



Color gamut volume and the maximum number of mutually discernible colors based on a Riemannian metric

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Abstract: For the calculation of the color gamut volume and the maximum number of mutually discernible colors, an algorithm based on a Riemannian metric and the densest packing of spheres is proposed. With this algorithm, the color gamut volume was calculated for the conditions of experiments reported in literature. Good agreement was found with the experimental findings of the color gamut volume as a function of the peak luminance. Using the new algorithm, the color gamut volume and the maximum number of mutually discernible colors was calculated for various sets of primary colors corresponding to display standards and various dynamic ranges. Comparisons were made with state-of-the-art methods which are based on the Euclidean metric in approximately uniform color spaces and a simple cubic lattice. It was found that the state-of-the-art methods underestimate the maximum number of mutually discernible colors. However, the relative differences decrease as the primary colors are more saturated. Based on the new algorithm the maximum number of mutually discernible colors was calculated for a range of peak retinal illuminance levels and various sets of primary colors. We found that, for a given set of primary colors, the maximum number of mutually discernible colors is proportional to the logarithm of the ratio of the peak retinal illuminance level and a fitting parameter.

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1. Introduction

The number of discernible colors has been investigated by many researchers, and very different values have been reported [1–3]. In general there are two sets of parameters that determine the maximum number of discernible colors: (i) the spectral and luminance range that can be produced by a physical process and (ii) the color difference threshold for the human eye to be able to separate two similar stimuli. The physical process is either emission of light (self-luminous colors that can be produced) or reflection of light by illuminated objects (surface or object colors). The space of colors is a 3-dimensional space. The set of all the colors that can be produced by a specific physical process is represented by a specific color solid in a color space [4]. The dissimilarity between two similar colors corresponds to a distance in color space. A just noticeable difference between two colors corresponds to the threshold distance. Because the color space is non-Euclidean, a Riemannian metric is required to measure the distance between two similar colors [5–7]. In this study we consider the maximum number of mutually discernible colors that can be positioned within a color solid. The proliferation of new display technologies with high dynamic range (HDR) and wide color gamut (WCG), e.g. light-emitting diode (LED) [8,9], organic light-emitting diode (OLED) [10,11], quantum-dot (QD) [12,13] and RGB-laser [14,15], drives the need to compare their performance, and an important quantity is the maximum number of displayable colors, with absolute or relative values. Previously, investigators estimated the color gamut volume (or simply 'color volume') and the maximum number of discernible colors of display standards using Euclidean color spaces [16–21]. In this paper the maximum

number of mutually discernible colors of a set of display standards is calculated based on a Riemannian color space with the line element that was introduced in Ref. [22] (abbreviated as new LE). The next section gives a brief overview of the state-of-the-art approach to determine the color volume and the maximum number of mutually discernible colors. Section 3 gives the outline of the Riemannian color space relevant for the calculations, and the lattice of points to represent mutually discernible colors in the space of colors. An algorithm to determine the color volume is presented in section 4. In Section 5 the effect of the peak luminance and various sets of primary colors on the maximum number of mutually discernible colors is presented and comparisons with the results of other investigators are made.

2. Related work on display color volumes

Wen (2005) [23] calculated the color volume of displays in the CIE 1976 ($L^*a^*b^*$) color space [24] (referred to as CIELAB). The CIELAB color space was used because it is perceptually more uniform than the CIE1931 [25] color space. However, the uniformity of the CIELAB color space is still poor. This is illustrated in section 3.5 with the MacAdam threshold ellipses [26,27] in the (a^*, b^*) plane. CIELAB requires normalisation to a reference white. For displays the peak white is chosen as the reference white. With this normalization the maximum lightness is equal to 100 [28] (p.82, 605). Consequently, displays with equal chromaticities of the primary colors but different peak white luminance levels have, according to CIELAB, the same color volume. However, it is known that the absolute luminance level has an important impact on the color appearance, e.g. the perceived colorfulness increases with the absolute luminance level of the stimulus, known as the Hunt effect [29,30], and the perceived contrast increases with the absolute luminance level of the stimulus, known as the Stevens effect [31]. For color volume calculations of HDR/WCG displays, Masaoka (2017) [32] proposed to use the CIELAB color space at luminance levels above the reference white luminance. According to this approach, it was shown that the calculated color volume increases linearly with the peak luminance in the range of 100 to 1000 cd m^{-2} (reference white was D65 [25](p.144, 177) at 100 cd m^{-2}). Using this extended CIELAB color space Jiang et al. (2018) [33] reported for HDR/WCG displays also a linear increase of the calculated color volume with the peak luminance in the range of 500 to 4000 cd m^{-2} (reference white was D65 at 200 cd m^{-2}).

Baek et al. (2018) [34] carried out psychophysical experiments to investigate the effect of the display-parameters peak luminance, black level and color gamut on the perceived color volume. The paired comparison method was used. The observers were asked to evaluate the 'richness' of two static side-by-side images shown on liquid crystal display (LCD) monitors. Two experimental set ups were used. The first set up was standard dynamic range (SDR), using a 27-inch LCD monitor with 1920×1080 pixels. The second set up was HDR, using a 65-inch HDR LCD TV with 3840×2160 pixels. SDR images (650×365 pixels) with different content and settings (peak luminance, black level and color gamut) were evaluated side-by-side by 20 observers. This was also done for the HDR images (1280×720 pixels) by 9 observers. They found that the perceptual color volume is affected by the peak luminance, black level and color gamut. The relationship between the perceptual color volume and calculated color volumes according to the CIELAB, $Q_aM b_M$ (CIECAM02) [35] and ICtCp [36–38] color spaces was investigated. For the SDR and HDR images they reported a strong correlation between the perceptual color volume and the calculated color volumes according to the color spaces $Q_aM b_M$ and ICtCp. For CIELAB the correlation was low.

Jiang et al. (2020) [39,40] (pp. 68-80) (abbreviated JI20) conducted psychophysical experiments to investigate the effect of the peak luminance of a display on the perceived color volume. Static images were shown on a Sony BVM-X300 monitor (30-inch OLED, 4096×2160 pixels, effective picture size: $663.5 \text{ mm} \times 349.9 \text{ mm}$, supports DCI-P3) at various peak luminance levels. The paired comparison method was used. The observers were asked to compare the

'colorfulness and detail' of pairs of images shown sequentially. A first image was shown during 15 sec, followed by 10 sec mid gray image and during the next 15 sec a second image was shown. The more colorful and detailed image was selected. Nine HDR images with different content were used, 54 pairs were shown to each observer, 21 observers participated in the experiment. The 9 HDR images included 3 natural images and 6 images with a calibrated diffuse white level. For the first experiment the calibrated diffuse white level was set at a constant level of 200 cd m⁻². For the second experiment the diffuse white level was set at 20% of the peak white level. For both experiments they found a good linear relationship between the perceptual color volume and the log scale of the peak luminance.

As stated in Refs. [41,42], the color appearance models CIECAM02 and CIECAM16 were developed under SDR conditions, while the ICtCp color space, with its perceptual quantizer function, was developed for HDR and WCG content. Dolby Laboratories (2018) [16] (abbreviated as DO18) calculated the number of discernible colors in the approximately uniform ICtCp space for Rec.2020 [43], DCI-P3 [44] and Rec.709 [45]. Poynton and LeHoty (2020) [19,20] (abbreviated as PL20) calculated, also in the ICtCp space, the number of discernible colors for Rec.1886 and P3-D65. (The chromaticities of the primary colors and reference white of Rec.1886 are identical to those of Rec.709. P3-D65 has the same primary colors as DCI-P3 but the reference white is D65). Different black levels and peak white levels were used in DO18 and PL20. The results of DO18 and PL20 are very similar. They found that the number of discernible colors is higher for (i) wider color gamuts at a given peak luminance level, and (ii) higher peak luminance levels for a given color gamut. Notice that the ICtCp color space deviates substantially from a uniform color space. This is illustrated in section 3.5 with the MacAdam threshold ellipses [26,27] in the (0.5 C_t , C_p) plane. Therefore, it seems appropriate to introduce a new method to estimate the number of discernible colors based on a Riemannian metric. [In the following sections of the paper the n -dimensional Euclidean space will be denoted as $\mathbb{R}^n$.]

