Each sequence in the columns of Table 3.1 can be generated and extended beyond infinity norm $s = 26$ using an ordinary generating function as given in Table 3.2.

Table 3.2: Table of sequences and ordinary generating functions in $\mathbb{Z}^7$.

$\#\,(\{\mathrm{Orb}\,(z)\})$	OEIS reference	Ordinary generating function $G(a_n; z)$
1	A000007	Not defined
14	A000012	$1/(1-z)$
84	A000012	$1/(1-z)$
128	A000012	$1/(1-z)$
168	A001477	$z^2/(1-z)^2$
280	A000012	$1/(1-z)$
448	A000012	$1/(1-z)$
560	A000012	$1/(1-z)$
…	…	…

$\#\left(\{\text{Orb}\,(\boldsymbol{z})\}\right)$	OEIS reference	Ordinary generating function $G(a_n; z)$
672	A000012	$1/(1-z)$
840	A005843	$2z^2/(1-z)^2$
896	A005843	$2z^2/(1-z)^2$
1680	A000217	$z^3/(1-z)^3$
2240	A005843	$2z^2/(1-z)^2$
2688	A008586	$4z^2/(1-z)^2$
3360	A008585	$3z^2/(1-z)^2$
4480	A005843	$2z^2/(1-z)^2$
5376	A045943	$3z^3/(1-z)^3$
6720	A115067	$z^2(4-z)/(1-z)^3$
8960	A001477	$z^2/(1-z)^2$
13440	A266398	$z^3(12-11z)/(1-z)^4$
17920	A045943	$3z^3/(1-z)^3$
20160	A045943	$3z^3/(1-z)^3$
26880	A266397	$z^3(9-5z)/(1-z)^4$
40320	A002412	$z^3(1+3z)/(1-z)^4$
53760	A102860	$16z^4/(1-z)^4$
80640	A266396	$z^4(10-9z)/(1-z)^5$
107520	A154286	$5z^5/(1-z)^5$
161280	A266395	$15z^5/(1-z)^5$
322560	A266387	$7z^6/(1-z)^6$
645120	A000579	$z^7/(1-z)^7$
RowSum	A000579	$1/(1-z)^7$

Consider the first non-zero mass in each column of Table 3.1. That non-zero mass in the specific column can be considered as a set containing the fundamental orbits because bijections can be formed between these orbits and those in the remaining part of the column.

Construct a set containing the sets of fundamental orbits for each orbit cardinality and call this the *backbone structure of* $\{0,1\} \times \mathbb{Z}^7$. The backbone structure of $\{0,1\} \times \mathbb{Z}^7$ contains 115 distinct sets, given in Table 3.3, representing 8 177 907 integer lattice points .

Table 3.3: Backbone structure of SI physics.

Cardinalities	1	14	84	128	168	280	448	560	672	840	896	1 680
Number of orbits	1	1	1	1	1	1	1	1	1	2	2	1
Number of lattice points	1	14	84	128	168	280	448	560	672	1 680	1 792	1 680

Cardinalities	2 240	2 688	3 360	4 480	5 376	6 720	8 960	13 440	17 920	20 160	26 880	40 320
Number of orbits	2	4	3	2	3	4	1	12	3	3	9	1
Number of lattice points	4 480	10 752	10 080	8 960	16 128	26 880	8 960	161 280	53 760	60 480	241 920	40 320

Cardinalities	53 760	80 640	107 520	161 280	322 560	645 120
Number of orbits	16	10	5	15	7	1
Number of lattice points	860 160	806 400	537 600	2 419 200	2 257 920	645 120

3.5.1 Cardinalities of orbits in sub-lattices of $\{0,1\} \times \mathbb{Z}^7$

The connectives between the sublattices of SI physics are given in Table 3.4

Table 3.4: Cardinalities of orbits in sub-lattices of $\{0,1\} \times \mathbb{Z}^7$.

Cardinalities	$\mathbb{Z}^0$	$\mathbb{Z}^1$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}^5$	$\mathbb{Z}^6$	$\mathbb{Z}^7$
1	1	0	0	0	0	0	0	0
14	0	1	0	0	0	0	0	0
84	0	0	1	0	0	0	0	0
128	0	0	0	0	0	0	0	1
168	0	0	1	0	0	0	0	0
280	0	0	0	1	0	0	0	0
448	0	0	0	0	0	0	1	0
560	0	0	0	0	0	1	0	0
672	0	0	0	0	0	1	0	0
840	0	0	0	1	0	0	0	0
896	0	0	0	0	0	0	0	1
1 680	0	0	0	1	0	0	0	0
2 240	0	0	0	0	1	0	0	0
2 688	0	0	0	0	0	0	1	1
3 360	0	0	0	0	0	1	0	0
4 480	0	0	0	0	0	0	0	1
5 376	0	0	0	0	0	0	0	1
6 720	0	0	0	0	1	1	1	0
8 960	0	0	0	0	0	0	1	0
13 440	0	0	0	0	1	1	1	1
17 920	0	0	0	0	0	0	0	1
20 160	0	0	0	0	0	1	0	0
26 880	0	0	0	0	0	0	1	1
40 320	0	0	0	0	0	1	1	0
53 760	0	0	0	0	0	0	1	1
80 640	0	0	0	0	0	1	1	1
107 520	0	0	0	0	0	0	0	1
161 280	0	0	0	0	0	0	1	1
322 560	0	0	0	0	0	0	1	1
645 120	0	0	0	0	0	0	0	1

3.5.2 Connectives between the sub-lattices of $\{0,1\} \times \mathbb{Z}^7$

Connectives between the sub-lattices of $\{0,1\} \times \mathbb{Z}^7$ are identified as orbits with the cardinalities 2 688, 6 720, 13 440, 26 880, 40 320, 53 760, 80 640, 161 280, 322 560. The connectives are possible based on the potential existence of bijections between the orbit representatives of different sub-lattices. The largest connective exists for the orbits having cardinality 13 440. It is a connective of the sub-lattices $\mathbb{Z}^4, \mathbb{Z}^5, \mathbb{Z}^6, \mathbb{Z}^7$.

From Appendix A, we find that the kinds of quantities belonging to that connective with cardinality 13 440 are: the electric constant $(0 \mid -3,-1,4,2,0,0,0)$, the first hyper-polarizability $(0 \mid -1,-2,7,3,0,0,0)$, and the second hyper-polarizability $(0 \mid -2,-3,10,4,0,0,0)$.

3.6 Equivalence relation $R_1 \cap R_2 \cap R_3$

The result of this intersection can be made available after approval by UGent TechTransfer in accordance with valorization project: P2021/066 – Mathematical Classification. The data are presently in the author's personal repository. The creation of an application giving public access to the research data is part of the future work.

3.7 Conclusion

In this chapter we succeeded to create a table that mathematically classifies the kinds of quantities based on the mathematical tools created in the previous chapter. It is formed by the equivalence relation $R_2 \cap R_3$ generating a Table 3.1, denoted 'Table of SI physics', in which the rows are representing the infinity norms $\ell_\infty = s$ of the orbit representative lattice points and the columns are the cardinalities $\#([w])$ of the orbits of the integer lattice $\{0,1\} \times \mathbb{Z}^7$.

This 'Table of SI physics' representing the mathematical classification of kinds of quantities is to the best of the author's knowledge completely new, except in the author's publication `https://www.sciencedirect.com/science/article/pii/S0016003222006792`.

We identify fundamental orbits as the first non-zero mass cells in the columns of Table 3.1.

Moreover, we connect the fundamental orbits to a backbone structure in orthant 1 that connects the lattice point $(0 \mid 0^7)$ with the lattice point $(0 \mid 7,6,5,4,3,2,1)$, and where the fundamental orbits are located on the Gödel's walk through orthant 1 of the integer lattice $\{0,1\} \times \mathbb{Z}^7$.

The 'Table of SI physics' is the answer to our first research question: *Is it possible to mathematically classify kinds of quantities?* The answer is *yes* and yields the Table 3.1. In the next chapter we will search for a method to classify quantity equations such that we can find an answer to the second research question: *Is it possible to mathematically select which quantity equations are 'laws of physics'?*

CHAPTER 4

Encoding and decoding physical quantities

Some engineering problems have unknown systems of partial differential equations, and in those problems, one relies on the state-of-the-art Buckingham theorem in which dimensionless products of kinds of quantities are formed from the set of variables that the engineer has identified. Application of the Buckingham theorem results in a reduction of the number k of dimensional variables in N dimensional units to a set of $(k-r)$ dimensionless products in which r is the rank of the $N \times k$ dimensional matrix. The engineer starts with a formal description of the problem postulating the equation

$$f(x_1, \ldots, x_k) = 0\,, \tag{4.1}$$

and using the theorem obtains a formal description of lower complexity, and thus reduces the computational load, through the equation

$$F(\pi_1, \ldots, \pi_{(k-r)}) = 0\,, \tag{4.2}$$

that can further be reduced to

$$\pi_1 = g(\pi_2, \pi_3, \ldots, \pi_{(k-r)})\,. \tag{4.3}$$

The problem is to make a good choice between the dependent and independent variables and subsequently their associated dimensionless products. The new method contains an automatic selection of the independent variables, unlike the Buckingham theorem in which the user must select the independent variables based on his experience or skill. This could lead to a false modeling of the engineering problem, requiring the re-iteration of all the calculations resulting in a waste of time and computing resources. The method is computing efficient as it is based on low complexity, high performing, and well-established computer algorithms of number theoretic functions. The computer efficacy of the method is very high as it maps, based on the similitude principle, for example 1 680 engineering problems to 1 canonical problem as given in the case of the first order partial derivative of the power density with respect to time. The computer efficacy depends on the choice of the kind of quantity and the number of base quantities. The computer efficacy ratios for kinds of quantities expressed in SI units and based on their orbit cardinalities are given in the finite set {28, 168, 256, 336, 560, 896, 1120, 1 344, 1 680, 1 792, 3 360, 4 480, 5 376, 6 720, 8 960, 10 752, 13 440, 17 920, 26 880, 35 840, 40 320, 53 760, 80 640, 107 520, 161 280, 215 040, 322 560, 645 120, 1 290 240}.

4.1 Method for encoding/decoding physical quantities

In this section we present the three steps of the machine-implementable encoding-decoding method for forming quantity equations from a selection of sets of dependent and independent variables, comprises the steps of:

- defining an input from a selection of a set or sets of dependent and independent variables, the input comprises a list of physical quantities;
- processing the input, comprising the steps of encoding and decoding of dimensionless groups in an integer lattice, using integer factorization techniques, thereby obtaining a system of quantity equations;
- presenting the system of quantity equations as output.

This *encoding-decoding method* is new, to the best of the author's knowledge.

A visual summary of the steps of the encoding-decoding method is given in Figure 4.1.

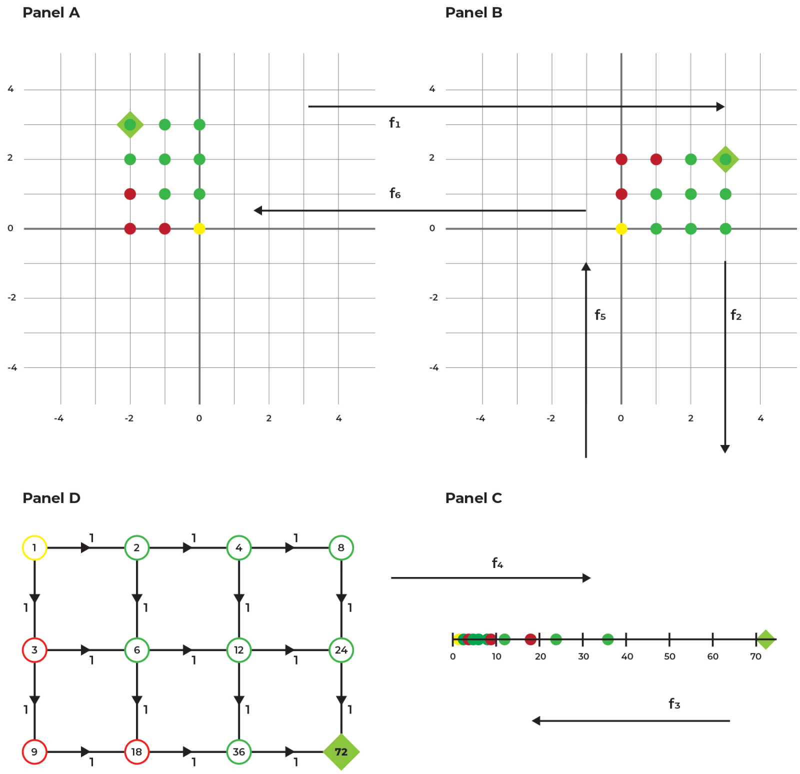

Figure 4.1: The encoding-decoding method is visually exemplified in $\mathbb{Z}^2$ for the lattice point $(-2, 3)$. The sequence of steps starts in quadrant II of panel A at the green square. The center of the square $(-2, 3)$ is mapped by a signed permutation f_1 to the green square with point $(3, 2)$ in quadrant I of panel B. The lattice point $(3, 2)$ is mapped by f_2 to the integer 72, being its signed Gödel number, shown as a green square in panel C. The Gödel number 72 has the divisors set $\{1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72\}$ that generates a lattice with a Hasse diagram, shown in panel D. The subset $\{2, 4, 6, 8, 12, 24, 36, 72\}$ comprises the signed Gödel numbers of orbit representatives shown as green dots in panel D. The divisors of 72 are subsequently mapped by f_4 on the $\mathbb{Z}$ axis as red and green dots in panel C. Each of these red and green dots represent integers that are subjected to a prime factorization and mapped by f_5 as red and green lattice points in quadrant I of panel B. Subsequently these red and green lattice points in panel B are mapped by the inverse of the signed permutation f_6 to the red and green lattice points in quadrant II of panel A. The encoding-decoding method generates the unknown red and green lattice points that become the arguments of the multivariate trial function $F(Q^1, \ldots, Q^{11}) = 0$.

4.1.1 Step one: input and selection of variables

The encoding-decoding method comprises the step of choosing a list of (physical) base quantities by selecting and ordering N base quantities, where $N \in \{1, \ldots, 7\}$. The base quantities may be chosen freely, but are preferably selected from the seven SI base quantities (JCGM, 2012) in the following order time, length, mass, electric current, thermodynamic temperature, amount of substance, and luminous intensity.

The encoding-decoding method allows for other choices to be made, resulting in a different meaning of the quantity equations. The encoding-decoding method mainly relates to the syntax of the quantity equations , while the semantics depends on the chosen base quantities and their ordering. The semantics also depends on the (physical) theory upon which the encoding-decoding method is applied. An integer lattice point $\boldsymbol{q}$ of $\{0,1\} \times \mathbb{Z}^N$ has the coordinates $\boldsymbol{q} = (q_0 \mid q_1, \ldots, q_N)$, where $q_0 \in \{0,1\}$ and $q_1, \ldots, q_N \in \mathbb{Z}$.

The orbit representative,denoted $\mathrm{Orb}\,(\boldsymbol{q})$, is found by taking the absolute value of the coordinates of the integer lattice point $\boldsymbol{q}$ obtaining the lattice point $(q_0 \mid |q_1|, \ldots, |q_N|)$, and sorting them by decreasing order, and renaming the coordinates such that $\mathrm{Orb}\,(\boldsymbol{q}) = (z_0 \mid z_1, \ldots, z_N)$, where $z_0 \in \{0,1\}$ and $z_1, \ldots, z_N \in \mathbb{Z}$, and $z_1 \geq z_2 \geq \ldots \geq z_N$.

In some cases, the encoding-decoding method comprises the step of selecting M physical quantities $Q^1, \ldots, Q^M$ to be analyzed, where $M \in \mathbb{N}_1$ and the step of comparing the M physical quantities to a kind of quantity in a database to determine the respective integer lattice point of $\{0,1\} \times \mathbb{Z}^N$.

The encoding-decoding method comprises the step of calculating for each of the M integer lattice points their respective orbit representative $\mathrm{Orb}\,(\boldsymbol{q}^m)$, where $m \in \{1, \ldots, M\}$. In some cases those orbit representatives will be identical e.g. with $N = 7$ one has $\mathrm{Orb}\,\big((0 \mid 0, 1, 0^5)\big) = \mathrm{Orb}\,\big((0 \mid 0, 0, -1, 0^4)\big) = (0 \mid 1, 0^6)$.

For each of the M orbit representatives $\mathrm{Orb}\,(\boldsymbol{q}^m)$ we calculate the degree:

$$d_m = \|(z_1^m, \ldots, z_N^m)\|_1 ,$$

being the respective 1-norms of the lattice points $(z_1^m, \ldots, z_N^m)$, where $m \in \{1, \ldots, M\}$.

Furthermore, we determine the largest degree:

$$d_s = \max\{d_1, \ldots, d_s, \ldots, d_M\} .$$

The orbit representative with the largest degree is denoted $\boldsymbol{y} = \mathrm{Orb}\,(\boldsymbol{q}^s)$ and calling its associated integer lattice point $\boldsymbol{q}^s$.

In some cases the largest degree can have a multiplicity higher than one. Assume that the multiplicity is b_m, then there will be b_m dependent variables $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{b_m}$ and the system is comparable to the control engineering system denoted MIMO (multiple-input multiple-output) instead of the common multivariate system denoted MISO (multiple-input single-output). The simplest system is the SISO (single-input single-output). The ranking of the dependent variables can be done by ordering the dependent variables based on the signed Gödel number. The most important dependent variable has the largest signed Gödel number $G_{max} = \max\{G(\boldsymbol{y}_i)\}$.

4.1.2 Step two: encoding, decoding, factorization, and additive partitioning

Step two comprises the encoding of each orbit representative $\mathrm{Orb}\,(\boldsymbol{q}^m)$ using the mapping:

$$G\colon \{0,1\} \times \mathbb{Z}^N \mapsto SG \subset 2\mathbb{Z}\,,$$
$$\mathrm{Orb}\,(\boldsymbol{q}^m) \mapsto G(\mathrm{Orb}\,(\boldsymbol{q}^m)) \coloneqq (-1)^{z_0^m}\, p_1^{z_1^m} \ldots p_N^{z_N^m}\,,$$

where p_n is the n-th prime number, $n \in \{1,\ldots,N\}$, and $m \in \{1,\ldots,M\}$. We call the encoding a *signed Gödel encoding* through its similarity with Gödel numbers (Gödel, 1986, p. 129). Note that for $N = 7$ we map the seven ISQ symbols (JCGM, 2012) to the first seven prime numbers that are $2, 3, 5, 7, 11, 13, 17$. A similar type of encoding has been proposed by Bhargava (1993) for checking the dimensional consistency in databases.

All the *orbit representatives* of the integer lattice points of $\{0,1\}\times\mathbb{Z}^N$ yield an *even* signed Gödel number because the prime number $p_1 = 2$ is always a factor of the signed Gödel number of an orbit representative. But, not all even integers of $2\mathbb{Z}$ are mapped to orbit representatives.

However, there exist a one-to-one relation between all the orbit representatives of $\{0,1\} \times \mathbb{Z}^N$ and a subset SG of the even integers $2\mathbb{Z}$. In the case that there are more than one dependent variable a *unique* ranking between the dependent variables is obtained based on the signed Gödel encoding of their orbit representative.

The most important dependent variable has the largest signed Gödel number G_{max}. The number of divisors of G_{max} is $\tau(G_{max})$.

Furthermore, the encoding-decoding method comprises the decoding of each H-factorization of the largest signed Gödel number G_{max}, where $G_{max} = \pm\prod_{h=1}^{H} F_h$, by calculating the prime factorization of each distinct factor F_h with the index $h \in \{1,\ldots,H\}$ according to the following mapping:

$$F_h\colon \mathbb{Z} \to \{0,1\} \times \mathbb{Z}^N\,,$$
$$F_h \coloneqq (-1)^{x_0^h} p_1^{x_1^h} \ldots p_N^{x_N^h} \mapsto (x_0^h \mid x_1^h, \ldots, x_N^h)\,.$$

The algebraic structure $(\mathbb{Z},\cdot)$ is mapped to the algebraic structure $\{0,1\} \times (\mathbb{Z}^N,+)$, where the operator $+$ of $\{0,1\}$ is the modular addition. The H-factorization of the integer k in distinct integer factors, given in Appendix D, is mapped to an additive partitioning of the orbit representative in vectors of the nonnegative orthant with sign $(+,+,\ldots,+)$ of $\{0,1\} \times \mathbb{Z}^N$ resulting in $(H+1)$-ary vector equations. The selected decoding is advantageous because it is based on the fundamental theorem of arithmetic: *the unique prime factorization of a natural number*.

The encoding-decoding method comprises the step of calculating the $(N+1)\times(N+1)$ signed permutation matrix $\mathbf{P}$ that maps the orbit representative $\boldsymbol{y}$ to the integer lattice point $\boldsymbol{q}^s$ of $\{0,1\}\times\mathbb{Z}^N$ such that $\boldsymbol{y}=\mathbf{P}\tilde{\boldsymbol{q}}^s$, where $\tilde{\boldsymbol{q}}^s$ is the transposed vector of the integer lattice point $\boldsymbol{q}^s$. Define $K=\dfrac{1}{2}\tau(G_{max})-1$ dimensionless quantities π_k from the $\mathbf{P}^{-1}$-mapped ternary vector equations and generate a positively homogeneous measurement model $u(\pi_1,\ldots,\pi_K)=0$.

4.1.3 Step three: output and visualization

Step three comprises the following actions:

- selecting the M integers $G(\mathrm{Orb}\,(\boldsymbol{q}^m))$, where $m\in\{1,\ldots,M\}$;
- ordering the fully ordered sets of the divisors of the M integers $G(\mathrm{Orb}\,(\boldsymbol{q}^m))$ in their subsets of equal degree;
- building a divisor lattice for each of the M integers $G(\mathrm{Orb}\,(\boldsymbol{q}^m))$ by stacking the subsets from low to high degree $d_m=\Omega(|G(\mathrm{Orb}\,(\boldsymbol{q}^m))|)$, where $\Omega(n)$ is a prime factor counting function;
- determine the multiplicity b_m of the largest degree, if $b_m\neq 1$, then the engineering problem has multiple dependent variables;
- taking the union of the $w_m=b_m$ division lattices.

The steps outlined above are functional to construct a Hasse diagram.

In some cases, the encoding-decoding method comprises the steps of:

- selecting the p-norm $\|\boldsymbol{q}\|_p=\left(\sum_{n=1}^{N}|q_n|^p\right)^{1/p}$;
- calculating the p-norm between all the lattice points generated as the union of the division lattices;
- visualizing the Euclidean graph for the p-norm of all the lattice points generated as the union of the division lattices.

The encoding-decoding method comprises the creation of a system of equations from the vector equations of the integer lattice $\{0,1\}\times\mathbb{Z}^N$, wherein the system of equations comprises algebraic equations, and/or ordinary differential equations, and/or partial differential equations, and/or integro-differential equations. Moreover, the encoding-decoding method comprises the labeling of the variables using a lexicon based on the present status of a compilation of published SI physical quantities. This supports the user in order to provide meaning to the equations generated by the computer.

The encoding-decoding method comprises the steps of:

- updating a global lexicon or dictionary with the output of the encoding-decoding method as described herein;
- analyzing the results through viewing information, visualizations, graphs, tables, and the like;
- optionally, providing extra information if the quantity equations are known.

This visualization allows a holistic approach to the engineering problem. After the results have been obtained, they can be added to the global lexicon or dictionary. Representatives of the lexicon of surveyed SI physical quantities in $\{0,1\} \times \mathbb{Z}^N$ are given in Appendix A. The end user can view information, visualizations, graphs, tables, etc... related to the quantity equation. If the quantity equation is known, extra information can be provided.

In some cases, the lexicon is based on another system of units than the SI database. Another system of units is the Metre-Kilogram-Second-Ampere (MKSA) system in which only four base quantities are used for the description of engineering problems.

Newell (2014) of the National Institute of Standards and Technology (NIST) proposed a seven base quantity system using: frequency, velocity, action, electric charge, heat capacity, amount of substance and luminous intensity. In this base system the kind of quantity energy has the coordinate $(0 \mid 1, 0, 1, 0, 0, 0, 0)$ with signed Gödel number six instead of the SI coordinate $(0 \mid -2, 2, 1, 0, 0, 0, 0)$ with signed Gödel number 180. The base quantity system proposed by Newell results in a smaller number for the kind of quantity energy, which in turn allow for the energy to become an independent variable in many engineering problems. Unfortunately, the base quantity system proposed by Newell was *not adopted* in 2019 by the international community.

4.2 Second order partial derivative of energy density with respect to time

The case study was inspired from an article published in IEEE Control Systems that highlight modeling and control problems in the energy transformation challenge that is taking place in our present society. The article of Parisini and Blaabjerg gives an introduction to the challenges in power electronics-dominated grids with large renewable generation capacity. Important changes to the present power grid, that has a fixed frequency and voltage, include an increase in renewable generation capacity (photovoltaic capacity and wind turbine capacity), a modernized grid structure, electric cars, and a higher degree of automation (Parisini & Blaabjerg, 2021). By introducing new controller structures with frequency dynamics up to kilohertz, power grid operation and stability must be more carefully studied (Parisini & Blaabjerg, 2021). Such a study of the future power grid requires a mathematical model that comprises wave phenomena, electromagnetic phenomena, electromechanic phenomena, and thermodynamic phenomena with their corresponding time constants and dynamics (Parisini & Blaabjerg, 2021, p.12). We embrace this mathematical modeling problem as a use case for the new encoding-decoding method for variables in engineering problems. Let instantaneous power $P(t)$ be defined as $P(t) = \dfrac{\mathrm{d}E(t)}{\mathrm{d}t}$, in which $E(t)$ is a time-varying energy. . .

If variations of the power grid have to be mathematically modeled we should consider in a first approximation $\dfrac{\mathrm{d}P(t)}{\mathrm{d}t}$. Instead of considering the energy $E(t)$ we model the energy density $W(\boldsymbol{r}, t)$ that has intensive properties. We consider only the time variable of the energy density but it should be obvious that there is also a spatial variable that should be considered to make the model complete. To illustrate the encoding-decoding method we consider only the second order partial derivative of the energy density with respect to time, and postulate the measurement model

$$F_1(\frac{\partial^2 W(\boldsymbol{r}, t)}{\partial t^2}, Q^2, \ldots, Q^M) = 0\,, \tag{4.4}$$

in which the Q^m are unknown physical quantities with $m \in \{2, \ldots, M\}$ with the value M unknown.

The quantity equation (4.4) is technically relevant for capturing the first order dynamics of the power grids, in which wind turbine power and photovoltaic power from solar farms, and millions of houses are connected to the classical power grid, in which the power comes from a steam plant, a gas turbine, or a nuclear power plant (Parisini & Blaabjerg, 2021). Higher order partial derivatives could be considered by the researcher. Such a modeling would include the results of the second order modeling because the orbit of the higher order partial derivatives of the energy density $W(\boldsymbol{r}, t)$ would have larger Gödel numbers but would have as divisor the Gödel number corresponding to the second order partial derivative of the energy density with respect to time.

The encoding-decoding method will search for $(M - 1)$ independent variables that are *effectively connected* to the second order partial derivative of the energy density with respect to time. It means that we postulate a MISO system from a control engineering point of view, but further we don't know the $(M - 1)$ independent variables. We select $N = 7$ to perform the modeling in compliance with the International System of Quantities. The dimension of the kind of quantity called 'second order partial derivative of the energy density with respect to time' is denoted by $\dim Q^1 = \mathsf{T}^{-4}\mathsf{L}^{-1}\mathsf{M}^1\mathsf{I}^0\Theta^0\mathsf{N}^0\mathsf{J}^0$ and represented by the integer lattice point $\boldsymbol{q}^1 = (0 \mid -4, -1, 1, 0, 0, 0, 0)$, and abbreviated using the Conway notation (N. J. A. Conway J.H. and. Sloane, 1998, p. 101) to $\boldsymbol{q}^1 = (0 \mid -4, -1, 1, 0^4)$ when needed.

4.2.1 Encoding

We encode the kind of quantity called the second order partial derivative of the energy density with respect to time $\dfrac{\partial^2 W(\boldsymbol{r},t)}{\partial t^2}$ by determining the orbit representative $\mathrm{Orb}\left((0 \mid -4,-1,1,0^4)\right) = (0 \mid 4,1^2,0^4)$ that is an integer lattice point of the positive orthant of $\{0,1\} \times \mathbb{Z}^7$. We map the orbit representative $\mathrm{Orb}\left(\boldsymbol{q}^{1}\right) = (0 \mid 4,1^2,0^4)$ to the integer $G(\mathrm{Orb}\left(\boldsymbol{q}^{1}\right))$ of the set of signed Gödel numbers denoted SG where $SG \subset 2\mathbb{Z}$ according to the mapping:

$$
\begin{aligned}
&G\colon \{0,1\} \times (\mathbb{Z}^7) \to SG \subset 2\mathbb{Z}\,, \\
&\mathrm{Orb}\,(\boldsymbol{q}^m) \mapsto G(\mathrm{Orb}\,(\boldsymbol{q}^m)) \coloneqq (-1)^{z_0^m} p_1^{z_1^m} \ldots p_7^{z_7^m}\,,
\end{aligned}
$$

of the orbit representative $\mathrm{Orb}\,(\boldsymbol{q}^m)$ in the lattice $\{0,1\} \times \mathbb{Z}^7$, in which z_n is the n-th power of the n-th prime number with $n \in \{1,\ldots,7\}$, such that

$$
G((0 \mid 4,1^2,0^4)) = (-1)^0 2^4 3^1 5^1 7^0 11^0 13^0 17^0 = 240\,.
$$

The number of positive divisors of $l \in \mathbb{N}_1$ is denoted $\tau(l)$. We find $\tau(240) = 20$. The fully ordered set $\{1,2,3,4,5,6,8,10,12,15,16,20,24,30,40,48,60,80,120,240\}$ is formed from the divisors of 240. The subset $\{2,4,6,8,12,16,24,30,48,60,120,240\}$ comprises the signed Gödel numbers of orbit representatives.

We perform the factorization of the number 240 in H *distinct* factors and obtain the following: nine factors in two distinct factors, twelve factors in three distinct factors, and three factors in four distinct factors.

The number of factorizations of a natural number from 1 to 10000 in distinct factors greater than 1 are tabulated in the On-Line Encyclopedia of Integer Sequences (OEIS) with identifier A045778 (Wilson, 2009).
The factorizations in distinct factors have been listed by Mathar at http://oeis.org/A045778/a045778.txt for natural numbers from 1 to 1500.

4.2.2 Decoding

We apply a decoding that maps each factor F_h of $G(\mathrm{Orb}\,(\boldsymbol{q}^m))$ to the positive orthant of the integer lattice $\{0,1\} \times \mathbb{Z}^7$:

$$
\begin{aligned}
&F_h\colon \mathbb{Z} \to \{0,1\} \times \mathbb{Z}^7\,, \\
&F_h \coloneqq (-1)^{x_0^h} p_1^{x_1^h} \ldots p_7^{x_7^h} \mapsto (x_0^h \mid x_1^h, \ldots, x_7^h)\,.
\end{aligned}
$$

This decoding creates a cluster of 20 integer lattice points in $\{0,1\} \times \mathbb{Z}^7$ corresponding to the number of divisors. The degree of the orbit representative $\boldsymbol{z} = \mathrm{Orb}\,(\boldsymbol{x})$ is calculated through the equation:

$$
\begin{aligned}
&\Omega(|G(\mathrm{Orb}\,(\boldsymbol{q}^m))|) = \deg(\mathrm{Orb}\,(\boldsymbol{q}^m)) = \deg(\boldsymbol{z}) \\
&= \|\boldsymbol{z}\|_1 = z_1 + z_2 + \ldots + z_7\,,
\end{aligned}
$$

and thus $\deg((0 \mid 4, 1^2, 0^4)) = 6$. The lattice points can be grouped into seven distinct sets S_d. These seven sets, ordered by decreasing degree d are:

$$
\begin{aligned}
S_6 &= \{(0 \mid 4,1,1,0^4)\},\\
S_5 &= \{(0 \mid 4,1,0^5),(0 \mid 4,0,1,0^4),(0 \mid 3,1,1,0^4)\},\\
S_4 &= \{(0 \mid 4,0^6),(0 \mid 3,1,0^5),(0 \mid 3,0,1,0^4),(0 \mid 2,1,1,0^4)\},\\
S_3 &= \{(0 \mid 3,0^6),(0 \mid 2,1,0^5),(0 \mid 2,0,1,0^4),(0 \mid 1,1,1,0^4)\},\\
S_2 &= \{(0 \mid 2,0^6),(0 \mid 1,1,0^5),(0 \mid 1,0,1,0^4),(0 \mid 0,1,1,0^4)\},\\
S_1 &= \{(0 \mid 1,0^6),(0 \mid 0,1,0^5),(0 \mid 0,0,1,0^4)\},\\
S_0 &= \{(0 \mid 0^7)\}\,.
\end{aligned}
$$

The Hasse diagram containing the seven sets is given in Figure 4.2.

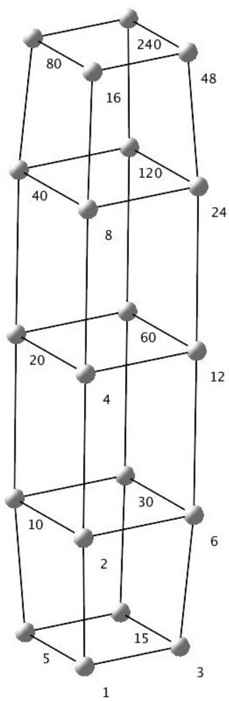

Figure 4.2: Hasse representation of the decoding of the orbit representative of the second order partial derivative of the energy density with respect to time given by Gödel number $G(\mathrm{Orb}\,(\boldsymbol{q}^1)) = (-1)^0 2^4 3^1 5^1 7^0 11^0 13^0 17^0 = 240$.

4.2.3 Additive partitioning

Each H-factorization yields an additive partitioning of the orbit representative $\mathrm{Orb}\left(\boldsymbol{q}^{1}\right)$ that is equivalent to a multiplicative $(H+1)$-ary equation. The 4-factorization yields three quinary equations:

$$240 = 2\times 3\times 4\times 10 = 2\times 3\times 5\times 8 = 2\times 4\times 5\times 6\,.$$

The decoding, followed by the partitioning, results in three vector equations:

$$(0 \mid 4,1,1,0^4) = (0 \mid 0^7) + (0 \mid 1,0^6) + (0 \mid 0,1,0^5) + (0 \mid 2,0^6) + (0 \mid 1,0,1,0^4); \tag{4.5}$$

$$(0 \mid 4,1,1,0^4) = (0 \mid 0^7) + (0 \mid 1,0^6) + (0 \mid 0,1,0^5) + (0 \mid 0,0,1,0^4) + (0 \mid 3,0^6); \tag{4.6}$$

$$(0 \mid 4,1,1,0^4) = (0 \mid 0^7) + (0 \mid 1,0^6) + (0 \mid 2,0^6) + (0 \mid 0,0,1,0^4) + (0 \mid 1,1,0^5)\,. \tag{4.7}$$

The order of occurrence of the vectors is not important due to the commutativity of the addition operator. Observe that we include $(0 \mid 0^7)$ in the additive partioning to highlight the existence of a dimensionless quantity in the quantity equations. The 3-factorization yields twelve quaternary equations:

$$\begin{aligned}240 &= 2\times 3\times 40 = 2\times 4\times 30 = 2\times 5\times 24 = 2\times 6\times 20 = 2\times 8\times 15 = 2\times 10\times 12\\ &= 3\times 4\times 20 = 3\times 5\times 16 = 3\times 8\times 10 = 4\times 5\times 12 = 4\times 6\times 10 = 5\times 6\times 8\,.\end{aligned}$$

The decoding, followed by the partitioning, results in twelve vector equations:

$$(0 \mid 4,1,1,0^4) = (0 \mid 0^7) + (0 \mid 1,0^6) + (0 \mid 0,1,0^5) + (0 \mid 3,0,1,0^4); \tag{4.8}$$

$$(0 \mid 4,1,1,0^4) = (0 \mid 0^7) + (0 \mid 1,0^6) + (0 \mid 2,0^6) + (0 \mid 1,1,1,0^4); \tag{4.9}$$

$$(0 \mid 4,1,1,0^4) = (0 \mid 0^7) + (0 \mid 1,0^6) + (0 \mid 0,0,1,0^4) + (0 \mid 3,1,0^5); \tag{4.10}$$

$$(0 \mid 4,1,1,0^4) = (0 \mid 0^7) + (0 \mid 1,0^6) + (0 \mid 1,1,0^5) + (0 \mid 2,0,1,0^4); \tag{4.11}$$

$$(0 \mid 4,1,1,0^4) = (0 \mid 0^7) + (0 \mid 1,0^6) + (0 \mid 3,0^6) + (0 \mid 0,1,1,0^4); \tag{4.12}$$

$$(0 \mid 4,1,1,0^4) = (0 \mid 0^7) + (0 \mid 1,0^6) + (0 \mid 1,0,1,0^4) + (0 \mid 2,1,0^5); \tag{4.13}$$

$$(0 \mid 4,1,1,0^4) = (0 \mid 0^7) + (0 \mid 0,1,0^5) + (0 \mid 2,0^6) + (0 \mid 2,0,1,0^4); \tag{4.14}$$

$$(0 \mid 4,1,1,0^4) = (0 \mid 0^7) + (0 \mid 0,1,0^5) + (0 \mid 0,0,1,0^4) + (0 \mid 4,0^6); \tag{4.15}$$

$$(0 \mid 4,1,1,0^4) = (0 \mid 0^7) + (0 \mid 0,1,0^5) + (0 \mid 3,0^6) + (0 \mid 1,0,1,0^4); \tag{4.16}$$

$$(0 \mid 4,1,1,0^4) = (0 \mid 0^7) + (0 \mid 2,0^6) + (0 \mid 0,0,1,0^4) + (0 \mid 2,1,0^5); \tag{4.17}$$

$$(0 \mid 4,1,1,0^4) = (0 \mid 0^7) + (0 \mid 2,0^6) + (0 \mid 1,1,0^5) + (0 \mid 1,0,1,0^4); \tag{4.18}$$

$$(0 \mid 4,1,1,0^4) = (0 \mid 0^7) + (0 \mid 0,0,1,0^4) + (0 \mid 1,1,0^5) + (0 \mid 3,0^6)\,. \tag{4.19}$$

The 2-factorization yields nine ternary equations:

$$\begin{aligned}240 &= 2\times 120 = 3\times 80 = 4\times 60 = 5\times 48 = 6\times 40 = 8\times 30 = 10\times 24\\ &= 12\times 20 = 15\times 16\,.\end{aligned}$$

The decoding, followed by the partitioning, results in nine vector equations:

$$(0 \mid 4,1,1,0^4) = (0 \mid 0^7) + (0 \mid 1,0^6) + (0 \mid 3,1,1,0^4); \tag{4.20}$$

$$(0 \mid 4,1,1,0^4) = (0 \mid 0^7) + (0 \mid 0,1,0^5) + (0 \mid 4,0,1,0^4); \tag{4.21}$$

$$(0 \mid 4,1,1,0^4) = (0 \mid 0^7) + (0 \mid 2,0^6) + (0 \mid 2,1,1,0^4); \tag{4.22}$$

$$(0 \mid 4,1,1,0^4) = (0 \mid 0^7) + (0 \mid 0,0,1,0^4) + (0 \mid 4,1,0^5); \tag{4.23}$$

$$(0 \mid 4,1,1,0^4) = (0 \mid 0^7) + (0 \mid 1,1,0^5) + (0 \mid 3,0,1,0^4); \tag{4.24}$$

$$(0 \mid 4,1,1,0^4) = (0 \mid 0^7) + (0 \mid 3,0^6) + (0 \mid 1,1,1,0^4); \tag{4.25}$$

$$(0 \mid 4,1,1,0^4) = (0 \mid 0^7) + (0 \mid 1,0,1,0^4) + (0 \mid 3,1,0^5); \tag{4.26}$$

$$(0 \mid 4,1,1,0^4) = (0 \mid 0^7) + (0 \mid 2,1,0^5) + (0 \mid 2,0,1,0^4); \tag{4.27}$$

$$(0 \mid 4,1,1,0^4) = (0 \mid 0^7) + (0 \mid 0,1,1,0^4) + (0 \mid 4,0^6)\,. \tag{4.28}$$

Those nine ternary vector equations are representing *parallelograms*. The equation (4.25) corresponds to the primorial factorization of $240 = 8 \times 30$ and the additive partitioning results in two vectors with the same degree $d = 3$. The equations (4.21),(4.23), and (4.28) are parallelograms in the form of a rectangle.

Observe that the nine *canonical* ternary equations contain *all* the 20 distinct integer lattice points. The cardinality of the orbit $\{\mathrm{Orb}\left(\boldsymbol{q}^{1}\right)\}$ with orbit representative $\mathrm{Orb}\left(\boldsymbol{q}^{1}\right) = (0 \mid 4, 1^2, 0^4)$, is calculated using equation (2.20) and expressed by the equation (4.29):

$$\#\left(\{\mathrm{Orb}\left(\boldsymbol{q}^{1}\right)\}\right) = \left(\frac{2^{(7-4)}7!}{1!2!4!}\right) = 840\,. \tag{4.29}$$

The encoding-decoding method achieves the modeling of 840 different engineering problems to be expressed in 1 structure containing 24 vector equations based on their geometrical properties in the integer lattice. The orbit representative $(0 \mid 4, 1^2, 0^4)$ can be subjected to 840 signed permutations. Each of these new lattice points represents another kind of quantity and thus another engineering problem having each its set of 24 equations. Hence, instead of solving $840 \times 24 = 20\,160$ vector equations in the computer to solve the 840 engineering problems, we reduce the problem to the task of solving only one time the 24 vector equations. To find the solution of one of the 840 problems we only multiply the 24 vector equations by the signed permutation matrix that maps the orbit representative to the specific lattice point representing the kind of quantity under study. This has a significant impact on computing time and power usage. Such a signed permutation matrix is exemplified in the next section.

4.2.4 Mapping

For each of the 840 vectors $\boldsymbol{y}$, representing the individual engineering problems, exists a unique signed permutation matrix $\mathbf{P_1}$ that maps the lattice point $\mathrm{Orb}\left(\boldsymbol{q}^{1}\right) = (0 \mid 4,1,1,0^4)$ to the lattice point $\boldsymbol{y}$. In this engineering problem, the orbit representative $\mathrm{Orb}\left(\boldsymbol{q}^{1}\right) = (0 \mid 4, 1^2, 0^4)$ is mapped on the lattice point $\boldsymbol{y} = (0 \mid -4, -1, 1, 0^4)$ by the 8×8 signed permutation matrix $\mathbf{P_1}$ (4.30):

$$\mathbf{P_1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.30}$$

Applying the matrix $\mathbf{P_1}$ to each of the 24 vector equations ((4.5) to (4.28)), yields a new set of 24 vector equations describing the modeling of the kind of quantity second order partial derivative of the energy density with respect to time.

The semantic interpretation of the vector equations relies on the know-how of the researcher. The researcher can connect to a database containing a lexicon/dictionary Appendix A of known SI quantities with their associated lattice point cluster. The vector equations and quantity equations for the second order partial derivative of the energy density with respect to time are given in Table 4.1 . The columns marked $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$, $\boldsymbol{x}_4$, and $\boldsymbol{x}_5$ list the lattice points in $\{0, 1\} \times \mathbb{Z}^7$ forming the canonical constellation for the second order partial derivative of the energy density with respect to time.

The symbols used in the last column, representing the quantity equations for the kind of quantity second order partial derivative of the energy density with respect to time, denoted with the heading $\frac{\partial^2 W}{\partial t^2}$, have the following semantic interpretation: $W(\boldsymbol{r}, t)$ energy density, $\frac{\partial W(\boldsymbol{r}, t)}{\partial t}$ first order partial derivative of energy density with respect to time, $\frac{\partial^2 W(\boldsymbol{r}, t)}{\partial t^2}$ second order partial derivative of the energy density with respect to time, s displacement, t time, ν_i specific frequency, ω angular frequency, $m(\boldsymbol{r}, t)$ mass, $k(\boldsymbol{r}, t)$ wavenumber, J action, V volume, and $f_i(\boldsymbol{\pi}_i)$ an unspecified function of a vector of dimensionless quantities $\boldsymbol{\pi}_i$.

Table 4.1: Vector equations and quantity equations for the kind of quantity second order partial derivative of the energy density with respect to time in the integer lattice $\{0,1\} \times \mathbb{Z}^7$.

ID	$\boldsymbol{x}_1$	$\boldsymbol{x}_2$	$\boldsymbol{x}_3$	$\boldsymbol{x}_4$	$\boldsymbol{x}_5$	$\dfrac{\partial^2 W(\boldsymbol{r},t)}{\partial t^2}$
1	$(0 \mid 0^7)$	$(0 \mid -1, 0^6)$	$(0 \mid 0, 0, 1, 0^4)$	$(0 \mid -2, 0^6)$	$(0 \mid -1, -1, 0^5)$	$f_1(\boldsymbol{\pi}_1)\nu_1 m(\boldsymbol{r},t)\nu_2^2 \dfrac{\partial k(\boldsymbol{r},t)}{\partial t}$
2	$(0 \mid 0^7)$	$(0 \mid -1, 0^6)$	$(0 \mid 0, 0, 1, 0^4)$	$(0 \mid 0, -1, 0^5)$	$(0 \mid -3, 0^6)$	$f_2(\boldsymbol{\pi}_2)\nu_1 m(\boldsymbol{r},t)k(\boldsymbol{r},t)\nu_2^3$
3	$(0 \mid 0^7)$	$(0 \mid -1, 0^6)$	$(0 \mid -2, 0^6)$	$(0 \mid 0, -1, 0^5)$	$(0 \mid -1, 0, 1, 0^4)$	$f_3(\boldsymbol{\pi}_3)\nu_1\nu_2^2 k(\boldsymbol{r},t)\dfrac{\partial m(\boldsymbol{r},t)}{\partial t}$
4	$(0 \mid 0^7)$	$(0 \mid -1, 0^6)$	$(0 \mid 0, 0, 1, 0^4)$	$(0 \mid 0, -3, -1, 0^4)$	$(0 \mid 0^7)$	$f_4(\boldsymbol{\pi}_4)\nu_1 m(\boldsymbol{r},t)\dfrac{\partial^3 k(\boldsymbol{r},t)}{\partial t^3}$
5	$(0 \mid 0^7)$	$(0 \mid 0, -1, 0^5)$	$(0 \mid 0, -2, 0^5)$	$(0 \mid 1, -1, -1, 0^4)$	$(0 \mid 0^7)$	$f_5(\boldsymbol{\pi}_5)\nu_1\nu_2^2(J/V)$
6	$(0 \mid 0^7)$	$(0 \mid 0, -1, 0^5)$	$(0 \mid 0, -1, 0^5)$	$(0 \mid -3, 0, 1, 0^4)$	$(0 \mid 0^7)$	$f_6(\boldsymbol{\pi}_6)\nu_1 k(\boldsymbol{r},t)\dfrac{\partial^3 m(\boldsymbol{r},t)}{\partial t^3}$
7	$(0 \mid 0^7)$	$(0 \mid -1, 0, 0, 0^4)$	$(0 \mid -1, 0, 1, 0^4)$	$(0 \mid -2, -1, 0^5)$	$(0 \mid 0^7)$	$f_7(\boldsymbol{\pi}_7)\nu_1 \dfrac{\partial^2 k(\boldsymbol{r},t)}{\partial t^2}\dfrac{\partial m(\boldsymbol{r},t)}{\partial t}$
8	$(0 \mid 0^7)$	$(0 \mid -1, 0^6)$	$(0 \mid -3, 0^6)$	$(0 \mid 0, -1, 1, 0^4)$	$(0 \mid 0^7)$	$f_8(\boldsymbol{\pi}_8)\nu_1\nu_2^3 \dfrac{\partial m(\boldsymbol{r},t)}{\partial \boldsymbol{r}}$
…	…	…	…	…	…	…

ID	$\boldsymbol{x}_1$	$\boldsymbol{x}_2$	$\boldsymbol{x}_3$	$\boldsymbol{x}_4$	$\boldsymbol{x}_5$	$\dfrac{\partial^2 W(\boldsymbol{r},t)}{\partial t^2}$
9	$(0 \mid 0^7)$	$(0 \mid -1, 0^6)$	$(0 \mid -1, -1, 0^5)$	$(0 \mid -2, 0, 1, 0^4)$	$(0 \mid 0^7)$	$f_9(\boldsymbol{\pi}_9)\nu_1 \dfrac{\partial k(\boldsymbol{r},t)}{\partial t} \dfrac{\partial^2 m(\boldsymbol{r},t)}{\partial t^2}$
10	$(0 \mid 0^7)$	$(0 \mid 0, 0, 1, 0^4)$	$(0 \mid -2, 0^6)$	$(0 \mid -2, -1, 0^5)$	$(0 \mid 0^7)$	$f_{10}(\boldsymbol{\pi}_{10}) m(\boldsymbol{r},t) \nu_1^2 \dfrac{\partial^2 k(\boldsymbol{r},t)}{\partial t^2}$
11	$(0 \mid 0^7)$	$(0 \mid 0, 0, 1, 0^4)$	$(0 \mid 3, -1, 0^5)$	$(0 \mid -4, 0^6)$	$(0 \mid 0^7)$	$f_{11}(\boldsymbol{\pi}_{11}) m(\boldsymbol{r},t) k(\boldsymbol{r},t) \nu_1^4$
12	$(0 \mid 0^7)$	$(0 \mid 0, 0, 1, 0^4)$	$(0 \mid -3, 0^6)$	$(0 \mid -1, -1, 0^5)$	$(0 \mid 0^7)$	$f_{12}(\boldsymbol{\pi}_{12}) m(\boldsymbol{r},t) \nu_1^3 \dfrac{\partial k(\boldsymbol{r},t)}{\partial t}$
13	$(0 \mid 0^7)$	$(0 \mid -2, 0, 0, 0^4)$	$(0 \mid 0, -1, 0^5)$	$(0 \mid -2, 0, 1, 0^4)$	$(0 \mid 0^7 4)$	$f_{13}(\boldsymbol{\pi}_{13}) \nu_1^2 k(\boldsymbol{r},t) \dfrac{\partial^2 m(\boldsymbol{r},t)}{\partial t^2}$
14	$(0 \mid 0^7)$	$(0 \mid -2, 0^6)$	$(0 \mid -1, 0, 1, 0^4)$	$(0 \mid -1, -1, 0, 0^4)$	$(0 \mid 0^7)$	$f_{14}(\boldsymbol{\pi}_{14}) \nu_1^2 \dfrac{\partial k(\boldsymbol{r},t)}{\partial t} \dfrac{\partial m(\boldsymbol{r},t)}{\partial t}$
15	$(0 \mid 0^7)$	$(0 \mid 0, -1, 0^5)$	$(0 \mid -1, 0, 1, 0^4)$	$(0 \mid -3, 0^6)$	$(0 \mid 0^7)$	$f_{15}(\boldsymbol{\pi}_{15}) k(\boldsymbol{r},t) \nu_1^3 \dfrac{\partial m(\boldsymbol{r},t)}{\partial t}$
16	$(0 \mid 0^7)$	$(0 \mid -1, 0^6)$	$(0 \mid -3, -1, 1, 0^4)$	$(0 \mid 0^7)$	$(0 \mid 0^7)$	$f_{16}(\boldsymbol{\pi}_{16}) \nu_1 \dfrac{\partial W(\boldsymbol{r},t)}{\partial t}$
17	$(0 \mid 0^7)$	$(0 \mid 0, 0, 1, 0^4)$	$(0 \mid -4, -1, 0^5)$	$(0 \mid 0^7)$	$(0 \mid 0^7)$	$f_{17}(\boldsymbol{\pi}_{17}) m(\boldsymbol{r},t) \dfrac{\partial^4 k(\boldsymbol{r},t)}{\partial t^4}$
18	$(0 \mid 0^7)$	$(0 \mid -2, 0^6)$	$(0 \mid -2, -1, 1, 0^4)$	$(0 \mid 0^7)$	$(0 \mid 0^7)$	$f_{18}(\boldsymbol{\pi}_{18}) \nu_1^2 W(\boldsymbol{r},t)$
...	...	...	...	...	...	...

ID	$\boldsymbol{x}_1$	$\boldsymbol{x}_2$	$\boldsymbol{x}_3$	$\boldsymbol{x}_4$	$\boldsymbol{x}_5$	$\dfrac{\partial^2 W(\boldsymbol{r},t)}{\partial t^2}$
19	$(0 \mid 0^7)$	$(0 \mid 0, -1, 0^5)$	$(0 \mid -4, 0, 1, 0^3)$	$(0 \mid 0^7)$	$(0 \mid 0^7)$	$f_{19}(\boldsymbol{\pi}_{19})k(\boldsymbol{r},t)\dfrac{\partial^4 m(\boldsymbol{r},t)}{\partial t^4}$
20	$(0 \mid 0^7)$	$(0 \mid -1, 0, 1, 0^4)$	$(0 \mid -3, -1, 0^5)$	$(0 \mid 0^7)$	$(0 \mid 0^7)$	$f_{20}(\boldsymbol{\pi}_{20})\dfrac{\partial m(\boldsymbol{r},t)}{\partial t}\dfrac{\partial^3 k(\boldsymbol{r},t)}{\partial t^3}$
21	$(0 \mid 0^7)$	$(1 \mid -3, 0^6)$	$(1 \mid -1, -1, 1, 0^4)$	$(0 \mid 0^7)$	$(0 \mid 0^7)$	$f_{21}(\boldsymbol{\pi}_{21})\omega_1^3\left(\dfrac{J}{V}\right)$
22	$(0 \mid 0^7)$	$(0 \mid -1, -1, 0^5)$	$(0 \mid -3, 0, 1, 0^4)$	$(0 \mid 0^7)$	$(0 \mid 0^7)$	$f_{22}(\boldsymbol{\pi}_{22})\dfrac{\partial k(\boldsymbol{r},t)}{\partial t}\dfrac{\partial^3 m(\boldsymbol{r},t)}{\partial t^3}$
23	$(0 \mid 0^7)$	$(0 \mid -2, 0, 1, 0^4)$	$(0 \mid -2, -1, 0^5)$	$(0 \mid 0^7)$	$(0 \mid 0^7)$	$f_{23}(\boldsymbol{\pi}_{23})\dfrac{\partial^2 m}{\partial t^2}\dfrac{\partial^2 k(\boldsymbol{r},t)}{\partial t^2}$
24	$(0 \mid 0^7)$	$(0 \mid 0, -1, 1, 0^4)$	$(-4, 0^6)$	$(0 \mid 0^7)$	$(0 \mid 0^7)$	$f_{24}(\boldsymbol{\pi}_{24})\nu_1^4\dfrac{\partial m(\boldsymbol{r},t)}{\partial \boldsymbol{r}}$

The equations with ID = 16 up to ID = 24 are visualized as parallelograms in Figure 4.3.

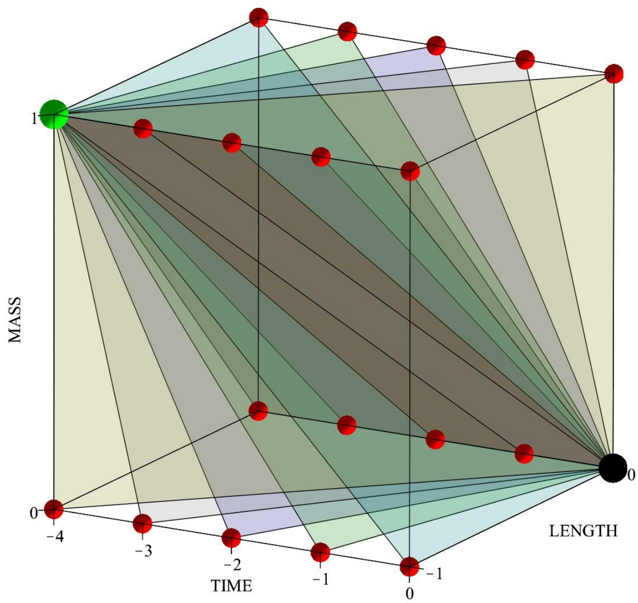

Figure 4.3: Set of nine canonical parallelograms, visualizing 2-factors of the second order partial derivative of the energy density with respect to time (green sphere), connecting 18 integer lattice points (red spheres) in the sub-lattice $\mathbb{Z}^3$ of the integer lattice $\{0, 1\} \times \mathbb{Z}^7$, in which the black sphere is the origin of the sub-lattice $\mathbb{Z}^3$.

Each of the 24 quantity equations can be rearranged to represent dimensionless quantities. Those unspecified functions of a vector of dimensionless quantities $f_i(\boldsymbol{\pi_i})$ can be studied and solved using numerical mathematical techniques.

The encoding-decoding method yields the mathematical model for $\dfrac{\partial^2 W(\boldsymbol{r}, t)}{\partial t^2}$ in the form of the measurement model $F(Q_1, \ldots, Q_{19}) = 0$ (4.31), in which the variables Q_m are ordered by decreasing divisor value from left to right:

$$F\left(\frac{\partial^2 W(\boldsymbol{r},t)}{\partial t^2}, \frac{\partial^4 m(\boldsymbol{r},t)}{\partial t^4}, \frac{\partial^4 k(\boldsymbol{r},t)}{\partial t^4}, \frac{\partial W(\boldsymbol{r},t)}{\partial t}, \nu^4, \frac{\partial^3 m(\boldsymbol{r},t)}{\partial t^3}, \frac{\partial^3 k(\boldsymbol{r},t)}{\partial t^3},\right.$$
$$W(\boldsymbol{r},t), \frac{\partial^2 m(\boldsymbol{r},t)}{\partial t^2}, \frac{\partial^2 k(\boldsymbol{r},t)}{\partial t^2}, \nu^3, \left(\frac{J}{V}\right), \frac{\partial m(\boldsymbol{r},t)}{\partial \boldsymbol{r}}, \frac{\partial k(\boldsymbol{r},t)}{\partial t}, \frac{\partial m(\boldsymbol{r},t)}{\partial t},$$
$$\left.\nu^2, k(\boldsymbol{r},t), m(\boldsymbol{r},t), \nu\right) = 0\,. \tag{4.31}$$

The dependent variable occurs in *all* of the dimensionless quantities and generates meaningful quantity equations on their own. Some of these quantity equations are of the 2-factors type. These quantity equations represent parallelograms in the integer lattice $\{0,1\} \times \mathbb{Z}^7$. The set of quantity equations of the 2-factors type generates nine dimensionless quantities:

$$f_{16}(\pi_{16}) \sim \frac{\left(\frac{\partial^2 W(\boldsymbol{r},t)}{\partial t^2}\right)}{\nu\left(\frac{\partial W(\boldsymbol{r},t)}{\partial t}\right)}; \qquad f_{17}(\pi_{17}) \sim \frac{\left(\frac{\partial^2 W(\boldsymbol{r},t)}{\partial t^2}\right)}{m(\boldsymbol{r},t)\frac{\partial^4 k(\boldsymbol{r},t)}{\partial t^4}};$$

$$f_{18}(\pi_{18}) \sim \frac{\left(\frac{\partial^2 W(\boldsymbol{r},t)}{\partial t^2}\right)}{\nu^2 W(\boldsymbol{r},t)}; \qquad f_{19}(\pi_{19}) \sim \frac{\left(\frac{\partial^2 W(\boldsymbol{r},t)}{\partial t^2}\right)}{k(\boldsymbol{r},t)\left(\frac{\partial^4 m(\boldsymbol{r},t)}{\partial t^4}\right)};$$

$$f_{20}(\pi_{20}) \sim \frac{\left(\frac{\partial^2 W(\boldsymbol{r},t)}{\partial t^2}\right)}{\left(\frac{\partial m(\boldsymbol{r},t)}{\partial t}\right)\left(\frac{\partial^3 k(\boldsymbol{r},t)}{\partial t^3}\right)}; \quad f_{21}(\pi_{21}) \sim \frac{\left(\frac{\partial^2 W(\boldsymbol{r},t)}{\partial t^2}\right)}{\nu^3\left(\frac{J}{V}\right)};$$

$$f_{22}(\pi_{22}) \sim \frac{\left(\frac{\partial^2 W(\boldsymbol{r},t)}{\partial t^2}\right)}{\left(\frac{\partial k(\boldsymbol{r},t)}{\partial t}\right)\left(\frac{\partial^3 m(\boldsymbol{r},t)}{\partial t^3}\right)}; \quad f_{23}(\pi_{23}) \sim \frac{\left(\frac{\partial^2 W(\boldsymbol{r},t)}{\partial t^2}\right)}{\left(\frac{\partial^2 m}{\partial t^2}\right)\left(\frac{\partial^2 k(\boldsymbol{r},t)}{\partial t^2}\right)};$$

$$f_{24}(\pi_{24}) \sim \frac{\left(\frac{\partial^2 W(\boldsymbol{r},t)}{\partial t^2}\right)}{\nu^4\left(\frac{\partial m(\boldsymbol{r},t)}{\partial \boldsymbol{r}}\right)}\,.$$

Observe that these nine dimensionless quantities contain all the variables of the measurement model $F(Q_1, \ldots, Q_{19}) = 0$ (4.31) and thus it is a *complete set*, according to the dimensional analysis terminology (Langhaar, 1946, p.461). The nine dimensionless quantities are the arguments of the dimensionless measurement model $u_1(\pi_{16}, \ldots, \pi_{24}) = 0$ forming the starting point for experiments.

Figure 4.4 shows the histogram of semi-perimeters of parallelograms for the kind of quantity second order partial derivative of the energy density with respect to time in the fundamental (7+1)-dimensional ellipsoid.

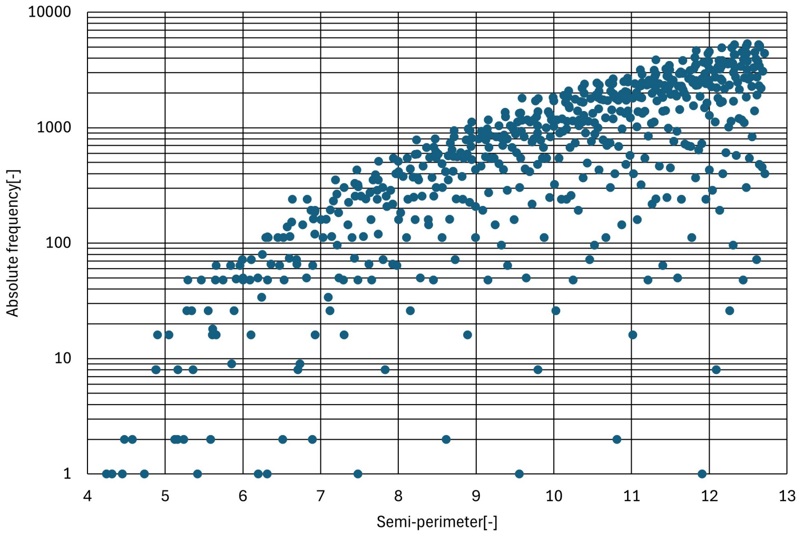

Figure 4.4: Histogram of semi-perimeters of parallelograms for the kind of quantity second order partial derivative of the energy density with respect to time in the fundamental (7+1)-dimensional ellipsoid.

We identify the canonical equations with ID = 16 to ID = 24 with the output of the the data generated by the Python code shown in Appendix V. The software code gives the input to a CSV file that is imported in Excel to create Figure 4.4 and the software code generates also a text file containing the coordinates of the unique parallelograms. The analysis of the data results in Table 4.2 that shows the semi-perimeters and the area squared A_p^2 of the respective parallelograms, and the inner products of their respective vectors $\boldsymbol{x} \cdot \boldsymbol{y}$.

Table 4.2: Geometric properties of the canonical quantity equations of the kind of quantity second order partial derivative of the energy density with respect to time.

ID	semi-perimeter	A_p^2	$\boldsymbol{x} \cdot \boldsymbol{y}$
16	4.316625	2	3
17	5.123106	17	0
18	4.44949	8	4
19	5.123106	17	0
20	4.576491	32	-1
21	4.732051	18	3
22	4.576491	11	3
23	4.472136	9	4
24	5.414214	32	0

We make the following observations from the data in Table 4.2:

- the quantity equation with ID = 24 is a unique rectangle;
- the quantity equations with ID = 16, ID = 18, ID = 21, and ID = 23 are unique parallelograms;
- the quantity equations with ID = 17 and ID = 19 are coupled rectangles because they have the same semi-perimeter;
- the quantity equations with ID = 20 and ID = 22 are coupled parallelograms because they have the same semi-perimeter.

Observe that vector equation ID = 21 contains two pseudo-tensor quantities: angular frequency to the third power ω^3 and angular momentum density $\left(\frac{L}{V}\right)$. The addition of these two vectors in $\{0,1\} \times \mathbb{Z}^7$ yields $(1 \mid -3,0,0,0,0,0,0) + (1 \mid -1,-1,1,0,0,0,0) = (0 \mid -4,-1,1,0,0,0,0)$. The choice of the pseudo-tensors instead of the tensors is based on the physical quantity data compiled in the lexicon. It is known that the angular momentum and the angular frequency are pseudo- tensors and are as such present in Appendix A. Observe that the addition of the pseudo-tensor representatives results in a tensor representative because the operator for the first coordinate is the modulo 2 addition. The semi-perimeter values of the quantity equations with ID = 17 and ID = 19 yield the following couplings:

$$f_{17}(\pi) m(t) \frac{\partial^4 k(t)}{\partial t^4} \sim f_{19}(\pi) k(t) \frac{\partial^4 m(t)}{\partial t^4} \,. \tag{4.32}$$

The coupling equation (4.32) is unknown to the author. The semi-perimeter values of the quantity equations with ID = 20 and ID = 22 yield the following coupling:

$$f_{20}(\pi) \frac{\partial m(t)}{\partial t} \frac{\partial^3 k(t)}{\partial t^3} \sim f_{22}(\pi) \frac{\partial k(t)}{\partial t} \frac{\partial^3 m(t)}{\partial t^3} \,. \tag{4.33}$$

The coupling equation (4.33) is unknown to the author. We *speculate* that the (4.32) and (4.33) are related to nuclear physics or astrophysics or non-steady state open systems due to the occurrence of the rate of change of mass $m(t)$.

4.3 Validation of the method

In this section we validate the encoding-decoding method by applying it to two cases. The first case is the period of the simple pendulum that occurs in many textbooks on dimensional analysis (Huntley, 1958; Palacios, 1964; Szirtes, 2007; Meinsma, 2019). The second case addresses the kind of quantity *energy*. The kind of quantity called energy comprises multiple physical quantities (e.g. heat, kinetic energy, potential energy…) (JCGM, 2012). We selected the kind of quantity energy because it has been studied extensively by researchers and it is a cornerstone of physics and engineering.

4.3.1 Simple pendulum

We refer to the simple pendulum example given by Meinsma (2019). A function $\phi(t)$, representing the angle between the pendulum cable and the vertical axis as function of time t is setup assuming that it depends on: time t, initial angle ϕ_0, mass m, cable length l, and gravitational acceleration g. We assume the same set of variables: $\phi(t), t, \phi_0, m, l, g$. We map the variables to the lattice points in $\{0,1\} \times \mathbb{Z}^7$ and find the set $\{\phi(t) \to (0 \mid 0^7), t \to (0 \mid 1,0,0,0^4), \phi_0 \to (0 \mid 0^7), m \to (0 \mid 0,0,1,0^4), l \to (0 \mid 0,1,0^5), g \to (0 \mid -2,1,0^5)\}$.

These lattice points are mapped to their respective orbit representatives resulting in the set $\{(0 \mid 0^7), (0 \mid 1,0^6), (0 \mid 2,1,0^5)\}$. The largest degree $d = 3$ occurs at the orbit representative $(0 \mid 2,1,0^5)$, which has a signed Gödel number $G((0 \mid 2,1,0^5)) = 12$, a cardinality:

$$\frac{2^{(7-5)}7!}{1!1!5!} = 168\,,$$

and is a *unique* orbit as shown in Table 3.1.

The divisors of 12 form the fully ordered set $1,2,3,4,6,12$. The subset $\{2,4,6,12\}$ comprises the signed Gödel numbers of orbit representatives. The factorization of 12 in 2 distinct factors is in two ways: $12 = 2 \times 6$ resulting in the vector equation $(0 \mid 2,1,0^5) = (0 \mid 1,0^6) + (0 \mid 1,1,0^5)$ and $12 = 3 \times 4$ resulting in the vector equation $(0 \mid 2,1,0^5) = (0 \mid 0,1,0^5) + (0 \mid 2,0^6)$. The mapping of the lattice points of the vector equations, using a 8×8 signed permutation matrix, yields the Table 4.3.

Table 4.3: Vector equations and quantity equations for the kind of quantity acceleration in $\{0\} \times \mathbb{Z}^7$.

ID	$\boldsymbol{x}_1$	$\boldsymbol{x}_2$	$\boldsymbol{x}_3$	$\boldsymbol{x}_4$	a
1	$(0 \mid 0^7)$	$(0 \mid -1, 0^6)$	$(0 \mid -1, 1, 0^5)$	$(0 \mid 0^7)$	$f_1(\boldsymbol{\pi}_3)\dfrac{\mathrm{d}}{\mathrm{d}t}v(t)$
2	$(0 \mid 0^7)$	$(0 \mid 0, 1, 0^5)$	$(0 \mid -2, 0^6)$	$(0 \mid 0^7)$	$f_2(\boldsymbol{\pi}_2)sf^2$

The row with ID = 1 is nothing else than the definition of the acceleration a as the first order derivative of a velocity v with respect to time t if $f_1(\boldsymbol{\pi}_3) = 1$. For the row with ID = 2 we use the definition of the period $T = 1/f$, in which f is a frequency. Let the acceleration be $a = g$ and the length be $s = l$. This yields the well-known period T of the simple pendulum:

$$T = \sqrt{(l/g) f_2(\boldsymbol{\pi}_2)} \,. \tag{4.34}$$

The dimensionless quantities are: $\pi_1 = \dfrac{a}{\left(\dfrac{\mathrm{d}}{\mathrm{d}t}v(t)\right)}$ and $\pi_2 = \dfrac{a}{sf^2}$ resulting in the dimensionless measurement model $u_2(\pi_1, \pi_2) = 0$.

4.3.2 Kind of quantity energy

The kind of quantity energy has the lattice coordinates $(0 \mid -2, 2, 1, 0^4)$ in the integer lattice $\{0, 1\} \times \mathbb{Z}^7$. The orbit representative of the kind of quantity energy is $(0 \mid 2^2, 1, 0^4)$ and has the signed Gödel number $G((0 \mid 2^2, 1, 0^4)) = 180$. The number of divisors of 180 is $\tau(180) = 18$. The divisor set is $\{1, 2, 3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 30, 36, 45, 60, 90, 180\}$.
The subset $\{2, 4, 6, 12, 30, 36, 60, 180\}$ comprises the signed Gödel numbers of orbit representatives. The orbit for the kind of quantity energy is $[(0 \mid 2^2, 1, 0^4)]$. It has Gödel number 180. We find from the OEIS A045778 sequence (Wilson, 2009) the factorizations for the number 180:

$$180 = 2 \times 3 \times 5 \times 6$$

The 4-factoring results in one equation that represents a 5-ary equation. By applying the Gödel decoding on the 4-factoring of 180, we find the additive partitioning of the orbit representative $(0 \mid 2, 2, 1, 0^4)$ in a 5-ary equation:

$$(0 \mid 2, 2, 1, 0^4) = (0 \mid 0^7) + (0 \mid 1, 0^6) + (0 \mid 0, 1, 0^5) + (0 \mid 0, 0, 1, 0^4) + (0 \mid 1, 1, 0^5) \,.$$

The factorizations are:

$$\begin{aligned} 180 &= 2 \times 3 \times 30 = 2 \times 5 \times 18 = 2 \times 6 \times 15 = 2 \times 9 \times 10 \\ &= 3 \times 4 \times 15 = 3 \times 5 \times 12 = 3 \times 6 \times 10 = 4 \times 5 \times 9 \end{aligned}$$

The 3-factoring results in eight equations that represent each a 4-ary equation. The additive partitioning of the orbit representative $(0 \mid 2,2,1,0^4)$ in 4-ary equations are:

$$(0 \mid 2,2,1,0^4) = (0 \mid 0^7) + (0 \mid 1,0^6) + (0 \mid 0,1,0^5) + (0 \mid 1,1,1,0^4) \tag{4.35}$$
$$(0 \mid 2,2,1,0^4) = (0 \mid 0^7) + (0 \mid 1,0^6) + (0 \mid 0,0,1,0^4) + (0 \mid 1,2,0^5) \tag{4.36}$$
$$(0 \mid 2,2,1,0^4) = (0 \mid 0^7) + (0 \mid 1,0^6) + (0 \mid 1,1,0^5) + (0 \mid 0,1,1,0^4) \tag{4.37}$$
$$(0 \mid 2,2,1,0^4) = (0 \mid 0^7) + (0 \mid 1,0^6) + (0 \mid 0,2,0^5) + (0 \mid 1,0,1,0^4) \tag{4.38}$$
$$(0 \mid 2,2,1,0^4) = (0 \mid 0^7) + (0 \mid 0,1,0^5) + (0 \mid 2,0^6) + (0 \mid 0,1,1,0^4) \tag{4.39}$$
$$(0 \mid 2,2,1,0^4) = (0 \mid 0^7) + (0 \mid 0,1,0^5) + (0 \mid 0,0,1,0^4) + (0 \mid 2,1,0^5) \tag{4.40}$$
$$(0 \mid 2,2,1,0^4) = (0 \mid 0^7) + (0 \mid 0,1,0^5) + (0 \mid 1,1,0^5) + (0 \mid 1,0,1,0^4) \tag{4.41}$$
$$(0 \mid 2,2,1,0^4) = (0 \mid 0^7) + (0 \mid 2,0^6) + (0 \mid 0,0,1,0^4) + (0 \mid 0,2,0^5) \tag{4.42}$$

The 2-factoring results also in eight equations that represent each a ternary equation.

$$\begin{aligned} 180 &= 2 \times 90 = 3 \times 60 = 4 \times 45 = 5 \times 36 = 6 \times 30 = 9 \times 20 \\ &= 10 \times 18 = 12 \times 15 \end{aligned}$$

The additive partitioning of the orbit representative $(0 \mid 2,2,1,0^4)$ in 3-ary equations are:

$$(0 \mid 2,2,1,0^4) = (0 \mid 0^7) + (0 \mid 1,0^6) + (0 \mid 1,2,1,0^4) \tag{4.43}$$
$$(0 \mid 2,2,1,0^4) = (0 \mid 0^7) + (0 \mid 0,1,0^5) + (0 \mid 2,1,1,0^4) \tag{4.44}$$
$$(0 \mid 2,2,1,0^4) = (0 \mid 0^7) + (0 \mid 2,0^6) + (0 \mid 0,2,1,0^4) \tag{4.45}$$
$$(0 \mid 2,2,1,0^4) = (0 \mid 0^7) + (0 \mid 0,0,1,0^4) + (0 \mid 2,2,0^5) \tag{4.46}$$
$$(0 \mid 2,2,1,0^4) = (0 \mid 0^7) + (0 \mid 1,1,0^5) + (0 \mid 1,1,1,0^4) \tag{4.47}$$
$$(0 \mid 2,2,1,0^4) = (0 \mid 0^7) + (0 \mid 0,2,0^5) + (0 \mid 2,0,1,0^4) \tag{4.48}$$
$$(0 \mid 2,2,1,0^4) = (0 \mid 0^7) + (0 \mid 1,0,1,0^4) + (0 \mid 1,2,0^5) \tag{4.49}$$
$$(0 \mid 2,2,1,0^4) = (0 \mid 0^7) + (0 \mid 2,1,0^5) + (0 \mid 0,1,1,0^4) \tag{4.50}$$

Hence, the orbit representative $(0 \mid 2,2,1,0^4)$ can be partitioned in 17 distinct terms. As this orbit is representative for the kind of quantity energy we conclude to the existence of 17 distinct forms of equations representing the kind of quantity energy.

The 8×8 signed permutation matrix $\mathbf{P_{energy}}$ transforms the orbit representative $(0 \mid 2,2,1,0^4)$ in the lattice point $(0 \mid -2,2,1,0^4)$ representing the quantity energy and is given below:

$$\mathbf{P_{energy}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.51}$$

We apply the matrix $\mathbf{P_{energy}}$ on the 17 quantity equations that represent the additive partitions of the orbit representative $(0 \mid 2, 2, 1, 0^4)$ and find the energy quantity equations given in Table 4.4 . The symbols used in the seventh column with heading E have the following interpretation: $E(\boldsymbol{r}, t)$ energy, s displacement, $\boldsymbol{r}$ position vector, d distance, t time, ν frequency, ω angular frequency, α angular acceleration, $m(\boldsymbol{r}, t)$ mass, A surface area, v speed, F force, J action, p linear momentum, a acceleration, $f_i(\boldsymbol{\pi}_i)$ unspecified function of a vector of dimensionless quantities, the index L refers to a path, and the index S refers to a surface.

Table 4.4: Vector equations and quantity equations of the kind of quantity energy in $\{0,1\} \times \mathbb{Z}^7$.

ID	$\boldsymbol{x}_1$	$\boldsymbol{x}_2$	$\boldsymbol{x}_3$	$\boldsymbol{x}_4$	$\boldsymbol{x}_5$	E
1	$(0 \mid 0^7)$	$(0 \mid 0, 1, 0^5)$	$(0 \mid -1, 0, 0, 0^4)$	$(0 \mid 0, 0, 1, 0^4)$	$(0 \mid -1, 1, 0, 0^4)$	$f_1(\boldsymbol{\pi}_1) s \nu m v$
2	(0^7)	$(0, 1, 0^5)$	$(-1, 0, 0, 0^4)$	$(-1, 1, 1, 0^4)$	(0^7)	$f_2(\boldsymbol{\pi}_2) s \nu p$
3	$(0 \mid 0^7)$	$(0 \mid 0, 1, 0^5)$	$(0 \mid 0, 0, 1, 0^4)$	$(0 \mid -2, 1, 0, 0^4)$	$(0 \mid 0^7)$	$f_3(\boldsymbol{\pi}_3) s m a$
4	$(0 \mid 0^7)$	$(0 \mid 0, 1, 0^5)$	$(0 \mid -1, 1, 0, 0^4)$	$(0 \mid -1, 0, 1, 0^4)$	$(0 \mid 0^7)$	$f_4(\boldsymbol{\pi}_4) s v \left(\frac{\partial m(\boldsymbol{r}, t)}{\partial t} \right)$
5	$(0 \mid 0^7)$	$(0 \mid 0, 1, 0^5)$	$(0 \mid -2, 0, 0, 0^4)$	$(0 \mid 0, 1, 1, 0^4)$	$(0 \mid 0^7)$	$f_5(\boldsymbol{\pi}_5) s \nu^2 \int_L m(\boldsymbol{r}, t) \, \mathrm{d}s$
6	$(0 \mid 0^7)$	$(0 \mid -1, 0, 0, 0^4)$	$(0 \mid 0, 2, 0^5)$	$(0 \mid -1, 0, 1, 0^4)$	$(0 \mid 0^7)$	$f_6(\boldsymbol{\pi}_6) \nu s^2 \left(\frac{\partial m(\boldsymbol{r}, t)}{\partial t} \right)$
7	$(0 \mid 0^7)$	$(0 \mid -1, 0, 0, 0^4)$	$(0 \mid 0, 0, 1, 0^4)$	$(0 \mid -1, 2, 0, 0^4)$	$(0 \mid 0^7)$	$f_7(\boldsymbol{\pi}_7) \nu m(\boldsymbol{r}, t) \left(\frac{\partial A(\boldsymbol{r}, t)}{\partial t} \right)$
8	$(0 \mid 0^7)$	$(0 \mid -1, 0, 0, 0^4)$	$(0 \mid -1, 1, 0, 0^4)$	$(0 \mid 0, 1, 1, 0^4)$	$(0 \mid 0^7)$	$f_8(\boldsymbol{\pi}_8) \nu v \int_L m(\boldsymbol{r}, t) \, \mathrm{d}s$
9	$(0 \mid 0^7)$	$(0 \mid 0, 2, 0^5)$	$(0 \mid 0, 0, 1, 0^4)$	$(0 \mid -2, 0, 0, 0^4)$	$(0 \mid 0^7)$	$f_9(\boldsymbol{\pi}_9) s^2 m \nu^2$
10	$(0 \mid 0^7)$	$(0 \mid 0, 1, 0^5)$	$(0 \mid -2, 1, 1, 0^4)$	$(0 \mid 0^7)$	$(0 \mid 0^7)$	$f_{10}(\boldsymbol{\pi}_{10}) \int_L \mathrm{d}\boldsymbol{s} \cdot \boldsymbol{F}(\boldsymbol{s})$
11	$(0 \mid 0^7)$	$(1 \mid -1, 0, 0, 0^4)$	$(1 \mid -1, 2, 1, 0^4)$	$(0 \mid 0^7)$	$(0 \mid 0^7)$	$f_{11}(\boldsymbol{\pi}_{11}) \omega J$
12	$(0 \mid 0^7)$	$(0 \mid 0, 2, 0^5)$	$(0 \mid -2, 0, 1, 0^4)$	$(0 \mid 0^7)$	$(0 \mid 0^7)$	$f_{12}(\boldsymbol{\pi}_{12}) s^2 \left(\frac{\partial^2 m(\boldsymbol{r}, t)}{\partial t^2} \right)$
…	…	…	…	…	…	…

ID	$\boldsymbol{x}_1$	$\boldsymbol{x}_2$	$\boldsymbol{x}_3$	$\boldsymbol{x}_4$	$\boldsymbol{x}_5$	E
13	$(0 \mid 0^7)$	$(0 \mid 0, 0, 1, 0^4)$	$(0 \mid -2, 2, 0, 0^4)$	$(0 \mid 0^7)$	$(0 \mid 0^7)$	$f_{13}(\boldsymbol{\pi}_{13})mv^2$
14	$(0 \mid 0^7)$	$(0 \mid -1, 1, 0, 0^4)$	$(0 \mid -1, 1, 1, 0^4)$	$(0 \mid 0^7)$	$(0 \mid 0^7)$	$f_{14}(\boldsymbol{\pi}_{14})vp$
15	$(0 \mid 0^7)$	$(0 \mid -2, 0, 0, 0^4)$	$(0 \mid 0, 2, 1, 0^4)$	$(0 \mid 0^7)$	$(0 \mid 0^7)$	$f_{15}(\boldsymbol{\pi}_{15})\alpha \int_m d^2 \,\mathrm{d}m$
16	$(0 \mid 0^7)$	$(0 \mid -2, 1, 0, 0^4)$	$(0 \mid 0, 1, 1, 0^4)$	$(0 \mid 0^7)$	$(0 \mid 0^7)$	$f_{16}(\boldsymbol{\pi}_{16})a \int_L m(\boldsymbol{r}, t) \,\mathrm{d}s$
17	$(0 \mid 0^7)$	$(0 \mid -1, 2, 0, 0^4)$	$(0 \mid -1, 0, 1, 0^4)$	$(0 \mid 0^7)$	$(0 \mid 0^7)$	$f_{17}(\boldsymbol{\pi}_{17})\left(\frac{\partial A(\boldsymbol{r}, t)}{\partial t}\right)\left(\frac{\partial m(\boldsymbol{r}, t)}{\partial t}\right)$

We discuss the 17 quantity equations of the kind of quantity energy:

4.3.2.1 Energy quantity equation ID = 1

The equation $E_1 = f_1(\pi_1) \cdot s \cdot \omega \cdot m \cdot v$ is a generic quantity equation factored in four kinds of quantities.

4.3.2.2 Energy quantity equation ID = 2

The equation $E_2 = f_2(\pi_2) \cdot s \cdot \omega \cdot p$ is a generic quantity equation factored in three kinds of quantities. When we compare E_2 and E_1 we see that the mass m and the velocity v are combined to the kind of quantity linear impulse p.

4.3.2.3 Energy quantity equation ID = 3

The equation $E_3 = f_3(\pi_3) \cdot s \cdot m \cdot a$ is a generic quantity equation factored in three kinds of quantities. When we compare E_3 and E_1 we see that the frequency ω has taken the form of a differential operator $\dfrac{\mathrm{d}}{\mathrm{d}t}$ acting on the quantity velocity v and resulting in the linear acceleration a.

4.3.2.4 Energy quantity equation ID = 4

The equation $E_4 = f_4(\pi_4) \cdot s \cdot v \cdot H_{e,\nu}$ is a generic quantity equation factored in three kinds of quantities. When we compare E_4 and E_1 we see that the frequency ω and the mass are combined to the spectral exposure in frequency $H_{e,\nu}$. This spectral exposure in frequency is the radiant exposure of a surface per unit frequency and thus expresses the number of joule per square meter per hertz. An alternative equation is $E_4 = f_4(\pi_4) \cdot s \cdot v \cdot \dfrac{\mathrm{d}m}{\mathrm{d}t}$ where the frequency ω has taken the form of a differential operator $\dfrac{\mathrm{d}}{\mathrm{d}t}$ acting on the quantity mass m and resulting in the rate of change of mass $\dfrac{\mathrm{d}m}{\mathrm{d}t}$. The first interpretation of the generic equation E_4 will apply in a radiative process while the second interpretation could occur in thermodynamics of open systems, where mass flows through the system boundary. Another interpretation of the second form of equation E_4 is when we consider nuclear processes.

4.3.2.5 Energy quantity equation ID = 5

The equation $E_5 = f_5(\pi_5) \cdot s \cdot \omega^2 \cdot \int m \, \mathrm{d}s$ is a generic quantity equation factored in three kinds of quantities. The integral $\int m \, \mathrm{d}s$ is known as the spectral linear momentum.

4.3.2.6 Energy quantity equation ID = 6

The equation $E_6 = f_6(\pi_6) \cdot \omega \cdot s^2 \cdot \dfrac{\partial m}{\partial t}$ is a generic quantity equation factored in three kinds of quantities. Observe the temporal dependency of mass $\dfrac{\partial m}{\partial t}$ that suggests an application in astrophysics, nuclear physics and open thermodynamic systems (Haberman, 1980, p.55).

4.3.2.7 Energy quantity equation ID = 7

The equation $E_7 = f_7(\pi_7) \cdot \omega \cdot m \cdot \dfrac{\partial A}{\partial t}$ is a generic quantity equation factored in three kinds of quantities. Observe the temporal dependency of a surface area $\dfrac{\partial A}{\partial t}$ that suggests an application in astrophysics and open thermodynamic systems (Haberman, 1980, p.55).

4.3.2.8 Energy quantity equation ID = 8

The equation $E_8 = f_8(\pi_8) \cdot \omega \cdot v \cdot \int m \, \mathrm{d}s$ is a generic quantity equation factored in three kinds of quantities. The term $\int m \, \mathrm{d}s$ relates the energy equation to a mass transport problem. This occurs in haulage problems in open pit mines. We expect v to be related to the velocity of the vehicle in the haulage process.

4.3.2.9 Energy quantity equation ID = 9

The equation $E_9 = f_9(\pi_9) \cdot s^2 \cdot m \cdot \omega^2$ is a generic quantity equation factored in three kinds of quantities. We recognize the energy equation of a classical harmonic oscillator $T + U = \dfrac{1}{2} m \omega_0^2 a^2$, with T kinetic energy, U potential energy, ω_0 angular velocity, and a the maximum displacement (Feynman, 1977b, p.21-5).

4.3.2.10 Energy quantity equation ID = 10

The equation $E_{10} = f_{10}(\pi_{10}) \cdot s \cdot F$ is a generic quantity equation factored in two kinds of quantities. It is well-known and is the generic quantity equation for the work $W = \int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s}$ when moving a test particle along a piece-wise smooth curve $C \subset U$ in a force field $\boldsymbol{F}$ where $\boldsymbol{F} : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$.

4.3.2.11 Energy quantity equation ID = 11

The equation $E_{11} = f_{11}(\pi_{11}) \cdot \omega \cdot J$ is a generic quantity equation factored in two kinds of quantities. It is obvious that we can link equation E_{11} to the equation $E = \pm\sqrt{n(n+1)} \cdot \hbar \cdot \omega$ of quantum mechanics where $\hbar$ is the reduced Planck constant and $n = 0, \dfrac{1}{2}, 1, \dfrac{3}{2} \ldots$. Equation E_{11} has the same form as the

Planck law for photons $E = \hbar \cdot \omega$ where we substitute $\hbar$ for the total angular momentum J.

4.3.2.12 Energy quantity equation ID = 12

The equation $E_{12} = f_{12}(\pi_{12}) \cdot s^2 \cdot \dfrac{\partial^2 m}{\partial t^2}$ is a generic quantity equation factored in two kinds of quantities.

4.3.2.13 Energy quantity equation ID = 13

The equation $E_{13} = f_{13}(\pi_{13}) m \cdot v^2$ is a generic quantity equation factored in two physical quantities. It is well-known and is the generic quantity equation for the kinetic energy $E = \dfrac{1}{2} m v^2$. It is also the generic form of the rest energy $E_0 = mc^2$ of a test particle with mass m.

4.3.2.14 Energy quantity equation ID = 14

The equation $E_{14} = f_{14}(\pi_{14}) \cdot v \cdot p$ is a generic quantity equation factored in two kinds of quantities. J. C. Maxwell (1871b) compared the energy equations E_{13} and E_{14} and expressed his preference for the form E_{14} because of its inner product form $\boldsymbol{v} \cdot \boldsymbol{p}$ that connects the velocity $\boldsymbol{v}$ with the linear momentum $\boldsymbol{p}$. The velocity $\boldsymbol{v}$ and the linear momentum $\boldsymbol{p}$ are vectors, while in the quantity equation E_{13} the factors are the scalars m and v^2. Maxwell considered the vectors of physical quantities more important than the square of the velocity (J. C. Maxwell, 1871b).

4.3.2.15 Energy quantity equation ID = 15

The equation $E_{15} = f_{15}(\pi_{15}) \cdot \alpha \cdot \int_m d^2 \mathrm{d}m$ is a generic quantity equation factored in two kinds of quantities. The term $\int_m d^2 \mathrm{d}m$ represents the mass moment of inertia and α is an angular acceleration. This kind of energy can be related to the rotational energy of objects like the wheels of a car.

4.3.2.16 Energy quantity equation ID = 16

The equation $E_{16} = f_{16}(\pi_{16}) \cdot a \int m \, \mathrm{d}s$ is a generic quantity equation factored in two kinds of quantities. We see a link with ID = 8 and expect a to be related to the acceleration of the vehicle in the haulage process.

4.3.2.17 Energy quantity equation ID = 17

The equation $E_{17} = f_{17}(\pi_{17}) \cdot \dfrac{\partial A}{\partial t} \cdot \dfrac{\partial m}{\partial t}$ is a generic quantity equation factored in two kinds of quantities. We have recently been able to associate this equation to living organisms belonging to the class of dissipative systems. It should also have a connection in the field of astrophysics concerning the mass loss rate of stars (Holzwarth & Jardine, 2007). Another application of the equation is found in the raindrop oscillations (Beard, 1984).

Figure 5.2 shows the histogram of semi-perimeters of parallelograms for the kind of quantity energy in the fundamental (7+1)-dimensional ellipsoid.

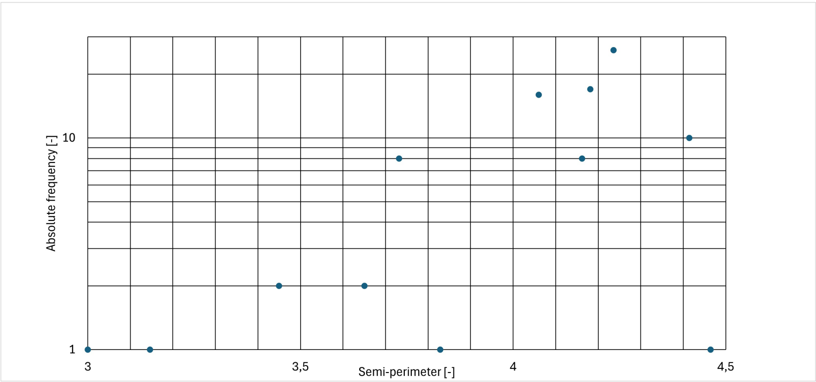

Figure 4.5: Histogram of semi-perimeters of parallelograms for the kind of quantity energy in the fundamental (7+1)-dimensional ellipsoid.

We identify the canonical equations with ID = 10 to ID = 17 with the output of the the data generated by the Python code shown in Appendix V. The software code gives the input to a CSV file that is imported in Excel to create Figure 5.2 and the software code generates also a text file containing the coordinates of the unique parallelograms. The analysis of the data results in Table 4.5 that shows the semi-perimeters and the area squared A_p^2 of the respective parallelograms, and the inner products of their respective vectors $\boldsymbol{x} \cdot \boldsymbol{y}$.

The Table 4.5 shows the semi-perimeters SP and the area squared A_p^2 of the respective parallelograms, and the inner products of their vectors $\boldsymbol{x} \cdot \boldsymbol{y}$.

Table 4.5: Geometric properties of the canonical quantity equations of the kind of quantity energy.

ID	SP	A_p^2	$\boldsymbol{x} \cdot \boldsymbol{y}$
10	3.44949	5	1
11	3.44949	5	1
12	4.236068	20	0
13	3.828427	8	0
14	3.146264	2	2
15	4.236068	20	0
16	3.650282	9	1
17	3.650282	9	1

We make the following observations:

- the quantity equation with ID = 13 is a unique rectangle;
- the quantity equation with ID = 14 is a unique parallelogram;
- the quantity equations with ID = 10 and ID = 11 are coupled parallelograms;
- the quantity equations with ID = 12 and ID = 15 are coupled rectangles; and,
- the quantity equations with ID = 16 and ID = 17 are coupled parallelograms.

The measurement model, in the form $F(Q_1,\ldots,Q_{17}) = 0$ is the result of the encoding and decoding method when applied to the general energy problem, in which the variables Q_m are ordered as function of decreasing divisor value of the kind of quantity $\{180, 90, 60, 45, 36, 30, 20, 18, 15, 12, 10, 9, 6, 5, 4, 3, 2, 1\}$, where E is associated to the divisor 180, F to the divisor 9, and so on until s is associated to the divisor 2:

$$F\left(E, F, J, \frac{\partial^2 m(\boldsymbol{r},t)}{\partial t^2}, v^2, p, \iint_S m(S)\,\mathrm{d}S, a, \frac{\partial m(\boldsymbol{r},t)}{\partial t}, \frac{\partial A(\boldsymbol{r},t)}{\partial t},\right.$$
$$\left.\int_L m(s)\,\mathrm{d}s, f^2, v, m, s^2, f, s\right) = 0\,. \tag{4.52}$$

The 2-factors quantity equations yield *eight* dimensionless quantities:

$$f_{10}(\pi_{10}) \sim \frac{E}{\int_L \boldsymbol{F(s)}\cdot\mathrm{d}\boldsymbol{s}}; \qquad f_{11}(\pi_{11}) \sim \frac{E}{Jf};$$
$$f_{12}(\pi_{12}) \sim \frac{E}{s^2\left(\dfrac{\partial^2 m(\boldsymbol{r},t)}{\partial t^2}\right)}; \qquad f_{13}(\pi_{13}) \sim \frac{E}{mv^2};$$
$$f_{14}(\pi_{14}) \sim \frac{E}{vp}; \qquad f_{15}(\pi_{15}) \sim \frac{E}{f^2\int_m d^2\mathrm{d}m};$$
$$f_{16}(\pi_{16}) \sim \frac{E}{a\int_L m(\boldsymbol{r},t)\,\mathrm{d}s}; \qquad f_{17}(\pi_{17}) \sim \frac{E}{\left(\dfrac{\partial A(\boldsymbol{r},t)}{\partial t}\right)\left(\dfrac{\partial m(\boldsymbol{r},t)}{\partial t}\right)}.$$

The eight dimensionless quantities are the arguments of the positively homogeneous dimensionless measurement model $u(\pi_{10},\ldots,\pi_{17}) = 0$. Observe that these eight dimensionless quantities contain *all* the variables of the dimensional measurement model $F(Q_1,\ldots,Q_{17}) = 0$ and thus it is a complete set of dimensionless quantities. There is no need to consider other quantity equations than the 2-factors quantity equations. A 2-factors quantity equation is a ternary equation and thus a h-factor quantity equation is an $(h+1)$-ary equation. A ternary equation has two fixed points $\boldsymbol{o}$ and $\boldsymbol{z}$ and two free points left $\boldsymbol{x}$ and $\boldsymbol{y}$. From combinatorics we have the situation that two points can be chosen out of two points and thus only one parallelogram can be formed from a ternary equation. In the case of a quaternary equation, where $h = 3$, we have the situation that two points can be chosen out of three points and thus $\binom{h}{2} = \binom{3}{2} = 3$ resulting in three parallelograms. It is obvious that for a $h+1$-ary equation we obtain a set of $\binom{h}{2}$ parallelograms. This shows the importance of parallelograms and thus ternary equations.

The geometric properties of the 2-factors quantity equations yield the following couplings:

$$f_{10}(\boldsymbol{\pi}_{10}) \int_L \boldsymbol{F(s)} \cdot \mathrm{d}\boldsymbol{s} \sim f_{11}(\boldsymbol{\pi}_{11})\nu J \,, \tag{4.53}$$

$$f_{12}(\boldsymbol{\pi}_{12})s^2 \left(\frac{\partial^2 m(\boldsymbol{r},t)}{\partial t^2}\right) \sim f_{15}(\boldsymbol{\pi}_{15})\nu^2 \iint_S m(\boldsymbol{r},t)\,\mathrm{d}S \,, \tag{4.54}$$

$$f_{16}(\boldsymbol{\pi}_{16})a \int_L m(\boldsymbol{r},t)\,\mathrm{d}s \sim f_{17}(\boldsymbol{\pi}_{17})\left(\frac{\partial A(\boldsymbol{r},t)}{\partial t}\right)\left(\frac{\partial m(\boldsymbol{r},t)}{\partial t}\right) . \tag{4.55}$$

4.4 Discussion on the encoding-decoding method

4.4.1 Which Q_m is the dependent variable?

The encoding-decoding method answers this question by associating a degree to each orbit representative of the kind of quantity. The orbit representative with the largest degree is the orbit representative of the dependent variable. Multiple dependent variables can occur in the event that the largest degree has a multiplicity $b_m > 1$. The ranking of the dependent variables can be done by ordering the dependent variables based on the signed Gödel number. The most important dependent variable has the largest signed Gödel number.

