A holistic approach to the composition of ternary relations

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Abstract

In this paper, we present a systematic approach to the study of the composition of ternary relations from the point of view of the degrees of freedom available when linking a **3**-tuple to two given **3**-tuples. We propose a way of enumerating all possible **4**-point compositions (one degree of freedom) and **5**-point compositions (two degrees of freedom) of ternary relations, and establish a correspondence between them. Furthermore, we identify the associative compositions and explore interesting mixed-associativity cases. Finally, we use the tools of projection and cylindrical extension to relate the **4**-point and **5**-point compositions of ternary relations to the **3**-point compositions of binary relations.

Keywords: Ternary relation, Relational composition, Associativity, Mixed-associativity

MSC Classification: 03E20 , 08A02

1 Introduction

In spite of the fact that human cognition is wired to grasp relationships between items, it has just been over a century since the theory of relations has been the focus of a thorough investigation, initiated in 1860 by De Morgan (De Morgan 1860), and carried on twenty years later by Peirce (Peirce 1880), who introduced the basic ideas underlying the theory of relations and established its fundamental principles. Peirce's work was further continued by (Schröder

1890). The peculiar and somewhat erratic course of the historical development of the subject did not allow for an extensive study of relations beyond the binary case.

The first type of higher-order relations that has been studied was the ternary relation of betweenness (Huntington 1917; Pitcher and Smiley 1942). Novák and Novotný laid the groundwork for the study of ternary relations in numerous studies that were published in the late 20th century (Novák and Novotný 1989a,b, 1992). Ternary relations have been found to be useful in a variety of scientific fields, for instance, in theoretical physics (e.g., in computational physics (Wolfram 2020)), in mathematics (e.g., in group theory (Cristea 2009)), in artificial intelligence (e.g., in qualitative spatial reasoning (Clementini and Billen 2006; Isli and Cohn 2000)), in computer science (e.g., in knowledge graphs (Zhang et al 2021), data structures (Alvarez-Garcia et al. 2017), string matching (Kim et al. 2017), video recognition (Shi et al. 2020), information modelling (Pourabdollah 2009; Powers 2003) and in social networks (Firouzkouhi et al. 2024) (also including quaternary and quinary relations)) and in biology (e.g., in phylogenetic modelling (Steel 2016)). The calculus of relations is concerned with the fundamental operations on relations, also called Peircean operations, such as forming the inverse of a relation, the composition of two relations, and the dual of a relation, as well as the Boolean operations of forming unions, intersections, and complements of relations. This subject has been widely studied in the binary case (Bělohlávek 2002; Davey and Priestley 2002; Fraisse 2000; Schmidt and Ströhlein 2012). However, less attention has been paid to the calculus of ternary relations and higher-order relations in general. By demonstrating how to create all possible compositions of ternary relations, this work aims to serve as a foundation for the study of ternary relations. Contrary to the binary case, two types of composition will emerge: we study each type, demonstrate connections between them, and establish links with the composition of binary relations. Since associativity is by far the most important property of the composition of binary relations, it should come as no surprise that our main focus in this paper will be on the associativity of the compositions of ternary relations identified, not limiting our attention to the standard associativity property, but also exploring the property of mixed-associativity.

This paper is organized as follows. In Section 2, we recall the necessary basic concepts and properties of binary and ternary relations. In Section 3, we explore the full extent of the notion of composition of ternary relations by introducing two types of composition. The first type consists of 324 different compositions, while the second type consists of 216 different compositions. For each type, we provide a convenient senary numeral system to generate all of them in a comprehensive manner. In Section 4, we study the associativity of the compositions introduced. We identify six associative compositions of the first type and twelve associative compositions of the second type. We also study the mixed-associativity property referring to a couple of compositions that verify an associativity-like property. In Section 5, we present the connections between the compositions of binary relations and the compositions of ternary relations by using the binary projections of ternary relations and the cylindrical extensions of binary relations. We end the paper with a brief conclusion in Section 6.

2 Preliminaries

In this section, we recall some basic notions on binary and ternary relations that will be needed in this paper. Throughout this paper, X represents a non-empty universe. An n-ary relation, with $n \in \mathbb{N}$, on X is a subset of X^n; if $n = 2$, then we talk about a binary relation, while if $n = 3$, then we talk about a ternary relation.

Consider the two 2-permutations ρ_0 and ρ_1 mapping any 2-tuple (x, y) as follows: $\rho_0(x, y) = (x, y)$ and $\rho_1(x, y) = (y, x)$. For a binary relation R and a 2-permutation ρ, the binary relation R^ρ is defined as: $R^\rho = \{\rho(x, y) \in X^2 \mid (x, y) \in R\}$. Note that $R^{\rho_0} = R$ and $R^{\rho_1} = R^t$, the *transpose* of R.

Similarly, we consider the ternary relations obtained by permutation. The six 3-permutations are listed according to the lexicographical order, mapping any 3-tuple (x, y, z) as follows:

$$\begin{array}{lll} \sigma_0(x,y,z) = (x,y,z) & \sigma_1(x,y,z) = (x,z,y) & \sigma_2(x,y,z) = (y,x,z) \\ \sigma_3(x,y,z) = (y,z,x) & \sigma_4(x,y,z) = (z,x,y) & \sigma_5(x,y,z) = (z,y,x)\,. \end{array}$$

For a ternary relation T and a 3-permutation σ, the ternary relation T^σ is defined as (Zedam, Barkat and De Baets 2018): $T^\sigma = \{\sigma(x, y, z) \in X^3 \mid (x, y, z) \in T\}$. Note that we can also write $T^\sigma = \{(x, y, z) \in X^3 \mid \sigma^{-1}(x, y, z) \in T\}$, with $\sigma_i^{-1} = \sigma_i$ for any $i \in \{0, 1, 2, 5\}$ and $\sigma_3^{-1} = \sigma_4$. Obviously, $T^{\sigma_0} = T$. For more details on ternary relations, we refer to (Bakri, Zedam, and De Baets 2021; Cristea 2009; Novák and Novotný 1989a, 1992; Zedam, Bakri and De Baets 2020; Zedam, Barkat and De Baets 2018).