3. Riemannian color space

3.1. Line element

The color volume calculations of the state-of-the-art methods, as described in the previous section, assume Euclidean color spaces. However, the 3-dimensional space of colors is not Euclidean [5–7]. A Riemannian metric is required to calculate metric properties such as length and volume [46] (pp.193-196). In general, a Riemannian manifold has curvature, which indicates how much the local geometric properties of the Riemannian manifold differ from the Euclidean space [47] (pp.164-174). In this section mathematical elements are provided to calculate the volume and number of discernible colors defined by a color solid and a Riemannian metric. Consider a rectangular Cartesian coordinate system (u_1, u_2, u_3) that represents a color coordinate system. The metric coefficients $h_{ij}(u_1, u_2, u_3)$ of the Riemannian metric $[h_{ij}]$ ($i, j = 1, 2, 3$) are smooth functions of the coordinates (u_1, u_2, u_3). The line element (element of length) $d\sigma$ between two neighboring coordinate points (u_1, u_2, u_3) and ($u_1 + du_1, u_2 + du_2, u_3 + du_3$) is given by the differential quadratic form:

$$(d\sigma)^2 = \begin{bmatrix} du_1 & du_2 & du_3 \end{bmatrix} [h_{ij}] \begin{bmatrix} du_1 & du_2 & du_3 \end{bmatrix}^T. \quad (1)$$

The Riemannian metric tensor $[h_{ij}]$ is scaled in such a way that a just noticeable difference for the human eye corresponds to a distance $d\sigma = 1$. However, a just noticeable difference and the actual magnitude of the threshold distance $d\sigma$ depend on the specific threshold criteria and the measurement conditions [22,48]. If one of the coordinates is kept constant then the line element reduces to the 2-dimensional case with e.g. only the coordinates u_1 and u_2 . Compared to other color difference metrics, a better correspondence was found between the experimental threshold ellipsoids and the Riemannian metric of the new LE. In Ref. [22] Table 6 we reported the

following dissimilarities (averaged over 4 data sets) between measured and calculated threshold ellipses/ellipsoids for CIELAB, CIEDE2000, ICtCp and the new LE: 0.529, 0.483, 0.422 and 0.352 respectively. In the following sections, the new LE [22] will be applied for the calculation of the maximum number of mutually discernible colors.

3.2. Color area and maximum number of mutually discernible colors in a 2-dimensional space

Consider a 2-dimensional rectangular Cartesian coordinate system (u_1, u_2) that represents e.g. the chromaticity plane. The area of an element is given by:

$$dA = \sqrt{h(u_1, u_2)} du_1 du_2, \quad (2)$$

with $h(u_1, u_2) = \det[h_{ij}]$ and $[h_{ij}]$ ($i, j = 1, 2$) the metric tensor [46] (pp.193-196). The function $\sqrt{h(u_1, u_2)}$ is called the 'scalar density' [46] (p.123). The area of a surface defined by the metric tensor $[h_{ij}]$ and a domain $\mathcal{B} = \{(u_1, u_2) : (u_1, u_2) \in \text{color solid}\}$ is given by:

$$A = \int_{\mathcal{B}} \sqrt{h(u_1, u_2)} du_1 du_2. \quad (3)$$

A coordinate transformation will in general modify the local scalar density $\sqrt{h(u_1, u_2)}$, however the integral in Eq. (3), integrated over the transformed domain, remains unchanged [5]. On the true color surface [5], the colors that have a just noticeable difference with a reference color are located on a circle with radius equal to the threshold distance $d\sigma$. To find the maximum number of mutually discernible colors, we consider first the densest packing of non-overlapping circles with radii equal to $\frac{d\sigma}{2}$ in $\mathbb{R}^2$. The packing density, denoted as p , is the fraction of the area filled by the circles. Therefore, in this case the packing density is given by:

$$p = \frac{\pi N (d\sigma)^2}{4A}, \quad (4)$$

with N the number of circles in the area A . It is known that the highest packing density in $\mathbb{R}^2$ is equal to $\frac{\pi}{\sqrt{12}} \approx 0.90690$, which can be realized with the hexagonal lattice packing [49]. This is shown in Fig. 1, the circles with radii equal to $\frac{d\sigma}{2}$ are centered in a hexagonal lattice with nearest-neighbor distance equal to $d\sigma$. From Eq. (4) it follows that the maximum number of mutually discernible colors on a plane of area A in $\mathbb{R}^2$ is given by:

$$N = \kappa \frac{A}{(d\sigma)^2}, \quad \kappa = \frac{2}{\sqrt{3}} \approx 1.1547. \quad (5)$$

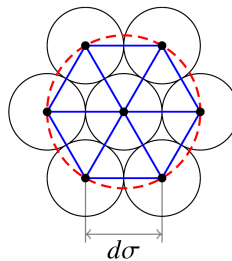


Fig. 1. Hexagonal lattice packing

However, in general the local geometric properties of surfaces with a Riemannian metric are different from the geometric properties of $\mathbb{R}^2$, because in general these surfaces are curved. This

might have an impact on the packing density. In Appendix A we give a motivation for a packing density equal to $\frac{\pi}{\sqrt{12}} \approx 0.90690$ for 2-dimensional surfaces with the Riemannian metric of the new LE.

3.3. Color volume and maximum number of mutually discernible colors in a 3-dimensional space

Consider a 3-dimensional rectangular Cartesian coordinate system (u_1, u_2, u_3) that represents the color space. The element of volume is given by:

$$dV = \sqrt{h(u_1, u_2, u_3)} du_1 du_2 du_3, \quad (6)$$

with $h(u_1, u_2, u_3) = \det[h_{ij}]$, $[h_{ij}]$ the metric tensor and $\sqrt{h(u_1, u_2, u_3)}$ the scalar density [46] (pp.123,193-196). The set of colors that can be produced by a specific physical process is represented by a specific color solid in the color space. The volume defined by the quadratic differential form Eq. (1) and domain $C = \{(u_1, u_2, u_3) : (u_1, u_2, u_3) \in \text{color solid}\}$ is given by:

$$V = \int_C \sqrt{h(u_1, u_2, u_3)} du_1 du_2 du_3. \quad (7)$$

A coordinate transformation will in general modify the local scalar density, while the volume integral in Eq. (7), integrated over the transformed color solid, remains unchanged [5]. We assume that the metric coefficients do not change appreciably over a distance corresponding to a just noticeable color difference [27]. In the true color space [5], the colors that have a just noticeable color difference with a reference color are located on a spherical surface with radius equal to the threshold distance $d\sigma$. To find the maximum number of mutually discernible colors in the true color space, we first consider the densest packing of spheres with radii equal to $\frac{d\sigma}{2}$ in $\mathbb{R}^3$. The packing density p is the fraction of the volume filled by the spheres. Therefore, in this case the packing density is given by:

$$p = \frac{\pi N (d\sigma)^3}{6 V}, \quad (8)$$

with N the number of spheres in the volume V . It is known that the highest packing density in $\mathbb{R}^3$ is equal to $\frac{\pi}{\sqrt{18}} \approx 0.74048$, which can be realized with the face-centered cubic (FCC) lattice packing [49]. Therefore, the spheres with radii $\frac{d\sigma}{2}$ are centered in a FCC lattice with nearest neighbor distance equal to $d\sigma$. From Eq. (8) it follows that the maximum number of mutually discernible colors in a volume V in $\mathbb{R}^3$ is given by:

$$N = \rho \frac{V}{(d\sigma)^3}, \quad \rho = \sqrt{2} \approx 1.4142. \quad (9)$$

The local geometric properties of the 3-dimensional color manifold with the Riemannian metric of the new LE are in general different from the geometric properties of $\mathbb{R}^3$. This might have an impact on the packing density. In Appendix A we give a motivation for a packing density equal to $\frac{\pi}{\sqrt{18}} \approx 0.74048$ for 3-dimensional manifolds with the Riemannian metric of the new LE. The impact of the packing density error on the uncertainty in the maximum number of mutually discernible colors is addressed in section 5.

Notice that: (i) The hexagonal close-packed (HCP) lattice results in an equal maximum number of mutually discernible colors (cf. Appendix B). (ii) Other investigators have used a simple cubic (SC) lattice ($\rho = 1$), because they are considering one discernible color per cube with edge length equal to one just noticeable color difference [2,17,28] (pp. 89-90) (cf. Appendix B). This results in an underestimation of the maximum number of mutually discernible colors by a factor 0.707.

3.4. Metric tensor

The Riemannian color space with the new LE [22] will be applied for the color volume calculations. The colors are specified in the color space with coordinates the retinal illuminance Y and the MacLeod-Boynton chromaticities l and s [50]. Therefore, the general coordinates u_1 , u_2 and u_3 are respectively replaced by Y , l and s . The new LE in the MacLeod-Boynton differential space is given by:

$$(d\sigma)^2 = \begin{bmatrix} dY & dl & ds \end{bmatrix} H_D \begin{bmatrix} dY \\ dl \\ ds \end{bmatrix}, \quad H_D = \begin{bmatrix} h_{11} & h_{12} & 0 \\ h_{12} & h_{22} & h_{23} \\ 0 & h_{23} & h_{33} \end{bmatrix} = \begin{bmatrix} \frac{g_{11}}{Y^2} & \frac{g_{12}}{Yl} & 0 \\ \frac{g_{12}}{Yl} & \frac{g_{22}}{l^2} & \frac{g_{23}}{ls} \\ 0 & \frac{g_{23}}{ls} & \frac{g_{33}}{s^2} \end{bmatrix}, \quad (10)$$

with H_D the metric tensor and

$$\begin{aligned} g_{11} &= \psi_A^{-2}(Y, Y_a), & g_{22} &= \psi_T^{-2}(l, l_a, Y), & g_{33} &= \psi_D^{-2}(s, s_a, Y), \\ g_{12} &= \sqrt{g_{11} g_{22}} k_{12} (l - l_a)^2, & g_{23} &= \sqrt{g_{22} g_{33}} \frac{k_{23} s}{\sqrt{1 + k_{23}^2 s^2}}. \end{aligned} \quad (11)$$

The functions $\psi_A(Y, Y_a)$, $\psi_T(l, l_a, Y)$, $\psi_D(s, s_a, Y)$ and the generic values of the model parameters are given in Ref. [22]. Y_a is the retinal illuminance of the adapting background and l_a and s_a are respectively the MacLeod-Boynton chromaticities l and s of the adapting background. For a given color solid C and $h(Y, l, s) = \det H_D$ the volume in the Riemannian color space can be calculated with Eq. (7).