4.4.2 What is the value of M ?

The encoding-decoding method gives the answer to this question by calculating the number of divisors of the signed Gödel number(s) of the orbit representative(s) of the dependent variable(s) $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{b_m}$. Hence, the number of variables is $M = \tau(\max\{G(\boldsymbol{y}_i)\}) - 1$, in which $\tau(\cdot)$ is the divisor function.

In the case study of $\left(\frac{\partial^2 W(\boldsymbol{r},t)}{\partial t^2}\right)$ we found $M = 19$. However, it does not mean that the modeling of the power grid is complete. Recall that we have not studied $\left(\frac{\partial^2 W(\boldsymbol{r},t)}{\partial \boldsymbol{r} \partial t}\right)$, in which $\boldsymbol{r}$ is the distance, t is the time and that it has the orbit representative $(0 \mid 3, 2, 1, 0^4)$ of degree $d = 6$ with Gödel number $G((0 \mid 3, 2, 1, 0^4)) = 360$ requiring a divisor lattice of $\tau(360) = 24$ vertices.

Observe that 240 is not dividing 360 and thus the divisor lattice of 240 is not a subset of the divisor lattice of 360. Parisini and Blaabjerg (2021) mentioned wave phenomena, electromagnetic phenomena, electro-mechanical phenomena, and thermodynamic phenomena that should be modeled.

The case study resides in the sub-lattice $\{0, 1\} \times \mathbb{Z}^3$. Hence, the study should be enlarged by considering the kind of quantity $\left(\dfrac{\partial^5 W(\boldsymbol{r}, t, I, T)}{\partial \boldsymbol{r} \partial t^2 \partial I \partial T}\right)$, in which I is the electric current, T the temperature and that it has the integer lattice point $(0 \mid -4, -2, 1, -1, -1, 0, 0)$ with orbit representative $(0 \mid 4, 2, 1, 1, 1, 0^2)$ and Gödel number $G((0 \mid 4, 2, 1, 1, 1, 0^2)) = 2^4 \cdot 3^2 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^0 \cdot 17^0 = 55\,440$. This kind of quantity should encompass all the phenomena stated by Parisini. That modeling requires a divisor lattice of $\tau(55\,440) = 120$ vertices. Remark that 240 and 360 are divisors of 55 440 and thus are vertices of the Hasse diagram of Gödel number 55 440.

4.4.3 Are the chosen Q_m effective?

The encoding-decoding method differs fundamentally from the *algebraic* approach of Classical Dimensional Analysis (CDA) and Modern Dimensional Analysis (MDA). The encoding-decoding method yields *less* dimensional quantities π_k when the following condition is satisfied:

$$n - \frac{1}{2}(M + 1) + 1 > r\,, \tag{4.56}$$

in which n is the number of unrelated kinds of quantities in the dimensional measurement model used in CDA or MDA, $M + 1 = \tau(G(\mathrm{Orb}\left(\boldsymbol{q}^1\right)))$ the number of divisors of the signed Gödel number of the orbit representative of the dependent variable Q_1, and r is the rank of the reduced dimensional matrix while satisfying the *complete set* condition of CDA and MDA.

The results of the encoding-decoding method can be compared with the results given by the state-of-the-art method (MDA) as exemplified in Appendix O.

The completeness of the set of dimensionless quantities is mainly due to the creation of the divisor lattice. The origin of this effectiveness is the use of a *divisor lattice* to find the dimensionless quantities instead of using *linear algebra*. Within the divisor lattice we find that the 2-factors are an *effective* selection of the ways to form dimensionless quantities π_k and find an agreement with the research of Palacios (1964, pp. 24-26) in the classification of equations in which fundamental constants occur and in those in which two physical quantities occur, and in equations involving more than two physical quantities.

Each dimensionless quantity generated by the encoding-decoding method has the lowest non-trivial form $\dfrac{Q_1}{Q_m Q_j}$ that represents geometrically a non-degenerate parallelogram in $\{0,1\} \times \mathbb{Z}^7$. It is effective for experimenters to create 2D-maps with axes Q_m versus Q_j .

It is very common in engineering and physics to find relations of the type $Q_m Q_j = C$, in which C is a constant.

The recommendation to place the *dependent variable* to the leftmost position in the dimensional matrix (Langhaar, 1946, p.39), (Szirtes, 2007, p.391) results in a loss of the connectivity of the dependent variable such that it appears in one dimensionless quantity π_1 .

4.5 Conclusion

In this chapter we find the existence of a kind of quantity that maps to the orbit representative $\boldsymbol{x} = (0 \mid N, N-1, \ldots, 1)$ of $\{0,1\} \times \mathbb{Z}^N$ of largest cardinality equal to the order $2^N N!$ of the integer sub-lattice $\{0\} \times \mathbb{Z}^N$. By selecting this specific integer lattice point, the smallest lattice can be created, on which all the symmetries of the lattice can act. The number of symmetries $2^N N!$ corresponds to the number of signed permutation matrices that can act on the lattice points. For $N = 7$ we find the orbit representative $\boldsymbol{x} = (0 \mid 7, 6, 5, 4, 3, 2, 1)$ of cardinality $2^7 7! = 645\,120$

We present an *encoding and decoding* method for creating positively homogeneous dimensionless measurement models. These models consists of dimensionless quantities and have a universal character. ‘Laws of physics’ are universal and should be positively homogeneous dimensionless measurement models.

We created such a model for the kind of quantity energy. Eight dimensionless quantities are the arguments of the positively homogeneous dimensionless measurement model $u(\pi_{10}, \ldots, \pi_{17}) = 0$ and are representing parallelograms in $\{0,1\} \times \mathbb{Z}^N$.

We found that some of these parallelograms have unique semi-perimeters when considering the histogram of semi-perimeters of the kind of quantity energy. A similar situation is observed for the kind of quantity second derivative of the energy density over time. The energy density behaves in the same way as the Lagrangian density in theoretical physics where it is at the basis of many physics’s theories.

The uniqueness of some parallelogram semi-perimeters can be considered as a ‘surprise’ and can be described as Shannon information. Normalizing the histogram results in a probability mass function for the discrete random variable, being the semi-perimeter of the parallelogram.

We find that the unique semi-perimeters have the lowest probability of occurring and thus have the largest information content $I(SP) = -\log_2(p(SP))$ of a kind of quantity E , in which $p(SP)$ is the probability of occurrence of a value SP for the semi-perimeter SP of the parallelogram $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$ representing the quantity equation $E = f(\boldsymbol{\pi})XY$.

We observe that some of the parallelograms with unique semi-perimeters correspond to known 'laws of physics' and that other parallelograms with unique semi-perimeters cannot be match by the author to 'laws of physics'. These cases are very interesting to be examined for designing experiments to test the correctness of the quantity equation.

The most striking example is the parallelogram with semi-perimeter equal to $1 + 2\sqrt{2}$ representing the quantity equation $E = mv^2$, being the prototype equation of the famous equation $E = mc^2$. This equation has an additional peculiarity that the parallelogram has the form of a rectangle. The number of known 'laws of physics' is small with respect to the infinity of ternary equations that can be formed theoretically. We formulate a conjecture that 'laws of physics' are represented by parallelograms having unique semi-perimeters in $\{0, 1\} \times \mathbb{Z}^N$, with the dimension N of the integer lattice properly chosen. The conjecture is clearly falsifiable because if someone finds a 'law of physics' represented by a parallelogram in $\{0, 1\} \times \mathbb{Z}^N$ having in the histogram of semi-perimeters for the kind of quantity under investigation an absolute frequency f larger than two then the conjecture is proven *wrong*. We cannot prove that the conjecture is *right* but we can prove that it is *not wrong* to the best of our knowledge.

CHAPTER 5

Dimensional exploration

In this chapter, we will search for additional properties of kinds of quantities and quantity equations that could lead to a faster selection of which quantity equations are ‘laws of physics’. We call this search for ’laws of physics’ dimensional exploration.

The term dimensional exploration originates from chapter 11 in the book of John Roche entitled ‘The Mathematics of Measurement, A Critical History’. Dimensional exploration is a method of enquiry to mathematical physics (Roche, 1998, p.208). This dissertation is a summary of the research we performed in this *new* field of applied mathematics (P. A. J. G. Chevalier, 2013, 2014, 2015a, 2015b). We will research which are the common mathematical properties of the parallelograms associated with ternary equations of kinds of quantities.

5.1 Histogram of N-dimensional confocal ellipsoids

We recall from the ternary quantity equations, where X and Y are variables, and Z is the kind of quantity under study, that each of these ternary quantity equations is represented by a parallelogram, as proven in Theorem (2.3.1), with two fixed points $\boldsymbol{o}$, the origin of the integer lattice and $\boldsymbol{z}$, the lattice point representing the kind of quantity under study. We will find that the fixed points are the foci of N-dimensional confocal ellipsoids. Recall that the value of the semi-perimeter SP of a parallelogram of a ternary equation of kinds of quantities is obtained by the equation:

$$SP = \sqrt{u} + \sqrt{v}\,,$$

with $u, v \in \mathbb{N}$ and expressed through the following equations:

$$u = \sum_{n=1}^{N} x_n^2 \qquad\qquad v = \sum_{n=1}^{N} y_n^2 = \sum_{n=1}^{N} (x_n - z_n)^2\,,$$

in which the integer lattice points have the coordinates $(x_0 \mid x_1, x_2, \ldots, x_N)$.

Let as introduction consider the case where $N = 2$ so that the N-dimensional confocal ellipsoid is a confocal ellipse. The semi-perimeter SP

of a parallelogram of a ternary equation of kinds of quantities is equal to a characteristic parameter of an ellipse, typically known as $2a$ in the normalized equation of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,,$$

in which it is known as the length of the major axis of the ellipse.

We are interested in the distribution of this parameter $2a = SP$ of the ellipses that include the parallelograms that we are studying. Observe that we are not counting the number of parallelograms but are counting the number of different ellipses.

Consider the lattice point $\boldsymbol{x} = (0 \mid s, s, \ldots, s)$ in $\{0,1\} \times \mathbb{Z}^N$ that is, for a chosen infinity norm s, the point with the largest norm $\|\boldsymbol{x}\|_2 = \sqrt{Ns^2}$ implying that the maximum value for the sum of the square roots of natural numbers is $SP_{max} = \sqrt{u} + \sqrt{v} = 2\sqrt{Ns^2}$ based on a centrally symmetric set, being the N-dimensional hypercube with center the origin $\boldsymbol{o}$. It is obvious that this parallelogram is a degenerated parallelogram. The minimum value for the sum of the square roots of natural numbers is $SP_{min} = 2$.

The fundamental theorem of arithmetic states that every integer greater than 1 can be represented as the product of prime numbers and thus $u = \prod_{i=1}^{\omega(u)} p_i^{\alpha_i}$ in which $\omega(u)$ is the number of distinct prime factors of u and each $\alpha_i = \nu_{p_i}(u)$, in which $\nu_{p_i}(u)$ is the p_i-adic order of u being the highest exponent ν of prime number p such that p^ν divides u . We have the equation:

$$\Omega(u) = \sum_{i=1}^{\omega(u)} \alpha_i = \sum_{i=1}^{\omega(u)} \nu_{p_i}(u) = \sum_{p|u} \nu_p(u)\,. \tag{5.1}$$

The values of $\Omega(u)$ are given in the sequence of the On-line Encyclopedia of Integer Sequences (OEIS) with identifier A001222 and give the number of prime factors of u with multiplicity. The ordered prime signature is a sorted list of the non-zero exponents α_i in the prime factorization that is denoted $\{\alpha_1, \ldots, \alpha_u\}$.

A square free number is a number that can be represented by the monomial $u = \prod_{i=1}^{n} p_i^1$ in which p_i is the i-th prime number. The square free numbers are the signed Gödel numbers of the orbit representatives with infinity norm $s = 1$. The square free numbers are listed in the OEIS sequence A005117 (N. J. A. Sloane, 2019). The first few square free numbers are: 1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30,. . . It is also known that the probability that a random natural number belongs to the sequence is $\dfrac{6}{\pi^2} = 0.608$ (N. J. A. Sloane, 2019) and thus also applicable to the Gödel numbers of the orbit representatives of $\{0,1\} \times \mathbb{Z}^N$.

Each of the square free numbers is an element of a sequence containing the square free numbers with the same prime signature. Table 5.1 shows the OEIS sequences of square free numbers having the same signature.

Table 5.1: OEIS sequences containing the square free numbers with the same prime signature.

OEIS sequence	Prime signature	First element of OEIS sequence
A000040	$\{1\}$	2
A006881	$\{1,1\}$	6
A007304	$\{1,1,1\}$	30
A046386	$\{1,1,1,1\}$	210
A046387	$\{1,1,1,1,1\}$	2310
A067885	$\{1,1,1,1,1,1\}$	30030
A046325	$\{1,1,1,1,1,1,1\}$	510510

The first element of each OEIS sequence of Table 5.1 corresponds to the Gödel number of an orbit representative of $\{0,1\} \times \mathbb{Z}^N$, where $1 \leq N \leq 7$, with infinity norm $s = 1$.

We identify two classes of N-dimensional confocal ellipsoids with respective generic equations:

$$SP = \sqrt{u} + \sqrt{v} = \text{irrational number}\,, \tag{5.2}$$

or

$$SP = \sqrt{u} + \sqrt{v} = k + g = n = \text{natural number}\,, \tag{5.3}$$

in which $k = \sqrt{u}$ and $g = \sqrt{v}$ with $n \in \mathbb{N}$.

The natural numbers u and v, with $u > 1$ and $v > 1$, can always be represented as the product of a square free number and a non-square free number (Square-free integer, 2024). We can write $u = a^2 b$ and $v = c^2 h$ in which b and h are square free and a^2 and c^2 are non-square free numbers.

Thus we write:

$$\begin{aligned} u &= p_1^{\alpha_1} \dots p_u^{\alpha_u}\,, \\ u &= (p_1^{\alpha_1 - 1} \dots p_u^{\alpha_u - 1})(p_1 \dots p_u)\,; \\ v &= p_1^{\gamma_1} \dots p_v^{\gamma_v}\,, \\ v &= (p_1^{\gamma_1 - 1} \dots p_v^{\gamma_v - 1})(p_1 \dots p_v)\,, \end{aligned}$$

in which $b = (p_1, \dots, p_u)$ and $h = (p_1, \dots, p_v)$ being square free numbers and $a^2 = (p_1^{\alpha_1 - 1}, \dots, p_u^{\alpha_u - 1})$ and $c^2 = (p_1^{\gamma_1 - 1}, \dots, p_v^{\gamma_v - 1})$ being non-square free numbers.

If $b = h$ we obtain the equation:

$$\sqrt{u} + \sqrt{v} = (a + c)\sqrt{b}\,. \tag{5.4}$$

We denote the absolute frequency in the histogram of the N-dimensional confocal ellipsoids by f. We plot the histogram of the sum of square roots of natural numbers in which $N = 7$ and $s = 3$ and find Figure 5.1.

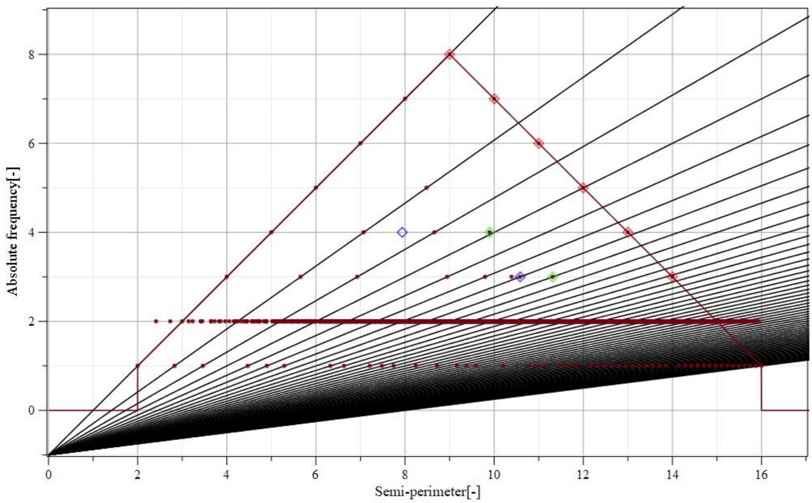

Figure 5.1: Histogram of the sum of square roots of natural numbers for $N = 7$ and $s = 3$.

A detailed analysis of the data from the histogram shows the linear relation $SP = (f + 1)\sqrt{r}$ with $r \in \mathbb{N}$ for the points on the black lines in Figure 5.1. The first inclined black line has the equation $SP = (f + 1)\sqrt{1}$.

We write:

$$\begin{aligned} SP &= \sqrt{u} + \sqrt{v} = (f + 1)\sqrt{r} \\ &= a\sqrt{b} + c\sqrt{h} = (f + 1)\sqrt{(r)}\,. \end{aligned}$$

If $\sqrt{b} = \sqrt{h} = \sqrt{r}$ then $a + c = f + 1$ and thus $a + c - 1 = f$.

Let us consider the number of compositions into positive integer parts of $(a + c)$. We have $C(a + c) = 2^{a+c-1} = 2^f$. The number of compositions of a positive integer n is given in the OEIS sequence $A011782(n)$.
The first few numbers are: $1, 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1\,024, \ldots$

If $b \neq h$ we obtain the equation:

$$SP = \sqrt{u} + \sqrt{v} = a\sqrt{b} + c\sqrt{h}\,. \tag{5.5}$$

If $u = v$ then we obtain a parallelogram in the form of a *rhombus*. We obtain the equation:

$$2\sqrt{u} = 2a\sqrt{b}\,, \tag{5.6}$$

that occurs when the frequency $f = 1$. We infer from the histogram of Figure 5.1 that the following equation is valid:

$$1 = \frac{2a\sqrt{b}}{\sqrt{b}} - 1\,, \tag{5.7}$$

yielding $a = 1$ for the parallelograms that are a rhombus. Hence, $u \in$ A005117, being the sequence of square free numbers. Inspection of the histogram in Figure 5.1 shows that the form $\sqrt{u}+\sqrt{v} = a\sqrt{b}+c\sqrt{h}$ occurs when the frequency $f = 2$ and that the form $\sqrt{u} + \sqrt{v} = (a + c)\sqrt{b}$ occurs for all cases in which $f > 2$.

The number of ways to satisfy the equation (5.3) is known in number theory as the problem of finding the decompositions of a natural number n with the constraint that the number of terms in the decomposition is two. This number of ways is known to be $n - 1$.

We infer a piece-wise linear function $f_1(SP)$ to describe the situation represented by equation (5.3):

$$\begin{cases} f_1(SP) = SP - 1\,, & SP \leq m_1\,, \\ f_1(SP) = -SP - 1 + 2m_1\,, & m_1 < SP < 2(m_1 - 1)\,, \end{cases}$$

in which $m_1 = 2 + \lfloor\big((2\sqrt{Ns^2} - 2\big)/2\rfloor$ is the maximum frequency in the histogram and $p \in \mathbb{N}$. Triangular shapes with unequal slopes are clearly present in the histogram. The positive slopes can all be represented by the equations:

$$f = \frac{SP}{\sqrt{b}} - 1\,, \tag{5.8}$$

in which $b \in$ A005117 and $SP \leq (m_1 + 1)$.

The maximum value b_{max} of the square free numbers is found by solving the equation:

$$3 = \frac{SP_{max}}{\sqrt{b_{max}}} - 1\,, \tag{5.9}$$

resulting in $b_{max} = \dfrac{(m_1 + 1)^2}{16}$ with the exception of the pairs (N, s) being (1,2), (1,6), (1,10), (2,10), (3,1), (3,6), (4,1), (4,3), (4,5), (4,7), (4,9), (5,1), (5,3), (5,10), (6,1), (6,6), (6,9), (7,1), (7,4), (7,7), (7,10), (8,5), (8,8) that all form squares for b_{max} when $N \in \{1, \ldots, 8\}$ and $s \in \{1, \ldots, 10\}$. No explanation exists for these exceptions on the value of b_{max}.

We find that all the semi-perimeters $SP(s)$ have the form $SP(s) = m(s)\sqrt{b}$ in which b is a square free number and the value of the multiplier $m(s)$ is always even.

We describe how the multiplier $m(s)$ varies as function of the infinity norm s and this for each square free number b.
The multiplier $m(s)$ of $\sqrt{2}$ takes two values:

$$
\begin{aligned}
&4, \\
&4((s-1)-\frac{1}{2}\lfloor s/7 \rfloor)\,.
\end{aligned}
$$

The multiplier $m(s)$ of $\sqrt{3}$ takes two values:

$$
\begin{aligned}
&4, \\
&4(\frac{3}{4}s-\frac{1}{2}-\frac{1}{4}(s \bmod 2))\,.
\end{aligned}
$$

The case of $\sqrt{5}$ is more complicated:

$$
\begin{aligned}
&4(\frac{1}{2}(s-1)), && s \leq 5\,, \\
&4(\frac{1}{2}s), && 6 \leq s \leq 10\,, \\
&4(\frac{1}{2}(s+1)), && 11 \leq s \leq 16\,, \\
&4(\frac{1}{2}(s+2)), && 17 \leq s \leq 23\,.
\end{aligned}
$$

The multiplier $m(s)$ of $\sqrt{6}$ takes two values:

$$
\begin{aligned}
&4, \\
&4(\frac{1}{2}(s-1)+\frac{1}{2}\lfloor s/13 \rfloor)\,.
\end{aligned}
$$

The multiplier $m(s)$ of $\sqrt{7}$ takes two values:

$$
\begin{aligned}
&4, \\
&4(\frac{1}{2}(s-1))\,.
\end{aligned}
$$

The multiplier $m(s)$ of $\sqrt{10}$ takes two values:

$$
\begin{aligned}
&4, \\
&4((\frac{1}{2}s-1)-\frac{1}{2}\lceil s/6 \rceil)\,.
\end{aligned}
$$

5.2 Minimum constant distance between hyperplanes

We search for an equation giving the number of integer lattice points on confocal N-dimensional ellipsoids. We consider the origin $\boldsymbol{o}$ and the lattice point under study $\boldsymbol{z}$ in which $\boldsymbol{o}, \boldsymbol{z} \in \mathbb{Z}^N$. These two lattice points are the foci of the confocal N- dimensional ellipsoids. Let the greatest common divisor $m = \gcd(z_1, \ldots, z_N)$ with $m \in \mathbb{Z}$ and put $\boldsymbol{z} = m\boldsymbol{v}$. We normalize the vector $\boldsymbol{v}$ and denote this normalized vector by $\boldsymbol{n} = \dfrac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$.

We consider a hyperplane defined by the normalized vector $\boldsymbol{n}$ and such that it contains the integer lattice point $\boldsymbol{v}$. We have then for this hyperplane the equation $\boldsymbol{vx} \cdot \boldsymbol{n} = 0$. We consider an integer lattice point $\boldsymbol{l}$ in the neighborhood of the lattice point $\boldsymbol{v}$ such that the Euclidean norm $\|\boldsymbol{vl}\| = 1$.

We assume that the inner product $\boldsymbol{vl} \cdot \boldsymbol{n} \neq 0$ then we have $\boldsymbol{vl} \cdot \boldsymbol{n} = Ka$ in which $K \in \mathbb{Z}$ and the constant distance between hyperplanes $a \in \mathbb{R}$. The values of K and a are unknown. We derive the following equations:

$$\boldsymbol{vl} \cdot \boldsymbol{n} = Ka \tag{5.10}$$

$$(\boldsymbol{l} - \boldsymbol{v}) \cdot \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} = Ka \tag{5.11}$$

$$\boldsymbol{l} \cdot \boldsymbol{v} - \boldsymbol{v}^2 = Ka\|\boldsymbol{v}\| \tag{5.12}$$

In $\mathbb{Z}^N$ we have $2d$ closest neighbors of $\boldsymbol{v}$ and thus we have the set $NB(\boldsymbol{v}) = \{\boldsymbol{v}+\boldsymbol{e}_1, \boldsymbol{v}-\boldsymbol{e}_1, \boldsymbol{v}+\boldsymbol{e}_2, \boldsymbol{v}-\boldsymbol{e}_2, \boldsymbol{v}+\boldsymbol{e}_3, \boldsymbol{v}-\boldsymbol{e}_3, \ldots, \boldsymbol{v}+\boldsymbol{e}_N, \boldsymbol{v}-\boldsymbol{e}_N\}$. Let $N = 3$ then we find 6 equations by substitution of the elements of $NB(\boldsymbol{v})$ for the vector $\boldsymbol{l}$ in equation (5.12) and thus we find:

$$\begin{aligned}
\boldsymbol{l}_1 \cdot \boldsymbol{v} - \boldsymbol{v}^2 &= (v_1 + 1)v_1 + v_2 v_2 + v_3 v_3 - (v_1^2 + v_2^2 + v_3^2) \\
&= v_1 = K_1 a\|\boldsymbol{v}\|, \\
\boldsymbol{l}_2 \cdot \boldsymbol{v} - \boldsymbol{v}^2 &= (v_1 - 1)v_1 + v_2 v_2 + v_3 v_3 - (v_1^2 + v_2^2 + v_3^2) \\
&= -v_1 = K_2 a\|\boldsymbol{v}\|, \\
\boldsymbol{l}_3 \cdot \boldsymbol{v} - \boldsymbol{v}^2 &= v_1 v_1 + (v_2 + 1)v_2 + v_3 v_3 - (v_1^2 + v_2^2 + v_3^2) \\
&= v_2 = K_3 a\|\boldsymbol{v}\|, \\
\boldsymbol{l}_4 \cdot \boldsymbol{v} - \boldsymbol{v}^2 &= v_1 v_1 + (v_2 - 1)v_2 + v_3 v_3 - (v_1^2 + v_2^2 + v_3^2) \\
&= -v_2 = K_4 a\|\boldsymbol{v}\|, \\
\boldsymbol{l}_5 \cdot \boldsymbol{v} - \boldsymbol{v}^2 &= v_1 v_1 + v_2 v_2 + (v_3 + 1)v_3 - (v_1^2 + v_2^2 + v_3^2) \\
&= v_3 = K_5 a\|\boldsymbol{v}\|, \\
\boldsymbol{l}_6 \cdot \boldsymbol{v} - \boldsymbol{v}^2 &= v_1 v_1 + v_2 v_2 + (v_3 - 1)v_3 - (v_1^2 + v_2^2 + v_3^2) \\
&= -v_3 = K_6 a\|\boldsymbol{v}\|.
\end{aligned}$$

We find the constraints:

$$K_2 = -K_1\,, \tag{5.13}$$

$$K_4 = -K_3\,, \tag{5.14}$$

$$K_6 = -K_5\,, \tag{5.15}$$

$$K_1 = v_1\left(\frac{1}{a\|\boldsymbol{v}\|}\right), \tag{5.16}$$

$$K_3 = v_2\left(\frac{1}{a\|\boldsymbol{v}\|}\right), \tag{5.17}$$

$$K_5 = v_3\left(\frac{1}{a\|\boldsymbol{v}\|}\right), \tag{5.18}$$

$$\frac{1}{a^2} = K_1^2 + K_3^2 + K_5^2\,, \tag{5.19}$$

$$\|\boldsymbol{v}\|^2 = \left(\frac{v_1}{K_1}\right)^2 \frac{1}{a^2}\,, \tag{5.20}$$

$$\|\boldsymbol{v}\|^2 = \left(\frac{v_2}{K_3}\right)^2 \frac{1}{a^2}\,, \tag{5.21}$$

$$\|\boldsymbol{v}\|^2 = \left(\frac{v_3}{K_5}\right)^2 \frac{1}{a^2}\,, \tag{5.22}$$

$$1 = \gcd(v_1, v_2, v_3)\,. \tag{5.23}$$

in which $K_1, K_3, K_5 \in \mathbb{Z}$ and $n_1, n_2, n_3 \in [-1, 1]$. Solutions are $v_1 = bK_1, v_2 = bK_3, v_3 = bK_5$ in which $b = \gcd(K_1, v_1) = \gcd(K_3, v_2) = \gcd(K_5, v_3)$ and then we have $a = \dfrac{b}{\|\boldsymbol{v}\|}$.

It is obvious that the minimum distance between hyperplanes is found for $b = 1$ and thus $a = \dfrac{1}{\|\boldsymbol{v}\|}$ and that remains valid in $\mathbb{Z}^N$. The above results are confirmed in Coxeter (1973, p.181).

5.3 Cross-section of hyperplanes, confocal N-dimensional ellipsoids and confocal N-dimensional hyperboloids

We define a family of parallel hyperplanes through the equation:

$$(\boldsymbol{x} - m\boldsymbol{v}) \cdot \boldsymbol{n} = \frac{b}{\|\boldsymbol{v}\|}\,, \tag{5.24}$$

in which $m, b \in \mathbb{Z}$ and $m = \gcd(z_1, \ldots, z_N)$. We define the confocal N-dimensional ellipsoids through the equation:

$$E = \|\boldsymbol{x}\| + \|\boldsymbol{x} - m\boldsymbol{v}\|\,. \tag{5.25}$$

We define a first set of confocal N- dimensional hyperboloids through the equation:

$$H_1 = \|\boldsymbol{x}\| - \|\boldsymbol{x} - m\boldsymbol{v}\| \,. \tag{5.26}$$

We define a second set of confocal N-dimensional hyperboloids through the equation:

$$H_2 = \|\boldsymbol{x} - m\boldsymbol{v}\| - \|\boldsymbol{x}\| \,. \tag{5.27}$$

Combining the N-dimensional ellipsoids and the first set of N-dimensional hyperboloids we find:

$$EH_1 = (\|\boldsymbol{x}\| + \|\boldsymbol{x} - m\boldsymbol{v}\|)(\|\boldsymbol{x}\| - \|\boldsymbol{x} - m\boldsymbol{v}\|) \,, \tag{5.28}$$

$$= \boldsymbol{x}\cdot\boldsymbol{x} - (\boldsymbol{x} - m\boldsymbol{v})\cdot(\boldsymbol{x} - m\boldsymbol{v}) \,, \tag{5.29}$$

$$= 2m\boldsymbol{x}\cdot\boldsymbol{v} - m^2\boldsymbol{v}\cdot\boldsymbol{v} \,, \tag{5.30}$$

in which $EH_1 \in \mathbb{Z}$ and $m\boldsymbol{v} = \boldsymbol{z}$. We infer that:

$$EH_1 + \|\boldsymbol{z}\|^2 = 2\boldsymbol{x}\cdot\boldsymbol{z} \,, \tag{5.31}$$

in which the left side of the equation is an integer resulting in the condition that $2\boldsymbol{x}\cdot\boldsymbol{z} = h \in 2\mathbb{Z}$. We modify the equation of the family of hyperplanes in the following way:

$$(\boldsymbol{x} - m\boldsymbol{v})\cdot\boldsymbol{n} = \frac{b}{\|\boldsymbol{v}\|} \,, \tag{5.32}$$

$$(\boldsymbol{x} - m\boldsymbol{v})\cdot\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} = \frac{b}{\|\boldsymbol{v}\|} \,, \tag{5.33}$$

$$\boldsymbol{x}\cdot m\boldsymbol{v} - m^2\boldsymbol{v}\cdot\boldsymbol{v} = mb \,, \tag{5.34}$$

$$2\boldsymbol{x}\cdot\boldsymbol{z} - 2\|\boldsymbol{z}\|^2 = 2mb \,, \tag{5.35}$$

$$2\boldsymbol{x}\cdot\boldsymbol{z} = 2\|\boldsymbol{z}\|^2 + 2mb = EH_1 + \|\boldsymbol{z}\|^2 = h \,, \tag{5.36}$$

expressing the condition for the lattice points $\boldsymbol{x}$ to be at the cross-section of the hyperplanes, the confocal N-dimensional ellipsoids and the confocal N-dimensional hyperboloids. We need to solve numerically the Diophantine equation $\boldsymbol{x}\cdot\boldsymbol{z} = g$ in which $g \in \mathbb{Z}$.

Data analysis of the outcome of the calculations shows that the physically relevant solutions obey the supplementary condition $\boldsymbol{x}\cdot(\boldsymbol{z}-\boldsymbol{x}) \geq 0$. This yields the equations:

$$\boldsymbol{x}\cdot\boldsymbol{y} = k \geq 0 \,, \tag{5.37}$$

$$\boldsymbol{x}\cdot(\boldsymbol{z} - \boldsymbol{x}) = k \,, \tag{5.38}$$

$$\boldsymbol{x}\cdot\boldsymbol{z} - \|\boldsymbol{x}\|^2 = k \,, \tag{5.39}$$

$$m = g - k = \|\boldsymbol{x}\|^2 \,, \tag{5.40}$$

in which $g > k$ and $g, k \in \mathbb{Z}_+$ and $m = \boldsymbol{x} \cdot \boldsymbol{x}$ is known as the norm and occurs in the theta series of a lattice (J. Conway et al., 1999, p.45,p.102):

$$\Theta_L(z) = \sum_{\boldsymbol{x} \in L} q^{\boldsymbol{x} \cdot \boldsymbol{x}} \tag{5.41}$$

$$= \sum_{m=0}^{\infty} N(m) q^m \,, \tag{5.42}$$

in which $q = e^{\pi i z}$, $\text{Im}(z) > \mathit{0}$ and $N(m)$ is the number of vectors of the lattice L of norm $m = (\|\boldsymbol{x}\|_2)^2$.

We can always split the integer lattice in two half-spaces based on Farkas'lemma. We denote the half-space containing the lattice point $\boldsymbol{z}$ as the cone C . There exists a dual cone $C^* = \{\boldsymbol{y} \in \mathbb{Z}^N \mid \boldsymbol{y} \cdot \boldsymbol{x} \geq 0 \,, \forall \boldsymbol{x} \in C\}$.

We apply this finding to the quantity equations representing energy in the lattice $\{0, 1\} \times \mathbb{Z}^4$. The kind of quantity energy is represented by the lattice point $\boldsymbol{z}$ with coordinates $(0 \mid -2, 2, 1, 0)$ and we search for solutions obeying the conditions $\boldsymbol{x} \cdot \boldsymbol{z} = g$, in which $g \in \mathbb{Z}$ and $\boldsymbol{x} \cdot (\boldsymbol{z} - \boldsymbol{x}) \geq 0$.

We let $g \in [1, 10]$ and the infinity norm $s \in [-4, 5]$. The solutions $\boldsymbol{x}$ have a norm $m = g - k$ and thus we expect $m \in [1, 10]$ but $\|\boldsymbol{x}\|_\infty = s$ resulting in the largest norm to be $m_{max} = 4 * 5^2 + 1 = 101$.

The results are given in Table5.2 and show the existence of 20 quantity equations in the sub-lattice $\{0\} \times \mathbb{Z}^4$. The symbols in the quantity equations are m is a mass, x is a length, I is a current, t is a time, a is an acceleration, v is a velocity, ω is an angular velocity, L is an angular momentum, p is a linear momentum, N is a diffusion constant, A is a magnetic vector potential, F is a force, H is a magnetic field, γ is a surface tension, xy is a surface area, q is an electric charge and $f(\pi)$ is an arbitrary function of a dimensionless quantity π .

Table 5.2: Equations of the kind of quantity energy in $\{0\} \times \mathbb{Z}^4$.

ID	$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{x} \cdot \boldsymbol{y}$	Quantity equation
1	$(0 \mid 0,0,0,0)$	$(0 \mid -2,2,1,0)$	0	$E \sim f(\pi) \cdot E_0$
2	$(0 \mid 0,0,1,0)$	$(0 \mid -2,2,0,0)$	0	$E \sim m \cdot v^2$
3	$(0 \mid -1,0,0,-1)$	$(0 \mid -1,2,1,1)$	0	$E \sim (\frac{1}{q}) \cdot (\frac{x^2 mI}{t})$
4	$(0 \mid -1,2,1,0)$	$(0 \mid -1,0,0,0)$	1	$E \sim L \cdot \omega$
5	$(0 \mid -1,0,1,-1)$	$(0 \mid -1,2,0,1)$	0	$E \sim (\frac{m}{q}) \cdot (\frac{Ix^2}{t})$
6	$(0 \mid -1,0,1,0)$	$(0 \mid -1,2,0,0)$	1	$E \sim (\frac{m}{t}) \cdot N$
7	$(0 \mid -1,1,-1,0)$	$(0 \mid -1,1,2,0)$	0	$E \sim (\frac{v}{m}) \cdot (vm^2)$
8	$(0 \mid 0,1,1,-1)$	$(0 \mid -2,1,0,1)$	0	$E \sim (\frac{m}{H}) \cdot (aI)$
9	$(0 \mid -2,0,0,0)$	$(0 \mid 0,2,1,0)$	0	$E \sim \omega^2 \cdot mx^2$
10	$(0 \mid -2,0,1,0)$	$(0 \mid 0,2,0,0)$	0	$E \sim \gamma \cdot xy$
11	$(0 \mid -1,1,1,-1)$	$(0 \mid -1,1,0,1)$	1	$E \sim (\frac{mx}{q}) \cdot (Iv)$
12	$(0 \mid -1,1,1,0)$	$(0 \mid -1,1,0,0)$	2	$E \sim p \cdot v$
13	$(0 \mid -1,1,1,1)$	$(0 \mid -1,1,0,-1)$	1	$E \sim (pI) \cdot (\frac{v}{I})$
14	$(0 \mid -2,1,0,-1)$	$(0 \mid 0,1,1,1)$	0	$E \sim (\frac{a}{I}) \cdot mxI$
15	$(0 \mid -2,1,0,0)$	$(0 \mid 0,1,1,0)$	1	$E \sim a \cdot mx$
16	$(0 \mid -1,2,0,-1)$	$(0 \mid -1,0,1,1)$	0	$E \sim (\frac{x^2}{q}) \cdot (\frac{mI}{t})$
17	$(0 \mid -2,1,1,-1)$	$(0 \mid 0,1,0,1)$	0	$E \sim A \cdot xI$
18	$(0 \mid -2,1,1,0)$	$(0 \mid 0,1,0,0)$	1	$E \sim F \cdot x$
19	$(0 \mid -2,1,1,1)$	$(0 \mid 0,1,0,-1)$	0	$E \sim (FI) \cdot \frac{1}{H}$
20	$(0 \mid -1,2,1,-1)$	$(0 \mid -1,0,0,1)$	0	$E \sim (\frac{L}{I}) \cdot (\frac{I}{t})$

Some quantity equations are in a well-known form but others require further investigation. The quantity equations for the pseudo quantities are obtained by replacing the first 0 in the $\boldsymbol{x}$ and $\boldsymbol{y}$ by a 1 .

5.4 Cross-dimensional properties of integer lattices

In this section, we investigate the distributions of *unique* semi-perimeters of parallelograms as function of the dimension N of the integer lattice $\mathbb{Z}^N$ and the infinity norm s .

5.4.1 Kind of quantity energy in an N-ball

We study the dependency of the number of unique semi-perimeters SP of parallelograms for the kind of quantity energy, having the coordinate $(0 \mid -2, 2, 1, 0^{N-3})$ in $\{0, 1\} \times \mathbb{Z}^N$. The kind of quantity energy E has an infinity norm of $s = 2$.

Table 5.3: Number of unique semi-perimeters of parallelograms, with boundary the N-ball B_N, for the kind of quantity energy as function of the hypercube parameters (s, N).

s	$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}^5$	$\mathbb{Z}^6$	$\mathbb{Z}^7$	$\mathbb{Z}^8$	$\mathbb{Z}^9$
1	0	2	2	2	2	2	2
2	0	10	10	8	8	8	8
3	0	18	18	14	14	14	14
4	50	26	24	**16**	16	16	16
5	40	24	24	16	16	16	16
6	40	24	24	16	16	16	16
7	40	24	24	16	16	16	16
8	40	24	24	16	16	16	16
9	40	24	24	16	16	16	16
10	40	24	24	16	16	16	16
11	40	24	24	16	16	16	16

Table 5.3 shows that when the infinity norm $s \leq 11$ we find as minimum dimension $d_{min} = 6$. The number of unique semi-perimeters of parallelograms is invariant for $N \geq 6$ and $s \geq 4$. The threshold point is shown in red in Table 5.3.

Table 5.3 shows that the kind of quantity energy should be represented by an 8-tuple of dimensionless quantities $(E1, E2, E3, E4, E5, E6, E7, E8)$ where the geometric representation of the dimensionless quantity E_i is a unique parallelogram.

Table 5.3 is valid for all the elements of the orbits $(0 \mid 2, 2, 1, 0^{N-3})$ and $(1 \mid 2, 2, 1, 0^{N-3})$. These orbits contain the following kinds of quantities: energy, body force density, etc.

5.4.2 Kind of quantity energy density in an N-ball

We study the dependency of the number of unique semi-perimeters SP of parallelograms for the kind of quantity energy density, having coordinate

$(0 \mid -2, -1, 1, 0^{N-3})$ in $\{0,1\} \times \mathbb{Z}^N$. The kind of quantity energy density W has an infinity norm of $s = 2$.

Table 5.4: Number of unique semi-perimeters of parallelograms, with boundary the N-ball B_N, for the kind of quantity energy density as function of the hypercube parameters (s, N).

s	$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}^5$	$\mathbb{Z}^6$	$\mathbb{Z}^7$	$\mathbb{Z}^8$	$\mathbb{Z}^9$
1	0	2	2	2	2	2	2
2	0	6	6	6	6	6	6
3	20	10	8	8	8	8	8
4	22	10	**10**	10	10	10	10
5	22	10	10	10	10	10	10
6	22	10	10	10	10	10	10
7	22	10	10	10	10	10	10
8	22	10	10	10	10	10	10
9	22	10	10	10	10	10	10
10	22	10	10	10	10	10	10
11	22	10	10	10	10	10	10

Table 5.4 shows that when the infinity norm $s \leq 11$ we find as minimum dimension $d_{min} = 5$. The number of unique semi-perimeters of parallelograms is invariant for $N \geq 5$ and $s \geq 4$. The threshold point is shown in red in Table 5.4.

Table 5.4 shows that the kind of quantity energy density should be represented by an 5-tuple of dimensionless quantities $(W1, W2, W3, W4, W5)$ where the geometric representation of the dimensionless quantity W_i is a unique parallelogram.

Table 5.4 is valid for all the elements of the orbits $(0 \mid 2, 1, 1, 0^{N-3})$ and $(1 \mid 2, 1, 1, 0^{N-3})$. These orbits contain the following kinds of quantities: energy density, stress-energy tensor, force, etc. The kind of quantity energy density contains the physical quantities energy density, Lagrangian density in (x,y,z,t), Hamilton density, radiant energy density, spectral exposure in wavelength, sound energy density, toughness, pressure Lamé's first parameter,Lamé's second parameter, P-wave modulus, tensile stress, sound pressure, modulus of elasticity, Young's modulus, shear modulus, modulus of rigidity, bulk modulus, compression modulus, normal stress, shear stress, and energy momentum tensor.

5.4.3 Number of distinct distances of lattice points from the origin

When a lattice point $\boldsymbol{z}$ is an element of an orbit $\mathrm{Orb}\,(\boldsymbol{a})$ and the Euclidean distance $\|\boldsymbol{z}\|_2$ of this lattice point to the origin of the lattice is known then all the other lattice points belonging to this orbit have the same Euclidean distance, because signed permutations are isometric. Thus the cardinality of the orbit corresponds to the absolute frequency of occurrence of this specific Euclidean distance. The value given in a specific cell of the tables of the elements of physics represents the number of distinct orbits. Each of these distinct orbits has a specific distance. We infer that each orbit can be associated to a unique distance and thus there exists a bijection between the set of orbits of the integer lattice and the set of square roots of natural numbers. Hence, the values of the cells in the tables of the elements of physics represent the number of distinct distances. The last column denoted *RowSum* can be generated by the equation:

$$a(s) = \binom{s+N-1}{N-1}, \tag{5.43}$$

in which $N \leq 7$. To find the number of distinct distances of lattice points from the origin within a centrally symmetric set, being the N-dimensional hypercube with center the origin $\boldsymbol{o}$ we sum the appropriate values of the column *RowSum*. We generalize the results and state without proof the Theorem 5.4.1:

Theorem 5.4.1. *Suppose $f(s,N)$ is the number of distinct distances of lattice points from the origin **o** within a centrally symmetric N-dimensional hypercube of size $2s$. Then $f(s,N)$ is given by the equation:*

$$f(s,N) = \sum_{i=0}^{s} \binom{i+N-1}{N-1}, \tag{5.44}$$

in which $2s$ is the size of the N-dimensional hypercube.

5.4.4 Dimensional invariance of the number of unique quantity equations

From the computer simulations we discover the existence of a threshold point (N,s) for the number of unique quantity equations of a kind of quantity. This result shows the existence of a meta-law similar to the *linguistic relativity* (Linguistic relativity, 2024).

5.5 Exploring the kind of quantity energy

5.5.1 Histogram of semi-perimeters of parallelograms for the kind of quantity energy

We consider parallelograms in which the general condition is $\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{z}$ with $\boldsymbol{x}$ and $\boldsymbol{y}$ lattice points of $\{0,1\} \times \mathbb{Z}^7$. We define the semi-perimeter of the parallelogram as $SP = \|\boldsymbol{x}\|_2 + \|\boldsymbol{y}\|_2 = \sqrt{u} + \sqrt{v}$. Figure 5.2 represents a histogram of semi-perimeters SP of parallelograms in semi-log format.

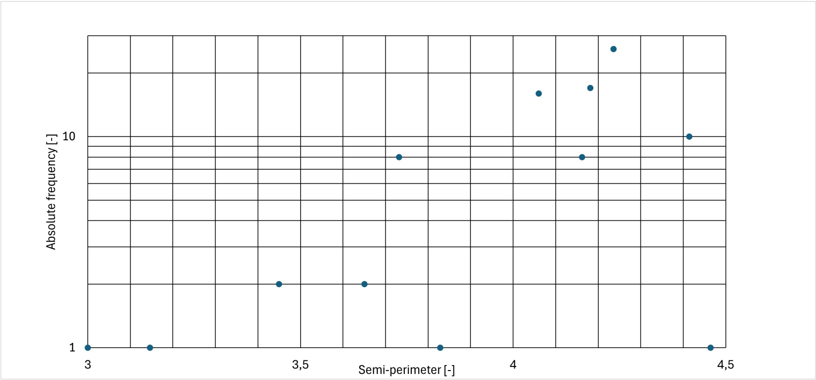

Figure 5.2: Histogram of semi-perimeters of parallelograms of $(0 \mid -2, 2, 1, 0^4)$ in the fundamental (7+1)-dimensional ellipsoid.

The semi-perimeters with absolute frequency $f = 1$ are given in Table 5.5

Table 5.5: Semi-perimeters SP of parallelograms with absolute frequency $f = 1$ of $(0 \mid -2, 2, 1, 0^4)$ in the fundamental (7+1)-dimensional ellipsoid.

ID	u	v	SP	$\boldsymbol{x}$	$\boldsymbol{y}$	$E \propto$
1	0	9	3.000	$(0 \mid 0^7)$	$(0 \mid -2, 2, 1, 0^4)$	E
2	2	3	3.146	$(0 \mid -1, 1, 0^5)$	$(0 \mid -1, 1, 1, 0^4)$	$v \cdot p$
3	1	8	3.828	$(0 \mid 0, 0, 1, 0^4)$	$(0 \mid -2, 2, 0^5)$	mv^2
4	1	12	4.464	$(0 \mid 0, 0, -1, 0^4)$	$(0 \mid -2, 2, 2, 0^4)$	$\frac{1}{m} \cdot p^2$

Analysis of the quantity equations confirms the universal character of the dimensionless equations:

$$f_{11}(\pi) = \frac{pv}{E} = \frac{\boldsymbol{p} \cdot \boldsymbol{v}}{E}\,, \tag{5.45}$$

$$f_{12}(\pi) = \frac{mv^2}{E}\,, \tag{5.46}$$

$$f_{13}(\pi) = \frac{\frac{1}{m} \cdot p^2}{E}\,, \tag{5.47}$$

that have been extensively proven in physics's experiments worldwide. We identify the product of velocity and linear impulse in ID = 2, kinetic energy and total energy in ID = 3, and kinetic energy of the Hamiltonian in ID = 4. The semi-perimeters with absolute frequency $f = 2$ are given in Table 5.6

Table 5.6: Semi-perimeters SP of parallelograms with absolute frequency $f = 2$ of $(0 \mid -2, 2, 1, 0^4)$ in the fundamental (7+1)-dimensional ellipsoid.

ID	u	v	SP	$\boldsymbol{x}$	$\boldsymbol{y}$	$E \propto$
1a	1	6	3.449	$(0 \mid -1, 0^6)$	$(0 \mid -1, 2, 1, 0^4)$	$\nu \cdot J$
1b	1	6	3.449	$(0 \mid 0, 1, 0^5)$	$(0 \mid -2, 1, 1, 0^4)$	$s \cdot F$
2a	2	5	3.650	$(0 \mid -1, 0, 1, 0^4)$	$(0 \mid -1, 2, 0^5)$	$\frac{\mathrm{d}m}{\mathrm{d}t} \cdot \frac{\mathrm{d}A}{\mathrm{d}t}$
2b	2	5	3.650	$(0 \mid 0, 1, 1, 0^4)$	$(0 \mid -2, 1, 0^5)$	$mr \cdot a$

Analysis of the quantity equations confirms the universal character of the dimensionless equations:

$$f_{21}(\pi) = \frac{\nu \cdot J}{E}\,, \tag{5.48}$$

$$f_{22}(\pi) = \frac{s \cdot F}{E}\,, \tag{5.49}$$

$$f_{23}(\pi) = \frac{\frac{\mathrm{d}m}{\mathrm{d}t} \cdot \frac{\mathrm{d}A}{\mathrm{d}t}}{E}\,, \tag{5.50}$$

$$f_{24}(\pi) = \frac{mr \cdot a}{E}\,, \tag{5.51}$$

that have been extensively proven in physics's experiments worldwide. We identify Planck's law in ID = 1a, work in ID = 1b, energy of dissipative systems in ID = 2a, potential energy in ID =2b.

Figure 5.2 shows the existence of extremes in the absolute frequencies of the semi-perimeter and more particularly the existence of the absolute frequency $f = 1$. It means that the absolute frequency $f = 1$ is the evidence for the existence of *unique* parallelograms.

Figure 5.2 shows that most of the quantity equations representing ternary equations of the kind of quantity energy have large frequencies $f > 2$. These quantity equations having the same semi-perimeter SP should be considered as strongly connected.

Inspection of the quantity equations in Table 5.5 with frequency $f = 1$ and in Table 5.6 with frequency $f = 2$ makes obvious that these quantity equations can be mapped to known energy equations.

We infer that the *energy laws* have a low probability based on Figure 5.2. We conjecture that the *energy laws* should maximize the information content $I(E) = -\log_2(p(E))$ of an *energy event* E, in which $p(E)$ is the probability of an energy quantity equation having a value $\sqrt{u} + \sqrt{v}$ for the semi-perimeter of the representing parallelogram, in which $u, v \in \mathbb{N}$.

5.5.2 Phase space of the kind of quantity energy

We aim to construct a discrete phase space for quantity equations of a physical quantity Z that can be represented by equations of the type $Z = f(\pi)XY$. We know that this equation represents a parallelogram in $\{0,1\} \times \mathbb{Z}^N$. A parallelogram is characterized by the Euclidean lengths of its sides $\|\boldsymbol{x}\|_2, \|\boldsymbol{y}\|_2$ and the angle between the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. The disadvantage of these geometric characteristics is that the values are real numbers. However, the calculation of the area squared A^2 of a parallelogram in $\{0,1\} \times \mathbb{Z}^N$ yields an integer. When considering the 1-norm of the lattice points we also obtain integers. Thus a discrete phase space could be constructed from phase space points P having the coordinates $(\|\boldsymbol{x}\|_1, \|\boldsymbol{y}\|_1, A^2)$. Each parallelogram can be mapped to a phase space point P. The discrete phase space is isomorphic to $\mathbb{N}_1^3$. It is obvious that some parallelograms will be mapped to the same phase space point P resulting in a certain multiplicity for that specific phase space point. Let $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{z}$ with $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \{0,1\} \times \mathbb{Z}^N$, then

$$\|\boldsymbol{y}\|_1 = \|\boldsymbol{x}\|_1 - \|\boldsymbol{z}\|_1 + 2v\,, \tag{5.52}$$

with $v \in \{0, \ldots, \|\boldsymbol{z}\|_1\}$. Figure 5.3 illustrates the phase space for 2-factors energy equations generated in the hypercube with infinity norm $\|\boldsymbol{x}\|_\infty = 3$ in the sub-lattice $\{0,1\} \times \mathbb{Z}^7$ and shows the existence of six planes labeled by the values of the parameter $v \in \{0, \ldots, \|\boldsymbol{z}\|_1\}$.

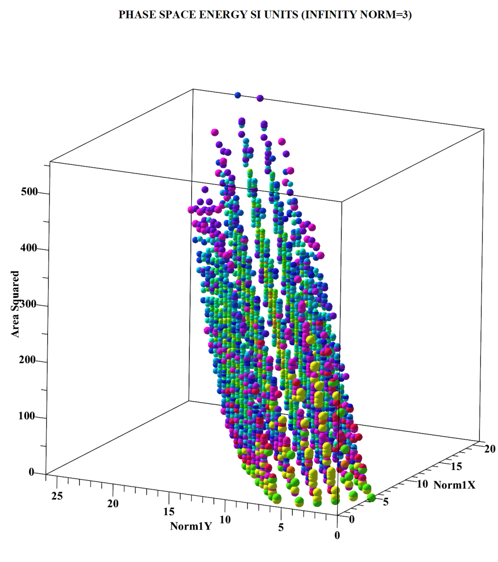

Figure 5.3: Discrete phase space of 2-factors energy equations.

The results of the phase space representation of quantity equations yields a theorem.

Theorem 5.5.1. Let $\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{z}$ with $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in\mathbb{Z}^N$ then $\|\boldsymbol{y}\|_1=\|\boldsymbol{x}\|_1-\|\boldsymbol{z}\|_1+2n$, with $n\in\ \{0,\ \ldots,\ \|\boldsymbol{z}\|_1\}$.

Proof. A graphical proof is given through the Figure 5.4.

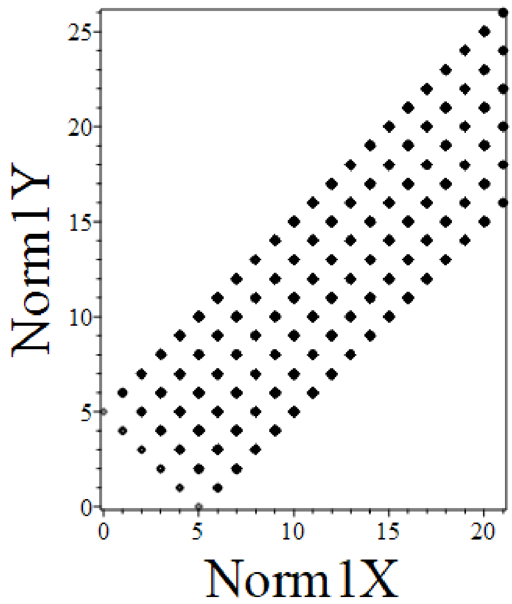

Figure 5.4: $\|\boldsymbol{y}\|_1$ versus $\|\boldsymbol{x}\|_1$ for the hypercube $\|\boldsymbol{x}\|_\infty=3$ in the integer lattice $\{0,1\}\times\mathbb{Z}^7$ for the physical quantity $\boldsymbol{z}=(-2,1,2,0,0,0,0)$.

Straight lines under an inclination of $45\,\text{deg}$ with the horizontal axis are observed and inspection results in the equation to be proven. □

The maxima for the respective 1-norms are given by the equations:

$$\max\|\boldsymbol{x}\|_1=N\cdot\max\|\boldsymbol{x}\|_\infty\,,$$
$$\max\|\boldsymbol{y}\|_1=N\cdot\max\|\boldsymbol{x}\|_\infty+\|\boldsymbol{z}\|_1\,,$$

in which N is the dimension of the integer lattice. Figure 5.3 illustrates the existence of six planes in the phase space of quantity equations. Due to the symmetry of the equation $Z=f(\pi)XY$ only three distinct planes remain. The distinct planes are characterized with the label n that takes values in the set $\{0,\ldots,\left\lfloor\frac{\|\boldsymbol{z}\|_1}{2}\right\rfloor\}$. The observed quantity equations are found at phase space points close to the equation $E=E$. The quantity equation $E=E$ has the phase space coordinates $(0, 5, 0)$. An illustration is given in Figure 5.5. The coloring in the phase space is according to the legend: $n = 0$ is red, $n = 1$ is green, and $n = 2$ is blue and these colorings refer to the three different quantity equations families.

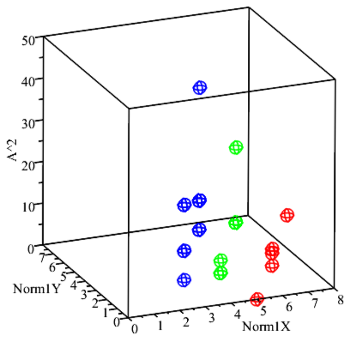

Figure 5.5: Phase space points of observed quantity equations for the hypercube $\|\boldsymbol{x}\|_\infty = 3$ in the integer lattice $\{0, 1\} \times \mathbb{Z}^7$ for the kind of quantity energy.

Table 5.7 gives the phase space coordinates $(\|\boldsymbol{x}\|_1, \|\boldsymbol{y}\|_1, A^2)$ of some known energy equations.

Table 5.7: Phase space coordinates of observed energy equations.

ID	phase space coordinate $(\|\boldsymbol{x}\|_1, \|\boldsymbol{y}\|_1, A^2)$	m	quantity equation
1	$(5,0,0)$	0	$E \sim E$
2	$(6,1,5)$	0	$E \sim \int P \cdot \mathrm{d}t$
3	$(6,1,8)$	0	$E \sim \frac{p^2}{m}$
4	$(6,1,9)$	0	$E \sim k \cdot T$
5	$(7,2,14)$	0	$E \sim q \cdot U$
6	$(4,1,5)$	1	$E \sim \int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s}$
7	$(4,1,5)$	1	$E \sim \hbar \cdot \omega$
8	$(4,1,8)$	1	$E \sim m \cdot v^2$
9	$(5,2,32)$	1	$E \sim m^2 \cdot \frac{G}{r}$
10	$(3,2,2)$	2	$E \sim \boldsymbol{p} \cdot \boldsymbol{v}$
11	$(3,2,9)$	2	$E \sim a \cdot mx$
12	$(3,2,20)$	2	$E \sim mx^2 \cdot \omega^2$
13	$(4,3,45)$	2	$E \sim \int p \cdot \mathrm{d}V$

Some quantity equations are mapped to the same phase space coordinates. This is an indication that the quantity equations can be coupled. Table 5.7 shows this for the phase space coordinate $(4,1,5)$ yielding the connection $\displaystyle\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} \sim \hbar \cdot \omega$. The separation of the energy equations in six distinct families is based *only* on geometric properties of their respective algebraic equation within a language of science formalism that in this specific case is the SI2019 convention among researchers.

5.6 Exploring the kind of quantity Newtonian constant of gravitation

The lattice point $(0 \mid -2,3,-1,0^4)$ representing the kind of quantity Newtonian constant of gravitation, denoted by G, is an element of the orbit $[0 \mid 3210^4]$ having cardinality 1680. The infinity norm of $\|(0 \mid -2,3,-1,0^4)\|_\infty = 3$. So, the orbit representative $(0 \mid 3,2,1,0^4)$ has the coordinates $3, 1680$ in the Table 3.1.

The orthant 81, with sign $(-,+,-,+,+,+,+)$, contains the lattice point $(0 \mid -2,3,-1,0^4)$ representing the physical quantity Newtonian constant of gravitation. It is the only lattice point from the lexicon Appendix A, being in orthant 81. In the same orbit $[(0 \mid 3,2,1,0^4)]$ we find the lattice point $(0 \mid -3,2,1,0^4)$ representing the physical quantity power denoted by P and located in the orthant 65 with sign $(-,+,+,+,+,+,+)$.

We expect the existence of a similitude between the kind of quantity power P and the kind of quantity Newtonian constant of gravitation G .

A signed permutation matrix $\mathbf{M}$ can be constructed such that the lattice point representing the kind of quantity power P can be mapped in the lattice point representing the kind of quantity Newtonian constant of gravitation G . The 8×8 signed permutation matrix $\mathbf{M}$ is defined by:

$$\mathbf{M}=\begin{bmatrix}1&0&0&0&0&0&0&0\\0&0&-1&0&0&0&0&0\\0&-1&0&0&0&0&0&0\\0&0&0&-1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix}$$

We verify that the operation of the signed permutation matrix $\mathbf{M}$ on the lattice point $(0 \mid -3,2,1,0^4)$ as column vector yields the lattice point $(0 \mid -2,3,-1,0^4)$.

$$\begin{bmatrix}0\\-2\\3\\-1\\0\\0\\0\\0\end{bmatrix}=\begin{bmatrix}1&0&0&0&0&0&0&0\\0&0&-1&0&0&0&0&0\\0&-1&0&0&0&0&0&0\\0&0&0&-1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}0\\-3\\2\\1\\0\\0\\0\\0\end{bmatrix} \tag{5.53}$$

Figure 5.6 shows the histogram of semi-perimeters of parallelograms for the kind of quantity Newtonian constant of gravitation in the fundamental (7+1)-dimensional ellipsoid. The semi-perimeter limit of the fundamental (7+1)-dimensional ellipsoid yields $SP \leq \frac{3}{2}\sqrt{14}$.

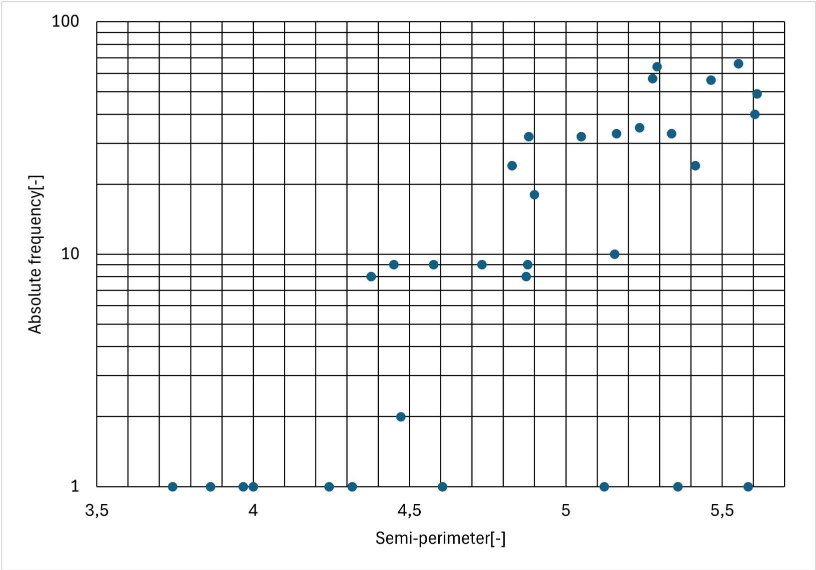

Figure 5.6: Histogram of semi-perimeters of parallelograms for the kind of quantity Newtonian constant of gravitation in the fundamental (7+1)-dimensional ellipsoid.

Figure 5.7 shows the histogram of semi-perimeters of parallelograms for the kind of quantity power in the fundamental (7+1)-dimensional ellipsoid.

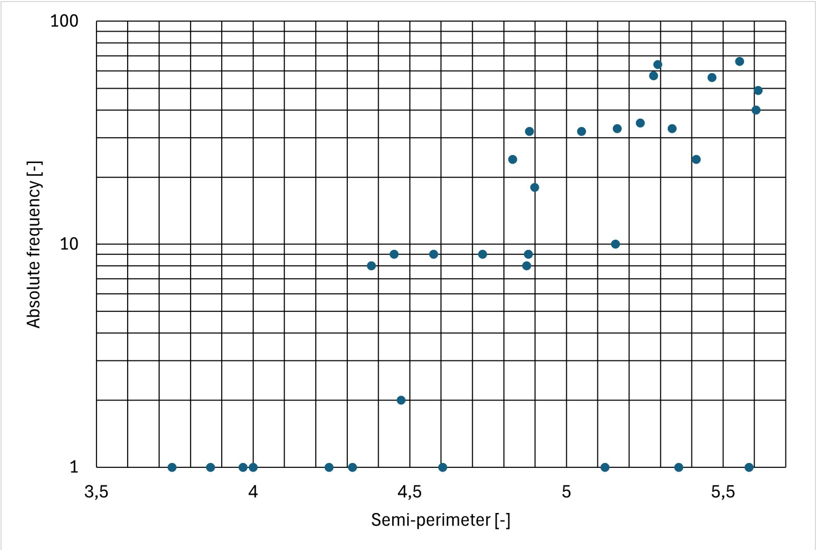

Figure 5.7: Histogram of semi-perimeters of parallelograms for the kind of quantity power in the fundamental (7+1)-dimensional ellipsoid.

Comparing the Figure 5.6 with the Figure 5.7 shows that they are *identical*. It confirms the fact that signed permutations are isometric.

We perform a systematic search for unique parallelogram perimeters for the parallelogram formed by $\boldsymbol{o}, \boldsymbol{x}, \boldsymbol{z}, \boldsymbol{y}, \boldsymbol{o}$ in which $\boldsymbol{z} = (0 \mid -2, 3, -1, 0^4)$ and we limit the search to lattice points in the hypercube with $\|\boldsymbol{z}\|_\infty <= 3 + 1$ because we are interested in the neighborhood of the kind of quantity Newtonian constant of gravitation and this is realized by increasing the infinity norm of the kind of quantity Newtonian constant of gravitation by 1.

The results of the search are given in Table 5.8. The symbols in the quantity equations of Table 5.8 are: m is a mass, r is a characteristic length, v is a velocity, ω is an angular velocity, t is a characteristic time, a is an acceleration, A is an area, V is a volume, G is the Newtonian constant of gravitation and $f_i(\pi_i)$ are functions of dimensionless parameters π_i .

Table 5.8: Unique parallelograms of $(0 \mid -2, 3, -1, 0^4)$ with $\|\boldsymbol{z}\|_\infty <= 4$.

ID	SP	$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{x} \cdot \boldsymbol{y}$	Quantity equation
1	3.741	$(0 \mid 0^7)$	$(0 \mid -2, 3, -1, 0^4)$	0	$G = f_1(\pi_1) \cdot 1 \cdot G$
2	3.863	$(0 \mid -1, 1, 0^5)$	$(0 \mid -1, 2, -1, 0^4)$	3	$G = f_2(\pi_2) \cdot v \cdot \frac{\partial^2 A(m,t)}{\partial m \partial t}$
3	3.968	$(0 \mid -1, 1, -1, 0^4)$	$(0 \mid -1, 2, 0^5)$	3	$G = f_3(\pi_3) \cdot \frac{v}{m} \cdot \frac{\mathrm{d}A}{\mathrm{d}t}$
4	4.000	$(0 \mid 0, 1, 0^5)$	$(0 \mid -2, 2, -1, 0^4)$	2	$G = f_4(\pi_4) \cdot r \cdot \frac{v^2}{m}$
5	4.243	$(0 \mid 0, 1, -1, 0^4)$	$(0 \mid -2, 2, 0^5)$	4	$G = f_5(\pi_5) \cdot \frac{r}{m} \cdot v^2$
6	4.316	$(0 \mid -1, 0^6)$	$(0 \mid -1, 3, -1, 0^4)$	1	$G = f_6(\pi_6) \cdot \frac{1}{t} \cdot \frac{\partial^2 V(m,t)}{\partial m \partial t}$
7	4.472	$(0 \mid -2, 1, 0^5)$	$(0 \mid 0, 2, -1, 0^4)$	2	$G = f_7(\pi_7) \cdot a \cdot \frac{r^2}{m}$
8	4.605	$(0 \mid 0, 0, -1, 0^4)$	$(0 \mid -2, 3, 0^5)$	0	$G = f_8(\pi_8) \cdot \frac{1}{m} \cdot \frac{\mathrm{d}^2 V}{\mathrm{d}t^2}$
9	5.123	$(0 \mid 0, 0, 1, 0^4)$	$(0 \mid -2, 3, -2, 0^4)$	-2	$G = f_9(\pi_9) \cdot m \cdot \frac{r^3 \omega^2}{m^2}$
…	…	…	…	…	…

ID	SP	$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{x} \cdot \boldsymbol{y}$	Quantity equation
10	5.359	$(0 \mid 1, 0^6)$	$(0 \mid -3, 3, -1, 0^4)$	-3	$G = f_{10}(\pi_{10}) \cdot t \cdot \frac{r^3\omega^3}{m}$
11	5.583	$(0 \mid 0, -1, 0^5)$	$(0 \mid -2, 4, -1, 0^4)$	-4	$G = f_{11}(\pi_{11}) \cdot \frac{1}{r} \cdot \frac{r^4\omega^2}{m}$

It is obvious from Table 5.8 that the quantity equations can be classified as function of the inner product $\boldsymbol{x} \cdot \boldsymbol{y}$. The equation with ID = 1 is a degenerated parallelogram. We find a rectangle for ID = 8 because $\boldsymbol{x} \cdot \boldsymbol{y} = 0$. This quantity equation is unknown to the author. Furthermore we observe that the inner product $\boldsymbol{x} \cdot \boldsymbol{y} >= 0$ for the equations in which ID ≤ 8 and $\boldsymbol{x} \cdot \boldsymbol{y} < 0$ for the ID > 8.

When the inner product $\boldsymbol{x} \cdot \boldsymbol{y} < 0$ then the parallelograms are becoming needle shaped.

When the inner product $\boldsymbol{x} \cdot \boldsymbol{y} >= 0$ then the lattice points $\boldsymbol{x}$ and $\boldsymbol{y}$ are located inside the $(7+1)$-hyperball, having a $(7+1)$-hypersphere generated by the lattice point $\boldsymbol{z} = (0 \mid -2, 3, -1, 0^4)$ through the equation (2.19).

5.6.1 Mapping of $P = \boldsymbol{F} \cdot \boldsymbol{v}$

It is known that the rate of change of energy with time, defining the kind of quantity power P, equals force times velocity (R. Feynman, 1997, p.69):

$$P = \boldsymbol{F} \cdot \boldsymbol{v}\,, \tag{5.54}$$

in which the vector $\boldsymbol{F}$ is a force and the vector $\boldsymbol{v}$ is a linear velocity.

We subject the parallelogram representing the equation 5.54 in $\{0, 1\} \times \mathbb{Z}^7$ to the signed permutation $\mathbf{M}$. We find the following mappings:

$$\begin{aligned} P &\mapsto G\,, \\ F &\mapsto Q_1\,, \\ v &\mapsto v\,, \end{aligned}$$

in which Q_1 is a kind of quantity that is not yet in Appendix A.

The unlabeled kind of quantity Q_1 is represented by the lattice point $(0 \mid -1, 2, -1, 0^4)$ being an element of the orbit $[0 \mid 21^2 0^4]$ with cardinality 840. We explore the neighborhood of the lattice point $(0 \mid -1, 2, -1, 0^4)$. The cardinality of the neighborhood of a lattice point in $\{0, 1\} \times \mathbb{Z}^7$ is $2 \times 3^7 = 4\,372$ based on the cardinality of a lattice point in a N-dimensional lattice as given in Table W.1. We restrict the exploration of the neighborhood of the lattice point $(0 \mid -1, 2, -1, 0^4)$ to the sub-lattice $\mathbb{Z}^3$ such that we have to analyze only 26 lattice points of the neighborhood of $(-1, 2, -1)$. The neighborhood of $(-1, 2, -1)$ in the sub-lattice $\mathbb{Z}^3$ is enumerated in the Table 5.9.

Table 5.9: $\mathbb{Z}^3$ neighborhood of $(-1, 2, -1)$.

ID	Lattice point	Euclidean distance	Physical quantity
1	$(-2, 1, -2)$	$\sqrt{3}$	unlabeled
2	$(1, 1, -2)$	$\sqrt{2}$	unlabeled
3	$(0, 1, -2)$	$\sqrt{3}$	unlabeled
4	$(-2, 1, -1)$	$\sqrt{2}$	unlabeled
5	$(-1, 1, -1)$	1	unlabeled
6	$(0, 1, -1)$	$\sqrt{2}$	unlabeled
7	$(-2, 1, 0)$	$\sqrt{3}$	acceleration
8	$(-1, 1, 0)$	$\sqrt{2}$	velocity
9	$(0, 1, 0)$	$\sqrt{3}$	length
10	$(-2, 2, -2)$	$\sqrt{2}$	unlabeled
11	$(-1, 2, -2)$	1	unlabeled
12	$(0, 2, -2)$	$\sqrt{2}$	unlabeled
13	$(-2, 2, -1)$	1	unlabeled
14	$(0, 2, -1)$	1	mass attenuation coefficient
15	$(-2, 2, 0)$	$\sqrt{2}$	velocity squared, specific energy, gravitational potential
16	$(-1, 2, 0)$	1	kinematic viscosity, quantum of circulation, areal velocity, diffusion constant
17	$(0, 2, 0)$	$\sqrt{2}$	area
18	$(-2, 3, -2)$	$\sqrt{3}$	unlabeled
19	$(-1, 3, -2)$	$\sqrt{2}$	unlabeled
20	$(0, 3, -2)$	$\sqrt{3}$	unlabeled
21	$(-2, 3, -1)$	$\sqrt{2}$	Newtonian constant of gravitation
22	$(-1, 3, -1)$	1	unlabeled
23	$(0, 3, -1)$	$\sqrt{2}$	specific volume
24	$(-2, 3, 0)$	$\sqrt{3}$	standard gravitational parameter
25	$(-1, 3, 0)$	$\sqrt{2}$	unlabeled
26	$(0, 3, 0)$	$\sqrt{3}$	volume

From the six lattice points at Euclidean distance of one we have two that have a label. The other four lattice points are not in the lexicon Appendix A and thus are unlabeled. We have the choice between ID = 14 representing the mass attenuation coefficient and ID = 16 that can represent the physical quantities kinematic viscosity, quantum of circulation, areal velocity, diffusion constant. We postulate the existence of the quantity equation:

$$G = f(\pi_1) \cdot Q_1 \cdot v\,, \tag{5.55}$$

in which G denotes the Newtonian constant of gravitation, Q_1 denotes the specific kinematic viscosity, v is the magnitude of a velocity vector $\boldsymbol{v}$ and $f(\pi_1)$ a function of a dimensionless quantity π_1 . The equation 5.55 is listed in Table 5.8 with ID = 2. The equation 5.55 is unknown to the author. Assume that $f(\pi_1) = 1$ and consider that $v \to c_0$ then we find that $Q_1 \to \dfrac{G}{c_0}$. The minimum value of Q_1 is $\dfrac{G}{c_0} = 2.22623 \times 10^{-19}\,\mathrm{m}^2\,\mathrm{kg}^{-1}\,\mathrm{s}^{-1}$ and the maximum value of Q_1 is ∞ . Observe that the minimum value of Q_1 has a very small magnitude indicating that the medium is almost in a state of *superfluidity* with a hint to superfluid vacuum theory (Sinha, Sivaram, & Sudarshan, 1976; Volovik, 2001; Wilczek, 2013; Huang, 2017).

Another well-known equation about the physical quantity power is $P = \frac{\mathrm{d}E}{\mathrm{d}t}$. We subject the parallelogram representing this equation in $\{0,1\} \times \mathbb{Z}^7$ to the signed permutation $\mathbf{M}$. We find the following mappings:

$$P \mapsto G\,,$$
$$E \mapsto Q_2\,,$$
$$\frac{\mathrm{d}}{\mathrm{d}t} \mapsto r\,,$$

in which Q_2 is a physical quantity that is not in Appendix A and r is a characteristic length. The unlabeled physical quantity Q_2 is represented by the lattice point $(0 \mid -2, 2, -1, 0^4)$ being an element of the orbit $[(0 \mid 2^2, 1, 0^4)]$ with cardinality 840. The neighborhood of $(-2, 2, -1)$ in the sub-lattice $\mathbb{Z}^3$ is enumerated in the Table 5.10.

Table 5.10: $\mathbb{Z}^3$ neighborhood of $(-2, 2, -1)$.

ID	Lattice point	Euclidean distance	Physical quantity
1	$(-3, 1, -2)$	$\sqrt{3}$	unlabeled
2	$(-2, 1, -2)$	$\sqrt{2}$	unlabeled
3	$(-1, 1, -2)$	$\sqrt{3}$	unlabeled
4	$(-3, 1, -1)$	$\sqrt{2}$	unlabeled
5	$(-2, 1, -1)$	1	specific acceleration
6	$(-1, 1, -1)$	$\sqrt{2}$	specific velocity
7	$(-3, 1, 0)$	$\sqrt{3}$	unlabeled
8	$(-2, 1, 0)$	$\sqrt{2}$	acceleration
9	$(-1, 1, 0)$	$\sqrt{3}$	velocity
10	$(-3, 2, -2)$	$\sqrt{2}$	unlabeled
11	$(-2, 2, -2)$	1	unlabeled
12	$(-1, 2, -2)$	$\sqrt{2}$	unlabeled
13	$(-3, 2, -1)$	1	unlabeled
14	$(-1, 2, -1)$	1	Q_1
15	$(-3, 2, 0)$	$\sqrt{2}$	unlabeled
16	$(-2, 2, 0)$	1	velocity squared
17	$(-1, 2, 0)$	$\sqrt{2}$	kinematic viscosity, quantum of circulation, areal velocity, diffusion constant
18	$(-3, 3, -2)$	$\sqrt{3}$	unlabeled
19	$(-2, 3, -2)$	$\sqrt{2}$	unlabeled
20	$(-1, 3, -2)$	$\sqrt{3}$	unlabeled
21	$(-3, 3, -1)$	$\sqrt{2}$	unlabeled
22	$(-2, 3, -1)$	1	Newtonian constant of gravitation
23	$(-1, 3, -1)$	$\sqrt{2}$	unlabeled
24	$(-3, 3, 0)$	$\sqrt{3}$	3rd time derivative of volume, third power of velocity
25	$(-2, 3, 0)$	$\sqrt{2}$	standard gravitational parameter
26	$(-1, 3, 0)$	$\sqrt{3}$	1st time derivative of volume

We postulate the existence of a quantity equation:

$$G = f(\pi_2) \cdot \int Q_2 \cdot \mathrm{d}r \,, \tag{5.56}$$

in which G denotes the Newtonian constant of gravitation, Q_2 denotes the specific gravitational potential, r denotes a characteristic length and $f(\pi_2)$ a function of a dimensionless quantity π_2. The equation 5.56 is listed in Table 5.8 with ID = 4. Observe that under the condition that $f(\pi_1) = f(\pi_2) = 1$ we find:

$$Q_2 = \frac{\mathrm{d}Q_1}{\mathrm{d}t} \,. \tag{5.57}$$

We previously studied the orthogonal decomposition of the lattice point $(0 \mid -3, 2, 1, 0^4)$ in Appendix G, representing the kind of quantity power, when embedded in the sub-lattice $\mathbb{Z}^3$. This orthogonal decomposition showed the existence of four *unique rectangles* with lattice points given in Table G.1.

We map those four rectangles P_i with $i = 1, 2, 3, 4$ to four new rectangles. The symbols in these mappings denote the following physical quantities: P is power, m is a mass, v is a velocity, t is a characteristic time, G is the Newtonian constant of gravity, V is a volume, p is a linear momentum, r is a characteristic length, ω is a characteristic angular frequency, ρ is a mass density.

5.6.2 Rectangle P_1

We map rectangle P_1 with the following prescription:

$$\begin{aligned} P &\mapsto G \,, \\ m &\mapsto \frac{1}{m} \,, \\ \frac{\mathrm{d}v^2}{\mathrm{d}t} &\mapsto \frac{\mathrm{d}^2 V}{\mathrm{d}t^2} \,. \end{aligned}$$

We obtain the quantity equation:

$$G = f(\pi_3) \cdot \frac{1}{m} \cdot \frac{\mathrm{d}^2 V}{\mathrm{d}t^2} \,. \tag{5.58}$$

The equation 5.58 is listed in Table 5.8 with ID = 8. It should be an important quantity equation due to the fact that its is a unique rectangle. However, it is unknown to the author. We have the intention to discuss it with professor Shirley Ho. Her team probably discovered a new physics law about the galaxy-halo connection (Delgado et al., 2022). We speculate that m is the mass of the galaxy and V is the volume of the halo.

5.6.3 Rectangle P_2

We map rectangle P_2 with the following prescription:

$$P \mapsto G\,,$$
$$p^2 \mapsto \left(\frac{v}{m}\right)^2,$$
$$\frac{\omega}{m} \mapsto mr\,.$$

We obtain the quantity equation:

$$G = f(\pi_4) \cdot \left(\frac{v}{m}\right)^2 \cdot (mr)\,. \tag{5.59}$$

The equation 5.59 is already known from the circular orbits of radius r in which a small body of mass m_1 is describing a circular motion with velocity v around a central body with mass m_2 .

In the framework of classical mechanics, the equilibrium of forces is expressed by stating that the sum of the centripetal force and the gravitational force is equal to zero. We obtain the equations:

$$\frac{G \cdot m_1 \cdot m_2}{r^2} = \frac{m_1 \cdot v^2}{r}\,,$$
$$G = \frac{m_1 \cdot v^2 \cdot r^2}{r \cdot m_1 \cdot m_2}\,,$$
$$G = \frac{v^2}{m_1 \cdot m_2} \cdot \frac{m_1 \cdot r^2}{r}\,.$$

When $m_1 = m_2 = m$ we obtain the equation:

$$G = (\frac{v}{m})^2 \cdot (mr)\,, \tag{5.60}$$
$$\tag{5.61}$$

and can infer that $f(\pi_4) = 1$ under the above assumptions.

5.6.4 Rectangle P_3

We map rectangle P_3 with the following prescription:

$$P \mapsto G\,,$$
$$r^2 \mapsto \nu^2\,,$$
$$m\nu^3 \mapsto \frac{1}{\rho}\,.$$

We obtain the quantity equation:

$$G = f(\pi_5) \cdot \nu^2 \cdot \frac{1}{\rho} \,. \tag{5.62}$$

The equation 5.62 is known in astrophysics when the characteristic frequency ν is identified with the Hubble constant H_0. The equation is:

$$G = \frac{1}{(8/3)\pi} \cdot H_0^2 \cdot \frac{1}{\rho} \,, \tag{5.63}$$

in which H_0 is the Hubble constant at the present time with value $H_0 = 2.1 \times 10^{-18}\,\mathrm{s}^{-1}$. The critical mass density ρ_{crit} of the universe has the value $\rho_{crit} = 7.9 \times 10^{-27}\,\mathrm{kg} \cdot \mathrm{m}^{-3}$. The dimensionless equation takes the following form $f(\pi_5) = \dfrac{1}{(8/3)\pi}$ and is a constant.

5.6.5 Rectangle P_4

$$\begin{aligned} &P \to G \,, \\ &\frac{1}{m} \cdot \frac{\mathrm{d}A}{\mathrm{d}t} \to F \,, \\ &(\frac{\mathrm{d}m}{\mathrm{d}t})^2 \to \frac{r^2}{m^2} \,, \end{aligned}$$

in which F is a force, A is a characteristic area, m is a mass and we obtain the quantity equation:

$$G = f(\pi_6) \cdot F \cdot \frac{r^2}{m^2} \,. \tag{5.64}$$

The equation 5.64 is the Newtonian attraction law between equal masses $m = m_1 = m_2$ at a distance r in the case that $f(\pi_6) = 1$. We have for the Newtonian attraction law the equation for the magnitude of the force $\boldsymbol{F}$:

$$F = G \frac{m_1 \cdot m_2}{r^2} \,. \tag{5.65}$$

Based on the importance of equation 5.64 for classical mechanics, we speculate that the quantity equation of rectangle P_4:

$$P(t) = f(\pi_0) \cdot \left(\frac{1}{m(t)} \cdot \frac{\mathrm{d}A(t)}{\mathrm{d}t} \right) \cdot \left(\frac{\mathrm{d}m(t)}{\mathrm{d}t} \right)^2 \,, \tag{5.66}$$

is also important. The term $f(\pi_0)$ is an unknown function of a dimensionless parameter π_0, $A(t)$ is the time-varying area of the surface of the object, $m(t)$ is the time-varying mass of the object, t is the time parameter and P is the power.

We *speculate* that the quantity equation 5.66 is related to variable stars where $P(t)$ corresponds to the instantaneous luminosity of the star, $m(t)$ is the time-varying mass of the star, and $A(r,t)$ is the time-varying surface area of the star while $\frac{\mathrm{d}m}{\mathrm{d}t}$ is representing the thermonuclear burning inside the star or the mass loss of the star.

We searched for that quantity equation in astrophysics literature (Percy, 2007), (Misner, Thorne, & Wheeler, 2017, p.688-699), (Landau & Lifchitz, 1967, p.400), (Kurth, 1972, p.127-134), and (Clayton, 2007, p.3-14) but have not yet found a similar equation.

5.7 Exploring electromagnetism

We have discussed several mathematical methods useful for discovering 'laws of physics'. We will apply these methods to the set of physical quantities known to Maxwell in the period 1855 to 1873 and discuss the results. We refer to chapter 10 in Maxwell's 'A Treatise on Electricity and Magnetism: Volume 2' (J. Maxwell, 1873). Maxwell considered the equations of dimensions, which we call quantity equations, for energy $\frac{\mathsf{L}^2\mathsf{M}^2}{\mathsf{T}^2}$ and energy density $\frac{\mathsf{M}}{\mathsf{LT}^2}$.

Maxwell considered also the relations of physical quantities in terms of integrals, more specific: time-integrals, line-integrals and surface-integrals. We use the SI2019 units instead of the $\mathsf{L}, \mathsf{M}, \mathsf{T}$ unit system of Maxwell. Table 5.11 gives the physical quantities considered by Maxwell. It contains 7 columns.

The first column gives the cardinality of the orbit that contains the physical quantity. The second column represents the name of the physical quantity. The third column indicates the infinity norm $\|z\|_\infty$ of the vertex. The fourth column lists the orbit that contains the physical quantity. The fifth column identifies the physical quantity by its integer lattice point in $\{0,1\} \times \mathbb{Z}^7$. The sixth column gives the value of the sum of the coordinates of the lattice point in $\{0,1\} \times \mathbb{Z}^7$.

Table 5.11: Physical quantities considered by Maxwell in 1873.

$\#$ (Orbit)	Physical quantity	$\lVert z \rVert_\infty$	Orbit	Vertex	$\mathrm{soc}\,(z)$
14	length	1	$[(0 \mid 1, 0^6)]$	$(0 \mid 0, 1, 0, 0, 0, 0, 0)$	1
14	mass	1	$[(0 \mid 1, 0^6)]$	$(0 \mid 0, 0, 1, 0, 0, 0, 0)$	1
14	time	1	$[(0 \mid 1, 0^6)]$	$(0 \mid 1, 0, 0, 0, 0, 0, 0)$	1
14	electric current	1	$[(0 \mid 1, 0^6)]$	$(0 \mid 0, 0, 0, 1, 0, 0, 0)$	1
14	magnetic potential difference	1	$[(0 \mid 1, 0^6)]$	$(0 \mid 0, 0, 0, 1, 0, 0, 0)$	1
14	magnetomotive force	1	$[(0 \mid 1, 0^6)]$	$(0 \mid 0, 0, 0, 1, 0, 0, 0)$	1
84	electric charge	1	$[(0 \mid 1^2, 0^5)]$	$(0 \mid 1, 0, 0, 1, 0, 0, 0)$	2
840	magnetic induction	2	$[(1 \mid 2, 1^2, 0^4)]$	$(1 \mid -2, 0, 1, -1, 0, 0, 0)$	-2
840	electrical displacement	2	$[(0 \mid 2, 1^2, 0^4)]$	$(0 \mid 1, -2, 0, 1, 0, 0, 0)$	0
2240	magnetic vector potential	2	$[(0 \mid 2, 1^3, 0^3)]$	$(0 \mid -2, 1, 1, -1, 0, 0, 0)$	-1
2240	electric field strength	3	$[(0 \mid 3, 1^3, 0^3)]$	$(0 \mid -3, 1, 1, -1, 0, 0, 0)$	-2
3360	permeability	2	$[(0 \mid 2^2, 1^2, 0^3)]$	$(0 \mid -2, 1, 1, -2, 0, 0, 0)$	-2
6720	electric potential difference	3	$[(0 \mid 3, 2, 1^2, 0^3)]$	$(0 \mid -3, 2, 1, -1, 0, 0, 0)$	-1
6720	electrical resistance	3	$[(0 \mid 3, 2^2, 1, 0^3)]$	$(0 \mid -3, 2, 1, -2, 0, 0, 0)$	-2
6720	electric capacitance	4	$[(0 \mid 4, 2^2, 1, 0^3)]$	$(0 \mid 4, -2, -1, 2, 0, 0, 0)$	3
13440	permittivity	4	$[(0 \mid 4, 3, 2, 1, 0^3)]$	$(0 \mid 4, -3, -1, 2, 0, 0, 0)$	2

The representative integer lattice points of $\{0,1\} \times \mathbb{Z}^7$ for the physical quantities $\boldsymbol{H}, \boldsymbol{B}, \boldsymbol{E}, \boldsymbol{D}$ in the SI2019 are:

$$\begin{aligned}
\operatorname{dex}([\boldsymbol{H}]) &= (1 \mid 0,-1,0,1,0,0,0)\,, \\
\operatorname{dex}([\boldsymbol{B}]) &= (1 \mid -2,0,1,-1,0,0,0)\,, \\
\operatorname{dex}([\boldsymbol{E}]) &= (0 \mid -3,1,1,-1,0,0,0)\,, \\
\operatorname{dex}([\boldsymbol{D}]) &= (0 \mid 1,-2,0,1,0,0,0).
\end{aligned}$$

5.7.1 Maxwell's equations

The integral and differential representation of Maxwell's equations are:

$$\oint_{L(S)} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{s} = -\frac{\partial}{\partial t} \iint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} \qquad \nabla \times \boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{B} \tag{5.67}$$

$$\oint_{L(S)} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{s} = \iint_S \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S} + \frac{\partial}{\partial t} \iint_S \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} \qquad \nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial}{\partial t}\boldsymbol{D} \tag{5.68}$$

$$\oiint_{S(V)} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = \iiint_V \rho_f \, \mathrm{d}V \qquad \nabla \cdot \boldsymbol{D} = \rho_f \tag{5.69}$$

$$\oiint_{S(V)} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = 0 \qquad \nabla \cdot \boldsymbol{B} = 0 \tag{5.70}$$

(5.71)

The constitutive equations are:

$$\boldsymbol{D} \cdot \epsilon_0 \boldsymbol{E} + \boldsymbol{P} = \epsilon \boldsymbol{E} \qquad \boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M}$$

We denote electric current I, electric charge q, electric charge density ρ_f, volume V, area S, time t, length s, velocity v, and electric current density J.

5.7.2 Semi-perimeters of parallelograms of *H*

The semi-perimeter limit of the fundamental (7+1)-dimensional ellipsoid yields $SP \leq \frac{3}{2}\sqrt{2}$. The histogram of semi-perimeters of the parallelograms of the lattice point $(1 \mid 0,-1,0,1,0^3)$, representing the kind of quantity magnetic field, in the fundamental (7+1)-dimensional ellipsoid is given in Figure 5.8.

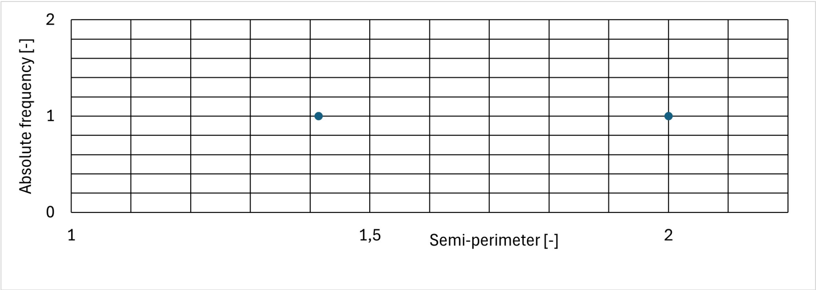

Figure 5.8: Histogram of semi-perimeters of parallelograms of the kind of quantity magnetic field $(1 \mid 0, -1, 0, 1, 0^3)$ in the fundamental (7+1)-dimensional ellipsoid.

We list the semi-perimeters SP of the parallelograms for the kind of quantity $\boldsymbol{H}$ having absolute frequencies $f = 1$ in the fundamental (7+1)-dimensional ellipsoid in Table 5.12.

Table 5.12: Parallelograms in the fundamental (7+1)-dimensional ellipsoid of the kind of quantity magnetic field $\boldsymbol{H}$.

ID	$\boldsymbol{x}$	$\boldsymbol{y}$	SP	f	Quantity equation
1	$(0 \mid 0^7)$	$(1 \mid 0, -1, 0, -1, 0^3)$	1.414	1	$H = f(\pi_1)\, H$
2	$(0 \mid 0, -1, 0^5)$	$(1 \mid 0, 0, 0, 1, 0^3)$	2.000	1	$H = f(\pi_2)\, \frac{I}{s}$

The parallelogram with ID = 1 is a degenerated parallelogram.
The parallelogram with ID = 2 is a *rectangle* representing the quantity equation:

$$H = f(\pi_1)\frac{\mathrm{d}I}{\mathrm{d}s}, \tag{5.72}$$

$$H \cdot \mathrm{d}s = f(\pi_1)\mathrm{d}I, \tag{5.73}$$

$$\oint_{L(S)} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{s} = f(\pi_1) I, \tag{5.74}$$

$$\oint_{L(S)} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{s} = f(\pi_1) \iint_S \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S}, \tag{5.75}$$

where I is the total electric current and $\mathrm{d}s$ is an elemental length. The transition from (5.73) to (5.74) consists of integrating the current of the closed path L. The equation (5.74) has the form of Ampère's law when considering the line integral of the field $\boldsymbol{H}$ over a closed path L. The dimensionless parameter $f(\pi_1)$ is known to appear as the number of turns N_t on the winding of a solenoid. The transition from (5.74) to (5.75) is the substitution of the definition of the electric current through a surface S using the equation:

$$I = \iint_S \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S}\,,$$

where $\boldsymbol{J}$ is the current density.

5.7.3 Semi-perimeters of parallelograms of *B*

The semi-perimeter limit of the fundamental (7+1)-dimensional ellipsoid yields $SP \leq \frac{3}{2}\sqrt{6}$. The histogram of semi-perimeters of the parallelograms of the lattice point $(1 \mid 0, -1, 0, 1, 0^3)$, representing the kind of quantity magnetic induction, in the fundamental (7+1)-dimensional ellipsoid is given in Figure 5.9.

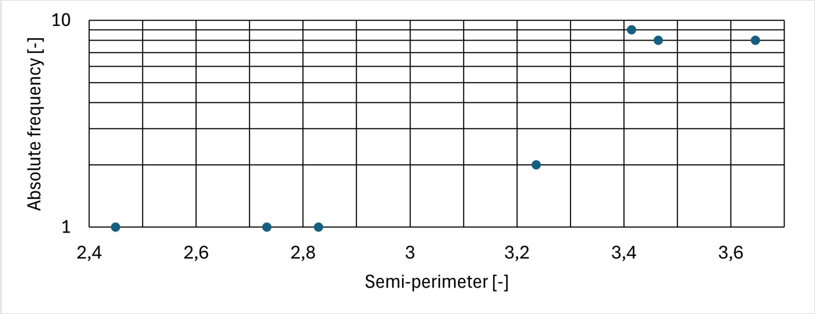

Figure 5.9: Histogram of semi-perimeters of parallelograms of the kind of quantity magnetic induction $(1 \mid -2, 0, 1, -1, 0, 0, 0)$ in the fundamental (7+1)-dimensional ellipsoid.

We list the semi-perimeters SP of the parallelograms for the kind of quantity $\boldsymbol{B}$ having absolute frequencies $f = 1$ or $f = 2$ in the fundamental (7+1)-dimensional ellipsoid in Table 5.13.

Table 5.13: Parallelograms in the fundamental (7+1)-dimensional ellipsoid of the kind of quantity magnetic induction $\boldsymbol{B}$.