3.5. Scalar density

In a Riemannian geometry the scalar density $\sqrt{h}$ varies as a function of the color coordinates, whereas in the Euclidean color spaces CIELAB [24] and ICtCp [36] the scalar density is assumed

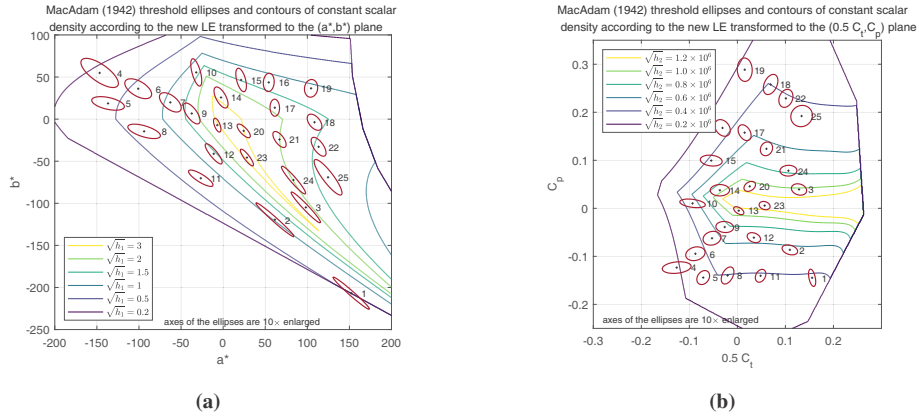


Fig. 2. The 25 MacAdam threshold ellipses [26,27]: **(a)** in the (a^*, b^*) plane of the CIELAB color space and **(b)** in the $(0.5 C_t, C_p)$ plane of the ICtCp color space. In an Euclidean color space the threshold loci are circles with equal radii; these figures show that both color spaces deviate substantially from an Euclidean color space. The scalar density calculated according the new LE (with $d\sigma = 1.55$ and equal-energy white (EEW) adapting background) in the (a^*, b^*) plane (constant $L^* = 70$) and the $(0.5 C_t, C_p)$ plane (constant $I = 0.436$) are denoted as $\sqrt{h_1}$ and $\sqrt{h_2}$, respectively. **(a)** Contours of constant $\sqrt{h_1}$ in the (a^*, b^*) plane. **(b)** Contours of constant $\sqrt{h_2}$ in the $(0.5 C_t, C_p)$ plane.

to be constant, being equal to respectively 1 and $720^2 \approx 0.52 \cdot 10^6$. The variation of the scalar density has a significant impact on the resulting volume V or area A of color solids. To illustrate this we quantified the degree of change of the scalar density of the new LE in the (a^*, b^*) and the $(0.5 C_t, C_p)$ coordinate systems. This is done by means of the Riemannian metric of the new LE [Eq. (10)] and the appropriate color space transformations. The results are shown in Fig. 2(a) and 2(b) as contours of constant value of respectively $\sqrt{h(a^*, b^*)}$ (denoted as $\sqrt{h_1}$) and $\sqrt{h(0.5 C_t, C_p)}$ (denoted as $\sqrt{h_2}$), together with the MacAdam threshold ellipses [26,27]. We find that in both chromaticity planes there is a substantial variation of the scalar density, according to the new LE. The scalar density varies by a factor of about 15 and 6 in respectively the (a^*, b^*) and the $(0.5 C_t, C_p)$ plane. Notice that the scalar density is proportional to the reciprocal of the threshold ellipse's area ($d\sigma = 1$).

4. Numerical calculation of the volume of a color solid

This section explains the numerical calculation of a color solid's volume V in a Riemannian space with metric tensor H_D [Eq. (10)] and scalar density $\sqrt{h}$, with $h = \det H_D$. First an additive color mixing model is formulated to determine the boundaries of the color solid in the color space, defined by a red, green and blue light beam of a generic color reproduction system. Using the additive color mixing model, each color coordinate within the color solid can be determined, which allows to calculate numerically the volume V of the color solid. From the calculated color volume the maximum number of discernible colors can be determined.

4.1. Additive color mixing model

Consider an additive color reproduction system using three light beams R , G and B with CIE1931 chromaticities (x_r, y_r) , (x_g, y_g) and (x_b, y_b) , respectively. The colors of the light beams R , G and B are called the primary colors. The additive color reproduction system makes a uniform light patch. The luminance of each beam is variable from zero to a maximum level with signals represented by the variables $c_r \in [0, 1]$, $c_g \in [0, 1]$ and $c_b \in [0, 1]$. For $c_r = c_g = c_b = 1$ the mixed beam has the peak retinal illuminance Y_p and the chromaticities (x_n, y_n) of a neutral color. The tristimulus values of the mixed light are denoted as X_{mix} , Y_{mix} and Z_{mix} . From the additivity of the tristimulus values it follows that:

$$\begin{bmatrix} X_{mix} \\ Y_{mix} \\ Z_{mix} \end{bmatrix} = \begin{bmatrix} X_r \\ Y_r \\ Z_r \end{bmatrix} + \begin{bmatrix} X_g \\ Y_g \\ Z_g \end{bmatrix} + \begin{bmatrix} X_b \\ Y_b \\ Z_b \end{bmatrix} = \begin{bmatrix} X_{r0} \\ Y_p \\ Z_{r0} \end{bmatrix} c_r k_r + \begin{bmatrix} X_{g0} \\ Y_p \\ Z_{g0} \end{bmatrix} c_g k_g + \begin{bmatrix} X_{b0} \\ Y_p \\ Z_{b0} \end{bmatrix} c_b k_b = \mathbf{M} \mathbf{K} \begin{bmatrix} c_r \\ c_g \\ c_b \end{bmatrix} Y_p, \quad (12)$$

with:

$$\mathbf{M} = \begin{bmatrix} \frac{x_r}{y_r} & \frac{x_g}{y_g} & \frac{x_b}{y_b} \\ 1 & 1 & 1 \\ \frac{z_r}{y_r} & \frac{z_g}{y_g} & \frac{z_b}{y_b} \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_r & 0 & 0 \\ 0 & k_g & 0 \\ 0 & 0 & k_b \end{bmatrix},$$

where k_r , k_g and k_b are the relative luminances of the primary colors, which must be determined. For $c_r = c_g = c_b = 1$ we have:

$$\begin{bmatrix} X_{mix} \\ Y_{mix} \\ Z_{mix} \end{bmatrix} = \begin{bmatrix} \frac{x_n}{y_n} \\ 1 \\ \frac{z_n}{y_n} \end{bmatrix} Y_p = \mathbf{M} \begin{bmatrix} k_r \\ k_g \\ k_b \end{bmatrix} Y_p \quad \text{and} \quad \begin{bmatrix} k_r \\ k_g \\ k_b \end{bmatrix} = \mathbf{M}^{-1} \begin{bmatrix} \frac{x_n}{y_n} \\ 1 \\ \frac{z_n}{y_n} \end{bmatrix}. \quad (13)$$

Notice that $k_r + k_g + k_b = 1$. For the MacLeod-Boynton chromaticities we obtain:

$$\begin{bmatrix} l_{mix} \\ m_{mix} \\ s_{mix} \end{bmatrix} = \mathbf{A} \mathbf{M} \mathbf{K} \begin{bmatrix} c_r \\ c_g \\ c_b \end{bmatrix} \frac{Y_p}{Y_{mix}}, \quad \mathbf{A} = \begin{bmatrix} 0.15514 & 0.54312 & -0.03286 \\ -0.15514 & 0.45684 & 0.03286 \\ 0 & 0 & 0.01608 \end{bmatrix}. \quad (14)$$

The appearance of colors depends on the adapting background. Colors with the same appearance under different adapting conditions can be predicted by a chromatic adaptation transform (CAT). In this model we transform colors observed under different adapting backgrounds to the equal energy white (EEW) background. For this we use the von Kries transformation [51] (pp.168-171) [52] (pp.21-23). Our motivation to do this is based on the analysis in [22] of the impact of the von Kries transformation on the predictability of measured threshold ellipses/ellipsoids, where we found higher similarities when using the von Kries transformation. The general von Kries CAT to transform colors under viewing condition a (e.g. adaptation to D65) to viewing condition b (e.g. adaptation to EEW) is described in Appendix C. From Eqs. (24) it follows that:

$$\begin{bmatrix} X_{mix}^b \\ Y_{mix}^b \\ Z_{mix}^b \end{bmatrix} = \mathbf{T} \mathbf{M} \mathbf{K} \begin{bmatrix} c_r \\ c_g \\ c_b \end{bmatrix} Y_p \quad \text{and} \quad \begin{bmatrix} l_{mix}^b \\ m_{mix}^b \\ s_{mix}^b \end{bmatrix} = \mathbf{B} \mathbf{M} \mathbf{K} \begin{bmatrix} c_r \\ c_g \\ c_b \end{bmatrix} \frac{Y_p}{Y_{mix}^b}. \quad (15)$$

4.2. Volume calculation

A set of triplets (Y, l, s) within a color solid C is numerically determined as follows. First a domain $\mathcal{D} = \{(Y, l, s) : Y \in [Y_A, Y_B], l \in [l_A, l_B], s \in [s_A, s_B]\}$ is determined such that $C \subset \mathcal{D}$. Then a sampling grid in $\mathcal{D}$ is defined as $\mathcal{S} = \{(Y_i, l_j, s_k) : Y_i = Y_A + i \delta Y, l_j = l_A + j \delta l, s_k = s_A + k \delta s\}$ with: $Y_A \leq Y_i \leq Y_B, l_A \leq l_j \leq l_B, s_A \leq s_k \leq s_B$, the indices $i, j, k \in \{0, 1, 2, 3, \dots\}$ and the sampling increments $\delta l, \delta s, \delta Y$. In the next step an algorithm is applied to select the coordinates of the sampling grid within the color solid C . This algorithm is based on the additive color mixing model explained in section 4.1. From Eq. (15) it follows that the mapping from the MacLeod-Boynton chromaticities to the signals c_r, c_g and c_b of the color mixer is given by:

$$\begin{bmatrix} c_r \\ c_g \\ c_b \end{bmatrix} = \mathbf{K}^{-1} \mathbf{M}^{-1} \mathbf{B}^{-1} \begin{bmatrix} l_{mix}^b \\ 1 - l_{mix}^b \\ s_{mix}^b \end{bmatrix} \frac{Y_{mix}^b}{Y_p}. \quad (16)$$

Equation (16) is evaluated for each triplet $(Y_{mix}^b, l_{mix}^b, s_{mix}^b) \in \mathcal{S}$. The triplets representing physical colors and for which $c_r \in [0, 1]$ and $c_g \in [0, 1]$ and $c_b \in [0, 1]$, are grid points within the color solid C . For sufficiently small sampling increments, the volume of the color solid C [Eq. (7)] is approximated by:

$$V \approx \sum_C \sqrt{h} \delta Y \delta l \delta s, \quad h = \det \mathbf{H}_D, \quad (17)$$

with the summation over all triplets (Y_i, l_j, s_k) within the color solid and $\mathbf{H}_D$ the metric tensor [Eq. (10)]. The impact of the finite cell size $\delta Y \times \delta l \times \delta s$ on the uncertainty in the color volume V is addressed in section 5.

4.3. Visualization of the boundaries of color solids

The color solids in the (Y, l, s) space corresponding to the primary colors and reference white of the display standards Rec.2020, DCI-P3, Rec.709 and SMPTE-C [53] (Appendix D, Table 1) were determined as described above. Adaptation to the reference white of the display standards was assumed (adaptation condition *a*). The corresponding colors of the primary colors for adaptation to EEW (adaptation condition *b*) were calculated, as explained in Appendix C. Figure 3 shows the boundaries of the calculated color solids at constant retinal illuminance levels Y in the (l, s) chromaticity plane for Y_p equal to 3000 td (under standard conditions this corresponds to ≈ 615 cd m^{-2}). Figure 4 shows the color solid in the (Y, l, s) color space for Rec.709 and Y_p equal to 3000 td, represented as a set of slices at retinal illuminance levels between 100 td (≈ 10 cd m^{-2}) and 2750 td (≈ 555 cd m^{-2}) [54] (pp.94-108).

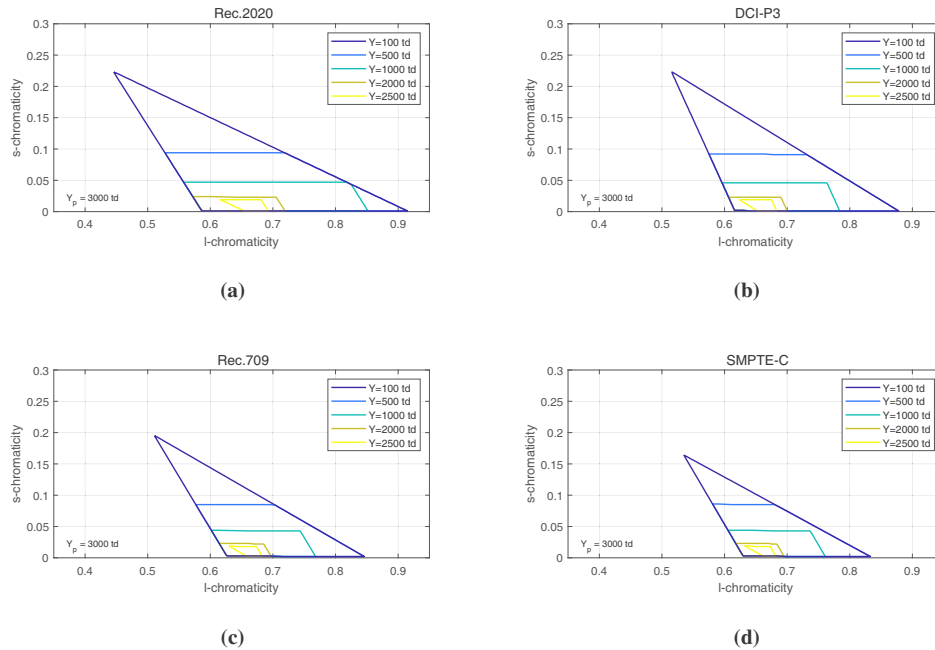


Fig. 3. Boundaries of the color solids at constant retinal illuminance levels Y for Rec.2020, DCI-P4, Rec.709 and SMPTE-C in the MacLeod-Boynton (l, s) chromaticity plane. The peak retinal illuminance Y_p is equal to 3000 td.

4.4. Range of luminance levels

An attention point, in particular for HDR displays, is the range of luminance levels. Foveal small field (2° field of view) color matching obeys the proportionality law of Grassmann [25] (p.118) in a wide range of retinal illuminance levels, however this range is limited. The lower bound is at about 1 td [55] (pp.116-117), which is a mesopic light level. The upper bound, is at a retinal illuminance level of about 8000 td [56,57], which is caused by a significant bleaching of cone pigments [25] (pp.374-379, 619-633) [55] (pp.116-117). This means that the color matching functions are only independent of the retinal illuminance in the range of 1 td to 8000 td. The lower and upper bounds correspond with luminance levels of respectively ≈ 0.03 cd m^{-2} and ≈ 1870 cd m^{-2} [54] (pp.94-108).

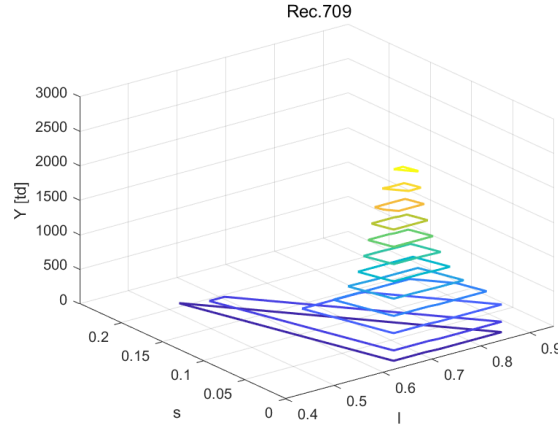


Fig. 4. Color solid in the (Y, l, s) color space for Rec.709 and $Y_p = 3000$ td, represented as a set of slices at constant retinal illuminance levels Y : 2750, 2500, 2250, 2000, 1750, 1500, 1250, 1000, 750, 500, 250 and 100 td.

5. Results and discussion

This section starts with an evaluation of the accuracy limitation of the proposed method, and is followed by a comparison with the results of other investigators, and an analysis of various sets of primary colors and peak luminance levels.

5.1. Accuracy limitations

For the numerical calculation of the color volumes V a finite cell size $\delta Y \times \delta l \times \delta s = 1 \times 0.001 \times 0.001$ was chosen. The quantization error of the color volume, caused by the finite cell size was estimated by calculating the color volumes with a substantial smaller cell size equal to $\delta Y \times \delta l \times \delta s = 0.5 \times 0.0005 \times 0.0005$. In the case of Rec.2020 and the peak illuminance levels of 400 td and 4000 td the relative color volume differences $\frac{\Delta V}{V}$ are equal to +0.29% and +0.46%, respectively. From this we conclude that the finite cell size $\delta Y \times \delta l \times \delta s = 1 \times 0.001 \times 0.001$ has a negligible impact on the error of the calculated color volume. The error on the packing density has also an impact on the error of the maximum number of mutually discernible colors N . A constant packing density equal to $\frac{\pi}{\sqrt{18}} \approx 0.74048$ for $K \in [0, 0.1]$ was assumed for the calculation of the maximum number of mutually discernible colors (cf. Appendix A). It is known that the upper bound of the packing density in the 3-dimensional spherical and hyperbolic manifolds is approximately equal to 0.77964 for $|K| \in [0, 0.1]$ (cf. Appendix A). Assuming that for $K \in [0, 0.1]$ the lowest lower bound of the packing density is equal to 0.616, which corresponds to the packing density of the laminated spherical codes of Hamkins and Zeger for $(K = 0.1)$ [58] (cf. Appendix A), then we have $-16.81\% \leq \frac{\Delta N}{N} \leq 5.29\%$.