ID	$\boldsymbol{x}$	$\boldsymbol{y}$	SP	f	Quantity equation
1	$(0 \mid 0^7)$	$(1 \mid -2, 0, 1, -1, 0^3)$	2.449	1	$B = f(\pi_1)\, B$
2	$(1 \mid -1, 0^6)$	$(0 \mid -1, 0, 1, -1, 0^3)$	2.732	1	$B = f(\pi_2)\, \omega \cdot \frac{m}{q}$
3	$(0 \mid -1, 0, 0, -1, 0^3)$	$(1 \mid -1, 0, 1, 0^4)$	2.828	1	$B = f_1(\pi_3)\, \frac{1}{q} \cdot m \cdot \omega$
4a	$(0 \mid 0, 0, 1, 0^4)$	$(1 \mid -2, 0, 0, -1, 0^3)$	3.236	2	$B = f_1(\pi_4)\, m \cdot \frac{\omega^2}{I}$
4b	$(0 \mid 0, 0, 0, -1, 0^3)$	$(1 \mid -2, 0, 1, 0^4)$	3.236	2	$B = f_2(\pi_4)\, \frac{1}{I} \frac{F}{l}$

The parallelogram with ID = 1 is a degenerated parallelogram.
The parallelogram with ID = 2 is representing the quantity equation:

$$B = f(\pi_2)\,\frac{1}{t}\frac{m}{q} \tag{5.76}$$

$$\int Bdt = f(\pi_2)\frac{m}{q} \tag{5.77}$$

$$\frac{\partial}{\partial x}\int B\,\mathrm{d}t = f(\pi_2)\frac{\partial}{\partial x}\left(\frac{m}{q}\right) \tag{5.78}$$

$$\frac{\partial}{\partial y}\int B\,\mathrm{d}t = f(\pi_2)\frac{\partial}{\partial y}\left(\frac{m}{q}\right) \tag{5.79}$$

$$\frac{\partial}{\partial z}\int B\,\mathrm{d}t = f(\pi_2)\frac{\partial}{\partial z}\left(\frac{m}{q}\right) \tag{5.80}$$

$$\int (\nabla\cdot\boldsymbol{B})\,\mathrm{d}t = f(\pi_2)(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z})\left(\frac{m}{q}\right) \tag{5.81}$$

$$\int (\nabla\cdot\boldsymbol{B})\,\mathrm{d}t = \frac{1}{2}f(\pi_2)\nabla\left(\frac{1}{\gamma}\right) \tag{5.82}$$

$$\nabla\cdot\boldsymbol{B} = 0\,, \tag{5.83}$$

where $f(\pi_2) = 1$. We identify the equation (5.83) with Maxwell's equation (5.70). The right hand side of the equation (5.82) is related to the gyro-magnetic ratio $\gamma = \frac{q}{2m}$. For an isolated electron we have $|\gamma_e| = g_e\frac{|-e|}{2m_e} = g_e\frac{\mu_\mathrm{B}}{\hbar}$ where μ_B is the Bohr magneton. The electron g-factor has been measured to twelve decimal places $g_e = 2.0023193043617(15)$. The gyro-magnetic ratio is constant but varies from one nucleus to another. We infer that *if* the gradient of the reciprocal of the gyromagnetic ratio $\nabla\left(\frac{1}{\gamma}\right) = 0$ then we find (5.83) $\int(\nabla\cdot\boldsymbol{B})\,\mathrm{d}t = 0$.
The parallelogram with ID = 3 contains the quantity $(0 \mid -1, 0, 0, -1, 0^3)$ which is identified as a reciprocal electric charge.
The parallelograms with ID = 4a and ID = 4b are *rectangles*. The parallelogram with ID = 4a is unknown to the author.
The parallelogram with ID = 4b is yielding the equation:

$$B = f_2(\pi_4)\,\frac{1}{I}\frac{F}{l} \tag{5.84}$$

$$l\times B = f_2(\pi_4)\,\frac{1}{I}F \tag{5.85}$$

$$\boldsymbol{F} = I\,l\times\boldsymbol{B}\,, \tag{5.86}$$

where $f_2(\pi_4) = 1$, $\boldsymbol{F}$ is a force acting on a current-carrying wire of length l.

5.7.4 Semi-perimeters of parallelograms of *E*

The semi-perimeter limit of the fundamental (7+1)-dimensional ellipsoid yields $SP \leq \frac{3}{2}\sqrt{12}$. The histogram of semi-perimeters of the parallelograms of the lattice point $(0 \mid -3, 1, 1, -1, 0^3)$, representing the kind of quantity electric field, in the fundamental (7+1)-dimensional ellipsoid is given in Figure 5.10.

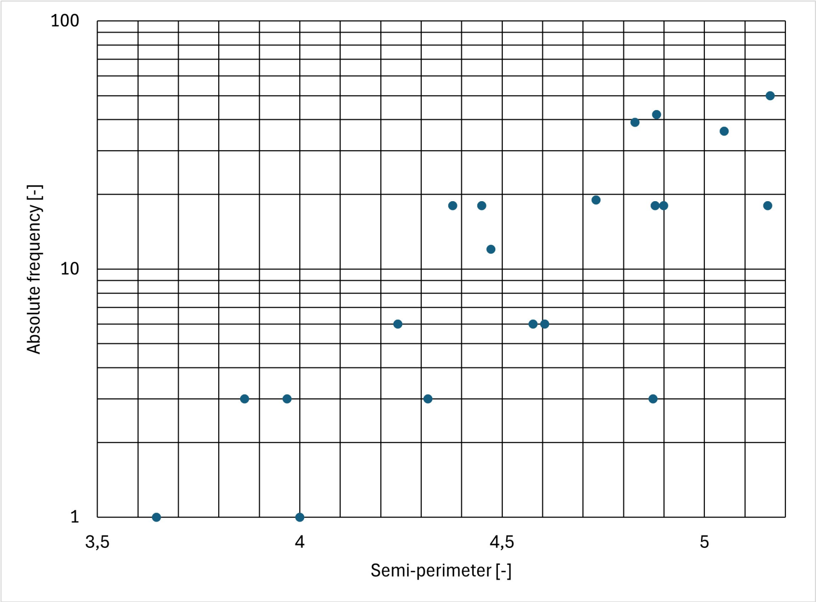

Figure 5.10: Histogram of semi-perimeters of parallelograms of the kind of quantity electric field $(0 \mid -3, 1, 1, -1, 0^3)$ in the fundamental (7+1)-dimensional ellipsoid.

We list the semi-perimeters SP of the parallelograms for the kind of quantity $\boldsymbol{E}$ having absolute frequencies $f = 1$ or $f = 2$ in the fundamental (7+1)-dimensional ellipsoid in Table 5.14.

Table 5.14: Parallelograms in the fundamental (7+1)-dimensional ellipsoid of the kind of quantity electric field $\boldsymbol{E}$.

ID	$\boldsymbol{x}$	$\boldsymbol{y}$	SP	f	Quantity equation
1	$(0 \mid 0^7)$	$(0 \mid -3, 1, 1, -1, 0^3)$	3.464	1	$E = f(\pi_1)\, E$
2	$(0 \mid -1, 0^6)$	$(0 \mid -2, 1, 1, -1, 0^3)$	3.645	1	$E = f(\pi_2) \dfrac{\partial}{\partial t} \cdot A$
3	$(0 \mid -2, 0^6)$	$(0 \mid -1, 1, 1, -1, 0^3)$	4.000	1	$E = f(\pi_3)\, \nu^2 \cdot \dfrac{mr}{q}$

The parallelogram with ID = 1 is a degenerated parallelogram.
The parallelogram with ID = 2 contains $(1 \mid -1, 0^6)$ representing a time derivative and the lattice point $(0 \mid -2, 1, 1, -1, 0^3)$ is the magnetic vector potential $\boldsymbol{A}$. We find the quantity equation:

$$E = f(\pi_2)\frac{\partial}{\partial t} \cdot A \tag{5.87}$$

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t}, \tag{5.88}$$

where $f(\pi_2) = -1$. From the definition $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ we obtain:

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{5.89}$$

which corresponds with Maxwell's equation (5.67).

The parallelogram with ID = 3 contains a frequency squared ν^2 and the quantity $(0 \mid -1, 1, 1, -1, 0^3)$ that is not in the lexicon Appendix A. The equation is unknown to the author.

5.7.5 Semi-perimeters of parallelograms of *D*

The semi-perimeter limit of the fundamental (7+1)-dimensional ellipsoid yields $SP \leq \frac{3}{2}\sqrt{6}$. The histogram of semi-perimeters of the parallelograms of the lattice point $(0 \mid 1, -2, 0, 1, 0^3)$, representing the kind of quantity electric displacement, is given in Figure 5.11.

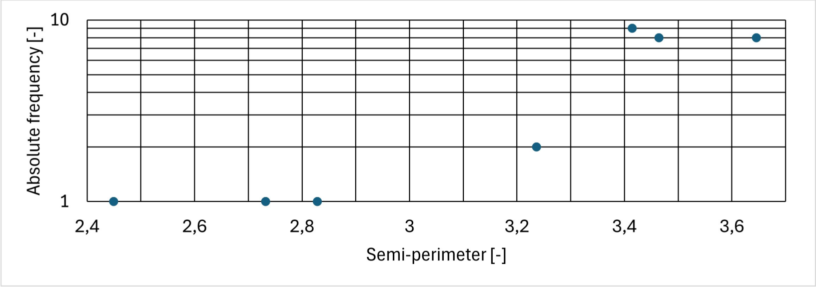

Figure 5.11: Histogram of semi-perimeters of parallelograms of the kind quantity electric displacement $(0 \mid 1, -2, 0, 1, 0^3)$ in the fundamental (7+1)-dimensional ellipsoid.

Observe that the semi-perimeter distribution of the parallelograms of the kinds of quantity $\boldsymbol{B}$ and $\boldsymbol{D}$ are identical. The origin of this is the fact that a signed permutation can be found that maps $\boldsymbol{D}$ in $\boldsymbol{B}$. The kind of quantity $\boldsymbol{B}$ is a pseudo-tensor while the kind of quantity $\boldsymbol{D}$ is a tensor. Hence, the Gödel numbers are equal but of opposite sign. We list the semi-perimeters SP of the parallelograms for the kind of quantity $\boldsymbol{D}$ having absolute frequencies $f = 1$ or $f = 2$ in the fundamental (7+1)-dimensional ellipsoid Table 5.15.

Table 5.15: Parallelograms in the fundamental (7+1)-dimensional ellipsoid of the kind of quantity electric displacement $\boldsymbol{D}$.

ID	$\boldsymbol{x}$	$\boldsymbol{y}$	SP	f	Quantity equation
1	$(0 \mid 0^7)$	$(0 \mid 1, -2, 0, 1, 0^3)$	2.449	1	$D = f(\pi_1)\, D$
2	$(0 \mid 0, -1, 0^5)$	$(0 \mid 1, -1, 0, 1, 0^3)$	2.732	1	$D = f(\pi_2) \dfrac{1}{s} \cdot \dfrac{q}{l}$
3	$(0 \mid 1, -1, 0^5)$	$(0 \mid 0, -1, 0, 1, 0^3)$	2.828	1	$D = f(\pi_3)\, \dfrac{1}{v} \cdot \dfrac{\mathrm{d}I}{\mathrm{d}x}$
4a	$(0 \mid 1, 0^6)$	$(0 \mid 0, -2, 0, 1, 0^3)$	3.236	2	$D = f_1(\pi_4)\, t \cdot \nabla \times H$
4b	$(0 \mid 0, 0, 0, 1, 0^3)$	$(0 \mid 1, -2, 0^5)$	3.236	2	$D = f_2(\pi_4)\, I \cdot \dfrac{t}{S}$

The parallelogram with ID = 1 is a degenerated parallelogram.
The parallelogram with ID = 2 is representing the quantity equation:

$$D = f(\pi_2)\frac{1}{s} \cdot \frac{q}{s} \tag{5.90}$$

$$D \cdot s^2 = q \tag{5.91}$$

$$D_i \cdot \mathrm{d}S_i = f_i(\pi_2)q,\ i = x, y, z \tag{5.92}$$

$$\oiint\limits_{S(V)} (D_x\,\mathrm{d}S_x + D_y\,\mathrm{d}S_y + D_z\,\mathrm{d}S_z) = (f_x(\pi_2) + f_y(\pi_2) + f_z(\pi_2))q \tag{5.93}$$

$$\oiint\limits_{S(V)} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = f(\pi_2) \iiint\limits_{V} \rho_f\,\mathrm{d}V \tag{5.94}$$

$$\oiint\limits_{S(V)} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = \iiint\limits_{V} \rho_f\,\mathrm{d}V\,, \tag{5.95}$$

where we put $f(\pi_2) = 1$.
The equation (5.95) is identified with Maxwell's equation (5.69). The parallelogram with ID = 3 contains the quantity $(0 \mid -1, 0, 0, -1, 0^3)$ which is identified as a reciprocal electric charge. The quantity Q_3 is not in the lexicon Appendix A.
The parallelogram with ID = 4a and ID = 4b are *rectangles*. The parallelogram with ID = 4a contains the quantity $(0 \mid 1, 0^6)$ which is identified as time and $(0 \mid 0, -2, 0, 1, 0^3)$ is found in the lexicon Appendix A as the curl of the magnetic field $\nabla \times H$. We have the equation:

$$D = f_1(\pi_4)t\nabla \times H \tag{5.96}$$

$$\frac{D}{t} = f_1(\pi_4)\nabla \times H \tag{5.97}$$

$$\frac{\partial \boldsymbol{D}}{\partial t} = f(\pi_4)\nabla \times \boldsymbol{H}\,. \tag{5.98}$$

The parallelogram with ID = 4b contains the lattice point $(0 \mid 0, -2, 0, 1, 0^3)$ which is found in the lexicon Appendix A as the electric current density j_f. We have the equation:

$$D = f_2(\pi_4)t\,j_f \tag{5.99}$$

$$\frac{D}{t} = f_2(\pi_4)j_f \tag{5.100}$$

$$\frac{\partial \boldsymbol{D}}{\partial t} = f_2(\pi_4)j_f\,. \tag{5.101}$$

The common part in both equations is $\frac{\partial \boldsymbol{D}}{\partial t}$ and thus we postulate the equation:

$$\frac{\partial \boldsymbol{D}}{\partial t} = f_1(\pi_4)\nabla \times \boldsymbol{H} + f_2(\pi_4) j_f \tag{5.102}$$

$$\frac{\partial \boldsymbol{D}}{\partial t} = \nabla \times \boldsymbol{H} - j_f \,, \tag{5.103}$$

where choosing $f_1(\pi_4) = 1$ and $f_2(\pi_4) = -1$ yields one of Maxwell's laws in differential form:

$$\nabla \times \boldsymbol{H} = j_f + \frac{\partial \boldsymbol{D}}{\partial t} \,.$$

Hence, we identify the equation (5.103) with Maxwell's equation (5.68) .

5.7.6 Encoding and decoding of *H*

The signed Gödel number of the orbit representative $\mathrm{Orb}\left((1 \mid 0, -1, 0, 1, 0^3)\right) = (1 \mid 1, 1, 0^5)$ is $G((1 \mid 1, 1, 0^5)) = -6$. The number of divisors of 6 is $\tau(6) = 4$. The divisor set is $\{1, 2, 3, 6\}$.
Table E.1 shows that for the orbit representative $(1 \mid 1^2, 0^5)$ we have $F_2 = 1$ and thus one ternary canonical equation exist for $\boldsymbol{H}$. The 2-factoring results in one equation that represent a 3-ary equation.

$$-6 = 2 \times (-3)$$

The additive partitioning of the orbit representative $(1 \mid 1, 1, 0^5)$ in a 3-ary equation is:

$$(1 \mid 1, 1, 0^5) = (0 \mid 0^7) + (0 \mid 1, 0^6) + (1 \mid 0, 1, 0^5) \tag{5.104}$$

The Hasse diagram is given in Figure 5.12.

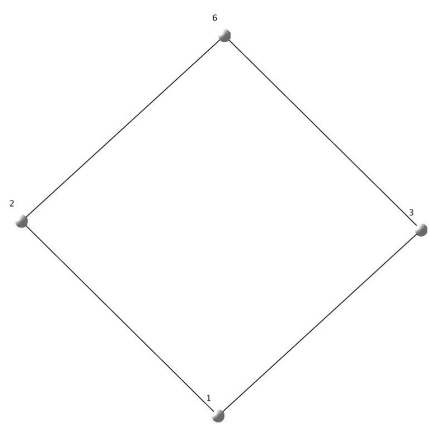

Figure 5.12: Hasse representation of the decoding of the orbit representative of the magnetic field strength given by Gödel number $G(\mathrm{Orb}\,(\boldsymbol{H})) = (-1)^1 2^1 3^1 5^0 7^0 11^0 13^0 17^0 = -6$.

The 8×8 signed permutation matrix $\mathbf{P_H}$ transforms the orbit representative $(1 \mid 1, 1, 0^5)$ in the lattice point $(1 \mid 0, -1, 0, 1, 0^3)$ representing the kind of quantity magnetic field and is given below:

$$\mathbf{P_H} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} . \tag{5.105}$$

We apply the matrix $\mathbf{P_H}$ on the one quantity equation that represent the additive partition of the orbit representative $(1 \mid 1, 1, 0^5)$ and find the magnetic field quantity equation given in Table 5.16.

Table 5.16: Vector equation and quantity equation of the kind of quantity magnetic field in $\{0, 1\} \times \mathbb{Z}^7$.

ID	$\boldsymbol{x}_1$	$\boldsymbol{x}_2$	$\boldsymbol{x}_3$	H
1	$(0 \mid 0^7)$	$(0 \mid 0, -1, 0^5)$	$(0 \mid 0, 0, 0, 1, 0^3)$	$f(\boldsymbol{\pi}_1)\frac{1}{r}I$

The symbols in Table 5.16 are r characteristic length, I electric current, and H magnetic field.

5.7.7 Encoding and decoding of *B*

The signed Gödel number of the orbit representative $\mathrm{Orb}\big((1 \mid -2, 0, 1, -1, 0^3)\big) = (1 \mid 2, 1^2, 0^4)$ is $G((1 \mid 2, 1^2, 0^4)) = -60$. The number of divisors of 60 is $\tau(60) = 12$. The divisor set is $\{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$.
Table E.1 shows that for the orbit representative $(1 \mid 2, 1^2, 0^4)$ we have $F_2 = 5$ and $F_3 = 3$.
Hence five ternary canonical equations and three quaternary equations exist for ***B***. The 3-factoring results in three equations that represent 4-ary equations.

$$60 = 2 \times 3 \times 10 = 2 \times 5 \times 6 = 3 \times 4 \times 5$$

The 2-factoring results in five equations that represent 3-ary equations.

$$60 = 2 \times 30 = 3 \times 20 = 4 \times 15 = 5 \times 12 = 6 \times 10$$

The additive partitioning of the orbit representative $(1 \mid 2, 1^2, 0^4)$ in 3-ary equations are:

$$(1 \mid 2, 1^2, 0^4) = (0 \mid 0^7) + (0 \mid 1, 0^6) + (1 \mid 1, 1, 1, 0^4) \tag{5.106}$$

$$(1 \mid 2, 1^2, 0^4) = (0 \mid 0^7) + (0 \mid 0, 1, 0^5) + (1 \mid 2, 0, 1, 0^4) \tag{5.107}$$

$$(1 \mid 2, 1^2, 0^4) = (0 \mid 0^7) + (0 \mid 2, 0^6) + (1 \mid 0, 1, 1, 0^4) \tag{5.108}$$

$$(1 \mid 2, 1^2, 0^4) = (0 \mid 0^7) + (0 \mid 0, 0, 1, 0^4) + (1 \mid 2, 1, 0^4) \tag{5.109}$$

$$(1 \mid 2, 1^2, 0^4) = (0 \mid 0^7) + (0 \mid 1, 1, 0^5) + (1 \mid 1, 0, 1, 0^4) \tag{5.110}$$

The Hasse diagram containing the three sets is given in Figure 5.13.

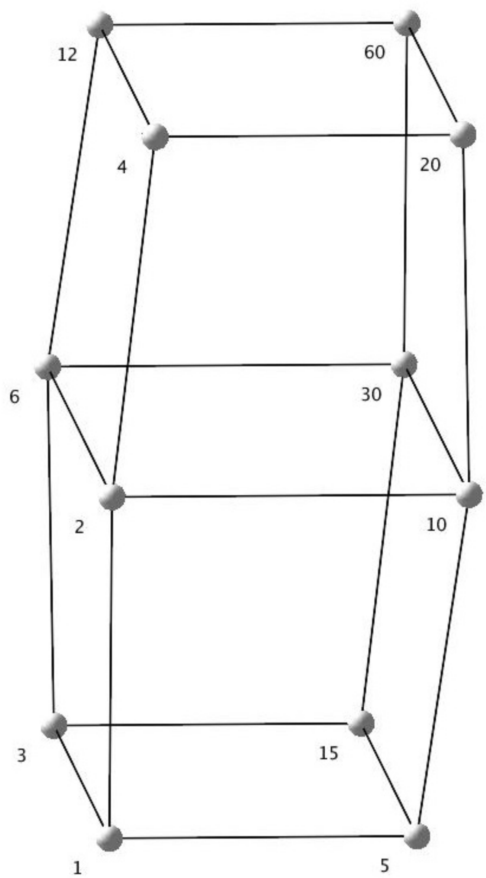

Figure 5.13: Hasse representation of the decoding of the orbit representative of the magnetic induction given by Gödel number $G(\mathrm{Orb}\,(\boldsymbol{B})) = (-1)^1 2^2 3^1 5^1 7^0 11^0 13^0 17^0 = -60\,.$

The 8×8 signed permutation matrix $\mathbf{P_B}$ transforms the orbit representative $(1 \mid 2, 1^2, 0^4)$ in the lattice point $(1 \mid -2, 0, 1, -1, 0^3)$ representing the kind of quantity magnetic induction and is given below:

$$\mathbf{P_B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} . \tag{5.111}$$

We apply the matrix $\mathbf{P_B}$ on the ternary quantity equations that represent the additive partition of the orbit representative $(1 \mid 2, 1^2, 0^4)$ and find the magnetic induction quantity equations given in Table 5.17.

Table 5.17: Vector equations and quantity equations of the kind of quantity magnetic induction in $\{0,1\} \times \mathbb{Z}^7$.

ID	$\boldsymbol{x}_1$	$\boldsymbol{x}_2$	$\boldsymbol{x}_3$	B
1	$(0 \mid 0^7)$	$(0 \mid -1, 0^6)$	$(1 \mid -1, 0, 1, -1, 0^3)$	$f(\boldsymbol{\pi}_1)\frac{1}{t}\frac{m}{q}$
2	$(0 \mid 0^7)$	$(0 \mid 0, 0, 1, 0^4)$	$(1 \mid -2, 0, 0, -1, 0^3)$	$f(\boldsymbol{\pi}_2)m\frac{1}{qt}$
3	$(0 \mid 0^7)$	$(0 \mid -2, 0^6)$	$(1 \mid 0, 0, 1, -1, 0^3)$	$f(\boldsymbol{\pi}_3)\omega^2\frac{m}{I}$
4	$(0 \mid 0^7)$	$(0 \mid 0, 0, 0, -1, 0^3)$	$(1 \mid -2, 0, 1, 0^4)$	$f(\boldsymbol{\pi}_4)\frac{1}{I}m\omega^2$
5	$(0 \mid 0^7)$	$(1 \mid -1, 0, 1, 0^4)$	$(0 \mid -1, 0, 0, -1, 0^3)$	$f(\boldsymbol{\pi}_5)m\omega\frac{1}{q}$

The symbols in Table 5.17 are t characteristic time, ω angular frequency, m mass, q electric charge, I electric current, and B magnetic induction.

5.7.8 Encoding and decoding of $\boldsymbol{E}$

The signed Gödel number of the orbit representative $\mathrm{Orb}\left((0 \mid -3, 1, 1, -1, 0^3)\right) = (0 \mid 3, 1^3, 0^3)$ is $G((0 \mid 3, 1^3, 0^3)) = 840$. The number of divisors of 840 is $\tau(840) = 32$.
The divisor set is $\{1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 14, 15, 20, 21, 24, 28, 30, 35, 40, 42, 56, 60, 70, 84, 105, 120, 140, 168, 210, 280, 420, 840\}$.

We find from the OEIS A045778 sequence (Wilson, 2009) the distinct factorizations for the integer 840. Table E.1 shows that for the orbit representative $(0 \mid 3, 1^3, 0^3)$ we have $F_2 = 15$, $F_3 = 29$, $F_4 = 13$ and $F_5 = 1$ and thus we have 15 ternary, 29 quaternary, 13 quinternary and 1 senary equations for $\boldsymbol{E}$. The 5-factoring results in one equation that represent a 6-ary equation.

$$840 = 2 \times 3 \times 4 \times 5 \times 7$$

The 4-factoring results in 13 equations that represent 5-ary equations.

$$\begin{aligned}840 &= 2 \times 3 \times 4 \times 35 = 2 \times 3 \times 5 \times 28 = 2 \times 3 \times 7 \times 20 = 2 \times 3 \times 10 \times 14 \\ &= 2 \times 4 \times 5 \times 21 = 2 \times 4 \times 7 \times 15 = 2 \times 5 \times 6 \times 14 = 2 \times 5 \times 7 \times 12 \\ &= 2 \times 6 \times 7 \times 10 = 3 \times 4 \times 5 \times 14 = 3 \times 4 \times 7 \times 10 = 3 \times 5 \times 7 \times 8 \\ &= 4 \times 5 \times 6 \times 7\end{aligned}$$

The additive partitioning of the orbit representative $(0 \mid 3, 1^3, 0^3)$ in 5-ary

equations are:

$$(0 \mid 3, 1^3, 0^3) = (0 \mid 0^7) + (0 \mid 1, 0^6) + (0 \mid 0, 1, 0^5) + (0 \mid 2, 0, 0, 0, 0^3) + (0 \mid 0, 0, 1, 1, 0^3) \quad (5.112)$$

$$(0 \mid 3, 1^3, 0^3) = (0 \mid 0^7) + (0 \mid 1, 0^6) + (0 \mid 0, 1, 0^5) + (0 \mid 0, 0, 1, 0^4) + (0 \mid 2, 0, 0, 1, 0^3) \quad (5.113)$$

$$(0 \mid 3, 1^3, 0^3) = (0 \mid 0^7) + (0 \mid 1, 0^6) + (0 \mid 0, 1, 0^5) + (0 \mid 0, 0, 0, 1, 0^3) + (0 \mid 2, 0, 1, 0^4) \quad (5.114)$$

$$(0 \mid 3, 1^3, 0^3) = (0 \mid 0^7) + (0 \mid 1, 0^6) + (0 \mid 0, 1, 0^5) + (0 \mid 1, 0, 1, 0, 0^3) + (0 \mid 1, 0, 0, 1, 0^3) \quad (5.115)$$

$$(0 \mid 3, 1^3, 0^3) = (0 \mid 0^7) + (0 \mid 1, 0^6) + (0 \mid 2, 0^6) + (0 \mid 0, 0, 1, 0^4) + (0 \mid 0, 1, 0, 1, 0^3) \quad (5.116)$$

$$(0 \mid 3, 1^3, 0^3) = (0 \mid 0^7) + (0 \mid 1, 0^6) + (0 \mid 2, 0^6) + (0 \mid 0, 0, 0, 1, 0^3) + (0 \mid 0, 1, 1, 0^4) \quad (5.117)$$

$$(0 \mid 3, 1^3, 0^3) = (0 \mid 0^7) + (0 \mid 1, 0^6) + (0 \mid 0, 0, 1, 0^4) + (0 \mid 1, 1, 0^5) + (0 \mid 1, 0, 0, 1, 0^3) \quad (5.118)$$

$$(0 \mid 3, 1^3, 0^3) = (0 \mid 0^7) + (0 \mid 1, 0^6) + (0 \mid 0, 0, 1, 0^4) + (0 \mid 0, 0, 0, 1, 0^3) + (0 \mid 2, 1, 0^5) \quad (5.119)$$

$$(0 \mid 3, 1^3, 0^3) = (0 \mid 0^7) + (0 \mid 1, 0^6) + (0 \mid 1, 1, 0^5) + (0 \mid 0, 0, 0, 1, 0^3) + (0 \mid 1, 0, 1, 0^4) \quad (5.120)$$

$$(0 \mid 3, 1^3, 0^3) = (0 \mid 0^7) + (0 \mid 0, 1, 0^5) + (0 \mid 2, 0^6) + (0 \mid 0, 0, 1, 0^4) + (0 \mid 1, 0, 0, 1, 0^3) \quad (5.121)$$

$$(0 \mid 3, 1^3, 0^3) = (0 \mid 0^7) + (0 \mid 0, 1, 0^5) + (0 \mid 2, 0^5) + (0 \mid 0, 0, 0, 1, 0^3) + (0 \mid 1, 0, 1, 0^4) \quad (5.122)$$

$$(0 \mid 3, 1^3, 0^3) = (0 \mid 0^7) + (0 \mid 0, 1, 0^5) + (0 \mid 0, 0, 1, 0^4) + (0 \mid 0, 0, 0, 1, 0^3) + (0 \mid 3, 0^6) \quad (5.123)$$

$$(0 \mid 3, 1^3, 0^3) = (0 \mid 0^7) + (0 \mid 2, 0^6) + (0 \mid 0, 0, 1, 0^4) + (0 \mid 1, 1, 0^5) + (0 \mid 0, 0, 0, 1, 0^3) \quad (5.124)$$

The 3-factoring results in 29 equations that represent 4-ary equations.

$$\begin{aligned}
840 &= 2 \times 3 \times 140 = 2 \times 4 \times 105 = 2 \times 5 \times 84 \\
&= 2 \times 6 \times 70 = 2 \times 7 \times 60 = 2 \times 10 \times 42 \\
&= 2 \times 12 \times 35 = 2 \times 14 \times 30 = 2 \times 15 \times 28 \\
&= 2 \times 20 \times 21 = 3 \times 4 \times 70 = 3 \times 5 \times 56 \\
&= 3 \times 7 \times 40 = 3 \times 8 \times 35 = 3 \times 10 \times 28 \\
&= 3 \times 14 \times 20 = 4 \times 5 \times 42 = 4 \times 6 \times 35 \\
&= 4 \times 7 \times 30 = 4 \times 10 \times 21 = 4 \times 14 \times 15 \\
&= 5 \times 6 \times 28 = 5 \times 7 \times 24 = 5 \times 8 \times 21 \\
&= 5 \times 12 \times 14 = 6 \times 7 \times 20 = 6 \times 10 \times 14 \\
&= 7 \times 8 \times 15 = 7 \times 10 \times 12
\end{aligned}$$

The additive partitioning of the orbit representative $(0 \mid 3, 1^3, 0^3)$ in 4-ary

equations are:

$$(0 \mid 3,1^3,0^3) = (0 \mid 0^7) + (0 \mid 1,0^6) + (0 \mid 0,1,0^5) + (0 \mid 2,0,1,1,0^3) \tag{5.125}$$
$$(0 \mid 3,1^3,0^3) = (0 \mid 0^7) + (0 \mid 1,0^6) + (0 \mid 2,0^6) + (0 \mid 0,1,1,1,0^3) \tag{5.126}$$
$$(0 \mid 3,1^3,0^3) = (0 \mid 0^7) + (0 \mid 1,0^6) + (0 \mid 0,0,1,0^4) + (0 \mid 2,1,0,1,0^3) \tag{5.127}$$
$$(0 \mid 3,1^3,0^3) = (0 \mid 0^7) + (0 \mid 1,0^6) + (0 \mid 1,1,0^5) + (0 \mid 1,0,1,1,0^3) \tag{5.128}$$
$$(0 \mid 3,1^3,0^3) = (0 \mid 0^7) + (0 \mid 1,0^6) + (0 \mid 0,0,0,1,0^3) + (0 \mid 2,1,1,0^4) \tag{5.129}$$
$$(0 \mid 3,1^3,0^3) = (0 \mid 0^7) + (0 \mid 1,0^6) + (0 \mid 1,0,1,0^4) + (0 \mid 2,1,0^5) \tag{5.130}$$
$$(0 \mid 3,1^3,0^3) = (0 \mid 0^7) + (0 \mid 1,0^6) + (0 \mid 2,1,0^5) + (0 \mid 0,0,1,1,0^3) \tag{5.131}$$
$$(0 \mid 3,1^3,0^3) = (0 \mid 0^7) + (0 \mid 1,0^6) + (0 \mid 1,0,0,1,0^3) + (0 \mid 1,1,1,0^4) \tag{5.132}$$
$$(0 \mid 3,1^3,0^3) = (0 \mid 0^7) + (0 \mid 1,0^6) + (0 \mid 0,1,1,0^4) + (0 \mid 2,0,0,1,0^3) \tag{5.133}$$
$$(0 \mid 3,1^3,0^3) = (0 \mid 0^7) + (0 \mid 1,0^6) + (0 \mid 2,0,1,0^4) + (0 \mid 0,1,0,1,0^3) \tag{5.134}$$
$$(0 \mid 3,1^3,0^3) = (0 \mid 0^7) + (0 \mid 0,1,0^5) + (0 \mid 2,0^5) + (0 \mid 1,0,1,1,0^3) \tag{5.135}$$
$$(0 \mid 3,1^3,0^3) = (0 \mid 0^7) + (0 \mid 0,1,0^5) + (0 \mid 0,0,1,0^4) + (0 \mid 3,0,0,1,0^3) \tag{5.136}$$
$$(0 \mid 3,1^3,0^3) = (0 \mid 0^7) + (0 \mid 0,1,0^5) + (0 \mid 0,0,0,1,0^3) + (0 \mid 3,0,1,0,0^3) \tag{5.137}$$
$$(0 \mid 3,1^3,0^3) = (0 \mid 0^7) + (0 \mid 0,1,0^5) + (0 \mid 3,0^6) + (0 \mid 0,0,1,1,0^3) \tag{5.138}$$
$$(0 \mid 3,1^3,0^3) = (0 \mid 0^7) + (0 \mid 0,1,0^5) + (0 \mid 1,0,1,0^4) + (0 \mid 2,0,0,1,0^3) \tag{5.139}$$
$$(0 \mid 3,1^3,0^3) = (0 \mid 0^7) + (0 \mid 0,1,0^5) + (0 \mid 1,0,0,1,0^3) + (0 \mid 2,0,1,0,0^3) \tag{5.140}$$
$$(0 \mid 3,1^3,0^3) = (0 \mid 0^7) + (0 \mid 2,0^6) + (0 \mid 0,0,1,0^4) + (0 \mid 1,1,0,1,0^3) \tag{5.141}$$
$$(0 \mid 3,1^3,0^3) = (0 \mid 0^7) + (0 \mid 2,0^6) + (0 \mid 1,1,0^5) + (0 \mid 0,0,1,1,0^3) \tag{5.142}$$
$$(0 \mid 3,1^3,0^3) = (0 \mid 0^7) + (0 \mid 2,0^6) + (0 \mid 0,0,0,1,0^3) + (0 \mid 1,1,1,0^4) \tag{5.143}$$
$$(0 \mid 3,1^3,0^3) = (0 \mid 0^7) + (0 \mid 2,0^6) + (0 \mid 1,0,1,0^4) + (0 \mid 0,1,0,1,0^3) \tag{5.144}$$
$$(0 \mid 3,1^3,0^3) = (0 \mid 0^7) + (0 \mid 2,0^6) + (0 \mid 1,0,0,1,0^3) + (0 \mid 0,1,1,0^4) \tag{5.145}$$
$$(0 \mid 3,1^3,0^3) = (0 \mid 0^7) + (0 \mid 0,0,1,0^4) + (0 \mid 1,1,0^5) + (0 \mid 2,0,0,1,0^3) \tag{5.146}$$
$$(0 \mid 3,1^3,0^3) = (0 \mid 0^7) + (0 \mid 0,0,1,0^4) + (0 \mid 0,0,0,1,0^3) + (0 \mid 3,1,0^5) \tag{5.147}$$
$$(0 \mid 3,1^3,0^3) = (0 \mid 0^7) + (0 \mid 0,0,1,0^4) + (0 \mid 3,0^6) + (0 \mid 0,1,0,1,0^3) \tag{5.148}$$
$$(0 \mid 3,1^3,0^3) = (0 \mid 0^7) + (0 \mid 0,0,1,0^4) + (0 \mid 2,1,0^5) + (0 \mid 1,0,0,1,0^3) \tag{5.149}$$
$$(0 \mid 3,1^3,0^3) = (0 \mid 0^7) + (0 \mid 1,1,0^5) + (0 \mid 0,0,0,1,0^3) + (0 \mid 2,0,1,0^4) \tag{5.150}$$
$$(0 \mid 3,1^3,0^3) = (0 \mid 0^7) + (0 \mid 1,1,0^5) + (0 \mid 1,0,1,0^4) + (0 \mid 1,0,0,1,0^4) \tag{5.151}$$
$$(0 \mid 3,1^3,0^3) = (0 \mid 0^7) + (0 \mid 0,0,0,1,0^3) + (0 \mid 3,0^6) + (0 \mid 0,1,1,0^4) \tag{5.152}$$
$$(0 \mid 3,1^3,0^3) = (0 \mid 0^7) + (0 \mid 0,0,0,1,0^3) + (0 \mid 1,0,1,0^4) + (0 \mid 2,1,0^5) \tag{5.153}$$

The 2-factoring results in 15 equations that represent 3-ary equations.

$$\begin{aligned} 840 &= 2\times 420 = 3\times 280 = 4\times 210 = 5\times 168 \\ &= 6\times 140 = 7\times 120 = 8\times 105 = 10\times 84 \\ &= 12\times 70 = 14\times 60 = 15\times 56 = 20\times 42 \\ &= 21\times 40 = 24\times 35 = 28\times 30 \end{aligned}$$

The additive partitioning of the orbit representative $(0 \mid 3,1^3,0^3)$ in 3-ary

equations are:

$$(0 \mid 3, 1^3, 0^3) = (0 \mid 0^7) + (0 \mid 1, 0^6) + (0 \mid 2, 1, 1, 1, 0^3) \tag{5.154}$$
$$(0 \mid 3, 1^3, 0^3) = (0 \mid 0^7) + (0 \mid 0, 1, 0^5) + (0 \mid 3, 0, 1, 1, 0^3) \tag{5.155}$$
$$(0 \mid 3, 1^3, 0^3) = (0 \mid 0^7) + (0 \mid 2, 0^6) + (0 \mid 1, 1, 1, 1, 0^3) \tag{5.156}$$
$$(0 \mid 3, 1^3, 0^3) = (0 \mid 0^7) + (0 \mid 0, 0, 1, 0^4) + (0 \mid 3, 1, 0, 1, 0^3) \tag{5.157}$$
$$(0 \mid 3, 1^3, 0^3) = (0 \mid 0^7) + (0 \mid 1, 1, 0^5) + (0 \mid 2, 0, 1, 1, 0^3) \tag{5.158}$$
$$(0 \mid 3, 1^3, 0^3) = (0 \mid 0^7) + (0 \mid 0, 0, 0, 1, 0^3) + (0 \mid 3, 1, 1, 0^4) \tag{5.159}$$
$$(0 \mid 3, 1^3, 0^3) = (0 \mid 0^7) + (0 \mid 3, 0^6) + (0 \mid 0, 1, 1, 1, 0^3) \tag{5.160}$$
$$(0 \mid 3, 1^3, 0^3) = (0 \mid 0^7) + (0 \mid 1, 0, 1, 0^4) + (0 \mid 2, 1, 0, 1, 0^3) \tag{5.161}$$
$$(0 \mid 3, 1^3, 0^3) = (0 \mid 0^7) + (0 \mid 2, 1, 0^5) + (0 \mid 1, 0, 1, 1, 0^3) \tag{5.162}$$
$$(0 \mid 3, 1^3, 0^3) = (0 \mid 0^7) + (0 \mid 1, 0, 0, 1, 0^3) + (0 \mid 2, 1, 1, 0^4) \tag{5.163}$$
$$(0 \mid 3, 1^3, 0^3) = (0 \mid 0^7) + (0 \mid 0, 1, 1, 0^4) + (0 \mid 3, 0, 0, 1, 0^3) \tag{5.164}$$
$$(0 \mid 3, 1^3, 0^3) = (0 \mid 0^7) + (0 \mid 2, 0, 1, 0^4) + (0 \mid 1, 1, 0, 1, 0^3) \tag{5.165}$$
$$(0 \mid 3, 1^3, 0^3) = (0 \mid 0^7) + (0 \mid 0, 1, 0, 1, 0^3) + (0 \mid 3, 0, 1, 0^4) \tag{5.166}$$
$$(0 \mid 3, 1^3, 0^3) = (0 \mid 0^7) + (0 \mid 3, 1, 0^5) + (0 \mid 0, 0, 1, 1, 0^3) \tag{5.167}$$
$$(0 \mid 3, 1^3, 0^3) = (0 \mid 0^7) + (0 \mid 2, 0, 0, 1, 0^3) + (0 \mid 1, 1, 1, 0^4) \tag{5.168}$$

The Hasse diagram is given in Figure 5.14.

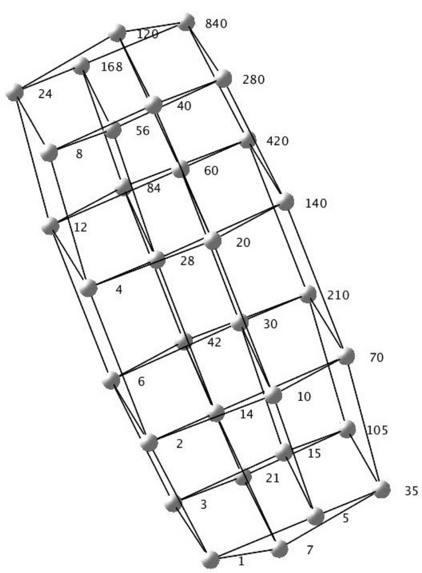

Figure 5.14: Hasse representation of the decoding of the orbit representative of the electric field given by Gödel number $G(\mathrm{Orb}\,(\boldsymbol{E})) = (-1)^0 2^3 3^1 5^1 7^1 11^0 13^0 17^0 = 840$.

The 8×8 signed permutation matrix $\mathbf{P_E}$ transforms the orbit representative $(0 \mid 3, 1^3, 0^3)$ in the lattice point $(0 \mid -3, 1, 1, -1, 0^3)$ representing the quantity electric field strength and is given below:

$$\mathbf{P_E} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} . \tag{5.169}$$

We apply the matrix $\mathbf{P_E}$ on the ternary quantity equations that represent the additive partition of the orbit representative $(0 \mid 3, 1^3, 0^3)$ and find the electric field strength quantity equations given in Table 5.18.

Table 5.18: Vector equations and quantity equations of the kind of quantity electric field strength in $\{0,1\} \times \mathbb{Z}^7$.

ID	$\boldsymbol{x}_1$	$\boldsymbol{x}_2$	$\boldsymbol{x}_3$	E
1	$(0 \mid 0^7)$	$(0 \mid -1, 0^6)$	$(0 \mid -2, 1, 1, -1, 0^3)$	$f(\boldsymbol{\pi}_1)\frac{1}{t}\frac{mr}{qt}$
2	$(0 \mid 0^7)$	$(0 \mid 0, 1, 0^5)$	$(0 \mid -3, 0, 1, -1, 0^3)$	$f(\boldsymbol{\pi}_2)r\frac{m}{qt^2}$
3	$(0 \mid 0^7)$	$(0 \mid 2, 0^6)$	$(0 \mid -1, 1, 1, -1, 0^3)$	$f(\boldsymbol{\pi}_3)t^2\frac{mr}{q}$
4	$(0 \mid 0^7)$	$(0 \mid 0, 0, 1, 0^4)$	$(0 \mid -3, 1, 0, -1, 0^3)$	$f(\boldsymbol{\pi}_4)m\frac{r}{qt^2}$
5	$(0 \mid 0^7)$	$(0 \mid -1, 1, 0^5)$	$(0 \mid -2, 0, 1, -1, 0^3)$	$f(\boldsymbol{\pi}_5)v\frac{m}{qt}$
6	$(0 \mid 0^7)$	$(0 \mid 0, 0, 0, -1, 0^3)$	$(0 \mid -3, 1, 1, 0^4)$	$f(\boldsymbol{\pi}_6)\frac{1}{I}\frac{mr}{t^3}$
7	$(0 \mid 0^7)$	$(0 \mid -3, 0^6)$	$(0 \mid 0, 1, 1, -1, 0^3)$	$f(\boldsymbol{\pi}_7)\frac{1}{t^3}\frac{mr}{I}$
8	$(0 \mid 0^7)$	$(0 \mid -1, 0, 1, 0^4)$	$(0 \mid -2, 1, 0, -1, 0^3)$	$f(\boldsymbol{\pi}_8)\frac{m}{t}\frac{r}{qt}$
9	$(0 \mid 0^7)$	$(0 \mid -2, 1, 0^5)$	$(0 \mid -1, 0, 1, -1, 0^3)$	$f(\boldsymbol{\pi}_9)a\frac{m}{q}$
...	...	...	...	...

ID	$\boldsymbol{x}_1$	$\boldsymbol{x}_2$	$\boldsymbol{x}_3$	E
10	$(0 \mid 0^7)$	$(0 \mid -1, 0, 0, -1, 0^3)$	$(0 \mid -2, 1, 1, 0^4)$	$f(\boldsymbol{\pi}_1 0)\frac{1}{q}F$
11	$(0 \mid 0^7)$	$(0 \mid 0, 1, 1, 0^4)$	$(0 \mid -3, 0, 0, -1, 0^3)$	$f(\boldsymbol{\pi}_1 1)mr\frac{1}{qt^2}$
12	$(0 \mid 0^7)$	$(0 \mid -2, 0, 1, 0^4)$	$(0 \mid -1, 1, 0, -1, 0^3)$	$f(\boldsymbol{\pi}_1 2)\frac{m}{t^2}\frac{r}{q}$
13	$(0 \mid 0^7)$	$(0 \mid 0, 1, 0, -1, 0^3)$	$(0 \mid -3, 0, 1, 0^4)$	$f(\boldsymbol{\pi}_1 3)\frac{r}{I}\frac{m}{t^3}$
14	$(0 \mid 0^7)$	$(0 \mid -3, 1, 0^5)$	$(0 \mid 0, 0, 1, -1, 0^3)$	$f(\boldsymbol{\pi}_1 4)\frac{r}{t^3}\frac{m}{I}$
15	$(0 \mid 0^7)$	$(0 \mid -2, 0, 0, -1, 0^3)$	$(0 \mid -1, 1, 1, 0^4)$	$f(\boldsymbol{\pi}_1 5)\frac{1}{qt}\frac{mr}{t}$

The symbols in Table 5.18 are a acceleration, F force, t characteristic time, r characteristic length, v velocity, m mass, q electric charge, I electric current, and E electric field.

5.7.9 Encoding and decoding of *D*

The signed Gödel number of the orbit representative $\mathrm{Orb}\left((0 \mid 1, -2, 0, 1, 0^3)\right) = (0 \mid 2, 1^2, 0^4)$ is $G((0 \mid 2, 1^2, 0^4)) = 60$. The number of divisors of 60 is $\tau(60) = 12$. The divisor set is $\{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$.
Table E.1 shows that for the orbit representative $(1 \mid 2, 1^2, 0^4)$ we have $F_2 = 5$ and $F_3 = 3$. Hence five ternary canonical equations and three quaternary equations exist for ***D***. The 3-factoring results in three equations that represent 4-ary equations.

$$60 = 2 \times 3 \times 10 = 2 \times 5 \times 6 = 3 \times 4 \times 5$$

The 2-factoring results in five equations that represent 3-ary equations.

$$60 = 2 \times 30 = 3 \times 20 = 4 \times 15 = 5 \times 12 = 6 \times 10$$

The additive partitioning of the orbit representative $(1 \mid 2, 1^2, 0^4)$ in 3-ary equations are:

$$(0 \mid 2, 1^2, 0^4) = (0 \mid 0^7) + (0 \mid 1, 0^6) + (0 \mid 1, 1, 1, 0^4) \tag{5.170}$$

$$(0 \mid 2, 1^2, 0^4) = (0 \mid 0^7) + (0 \mid 0, 1, 0^5) + (0 \mid 2, 0, 1, 0^4) \tag{5.171}$$

$$(0 \mid 2, 1^2, 0^4) = (0 \mid 0^7) + (0 \mid 2, 0^6) + (0 \mid 0, 1, 1, 0^4) \tag{5.172}$$

$$(0 \mid 2, 1^2, 0^4) = (0 \mid 0^7) + (0 \mid 0, 0, 1, 0^4) + (0 \mid 2, 1, 0^4) \tag{5.173}$$

$$(0 \mid 2, 1^2, 0^4) = (0 \mid 0^7) + (0 \mid 1, 1, 0^5) + (0 \mid 1, 0, 1, 0^4) \tag{5.174}$$

The Hasse diagram is given in Figure 5.15.

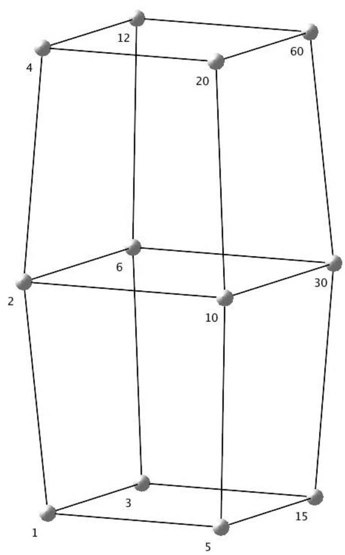

Figure 5.15: Hasse representation of the decoding of the orbit representative of the electric displacement given by Gödel number $G(\mathrm{Orb}\,(\boldsymbol{D})) = (-1)^0 2^2 3^1 5^1 7^0 11^0 13^0 17^0 = 60\,.$

The 8×8 signed permutation matrix $\mathbf{P_{energy}}$ transforms the orbit representative $(0 \mid 2, 1^2, 0^4)$ in the lattice point $(0 \mid 1, -2, 0, 1, 0^3)$ representing the quantity energy and is given below:

$$\mathbf{P_D} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5.175}$$

We apply the matrix $\mathbf{P_D}$ on the ternary quantity equations that represent the additive partition of the orbit representative $(0 \mid 3, 1^3, 0^3)$ and find the electrical displacement quantity equations given in Table 5.19.

Table 5.19: Vector equations and quantity equations of the kind of quantity electrical displacement in $\{0,1\} \times \mathbb{Z}^7$.

ID	$\boldsymbol{x}_1$	$\boldsymbol{x}_2$	$\boldsymbol{x}_3$	D
1	$(0 \mid 0^7)$	$(0 \mid 0, -1, 0^5)$	$(0 \mid 1, -1, 0, 1, 0^3)$	$f(\boldsymbol{\pi}_1)\frac{1}{r}\frac{q}{r}$
2	$(0 \mid 0^7)$	$(0 \mid 1, 0^6)$	$(0 \mid 0, -2, 0, 1, 0^3)$	$f(\boldsymbol{\pi}_2)t\frac{I}{r^2}$
3	$(0 \mid 0^7)$	$(0 \mid 0, -2, 0^5)$	$(0 \mid 0, 1, -1, 0^4)$	$f(\boldsymbol{\pi}_3)\frac{1}{r^2}\frac{r}{m}$
4	$(0 \mid 0^7)$	$(0 \mid 0, 0, 0, 1, 0^3)$	$(0 \mid 1, -2, 0^4)$	$f(\boldsymbol{\pi}_4)I\frac{t}{r^2}$
5	$(0 \mid 0^7)$	$(0 \mid 1, -1, 0^5)$	$(0 \mid 0, -1, 0, 1, 0^3)$	$f(\boldsymbol{\pi}_5)\frac{1}{v}\frac{I}{r}$

The symbols in Table 5.19 are t characteristic time, r characteristic length, v velocity, m mass, q electric charge, I electric current, and D electrical displacement.

5.7.10 Discussion of the exploration of electromagnetism

The representative of the kind of quantity electric field has the largest Gödel number $G((0 \mid 3, 1, 1, 1, 0^4)) = 840$. The divisors of 840 are 1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 14, 15, 20, 21, 24, 28, 30, 35, 40, 42, 56, 60, 70, 84, 105, 120, 140, 168, 210, 280, 420, 840. The divisors -6, -60, 60, and 840 are elements of this set when considering their absolute values.

Hence we expect the following factoring as the ternary connections between $\boldsymbol{E}$ and respectively $\boldsymbol{D}$, $\boldsymbol{B}$, and $\boldsymbol{H}$:

$$840 = 14 \times 60 \tag{5.176}$$

$$840 = (-14) \times (-60) \tag{5.177}$$

$$840 = (-140) \times (-6)\,. \tag{5.178}$$

We retrieve all the Maxwell equations. Table 5.20 shows the correspondence between the Maxwell's equations and the quantity equations generated by the semi-perimeter method.

Table 5.20: Correspondence of Maxwell's equations with quantity equations.

Maxwell equation reference	Quantity equation reference
(5.67)	(5.89)
(5.68)	(5.96)+(5.99)
(5.69)	(5.95)
(5.70)	(5.76)

5.8 Non-linear dimensional reduction

In this section we consider techniques to visualize the N-dimensional lattice information to lower dimensional lattices and preferably to 3D spaces.

We propose a non-linear dimensional reduction method that we denote *transposition method*. This transposition method, that seems trivial, is new to the best of our knowledge. It is a powerful method showing the connection between the perimeter of a parallelogram and the area of a parallelogram.

5.8.1 Transposition method

We call this technique the transposition method because it transforms a $N \times M$ matrix in a $M \times N$ matrix by applying the transpose operation on a matrix $\mathbf{A}$ resulting in $\mathbf{A}^{tr}$. Let us consider the parallelograms $oxzyo$ of $\{0,1\} \times \mathbb{Z}^7$ used in our study of the SI 'laws of physics'. Let the coordinates of the points $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ be arranged in columns of a matrix $\mathbf{A}$. We find:

$$\mathbf{A} = \begin{bmatrix} X^0 & Y^0 & Z^0 \\ X^1 & Y^1 & Z^1 \\ X^2 & Y^2 & Z^2 \\ X^3 & Y^3 & Z^3 \\ X^4 & Y^4 & Z^4 \\ X^5 & Y^5 & Z^5 \\ X^6 & Y^6 & Z^6 \\ X^7 & Y^7 & Z^7 \end{bmatrix}, \tag{5.179}$$

and its transposed matrix

$$\mathbf{A^{tr}} = \begin{bmatrix} X^0 & X^1 & X^2 & X^3 & X^4 & X^5 & X^6 & X^7 \\ Y^0 & Y^1 & Y^2 & Y^3 & Y^4 & Y^5 & Y^6 & Y^7 \\ Z^0 & Z^1 & Z^2 & Z^3 & Z^4 & Z^5 & Z^6 & Z^7 \end{bmatrix}. \tag{5.180}$$

Observe that the three lattice points $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ from $\{0,1\} \times \mathbb{Z}^7$ have been mapped to eight lattice points with coordinates in $\mathbb{Z}^3$. The original parallelogram in $\{0,1\} \times \mathbb{Z}^7$ is now an object of eight connected points of $\mathbb{Z}^3$. In addition, we have the condition on each set of respective coordinates of the lattice points that:

$$X^i + Y^i - Z^i = 0\,, \tag{5.181}$$

in which $i = 1, \ldots, 7$. This equation represents a plane a through the origin $\boldsymbol{o}$ of $\mathbb{Z}^3$. The lattice points coincident with the plane a can be mapped through a bijection to the hexagonal lattice A_2 (J. Conway et al., 1999, p.110). The vectors spanning the plane a are $\boldsymbol{b}_1 = (1,0,1)$ and $\boldsymbol{b}_2 = (0,1,1)$. We define a shift vector $\boldsymbol{s} = \boldsymbol{b}_2 - \boldsymbol{b}_1 = (-1,1,0)$ that will be useful for calculating the new positions of lattice points in the plane a. Figure 5.16 shows the lattice points in green, red and blue of the plane a in $\mathbb{Z}^3$ that could be combined to form any type of quantity equations of the kind of quantity energy. The prescription to form a quantity equation of the parallelogram type, is to connect the lattice points in the order red, green and blue. The advantage is that this graphical representation shows *all possible combinations* in one plane.

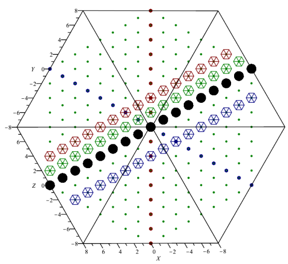

Figure 5.16: Energy template.

Einstein's equation $E = mc^2$ projected from $\{0, 1\} \times \mathbb{Z}^7$ to the plane a is shown in Figure 5.17.

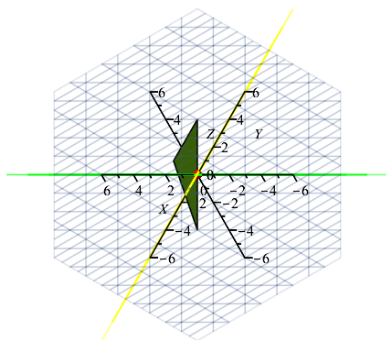

Figure 5.17: Einstein's equation $E = mc^2$.

5.8.1.1 Kind of quantity energy

Figure 5.18 shows the network of the kind of quantity energy in the plane a in $\mathbb{Z}^3$.

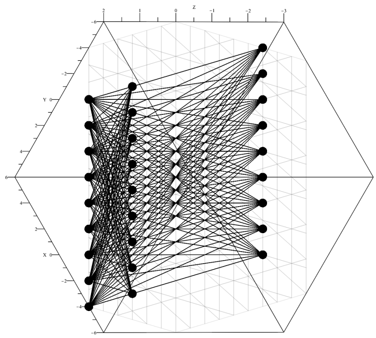

Figure 5.18: Network of quantity equations of the parallelogram type of the kind of quantity energy.

5.8.1.2 Kind of quantity energy density

Figure 5.19 shows the network of the kind of quantity energy density in the plane a in $\mathbb{Z}^3$.

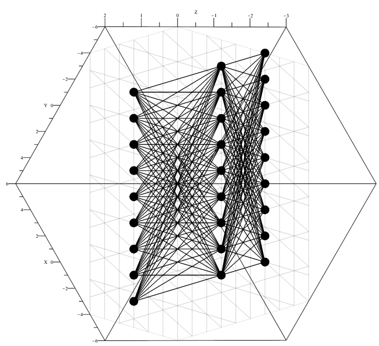

Figure 5.19: Network of quantity equations of the parallelogram type of the kind of quantity of energy density.

5.8.1.3 Kind of quantity first time derivative of energy density

Figure 5.20 shows the network of the kind of quantity first time derivative of the energy density in the plane a in $\mathbb{Z}^3$.

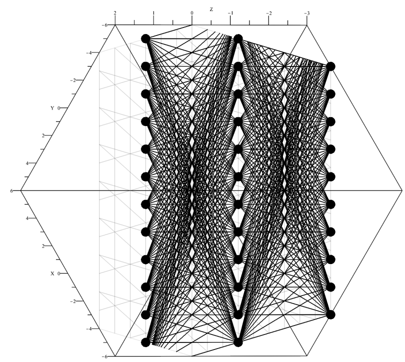

Figure 5.20: Network of quantity equations of the parallelogram type of the kind of quantity first time derivative of energy density.

5.8.1.4 Kind of quantity second time derivative of energy density

Figure 5.21 shows the network of the kind of quantity second time derivative of the energy density in the plane a in $\mathbb{Z}^3$.

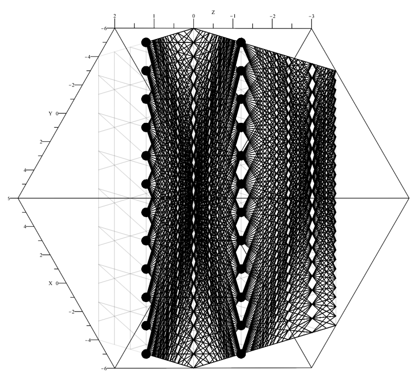

Figure 5.21: Network of quantity equations of the parallelogram type of the kind of quantity second time derivative of energy density.

5.9 Dimensionless quantities revisited

The number of dimensionless quantities is infinite however some are more important than others. Each dimensionless quantity can be represented by a parallelogram in an N-dimensional integer lattice. We showed that unique parallelograms are a first selection filter to find the more important dimensionless quantities.

We consider a well-known dimensionless quantity denoted as Reynolds number Re and discuss it in the framework that we created. Other well-known dimensionless quantities from fluid mechanics, heat transfer, hydrodynamics, aerodynamics can be treated in the same way as given in this section.

5.9.1 Reynolds number

Unmanned Aircraft Systems (UAS) are becoming ubiquitous in the daily information. An unmanned aerial vehicle (UAV) and a micro air vehicle (MAV) have typically a Reynolds number between 10 000 and 100 000 (Winslow, Otsuka, Govindarajan, & Chopra, 2018) because it is flying at a low speed and has a small characteristic dimension. An airliner, like a Boeing 747, has a Reynolds number of 1×10^7 to 1×10^8 (Decuypere, 2011, p.20-21).

The Reynolds number is a dimensionless quantity defined by the equation:

$$\mathrm{Re} = \frac{v \cdot D}{\nu}\,, \tag{5.182}$$

where Re is the Reynolds number, v is a velocity, D a characteristic length, and ν is the kinematic viscosity of the medium. Reynolds numbers are found in the range $[1 \times 10^{-6} - 1 \times 10^12]$. The onset of turbulent flow occurs approximately at $Re = 2300$ for pipe flow. The characteristic length for an aircraft can be the length of the fuselage or the mean aerodynamic chord length of the airfoil. In the latter case it is called the chord-based Reynolds number. The Reynolds number has a strong effect on the conventional airfoil performance (Winslow et al., 2018).

In the present framework we rewrite the equation to the form:

$$\nu = \left(\frac{1}{\mathrm{Re}}\right) \cdot vD\,. \tag{5.183}$$

This equation is of the form $Z = f(\pi)XY$ that represents a parallelogram in the integer lattice $\{0,1\} \times \mathbb{Z}^7$. We have the vector equation:

$$(0 \mid -1, 2, 0^5) = (0 \mid 0^7) + (0 \mid -1, 1, 0^5) + (0 \mid 0, 1, 0^5)\,, \tag{5.184}$$

where $z = (0 \mid -1, 2, 0^5)$ is the lattice point representing the kind of quantity kinematic viscosity. The orbit representative is the lattice point $(0 \mid 2, 1, 0^5)$. The cardinality of the orbit representative is 168. The signed Gödel number is 12. The number of divisors of 12 is $\tau(12) = 6$. The divisor set is $\{1, 2, 3, 4, 6, 12\}$.
The subset $\{2, 4, 6, 12\}$ comprises the signed Gödel numbers of orbit representatives. The degree of the orbit representative is $\deg((0 \mid 2, 1, 0^5)) = 3$. The lattice points can be grouped into four distinct sets S_d. These four sets,

ordered by decreasing degree d are:

$$S_3 = \{(0 \mid 2, 1, 0^5)\},$$
$$S_2 = \{(0 \mid 2, 0^6), (0 \mid 1, 1, 0^5)\},$$
$$S_1 = \{(0 \mid 1, 0^6), (0 \mid 0, 1, 0^5)\},$$
$$S_0 = \{(0 \mid 0^7)\}\,.$$

The Hasse diagram containing the four sets is given in Figure 5.22.

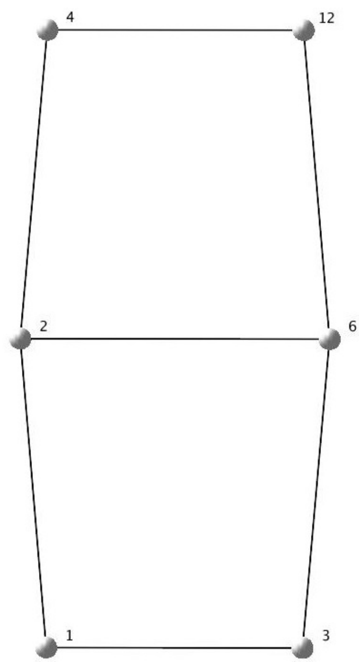

Figure 5.22: Hasse representation of the decoding of the orbit representative of kinematic viscosity given by Gödel number $G(\mathrm{Orb}\,(\boldsymbol{z})) = (-1)^0 2^2 3^1 5^0 7^0 11^0 13^0 17^0 = 12\,.$

The Reynolds number is represented by the closed path $\{1, 2, 12, 6, 1\}$ and by the parallelogram in $\{0, 1\} \times \mathbb{Z}^7$ with integer lattice points $\{(0 \mid 0^7), (0 \mid -1, 1, 0^5), (0 \mid -1, 2, 0^5), (0 \mid 0, 1, 0^5)\}\,.$

The semi-perimeter SP of the parallelogram representing the Reynolds number is shown as a red square and is equal to $SP = \sqrt{2} + \sqrt{1} = 2.4214213562\,.$ It is a *unique* semi-perimeter in the semi-log histogram given in Figure 5.23.

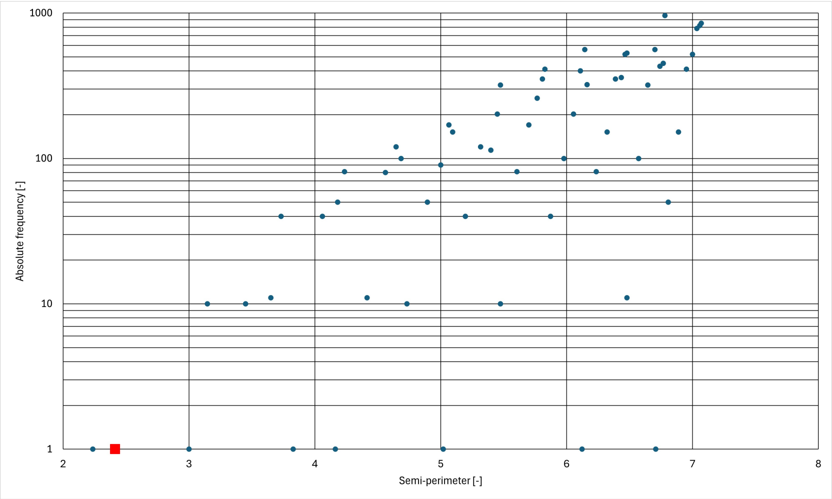

Figure 5.23: Histogram of semi-perimeters of the kind of quantity kinematic viscosity inside a 7-ball with $s \leq 4$.

The first point with frequency 1 is a degenerated parallelogram. The other dimensionless quantities with frequency 1 having a larger semi-perimeter than the parallelogram representing the Reynolds number are unknown to the author and could be explored in future work.

5.10 Conclusion

We show in this chapter the potential of the dimensional exploration technique. We elaborate on the histogram of semi-perimeters of parallelograms. We found that the parallelograms are inscribed in $N+1$-dimensional confocal ellipsoids and that the special parallelograms that are rectangles are inscribed in $N+1$-dimensional hyperspheres. We define a fundamental $(N+1)$-dimensional ellipsoid for a kind of quantity. We find cross-dimensional properties that show the existence of a threshold point (N, s) for the number of unique quantity equations of a kind of quantity. This result shows the existence of a meta-law similar to the *linguistic relativity* (Linguistic relativity, 2024). We explore the kind of quantities:

- energy
- Newtonian constant of gravitation
- electromagnetism

We retrieve all the Maxwell equations using the dimensional exploration technique. We show a dimensional reduction technique, that we call the transposition method, which maps all the ternary equations of a kind of quantity on a plane in $\mathbb{Z}^3$.

CHAPTER 6

Conclusion and future work

The 46 years of reflection on the research questions has resulted in a new point of view on science in general and more specifically on physics and applied physics. We showed that the universality of 'laws of physics' depends on the number of *letters* of the alphabet that we use to write the equations of these laws. Our results suggest a connection with *linguistic relativity*. This connection should be explored in future work. We formulate a conjecture that 'laws of physics' are represented by parallelograms having unique semi-perimeters in $\{0, 1\} \times \mathbb{Z}^N$, with the dimension N of the integer lattice properly chosen. The conjecture can be rephrased in terms of information content $I(SP) = -\log_2(p(SP))$ of a kind of quantity Z, in which $p(SP)$ is the probability of occurrence of a value SP for the semi-perimeter SP of the parallelogram $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$ representing the quantity equation $Z = f(\boldsymbol{\pi})XY$. We conjecture that each 'law of physics' must have maximum information content. The conjecture is clearly falsifiable because if someone finds a 'law of physics' represented by a parallelogram in $\{0, 1\} \times \mathbb{Z}^N$ having in the histogram of semi-perimeters, for the kind of quantity under investigation, a semi-perimeter value SP with an absolute frequency $f > 2$ then the conjecture is proven *wrong*. We cannot prove that the conjecture is *right* but we can prove that it is *not wrong* to the best of our knowledge.

6.1 Conclusion

We show the mapping of SI kinds of quantities onto the integer lattice points of $\{0, 1\} \times \mathbb{Z}^7$. We demonstrate that quantity equations created in dimensional analysis can be classified using the geometric properties of these equations.

Parallelograms are the fundamental geometric form of all kinds of quantity equations because each non-trivial quantity equation can be reduced to the algebraic form $Z = f(\boldsymbol{\pi})XY$. This equivalence is the basis for a computer search of parallelograms formed by integer lattice points of $\{0, 1\} \times \mathbb{Z}^7$.

We prove that ternary relations between kinds of quantities are classified in four distinct two-coloring patterns of $\{0, 1\} \times \mathbb{Z}^7$.

We create a 3D-phase space of parallelograms with coordinates: the $\|\boldsymbol{x}\|_1$, the $\|\boldsymbol{y}\|_1$, and the area of the parallelogram squared A^2. The multiplicity of each *parallelogram state* is attached as an attribute to the three phase space coordinates.

We show that integer lattice points, representing the kinds of quantities, are partitioned in orbits with representative lattice points in $\{0,1\} \times \mathbb{Z}_+^7$.

We construct, for the SI2019 quantities, a 'Table of SI physics', that results in the mathematical classification of the SI2019 kinds of quantities. The 'Table of SI physics' is created from an equivalence relation, in which the rows are representing the infinity norms $\|\boldsymbol{z}\|_\infty = s$ of the orbit representative lattice points and the columns are the cardinalities $\#\left([w]\right)$ of the orbits of the integer lattice $\{0,1\} \times \mathbb{Z}^7$.

A similar mathematical classification can be done for kinds of quantities described using another convention than the SI2019 in which other base quantities are used. The sets that are generated are a property of the N-dimensional integer lattices selected to describe the kinds of quantities.

We show that the encoding of the orbit representative can be done using the signed Gödel encoding scheme. This encoding of the orbit representatives generates a partial order between the kinds of quantities.

We show that the divisibility relation $n|m$ applied to the Gödel numbers creates the fundamental sub-lattice of the SI kinds of quantities. We discover that this fundamental sub-lattice has 40 320 vertices, that is the divisor graph of the Gödel encoding of the lattice point $(0 \mid 7,6,5,4,3,2,1)$ that generates the $2^7\,7!$ symmetries of $\{0\} \times \mathbb{Z}^7$. This SI fundamental sub-lattice will help engineers and physicists in selecting the relevant SI kinds of quantities for the design of experiments (DoE).

We study constellations of lattice points $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \{0,1\} \times \mathbb{Z}^7$ forming parallelograms represented by the vector equation $\boldsymbol{x} + \boldsymbol{y} - \boldsymbol{z} = \boldsymbol{o}$ and show the existence of *unique* semi-perimeters of parallelograms.

Cross-dimensional properties of integer lattices show that the kind of quantity energy is represented by a positively homogeneous dimensionless measurement model $E(\pi_1, \ldots, \pi_8) = 0$ of *eight* dimensionless quantities and the kind of quantity energy density is represented by a positively homogeneous dimensionless measurement model $W(\pi_1, \ldots, \pi_5) = 0$ of *five* dimensionless quantities. The dimensionless quantities represent geometrically parallelograms. Some of these parallelograms are unique. For the kind of quantity energy we find *six* unique non-degenerated parallelograms in the fundamental $N+1$-dimensional confocal ellipsoid of the kind of quantity energy.

The histograms of the semi-perimeters of the parallelograms are classified according to the orbit representatives. We find that the number of orbit representatives $G\big(\mathrm{Orb}\,(\boldsymbol{z})\big)$ of $\{0\} \times \mathbb{Z}^7$ amounts to 3 432 *within* the hypercube with infinity norm $\|\boldsymbol{z}\|_\infty \leq 7$. Only 0.22 % of these orbits have been studied by researchers.

We select 115 orbits from the 3 432 orbits of the hypercube with infinity norm $\|\boldsymbol{z}\|_\infty \leq 7$ to create the *backbone* of the SI physics. They represent 8 177 907 integer lattice points. Those 115 orbit representatives are on the Gödel walk through $\{0,1\} \times \mathbb{Z}^7$ beginning at the origin $\boldsymbol{o}$ and ending with the lattice point $(0 \mid 7,6,5,4,3,2,1)$.

The histograms of the perimeters of the parallelograms are invariant under signed permutation matrices that are the elements of the automorphism group $\mathrm{Aut}(\mathbb{Z}^{\mathrm{N}})$. Addition of the condition $\boldsymbol{x} \cdot \boldsymbol{y} = 0$ to the parallelogram equation restricts the parallelograms to rectangles.

We find that the rectangles are defined by the solutions of the equation $(\boldsymbol{x} - \frac{\boldsymbol{z}}{2})^2 = (\frac{\boldsymbol{z}}{2})^2$ of a $(N+1)$-dimensional hypersphere. This $(N+1)$-dimensional hypersphere is the geometrical representation of an $(N+1)$-dimensional Helmholtz differential equation of a unique scalar wave function associated to the orbit representative $G\big(\mathrm{Orb}\,(\boldsymbol{z})\big)$ in $\{0,1\} \times \mathbb{Z}^N$. The solutions of the differential equation are given by the lattice points of the integer lattice that are incident on the $(N+1)$-dimensional hypersphere.

We enumerate for $N = 7$ the number of rectangles formed by the orbit representatives of $\{0,1\} \times \mathbb{Z}^7$ as a discrete function of the infinity norm, given in the On-line Encyclopedia of Integer Sequences (OEIS) with identifier A240934 and those with a unique perimeter, given in the OEIS integer sequence A247557.

For $\|\boldsymbol{z}\|_\infty \leq 10$, we find in $\mathbb{Z}^7_+$ a total of 7 747 unique rectangles out of 6 510 466 998 rectangles, obtained using the supercomputer of Ghent University. These unique rectangles form the *atlas* of canonical ternary quantity equations of SI physics.

We show an efficient and effective encoding and decoding method for generating a dimensional measurement model $F(Q_1, \ldots, Q_M) = 0$, in which $M+1 = \tau\Big(G\big(\mathrm{Orb}\,(\boldsymbol{z})\big)\Big)$ and $u(\pi_1, \ldots, \pi_K) = 0$ being a positively homogeneous dimensionless measurement model applicable to engineering problems.

The new method has the advantage over classical dimensional analysis (CDA) and modern dimensional analysis (MDA) in requiring fewer dimensionless quantities π_k as arguments of the dimensionless measurement model $u(\pi_1, \ldots, \pi_K) = 0$ when modeling real-world problems. The number of dimensionless quantities obtained using the encoding and decoding method is:

$$K = \frac{1}{2}\tau\Big(G\big(\mathrm{Orb}\,(\boldsymbol{z})\big)\Big) - 1. \tag{6.1}$$

The efficiency of the encoding and decoding method is based on low complexity, high performing, and well-established computer algorithms of number theoretic functions.

The effectiveness of the encoding and decoding method is demonstrated in generating complete sets of dimensionless quantities of lower or equal cardinality than those of classical and modern dimensional analysis.

The validation of the encoding and decoding method is illustrated by the cases of the simple pendulum and the kind of quantity energy, in which this new method retrieves well-known equations of physics.

We use our methods to perform a dimensional exploration of electromagnetism and find the quantity equations which represent the four Maxwell equations.

6.2 Future work

This dissertation shows that the knowledge about the kinds of quantities and about their constellations is far from being understood and that large hyper-volumes of $\{0,1\} \times \mathbb{Z}^7$ are still to be explored. Future work will encompass expanding Appendix A by further compilation of SI physical quantities and the automation of the encoding and decoding method. Applying this new method to the 3 432 orbit representatives of $\{0,1\} \times \mathbb{Z}^7$, in the hypercube with infinity norm $\|\boldsymbol{z}\|_\infty \leq 7$, will create division lattices that can be stored as a database in a repository managed by UGent TechTransfer under valorization project: P2021/066 – Mathematical Classification. Queries on that database could help engineers and scientists to find an unknown 'law of physics' and help them in the design of experiments (DoE) to validate this new 'law of physics'.

APPENDIX A

Lexicon of physical quantities in $\{0,1\} \times \mathbb{Z}^7$

Table A.1 is our compilation of physical quantities that can be found in scientific literature (IUPAP, 1978), (Cohen et al., 1987), (Cohen et al., 2007), (Szirtes, 2007, p.57-59), (Feynman, 1977a; R. P. Feynman et al., 1971, 1966),(Butcher & Cotter, 1998, p.26-27) and, (List of Physical quantities, 2024) .

There are 139 distinct lattice points representing the kinds of quantities of the physical quantities compiled by the author.

Table A.1 contains ten columns.

The first column gives the cardinality of the orbit that contains the kind of quantity.
The second column indicates the infinity norm $\|z\|_\infty$ of the lattice point.
The third column gives the value of $\|z\|_1$ of the coordinates of the lattice point in $\{0,1\} \times \mathbb{Z}^7$.
The fourth column gives the value of $\|z\|_2^2$ of the coordinates of the lattice point in $\{0,1\} \times \mathbb{Z}^7$.
The fifth column gives the designation of the orthant containing the lattice point in $\{0,1\} \times \mathbb{Z}^7$. There are $2^7 = 128$ orthants in $\{0,1\} \times \mathbb{Z}^7$. Only 12 orthants of the 128 orthants are populated by published kinds of quantities: 1, 3, 5, 17, 33, 49, 65, 69, 71, 73, 81 and, 97.
The sixth column represents the name of the physical quantity.
The seventh column lists the orbit that contains the kind of quantity.
The eighth column lists the Gödel number of the orbit representative lattice point.
The ninth column identifies the kind of quantity by its integer lattice point in $\{0,1\} \times \mathbb{Z}^7$. The coordinates $(z_0 \mid z_1, \ldots, z_7)$ corresponds to the dimensional exponents in the following order $(0 \mid \alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta)$.
The tenth column gives the value of the sum of the coordinates $\operatorname{soc}(z)$ of the lattice point in $\{0,1\} \times \mathbb{Z}^7$.

Table A.1: Lexicon of physical quantities in $\{0,1\} \times \mathbb{Z}^7$.

# (orbit)	$\|z\|_\infty$	$\|z\|_1$	$\|z\|_2^2$	Orthant	Physical quantity	$\mathrm{Orb}\,(z)$	$G(\mathrm{Orb}\,(z))$	z	$\mathrm{soc}\,(z)$
1	0	0	0	1	plane angle	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	solid angle	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	linear strain	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	shear strain	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	bulk strain	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	tensile strain	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	relative elongation	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	refractive index	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	electric susceptibility, 1st-order scalar susceptibility	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	mass ratio, mass fraction	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	electroweak charge of a nucleus	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	fine-structure constant (α_e)	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	(α_w)	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	(α_s)	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	(α_G)	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	redshift	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	ratio of pressure to energy density	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	Poisson's ratio	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	relative density, specific gravity	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	relative permeability	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	relative permittivity, dielectric constant	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	optical depth, optical thickness	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	spectral optical depth	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	hemispherical emissivity	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	spectral hemispherical emissivity	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	directional emissivity	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	spectral directional emissivity	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	hemispherical absorptance	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	spectral hemispherical absorptance	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	directional absorptance	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	spectral directional absorptance	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	hemispherical reflectance	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	spectral hemispherical reflectance	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
...	...	...	...	...	...	...	...	...	...

# (orbit)	$\|z\|_\infty$	$\|z\|_1$	$\|z\|_2^2$	Orthant	Physical quantity	Orb (z)	$G(\mathrm{Orb}\,(z))$	z	soc (z)
1	0	0	0	1	directional reflectance	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	spectral directional reflectance	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	hemispherical transmittance	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	spectral hemispherical transmittance	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	directional transmittance	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	spectral directional transmittance	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	relative gradient, field index of accelerator guide fields	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	number of expected scattering events	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	rapidity	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	pseudo rapidity	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	Lorentz factor	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	Mach number	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
1	0	0	0	1	g-factor	$[(0 \mid 0^7)]$	1	$(0 \mid 0,0,0,0,0,0,0)$	0
14	1	1	1	65	angular velocity	$[(1 \mid 1,0^6)]$	-2	$(1 \mid -1,0,0,0,0,0,0)$	-1
14	1	1	1	65	vorticity	$[(1 \mid 1,0^6)]$	-2	$(1 \mid -1,0,0,0,0,0,0)$	-1
14	1	1	1	65	amount of circulation	$[(1 \mid 1,0^6)]$	-2	$(1 \mid -1,0,0,0,0,0,0)$	-1
14	1	1	1	1	length	$[(0 \mid 1,0^6)]$	2	$(0 \mid 0,1,0,0,0,0,0)$	1
14	1	1	1	1	proper length, rest length	$[(0 \mid 1,0^6)]$	2	$(0 \mid 0,1,0,0,0,0,0)$	1
14	1	1	1	1	proper distance	$[(0 \mid 1,0^6)]$	2	$(0 \mid 0,1,0,0,0,0,0)$	1
14	1	1	1	1	comoving distance	$[(0 \mid 1,0^6)]$	2	$(0 \mid 0,1,0,0,0,0,0)$	1
14	1	1	1	1	length contraction	$[(0 \mid 1,0^6)]$	2	$(0 \mid 0,1,0,0,0,0,0)$	1
14	1	1	1	1	height	$[(0 \mid 1,0^6)]$	2	$(0 \mid 0,1,0,0,0,0,0)$	1
14	1	1	1	1	breadth	$[(0 \mid 1,0^6)]$	2	$(0 \mid 0,1,0,0,0,0,0)$	1
14	1	1	1	1	thickness	$[(0 \mid 1,0^6)]$	2	$(0 \mid 0,1,0,0,0,0,0)$	1
14	1	1	1	1	distance	$[(0 \mid 1,0^6)]$	2	$(0 \mid 0,1,0,0,0,0,0)$	1
14	1	1	1	1	radius	$[(0 \mid 1,0^6)]$	2	$(0 \mid 0,1,0,0,0,0,0)$	1
14	1	1	1	1	radius excess	$[(0 \mid 1,0^6)]$	2	$(0 \mid 0,1,0,0,0,0,0)$	1
14	1	1	1	1	Bohr radius	$[(0 \mid 1,0^6)]$	2	$(0 \mid 0,1,0,0,0,0,0)$	1
14	1	1	1	1	diameter	$[(0 \mid 1,0^6)]$	2	$(0 \mid 0,1,0,0,0,0,0)$	1
14	1	1	1	1	optical path length	$[(0 \mid 1,0^6)]$	2	$(0 \mid 0,1,0,0,0,0,0)$	1
14	1	1	1	1	optical path difference	$[(0 \mid 1,0^6)]$	2	$(0 \mid 0,1,0,0,0,0,0)$	1
14	1	1	1	1	mean free path	$[(0 \mid 1,0^6)]$	2	$(0 \mid 0,1,0,0,0,0,0)$	1
14	1	1	1	1	persistence length	$[(0 \mid 1,0^6)]$	2	$(0 \mid 0,1,0,0,0,0,0)$	1
14	1	1	1	1	penetration depth (of skin effect)	$[(0 \mid 1,0^6)]$	2	$(0 \mid 0,1,0,0,0,0,0)$	1
14	1	1	1	1	effective focal length	$[(0 \mid 1,0^6)]$	2	$(0 \mid 0,1,0,0,0,0,0)$	1
...	...	...	...	...	...	...	...	...	...

# (orbit)	$\|z\|_\infty$	$\|z\|_1$	$\|z\|_2^2$	Orthant	Physical quantity	Orb (z)	$G(\text{Orb}\,(z))$	z	soc (z)
14	1	1	1	1	length of arc	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0, 1, 0, 0, 0, 0, 0)$	1
14	1	1	1	1	Planck length	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0, 1, 0, 0, 0, 0, 0)$	1
14	1	1	1	1	Debye length	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0, 1, 0, 0, 0, 0, 0)$	1
14	1	1	1	1	wavelength	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0, 1, 0, 0, 0, 0, 0)$	1
14	1	1	1	1	Compton wavelength	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0, 1, 0, 0, 0, 0, 0)$	1
14	1	1	1	1	relaxation length	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0, 1, 0, 0, 0, 0, 0)$	1
14	1	1	1	1	luminosity distance	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0, 1, 0, 0, 0, 0, 0)$	1
14	1	1	1	1	mass	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0, 0, 1, 0, 0, 0, 0)$	1
14	1	1	1	1	invariant mass	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0, 0, 1, 0, 0, 0, 0)$	1
14	1	1	1	1	rest mass	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0, 0, 1, 0, 0, 0, 0)$	1
14	1	1	1	1	intrinsic mass	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0, 0, 1, 0, 0, 0, 0)$	1
14	1	1	1	1	proper mass	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0, 0, 1, 0, 0, 0, 0)$	1
14	1	1	1	1	reduced mass	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0, 0, 1, 0, 0, 0, 0)$	1
14	1	1	1	1	effective mass	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0, 0, 1, 0, 0, 0, 0)$	1
14	1	1	1	1	Planck mass	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0, 0, 1, 0, 0, 0, 0)$	1
14	1	1	1	1	time	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 1, 0, 0, 0, 0, 0, 0)$	1
14	1	1	1	1	coordinate time	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 1, 0, 0, 0, 0, 0, 0)$	1
14	1	1	1	1	period	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 1, 0, 0, 0, 0, 0, 0)$	1
14	1	1	1	1	orbital period	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 1, 0, 0, 0, 0, 0, 0)$	1
14	1	1	1	1	relaxation time	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 1, 0, 0, 0, 0, 0, 0)$	1
14	1	1	1	1	time constant	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 1, 0, 0, 0, 0, 0, 0)$	1
14	1	1	1	1	time interval	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 1, 0, 0, 0, 0, 0, 0)$	1
14	1	1	1	1	proper time	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 1, 0, 0, 0, 0, 0, 0)$	1
14	1	1	1	1	time dilation	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 1, 0, 0, 0, 0, 0, 0)$	1
14	1	1	1	1	Planck time	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 1, 0, 0, 0, 0, 0, 0)$	1
14	1	1	1	1	half-life	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 1, 0, 0, 0, 0, 0, 0)$	1
14	1	1	1	1	mean life, lifetime	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 1, 0, 0, 0, 0, 0, 0)$	1
14	1	1	1	1	specific impulse	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 1, 0, 0, 0, 0, 0, 0)$	1
14	1	1	1	1	electric current	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0, 0, 0, 1, 0, 0, 0)$	1
14	1	1	1	1	magnetic potential difference	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0, 0, 0, 1, 0, 0, 0)$	1
14	1	1	1	1	magnetomotive force	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0, 0, 0, 1, 0, 0, 0)$	1
14	1	1	1	1	thermodynamic temperature	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0, 0, 0, 0, 1, 0, 0)$	1
14	1	1	1	1	Planck temperature	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0, 0, 0, 0, 1, 0, 0)$	1
14	1	1	1	1	Curie temperature	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0, 0, 0, 0, 1, 0, 0)$	1
14	1	1	1	1	amount of substance	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0, 0, 0, 0, 0, 1, 0)$	1
...	...	...	...	...	...	...	...	...	...

# (orbit)	$\|z\|_\infty$	$\|z\|_1$	$\|z\|_2^2$	Orthant	Physical quantity	Orb (z)	$G(\text{Orb}(z))$	z	soc (z)
14	1	1	1	1	luminous intensity	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0,0,0,0,0,0,1)$	1
14	1	1	1	1	luminous flux	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0,0,0,0,0,0,1)$	1
14	1	1	1	3	Avogadro constant	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0,0,0,0,0,-1,0)$	-1
14	1	1	1	5	thermal expansion coefficient	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0,0,0,0,-1,0,0)$	-1
14	1	1	1	65	frequency	$[(0 \mid 1, 0^6)]$	2	$(0 \mid -1,0,0,0,0,0,0)$	-1
14	1	1	1	65	plasma frequency	$[(0 \mid 1, 0^6)]$	2	$(0 \mid -1,0,0,0,0,0,0)$	-1
14	1	1	1	65	Hubble constant	$[(0 \mid 1, 0^6)]$	2	$(0 \mid -1,0,0,0,0,0,0)$	-1
14	1	1	1	65	angular frequency	$[(0 \mid 1, 0^6)]$	2	$(0 \mid -1,0,0,0,0,0,0)$	-1
14	1	1	1	65	circular frequency	$[(0 \mid 1, 0^6)]$	2	$(0 \mid -1,0,0,0,0,0,0)$	-1
14	1	1	1	65	cutoff frequency	$[(0 \mid 1, 0^6)]$	2	$(0 \mid -1,0,0,0,0,0,0)$	-1
14	1	1	1	65	activity	$[(0 \mid 1, 0^6)]$	2	$(0 \mid -1,0,0,0,0,0,0)$	-1
14	1	1	1	65	specific material permeability	$[(0 \mid 1, 0^6)]$	2	$(0 \mid -1,0,0,0,0,0,0)$	-1
14	1	1	1	65	decay (rate) constant, disintegration (rate) constant	$[(0 \mid 1, 0^6)]$	2	$(0 \mid -1,0,0,0,0,0,0)$	-1
14	1	1	1	65	strain rate	$[(0 \mid 1, 0^6)]$	2	$(0 \mid -1,0,0,0,0,0,0)$	-1
14	1	1	1	33	curl operator	$[(1 \mid 1, 0^6)]$	-2	$(1 \mid 0,-1,0,0,0,0,0)$	-1
14	1	1	1	33	div operator	$[(1 \mid 1, 0^6)]$	2	$(0 \mid 0,-1,0,0,0,0,0)$	-1
14	1	1	1	33	wavenumber	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0,-1,0,0,0,0,0)$	-1
14	1	1	1	33	optical power	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0,-1,0,0,0,0,0)$	-1
14	1	1	1	33	curvature	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0,-1,0,0,0,0,0)$	-1
14	1	1	1	33	spatial frequency	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0,-1,0,0,0,0,0)$	-1
14	1	1	1	33	absorption coefficient	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0,-1,0,0,0,0,0)$	-1
14	1	1	1	33	linear attenuation coefficient	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0,-1,0,0,0,0,0)$	-1
14	1	1	1	33	hemispherical attenuation coefficient	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0,-1,0,0,0,0,0)$	-1
14	1	1	1	33	spectral hemispherical attenuation coefficient	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0,-1,0,0,0,0,0)$	-1
14	1	1	1	33	directional attenuation coefficient	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0,-1,0,0,0,0,0)$	-1
14	1	1	1	33	spectral directional attenuation coefficient	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0,-1,0,0,0,0,0)$	-1
14	1	1	1	33	laser gain	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0,-1,0,0,0,0,0)$	-1
14	1	1	1	33	rotational constant	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0,-1,0,0,0,0,0)$	-1
14	1	1	1	33	Rydberg constant	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0,-1,0,0,0,0,0)$	-1
14	2	2	4	1	area	$[(0 \mid 2, 0^6)]$	4	$(0 \mid 0,2,0,0,0,0,0)$	2
14	2	2	4	1	Thomson cross section	$[(0 \mid 2, 0^6)]$	4	$(0 \mid 0,2,0,0,0,0,0)$	2
14	2	2	4	1	atomic attenuation coefficient	$[(0 \mid 2, 0^6)]$	4	$(0 \mid 0,2,0,0,0,0,0)$	2
14	2	2	4	65	angular acceleration	$[(0 \mid 2, 0^6)]$	4	$(0 \mid -2,0,0,0,0,0,0)$	-2
14	2	2	4	33	spacetime curvature	$[(0 \mid 2, 0^6)]$	4	$(0 \mid 0,-2,0,0,0,0,0)$	-2
14	2	2	4	33	scalar curvature	$[(0 \mid 2, 0^6)]$	4	$(0 \mid 0,-2,0,0,0,0,0)$	-2
...	...	...	...	...	...	...	...	...	...