5.2. Comparison with previous work

The reported results of the psychophysical experiments by Baek et al. (2018) [34] are interesting but limited to correlations between the perceptual color volume and the calculated color volumes according to different color spaces. These results do not allow to make a comparisons with our color volume model. Fortunately, this is not the case for the results of the psychophysical experiments reported in JI20. We calculated the color volume V according to Eq. (17) and the Riemannian metric of the new LE, for conditions similar to experiment 1 and 2 in JI20 with P3-D65 chromaticities. In experiment 1 the peak luminances of the images are clipped at 200, 300, 500 and 1000 cd m^{-2} and the diffuse white is at a constant level of 200 cd m^{-2} . We assume

adaptation at a luminance level of 20% of the diffuse white level, being 40 cd m^{-2} . In experiment 2 the images are filtered with neutral density (ND) filters. The unfiltered images have a peak luminance of 1000 cd m^{-2} and a diffuse white of 200 cd m^{-2} . The fractional transmittances of the ND filters equal 0.35, 0.5 and 0.7, result in peak luminance levels of respectively 350, 500 and 700 cd m^{-2} , and diffuse white levels of respectively 70, 100 and 140 cd m^{-2} . We assume adaptation at a luminance level of 20% of the respective diffuse white levels, being 4% of the peak luminance levels. The black level is set at 10 td. The calculated volume V as a function of the peak luminance Y_p is shown in Fig. 5(a). For both experiments we find very good correlations between the calculated color volumes and the logarithmic trendlines; the coefficient of determination R^2 [59] (pp. 644-646) is 0.999 and 1.000 for experiment 1 and 2, respectively. The average perceptual color volumes of the images of the psychophysical experiments in JI20 have also a good correlation with logarithmic trendlines; the value of R^2 is 0.974 and 1.000 for respectively experiment 1 and 2 (cf. Fig. 4 and Table 1 in [39]). However, the slopes of the curves $V(Y_p)$ as calculated (cf. Figure 5(a)) and measured (cf. Fig. 4 in [39]) are not in accordance. It can be verified that the calculated difference (following our model) between experiment 1 and experiment 2 in JI20 is entirely due to the difference in adaptation levels and the loss of detail due to scattering in the eye, whereas according to the QMh model [40] (pp. 58-59) there is little difference between the two experiments. The discrepancy with the experimental curves can be understood qualitatively by referring to Fig. 6 in Ref. [39]. The highlights contribute relatively little to the effective 'perceived' volume, whereas our calculated volume does take into account the full-blown color solid for the peak luminance.

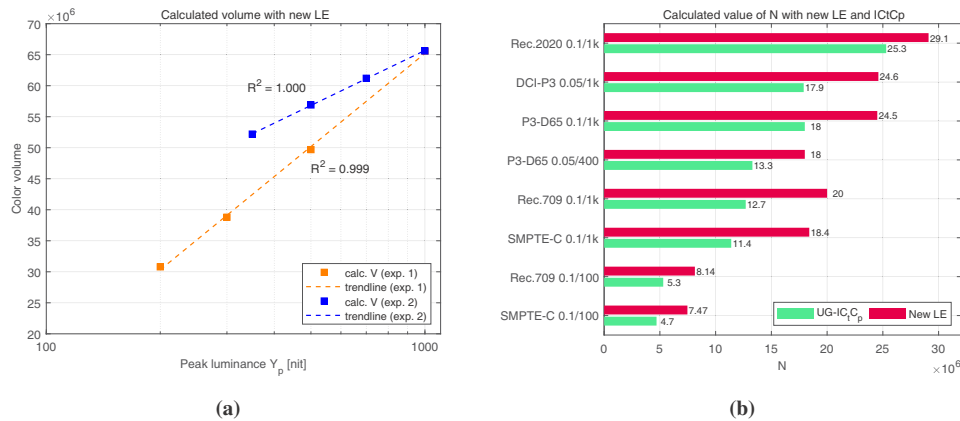


Fig. 5. (a) Semi-logarithmic plot of the calculated color volume as a function of the peak luminance levels 200, 300, 500 and 1000 cd m^{-2} and constant adapting background of 40 cd m^{-2} (orange squares); the logarithmic trendline (orange dashed line) calculated with MS Excel has the value $R^2 = 0.999$. (R^2 is the coefficient of determination [59] (pp.644-646)). Semi-logarithmic plot of the calculated color volume as a function of the peak luminance levels 350, 500, 700 and 1000 cd m^{-2} and an adapting background of 4% of the peak luminance levels (blue squares); the logarithmic trendline (blue dashed line) calculated with MS Excel has the value $R^2 = 1.000$. **(b)** Calculated value of N with the Riemannian metric of the new LE (red bars) and with the Euclidean ICtCp metric (green bars). The minimum and peak luminance levels [cd m^{-2}] are indicated on the vertical axis.

5.3. Analysis of various sets of primary colors and peak luminance levels

For the primary colors of Rec.2020, DCI-P3, P3-D65, Rec.709 and SMPTE-C and various peak retinal illuminance levels (denoted 'test cases') the volume V was calculated according to Eq. (17)

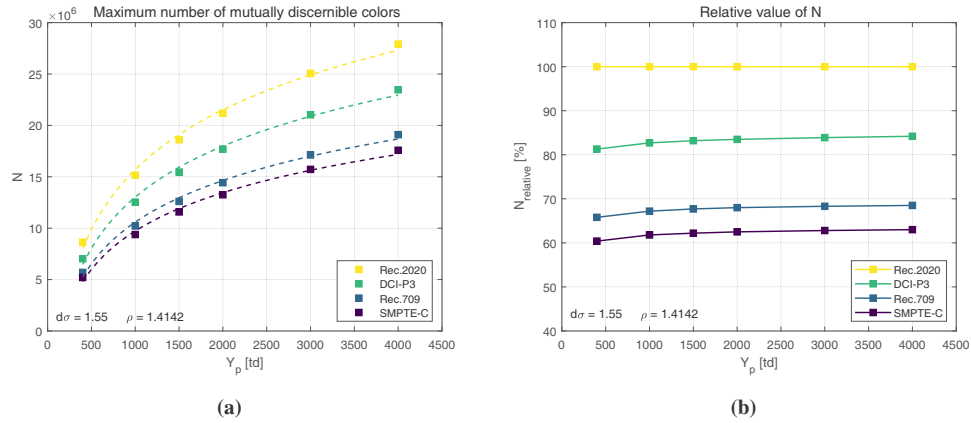


Fig. 6. (a) Maximum number of mutually discernible colors N as a function of the peak retinal illuminance Y_p [td] for Rec.2020, DCI-P3, Rec.709 and SMPTE-C (square markers). The logarithmic trendlines $N = a \log(Y_p) + b$ (dashed lines) calculated with MS Excel have in the 4 cases the value $R^2 \approx 0.994$, with R^2 the coefficient of determination [59] (pp.644-646). (b) For each set of primary colors we have a nearly constant ratio of $N(Y_p)$ and the N value of Rec.2020 at the same Y_p level (square markers connected with solid lines).

and the Riemannian metric of the new LE with $\delta Y = 1$ td and $\delta l = \delta s = 0.001$. The background and the black level are set at a retinal illuminance equal to 1 td. For the different test cases, the maximum number of mutually discernible colors N was calculated with Eq. (9) for $\rho = \sqrt{2}$ and $d\sigma = 1.55$. This choice for the threshold distance is based on our finding in [22] that an optimized similarity between the MacAdam threshold ellipses [26,27] and threshold ellipses according to the new LE was found for $d\sigma = 1.55$. The results for the different test cases are shown in Fig. 5(b). For the same test cases, and using the algorithm of section 4.2 to determine the color solids, the number of discernible colors N was calculated in the Euclidean ICtCp color space as a comparison. In this case we have $N = 720^3 V$, with V the volume of the color solid in the $(I, 0.5 C_l, C_p)$ color space, because a SC lattice is used and the color difference metric is defined as the Euclidean distance $\Delta E_{ICtCp} = 720 \sqrt{\Delta I^2 + 0.25 \Delta C_l^2 + \Delta C_p^2}$. When the viewer has an adaptation state corresponding to the highest visual sensitivity, then a just noticeable difference is equivalent to ΔE_{ICtCp} equal to one [36]. The results, denoted as UG-ICtCp, are also shown in Fig. 5(b). In comparison with the ICtCp space, N calculated with the new LE is larger for each test case. The differences are mainly caused by the higher number of lattice points in the same volume V for the FCC lattice ($\rho = \sqrt{2}$) compared to a SC lattice ($\rho = 1$). Obviously, the maximum number of mutually discernible colors is larger for the test cases with a wider color gamut and a wider dynamic range. In addition, for the WCG cases, P3 and Rec.2020, the relative differences of N , calculated according to ICtCp and the new LE, are respectively $\approx 36\%$ and $\approx 15\%$, while for the small gamuts Rec.709 and SMPTE-C the relative differences are $\approx 60\%$. These differences are caused by the impact of the scalar density $\sqrt{h}$. For the Euclidean ICtCp metric the scalar density is independent of the color coordinates. However, in case of the Riemannian metric of the new LE, the scalar density depends on the color coordinates. Specifically, as the color gamut becomes larger, more regions of the color volume with lower scalar density contribute to the color volume V (cf. section 3.5).