$\#\,(\mathrm{orbit})$	$\|z\|_\infty$	$\|z\|_1$	$\|z\|_2^2$	Orthant	Physical quantity	$\mathrm{Orb}\,(z)$	$G(\mathrm{Orb}\,(z))$	z	$\mathrm{soc}\,(z)$
14	2	2	4	33	Ricci curvature tensor	$[(0 \mid 2, 0^6)]$	4	$(0 \mid 0, -2, 0, 0, 0, 0, 0)$	-2
14	2	2	4	33	Einstein tensor	$[(0 \mid 2, 0^6)]$	4	$(0 \mid 0, -2, 0, 0, 0, 0, 0)$	-2
14	2	2	4	33	cosmological constant	$[(0 \mid 2, 0^6)]$	4	$(0 \mid 0, -2, 0, 0, 0, 0, 0)$	-2
14	2	2	4	33	integrated luminosity	$[(0 \mid 2, 0^6)]$	4	$(0 \mid 0, -2, 0, 0, 0, 0, 0)$	-2
14	2	2	4	33	particle fluence	$[(0 \mid 2, 0^6)]$	4	$(0 \mid 0, -2, 0, 0, 0, 0, 0)$	-2
14	2	2	4	33	fuel efficiency	$[(0 \mid 2, 0^6)]$	4	$(0 \mid 0, -2, 0, 0, 0, 0, 0)$	-2
14	3	3	9	1	volume	$[(0 \mid 3, 0^6)]$	8	$(0 \mid 0, 3, 0, 0, 0, 0, 0)$	3
14	3	3	9	65	angular jerk	$[(0 \mid 3, 0^6)]$	8	$(0 \mid -3, 0, 0, 0, 0, 0, 0)$	-3
14	3	3	9	33	Loschmidt constant	$[(0 \mid 3, 0^6)]$	8	$(0 \mid 0, -3, 0, 0, 0, 0, 0)$	-3
14	3	3	9	33	number density	$[(0 \mid 3, 0^6)]$	8	$(0 \mid 0, -3, 0, 0, 0, 0, 0)$	-3
14	3	3	9	33	ion number density	$[(0 \mid 3, 0^6)]$	8	$(0 \mid 0, -3, 0, 0, 0, 0, 0)$	-3
14	4	4	16	1	second moment of area	$[(0 \mid 4, 0^6)]$	16	$(0 \mid 0, 4, 0, 0, 0, 0, 0)$	4
84	1	2	2	1	magnetic pole strength	$[(1 \mid 1^2, 0^5)]$	-6	$(1 \mid 0, 1, 0, 1, 0, 0, 0)$	2
84	1	2	2	33	magnetic field strength	$[(1 \mid 1^2, 0^5)]$	-6	$(1 \mid 0, -1, 0, 1, 0, 0, 0)$	0
84	1	2	2	33	magnetization	$[(1 \mid 1^2, 0^5)]$	-6	$(1 \mid 0, -1, 0, 1, 0, 0, 0)$	0
84	1	2	2	1	electric charge	$[(0 \mid 1^2, 0^5)]$	6	$(0 \mid 1, 0, 0, 1, 0, 0, 0)$	2
84	1	2	2	1	electric flux	$[(0 \mid 1^2, 0^5)]$	6	$(0 \mid 1, 0, 0, 1, 0, 0, 0)$	2
84	1	2	2	1	second radiation constant	$[(0 \mid 1^2, 0^5)]$	6	$(0 \mid 0, 1, 0, 0, 1, 0, 0)$	2
84	1	2	2	1	luminous energy	$[(0 \mid 1^2, 0^5)]$	6	$(0 \mid 1, 0, 0, 0, 0, 0, 1)$	2
84	1	2	2	1	quantity of light	$[(0 \mid 1^2, 0^5)]$	6	$(0 \mid 1, 0, 0, 0, 0, 0, 1)$	2
84	1	2	2	3	molar mass	$[(0 \mid 1^2, 0^5)]$	6	$(0 \mid 0, 1, 0, 0, 0, -1, 0)$	0
84	1	2	2	65	linear velocity	$[(0 \mid 1^2, 0^5)]$	6	$(0 \mid -1, 1, 0, 0, 0, 0, 0)$	0
84	1	2	2	65	peculiar velocity	$[(0 \mid 1^2, 0^5)]$	6	$(0 \mid -1, 1, 0, 0, 0, 0, 0)$	0
84	1	2	2	65	recessional velocity	$[(0 \mid 1^2, 0^5)]$	6	$(0 \mid -1, 1, 0, 0, 0, 0, 0)$	0
84	1	2	2	65	velocity of sound	$[(0 \mid 1^2, 0^5)]$	6	$(0 \mid -1, 1, 0, 0, 0, 0, 0)$	0
84	1	2	2	65	group velocity	$[(0 \mid 1^2, 0^5)]$	6	$(0 \mid -1, 1, 0, 0, 0, 0, 0)$	0
84	1	2	2	65	volumetric flux	$[(0 \mid 1^2, 0^5)]$	6	$(0 \mid -1, 1, 0, 0, 0, 0, 0)$	0
84	1	2	2	65	speed	$[(0 \mid 1^2, 0^5)]$	6	$(0 \mid -1, 1, 0, 0, 0, 0, 0)$	0
84	1	2	2	65	speed of light in vacuum	$[(0 \mid 1^2, 0^5)]$	6	$(0 \mid -1, 1, 0, 0, 0, 0, 0)$	0
84	1	2	2	65	catalytic activity	$[(0 \mid 1^2, 0^5)]$	6	$(0 \mid -1, 0, 0, 0, 0, 1, 0)$	0
84	1	2	2	65	mass flow rate	$[(0 \mid 1^2, 0^5)]$	6	$(0 \mid -1, 0, 1, 0, 0, 0, 0)$	0
84	1	2	2	65	mechanical impedance	$[(0 \mid 1^2, 0^5)]$	6	$(0 \mid -1, 0, 1, 0, 0, 0, 0)$	0
84	1	2	2	65	spectral exposure in frequency	$[(0 \mid 1^2, 0^5)]$	6	$(0 \mid -1, 0, 1, 0, 0, 0, 0)$	0
84	1	2	2	73	reciprocal electric charge	$[(0 \mid 1^2, 0^5)]$	6	$(0 \mid -1, 0, 0, -1, 0, 0, 0)$	-2
84	1	2	2	17	molality	$[(0 \mid 1^2, 0^5)]$	6	$(0 \mid 0, 0, -1, 0, 0, 1, 0)$	0
…	…	…	…	…	…	…	…	…	…

# (orbit)	$\|z\|_\infty$	$\|z\|_1$	$\|z\|_2^2$	Orthant	Physical quantity	$\mathrm{Orb}\,(z)$	$G(\mathrm{Orb}\,(z))$	z	$\mathrm{soc}\,(z)$
84	1	2	2	33	material permeance	$[(0 \mid 1^2, 0^5)]$	6	$(0 \mid 1, -1, 0, 0, 0, 0, 0)$	0
84	1	2	2	33	temperature gradient	$[(0 \mid 1^2, 0^5)]$	6	$(0 \mid 0, -1, 0, 0, 1, 0, 0)$	0
84	1	2	2	33	linear mass density	$[(0 \mid 1^2, 0^5)]$	6	$(0 \mid 0, -1, 1, 0, 0, 0, 0)$	0
84	1	2	2	33	specific radius excess	$[(0 \mid 1^2, 0^5)]$	6	$(0 \mid 0, 1, -1, 0, 0, 0, 0)$	0
84	1	2	2	1	spectral linear momentum	$[(0 \mid 1^2, 0^5)]$	6	$(0 \mid 0, 1, 1, 0, 0, 0, 0)$	2
84	1	2	2	1	absement	$[(0 \mid 1^2, 0^5)]$	6	$(0 \mid 1, 1, 0, 0, 0, 0, 0)$	2
84	2	4	8	65	absorbed dose of ionising radiation	$[(0 \mid 2^2, 0^5)]$	36	$(0 \mid -2, 2, 0, 0, 0, 0, 0)$	0
84	2	4	8	65	equivalent dose of ionising radiation	$[(0 \mid 2^2, 0^5)]$	36	$(0 \mid -2, 2, 0, 0, 0, 0, 0)$	0
84	2	4	8	65	specific energy	$[(0 \mid 2^2, 0^5)]$	36	$(0 \mid -2, 2, 0, 0, 0, 0, 0)$	0
84	2	4	8	65	specific enthalpy	$[(0 \mid 2^2, 0^5)]$	36	$(0 \mid -2, 2, 0, 0, 0, 0, 0)$	0
84	2	4	8	65	gravitational potential	$[(0 \mid 2^2, 0^5)]$	36	$(0 \mid -2, 2, 0, 0, 0, 0, 0)$	0
84	2	4	8	65	velocity squared	$[(0 \mid 2^2, 0^5)]$	36	$(0 \mid -2, 2, 0, 0, 0, 0, 0)$	0
84	2	4	8	65	specific stiffness	$[(0 \mid 2^2, 0^5)]$	36	$(0 \mid -2, 2, 0, 0, 0, 0, 0)$	0
168	2	3	5	1	magnetic dipole moment	$[(1 \mid 2, 1, 0^5)]$	-12	$(1 \mid 0, 2, 0, 1, 0, 0, 0)$	3
168	2	3	5	1	electromagnetic moment	$[(1 \mid 2, 1, 0^5)]$	-12	$(1 \mid 0, 2, 0, 1, 0, 0, 0)$	3
168	2	3	5	1	Bohr magneton	$[(1 \mid 2, 1, 0^5)]$	-12	$(1 \mid 0, 2, 0, 1, 0, 0, 0)$	3
168	2	3	5	1	moment of inertia	$[(0 \mid 2, 1, 0^5)]$	12	$(0 \mid 0, 2, 1, 0, 0, 0, 0)$	3
168	2	3	5	17	mass attenuation coefficient	$[(0 \mid 2, 1, 0^5)]$	12	$(0 \mid 0, 2, -1, 0, 0, 0, 0)$	1
168	2	3	5	33	area density, surface density, mass thickness	$[(0 \mid 2, 1, 0^5)]$	12	$(0 \mid 0, -2, 1, 0, 0, 0, 0)$	-1
168	2	3	5	97	instantaneous luminosity	$[(0 \mid 2, 1, 0^5)]$	12	$(0 \mid -1, -2, 0, 0, 0, 0, 0)$	-3
168	2	3	5	65	areal velocity	$[(0 \mid 2, 1, 0^5)]$	12	$(0 \mid -1, 2, 0, 0, 0, 0, 0)$	1
168	2	3	5	65	specific orbital angular momentum, specific angular momentum	$[(0 \mid 2, 1, 0^5)]$	12	$(0 \mid -1, 2, 0, 0, 0, 0, 0)$	1
168	2	3	5	65	diffusion constant, diffusion coefficient	$[(0 \mid 2, 1, 0^5)]$	12	$(0 \mid -1, 2, 0, 0, 0, 0, 0)$	1
168	2	3	5	65	thermal diffusivity	$[(0 \mid 2, 1, 0^5)]$	12	$(0 \mid -1, 2, 0, 0, 0, 0, 0)$	1
168	2	3	5	65	kinematic viscosity	$[(0 \mid 2, 1, 0^5)]$	12	$(0 \mid -1, 2, 0, 0, 0, 0, 0)$	1
168	2	3	5	65	quantum of circulation	$[(1 \mid 2, 1, 0^5)]$	12	$(1 \mid -1, 2, 0, 0, 0, 0, 0)$	1
168	2	3	5	97	particle fluence rate	$[(0 \mid 2, 1, 0^5)]$	12	$(0 \mid -1, -2, 0, 0, 0, 0, 0)$	-3
168	2	3	5	97	instantaneous luminosity	$[(0 \mid 2, 1, 0^5)]$	12	$(0 \mid -1, -2, 0, 0, 0, 0, 0)$	-3
168	2	3	5	97	particle flux density	$[(0 \mid 2, 1, 0^5)]$	12	$(0 \mid -1, -2, 0, 0, 0, 0, 0)$	-3
168	2	3	5	33	electric current density	$[(0 \mid 2, 1, 0^5)]$	12	$(0 \mid 0, -2, 0, 1, 0, 0, 0)$	-1
168	2	3	5	33	curl of magnetic field	$[(0 \mid 2, 1, 0^5)]$	12	$(0 \mid 0, -2, 0, 1, 0, 0, 0)$	-1
168	2	3	5	33	gradient of magnetic field	$[(1 \mid 2, 1, 0^5)]$	-12	$(1 \mid 0, -2, 0, 1, 0, 0, 0)$	-1
168	2	3	5	33	surface current density	$[(0 \mid 2, 1, 0^5)]$	12	$(0 \mid 0, -2, 0, 1, 0, 0, 0)$	-1
168	2	3	5	65	linear acceleration	$[(0 \mid 2, 1, 0^5)]$	12	$(0 \mid -2, 1, 0, 0, 0, 0, 0)$	-1
168	2	3	5	3	molar attenuation coefficient	$[(0 \mid 2, 1, 0^5)]$	12	$(0 \mid 0, 2, 0, 0, 0, -1, 0)$	1
…	…	…	…	…	…	…	…	…	…

# (orbit)	$\|z\|_\infty$	$\|z\|_1$	$\|z\|_2^2$	Orthant	Physical quantity	Orb (z)	$G(\mathrm{Orb}\,(z))$	z	soc (z)
168	2	3	5	33	luminance	$[(0 \mid 2,1,0^5)]$	12	$(0 \mid 0,-2,0,0,0,0,1)$	-1
168	2	3	5	33	illuminance	$[(0 \mid 2,1,0^5)]$	12	$(0 \mid 0,-2,0,0,0,0,1)$	-1
168	2	3	5	33	luminous emittance	$[(0 \mid 2,1,0^5)]$	12	$(0 \mid 0,-2,0,0,0,0,1)$	-1
168	2	3	5	33	photometric irradiance	$[(0 \mid 2,1,0^5)]$	12	$(0 \mid 0,-2,0,0,0,0,1)$	-1
168	2	3	5	65	radiant exposure	$[(0 \mid 2,1,0^5)]$	12	$(0 \mid -2,1,0,0,0,0,0)$	-1
168	2	3	5	65	spectral radiance in frequency	$[(0 \mid 2,1,0^5)]$	12	$(0 \mid -2,0,1,0,0,0,0)$	-1
168	2	3	5	65	spectral radiosity in frequency	$[(0 \mid 2,1,0^5)]$	12	$(0 \mid -2,0,1,0,0,0,0)$	-1
168	2	3	5	65	spectral exitance in frequency	$[(0 \mid 2,1,0^5)]$	12	$(0 \mid -2,0,1,0,0,0,0)$	-1
168	2	3	5	65	surface tension	$[(0 \mid 2,1,0^5)]$	12	$(0 \mid -2,0,1,0,0,0,0)$	-1
168	2	3	5	65	stiffness	$[(0 \mid 2,1,0^5)]$	12	$(0 \mid -2,0,1,0,0,0,0)$	-1
168	2	3	5	65	spring constant	$[(0 \mid 2,1,0^5)]$	12	$(0 \mid -2,0,1,0,0,0,0)$	-1
168	2	3	5	17	compliance	$[(0 \mid 2,1,0^5)]$	12	$(0 \mid 2,0,-1,0,0,0,0)$	1
168	2	3	5	17	electron mobility	$[(0 \mid 2,1,0^5)]$	12	$(0 \mid 2,0,-1,1,0,0,0)$	2
168	3	4	10	17	specific volume	$[(0 \mid 3,1,0^5)]$	24	$(0 \mid 0,3,-1,0,0,0,0)$	2
168	3	4	10	33	volumetric mass density, specific mass	$[(0 \mid 3,1,0^5)]$	24	$(0 \mid 0,-3,1,0,0,0,0)$	-2
168	3	4	10	65	volume rate of flow, volumetric flow rate	$[(0 \mid 3,1,0^5)]$	24	$(0 \mid -1,3,0,0,0,0,0)$	2
168	3	4	10	97	rate of nucleation	$[(0 \mid 3,1,0^5)]$	24	$(0 \mid -1,-3,0,0,0,0,0)$	-4
168	3	4	10	33	electromagnetic local density of states of vacuum	$[(0 \mid 3,1,0^5)]$	24	$(0 \mid 1,-3,0,0,0,0,0)$	-2
168	3	4	10	3	molar volume	$[(0 \mid 3,1,0^5)]$	24	$(0 \mid 0,3,0,0,0,-1,0)$	2
168	3	4	10	33	amount of substance concentration, molarity	$[(0 \mid 3,1,0^5)]$	24	$(0 \mid 0,-3,0,0,0,1,0)$	-2
168	3	4	10	65	jerk	$[(0 \mid 3,1,0^5)]$	24	$(0 \mid -3,1,0,0,0,0,0)$	-2
168	3	4	10	65	rate of energy transfer per unit area	$[(0 \mid 3,1,0^5)]$	24	$(0 \mid -3,0,1,0,0,0,0)$	-2
168	3	4	10	65	heat flux density, energy flux density	$[(0 \mid 3,1,0^5)]$	24	$(0 \mid -3,0,1,0,0,0,0)$	-2
168	3	4	10	65	Poynting vector	$[(0 \mid 3,1,0^5)]$	24	$(0 \mid -3,0,1,0,0,0,0)$	-2
168	3	4	10	65	radiative flux	$[(0 \mid 3,1,0^5)]$	24	$(0 \mid -3,0,1,0,0,0,0)$	-2
168	3	4	10	65	thermal emittance	$[(0 \mid 3,1,0^5)]$	24	$(0 \mid -3,0,1,0,0,0,0)$	-2
168	3	4	10	65	sound intensity	$[(0 \mid 3,1,0^5)]$	24	$(0 \mid -3,0,1,0,0,0,0)$	-2
168	3	4	10	65	radiance	$[(0 \mid 3,1,0^5)]$	24	$(0 \mid -3,0,1,0,0,0,0)$	-2
168	3	4	10	65	irradiance, heat flux density	$[(0 \mid 3,1,0^5)]$	24	$(0 \mid -3,0,1,0,0,0,0)$	-2
168	3	4	10	65	radiant exitance	$[(0 \mid 3,1,0^5)]$	24	$(0 \mid -3,0,1,0,0,0,0)$	-2
168	3	4	10	65	radiant emittance	$[(0 \mid 3,1,0^5)]$	24	$(0 \mid -3,0,1,0,0,0,0)$	-2
168	3	4	10	65	radiosity	$[(0 \mid 3,1,0^5)]$	24	$(0 \mid -3,0,1,0,0,0,0)$	-2
168	3	4	10	65	energy fluence rate	$[(0 \mid 3,1,0^5)]$	24	$(0 \mid -3,0,1,0,0,0,0)$	-2
168	3	5	13	65	standard gravitational parameter of a celestial body	$[(0 \mid 3,2,0^5)]$	72	$(0 \mid -2,3,0,0,0,0,0,0)$	1
168	3	5	13	65	atomic polarizability	$[(0 \mid 3,2,0^5)]$	72	$(0 \mid -2,3,0,0,0,0,0,0)$	1
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# (orbit)	$\|z\|_\infty$	$\|z\|_1$	$\|z\|_2^2$	Orthant	Physical quantity	$\mathrm{Orb}(z)$	$G(\mathrm{Orb}(z))$	z	soc (z)
168	3	5	13	65	absorbed dose rate	$[(0 \mid 3, 2, 0^5)]$	72	$(0 \mid -3, 2, 0, 0, 0, 0, 0, 0)$	-1
168	3	5	13	65	dose equivalent rate	$[(0 \mid 3, 2, 0^5)]$	72	$(0 \mid -3, 2, 0, 0, 0, 0, 0, 0)$	-1
168	3	5	13	65	specific power, power loss per unit of mass	$[(0 \mid 3, 2, 0^5)]$	72	$(0 \mid -3, 2, 0, 0, 0, 0, 0, 0)$	-1
168	4	5	17	65	jounce, snap	$[(0 \mid 4, 1, 0^5)]$	48	$(0 \mid -4, 1, 0, 0, 0, 0, 0)$	-3
168	5	6	26	65	crackle	$[(0 \mid 5, 1, 0^5)]$	96	$(0 \mid -5, 1, 0, 0, 0, 0, 0, 0)$	-4
168	6	7	37	65	pop	$[(0 \mid 6, 1, 0^5)]$	192	$(0 \mid -6, 1, 0, 0, 0, 0, 0, 0)$	-5
280	1	3	3	1	electric dipole moment	$[(0 \mid 1^3, 0^4)]$	30	$(0 \mid 1, 1, 0, 1, 0, 0, 0)$	3
280	1	3	3	3	Faraday constant	$[(0 \mid 1^3, 0^4)]$	30	$(0 \mid 1, 0, 0, 1, 0, -1, 0)$	1
280	1	3	3	65	linear momentum	$[(0 \mid 1^3, 0^4)]$	30	$(0 \mid -1, 1, 1, 0, 0, 0, 0)$	1
280	1	3	3	65	potential momentum	$[(0 \mid 1^3, 0^4)]$	30	$(0 \mid -1, 1, 1, 0, 0, 0, 0)$	1
280	1	3	3	65	linear impulse	$[(0 \mid 1^3, 0^4)]$	30	$(0 \mid -1, 1, 1, 0, 0, 0, 0)$	1
280	1	3	3	17	fluidity	$[(0 \mid 1^3, 0^4)]$	30	$(0 \mid 1, 1, -1, 0, 0, 0, 0)$	1
280	1	3	3	17	gyromagnetic ratio	$[(0 \mid 1^3, 0^4)]$	30	$(0 \mid 1, 0, -1, 1, 0, 0, 0)$	1
280	1	3	3	73	mass to charge ratio	$[(0 \mid 1^3, 0^4)]$	30	$(0 \mid -1, 0, 1, -1, 0, 0, 0)$	-1
280	1	3	3	17	exposure to x- and γ rays	$[(0 \mid 1^3, 0^4)]$	30	$(0 \mid 1, 0, -1, 1, 0, 0, 0)$	1
280	1	3	3	97	dynamic viscosity	$[(0 \mid 1^3, 0^4)]$	30	$(0 \mid -1, -1, 1, 0, 0, 0, 0)$	-1
280	1	3	3	97	angular momentum density	$[(0 \mid 1^3, 0^4)]$	30	$(0 \mid -1, -1, 1, 0, 0, 0, 0)$	-1
280	1	3	3	97	Lagrangian density in (x,y,z)	$[(0 \mid 1^3, 0^4)]$	30	$(0 \mid -1, -1, 1, 0, 0, 0, 0)$	-1
280	1	3	3	33	linear charge density	$[(0 \mid 1^3, 0^4)]$	30	$(0 \mid 1, -1, 0, 1, 0, 0, 0)$	1
560	1	4	4	49	transport diffusion coefficient	$[(0 \mid 1^4, 0^3)]$	210	$(0 \mid 1, -1, -1, 0, 0, 1, 0)$	0
840	2	4	6	73	magnetic flux density	$[(1 \mid 2, 1^2, 0^4)]$	-60	$(1 \mid -2, 0, 1, -1, 0, 0, 0)$	-2
840	2	4	6	73	magnetic induction	$[(1 \mid 2, 1^2, 0^4)]$	-60	$(1 \mid -2, 0, 1, -1, 0, 0, 0)$	-2
840	2	4	6	73	magnetic polarization	$[(1 \mid 2, 1^2, 0^4)]$	-60	$(1 \mid -2, 0, 1, -1, 0, 0, 0)$	-2
840	2	4	6	97	stress energy momentum pseudotensor	$[(1 \mid 2, 1^2, 0^4)]$	-60	$(1 \mid -2, -1, 1, 0, 0, 0, 0)$	-2
840	2	4	6	65	angular momentum	$[(1 \mid 2, 1^2, 0^4)]$	-60	$(1 \mid -1, 2, 1, 0, 0, 0, 0)$	2
840	2	4	6	65	moment of momentum	$[(1 \mid 2, 1^2, 0^4)]$	-60	$(1 \mid -1, 2, 1, 0, 0, 0, 0)$	2
840	2	4	6	65	Planck constant	$[(1 \mid 2, 1^2, 0^4)]$	60	$(1 \mid -1, 2, 1, 0, 0, 0, 0)$	2
840	2	4	6	65	action	$[(1 \mid 2, 1^2, 0^4)]$	60	$(1 \mid -1, 2, 1, 0, 0, 0, 0)$	2
840	2	4	6	65	spin	$[(1 \mid 2, 1^2, 0^4)]$	60	$(1 \mid -1, 2, 1, 0, 0, 0, 0)$	2
840	2	4	6	97	acoustic impedance	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -1, -2, 1, 0, 0, 0, 0)$	-2
840	2	4	6	97	mass flux	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -1, -2, 1, 0, 0, 0, 0)$	-2
840	2	4	6	97	linear impulse density	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -1, -2, 1, 0, 0, 0, 0)$	-2
840	2	4	6	1	electrical quadrupole moment	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid 1, 2, 0, 1, 0, 0, 0)$	4
840	2	4	6	33	surface charge density	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid 1, -2, 0, 1, 0, 0, 0)$	0
840	2	4	6	33	dielectric polarization	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid 1, -2, 0, 1, 0, 0, 0)$	0
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# (orbit)	$\|z\|_\infty$	$\|z\|_1$	$\|z\|_2^2$	Orthant	Physical quantity	Orb (z)	$G(\mathrm{Orb}\,(z))$	z	soc (z)
840	2	4	6	33	electric displacement field	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid 1, -2, 0, 1, 0, 0, 0)$	0
840	2	4	6	33	electric flux density	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid 1, -2, 0, 1, 0, 0, 0)$	0
840	2	4	6	33	polarization density	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid 1, -2, 0, 1, 0, 0, 0)$	0
840	2	4	6	33	luminous exposure	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid 1, -2, 0, 0, 0, 0, 1)$	0
840	2	4	6	65	centrifugal force	$[(1 \mid 2, 1^2, 0^4)]$	-60	$(1 \mid -2, 1, 1, 0, 0, 0, 0)$	0
840	2	4	6	65	Coriolis force	$[(1 \mid 2, 1^2, 0^4)]$	-60	$(1 \mid -2, 1, 1, 0, 0, 0, 0)$	0
840	2	4	6	65	Euler force	$[(1 \mid 2, 1^2, 0^4)]$	-60	$(1 \mid -2, 1, 1, 0, 0, 0, 0)$	0
840	2	4	6	65	force	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, 1, 1, 0, 0, 0, 0)$	0
840	2	4	6	65	drag	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, 1, 1, 0, 0, 0, 0)$	0
840	2	4	6	65	lift	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, 1, 1, 0, 0, 0, 0)$	0
840	2	4	6	65	aerodynamic force	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, 1, 1, 0, 0, 0, 0)$	0
840	2	4	6	65	hydrodynamic force	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, 1, 1, 0, 0, 0, 0)$	0
840	2	4	6	65	energy line density	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, 1, 1, 0, 0, 0, 0)$	0
840	2	4	6	65	weight	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, 1, 1, 0, 0, 0, 0)$	0
840	2	4	6	65	energy deposit	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, 1, 1, 0, 0, 0, 0)$	0
840	2	4	6	65	Einstein gravitational constant	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid 2, -1, -1, 0, 0, 0, 0)$	0
840	2	4	6	17	compressibility	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid 2, 1, -1, 0, 0, 0, 0)$	2
840	2	4	6	97	energy density	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, -1, 1, 0, 0, 0, 0)$	-2
840	2	4	6	97	Lagrangian density in (x,y,z,t)	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, -1, 1, 0, 0, 0, 0)$	-2
840	2	4	6	97	Hamilton density	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, -1, 1, 0, 0, 0, 0)$	-2
840	2	4	6	97	radiant energy density	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, -1, 1, 0, 0, 0, 0)$	-2
840	2	4	6	97	spectral exposure in wavelength	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, -1, 1, 0, 0, 0, 0)$	-2
840	2	4	6	97	sound energy density	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, -1, 1, 0, 0, 0, 0)$	-2
840	2	4	6	97	toughness	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, -1, 1, 0, 0, 0, 0)$	-2
840	2	4	6	97	pressure	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, -1, 1, 0, 0, 0, 0)$	-2
840	2	4	6	97	Lamé’s first parameter	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, -1, 1, 0, 0, 0, 0)$	-2
840	2	4	6	97	Lamé’s second parameter	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, -1, 1, 0, 0, 0, 0)$	-2
840	2	4	6	97	P-wave modulus	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, -1, 1, 0, 0, 0, 0)$	-2
840	2	4	6	97	tensile stress	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, -1, 1, 0, 0, 0, 0)$	-2
840	2	4	6	97	sound pressure	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, -1, 1, 0, 0, 0, 0)$	-2
840	2	4	6	97	modulus of elasticity	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, -1, 1, 0, 0, 0)$	-2
840	2	4	6	97	Young’s modulus	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, -1, 1, 0, 0, 0, 0)$	-2
840	2	4	6	97	shear modulus, modulus of rigidity	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, -1, 1, 0, 0, 0, 0)$	-2
840	2	4	6	97	bulk modulus	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, -1, 1, 0, 0, 0, 0)$	-2
840	2	4	6	97	compression modulus	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, -1, 1, 0, 0, 0, 0)$	-2
…	…	…	…	…	…	…	…	…	…

# (orbit)	$\|z\|_\infty$	$\|z\|_1$	$\|z\|_2^2$	Orthant	Physical quantity	Orb (z)	$G(\text{Orb}\,(z))$	z	soc (z)
840	2	4	6	97	normal stress	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, -1, 1, 0, 0, 0, 0)$	-2
840	2	4	6	97	shear stress	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, -1, 1, 0, 0, 0, 0)$	-2
840	2	4	6	97	energy momentum tensor	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, -1, 1, 0, 0, 0, 0)$	-2
840	2	5	9	65	torque	$[(1 \mid 2^2, 1, 0^4)]$	-180	$(1 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	moment of force	$[(1 \mid 2^2, 1, 0^4)]$	-180	$(1 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	energy	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	total energy	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	rest energy	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	elastic energy	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	electrical energy	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	electrostatic energy	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	internal energy	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	radiant energy, electromagnetic energy	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	spectral flux in frequency	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	spectral intensity in frequency	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	potential energy	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	gravitational potential energy	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	kinetic energy	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	work	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	thermodynamic work	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	enthalpy	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	chemical potential	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	Gibbs free energy	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	Fermi energy	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	availability	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	exergy	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	Lagrange function	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	Hamilton function	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	Hartree energy	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	ionization energy	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	electron affinity	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	electronegativity	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	dissociation energy	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	quadrupole interaction energy tensor	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	65	level width	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
…	…	…	…	…	…	…	…	…	…

# (orbit)	$\|z\|_\infty$	$\|z\|_1$	$\|z\|_2^2$	Orthant	Physical quantity	Orb (z)	$G(\text{Orb}(z))$	z	soc (z)
840	2	5	9	65	disintegration energy	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
840	2	5	9	97	body force density	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, -2, 1, 0, 0, 0, 0)$	-3
840	2	5	9	69	specific heat capacity	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 0, 0, -1, 0, 0)$	-1
840	2	5	9	69	specific entropy	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 0, 0, -1, 0, 0)$	-1
840	3	5	11	33	electric charge density	$[(0 \mid 3, 1^2, 0^4)]$	120	$(0 \mid 1, -3, 0, 1, 0, 0, 0)$	-1
840	3	5	11	33	volume charge density	$[(0 \mid 3, 1^2, 0^4)]$	120	$(0 \mid 1, -3, 0, 1, 0, 0, 0)$	-1
840	3	5	11	33	divergence of the electric displacement	$[(0 \mid 3, 1^2, 0^4)]$	120	$(0 \mid 1, -3, 0, 1, 0, 0, 0)$	-1
840	3	5	11	97	time derivative of mass density	$[(0 \mid 3, 1^2, 0^4)]$	120	$(0 \mid -1, -3, 1, 0, 0, 0, 0)$	-3
840	3	5	11	97	catalytic activity concentration	$[(0 \mid 3, 1^2, 0^4)]$	120	$(0 \mid -1, -3, 0, 0, 0, 1, 0)$	-3
840	3	5	11	97	reaction rate	$[(0 \mid 3, 1^2, 0^4)]$	120	$(0 \mid -1, -3, 0, 0, 0, 1, 0)$	-3
840	3	5	11	67	catalytic efficiency	$[(0 \mid 3, 1^2, 0^4)]$	120	$(0 \mid -1, 3, 0, 0, 0, -1, 0)$	1
840	3	5	11	33	luminous energy density	$[(0 \mid 3, 1^2, 0^4)]$	120	$(0 \mid 1, -3, 0, 0, 0, 0, 1)$	-1
840	3	5	11	65	yank	$[(0 \mid 3, 1^2, 0^4)]$	120	$(0 \mid -3, 1, 1, 0, 0, 0, 0)$	-1
840	3	5	11	65	spectral power	$[(0 \mid 3, 1^2, 0^4)]$	120	$(0 \mid -3, 1, 1, 0, 0, 0, 0)$	-1
840	3	5	11	65	spectral intensity in wavelength	$[(0 \mid 3, 1^2, 0^4)]$	120	$(0 \mid -3, 1, 1, 0, 0, 0, 0)$	-1
840	3	5	11	65	spectral flux in wavelength	$[(0 \mid 3, 1^2, 0^4)]$	120	$(0 \mid -3, 1, 1, 0, 0, 0, 0)$	-1
840	3	5	11	97	spectral flux density	$[(0 \mid 3, 1^2, 0^4)]$	120	$(0 \mid -3, -1, 1, 0, 0, 0, 0)$	-3
840	3	5	11	97	spectral exitance in wavelength	$[(0 \mid 3, 1^2, 0^4)]$	120	$(0 \mid -3, -1, 1, 0, 0, 0, 0)$	-3
840	3	5	11	97	spectral radiance in wavelength	$[(0 \mid 3, 1^2, 0^4)]$	120	$(0 \mid -3, -1, 1, 0, 0, 0, 0)$	-3
840	3	5	11	97	spectral irradiance, power density	$[(0 \mid 3, 1^2, 0^4)]$	120	$(0 \mid -3, -1, 1, 0, 0, 0, 0)$	-3
840	3	5	11	97	time derivative of (mass density times specific energy)	$[(0 \mid 3, 1^2, 0^4)]$	120	$(0 \mid -3, -1, 1, 0, 0, 0, 0)$	-3
840	3	5	11	73	electric field gradient tensor	$[(0 \mid 3, 1^2, 0^4)]$	120	$(0 \mid -3, 0, 1, -1, 0, 0, 0)$	-3
840	3	5	11	73	curl of electric field	$[(1 \mid 3, 1^2, 0^4)]$	-120	$(1 \mid -3, 0, 1, -1, 0, 0, 0)$	-3
840	3	5	11	73	time derivative of magnetic induction	$[(1 \mid 3, 1^2, 0^4)]$	-120	$(1 \mid -3, 0, 1, -1, 0, 0, 0)$	-3
840	3	5	11	73	heat transfer coefficient	$[(0 \mid 3, 1^2, 0^4)]$	120	$(0 \mid -3, 0, 1, 0, -1, 0, 0)$	-3
840	3	5	11	17	thermal insulation	$[(0 \mid 3, 1^2, 0^4)]$	120	$(0 \mid 3, 0, -1, 0, 1, 0, 0)$	3
840	3	5	11	49	thermal distortion	$[(0 \mid 3, 1^2, 0^4)]$	120	$(0 \mid 3, -1, -1, 0, 0, 0, 0)$	1
840	4	10	36	49	Fermi constant	$[(0 \mid 4^2, 2, 0^4)]$	32 400	$(0 \mid 4, -4, -2, 0, 0, 0, 0)$	-2
840	6	12	54	65	3D phase space volume	$[(0 \mid 6, 3^2, 0^4)]$	216 000	$(0 \mid -3, 6, 3, 0, 0, 0, 0)$	6
1 680	3	6	14	81	Newtonian constant of gravitation	$[(0 \mid 3, 2, 1, 0^4)]$	360	$(0 \mid -2, 3, -1, 0, 0, 0, 0)$	0
1 680	3	6	14	65	radiant intensity	$[(0 \mid 3, 2, 1, 0^4)]$	360	$(0 \mid -3, 2, 1, 0, 0, 0, 0)$	0
1 680	3	6	14	65	radiant flux	$[(0 \mid 3, 2, 1, 0^4)]$	360	$(0 \mid -3, 2, 1, 0, 0, 0, 0)$	0
1 680	3	6	14	65	power	$[(0 \mid 3, 2, 1, 0^4)]$	360	$(0 \mid -3, 2, 1, 0, 0, 0, 0)$	0
1 680	3	6	14	65	electric power	$[(0 \mid 3, 2, 1, 0^4)]$	360	$(0 \mid -3, 2, 1, 0, 0, 0, 0)$	0
1 680	3	6	14	65	sound energy flux	$[(0 \mid 3, 2, 1, 0^4)]$	360	$(0 \mid -3, 2, 1, 0, 0, 0, 0)$	0
…	…	…	…	…	…	…	…	…	…

# (orbit)	$\|z\|_\infty$	$\|z\|_1$	$\|z\|_2^2$	Orthant	Physical quantity	Orb (z)	$G(\text{Orb}(z))$	z	soc (z)
1 680	3	6	14	65	sound power	$[(0 \mid 3, 2, 1, 0^4)]$	360	$(0 \mid -3, 2, 1, 0, 0, 0, 0)$	0
1 680	3	6	14	65	bolometric luminosity	$[(0 \mid 3, 2, 1, 0^4)]$	360	$(0 \mid -3, 2, 1, 0, 0, 0, 0)$	0
1 680	4	6	18	97	second order partial derivative of energy density with respect to time	$[(0 \mid 4, 1^2, 0^4)]$	240	$(0 \mid -4, -1, 1, 0, 0, 0, 0)$	-4
1 680	4	7	21	17	electric polarizability	$[(0 \mid 4, 2, 1, 0^4)]$	720	$(0 \mid 4, 0, -1, 2, 0, 0, 0)$	5
1 680	4	8	26	65	first radiation constant	$[(0 \mid 4, 3, 1, 0^4)]$	720	$(0 \mid -3, 4, 1, 0, 0, 0, 0)$	2
1 680	4	8	26	69	Stefan-Boltzmann constant	$[(0 \mid 4, 3, 1, 0^4)]$	720	$(0 \mid -3, 0, 1, 0, -4, 0, 0)$	-6
2 240	2	5	7	65	molar Planck constant	$[(0 \mid 2, 1^3, 0^3)]$	420	$(0 \mid -1, 2, 1, 0, 0, -1, 0)$	1
2 240	2	5	7	73	magnetic vector potential, magnetostatic potential	$[(0 \mid 2, 1^3, 0^3)]$	420	$(0 \mid -2, 1, 1, -1, 0, 0, 0)$	-1
2 240	2	5	7	73	magnetic rigidity	$[(0 \mid 2, 1^3, 0^3)]$	420	$(0 \mid -2, 1, 1, -1, 0, 0, 0)$	-1
2 240	2	5	7	105	divergence of magnetic induction	$[(1 \mid 2, 1^3, 0^3)]$	-420	$(1 \mid -2, -1, 1, -1, 0, 0, 0)$	-3
2 240	2	7	13	73	inductance	$[(0 \mid 2^3, 1, 0^3)]$	6 300	$(0 \mid -2, 2, 1, -2, 0, 0, 0)$	-1
2 240	2	7	13	73	self-inductance	$[(0 \mid 2^3, 1, 0^3)]$	6 300	$(0 \mid -2, 2, 1, -2, 0, 0, 0)$	-1
2 240	2	7	13	73	magnetic permeance	$[(0 \mid 2^3, 1, 0^3)]$	6 300	$(0 \mid -2, 2, 1, -2, 0, 0, 0)$	-1
2 240	2	7	13	73	mutual inductance	$[(0 \mid 2^3, 1, 0^3)]$	6 300	$(0 \mid -2, 2, 1, -2, 0, 0, 0)$	-1
2 240	2	7	13	17	magnetizability	$[(0 \mid 2^3, 1, 0^3)]$	6 300	$(0 \mid 2, 2, -1, 2, 0, 0, 0)$	5
2 240	2	7	13	49	magnetic reluctance	$[(0 \mid 2^3, 1, 0^3)]$	6 300	$(0 \mid 2, -2, -1, 2, 0, 0, 0)$	1
2 240	3	6	12	69	thermal conductivity	$[(0 \mid 3, 1^3, 0^3)]$	840	$(0 \mid -3, 1, 1, 0, -1, 0, 0)$	-2
2 240	3	6	12	73	electric field strength	$[(0 \mid 3, 1^3, 0^3)]$	840	$(0 \mid -3, 1, 1, -1, 0, 0, 0)$	-2
2 240	3	6	12	73	electrostatic field strength	$[(0 \mid 3, 1^3, 0^3)]$	840	$(0 \mid -3, 1, 1, -1, 0, 0, 0)$	-2
2 240	3	6	12	49	first hyper-susceptibility, 2nd-order scalar susceptibility	$[(0 \mid 3, 1^3, 0^3)]$	840	$(0 \mid 3, -1, -1, 1, 0, 0, 0)$	2
2 240	3	6	12	49	thermal resistivity	$[(0 \mid 3, 1^3, 0^3)]$	840	$(0 \mid 3, -1, -1, 0, 1, 0, 0)$	2
2 240	6	12	48	73	electric field strength squared	$[(0 \mid 6, 2^3, 0^3)]$	705 600	$(0 \mid -6, 2, 2, -2, 0, 0, 0)$	-4
2 240	6	12	48	49	second hyper-susceptibility, 3rd-order scalar susceptibility	$[(0 \mid 6, 2^3, 0^3)]$	705 600	$(0 \mid 6, -2, -2, 2, 0, 0, 0)$	4
2 240	9	12	48	73	electric field strength to the 3rd power	$[(0 \mid 9, 3^3, 0^3)]$	592 704 000	$(0 \mid -9, 3, 3, -3, 0, 0, 0)$	-6
2 240	12	24	192	73	electric field strength to the 4th power	$[(0 \mid (12), 4^3, 0^3)]$	497 871 360 000	$(0 \mid -12, 4, 4, -4, 0, 0, 0)$	-8
3 360	2	6	10	67	electrochemical potential	$[(0 \mid 2^2, 1^2, 0^3)]$	1 260	$(0 \mid -2, 2, 1, 0, 0, -1, 0)$	0
3 360	2	6	10	67	molar energy	$[(0 \mid 2^2, 1^2, 0^3)]$	1 260	$(0 \mid -2, 2, 1, 0, 0, -1, 0)$	0
3 360	2	6	10	67	activation energy	$[(0 \mid 2^2, 1^2, 0^3)]$	1 260	$(0 \mid -2, 2, 1, 0, 0, -1, 0)$	0
3 360	2	6	10	69	entropy	$[(0 \mid 2^2, 1^2, 0^3)]$	1 260	$(0 \mid -2, 2, 1, 0, -1, 0, 0)$	0
3 360	2	6	10	69	heat capacity	$[(0 \mid 2^2, 1^2, 0^3)]$	1 260	$(0 \mid -2, 2, 1, 0, -1, 0, 0)$	0
3 360	2	6	10	69	Boltzmann constant	$[(0 \mid 2^2, 1^2, 0^3)]$	1 260	$(0 \mid -2, 2, 1, 0, -1, 0, 0)$	0
3 360	2	6	10	73	magnetic flux quantum	$[(0 \mid 2^2, 1^2, 0^3)]$	1 260	$(0 \mid -2, 2, 1, -1, 0, 0, 0)$	0
3 360	2	6	10	73	magnetic flux	$[(0 \mid 2^2, 1^2, 0^3)]$	1 260	$(0 \mid -2, 2, 1, -1, 0, 0, 0)$	0
3 360	2	6	10	73	magnetic constant	$[(0 \mid 2^2, 1^2, 0^3)]$	1 260	$(0 \mid -2, 1, 1, -2, 0, 0, 0)$	-2
3 360	2	6	10	73	magnetic permeability	$[(0 \mid 2^2, 1^2, 0^3)]$	1 260	$(0 \mid -2, 1, 1, -2, 0, 0, 0)$	-2
…	…	…	…	…	…	…	…	…	…

$\#\,(\mathrm{orbit})$	$\lVert z\rVert_\infty$	$\lVert z\rVert_1$	$\lVert z\rVert_2^2$	Orthant	Physical quantity	$\mathrm{Orb}\,(z)$	$G(\mathrm{Orb}\,(z))$	z	$\mathrm{soc}\,(z)$
3 360	2	6	10	49	magnetic susceptibility	$[(0 \mid 2^2, 1^2, 0^3)]$	1 260	$(0 \mid -2, -1, -1, 2, 0, 0, 0)$	-2
3 360	2	6	10	49	Josephson constant	$[(0 \mid 2^2, 1^2, 0^3)]$	1 260	$(0 \mid 2, -2, -1, 1, 0, 0, 0)$	0
3 360	3	8	20	73	specific resistance	$[(0 \mid 3^2, 1^2, 0^3)]$	7 560	$(0 \mid -3, 3, 1, -1, 0, 0, 0)$	0
6 720	2	7	11	71	molar heat capacity, molar entropy	$[(0 \mid 2^2, 1^3, 0^2)]$	13 860	$(0 \mid -2, 2, 1, 0, -1, -1, 0)$	-1
6 720	2	7	11	71	molar gas constant	$[(0 \mid 2^2, 1^3, 0^2)]$	13 860	$(0 \mid -2, 2, 1, 0, -1, -1, 0)$	-1
6 720	2	7	11	71	molar entropy	$[(0 \mid 2^2, 1^3, 0^2)]$	13 860	$(0 \mid -2, 2, 1, 0, -1, -1, 0)$	-1
6 720	3	7	15	73	electric potential difference	$[(0 \mid 3, 2, 1^2, 0^3)]$	2 520	$(0 \mid -3, 2, 1, -1, 0, 0, 0)$	-1
6 720	3	7	15	73	electric potential	$[(0 \mid 3, 2, 1^2, 0^3)]$	2 520	$(0 \mid -3, 2, 1, -1, 0, 0, 0)$	-1
6 720	3	7	15	73	dipole potential	$[(0 \mid 3, 2, 1^2, 0^3)]$	2 520	$(0 \mid -3, 2, 1, -1, 0, 0, 0)$	-1
6 720	3	7	15	73	voltage	$[(0 \mid 3, 2, 1^2, 0^3)]$	2 520	$(0 \mid -3, 2, 1, -1, 0, 0, 0)$	-1
6 720	3	7	15	73	electromotive force	$[(0 \mid 3, 2, 1^2, 0^3)]$	2 520	$(0 \mid -3, 2, 1, -1, 0, 0, 0)$	-1
6 720	3	7	15	49	responsivity	$[(0 \mid 3, 2, 1^2, 0^3)]$	2 520	$(0 \mid 3, -2, -1, 1, 0, 0, 0)$	1
6 720	3	7	15	73	thermal conductance	$[(0 \mid 3, 2, 1^2, 0^3)]$	2 520	$(0 \mid -3, 2, 1, 0, -1, 0, 0)$	-1
6 720	3	7	15	49	thermal resistance	$[(0 \mid 3, 2, 1^2, 0^3)]$	2 520	$(0 \mid 3, -2, -1, 0, 1, 0, 0)$	1
6 720	3	7	15	49	luminous efficacy	$[(0 \mid 3, 2, 1^2, 0^3)]$	2 520	$(0 \mid 3, -2, -1, 0, 0, 0, 1)$	1
6 720	3	7	15	19	molar conductivity	$[(0 \mid 3, 2, 1^2, 0^3)]$	2 520	$(0 \mid 3, 0, -1, 2, 0, -1, 0)$	3
6 720	3	8	18	73	electrical resistance	$[(0 \mid 3, 2^2, 1, 0^3)]$	12 600	$(0 \mid -3, 2, 1, -2, 0, 0, 0)$	-2
6 720	3	8	18	73	reactance	$[(0 \mid 3, 2^2, 1, 0^3)]$	12 600	$(0 \mid -3, 2, 1, -2, 0, 0, 0)$	-2
6 720	3	8	18	73	impedance	$[(0 \mid 3, 2^2, 1, 0^3)]$	12 600	$(0 \mid -3, 2, 1, -2, 0, 0, 0)$	-2
6 720	3	8	18	73	characteristic impedance of vacuum	$[(0 \mid 3, 2^2, 1, 0^3)]$	12 600	$(0 \mid -3, 2, 1, -2, 0, 0, 0)$	-2
6 720	3	8	18	73	von Klitzing constant	$[(0 \mid 3, 2^2, 1, 0^3)]$	12 600	$(0 \mid -3, 2, 1, -2, 0, 0, 0)$	-2
6 720	3	8	18	49	electrical conductance	$[(0 \mid 3, 2^2, 1, 0^3)]$	12 600	$(0 \mid 3, -2, -1, 2, 0, 0, 0)$	2
6 720	3	8	18	49	admittance	$[(0 \mid 3, 2^2, 1, 0^3)]$	12 600	$(0 \mid 3, -2, -1, 2, 0, 0, 0)$	2
6 720	3	8	18	49	susceptance	$[(0 \mid 3, 2^2, 1, 0^3)]$	12 600	$(0 \mid 3, -2, -1, 2, 0, 0, 0)$	2
6 720	3	9	23	73	electrical resistivity	$[(0 \mid 3^2, 2, 1, 0^3)]$	37 800	$(0 \mid -3, 3, 1, -2, 0, 0, 0)$	-1
6 720	3	9	23	49	electrical conductivity	$[(0 \mid 3^2, 2, 1, 0^3)]$	37 800	$(0 \mid 3, -3, -1, 2, 0, 0, 0)$	1
6 720	4	9	25	49	electric capacitance	$[(0 \mid 4, 2^2, 1, 0^3)]$	25 200	$(0 \mid 4, -2, -1, 2, 0, 0, 0)$	3
6 720	4	9	25	49	mutual capacitance	$[(0 \mid 4, 2^2, 1, 0^3)]$	25 200	$(0 \mid 4, -2, -1, 2, 0, 0, 0)$	3
6 720	4	9	25	49	electric elastance	$[(0 \mid 4, 2^2, 1, 0^3)]$	25 200	$(0 \mid -4, 2, 1, -2, 0, 0, 0)$	-3
6 720	4	9	27	73	electrical mobility	$[(0 \mid 4, 3, 1^2, 0^3)]$	15 120	$(0 \mid -4, 3, 1, -1, 0, 0, 0)$	-1
13 440	4	10	30	49	electric constant	$[(0 \mid 4, 3, 2, 1, 0^3)]$	75 600	$(0 \mid 4, -3, -1, 2, 0, 0, 0)$	2
13 440	4	10	30	49	permittivity	$[(0 \mid 4, 3, 2, 1, 0^3)]$	75 600	$(0 \mid 4, -3, -1, 2, 0, 0, 0)$	2
13 440	7	13	63	49	first hyper-polarizability	$[(0 \mid 7, 3, 2, 1, 0^3)]$	604 800	$(0 \mid 7, -1, -2, 3, 0, 0, 0)$	7
13 440	10	19	129	49	second hyper-polarizability	$[(0 \mid (10), 4, 3, 2, 0^3)]$	508 032 000	$(0 \mid 10, -2, -3, 4, 0, 0, 0)$	9

APPENDIX B

Integer lattice dimensions $N = 1, 2, 3, 4, 5, 6, 7$

In this chapter we study the lattice $\mathbb{Z}^N$ where the dimension N complies with $1 \leq N \leq 7$. The mathematical properties of these integer lattices are at the basis for orbits of the Cartesian products $\{0, 1\} \times \mathbb{Z}^N$. For real world problems, engineers use the SI system given in Appendix Q in which the number of base quantities is $N = 7$. Most theoretical physicists use a time, length, and mass base quantity system in which $N = 3$ but there is not really a consensus as shown in Duff et al. (2002). Problems with electrostatic and electromagnetic units occurring in $N = 3$ were solved by adding the electric current as a fourth base quantity. A consensus exists among scientists for the case $N = 4$.

B.1 Dimension $N = 1$

We are free to select one basis lattice points of $\mathbb{Z}$ and define:

$$\boldsymbol{e}_1 \doteq \mathrm{dex}\,([\mathsf{T}]_\sim) = (1) = (1)\,,$$

with $\boldsymbol{e}_1 \in \mathbb{Z}$. A lattice can be created based on the relational operator $\leq$.

B.2 Dimension $N = 2$

We are free to select two basis lattice points of $\mathbb{Z}^2$ and define:

$$\boldsymbol{e}_1 \doteq \mathrm{dex}\,([\mathsf{T}]_\sim) = (1, 0)\,,$$
$$\boldsymbol{e}_2 \doteq \mathrm{dex}\,([\mathsf{L}]_\sim) = (0, 1)\,,$$

with $\boldsymbol{e}_i \in \mathbb{Z}^2$. In this system a kind of quantity Q has the dimension:

B.2.1 The structure constants of the integer lattice $\mathbb{Z}^2$

The integer lattice $\mathbb{Z}^2$, as an incidence structure, has the structure constants given in Table B.1 and denoted as $b^{(2)}_{kl}$.

Table B.1: Structure constants for integer lattice $\mathbb{Z}^2$: $b_{kl}^{(2)}$.

$b_{kl}^{(2)}$	$l = 0$	1	2
$k = 0$	1	4	4
1	2	1	2
2	4	4	1

B.2.2 Enumeration of the orbit cardinalities in $\mathbb{Z}^2$

The cardinalities of the orbits of $\mathbb{Z}^2$ are 1, 4, 8.

B.2.3 Enumeration of ternary quantity equations in $\mathbb{Z}^2$

We are interested in the additive relations between the orbit of the integer lattice $\boldsymbol{x}$, denoted $\mathrm{Orb}\,(\boldsymbol{x})$, the orbit of the integer lattice $\boldsymbol{y}$, denoted $\mathrm{Orb}\,(\boldsymbol{x})$, and the orbit of the integer lattice $\boldsymbol{z}$, denoted $\mathrm{Orb}\,(\boldsymbol{z})$. Each of these orbits has a cardinality. We investigate the addition of the cardinality of the orbits of $\boldsymbol{x}$ and $\boldsymbol{y}$ and compare it with the cardinality of the orbit of $\boldsymbol{z}$. If the cardinalities add-up then we expect that an additive partioning in two parts exists. This could reveal some unknown structure between the orbit representatives.

A classical example in the enumeration methods is the case of two dices where the number of ways have to be calculated to obtain the sum of the eyes of the dices. A generating function is used to mathematically model the throwing of the dice. The generating function for one die is $f(x) = x^1 + x^2 + x^3 + x^4 + x^5 + x^6$ and for two dice we have to square $f(x)^2$ and read the coefficients of the term with exponent equal to the sum of the eyes of the dice to obtain the number of ways to obtain that sum. We find $f(x)^2 = x^12 + 2x^11 + 3x^10 + 4x^9 + 5x^8 + 6x^7 + 5x^6 + 4x^5 + 3x^4 + 2x^3 + x^2$. The number of ways to obtain the sum = 9 is 4. We will use this method to find the number of ways to have $\#\,(\mathrm{Orb}\,(\boldsymbol{x})) + \#\,(\mathrm{Orb}\,(\boldsymbol{y}))$. We create a generating function represented by the polynomial $g(z; 2)$ in the indeterminate z for which the exponents take the values of the cardinalities of the orbits of $\mathbb{Z}^2$:

$$g(z; 2) = z^8 + z^4 + z\,.$$

We expand the generating polynomial $g(z; 2)^2$ in its terms and find:

$$g(z; 2)^2 = z^{16} + 2z^{12} + 2z^9 + z^8 + 2z^5 + z^2\,.$$

Observe that by adding cardinalities we could generate cardinalities that are *not* belonging to the set of cardinalities of $\mathbb{Z}^2$, being $\{1, 4, 8\}$. The *closure condition* of addition of cardinalities is applicable and thus the polynomial is restricted, denoted $g|_M(z; 2)^2$, in which $M = \{z^8, z^4, z\}$. The *restricted* generating polynomial is:

$$g|_M(z; 2)^2 = z^8\,,$$

because only the term z^8 has an exponent being an element of the set $\{1, 4, 8\}$.

The coefficient one in front of z^8 indicates the occurrence of degenerated relations between quantities and this occurs when the sum of the cardinalities is 8.

The green cell in Figure B.1 shows the combinations of the cardinalities of the kinds of quantities $\boldsymbol{x}$ and $\boldsymbol{y}$ resulting in a unique ternary kind of quantity equation xy of $\mathbb{Z}^2$.

	1	4	8
1	2	5	9
4	5	8	12
8	9	12	16

Figure B.1: Ternary relations between orbits of unequal cardinalities in $\mathbb{Z}^2$.

We can summarize the above results about the multiplicities m of the ternary relations by defining a generating function $v(z; 2)$ as:

$$v(z; 2) = z \,, \tag{B.1}$$

in which the coefficient in the generating polynomial $v(z; 2)$ yields the number of sums of cardinalities in ternary equations of multiplicity m. The multiplicity is given by the exponent of the indeterminate z. There is *one* distinct sum of cardinalities for ternary equations $f(\pi) = \dfrac{z}{xy}$ in $\mathbb{Z}^2$.

B.2.4 Elements of physics in $\mathbb{Z}^2$

We classify the orbits of $\mathbb{Z}^2$ in a table in which the rows are representing the infinity norm $\ell_\infty = s$ and the columns are the elements of the finite set of the cardinalities $\#([w])$ of the orbits. Thus, each orbit $[w]$ of $\mathbb{Z}^2$ is mapped to a cell in the Table B.2. We call this cell an element of physics. Each element of physics has a coordinate $(s, \#([w]))$. The value of the cell gives the number of orbits with the properties s and $\#([w])$. The number of distinct orbits with the specific properties are given in each cell of the Table B.2.

Table B.2: Elements of physics in $\mathbb{Z}^2$.

$\|z\|_\infty$	1	4	8	RowSum
0	1	0	0	1
1	0	2	0	2
2	0	2	1	3
3	0	2	2	4
4	0	2	3	5
5	0	2	4	6
6	0	2	5	7
7	0	2	6	8
8	0	2	7	9
9	0	2	8	10
10	0	2	9	11
11	0	2	10	12
12	0	2	11	13
13	0	2	12	14
14	0	2	13	15
15	0	2	14	16
16	0	2	15	17
17	0	2	16	18
18	0	2	17	19
19	0	2	18	20
20	0	2	19	21
21	0	2	20	22
22	0	2	21	23
23	0	2	22	24
24	0	2	23	25
25	0	2	24	26
26	0	2	25	27

Each number in the column RowSum is obtained by summing the row values for the corresponding cardinalities. We consider the sequence of numbers in the RowSum and identify this sequence as the integer sequence A000027 in N. J. A. Sloane (2006) in which n is substituted by the infinity norm s. The sequence A000027 represents the positive integers. Hence, the column denoted RowSum in Table B.2 is obtained using the equation $a(s) = s + 1 = \binom{s+2-1}{2-1}$. The sequences of integers in the columns of Table B.2 can be referenced to sequences in the On-line Encyclopedia of Integer Sequences (OEIS). Each sequence in the columns can be generated and extended using an ordinary generating function as given in Table B.3.

Table B.3: Table of $\mathbb{Z}^2$-physics sequences and ordinary generating functions.

$\#(\{\mathrm{Orb}(z)\})$	OEIS Reference	Ordinary generating function $G(a_n; z)$
1	A000007	Not defined
...	...	...

$\#(\{\mathrm{Orb}(z)\})$	OEIS Reference	Ordinary generating function $G(a_n;z)$
4	2 × A057427	$2z/(1-z)$
8	A001477 + 1 zero	$z^2/(z-1)^2$
RowSum	A001477 - 1 zero	$1/(z-1)^2$

From the Table B.2 we derive the backbone structure of $\mathbb{Z}^2$ and obtain four orbits representing 17 integer lattice points given in Table B.4.

Table B.4: Backbone structure of physics in $\mathbb{Z}^2$.

Cardinalities	1	4	8	Total
Orbits	1	2	1	4
Lattice points	1	8	8	17

B.3 Dimension $N = 3$

We are free to select three basis lattice points of $\mathbb{Z}^3$ and define:

$$\begin{aligned}\boldsymbol{e}_1 &\doteq \mathrm{dex}\,([\mathsf{T}]_\sim) = (1,0,0)\,,\\ \boldsymbol{e}_2 &\doteq \mathrm{dex}\,([\mathsf{L}]_\sim) = (0,1,0)\,,\\ \boldsymbol{e}_3 &\doteq \mathrm{dex}\,([\mathsf{M}]_\sim) = (0,0,1)\,,\end{aligned}$$

with $\boldsymbol{e}_i \in \mathbb{Z}^3$.

B.3.1 The structure constants of the integer lattice $\mathbb{Z}^3$

Table B.5 enumerates the structure constants $b_{kl}^{(3)}$ for the integer lattice $\mathbb{Z}^3$.

Table B.5: Structure constants $b_{kl}^{(3)}$.

$b_{kl}^{(3)}$	$l=0$	1	2	3
$k=0$	1	6	12	8
1	2	1	4	4
2	4	4	1	2
3	8	12	6	1

B.3.2 Enumeration of the orbit cardinalities in $\mathbb{Z}^3$

The cardinalities of the orbits $\#([w])$ of $\mathbb{Z}^3$ are 1, 6, 8, 12, 24, 48. A classification of the non-trivial orbits is given in Figure B.2 for lattice points with an infinity norm $s = 3$. The color codes are:

- Aquamarine: $\#([w]) = 6$;
- Black: $\#([w]) = 8$;
- Blue: $\#([w]) = 12$;
- Brown: $\#([w]) = 24$;
- Coral: $\#([w]) = 24$;
- Cyan: $\#([w]) = 24$;
- Gold: $\#([w]) = 24$;
- Grey: $\#([w]) = 24$;
- Green: $\#([w]) = 24$; and,
- Maroon: $\#([w]) = 48$.

Observe that $\#([w]) = 24$ has a multiplicity of six and thus six independent orbits exist with the same cardinality. The orbits with the same cardinality cannot be mapped in each other through a signed permutation. This is the reason to use the terminology *independent*.

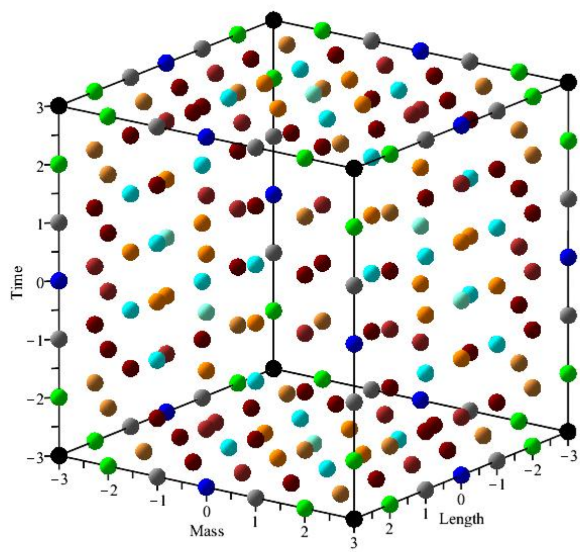

Figure B.2: Classification of orbit cardinalities in $\mathbb{Z}^3$ for $s = 3$.

B.3.3 Enumeration of ternary quantity equations in $\mathbb{Z}^3$

We create a generating polynomial $g(z;3)$ in the indeterminate z for which the exponents take the values of the cardinalities of the orbits of $\mathbb{Z}^3$:

$$g(z;3) = z^{48} + z^{24} + z^{12} + z^8 + z^6 + z\,.$$

We are interested in the ternary relations between the quantities. We therefore expand the generating polynomial $g(z;3)^2$ in its terms and find:

$$\begin{aligned} &g(z;3)^2 = \\ &z^{96} + 2z^{72} + 2z^{60} + 2z^{56} + 2z^{54} + 2z^{49} + z^{48} + 2z^{36} + 2z^{32} \\ &+ 2z^{30} + 2z^{25} + z^{24} + 2z^{20} + 2z^{18} + z^{16} + 2z^{14} \\ &+ 2z^{13} + z^{12} + 2z^9 + 2z^7 + z^2\,. \end{aligned}$$

Observe that by adding cardinalities we could generate cardinalities that are not belonging to the set of cardinalities of $\mathbb{Z}^3$. Hence, the closure condition of addition of cardinalities is applicable and thus the polynomial is restricted, denoted $g|_M(z;3)^2$, in which $M = \{z^{48}, z^{24}, z^{12}, z^8, z^6, z\}$. The restricted generating polynomial is:

$$g|_M(z;3)^2 = z^{48} + z^{24} + z^{12}\,.$$

The green cells in Figure B.3 show the sum of the cardinalities of the orbit $\mathrm{Orb}\,(\boldsymbol{x})$ and the cardinality of the orbit $\mathrm{Orb}\,(\boldsymbol{y})$ reulting a cardinality of the orbit $\mathrm{Orb}\,(\boldsymbol{z})$. Only the values of the set $\{12, 24, 48\}$ are valid cadinalities of the integer lattice $\mathbb{Z}^3$.

	1	6	8	12	24	48
1	2	7	9	13	25	49
6	7	**12**	14	18	30	54
8	9	14	16	20	32	56
12	13	18	20	**24**	36	60
24	25	30	32	36	**48**	72
48	49	54	56	60	72	96

Figure B.3: Valid sums of orbit cardinalities in $\mathbb{Z}^3$.

The coefficient one occurs for the sum of the cardinalities being 12, 24, 48. We expect the occurrence of degenerated relations between quantities when one of these three numbers occur. We can summarize the above results reflecting the multiplicities m of the ternary relations by defining a generating polynomial $v(z;3)$ as:

$$v(z;3) = 3z\,, \tag{B.2}$$

in which the coefficient in the generating polynomial $v(z;3)$ gives the number of sums of cardinalities in ternary equations of multiplicity m. The multiplicity is given by the exponent of the indeterminate z. Observe that there are three distinct sums of cardinalities for ternary dimensionless quantity equations $f(\pi) = \dfrac{z}{xy}$ in $\mathbb{Z}^3$.

B.3.4 Elements of physics in $\mathbb{Z}^3$

Consider a table in which the rows are representing the infinity norm $\ell_\infty = s$ and the columns are the cardinalities $\#([w])$ of the orbits of $\mathbb{Z}^3$. We write in each cell of the Table B.6 the number of orbits that have the cell coordinates $(s, \#([w]))$ and call this the mass m of the element. We call this cell an element of physics in the $\mathbb{Z}^3$ framework.

Hence, each element of physics is a set of orbits of $\mathbb{Z}^3$ and each orbit is a set of integer lattice points of $\mathbb{Z}^3$. Thus an element of physics is a set of sets of integer lattice points of $\mathbb{Z}^3$. These masses are given in each cell of the Table B.6. We have limited the number of rows in the Table B.6 to infinity norms $s \leq 26$ to encompass the 26 spacetime dimensions of bosonic string theory. A space-time volume element in that theory would be represented by the integer lattice point $(1, 25, 0)$ that is an element of the orbit $[(25, 1, 0)]$. The orbit $[(25, 1, 0)]$ is an element of the set defined by the coordinates $(25, 24)$ having a mass $m = 72$ in the Table B.6 and indicating the existence of $\dbinom{72}{2}$ bijections between the orbits of cardinality $\#([w]) = 24$ for the given infinity norm $s = 25$.

Table B.6: Elements of physics in $\mathbb{Z}^3$.

$\|z\|_\infty$	1	6	8	12	24	48	RowSum
0	1	0	0	0	0	0	1
1	0	1	1	1	0	0	3
2	0	1	1	1	3	0	6
3	0	1	1	1	6	1	10
4	0	1	1	1	9	3	15
5	0	1	1	1	12	6	21
6	0	1	1	1	15	10	28
7	0	1	1	1	18	15	36
8	0	1	1	1	21	21	45
9	0	1	1	1	24	28	55
10	0	1	1	1	27	36	66
11	0	1	1	1	30	45	78
12	0	1	1	1	33	55	91
13	0	1	1	1	36	66	105
14	0	1	1	1	39	78	120
15	0	1	1	1	42	91	136
16	0	1	1	1	45	105	153
17	0	1	1	1	48	120	171
18	0	1	1	1	51	136	190
19	0	1	1	1	54	153	210
20	0	1	1	1	57	171	231
21	0	1	1	1	60	190	253
22	0	1	1	1	63	210	276
23	0	1	1	1	66	231	300
24	0	1	1	1	69	253	325
25	0	1	1	1	72	276	351
26	0	1	1	1	75	300	378

Table B.6 contains elements with mass $m = 1$ indicating the existence of unique orbits in $\mathbb{Z}^3$. Uniqueness occurs for all $s \geq 1$ when the orbit cardinality is $6, 8, 12$. We observe that the orbit cardinalities which are equal to the values of the structure constants $b_{01}^{(3)}, b_{02}^{(3)}, b_{03}^{(3)}$ in the integer lattice $\mathbb{Z}^3$ have a constant mass $m = 1$. We suspect a deeper relation between the orbit cardinalities and the structure constants.

Uniqueness occurs for the orbit $[(0, 0, 0)]$ resulting in cardinality $\#([w]) = 1$ and this in a trivial way. Uniqueness occurs for all orbits of the types $n \times [(1, 0, 0)]$, $n \times [(1, 1, 1)]$, and $n \times [(1, 1, 0)]$ in which $n \in \mathbb{N}_1$ resulting in the respective cardinalities 6, 8, and 12. Uniqueness occurs for the specific orbit $[(3, 2, 1)]$ resulting in cardinality $\#([w]) = 48$.

The lattice points $(1, 0, 0)$, $(1, 1, 0)$ and $(1, 1, 1)$ are known as the successive minima (Davenport, 2005) of the lattice $\mathbb{Z}^3$. Those lattice points are not unique but the Euclidean norm of the lattice points is unique. We obtain the inequalities $0 < 1 < \sqrt{2} < \sqrt{3}$. Those results derive from Minkowski's first fundamental theorem (Davenport, 2005). We remark that the lattice points $(1, 0, 0)$, $(1, 1, 0)$ and $(1, 1, 1)$ are forming a basis for $\mathbb{Z}^3$ and it is known that the line spanned from the origin $\boldsymbol{o}$ to the lattice point being the sum of the basis vectors $\sum_{n=1}^{3} \boldsymbol{e}_n = (1,0,0) + (1,1,0) + (1,1,1) = (3,2,1)$ is *invariant* and has the complement subspace $V = \{(x_1, x_2, x_3) | x_1 + x_2 + x_3 = 0\}$. The sum of these lattice points results in the lattice point $(3, 2, 1)$ that is the representative of a unique orbit.

Remark that the masses of the cells for the cardinalities $6, 8, 12$ are given by the generating function $\dfrac{z}{1-z}$. The integer sequence is know as A000012 in the OEIS (N. J. A. Sloane, 2006).

The masses for the cardinality 24 are given by the generating function $\dfrac{3z^2}{(z-1)^2}$. The integer sequence is know as A00858 in the OEIS (N. J. A. Sloane, 2006).

The masses for the cardinality 48 are given by the generating function $\dfrac{z^3}{(1-z)^3}$. This sequence was found by Simon Plouffe in his 1992 dissertation and is known as A000217 in the OEIS (N. J. A. Sloane, 2006).

Observe that a similar sequence with different offset is found in the column RowSum in the Table B.6. Each term of the sequence A000217 (N. J. A. Sloane, 2006) in which n is substituted by $n+1$ is obtained by associating $n+1$ to the infinity norm s and taking the horizontal sums in the Table B.6. The sequence A000217 represents the binomial coefficients $\binom{n+1}{2}$ also known as the triangular numbers. Hence, the column denoted RowSum in Table B.6 is obtained using the equation $a(s) = \binom{s+3-1}{3-1}$.

The sequences of integers in the columns of Table B.6 can be referenced to sequences in the On-line Encyclopedia of Integer Sequences (OEIS). The references are: A000007 (N. J. A. Sloane, 1995a), A057427 (Bottomley, 2000), A008525 (N. J. A. Sloane, 1995b). Each sequence in the columns can be generated and extended using an ordinary generating function as given in Table B.7.

Table B.7: Table of sequences and ordinary generating functions in $\mathbb{Z}^3$.

$\#(\{\text{Orb}(\boldsymbol{z})\})$	OEIS Reference	Ordinary generating function $G(a_n; z)$
1	A000007	Not defined
6	A057427	$z/(1-z)$
8	A057247	$z/(1-z)$
12	A057247	$z/(1-z)$
24	A008585(n-1)	$3z^2/(1-z)^2$
48	A000217(n-2)	$z^3/(1-z)^3$
RowSum	A000217(n+1)	$1/(1-z)^3$

From the Table B.6 we derive the backbone structure of $\mathbb{Z}^3$ and obtain 8 distinct sets representing 147 integer lattice points given in Table B.8.

Table B.8: Backbone structure of physics in $\mathbb{Z}^3$.

Cardinalities	1	6	8	12	24	48
Number of orbits	1	1	1	1	3	1
Number of lattice points	1	6	8	12	72	48

The backbone of $\mathbb{Z}^3$ is the path formed by the following integer lattice points, ordered in increasing Gödel number: $(0|0,0,0)$, $(0|1,0,0)$, $(0|1,1,0)$, $(0|2,1,0)$, $(0|1,1,1)$, $(0|2,1,1)$, $(0|2,2,1)$, and $(0|3,2,1)$.

B.4 Dimension $N = 4$

We are free to select four basis lattice points of $\mathbb{Z}^4$ and define:

$$\begin{aligned} \boldsymbol{e}_1 &\doteq \operatorname{dex}([\mathsf{T}]_\sim) = (1,0,0,0) = (1,0^3)\,, \\ \boldsymbol{e}_2 &\doteq \operatorname{dex}([\mathsf{L}]_\sim) = (0,1,0,0) = (0,1,0^2)\,, \\ \boldsymbol{e}_3 &\doteq \operatorname{dex}([\mathsf{M}]_\sim) = (0,0,1,0) = (0^2,1,0)\,, \\ \boldsymbol{e}_4 &\doteq \operatorname{dex}([\mathsf{I}]_\sim) = (0,0,0,1) = (0^3,1)\,, \end{aligned}$$

with $\boldsymbol{e}_i \in \mathbb{Z}^4$.

B.4.1 The structure constants of the integer lattice $\mathbb{Z}^4$

Table B.9 enumerates the structure constants $b_{kl}^{(4)}$ for the integer lattice $\mathbb{Z}^4$.

Table B.9: Structure constants $b_{kl}^{(4)}$.

$b_{kl}^{(4)}$	$l = 0$	1	2	3	4
$k = 0$	1	8	24	32	16
1	2	1	6	12	8
2	4	4	1	4	4
3	8	12	6	1	2
4	16	32	24	8	1

B.4.2 Enumeration of the orbit cardinalities in $\mathbb{Z}^4$

The cardinalities of the orbits of $\mathbb{Z}^4$ are 1, 8, 16, 24, 32, 48, 64, 96, 192, 384.

B.4.3 Enumeration of ternary quantity equations in $\mathbb{Z}^4$

We create a generating polynomial $g(z;4)$ in the indeterminate z for which the exponents take the values of the cardinalities of the orbits of $\mathbb{Z}^4$:

$$g(z;4) = \\ z^{384} + z^{192} + z^{96} + z^{64} + z^{48} + z^{32} + z^{24} + z^{16} + z^{8} + z\,.$$

We are interested in the ternary relations between the quantities. We therefore expand the generating polynomial $g(z;4)^2$ in its terms and find:

$$\begin{aligned} &g(z;4)^2 = \\ &z^{768} + 2z^{576} + 2z^{480} + 2z^{448} + 2z^{432} + 2z^{416} + 2z^{408} + 2z^{400} \\ &+ 2z^{392} + 2z^{385} + z^{384} + 2z^{288} + 2z^{256} + 2z^{240} + 2z^{224} + 2z^{216} \\ &+ 2z^{208} + 2z^{200} + 2z^{193} + z^{192} + 2z^{160} + 2z^{144} + 3z^{128} \\ &+ 2z^{120} + 4z^{112} + 2z^{104} + 2z^{97} + 3z^{96} + 2z^{88} + 4z^{80} \\ &+ 4z^{72} + 2z^{65} + 3z^{64} + 4z^{56} + 2z^{49} + 3z^{48} + 4z^{40} \\ &+ 2z^{33} + 3z^{32} + 2z^{25} + 2z^{24} + 2z^{17} + z^{16} + 2z^{9} + z^{2}\,. \end{aligned}$$

Observe that by adding cardinalities we could generate cardinalities that are not belonging to the set of cardinalities of $\mathbb{Z}^4$. Hence, the closure condition of addition of cardinalities is applicable and thus the polynomial is restricted, denoted by $g|_M(z;4)^2$, in which $M = \{z^{384}, z^{192}, z^{96}, z^{64}, z^{48}, z^{32}, z^{24}, z^{16}, z^{8}, z\}$. The restricted generating polynomial is:

$$g|_M(z;4)^2 = z^{384} + z^{192} + 3z^{96} + 3z^{64} + 3z^{48} + 3z^{32} + 2z^{24} + z^{16}\,.$$

The coefficient three occurs for the sum of the cardinalities yielding 32, 48, 64, 96.

The coefficient two occurs for the sum of the cardinalities being 24 . We expect unique relations when the sum of cardinalities is equal to 24. The green cells in Figure B.4 show the combinations of the cardinalities of the kinds of quantities $\boldsymbol{x}$ and $\boldsymbol{y}$ resulting in ternary quantity equations xy of $\mathbb{Z}^4$.

	1	8	16	24	32	48	64	96	192	384
1	2	9	17	25	33	49	65	97	193	385
8	9	**16**	**24**	**32**	40	56	72	104	200	392
16	17	**24**	**32**	40	**48**	**64**	80	112	208	400
24	25	**32**	40	**48**	56	72	88	120	216	408
32	33	40	**48**	56	**64**	80	**96**	128	224	416
48	49	56	**64**	72	80	**96**	112	144	240	432
64	65	72	80	88	**96**	112	128	160	256	448
96	97	104	112	120	128	144	160	**192**	288	480
192	193	200	208	216	224	240	256	288	**384**	576
384	385	392	400	408	416	432	448	480	576	768

Figure B.4: Ternary relations between orbits of unequal cardinalities in $\mathbb{Z}^4$.

The coefficient one occurs for the sum of the cardinalities being 16, 192, 384. We expect the occurrence of degenerated relations between quantities when one of these three numbers occur. We can summarize the above results reflecting the multiplicities m of the ternary relations by defining a generating polynomial $v(z;4)$ as:

$$v(z;4) = 4z^3 + z^2 + 3z\,, \tag{B.3}$$

in which the coefficient in the generating polynomial $v(z;4)$ gives the number of sums of cardinalities in ternary equations of multiplicity m . The multiplicity is given by the exponent of the indeterminate z . Observe that there are eight distinct sums of cardinalities for ternary quantity equations $f(\pi) = \dfrac{z}{xy}$ in $\mathbb{Z}^4$.

B.4.4 Elements of physics in $\mathbb{Z}^4$

We classify the orbits of $\mathbb{Z}^4$ in Table B.10 in which the rows are representing the infinity norm $\ell_\infty = s$ and the columns are the finite set of the cardinalities $\#([w])$ of the orbits. Thus, each orbit $[w]$ of $\mathbb{Z}^4$ is mapped to a cell in Table B.10. We call this cell an element of physics. Hence, each element of physics has a coordinate $(s, \#([w]))$. We can count the number of orbits, and denote it by m and call this the mass of the element. Hence, we write in each cell with coordinates $(s, \#([w]))$ the value of the mass m of the element. These masses are given in each cell of the Table B.10.

Table B.10: Elements of physics in $\mathbb{Z}^4$.

$\|z\|_\infty$	1	8	16	24	32	48	64	96	192	384	RowSum
0	1	0	0	0	0	0	0	0	0	0	1
1	0	1	1	1	1	0	0	0	0	0	4
2	0	1	1	1	1	1	2	3	0	0	10
3	0	1	1	1	1	2	4	6	4	0	20
4	0	1	1	1	1	3	6	9	12	1	35
5	0	1	1	1	1	4	8	12	24	4	56
6	0	1	1	1	1	5	10	15	40	10	84
7	0	1	1	1	1	6	12	18	60	20	120
8	0	1	1	1	1	7	14	21	84	35	165
9	0	1	1	1	1	8	16	24	112	56	220
10	0	1	1	1	1	9	18	27	144	84	286
11	0	1	1	1	1	10	20	30	180	120	364
12	0	1	1	1	1	11	22	33	220	165	455
13	0	1	1	1	1	12	24	36	264	220	560
14	0	1	1	1	1	13	26	39	312	286	680
15	0	1	1	1	1	14	28	42	364	364	816
16	0	1	1	1	1	15	30	45	420	455	969
17	0	1	1	1	1	16	32	48	480	560	1140
18	0	1	1	1	1	17	34	51	544	680	1330
19	0	1	1	1	1	18	36	54	612	816	1540
20	0	1	1	1	1	19	38	57	684	969	1771
21	0	1	1	1	1	20	40	60	760	1140	2024
22	0	1	1	1	1	21	42	63	840	1330	2300
23	0	1	1	1	1	22	44	66	924	1540	2600
24	0	1	1	1	1	23	46	69	1012	1771	2925
25	0	1	1	1	1	24	48	72	1104	2024	3276
26	0	1	1	1	1	25	50	75	1200	2300	3654

Remark that the masses of the cells for the cardinalities $8, 16, 24$ are given by the generating function $\frac{z}{1-z}$. The integer sequence is similar to A000012 in the OEIS (N. J. A. Sloane, 2006). We observe a link between the structure constants $b_{01}^{(4)}, b_{02}^{(4)}, b_{03}^{(4)}, b_{04}^{(4)}$ and the first 4 non-trivial orbit cardinalities $8, 16, 24, 32$. The orbit cardinalities equal to the structure constants $b_{01}^{(4)}, b_{02}^{(4)}, b_{03}^{(4)}, b_{04}^{(4)}$ have a constant mass $m = 1$. The masses for the cardinality 48 are given by the generating function $\frac{1}{(z-1)^2}$. The integer sequence is know as A000027 in the OEIS (N. J. A. Sloane, 2006). The masses for the cardinality 64 are given by the generating function $\frac{2z^2}{(1-z)^2}$. The integer sequence is similar to A005843 in the OEIS (N. J. A. Sloane, 2006) but with an offset. The masses for the cardinality 96 are given by the generating function $\frac{3z^2}{(1-z)^2}$. The integer sequence is know as A00858 in the OEIS (N. J. A. Sloane, 2006). The masses for the cardinality 192 are given by the generating function $\frac{4z^3}{(1-z)^3}$. The integer sequence is know as A046092 in the

OEIS (N. J. A. Sloane, 2006). The masses for the cardinality 384 are given by the generating function $\dfrac{z^4}{(1-z)^4}$. The integer sequence is similar to A046092 in the OEIS (N. J. A. Sloane, 2006). The RowSum is given by the generating function $\dfrac{1}{(1-z)^4}$. The integer sequence is know as A046092 in the OEIS (N. J. A. Sloane, 2006). The OEIS sequence A000292 (N. J. A. Sloane, 2006) represents the binomial coefficients $\binom{n+2}{3}$. The horizontal sum in the Table B.10 generates the numbers of the sequence A000292 if n is substituted by $s+1$. The column denoted RowSum in Table B.10 can also be obtained using the equation $a(s) = \binom{s+4-1}{4-1}$. The sequences of integers in the columns of Table B.10 can be referenced to sequences in the On-line Encyclopedia of Integer Sequences (OEIS). Each sequence in the columns can be generated and extended using an ordinary generating function as given in Table B.11.

Table B.11: Table of sequences and ordinary generating functions in $\mathbb{Z}^4$.

$\#(\{\mathrm{Orb}(z)\})$	OEIS reference	Ordinary generating function $G(a_n; z)$
1	A000007	Not defined
8	A057427	$z/(1-z)$
16	A057427	$z/(1-z)$
24	A057427	$z/(1-z)$
32	A057427	$z/(1-z)$
48	A001477(n-1)	$z^2/(z-1)^2$
64	A005843(n-1)	$2z^2/(z-1)^2$
96	A008585(n-1)	$3z^2/(z-1)^2$
192	A046092(n-2)	$-4z^3/(z-1)^3$
384	A000292(n-3)	$z^4/(z-1)^4$
RowSum	A000292(n+1)	$1/(z-1)^4$

From the Table B.10 we derive the backbone structure of $\mathbb{Z}^4$ and obtain 16 orbits representing 1 697 integer lattice points given in Table B.12.

Table B.12: Backbone of physics in $\mathbb{Z}^4$.

Cardinalities	1	8	16	24	32	48	64	96	192	384
Orbits	1	1	1	1	1	1	2	3	4	1
Lattice points	1	8	16	24	32	48	128	288	768	384

B.5 Dimension $N = 5$

We are free to select five basis lattice points of $\mathbb{Z}^5$ and define:

$$\begin{aligned}
\boldsymbol{e}_1 &\doteq \mathrm{dex}\,([\mathsf{T}]_\sim) = (1,0,0,0,0) = (1,0^4)\,,\\
\boldsymbol{e}_2 &\doteq \mathrm{dex}\,([\mathsf{L}]_\sim) = (0,1,0,0,0) = (0,1,0^3)\,,\\
\boldsymbol{e}_3 &\doteq \mathrm{dex}\,([\mathsf{M}]_\sim) = (0,0,1,0,0) = (0^2,1,0^2)\,,\\
\boldsymbol{e}_4 &\doteq \mathrm{dex}\,([\mathsf{I}]_\sim) = (0,0,0,1,0) = (0^3,1,0)\,,\\
\boldsymbol{e}_5 &\doteq \mathrm{dex}\,([\Theta]_\sim) = (0,0,0,0,1) = (0^4,1)\,,
\end{aligned}$$

with $\boldsymbol{e}_i \in \mathbb{Z}^5$.

B.5.1 The structure constants of the integer lattice $\mathbb{Z}^5$

Table B.13 enumerates the structure constants $b_{kl}^{(5)}$ for the integer lattice $\mathbb{Z}^5$.

Table B.13: Structure constants $b_{kl}^{(5)}$.

$b_{kl}^{(5)}$	$l = 0$	1	2	3	4	5
$k = 0$	1	10	40	80	80	32
1	2	1	8	24	32	16
2	4	4	1	6	12	8
3	8	12	6	1	4	4
4	16	32	24	8	1	2
5	32	80	80	40	10	1

B.5.2 Enumeration of the orbit cardinalities in $\mathbb{Z}^5$

The cardinalities of the orbits of $\mathbb{Z}^5$ are 1, 10, 32, 40, 80, 160, 240, 320, 480, 640, 960, 1 920, 3 840.

B.5.3 Enumeration of ternary quantity equations in $\mathbb{Z}^5$

We create a generating polynomial $g(z;5)$ in the indeterminate z for which the exponents take the values of the cardinalities of the orbits of $\mathbb{Z}^5$:

$$\begin{aligned}
&g(z;5) = \\
&z^{3840} + z^{1920} + z^{960} + z^{640} + z^{480} + z^{320} + z^{240} \\
&+ z^{160} + z^{80} + z^{40} + z^{32} + z^{10} + z\,.
\end{aligned}$$

We are interested in the ternary relations between the quantities. We therefore expand the generating polynomial $g(z;5)^2$ in its terms and find:

$$
\begin{aligned}
&g(z;5)^2 = \\
&z^{7680} + 2z^{5760} + 2z^{4800} + 2z^{4480} + 2z^{4320} + 2z^{4160} + 2z^{4080} + 2z^{4000} \\
&+ 2z^{3920} + 2z^{3880} + 2z^{3872} + 2z^{3850} + 2z^{3841} + z^{3840} + 2z^{2880} + 2z^{2560} \\
&+ 2z^{2400} + 2z^{2240} + 2z^{2160} + 2z^{2080} + 2z^{2000} + 2z^{1960} + 2z^{1952} + 2z^{1930} \\
&+ 2z^{1921} + z^{1920} + 2z^{1600} + 2z^{1440} + 3z^{1280} + 2z^{1200} + 4z^{1120} + 2z^{1040} \\
&+ 2z^{1000} + 2z^{992} + 2z^{970} + 2z^{961} + 3z^{960} + 2z^{880} + 4z^{800} + 4z^{720} + 2z^{680} \\
&+ 2z^{672} + 2z^{650} + 2z^{641} + 3z^{640} + 4z^{560} + 2z^{520} + 2z^{512} + 2z^{490} + 2z^{481} \\
&+ 3z^{480} + 4z^{400} + 2z^{360} + 2z^{352} + 2z^{330} + 2z^{321} + 3z^{320} + 2z^{280} + 2z^{272} \\
&+ 2z^{250} + 2z^{241} + 2z^{240} + 2z^{200} + 2z^{192} + 2z^{170} + 2z^{161} + z^{160} + 2z^{120} \\
&+ 2z^{112} + 2z^{90} + 2z^{81} + z^{80} + 2z^{72} + z^{64} + 2z^{50} + 2z^{42} + 2z^{41} + 2z^{33} \\
&+ z^{20} + 2z^{11} + z^{2}\,.
\end{aligned}
$$

Observe that by adding cardinalities we could generate cardinalities that are not belonging to the set of cardinalities of $\mathbb{Z}^5$. Hence, the closure condition of addition of cardinalities is applicable and thus the polynomial is restricted, denoted by $g|_M(z;5)^2$, in which $M = \{z^{3840}, z^{1920}, z^{960}, z^{640}, z^{480}, z^{320}, z^{240}, z^{160}, z^{80}, z^{40}, z^{32}, z^{10}, z\}$. The restricted generating polynomial is:

$$g|_M(z;5)^2 = z^{3840} + z^{1920} + 3z^{960} + 3z^{640} + 3z^{480} + 3z^{320} + 2z^{240} + z^{160} + z^{80}\,.$$

The coefficient three occurs for the sum of the cardinalities being 320, 480, 640, and 960. The coefficient two occurs for the sum of the cardinalities yielding 240 and thus we expect that unique relations occur when the sum is 240. The green cells in Figure B.5 show the combinations of the cardinalities of the quantities ***x*** and ***y*** resulting in ternary quantity equation xy of $\mathbb{Z}^5$.

	1	10	32	40	80	160	240	320	480	640	960	1920	3840
1	2	11	33	41	81	161	241	321	481	641	961	1921	3841
10	11	20	42	50	90	170	250	330	490	650	970	1930	3850
32	33	42	64	72	112	192	272	352	512	672	992	1952	3872
40	41	50	72	**80**	120	200	280	360	520	680	1000	1960	3880
80	81	90	112	120	**160**	**240**	**320**	400	560	720	1040	2000	3920
160	161	170	192	200	**240**	**320**	400	**480**	**640**	800	1120	2080	4000
240	241	250	272	280	**320**	400	**480**	560	720	880	1200	2160	4080
320	321	330	352	360	400	**480**	560	**640**	800	**960**	1280	2240	4160
480	481	490	512	520	560	**640**	720	800	**960**	1120	1440	2400	4320
640	641	650	672	680	720	800	880	**960**	1120	1280	1600	2560	4480
960	961	970	992	1000	1040	1120	1200	1280	1440	1600	**1920**	2880	4800
1920	1921	1930	1952	1960	2000	2080	2160	2240	2400	2560	2880	**3840**	5760
3840	3841	3850	3872	3880	3920	4000	4080	4160	4320	4480	4800	5760	7680

Figure B.5: Ternary relations between orbits of unequal cardinalities in $\mathbb{Z}^5$.

The coefficient one occurs for the sum of the cardinalities being 80, 160, 1 920, 3 840. We expect the occurrence of degenerated relations between quantities when one of these four numbers occur. We can summarize the above results reflecting the multiplicities m of the ternary relations by defining a generating polynomial $v(z;5)$ as:

$$v(z;5) = 4z^3 + z^2 + 4z\,. \tag{B.4}$$

The multiplicity is given by the exponent of the indeterminate z. Observe that there are nine distinct sums of cardinalities for ternary quantity equations $f(\pi) = \dfrac{z}{xy}$ in $\mathbb{Z}^5$.

B.5.4 Elements of physics in $\mathbb{Z}^5$

We classify the orbits of $\mathbb{Z}^5$ in Table B.14 in which the rows are representing the infinity norm $\ell_\infty = s$ and the columns are the finite set of the cardinalities $\#([w])$ of the orbits. Thus, each orbit $[w]$ of $\mathbb{Z}^5$ is mapped to a cell in Table B.14. We call this cell an element of physics. Hence, each element of physics has a coordinate $(s, \#([w]))$. We can count the number of orbits, and denote it by m and call this the mass of the element. Hence, we write in each cell with coordinates $(s, \#([w]))$ the value of the mass m of the element. These masses are given in Table B.14.

Table B.14: Elements of physics in $\mathbb{Z}^5$.

$\|\|z\|\|_\infty$	1	10	32	40	80	160	240	320	480	640	960	1920	3840	RowSum
0	1	0	0	0	0	0	0	0	0	0	0	0	0	1
1	0	1	1	1	2	0	0	0	0	0	0	0	0	5
2	0	1	1	1	3	2	2	4	1	0	0	0	0	15
3	0	1	1	1	4	4	4	8	3	3	6	0	0	35
4	0	1	1	1	5	6	6	12	6	9	18	5	0	70
5	0	1	1	1	6	8	8	16	10	18	36	20	1	126
6	0	1	1	1	7	10	10	20	15	30	60	50	5	210
7	0	1	1	1	8	12	12	24	21	45	90	100	15	330
8	0	1	1	1	9	14	14	28	28	63	126	175	35	495
9	0	1	1	1	10	16	16	32	36	84	168	280	70	715
10	0	1	1	1	11	18	18	36	45	108	216	420	126	1001
11	0	1	1	1	12	20	20	40	55	135	270	600	210	1365
12	0	1	1	1	13	22	22	44	66	165	330	825	330	1820
13	0	1	1	1	14	24	24	48	78	198	396	1100	495	2380
14	0	1	1	1	15	26	26	52	91	234	468	1430	715	3060
15	0	1	1	1	16	28	28	56	105	273	546	1820	1001	3876
16	0	1	1	1	17	30	30	60	120	315	630	2275	1365	4845
17	0	1	1	1	18	32	32	64	136	360	720	2800	1820	5985
18	0	1	1	1	19	34	34	68	153	408	816	3400	2380	7315
19	0	1	1	1	20	36	36	72	171	459	918	4080	3060	8855
20	0	1	1	1	21	38	38	76	190	513	1026	4845	3876	10626
21	0	1	1	1	22	40	40	80	210	570	1140	5700	4845	12650
22	0	1	1	1	23	42	42	84	231	630	1260	6650	5985	14950
23	0	1	1	1	24	44	44	88	253	693	1386	7700	7315	17550
24	0	1	1	1	25	46	46	92	276	759	1518	8855	8855	20475
25	0	1	1	1	26	48	48	96	300	828	1656	10120	10626	23751
26	0	1	1	1	27	50	50	100	325	900	1800	11500	12650	27405

Observe that each term of the OEIS integer sequence A000332 (N. J. A. Sloane, 2006) in which n is substituted by $n+4$ is obtained by associating $n+4$ to the infinity norm s and taking the sum of the masses in each row of the Table B.14. The OEIS integer sequence A000332 represent the binomial coefficients $\begin{pmatrix} n \\ 5-1 \end{pmatrix}$. The column denoted RowSum in Table B.14 is obtained using the equation $a(s) = \begin{pmatrix} s+5-1 \\ 5-1 \end{pmatrix}$. The sequences of integers in the columns of Table B.14 can be referenced to sequences in the On-line Encyclopedia of Integer Sequences (OEIS). Each sequence in the columns can be generated and extended using an ordinary generating function as given in Table B.15.

Table B.15: Table of sequences and ordinary generating functions in $\mathbb{Z}^5$.

$\#(\{\mathrm{Orb}(z)\})$	OEIS reference	Ordinary generating function $G(a_n; z)$
1	A000007	Not defined
10	A057427	$z/(1-z)$
32	A057427	$z/(1-z)$
40	A057427	$z/(1-z)$
80	A087156	$z(2-z)/(1-z)^2$
160	A005843(n-1)	$2z^2/(1-z)^2$
240	A005843(n-1)	$2z^2/(1-z)^2$
320	2 × (A005843(n-1))	$4z^2/(1-z)^2$
480	A000217(n-1)	$z^2/(1-z)^3$
640	A056115(n-2)	$3z^3/(1-z)^3$
960	2 × (A056115(n-2)	$6z^3/(1-z)^3$
1920	5 × (A210440(n-3)	$5z^4/(z-1)^4$

The RowSum column has the OEIS reference A000332(n+4) with ordinary generating function $1/(1-z)^5$.

From the Table B.14 we derive the backbone structure of $\mathbb{Z}^5$ and obtain 30 orbits representing 23 923 integer lattice points given in Table B.16.

Table B.16: Backbone structure of physics in $\mathbb{Z}^5$.

Cardinality	1	10	32	40	80	160	240	320	480	640	960	1 920	3 840
Orbits	1	1	1	1	2	2	2	4	1	3	6	5	1
Lattice points	1	10	32	40	160	320	480	1 280	480	1 920	5 760	9 600	3840

B.6 Dimension $N = 6$

We are free to select six basis lattice points of $\mathbb{Z}^6$ and define:

$$\begin{aligned}
\boldsymbol{e}_1 &\doteq \operatorname{dex}([\mathsf{T}]_\sim) = (1,0,0,0,0,0)\,,\\
\boldsymbol{e}_2 &\doteq \operatorname{dex}([\mathsf{L}]_\sim) = (0,1,0,0,0,0)\,,\\
\boldsymbol{e}_3 &\doteq \operatorname{dex}([\mathsf{M}]_\sim) = (0,0,1,0,0,0)\,,\\
\boldsymbol{e}_4 &\doteq \operatorname{dex}([\mathsf{I}]_\sim) = (0,0,0,1,0,0)\,,\\
\boldsymbol{e}_5 &\doteq \operatorname{dex}([\Theta]_\sim) = (0,0,0,0,1,0)\,,\\
\boldsymbol{e}_6 &\doteq \operatorname{dex}([\mathsf{N}]_\sim) = (0,0,0,0,0,1)\,,
\end{aligned}$$

with $\boldsymbol{e}_i \in \mathbb{Z}^6$.

B.6.1 The structure constants of the integer lattice $\mathbb{Z}^6$

Table B.17 enumerates the structure constants $b_{kl}^{(6)}$ for the integer lattice $\mathbb{Z}^6$.

Table B.17: Structure constants $b_{kl}^{(6)}$.

$b_{kl}^{(6)}$	$l = 0$	1	2	3	4	5	6
$k = 0$	1	12	60	160	240	192	64
1	2	1	10	40	80	80	32
2	4	4	1	8	24	32	16
3	8	12	6	1	6	12	8
4	16	32	24	8	1	4	4
5	32	80	80	40	10	1	2
6	64	192	240	160	60	12	1

B.6.2 Enumeration of the orbit cardinalities in $\mathbb{Z}^6$

The cardinalities of the orbits of $\mathbb{Z}^6$ are 1, 12, 60, 64, 120, 160, 192 , 240, 384, 480, 960, 1 280, 1 440, 1 920, 2 880, 3 840, 5 760, 7 680, 11 520, 23 040, 46 080.

B.6.3 Enumeration of ternary quantity equations in $\mathbb{Z}^6$

We create a generating polynomial $g(z;6)$ in the indeterminate z for which the exponents take the values of the cardinalities of the orbits of $\mathbb{Z}^6$:

$$
\begin{aligned}
&g(z;6) = \\
&z^{46080} + z^{23040} + z^{11520} + z^{7680} + z^{5760} + z^{3840} + z^{2880} \\
&+ z^{1920} + z^{1440} + z^{1280} + z^{960} + z^{480} + z^{384} + z^{240} + z^{192} \\
&+ z^{160} + z^{120} + z^{64} + z^{60} + z^{12} + z\,.
\end{aligned}
$$

We are interested in the ternary relations between the kinds of quantities. The number of ways to create a sum of cardinalities can be found by expanding the generating polynomial $g(z;6)^2$ in its terms. We find:

$$
\begin{aligned}
&g(z;6)^2 = \\
&z^{92160} + 2z^{69120} + 2z^{57600} + 2z^{53760} + 2z^{51840} + 2z^{49920} + 2z^{48960} \\
&+ 2z^{48000} + 2z^{47520} + 2z^{47360} + 2z^{47040} + 2z^{46560} + 2z^{46464} + 2z^{46320} \\
&+ 2z^{46272} + 2z^{46240} + 2z^{46200} + 2z^{46144} + 2z^{46140} + 2z^{46092} + 2z^{46081} \\
&+ z^{46080} + 2z^{34560} + 2z^{30720} + 2z^{28800} + 2z^{26880} + 2z^{25920} + 2z^{24960} \\
&+ 2z^{24480} + 2z^{24320} + 2z^{24000} + 2z^{23520} + 2z^{23424} + 2z^{23280} + 2z^{23232} \\
&+ 2z^{23200} + 2z^{23160} + 2z^{23104} + 2z^{23100} + 2z^{23052} + 2z^{23041} + z^{23040} \\
&+ 2z^{19200} + 2z^{17280} + 3z^{15360} + 2z^{14400} + 4z^{13440} + 2z^{12960} + 2z^{12800} \\
&+ 2z^{12480} + 2z^{12000} + 2z^{11904} + 2z^{11760} + 2z^{11712} + 2z^{11680} + 2z^{11640} \\
&+ 2z^{11584} + 2z^{11580} + 2z^{11532} + 2z^{11521} + 3z^{11520} + 2z^{10560} + 4z^{9600} \\
&+ 2z^{9120} + 2z^{8960} + 4z^{8640} + 2z^{8160} + 2z^{8064} + 2z^{7920} + 2z^{7872} + 2z^{7840} \\
&+ 2z^{7800} + 2z^{7744} + 2z^{7740} + 2z^{7692} + 2z^{7681} + 3z^{7680} + 2z^{7200} + 2z^{7040} \\
&+ 4z^{6720} + 2z^{6240} + 2z^{6144} + 2z^{6000} + 2z^{5952} + 2z^{5920} + 2z^{5880} + 2z^{5824} \\
&+ 2z^{5820} + 2z^{5772} + 2z^{5761} + 3z^{5760} + 2z^{5280} + 2z^{5120} + 4z^{4800} + 4z^{4320} \\
&+ 2z^{4224} + 2z^{4160} + 2z^{4080} + 2z^{4032} + 2z^{4000} + 2z^{3960} + 2z^{3904} + 2z^{3900} \\
&+ 2z^{3852} + 2z^{3841} + 3z^{3840} + 4z^{3360} + 2z^{3264} + 2z^{3200} + 2z^{3120} + 2z^{3072} \\
&+ 2z^{3040} + 2z^{3000} + 2z^{2944} + 2z^{2940} + 2z^{2892} + 2z^{2881} + 3z^{2880} + 2z^{2720} \\
&+ z^{2560} + 4z^{2400} + 2z^{2304} + 2z^{2240} + 2z^{2160} + 2z^{2112} + 2z^{2080} + 2z^{2040} \\
&+ 2z^{1984} + 2z^{1980} + 2z^{1932} + 2z^{1921} + 3z^{1920} + 2z^{1824} + 2z^{1760} + 2z^{1680} \\
&+ 2z^{1664} + 2z^{1632} + 2z^{1600} + 2z^{1560} + 2z^{1520} + 2z^{1504} + 2z^{1500} + 2z^{1472} \\
&+ 2z^{1452} + 2z^{1441} + 4z^{1440} + 2z^{1400} + 4z^{1344} + 2z^{1340} + 2z^{1292} + 2z^{1281} \\
&+ 2z^{1200} + 2z^{1152} + 2z^{1120} + 2z^{1080} + 2z^{1024} + 2z^{1020} + 2z^{972} + 2z^{961} \\
&+ z^{960} + 2z^{864} + z^{768} + 2z^{720} + 2z^{672} + 2z^{640} + 2z^{624} + 2z^{600} + 2z^{576} + 4z^{544} \\
&+ 2z^{540} + 2z^{504} + 2z^{492} + 2z^{481} + z^{480} + 2z^{448} + 2z^{444} + 2z^{432} + 2z^{400} + 2z^{396} \\
&+ 2z^{385} + z^{384} + 2z^{360} + 2z^{352} + z^{320} + 2z^{312} + 2z^{304} + 2z^{300} + 2z^{280} + 2z^{256} \\
&+ 4z^{252} + 2z^{241} + z^{240} + 2z^{224} + 2z^{220} + 2z^{204} + 2z^{193} + 2z^{184} + 2z^{180} + 2z^{172} \\
&+ 2z^{161} + 2z^{132} + z^{128} + 2z^{124} + 2z^{121} + z^{120} + 2z^{76} + 2z^{72} + 2z^{65} + 2z^{61} \\
&+ z^{24} + 2z^{13} + z^{2} .
\end{aligned}
$$

The largest coefficient is *four* and occurs for the sum of the cardinalities being 252, 544, 1 344, 1 440, 2 400, 3 360, 4 320, 4 800, 6 720, 8 640, 9 600, 13 440.

The coefficient *three* occurs for the sum of the cardinalities being 1 920, 2 880, 3 840, 5 760, 7 680, 11 520, 15 360.

The coefficient *two* occurs for the sum of the cardinalities being 13, 61, 65, 72, 76, 121, 124, 132, 161, 172, 180, 184, 193, 204, 220, 224, 241, 256, 280, 300, 304, 312, 352, 360, 385, 396, 400, 432, 444, 448, 481, 492, 504, 540, 576, 600, 624, 640, 672, 720, 864, 961, 972, 1 020, 1 024, 1 080, 1 120, 1 152, 1 200, 1 281, 1 292, 1 340, 1 400, 1 441, 1 452, 1 472, 1 500, 1 504, 1 520, 1 560, 1 600, 1 632, 1 664, 1 680, 1 760, 1 824, 1 921, 1 932, 1 980, 1 984, 2 040, 2 080, 2 112, 2 160, 2 240, 2 304, 2 720, 2 881, 2 892, 2 940, 2 944, 3 000, 3 040, 3 072, 3 120, 3 200, 3 264, 3 841, 3 852, 3 900, 3 904, 3 960, 4 000, 4 032, 4 080, 4 160, 4 224, 5 120, 5 280, 5 761, 5 772, 5 820, 5 824, 5 880, 5 920, 5 952, 6 000, 6 144, 6 240, 7 040, 7 200, 7 681, 7 692, 7 740, 7 744, 7 800, 7 840, 7 872, 7 920, 8 064, 8 160, 8 960, 9 120, 10 560, 11 521, 11 532, 11 580, 11 584, 11 640, 11 680, 11 712, 11 760, 11 904, 12 000, 12 480, 12 800, 12 960, 14 400, 17 280, 19 200, 23 041, 23 052, 23 100, 23 104, 23 160, 23 200, 23 232, 23 280, 23 424, 23 520, 24 000, 24 320, 24 480, 24 960, 25 920, 26 880, 28 800, 30 720, 34 560, 46 081, 46 092, 46 140, 46 144, 46 200, 46 240, 46 272, 46 320, 46 464, 46 560, 47 040, 47 360, 47 520, 48 000, 48 960, 49 920, 51 840, 53 760, 57 600, 69 120.

Observe that by adding cardinalities we could generate cardinalities that are not belonging to the set of cardinalities of $\mathbb{Z}^6$. Hence, the closure condition of addition of cardinalities is applicable.

We expect the occurrence of unique relations between quantities when one of these 179 numbers occurs.

The green cells in Figure B.6 show the valid combinations of the cardinalities of the quantities ***x*** and ***y*** resulting in ternary quantity equation $z = xy$ of $\mathbb{Z}^6$. The forbidden cardinalities are the white cells.

	1	12	60	64	120	160	192	240	384	480	960	1280	1440	1920	2880	3840	5760	7680	11520	23040	46080
1	2	13	61	65	121	161	193	241	385	481	961	1281	1441	1921	2881	3841	5761	7681	11521	23041	46081
12	13	24	72	76	132	172	204	252	396	492	972	1292	1452	1932	2892	3852	5772	7692	11532	23052	46092
60	61	72	**120**	124	180	220	252	300	444	540	1020	1340	1500	1980	2940	3900	5820	7740	11580	23100	46140
64	65	76	124	128	184	224	256	304	448	544	1024	1344	1504	1984	2944	3904	5824	7744	11584	23104	46144
120	121	132	180	184	**240**	280	312	360	504	600	1080	1400	1560	2040	3000	3960	5880	7800	11640	23160	46200
160	161	172	220	224	280	320	352	400	544	640	1120	**1440**	1600	2080	3040	4000	5920	7840	11680	23200	46240
192	193	204	252	256	312	352	**384**	432	576	672	1152	1472	1632	2112	3072	4032	5952	7872	11712	23232	46272
240	241	252	300	304	360	400	432	**480**	624	720	1200	1520	1680	2160	3120	4080	6000	7920	11760	23280	46320
384	385	396	444	448	504	544	576	624	768	864	1344	1664	1824	2304	3264	4224	6144	8064	11904	23424	46464
480	481	492	540	544	600	640	672	720	864	**960**	**1440**	1760	**1920**	2400	3360	4320	6240	8160	12000	23520	46560
960	961	972	1020	1024	1080	1120	1152	1200	1344	**1440**	**1920**	2240	2400	**2880**	**3840**	4800	6720	8640	12480	24000	47040
1280	1281	1292	1340	1344	1400	**1440**	1472	1520	1664	1760	2240	2560	2720	3200	4160	5120	7040	8960	12800	24320	47360
1440	1441	1452	1500	1504	1560	1600	1632	1680	1824	**1920**	2400	2720	**2880**	3360	4320	5280	7200	9120	12960	24480	47520
1920	1921	1932	1980	1984	2040	2080	2112	2160	2304	2400	**2880**	3200	3360	**3840**	4800	**5760**	**7680**	9600	13440	24960	48000
2880	2881	2892	2940	2944	3000	3040	3072	3120	3264	3360	**3840**	4160	4320	4800	**5760**	6720	8640	10560	14400	25920	48960
3840	3841	3852	3900	3904	3960	4000	4032	4080	4224	4320	4800	5120	5280	**5760**	6720	**7680**	9600	**11520**	15360	26880	49920
5760	5761	5772	5820	5824	5880	5920	5952	6000	6144	6240	6720	7040	7200	**7680**	8640	9600	**11520**	13440	17280	28800	51840
7680	7681	7692	7740	7744	7800	7840	7872	7920	8064	8160	8640	8960	9120	9600	10560	**11520**	13440	15360	19200	30720	53760
11520	11521	11532	11580	11584	11640	11680	11712	11760	11904	12000	12480	12800	12960	13440	14400	15360	17280	19200	**23040**	34560	57600
23040	23041	23052	23100	23104	23160	23200	23232	23280	23424	23520	24000	24320	24480	24960	25920	26880	28800	30720	34560	**46080**	69120
46080	46081	46092	46140	46144	46200	46240	46272	46320	46464	46560	47040	47360	47520	48000	48960	49920	51840	53760	57600	69120	92160

Figure B.6: Green cells representing unique relations based on the sums of the cardinalities in $\mathbb{Z}^6$.

The coefficient *one* occurs for the sum of the cardinalities being 2, 24, 120, 128, 240, 320, 384, 480, 768, 960, 2 560, 23 040, 46 080, 92 160. We expect the occurrence of degenerated relations between quantities when one of these 14 numbers occurs. We can summarize the above results reflecting the multiplicities m of the ternary relations by defining a generating polynomial $v(z; 6)$ as:

$$v(z; 6) = 12z^4 + 7z^3 + 179z^2 + 13z\,, \qquad \text{(B.5)}$$

The multiplicity is given by the exponent of the indeterminate z. Observe that there are 211 distinct sums of cardinalities for ternary quantity equations

$$f(\pi) = \frac{z}{xy} \text{ in } \mathbb{Z}^6 .$$

B.6.4 Elements of physics in $\mathbb{Z}^6$

We classify the orbits of $\mathbb{Z}^6$ in Table B.18 in which the rows are representing the infinity norm $\ell_\infty = s$ and the columns are the finite set of the cardinalities $\#([w])$ of the orbits. Thus, each orbit $[w]$ of $\mathbb{Z}^6$ is mapped to a cell in Table B.18. We call this cell an element of physics. Hence, each element of physics has a coordinate $(s, \#([w]))$. We can count the number of orbits, and denote it by m and call this the mass of the element. Hence, we write in each cell with coordinates $(s, \#([w]))$ the value of the mass m of the element. These masses are given in each cell of the Table B.18.

Table B.18: Elements of physics in $\mathbb{Z}^6$.

$\|z\|_\infty$	1	12	60	64	120	160	192	240	384	480	960	1280	1440	1920	2880	3840	5760	7680	11520	23040	46080	RowSum
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
1	0	1	1	1	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	6
2	0	1	1	1	1	1	1	1	2	2	6	1	1	2	0	0	0	0	0	0	0	21
3	0	1	1	1	2	1	1	1	4	4	13	2	2	7	3	9	4	0	0	0	0	56
4	0	1	1	1	3	1	1	1	6	6	21	3	3	15	9	27	13	4	10	0	0	126
5	0	1	1	1	4	1	1	1	8	8	30	4	4	26	18	54	28	16	40	6	0	252
6	0	1	1	1	5	1	1	1	10	10	40	5	5	40	30	90	50	40	100	30	1	462
7	0	1	1	1	6	1	1	1	12	12	51	6	6	57	45	135	80	80	200	90	6	792
8	0	1	1	1	7	1	1	1	14	14	63	7	7	77	63	189	119	140	350	210	21	1287
9	0	1	1	1	8	1	1	1	16	16	76	8	8	100	84	252	168	224	560	420	56	2002
10	0	1	1	1	9	1	1	1	18	18	90	9	9	126	108	324	228	336	840	756	126	3003
11	0	1	1	1	10	1	1	1	20	20	105	10	10	155	135	405	300	480	1200	1260	252	4368
12	0	1	1	1	11	1	1	1	22	22	121	11	11	187	165	495	385	660	1650	1980	462	6188
13	0	1	1	1	12	1	1	1	24	24	138	12	12	222	198	594	484	880	2200	2970	792	8568
14	0	1	1	1	13	1	1	1	26	26	156	13	13	260	234	702	598	1144	2860	4290	1287	11628
15	0	1	1	1	14	1	1	1	28	28	175	14	14	301	273	819	728	1456	3640	6006	2002	15504
16	0	1	1	1	15	1	1	1	30	30	195	15	15	345	315	945	875	1820	4550	8190	3003	20349
17	0	1	1	1	16	1	1	1	32	32	216	16	16	392	360	1080	1040	2240	5600	10920	4368	26334
18	0	1	1	1	17	1	1	1	34	34	238	17	17	442	408	1224	1224	2720	6800	14280	6188	33649
19	0	1	1	1	18	1	1	1	36	36	261	18	18	495	459	1377	1428	3264	8160	18360	8568	42504
20	0	1	1	1	19	1	1	1	38	38	285	19	19	551	513	1539	1653	3876	9690	23256	11628	53130
21	0	1	1	1	20	1	1	1	40	40	310	20	20	610	570	1710	1900	4560	11400	29070	15504	65780
22	0	1	1	1	21	1	1	1	42	42	336	21	21	672	630	1890	2170	5320	13300	35910	20349	80730
23	0	1	1	1	22	1	1	1	44	44	363	22	22	737	693	2079	2464	6160	15400	43890	26334	98280
24	0	1	1	1	23	1	1	1	46	46	391	23	23	805	759	2277	2783	7084	17710	53130	33649	118755
25	0	1	1	1	24	1	1	1	48	48	420	24	24	876	828	2484	3128	8096	20240	63756	42504	142506
26	0	1	1	1	25	1	1	1	50	50	450	25	25	950	900	2700	3500	9200	23000	75900	53130	169911

Observe that each term of the OEIS sequence A000389 (N. J. A. Sloane, 2006) in which n is substituted by $n+5$ is obtained by associating $n+5$ to the infinity norm s and taking the sum of the masses in each row of the Table B.18. The OEIS sequence A000389 represent the binomial coefficients $\binom{n}{6-1}$. Hence, the column denoted RowSum in Table B.18 is obtained using the equation $a(s) = \binom{s+6-1}{6-1}$. The sequences of integers in the columns of Table B.18 can be referenced to sequences in the On-line Encyclopedia of Integer Sequences (OEIS). Each sequence in the columns can be generated and extended using an ordinary generating function as given in Table B.19.

Table B.19: Table of sequences and ordinary generating functions in $\mathbb{Z}^6$.

$\#(\{\mathrm{Orb}\,(z)\})$	OEIS reference	Ordinary generating function $G(a_n; z)$
1	A000007	Not defined
12	A057427	$z/(1-z)$
60	A057427	$z/(1-z)$
64	A057427	$z/(1-z)$
120	A001477	$z/(z-1)^2$
160	A057427	$z/(1-z)$
192	A057427	$z/(1-z)$
240	A057427	$z/(1-z)$
384	A005843 + 1 zero	$2z^2/(z-1)^2$
480	A005843 + 1 zero	$2z^2/(z-1)^2$
960	A056115 + 1 zero	$z^2(5z-6)/(z-1)^3$
1280	A001477 + 1 zero	$z^2/(z-1)^2$
1440	A001477 + 1 zero	$z^2/(z-1)^2$
1920	A005449 + 1 zero	$z^2(z+2)/(1-z)^3$
2880	A045943 + 2 zeros	$3z^3/(1-z)^3$
3840	A027468 + 2 zeros	$9z^3/(1-z)^3$
5760	A060488 + 3 zeros	$(-3z^4+4z^3)/(z-1)^4$
7680	A210440 + 3 zeros	$4z^4/(z-1)^4$
11520	10 × (A210440+ 3 zeros)	$10z^4/(z-1)^4$
23040	6 × (A000332 + 1 zero)	$6z^5/(1-z)^5$
46080	A000389 + 1 zero	$z^6/(z-1)^6$
RowSum	A000389 - 5 zeros	$1/(z-1)^6$

From the Table B.18 we derive the backbone structure of $\mathbb{Z}^6$ and obtain 59 orbits representing 411 377 integer lattice points given in Table B.20.

Table B.20: Backbone structure of physics in $\mathbb{Z}^6$.

Cardinalities	1	12	60	64	120	160	192	240	384	480	960
Orbits	1	1	1	1	1	1	1	1	2	2	6
Lattice points	1	12	60	64	120	160	192	240	768	960	5 760

Cardinalities	1 280	1 440	1 920	2 880	3 840	5 760	7 680	11 520	23 040	46 080	Total
Orbits	1	1	2	3	9	4	4	10	6	1	59
Lattice points	1 280	1 440	3 840	8 640	34 560	23 040	30 720	115 200	138 240	46 080	411 377

B.7 Dimension $N = 7$

We are free to select seven basis lattice points of $\mathbb{Z}^7$ and define:

$$\begin{aligned}
\boldsymbol{e}_1 &\doteq \operatorname{dex}([\mathsf{T}]_\sim) = (1,0,0,0,0,0,0) = (1,0^6)\,,\\
\boldsymbol{e}_2 &\doteq \operatorname{dex}([\mathsf{L}]_\sim) = (0,1,0,0,0,0,0) = (0,1,0^5)\,,\\
\boldsymbol{e}_3 &\doteq \operatorname{dex}([\mathsf{M}]_\sim) = (0,0,1,0,0,0,0) = (0^2,1,0^4)\,,\\
\boldsymbol{e}_4 &\doteq \operatorname{dex}([\mathsf{I}]_\sim) = (0,0,0,1,0,0,0) = (0^3,1,0^3)\,,\\
\boldsymbol{e}_5 &\doteq \operatorname{dex}([\Theta]_\sim) = (0,0,0,0,1,0,0) = (0^4,1,0^2)\,,\\
\boldsymbol{e}_6 &\doteq \operatorname{dex}([\mathsf{N}]_\sim) = (0,0,0,0,0,1,0) = (0^5,1,0)\,,\\
\boldsymbol{e}_7 &\doteq \operatorname{dex}([\mathsf{J}]_\sim) = (0,0,0,0,0,0,1) = (0^6,1)
\end{aligned}$$

with $\boldsymbol{e}_i \in \mathbb{Z}^7$.

The majority of the engineers formulate engineering problems using physical quantities defined in seven dimensions and this is by convention the standard worldwide and known as *SI*. We denote the set of SI base quantities as $B_q = \{\mathsf{T}, \mathsf{L}, \mathsf{M}, \mathsf{I}, \Theta, \mathsf{N}, \mathsf{J}\}$. The cardinality of B_q is finite and has the value $\#(B_q) = 7$. The power set $\mathcal{P}(B_q)$ is the collection of all subsets of B_q and has cardinality $\#(\mathcal{P}(B_q)) = 2^7 = 128$. One of these subsets was discussed in B.3 and is denoted the LMT framework. The automorphism group of the 7-dimensional cubic lattice $\mathrm{Aut}(\mathbb{Z}^7)$ contains all permutations and sign changes of the 7 coordinates and has order $2^7\,7! = 645120$. Each signed permutation matrix is an orthogonal matrix (J. Conway et al., 1999). The additive partition of $N = 7$ is:

$$\begin{aligned}
partition(7) = [&[1,1,1,1,1,1,1],[1,1,1,1,1,2],[1,1,1,2,2],[1,2,2,2],\\
&[1,1,1,1,3],[1,1,2,3],[2,2,3],[1,3,3],[1,1,1,4],[1,2,4],\\
&[3,4],[1,1,5],[2,5],[1,6],[7]]\,.
\end{aligned}$$

B.7.1 The structure constants of the integer lattice $\mathbb{Z}^7$

Table B.21 enumerates the structure constants $b_{kl}^{(7)}$ for the integer lattice $\mathbb{Z}^7$.

Table B.21: Structure constants $b_{kl}^{(7)}$.

$b_{kl}^{(7)}$	$l=0$	1	2	3	4	5	6	7
$k=0$	1	14	84	280	560	672	448	128
1	2	1	12	60	160	240	192	64
2	4	4	1	10	40	80	80	32
3	8	12	6	1	8	24	32	16
4	16	32	24	8	1	6	12	8
5	32	80	80	40	10	1	4	4
6	64	192	240	160	60	12	1	2
7	128	448	672	560	280	84	14	1

B.7.2 Enumeration of the orbit cardinalities in $\mathbb{Z}^7$

The cardinalities of the orbits of $\mathbb{Z}^7$ are 1, 14, 84, 128, 168, 280, 448, 560, 672, 840, 896, 1 680, 2 240, 2 688, 3 360, 4 480, 5 376, 6 720, 8 960, 13 440, 17 920, 20 160, 26 880, 40 320, 53 760, 80 640, 107 520, 161 280, 322 560, 645 120.

B.7.3 Enumeration of ternary quantity equations in the sub-lattice $\{0\} \times \mathbb{Z}^7$

Each orbit representative of $\{0\} \times \mathbb{Z}^7$ is characterized by two integers that are the Gödel number of the orbit representative and the cardinality of its orbit. We know that the Gödel number has multiplicative properties while the cardinalities of the orbit have additive properties. Both properties can be combined in a number $\psi(\boldsymbol{z})$ defined by:

$$\psi(\boldsymbol{z}) = G(\mathrm{Orb}\,(\boldsymbol{z})) \cdot \exp(\#\,(\mathrm{Orb}\,(\boldsymbol{z})))\,, \tag{B.6}$$

where $\mathrm{Orb}\,(\boldsymbol{z})$ is the orbit representative of the lattice point $(z_0 \mid z_1, \ldots, z_7)$. Assume that $G(\mathrm{Orb}\,(\boldsymbol{x}))$ and $G(\mathrm{Orb}\,(\boldsymbol{y})))$ are divisors of $G(\mathrm{Orb}\,(\boldsymbol{z}))$ then we can write:

$$\begin{aligned}\psi(\boldsymbol{z}) =& G(\mathrm{Orb}\,(\boldsymbol{x})) \cdot G(\mathrm{Orb}\,(\boldsymbol{y})) \\ & \cdot \exp(\#\,(\mathrm{Orb}\,(\boldsymbol{x}))) \cdot \exp(\#\,(\mathrm{Orb}\,(\boldsymbol{y})))\,,\end{aligned}$$

where $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$ and $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \{0\} \times \mathbb{Z}^7$.

Consider the number of divisors of $G(\mathrm{Orb}\,(\boldsymbol{z}))$ that is denoted by $\tau(G(\mathrm{Orb}\,(\boldsymbol{z})))$ then it is obvious that we can consider $\tau(G(\mathrm{Orb}\,(\boldsymbol{z})))$ additions of the respective cardinalities $\#\,(G(\mathrm{Orb}\,(\boldsymbol{x})))$ and $\#\,(G(\mathrm{Orb}\,(\boldsymbol{y})))$ of the pairs of divisors of $G(\mathrm{Orb}\,(\boldsymbol{z}))$. We created a Maple(Maplesoft, 2018) file to analyze the histogram of the possible outcomes of the additions of the respective cardinalities. As example we consider the kind of quantity with lattice point $(0|3, 2, 1, 0^4)$ having Gödel number 360. The number of divisors $\tau(360) = 24$. Each of the divisors considered as a Gödel number can be mapped to an orbit representative. This orbit representative has an orbit with some cardinality. Consider the plot of the sum of the cardinalities versus the divisors of $(0|3, 2, 1, 0^4)$ then we obtain the Figure B.7.

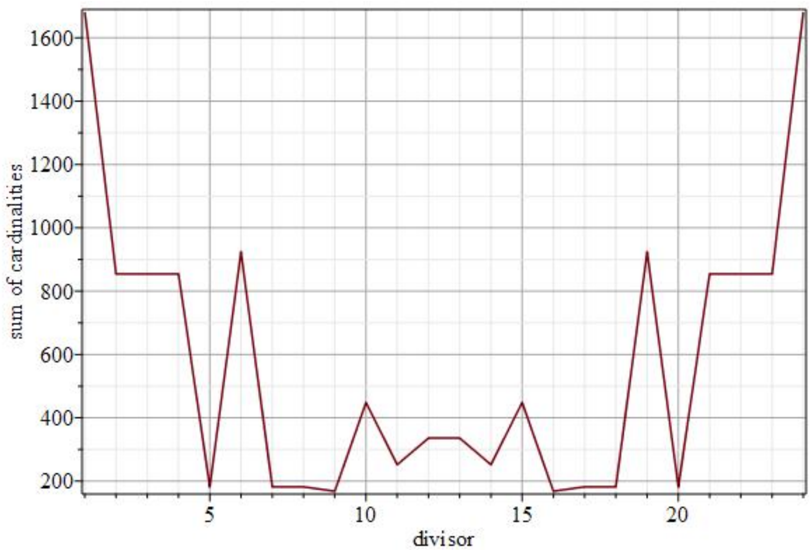

Figure B.7: Sum of cardinalities versus divisors of $(0|3, 2, 1, 0^4)$.

The histogram of the sum of cardinalities of the orbits generated from the divisors of 360 is given in Figure B.8.

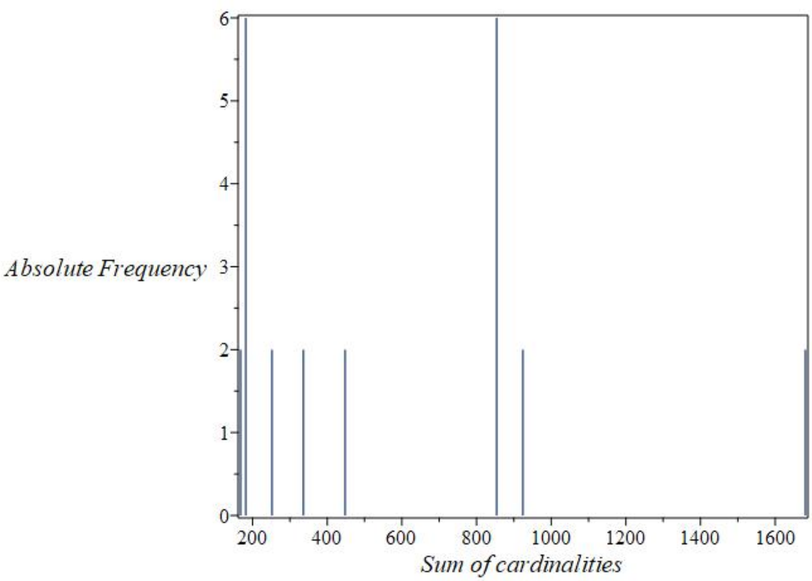

Figure B.8: Histogram of the sum of cardinalities of the orbits generated from the divisors of 360.

It reveals the existence of *unique pairs* of divisors of $G(\mathrm{Orb}\,(\boldsymbol{z}))$.

This observation leads to the study of the enumeration of the ternary equations. A classical example in the enumeration methods is the case of two dices where the number of ways have to be calculated to obtain the sum of the eyes of the dices. The technique of generating functions is used. The generating function for one die is $f(x) = x^1+x^2+x^3+x^4+x^5+x^6$ and for two dice we have to square $f(x)^2$ and read the coefficients of the term with exponent equal to the sum of the eyes of the dice to obtain the number of ways to obtain that sum. We find $f(x)^2 = x^{12}+2x^{11}+3x^{10}+4x^9+5x^8+6x^7+5x^6+4x^5+3x^4+2x^3+x^2$. The number of ways to obtain the sum equal to nine is four. We will use this method to find the number of ways to have $\#\,(\mathrm{Orb}\,(\boldsymbol{x})) + \#\,(\mathrm{Orb}\,(\boldsymbol{y}))$.

We found previously 30 distinct cardinalities in the integer lattice $\mathbb{Z}^7$. Consider a first hypothetical die having 30 facets where on each facet only one of the cardinalities is given and a second hypothetical die that is identical to the first die. After rolling the dice we add the cardinalities of the facets. By doing so we combine two orbit representatives $\boldsymbol{x}$ and $\boldsymbol{y}$ each having a cardinality.

We are interested in the number of ways one can generate the sum of the respective cardinalities. To answer this question we use the standard technique of generating functions. We create a generating polynomial $g(z;7)$ in the indeterminate z for which the exponents take the values of the cardinalities of the orbits of $\mathbb{Z}^7$:

$$\begin{aligned}
&g(z;7) = \\
&z^{645120} + z^{322560} + z^{161280} + z^{107520} + z^{80640} + z^{53760} + z^{40320} \\
&+ z^{26880} + z^{20160} + z^{17920} + z^{13440} + z^{8960} + z^{6720} + z^{5376} \\
&+ z^{4480} + z^{3360} + z^{2688} + z^{2240} + z^{1680} + z^{896} + z^{840} + z^{672} \\
&+ z^{560} + z^{448} + z^{280} + z^{168} + z^{128} + z^{84} + z^{14} + z.
\end{aligned}$$

We are interested in the ternary relations between the quantities. We therefore expand the generating polynomial $g_7(z)^2$ in its terms and find:

$$
\begin{aligned}
&g(z;7)^2 = \\
&z^{1290240} + 2z^{967680} + 2z^{806400} + 2z^{752640} + 2z^{725760} + 2z^{698880} + 2z^{685440} + 2z^{672000} \\
&+ 2z^{665280} + 2z^{663040} + 2z^{658560} + 2z^{654080} + 2z^{651840} + 2z^{650496} + 2z^{649600} + 2z^{648480} \\
&+ 2z^{647808} + 2z^{647360} + 2z^{646800} + 2z^{646016} + 2z^{645960} + 2z^{645792} + 2z^{645680} + 2z^{645568} \\
&+ 2z^{645400} + 2z^{645288} + 2z^{645248} + 2z^{645204} + 2z^{645134} + 2z^{645121} + z^{645120} + 2z^{483840} \\
&+ 2z^{430080} + 2z^{403200} + 2z^{376320} + 2z^{362880} + 2z^{349440} + 2z^{342720} + 2z^{340480} + 2z^{336000} \\
&+ 2z^{331520} + 2z^{329280} + 2z^{327936} + 2z^{327040} + 2z^{325920} + 2z^{325248} + 2z^{324800} + 2z^{324240} \\
&+ 2z^{323456} + 2z^{323400} + 2z^{323232} + 2z^{323120} + 2z^{323008} + 2z^{322840} + 2z^{322728} + 2z^{322688} \\
&+ 2z^{322644} + 2z^{322574} + 2z^{322561} + z^{322560} + 2z^{268800} + 2z^{241920} + 3z^{215040} + 2z^{201600} \\
&+ 4z^{188160} + 2z^{181440} + 2z^{179200} + 2z^{174720} + 2z^{170240} + 2z^{168000} + 2z^{166656} + 2z^{165760} \\
&+ 2z^{164640} + 2z^{163968} + 2z^{163520} + 2z^{162960} + 2z^{162176} + 2z^{162120} + 2z^{161952} + 2z^{161840} \\
&+ 2z^{161728} + 2z^{161560} + 2z^{161448} + 2z^{161408} + 2z^{161364} + 2z^{161294} + 2z^{161281} + 3z^{161280} \\
&+ 2z^{147840} + 4z^{134400} + 2z^{127680} + 2z^{125440} + 4z^{120960} + 2z^{116480} + 2z^{114240} + 2z^{112896} \\
&+ 2z^{112000} + 2z^{110880} + 2z^{110208} + 2z^{109760} + 2z^{109200} + 2z^{108416} + 2z^{108360} + 2z^{108192} \\
&+ 2z^{108080} + 2z^{107968} + 2z^{107800} + 2z^{107688} + 2z^{107648} + 2z^{107604} + 2z^{107534} + 2z^{107521} \\
&+ 3z^{107520} + 2z^{100800} + 2z^{98560} + 4z^{94080} + 2z^{89600} + 2z^{87360} + 2z^{86016} + 2z^{85120} \\
&+ 2z^{84000} + 2z^{83328} + 2z^{82880} + 2z^{82320} + 2z^{81536} + 2z^{81480} + 2z^{81312} + 2z^{81200} \\
&+ 2z^{81088} + 2z^{80920} + 2z^{80808} + 2z^{80768} + 2z^{80724} + 2z^{80654} + 2z^{80641} + 3z^{80640} + 2z^{73920} \\
&+ 2z^{71680} + 4z^{67200} + 2z^{62720} + 4z^{60480} + 2z^{59136} + 4z^{58240} + 2z^{57120} + 2z^{56448} + 2z^{56000} \\
&+ 2z^{55440} + 2z^{54656} + 2z^{54600} + 2z^{54432} + 2z^{54320} + 2z^{54208} + 2z^{54040} + 2z^{53928} + 2z^{53888} \\
&+ 2z^{53844} + 2z^{53774} + 2z^{53761} + 3z^{53760} + 2z^{49280} + 4z^{47040} + 2z^{45696} + 4z^{44800} + 2z^{43680} \\
&+ 2z^{43008} + 2z^{42560} + 2z^{42000} + 2z^{41216} + 2z^{41160} + 2z^{40992} + 2z^{40880} + 2z^{40768} + 2z^{40600} \\
&+ 2z^{40488} + 2z^{40448} + 2z^{40404} + 2z^{40334} + 2z^{40321} + 3z^{40320} + 2z^{38080} + 3z^{35840} + 4z^{33600} \\
&+ 2z^{32256} + 4z^{31360} + 2z^{30240} + 2z^{29568} + 4z^{29120} + 2z^{28560} + 2z^{27776} + 2z^{27720} + 2z^{27552} \\
&+ 2z^{27440} + 2z^{27328} + 2z^{27160} + 2z^{27048} + 2z^{27008} + 2z^{26964} + 2z^{26894} + 2z^{26881} + 5z^{26880} \\
&+ 2z^{25536} + 4z^{24640} + 2z^{23520} + 2z^{23296} + 2z^{22848} + 6z^{22400} + 2z^{21840} + 2z^{21280} + 2z^{21056} \\
&+ 2z^{21000} + 2z^{20832} + 2z^{20720} + 4z^{20608} + 2z^{20440} + 2z^{20328} + 2z^{20288} + 2z^{20244} + 2z^{20174} \\
&+ 2z^{20161} + 4z^{20160} + 2z^{19600} + 4z^{18816} + 2z^{18760} + 2z^{18592} + 2z^{18480} + 2z^{18368} + 2z^{18200} \\
&+ 2z^{18088} + 2z^{18048} + 2z^{18004} + 2z^{17934} + 2z^{17921} + 3z^{17920} + 2z^{16800} + 2z^{16128} + 4z^{15680} \\
&+ 2z^{15120} + 4z^{14336} + 2z^{14280} + 2z^{14112} + 2z^{14000} + 2z^{13888} + 2z^{13720} + 2z^{13608} + 2z^{13568} \\
&+ 2z^{13524} + 2z^{13454} + 2z^{13441} + 3z^{13440} + 2z^{12320} + 2z^{12096} + 2z^{11648} + 4z^{11200} + z^{10752} \\
&+ 2z^{10640} + 2z^{10080} + 4z^{9856} + 2z^{9800} + 2z^{9632} + 2z^{9520} + 4z^{9408} + 2z^{9240} + 2z^{9128} \\
&+ 2z^{9088} + 2z^{9044} + 2z^{8974} + 2z^{8961} + 3z^{8960} + 2z^{8736} + 2z^{8400} + 2z^{8064} + 2z^{7840} \\
&+ 4z^{7616} + 2z^{7560} + 2z^{7392} + 2z^{7280} + 4z^{7168} + 2z^{7056} + 2z^{7000} + 2z^{6888} + 2z^{6848} \\
&+ 2z^{6804} + 2z^{6734} + 2z^{6721} + 3z^{6720} + 2z^{6272} + 2z^{6216} + 2z^{6160} + 4z^{6048} + 2z^{5936} \\
&+ 2z^{5824} + 2z^{5656} + 2z^{5600} + 2z^{5544} + 2z^{5504} + 2z^{5460} + 2z^{5390} + 2z^{5377} + 3z^{5376} \\
&+ 2z^{5320} + 2z^{5152} + 4z^{5040} + 4z^{4928} + 2z^{4760} + 2z^{4648} + 2z^{4608} + 2z^{4564} + 2z^{4494} \\
&+ 2z^{4481} + z^{4480} + 2z^{4368} + 2z^{4256} + 2z^{4200} + 2z^{4032} + 4z^{3920} + 2z^{3808} + 2z^{3640} + 2z^{3584} \\
&+ 4z^{3528} + 2z^{3488} + 2z^{3444} + 2z^{3374} + 2z^{3361} + 3z^{3360} + 2z^{3248} + 4z^{3136} + 2z^{3080} + 2z^{2968} \\
&+ 2z^{2912} + 2z^{2856} + 2z^{2816} + 2z^{2800} + 2z^{2772} + 2z^{2702} + 2z^{2689} + 2z^{2688} + 2z^{2576} + 4z^{2520} \\
&+ 2z^{2408} + 2z^{2368} + 2z^{2352} + 2z^{2324} + 2z^{2254} + 2z^{2241} + 2z^{2240} + 2z^{2128} + 2z^{1960} + 2z^{1848} \\
&+ 2z^{1808} + z^{1792} + 2z^{1764} + 2z^{1736} + 2z^{1694} + 2z^{1681} + z^{1680} + 2z^{1568} + 2z^{1512} + 2z^{1456} \\
&+ 2z^{1400} + 3z^{1344} + 2z^{1288} + 2z^{1232} + 2z^{1176} + 5z^{1120} + 2z^{1064} + 2z^{1024} + 4z^{1008} + 2z^{980} \\
&+ 2z^{968} + 2z^{952} + 2z^{924} + 2z^{910} + 2z^{897} + z^{896} + 2z^{854} + 2z^{841} + 4z^{840} + 2z^{800} + 2z^{756} \\
&+ 4z^{728} + 2z^{688} + 2z^{686} + 2z^{673} + 2z^{644} + 2z^{616} + 2z^{576} + 2z^{574} + 2z^{561} + z^{560} + 2z^{532} \\
&+ 2z^{462} + 2z^{449} + 2z^{448} + 2z^{408} + 2z^{364} + z^{336} + 2z^{296} + 2z^{294} + 2z^{281} + z^{256} + 2z^{252} \\
&+ 2z^{212} + 2z^{182} + 2z^{169} + z^{168} + 2z^{142} + 2z^{129} + 2z^{98} + 2z^{85} + z^{28} + 2z^{15} + z^2
\end{aligned}
$$

The largest coefficient is six and occurs only for the sum of the cardinalities being: 22400.

The coefficient five occurs for the sum of the cardinalities being: 1120 and 26880.

The coefficient four occurs for the sum of the cardinalities being:
728, 840, 1008, 2520, 3136, 3528, 3920, 4928, 5040, 6048, 7168, 7616, 9408, 9856, 11200, 14336, 15680, 18816, 20160, 20608, 24640, 29120, 31360, 33600, 44800, 47040, 58240, 60480 , 67200 , 94080 , 120960, 134400, 188160.

The coefficient three occurs for the sum of the cardinalities being:
1344, 3360, 5376, 6720, 8960, 13440, 17920, 35840, 40320, 53760, 80640, 107520, 161280, 215040.

The coefficient two occurs for the cardinalities being:
15, 85, 98, 129, 142, 169, 182, 212, 252, 281, 294, 296, 364, 408, 448, 449, 462, 532, 561, 574, 576, 616, 644, 673, 686, 688, 756, 800, 841, 854, 897, 910, 924, 952, 968, 980, 1024, 1064, 1176, 1232, 1288, 1400, 1456, 1512, 1568, 1681, 1694, 1736, 1764, 1808, 1848, 1960, 2128, 2240, 2241, 2254, 2324, 2352, 2368, 2408, 2576, 2688, 2689, 2702, 2772, 2800, 2816, 2856, 2912, 2968, 3080, 3248, 3361, 3374, 3444, 3488, 3584, 3640, 3808, 4032, 4200, 4256, 4368, 4481, 4494, 4564, 4608, 4648, 4760, 5152, 5320, 5377, 5390, 5460, 5504, 5544, 5600, 5656, 5824, 5936, 6160, 6216, 6272, 6721, 6734, 6804, 6848, 6888, 7000, 7056, 7280, 7392, 7560, 7840,8064, 8400, 8736, 8961, 8974, 9044, 9088, 9128, 9240, 9520, 9632, 9800, 10080, 10640, 11648, 12096, 12320, 13441, 13454, 13524, 13568, 13608, 13720, 13888, 14000, 14112, 14280, 15120, 16128, 16800, 17921, 17934, 18004, 18048, 18088, 18200, 18368, 18480, 18592, 18760, 19600, 20161, 20174, 20244, 20288, 20328, 20440, 20720, 20832, 21000, 21056, 21280, 21840, 22848, 23296, 23520, 25536, 26881, 26894, 26964, 27008, 27048, 27160, 27328, 27440, 27552, 27720, 27776, 28560, 29568, 30240, 32256, 38080, 40321, 40334, 40404, 40448, 40488, 40600, 40768, 40880, 40992, 41160, 41216, 42000, 42560, 43008, 43680, 45696, 49280, 53761, 53774, 53844, 53888, 53928, 54040, 54208, 54320, 54432, 54600, 54656, 55440, 56000, 56448, 57120, 59136, 62720, 71680, 73920, 80641, 80654, 80724, 80768, 80808, 80920, 81088, 81200, 81312, 81480, 81536, 82320, 82880, 83328, 84000, 85120, 86016, 87360, 89600, 98560, 100800, 107521, 107534, 107604, 107648, 107688, 107800, 107968, 108080, 108192, 108360, 108416, 109200, 109760, 110208, 110880, 112000, 112896, 114240, 116480, 125440, 127680, 147840, 161281, 161294, 161364, 161408, 161448, 161560, 161728, 161840, 161952, 162120, 162176, 162960, 163520, 163968, 164640, 165760, 166656, 168000, 170240, 174720, 179200, 181440, 201600, 241920, 268800, 322561, 322574, 322644, 322688, 322728, 322840, 323008, 323120, 323232, 323400, 323456, 324240, 324800, 325248, 325920, 327040, 327936, 329280, 331520, 336000, 340480, 342720, 349440, 362880, 376320, 403200, 430080, 483840, 645121, 645134, 645204, 645248, 645288, 645400, 645568, 645680, 645792, 645960, 646016, 646800, 647360, 647808, 648480, 649600, 650496, 651840, 654080, 658560, 663040, 665280, 672000, 685440, 725760, 752640, 806400, 967680.

The above sums of cardinalities occur for *unique relations* between quantities when one of these 348 numbers occur.

Example B.7.1. *The quantity mass m has cardinality 14 and velocity squared v^2 has cardinality 84. The quantity equation mv^2 has cardinality 98 and thus is a unique quantity equation. Figure* **??** *gives the confirmation of the uniqueness of the quantity equation based on the value of the semi-perimeter of parallelogram representing the quantity equation $E = mv^2$.*

Example B.7.2. *The quantity linear momentum p has cardinality 280 and the quantity velocity v has cardinality 84. The quantity equation pv has cardinality 364 and thus is a unique quantity equation. This is confirmed in the histogram of parallelograms of the quantity energy E.*

Observe that the coefficient one occurs when the sum of the cardinalities is:
2, 28, 168, 256, 336, 560, 896, 1680, 1792, 4480, 10752, 322560, 1290240. We expect the occurrence of degenerated relations between quantities when one of these 13 numbers occur.

Example B.7.3. *The quantity volume V has cardinality 14. Consider the quantity equation $V \cdot V$ that has cardinality 28. Looking up in the table of sums of cardinalities we find that it is a degenerated quantity equation because it corresponds to a degenerated parallelogram in $\mathbb{Z}^7$ in the histogram of parallelograms of the quantity volume squared V^2.*

Example B.7.4. *The quantity energy E has cardinality 840. Consider the quantity equation $E \cdot E$ that has cardinality 1680. Looking up in the table of sums of cardinalities we find that it is a degenerated quantity equation because it corresponds to a degenerated parallelogram in $\mathbb{Z}^7$ in the histogram of parallelograms of the quantity energy squared E^2.*

Consider the set of cardinalities of the orbits of $\mathbb{Z}^7$ and denote it C7. Let us create the Cartesian product $C7 \times C7$ and define the map $S7 : C7 \times C7 \to \mathbb{N} : S7(x, y) = x + y$ where $(x, y) \in C7 \times C7$ resulting in the table given in Figure B.9:

	1	14	84	128	168	280	448	560	672	840	896	1680	2240	2688	3360	4480	5376	6720	8960	13440	17920	20160	26880	40320	53760	80640	107520	161280	322560	645120
1	**2**	15	85	129	169	281	449	561	673	841	897	1681	2241	2689	3361	4481	5377	6721	8961	13441	17921	20161	26881	40321	53761	80641	107521	161281	322561	645121
14	15	**28**	98	142	182	294	462	574	686	854	910	1694	2254	2702	3374	4494	5390	6734	8974	13454	17934	20174	26894	40334	53774	80654	107534	161294	322574	645134
84	85	98	**168**	212	252	364	532	644	756	924	980	1764	2324	2772	3444	4564	5460	6804	9044	13524	18004	20244	26964	40404	53844	80724	107604	161364	322644	645204
128	129	142	212	**256**	296	408	576	688	800	968	1024	1808	2368	2816	3488	4608	5504	6848	9088	13568	18048	20288	27008	40448	53888	80768	107648	161408	322688	645248
168	169	182	252	296	**336**	448	616	728	840	1008	1064	1848	2408	2856	3528	4648	5544	6888	9128	13608	18088	20328	27048	40488	53928	80808	107688	161448	322728	645288
280	281	294	364	408	448	**560**	728	840	952	1120	1176	1960	2520	2968	3640	4760	5656	7000	9240	13720	18200	20440	27160	40600	54040	80920	107800	161560	322840	645400
448	449	462	532	576	616	728	**896**	1008	1120	1288	1344	2128	2688	3136	3808	4928	5824	7168	9408	13888	18368	20608	27328	40768	54208	81088	107968	161728	323008	645568
560	561	574	644	688	728	840	1008	1120	1232	1400	1456	2240	2800	3248	3920	5040	5936	7280	9520	14000	18480	20720	27440	40880	54320	81200	108080	161840	323120	645680
672	673	686	756	800	840	952	1120	1232	1344	1512	1568	2352	2912	3360	4032	5152	6048	7392	9632	14112	18592	20832	27552	40992	54432	81312	108192	161952	323232	645792
840	841	854	924	968	1008	1120	1288	1400	1512	**1680**	1736	2520	3080	3528	4200	5320	6216	7560	9800	14280	18760	21000	27720	41160	54600	81480	108360	162120	323400	645960
896	897	910	980	1024	1064	1176	1344	1456	1568	1736	**1792**	2576	3136	3584	4256	5376	6272	7616	9856	14336	18816	21056	27776	41216	54656	81536	108416	162176	323456	646016
1680	1681	1694	1764	1808	1848	1960	2128	2240	2352	2520	2576	3360	3920	4368	5040	6160	7056	8400	10640	15120	19600	21840	28560	42000	55440	82320	109200	162960	324240	646800
2240	2241	2254	2324	2368	2408	2520	2688	2800	2912	3080	3136	3920	**4480**	4928	5600	6720	7616	8960	11200	15680	20160	22400	29120	42560	56000	82880	109760	163520	324800	647360
2688	2689	2702	2772	2816	2856	2968	3136	3248	3360	3528	3584	4368	4928	5376	6048	7168	8064	9408	11648	16128	20608	22848	29568	43008	56448	83328	110208	163968	325248	647808
3360	3361	3374	3444	3488	3528	3640	3808	3920	4032	4200	4256	5040	5600	6048	6720	7840	8736	10080	12320	16800	21280	23520	30240	43680	57120	84000	110880	164640	325920	648480
4480	4481	4494	4564	4608	4648	4760	4928	5040	5152	5320	5376	6160	6720	7168	7840	8960	9856	11200	13440	17920	22400	24640	31360	44800	58240	85120	112000	165760	327040	649600
5376	5377	5390	5460	5504	5544	5656	5824	5936	6048	6216	6272	7056	7616	8064	8736	9856	**10752**	12096	14336	18816	23296	25536	32256	45696	59136	86016	112896	166656	327936	650496
6720	6721	6734	6804	6848	6888	7000	7168	7280	7392	7560	7616	8400	8960	9408	10080	11200	12096	13440	15680	20160	24640	26880	33600	47040	60480	87360	114240	168000	329280	651840
8960	8961	8974	9044	9088	9128	9240	9408	9520	9632	9800	9856	10640	11200	11648	12320	13440	14336	15680	17920	22400	26880	29120	35840	49280	62720	89600	116480	170240	331520	654080
13440	13441	13454	13524	13568	13608	13720	13888	14000	14112	14280	14336	15120	15680	16128	16800	17920	18816	20160	22400	26880	31360	33600	40320	53760	67200	94080	120960	174720	336000	658560
17920	17921	17934	18004	18048	18088	18200	18368	18480	18592	18760	18816	19600	20160	20608	21280	22400	23296	24640	26880	31360	35840	38080	44800	58240	71680	98560	125440	179200	340480	663040
20160	20161	20174	20244	20288	20328	20440	20608	20720	20832	21000	21056	21840	22400	22848	23520	24640	25536	26880	29120	33600	38080	40320	47040	60480	73920	100800	127680	181440	342720	665280
26880	26881	26894	26964	27008	27048	27160	27328	27440	27552	27720	27776	28560	29120	29568	30240	31360	32256	33600	35840	40320	44800	47040	53760	67200	80640	107520	134400	188160	349440	672000
40320	40321	40334	40404	40448	40488	40600	40768	40880	40992	41160	41216	42000	42560	43008	43680	44800	45696	47040	49280	53760	58240	60480	67200	80640	94080	120960	147840	201600	362880	685440
53760	53761	53774	53844	53888	53928	54040	54208	54320	54432	54600	54656	55440	56000	56448	57120	58240	59136	60480	62720	67200	71680	73920	80640	94080	107520	134400	161280	215040	376320	698880
80640	80641	80654	80724	80768	80808	80920	81088	81200	81312	81480	81536	82320	82880	83328	84000	85120	86016	87360	89600	94080	98560	100800	107520	120960	134400	161280	188160	241920	403200	725760
107520	107521	107534	107604	107648	107688	107800	107968	108080	108192	108360	108416	109200	109760	110208	110880	112000	112896	114240	116480	120960	125440	127680	134400	147840	161280	188160	215040	268800	430080	752640
161280	161281	161294	161364	161408	161448	161560	161728	161840	161952	162120	162176	162960	163520	163968	164640	165760	166656	168000	170240	174720	179200	181440	188160	201600	215040	241920	268800	**322560**	483840	806400
322560	322561	322574	322644	322688	322728	322840	323008	323120	323232	323400	323456	324240	324800	325248	325920	327040	327936	329280	331520	336000	340480	342720	349440	362880	376320	403200	430080	483840	645120	967680
645120	645121	645134	645204	645248	645288	645400	645568	645680	645792	645960	646016	646800	647360	647808	648480	649600	650496	651840	654080	658560	663040	665280	672000	685440	698880	725760	752640	806400	967680	**1290240**

Figure B.9: Table of sums of cardinalities of the orbits of $\mathbb{Z}^7$

We can summarize the above results reflecting the multiplicities m of the ternary relations by defining a generating polynomial $v(z;7)$ as:

$$v(z;7) = z^6 + 2z^5 + 33z^4 + 14z^3 + 348z^2 + 13z\,, \tag{B.7}$$

where the coefficient in the polynomial $v(z;7)$ gives the number of sums of cardinalities in ternary equations of multiplicity m. The multiplicity is given by the exponent of the indeterminate z. We conclude that there are 411 distinct sums of cardinalities for ternary dimensionless quantity equations $f(\pi) = \frac{z}{xy}$ in $\mathbb{Z}^7$.

B.7.4 Enumeration of the hypercube HC_7^3

The Table B.22 of the hypercube HC_7^3 consists of 8 columns. The first column contains the values of the infinity norm $\|z\|_\infty$. The second column is the row identifier for each infinity norm. The third column gives the representative of the orbit. The fourth column contains the sum of the absolute value of the coordinates of the lattice points being elements of the orbit that is exclusively the total degree of the monomial associated with the orbit representative. The fifth column gives the parity of the representative of the orbit. The sixth column gives the ℓ_1-norm of the representative of the orbit. The seventh column gives the cardinality of the orbit. The eighth column gives the Gödel number of the representative of the orbit. The ordering of the orbits is based on graded reverse lex order (Cox et al., 1997). Hence, the N-sphere shells can be considered as the union of some orbits of $\mathbb{Z}^7$.

Table B.22: Partitions of the hypercube HC_7^3 of $\{0\} \times \mathbb{Z}^7$.

Infinity norm	ID	Orbit	Degree	psoc (z)	$\|z\|_1$	$\#(vertices)$	Gödel number
0	1	$[(0 \mid 0^7)]$	0	0	0	1	1
1	1	$[(0 \mid 1, 0^6)]$	1	1	1	14	2
1	2	$[(0 \mid 1^2, 0^5)]$	2	0	2	84	6
1	3	$[(0 \mid 1^3, 0^4)]$	3	1	3	280	30
1	4	$[(0 \mid 1^4, 0^3)]$	4	0	4	560	210
1	5	$[(0 \mid 1^5, 0^2)]$	5	1	5	672	2 310
1	6	$[(0 \mid 1^6, 0)]$	6	0	6	448	30 030
1	7	$[(0 \mid 1^7)]$	7	1	7	128	510 510
2	1	$[(0 \mid 2, 0^6)]$	2	0	4	14	4
2	2	$[(0 \mid 2, 1, 0^5)]$	3	1	5	168	12
2	3	$[(0 \mid 2, 1^2, 0^4)]$	4	0	6	840	60
2	4	$[(0 \mid 2^2, 0^5)]$	4	0	8	84	36
2	5	$[(0 \mid 2, 1^3, 0^3)]$	5	1	7	2 240	420
2	6	$[(0 \mid 2^2, 1, 0^4)]$	5	1	9	840	180
2	7	$[(0 \mid 2, 1^4, 0^2)]$	6	0	8	3 360	4 620
2	8	$[(0 \mid 2^2, 1^2, 0^3)]$	6	0	10	3 360	1 260
2	9	$[(0 \mid 2^3, 0^4)]$	6	0	12	280	900
2	10	$[(0 \mid 2, 1^5, 0)]$	7	1	9	2 688	60 060
2	11	$[(0 \mid 2^2, 1^3, 0^2)]$	7	1	11	6 720	13 860
2	12	$[(0 \mid 2^3, 1, 0^3)]$	7	1	13	2 240	6 300
2	13	$[(0 \mid 2, 1^6)]$	8	0	10	896	1 021 020
2	14	$[(0 \mid 2^2, 1^4, 0)]$	8	0	12	6 720	180 180
…	…	…	…	…	…	…	…

Infinity norm	ID	Orbit	Degree	psoc (z)	$\|z\|_1$	$\#(vertices)$	Gödel number
2	15	$[(0 \mid 2^3, 1^2, 0^2)]$	8	0	14	6 720	69 300
2	16	$[(0 \mid 2^4, 0^3)]$	8	0	16	560	44 100
2	17	$[(0 \mid 2^2, 1^5)]$	9	1	13	2 688	3 063 060
2	18	$[(0 \mid 2^3, 1^3, 0)]$	9	1	15	8 960	900 900
2	19	$[(0 \mid 2^4, 1, 0^2)]$	9	1	17	3 360	485 100
2	20	$[(0 \mid 2^3, 1^4)]$	10	0	16	4 480	15 315 300
2	21	$[(0 \mid 2^4, 1^2, 0)]$	10	0	18	6 720	6 306 300
2	22	$[(0 \mid 2^5, 0^2)]$	10	0	20	672	5 336 100
2	23	$[(0 \mid 2^4, 1^3)]$	11	1	19	4 480	107 207 100
2	24	$[(0 \mid 2^5, 1, 0)]$	11	1	21	2 688	69 369 300
2	25	$[(0 \mid 2^5, 1^2)]$	12	0	22	2 688	1 179 278 100
2	26	$[(0 \mid 2^6, 0)]$	12	0	24	448	901 800 900
2	27	$[(0 \mid 2^6, 1)]$	13	1	25	896	15 330 615 300
2	28	$[(0 \mid 2^7)]$	14	0	28	128	260 620 460 100
3	1	$[(0 \mid 3, 0^6)]$	3	1	9	14	8
3	2	$[(0 \mid 3, 1, 0^5)]$	4	0	10	168	24
3	3	$[(0 \mid 3, 1^2, 0^4)]$	5	1	11	840	120
3	4	$[(0 \mid 3, 2, 0^5)]$	5	1	13	168	72
3	5	$[(0 \mid 3, 1^3, 0^3)]$	6	0	12	2 240	840
3	6	$[(0 \mid 3, 2, 1, 0^4)]$	6	0	14	1 680	360
3	7	$[(0 \mid 3^2, 0^5)]$	6	0	18	84	216
3	8	$[(0 \mid 3, 1^4, 0^2)]$	7	1	13	3 360	9 240
3	9	$[(0 \mid 3, 2, 1^2, 0^3)]$	7	1	15	6 720	2 520
3	10	$[(0 \mid 3, 2^2, 0^4)]$	7	1	17	840	1 800
…	…	…	…	…	…	…	…

Infinity norm	ID	Orbit	Degree	psoc (z)	$\|z\|_1$	$\#(vertices)$	Gödel number
3	11	$[(0 \mid 3^2, 1, 0^4)]$	7	1	19	840	1 080
3	12	$[(0 \mid 3, 1^5, 0)]$	8	0	14	2 688	120 120
3	13	$[(0 \mid 3, 2, 1^3, 0^2)]$	8	0	16	13 440	27 720
3	14	$[(0 \mid 3, 2^2, 1, 0^3)]$	8	0	18	6 720	12 600
3	15	$[(0 \mid 3^2, 1^2, 0^3)]$	8	0	20	3 360	7 560
3	16	$[(0 \mid 3^2, 2, 0^4)]$	8	0	22	840	5 400
3	17	$[(0 \mid 3, 1^6)]$	9	1	15	896	2 042 040
3	18	$[(0 \mid 3, 2, 1^4, 0)]$	9	1	17	13 440	360 360
3	19	$[(0 \mid 3, 2^2, 1^2, 0^2)]$	9	1	19	20 160	138 600
3	20	$[(0 \mid 3, 2^3, 0^3)]$	9	1	21	2 240	88 200
3	21	$[(0 \mid 3^2, 1^3, 0^2)]$	9	1	21	6 720	83 160
3	22	$[(0 \mid 3^2, 2, 1, 0^3)]$	9	1	23	6 720	37 800
3	23	$[(0 \mid 3^3, 0^4)]$	9	1	27	280	27 000
3	24	$[(0 \mid 3, 2, 1^5)]$	10	0	18	5 376	6 126 120
3	25	$[(0 \mid 3, 2^2, 1^3, 0)]$	10	0	20	26 880	1 801 800
3	26	$[(0 \mid 3, 2^3, 1, 0^2)]$	10	0	22	13 440	970 200
3	27	$[(0 \mid 3^2, 1^4, 0)]$	10	0	22	6 720	1 081 080
3	28	$[(0 \mid 3^2, 2, 1^2, 0^2)]$	10	0	24	20 160	415 800
3	29	$[(0 \mid 3^2, 2^2, 0^3)]$	10	0	26	3 360	264 600
3	30	$[(0 \mid 3^3, 1, 0^3)]$	10	0	28	2 240	189 000
3	31	$[(0 \mid 3, 2^2, 1^4)]$	11	1	21	13 440	30 630 600
3	32	$[(0 \mid 3, 2^3, 1^2, 0)]$	11	1	23	26 880	12 612 600
3	33	$[(0 \mid 3, 2^4, 0^2)]$	11	1	25	3 360	10 672 200
3	34	$[(0 \mid 3^2, 1^5)]$	11	1	23	2 688	18 378 360
...	...	...	...	...	...	...	...

Infinity norm	ID	Orbit	Degree	psoc (z)	$\|z\|_1$	$\#(vertices)$	Gödel number
3	35	$[(0 \mid 3^2, 2, 1^3, 0)]$	11	1	25	26 880	5 405 400
3	36	$[(0 \mid 3^2, 2^2, 1, 0^2)]$	11	1	27	20 160	2 910 600
3	37	$[(0 \mid 3^3, 1^2, 0^2)]$	11	1	29	6 720	2 079 000
3	38	$[(0 \mid 3^3, 2, 0^3)]$	11	1	31	2 240	1 323 000
3	39	$[(0 \mid 3, 2^3, 1^3)]$	12	0	24	17 920	214 414 200
3	40	$[(0 \mid 3, 2^4, 1, 0)]$	12	0	26	13 440	138 738 600
3	41	$[(0 \mid 3^2, 2, 1^4)]$	12	0	26	13 440	91 891 800
3	42	$[(0 \mid 3^2, 2^2, 1^2, 0)]$	12	0	28	40 320	37 837 800
3	43	$[(0 \mid 3^2, 2^3, 0^2)]$	12	0	30	6 720	32 016 600
3	44	$[(0 \mid 3^3, 1^3, 0)]$	12	0	30	8 960	27 027 000
3	45	$[(0 \mid 3^3, 2, 1, 0^2)]$	12	0	32	13 440	14 553 000
3	46	$[(0 \mid 3^4, 0^3)]$	12	0	36	560	9 261 000
3	47	$[(0 \mid 3, 2^4, 1^2)]$	13	1	27	13 440	2 358 556 200
3	48	$[(0 \mid 3, 2^5, 0)]$	13	1	29	2 688	1 803 601 800
3	49	$[(0 \mid 3^2, 2^2, 1^3)]$	13	1	29	26 880	643 242 600
3	50	$[(0 \mid 3^2, 2^3, 1, 0)]$	13	1	31	26 880	416 215 800
3	51	$[(0 \mid 3^3, 1^4)]$	13	1	31	4 480	459 459 000
3	52	$[(0 \mid 3^3, 2, 1^2, 0)]$	13	1	33	26 880	189 189 000
3	53	$[(0 \mid 3^3, 2^2, 0^2)]$	13	1	35	6 720	160 083 000
3	54	$[(0 \mid 3^4, 1, 0^2)]$	13	1	37	3 360	101 871 000
3	55	$[(0 \mid 3, 2^5, 1)]$	14	0	30	5 376	30 661 260 600
3	56	$[(0 \mid 3^2, 2^3, 1^2)]$	14	0	32	26 880	7 075 668 600
3	57	$[(0 \mid 3^2, 2^4, 0)]$	14	0	34	6 720	5 410 805 400
3	58	$[(0 \mid 3^3, 2, 1^3)]$	14	0	34	17 920	3 216 213 000
…	…	…	…	…	…	…	…

Infinity norm	ID	Orbit	Degree	psoc (z)	$\|z\|_1$	$\#\,(vertices)$	Gödel number
3	59	$[(0 \mid 3^3, 2^2, 1, 0)]$	14	0	36	26 880	2 081 079 000
3	60	$[(0 \mid 3^4, 1^2, 0)]$	14	0	38	6 720	1 324 323 000
3	61	$[(0 \mid 3^4, 2, 0^2)]$	14	0	40	3 360	1 120 581 000
3	62	$[(0 \mid 3, 2^6)]$	15	1	33	896	521 240 920 200
3	63	$[(0 \mid 3^2, 2^4, 1)]$	15	1	35	13 440	91 983 691 800
3	64	$[(0 \mid 3^3, 2^2, 1^2)]$	15	1	37	26 880	35 378 343 000
3	65	$[(0 \mid 3^3, 2^3, 0)]$	15	1	39	8 960	27 054 027 000
3	66	$[(0 \mid 3^4, 1^3)]$	15	1	39	4 480	22 513 491 000
3	67	$[(0 \mid 3^4, 2, 1, 0)]$	15	1	41	13 440	14 567 553 000
3	68	$[(0 \mid 3^5, 0^2)]$	15	1	45	672	12 326 391 000
3	69	$[(0 \mid 3^2, 2^5)]$	16	0	38	2,688	1 563 722 760 600
3	70	$[(0 \mid 3^3, 2^3, 1)]$	16	0	40	17 920	459 918 459 000
3	71	$[(0 \mid 3^4, 2, 1^2)]$	16	0	42	13 440	247 648 401 000
3	72	$[(0 \mid 3^4, 2^2, 0)]$	16	0	44	6 720	189 378 189 000
3	73	$[(0 \mid 3^5, 1, 0)]$	16	0	46	2 688	160 243 083 000
3	74	$[(0 \mid 3^3, 2^4)]$	17	1	43	4 480	7 818 613 803 000
3	75	$[(0 \mid 3^4, 2^2, 1)]$	17	1	45	13 440	3 219 429 213 000
3	76	$[(0 \mid 3^5, 1^2)]$	17	1	47	2 688	2 724 132 411 000
3	77	$[(0 \mid 3^5, 2, 0)]$	17	1	49	2 688	2 083 160 079 000
3	78	$[(0 \mid 3^4, 2^3)]$	18	0	48	4 480	54 730 296 621 000
3	79	$[(0 \mid 3^5, 2, 1)]$	18	0	50	5 376	35 413 721 343 000
3	80	$[(0 \mid 3^6, 0)]$	18	0	54	448	27 081 081 027 000
3	81	$[(0 \mid 3^5, 2^2)]$	19	1	53	2 688	602 033 262 831 000
3	82	$[(0 \mid 3^6, 1)]$	19	1	55	896	460 378 377 459 000
...	...	...	...	...	...	...	...

Infinity norm	ID	Orbit	Degree	psoc (z)	$\|z\|_1$	$\#\,(vertices)$	Gödel number
3	83	$[(0 \mid 3^6, 2)]$	20	0	58	896	7 826 432 416 803 000
3	84	$[(0 \mid 3^7)]$	21	1	63	128	133 049 351 085 651 000

B.7.5 Partitioning of seven-dimensional hyperspheres in orbits of $\{0\} \times \mathbb{Z}^7$

Table B.23 gives the relation between the sequence A008451 and the partitioning of seven-dimensional hyperspheres in orbits of orbits of $\{0\} \times \mathbb{Z}^7$.

Observe that each term of the sequence A008451 (N. J. A. Sloane, 2006) given by $r_7(m)$ = 1, 14, 84, 280, 574, 840, 1288, 2368, 3444, 3542, 4424, 7560, 9240, 8456, 11088, 16576, 18494, 17808, 19740, 27720, 34440, 29456, 31304, 49728, 52808, 43414, 52248, 68320, 74048, 68376, 71120, 99456, 110964, 89936, 94864, 136080 ... is obtained by the addition of the cardinalities of orbits of $\{0\} \times \mathbb{Z}^7$.

Table B.23: Partitioning of seven-dimensional hyperspheres in orbits of $\{0\} \times \mathbb{Z}^7$

m	Disjunct union of orbits of $\{0\} \times \mathbb{Z}^7$	$r_7(m)$
0	$[(0 \mid 0^7)]$	1
1	$[(0 \mid 1, 0^6)]$	14
2	$[(0 \mid 1^2, 0^5)]$	84
3	$[(0 \mid 1^3, 0^4)]$	280
4	$[(0 \mid 1^4, 0^3)] \cup [(0 \mid 2, 0^6)]$	574
5	$[(0 \mid 1^5, 0^2)] \cup [(0 \mid 2, 1, 0^5)]$	840
6	$[(0 \mid 1^6, 0)] \cup [(0 \mid 2, 1^2, 0^4)]$	1 288
7	$[(0 \mid 1^7)] \cup [(0 \mid 2, 1^3, 0^3)]$	2 368
8	$[(0 \mid 2^2, 0^5)] \cup [(0 \mid 2, 1^4, 0^2)]$	3 444
9	$[(0 \mid 2^2, 1, 0^4)] \cup [(0 \mid 2, 1^5, 0)] \cup [(0 \mid 3, 0^6)]$	3 542
10	$[(0 \mid 2^2, 1^2, 0^3)] \cup [(0 \mid 2, 1^6)] \cup [(0 \mid 3, 1, 0^5)]$	4 424
11	$[(0 \mid 2^2, 1^3, 0^2)] \cup [0(\mid 3, 1^2, 0^4)]$	7 560
12	$[(0 \mid 2^3, 0^4)] \cup [(0 \mid 2^2, 1^4, 0)] \cup [(0 \mid 3, 1^3, 0^3)]$	9 240
13	$[(0 \mid 2^3, 1, 0^3)] \cup [(0 \mid 2^2, 1^5)] \cup [(0 \mid 3, 2, 0^5)] \cup [(0 \mid 3, 1^4, 0^2)]$	8 456
14	$[(0 \mid 2^3, 1^2, 0^2)] \cup [(0 \mid 3, 2, 1, 0^4)] \cup [(0 \mid 3, 1^5, 0)]$	11 088
15	$[(0 \mid 2^3, 1^3, 0)] \cup [(0 \mid 3, 2, 1^2, 0^3)] \cup [(0 \mid 3, 1^6)]$	16 576
16	$[(0 \mid 2^4, 0^3)] \cup [(0 \mid 2^3, 1^4)] \cup [(0 \mid 3, 2, 1^3, 0^2)] \cup [(0 \mid 4, 0^6)]$	18 494
17	$[(0 \mid 2^4, 1, 0^2)] \cup [(0 \mid 3, 2^2, 0^4)] \cup [(0 \mid 3, 2, 1^4, 0)] \cup [(0 \mid 4, 1, 0^5)]$	17 808
18	$[(0 \mid 2^4, 1^2, 0)] \cup [(0 \mid 3^2, 0^5)] \cup [(0 \mid 3, 2^2, 1, 0^3)] \cup [(0 \mid 3, 2, 1^5)] \cup [(0 \mid 4, 1^2, 0^4)]$	19 740
19	$[(0 \mid 2^4, 1^3)] \cup [(0 \mid 3^2, 1, 0^4)] \cup [(0 \mid 3, 2^2, 1^2, 0^2)] \cup [(0 \mid 4, 1^3, 0^3)]$	27 720
20	$[(0 \mid 2^5, 0^2)] \cup [(0 \mid 3^2, 1^2, 0^3)] \cup [(0 \mid 3, 2^2, 1^3, 0)] \cup [(0 \mid 4, 1^4, 0^2)] \cup [(0 \mid 4, 2, 0^5)]$	34 440
21	$[(0 \mid 2^5, 1, 0)] \cup [(0 \mid 3, 2^3, 0^3)] \cup [(0 \mid 3^2, 1^3, 0^2)] \cup [(0 \mid 3, 2^2, 1^4)] \cup [(0 \mid 4, 1^5, 0)] \cup [(0 \mid 4, 2, 1, 0^4)]$	29 456
22	$[(0 \mid 2^5, 1^2)] \cup [(0 \mid 3^2, 2, 0^4)] \cup [(0 \mid 3, 2^3, 1, 0^2)] \cup [(0 \mid 3^2, 1^4, 0)] \cup [(0 \mid 4, 1^6)] \cup [(0 \mid 4, 2, 1^2, 0^3)]$	31 304
...	...	...

m	Disjunct union of orbits of $\{0\} \times \mathbb{Z}^7$	$r_7(m)$
23	$[(0 \mid 3^2, 2, 1, 0^3)] \cup [(0 \mid 3, 2^3, 1^2, 0)] \cup [(0 \mid 3^2, 1^5)] \cup [(0 \mid 4, 2, 1^3, 0^2)]$	49 728
24	$[(0 \mid 2^6, 0)] \cup [(0 \mid 3^2, 2, 1^2, 0^2)] \cup [(0 \mid 3, 2^3, 1^3)] \cup [(0 \mid 4, 2, 1^4, 0)] \cup [(0 \mid 4, 2^2, 0^4)]$	52 808
25	$[(0 \mid 2^6, 1)] \cup [(0 \mid 3, 2^4, 0^2)] \cup [(0 \mid 3^2, 2, 1^3, 0)] \cup [(0 \mid 4, 2, 1^5)] \cup [(0 \mid 4, 2^2, 1, 0^3)]$ $\cup[(0 \mid 4, 3, 0^5)] \cup [(0 \mid 5, 0^6)]$	43 414
26	$[(0 \mid 3^2, 2^2, 0^3)] \cup [(0 \mid 3, 2^4, 1, 0)] \cup [(0 \mid 3^2, 2, 1^4)] \cup [(0 \mid 4, 2^2, 1^2, 0^2)] \cup [(0 \mid 4, 3, 1, 0^4)]$ $\cup[(0 \mid 5, 1, 0^5)]$	52 248
27	$[(0 \mid 3^3, 0^4)] \cup [(0 \mid 3^2, 2^2, 1, 0^2)] \cup [(0 \mid 3, 2^4, 1^2)] \cup [(0 \mid 4, 2^2, 1^3, 0)] \cup [(0 \mid 4, 3, 1^2, 0^3)]$ $\cup[(0 \mid 5, 1^2, 0^4)]$	68 320
28	$[(0 \mid 2^7)] \cup [(0 \mid 3^3, 1, 0^3)] \cup [(0 \mid 3^2, 2^2, 1^2, 0)] \cup [(0 \mid 4, 2^2, 1^4)] \cup [(0 \mid 4, 2^3, 0^3)]$ $\cup[(0 \mid 4, 3, 1^3, 0^2)] \cup [(0 \mid 5, 1^3, 0^3)]$	74 048
29	$[(0 \mid 3^3, 1^2, 0^2)] \cup [(0 \mid 3, 2^5, 0)] \cup [(0 \mid 3^2, 2^2, 1^3)] \cup [(0 \mid 4, 2^3, 1, 0^2)] \cup [(0 \mid 4, 3, 1^4, 0)]$ $\cup[(0 \mid 4, 3, 2, 0^4)] \cup [(0 \mid 5, 1^4, 0^2)] \cup [(0 \mid 5, 2, 0^5)]$	68 376
30	$[(0 \mid 3^2, 2^3, 0^2)] \cup [(0 \mid 3^3, 1^3, 0)] \cup [(0 \mid 3, 2^5, 1)] \cup [(0 \mid 4, 2^3, 1^2, 0)] \cup [(0 \mid 4, 3, 1^5)]$ $\cup[(0 \mid 4, 3, 2, 1, 0^3)] \cup [(0 \mid 5, 1^5, 0)] \cup [(0 \mid 5, 2, 1, 0^4)]$	71 120
31	$[(0 \mid 3^3, 2, 0^3)] \cup [(0 \mid 3^2, 2^3, 1, 0)] \cup [(0 \mid 3^3, 1^4)] \cup [(0 \mid 4, 2^3, 1^3)] \cup [(0 \mid 4, 3, 2, 1^2, 0^2)]$ $\cup[(0 \mid 5, 1^6)] \cup [(0 \mid 5, 2, 1^2, 0^3)]$	99 456
32	$[(0 \mid 3^3, 2, 1, 0^2)] \cup [(0 \mid 3^2, 2^3, 1^2)] \cup [(0 \mid 4, 2^4, 0^2)] \cup [(0 \mid 4^2, 0^5)] \cup [(0 \mid 4, 3, 2, 1^3, 0)]$ $\cup[(0 \mid 5, 2, 1^3, 0^2)]$	110 964
33	$[(0 \mid 3^3, 2, 1^2, 0)] \cup [(0 \mid 3, 2^6)] \cup [(0 \mid 4, 2^4, 1, 0)] \cup [(0 \mid 4^2, 1, 0^4)] \cup [(0 \mid 4, 3, 2, 1^4)]$ $\cup[(0 \mid 4, 3, 2^2, 0^3)] \cup [(0 \mid 5, 2, 1^4, 0)] \cup [(0 \mid 5, 2^2, 0^4)]$	89 936
34	$[(0 \mid 3^2, 2^4, 0)] \cup [(0 \mid 3^3, 2, 1^3)] \cup [(0 \mid 4, 2^4, 1^2)] \cup [(0 \mid 4, 3, 2^2, 1, 0^2)] \cup [(0 \mid 4, 3^2, 0^4)]$ $\cup[(0 \mid 4^2, 1^2, 0^3)] \cup [(0 \mid 5, 2, 1^5)] \cup [(0 \mid 5, 2^2, 1, 0^3)] \cup [(0 \mid 5, 3, 0^5)]$	94 864
35	$[(0 \mid 3^3, 2^2, 0^2)] \cup [(0 \mid 3^2, 2^4, 1)] \cup [(0 \mid 4, 3^2, 1, 0^3)] \cup [(0 \mid 4^2, 1^3, 0^2)] \cup [(0 \mid 4, 3, 2^2, 1^2, 0)]$ $\cup[(0 \mid 5, 2^2, 1^2, 0^2)] \cup [(0 \mid 5, 3, 1, 0^4)]$	136 080

B.7.6 Number of orbits as function of the infinity norm

The number of orbits in $\mathbb{Z}^7$ for a hypercube with infinity norm $\ell_\infty = s$ is $\#\left(HC_7^s\right) = \binom{s+7-1}{7-1}$. We find the truncated OEIS integer sequence A000579 (N. J. A. Sloane, 2015) for $s \in [1..10]$: $a(s)$= 1, 7, 28, 84, 210, 462, 924, 1 716, 3 003, 5 005, 8 008.

B.7.7 Properties of the 0-skeletons of the hypercubes HC_7^s

We denote the 0-skeletons of the hypercubes HC_7^s as $[a]$. The number of orbits (Table B.24) with infinity norm $\|z\|_\infty \leq s$ in $\mathbb{Z}^7$ is the result from the application of lemma 4 (Cox et al., 1997).

Table B.24: Properties of the 0-skeletons of the hypercubes HC_7^s in $\mathbb{Z}^7$ for $s \leq 10$.

$\|z\|_\infty = s$	$\#([a])$	Cumulative of $\#([a])$	$\#\left(HC_7^s\right)$	Cumulative of $\#\left(HC_7^s\right)$
0	1	1	1	1
1	2 186	2 187	7	8
2	75 938	78 125	28	36
3	745 418	823 543	84	120
4	3 959 426	4 782 969	210	330
5	14 704 202	19 487 171	462	792
6	43 261 346	62 748 517	924	1 716
7	108 110 858	170 859 375	1 716	3 432
8	239 479 298	410 338 673	3 003	6 435
9	483 533 066	893 871 739	5 005	11 440
10	907 216 802	1 801 088 541	8 008	19 448

The second column of Table B.24 shows the number of integer lattice points in the 0-skeleton of the hypercubes HC_7^s. The third column gives the cumulated number of integer lattice points. The fourth and fifth columns have a similar meaning but are expressing the number of orbits in each hypercube HC_7^s. The fourth column is obtained using the equation $a(s) = \binom{s+7-1}{7-1}$. To find the number of orbits in $\{0,1\} \times \mathbb{Z}^7$ we multiply the number of orbits in $\mathbb{Z}^7$ with 2.

B.7.8 Number of unique rectangles

We determine the histogram of non-degenerated unique rectangles formed by four lattice points $\boldsymbol{o}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ in $\mathbb{Z}_+^7$ as discrete function of the infinity norm $\|z\|_\infty = s$. We define a sample space Ω consisting of one orthant of the 7-cube with infinity norm $\|z\|_\infty = s$, with $s \in \mathbb{N}$ and search for the event of a unique rectangle perimeter p.

Table B.25 gives the result of the search for rectangles. We find in the 7D-hypercube where $\|z\|_\infty \leq 10$ and $z \in \mathbb{Z}_+^7$, a total of 7 747 unique rectangles UR out of 6 510 466 998 rectangles R. We searched the sample space Ω using a brute force method making use of the supercomputer of Ghent University. The unique rectangles represent unique dimensionless quantity equations of the type $f(\pi) = \left(\frac{z}{xy}\right)$ for the selected kind of quantity z.

Table B.25: Histogram of rectangles in $\mathbb{Z}_+^7$ as discrete function of the infinity norm $\|z\|_\infty = s$.

$\|z\|_\infty = s$	Unique rectangles UR	Rectangles R	$\frac{UR}{R}$
1	1	120	8.33×10^{-3}
2	7	7196	9.73×10^{-4}
3	26	162554	1.60×10^{-4}
4	79	1341957	5.89×10^{-5}
5	182	9255603	1.97×10^{-5}
6	333	40532530	8.22×10^{-6}
7	693	168302117	4.12×10^{-6}
8	1180	523421602	2.25×10^{-6}
9	1999	1637895896	1.22×10^{-6}
10	3247	4129547423	7.86×10^{-7}

We show in Table B.25 that the ratio of the number of unique rectangles UR to the number of rectangles R is decreasing for increasing infinity norm $\|z\|_\infty = s$ and that for $s = 10$ the ratio is 7.86×10^{-7}. The number of rectangles is defined by the radius squared of the $N+1$-dimensional hypersphere. How larger the values of the infinity norm s how larger the number of lattice points incident on the $N+1$- dimensional hypersphere. The unique rectangles are generated by $N+1$-dimensional confocal ellipsoids that intersect the $N+1$-dimensional hypersphere in only two lattice points x and y. The increase in the amount of rectangles is much higher than the increase of the occurrence of unique rectangles.

B.7.9 Properties of the unit 7-sphere

For $N = 7$ we find $S_7 = \frac{16\pi^3}{15}$.

For a unit 7-sphere we have $V_7 = \frac{16\pi^3}{105}$.

B.7.10 Successive Gödel numbers of orbits

Table B.26 contains in the first column the row identifier ID. In the second column we list the vertices in the order of appearance in the Gödel walk. The third column gives the value of the Gödel number up to the number 100. The fourth column shows the dimension N of $\mathbb{Z}^N \times \{0\}^{7-N}$ in which the lattice point is embedded. The fifth column indicates to which hypercube HC_7^s the lattice point belongs. The sixth column shows the orbit containing the lattice point.

Table B.26: Successive Gödel numbers in sub-lattice $\{0\} \times \mathbb{Z}^7$.

ID	Lattice point	Gödel number	Sub-lattice dimension	$\|z\|_\infty = s$	Orbit
1	$(0 \mid 0,0,0,0,0,0,0)$	1	0	0	$[(0 \mid 0^7)]$
2	$(0 \mid 1,0,0,0,0,0,0)$	2	1	1	$[(0 \mid 1,0^6)]$
3	$(0 \mid 0,1,0,0,0,0,0)$	3	2	1	$[(0 \mid 1,0^6)]$
4	$(0 \mid 2,0,0,0,0,0,0)$	4	1	2	$[(0 \mid 2,0^6)]$
5	$(0 \mid 0,0,1,0,0,0,0)$	5	3	1	$[(0 \mid 1,0^6)]$
6	$(0 \mid 1,1,0,0,0,0,0)$	6	2	1	$[(0 \mid 1^2,0^5)]$
7	$(0 \mid 0,0,0,1,0,0,0)$	7	4	1	$[(0 \mid 1,0^6)]$
8	$(0 \mid 3,0,0,0,0,0,0)$	8	1	3	$[(0 \mid 3,0^6)]$
9	$(0 \mid 0,2,0,0,0,0,0)$	9	2	2	$[(0 \mid 2,0^6)]$
10	$(0 \mid 1,0,1,0,0,0,0)$	10	3	1	$[(0 \mid 1^2,0^5)]$
11	$(0 \mid 0,0,0,0,1,0,0)$	11	5	1	$[(0 \mid 1,0^6)]$
12	$(0 \mid 2,1,0,0,0,0,0)$	12	2	2	$[(0 \mid 2,1,0^5)]$
13	$(0 \mid 0,0,0,0,0,1,0)$	13	6	1	$[(0 \mid 1,0^6)]$
14	$(0 \mid 1,0,0,1,0,0,0)$	14	4	1	$[(0 \mid 1^2,0^5)]$
15	$(0 \mid 0,1,1,0,0,0,0)$	15	3	1	$[(0 \mid 1^2,0^5)]$
16	$(0 \mid 4,0,0,0,0,0,0)$	16	1	4	$[(0 \mid 4,0^6)]$
17	$(0 \mid 0,0,0,0,0,0,1)$	17	7	1	$[(0 \mid 1,0^6)]$
18	$(0 \mid 1,2,0,0,0,0,0)$	18	2	2	$[(0 \mid 2,1,0^5)]$
19	$(0 \mid 2,0,1,0,0,0,0)$	20	3	2	$[(0 \mid 2,1,0^5)]$
20	$(0 \mid 0,1,0,1,0,0,0)$	21	4	1	$[(0 \mid 1^2,0^5)]$
21	$(0 \mid 1,0,0,0,1,0,0)$	22	5	1	$[(0 \mid 1^2,0^5)]$
22	$(0 \mid 3,1,0,0,0,0,0)$	24	2	3	$[(0 \mid 3,1,0^5)]$
23	$(0 \mid 0,0,2,0,0,0,0)$	25	3	2	$[(0 \mid 2,0^6)]$
24	$(0 \mid 1,0,0,0,0,1,0)$	26	6	1	$[(0 \mid 1^2,0^5)]$
25	$(0 \mid 0,3,0,0,0,0,0)$	27	2	3	$[(0 \mid 3,0^6)]$
26	$(0 \mid 2,0,0,1,0,0,0)$	28	4	2	$[(0 \mid 2,1,0^5)]$
27	$(0 \mid 1,1,1,0,0,0,0)$	30	3	1	$[(0 \mid 1^3,0^4)]$
28	$(0 \mid 5,0,0,0,0,0,0)$	32	1	5	$[(0 \mid 5,0^6)]$
29	$(0 \mid 0,1,0,0,1,0,0)$	33	5	1	$[(0 \mid 1^2,0^5)]$
30	$(0 \mid 1,0,0,0,0,0,1)$	34	7	1	$[(0 \mid 1^2,0^5)]$
31	$(0 \mid 0,0,1,1,0,0,0)$	35	4	1	$[(0 \mid 1^2,0^5)]$
32	$(0 \mid 2,2,0,0,0,0,0)$	36	2	2	$[(0 \mid 2^2,0^5)]$
33	$(0 \mid 0,1,0,0,0,1,0)$	39	6	1	$[(0 \mid 1^2,0^5)]$
34	$(0 \mid 3,0,1,0,0,0,0)$	40	3	3	$[(0 \mid 3,1,0^5)]$
35	$(0 \mid 1,1,0,1,0,0,0)$	42	4	1	$[(0 \mid 1^3,0^4)]$
36	$(0 \mid 2,0,0,0,1,0,0)$	44	5	2	$[(0 \mid 2,1,0^5)]$
37	$(0 \mid 0,2,1,0,0,0,0)$	45	3	2	$[(0 \mid 2,1,0^5)]$
38	$(0 \mid 4,1,0,0,0,0,0)$	48	2	4	$[(0 \mid 4,1,0^5)]$
39	$(0 \mid 0,0,0,2,0,0,0)$	49	4	2	$[(0 \mid 2,0^6)]$
40	$(0 \mid 1,0,2,0,0,0,0)$	50	3	2	$[(0 \mid 2,1,0^5)]$
41	$(0 \mid 0,1,0,0,0,0,1)$	51	7	1	$[(0 \mid 1^2,0^5)]$
42	$(0 \mid 2,0,0,0,0,1,0)$	52	6	2	$[(0 \mid 2,1,0^5)]$
43	$(0 \mid 1,3,0,0,0,0,0)$	54	2	3	$[(0 \mid 3,1,0^5)]$
44	$(0 \mid 0,0,1,0,1,0,0)$	55	5	1	$[(0 \mid 1^2,0^5)]$
45	$(0 \mid 3,0,0,1,0,0,0)$	56	4	3	$[(0 \mid 3,1,0^5)]$
46	$(0 \mid 2,1,1,0,0,0,0)$	60	3	2	$[(0 \mid 2,1^2,0^4)]$
47	$(0 \mid 0,2,0,1,0,0,0)$	63	4	2	$[(0 \mid 2,1,0^5)]$
…	…	…	…	…	…

ID	Lattice point	Gödel number	Sub-lattice dimension	$\|z\|_\infty = s$	Orbit
48	$(0 \mid 6,0,0,0,0,0,0)$	64	1	6	$[(0 \mid 6, 0^6)]$
49	$(0 \mid 0,0,1,0,0,1,0)$	65	6	1	$[(0 \mid 1^2, 0^5)]$
50	$(0 \mid 1,1,0,0,1,0,0)$	66	5	1	$[(0 \mid 1^3, 0^4)]$
51	$(0 \mid 2,0,0,0,0,0,1)$	68	7	2	$[(0 \mid 2, 1, 0^5)]$
52	$(0 \mid 1,0,1,1,0,0,0)$	70	4	1	$[(0 \mid 1^3, 0^4)]$
53	$(0 \mid 3,2,0,0,0,0,0)$	72	2	3	$[(0 \mid 3, 2, 0^5)]$
54	$(0 \mid 0,1,2,0,0,0,0)$	75	3	2	$[(0 \mid 2, 1, 0^5)]$
55	$(0 \mid 0,0,0,1,1,0,0)$	77	5	1	$[(0 \mid 1^2, 0^5)]$
56	$(0 \mid 1,1,0,0,0,1,0)$	78	6	1	$[(0 \mid 1^3, 0^4)]$
57	$(0 \mid 4,0,1,0,0,0,0)$	80	3	4	$[(0 \mid 4, 1, 0^5)]$
58	$(0 \mid 0,4,0,0,0,0,0)$	81	2	4	$[(0 \mid 4, 0^6)]$
59	$(0 \mid 2,1,0,1,0,0,0)$	84	4	2	$[(0 \mid 2, 1^2, 0^4)]$
60	$(0 \mid 0,0,1,0,0,0,1)$	85	7	1	$[(0 \mid 1^2, 0^5)]$
61	$(0 \mid 3,0,0,0,1,0,0)$	88	5	3	$[(0 \mid 3, 1, 0^5)]$
62	$(0 \mid 1,2,1,0,0,0,0)$	90	3	2	$[(0 \mid 2, 1^2, 0^4)]$
63	$(0 \mid 0,0,0,1,0,1,0)$	91	6	1	$[(0 \mid 1^2, 0^5)]$
64	$(0 \mid 5,1,0,0,0,0,0)$	96	2	5	$[(0 \mid 5, 1, 0^5)]$
65	$(0 \mid 1,0,0,2,0,0,0)$	98	4	2	$[(0 \mid 2, 1, 0^5)]$
66	$(0 \mid 0,2,0,0,1,0,0)$	99	5	2	$[(0 \mid 2, 1, 0^5)]$
67	$(0 \mid 2,0,2,0,0,0,0)$	100	3	2	$[(0 \mid 2^2, 0^5)]$

APPENDIX C

Algorithm for calculating the histogram of perimeters of a 3-cycles

We give a brute-force algorithm in pseudocode to calculate the histogram of perimeters of a 3-cycles.

Calculate for each integer lattice point $\boldsymbol{x}$ of a N-dimensional lattice the following:

1. $d(\boldsymbol{o},\boldsymbol{z})$, the Euclidean distance from $\boldsymbol{o}$ to the lattice point $\boldsymbol{z}$, with coordinates $(z_1,\ldots,z_N)$,
2. $d(\boldsymbol{x},\boldsymbol{o})$, the Euclidean distance from $\boldsymbol{x}$ to the origin $\boldsymbol{o}$,
3. the cosine of the angle between $\boldsymbol{x}$ and $\boldsymbol{z}$,
4. $2a = d(\boldsymbol{z},\boldsymbol{x}) + d(\boldsymbol{x},\boldsymbol{o})$, that is a characteristic of an N-ellipsoid,
5. the perimeter of the 3-cycle $p_t = d(\boldsymbol{o},\boldsymbol{z}) + d(\boldsymbol{z},\boldsymbol{x}) + d(\boldsymbol{x},\boldsymbol{o})$,
6. store these results in a data structure allowing sorting by perimeter,
7. query the data structure to obtain the number of lattice points $\boldsymbol{x}$ generating the same triangle perimeter,
8. find for each triangle perimeter p_t the number of lattice points corresponding to this triangle perimeter and record the histogram,
9. select the set of vertices having the same perimeter starting with the shortest 3-cycle perimeter,
10. calculate for each of these vertices the complementary vertices and write them in adjacent rows creating a listing of increasing perimeters.

Appendix D

Algorithm for the H-factorization of the integer k

Execute the following steps:

1. Calculate the orbit representative $\boldsymbol{z} = \mathrm{Orb}\,(\boldsymbol{q})$ of the kind of quantity $\boldsymbol{q}$;
2. calculate using the mapping $G\colon \{0,1\} \times \mathbb{Z}^N \to SG \subset 2\mathbb{Z}$ the Gödel number $G(\boldsymbol{z})$;
3. if the Gödel number is ≤ 1500 then;
4. open integer sequence OEIS A045778 (Wilson, 2009) or Appendix E and identify the row corresponding to the Gödel number and record the corresponding factorization;
5. else
6. perform the factorization of the Gödel number in *distinct integer factors*;
7. calculate using the inverse Gödel encoding the additive partitions of the orbit representative;
8. apply the appropriate signed permutation to transform the orbit representative in the quantity under investigation;
9. generate a table of quantity equations for the kind of quantity under study.

APPENDIX E

Canonical factorization for Gödel numbers ≤ 1500 in F_h distinct factors for selected orbits

Table E.1: Canonical factorization for a Gödel number ≤ 1500 in F_h distinct factors for selected orbits.

Orbit	Cardinality	Gödel number	F_2	F_3	F_4	F_5
$[(0 \mid 0^7)]$	1	1	0	0	0	0
$[(0 \mid 1, 0^6)]$	14	2	0	0	0	0
$[(0 \mid 2, 0^6)]$	14	4	0	0	0	0
$[(0 \mid 1^2, 0^5)]$	84	6	1	0	0	0
$[(0 \mid 3, 0^6)]$	14	8	1	0	0	0
$[(0 \mid 2, 1, 0^5)]$	168	12	2	0	0	0
$[(0 \mid 3, 1, 0^5)]$	168	24	3	1	0	0
$[(0 \mid 1^3, 0^4)]$	280	30	3	1	0	0
$[(0 \mid 2^2, 0^5)]$	84	36	3	1	0	0
$[(0 \mid 2, 1^2, 0^4)]$	840	60	5	3	0	0
$[(0 \mid 3, 2, 0^5)]$	168	72	5	3	0	0
$[(0 \mid 3, 1^2, 0^4)]$	840	120	7	7	1	0
$[(0 \mid 2^2, 1, 0^4)]$	840	180	8	8	1	0
$[(0 \mid 1^4, 0^3)]$	560	210	7	6	1	0
$[(0 \mid 3^2, 0^5)]$	84	216	7	8	1	0
$[(0 \mid 4, 1^2, 0^4)]$	840	240	9	12	3	0
$[(0 \mid 3, 2, 1, 0^4)]$	1680	360	11	17	5	0
$[(0 \mid 2, 1^3, 0^3)]$	2240	420	11	15	4	0
$[(0 \mid 3, 1^3, 0^3)]$	2240	840	15	29	13	1
$[(0 \mid 2^3, 0^4)]$	280	900	12	20	7	0
$[(0 \mid 3^2, 1, 0^4)]$	840	1080	15	33	17	1
$[(0 \mid 2^2, 1^2, 0^3)]$	3360	1260	17	35	16	1

APPENDIX F

Rectangles of the lattice point $(0 \mid -2, 2, 1, 0^4)$

We solve the equation (2.19) for the orbit $[(0 \mid 2^2, 1, 0^4)]$, that represents the kind of quantity *energy*. Table F.1 enumerates the 60 pairs of orthogonal vertices of $\{0\} \times \mathbb{Z}^7$ resulting in the vertex $\boldsymbol{E} = \boldsymbol{z} = (0 \mid -2, 2, 1, 0^4)$. Table F.1 contains 5 columns. The first column is the row identifier. The second column represents the semi-perimeter SP of the rectangle . The third column contains vertex $\boldsymbol{x}$. The fourth column contains the vertex $\boldsymbol{y}$. The fifth column contains the square of the rectangle area A_r^2 . The squares of the areas of the rectangles are all even.

We observe that the vertices with ID = 1 , ID = 35 and ID = 36 were already found through the two-factoring of the orbit representative $[0 \mid 2^2, 1, 0^4]$.

Table F.1: Orthogonal decomposition of the vertex $(0 \mid -2, 2, 1, 0^4)$ in $\{0\} \times \mathbb{Z}^7$.

ID	SP	$\boldsymbol{x}$	$\boldsymbol{y}$	A_r^2
1	3.828	$(\mathbf{0} \mid \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$	$(\mathbf{0} \mid -\mathbf{2}, \mathbf{2}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$	8
2	4.060	$(0 \mid -1, 0, 0, -1, 0, 0, 0)$	$(0 \mid -1, 2, 1, 1, 0, 0, 0)$	14
3	4.060	$(0 \mid -1, 0, 0, 0, -1, 0, 0)$	$(0 \mid -1, 2, 1, 0, 1, 0, 0)$	14
4	4.060	$(0 \mid -1, 0, 0, 0, 0, -1, 0)$	$(0 \mid -1, 2, 1, 0, 0, 1, 0)$	14
5	4.060	$(0 \mid -1, 0, 0, 0, 0, 0, -1)$	$(0 \mid -1, 2, 1, 0, 0, 0, 1)$	14
6	4.060	$(0 \mid -1, 0, 0, 0, 0, 0, 1)$	$(0 \mid -1, 2, 1, 0, 0, 0, -1)$	14
7	4.060	$(0 \mid -1, 0, 0, 0, 0, 1, 0)$	$(0 \mid -1, 2, 1, 0, 0, -1, 0)$	14
8	4.060	$(0 \mid -1, 0, 0, 0, 1, 0, 0)$	$(0 \mid -1, 2, 1, 0, -1, 0, 0)$	14
9	4.060	$(0 \mid -1, 0, 0, 1, 0, 0, 0)$	$(0 \mid -1, 2, 1, -1, 0, 0, 0)$	14
10	4.060	$(0 \mid 0, 1, 0, -1, 0, 0, 0)$	$(0 \mid -2, 1, 1, 1, 0, 0, 0)$	14
11	4.060	$(0 \mid 0, 1, 0, 0, -1, 0, 0)$	$(0 \mid -2, 1, 1, 0, 1, 0, 0)$	14
12	4.060	$(0 \mid 0, 1, 0, 0, 0, -1, 0)$	$(0 \mid -2, 1, 1, 0, 0, 1, 0)$	14
13	4.060	$(0 \mid 0, 1, 0, 0, 0, 0, -1)$	$(0 \mid -2, 1, 1, 0, 0, 0, 1)$	14
14	4.060	$(0 \mid 0, 1, 0, 0, 0, 0, 1)$	$(0 \mid -2, 1, 1, 0, 0, 0, -1)$	14
15	4.060	$(0 \mid 0, 1, 0, 0, 0, 1, 0)$	$(0 \mid -2, 1, 1, 0, 0, -1, 0)$	14
16	4.060	$(0 \mid 0, 1, 0, 0, 1, 0, 0)$	$(0 \mid -2, 1, 1, 0, -1, 0, 0)$	14
17	4.060	$(0 \mid 0, 1, 0, 1, 0, 0, 0)$	$(0 \mid -2, 1, 1, -1, 0, 0, 0)$	14
18	4.182	$(0 \mid -1, 0, 1, -1, 0, 0, 0)$	$(0 \mid -1, 2, 0, 1, 0, 0, 0)$	18
19	4.182	$(0 \mid -1, 0, 1, 0, -1, 0, 0)$	$(0 \mid -1, 2, 0, 0, 1, 0, 0)$	18
20	4.182	$(0 \mid -1, 0, 1, 0, 0, -1, 0)$	$(0 \mid -1, 2, 0, 0, 0, 1, 0)$	18
21	4.182	$(0 \mid -1, 0, 1, 0, 0, 0, -1)$	$(0 \mid -1, 2, 0, 0, 0, 0, 1)$	18
22	4.182	$(0 \mid -1, 0, 1, 0, 0, 0, 1)$	$(0 \mid -1, 2, 0, 0, 0, 0, -1)$	18
23	4.182	$(0 \mid -1, 0, 1, 0, 0, 1, 0)$	$(0 \mid -1, 2, 0, 0, 0, -1, 0)$	18
24	4.182	$(0 \mid -1, 0, 1, 0, 1, 0, 0)$	$(0 \mid -1, 2, 0, 0, -1, 0, 0)$	18
25	4.182	$(0 \mid -1, 0, 1, 1, 0, 0, 0)$	$(0 \mid -1, 2, 0, -1, 0, 0, 0)$	18
...	...	...	...	...