The volume V according to Eq. (17) and the Riemannian metric of the new LE was calculated for Rec.2020, DCI-P3, Rec.709 and SMPTE-C at the peak retinal illuminance levels $Y_p = 400, 1000, 1500, 2000, 3000$ and 4000 td. The background and the black level are set at a retinal illuminance equal to 1 td. With Eq. (9) the maximum number of mutually discernible colors N

was determined for $\rho = \sqrt{2}$ and $d\sigma = 1.55$. The results are shown in Fig. 6(a). This figure shows for equal levels of Y_p , the expected ranking SMPTE-C, Rec.709, DCI-P3 and Rec.2020 from the lowest to the highest value of N . For each set of primary colors and $Y_p \in [400 \text{ td}, 4000 \text{ td}]$, we find that:

$$N \propto \log \frac{Y_p}{Y_0}, \quad (18)$$

with Y_0 a fitting parameter with unit td. The values of N relative to the maximum number of mutually discernible colors of Rec.2020 at the same Y_p level are shown in Fig. 6(b). This figure shows a nearly constant relative N value of about 62%, 68% and 83% for respectively SMPTE-C, Rec.709 and DCI-P3.

6. Conclusions

We calculated the color volume V of the color solid and the maximum number of mutually discernible colors N for various sets of primary colors corresponding to display standards and various dynamic ranges. Our color volume calculations are based on the Riemannian metric proposed in Ref. [22], whereas state-of-the-art methods are based on an Euclidean metric in approximately uniform color spaces. In the 3-dimensional color manifold the colors that have a just noticeable color difference with a reference color are located on a spherical surface with radius equal to the threshold distance $d\sigma$. For the determination of the maximum number of mutually discernible colors N the densest packing of spheres with radii equal to $\frac{d\sigma}{2}$ is considered in the 3-dimensional color manifold with the above mentioned Riemannian metric and volume V . We motivated that the face-centered cubic (FCC) lattice packing is an acceptable approximation, i.e. the spheres with radius $\frac{d\sigma}{2}$ are centered in a FCC lattice with nearest neighbor distance equal to $d\sigma$. This FCC lattice has the highest lattice point density equal to $\frac{\sqrt{2}}{(d\sigma)^3}$ and corresponds with the maximum number of mutually discernible colors. The state-of-the-art methods use a simple cubic (SC) lattice with point density equal to $\frac{1}{(d\sigma)^3}$, causing an underestimation of N by a factor of 0.707, for the same color volume. One has to take into account the high sensitivity of N to the magnitude of the threshold distance $d\sigma$. The latter depends on the threshold criteria and the measurement conditions [22,48]. Based on the Riemannian metric, we calculated the color volume V as a function of the peak luminance according to the experimental conditions of Jiang et al. (2020) [39]. We found for both experiments very good correlations between the calculated color volumes and the logarithmic trendlines, which corresponds to the findings of Jiang et al. (2020) [39]. For the primary colors of Rec.2020, DCI-P3, P3-D65, Rec.709 and SMPTE-C and various dynamic ranges, we compared the calculated number of discernible colors N according to the Riemannian metric of the new LE and the FCC lattice, with the calculated values of N according to the Euclidean metric of the ICtCp color space and the SC lattice. In each case we find a higher number of discernible colors according to the Riemannian metric and the FCC lattice. The relative differences decrease as the primary colors are more saturated, because the scalar density decreases for more saturated chromaticities. Based on the Riemannian metric of the new LE we calculated, for the primary colors of Rec.2020, DCI-P3, Rec.709 and SMPTE-C, the maximum number of mutually discernible colors N for the peak retinal illuminance levels in the range of 400 td to 4000 td. We find, for a given set of primary colors, that N is proportional to the logarithm of the ratio of the peak retinal illuminance level and a fitting parameter with unit td. For the primary colors of DCI-P3, Rec.709 and SMPTE-C, the ratio of the number of discernible colors to those of Rec.2020 at the same peak retinal illuminance Y_p is nearly independent of Y_p . The relative number of discernible colors of DCI-P3, Rec.709 and SMPTE-C are respectively 83%, 68% and 62% of those of Rec.2020.

Appendix A: curvature and packing density

First we consider the spaces of zero curvature $\mathbb{R}^2$ and $\mathbb{R}^3$. It is known that the densest packing of equal circles in $\mathbb{R}^2$ is equal to $p_2 \equiv \frac{\pi}{\sqrt{12}} \approx 0.90690$ [49] and that the densest packing of equal spheres in $\mathbb{R}^3$ has a packing density equal to $p_3 \equiv \frac{\pi}{\sqrt{18}} \approx 0.74048$ [49]. The curvature of a Riemannian manifold indicates how much the geometry of the manifold differs locally from the Euclidean geometry, and is determined by the metric coefficients of the Riemannian metric and their derivatives. Consequently, the curvature might have an impact on the packing density. In two dimensions, with coordinates (u_1, u_2) , the curvature of a surface is a scalar function $K(u_1, u_2)$, called the Gaussian curvature [60] (pp. 8-11). The complete specification of the local curvature of a 3-dimensional manifold is far more complicated [60] (pp. 131-135). However, a mean local curvature can be determined at each point with coordinates (u_1, u_2, u_3) of a 3-dimensional manifold, called the scalar curvature $S(u_1, u_2, u_3)$ [61]. The 3-dimensional color manifold has a variable curvature; at some points of the color manifold it is possible to have a positive scalar curvature and at other points to have a zero or negative scalar curvature. For spaces with a variable curvature there is no general mathematical solution for the best packing density. Furthermore, for the particular 3-dimensional manifolds of constant curvature, viz. the spherical space $\mathbb{S}^3$ ($S = \frac{6}{r^2}$, with r the radius of the 3-sphere), and the hyperbolic space $\mathbb{H}^3$ ($S = -\frac{6}{r^2}$), there are also no general mathematical solutions for the best packing of equal spheres. Similarly, in two dimensions there are no general mathematical solutions for the best packing of equal circles on a surface with variable Gaussian curvature, and on the particular surfaces of constant Gaussian curvature, viz. the 2-sphere $\mathbb{S}^2$ ($K = \frac{1}{r^2}$, with r the radius of the 2-sphere), and the hyperbolic surface $\mathbb{H}^2$ ($K = -\frac{1}{r^2}$). However, based on a theorem of L. Fejes Tóth [62], which was extended to dimensions greater than two by Coxeter and Böröczky [62,63], it is possible to calculate an upper bound for the packing density of equal circles in $\mathbb{S}^2$ and $\mathbb{H}^2$, and equal spheres in $\mathbb{S}^3$ and $\mathbb{H}^3$. In Fig. 7 the upper bounds in $\mathbb{S}^2$ and $\mathbb{H}^2$ (called the Fejes Tóth upper bounds) and in $\mathbb{S}^3$ and $\mathbb{H}^3$ (called the Coxeter upper bounds) are shown in the range $|K| \in [0, 0.5]$. For the manifolds $\mathbb{S}^2$ and $\mathbb{H}^2$, and $|K| \rightarrow 0$ the upper bound equals $\frac{\pi}{\sqrt{12}}$, which is equal to the best packing density in $\mathbb{R}^2$. For the manifolds $\mathbb{S}^3$ and $\mathbb{H}^3$, and $|K| = \pm \frac{1}{r^2} \rightarrow 0$ the upper bound is

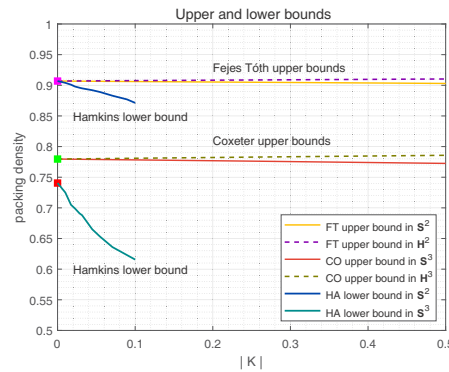


Fig. 7. Fejes Tóth (FT) upper bound in $\mathbb{S}^2$ (amber solid line) and $\mathbb{H}^2$ (violet dashed line), Coxeter (CO) upper bound in $\mathbb{S}^3$ (vermillion solid line) and $\mathbb{H}^3$ (olive dashed line). Hamkins (HA) lower bound in $\mathbb{S}^2$ (blue line) and in $\mathbb{S}^3$ (cyan line) for $K \in [0, 0.1]$. The FT upper bound for $|K| \rightarrow 0$ is equal to $\frac{\pi}{\sqrt{12}} \approx 0.90690$ (square magenta marker). The CO upper bound for $|K| \rightarrow 0$ is equal to 0.77964 (square green marker). The packing density of the densest packing of equal spheres in $\mathbb{R}^3$ is equal to $\frac{\pi}{\sqrt{18}} \approx 0.74048$ (square red marker).

equal to 0.77964, which is larger than the best packing density of equal spheres in $\mathbb{R}^3$ [64]. For the color manifold with the Riemannian metric of the new LE the situation is quite different, viz. the curvature is variable and in the range $|K| \in [0, 0.1]$. On the other hand spherical codes have been designed in communication theory [58] which form a lower bound of the packing density in $\mathbb{S}^2$ and $\mathbb{S}^3$. Unfortunately no such data has been found for the $\mathbb{H}^2$ and $\mathbb{H}^3$ spaces. Both lower bounds have been reproduced for $K \in [0, 0.1]$ in Fig. 7, referred to as Hamkins lower bounds. In the 2-dimensional case and for $|K| \rightarrow 0$ there is little room for variations, whereas for the 3-dimensional case a wider gap is available. Since in a large part of the color volume the value of $|K| < 0.01$ we set the packing density for the 2-dimensional case equal to the limiting value $p_2 \approx 0.90690$ and for the 3-dimensional case equal to $p_3 \approx 0.74048$. Notice that in $\mathbb{S}^2$ for $K \in [0, 0.1]$ the lowest value of the lower bound is 3.92% lower than p_2 . In $\mathbb{S}^3$ for $K \in [0, 0.1]$ the lowest value of the lower bound is about 16.81% lower than p_3 and the upper bound is about 5.29% higher than p_3 .