ID	SP	$\boldsymbol{x}$	$\boldsymbol{y}$	A_r^2
26	4.182	$(\mathbf{0} \mid -\mathbf{1}, \mathbf{1}, -\mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$	$(\mathbf{0} \mid -\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$	18
27	4.182	$(0 \mid -2, 1, 0, -1, 0, 0, 0)$	$(0 \mid 0, 1, 1, 1, 0, 0, 0)$	18
28	4.182	$(0 \mid -2, 1, 0, 0, -1, 0, 0)$	$(0 \mid 0, 1, 1, 0, 1, 0, 0)$	18
29	4.182	$(0 \mid -2, 1, 0, 0, 0, -1, 0)$	$(0 \mid 0, 1, 1, 0, 0, 1, 0)$	18
30	4.182	$(0 \mid -2, 1, 0, 0, 0, 0, -1)$	$(0 \mid 0, 1, 1, 0, 0, 0, 1)$	18
31	4.182	$(0 \mid -2, 1, 0, 0, 0, 0, 1)$	$(0 \mid 0, 1, 1, 0, 0, 0, -1)$	18
32	4.182	$(0 \mid -2, 1, 0, 0, 0, 1, 0)$	$(0 \mid 0, 1, 1, 0, 0, -1, 0)$	18
33	4.182	$(0 \mid -2, 1, 0, 0, 1, 0, 0)$	$(0 \mid 0, 1, 1, 0, -1, 0, 0)$	18
34	4.182	$(0 \mid -2, 1, 0, 1, 0, 0, 0)$	$(0 \mid 0, 1, 1, -1, 0, 0, 0)$	18
35	4.236	$(\mathbf{0} \mid -\mathbf{2}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$	$(\mathbf{0} \mid \mathbf{0}, \mathbf{2}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$	20
36	4.236	$(\mathbf{0} \mid -\mathbf{2}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$	$(\mathbf{0} \mid \mathbf{0}, \mathbf{2}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$	20
37	4.236	$(0 \mid -1, 1, 0, -1, -1, 0, 0)$	$(0 \mid -1, 1, 1, 1, 1, 0, 0)$	20
38	4.236	$(0 \mid -1, 1, 0, -1, 0, -1, 0)$	$(0 \mid -1, 1, 1, 1, 0, 1, 0)$	20
39	4.236	$(0 \mid -1, 1, 0, -1, 0, 0, -1)$	$(0 \mid -1, 1, 1, 1, 0, 0, 1)$	20
40	4.236	$(0 \mid -1, 1, 0, -1, 0, 0, 1)$	$(0 \mid -1, 1, 1, 1, 0, 0, -1)$	20
41	4.236	$(0 \mid -1, 1, 0, -1, 0, 1, 0)$	$(0 \mid -1, 1, 1, 1, 0, -1, 0)$	20
42	4.236	$(0 \mid -1, 1, 0, -1, 1, 0, 0)$	$(0 \mid -1, 1, 1, 1, -1, 0, 0)$	20
43	4.236	$(0 \mid -1, 1, 0, 0, -1, -1, 0)$	$(0 \mid -1, 1, 1, 0, 1, 1, 0)$	20
44	4.236	$(0 \mid -1, 1, 0, 0, -1, 0, -1)$	$(0 \mid -1, 1, 1, 0, 1, 0, 1)$	20
45	4.236	$(0 \mid -1, 1, 0, 0, -1, 0, 1)$	$(0 \mid -1, 1, 1, 0, 1, 0, -1)$	20
46	4.236	$(0 \mid -1, 1, 0, 0, -1, 1, 0)$	$(0 \mid -1, 1, 1, 0, 1, -1, 0)$	20
47	4.236	$(0 \mid -1, 1, 0, 0, 0, -1, -1)$	$(0 \mid -1, 1, 1, 0, 0, 1, 1)$	20
48	4.236	$(0 \mid -1, 1, 0, 0, 0, -1, 1)$	$(0 \mid -1, 1, 1, 0, 0, 1, -1)$	20
49	4.236	$(0 \mid -1, 1, 0, 0, 0, 1, -1)$	$(0 \mid -1, 1, 1, 0, 0, -1, 1)$	20
50	4.236	$(0 \mid -1, 1, 0, 0, 0, 1, 1)$	$(0 \mid -1, 1, 1, 0, 0, -1, -1)$	20
51	4.236	$(0 \mid -1, 1, 0, 0, 1, -1, 0)$	$(0 \mid -1, 1, 1, 0, -1, 1, 0)$	20
52	4.236	$(0 \mid -1, 1, 0, 0, 1, 0, -1)$	$(0 \mid -1, 1, 1, 0, -1, 0, 1)$	20
53	4.236	$(0 \mid -1, 1, 0, 0, 1, 0, 1)$	$(0 \mid -1, 1, 1, 0, -1, 0, -1)$	20
54	4.236	$(0 \mid -1, 1, 0, 0, 1, 1, 0)$	$(0 \mid -1, 1, 1, 0, -1, -1, 0)$	20
55	4.236	$(0 \mid -1, 1, 0, 1, -1, 0, 0)$	$(0 \mid -1, 1, 1, -1, 1, 0, 0)$	20
56	4.236	$(0 \mid -1, 1, 0, 1, 0, -1, 0)$	$(0 \mid -1, 1, 1, -1, 0, 1, 0)$	20
57	4.236	$(0 \mid -1, 1, 0, 1, 0, 0, -1)$	$(0 \mid -1, 1, 1, -1, 0, 0, 1)$	20
58	4.236	$(0 \mid -1, 1, 0, 1, 0, 0, 1)$	$(0 \mid -1, 1, 1, -1, 0, 0, -1)$	20
59	4.236	$(0 \mid -1, 1, 0, 1, 0, 1, 0)$	$(0 \mid -1, 1, 1, -1, 0, -1, 0)$	20
60	4.236	$(0 \mid -1, 1, 0, 1, 1, 0, 0)$	$(0 \mid -1, 1, 1, -1, -1, 0, 0)$	20

We find in total 60 non-degenerated rectangles for the vertex $\boldsymbol{z} = (-2, 2, 1, 0^4)$ with the 4 vertices of each rectangle incident on the same 7D-hypersphere $(\boldsymbol{x} - \frac{\boldsymbol{z}}{2})^2 = (\frac{\boldsymbol{z}}{2})^2$. There are no other rectangles for the vertex $\boldsymbol{z} = (0 \mid -2, 2, 1, 0^4)$ in $\{0, 1\} \times \mathbb{Z}^7$ than those listed in Table F.1. Observe that there is a *bijection* between the perimeters of the rectangles and the squares of the areas of the rectangles. The bijection is given by the equation:

$$A_r^2 = 20.08 SP^2 - 132.49 SP + 220.94\,. \tag{F.1}$$

Figure F.1 shows the polynomial relation of order 2.

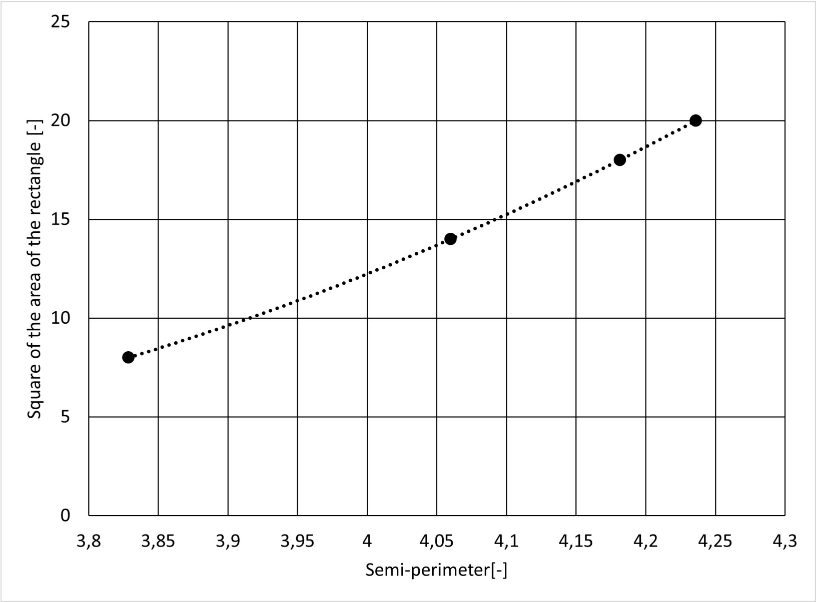

Figure F.1: Relation between the square of the areas of the rectangles A_r^2 and the semi-perimeter of the rectangles SP in $\{0,1\} \times \mathbb{Z}^7$ for the lattice point $(0 \mid -2, 2, 1, 0^4)$.

The multiplicities f for the square of the areas of the rectangles are $f = 1$ for $A_r^2 = 8$, $f = 16$ for $A_r^2 = 14$, $f = 17$ for $A_r^2 = 18$, and $f = 26$ for $A_r^2 = 20$. This yields four sets of isomorphic rectangles within the 7D-hypersphere for the kind of quantity energy. The rectangle with ID = 2 suggests the existence of a quantity equation:

$$E = f(\pi)\frac{1}{q} \times mA\frac{\mathrm{d}I}{\mathrm{d}t}, \tag{F.2}$$

in which q is an electric charge, m a mass, A an area and $\frac{\mathrm{d}I}{\mathrm{d}t}$ the rate of change of the electric current and $f(\pi)$ a dimensionless function. We have not found this type of equation in the literature. The rectangle with ID = 1 represents the quantity equation $E = f(\pi)mv^2$. It is a unique rectangle having the smallest semi-perimeter $SP = 3.828$ and the smallest square of the area of the rectangle $A_r^2 = 8$.

Table F.2 lists the four pairs having in column 1 the respective indices 1, 26, 35 and 36 that are embedded in the sub-lattice $\mathbb{Z}^3$. We find that these rectangles in the 7D-hypersphere have the semi-perimeter values 3.828 4.182 and 4.236 . By counting the number of equal perimeters in Table F.1 we find that the absolute frequency of the rectangle perimeters is respectively 1, 16, 17, and 26. Column 5 of Table F.2 gives the rank of the 2×7 matrix. We observe that the four orthogonal pairs have rank 2 and thus are linearly independent. The quantity equations in column 6 contain the physical quantities p linear impulse, m mass, s distance, E energy, t a characteristic time, v speed, ν frequency and $f_i(\pi)$ a dimensionless function.

Table F.2: Quantity equations of orthogonal integer lattice points of the kind of quantity energy.

ID	SP	$\boldsymbol{x}$	$\boldsymbol{y}$	Rank	Quantity equation
1	3.828	$(0 \mid 0, 0, 1, 0^4)$	$(0 \mid -2, 2, 0^5)$	2	$E = f_1(\pi) m v^2$
26	4.182	$(0 \mid -1, 1, -1, 0^4)$	$(0 \mid -1, 1, 2, 0^4)$	2	$E = f_2(\pi) \left(\frac{v}{m}\right) \left(\frac{p^2}{v}\right)$
35	4.236	$(0 \mid 0, 2, 1, 0^4)$	$(0 \mid -2, 0^6)$	2	$E = f_3(\pi) m s^2 \nu^2$
36	4.236	$(0 \mid -2, 0, 1, 0^4)$	$(0 \mid 0, 2, 0^5)$	2	$E = f_4(\pi) \frac{\mathrm{d}^2 m}{\mathrm{d}t^2} s^2$

The four rectangles representing ternary quantity equations in the sub-lattice $\mathbb{Z}^3$ for the integer lattice point $z = (0 \mid -2, 2, 1, 0^4)$ are shown in (Fig. F.2).

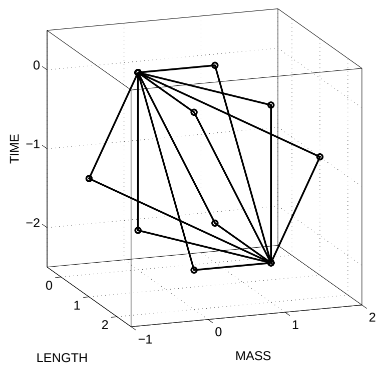

Figure F.2: Rectangles embedded in the sub-lattice $\mathbb{Z}^3$ representing ternary quantity equations for $z = (0 \mid -2, 2, 1, 0^4)$.

The rectangles with ID = 36 and ID = 36 are coupled because they have the same perimeter.

APPENDIX G

Rectangles of the lattice point $(0 \mid -3, 2, 1, 0^4)$

We denote the lattice point for the kind of quantity power by $\boldsymbol{P} = (0 \mid -3, 2, 1, 0^4)$. We consider the sub-lattice $\{0\} \times \mathbb{Z}^3$. This lattice point is an element of the orbit $[(0 \mid 3, 2, 1)]$ that has the same cardinality as the automorphism group $\mathrm{Aut}(\mathbb{Z}^3)$ being $2^3 3! = 48$. The sequential orthogonal decomposition of integer lattice point $\boldsymbol{P} = (0 \mid -3, 2, 1)$ representing the quantity power P is a 3D geometric graph represented in Figure G.1. The integer lattice point $\boldsymbol{P} = (0 \mid -3, 2, 1)$ is represented by the sphere in Figure G.1.

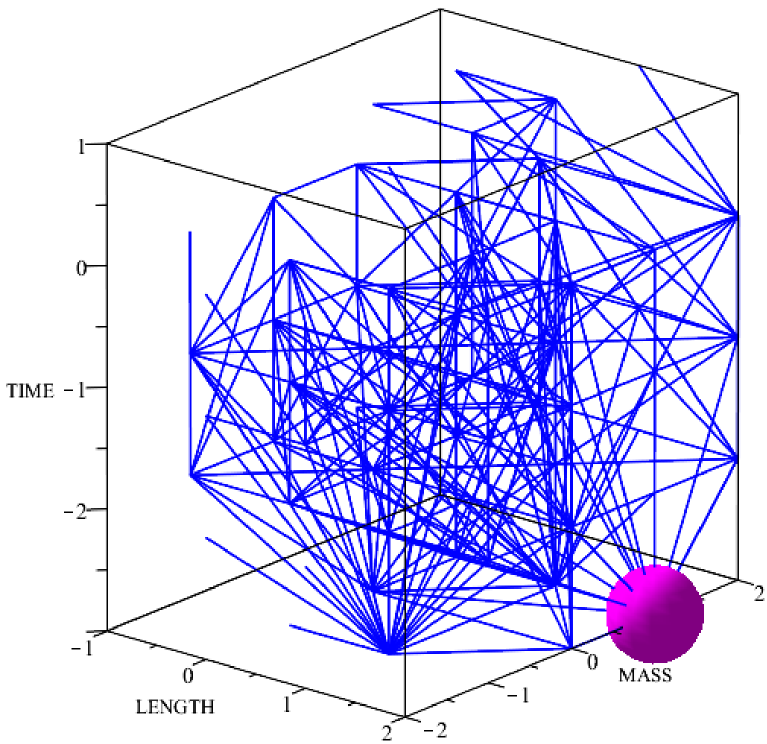

Figure G.1: 3D-geometric graph of the sequential orthogonal decomposition of the integer lattice point $\boldsymbol{P} = (0 \mid -3, 2, 1)$ that represents the kind of quantity power P .

In this 3D-geometric graph we find rectangles as shown in Figure G.2. We visualize the lattice points, which are part of the orthogonal decomposition, by given them a colored sphere and radius based on the cardinalities of the orbits of $\mathbb{Z}^3$.

We have the following coloring code: black = 1, blue = 6, green = 8, cyan = 12, red = 24 and magenta = 48. The 3D visualization is similar to that of an alloy consisting of 6 distinct atoms.

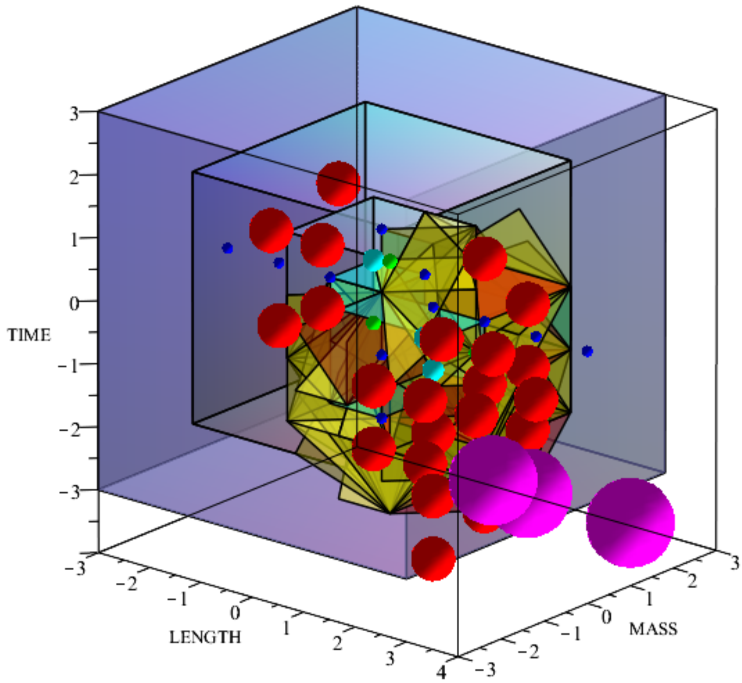

Figure G.2: Rectangles after sequential orthogonal decomposition of integer lattice point $\boldsymbol{P} = (0 \mid -3, 2, 1)$ representing the kind of quantity power P .

We calculate the perimeter histogram of rectangles of the integer lattice point $\boldsymbol{P} = (0 \mid -3, 2, 1)$ representing the kind of quantity power P and give the results in Figure G.3.

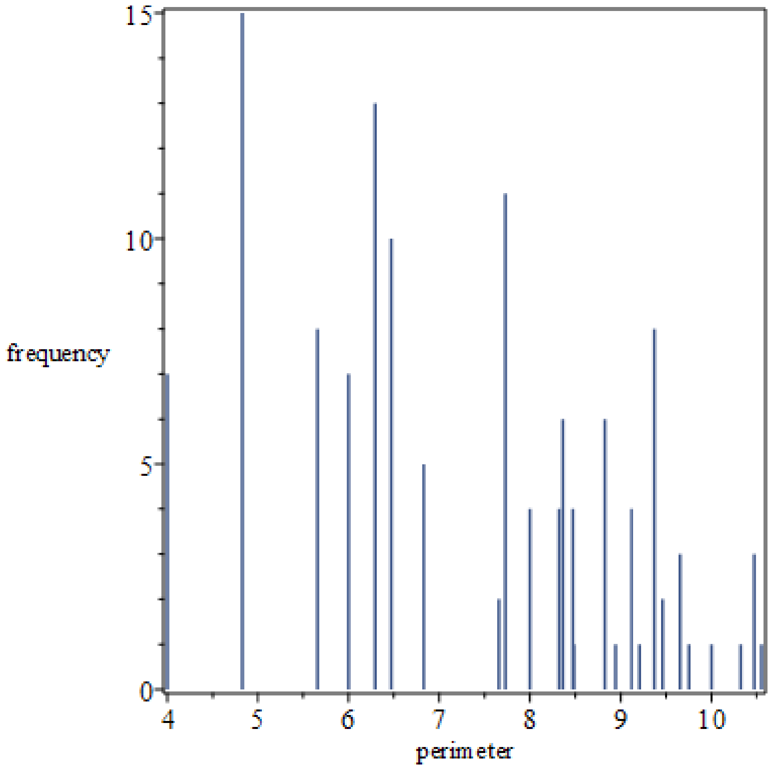

Figure G.3: Perimeter histogram of rectangles for the kind of quantity power $\boldsymbol{P} = (0 \mid -3, 2, 1)$.

We find from the numeric data of the histogram 26 different perimeters of rectangles of which seven perimeters are unique. Four of these unique rectangles are related to the kind of quantity power P and are given in the Table G.1. The symbols in Table G.1 have the following interpretation: $f_i(\pi_i)$ are functions of dimensionless variables π_i, v is a velocity, r is a characteristic distance, m is a mass, ω is an angular velocity, p is a linear impulse and t is a time.

Table G.1: Quantity equations of unique rectangles for the kind of quantity power in $\{0\} \times \mathbb{Z}^3$.

ID	Perimeter	***x***	***y***	Quantity equation
1	9.211	$(0 \mid 0, 0, 1)$	$(0 \mid -3, 2, 0)$	$P_1 = f_1(\pi_1) \cdot m \cdot \frac{\mathrm{d}v^2}{\mathrm{d}t}$
2	9.756	$(0 \mid -2, 2, 2)$	$(0 \mid -1, 0, -1)$	$P_2 = f_2(\pi_2) \cdot p^2 \cdot \frac{\omega}{m}$
3	10.324	$(0 \mid 0, 2, 0)$	$(0 \mid -3, 0, 1)$	$P_3 = f_3(\pi_3) \cdot r^2 \cdot m\omega^3$
7	10.558	$(0 \mid -1, 2, -1)$	$(0 \mid -2, 0, 2)$	$P_4 = f_4(\pi_4) \cdot (\frac{r^2\omega}{m}) \cdot (\frac{\mathrm{d}m}{\mathrm{d}t})^2$

We denote for further reference the rectangles with ID = 1, ID = 2, ID = 3, and ID = 7 respectively P_1, P_2, P_3, and P_4 .

Appendix H

Rectangles of the integer lattice point $(0 \mid -4, -1, 1, 0^4)$

The integer lattice point $(0 \mid -4, -1, 1, 0^4)$ represents the kind of quantity second order partial derivative of the energy density with respect to time $\dfrac{\partial^2 \mathcal{L}}{\partial t^2}$. The distribution of semi-perimeters SP of the rectangles is a *finite* list given in Table H.1.

Table H.1: Distribution of semi-perimeters of rectangles for the kind of quantity of the second order partial derivative of the energy density with respect to time.

ID	SP	Frequency
1	4.243	1
2	5.123	2
3	5.414	1
4	5.605	16
5	5.742	48
6	5.842	64
7	5.914	49
8	5.962	64
9	5.991	72
10	6.000	25

Observe that the first semi-perimeter and the third semi-perimeter are *unique*. The first semi-perimeter corresponds to a degenerated rectangle. The third semi-perimeter indicates that a unique rectangle exists. We enumerate in Table H.2 the rectangles. Table H.2 contains four columns. The first column is the row identifier. The second column contains lattice point $\boldsymbol{x}$. The third column contains the lattice point $\boldsymbol{y}$. The fourth column represents the semi-perimeter SP of the rectangle. We find in total 340 non-degenerated rectangles for the lattice point $\boldsymbol{z} = (0 \mid -4, -1, 1, 0^4)$ with the four lattice points of each rectangle incident on the same 7D-hypersphere $(\boldsymbol{x} - \dfrac{\boldsymbol{z}}{2})^2 = (\dfrac{\boldsymbol{z}}{2})^2$. The rectangle with ID = 4 in Table H.2 is a rectangle with a unique semi-perimeter. We can write the quantity equation for this rectangle as:

$$\frac{\partial^2 \mathcal{L}(x, \dot{x}, \tau)}{\partial \tau^2} = f(\pi) \frac{\partial^4 \mu(x, \tau)}{\partial \tau^4}, \tag{H.1}$$

in which $\mu(x,\tau)$ is a variable linear mass density, $f(\pi)$ is a function of a dimensionless quantity π and τ is a characteristic time. The rectangles with ID = 2 and ID = 3 form a system of 2 quantity equations. We write the equations in the following way:

$$\frac{\partial^2 \mathcal{L}(x,\dot{x},\tau)}{\partial \tau^2} = f_1(\pi_1)k(x,\tau)\frac{\partial^4 M(x,\tau)}{\partial \tau^4}$$
$$\frac{\partial^2 \mathcal{L}(x,\dot{x},\tau)}{\partial \tau^2} = f_2(\pi_2)M(x,\tau)\frac{\partial^4 k(x,\tau)}{\partial \tau^4}$$

in which $k(x,\tau)$ is a wave vector, $M(x,\tau)$ is a mass and $f_i(\pi_i)$ are functions of dimensionless quantities π_i . We combine the two quantity equations to the equation:

$$f_1(\pi_1)k(x,\tau)\frac{\partial^4 M(x,\tau)}{\partial \tau^4} - f_2(\pi_2)M(x,\tau)\frac{\partial^4 k(x,\tau)}{\partial \tau^4} = 0. \tag{H.2}$$

The rectangle with ID = 319 in Table H.2 describes a rectangle located in the sublattice $\mathbb{Z}^3$. Its quantity equation can be described as:

$$\frac{\partial^2 \mathcal{L}}{\partial \tau^2} = f(\pi)\left(\frac{1}{Mx^2t^2}\right)M^2 a\,, \tag{H.3}$$

in which M is a mass, x a characteristic length, τ is a characteristic time and a an acceleration.

Table H.2: Rectangles resulting in the kind of quantity 2nd time derivative of the energy density $\dfrac{\partial^2 \mathcal{L}}{\partial t^2}$.

ID	***x***	***y***	SP
1	$(0 \mid -4,-1,1,0,0,0,0)$	$(0 \mid 0,0,0,0,0,0,0)$	4.243
2	$(0 \mid 0,-1,0,0,0,0,0)$	$(0 \mid -4,0,1,0,0,0,0)$	5.123
3	$(0 \mid -4,-1,0,0,0,0,0)$	$(0 \mid 0,0,1,0,0,0,0)$	5.123
4	$(0 \mid 0,-1,1,0,0,0,0)$	$(0 \mid -4,0,0,0,0,0,0,)$	5.414
5	$(0 \mid -3,-2,1,0,0,-1,0)$	$(0 \mid -1,1,0,0,0,1,0)$	5.605
6	$(0 \mid -3,-2,1,0,0,0,-1)$	$(0 \mid -1,1,0,0,0,0,1)$	5.605
7	$(0 \mid -3,-2,1,0,0,0,1)$	$(0 \mid -1,1,0,0,0,0,-1)$	5.605
8	$(0 \mid -3,-2,1,0,-1,0,0)$	$(0 \mid -1,1,0,0,1,0,0)$	5.605
9	$(0 \mid -3,-2,1,1,0,0,0)$	$(0 \mid -1,1,0,-1,0,0,0)$	5.605
10	$(0 \mid -3,-2,1,-1,0,0,0)$	$(0 \mid -1,1,0,1,0,0,0)$	5.605
11	$(0 \mid -1,0,-1,-1,0,0,0)$	$(0 \mid -3,-1,2,1,0,0,0)$	5.605
12	$(0 \mid -3,-2,1,0,0,1,0)$	$(0 \mid -1,1,0,0,0,-1,0)$	5.605
13	$(0 \mid -1,0,-1,0,0,0,1)$	$(0 \mid -3,-1,2,0,0,0,-1)$	5.605
14	$(0 \mid -1,0,-1,0,0,1,0)$	$(0 \mid -3,-1,2,0,0,-1,0)$	5.605
15	$(0 \mid -1,0,-1,0,0,-1,0)$	$(0 \mid -3,-1,2,0,0,1,0)$	5.605
16	$(0 \mid -1,0,-1,0,0,0,-1)$	$(0 \mid -3,-1,2,0,0,0,1)$	5.605
17	$(0 \mid -1,0,-1,0,-1,0,0)$	$(0 \mid -3,-1,2,0,1,0,0)$	5.605
...	...	...	...

ID	$\boldsymbol{x}$	$\boldsymbol{y}$	SP
18	$(0 \mid -3,-2,1,0,1,0,0)$	$(0 \mid -1,1,0,0,-1,0,0)$	5.605
19	$(0 \mid -1,0,-1,0,1,0,0)$	$(0 \mid -3,-1,2,0,-1,0,0)$	5.605
20	$(0 \mid -1,0,-1,1,0,0,0)$	$(0 \mid -3,-1,2,-1,0,0,0)$	5.605
21	$(0 \mid -3,-1,1,-1,1,-1,0)$	$(0 \mid -1,0,0,1,-1,1,0)$	5.742
22	$(0 \mid -3,-1,1,-1,1,0,-1)$	$(0 \mid -1,0,0,1,-1,0,1)$	5.742
23	$(0 \mid -3,-1,1,-1,0,1,-1)$	$(0 \mid -1,0,0,1,0,-1,1)$	5.742
24	$(0 \mid -3,-1,1,-1,0,1,1)$	$(0 \mid -1,0,0,1,0,-1,-1)$	5.742
25	$(0 \mid -3,-1,1,1,1,1,0)$	$(0 \mid -1,0,0,-1,-1,-1,0)$	5.742
26	$(0 \mid -3,-1,1,1,1,0,1)$	$(0 \mid -1,0,0,-1,-1,0,-1)$	5.742
27	$(0 \mid -3,-1,1,-1,1,0,1)$	$(0 \mid -1,0,0,1,-1,0,-1)$	5.742
28	$(0 \mid -3,-1,1,-1,1,1,0)$	$(0 \mid -1,0,0,1,-1,-1,0)$	5.742
29	$(0 \mid -3,-1,1,-1,0,-1,1)$	$(0 \mid -1,0,0,1,0,1,-1)$	5.742
30	$(0 \mid -3,-1,1,-1,-1,-1,0)$	$(0 \mid -1,0,0,1,1,1,0)$	5.742
31	$(0 \mid -3,-1,1,-1,-1,0,-1)$	$(0 \mid -1,0,0,1,1,0,1)$	5.742
32	$(0 \mid -3,-2,0,0,0,0,1)$	$(0 \mid -1,1,1,0,0,0,-1)$	5.742
33	$(0 \mid -3,-2,0,0,0,1,0)$	$(0 \mid -1,1,1,0,0,-1,0)$	5.742
34	$(0 \mid -3,-2,0,,0,1,0,0)$	$(0 \mid -1,1,1,0,-1,0,0)$	5.742
35	$(0 \mid -3,-1,1,-1,0,-1,-1)$	$(0 \mid -1,0,0,1,0,1,1)$	5.742
36	$(0 \mid -3,-1,1,-1,-1,0,1)$	$(0 \mid -1,0,0,1,1,0,-1)$	5.742
37	$(0 \mid -3,-1,1,-1,-1,1,0)$	$(0 \mid -1,0,0,1,1,-1,0)$	5.742
38	$(0 \mid -3,-1,1,1,1,0,-1)$	$(0 \mid -1,0,0,-1,-1,0,1)$	5.742
39	$(0 \mid -3,-1,1,0,1,1,-1)$	$(0 \mid -1,0,0,0,-1,-1,1)$	5.742
40	$(0 \mid -3,-1,1,0,1,1,1)$	$(0 \mid -1,0,0,0,-1,-1,-1)$	5.742
41	$(0 \mid -3,-1,1,0,1,-1,-1)$	$(0 \mid -1,0,0,0,-1,1,1)$	5.742
42	$(0 \mid -3,-1,1,0,1,-1,1)$	$(0 \mid -1,0,0,0,-1,1,-1)$	5.742
43	$(0 \mid -3,-1,1,1,-1,0,-1)$	$(0 \mid -1,0,0,-1,1,0,1)$	5.742
44	$(0 \mid -3,-1,1,1,-1,-1,0)$	$(0 \mid -1,0,0,-1,1,1,0)$	5.742
45	$(0 \mid -3,-1,1,1,-1,1,0)$	$(0 \mid -1,0,0,-1,1,-1,0)$	5.742
46	$(0 \mid -3,-1,1,1,-1,0,1)$	$(0 \mid -1,0,0,-1,1,0,-1)$	5.742
47	$(0 \mid -3,-1,1,0,-1,1,1)$	$(0 \mid -1,0,0,0,1,-1,-1)$	5.742
48	$(0 \mid -3,-2,0,1,0,0,0)$	$(0 \mid -1,1,1,-1,0,0,0)$	5.742
49	$(0 \mid -3,-1,1,0,-1,-1,-1)$	$(0 \mid -1,0,0,0,1,1,1)$	5.742
50	$(0 \mid -3,-1,1,1,1,-1,0)$	$(0 \mid -1,0,0,-1,-1,1,0)$	5.742
51	$(0 \mid -3,-1,1,1,0,1,1)$	$(0 \mid -1,0,0,-1,0,-1,-1)$	5.742
52	$(0 \mid -3,-1,1,1,0,-1,1)$	$(0 \mid -1,0,0,-1,0,1,-1)$	5.742
53	$(0 \mid -3,-1,1,1,0,-1,-1)$	$(0 \mid -1,0,0,-1,0,1,1)$	5.742
54	$(0 \mid -3,-1,1,0,-1,-1,1)$	$(0 \mid -1,0,0,0,1,1,-1)$	5.742
55	$(0 \mid -3,-1,1,1,0,1,-1)$	$(0 \mid -1,0,0,-1,0,-1,1)$	5.742
56	$(0 \mid -1,-1,-1,0,0,1,0)$	$(0 \mid -3,0,2,0,0,-1,0)$	5.742
57	$(0 \mid -1,-1,-1,0,0,0,1)$	$(0 \mid -3,0,2,0,0,0,-1)$	5.742
58	$(0 \mid -1,-1,-1,1,0,0,0)$	$(0 \mid -3,0,2,-1,0,0,0)$	5.742
59	$(0 \mid -1,-1,-1,0,1,0,0)$	$(0 \mid -3,0,2,0,-1,0,0)$	5.742
60	$(0 \mid -1,-1,-1,0,0,0,-1)$	$(0 \mid -3,0,2,0,0,0,1)$	5.742
61	$(0 \mid -1,-1,-1,-1,0,0,0)$	$(0 \mid -3,0,2,1,0,0,0)$	5.742
62	$(0 \mid -3,-1,1,,0,-1,1,-1)$	$(0 \mid -1,0,0,0,1,-1,1)$	5.742
63	$(0 \mid -1,-1,-1,0,0,-1,0)$	$(0 \mid -3,0,2,0,0,1,0)$	5.742
64	$(0 \mid -1,-1,-1,0,-1,0,0)$	$(0 \mid -3,0,2,0,1,0,0)$	5.742
65	$(0 \mid -3,-2,0,0,0,-1,0)$	$(0 \mid -1,1,1,0,0,1,0)$	5.742
66	$(0 \mid -3,-2,0,-1,0,0,0)$	$(0 \mid -1,1,1,1,0,0,0)$	5.742
67	$(0 \mid -3,-2,0,0,-1,0,0)$	$(0 \mid -1,1,1,0,1,0,0)$	5.742
68	$(0 \mid -3,-2,0,0,0,0,-1)$	$(0 \mid -1,1,1,0,0,0,1)$	5.742
69	$(0 \mid -3,-1,0,0,1,1,-1)$	$(0 \mid -1,0,1,0,-1,-1,1)$	5.842
70	$(0 \mid -1,-1,0,1,0,1,1)$	$(0 \mid -3,0,1,-1,0,-1,-1)$	5.842
71	$(0 \mid -1,-1,0,1,0,1,-1)$	$(0 \mid -3,0,1,-1,0,-1,1)$	5.842
…	…	…	…

ID	$\boldsymbol{x}$	$\boldsymbol{y}$	SP
72	$(0 \mid -3,-1,0,0,1,1,1)$	$(0 \mid -1,0,1,0,-1,-1,-1)$	5.842
73	$(0 \mid -1,-1,0,-1,-1,-1,0)$	$(0 \mid -3,0,1,1,1,1,0)$	5.842
74	$(0 \mid -1,-1,0,1,0,-1,1)$	$(0 \mid -3,0,1,-1,0,1,-1)$	5.842
75	$(0 \mid -1,-1,0,1,1,1,0)$	$(0 \mid -3,0,1,-1,-1,-1,0)$	5.842
76	$(0 \mid -1,-1,0,1,1,0,1)$	$(0 \mid -3,0,1,-1,-1,0,-1)$	5.842
77	$(0 \mid -1,-1,0,1,-1,0,1)$	$(0 \mid -3,0,1,-1,1,0,-1)$	5.842
78	$(0 \mid -1,-1,0,0,-1,-1,1)$	$(0 \mid -3,0,1,0,1,1,-1)$	5.842
79	$(0 \mid -3,-1,0,0,1,-1,-1)$	$(0 \mid -1,0,1,0,-1,1,1)$	5.842
80	$(0 \mid -1,-1,0,1,1,0,-1)$	$(0 \mid -3,0,1,-1,-1,0,1)$	5.842
81	$(0 \mid -1,-1,0,1,1,-1,0)$	$(0 \mid -3,0,1,-1,-1,1,0)$	5.842
82	$(0 \mid -3,-1,0,0,1,-1,1)$	$(0 \mid -1,0,1,0,-1,1,-1)$	5.842
83	$(0 \mid -1,-1,0,1,-1,1,0)$	$(0 \mid -3,0,1,-1,1,-1,0)$	5.842
84	$(0 \mid -3,-1,0,1,1,1,0)$	$(0 \mid -1,0,1,,-1,-1,-1,0)$	5.842
85	$(0 \mid -1,-1,0,1,0,-1,-1)$	$(0 \mid -3,0,1,-1,0,1,1)$	5.842
86	$(0 \mid -3,-1,0,1,1,-1,0)$	$(0 \mid -1,0,1,-1,-1,1,0)$	5.842
87	$(0 \mid -1,-1,0,-1,-1,0,1)$	$(0 \mid -3,0,1,1,1,0,-1)$	5.842
88	$(0 \mid -3,-1,0,1,-1,-1,0)$	$(0 \mid -1,0,1,,-1,1,1,0)$	5.842
89	$(0 \mid -3,-1,0,1,0,-1,1)$	$(0 \mid -1,0,1,-1,0,1,-1)$	5.842
90	$(0 \mid -3,-1,0,1,0,-1,-1)$	$(0 \mid -1,0,1,-1,0,1,1)$	5.842
91	$(0 \mid -3,-1,0,1,0,1,1)$	$(0 \mid -1,0,1,-1,0,-1,-1)$	5.842
92	$(0 \mid -3,-1,0,1,0,1,-1)$	$(0 \mid -1,0,1,-1,0,-1,1)$	5.842
93	$(0 \mid -3,-1,0,1,-1,0,-1)$	$(0 \mid -1,0,1,-1,1,0,1)$	5.842
94	$(0 \mid -3,-1,0,1,-1,0,1)$	$(0 \mid -1,0,1,-1,1,0,-1)$	5.842
95	$(0 \mid -3,-1,0,1,1,0,1)$	$(0 \mid -1,0,1,-1,-1,0,-1)$	5.842
96	$(0 \mid -3,-1,0,1,1,0,-1)$	$(0 \mid -1,0,1,-1,-1,0,1)$	5.842
97	$(0 \mid -1,-1,0,-1,0,-1,-1)$	$(0 \mid -3,0,1,1,0,1,1)$	5.842
98	$(0 \mid -1,-1,0,-1,-1,1,0)$	$(0 \mid -3,0,1,1,1,-1,0)$	5.842
99	$(0 \mid -3,-1,0,1,-1,1,0)$	$(0 \mid -1,0,1,-1,1,-1,0)$	5.842
100	$(0 \mid -1,-1,0,-1,-1,0,-1)$	$(0 \mid -3,0,1,1,1,0,1)$	5.842
101	$(0 \mid -3,-1,0,-1,-1,-1,0)$	$(0 \mid -1,0,1,1,1,1,0)$	5.842
102	$(0 \mid -3,-1,0,-1,-1,0,-1)$	$(0 \mid -1,0,1,1,1,0,1)$	5.842
103	$(0 \mid -1,-1,0,-1,0,1,-1)$	$(0 \mid -3,0,1,1,0,-1,1)$	5.842
104	$(0 \mid -1,-1,0,-1,0,-1,1)$	$(0 \mid -3,0,1,1,0,1,-1)$	5.842
105	$(0 \mid -3,-1,0,-1,0,-1,-1)$	$(0 \mid -1,0,1,1,0,1,1)$	5.842
106	$(0 \mid -3,-1,0,-1,0,-1,1)$	$(0 \mid -1,0,1,1,0,1,-1)$	5.842
107	$(0 \mid -3,-1,0,-1,-1,0,1)$	$(0 \mid -1,0,1,1,1,0,-1)$	5.842
108	$(0 \mid -3,-1,0,-1,-1,1,0)$	$(0 \mid -1,0,1,1,1,-1,0)$	5.842
109	$(0 \mid -1,-1,0,0,-1,1,1)$	$(0 \mid -3,0,1,0,1,-1,-1)$	5.842
110	$(0 \mid -1,-1,0,-1,1,1,0)$	$(0 \mid -3,0,1,1,-1,-1,0)$	5.842
111	$(0 \mid -1,-1,0,0,-1,-1,-1)$	$(0 \mid -3,0,1,0,1,1,1)$	5.842
112	$(0 \mid -1,-1,0,0,-1,1,-1)$	$(0 \mid -3,0,1,0,1,-1,1)$	5.842
113	$(0 \mid -1,-1,0,-1,1,-1,0)$	$(0 \mid -3,0,1,1,-1,1,0)$	5.842
114	$(0 \mid -1,-1,0,-1,0,1,1)$	$(0 \mid -3,0,1,1,0,-1,-1)$	5.842
115	$(0 \mid -1,-1,0,-1,1,0,1)$	$(0 \mid -3,0,1,1,-1,0,-1)$	5.842
116	$(0 \mid -1,-1,0,-1,1,0,-1)$	$(0 \mid -3,0,1,1,-1,0,1)$	5.842
117	$(0 \mid -3,-1,0,0,-1,-1,1)$	$(0 \mid -1,0,1,0,1,1,-1)$	5.842
118	$(0 \mid -1,-1,0,0,1,1,-1)$	$(0 \mid -3,0,1,0,-1,-1,1)$	5.842
119	$(0 \mid -3,-1,0,-1,1,1,0)$	$(0 \mid -1,0,1,1,-1,-1,0)$	5.842
120	$(0 \mid -3,-1,0,0,-1,-1,-1)$	$(0 \mid -1,0,1,0,1,1,1)$	5.842
121	$(0 \mid -3,-1,0,0,-1,1,-1)$	$(0 \mid -1,0,1,0,1,-1,1)$	5.842
122	$(0 \mid -1,-1,0,1,-1,-1,0)$	$(0 \mid -3,0,1,-1,1,1,0)$	5.842
123	$(0 \mid -1,-1,0,1,-1,0,-1)$	$(0 \mid -3,0,1,-1,1,0,1)$	5.842
124	$(0 \mid -3,-1,0,0,-1,1,1)$	$(0 \mid -1,0,1,0,1,-1,-1)$	5.842
125	$(0 \mid -1,-1,0,0,1,1,1)$	$(0 \mid -3,0,1,0,-1,-1,-1)$	5.842
…	…	…	…

ID	$\boldsymbol{x}$	$\boldsymbol{y}$	SP
126	$(0 \mid -1,-1,0,0,1,-1,-1)$	$(0 \mid -3,0,1,0,-1,1,1)$	5.842
127	$(0 \mid -3,-1,0,-1,1,-1,0)$	$(0 \mid -1,0,1,1,-1,1,0)$	5.842
128	$(0 \mid -3,-1,0,-1,0,1,-1)$	$(0 \mid -1,0,1,1,0,-1,1)$	5.842
129	$(0 \mid -3,-1,0,-1,0,1,1)$	$(0 \mid -1,0,1,1,0,-1,-1)$	5.842
130	$(0 \mid -3,-1,0,-1,1,0,1)$	$(0 \mid -1,0,1,1,-1,0,-1)$	5.842
131	$(0 \mid -1,-1,0,0,1,-1,1)$	$(0 \mid -3,0,1,0,-1,1,-1)$	5.842
132	$(0 \mid -3,-1,0,-1,1,0,-1)$	$(0 \mid -1,0,1,1,-1,0,1)$	5.842
133	$(0 \mid -1,-1,1,-1,0,-1,-1)$	$(0 \mid -3,0,0,1,0,1,1)$	5.914
134	$(0 \mid -1,-1,1,-1,0,-1,1)$	$(0 \mid -3,0,0,1,0,1,-1)$	5.914
135	$(0 \mid -1,-1,1,-1,-1,0,-1)$	$(0 \mid -3,0,0,1,1,0,1)$	5.914
136	$(0 \mid -1,-1,1,-1,-1,0,1)$	$(0 \mid -3,0,0,1,1,0,-1)$	5.914
137	$(0 \mid -1,-1,1,-1,-1,1,0)$	$(0 \mid -3,0,0,1,1,-1,0)$	5.914
138	$(0 \mid -1,-1,1,-1,1,-1,0)$	$(0 \mid -3,0,0,1,-1,1,0)$	5.914
139	$(0 \mid -1,-1,1,-1,0,1,1)$	$(0 \mid -3,0,0,1,0,-1,-1)$	5.914
140	$(0 \mid -1,-2,0,0,0,-1,0)$	$(0 \mid -3,1,1,0,0,1,0)$	5.914
141	$(0 \mid -1,-2,0,0,-1,0,0)$	$(0 \mid -3,1,1,0,1,0,0)$	5.914
142	$(0 \mid -1,-1,1,-1,0,1,-1)$	$(0 \mid -3,0,0,1,0,-1,1)$	5.914
143	$(0 \mid -1,-1,1,1,1,0,-1)$	$(0 \mid -3,0,0,-1,-1,0,1)$	5.914
144	$(0 \mid -1,-1,1,1,1,-1,0)$	$(0 \mid -3,0,0,-1,-1,1,0)$	5.914
145	$(0 \mid -1,-1,1,1,1,0,1)$	$(0 \mid -3,0,0,-1,-1,0,-1)$	5.914
146	$(0 \mid -1,-1,1,1,1,1,0)$	$(0 \mid -3,0,0,-1,-1,-1,0)$	5.914
147	$(0 \mid -1,-1,1,1,-1,0,1)$	$(0 \mid -3,0,0,-1,1,0,-1)$	5.914
148	$(0 \mid -1,-2,0,-1,0,0,0)$	$(0 \mid -3,1,1,1,0,0,0)$	5.914
149	$(0 \mid -1,-1,1,-1,-1,-1,0)$	$(0 \mid -3,0,0,1,1,1,0)$	5.914
150	$(0 \mid -1,-1,1,1,0,-1,1)$	$(0 \mid -3,0,0,-1,0,1,-1)$	5.914
151	$(0 \mid -1,-1,1,1,0,1,1)$	$(0 \mid -3,0,0,-1,0,-1,-1)$	5.914
152	$(0 \mid -1,-1,1,1,0,1,-1)$	$(0 \mid -3,0,0,-1,0,-1,1)$	5.914
153	$(0 \mid -1,-1,1,1,0,-1,-1)$	$(0 \mid -3,0,0,-1,0,1,1)$	5.914
154	$(0 \mid -1,-1,1,0,1,1,-1)$	$(0 \mid -3,0,0,0,-1,-1,1)$	5.914
155	$(0 \mid -1,-1,1,0,1,-1,1)$	$(0 \mid -3,0,0,0,-1,1,-1)$	5.914
156	$(0 \mid -1,-1,1,0,-1,1,1)$	$(0 \mid -3,0,0,0,1,-1,-1)$	5.914
157	$(0 \mid -1,-1,1,0,1,-1,-1)$	$(0 \mid -3,0,0,0,-1,1,1)$	5.914
158	$(0 \mid -1,-1,1,0,1,1,1)$	$(0 \mid -3,0,0,0,-1,-1,-1)$	5.914
159	$(0 \mid -1,-2,0,1,0,0,0)$	$(0 \mid -3,1,1,-1,0,0,0)$	5.914
160	$(0 \mid -1,-1,1,1,-1,-1,0)$	$(0 \mid -3,0,0,-1,1,1,0)$	5.914
161	$(0 \mid -1,-1,1,1,-1,1,0)$	$(0 \mid -3,0,0,-1,1,-1,0)$	5.914
162	$(0 \mid -1,-1,1,1,-1,0,-1)$	$(0 \mid -3,0,0,-1,1,0,1)$	5.914
163	$(0 \mid -1,-2,0,0,0,0,-1)$	$(0 \mid -3,1,1,0,0,0,1)$	5.914
164	$(0 \mid -1,-2,0,0,0,0,1)$	$(0 \mid -3,1,1,0,0,0,-1)$	5.914
165	$(0 \mid -1,-1,1,-1,1,1,0)$	$(0 \mid -3,0,0,1,-1,-1,0)$	5.914
166	$(0 \mid -1,-1,1,-1,1,0,-1)$	$(0 \mid -3,0,0,1,-1,0,1)$	5.914
167	$(0 \mid -1,-1,1,-1,1,0,1)$	$(0 \mid -3,0,0,1,-1,0,-1)$	5.914
168	$(0 \mid -1,-1,1,0,-1,-1,1)$	$(0 \mid -3,0,0,0,1,1,-1)$	5.914
169	$(0 \mid -1,-1,1,0,-1,1,-1)$	$(0 \mid -3,0,0,0,1,-1,1)$	5.914
170	$(0 \mid -1,-1,1,0,-1,-1,-1)$	$(0 \mid -3,0,0,0,1,1,1)$	5.914
171	$(0 \mid -1,-2,0,0,0,1,0)$	$(0 \mid -3,1,1,0,0,-1,0)$	5.914
172	$(0 \mid -1,-2,0,0,1,0,0)$	$(0 \mid -3,1,1,0,-1,0,0)$	5.914
173	$(0 \mid -3,-1,-1,0,1,0,0)$	$(0 \mid -1,0,2,0,-1,0,0)$	5.914
174	$(0 \mid -3,-1,-1,0,0,1,0)$	$(0 \mid -1,0,2,0,0,-1,0)$	5.914
175	$(0 \mid -3,-1,-1,-1,0,0,0)$	$(0 \mid -1,0,2,1,0,0,0)$	5.914
176	$(0 \mid -2,1,-1,0,0,0,0)$	$(0 \mid -2,-2,2,0,0,0,0)$	5.914
177	$(0 \mid -3,-1,-1,0,0,0,1)$	$(0 \mid -1,0,2,0,0,0,-1)$	5.914
178	$(0 \mid -3,-1,-1,0,-1,0,0)$	$(0 \mid -1,0,2,0,1,0,0)$	5.914
179	$(0 \mid -3,-1,-1,1,0,0,0)$	$(0 \mid -1,0,2,-1,0,0,0)$	5.914
…	…	…	…

ID	$\boldsymbol{x}$	$\boldsymbol{y}$	SP
180	$(0 \mid -3,-1,-1,0,0,0,-1)$	$(0 \mid -1,0,2,0,0,0,1)$	5.914
181	$(0 \mid -3,-1,-1,0,0,-1,0)$	$(0 \mid -1,0,2,0,0,1,0)$	5.914
182	$(0 \mid -2,-2,1,0,1,0,-1)$	$(0 \mid -2,1,0,0,-1,0,1)$	5.962
183	$(0 \mid -2,-2,1,0,0,1,1)$	$(0 \mid -2,1,0,0,0,-1,-1)$	5.962
184	$(0 \mid -2,-2,1,0,1,-1,0)$	$(0 \mid -2,1,0,0,-1,1,0)$	5.962
185	$(0 \mid -2,0,-1,-1,0,0,1)$	$(0 \mid -2,-1,2,1,0,0,-1)$	5.962
186	$(0 \mid -3,0,-1,-1,0,0,0)$	$(0 \mid -1,-1,2,1,0,0,0)$	5.962
187	$(0 \mid -2,-2,1,0,1,0,1)$	$(0 \mid -2,1,0,0,-1,0,-1)$	5.962
188	$(0 \mid -2,-2,1,0,1,1,0)$	$(0 \mid -2,1,0,0,-1,-1,0)$	5.962
189	$(0 \mid -2,-2,1,0,0,1,-1)$	$(0 \mid -2,1,0,0,0,-1,1)$	5.962
190	$(0 \mid -1,-2,1,0,0,0,-1)$	$(0 \mid -3,1,0,0,0,0,1)$	5.962
191	$(0 \mid -1,-2,1,0,0,0,1)$	$(0 \mid -3,1,0,0,0,0,-1)$	5.962
192	$(0 \mid -2,-2,1,0,-1,1,0)$	$(0 \mid -2,1,0,0,1,-1,0)$	5.962
193	$(0 \mid -1,-2,1,0,0,-1,0)$	$(0 \mid -3,1,0,0,0,1,0)$	5.962
194	$(0 \mid -2,0,-1,-1,0,1,0)$	$(0 \mid -2,-1,2,1,0,-1,0)$	5.962
195	$(0 \mid -2,-2,1,0,0,-1,1)$	$(0 \mid -2,1,0,0,0,1,-1)$	5.962
196	$(0 \mid -2,-2,1,0,0,-1,-1)$	$(0 \mid -2,1,0,0,0,1,1)$	5.962
197	$(0 \mid -1,-2,1,0,-1,0,0)$	$(0 \mid -3,1,0,0,1,0,0)$	5.962
198	$(0 \mid -2,-2,1,1,0,0,1)$	$(0 \mid -2,1,0,-1,0,0,-1)$	5.962
199	$(0 \mid -2,-2,1,1,0,1,0)$	$(0 \mid -2,1,0,-1,0,-1,0)$	5.962
200	$(0 \mid -2,-2,1,1,0,-1,0)$	$(0 \mid -2,1,0,-1,0,1,0)$	5.962
201	$(0 \mid -2,-2,1,1,0,0,-1)$	$(0 \mid -2,1,0,-1,0,0,1)$	5.962
202	$(0 \mid -2,0,-1,-1,0,0,-1)$	$(0 \mid -2,-1,2,1,0,0,1)$	5.962
203	$(0 \mid -2,-2,1,1,1,0,0)$	$(0 \mid -2,1,0,-1,-1,0,0)$	5.962
204	$(0 \mid -2,0,-1,-1,-1,0,0)$	$(0 \mid -2,-1,2,1,1,0,0)$	5.962
205	$(0 \mid -2,0,-1,-1,0,-1,0)$	$(0 \mid -2,-1,2,1,0,1,0)$	5.962
206	$(0 \mid -3,0,-1,0,0,0,-1)$	$(0 \mid -1,-1,2,0,0,0,1)$	5.962
207	$(0 \mid -3,0,-1,0,0,0,1)$	$(0 \mid -1,-1,2,0,0,0,-1)$	5.962
208	$(0 \mid -3,0,-1,0,-1,0,0)$	$(0 \mid -1,-1,2,0,1,0,0)$	5.962
209	$(0 \mid -3,0,-1,0,0,-1,0)$	$(0 \mid -1,-1,2,0,0,1,0)$	5.962
210	$(0 \mid -3,0,-1,1,0,0,0)$	$(0 \mid -1,-1,2,-1,0,0,0)$	5.962
211	$(0 \mid -2,-2,1,1,-1,0,0)$	$(0 \mid -2,1,0,-1,1,0,0)$	5.962
212	$(0 \mid -3,0,-1,0,0,1,0)$	$(0 \mid -1,-1,2,0,0,-1,0)$	5.962
213	$(0 \mid -3,0,-1,0,1,0,0)$	$(0 \mid -1,-1,2,0,-1,0,0)$	5.962
214	$(0 \mid -1,-2,1,1,0,0,0)$	$(0 \mid -3,1,0,-1,0,0,0)$	5.962
215	$(0 \mid -2,0,-1,1,-1,0,0)$	$(0 \mid -2,-1,2,-1,1,0,0)$	5.962
216	$(0 \mid -2,-2,1,-1,0,0,1)$	$(0 \mid -2,1,0,1,0,0,-1)$	5.962
217	$(0 \mid -2,-2,1,-1,0,1,0)$	$(0 \mid -2,1,0,1,0,-1,0)$	5.962
218	$(0 \mid -2,0,-1,0,1,0,-1)$	$(0 \mid -2,-1,2,0,-1,0,1)$	5.962
219	$(0 \mid -2,0,-1,0,1,-1,0)$	$(0 \mid -2,-1,2,0,-1,1,0)$	5.962
220	$(0 \mid -2,0,-1,0,1,1,0)$	$(0 \mid -2,-1,2,0,-1,-1,0)$	5.962
221	$(0 \mid -2,0,-1,0,1,0,1)$	$(0 \mid -2,-1,2,0,-1,0,-1)$	5.962
222	$(0 \mid -2,0,-1,,1,0,0,1)$	$(0 \mid -2,-1,2,-1,0,0,-1)$	5.962
223	$(0 \mid -2,-2,1,-1,-1,0,0)$	$(0 \mid -2,1,0,1,1,0,0)$	5.962
224	$(0 \mid -2,0,-1,1,1,0,0)$	$(0 \mid -2,-1,2,-1,-1,0,0)$	5.962
225	$(0 \mid -2,0,-1,1,0,1,0)$	$(0 \mid -2,-1,2,-1,0,-1,0)$	5.962
226	$(0 \mid -2,-2,1,-1,0,-1,0)$	$(0 \mid -2,1,0,1,0,1,0)$	5.962
227	$(0 \mid -2,-2,1,-1,0,0,-1)$	$(0 \mid -2,1,0,1,0,0,1)$	5.962
228	$(0 \mid -2,0,-1,1,0,0,-1)$	$(0 \mid -2,-1,2,-1,0,0,1)$	5.962
229	$(0 \mid -2,0,-1,1,0,-1,0)$	$(0 \mid -2,-1,2,-1,0,1,0)$	5.962
230	$(0 \mid -2,0,-1,0,-1,-1,0)$	$(0 \mid -2,-1,2,,0,1,1,0)$	5.962
231	$(0 \mid -2,0,-1,-1,1,0,0)$	$(0 \mid -2,-1,2,1,-1,0,0)$	5.962
232	$(0 \mid -2,-1,0,-1,0,1)$	$(0 \mid -2,-1,2,0,1,0,-1)$	5.962
233	$(0 \mid -2,0,-1,0,-1,0,-1)$	$(0 \mid -2,-1,2,0,1,0,1)$	5.962
…	…	…	…

ID	$\boldsymbol{x}$	$\boldsymbol{y}$	SP
234	$(0 \mid -2,-2,1,0,-1,0,-1)$	$(0 \mid -2,1,0,0,1,0,1)$	5.962
235	$(0 \mid -2,-2,1,0,-1,0,1)$	$(0 \mid -2,1,0,0,1,0,-1)$	5.962
236	$(0 \mid -1,-2,1,0,0,1,0)$	$(0 \mid -3,1,0,0,0,-1,0)$	5.962
237	$(0 \mid -2,-2,1,0,-1,-1,0)$	$(0 \mid -2,1,0,0,1,1,0)$	5.962
238	$(0 \mid -1,-2,1,-1,0,0,0)$	$(0 \mid -3,1,0,1,0,0,0)$	5.962
239	$(0 \mid -2,-2,1,-1,1,0,0)$	$(0 \mid -2,1,0,1,-1,0,0)$	5.962
240	$(0 \mid -2,0,-1,0,0,1,1)$	$(0 \mid -2,-1,2,0,0,-1,-1)$	5.962
241	$(0 \mid -2,0,-1,,0,0,1,-1)$	$(0 \mid -2,-1,2,0,0,-1,1)$	5.962
242	$(0 \mid -2,0,-1,0,-1,1,0)$	$(0 \mid -2,-1,2,0,1,-1,0)$	5.962
243	$(0 \mid -1,-2,1,0,1,0,0)$	$(0 \mid -3,1,0,0,-1,0,0)$	5.962
244	$(0 \mid -2,0,-1,0,0,-1,1)$	$(0 \mid -2,-1,2,0,0,1,-1)$	5.962
245	$(0 \mid -2,0,-1,0,0,-1,-1)$	$(0 \mid -2,-1,2,0,0,1,1)$	5.962
246	$(0 \mid -2,-1,1,-1,-1,1,1)$	$(0 \mid -2,0,0,1,1,-1,-1)$	5.991
247	$(0 \mid -2,-1,-1,0,-1,0,1)$	$(0 \mid -2,0,2,0,1,0,-1)$	5.991
248	$(0 \mid -2,-1,-1,0,-1,0,-1)$	$(0 \mid -2,0,2,0,1,0,1)$	5.991
249	$(0 \mid -2,-1,-1,0,-1,-1,0)$	$(0 \mid -2,0,2,0,1,1,0)$	5.991
250	$(0 \mid -2,-1,-1,0,0,-1,1)$	$(0 \mid -2,0,2,0,0,1,-1)$	5.991
251	$(0 \mid -2,-1,-1,0,0,-1,-1)$	$(0 \mid -2,0,2,0,0,1,1)$	5.991
252	$(0 \mid -2,-1,-1,0,-1,1,0)$	$(0 \mid -2,0,2,0,1,-1,0)$	5.991
253	$(0 \mid -2,-1,-1,-1,0,0,-1)$	$(0 \mid -2,0,2,1,0,0,1)$	5.991
254	$(0 \mid -2,-1,-1,-1,0,-1,0)$	$(0 \mid -2,0,2,1,0,1,0)$	5.991
255	$(0 \mid -2,-1,-1,-1,-1,0,0)$	$(0 \mid -2,0,2,1,1,0,0)$	5.991
256	$(0 \mid -2,-1,-1,-1,1,0,0)$	$(0 \mid -2,0,2,1,-1,0,0)$	5.991
257	$(0 \mid -2,-1,-1,-1,0,1,0)$	$(0 \mid -2,0,2,1,0,-1,0)$	5.991
258	$(0 \mid -2,-1,-1,-1,0,0,1)$	$(0 \mid -2,0,2,1,0,0,-1)$	5.991
259	$(0 \mid -2,-1,-1,,1,0,0,-1)$	$(0 \mid -2,0,2,-1,0,0,1)$	5.991
260	$(0 \mid -2,-1,-1,,1,0,-1,0)$	$(0 \mid -2,0,2,-1,0,1,0)$	5.991
261	$(0 \mid -2,-1,-1,,1,-1,0,0)$	$(0 \mid -2,0,2,-1,1,0,0)$	5.991
262	$(0 \mid -2,-1,-1,,1,1,0,0)$	$(0 \mid -2,0,2,-1,-1,0,0)$	5.991
263	$(0 \mid -2,-1,-1,,1,0,1,0)$	$(0 \mid -2,0,2,-1,0,-1,0)$	5.991
264	$(0 \mid -2,-1,-1,,1,0,0,1)$	$(0 \mid -2,0,2,-1,0,0,-1)$	5.991
265	$(0 \mid -2,-1,-1,,0,1,-1,0)$	$(0 \mid -2,0,2,0,-1,1,0)$	5.991
266	$(0 \mid -2,-1,-1,,0,0,1,1)$	$(0 \mid -2,0,2,0,0,-1,-1)$	5.991
267	$(0 \mid -2,-1,-1,,0,0,1,-1)$	$(0 \mid -2,0,2,0,0,-1,1)$	5.991
268	$(0 \mid -2,-1,-1,0,1,1,0)$	$(0 \mid -2,0,2,0,-1,-1,0)$	5.991
269	$(0 \mid -2,-1,-1,,0,1,0,1)$	$(0 \mid -2,0,2,0,-1,0,-1)$	5.991
270	$(0 \mid -2,-1,-1,0,1,0,-1)$	$(0 \mid -2,0,2,0,-1,0,1)$	5.991
271	$(0 \mid -2,-1,1,0,0,0,2)$	$(0 \mid -2,0,0,0,0,0,-2)$	5.991
272	$(0 \mid -2,-1,1,0,0,2,0)$	$(0 \mid -2,0,0,0,0,-2,0)$	5.991
273	$(0 \mid -2,-2,0,1,1,0,0)$	$(0 \mid -2,1,1,-1,-1,0,0)$	5.991
274	$(0 \mid -2,-1,1,0,0,0,-2)$	$(0 \mid -2,0,0,0,0,0,2)$	5.991
275	$(0 \mid -2,-2,0,1,0,0,1)$	$(0 \mid -2,1,1,-1,0,0,-1)$	5.991
276	$(0 \mid -2,-2,0,1,0,1,0)$	$(0 \mid -2,1,1,-1,0,-1,0)$	5.991
277	$(0 \mid -2,-1,1,0,0,-2,0)$	$(0 \mid -2,0,0,0,0,2,0)$	5.991
278	$(0 \mid -2,-1,1,1,-1,1,1)$	$(0 \mid -2,0,0,-1,1,-1,-1)$	5.991
279	$(0 \mid -2,-1,1,1,1,-1,-1)$	$(0 \mid -2,0,0,-1,-1,1,1)$	5.991
280	$(0 \mid -2,-1,1,1,1,-1,1)$	$(0 \mid -2,0,0,-1,-1,1,-1)$	5.991
281	$(0 \mid -2,-1,1,1,-1,1,-1)$	$(0 \mid -2,0,0,-1,1,-1,1)$	5.991
282	$(0 \mid -2,-1,1,0,2,0,0)$	$(0 \mid -2,0,0,0,-2,0,0)$	5.991
283	$(0 \mid -2,-1,1,1,-1,-1,-1)$	$(0 \mid -2,0,0,-1,1,1,1)$	5.991
284	$(0 \mid -2,-1,1,1,-1,-1,1)$	$(0 \mid -2,0,0,-1,1,1,-1)$	5.991
285	$(0 \mid -2,-2,0,1,0,0,-1)$	$(0 \mid -2,1,1,-1,0,0,1)$	5.991
286	$(0 \mid -2,-2,0,0,1,0,-1)$	$(0 \mid -2,1,1,0,-1,0,1)$	5.991
287	$(0 \mid -2,-2,0,0,1,0,1)$	$(0 \mid -2,1,1,0,-1,0,-1)$	5.991
...	...	...	...

ID	$\boldsymbol{x}$	$\boldsymbol{y}$	SP
288	$(0 \mid -2,-2,0,0,1,1,0)$	$(0 \mid -2,1,1,0,-1,-1,0)$	5.991
289	$(0 \mid -2,-2,0,0,1,-1,0)$	$(0 \mid -2,1,1,0,-1,1,0)$	5.991
290	$(0 \mid -2,-2,0,0,0,-1,1)$	$(0 \mid -2,1,1,,0,0,1,-1)$	5.991
291	$(0 \mid -2,-2,0,0,0,1,-1)$	$(0 \mid -2,1,1,0,0,-1,1)$	5.991
292	$(0 \mid -2,-2,0,0,0,1,1)$	$(0 \mid -2,1,1,0,0,-1,-1)$	5.991
293	$(0 \mid -2,-1,1,0,-2,0,0)$	$(0 \mid -2,0,0,0,2,0,0)$	5.991
294	$(0 \mid -2,-2,0,1,-1,0,0)$	$(0 \mid -2,1,1,-1,1,0,0)$	5.991
295	$(0 \mid -2,-2,0,1,0,-1,0)$	$(0 \mid -2,1,1,-1,0,1,0)$	5.991
295	$(0 \mid -2,-1,1,-1,1,1,1)$	$(0 \mid -2,0,0,1,-1,-1,-1)$	5.991
296	$(0 \mid -2,-1,1,-1,1,-1,-1)$	$(0 \mid -2,0,0,1,-1,1,1)$	5.991
297	$(0 \mid -2,-1,1,-1,1,-1,1)$	$(0 \mid -2,0,0,1,-1,1,-1)$	5.991
298	$(0 \mid -2,-1,1,-1,1,1,-1)$	$(0 \mid -2,0,0,1,-1,-1,1)$	5.991
299	$(0 \mid -2,-2,0,0,-1,0,-1)$	$(0 \mid -2,1,1,0,1,0,1)$	5.991
300	$(0 \mid -2,-1,1,2,0,0,0)$	$(0 \mid -2,0,0,-2,0,0,0)$	5.991
301	$(0 \mid -2,-2,0,0,-1,-1,0)$	$(0 \mid -2,1,1,0,1,1,0)$	5.991
302	$(0 \mid -2,-1,1,-2,0,0,0)$	$(0 \mid -2,0,0,2,0,0,0)$	5.991
303	$(0 \mid -2,-2,0,-1,1,0,0)$	$(0 \mid -2,1,1,1,-1,0,0)$	5.991
304	$(0 \mid -2,-1,1,-1,-1,1,-1)$	$(0 \mid -2,0,0,1,1,-1,1)$	5.991
305	$(0 \mid -2,-1,1,-1,-1,-1,1)$	$(0 \mid -2,0,0,1,1,1,-1)$	5.991
306	$(0 \mid -2,-1,1,-1,-1,-1,-1)$	$(0 \mid -2,0,0,1,1,1,1)$	5.991
307	$(0 \mid -2,-2,0,0,-1,0,1)$	$(0 \mid -2,1,1,0,1,0,-1)$	5.991
308	$(0 \mid -2,-2,0,0,-1,1,0)$	$(0 \mid -2,1,1,0,1,-1,0)$	5.991
309	$(0 \mid -2,-2,0,-1,0,-1,0)$	$(0 \mid -2,1,1,1,0,1,0)$	5.991
310	$(0 \mid -2,-2,0,-1,0,0,-1)$	$(0 \mid -2,1,1,1,0,0,1)$	5.991
311	$(0 \mid -2,-1,1,1,1,1,1)$	$(0 \mid -2,0,0,-1,-1,-1,-1)$	5.991
312	$(0 \mid -2,-2,0,-1,-1,0,0)$	$(0 \mid -2,1,1,1,1,0,0)$	5.991
313	$(0 \mid -2,-1,1,1,1,1,-1)$	$(0 \mid -2,0,0,-1,-1,-1,1)$	5.991
314	$(0 \mid -2,-2,0,-1,0,0,1)$	$(0 \mid -2,1,1,1,0,0,-1)$	5.991
315	$(0 \mid -2,-2,0,0,0,-1,-1)$	$(0 \mid -2,1,1,0,0,1,1)$	5.991
316	$(0 \mid -2,-2,0,-1,0,1,0)$	$(0 \mid -2,1,1,1,0,-1,0)$	5.991
317	$(0 \mid -2,-1,0,-1,-1,-1,1)$	$(0 \mid -2,0,1,1,1,1,-1)$	6
318	$(0 \mid -2,-1,0,2,0,0,0)$	$(0 \mid -2,0,1,-2,0,0,0)$	6
319	$(0 \mid -2,-2,-1,0,0,0,0)$	$(0 \mid -2,1,2,0,0,0,0)$	6
320	$(0 \mid -2,-1,0,-2,0,0,0)$	$(0 \mid -2,0,1,2,0,0,0)$	6
321	$(0 \mid -2,-1,0,0,0,0,2)$	$(0 \mid -2,0,1,0,0,0,-2)$	6
322	$(0 \mid -2,-1,0,-1,-1,-1,-1)$	$(0 \mid -2,0,1,1,1,1,1)$	6
323	$(0 \mid -2,-1,0,-1,-1,1,-1)$	$(0 \mid -2,0,1,1,1,-1,1)$	6
324	$(0 \mid -2,-1,0,1,-1,-1,1)$	$(0 \mid -2,0,1,-1,1,1,-1)$	6
325	$(0 \mid -2,-1,0,0,-2,0,0)$	$(0 \mid -2,0,1,0,2,0,0)$	6
326	$(0 \mid -2,-1,0,-1,1,1,-1)$	$(0 \mid -2,0,1,1,-1,-1,1)$	6
327	$(0 \mid -2,-1,0,-1,1,1,1)$	$(0 \mid -2,0,1,1,-1,-1,-1)$	6
328	$(0 \mid -2,-1,0,1,-1,-1,-1)$	$(0 \mid -2,0,1,-1,1,1,1)$	6
329	$(0 \mid -2,-1,0,0,0,-2,0)$	$(0 \mid -2,0,1,0,0,2,0)$	6
330	$(0 \mid -2,-1,0,0,0,0,-2)$	$(0 \mid -2,0,1,0,0,0,2)$	6
331	$(0 \mid -2,-1,0,0,2,0,0)$	$(0 \mid -2,0,1,,0,-2,0,0)$	6
332	$(0 \mid -2,-1,0,0,0,2,0)$	$(0 \mid -2,0,1,0,0,-2,0)$	6
333	$(0 \mid -2,-1,0,1,1,1,-1)$	$(0 \mid -2,0,1,-1,-1,-1,1)$	6
334	$(0 \mid -2,-1,0,1,1,-1,1)$	$(0 \mid -2,0,1,-1,-1,1,-1)$	6
335	$(0 \mid -2,-1,0,-1,-1,1,1)$	$(0 \mid -2,0,1,1,1,-1,-1)$	6
336	$(0 \mid -2,-1,0,1,1,1,1)$	$(0 \mid -2,0,1,-1,-1,-1,-1)$	6
337	$(0 \mid -2,-1,0,1,1,-1,-1)$	$(0 \mid -2,0,1,-1,-1,1,1)$	6
338	$(0 \mid -2,-1,0,-1,1,-1,-1)$	$(0 \mid -2,0,1,1,-1,1,1)$	6
339	$(0 \mid -2,-1,0,-1,1,-1,1)$	$(0 \mid -2,0,1,1,-1,1,-1)$	6
340	$(0 \mid -2,-1,0,1,-1,1,1)$	$(0 \mid -2,0,1,-1,1,-1,-1)$	6
…	…	…	…

ID	$\boldsymbol{x}$	$\boldsymbol{y}$	SP
341	$(0 \mid -2, -1, 0, 1, -1, 1, -1)$	$(0 \mid -2, 0, 1, -1, 1, -1, 1)$	6

APPENDIX I

Unique parallelograms of the lattice point $(0 \mid -4, -1, 1, 0^4)$

An alternative dimensional exploration method for the partial second derivative of the energy density with respect to time is to search for the multiplicities of the parallelogram semi-perimeters SP.

Figure I.1 shows the histogram of semi-perimeters of parallelograms for the kind of quantity of the partial second derivative of the energy density with respect to time in the fundamental ellipsoid.

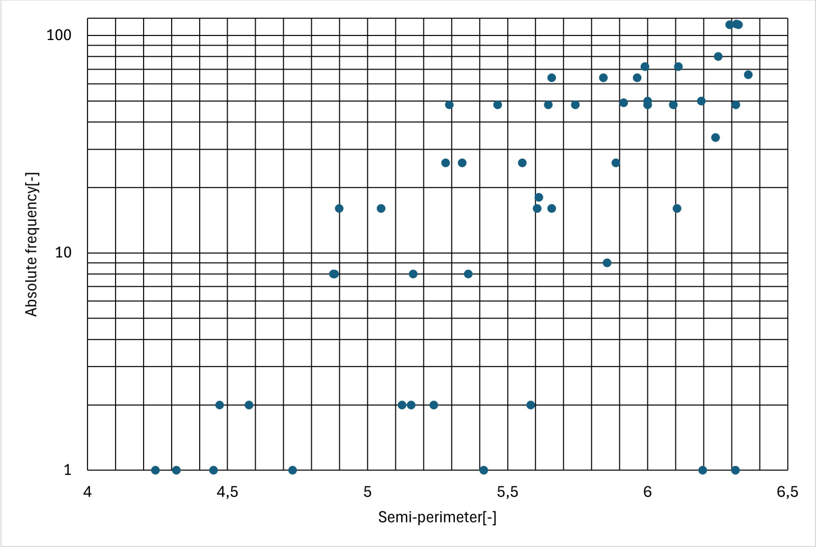

Figure I.1: Histogram of semi-perimeters of parallelograms, where $s \leq 4$ for the kind of quantity of the partial second derivative of the energy density with respect to time in the fundamental ellipsoid.

The unique parallelograms for $(0 \mid -4, -1, 1, 0^4)$ where $s \leq 5$ are given in Table I.1. The symbols used in the column quantity equation have the following interpretation: W energy density, $\dfrac{\partial W}{\partial t}$ 1st time derivative of energy density, $\dfrac{\partial^2 W}{\partial t^2}$ 2nd time derivative of energy density, x position, distance, L

characteristic length, τ proper time, ω_i angular frequency, M mass, A area, k magnitude of wavevector, S action, V volume and $f_i(\pi)$ function of dimensionless variable π . Observe that the lattice points of the unique parallelograms are located in the sub-lattice $\mathbb{Z}^3$.

Table I.1: Unique parallelograms for the kind of quantity of the partial second derivative of the energy density with respect to time in the fundamental ellipsoid.

ID	SP	$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{x} \cdot \boldsymbol{y}$	$\frac{\partial^2 W}{\partial t^2}$
1	4.243	$(0 \mid -4, -1, 1, 0^4)$	$(0 \mid 0^7)$	0	$\frac{\partial^2 W}{\partial t^2} \sim f_1(\pi) \frac{\partial^2 W}{\partial t^2}$
2	4.317	$(0 \mid -3, -1, 1, 0^4)$	$(0 \mid -1, 0^6)$	3	$\frac{\partial^2 W}{\partial t^2} \sim f_2(\pi) \omega_1 \frac{\partial W}{\partial t}$
3	4.449	$(0 \mid -2, -1, 1, 0^4)$	$(0 \mid -2, 0^6)$	4	$\frac{\partial^2 W}{\partial t^2} \sim f_3(\pi) \omega_1^2 W$
4	4.472	$(0 \mid -2, -1, 0^5)$	$(0 \mid -2, 0, 1, 0^4)$	4	$\frac{\partial^2 W}{\partial t^2} \sim f_4(\pi) \frac{\partial^2 k}{\partial \tau^2} \frac{\partial^2 M}{\partial t^2}$
5	4.732	$(0 \mid -1, -1, 1, 0^4)$	$(0 \mid -3, 0^6)$	3	$\frac{\partial^2 W}{\partial t^2} \sim f_5(\pi) \omega_1^3 \left(\frac{S}{V}\right)$
6	5.414	$(0 \mid 0, -1, 1, 0^4)$	$(0 \mid -4, 0^6)$	0	$\frac{\partial^2 W}{\partial t^2} \sim f_6(\pi) \omega_1^4 \frac{\partial M}{\partial x}$
7	6.196	$(0 \mid -5, -1, 1, 0^4)$	$(0 \mid 1, 0^6)$	-5	$\frac{\partial^2 W}{\partial t^2} \sim f_7(\pi) \int \frac{\partial^3 W}{\partial t^3} \, \mathrm{d}\tau$
8	6.313	$(0 \mid -4, -2, 2, 0^4)$	$(0 \mid 0, 1, -1, 0^4)$	-4	$\frac{\partial^2 W}{\partial t^2} \sim f_8(\pi) \left(\frac{L}{M}\right) W^2$
9	6.317	$(0 \mid -1, -2, 2, 0^4)$	$(0 \mid -3, 1, -1, 0^4)$	-1	$\frac{\partial^2 W}{\partial t^2} \sim f_9(\pi) \left(\frac{M^2}{A\tau}\right) \left(\frac{L}{M\tau^3}\right)$

I.1 Equation ID = 4

The first non-trivial parallelogram is the one with identifier ID = 4. It represents the quantity equation where $W = \mathcal{L}(x, \dot{x}, \tau)$:

$$\frac{\partial^2 \mathcal{L}(x, \dot{x}, \tau)}{\partial \tau^2} = f(\pi)(\frac{1}{x\tau^2})(\frac{M}{\tau^2}), \tag{I.1}$$

in which $\mathcal{L}(x, \dot{x}, \tau)$ is the Lagrange density. An alternative quantity equation in terms of frequencies is:

$$\frac{\partial^2 \mathcal{L}(x, \dot{x}, \tau)}{\partial \tau^2} = f(\pi)(\frac{\nu_1^2}{x})(M\nu_2^2), \tag{I.2}$$

in which ν_1 and ν_2 are characteristic frequencies.

I.2 Equation ID = 6

Identifier ID = 6 gives a quantity equation, with $W = T(\tau)$ under the assumption that the contracted energy-momentum tensor $= T(\tau)$ is a function only of the time parameter τ:

$$\frac{\mathrm{d}^2 T}{\mathrm{d}\tau^2} = f_6(\pi)\left(\frac{M}{L}\right)\left(\frac{1}{\tau^4}\right). \tag{I.3}$$

Observe that this parallelogram is a *unique rectangle*. We expect that this quantity equation is a 'law of physics'. The term $(\frac{M}{L})$ refers to a linear mass density.

I.2.1 Constant linear mass density

Consider the special case where the linear mass density is constant. We write the second order ordinary differential equation in an autonomous system of first order differential equation given by:

$$\frac{\mathrm{d}^2 T}{\mathrm{d}\tau^2} = \frac{\mathrm{d}P}{\mathrm{d}\tau} = f_6(\pi)\left(\frac{M}{L}\right)\tau^{-4}, \tag{I.4}$$

in which P is respectively the instantaneous power density expressed in $\mathrm{W} \cdot \mathrm{m}^{-3}$ and $\left(\frac{M}{L}\right)$ the linear mass density expressed in $\mathrm{kg} \cdot \mathrm{m}^{-1}$ of the open/-closed system under study. Integration of the equation, under the assumption that $\left(\frac{M}{L}\right)$ is constant, results in:

$$T(\tau) = \frac{1}{6} f_6(\pi)\left(\frac{M}{L}\right)\tau^{-2} + C_1\tau + C_2, \tag{I.5}$$

in which C_1 and C_2 are arbitrary constants. We plot this function under the conditions: $f_6(\pi)\left(\frac{M}{L}\right) = 1, C_2 = 1, 0 < \tau \leq 5$ with $C_1 = 0.15, C_1 = 0, C_1 = -0.15$ and obtain the Figure I.2.

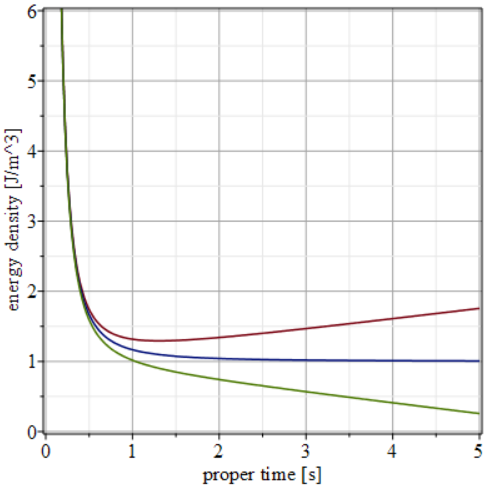

Figure I.2: Energy density with constant linear mass density and $f_6(\pi)\left(\frac{M}{L}\right) = 1, C_2 = 1, 0 < \tau \leq 5$ and $C_1 = 0.15$ red line, $C_1 = 0$ blue line and $C_1 = -0.15$ green line.

A curve of similar shape as the green line is observed in astrophysics where the average energy density of the universe is mapped versus the age of the universe (Krauss, 2012, p.151).

I.2.2 Constant angular frequency and spherical volume

Consider the angular frequency ω_1 as a constant in the quantity equation of identifier ID = 6 and consider that the length L is interpreted as a radial variable r . We write the following partial differential equation:

$$\frac{\partial^2 T(r,t)}{\partial t^2} = f_6(\pi)\omega_1^4 \frac{\partial M(r,t)}{\partial r} . \tag{I.6}$$

Consider a sphere of radius r with volume $V = \frac{4}{3}\pi r^3$ and assume that $T(r,t) = \frac{M(r,t)}{V}c^2$. We obtain the following parabolic partial differential equation:

$$\frac{\partial^2 T(r,t)}{\partial t^2} = A \frac{\partial (r^3 T(r,t))}{\partial r} . \tag{I.7}$$

where $A = \dfrac{4\pi f_6(\pi)\omega_1^4}{3c^2} > 0$. The parabolic differential equation is a variant on the heat equation and thus we use the separation of variables method and write $T(r,t) = R(r)U(t)$.

I.2.3 Constant angular frequency and cubic volume

Consider a cube with volume $V = L_x L_y L_z$. Assume that the volume V is constant. We consider that the box has a constant mass M but that the mass M varies over the direction x. We obtain the parabolic differential equation:

$$\frac{\partial^2 T(x,t)}{\partial t^2} = B\frac{\partial T(x,t)}{\partial x}\,, \tag{I.8}$$

in which $B = \dfrac{L_x L_y L_z f_6(\pi)\omega_1^4}{c^2} > 0$. We use the separation of variables and search for a solution of the type $T(x,t) = X(x)U(t)$. Using Maplesoft (2018) we find:

$$T(x,t) = W_0 \exp(-kx)[W_1 \exp(i\sqrt{kB}t) + W_2 \exp(-i\sqrt{kB}t)], \tag{I.9}$$

in which $W_0, W_1, W_2 > 0$.

I.2.3.1 Case: $k = 0$

We find $T(x,t) = W_0[W_1 + W_2]$ representing a constant energy density,

I.2.3.2 Case: $k > 0, x \in \mathbb{R}^+$

The boundary conditions are:

$$T(0,t) = W_0[W_1 \exp(i\sqrt{kB}t) + W_2 \exp(-i\sqrt{kB}t)]\,, \tag{I.10}$$

$$T(L_x,t) = W_0 \exp(-kL_x)[W_1 \exp(i\sqrt{kB}t) + W_2 \exp(-i\sqrt{kB}t)]\,, \tag{I.11}$$

$$T(x,0) = W_0 \exp(-kx)[W_1 + W_2]\,, \tag{I.12}$$

$$T(0,0) = W_0[W_1 + W_2]\,, \tag{I.13}$$

$$T(L_x,0) = W_0 \exp(-kL_x)[W_1 + W_2]\,. \tag{I.14}$$

(I.15)

Let $T_0 = W_0[W_1 + W_2]$ and put $W_1 = W_2$ then $T_0 = 2W_0W_1$.

$$T(0,t) = T_0 \cos\sqrt{kB}t\,, \tag{I.16}$$

$$T(L_x,t) = T_0 \exp(-kL_x)\cos\sqrt{kB}t\,, \tag{I.17}$$

$$T(0,0) = T_0\,, \tag{I.18}$$

$$T(L_x,0) = T_0 \exp(-kL_x)\,. \tag{I.19}$$

(I.20)

We write under the above assumptions:

$$T(x,t) = T_0 \exp(-kx) \cos\sqrt{kB}t\,. \tag{I.21}$$

We plot this function under the conditions: $T_0 = 1, k = 1, B = 1, 0 < t < 2\pi$ and obtain the Figure I.3.

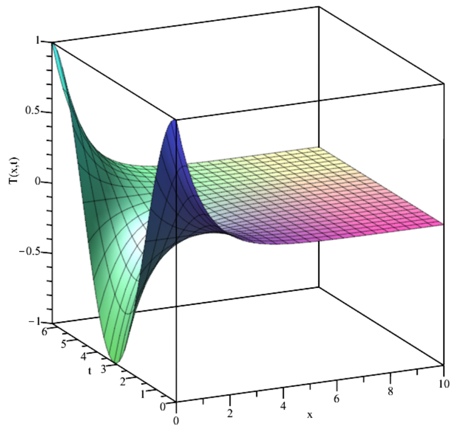

Figure I.3: Longitudinal energy density in a constant cubic volume for $k > 0, x \in \mathbb{R}^+, 0 < t < 2\pi$.

Observe that the time-averaged energy density $\frac{1}{2\pi}\int_0^{2\pi} T(x,t)\mathrm{d}t = 0$ when averaging over 1 period.

I.2.3.3 Case: $k < 0, x \in \mathbb{R}^-$

Set $k = -\kappa$ then we find:

$$T(x,t) = W_0 \exp(\kappa x)[W_1 \exp(-\sqrt{\kappa B}t) + W_2 \exp(\sqrt{\kappa B}t)]\,. \tag{I.22}$$

The boundary conditions are:

$$T(0,t) = W_0[W_1 \exp(-\sqrt{\kappa B}t) + W_2 \exp(\sqrt{\kappa B}t)]\,, \tag{I.23}$$

$$T(-L_x,t) = W_0 \exp(-\kappa L_x)[W_1 \exp(-\sqrt{\kappa B}t) + W_2 \exp(\sqrt{\kappa B}t)] \tag{I.24}$$

$$T(-x,0) = W_0 \exp(-\kappa x)[W_1 + W_2]\,, \tag{I.25}$$

$$T(0,0) = W_0[W_1 + W_2]\,, \tag{I.26}$$

$$T(-L_x,0) = W_0 \exp(-\kappa L_x)[W_1 + W_2]\,. \tag{I.27}$$

(I.28)

Let $T_0 = W_0[W_1 + W_2]$ and put $W_1 = W_2$ then $T_0 = 2W_0W_1$. We write under the above assumptions:

$$T(x,t) = T_0 \exp(\kappa x) \cosh \sqrt{\kappa B} t\,. \tag{I.29}$$

We plot this function under the conditions: $T_0 = 1, \kappa = 1, B = 1, 0 < t < 1$ and obtain the Figure I.4.

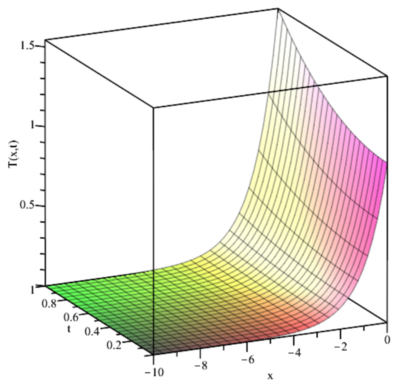

Figure I.4: Longitudinal energy density in a constant cubic volume for $k = -\kappa < 0,\ x \in \mathbb{R}^-,\ 0 < t < 1$.

I.2.3.4 Case: $k = i$

It is obvious that the energy density $T(x,t)$ becomes a complex function for the case $k = i$. Hence, we expect that a physically meaningful equation will be retrieved by taking the real part of the complex function $T(x,t)$. We write the equation:

$$T(x,t) = W_0 \exp(-ix)[W_1 \exp(i\sqrt{iB}t) + W_2 \exp(-i\sqrt{iB}t)]\,, \tag{I.30}$$

in which $i\sqrt{iB} = \frac{\sqrt{2}}{2}(-\operatorname{sgn} B\sqrt{|B|} + i\sqrt{|B|})$ and $-i\sqrt{iB} = \frac{\sqrt{2}}{2}(\operatorname{sgn} B\sqrt{|B|} - i\sqrt{|B|})$. We plot the real part of the energy density $\Re(T(x,t))$ where $W_0 = 1, W_1 = 1, B = 1$ and obtain the Figure I.5.

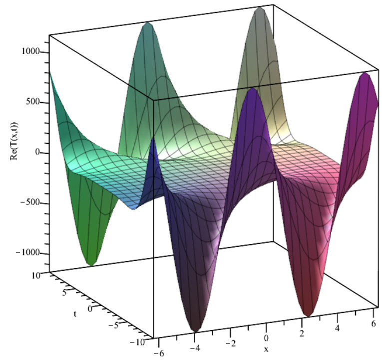

Figure I.5: Real part of the longitudinal energy density in a constant cubic volume for $k = i, -2\pi \leq x \leq 2\pi, -10 \leq t \leq 10$.

We plot the imaginary part of the energy density $\Im(T(x,t))$ where $W_0 = 1, W_1 = 1, B = 1$ and obtain the Figure I.6.

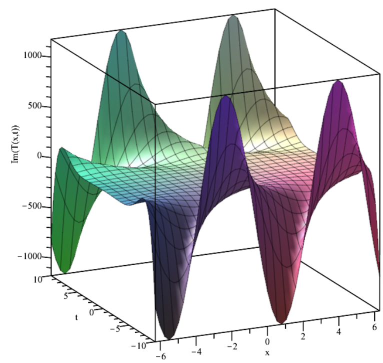

Figure I.6: Imaginary part of the longitudinal energy density in a constant cubic volume for $k = i, -2\pi \leq x \leq 2\pi, -10 \leq t \leq 10$.

We plot the modulus of the energy density $|T(x,t)|$ where $W_0 = 1, W_1 = 1, B = 1$ and obtain the Figure I.7.

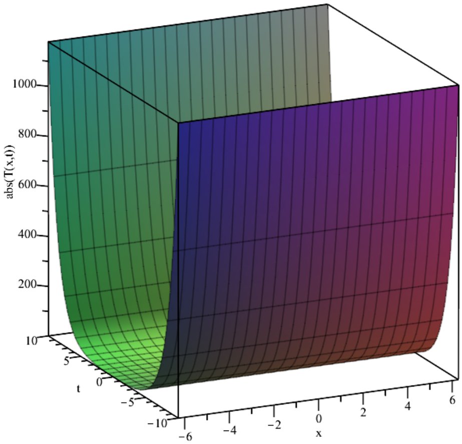

Figure I.7: Modulus of the longitudinal energy density in a constant cubic volume for $k = i,\ -2\pi \le x \le 2\pi, -10 \le t \le 10\,.$

We plot the argument of the energy density $\arg T(x,t)$ where $W_0 = 1, W_1 = 1, B = 1$ and obtain the Figure I.8.

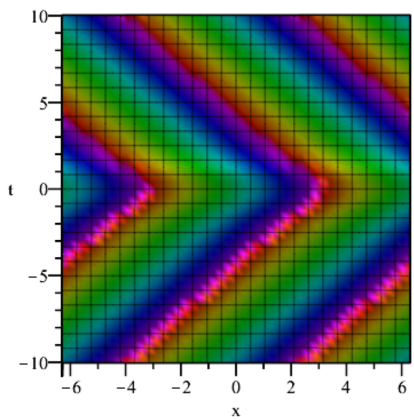

Figure I.8: Argument of the longitudinal energy density in a constant cubic volume for $k = i,\ -2\pi \le x \le 2\pi, -10 \le t \le 10\,.$

The values of the argument vary between $-\pi$ represented by blue and $+\pi$ represented by red.

I.3 Equation ID = 8

Assume that the energy density W is a function only of the time parameter t such that we can replace the partial derivative by a total derivative. We have then the equation:

$$\frac{\mathrm{d}^2 W}{\mathrm{d}t^2} = f_8(\pi)\left(\frac{L}{M}\right)W^2\,, \tag{I.31}$$

that can be identified as a non-linear ordinary differential equation of the type Emden-Fowler (Zaitsev & Polyanin, 2002, p. 306):

$$y'' = Ax^n y^m\,, \tag{I.32}$$

where $n = 0$, $m = 2$ and $A = \phi(a,b)$. The parametric form of the solution is:

$$x = f_1(\tau, C_1, C_2, a)$$
$$y = f_2(\tau, C_1, C_2, b)$$

in which τ is a parameter, C_1 and C_2 are arbitrary constants, and f_1 and f_2 are some functions. The solution to this special case is given in (Zaitsev & Polyanin, 2002, p. 308):

$$t = aC_1^{-1}\tau$$
$$W = bC_1^2 W_p$$
$$A = \pm 6a^{-2}b^{-1}$$

in which W_p is the Weiertstrass-P function and the parameter τ is defined in implicit form as:

$$\tau = \int \frac{\mathrm{d}W_p}{\sqrt{\pm(4W_p^3 - 1)}} - C_2\,. \tag{I.33}$$

The positive sign corresponds with a special case of the elliptic Weierstrass function: $W_p = W_p(\tau + C_2, 0, 1)$. The $WeierstrassP(z, g_2, g_3)$ function is given by:

$$WeierstrassP(z, g_2, g_3) = \frac{1}{z^2} + \sum_{\omega}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right), \tag{I.34}$$

in which $\omega = 2M\omega_1 + 2N\omega_2$ and $z \in \mathbb{C}$. The summation is over all the pairs (M, N) except $M = N = 0$ where $M, N \in \mathbb{Z}$. The quantities g_2 and g_3 are known as invariants and are defined as:

$$g_2 = 60\left(\sum_{\omega}\frac{1}{\omega^4}\right), \tag{I.35}$$

$$g_3 = 140\left(\sum_{\omega}\frac{1}{\omega^6}\right). \tag{I.36}$$

The elliptic Weierstrass function $W_p = W_p(\tau + C_2, 0, 1)$ is given in Abramowitz and Stegun (1964, p. 640) at paragraph 18.6.4. It is known as the equianharmonic case (Abramowitz & Stegun, 1964, p. 652) and is described in paragraph 18.13 where $(g_2 = 0, g_3 = 1)$. Comparing the respective terms of the equation ID = 8 and the Emden-Fowler equation we find that:

$$A = \pm\left(\frac{6}{a^2 b}\right) = f_8(\pi)\left(\frac{L}{M}\right).$$

A first integral to equation ID = 8 is known in the form (Zaitsev & Polyanin, 2002, p. 312):

$$\left(\frac{\mathrm{d}W}{\mathrm{d}t}\right)^2 - \left(\frac{2}{3}\right) f_8(\pi)\left(\frac{L}{M}\right) W^3 = C\,. \tag{I.37}$$

This can be obtained by transforming the second order nonlinear ordinary differential equation in an autonomous system of first order differential equation given by:

$$\begin{cases} \dfrac{\mathrm{d}W}{\mathrm{d}t} = P\,, \\ \dfrac{\mathrm{d}P}{\mathrm{d}t} = f_8(\pi)\left(\dfrac{L}{M}\right) W^2\,. \end{cases} \tag{I.38}$$

in which P is respectively the instantaneous power density expressed in $\mathrm{W} \cdot \mathrm{m}^{-3}$ and $\dfrac{M}{L}$ the linear mass density expressed in $\mathrm{kg} \cdot \mathrm{m}^{-1}$ of the open/closed system under study. The Jacobian $\mathbf{J}(W, P)$ is given by:

$$\mathbf{J}(W, P) = \begin{bmatrix} 0 & 1 \\ 2 f_8(\pi)\left(\dfrac{L}{M}\right) W & 0 \end{bmatrix}, \tag{I.39}$$

and the eigenvalues are:

$$\lambda_1 = \frac{\sqrt{2}\sqrt{M f_8(\pi) L W}}{M}\,, \tag{I.40}$$

$$\lambda_2 = -\frac{\sqrt{2}\sqrt{M f_8(\pi) L W}}{M}\,. \tag{I.41}$$

We find two real eigenvalues if $f_8(\pi) W > 0$ and two complex conjugate eigenvalues if $f_8(\pi) W < 0$ where we assume that $M, L > 0$. The critical point of the autonomous system is given by:

$$\begin{cases} \dfrac{\mathrm{d}W}{\mathrm{d}t} = 0 = P\,, \\ \dfrac{\mathrm{d}P}{\mathrm{d}t} = 0 = f_8(\pi)\left(\dfrac{L}{M}\right) W^2\,, \end{cases} \tag{I.42}$$

and thus we have a non-degenerate critical point $(W, P) = (0, 0)$. The critical point $(W, P) = (0, 0)$ is hyperbolic when $f_8(\pi)W > 0$ and non-hyperbolic when $f_8(\pi)W < 0$. In the case of the real eigenvalues we find a positive and a negative eigenvalue and thus the critical point is a saddle point. This can also be seen by inspecting the determinant of the Jacobian $\det \mathbf{J} = -2f_8(\pi)\left(\frac{L}{M}\right)W$. We find a saddle point when $\det \mathbf{J} < 0$. By dividing the equations we eliminate the time parameter $\mathrm{d}t$ and find:

$$\frac{\mathrm{d}W}{\mathrm{d}P} = \frac{P}{f_8(\pi)(\frac{L}{M})W^2} \,. \tag{I.43}$$

We integrate in the WP phase plane the equation between the points (W_2, P_2) and (W_1, P_1):

$$\int_{W(t_1)}^{W(t_2)} f_8(\pi)\left(\frac{L}{M}\right)W^2 \,\mathrm{d}W = \int_{P(t_1)}^{P(t_2)} P \,\mathrm{d}P \,, \tag{I.44}$$

resulting in the equation:

$$\frac{1}{3}f_8(\pi)\left(\frac{L}{M}\right)(W(t_2)^3 - W(t_1)^3) = \frac{1}{2}(P(t_2)^2 - P(t_1)^2) \,. \tag{I.45}$$

Solving for the function $f_8(\pi)$ of dimensionless parameter π, we find:

$$f_8(\pi) = \frac{3}{2}\frac{M}{L}\frac{(P(t_2)^2 - P(t_1)^2)}{(W(t_2)^3 - W(t_1)^3)} \,. \tag{I.46}$$

The equation $f_8(\pi)$ can be determined experimentally by measuring the pairs $(W(t_i), P(t_i))$ for the different states i of the system under test. The presence of the term $\frac{M}{L}$ suggest that the system under test should have cylindrical properties. The fact that this equation is derived from a parallelogram with a unique semi-perimeter should give the equation a universal character.

We explore the solution in the case $C = 0$, $f_8(Pi) = 1$, $t_1 = 4.34 \times 10^{17}$ s, $W(t_1) = W_0 = 7.65 \times 10^{-10}\,\mathrm{J \cdot m^{-3}}$, $P(t_1) = P_0 = -7.06 \times 10^{-28}\,\mathrm{W \cdot m^{-3}}$ and $\frac{M}{L} = 5.99 \times 10^{26}\,\mathrm{kg \cdot m^{-1}}$ and take the power density P as a variable parameter. The results of the phase plane analysis are given in Figure I.9.

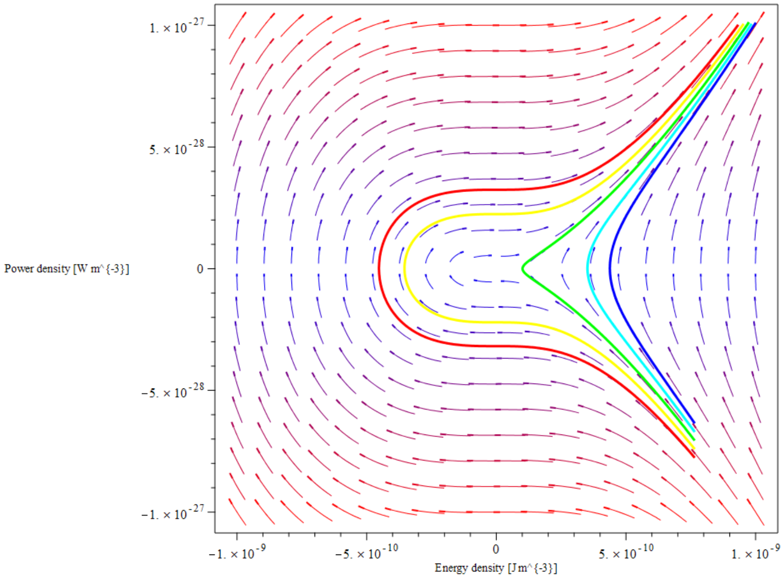

Figure I.9: Phase plane analysis of equation ID = 8 giving the power density versus the energy density for a hypothetical system having the parameters: Blue line $P = 0.9P_0$, Cyan line $P = 0.95P_0$, Green line $P = 1.0P_0$, Yellow line $P = 1.05P_0$, Red line $P = 1.1P_0$ and $W = W_0$ with a the time interval from $t_1 = 4.34 \times 10^{17}$ s to $t = 94.2 \times 10^{17}$ s .

Appendix J

Maximum number of distinct parallelogram semi-perimeters as function of N and s

We give in Table J.1 the number of distinct parallelogram semi-perimeters as function of the integer lattice dimension N and the infinity norm s.

Table J.1: Maximum number of distinct parallelogram semi-perimeters as function of the infinity norm s where $2 \leq N \leq 7$.

$s \vert N$	2	3	4	5	6	7
0	1	1	1	1	1	1
1	6	10	15	21	28	36
2	45	91	153	231	325	435
3	190	406	703	1 081	1 540	2 080
4	561	1 225	2 145	3 321	4 753	6 441
5	1 326	2 926	5 151	8 001	11 476	15 576
6	2 701	5 995	10 585	16 471	23 653	32 131
7	4 950	11 026	19 503	30 381	43 660	59 340
8	8 385	18 721	33 153	51 681	74 305	101 025
9	13 699	29 890	52 975	82 621	118 828	161 596
10	20 301	45 451	80 601	125 751	180 901	246 051
11	29 646	66 430	117 855	183 921	264 628	359 976
12	41 905	93 961	166 753	260 281	374 545	509 545
13	57 630	129 286	229 503	358 281	515 620	701 520
14	77 421	173 755	308 505	481 671	693 253	943 251

Appendix K

Derived and fundamental physical constants

Table K.1: Derived and fundamental physical constants.

Physical quantity	Physical constant	Value	SI units
permeability of vacuum	μ_0	$1.25663706212 \times 10^{-6}$	$\mathrm{s}^{-2} \cdot \mathrm{m} \cdot \mathrm{kg} \cdot \mathrm{A}^{-2}$
permittivity of vacuum	$\frac{1}{\mu_0 c_0^2}$	$8.8541878128 \times 10{-}12$	$\mathrm{s}^{4} \cdot \mathrm{m}^{-3} \cdot \mathrm{kg}^{-1} \cdot \mathrm{A}^{2}$
Velocity	c_0	299792458	$\mathrm{s}^{-1} \cdot \mathrm{m}$
Elementary charge	e	$1.602176634 \times 10^{-19}$	$\mathrm{s} \cdot \mathrm{A}$
Planck constant	h	$6.62607015 \times 10^{-34}$	$\mathrm{s}^{-1} \cdot \mathrm{m}^{2} \cdot \mathrm{kg}$
Newtonian constant of gravitation	G	6.67430×10^{-11}	$\mathrm{s}^{-2} \cdot \mathrm{m}^{3} \cdot \mathrm{kg}^{-1}$
Length	$\sqrt{\frac{hG}{c_0^3}}$	4.05128×10^{-35}	m
Mass	$\sqrt{\frac{hc_0}{G}}$	5.48860×10^{-8}	kg
Time	$\sqrt{\frac{hG}{c_0^5}}$	1.35136×10^{-43}	s
Length $\times$ Time	$\frac{hG}{c_0}$	5.47475×10^{-78}	$\mathrm{s} \cdot \mathrm{m}$
Length $\times$ Mass	$\frac{h}{c_0}$	$2.21021909 \times 10^{-42}$	$\mathrm{m} \cdot \mathrm{kg}$
...	...	...	...

Physical quantity	Physical constant	Value	SI units
Mass × Time	$\frac{h}{c_0^2}$	$7.37249732 \times 10^{-51}$	$\mathrm{s} \cdot \mathrm{kg}$
Mass / Time	$\frac{c_0^3}{G}$	$4.03711110 \times 10^{35}$	$\mathrm{kg} \cdot \mathrm{s}^{-1}$
Length / Mass	$\frac{G}{c_0^2}$	7.38126×10^{-28}	$\mathrm{m} \cdot \mathrm{kg}^{-1}$
Linear impulse	$\sqrt{\frac{hc_0^3}{G}}$	16.35548266	$\mathrm{s}^{-1} \cdot \mathrm{m} \cdot \mathrm{kg}$
Energy	$\sqrt{\frac{hc_0^5}{G}}$	4903250350	$\mathrm{s}^{-2} \cdot \mathrm{m}^2 \cdot \mathrm{kg}$
Energy density	$\frac{c_0^7}{hG^2}$	7.37405×10^{112}	$\mathrm{s}^{-2} \cdot \mathrm{m}^{-1} \cdot \mathrm{kg}$
Force(Tension)	$\frac{c_0^4}{G}$	1.21030×10^{44}	$\mathrm{s}^{-2} \cdot \mathrm{m} \cdot \mathrm{kg}$
Power	$\frac{c_0^5}{G}$	3.62837×10^{52}	$\mathrm{s}^{-3} \cdot \mathrm{m}^2 \cdot \mathrm{kg}$
Frequency	$\sqrt{\frac{c_0^5}{hG}}$	7.399941×10^{42}	s^{-1}
Electrical potential	$\sqrt{\frac{hc_0^5}{Ge^2}}$	3.06037×10^{28}	V
Electrical capacitance	$\sqrt{\frac{Ge^4}{hc_0^5}}$	5.23524×10^{-48}	F
...	...	...	...

Physical quantity	Physical constant	Value	SI units
von Klitzing constant	$\frac{h}{e^2}$	25812.807459305	Ω
Magnetic flux	$\frac{h}{e}$	$4.13566770 \times 10^{-15}$	T
Magnetic induction	$\frac{c_0^3}{eG}$	$2.519766553 \times 10^{54}$	T
Inductance	$\sqrt{\frac{h^3 G}{c_0^5 e^4}}$	3.48825×10^{-39}	H
Dynamic viscosity	$\sqrt{\frac{c_0^9}{hG^3}}$	9.96502×10^{69}	$\mathrm{s}^{-1} \cdot \mathrm{m}^{-1} \cdot \mathrm{kg}$

APPENDIX L

Parametric quantity equation for the kind of quantity energy

What quantity equations can be formed for the kind of quantity energy when energy E is expressed as a power r of the mass m of elementary particles, with $m \neq 0$, while the proportionality constant is a monomial containing the fundamental constants G, h, c_0 ?

Application of classical dimensional analysis (CDA) to answer the question, results in a parametric quantity equation:

$$E(r) \propto m^r \sqrt{h^{1-r} c_0^{5-r} G^{r-1}}, \tag{L.1}$$

with $r \in \mathbb{Z}$ being the parameter. The parametric quantity equation represents parallelograms with one side being a multiple of the vector $\boldsymbol{m}$ in $\{0, 1\} \times \mathbb{Z}^7$.

Consider the exponents of the fundamental constants:

$$r = -5 \qquad E(-5) \propto m^{-5}\frac{h^3 c_0^5}{G^3}, \tag{L.2}$$

$$r = -4 \qquad E(-4) \propto m^{-4}\sqrt{\frac{h^5 c_0^9}{G^5}}, \tag{L.3}$$

$$r = -3 \qquad E(-3) \propto m^{-3}\frac{h^2 c_0^4}{G^2}, \tag{L.4}$$

$$r = -2 \qquad E(-2) \propto m^{-2}\sqrt{\frac{h^3 c_0^7}{G^3}}, \tag{L.5}$$

$$r = -1 \qquad E(-1) \propto m^{-1}\frac{h c_0^3}{G}, \tag{L.6}$$

$$r = 0 \qquad E(0) \propto \sqrt{\frac{h c_0^5}{G}}, \tag{L.7}$$

$$r = 1 \qquad E(1) \propto m c_0^2, \tag{L.8}$$

$$r = 2 \qquad E(2) \propto m^2\sqrt{\frac{G c_0^3}{h}}, \tag{L.9}$$

$$r = 3 \qquad E(3) \propto m^3\frac{G c_0}{h}, \tag{L.10}$$

$$r = 4 \qquad E(4) \propto m^4\sqrt{\frac{c_0 G^3}{h^3}}, \tag{L.11}$$

$$r = 5 \qquad E(5) \propto m^5\frac{G^2}{h^2}. \tag{L.12}$$

The quantity equations in which $r = 0$ and $r = 1$ are known from the scientific literature. The other quantity equations are unknown to the author. To calculate the proportionality constants we use the following numerical values of the universal constants (Tiesinga, Mohr, Newell, & Taylor, 2021):

$$\begin{aligned} c_0 &= 299\,792\,458\,\mathrm{s}^{-1}\cdot\mathrm{m}, \\ h &= 6.626\,070\,15\times 10^{-34}\,\mathrm{s}^{-1}\cdot\mathrm{m}^2\cdot\mathrm{kg}, \\ G &= 6.674\,30\times 10^{-11}\,\mathrm{s}^{-2}\cdot\mathrm{m}^3\cdot\mathrm{kg}^{-1}. \end{aligned}$$

The proportionality constants are:

$$\sqrt{\frac{hc_0^5}{G}} = 4\,903\,169\,538\,\mathrm{m}^2 \cdot \mathrm{kg}^1 \cdot \mathrm{s}^{-2}, \tag{L.13}$$

$$c_0^2 = 89\,875\,517\,873\,681\,800\,\mathrm{m}^2 \cdot \mathrm{s}^{-2}, \tag{L.14}$$

$$\sqrt{\frac{Gc_0^3}{h}} = 1.647\,43 \times 10^{24}\,\mathrm{m}^2 \cdot \mathrm{kg}^{-1} \cdot \mathrm{s}^{-2}, \tag{L.15}$$

$$\frac{Gc_0}{h} = 3.019\,75 \times 10^{31}\,\mathrm{m}^2 \cdot \mathrm{kg}^{-2} \cdot \mathrm{s}^{-2}, \tag{L.16}$$

$$\sqrt{\frac{c_0 G^3}{h^3}} = 5.535\,22 \times 10^{38}\,\mathrm{m}^2 \cdot \mathrm{kg}^{-3} \cdot \mathrm{s}^{-2}, \tag{L.17}$$

$$\frac{G^2}{h^2} = 1.014\,61 \times 10^{46}\,\mathrm{m}^2 \cdot \mathrm{kg}^{-4} \cdot \mathrm{s}^{-2}. \tag{L.18}$$

Figure L.1 shows the cases in which $r \in \{0, \ldots, 5\}$. The cases in which $r \in \{-1, \ldots, -5\}$ are probably for hypothetical elementary particles that are heavier than the Planck mass $m_P \propto \sqrt{\frac{hc_0}{G}}$.

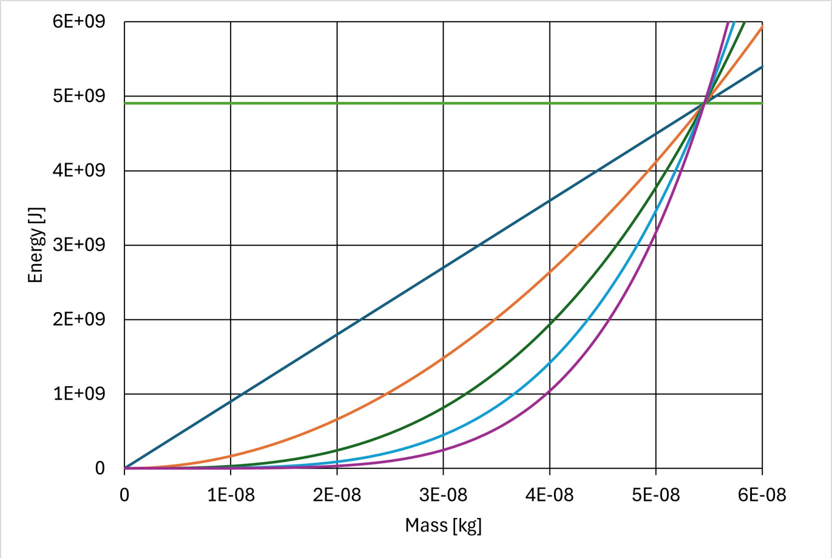

Figure L.1: Kind of quantity energy E versus mass m where the constant energy is intersecting the other graphs in the point where the mass is $m_P = \sqrt{\frac{hc_0}{G}}$.

Observe that the proportionality constant in the parametric quantity equation becomes independent of the speed of light in vacuum when $r = 5$ and that it becomes simultaneously independent of h and G when $r = 1$ leading to Einstein's famous equation. The parametric quantity equations result in the same energy E when the mass m equals the Planck mass $m_P = \sqrt{\frac{hc_0}{G}}$. We can consolidate the different cases into one dimensionless quantity equation by considering the ratio $E(r)/E(1)$. We obtain the dimensionless parametric equation:

$$\frac{E(r)}{mc_0^2} \propto \left(\sqrt{\frac{hc_0}{Gm^2}}\right)^{1-r}, \tag{L.19}$$

with $r \in \mathbb{N}_0$ being the parameter. Let us postulate the equation $E = mc_0^2 \sum_{r=1}^{\infty} E(r)$. We obtain an infinite series:

$$E = mc_0^2(1 + \left(\sqrt{\frac{Gm^2}{hc_0}}\right) + \left(\sqrt{\frac{Gm^2}{hc_0}}\right)^2 + \ldots), \tag{L.20}$$

$$E = \frac{mc_0^2}{1 - \left(\sqrt{\frac{Gm^2}{hc_0}}\right)}. \tag{L.21}$$

If the mass m is equal to the mass of an electron $m_e = 9.10938215 \times 10^{-31}\,\mathrm{kg}$ then we find

$$\sqrt{\frac{Gm_e^2}{hc_0}} = 4.185462 \times 10^{-23}. \tag{L.22}$$

and thus this hypothetical correction to the rest energy of an electron is unobservable using the present state of technology. Let the mass m increase up to the Planck mass m_P then $\lim_{m \to m_P} E = +\infty$.

From dimensional analysis we can postulate that the quantity equation $E = F \cdot s$ is valid in which E is a kind of quantity energy, F is a kind of quantity force and, s is a kind of quantity distance or length. We construct a table with as rows kinds of energy, kinds of length and kinds of force.

Table L.1: Some quantity equations between kinds of quantity energy, length, and force.

parameter r	0	1	5
Energy	$\sqrt{\frac{hc_0^5}{G}}$	mc_0^2	$\frac{m^5G^2}{h^2}$
Length	$\sqrt{\frac{Gh}{c_0^3}}$	$\frac{h}{mc_0}$	$\frac{h^2}{Gm^3}$
Force	$\frac{c_0^4}{G}$	$\frac{m^2c_0^3}{h}$	$\frac{m^8G^3}{h^4}$

Force should be interpreted as a *line energy density*. Hence, dividing the top row by the middle row gives the third row representing a kind of quantity force. The magnitude of the length or distance is increasing from left column to the right column for a mass $m \leq m_P$.

APPENDIX M

Algorithm for creating the atlas of unique ternary quantity equations in a N-dimensional integer lattice

Algorithm 1 Atlas of unique ternary quantity equations.

for N from 1 to d_{max} **do**
  for s from 0 to s_{max} **do**
    Create a list of the representative lattice points $\boldsymbol{z}_r$ of the orbits
    for i from 1 to the number of elements in the list of the representative lattice points $\boldsymbol{z}_r$ of the orbits **do**
      Calculate for $\boldsymbol{z}_{r_i}$ the histogram of the perimeter of the parallelograms formed by $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{z}_{r_i}$
      Filter for $\boldsymbol{z}_{r_i}$ the parallelograms that have a unique perimeter and associate the j-th unique lattice point $(x)_{u_j}$ with the representative lattice point $\boldsymbol{z}_{r_i}$
      Accumulate the string $\boldsymbol{z}_{r_i}, \boldsymbol{x}_{u_1}, \boldsymbol{x}_{u_2}, \ldots$
    end for
    Generate a table of unique ternary quantity equations in N-dimensional lattice
    Connect the representative lattice points $\boldsymbol{z}_r$ through the unique ternary quantity equations and form a N-dimensional network
  end for
end for

Appendix N

Orthants of $\{0, 1\} \times \mathbb{Z}^7$

Table N.1 contains the sign characteristic $\{-, +\}$ of the orthants of $\{0\} \times \mathbb{Z}^7$ and $1 \times \mathbb{Z}^7$. The first column is the orthant identifier. The other columns represent the coordinate sign of a lattice point located in the orthant of $\{0\} \times \mathbb{Z}^7$ or $1 \times \mathbb{Z}^7$. The non-negative orthant, having ID = 1, of the $\{0\} \times \mathbb{Z}^7$ or $1 \times \mathbb{Z}^7$ has the sign $(+, +, +, +, +, +, +)$. The sign $(+, +, +, +, +, +, +)$ indicates that all the coordinates z_i, where $i \in \{1, \ldots, 7\}$, of the lattice point $\boldsymbol{z}$ are non-negative integers.

Table N.1: Orthants of $\{0, 1\} \times \mathbb{Z}^7$.

ID	z_1	z_2	z_3	z_4	z_5	z_6	z_7
1	+	+	+	+	+	+	+
2	+	+	+	+	+	+	-
3	+	+	+	+	+	-	+
4	+	+	+	+	+	-	-
5	+	+	+	+	-	+	+
6	+	+	+	+	-	+	-
7	+	+	+	+	-	-	+
8	+	+	+	+	-	-	-
9	+	+	+	-	+	+	+
10	+	+	+	-	+	+	-
11	+	+	+	-	+	-	+
12	+	+	+	-	+	-	-
13	+	+	+	-	-	+	+
14	+	+	+	-	-	+	-
15	+	+	+	-	-	-	+
16	+	+	+	-	-	-	-
17	+	+	-	+	+	+	+
18	+	+	-	+	+	+	-
19	+	+	-	+	+	-	+
20	+	+	-	+	+	-	-
21	+	+	-	+	-	+	+
22	+	+	-	+	-	+	-
23	+	+	-	+	-	-	+
24	+	+	-	+	-	-	-
25	+	+	-	-	+	+	+
26	+	+	-	-	+	+	-
27	+	+	-	-	+	-	+
28	+	+	-	-	+	-	-
29	+	+	-	-	-	+	+
30	+	+	-	-	-	+	-
31	+	+	-	-	-	-	+
32	+	+	-	-	-	-	-
33	+	-	+	+	+	+	+
…	…	…	…	…	…	…	…

ID	z_1	z_2	z_3	z_4	z_5	z_6	z_7
34	+	-	+	+	+	+	-
35	+	-	+	+	+	-	+
36	+	-	+	+	+	-	-
37	+	-	+	+	-	+	+
38	+	-	+	+	-	+	-
39	+	-	+	+	-	-	+
40	+	-	+	+	-	-	-
41	+	-	+	-	+	+	+
42	+	-	+	-	+	+	-
43	+	-	+	-	+	-	+
44	+	-	+	-	+	-	-
45	+	-	+	-	-	+	+
46	+	-	+	-	-	+	-
47	+	-	+	-	-	-	+
48	+	-	+	-	-	-	-
49	+	-	-	+	+	+	+
50	+	-	-	+	+	+	-
51	+	-	-	+	+	-	+
52	+	-	-	+	+	-	-
53	+	-	-	+	-	+	+
54	+	-	-	+	-	+	-
55	+	-	-	+	-	-	+
56	+	-	-	+	-	-	-
57	+	-	-	-	+	+	+
58	+	-	-	-	+	+	-
59	+	-	-	-	+	-	+
60	+	-	-	-	+	-	-
61	+	-	-	-	-	+	+
62	+	-	-	-	-	+	-
63	+	-	-	-	-	-	+
64	+	-	-	-	-	-	-
65	-	+	+	+	+	+	+
66	-	+	+	+	+	+	-
67	-	+	+	+	+	-	+
68	-	+	+	+	+	-	-
69	-	+	+	+	-	+	+
70	-	+	+	+	-	+	-
71	-	+	+	+	-	-	+
72	-	+	+	+	-	-	-
73	-	+	+	-	+	+	+
74	-	+	+	-	+	+	-
75	-	+	+	-	+	-	+
76	-	+	+	-	+	-	-
77	-	+	+	-	-	+	+
78	-	+	+	-	-	+	-
79	-	+	+	-	-	-	+
80	-	+	+	-	-	-	-
81	-	+	-	+	+	+	+
82	-	+	-	+	+	+	-
83	-	+	-	+	+	-	+
84	-	+	-	+	+	-	-
85	-	+	-	+	-	+	+
86	-	+	-	+	-	+	-
87	-	+	-	+	-	-	+
88	-	+	-	+	-	-	-
89	-	+	-	-	+	+	+
90	-	+	-	-	+	+	-
...	...	...	...	...	...	...	...

ID	z_1	z_2	z_3	z_4	z_5	z_6	z_7
91	-	+	-	-	+	-	+
92	-	+	-	-	+	-	-
93	-	+	-	-	-	+	+
94	-	+	-	-	-	+	-
95	-	+	-	-	-	-	+
96	-	+	-	-	-	-	-
97	-	-	+	+	+	+	+
98	-	-	+	+	+	+	-
99	-	-	+	+	+	-	+
100	-	-	+	+	+	-	-
101	-	-	+	+	-	+	+
102	-	-	+	+	-	+	-
103	-	-	+	+	-	-	+
104	-	-	+	+	-	-	-
105	-	-	+	-	+	+	+
106	-	-	+	-	+	+	-
107	-	-	+	-	+	-	+
108	-	-	+	-	+	-	-
109	-	-	+	-	-	+	+
110	-	-	+	-	-	+	-
111	-	-	+	-	-	-	+
112	-	-	+	-	-	-	-
113	-	-	-	+	+	+	+
114	-	-	-	+	+	+	-
115	-	-	-	+	+	-	+
116	-	-	-	+	+	-	-
117	-	-	-	+	-	+	+
118	-	-	-	+	-	+	-
119	-	-	-	+	-	-	+
120	-	-	-	+	-	-	-
121	-	-	-	-	+	+	+
122	-	-	-	-	+	+	-
123	-	-	-	-	+	-	+
124	-	-	-	-	+	-	-
125	-	-	-	-	-	+	+
126	-	-	-	-	-	+	-
127	-	-	-	-	-	-	+
128	-	-	-	-	-	-	-

APPENDIX O

Examples of modern dimensional analysis (MDA)

O.1 MDA applied to the second order partial derivative of energy density with respect to time

The case study was inspired from an article published in IEEE Control Systems that highlight modeling and control problems in the energy transformation challenge that is taking place in our present society. The article of Parisini and Blaabjerg (Parisini & Blaabjerg, 2021) gives an introduction to the challenges in power electronics-dominated grids with large renewable generation capacity. We study the kind of quantity second order partial derivative of energy density with respect to time, denoted $\dfrac{\partial^2 W(\boldsymbol{r},t)}{\partial t^2}$.

The symbols used in this section have the following semantic interpretation: $W(\boldsymbol{r},t)$ energy density, $\dfrac{\partial W(\boldsymbol{r},t)}{\partial t}$ first order partial derivative of energy density with respect to time, $\dfrac{\partial^2 W(\boldsymbol{r},t)}{\partial t^2}$ second order partial derivative of the energy density with respect to time, s displacement, t time, ν_i specific frequency, ω angular frequency, $m(\boldsymbol{r},t)$ mass, $k(\boldsymbol{r},t)$ wavenumber, J action, V volume, and $f_i(\boldsymbol{\pi}_i)$ an unspecified function of a vector of dimensionless quantities $\boldsymbol{\pi}_i$. Assume that the expert researcher finds the dimensional measurement model:

$$
\begin{aligned}
&F_1\Bigg(\frac{\partial^2 W(\boldsymbol{r},t)}{\partial t^2}, \frac{\partial W(\boldsymbol{r},t)}{\partial t}, \frac{\partial^4 m(\boldsymbol{r},t)}{\partial t^4}, W(\boldsymbol{r},t), \frac{\partial^4 k(\boldsymbol{r},t)}{\partial t^4},\\
&k(\boldsymbol{r},t)\frac{\partial m(\boldsymbol{r},t)}{\partial t}, \frac{\partial^3 k(\boldsymbol{r},t)}{\partial t^3}, \frac{\partial^2 m(\boldsymbol{r},t)}{\partial t^2}, \nu^4, \frac{\partial m(\boldsymbol{r},t)}{\partial \boldsymbol{r}}, (J/V),\\
&\frac{\partial^2 k(\boldsymbol{r},t)}{\partial t^2}, \frac{\partial m(\boldsymbol{r},t)}{\partial t}, \nu^3, \frac{\partial k(\boldsymbol{r},t)}{\partial t}, m(\boldsymbol{r},t), \nu^2, k(\boldsymbol{r},t), \nu\Bigg) = 0\,.
\end{aligned}
\tag{O.1}
$$

Remark that equation (O.1) is obviously *difficult* to *guess* (R. Feynman, 2017) by the researcher as a start of a dimensional analysis.

To apply the MDA method, this dimensional measurement model of 19 arguments (O.1) will be transformed to a *reduced* new dimensionless measurement model of 16 arguments (O.1):

$$G_1\Bigg(\frac{\partial^2 W(\boldsymbol{r},t)}{\partial t^2}, \frac{\partial W(\boldsymbol{r},t)}{\partial t}, \frac{\partial^4 m(\boldsymbol{r},t)}{\partial t^4}, W(\boldsymbol{r},t), \frac{\partial^4 k(\boldsymbol{r},t)}{\partial t^4},$$
$$k(\boldsymbol{r},t)\frac{\partial m(\boldsymbol{r},t)}{\partial t}, \frac{\partial^3 k(\boldsymbol{r},t)}{\partial t^3}, \frac{\partial^2 m(\boldsymbol{r},t)}{\partial t^2}, \frac{\partial m(\boldsymbol{r},t)}{\partial \boldsymbol{r}}, (J/V), \tag{O.2}$$
$$\frac{\partial^2 k(\boldsymbol{r},t)}{\partial t^2}, \frac{\partial m(\boldsymbol{r},t)}{\partial t}, \frac{\partial k(\boldsymbol{r},t)}{\partial t}, m(\boldsymbol{r},t), k(\boldsymbol{r},t), \nu\Bigg) = 0\,.$$

by removal of the variables ν^2, ν^3, ν^4 because these variables are depending on ν .

The dependent variable $\dfrac{\partial^2 W(\boldsymbol{r},t)}{\partial t^2}$ is on the leftmost coordinate position in the dimensional measurement model $G_1(Q^1,\dots,Q^{16}) = 0$. We recall the definition of the dimensional set (1.12) and the fundamental formula (1.13).

We infer from equation (O.2) a *reduced* 3×16-dimensional matrix $\mathbf{RD}_1$ (O.3):

$$\mathbf{RD}_1 = \left[\begin{array}{ccccccccccccc|ccc} -1 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ -4 & -3 & -4 & -2 & -4 & -3 & -1 & -3 & -2 & 0 & -2 & -1 & -1 & 0 & 0 & -1 \end{array}\right], \tag{O.3}$$

where the rightmost sub-matrix is identified as $\mathbf{A_1}$ and the leftmost is identified as $\mathbf{B_1}$. The rank of the matrix $\mathbf{RD}_1$ is 3 and thus we find 16 - 3 = 13 dimensionless quantities using the MDA matrix method of Szirtes (2007).

The dimensional set $\mathbf{DS_1}$ for the kind of quantity called second order partial derivative of the energy density with respect to time is:

$$\mathbf{DS_1} = \left[\begin{array}{ccccccccccccc|ccc} -1 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ -4 & -3 & -4 & -2 & -4 & -3 & -1 & -3 & -2 & 0 & -2 & -1 & -1 & 0 & 0 & -1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -4 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \end{array}\right] \tag{O.4}$$

The resulting 13 dimensionless quantities are:

$$\pi_1 = \frac{\left(\dfrac{\partial^2 W(\boldsymbol{r},t)}{\partial t^2}\right)}{m(\boldsymbol{r},t)k(\boldsymbol{r},t)\nu^4}; \quad \pi_2 = \frac{\left(\dfrac{\partial W(\boldsymbol{r},t)}{\partial t}\right)}{m(\boldsymbol{r},t)k(\boldsymbol{r},t)\nu^3}; \quad \pi_3 = \frac{\left(\dfrac{\partial^4 m(\boldsymbol{r},t)}{\partial t^4}\right)}{m(\boldsymbol{r},t)\nu^4};$$

$$\pi_4 = \frac{W(\boldsymbol{r},t)}{m(\boldsymbol{r},t)k(\boldsymbol{r},t)\nu^2}; \quad \pi_5 = \frac{\left(\dfrac{\partial^4 k(\boldsymbol{r},t)}{\partial t^4}\right)}{k(\boldsymbol{r},t)\nu^4}; \quad \pi_6 = \frac{k(\boldsymbol{r},t)\left(\dfrac{\partial m(\boldsymbol{r},t)}{\partial t}\right)}{m(\boldsymbol{r},t)\nu^3};$$

$$\pi_7 = \frac{\left(\dfrac{\partial^3 k(\boldsymbol{r},t)}{\partial t^3}\right)}{m(\boldsymbol{r},t)k(\boldsymbol{r},t)\nu}; \quad \pi_8 = \frac{\left(\dfrac{\partial^2 m(\boldsymbol{r},t)}{\partial t^2}\right)}{k(\boldsymbol{r},t)\nu^3}; \quad \pi_9 = \frac{\left(\dfrac{\partial m(\boldsymbol{r},t)}{\partial \boldsymbol{r}}\right)}{m((\boldsymbol{r},t)\nu^2};$$

$$\pi_{10} = \frac{(J/V)}{m(\boldsymbol{r},t)k(\boldsymbol{r},t)}; \quad \pi_{11} = \frac{\left(\dfrac{\partial^2 k(\boldsymbol{r},t)}{\partial t^2}\right)}{k(\boldsymbol{r},t)\nu^2}; \quad \pi_{12} = \frac{\left(\dfrac{\partial m(\boldsymbol{r},t)}{\partial t}\right)}{m(\boldsymbol{r},t)\nu};$$

$$\pi_{13} = \frac{\left(\dfrac{\partial k(\boldsymbol{r},t)}{\partial t}\right)}{k(\boldsymbol{r},t)\nu}.$$

A dimensionless measurement model for the general problem of modeling the second order partial derivative of the energy density with respect to time is: $g_1(\pi_1,\ldots,\pi_{13}) = 0$. Remark that the dependent variable $\dfrac{\partial^2 W(\boldsymbol{r},t)}{\partial t^2}$ is occurring only in π_1.

O.2 MDA applied to the simple pendulum

We refer to the simple pendulum example given by Meinsma (2019). A function $\phi(t)$, representing the angle between the pendulum cable and the vertical axis as function of time t is setup assuming that it depends on: time t, initial angle ϕ_0, mass m, cable length l, and gravitational acceleration g. In MDA using the SI framework one postulates the dimensional measurement model (O.5)

$$G_2(\phi(t), t, \phi_0, m, l, g) = 0. \tag{O.5}$$

The reduced dimensional matrix is $\mathbf{RD}_2$ (O.6):

$$\mathbf{RD}_2 = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -2 \end{array}\right], \tag{O.6}$$

where the rightmost sub-matrix is identified as $\mathbf{A_2}$ and the leftmost is identified as $\mathbf{B_2}$. The rank of the matrix $\mathbf{RD}_2$ is 3 and thus we find 6 - 3 = 3 dimensionless quantities using the MDA matrix method of Szirtes (2007). The dimensional set for the simple pendulum is $\mathbf{DS_2}$:

$$\mathbf{DS_2} = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -2 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \tag{O.7}$$

The three dimensionless quantities are: $\pi_1 = \phi(t)$, $\pi_2 = t\sqrt{g/l}$, and $\pi_3 = \phi_0$ resulting in the dimensionless measurement model $g_2(\pi_1, \pi_2, \pi_3) = 0$.

O.3 MDA applied to the kind of quantity energy

Assume that the expert researcher finds the dimensional measurement model O.8 where the symbols used in the dimensional measurement model have the following interpretation: $E(\boldsymbol{r}, t)$ energy, s displacement, $\boldsymbol{r}$ position vector, t time, ν frequency, $m(\boldsymbol{r}, t)$ mass, A surface area, v speed, F force, J action, p linear momentum, a acceleration, $f_i(\boldsymbol{\pi}_i)$ unspecified function of a vector of dimensionless quantities, the index L refers to a path, and the index S refers to a surface.

$$\begin{aligned} &F_3\Bigg(E, F, J, \frac{\partial^2 m(\boldsymbol{r}, t)}{\partial t^2}, v^2, p, \iint_S m(S)\,\mathrm{d}S, a, \frac{\partial m(\boldsymbol{r}, t)}{\partial t}, \frac{\partial A(\boldsymbol{r}, t)}{\partial t}, \\ &\int_L m(s)\,\mathrm{d}s, \nu^2, v, m, s^2, \nu, s\Bigg) = 0\,. \end{aligned} \tag{O.8}$$

To apply the MDA method this dimensional measurement model of 17 arguments (O.8) will be transformed to a new dimensional measurement model of 14 arguments (O.9):

$$\begin{aligned} &G_3\Bigg(E, F, J, \frac{\partial^2 m(\boldsymbol{r}, t)}{\partial t^2}, p, \iint_S m(S)\,\mathrm{d}S, a, \\ &\frac{\partial m(\boldsymbol{r}, t)}{\partial t}, \frac{\partial A(\boldsymbol{r}, t)}{\partial t}, \int_L m(s)\,\mathrm{d}s, v, m, \nu, s\Bigg) = 0\,. \end{aligned} \tag{O.9}$$

We infer from equation (O.9) a reduced 3×14-dimensional matrix $\mathbf{RD}_3$ (O.10) :

$$\mathbf{RD}_3 = \left[\begin{array}{rrrrrrrrrrr|rrr} 2 & 1 & 2 & 0 & 1 & 2 & 1 & 0 & 2 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ -2 & -2 & -1 & -2 & -1 & 0 & -2 & -1 & -1 & 0 & -1 & 0 & -1 & 0 \end{array}\right]. \tag{O.10}$$

The rank of the matrix $\mathbf{RD}_3$ is 3 and thus we find 14-3 = 11 dimensionless quantities using the MDA matrix method of Szirtes (2007). The dimensional set for the kind of quantity energy is $\mathbf{DS_3}$:

$$\mathbf{DS_3} = \left[\begin{array}{rrrrrrrrrrr|rrr} 2 & 1 & 2 & 0 & 1 & 2 & 1 & 0 & 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 \\ -2 & -2 & -1 & -2 & -1 & 0 & -2 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & -2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \end{array}\right] \tag{O.11}$$

We form the following 11 dimensionless quantities:

$$\pi_1 = \frac{E}{m\nu^2 s^2}; \qquad \pi_2 = \frac{F}{m\nu^2 s}; \qquad \pi_3 = \frac{J}{m\nu s^2}; \quad \pi_4 = \frac{\left(\dfrac{\partial^2 m(\boldsymbol{r},t)}{\partial t^2}\right)}{m f^2};$$

$$\pi_5 = \frac{p}{m\nu s}; \qquad \pi_6 = \frac{\iint_S m(S)\,\mathrm{d}S}{m s^2}; \quad \pi_7 = \frac{a}{\nu^2 s}; \qquad \pi_8 = \frac{\left(\dfrac{\partial m(\boldsymbol{r},t)}{\partial t}\right)}{m\nu};$$

$$\pi_9 = \frac{\left(\dfrac{\partial A(\boldsymbol{r},t)}{\partial t}\right)}{\nu s^2}; \quad \pi_{10} = \frac{\int_L m(s)\,\mathrm{d}s}{m s}; \qquad \pi_{11} = \frac{v}{\nu s}.$$

A dimensionless measurement model for the general problem of modeling the kind of quantity energy is: $g_3(\pi_1, \ldots, \pi_{11}) = 0$. Observe that the dependent variable the kind of quantity energy E is occurring *only* in one dimensionless quantity π_1.

APPENDIX P

The search for the mathematical classification of kinds of quantities and quantity equations

P.1 Research years 1978-2014

This dissertation is the result of research which started in 1978, in my first year as engineering student at Ghent University, with a question from the book 'ESSENTIALS OF PHYSICS: A Text for Students of Science and Engineering, 2nd Edition' from Borowitz and Beiser (Borowitz & Beiser, 1971, p.173) and, with the additional information that G is the Newtonian constant of gravitation we quote:

> We shall learn later that a very important constant of nature is Planck's constant h, whose value is $h = 6.63 \times 10^{-34}\,\mathrm{J \cdot s}$. Construct an expression involving G, h and the velocity of light, $c = 3.0 \times 10^{8}\,\mathrm{ms}^{-1}$, which has the *dimension of a length*. What is its value? (No one knows its significance.)

This triggered my search for structure among the physical quantities. The answer was the length $L = \sqrt{\dfrac{Gh}{c_0^3}} = 4.05 \times 10^{-35}\,\mathrm{m}$ which is proportional to the Planck length $L_P = \sqrt{\dfrac{Gh}{2\pi c_0^3}}$ (Borowitz & Beiser, 1971, p.552).

I rephrased the question by replacing *dimension of length* by *dimension of mass, force, time, velocity, energy* ... After enumerating these results, some graphical structure, which I couldn't explain, started to appear. I expanded further the question by adding the physical quantity mass m and the elementary charge e to the set of universal constants: the Newtonian constant of gravitation G, Planck's constant h, and the speed of light in vacuum c_0. Solving the problem for the physical quantity energy E using m, G, h, c_0 resulted in a parametric equation of the form:

$$E = m^r \sqrt{h^{1-r} c_0^{5-r} G^{r-1}}, \tag{P.1}$$

with $r \in \mathbb{Z}$ being the parameter. A discussion of the parametric equation can be found in Appendix L. I explored further those quantity equations trying to put them in a 'lattice' by combining $h, c_0, m_0, G, e, k, T_0$. During my education at the Ghent University, I learned about dimensionless products and read the book 'Dimensional analysis and theory of models' (Langhaar, 1951). I applied the 'Buckingham π theorem' to various engineering problems during my study at the Ghent University.

In 1983, after submitting to Prof. K. Heyde (Ghent University) my first approach to a *master equation* based on dimensional analysis, he encouraged me to read 'Toward a constructive physics (SLAC-PUB-3116 June 1983)' (Noyes, Gefwert, & Manthey, 1983). This document opened a combinatorial approach to physics that I tried to couple with what I learned from dimensional analysis.

From 1984 till 2008, I made notes about ideas on the mathematical structure of physical quantities. In November 2004, I started considering classes of kinds of quantities based on the sum of their coordinates. I started in 2008 to compile all my ideas and submitted a document to Prof. H. Verschelde (Ghent University). He suggested to split the text in at least 3 subjects and to choose one of these for further research. Encouraged by Prof. H. Ferdinande (Ghent University), I continued my research restricting it to the 'Mathematical classification of physical quantities'. The mathematical classification was based on hyperplanes and hyperspheres. During that research, I read the book Regular Polytopes from Coxeter (Coxeter, 1973).

I submitted a document in August 2010 about the mathematical classification of physical quantities to Prof. F. Brackx (Ghent University) whose first reaction was 'intriguing'. He suggested to focus on the histograms of the perimeter of parallelograms in $\mathbb{Z}^N$ and he organized a meeting with his colleagues Prof. H. De Schepper (Ghent University) and then assistant Prof. H. De Bie (Ghent University) to discuss the topics in more detail. The birth of the theorem related to the geometric representation of quantity equations as parallelograms in $\mathbb{Z}^N$ occurred in that period. They supported me in how to write a scientific paper and how to formulate mathematical statements for theorems.

I tried to publish my research but all the journals refused. I was an independent researcher and not a PhD student. Prof. F. Brackx became emeritus professor and Prof. H. De Schepper suggested to contact the University of Brussels. I reached out to Prof. I. Veretennicoff who put me in contact with Prof. Ph. Cara (University of Brussels). They gave me a lot of valuable advices but stressed the fact that I needed to publish and should try to get funding for my research. One possibility was to get a substantial amount of international endorsement for my research to make my research visible.

I reached out for endorsement from Prof. M. Aigner (University Berlin), Prof. J-P. Antoine (Catholic University of Louvain), Dr. K. Brading (University of Notre Dame), Prof. em. M. Deza (CNRS), Prof. M. Rees (Cambridge University), Prof. R. S. Sirohi (Tezpur University), Prof. N. J. A. Sloane (Cornell University and AT&T) and Prof. D. Zeilberger (Rutger University), but all these efforts failed and I got no funding. I tried in October 2011 to publish my research in the journal 'Foundations of Physics' under the title 'On the geometry of the laws of physics' but it was rejected. I tried to publish in November

2011 in the journal ‘Annals of Physics’ under the title ‘On the discrete geometry of the laws of physics’ where it was also rejected. From the comments and encouragements of these scientific authorities, I continued my quest.

I contacted in August 2013 Prof. H. Steendam (Ghent University) and forwarded my rejected article ‘On the discrete geometry of physical quantities’ submitted in October 2012 to the Journal of Geometry and Physics. I gave a presentation with title ‘The codification of SI physical quantities, An information-theoretic approach’ to her research group.

P.2 Academic year 2014-2015

My daughter Amélie (Ghent University), working as teaching assistant, suggested to talk with Prof. dr. ir. J. Vierendeels† (Ghent University) who introduces engineers to dimensional analysis.

After two sessions of discussions, Prof. dr. ir. J. Vierendeels† agreed to become my supervisor for this PhD dissertation and the PhD application was approved by the faculty dean Prof. dr. ir.R. Van de Walle.

I enrolled as doctoral student on 21/11/2014 at Ghent University in the Faculty of Engineering and Architecture. It was in December 2014 that the ‘Table of SI physics’ was created. We proof that dimensionally equivalent physical SI2018 quantities are mapped to integer lattice points of Z7. We proof that integer lattice points are partitioned, using signed permutations as equivalence relation, in *leader classes* with representative lattice point in $\mathbb{Z}^7_+$. The cardinality of each leader class is an element of a finite set of 30 distinct cardinalities representative for the symmetries of the leader class.

We have constructed a table of the *elements* of physics by combining the infinity norm $\ell_\infty = s$ in the rows and the distinct cardinalities $\#([w])$ in the columns.

We show that the encoding of physical quantities up to a signed permutation can be done using the Gödel encoding scheme.

We show that each leader class has a *unique Gödel number* that generates a partial order between the physical quantities.

We show that each leader class has a *unique 7D-hypersphere* defining rectangles formed by four lattice points $\boldsymbol{o}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ in $\mathbb{Z}^7_+$ in which $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$. The resulting rectangles are the geometric representation of the *realizable binary form equations* $[z] = f(\pi)[x][y]$ for the selected physical quantity $[z]$. The lattice points incident on the 7D-hypersphere are forming *rectangles*.

We show that the perimeter distribution of the resulting rectangles shows the occurrence of *unique* rectangles that can be associated with *unique relations* between physical quantities.

We have applied the *hypersphere method* to the electric displacement $\boldsymbol{D}$ and retrieve Maxwell’s integral equation for $\boldsymbol{D}$.

We show that the divisibility relation $n|m$ applied on the Gödel numbers creates the *core lattice* of the physical quantities up to a signed permutation.

We find, using the supercomputer of Ghent University, in $\mathbb{Z}^7_+$ in which $\|z\|_\infty \leq 10$, a total of *7 747 unique rectangles* out of 6 510 466 998 rectangles.

I gave in March 2014 a presentation ‘On a Mathematical Method for Discovering Relations Between Physical Quantities: a Photonics case Study’ at the ICOL2014 conference in Dehradun (India).

I tried in 2014, without success, to publish in the Asian Journal of Physics the article ‘On a Mathematical Method for Discovering Relations Between Physical Quantities: Maxwell’s equations revisited’.

P.3 Academic year 2015-2016

After the passing away of the late Prof. dr. ir. J. Vierendeels†, the University of Ghent appointed Prof. dr. Denis Constales as my supervisor for this PhD. An intensive interaction started on the fundamentals of my research. The focus of the research was now on the mathematics of the classification of kinds of quantities and quantity equations. We have published in the Online Integer Sequences Encyclopedia (OEIS) the following sequences as result of dimensional explorations in the integer lattice $\mathbb{Z}^N$ for $N = 0 \ldots 17$. http://oeis.org/A128891 ; The constant A128891 and the constant of A128892 are connected by the equation $\sum_{N>=0} S_N - 2 * \pi * \sum N >= 0V_N = 2$, in which S_N and V_N are respectively the area and volume of a N-dimensional sphere of unit radius. http://oeis.org/A128892 ; The constant is equal to $\sum_{N>=0} S_N$, in which S_N is the area of an N-dimensional sphere of unit radius. This constant and the constant of A128891 are connected by the equation $\sum_{N>=0} S_N - 2 * \pi * \sum N >= 0V_N = 2$, in which V_N is the volume of an N-dimensional sphere of unit radius. http://oeis.org/A266387; Number of orbits of $\mathbb{Z}^7$ as function of the infinity norm n of the representative lattice point of the orbit, when the cardinality of the orbit is equal to 322560. http://oeis.org/A266398; Number of orbits of $\mathbb{Z}^7$ as function of the infinity norm n of the representative lattice point of the orbit, when the cardinality of the orbit is equal to 13440. http://oeis.org/A266397; Number of orbits of $\mathbb{Z}^7$ as function of the infinity norm n of the representative lattice point of the orbit, when the cardinality of the orbit is equal to 26880. http://oeis.org/A266396; Number of orbits of $\mathbb{Z}^7$ as function of the infinity norm n of the representative lattice point of the orbit, when the cardinality of the orbit is equal to 80640. http://oeis.org/A266395; Number of orbits of $\mathbb{Z}^7$ as function of the infinity norm n of the representative lattice point of the orbit, when the cardinality of the orbit is equal to 161280. http://oeis.org/A008586; The number of orbits of $\mathbb{Z}^7$ as function of the infinity norm n of the representative lattice point of the orbit, when the cardinality of the orbit is equal to 2688. http://oeis.org/A001477; The number of orbits of $\mathbb{Z}^7$ as function of the infinity norm n of the representative lattice point of the orbit, when the cardinality of the orbit is equal to 8960 or 168. http://oeis.org/A045943 ; Number of orbits of $\mathbb{Z}^7$ as function of the infinity norm $(n + 1)$ of the representative integer lattice point of the orbit, when the cardinality of the orbit is equal to 5376 or 17920 or 20160. http://oeis.org/A102860 ; Number of orbits of $\mathbb{Z}^7$ as function of the infinity norm $(n+2)$ of the representative integer lattice point of the orbit, when the cardinality of the orbit is equal to 53760. http://oeis.org/A154286 ; Number of orbits of $\mathbb{Z}^7$ as function of the infinity norm $(n+4)$ of the representative integer lattice point of the orbit, when the cardinality of the orbit is equal to 107520. http://oeis.org/A000579 ; Number of orbits of $\mathbb{Z}^7$ as function of the infinity norm n of the representative integer lattice point of the orbit, when the cardinality of the orbit is equal to 645120. http://oeis.org/A115067 ; Number of orbits of $\mathbb{Z}^7$ as function of the infinity norm n of the representative integer lattice point of the orbit, when the cardinality of the orbit is equal to 6720. http://oeis.org/A002412 ; Number of orbits of $\mathbb{Z}^7$ as function of the infinity norm $(n+1)$ of the representative integer lattice point of the orbit, when the cardinality of the orbit is equal to 40320. http://oeis.org/A270950 ; Number of distinct cardinalities of orbits of lattice points under the automorphism group of the N-dimensional integer lattice. A270950 sequence is a(N)=1, 1, 2, 5, 9, 12, 20, 29, 40, 53, 76, 99, 132, 172, 216, 270, 341, 750,... For $N = 0$ the a(0)=1. For $N = 3$ we have the following distinct cardinalities of the orbits 6, 8, 12, 24, 48 and thus a(3)=5. For $n = 4$ we have the distinct cardinalities of the orbits 8, 16, 24, 32, 48, 64, 96, 192, 384 and thus a(4)=9. For $N = 5$ we have the distinct cardinalities of the orbits 10, 32, 40, 160, 240, 320, 480, 640, 960, 1920, 3840 and thus a(5)=12.

We define the subset of the hypercubic shell as the set consisting of all elements xz (with x an element of the automorphism group $\mathrm{Aut}(\mathbb{Z}^{\mathrm{N}})$ and call it an orbit of z under $\mathrm{Aut}(\mathbb{Z}^{\mathrm{N}})$.

We are interested in the cardinalities of these orbits as function of the dimension N of the integer lattice. The number of orbits for a selected hypercubic shell in the integer lattice $\mathbb{Z}^N$ with infinity norm s is given by $\binom{s+(N-1)}{(N-1)}$. Let q_i be the number of characters of type i of the alphabet A . Suppose that the characters occurring in the N-tuple z are subjected to signed permutations and that we denote the set containing the generated lattice points as $[z]$. The cardinality of the set $[z]$ is given by the equation: $\#([z]) = \dfrac{2^{q-q_0} q!}{q_0! q_1! q_2! \ldots q_s!}$. The number of distinct cardinalities $\#([z])$ depends on the infinity norm s and on the dimension N of the integer lattice. The maximum number of possible cardinalities is obtained by setting $s = N$. What we count now is the number of distinct cardinalities as function of the dimension N of the integer lattice.

P.4 Academic year 2016-2017

We have discovered that in each integer lattice $\mathbb{Z}^N$ their exist a lattice point with coordinates $(N, N-1, N-2, \ldots 1)$ that is the representative of an orbit of $\mathbb{Z}^N$

having a cardinality equal to the order of the automorphism group $\mathrm{Aut}(\mathbb{Z}^{\mathrm{N}})$ being $2^N N!$.

We have selected the lattice point with coordinates $(-3, 2, 1)$ of $\mathbb{Z}^3$ belonging to the orbit $[(3, 2, 1)]$ and have performed a sequential orthogonal decomposition in $\mathbb{Z}^3$. The decomposition reveals the existence of a quintenary alloy structure in $\mathbb{Z}^3$ in which the lattice points receive a 5-coloring. The lattice point $(-3, 2, 1)$ can be identified in a (Length, Mass, Time) coordinate system as power.

We have observed that the Newtonian constant of gravitation G is also an element of the orbit $[(3, 2, 1)]$. Unique rectangles have been identified in this structure and have been enumerated in Maplesoft (2018).

We have started writing the PhD dissertation and have presently a manuscript of 200+ pages highlighting the exploration of the integer lattice in dimensions $N = 2$ up to $N = 7$. The dissertation starts with a detailed review/critique of the BIPM SI documents and definitions.

Two new methods are discovered to generated the coefficients of the OEIS sequence A008451 in which we discovered the condition from the norms of the lattice points $\ell_\infty = (\ell_2)^2$ in $\mathbb{Z}^7$.

We created an algorithm that, based on the cardinality of the physical quantities in a $\mathbb{Z}^N$ representation, can determine the number of coupled n-ary quantity equations of the type $z = f(\pi)x_1 \ldots x_{N-1}$ in which $x_1, x_2, \ldots x_{N-1}$ are arbitrary quantities.

We have performed the enumeration of all the p-Sylow subgroups of the automorphism group $\mathrm{Aut}(\mathbb{Z}^{\mathrm{N}})$ for $N = 2$ up to $N = 7$ and compared these subgroups to the orbits of $\mathbb{Z}^N$.

P.5 Academic year 2017-2018

We continued writing the PhD dissertation and have presently a manuscript of 256 pages including appendices.

We communicated with Dr.J.Roche and Prof. G. Peruzzi to find out if Maxwell's publication of 1871 was influenced by the publication in 1869 of the periodic table of chemical elements by D. Mendeleev. Both experts confirmed that there has been no communication between Maxwell and Mendeleev.

We calculated the 'Tables of physics' for the integer lattices with dimensions $N = 2$ up to $N = 7$.

We describe an algorithm for creating the atlas Appendix M of unique ternary quantity equations in a N-dimensional integer lattice. The unique ternary quantity equations are forming parallelograms in the N-dimensional integer lattice. The implementation of the algorithm is planned for the academic year 2018-2019. It will run on the supercomputer of the UGent. The resulting atlas will be used as look-up table for an app that will allow scientists to explore the unique quantity equations and setup new experiments for testing the ‘laws of physics’. The app will be protected by license/patent and will be an integral part of the PhD.

I agreed with Prof. J. Vierendeels to first deposit a patent before publishing a paper. A patent and a publication in an A-journal are the minimum requirements for the PhD.

A conjecture has been put forward related to cross-dimensional properties of parallelograms in the N-dimensional integer lattice:

A meaningful quantity equation $z = xy$ between physical quantities x, y, z is a local law of physics if and only if its corresponding parallelogram in $\mathbb{Z}^N$ has a unique perimeter that remains unique when the dimension N of the integer lattice $\mathbb{Z}^N$ tends to infinity. This conjecture cannot yet be proven but computer simulations up to $N = 11$ indicate the existence of a minimum dimension N_{min} needed to describe the ‘laws of physics’ in such a way that the number of laws becomes invariant of the selected dimension N . This discovery gives an answer to an unsolved problem of dimensional analysis: *How many dimensions are needed for describing physical phenomena?*

For the quantity action we found that $N_{min} = 6$, for the quantity energy we found that $N_{min} = 7$, for the quantity energy squared we found that $N_{min} = 7$ and for the quantity power we found that $N_{min} = 6$.

We plan for 2018-2019 to expand the table by running the calculations on the supercomputer and this up to dimension $N = 29$.

We also plan for 2018-2019 to solve the problem of “equality” between energy and moment of force. The action plan is to add the tensor type (p, q) and the tensor order $(p+q)$ to the classification of physical quantities. By doing this we can distinguish energy, which is a scalar from moment of force, that is a vector. A potential solution is to use Hurwitz quaternionic integers $(a + ip + jq + k(p+q))$ as dimensional exponents instead of the classical approach using integers. In that case we will operate in a lattice that is isomorphic with $\mathbb{Z}^4 N$.

P.6 Academic year 2018-2019

We continued writing the PhD dissertation and have presently a manuscript of 252 pages including appendices. We reorganized the physical quantities in two families by considering the tensors and the pseudo-tensors. The mathematical description of the physical quantities is now done in $\{0, 1\} \times \mathbb{Z}^N$. The first co-ordinate in the lattice is calculated modulo 2. Pseudo-tensors have the first coordinate equal to 1 while the tensors have co-ordinate zero. This reorganization results in a change in the Gödel coding. Tensors have an odd Gödel code while the pseudo-tensors have an even Gödel code. This reorganization solves the problem of “equality” between energy and moment of force.

We found an equation for the number of distinct parallelogram perimeters in a lattice of dimension N as function of the infinity norm and estimated the required memory to store this data for the dimension $N = 7$ and infinity norm $s = 10$. The amount of data of distinct perimeters is 9127 MB.

We tabulated the number of distinct perimeters and the largest perimeter for dimensions $N = 2$ to $N = 8$.

We started with solving the problem of the number of lattice points incident of confocal N-ellipsoids. The solution to this problem seems not yet to have been found. We prove that the lattice points on those confocal N-ellipsoids are located on hyperplanes separated by a constant distance. The case of parallel hyperplanes was already discussed by Coxeter in 1973 (Coxeter, 1973, p.181). We hope to find an equation in 2019-2020.

We studied the geometry of the integrals of motion that occur in classical mechanics and this for dimension $N = 3$. A directed 3D-graph representing the connections between the integrals of motion was found. We plan to expand the graph by adding the electrical charge and connect the graph to the Noether theorem.

The survey of cross-dimensional properties has been extended by tabulating the data for the physical quantities: action, energy, energy squared, power, stress-energy tensor.

The perimeter distribution of the parallelograms shows the existence of unique perimeters. Normalization of those histograms results in associating probabilities to the occurrence of a given perimeter. Those unique perimeters have the lowest probability and thus result in the terminology of Shannon in the largest information content. Hence, parallelograms with unique perimeters have the largest information entropy. They require the largest number of bits to be described. We put forward the conjecture: 'Laws of Physics are those quantity equations between kinds of quantities that maximize the information content'.

The geometrical dimensional analysis is being applied to the energy density, its 1st and 2nd time derivative. The motivation for this detailed analysis is that most of the theoretical results in physics are based on variations of the action leading to Lagrangian densities.

We focused on the lattice point $(1 \mid -4, -1, 1, 0)$ and generated the relevant quantity equations. We discovered an intriguing equation relating the second derivative of the energy density to the square of the energy density. This equation is a special case of a non-linear ordinary differential equation of the type Emden-Fowler. We performed a phase plane analysis of this 'universal equation' and entered values known from cosmology into the parameters of the differential equation.

We updated Appendix A that gives the classification of published quantities in $\{0, 1\} \times \mathbb{Z}^7$.

P.7 Academic year 2019-2020

We continued writing the PhD dissertation and have presently a manuscript of 287 pages including appendices.

Brainstorm sessions with the supervisor prof. dr. Denis Constales took place on a weekly basis.

The physical brainstorm sessions have been replaced by Skype meetings in accordance with the Covid-19 guidelines of the UGent.

We observed that the magnetic vector potential and the electric potential have a sum of coordinates denoted by $\mathrm{soc}\,(\boldsymbol{x})$ equal to -1. It is known that those physical quantities are *non-observables*. Is the value -1 a coincidence or not? A hypothesis is formulated that non-observable quantities have a sum of coordinates of the vertex $\mathrm{soc}\,(\boldsymbol{x}) < \mathit{0}$. This needs further research. For energy and the linear impulse, we find $\mathrm{soc}\,(\boldsymbol{x}) = \mathit{1}$ and for the quantity force the $\mathrm{soc}\,(\boldsymbol{x}) = \mathit{0}$. The value $\mathrm{soc}\,(\boldsymbol{x})$ is one of the columns of the lexicon of physical quantities in $\{0,1\} \times \mathbb{Z}^7$, given in Appendix A. The values of $\mathrm{soc}\,(\boldsymbol{x})$ are in the set $\{-6,-5,-4,-3,-2,-1,0,1,2,3,4,5,7,9\}$. The $\mathrm{soc}\,(\boldsymbol{x}) = n$ is nothing else than the equation of a hyperplane in $\mathbb{Z}^7$ containing lattice points having a common, yet to be found, property.

We explored De Bruijn sequences as a tool for the classification of the physical quantities. We presently believe that this track will not add relevant information to the classification.

The semi-perimeter of the parallelograms that we study in $\mathbb{Z}^N$ are mappings of the type $(u,v) \to \sqrt{u} + \sqrt{v}$ in which u and v are natural numbers. This is a domain of mathematics that is poorly explored.

We wrote a program in Maplesoft (2018) to calculate the distribution of compositions of 2 square roots with restrictions, in which the restriction is coming from the maximum infinity norm of the considered integer lattice points.

The study of these mappings revealed the existence of superposed triangular shapes when mapping the frequency of occurrence of the values $P = \sqrt{u} + \sqrt{v}$. We found nothing in the mathematical literature about these triangular shapes except that when the obtained sum is equal to P then the frequency is given by the formula $P - 1$. From observing the pattern of dots, we inferred the following numerical relation for the (P, f) coordinates of the distribution: $P_k = (f+1)\sqrt{k}$ in which $k = 1, 2, \ldots m$ is determining the slope of the straight lines crossing the superposition of triangular distributions, f is representing the frequency of occurrence of the sum of square roots given by P_k. Outliers have been detected with respect to the above equation and these need further investigations. The equation is very useful for searching in a lookup table of parallelogram perimeters because it allows to generate the list of all perimeters of parallelograms that have the same frequency of occurrence f without having to calculate the other perimeters. We reported in the manuscript the case for dimension $N = 7$ and infinity norm $s = 10$.

We also found a connection to the square-free numbers b and d and discovered that when the sum P can be represented as $P = \sqrt{u} + \sqrt{v} = a\sqrt{b} + c\sqrt{d}$ then the frequency $f = 2$. The equation $P = \sqrt{u} + \sqrt{v} = (a+c)\sqrt{b}$ occurs for all $f > 2$. The frequency $f = 1$ occurs when $u = v$ and the parallelogram is a rhombus.

We suspect the existence of a connection between the histograms of perimeters and these triangular distributions, but we cannot at the present stage of the research give a proof for that.

We explore the connectivity of the lattice points of the orbit representatives for the case of classical electromagnetism. The connectivity was reported using graph theory. The vertices in the graph were representing the Gödel number of the respective orbit representative. We observed that the factorization of the Gödel number of the electrical field strength results in the generation of quantity equations. We could generate in that way the Lorentz force quantity equation.

We (re)discovered that the sum of the $L1$-norms of the integer lattice points is invariant for signed permutations. We found the equation: $\|\boldsymbol{X}\|_1 + \|\boldsymbol{Y}\|_1 =$ invariant, in which $\boldsymbol{X}, \boldsymbol{Y}$ are the vectors of the integer lattice points forming the parallelogram $\boldsymbol{X} + \boldsymbol{Y} = \boldsymbol{Z}$.

We found that the parallelogram $\boldsymbol{O}, \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ in $\mathbb{Z}^7$ could be represented by a 7 x 3 matrix. Transposition of this matrix to a 7 x 3 matrix allows to map the four seven-dimensional lattice points to seven three-dimensional lattice points. The mapping resulted in constellations of integer lattice points that are laying in a plane equivalent to the A2 lattice (N. J. A. Conway J.H. and. Sloane, 1998, p.108) having as base vectors $(0, 1, 1)$ and $(1, 0, 1)$. Software was written in Maplesoft (2018) to study the properties of these constellations.

A new quantity equation was discovered having a hexagon pattern in the A2 plane. The quantity equation is: $Q = \left(\frac{mt}{n\theta}\right) \times \left(\frac{nl}{tI}\right)$ in which m is mass, l is length, t is a characteristic time, θ is temperature, I is electrical current, n is amount of substance and Q is an unknown physical quantity. The smallest triangle around the origin in the A2 plane represents the quantity velocity. It is remarkable that velocity is represented by the quantity equation: $v = \frac{l}{m} \times \frac{m}{t}$ in which m is mass, l is length, t is characteristic time. A surprise from the energy constellations in A2 is the differential equation:

$$t\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right) + h\nu = E\,,$$

in which E is energy, t is a characteristic time, h is the Planck constant and ν is a characteristic frequency.

Combining energy quantity equations having the same perimeter resulted in the (re)discovery of the following differential equation:

$$8\pi r\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}m}{\mathrm{d}t}\right) + \beta mar - E = 0,$$

in which r is a radius, m is a mass, β is a coupling parameter, a is an acceleration and E is an energy. This differential equation will have to be studied in the sequel of the research.

We have added in the lexicon Appendix A the column orthant identifier and thus mapped the physical quantities to their respective orthant of $Z_2 \times \mathbb{Z}^7$.

We decided to explore the orthants of $\{0,1\} \times \mathbb{Z}^7$ and started with orthant 49 that contains the Newtonian constant of gravitation G with lattice point $(0 \mid -2,3,-1,0,0,0,0)$ and that is an element of the orbit $[3210^4]$ having cardinality 1680. In the same orbit we find the quantity power P with lattice point $(0 \mid -3,2,1,0,0,0,0)$ located in orthant 17. We explored the mapping of power to the Newtonian constant of gravitation using and 8x8 signed permutation. Mapping of known equations of power to G resulted in "unknown" and "known" equations involving G and vice-versa in "unknown" and "known" equations of power P. We are presently exploring this subject.

We rearranged Appendix A and made it a lexicon of physical quantities in $\{0,1\} \times \mathbb{Z}^7$. The lexicon is ordered in columns with the sorting key: cardinality of the orbit, infinity norm, Euclidean norm 1, orthant ID of $\mathbb{Z}^7$, physical quantity name, ID of the orbit, Conway code of the orbit, vertex of $\{0,1\} \times \mathbb{Z}^7$, sum of coordinates of the vertex $\mathrm{soc}\,(\boldsymbol{x})$.

P.8 Academic year 2020-2021

We continued writing the PhD dissertation and have presently a manuscript of 239 pages without appendices. The appendices A to S range from page 241 to page 361. The content and structure of the PhD will be reviewed next academic year and cleaned up.

Brainstorm sessions with the supervisor prof. dr. Denis Constales took place on a weekly basis. The physical brainstorm sessions have been replaced by Skype meetings in accordance with the Covid-19 guidelines of the UGent.

The properties of the automorphism group $\mathrm{Aut}(\mathbb{Z}^3)$ have been explored and this resulted in a list of signed permutation matrices with identification of the SmallGroup(48,48). For $\mathbb{Z}^4$ the SmallGroup(384,5602) has been identified. We started using the GAP software to explore the algebraic groups. Prof. Constales wrote a macro for these structure calculations and explored this up to integer lattice dimension $N = 10$ with result: ((C2 x C2 x C2 x C2 x C2 x C2 x C2 x C2 x C2):A10):(C2 x C2).

We considered the use of MAGMA but stopped exploring this path.

We discussed the articles of Fleischmann concerning the axioms of quantity calculus.

A random integer matrix for SI physical quantities was formed and has been studied in which the integers have values reaching from -4 to +10 corresponding to the known range of dimensional exponents. We explored the application of the LLL-algorithm. A random base for physical quantities instead of the SI-base was explored and the histogram of the hyper-volumes of parallelotopes in $\mathbb{Z}^7$ formed by the random base vectors was created. We applied the method to the case of the physical quantity 'energy'. We focused on parallelotopes with hypervolume = 1 that contain the physical quantity 'energy' as one of the base vectors. These parallelotopes are mathematical alternatives to the SI-base system that has a hypervolume = 1 . A similar analysis was performed on the physical quantity ‘Lagrangian density’.

A potential connection between the Collatz conjecture and the construction of the Gödel path was studied by proposing an algorithm like the Collatz process for creating the Gödel path in dimension N.

The cumulative distribution of the parallelogram perimeters for the physical quantity ‘energy’ has been calculated and studied in dimension $N = 3$ with infinity norm $s = 10$. The number of parallelograms with parallelogram perimeter frequency = 1 and frequency = 2 have been calculated. A similar calculation was done for dimension $N = 4$.

We studied the potential existence of selection rules applicable to ‘laws of physics’. We created a state-space representation of ‘energy laws’ in $\mathbb{Z}^3$ using the parameters: perimeter of the parallelogram, area of the parallelogram and cosine of the angle between the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. We discovered a boundary curve representing a hyperbola for the scatter map of the area of a parallelogram versus the perimeter of a parallelogram for the case of energy in $\mathbb{Z}^3$ with infinity norm $s = 10$. We further studied the relation between area, area squared and the perimeter of a parallelogram. The square of the area of the parallelograms in $\mathbb{Z}^3$ is representing a cylinder. This study revealed that the ‘laws of physics’ in $\mathbb{Z}^3$ have constraints determined by the intersection of quadrics, specifically confocal ellipsoids, and cylinders. This was exemplified for the physical quantity ‘energy’.

We studied the state diagrams of parallelograms representing ‘laws of physics’. We observed the existence of a lot of ‘needle shaped parallelograms’. We observed in the state diagram ‘streamlines’ in which the distance between streamlines was changing. This change of distance between the lines of the flow would indicate a reduction of ‘velocity’ in the state space. It is not clear if this is a hint towards ‘evolving laws of physics’. Some theoretical physicists at the Perimeter Institute consider this scenario.

Doctoral Schools for ‘writing skills’ and ‘presentation skills’ have been successfully followed. The learnings of the course on writing skills resulted in the rework of the text of the introduction chapter of the thesis. The learnings of the course on presentation skills resulted in the text for an elevator pitch.

A video and presentation of the PhD topic for a layman’s audience have been made. This is intended as exercise for a TEDx Ghent presentation.

A new section named ‘Phase space of quantity equations’ was written in the PhD text. This section contains a theorem related to the 1-norms of the vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ of $\mathbb{Z}^N$ and a phase space representation of energy equations generated in a hypercube $\|\boldsymbol{x}\|_\infty = 3$ in the integer lattice $\mathbb{Z}^7$.

The information content according to Shannon is used in the analysis of the phase space. The analysis shows the occurrence of maxima in the information content. These maxima are related to the parallelograms having a unique perimeter. The probability that a parallelogram has a unique perimeter is low. Our study shows that ‘laws of physics’ have a high information content. Hence, our conjecture is that all observable ‘laws of physics’ must have a high information content. This is a major discovery. We have presently no means to prove this ‘meta-law’. We observe the existence of 6 planes in the phase space for the physical quantity ‘energy’. An article published by Robbert Dijkgraaf, Director of the Institute of Advanced Study in Princeton (US) drew our attention. He made the following statement: Instead of studying a natural phenomenon, and subsequently discovering a law of nature, one could first design a new law and then reverse engineer a system that actually displays the phenomena described by the law. Our study gives the tools to find unknown laws or to find unknown relations between existing laws. We could maybe invite Robbert Dijkgraaf in the PhD evaluation committee. We plan to contact him next academic year.

A procedure for generating candidate ‘laws of physics’ in $\mathbb{Z}^7$ was written in Maplesoft (2018). The procedure needs only the integer lattice coordinates of the physical quantity. It was exemplified for the quantity ‘energy’ having the coordinates $(-2, 1, 2, 0, 0, 0, 0)$.

We created a lexicon containing all published physical quantities known to us. An SI equation generator was written in Python. This is the start of the end-user software based on the research results of this PhD. We identified an ‘unknown’ system of equations:

$$E = \left(\frac{\mathrm{d}m}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}A}{\mathrm{d}t}\right),$$
$$E = mr\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right),$$

using our SI-equation generator. I cannot presently solve it in analytical form.

A draft patent with title ‘Method for the mathematical classification of dimensionless quantity equations as precursors for conjectures of laws of science’ has been submitted to UGent TechTransfer and is registered as valorization project: P2021/066 – Mathematical Classification.

P.9 Academic year 2021-2022

We continued with editing the patent application, under valorization project: P2021/066 – Mathematical Classification. A paid search for prior art using the “Dienst voor Intellectuele Eigendom” service was ordered. The list of prior art patents was reviewed from FOD Economie and transferred to UGent TechTransfer. The draft text (version 2021-08-27) was submitted to a patent attorney firm (Mr. David Lesthaeghe) for review and comments.

A draft of the patent application (20210923 UG-109-PDR Draft (cc) kb.docx) was generated and discussed by end of September 2021.

The intellectual property of the research from 1978 to 2021 was transferred from Mr. Philippe Chevalier to the UGent on 2021-09-29. An NDA was signed between the parties.

Further discussions and modifications on the text of the patent application were done and finalized by 2021-10-18.

The patent application submitted by UGent TechTransfer was done on 2021-11-15 and is registered under the number US 17/454,953 with the title: Machine-implementable method and system for encoding/decoding variables in engineering problems.

After submission of the patent application I started searching for an adequate journal to publish results of my doctoral research. Potential journals were SIAM REVIEW, Applied Mathematical Modelling (AMM). After agreement with Prof. Denis Constales we selected AMM.

I searched for an equation for the number of distinct distances from the origin for lattice points in a hypercube of dimension d with infinity norm $\leq s$. This problem is similar to a problem posed by Pál Erdős. The equation was found and becomes a part of the manuscript.

A draft article for AMM was ready by 2021-12-13 with title: Method for encoding and decoding variables in engineering problems. Several updates were made after discussions with my supervisor.

A plagiarism screening of the final text was done and then submitted on 2022-02-08 to Elsevier Applied Mathematical Modelling.

Hereafter, I studied the properties of projections of parallelograms in 7D to 3D on the SI reference plane having equation $x + y - z = 0$. I also discussed results of PCA reduction from 7D to 3D that were discussed with Prof. H. Verschelde in 2008.

We received on 2022-02-15 the message from Elsevier that the article was rejected. Elsevier suggested to transfer the article to the European Journal of Operational Research. We submitted the article on the 2022-02-21.

We further studied the spatial distribution of orbits of the lattice point $(4, 1, 1)$ in $\mathbb{Z}^3$. The lattice point $(4, 1, 1)$ is the representative of the orbit containing the second partial derivative of the energy density with respect to time. It is a key quantity in the modelling of future power grids that combine nuclear power plants, windmill farms and solar energy farms. The mathematical model is not well understood, and our patented method could offer a solution.

The submitted article to the European Journal of Operational Research could not be fitted in one of the topics and thus we retracted the submission.

We adapted the article to submit it to Elsevier Journal of the Franklin Institute on 2022-02-25.

We continued writing the PhD dissertation and discussed the Table of Content. We have presently a manuscript of 143 pages without appendices. The appendices A to S range from page 145 to page 255.

Brainstorm sessions with the supervisor prof. dr. Denis Constales took place on a weekly basis.

The physical brainstorm sessions have been replaced by Skype meetings in accordance with the Covid-19 guidelines of the UGent.

The article was accepted by the Editor of the Journal of the Franklin Institute and is presently undergoing a peer review, based on the editorial manager status date of June 17th, 2022.

Further work was done on the OEIS sequence A270950 (published on 2016-04-07) to try to find an equation or method to calculate the number of distinct cardinalities of the orbits of an integer lattice $\mathbb{Z}^N$ instead of using a brute force method.

We recently succeeded in finding a method implemented in Maplesoft (2018) to find the number of distinct cardinalities as well as the cardinality values. Those values are important in the classification process of databases. Multi-dimensional data points and their relations have intrinsic properties that have been studied in this doctoral research. We believe that these results could be useful for machine learning (ML) techniques (SVM, NN) and artificial intelligence (AI). Our research can tell a data analyst how many distinct types of classes that should be considered for classifying a 20-dimensional database. The number of distinct types of classes for a 20-dimensional database is 811. This knowledge could speed up classification problems by using the cardinality of the class instead of using the Euclidean norm of the data vector. To the best of my knowledge, the attribute of cardinality (i.e. coloring) combined with the infinity norm of the data point is a new mathematical classification scheme. The advantage is a *natural* partitioning of the integer lattice based on the orbits of $\mathbb{Z}^N$, instead of the standard filtering based on *features*, that could be biased by the data scientist. We found that for $N = 55$ the number of orbits (colorings) is equal to 153 095. The closest connection to the set of cardinalities are the incidence structure constants of $\mathbb{Z}^N$. The set of all incidence structure constants related to lattice points are a subset of the finite set of distinct cardinalities of $\mathbb{Z}^N$. The extension of the terms up to dimension $N = 50$ in the integer sequence A270950 has been submitted for approval to the OEIS. Interesting facts are:

- For theoretical physicists in $\mathbb{Z}^3$: number of distinct cardinalities = 1+ a(3) = 6;
- For relativistic physics (3+1D) in $\mathbb{Z}^4$: number of distinct cardinalities = 1+ a(4) = 10;
- For engineering physics in $\mathbb{Z}^7$: number of distinct cardinalities = 1+ a(7) = 30;
- For superstring theory in $\mathbb{Z}^{10}$: number of distinct cardinalities = 1+ a(10) = 77;
- For M-theory in $\mathbb{Z}^{11}$: number of distinct cardinalities = 1+ a(11) = 100;
- For bosonic string theorists in $\mathbb{Z}^{26}$: number of distinct cardinalities = 1 + a(26) = 2509.

A generating function for the OEIS integer sequence A270950 has not yet been found.

We plan for the academic year 2022-2023 to finalize the PhD dissertation and to defend the research results in May or June 2023.

P.10 Academic year 2022-2023

We submitted a manuscript with the title: ‘Method for encoding and decoding variables in research problems’ to the Journal of the Franklin Institute. Updates have been made after discussion with the peer reviewers. We created the backbone structure of SI physics (infinity norm < 10 and dimension $N = 7$). This backbone structure is a Hasse diagram of 19 448 orbits and contains all the representatives of the SI kinds of quantities known as per today by the scientists. The OEIS approved the publication of an extension to the integer sequence A270950 by adding the values from dimension $N = 17$ up to dimension $N = 51$. An unknown connection was discovered using the partition of dimension 'N'. These partitions create a base set of cardinalities. Each of these cardinalities can be subjected to the process of prime factorization. The prime factorization yields the exponents of the primes that form lattice points in a new integer lattice of dimension N. These lattice points become elements of a set A. The unique summands of a specific partition of N give the multipliers of the base vector $(1, 0^N)$ that need to be subtracted from the specific partition representative element of set A. The cardinality of set A increases until all the specific partitions of N have been processed. This augmented set A^* has the correct cardinality. This method has been implemented in a software code that finds the number of distinct cardinalities of orbits of lattice points as well as the values of the cardinalities in a much faster way than the brute force technique using histograms. The discovered cardinalities are: 0, 1, 2, 5, 9, 12, 20, 29, 40, 53, 76, 99, 132, 172, 216, 270, 341, 424, 532, 660, 810, 983, 1 210, 1 446, 1 750, 2 111, 2 508, 2 975, 3 569, 4 197, 4 948, 5 807, 6 817, 7 963, 9 351, 10 863, 12 604, 14 598, 16 892, 19 439, 22 472, 25 780, 29 588, 33 892, 38 800, 44 206, 50 463, 57 297, 65 086, 73 919, 83 842, 94 510. Each of these integers indicate the number of “distinct colors” that can be used to color a N-dimensional integer lattice based on the automorphism of the N-dimensional integer lattice or its isomorphic structures. After discussion with the publisher, the title of the manuscript was changed to: ‘Method for encoding and decoding variables in engineering problems’ and accepted for publication https://doi.org/10.1016/j.jfranklin.2022.09.035. The manuscript presents for the first time in 108 years a totally new method to create a dimensional measurement model and find a dimensionless measurement model that eliminates the limitations of the famous Buckingham π theorem of 1914. Dimensional analysis can be used in those cases, in which the system of equations describing the problem to solve is unknown. The setup of the dimensional measurement model $F(Q_1, \ldots, Q_M) = 0$ relies on the expertise of the researcher. The researcher is confronted with the questions: which Q_m is the dependent variable; what is the value of M; are the chosen Q_m effective. The new encoding-decoding method has the goal to answer these three questions and belongs to dimensional exploration techniques that can help in discovering the governing equations. This new method is based on low complexity, high performing, and well-established computer algorithms of number theoretic functions. The encoding-decoding method is exemplified on a real-world problem by searching for the positively homogeneous dimensionless measurement model that model wave phenomena, electromagnetic phenomena, electromechanical phenomena, and thermodynamic phenomena of the future power grids. The temporal variation of the power density is considered in its form of the kind of quantity called second order partial derivative of the energy density with respect to time denoted $\dfrac{\partial^2 W(\boldsymbol{r},t)}{\partial t^2}$. The new method generates a dimensional measurement model $F\left(\dfrac{\partial^2 W(\boldsymbol{r},t)}{\partial t^2}, Q_2, \ldots, Q_19\right) = 0$ and a positively homogeneous dimensionless measurement model $u(\pi_1, \ldots, \pi_9) = 0$ for the design of experiments. The validation of this new method is performed through its application in two cases: the simple pendulum and the kind of quantity energy. The efficiency, effectiveness, and completeness of the encoding-decoding method are compared with classical and modern dimensional analysis. The new method has the advantage over those state-of-the-art methods in requiring less dimensionless quantities π_k as arguments of the dimensionless measurement model $u(\pi_1, \ldots, \pi_K) = 0$ when modeling real-world problems. The encoding-decoding method is based on lattice theory while classical and modern dimensional analysis are based on linear algebra. The manuscript discloses for the first time the mathematical classification of SI kinds of quantities and by this it answers the open research question of James Clerk Maxwell dating from 1871. This mathematical classification of kinds of quantities is similar to the work done by Dmitri Mendeleev for chemistry in 1869.

The Hasse diagram for generic electro-mechanical systems was created from the orbit representative lattice point $(0 \mid 4, 2, 1, 1, 1)$ in $\{0, 1\} \times \mathbb{Z}^5$. The orbit representative has Gödel number 55 440 and has 120 divisors. This model can be used in projects related to Designer Centric Computational Design Synthesis (DCCDS) and for future power grid models. A potential customer for the UGent TechTransfer could be Elia.

Discussions of applications of the patent took place with one of the Deloitte directors from Toronto.

The backbone of TLM-physics was visualized in 3D showing the 8 different sets, represented by 8 orbit representatives and with a total of 147 vertices.

We explored the ‘Table of SI physics’ from the point of view of equivalence relations and are working to expand it to the intersection of 3 equivalence relations: infinity norm, orbit cardinality, and $||z||_2^2$ value of the orbit representative.

Research teams from Princeton (Miles Cranmer) and the Flatiron Institute (Shirley Ho) have clues for the existence of a new fundamental law applicable to the universe. They found this by using PySR. PySR uses symbolic regression applied on large astrophysical databases.

We are studying the publications and looking for the unknown equation related to the mass density parameter Ω_m of the universe. We found two candidate quantity equations for the mass density in our catalog and have the intention to discuss the topic with the research team in Princeton and from the Flatiron Institute.

The patent application submitted on 2021-11-15 has been published under the number US20230153487A1.

We further continue with improving the PhD manuscript and intend to submit the PhD manuscript in the academic year 2023-2024.

P.11 Academic year 2023-2024

We continued with improving the PhD manuscript in the academic year 2023-2024. We created a three-dimensional 'Table of SI physics' based on the intersection of three equivalence relations: infinity norm, orbit cardinality, and $\|\boldsymbol{z}\|_2^2$ value of the orbit representative. We succeeded in solving the 'unknown' system of differential equations, discovered in Academic year 2020-2021:

$$E = \left(\frac{\mathrm{d}m}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}A}{\mathrm{d}t}\right),$$

$$E = mr\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right).$$

Appendix Q

SI base unit definitions

We list the SI base unit definitions (BIPM, 2019).

Q.1 The second

We quote (BIPM, 2019):

> The second, symbol s, is the SI unit of time. It is defined by taking the fixed numerical value of the caesium frequency, $\Delta\nu_{\mathrm{Cs}}$, the unperturbed ground-state hyperfine transition frequency of the caesium 133 atom, to be 9 192 631 770 when expressed in the unit Hz, which is equal to s^{-1} .

$$1\,\mathrm{s} = \frac{9\,192\,631\,770}{\Delta\nu_{\mathrm{Cs}}} \tag{Q.1}$$

The time standard of the SI is based on a $^{133}\mathrm{Cs}$ atomic clock operating in the microwave regime that was demonstrated by Louis Essen in 1955 (Gill, 2005). It has continuously been improved and eventually adopted by the CIPM. These clocks have an accuracy of $1\mathrm{e}{-16}\mathrm{s}$. Further improvement of the accuracy by two orders of magnitude can be achieved by using optical atomic clocks (Godun, 2017). Chou, Hume, Rosenband, and Wineland (2010) have demonstrated relativistic time dilation for two optical atomic clocks moving at relative speeds $\mathrm{v} < 10\,\mathrm{ms}^{-1}$ by comparing the tick rates of the two identical clocks.

Q.2 The metre

We quote (BIPM, 2019):

> The metre, symbol m, is the SI unit of length. It is defined by taking the fixed numerical value of the speed of light in vacuum, c, to be 299 792 458 when expressed in the unit $\mathrm{m\,s}^{-1}$, where the second is defined in terms of the caesium frequency $\Delta\nu_{\mathrm{Cs}}$.

$$1\,\mathrm{m} = \frac{9\,192\,631\,770}{299\,792\,458}\frac{c}{\Delta\nu_{\mathrm{Cs}}} \tag{Q.2}$$

Q.3 The kilogram

We quote (BIPM, 2019):

> The kilogram, symbol kg, is the SI unit of mass. It is defined by taking the fixed numerical value of the Planck constant, h, to be $6.62607015 \times 10^{-34}$ when expressed in the unit J s, which is equal to $\mathrm{kg\,m^2\,s^{-1}}$, where the metre and the second are defined in terms of c and $\Delta\nu_{\mathrm{Cs}}$.

$$1\,\mathrm{kg} = \frac{(299\,792\,458)^2}{(6.626\,070\,15 \times 10^{-34})(9\,192\,631\,770)} \frac{h\Delta\nu_{\mathrm{Cs}}}{c^2} \tag{Q.3}$$

Q.4 The ampere

We quote (BIPM, 2019):

> The ampere, symbol A, is the SI unit of electric current. It is defined by taking the fixed numerical value of the elementary charge, e, to be $1.602176634 \times 10^{-19}$ when expressed in the unit C, which is equal to A s, where the second is defined in terms of $\Delta\nu_{Cs}$.

$$1\,\mathrm{A} = \frac{1}{(9\,192\,631\,770)(1.602\,176\,634 \times 10^{-19})} \Delta\nu_{\mathrm{Cs}}\mathrm{e} \tag{Q.4}$$

Q.5 The kelvin

We quote (BIPM, 2019):

> The kelvin, symbol K, is the SI unit of thermodynamic temperature. It is defined by taking the fixed numerical value of the Boltzmann constant, k, to be 1.380649×10^{-23} when expressed in the unit $\mathrm{J\,K^{-1}}$, which is equal to $\mathrm{kg\,m^2\,s^{-2}\,K^{-1}}$, where the kilogram, metre and second are defined in terms of h, c and $\Delta\nu_{\mathrm{Cs}}$.

$$1\,\mathrm{K} = \frac{1.380\,649 \times 10^{-23}}{(6.626\,070\,15 \times 10^{-34})(9\,192\,631\,770)} \frac{\Delta\nu_{\mathrm{Cs}}h}{k} \tag{Q.5}$$

Q.6 The mole

We quote (BIPM, 2019):

> The mole, symbol mol, is the SI unit of amount of substance. One mole contains exactly $6.02214076 \times 10^{23}$ elementary entities. This number is the fixed numerical value of the Avogadro constant, N_{A},

when expressed in the unit mol^{-1} and is called the Avogadro number. The amount of substance, symbol n, of a system is a measure of the number of specified elementary entities. An elementary entity may be an atom, a molecule, an ion, an electron, any other particle or specified group of particles.

$$1\,\mathrm{mol} = \left(\frac{6.022\,140\,76 \times 10^{23}}{N_\mathrm{A}}\right) \tag{Q.6}$$

Q.7 The candela

We quote (BIPM, 2019):

The candela, symbol cd, is the SI unit of luminous intensity in a given direction. It is defined by taking the fixed numerical value of the luminous efficacy of monochromatic radiation of frequency $540 \times 10^{12}\mathrm{Hz}$, K_cd, to be 683 when expressed in the unit $\mathrm{lm\,W^{-1}}$, which is equal to $\mathrm{cd\,sr\,W^{-1}}$, or $\mathrm{cd\,sr\,kg^{-1}m^{-2}s^{3}}$, where the kilogram, metre and second are defined in terms of h, c and $\Delta\nu_{Cs}$.

$$1\,\mathrm{cd} = \frac{1}{(6.626\,070\,15 \times 10^{-34})(9\,192\,631\,770)^2 683}(\Delta\nu_\mathrm{Cs})^2 h K_\mathrm{cd} \tag{Q.7}$$

APPENDIX R

Named units derived from SI base units

Special names are assigned by the SI to derived units. The list is given in Table R.1.

Table R.1: Named units derived from SI base units.

Name	Symbol	Physical quantity	$\mathrm{Orb}\,(z)$	$G(\mathrm{Orb}\,(z))$	Lattice point	$\mathrm{soc}\,(z)$
radian	rad	angle	$[(0 \mid 0^7)]$	0	$(0 \mid 0, 0, 0, 0, 0, 0, 0)$	0
steradian	sr	solid angle	$[(0 \mid 0^7)]$	0	$(0 \mid 0, 0, 0, 0, 0, 0, 0)$	0
hertz	Hz	frequency	$[(0 \mid 1, 0^6)]$	2	$(0 \mid -1, 0, 0, 0, 0, 0, 0)$	-1
becquerel	Bq	radioactivity (decays per unit time)	$[(0 \mid 1, 0^6)]$	2	$(0 \mid -1, 0, 0, 0, 0, 0, 0)$	-1
degree Celsius	řC	temperature relative to 273.15 K	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0, 0, 0, 0, 1, 0, 0)$	1
lumen	lm	luminous flux	$[(0 \mid 1, 0^6)]$	2	$(0 \mid 0, 0, 0, 0, 0, 0, 1)$	1
katal	kat	catalytic activity	$[(0 \mid 1^2, 0^5)]$	6	$(0 \mid -1, 0, 0, 0, 0, 1, 0)$	0
coulomb	C	electric charge	$[(0 \mid 1^2, 0^5)]$	6	$(0 \mid 1, 0, 0, 1, 0, 0, 0)$	2
lux	lx	illuminance, luminance	$[(0 \mid 2, 1, 0^5)]$	12	$(0 \mid 0, -2, 0, 0, 0, 0, 1)$	-1
gray	Gy	absorbed dose (of ionizing radiation)	$[(0 \mid 2^2, 0^5)]$	36	$(0 \mid -2, 2, 0, 0, 0, 0, 0)$	0
sievert	Sv	equivalent dose (of ionizing radiation)	$[(0 \mid 2^2, 0^5)]$	36	$(0 \mid -2, 2, 0, 0, 0, 0, 0)$	0
tesla	T	magnetic induction, magnetic flux density	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, 0, 1, -1, 0, 0, 0)$	-2
pascal	Pa	pressure, stress	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, -1, 1, 0, 0, 0, 0)$	-2
newton	N	force, weight	$[(0 \mid 2, 1^2, 0^4)]$	60	$(0 \mid -2, 1, 1, 0, 0, 0, 0)$	0
joule	J	energy, work, heat	$[(0 \mid 2^2, 1, 0^4)]$	180	$(0 \mid -2, 2, 1, 0, 0, 0, 0)$	1
watt	W	power, radiant flux	$[(0 \mid 3, 2, 1, 0^4)]$	360	$(0 \mid -3, 2, 1, 0, 0, 0, 0)$	0
weber	Wb	magnetic flux	$[(0 \mid 2^2, 1^2, 0^3)]$	1260	$(0 \mid -2, 2, 1, -1, 0, 0, 0)$	0
volt	V	voltage, electrical potential difference, electromotive force	$[(0 \mid 3, 2, 1^2, 0^3)]$	2520	$(0 \mid -3, 2, 1, -1, 0, 0, 0)$	-1
henry	H	electrical inductance	$[(0 \mid 2^3, 1, 0^3)]$	6300	$(0 \mid -2, 2, 1, -2, 0, 0, 0)$	-1
ohm	Ω	electrical resistance, impedance, reactance	$[(0 \mid 3, 2^2, 1, 0^3]$	12600	$(0 \mid -3, 2, 1, -2, 0, 0, 0)$	-2
siemens	S	electrical conductance	$[(0 \mid 3, 2^2, 1, 0^3)]$	12600	$(0 \mid 3, -2, -1, 2, 0, 0, 0)$	2
farad	F	electrical capacitance	$[(0 \mid 4, 2^2, 1, 0^3)]$	25200	$(0 \mid 4, -2, -1, 2, 0, 0, 0)$	3

APPENDIX S

General relativity and SI quantities

The definitions of the base units of the SI are locally applicable and we quote the SI brochure (BIPM, 2006, p.107):

> The definitions of the base units of the SI were adopted in a context that takes no account of relativistic effects. When such account is taken, it is clear that the definitions apply only in a small spatial domain sharing the motion of the standards that realize them. These units are known as proper units; they are realized from local experiments in which the relativistic effects that need to be taken into account are those of special relativity. The constants of physics are local quantities with their values expressed in proper units. Physical realizations of the definition of a unit are usually compared locally. For frequency standards, however, it is possible to make such comparisons at a distance by means of electromagnetic signals. To interpret the results the theory of general relativity is required since it predicts, among other things, a relative frequency shift between standards of about 1 part in 10^{16} per metre of altitude difference at the surface of the Earth. Effects of this magnitude cannot be neglected when comparing the best frequency standards.

Guinot (1997, 2010, 2011) discusses the proper units. We quote the explanation in the SI brochure (BIPM, 2006, p.107) on the question of proper units:

> The question of proper units is addressed in Resolution A4 adopted by the XXIst General Assembly of the International Astronomical Union (IAU) in 1991 and by the report of the CCDS Working Group on the Application of General Relativity to Metrology (Metrologia, 1997, 34, 261-290).

Chou et al. (2010) have demonstrated gravitational time dilation for two optical atomic clocks that change in height by $\mathrm{h} < 1\,\mathrm{m}$ by comparing the tick rates of the two identical clocks. The resolutions of the International Astronomical Union taken in 1991 have been amended and Soffel et al. (2003) gives the new resolutions for astrometry, celestial mechanics and metrology in the relativistic framework. The physical observables in general relativity are represented mathematically by functionals of fields that satisfy the Einstein field equations

(Woit, 2017). We may wonder what happens with the 'laws of physics' in curved spacetime. We quote (Misner et al., 2017):

> The answer is simple: *in any and every local Lorentz frame, anywhere and anytime in the universe, all the (nongravitational) laws of physics must take on their familiar special-relativistic form.*

The classical Einstein field equations, i.e. without cosmological constant Λ, can be written, using the Landau-Lifshitz Spacelike Convention (Misner et al., 2017), in the form (Landau & Lifchitz, 1970, p.363):

$$G_{\mu\nu} = \frac{8\pi G}{c_0^4} T_{\mu\nu}\,, \tag{S.1}$$

where G is the Newtonian constant of gravitation, c_0 the speed of light in vacuum, $T_{\mu\nu}$ the energy-momentum tensor and

$$G_{\mu\nu} \doteq R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{S.2}$$

where $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor, $g_{\mu\nu}$ is the metric tensor and R is the scalar curvature of spacetime.

Tensor contraction of Einstein's field equations yields the scalar equation:

$$-R = \frac{8\pi G}{c_0^4} T, \tag{S.3}$$

where $T = T^{\mu}_{\mu} = g^{\mu\nu} T_{\mu\nu} = \rho g^{\mu\nu} u_\mu u_\nu = \rho c_0^2$ and ρ is the relativistic mass density and u_μ is the μ-th component of the velocity.

Observe that the term $\dfrac{c_0^4}{G}$ has the dimension of a force or a tension and should be better considered as a constant energy per unit of length.