Appendix B: density of lattice points

In a 3D lattice Λ the set of points with position vectors $\vec{w}$ is given by:

$$\vec{w} = n_1 \vec{b}_1 + n_2 \vec{b}_2 + n_3 \vec{b}_3, \quad (19)$$

with n_1, n_2, n_3 integers and $\vec{b}_1, \vec{b}_2, \vec{b}_3$ the basis vectors of the lattice Λ . The density of points of the lattice Λ is given by:

$$\mathcal{D} = \frac{1}{|\det W|}, \quad (20)$$

with W the matrix whose columns are the basis vectors [65]. For the simple cubic (SC) lattice, the face-centered cubic lattice (FCC) and the hexagonal close-packed lattice (HCP) the matrices W and the densities $\mathcal{D}$ are given by: $W_{SC} = a \text{diag}(1, 1, 1)$, $\mathcal{D}_{SC} = \frac{1}{a^3}$,

$$W_{FCC} = \frac{a}{2} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \mathcal{D}_{FCC} = \frac{4}{a^3}, \quad W_{HCP} = b \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} \\ 0 & 0 & \frac{\sqrt{6}}{3} \end{bmatrix}, \quad \mathcal{D}_{HCP} = \frac{\sqrt{2}}{b^3}, \quad (21)$$

with a the edge length of the cubic unit cell and b the edge length of the hexagonal unit cell. The nearest-neighbor distance for the SC, FCC and HCP lattice are respectively a , $\frac{a}{\sqrt{2}}$ and b [66] (pp.13-17, 23-24) [67] (pp.131-135). For a hexagonal lattice in 2D we have:

$$W_H = b \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{bmatrix}, \quad \mathcal{D}_H = \frac{2}{\sqrt{3} b^2}, \quad \text{with } b \text{ the nearest-neighbor distance.} \quad (22)$$

Appendix C: chromatic adaptation transform

A pair of corresponding colors have the same appearance under adaptation to different lighting conditions. The von Kries rule is a chromatic adaptation transform (CAT) which allows to calculate corresponding colors [51] (pp.168-171) [52] (pp. 21-23). Suppose that in viewing condition a the observer is adapted to a white background with CIE tristimulus values (X_w^a, Y_w^a, Z_w^a) and in condition b to another white background with CIE tristimulus values (X_w^b, Y_w^b, Z_w^b) . The cone signals of the test field in adaptive condition a and b are respectively denoted as (L^a, M^a, S^a)

and (L^b, M^b, S^b) . According to the von Kries rule and complete adaptation we have:

$$\begin{bmatrix} L^b \\ M^b \\ S^b \end{bmatrix} = \begin{bmatrix} \frac{L_w^b}{L_w^a} & 0 & 0 \\ 0 & \frac{M_w^b}{M_w^a} & 0 \\ 0 & 0 & \frac{S_w^b}{S_w^a} \end{bmatrix} \begin{bmatrix} L^a \\ M^a \\ S^a \end{bmatrix} \triangleq V_{ab} \begin{bmatrix} L^a \\ M^a \\ S^a \end{bmatrix}, \quad \begin{bmatrix} L_w^\epsilon \\ M_w^\epsilon \\ S_w^\epsilon \end{bmatrix} = A \begin{bmatrix} X_w^\epsilon \\ Y_w^\epsilon \\ Z_w^\epsilon \end{bmatrix}, \quad \begin{bmatrix} L^\epsilon \\ M^\epsilon \\ S^\epsilon \end{bmatrix} = A \begin{bmatrix} X^\epsilon \\ Y^\epsilon \\ Z^\epsilon \end{bmatrix}, \quad (23)$$

with either $\epsilon = a$ or $\epsilon = b$ and matrix A as in Eq. (14). The tristimulus values of the pair of corresponding colors under viewing condition a and b are respectively (X^a, Y^a, Z^a) and (X^b, Y^b, Z^b) . From Eq. (23) it follows that:

$$\begin{bmatrix} X^b \\ Y^b \\ Z^b \end{bmatrix} = T \begin{bmatrix} X^a \\ Y^a \\ Z^a \end{bmatrix} = T \begin{bmatrix} \frac{x^a}{y^a} \\ 1 \\ \frac{z^a}{y^a} \end{bmatrix} Y^a, \quad \text{and} \quad \begin{bmatrix} l^b \\ m^b \\ s^b \end{bmatrix} = B \begin{bmatrix} X^a \\ Y^a \\ Z^a \end{bmatrix} \frac{1}{Y^b} = B \begin{bmatrix} \frac{x^a}{y^a} \\ 1 \\ \frac{z^a}{y^a} \end{bmatrix} \frac{Y^a}{Y^b}, \quad (24)$$

with $T = A^{-1} V_{ab} A$ and $B = V_{ab} A$.

Appendix D: chromaticities primary colors and reference white

Table 1. CIE1931 chromaticities of the primary colors and reference white for Rec.2020 [43], DCI-P3 [44], Rec.709 [45] and SMPTE-C [53].

	Rec.2020		DCI-P3		Rec.709		SMPTE-C	
	x	y	x	y	x	y	x	y
red	0.708	0.292	0.680	0.320	0.640	0.330	0.630	0.340
green	0.170	0.797	0.265	0.690	0.300	0.600	0.310	0.595
blue	0.131	0.046	0.150	0.060	0.150	0.060	0.155	0.070
ref. white	0.3127	0.3290	0.3140	0.3510	0.3127	0.3290	0.3127	0.3290

Disclosures. The authors declare no conflicts of interest.

Data availability. Data underlying the results presented in this paper are not publicly available at this time but may be obtained from the authors upon reasonable request.

References

1. M. R. Pointer and G. G. Attridge, "The number of discernible colours," *Color Res. Appl.* **23**(1), 52–54 (1998).
2. K. Masaoka, R. S. Berns, M. D. Fairchild, and F. M. Abed, "Number of discernible object colors is a conundrum," *J. Opt. Soc. Am. A* **30**(2), 264–277 (2013).
3. R. G. Kuehni, "How many object colors can we distinguish?" *Color Res. Appl.* **41**(5), 439–444 (2016).
4. E. Perales, F. Martínez-Verdú, V. Viqueira Pérez, M. J. Luque, and P. Capilla, "Computing the number of distinguishable colors under several illuminants and light sources," *CGIV 2006 - 3rd European Conference on Colour in Graphics, Imaging, and Vision, Final Program and Proceedings* pp. 414–419 (2006).
5. L. Silberstein, "Investigations on the intrinsic properties of the color domain. II," *J. Opt. Soc. Am.* **33**(1), 1–10 (1943).
6. D. L. MacAdam, "On the geometry of color space," *J. Franklin Inst.* **238**(3), 195–210 (1944).
7. I. Farup, "Hyperbolic geometry for colour metrics," *Opt. Express* **22**(10), 12369–12378 (2014).
8. R. Zhu, H. Chen, and S.-T. Wu, "Achieving 12-bit perceptual quantizer curve with liquid crystal display," *Opt. Express* **25**(10), 10939–10946 (2017).
9. G. Biwa, A. Aoyagi, M. Doi, K. Tomoda, A. Yasuda, and H. Kadota, "Technologies for the Crystal LED display system," *J. Soc. Inf. Disp.* **29**(6), 435–445 (2021).
10. B. Geffroy, P. le Roy, and C. Prat, "Organic light-emitting diode (OLED) technology: materials, devices and display technologies," *Polym. Int.* **55**(6), 572–582 (2006).
11. Y. Takita, K. Takeda, N. Hashimoto, S. Nomura, T. Suzuki, H. Nakashima, S. Uesaka, S. Seo, and S. Yamazaki, "Highly efficient deep-blue fluorescent dopant for achieving low-power OLED display satisfying BT.2020 chromaticity," *J. Soc. Inf. Disp.* **26**(2), 55–63 (2018).