Its value is huge: $\dfrac{c_0^4}{G} = 1.21030 \times 10^{44}\,\mathrm{N}$ and thus its reciprocal in the Einstein field equations is very small $\dfrac{G}{c_0^4} = 8.26241 \times 10^{-45}\,\mathrm{N}^{-1}$. The dimension of the Einstein tensor $G_{\mu\nu}$ is expressed in m^{-2} within the SI units. If we rewrite the Einstein field equations in the following way:

$$\frac{1}{8\pi} \frac{c_0^4}{G} G_{\mu\nu} = T_{\mu\nu}\,, \tag{S.4}$$

then we find dimensionally on the left side a pressure or stress that is the ratio of a *constant force* or *constant tension* to a variable area and on the right side an energy density. The ratio of the pressure to the energy density yields the dimensionless quantity:

$$w = \frac{\left(R \dfrac{c_0^4}{8\pi G}\right)}{T} = -1. \tag{S.5}$$

We can formulate a dimensionless equation expressing the ratio of pressure to energy density:

$$\frac{1}{8\pi}\frac{c_0^4}{G}\left(\frac{G_{\mu\nu}}{T_{\mu\nu}}\right) = 1 \qquad \text{(S.6)}$$

Observe that the constant force $\frac{c_0^4}{G}$ is dimensionally similar to the force at the origin of the Schwinger effect (Schwinger, 1951), known from quantum electrodynamics as $\frac{m^2c^3}{\hbar}$ when the mass m is equal to the mass of an electron. By equating the forces and solve for the mass m we obtain: $m = \sqrt{\frac{\hbar c_0}{G}}$ that is proportional to the Planck mass m_P. Hence, we *speculate*, based on dimensional considerations, to rewrite Einstein's field equations in the form:

$$\frac{1}{8\pi}\frac{m^2c_0^3}{\hbar}G_{\mu\nu} = T_{\mu\nu}\,, \qquad \text{(S.7)}$$

where the mass m is to be considered as a *parameter* that varies from $m = 0$ to $m = \sqrt{\frac{\hbar c_0}{G}}$. Unfortunately, this speculation is, according to standard physics textbooks, forbidden because general relativity is not valid for dimensions $\sim \frac{\hbar}{mc_0}$ (Landau & Lifchitz, 1970, p.393).

In the framework of ΛCDM (Lambda Cold Dark Matter) we have as Einstein field equations (Stephani, Kramer, MacCallum, Hoenselaers, & Herlt, 2003):

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c_0^4}T_{\mu\nu}\,, \qquad \text{(S.8)}$$

where Λ is the cosmological constant. This framework is presently known as the standard cosmological model. Tensor contraction of the standard cosmological model field equations yields the scalar equation:

$$-R + 4\Lambda = \frac{8\pi G}{c_0^4}T, \qquad \text{(S.9)}$$

where $T = T^\mu_\mu$. The ratio of the pressure to the energy density yields the dimensionless quantity:

$$w = \frac{\left((R-4\Lambda)\frac{c_0^4}{8\pi G}\right)}{T} = -1. \qquad \text{(S.10)}$$

Appendix T

Quantum mechanics and SI quantities

Physical observables in quantum mechanics are represented mathematically by self-adjoint linear operators on Hilbert spaces $\mathcal{H}$. Two well-known physical quantities are position and linear momentum. To these physical quantities correspond respectively a position operator $\hat{x}$ and a momentum operator $\hat{p}_x$ as physical observables. An important physical constant occurring in quantum mechanics is the reduced Planck constant $\hbar = \dfrac{h}{2\pi}$ where h is the Planck constant.

The international vocabulary of metrology VIM (BIPM et al., 2012) doesn't contain the concept or wording observable. It seems that the meaning of the word observable depends of the physical theory that is used for describing nature. Bastin and Kilmister (1952, p.560) define an observable as the result of a physical observation. They propose an axiom (Bastin & Kilmister, 1952, p.560), and we quote:

> *Axiom* 1. An observable can be represented by an ordered set of n real numbers, or 'vector', in the sense of modern algebra, over the field, R, of real numbers.

Bastin and Kilmister define the value of an observable ϕ as a set of values ϕ_α where $\alpha = 1, \ldots, n$. They summarize by stating that an observable is completely specified when one is given and we quote (Bastin & Kilmister, 1952, p.560):

(i) the experimental method of determination,
(ii) the method of calculation from experimental results,
(iii) the value.

They note that in order that the value of an observable may represent completely the result of an experiment, some standard of comparison is required and that this standard must be included in the value of the vector (Bastin & Kilmister, 1952, p.560).

Appendix U

Maple code for cardinalities of orbits of $\{0,1\} \times \mathbb{Z}^N$

The only method to find the number of distinct cardinalities of an integer lattice $\{0,1\} \times \mathbb{Z}^N$ was through a brute-force approach. We developed a new method to find the number of distinct cardinalties of orbits of $\{0,1\} \times \mathbb{Z}^N$ and to list the values of the cardinalities. The results have been published in the OEIS A270950 integer sequence. Listing U.1 shows a Maplesoft (2018) procedure to calculate the number of cardinalities of the orbits of an N-dimensional integer lattice. The procedure generates the numbers of OEIS sequence A270950 (P. A. Chevalier, 2016) for each dimension individually by calling the procedure. For example typing A270950(7) results in the value 29. It means that there are (29+1) distinct cardinalities of orbits in Z^7 .

Listing U.1: Maple code for calculation of the number of cardinalities of the orbits of an N-dimensional integer lattice

```
restart=with(combinat), with(numtheory), with(ListTools):
A270950:=proc(N::integer)::integer;
        local CardinalitiesSet,bvector,nom,k,ap,dn,i,u,npk,x,nx,ppow,q,j,L;
        description "Number of Distinct Cardinalities of Orbits in the integer lattice Z^N OEIS A270950";
CardinalitiesSet:={[seq(0,i=1..N)]}:# initialization with o-vector
bvector:=[1,seq(0,i=1..(N-1))]:#[1,0,0,...,0]
nom:=2^N*N!:#order of integer lattice Z^N
ap:=partition(N):#additive partition of dimension N
for k from 1 to numbpart(N) do
dn:= 1:#initialize denominator of multinomial
u[k]:= MakeUnique(ap[k]);#unique summands
npk:= numelems(ap[k]);#number of summands in the partition
        for i from 1 to npk do
           dn:=dn*ap[k][i]!;
        end do:
x:=ifactors(nom/dn);
nx:=numelems(x[2]);#number of elements in factoring
        for i from 1 to N do
          ppow[i]:=0;#prime power initialization
          q[k]:=[op(q[k]),ppow[i]];#re-initialization lattice vectors for each partition k
        end do:
        for j from 1 to nx do
            for i from 1 to N do
                    if x[2][j][1]=ithprime(i) then
                           ppow[i]:=x[2][j][2];
                           q[k]:=[seq(ppow[i], i =1..N)];#fundamental vectors mapped from original vectors with no zeros (z=0)
                        end if;
                end do:
        end do:
        for L from 1 to numelems(u[k]) do
            CardinalitiesSet:={op(CardinalitiesSet),q[k],q[k]-u[k][L]*bvector};
        end do:
end do:
return(numelems(CardinalitiesSet)-1):#origin is  giving cardinality 1 (trivial) and thus is subtracted
end proc:
```

Appendix V

Python code for semi-perimeter histogram

Listing V.1 shows a Python code to generate the semi-perimeter histogram of an integer lattice point. The program is exemplified for $N = 6$ and $s = 4$ and $z = (0 \mid -2, 2, 1, 0, 0, 0)$. The run of the code gives the result CELL 16. It shows that $16/2 = 8$ independent quantity equations exist for the kind of quantity energy in an integer lattice with dimension $N = 6$ and an infinity norm $s = 4$.

Listing V.1: Python code for semi-perimeter histogram

```
import time
import datetime
import math
import numpy as np
import csv
from joblib import Parallel, delayed
#import datetime
#import sys
#from collections import Counter
import pandas as pd
pd.options.display.max_rows = 100
#import matplotlib.pyplot as plt
#import unicodedata
#import pylab as pl
#plt.close("all")
#%matplotlib inline
np.set_printoptions(threshold=np.inf)
N = 6
# z is the target point
z = (-2,2,1,0,0,0)
# s is the max absolute value we use as a coordinate
s = 4
freq=dict()
A = np.zeros((N*s**2+1,N*s**2+1),dtype=int) #empty array

def do_one_process(x0):
cur_A = np.zeros((N*s**2+1,N*s**2+1),dtype=int) #empty array
y0 = z[0] - x0
if abs(y0) <= s:
for x1 in range(-s, s+1):
y1 = z[1] - x1
```

```
if abs(y1) <= s:
for x2 in range(-s, s+1):
y2 = z[2] - x2
if abs(y2) <= s:
for x3 in range(-s, s+1):
y3 = z[3] - x3
if abs(y3) <= s:
for x4 in range(-s, s+1):
y4 = z[4] - x4
if abs(y4) <= s:
for x5 in range(-s, s+1):
y5 = z[5] - x5
#xy = x0*y0+x1*y1+x2*y2+x3*y3+x4*y4+x5*y5
t1 = (x0-0.5*z[0])**2+(x1-0.5*z[1])**2+(x2-0.5*z[2])**2+(x3-0.5*z[3])**2+(x4-0.5*z[4])**2+(x5-0.5*z[5])**2
t2 = t1-(9/4)*(z[0]**2+z[1]**2+z[2]**2+z[3]**2+z[4]**2+z[5]**2)
if t2 <= 0:
u = x0**2 + x1**2 + x2**2 + x3**2 + x4**2 + x5**2
v = y0**2 + y1**2 + y2**2 + y3**2 + y4**2 + y5**2
cur_A[u][v] = cur_A[u][v] + 1
return cur_A

print("starting calculation")
start_time = time.time()
results = Parallel(n_jobs=8)(delayed(do_one_process)(x0) for x0 in range(-s,s+1))
print("merging results")
for partial_A in results:
A = np.add(A, partial_A)
stop_time = time.time()
print(f"Done with calculation, took: {datetime.timedelta(seconds=stop_time-start_time)} s")
print("start writing file")
start_time = time.time()
with open('sp6D_s_'+str(s)+'_2Dhistogram_N-Ball.csv','w',newline='') as csvfile:
```

```
writer = csv.writer(csvfile)
out_list = []
head = ["u","v","sp","freq"]
writer.writerow(head)
for u in range(0,N*s**2+1):
for v in range(0,N*s**2+1):
if A[u][v] != 0.0:
freq[(u,v)] = A[u][v]
sp = math.sqrt(u)+math.sqrt(v)
out_list = [u,v,sp,A[u][v]]
writer.writerow(out_list)
stop_time = time.time()
print(f"Done with writing file , took: {datetime.timedelta(seconds=stop_time-start_time)} s")
df = pd.DataFrame(data=list(freq.items()), columns=['CELL','FREQUENCY'])
#print(pd.options.display.max_rows)
#dfsorted=df.sort_values(by='FREQUENCY').head(40)
#print(dfsorted)
#df.plot()
#plt.show()
counts=df.groupby(by='FREQUENCY').count()
#print(counts)
#print(counts.size)
#for i in range(len(counts)):
#print(f"i:{i}, hist: {0.5*counts.iloc[i]}")
#print(0.5*counts.iloc[1])
#print(0.5*counts.iloc[2])
#print(0.5*counts.iloc[3])

#Dprint("aggregated histogram")
#print(0.5*counts.agg('sum'))
print(counts.iloc[0])
```

APPENDIX W

Mathematical toolbox

In this part of the dissertation we gather objects of the mathematical language. The objects will originate from disparate mathematical fields as discrete mathematics, group theory, lattice theory, combinatorics, algebraic geometry, tensor calculus, graded algebras, number theory, etc. to solve the major research question: 'A mathematical classification of dimensionless quantity equations of physics'. We will restrict the selection of the mathematical objects to the bare minimum. We will not provide proofs for the quoted theorems as these can be found in the mathematics literature. We will mainly quote the texts describing the used objects.

W.1 Number theory

We quote Coppel with respect to the role of prime numbers (Coppel, 2009, p. 88):

> The significance of the primes is that, as far as multiplication is concerned, they are the 'atoms' and the composite integers are the 'molecules'.

The fundamental theorem of arithmetic is (Coppel, 2009, p. 88):

> *Proposition 7* If $a \in \mathbb{N}$ and $a \neq 1$, then a can be represented as a product of finitely many primes. Moreover, the representation is unique, except for the order of the factors.

W.1.1 Relations between natural numbers inducing lattice structures

Consider the set of the natural numbers $\mathbb{N}$, the binary operation $\times$ representing the multiplication of two natural numbers and the relation divisibility between the natural numbers. We write x divides z with notation $x|z$ if there exists a natural number y such that $z = x \times y$. The relations $\leq$ and $|$ between natural numbers induce a lattice structure (Birkhoff, 1967).

W.2 Set theory

Let X denote a set with n distinct elements.

W.2.1 Relation

An n-ary relation R over sets $X_1, \ldots, X_n$ is a subset of the Cartesian product $X_1 \times \ldots \times X_n$.

W.2.2 Arity of a relation

The arity of a relation is the dimension of the domain in the corresponding Cartesian product $X_1 \times \ldots \times X_n$. The relation consisting of the set of triples (X, Y, Z) has is a 3-arity relation and we therefore call the equation $Z = XY$ a ternary relation.

W.2.3 Partition of a set

A partition of a set X is a disjoint collection of non-empty subsets of X whose union is X (Halmos, 1998, p. 28).

W.2.4 Equivalence relation

A relation R in X is reflexive when xRx for every $x \in X$. A relation R in X is symmetric if xRy implies yRx for every $x, y \in X$. A relation R in X is transitive if xRy and yRz implies xRz for every $x, y, z \in X$. A relation in a set is an equivalence relation if it is reflexive, symmetric, and transitive. If R is an equivalence relation in X, and if $x \in X$, the equivalence class of x with respect to R is the set of all those elements y in X for which xRy . The equivalence class of x with respect to R is denoted x/R . We denote X/R as the set of all equivalence classes with respect to R and pronounce it as "X modulo R" (Halmos, 1998, p. 27-28).

W.2.5 Bell number

The Bell number B_n represents the number of equivalence relations on the set X of n elements. The Bell numbers satisfy a recurrence relation:

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k \ . \tag{W.1}$$

The Bell numbers are listed in the on-line encyclopedia of integer sequences OEIS A000110 (N. J. Sloane, 2018). Consider the set of the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$, the binary operation $\times$ representing the multiplication of two natural numbers and the relation divisibility between the natural numbers.

We write x divides z with notation $x|z$ if there exists a natural number y such that $z = x \times y$. The relations $\leq$ and $|$ between natural numbers induce a lattice structure (Birkhoff, 1967).

W.3 Analysis

W.3.1 N-sphere

Here we use the geometrical definitions and not the topological definitions of N-spheres. The volume of a N-sphere of radius ρ is (J. Conway et al., 1999):

$$V = V_N \rho^N , \tag{W.2}$$

in which V_N is the volume of a N-sphere of radius $\rho = 1$ (J. Conway et al., 1999):

$$V_N = \frac{2^N \pi^{(N-1)/2} ((N-1)/2)!}{N!} . \tag{W.3}$$

Alternatively, the volume of a N-sphere of radius ρ is:

$$V = \frac{S_N \rho^N}{N} . \tag{W.4}$$

The sum of the volumes of N-spheres of *radius* $\rho = \mathbf{1}$ for $N \to \infty$ is finite. We find (Weisstein, 2015a):

$$\sum_{N=0}^{\infty} V_N = e^{\pi} [1 + \text{erf}(\sqrt{\pi})]) = \mathbf{45.999326089382855}, \tag{W.5}$$

in which

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2) \mathrm{d}t, \tag{W.6}$$

is the error function and of which the decimal expansion can be found in OEIS sequence A128891 (Deléham, 2007a). The surface area of a N-sphere of radius ρ is (J. Conway et al., 1999):

$$S = S_N \rho^{N-1} , \tag{W.7}$$

in which $S_N = N V_N$ is the area of a N-sphere of radius $\rho = 1$. Alternatively the surface area of a N-sphere of radius $R = 1$ is:

$$S_N = \frac{2 \left(\sqrt{\pi}\right)^N}{\Gamma \left(\dfrac{N}{2}\right)} , \tag{W.8}$$

in which the gamma function is given by

$$\Gamma(m) = 2 \int_0^{+\infty} r^{2m-1} \mathrm{e}^{-r^2} \, \mathrm{d}r. \tag{W.9}$$

Observe that sum of the areas of N-spheres of *radius* $\rho = \mathbf{1}$ for $N \to \infty$ is finite. We find:

$$\sum_{N=0}^{\infty} S_N = 2(1 + \pi e^{\pi}[1 + \operatorname{erf}(\sqrt{\pi})]) = \mathbf{291.0222898249729}, \quad \text{(W.10)}$$

of which the decimal expansion can be found in OEIS sequence A128892 (Deléham, 2007a). We find the following relation between the sum of the areas and the sum of volumes of N-spheres of radius $\rho = 1$:

$$\sum_{N=0}^{\infty} S_N - 2\pi \sum_{N=0}^{\infty} V_N = 2. \quad \text{(W.11)}$$

The histogram of normalized area of N-sphere of radius $\rho = 1$ is given in Figure W.1.

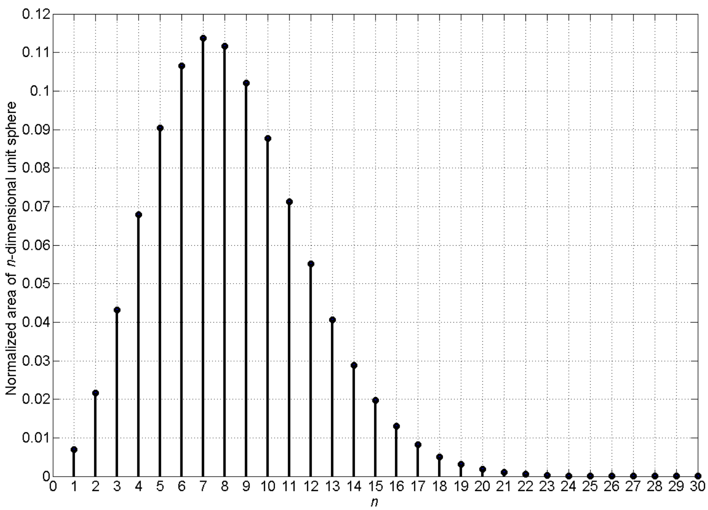

Figure W.1: Histogram of normalized area of N-sphere of radius $\rho = 1$.

The histogram of normalized volume of N-sphere of radius $\rho = 1$ is given in Figure W.2.

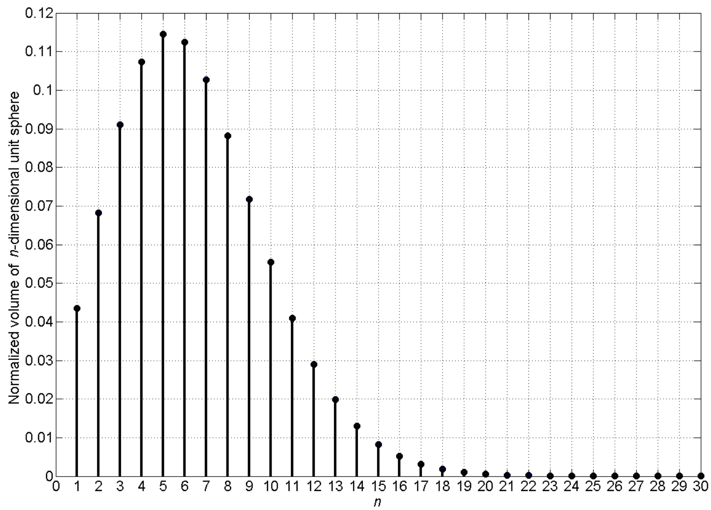

Figure W.2: Histogram of normalized volume of N-sphere of radius $\rho = 1$.

Observe that the largest area is obtained for dimension $N = 7$ and the largest volume is obtained for dimension $N = 5$.

W.3.2 N-dimensional discrete Fourier transform

We define the N-dimensional Fourier transform relations for $\boldsymbol{k}, \boldsymbol{x} \in \mathbb{R}^N$ as:

$$F(\boldsymbol{x}) = \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{N} f(\boldsymbol{k}) \mathrm{e}^{-2\pi i \boldsymbol{k}\cdot\boldsymbol{x}} \, \mathrm{d}^N \boldsymbol{k} \tag{W.12}$$

and

$$f(\boldsymbol{k}) = \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{N} F(\boldsymbol{x}) \mathrm{e}^{+2\pi i \boldsymbol{k}\cdot\boldsymbol{x}} \, \mathrm{d}^N \boldsymbol{x} \tag{W.13}$$

From the article of Lohmann et al. (1997) and the article of Foadi and Evans (2008) we construct a method that relates the N-sphere to a N-dimensional Helmholtz wave equation. We recall that the N-dimensional integer lattice $\mathbb{Z}^N$ can be represented as:

$$Q(\boldsymbol{r}) = \sum_{k_1=-\infty}^{+\infty} \cdots \sum_{k_N=-\infty}^{+\infty} \delta^N (\boldsymbol{r} - \sum_{i=1}^{N} k_i \boldsymbol{e}_i), \tag{W.14}$$

in which δ^N is the N-Dirac distribution. The Fourier transform of $Q(\boldsymbol{r})$ is the N-dimensional integral:

$$q(\boldsymbol{u}) = \int_{allspace} Q(\boldsymbol{r}) \times \exp\left[2\pi i \boldsymbol{u} \cdot \boldsymbol{r}\right] \mathrm{d}\boldsymbol{r} \tag{W.15}$$

It can be shown that (Foadi & Evans, 2008):

$$q(\boldsymbol{u}) = \sum_{k_1=-\infty}^{+\infty} \exp(2\pi i k_1 \boldsymbol{e}_1 \cdot \boldsymbol{u}) \ldots \sum_{k_N=-\infty}^{+\infty} \exp(2\pi i k_N \boldsymbol{e}_N \cdot \boldsymbol{u}) \tag{W.16}$$

and that we have:

$$\sum_{k_j=-\infty}^{+\infty} \exp(2\pi i k_j \boldsymbol{e}_j) = \sum_{m_j=-\infty}^{+\infty} \delta(\boldsymbol{e}_j \cdot \boldsymbol{u} - m_j) \tag{W.17}$$

in which $j = 1, \ldots, N$. The Fourier transform $q(\boldsymbol{u})$ has values different from 0 in the u-space in which $\boldsymbol{e}_j \cdot \boldsymbol{u} = m_j$ in which m_j are integers. The u-space is known as reciprocal space. We define in the u-space a reciprocal basis $\boldsymbol{e}_1^*, \ldots, \boldsymbol{e}_N^*$ and write:

$$q(\boldsymbol{u}) = \sum_{m_1=-\infty}^{+\infty} \delta(\boldsymbol{e}_1 \cdot \boldsymbol{u} - m_1 \boldsymbol{e}_1 \cdot \boldsymbol{e}_1^*) \ldots \sum_{m_N=-\infty}^{+\infty} \delta(\boldsymbol{e}_N \cdot \boldsymbol{u} - m_N \boldsymbol{e}_N \cdot \boldsymbol{e}_N^*) \tag{W.18}$$

that can be transformed to:

$$q(\boldsymbol{u}) = \sum_{m_1=-\infty}^{+\infty} \ldots \sum_{m_N=-\infty}^{+\infty} \delta^N(\boldsymbol{e}_1 \cdot (\boldsymbol{u} - m_1 \boldsymbol{e}_1^*), \ldots, \boldsymbol{e}_N \cdot (\boldsymbol{u} - m_N \boldsymbol{e}_N^*)) \tag{W.19}$$

The inner products under the N-Dirac delta can be considered as a linear transformation represented by a $N \times N$ matrix $\mathbf{T}$ on a lattice point $\boldsymbol{u} - \sum_{j=1}^{N} m_j \boldsymbol{e}_j^*$ with $det(\mathbf{T}) = V$ in which V is the cell volume. We find that (Foadi & Evans, 2008):

$$q(\boldsymbol{u}) = \frac{1}{V} \sum_{m_1=-\infty}^{+\infty} \ldots \sum_{m_N=-\infty}^{+\infty} \delta^N(\boldsymbol{u}^* - \boldsymbol{u}^*_{m_1,\ldots,m_N}) \tag{W.20}$$

W.4 Lattice theory

W.4.1 Gram matrix

Basic definitions for lattice packings are found in J. Conway et al. (1999). We recall some of these definitions. Let us consider N vectors in $\mathbb{R}^N$, then we can form a $N \times M$ matrix $\mathbf{M}$, called the generator matrix for the lattice represented by (J. Conway et al., 1999, p.4):

$$\mathbf{M} = \begin{bmatrix} v_{11} & v_{12} & \dots & v_{1M} \\ v_{21} & v_{22} & \dots & v_{2M} \\ \dots & \dots & \dots & \dots \\ v_{N1} & v_{N2} & \dots & v_{NM} \end{bmatrix}, \tag{W.21}$$

in which $M \geq N$. The Gram matrix for the lattice L is the matrix obtained by the product $\mathbf{A} = \mathbf{M} \cdot \mathbf{M}^{tr}$, in which $\mathbf{M}^{\mathrm{T}}$ is the transposed generator matrix. The elements of the matrix $\mathbf{A}$ are inner products $\boldsymbol{v}_i \cdot \boldsymbol{v}_j$. The determinant of a lattice L is the determinant of the matrix A and thus $\det L = \det \mathbf{A}$ (J. Conway et al., 1999, p.4).

W.4.2 Equivalent lattices

If one lattice can be obtained from another by (possibly) a rotation, reflection and change of scale we say they are equivalent, or similar, written $\simeq$. Two generator matrices $\mathbf{M_1}$ and $\mathbf{M_2}$ define equivalent lattices if and only if they are related by

$$\mathbf{M_2} = c \cdot \mathbf{U} \cdot \mathbf{M_1} \cdot \mathbf{B}, \tag{W.22}$$

in which c is a nonzero constant, $\mathbf{U}$ is a matrix with integer entries and determinant ± 1, and $\mathbf{B}$ is a real orthogonal matrix, with $\mathbf{B} \cdot \mathbf{B}^{tr} = 1$ (J. Conway et al., 1999, p.10). The corresponding Gram matrices $\mathbf{A_1}$ and $\mathbf{A_2}$ are related by (J. Conway et al., 1999, p.10)

$$\mathbf{A_2} = c^2 \cdot \mathbf{U} \cdot \mathbf{A_1} \cdot \mathbf{U}^{tr}. \tag{W.23}$$

W.4.3 Congruent lattices

If $\mathbf{U}$ has determinant ± 1 and $c = 1$ then $\mathbf{M_1}$ and $\mathbf{M_2}$ are congruent lattices. They are directly congruent if $\det \mathbf{U} = +1$ (J. Conway et al., 1999, p.10).

W.4.4 Dual lattice

A dual lattice L^* of a lattice L is given by $L^* = \{\boldsymbol{x} \in \mathbb{R}^N : \boldsymbol{x} \cdot \boldsymbol{u} \in \mathbb{Z}, \forall \boldsymbol{u} \in L\}$ (J. Conway et al., 1999, p.10).

W.4.5 Equivalent bases

Consider two lattice bases $\mathbf{B}_1$ and $\mathbf{B}_2$, in which $\mathbf{B}_1, \mathbf{B}_2 \in \mathbb{R}^{M \times N}$. The lattice bases $\mathbf{B}_1$ and $\mathbf{B}_2$ are *equivalent* if and only if there exists a unimodular matrix $\mathbf{U} \in \mathbb{Z}^{N \times N}$ such that $\det \mathbf{U} = \pm 1$ and that $\mathbf{B}_2 = \mathbf{B}_1 \cdot \mathbf{U}$. Under these conditions we find that the two lattice bases $\mathbf{B}_1$ and $\mathbf{B}_2$ generate the same lattice (Micciancio & Goldwasser, 2002, p.4).

W.4.6 Rescaling of lattices

The coordinates in lattices are most of the time real numbers. We are interested in the lattices with general integer coordinates. For that purpose we have to rescale lattices. We herewith rephrase the remarks of Conway on that topic (J. Conway et al., 1999, p.105): It is often necessary to rescale a lattice, changing L to $L' = cL = \{cx : x \in L\}$ for some constant $c \in \mathbb{R}$. The parameters of L and L' are related as follows: $M' = cM$, $A' = c^2 A$, $\det' = c^{2N} \det$, $(\text{minimal norm})' = c^2(\text{minimal norm})$, $\rho' = c\rho$, $R' = cR$. The dual lattice is related as follows $(L')^* = c^{-1}L^*$.

W.4.7 Lattice constellation of $\mathbb{Z}^N$

A set of lattice points is called a lattice constellation (Forney & Ungerboeck, 1998). The simplest constellation consists of 3 integer lattice points of which one is the origin of $\mathbb{Z}^N$.

We define a binary relation R_2 in $\mathbb{Z}^N \times \mathbb{Z}^N$ such that:

$$\forall(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{Z}^N \times \mathbb{Z}^N \ : (\boldsymbol{x},\boldsymbol{y}) \in R_2 \iff \boldsymbol{x} - \boldsymbol{y} = \boldsymbol{o},$$

We denote this constellation as a degenerate parallelogram.

We define a ternary relation R_3 in $\mathbb{Z}^N \times \mathbb{Z}^N \times \mathbb{Z}^N$ as a fundamental relation between four integer lattice points of which one is the origin of $\mathbb{Z}^N$.

$$\forall(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \in \mathbb{Z}^N \times \mathbb{Z}^N \times \mathbb{Z}^N \ : (\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \in R_3 \iff \boldsymbol{x} + \boldsymbol{y} - \boldsymbol{z} = \boldsymbol{o},$$

We denote this constellation as a non-degenerate parallelogram.

W.4.8 The N-dimensional integer lattice

The properties of the N-dimensional integer lattice are described elsewhere in the literature (J. Conway et al., 1999). We will recall briefly the properties that are useful for the purpose of structuring the quantities. Every lattice point is expressed in a unique way as the linear combination $\boldsymbol{x} = x_1\boldsymbol{e}_1 + \ldots + x_N\boldsymbol{e}_N$, in which the coefficients x_n are called the coordinates of $\boldsymbol{x}$. The basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_N$, that generates the integer lattice $\mathbb{Z}^N$ is orthonormal.

W.4.9 ℓ_p-norms

We define the ℓ_1-norm of $\boldsymbol{x}$ in $\mathbb{Z}^N$ by the expression:

$$N(\boldsymbol{x}) := \|\boldsymbol{x}\|_1 = a_{nm}X^n X^m \,, \tag{W.24}$$

in which we use the Einstein summation convention (n,m range from 1 to N). We define the ℓ_2-norm (known as the Euclidean norm) of $\boldsymbol{x}$ in $\mathbb{Z}^N$ by the expression:

$$\|\boldsymbol{x}\|_2 := \sqrt{a_{nm}X^n X^m} \tag{W.25}$$

We define the ℓ_∞-norm of $\boldsymbol{x}$ in $\mathbb{Z}^N$ by the expression:

$$\|\boldsymbol{x}\|_\infty := \max\{|x_1|, \dots, |x_N|\}, \tag{W.26}$$

that is also known as the Chebyshev norm, infinity norm, supremum norm, uniform norm, and maximum norm.

W.4.10 Metric space

Let $\boldsymbol{x}, \boldsymbol{y}$ be lattice points of $\mathbb{Z}^N$. The ℓ_2-distance (Euclidean distance) between the lattice points $\boldsymbol{x}, \boldsymbol{y}$ is defined by:

$$d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|_2 = \sqrt{(X_n - Y_m)(X^n - Y^m)}\,, \tag{W.27}$$

in which $\boldsymbol{x} - \boldsymbol{y} = (X^1 - Y^1, \dots, X^N - Y^N)$ if $\boldsymbol{x} = (X^1, \dots, X^N)$ and $\boldsymbol{y} = (Y^1, \dots, Y^N)$. We call two integer lattice points neighbors if their ℓ_2-distance is 1.

W.4.11 Low-dimensional lattices

A detailed study of low-dimensional lattices has been performed (J. Conway & Sloane, 1988b), (J. Conway & Sloane, 1988a), (J. H. Conway & Sloane, 1988), (J. Conway & Sloane, 1988c), (J. Conway & Sloane, 1989a), (J. Conway & Sloane, 1989b), (J. H. Conway & Sloane, 1992), and (J. H. Conway & Sloane, 1997).

W.4.12 Abstract connectives

Decomposition theorem in the theory of abstract connectives (see Klein 1932). A "lattice" is a domain of individuals in which two commutative and associative operations $\cup$ and $\cap$ are defined. If $a \cup b = b$ than a is called a part of b. Under the assumptions that the operation $\cap$ is distributive with respect to $\cup$ that further, a unit element e exists for $\cup$ and that every element has only finitely many parts, an element a of the lattice can be represented in at most one way in the form $a = p_1 \cup p_2 \cup \dots \cup p_n$ in which the p_i are primary elements over different prime elements. Here p is called a prime element if it has no proper part other than e, and q is called a primary element over p if p is the only prime element included in q. If in addition it is assumed that every non-primary element $a \neq e$ can be decomposed into two elements distinct from a, the existence of a representation in the form $a = p_1 \cup p_2 \cup \dots \cup p_n$ also follows for every element of the lattice.

W.4.13 Visible integer lattice points in $\mathbb{Z}^N$

Two lattice points $\boldsymbol{x} = (x_1, \dots, x_N)$ and $\boldsymbol{y} = (y_1, \dots, y_N)$ are mutually visible if the line segment joining them contains no further lattice points. The probability that an integer lattice point of $\mathbb{Z}^N$ is visible from the origin is $\dfrac{1}{\zeta(N)}$

in which $\zeta(N)$ is the Riemann zeta function (Weisstein, 2015b). We form the joint probability density distribution $P(N) = \dfrac{S_N}{\zeta(N) \times \sum_{N=0}^{\infty} S_N}$ of normalized area of N-sphere of radius $\rho = 1$ to express the degree of discrimination on the unit hypersphere between lattice points that are generated through the normalisation of the lattice point coordinates. Hence, two lattice points that are collinear with the origin are mapped on the same lattice point of the unit N-sphere and thus cannot be discriminated. The joint probability density distribution of normalized area of N-sphere of radius $\rho = 1$ is given in Figure W.3. The probability distribution $P(N)$ suggests to represent quantities in $\mathbb{Z}^7$.

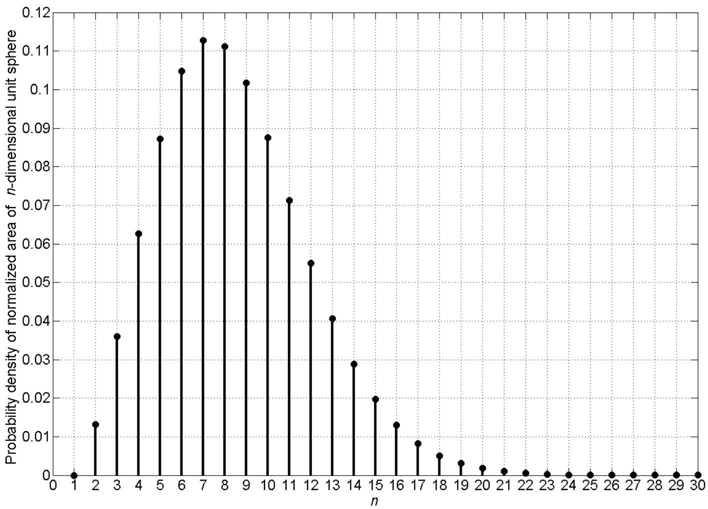

Figure W.3: Joint probability distribution of normalized area of N-sphere of radius $\rho = 1$.

W.4.14 Invariants in $\mathbb{Z}^N$

The line spanned by the sum of the basis vectors $\sum_{n=1}^{N} \boldsymbol{e}_n$ is invariant and has the complement subspace $V = \{(x_1, \ldots, x_N) \mid x_1 + \ldots + x_N = 0\}$.

W.4.15 Relations between natural numbers inducing lattice structures

Consider the set of the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$, the binary operation $\times$ representing the multiplication of two natural numbers and the relation divisibility between the natural numbers. We write ‘x divides z’ with notation

$x|z$ if there exists a natural number y such that $z = x \times y$. The relations $\leq$ and $|$ between natural numbers induce a lattice structure (Birkhoff, 1967).

W.4.16 Hyperplanes

We assign to each lattice point $\boldsymbol{x}$ of $\mathbb{Z}^N$ a hyperplane $H_{\boldsymbol{x}}$. A set $H_{\boldsymbol{x}}$ in $\mathbb{Z}^N$ is a hyperplane if and only if there exist scalars $C_0, C_1, \ldots, C_N$, in which not all $C_1, \ldots, C_N$ are zero, such that $H_{\boldsymbol{x}} = \{(X^1, \ldots, X^N) \mid C_0 + C_1 X^1 + \ldots + C_N X^N = 0\}$ (Webster, 1994). Consider now the lattice point $\boldsymbol{y} = (Y^1, \ldots, Y^N)$ and select its associated hyperplane $H_{\boldsymbol{y}}$ that contains the lattice point $\boldsymbol{o}$. The lattice point $\boldsymbol{x}$ is incident on the hyperplane $H_{\boldsymbol{y}}$ when it satisfies the equation $Y^i X_i = 0$. The distance between the lattice point $\boldsymbol{z}$ and the hyperplane $H_{\boldsymbol{y}}$, measured along the perpendicular, is the projection of $\boldsymbol{oz}$ in the direction of $\boldsymbol{oy}$ that is given by the equation:

$$\frac{\boldsymbol{z} \cdot \boldsymbol{y}}{\|\boldsymbol{y}\|_2} = \frac{\sum_{i=1}^N Z_i Y_i}{\sqrt{\sum_{i=1}^N Y_i Y_i}}. \tag{W.28}$$

Let the lattice point $\boldsymbol{x}_p$ be the image of $\boldsymbol{x}$ by reflection in the hyperplane $H_{\boldsymbol{y}}$. Consider the lattice point $\boldsymbol{z}$ satisfying $\boldsymbol{z} = \boldsymbol{x} - \boldsymbol{x}_p$, then the line $\boldsymbol{oz}$ is parallel to the line $\boldsymbol{oy}$. We define now a general reflection in the hyperplane $H_{\boldsymbol{y}}$ by the equation (Coxeter, 1973):

$$\boldsymbol{x} - \boldsymbol{x}_p = 2\frac{\boldsymbol{x} \cdot \boldsymbol{y}}{\boldsymbol{y} \cdot \boldsymbol{y}}\boldsymbol{y}. \tag{W.29}$$

We call the lattice point $\boldsymbol{y}$ the root of the reflecting hyperplane $H_{\boldsymbol{y}}$ (J. Conway et al., 1999).

W.4.17 Root system

The root system for the Lie algebra B_N (Carter, 2005) has the basis $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N$ defined by $\boldsymbol{a}_1 = \boldsymbol{e}_1 - \boldsymbol{e}_2, \boldsymbol{a}_2 = \boldsymbol{e}_2 - \boldsymbol{e}_3, \ldots, \boldsymbol{a}_{N-1} = \boldsymbol{e}_{N-1} - \boldsymbol{e}_N, \boldsymbol{a}_N = \boldsymbol{e}_N$. This root system generates the $\mathbb{Z}^N$ integer lattice as a root lattice (J. Conway et al., 1999) by reflections in the hyperplanes associated with the roots. The reflections are characterized by signed permutation matrices (Carter, 2005).

W.4.18 OEIS A008451

We find in the OEIS (N. J. A. Sloane, 2006) the sequence A008451 given by $r_7(m) = 1$, 14, 84, 280, 574, 840, 1288, 2368, 3444, 3542, 4424, 7560, 9240, 8456, 11088, 16576, 18494, 17808, 19740, 27720, 34440, 29456, 31304, 49728, 52808, 43414, 52248, 68320, 74048, 68376, 71120, 99456, 110964, 89936, 94864, 136080 .... The sequence represents the number of ways of writing a positive integer m as a sum of seven integral squares and is defined by:

$$\Theta_{\mathbb{Z}^7}(z) = \sum_{m=0}^{\infty} r_7(m) q^m,$$

in which $q = e^{\pi i z}$ and $m = \boldsymbol{x} \cdot \boldsymbol{x} = (\|\boldsymbol{x}\|_2)^2$ is the norm of the lattice point (J. Conway et al., 1999). Each term of the theta series $\Theta_{\mathbb{Z}^7}(z)$ of the integer lattice $\mathbb{Z}^7$ can be written as a sum of frequencies of cardinalities of the orbits of $\mathbb{Z}^7$ for a chosen infinity norm $\|\boldsymbol{x}\|_\infty$.

W.4.19 Covariant and contra-variant bases for a vector space

We select N linearly independent lattice points $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_N$ of $\mathbb{Z}^N$. The $\boldsymbol{e}_n$s form a covariant basis (Coxeter, 1973) for the integer lattice in $\mathbb{Z}^N$. Every lattice point is expressed in a unique way as the linear combination: $\boldsymbol{x} = X^1\boldsymbol{e}_1 + \ldots + X^N\boldsymbol{e}_N$. The inner product is defined as the equation: $\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{n=1}^{N} \sum_{m=1}^{N} a_{nm} X^n Y^m$ in which $a_{nm} = a_{mn}$. Consider N lattice points $\boldsymbol{e}^n$ satisfying the equation $\boldsymbol{e}^n = \sum_{m=1}^{N} a^{nm}\boldsymbol{e}_m$. This contravariant basis spans the space $\mathbb{Z}^N$ resulting in the equations $\sum_{n=1}^{N} a_{nm}\boldsymbol{e}^n = \sum_{n=1}^{N} \sum_{k=1}^{N} a_{nm} a^{nk}\boldsymbol{e}_k = \sum_{m=1}^{N} \delta_m^k \boldsymbol{e}_k = \boldsymbol{e}_m$.

A lattice point $\boldsymbol{x}$ has covariant components X_n, such that $\boldsymbol{x} = \sum_{n=1}^{N} X_n \boldsymbol{e}^n$. These components are related to the contravariant components by the equations: $X^m = \sum_{n=1}^{N} a^{nm} X_n$ and $X_n = \sum_{m=1}^{N} a_{nm} X^m$. With this notation the inner product is represented as $\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{n=1}^{N} X^n Y_n = \sum_{m=1}^{N} X_m Y^m$. Observe that, since $\boldsymbol{e}^n \cdot \boldsymbol{e}_m = \sum_{n=1}^{N} a^{nk}\boldsymbol{e}_k \cdot \boldsymbol{e}_m = \sum_{n=1}^{N} a^{nk} a_{mk} = \delta_m^n$, each $\boldsymbol{e}^n$ is orthogonal to every $\boldsymbol{e}_m$ except $\boldsymbol{e}_n$. Hence, $\boldsymbol{e}^n$ is perpendicular to the $(N-1)$-dimensional space spanned by $(N-1)$ $\boldsymbol{e}_m$. We obtain that $\boldsymbol{e}^n \cdot \boldsymbol{e}_n = 1$.

W.5 Algebra

W.5.1 Signed permutation matrices

We denote the set of $N \times N$ matrices with entries in $\mathbb{F}_3 = \{0, 1, -1\}$ by $\mathrm{Mat_N}(\mathbb{F}_3)$ that forms the full monomial group over $\mathbb{F}_3$ of order $(3-1)^N N!$

(J. Conway et al., 1999, p.77). These matrices are called signed permutation matrices or $N \times N$ monomial matrices (Suprunenko, 2016). The field $\mathbb{F}_3$ is the Galois field of order 3.

W.5.2 Automorphism group of $\mathbb{Z}^N$

The automorphism group of $\mathbb{Z}^N$, denoted $\mathrm{Aut}(\mathbb{Z}^{\mathrm{N}})$, consists of all signed permutation matrices acting on the integer lattice points and has order $2^N N!$, and is the Weyl group of root system B_N (J. Conway et al., 1999; Carter, 2005). The automorphism group $\mathrm{Aut}(\Lambda)$ of a lattice Λ is the set of distance-preserving transformations (or isometries) of the space that fix the origin and take the lattice to itself (J. Conway et al., 1999, p.90).

W.5.3 Orbit of an integer lattice point

The orbits of $\mathbb{Z}^N$ form a partition of $\mathbb{Z}^N$. The concept of orbit of a lattice is used in signal processing (Rault & Guillemot, 1997, 2001; Bruhn et al., 2008), in which it is also known as absolute leader class or equivalence class. An orbit is the set of lattice points of $\mathbb{Z}^N$ that are mapped in each other through a signed permutation. The orbit of the integer lattice point $\boldsymbol{z}$ under the automorphism group $G = \mathrm{Aut}(\mathbb{Z}^{\mathrm{N}})$ is denoted by $Orb_G(\boldsymbol{z}) = \{g\boldsymbol{z} : g \in G\}$. We note an orbit of $\mathbb{Z}^N$ as $[(X_1, \ldots, X_N)]$, in which $(X_1, \ldots, X_N)$ are the coordinates of the representative lattice point. Each orbit represents a set of lattice points that are symmetric about the origin $\boldsymbol{o}$. The cardinality of an orbit is calculated using elementary combinatorics. Let $A = \{0, 1, 2, \ldots, s\}$ be a totally ordered alphabet. The representative of an orbit is a word w constructed from the alphabet A. The words w have a length N that corresponds to the dimension of $\mathbb{Z}^N$. Let a_i be the number of characters of type i of the alphabet A. Suppose that the characters are subjected to a signed permutation, then the cardinality of an orbit is given by the equation:

$$\#\,(w) = 2^{N-a_0} \frac{N!}{a_0! a_1! a_2! \ldots a_s!}.$$

The number of integer lattice points in each orbit is equal to the cardinality of w. The representative lattice point has all its coordinates being non-negative integers. The coordinates are arranged in graded reverse lexicographical order (Cox et al., 1997). The union of orbits with norm $N(\boldsymbol{f})$ is forming a lattice shell (Vasilache, Dumitrescu, & Tăbuş, 2002; Vasilache & Tăbuş, 2003; Bruhn et al., 2008). The N-dimensional integer lattice $\mathbb{Z}^N$ is partitioned in $\binom{s+(N-1)}{(N-1)}$ orbits.

W.5.4 Group properties

Basic notions of group properties can be found in Lang (2005, Chapter 1). We recall some definitions, properties and theorems without proof that are relevant for this dissertation. We quote Lang (Lang, 2005):

> A *monoid* is a set G, with a law of composition which is associative, and having a unit element (so that in particular, G is not empty).

We quote Lang (Lang, 2005):

> Let G be a group. A *subgroup* H of G is a subset of G containing the unit element, and such that H is closed under the law of composition and inverse (i.e. it is a sub-monoid, such that if $x \in H$ then $x^{-1} \in H$).

We quote Lang (Lang, 2005)::

> By a *sub-monoid* of G, we shall mean a subset H of G containing the unit element e, and such that, if $x, y \in H$ then $xy \in H$ (we say that H is *closed* under the law of composition).

A collection F of subsets of a given set S is called a family of sets over S. Let G operate on a set S. Let $s \in S$. The subset of S consisting of all elements xs with $x \in G$ is denoted by Gs, and is called the orbit of s under G. Given elements $x_1, x_2, \ldots, x_n \in G$, these elements generate a cyclic subgroup $\langle x_1, x_2, \ldots, x_n \rangle$, namely the set of all elements of G of the form

$$x_{i_1}^{k_1} \ldots x_{i_r}^{k_r}, \tag{W.30}$$

with $k_1, \ldots, k_r \in \mathbb{Z}$. The automorphism group of $\mathbb{Z}^N$, denoted $\mathrm{Aut}(\mathbb{Z}^{\mathrm{N}})$, consists of all signed permutation $N \times N$ matrices acting on the integer lattice points and is the Weyl group of root system B_N (J. Conway et al., 1999; Carter, 2005). Here we have $G = \mathrm{Aut}(\mathbb{Z}^{\mathrm{N}})$ being a finite group, the composition law is the matrix multiplication and the unit element is I_N. The orbit of the integer lattice point $\boldsymbol{z}$ under the group $G = \mathrm{Aut}(\mathbb{Z}^{\mathrm{N}})$ is denoted by $Orb_G(\boldsymbol{z}) = \{g\boldsymbol{z} : g \in G\}$. The orbits of $\mathbb{Z}^N$ form a partition of $\mathbb{Z}^N$. We quote Lang (Lang, 2005):

> **Proposition W.5.1** (Lagrange's theorem)**.** *Let G be a group and H a subgroup. Then*
>
> $$(G : H)(H : 1) = (G : 1),$$
>
> *in the sense that if two of these indices are finite, so is the third and equality holds as stated. If $(G : 1)$ is finite, the order of H*

divides the order of G. More generally, let H,K be subgroups of G and let $H \supset K$. Let $\{x_i\}$ be a set of (left) coset representatives of K in H and let $\{y_j\}$ be a set of coset representatives of H in G. Then we contend that $\{y_j x_i\}$ is a set of coset representatives of K in G.

The converse of the Lagrange theorem is not true and thus there may be some q dividing the order of G for which G has no subgroup of order q. Sylow proved that the converse of Lagrange's theorem is true when $q = p^k$ and p is a prime number.

Theorem W.5.1 (Theorem A)**.** *Let $\#(G) = m \cdot p^k$, in which p is a prime, $k \geq 1$ and p doesn't divide m. For $i = 1, \ldots, k$,*

(a) G contains at least one subgroup of order p^i,

(b) if $i < k$, every such subgroup is normally contained in a subgroup of order p^{i+1}.

Definition W.5.1. *Let p be a prime. A Sylow p-subgroup of a finite group G is a subgroup H of G such that $\#(H)$ is the highest power of p dividing $\#(G)$.*

Corollary W.5.1 (Corollary to Theorem A)**.** *If the prime p divides the order of a finite group G and H is a p-subgroup of G then H is contained in at least one Sylow p-subgroup of G.*

Theorem W.5.2 (Sylow's First Theorem)**.** *If the prime p divides the order of a finite group G, then G has at least one Sylow p-subgroup.*

Theorem W.5.3 (Sylow's Second Theorem)**.** *For each prime p dividing the order of a finite group G, all Sylow p-subgroups of G are conjugate to each other.*

Theorem W.5.4 (Sylow's Third Theorem)**.** *For each prime p dividing the order of a finite group G, the number of Sylow p-subgroups of G is congruent to 1 modulo p and divides $\#(G)$.*

W.6 Euclidean geometry

W.6.1 Polytopes

A polytope P is a geometrical figure bounded by portions of lines, planes, or hyperplanes (Coxeter, 1973, p.vi). Alternatively, a polytope P is the convex hull of a set of finitely many points in $\mathbb{R}^N$ (Barvinok, Lee, & Novik, 2013). A polytope is called a polygon in two dimensions, a polyhedron in three dimensions, and a polychoron in four dimensions (C. McMullen, 2008, p.66). A polytope includes its interior region and should be considered as *solid* object.

W.6.2 k-skeleton of an d-polytope

A k-skeleton of a d-polytope P consists of all i-polytope elements of dimension up to k and is denoted $\mathrm{skel}_k(P)$ (P. McMullen & Schulte, 2002).

W.6.3 Simplex family

A N-simplex is the convex hull on $N+1$ vertices.

W.6.4 Cross-polytope family

W.6.5 Hypercube family

A hypercube with inner radius s, being the (minimal) distance from center of the polytope to the boundary, has side length $2s$, so the content, being the N-dimensional volume, of an N-dimensional hypercube is $V = 2^N \times s^N$ (Emert & Nelson, 1997). The surface of an N-dimensional hypercube has $2n$ faces, each an $(N-1)$-dimensional hypercube. An N-dimensional hypercube has surface content $A = 2N \times 2^{N-1} \times s^{N-1} = N \times 2^N \times s^{N-1} = \dfrac{\mathrm{d}V}{\mathrm{d}s}$ (Emert & Nelson, 1997).

W.6.6 N-parallelotope

A N-parallelotope is the generalization of a parallelepiped for dimensions $N > 3$. A parallelipiped is a 3-parallelotope.

W.6.7 Parallelogon

A parallelogon is a polygon with parallel opposite sides that can tile a plane by translation. A parallelogon can have only four or six sides. A parallelogon with four sides is called a parallelogram. The general parallelogram can take three special forms: rectangle, rhombus, and square.

W.6.8 Centrally symmetric set

Consider a lattice point $\boldsymbol{q}_0$ and a lattice points $\boldsymbol{q}$, which have the property $\boldsymbol{q}_0 + \boldsymbol{q} \in A \Leftrightarrow \boldsymbol{q}_0 - \boldsymbol{q} \in A$ then we call A a centrally symmetric set. In the remainder of the dissertation we will assume that $\boldsymbol{q}_0 = \boldsymbol{o}$ is the origin of $\mathbb{Z}^N$.

W.6.9 Centrally symmetric polytopes

A polytope $P \subset \mathbb{R}^N$ is centrally symmetric if $P = -P$ (Barvinok et al., 2013). A centrally symmetric polytope P is called k-neighborly if the convex hull of every set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ of k vertices of P, not containing a pair of antipodal vertices $\boldsymbol{v}_i = \boldsymbol{v}_j$, is a face of P (Barvinok et al., 2013).

W.6.10 Face of a convex body

A face F of a convex body is the intersection of the body with a supporting affine hyperplane (Barvinok et al., 2013). Let $f_k(P)$ denote the number of k-dimensional faces of a polytope.

W.6.11 Integer lattice polytopes

A N-dimensional integer lattice polytope P_N is the convex hull of a set of finitely many lattice points in $\mathbb{Z}^N$ (Coxeter, 1973). An infinity normed N-polytope P_N^s of infinity norm s is a subset of $\mathbb{Z}^N$ with the following property $P_N^s = \{\boldsymbol{x} \in \mathbb{Z}^N \mid \|\boldsymbol{x}\|_\infty = s\}$. We characterize the infinity normed N-polytope P_N^s by the parameters N and s, in which N represents the dimension of the integer lattice and s represents the value of the infinity norm $\|\boldsymbol{x}\|_\infty = s$. The integer convex hull $I(K)$ of a convex body K in $\mathbb{R}^N$ is defined as the convex hull of $K \cap \mathbb{Z}^N$ (Bárány, 2003).

W.6.12 Equivalent lattice polytopes

Two lattice polytopes $P, Q \in \mathbb{Z}^N$ are said to be equivalent if one can be carried to the other by a lattice preserving affine transformation $T : \mathbb{R}^N \to \mathbb{R}^N$ such that $TP = Q$ (Bárány, 2003). Equivalent polytopes have the same volume. How many equivalence classes $O_N(V)$ of integer lattice polytopes are there in dimension N of volume at most V? The question, formulated by Arnold results in the theorem (Bárány, 2003):

Theorem W.6.1. *When $N \geq 2$ and $V \to \infty$, then*

$$V^{\left(\frac{N-1}{N+1}\right)} \ll \log(O_N(V)) \ll V^{\left(\frac{N-1}{N+1}\right)} \log(V) \tag{W.31}$$

W.6.13 Cardinality of the 0-skeleton of the hypercube HC_N^s

Theorem W.6.2. *Let $\text{skel}_0(HC_N^s)$ be the set of the vertices of a centrally symmetric N-dimensional hypercube of edge-length $2s$ then the cardinality of HC_N^s is $(2s+1)^N$.*

Proof. For $N = 0$ the result is trivial.
For $N = 1$ we have the set $\text{skel}_0(HC_1^s) = \{-s, \ldots, 0, \ldots, s\}$ with edge-length $2s$. Let us denote the cardinality of the set S by $\#(S)$ then $\#(\text{skel}_0(HC_1^s)) = 2s + 1$.
For $N = 2$ we have to increase the dimension N by 1, which corresponds to calculate the Cartesian product of the sets $\text{skel}_0(HC_1^s) \times \text{skel}_0(HC_1^s) = \text{skel}_0(HC_2^s)$.
It is a property of cardinal numbers that: $\#(\text{skel}_0(HC_2^s)) = \#(\text{skel}_0(HC_1^s)) \times \#(\text{skel}_0(HC_1^s)) = \#(\text{skel}_0(HC_1^s)) \cdot \#(\text{skel}_0(HC_1^s)) =$

$(2s+1)^2$. Assume that $\#\left(\text{skel}_0(HC^s_{N-1})\right) = (2s+1)^{N-1}$. Then $\#\left(\text{skel}_0(HC^s_N)\right) = \#\left(\text{skel}_0(HC^s_{N-1}\right) \cdot \#\left(\text{skel}_0(HC^s_1)\right) = (2s+1)^{N-1} \cdot (2s+1) = (2s+1)^N$ $\square$. $\square$

We label the N-dimensional hypercube HC^s_N by the parameters N and s, in which N represents the dimension of the integer lattice and $2s$ represents the edge length of the hypercube. Remark that this labeling is unlike the classic unit hypercube which is not centrally symmetric. An orbit of the N-dimensional hypercube HC^s_N is the set of lattice points of $\mathbb{Z}^N$ that are equivalent. An orbit of HC^s_N is noted as $[(X_1, \ldots, X_N)]$ in which $(X_1, \ldots, X_N)$ are the coordinates of the representative lattice point. Each orbit forms a set of lattice points that are symmetric about the origin. Observe that each N-dimensional hypercube HC^s_N in $\mathbb{Z}^N$ represents a centrally symmetric integer lattice polytope (Coxeter, 1973; Grünbaum, 2003; Ziegler, 2005; Deza, Grishukhin, & Shtogrin, 2004).

W.6.14 Integer lattice as an incidence structure

Basic notions of discrete objects of N-dimensional integer lattice can be found in Voss (1993); Buekenhout (1995). An incidence structure $\Sigma = [E, I]$ is given by a set $E = \bigcup_{i=0}^N E_i$ of sets E_i and a binary, reflexive, symmetrical, and intransitive incidence relation $I \subset \bigcup_{i,k} E_i \times E_k$. The sets E_i are pairwise disjoint. The elements of the set E_i are called i-dimensional. The number N is the dimension of the incidence structure Σ. In a finite incidence structure $\Sigma = [E, I]$, there are non-negative numbers $b_{kl}(e)$, called structure constants, for each k-dimensional element $e \in E_k$ which describe the number of l-dimensional elements $e' \in E_l$ with $(e, e') \in I$. The elements of I are called flags. The number of elements of class E_k is denoted by a_k. For homogeneous incidence structures we have $a_k b_{kl} = a_l b_{lk}$. The Euler characteristic $\chi^{(N)}$ of a finite N-dimensional homogeneous structure is given by

$$\chi^{(N)} = \sum_{n=0}^{N} (-1)^n a_n = a_l \sum_{n=0}^{N} (-1)^n \frac{b_{ln}}{b_{nl}}, \tag{W.32}$$

in which $0 \leq l \leq N$. The equation (W.32) represents a system of $N+1$ non-linear Diophantine equations. For N-dimensional incidence structures we have the principle of double counting expressed by the equation

$$\sum_{e \in E_n} b_{nl}(e) = \sum_{e' \in E_l} b_{ln}(e'), \tag{W.33}$$

in which $0 \leq k, l \leq N$. We focus on discrete objects of the N-dimensional lattice that are attached to the origin $\boldsymbol{o}$ of $\mathbb{Z}^N$. The number of k-dimensional objects, that are attached to the origin $\boldsymbol{o}$ as a zero-dimensional element can be represented as a N-tuple of one's and zeros in which the ones occur k-times. There are $C(N, k)$ different tuples of this kind when we take only positive coordinate values into account. If we consider that the ones can have positive or negative signs we find the equation:

$$b_{0k}^{(N)} = 2^k C(N,k)\,, \tag{W.34}$$

in which $b_{0k}^{(N)}$ a structure constant of the incidence structure $\mathbb{Z}^N$. The number $b_{kl}^{(N)}$ is given in $\mathbb{Z}^N$ for $k < l$ by all possibilities of $(N-k)$ tuples with $(l-k)$ signed ones and $(N-l)$ zeros so that:

$$b_{kl}^{(N)} = 2^{(l-k)} C(N-k, l-k) = 2^{(l-k)} C(N-k, N-l). \tag{W.35}$$

The number $b_{kl}^{(N)}$ for $k > l$ is the number of all l-dimensional elements attached to a k-dimensional element in $\mathbb{Z}^N$. This number is independent of the dimension N .

$$b_{kl}^{(N)} = b_{N-k,N-l}^{(N)} = 2^{(k-l)} C(k, k-l) = 2^{(k-l)} C(k,l). \tag{W.36}$$

When $k = l$ then $b_{kl}^{(N)} = 1$. A neighborhood structure in $\mathbb{Z}^N$ is a geometric graph $[P, NB]$ in which the relation NB is irreflexive and symmetric and P is a set of lattice points. The Table W.1 gives the number of lattice point in the neighborhood NB of a lattice point of the N-dimensional lattice. The cardinality is found through the equation $\#\,(NB)\,(N) = 3^N - 1$.

Table W.1: Cardinality of the neighborhood of a lattice point in N-dimensional lattice.

Dimension N	Cardinality of neighborhood NB
0	0
1	2
2	8
3	26
4	80
5	242
6	728
7	**2 186**
8	6 560

The connection between two lattice points in a neighborhood structure is naturally combined with the notion of distance between lattice points. We expect that two neighbored lattice points have a small distance from one another. We introduce a metric in the lattice point set P by a real valued distance $d(\boldsymbol{p},\boldsymbol{q})$ for any two lattice points $\boldsymbol{p},\boldsymbol{q} \in P$ with the following properties:

$$d(\boldsymbol{p},\boldsymbol{p}) = 0 \tag{W.37}$$

$$d(\boldsymbol{p},\boldsymbol{q}) = d(\boldsymbol{q},\boldsymbol{p}) \tag{W.38}$$

$$d(\boldsymbol{p},\boldsymbol{q}) > 0, \forall \boldsymbol{p} \neq \boldsymbol{q} \tag{W.39}$$

$$d(\boldsymbol{p},\boldsymbol{q}) + d(\boldsymbol{q},\boldsymbol{r}) \geq d(\boldsymbol{p},\boldsymbol{r}) \tag{W.40}$$

Let $l(\boldsymbol{p},\boldsymbol{q})$ be the shortest path length between the lattice points $\boldsymbol{p}$ and $\boldsymbol{q}$ in a given neighborhood structure $[P, NB]$. It is obvious that there may be several different paths with the same minimal path length between lattice points. Geometries on neighborhood structures $[P, NB]$ are characterized by non-Euclidean properties. Dijkstra (1959); Dantzig (1966) developed algorithms for the determination of shortest paths.

W.6.15 Discrete point sets

Neighborhood structures $[P, NB]$ are loop less undirected graphs without multiple edges (Voss, 1993). The sets of lattice points investigated in $\mathbb{Z}^N$ are almost finite sets and the neighborhood of an integer lattice point contains only a finite number $\nu = \nu(P)$ of other integer lattice points. The neighbors $q \in NB(p)$ of each lattice point $p \in P$ are ordered cyclically as $< q_1, q_2, \ldots, q_N >$. Because of this ordering, we imprint an orientation on the neighborhood structure. Each directed edge (q_i, q_{i+1}) defines unambiguously a closed path in P which is denoted as a mesh. Point, edges and meshes represent an oriented neighborhood structure (Voss, 1993). Assume that all lattice points $p \in P$ have the same number $\nu = \nu(P)$ of neighbors and that all meshes m have the same length $\lambda = \lambda(m)$, the neighborhood structure is a homogeneous neighborhood structure.

W.7 Algebraic geometry

We define a monomial as an algebraic expression with a single term that can have multiple variables and various degrees:

Definition W.7.1. *A monomial m in $u_1, u_2, \ldots, u_N$ is a product of the form:*

$$m = \prod_{n=1}^{N} u_n^{X_n} \ ,$$

in which all the exponents $X_n \in \mathbb{Z}$ and $u_n \in \mathbb{U}$. The total degree deg *of this monomial is the sum $X_1 + \ldots + X_N$.*

From the N-tuple of non-negative integer exponents $(X_1, \ldots, X_N) \in \mathbb{Z}_+^N$ a monomial (Cox et al., 1997) is constructed one-to-one of the form $m = \prod_{n=1}^{N} u_n^{X_n}$. Its form is similar but not identical to equation (W.7.1) because equation (W.7.1) allows *negative* integer exponents. It means that a lot of results known from the commutative module of monomials are applicable to the classification of the kinds of quantities. The number of monomials of total degree $X_1 + \ldots + X_N \leq s$ in $k[x_1, \ldots, x_N]$ is $\begin{pmatrix} N+s \\ s \end{pmatrix}$ (Cox et al., 1997, p.438).

W.8 Analytical geometry

W.9 Confocal ellipses and hyperbolas

Observe that the parallelograms have the lattice points ***o*** and ***z*** as foci of an ellipse that has the lattice points ***x*** and ***y*** incident of it. From the definition of an ellipse we have $2a = \sqrt{u} + \sqrt{v}$. When exploring the distribution of parallelogram perimeters we find equal perimeters. Counting those equal perimeters is similar to finding the number of lattice points on a chosen confocal N-ellipsoid. Let for the moment $N = 2$ and let a be the value of the semi-major axis of the ellipse, b the semi-minor axis and e the eccentricity of the ellipse. We have for a family of confocal ellipses the equation (Abramowitz & Stegun, 1964, p.752):

$$\frac{x^2}{\xi^2} + \frac{y^2}{\xi^2 - 1} = f^2 \,, \tag{W.41}$$

in which $(1 < \xi < \infty)$ and $2f$ is the distance between the foci of the ellipse, with center equal to $(0, 0)$. The definition of the parameter ξ is:

$$\xi = \frac{\sqrt{u} + \sqrt{v}}{2f} \,. \tag{W.42}$$

We find (Abramowitz & Stegun, 1964, p.752) that:

$$a = f\xi \,,$$
$$b = f\sqrt{\xi^2 - 1} \,,$$
$$e = \frac{f}{a} \,.$$

The lattice points ***o*** and ***z*** can also be the foci of a family of hyperbolas with equation (Abramowitz & Stegun, 1964, p.752):

$$\frac{x^2}{\eta^2} - \frac{y^2}{1 - \eta^2} = f^2 \,, \tag{W.43}$$

in which $(-1 < \eta < 1)$ and $2f$ is the distance between the foci of the ellipse, with center equal to $(0, 0)$. The definition of the parameter η is:

$$\eta = \frac{\sqrt{u} - \sqrt{v}}{2f} \,. \tag{W.44}$$

The relation between Cartesian and elliptical coordinates is given by the equations (Abramowitz & Stegun, 1964, p.752):

$$x = f\xi\eta \,,$$
$$y = f\sqrt{(\xi^2 - 1)(1 - \eta^2)} \,.$$

Let us now consider $N = 3$ and let system of confocal ellipses and hyperbolas revolve around the axis ***oz*** then we obtain a prolate spheroid with respective equations (Abramowitz & Stegun, 1964, p.752):

$$\frac{x^2}{\xi^2} + \frac{r^2}{\xi^2 - 1} = f^2 \,,$$
$$\frac{x^2}{\eta^2} - \frac{r^2}{1 - \eta^2} = f^2 \,,$$

with $y = r\cos\phi$, $z = r\sin\phi$ and $(0 \leq \phi \leq 2\pi)$ in which ξ, η, ϕ are prolate spheroidal coordinates. The relation between Cartesian and prolate spheroidal coordinates is given by the equations (Abramowitz & Stegun, 1964, p.752):

$$x = f\xi\eta \,,$$
$$y = f\sqrt{(\xi^2 - 1)(1 - \eta^2)}\cos\phi \,,$$
$$z = f\sqrt{(\xi^2 - 1)(1 - \eta^2)}\sin\phi \,.$$

W.9.1 Perimeter of a triangle

Let p_t be the perimeter of the triangle formed by the 3-cycle ***ozxo***. The value of the perimeter p_t is obtained by the equation $p_t = \sqrt{u} + \sqrt{v} + \sqrt{w}$ with $u, v, w \in \mathbb{Z}_+$ and expressed through the following equations:

$$u = \sum_{n=1}^{N} x_n^2 \,, \qquad v = \sum_{n=1}^{N} (x_n - z_n)^2 = \sum_{n=1}^{N} y_n^2 \,, \qquad w = \sum_{n=1}^{N} z_n^2 \,.$$

An algorithm in pseudocode to calculate the histogram of perimeters of triangles for the 3-cycle ***ozxo*** is given in [Appendix C].

W.9.2 Area of a triangle

Let A_t be the area of a triangle formed by the 3-cycle ***ozxo***. The area of the triangle defined by the lattice points ***ozx*** is given by the equation, see Abramowitz and Stegun (1964), $A_t = \frac{1}{2}hb$, in which h is the height of the triangle that corresponds to the distance from the lattice point ***x*** to the axis ***oz*** and b is the base of the triangle and corresponds to $\|\boldsymbol{z}\|_2$. We call α the angle between ***z*** and ***x***. From the inner product of ***x*** and ***z*** we have

$$\boldsymbol{x} \cdot \boldsymbol{z} = \|\boldsymbol{x}\|_2 \|\boldsymbol{z}\|_2 \cos\alpha \,. \tag{W.45}$$

We call β the angle between ***z*** and ***y***. From the inner product of ***y*** and ***z*** we have

$$\boldsymbol{y} \cdot \boldsymbol{z} = \|\boldsymbol{y}\|_2 \|\boldsymbol{z}\|_2 \cos\beta \,. \tag{W.46}$$

We call γ the angle between $\boldsymbol{x}$ and $\boldsymbol{y}$. From the inner product of $\boldsymbol{x}$ and $\boldsymbol{z}$ we have

$$\boldsymbol{x} \cdot \boldsymbol{y} = \|\boldsymbol{x}\|_2 \|\boldsymbol{y}\|_2 \cos \gamma \,. \tag{W.47}$$

We have by definition that the height h of the triangle $\boldsymbol{ozxo}$ is given by the equation

$$h = \|\boldsymbol{x}\|_2 \sin \alpha \,. \tag{W.48}$$

We rewrite the terms in the following way

$$\cos \alpha = \frac{\boldsymbol{x} \cdot \boldsymbol{z}}{\|\boldsymbol{x}\|_2 \|\boldsymbol{z}\|_2} \,, \tag{W.49}$$

$$\sin \alpha = \frac{h}{\|\boldsymbol{x}\|_2} \,. \tag{W.50}$$

From elementary goniometry, see Abramowitz and Stegun (1964), we have:

$$\cos^2 \alpha + \sin^2 \alpha = 1 = \frac{(\sum_{n=1}^{N} z_n x_n)^2 + h^2 \|\boldsymbol{z}\|_2^2}{\|\boldsymbol{x}\|_2^2 \|\boldsymbol{z}\|_2^2} = \frac{(\sum_{n=1}^{N} z_n x_n)^2 + 4A_t^2}{\|\boldsymbol{x}\|_2^2 \|\boldsymbol{z}\|_2^2} \,. \tag{W.51}$$

We rewrite the equation (W.51) by moving the denominator to the left side of the equation and rearrange the terms to a quadratic form $Q(\boldsymbol{x})$ given by the equation:

$$Q(\boldsymbol{x}) = (\sum_{n=1}^{N} (z_n)^2)(\sum_{n=1}^{N} (x_n)^2) - (\sum_{n=1}^{N} ((z_n)(x_n)))^2 - 4A_t^2 = 0 \,,$$

that is easily transformed to a matrix equation given by:

$$Q(\mathbf{X}) = \mathbf{X}^{tr} \mathbf{M} \mathbf{X} \,,$$

and in which $\mathbf{X}^{tr}$ is a $1 \times (N+1)$ matrix, $\mathbf{M}$ is a symmetric $(N+1) \times (N+1)$ matrix and $\mathbf{X}$ is a $(N+1) \times 1$ matrix. The term $A_p^2 = 4A_t^2$ is the square of the area of the parallelogram formed by the 4-cycle $\boldsymbol{oyzxo}$ and is always a positive integer for non-degenerated parallelograms. The square of the area of the parallelogram has the property $A_p^2 \geq 1$, when degenerated parallelograms are excluded. The parallelogram in which $A_p^2 = 1$ is a fundamental parallelogram of $\mathbb{Z}^N$.

The matrix $\mathbf{M}$ has the following structure:

$$\mathbf{M} = \begin{bmatrix} (\sum_{n=1}^{N} z_n^2) - z_1^2 & -z_1 z_2 & \dots & -z_1 z_N & 0 \\ -z_1 z_2 & (\sum_{n=1}^{N} z_n^2) - z_2^2 & \dots & -z_2 z_N & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ -z_1 z_N & -z_2 z_N & \dots & (\sum_{n=1}^{N} z_n^2) - z_N^2 & 0 \\ 0 & 0 & \dots & 0 & -A_p^2 \,. \end{bmatrix} \tag{W.52}$$

The matrix $\mathbf{M}$ has $\operatorname{rank}\mathbf{M} = N + 1$ because $-A_p^2 \neq 0$. The first sub-matrix of $\mathbf{M}$ has rank N.

The matrix equation $Q(\mathbf{X})$ represents N-cylinders. The eigenvalues $\mathbf{E}$ of the matrix $\mathbf{M}$ are:

$$\mathbf{E} = \begin{bmatrix} 0 \\ -A_p^2 \\ (\sum_{n=1}^{N} z_n^2) \\ (\sum_{n=1}^{N} z_n^2) \\ (\sum_{n=1}^{N} z_n^2) \\ \vdots \\ (\sum_{n=1}^{N} z_n^2) \,. \end{bmatrix} \tag{W.53}$$

The first eigenvalue is $\lambda_1 = 0$ and the second eigenvalue is $\lambda_2 = -A_p^2$. The other $N - 1$ eigenvalues are identical with value $\lambda_i = \|\boldsymbol{z}\|_2^2$.

W.10 Discrete Mathematics

W.10.1 Generating function

The ordinary generating function (OGF) of a sequence a_n is (McBride, 1971), (Lando, 2003),(Generating function, 2024):

$$G(a_n; x) = \sum_{n=0}^{\infty} a_n x^n \,. \tag{W.54}$$

W.11 Graph theory

W.11.1 Paths, walks and cycles in a N-dimensional integer lattice

A path in $\mathbb{Z}^N$ is a non-empty graph $P = (V, E)$ of the form $V = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_k\}$ and $E = \{\boldsymbol{x}_0\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{k-1}\boldsymbol{x}_k\}$ in which the $\boldsymbol{x}_i$ are all distinct (Diestel, 2000). As we will connect lattice points in the integer lattice forming parallelograms, we use the term k-cycle from graph theory (Diestel, 2000), in which the k-cycle is a simple graph of length k, i.e., consisting of k vertices and k edges and represented by a sequence of consecutive vertices $\boldsymbol{x}_0 \ldots \boldsymbol{x}_{k-1}\boldsymbol{x}_0$. We are considering more specifically geometric graphs that are embeddings of a graph in a metric space $\mathbb{Z}^N$ that maps vertices to integer lattice points and edges to line segments connecting pairs of integer lattice points (Wilkinson, Anand, &

Grossman, 2006). Equations between quantities are represented by paths in $\mathbb{Z}^N$. Dimensional products are represented by cycles in $\mathbb{Z}^N$. A walk of length k in $\mathbb{Z}^N$ is a non-empty alternating sequence $\boldsymbol{v}_0 e_0 \boldsymbol{v}_1 e_1 \ldots e_{k-1} \boldsymbol{v}_k$ of vertices $\boldsymbol{v}_i$ and edges e_i in $\mathbb{Z}^N$ such that $e_i = \{\boldsymbol{v}_i, \boldsymbol{v}_{i+1}\}$ for all $i < k$.

W.12 Probability and Statistics

W.12.1 Probability mass function

A probability mass function is a function that gives the probability that a discrete random variable is exactly equal to some value.

APPENDIX X

Publications

This chapter presents an overview of the publications related to the research topic.

X.1 Patent application US 2023/0153487 A1

The intellectual property of the research from 1978 to 2021 was transferred from Mr. Philippe A.J.G. Chevalier to the Ghent University on 2021-09-29. A non-disclosure agreement was signed between the parties. A patent application has been submitted on the 15th of November 2021 under the title: MACHINE-IMPLEMENTABLE METHOD AND SYSTEM FOR ENCODING/DECODING VARIABLES IN ENGINEERING PROBLEMS and published at the USPTO on the 18th of May 2023. We present the abstract of the publication.

The invention is disclosing a machine-implementable method and system for the selection of a set of independent variables to form quantity equations for engineering problems of a kind. The method includes the encoding and decoding of dimensionless groups in an integer lattice. A preferred embodiment of the invention considers an integer lattice given by the cartesian product $\{0,1\} \times \mathbb{Z}^7$. The result of the method gives a system of quantity equations in the independent variables. The Hasse diagram containing the seven sets is given in Figure X.1.

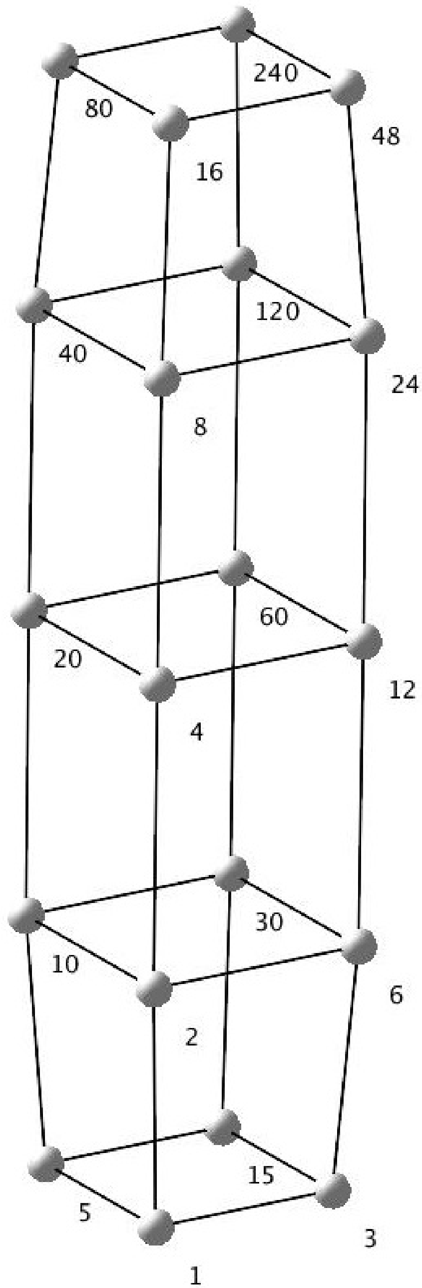

Figure X.1: Hasse representation of the decoding of the orbit representative of the second order partial derivative of the energy density with respect to time given by Gödel number $G(\mathrm{Orb}\left(\boldsymbol{q}^{1}\right)) = (-1)^0 2^4 3^1 5^1 7^0 11^0 13^0 17^0 = 240\,.$

X.2 Journal of the Franklin Institute

A publication of part of the research of this dissertation was done in the Journal of the Franklin Institute with title: Method for encoding and decoding variables in engineering problems `https://www.sciencedirect.com/science/article/pii/S0016003222006792`. We present the abstract of the publication (P. Chevalier & Constales, 2022).

Dimensional analysis can be used in those cases, in which the system of equations describing the problem to solve is unknown. The setup of the dimensional measurement model $F(Q^1, \ldots, Q^M) = 0$ relies on the expertise of the researcher. The researcher is confronted with the questions: which Q^m is the dependent variable; what is the value of M ; are the chosen Q^m effective. The new encoding-decoding method disclosed has the goal to answer these three questions and belongs to dimensional exploration techniques that can help in discovering the governing equations. This new method is based on low complexity, high performing, and well-established computer algorithms of number theoretic functions. The encoding-decoding method is exemplified on a real-world problem by searching for the positively homogeneous dimensionless measurement model that models wave phenomena, electromagnetic phenomena, electromechanical phenomena, and thermodynamic phenomena of the future power grids. The temporal variation of the power density is considered in its form of the kind of quantity called second order partial derivative of the energy density with respect to time denoted $\dfrac{\partial^2 W(\boldsymbol{r},t)}{\partial t^2}$. The new method generates a dimensional measurement model $F(\dfrac{\partial^2 W(\boldsymbol{r},t)}{\partial t^2}, Q_2, \ldots, Q_{19}) = 0$ and a positively homogeneous dimensionless measurement model $u(\pi_1, \ldots, \pi_9) = 0$ for the design of experiments. The validation of this new method is performed through its application on two cases: the simple pendulum and the kind of quantity energy. The efficiency, effectiveness, and completeness of the encoding-decoding method are compared with classical and modern dimensional analysis. The new method has the advantage over those state-of-the-art methods in requiring less dimensionless quantities π_k as arguments of the dimensionless measurement model $u(\pi_1, \ldots, \pi_K) = 0$ when modeling real-world problems. The encoding-decoding method is based on lattice theory while classical and modern dimensional analysis are based on linear algebra.

X.3 Contributions to the OEIS

The research on a mathematical classification of dimensionless quantity equations has resulted in the discovery of new integer sequences and new connections to existing sequences. We list the sequences in increasing OEIS identifier.

X.3.1 Comment to integer sequence A000579

The comment on the sequence (N. Sloane, 1973a) was published on Dec 28, 2015. The publication description is: number of orbits of $\mathbb{Z}^7$ as function of the infinity norm n of the representative integer lattice point of the orbit, when the cardinality of the orbit is equal to 645 120. A000579 sequence is a(n)= 0, 0, 0, 0, 0, 0, 1, 7, 28, 84, 210, 462, 924, 1 716, 3 003, 5 005, 8 008, 12 376, 18 564, 27 132, 38 760, 54 264, 74 613, 100 947, 134 596, 177 100, 230 230, 296 010, 376 740, 475 020, 593 775, 736 281, 906 192, 1 107 568, 1 344 904, 1 623 160, 1 947 792, 2 324 784, 2 760 681, 3 262 623... The generating function using the *guessgf* command of the *gfun* package of Maplesoft (2018) is:

$$g(z; A000579) = \frac{z^6}{(1-z)^7}. \tag{X.1}$$

X.3.2 Comment to integer sequence A001477

The comment on the sequence (N. Sloane, 2010) was published on Dec 29, 2015. The publication description is: the number of orbits of $\mathbb{Z}^7$ as function of the infinity norm n of the representative lattice point of the orbit, when the cardinality of the orbit is equal to 8 960 or 168. A001477 sequence is a(n)= 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77... The generating function using the *guessgf* command of the *gfun* package of Maplesoft (2018) is:

$$g(z; A001477) = \frac{z}{(1-z)^2}. \tag{X.2}$$

X.3.3 Comment to integer sequence A002412

The comment on the sequence (N. Sloane, 1973b) was published on Dec 28, 2015. The publication description is: number of orbits of $\mathbb{Z}^7$ as function of the infinity norm $(n+1)$ of the representative integer lattice point of the orbit, when the cardinality of the orbit is equal to 40 320. A002412 sequence is a(n)= 0, 1, 7, 22, 50, 95, 161, 252, 372, 525, 715, 946, 1 222, 1 547, 1 925, 2 360, 2 856, 3 417, 4 047, 4 750, 5 530, 6 391, 7 337, 8 372, 9 500, 10 725, 12 051, 13 482, 15 022, 16 675, 18 445, 20 336, 22 352, 24 497, 26 775, 29 190, 31 746, 34 447, 37 297, 40 300... The generating function using the *guessgf* command of the *gfun* package of Maplesoft (2018) is:

$$g(z; A002412) = \frac{3z^2+z}{(1-z)^4}. \tag{X.3}$$

X.3.4 Comment to integer sequence A008586

The comment on the sequence (N. Sloane, n.d.) was published on Dec 29, 2015. The publication description is: the number of orbits of $\mathbb{Z}^7$ as function of the infinity norm n of the representative lattice point of the orbit, when the cardinality of the orbit is equal to 2 688. A008586 sequence is a(n)= 0, 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56, 60, 64, 68, 72, 76, 80, 84, 88, 92, 96, 100, 104, 108, 112, 116, 120,124, 128, 132, 136, 140, 144, 148, 152, 156, 160, 164, 168, 172, 176, 180, 184, 188, 192, 196, 200, 204, 208, 212, 216, 220, 224, 228... The generating function using the *guessgf* command of the *gfun* package of Maplesoft (2018) is:

$$g(z; A008586) = \frac{4z}{(1-z)^2}. \tag{X.4}$$

X.3.5 Comment to integer sequence A045943

The comment on the sequence (Guy, n.d.) was published on Dec 28, 2015. The publication description is: number of orbits of $\mathbb{Z}^7$ as function of the infinity norm $(n+1)$ of the representative integer lattice point of the orbit, when the cardinality of the orbit is equal to 5 376 or 17 920 or 20 160. A045943 sequence is a(n)= 0, 3, 9, 18, 30, 45, 63, 84, 108, 135, 165, 198, 234, 273, 315, 360, 408, 459, 513, 570, 630, 693, 759, 828, 900, 975, 1 053, 1 134, 1 218, 1 305, 1 395, 1 488, 1 584, 1 683, 1 785, 1 890, 1 998, 2 109, 2 223, 2 340, 2 460, 2 583, 2 709, 2 838, 2 970, 3 105, 3 243, 3 384, 3 528... The generating function using the *guessgf* command of the *gfun* package of Maplesoft (2018) is:

$$g(z; A045943) = \frac{3z}{(1-z)^3}. \tag{X.5}$$

X.3.6 Comment to integer sequence A102860

The comment on the sequence (Lajos, 2005) was published on Dec 28, 2015. The publication description is: number of orbits of $\mathbb{Z}^7$ as function of the infinity norm $(n+2)$ of the representative integer lattice point of the orbit, when the cardinality of the orbit is equal to 53 760. A102860 sequence is a(n)= 0, 16, 64, 160, 320, 560, 896, 1 344, 1 920, 2 640, 3 520, 4 576, 5 824, 7 280, 8 960, 10 880, 13 056, 15 504, 18 240, 21 280, 24 640, 28 336, 32 384, 36 800, 41 600, 46 800, 52 416, 58 464, 64 960, 71 920, 79 360, 87 296, 95 744, 104 720, 114 240, 124 320, 134 976, 146 224... The generating function using the *guessgf* command of the *gfun* package of Maplesoft (2018) is:

$$g(z; A102860) = \frac{16z}{(1-z)^4}. \tag{X.6}$$

X.3.7 Comment to integer sequence A115067

The comment on the sequence (Bagula, 2006) was published on Dec 28, 2015. The publication description is: number of orbits of $\mathbb{Z}^7$ as function of the infinity norm n of the representative integer lattice point of the orbit, when the cardinality of the orbit is equal to 6 720. A115067 sequence is a(n)= 0, 4, 11, 21, 34, 50, 69, 91, 116, 144, 175, 209, 246, 286, 329, 375, 424, 476, 531, 589, 650, 714, 781, 851, 924, 1 000, 1 079, 1 161, 1 246, 1 334, 1 425, 1 519, 1 616, 1 716, 1 819, 1 925, 2 034, 2 146, 2 261, 2 379, 2 500, 2 624, 2 751, 2 881, 3 014, 3 150, 3 289, 3 431, 3 576... The generating function using the *guessgf* command of the *gfun* package of Maplesoft (2018) is:

$$g(z; A115067) = \frac{z(4-z)}{(1-z)^3}. \tag{X.7}$$

X.3.8 Comment to integer sequence A128891

The comment on the sequence (Deléham, 2007a) was published on Dec 17, 2015. The publication description is: A128891 and A128892 are connected by the equation:

$$\sum_{n>=0} S_n - 2\pi \sum_{n>=0} V_n = 2\,, \tag{X.8}$$

in which S_n and V_n are respectively the area and volume of a n-dimensional sphere of unit radius.

A128891 sequence is a(n)= 4, 5, 9, 9, 9, 3, 2, 6, 0, 8, 9, 3, 8, 2, 8, 5, 5, 3, 6, 6, 2, 7, 4, 0, 5, 3, 1, 8, 2, 7, 0, 0, 1, 9, 7, 5, 5, 2, 8, 6, 4, 7, 9, 6, 1, 8, 3, 5, 8, 7, 4, 1, 3, 2, 9, 5, 7, 7, 7, 0, 3, 0, 2, 1, 3, 1, 8, 1, 6, 6, 0, 9, 5, 6, 6, 4, 6, 7, 8, 1, 1, 2, 3, 0, 1, 9, 8, 5, 7, 1, 7, 7, 7, 7, 3, 7, 8, 5, 2, 0, 0, 4, 3, 7, 5...

The formula for number represented by the digits of sequence A128891 is given by Jean-Francois Alcover and is:

$$\sum_{n>=0} \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}. \tag{X.9}$$

X.3.9 Comment to integer sequence A128892

The comment on the sequence (Deléham, 2007b) was published on Dec 17, 2015. The publication description is: the constant is equal to $\sum_{n>=0} S_n$, in which S_n is the area of an n-dimensional sphere of unit radius. This constant and the constant of A128891 are connected by the equation:

$$\sum_{n>=0} S_n - 2\pi \sum_{n>=0} V_n = 2\,, \tag{X.10}$$

in which V_n is the volume of an n-dimensional sphere of unit radius.

A128892 sequence is a(n)= 2, 9, 1, 0, 2, 2, 2, 8, 9, 8, 2, 4, 9, 7, 2, 9, 8, 2, 4, 4, 8, 4, 0, 8, 9, 3, 0, 0, 0, 4, 0, 6, 3, 6, 9, 2, 1, 1, 5, 3, 9, 7, 8, 3, 8, 6, 2, 3, 8, 8, 5, 0, 4, 9, 2, 6, 1, 3, 9, 9, 5, 9, 0, 3, 4, 2, 2, 1, 8, 5, 4, 8, 1, 3, 5, 4, 9, 0, 8, 1, 0, 4, 9, 5, 3, 5, 2, 1, 2, 8, 0, 3, 0, 3, 7, 6, 3, 1, 1, 3, 4, 0, 6, 4, 7... in which n runs from 3 to 101.

The formula for the constant is $2(1 + \pi e^{\pi}(1 + \mathrm{erf}(\sqrt{\pi})))$.

X.3.10 Comment to integer sequence A154286

The comment on the sequence (Luschny, 2009) was published on Dec 28, 2015. The publication description is: number of orbits of $\mathbb{Z}^7$ as function of the infinity norm $(n+4)$ of the representative integer lattice point of the orbit, when the cardinality of the orbit is equal to 107 520. A154286 sequence is $a(n) = 5, 25, 75, 175, 350, 630, 1\,050, 1\,650, 2\,475, 3\,575, 5\,005, 6\,825, 9\,100, 11\,900, 15\,300, 19\,380, 24\,225, 29\,925, 36\,575, 44\,275, 53\,130, 63\,250, 74\,750, 87\,750, 102\,375, 118\,755, 137\,025, 157\,325, 179\,800, 204\,600, 231\,880, 261\,800, 294\,525, 330\,225, 369\,075, 411\,255\ldots$

The generating function using the *guessgf* command of the *gfun* package of Maplesoft (2018) is:

$$g(z; A154286) = \frac{5}{(1-z)^5}. \tag{X.11}$$

X.3.11 Integer sequence A240934

The sequence (P. A. Chevalier, 2014a) was published on Aug 3, 2014. The publication description is: number of rectangles formed by the absolute leader classes of the seven dimensional integer lattice as function of the infinity norm n, in which the rectangles have one common lattice point being the origin of the seven dimensional integer lattice.

A240934 sequence is $a(n) = 120, 7\,196, 162\,554, 1\,341\,957, 9\,255\,603, 40\,532\,530, 168\,302\,117, 523\,421\,602, 1\,637\,895\,896, 4\,129\,547\,423\ldots$

A closed form for the generating function is presently unknown and generates a FAIL response when using the *guessgf* command of the *gfun* package in Maplesoft (2018).

X.3.12 Integer sequence A247557

The sequence (P. A. Chevalier, 2014b) was published on Oct 24, 2014. The publication description is: number of rectangles formed by the absolute leader classes of the seven-dimensional integer lattice as a function of the infinity norm n and having a unique perimeter, in which the rectangles have one common lattice point being the origin of the seven-dimensional integer lattice.

A247557 sequence is $a(n) = 1, 7, 26, 79, 182, 333, 693, 1\,180, 1\,999, 3\,247\ldots$

A closed form for the generating function is presently unknown and generates a FAIL response when using the *guessgf* command of the *gfun* package in Maplesoft (2018).

X.3.13 Integer sequence A266387

The sequence (P. A. Chevalier, 2015b) was published on Dec 28, 2015. The publication description is: number of orbits of $\mathbb{Z}^7$ as function of the infinity norm n of the representative lattice point of the orbit, when the cardinality of the orbit is equal to 322 560.
A266387 sequence is $a(n) = 0, 0, 0, 0, 0, 7, 42, 147, 392, 882, 1\,764, 3\,234, 5\,544, 9\,009, 14\,014, 21\,021, 30\,576, 43\,316, 59\,976, 81\,396, 108\,528, 142\,443, 184\,338, 235\,543, 297\,528, 371\,910, 460\,460, 565\,110, 687\,960, 831\,285, 997\,542, 1\,189\,377, 1\,409\,632, 1\,661\,352, 1\,947\,792, 2\,272\,424, 2\,638\,944, 3\,051\,279\ldots$
The generating function using the *guessgf* command of the *gfun* package of Maplesoft (2018) is:

$$g(z; A266387) = \frac{7z^5}{(1-z)^6}. \tag{X.12}$$

X.3.14 Integer sequence A266395

The sequence (P. A. Chevalier, 2015c) was published on Dec 29, 2015. The publication description is: number of orbits of $\mathbb{Z}^7$ as function of the infinity norm n of the representative lattice point of the orbit, when the cardinality of the orbit is equal to 161 280.
A266395 sequence is $a(n) = 0, 0, 0, 0, 15, 75, 225, 525, 1\,050, 1\,890, 3\,150, 4\,950, 7\,425, 10\,725, 15\,015, 20\,475, 27\,300, 35\,700, 45\,900, 58\,140, 72\,675, 89\,775, 109\,725, 132\,825, 159\,390, 189\,750, 224\,250, 263\,250, 307\,125, 356\,265, 411\,075, 471\,975, 539\,400, 613\,800, 695\,640, 785\,400, 883\,575, 990\,675, 1\,107\,225\ldots$
The generating function using the *guessgf* command of the *gfun* package of Maplesoft (2018) is:

$$g(z; A266395) = \frac{15z^4}{(1-z)^5}. \tag{X.13}$$

X.3.15 Integer sequence A266396

The sequence (P. A. Chevalier, 2015a) was published on Dec 29, 2015. The publication description is: number of orbits of $\mathbb{Z}^7$ as function of the infinity norm n of the representative lattice point of the orbit, when the cardinality of the orbit is equal to 80 640.
A266396 sequence is $a(n) = 0, 0, 0, 10, 41, 105, 215, 385, 630, 966, 1\,410, 1\,980, 2\,695, 3\,575, 4\,641, 5\,915, 7\,420, 9\,180, 11\,220, 13\,566, 16\,245, 19\,285, 22\,715,$

26 565, 30 866, 35 650, 40 950, 46 800, 53 235, 60 291, 68 005, 76 415, 85 560, 95 480, 106 216, 117 810, 130 305, 143 745, 158 175, 173 641, 190 190 . . .
The generating function using the *guessgf* command of the *gfun* package of Maplesoft (2018) is:

$$g(z; A266396) = \frac{z^3(9z-10)}{(z-1)^5}. \tag{X.14}$$

X.3.16 Integer sequence A266397

The sequence (P. A. Chevalier, 2015d) was published on Dec 29, 2015. The publication description is: number of orbits of $\mathbb{Z}^7$ as function of the infinity norm n of the representative lattice point of the orbit, when the cardinality of the orbit is equal to 26 880.
A266397 sequence is a(n)= 0, 0, 9, 31, 70, 130, 215, 329, 476, 660, 885, 1 155, 1 474, 1 846, 2 275, 2 765, 3 320, 3 944, 4 641, 5 415, 6 270, 7 210, 8 239, 9 361, 10 580, 11 900, 13 325, 14 859, 16 506, 18 270, 20 155, 22 165, 24 304, 26 576, 28 985, 31 535, 34 230, 37 074, 40 071, 43 225, 46 540, 50 020, 53 669. . . The generating function using the *guessgf* command of the *gfun* package of Maplesoft (2018) is:

$$g(z; A266397) = \frac{-5z^3+9z^2}{(1-z)^4}. \tag{X.15}$$

X.3.17 Integer sequence A266398

The sequence (P. A. Chevalier, 2015e) was published on Dec 29, 2015. The publication description is: number of orbits of $\mathbb{Z}^7$ as function of the infinity norm n of the representative lattice point of the orbit, when the cardinality of the orbit is equal to 13440.
A266398 sequence is a(n)= 0, 0, 12, 37, 76, 130, 200, 287, 392, 516, 660, 825, 1 012, 1 222, 1 456, 1 715, 2 000, 2 312, 2 652, 3 021, 3 420, 3 850, 4 312, 4 807, 5 336, 5 900, 6 500, 7 137, 7 812, 8 526, 9 280, 10 075, 10 912, 11 792, 12 716, 13 685, 14 700, 15 762, 16 872, 18 031, 19 240, 20 500, 21 812, 23 177. . . The generating function using the *guessgf* command of the *gfun* package of Maplesoft (2018) is:

$$g(z; A266398) = \frac{-11z^3+12z^2}{(1-z)^4}. \tag{X.16}$$

X.3.18 Integer sequence A270950

The sequence (P. A. Chevalier, 2016) was published on 26 March 2016 and corrected/extended on 24 June 2022. The publication description is: number of distinct cardinalities of orbits of the n-dimensional integer lattice. A270950 sequence is a(n)= 1, 1, 2, 5, 9, 12, 20, 29, 40, 53, 76, 99, 132, 172, 216, 270, 341, 424, 532, 660, 810, 983, 1 210, 1 446, 1 750, 2 111, 2 508, 2 975, 3 569, 4 197, 4 948, 5 807, 6 817, 7 963, 9 351, 10 863, 12 604, 14 598, 16 892, 19 439, 22 472, 25 780, 29 588, 33 892, 38 800, 44 206, 50 463, 57 297, 65 086, 73 919, 83 842, 94 510...

A Maple procedure to calculate the values of the sequence is given in Appendix U. For $n = 0$ the $a(0) = 1$. For $n = 3$ we have the following distinct cardinalities of the orbits 6, 8, 12, 24, 48 and thus $a(3) = 5$. For $n = 4$ we have the distinct cardinalities of the orbits 8, 16, 24, 32, 48, 64, 96, 192, 384 and thus $a(4) = 9$. For $n = 5$ we have the distinct cardinalities of the orbits 10, 32, 40, 160, 240, 320, 480, 640, 960, 1 920, 3 840 and thus $a(5) = 12$.

A closed form for the generating function is presently unknown and generates a FAIL response when using the *guessgf* command of the *gfun* package in Maplesoft (2020).

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