12. Z. Luo, Y. Chen, and S.-T. Wu, "Wide color gamut LCD with a quantum dot backlight," *Opt. Express* **21**(22), 26269–26284 (2013).
13. S. Xu, X. Lin, X. Zhang, E. Chen, Y. Ye, T. Guo, and Q. F. Yan, "Quantum-dot color conversion towards white-balanced healthy displays: Theoretical and simulation study," *J. Soc. Inf. Disp.* **30**(11), 837–844 (2022).
14. P. Janssens and K. Malfait, "Future prospects of high-end laser projectors," in *Emerging Liquid Crystal Technologies IV*, vol. 7232 L.-C. Chien and M. H. Wu, eds., International Society for Optics and Photonics (SPIE, 2009), p. 72320Y.
15. L. Zhu, G. Wang, Y. Yang, B. Yao, C. Gu, and L. Xu, "Six-primary-laser projection display system: demonstration and stereo color gamut measurement," *Opt. Express* **29**(26), 43885–43898 (2021).
16. Dolby, "Perceptual Color Volume - Measuring the Distinguishable Colors of HDR and WCG Displays," White Paper - Version 7.1 Dolby Laboratories, Inc. (2018).
17. E. Smith, R. L. Heckaman, K. Lang, J. Penczek, and J. Bergquist, "Measuring the color capability of modern display systems," *J. Soc. Inf. Disp.* **28**(6), 548–556 (2020).
18. E. Smith, R. L. Heckaman, K. Lang, J. Penczek, and J. Bergquist, "Evaluating Display Color Capability," *Inf. Disp.* **36**(5), 9–15 (2020).
19. C. Poynton and D. A. LeHoty, "50-1: color/white light output, luminance contours, and colour volume," *SID Symposium Digest of Technical Papers* **51**(1), 730–733 (2020).
20. D. A. LeHoty and C. Poynton, "Comparing Displays Using Luma Contours and Color Volumes," *Inf. Disp.* **36**(5), 16–21 (2020).
21. K. Masaoka, F. Jiang, M. D. Fairchild, and R. L. Heckaman, "Analysis of color volume of multi-chromatic displays using gamut rings," *J. Soc. Inf. Disp.* **28**(3), 273–286 (2020).
22. P. Candry, P. De Visschere, and K. Neyts, "Line element for the perceptual color space," *Opt. Express* **30**(20), 36307–36331 (2022).
23. S. Wen, "Design of relative primary luminances for four-primary displays," *Displays* **26**(4-5), 171–176 (2005).
24. A. R. Robertson, "The CIE 1976 Color-Difference Formulae," *Color Res. Appl.* **2**(1), 7–11 (1977).
25. G. Wyszecki and W. S. Stiles, *Color science: Concepts and methods, quantitative data and formulae*, 2nd Edition (John Wiley & Sons., New York, 1982).
26. D. L. MacAdam, "Visual sensitivities to color differences in daylight," *J. Opt. Soc. Am.* **32**(5), 247–274 (1942).
27. L. Silberstein and D. L. MacAdam, "The distribution of color matchings around a color center," *J. Opt. Soc. Am.* **35**(1), 32–39 (1945).
28. IDMS, "Information Display Measurements Standard, Version 1.1a," <https://www.icdm-sid.org> (2022). [Online; accessed 27-December-2022].
29. R. W. G. Hunt, "The Effects of Daylight and Tungsten Light-Adaptation on Color Perception," *J. Opt. Soc. Am.* **40**(6), 362–371 (1950).
30. R. W. G. Hunt, "Light and Dark Adaptation and the Perception of Color," *J. Opt. Soc. Am.* **42**(3), 190–199 (1952).
31. J. C. Stevens and S. S. Stevens, "Brightness function: effects of adaptation," *J. Opt. Soc. Am.* **53**(3), 375–385 (1963).
32. K. Masaoka, "CIELAB-metric color volume for HDR displays," *IDW* 17 pp. 1040–1043 (2017).
33. F. Jiang, M. Fairchild, and K. Masaoka, "Estimation of HDR WCG display color gamut volume," *Color. Imaging Conf.* **26**(1), 344–349 (2018).
34. Y. S. Baek, Y. Kwak, S. Seo, and J. Lee, "78-4: visual evaluation of displays 3D color gamut volume," *SID Symp. Dig. Tech. Pap.* **49**(1), 1052–1055 (2018).
35. N. Moroney, M. Fairchild, R. Hunt, I. Changjun, M. Luo, and T. Newman, "The CIECAM02 color appearance model," *IS & T/SID Color Imaging Conf* **10**, 23–27 (2002).
36. ITU, "Recommendation ITU-R BT.2124-0 - Objective metric for the assessment of the potential visibility of colour differences in television," ITU (2019).
37. T. Lu, F. Pu, P. Yin, T. Chen, W. Husak, J. Pytlarz, R. Atkins, J. Froehlich, and G.-M. Su., "ITP colour space and its compression performance for high dynamic range and wide colour gamut video distribution," *ZTE Communications* **14**, 32–38 (2016).
38. J. Pytlarz, E. Pieri, and R. Atkins, "Objectively evaluating high-dynamic-range and wide-color-gamut color differences," *SMPTE Mot. Imag. J* **126**(2), 27–32 (2017).
39. F. Jiang, M. Fairchild, and K. Masaoka, "Effect of peak luminance on perceptual color gamut volume," *Color. Imaging Conf.* **28**(1), 13–18 (2020).
40. F. Jiang, *High Dynamic Range (HDR) Display Perception* (PhD Thesis, Rochester Institute of Technology, 2021).
41. M. Safdar, G. Cui, Y. J. Kim, and M. R. Luo, "Perceptually uniform color space for image signals including high dynamic range and wide gamut," *Opt. Express* **25**(13), 15131–15151 (2017).
42. L. Xu, Q. Ye, and M. R. Luo, "Estimation of the perceptual color gamut on displays," *Opt. Express* **30**(24), 43872–43887 (2022).
43. ITU, "Recommendation ITU-R BT.2020-2 - Parameter values for ultra-high definition television systems for production and international programme exchange," ITU (2015).
44. SMPTE, "RP 431-2:2011 - SMPTE Recommended Practice - D-Cinema Quality — Reference Projector and Environment," RP 431-2:2011 (2011).
45. ITU, "Recommendation ITU-R BT.709-6 - Parameter values for the HDTV standards for production and international programme exchange," ITU (2015).

46. I. S. Sokolnikoff, *Tensor Analysis: Theory and Applications* (John Wiley & Sons, Inc., 1951).
47. A. D. Aleksandrov, A. N. Kolmogorov, and M. A. Lavrent'ev, *Mathematics: Its Contents, Methods and Meaning*, vol. 3 (The MIT Press, 1963).
48. Y. Ohno and P. Blattner, "Chromaticity Difference Specification for Light Sources," CIE Technical Note 001 (2014).
49. M. S. Viazovska, "The sphere packing problem in dimension 8," *Annals Math.* **185**(3), 991–1015 (2017).
50. D. I. A. MacLeod and R. M. Boynton, "Chromaticity diagram showing cone excitation by stimuli of equal luminance," *J. Opt. Soc. Am.* **69**(8), 1183–1186 (1979).
51. M. D. Fairchild, *Color Appearance Models*, 2nd Edition (John Wiley & Sons, Ltd., 2005).
52. R. S. Berns, *Billmeyer and Saltzman's Principles of Color Technology*, 4th Edition (John Wiley & Sons, Inc., 2019).
53. SMPTE, "RP 145:2004 - SMPTE Recommended Practice - SMPTE C Color Monitor Colorimetry," RP 145:2004 pp. 1 (2004).
54. Y. L. Grand, *Light, Colour and Vision*, 2nd Edition (Chapman and Hall Ltd., 1968).
55. V. C. Smith and J. Pokorny, "Chapter 3 - Color Matching and Color Discrimination," in *The Science of Color* (2nd Edition), S. K. Shevell, ed. (Elsevier Science Ltd, Amsterdam, 2003), pp. 103–148.
56. G. Wyszecki, "Color matching at moderate to high levels of retinal illuminance: A pilot study," *Vision Res.* **18**(3), 341–346 (1978).
57. G. Wyszecki and W. Stiles, "High-level trichromatic color matching and the pigment-bleaching hypothesis," *Vision Res.* **20**(1), 23–37 (1980).
58. J. Hamkins and K. Zeger, "Asymptotically dense spherical codes.II. laminated spherical codes," *IEEE Trans. Inf. Theory* **43**(6), 1786–1798 (1997).
59. R. L. Ott and M. Longnecker, *An Introduction to Statistical Methods and Data Analysis*, 7th Edition (Cengage Learning, 1988).
60. S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley & Sons, Inc., 1972).
61. J. Rosenberg and D. Wraith, "Positive Scalar Curvature and Applications," *Snapshots of modern mathematics from Oberwolfach* pp. 1–12 (2009).
62. O. R. Musin, "Five essays on the geometry of László Fejes Tóth," *arXiv*, arXiv:1604.02776 (2016).
63. K. Böröczky, "Packing of spheres in spaces of constant curvature," *Acta Math. Acad. Sci. Hungarica* **32**(3-4), 243–261 (1978).
64. J. H. Conway and S. Torquato, "Packing, tiling, and covering with tetrahedra," *Proc. Natl. Acad. Sci. U. S. A.* **103**(28), 10612–10617 (2006).
65. E. Dubois, "The sampling and reconstruction of time-varying imagery with application in video systems," *Proc. IEEE* **73**(4), 502–522 (1985).
66. C. Kittel, *Introduction to Solid State Physics*, 5th Edition (John Wiley & Sons, Inc., 1976).
67. D. E. Sands, *Introduction to Crystallography* (Dover Publications, Inc., 1993).