3 Compositions of ternary relations

In this section, we lay bare all possible compositions of ternary relations. First, we revisit the composition of binary relations to serve as a source of inspiration.

3.1 Compositions of binary relations

The composition of two binary relations R and S on X is defined as follows (Peirce 1880):

$$R \circ S = \{(x, z) \in X^2 \mid (\exists t \in X)((x, t) \in R \land (t, z) \in S)\}.$$

One could imagine other definitions of the composition of binary relations:

$$\begin{aligned} R \circ_1 S &= \{(x, z) \in X^2 \mid (\exists t \in X)((t, x) \in R \land (t, z) \in S)\}; \\ R \circ_2 S &= \{(x, z) \in X^2 \mid (\exists t \in X)((x, t) \in R \land (z, t) \in S)\}; \\ R \circ_3 S &= \{(x, z) \in X^2 \mid (\exists t \in X)((t, x) \in R \land (z, t) \in S)\}; \\ R \circ_4 S &= \{(x, z) \in X^2 \mid (\exists t \in X)((z, t) \in R \land (t, x) \in S)\}; \\ R \circ_5 S &= \{(x, z) \in X^2 \mid (\exists t \in X)((t, z) \in R \land (t, x) \in S)\}; \\ R \circ_6 S &= \{(x, z) \in X^2 \mid (\exists t \in X)((z, t) \in R \land (x, t) \in S)\}; \\ R \circ_7 S &= \{(x, z) \in X^2 \mid (\exists t \in X)((t, z) \in R \land (x, t) \in S)\}. \end{aligned}$$

As all of these compositions link two 2-tuples while allowing for a degree of freedom, we refer to them as 3-*point compositions*.

Any of the above 3-point compositions is determined by three 2-permutations ρ_i, ρ_j, ρ_k, $i, j, k \in \{0, 1\}$, as explained next. First of all, let us fix two (possibly identical) 2-permutations

ρ_i and ρ_j. If we say that a 2-tuple $(x, z) \in X^2$ belongs to some composition of two binary relations R and S on X if there exists an element $t \in X$ such that $\rho_i(x, t) \in R$ and $\rho_j(z, t) \in S$, then this allows to retrieve the first four compositions above. If, additionally, we allow to permute the 2-tuple (x, z), then we can also retrieve the last four compositions (after a proper renaming of variables). This view allows to develop the following enumeration scheme. For any $q \in \{0, \ldots, 7\}$, with $\circ_0 := \circ$ the basic composition, the composition $R\circ_q S$ can be written as:

$$R\circ_q S = \{\rho_k(x, z) \in X^2 \mid (\exists t \in X)(\rho_i(x, t) \in R \wedge \rho_j(t, z) \in S)\} , \tag{1}$$

with $q = (kji)_2 = 4k + 2j + i$ and $i, j, k \in \{0, 1\}$. Hence, the compositions $\circ_q$, $q = 1, \ldots, 7$, can be expressed in terms of the basic composition $\circ_0$ as follows:

$$R\circ_q S = (R^{\rho_i} \circ_0 S^{\rho_j})^{\rho_k} .$$

Note that for a given 3-point composition $\circ_q$, with $q = (kji)_2$, there exists a 3-point composition $\circ_{q'}$ such that for any binary relations R and S on X, it holds that

$$R\circ_q S = S\circ_{q'} R ,$$

with $q' = (\bar{k}\bar{i}\bar{j})_2$, $\bar{0} = 1$ and $\bar{1} = 0$.

3.2 Compositions of ternary relations

In the literature, various compositions of two ternary relations S and T have been proposed, each with its own motivation. A first example is the composition $\circ_B$ (used in the definition of the transitivity of betweenness relations (Pitcher and Smiley 1942)) defined as:

$$S\circ_B T = \{(x, y, z) \in X^3 \mid (\exists t \in X)((x, y, t) \in S \wedge (x, t, z) \in T)\} . \tag{2}$$

A 3-tuple belongs to this composition if there exist two 3-tuples that each share two components with the given tuple, of which exactly one in common, and one common degree of freedom t. We will refer to such a composition as a 4-point composition. A second example is the composition $\circ_{c_1}$ (derived from the composition of a ternary relation with a binary relation in (Zedam, Barkat and De Baets 2018)) defined as:

$$S\circ_{c_1} T = \{(x, y, z) \in X^3 \mid (\exists (s, t) \in X^2)((x, y, t) \in S \wedge (s, t, z) \in T)\} . \tag{3}$$

A 3-tuple belongs to this composition if there exist two 3-tuples of which one shares two components with the given tuple, while the other one contains the third component, complemented by two degrees of freedom s and t, of which one in common. We will refer to such a composition as a 5-point composition. In Subsection 3.2.1, we will identify all 4-point compositions of ternary relations and study their properties, while in Subsection 3.2.2, we will do the same for the 5-point compositions.

3.2.1 Four-point compositions of ternary relations

The reasoning underlying the 3-point compositions of binary relations can be naturally extended to enumerate the 4-point compositions of ternary relations. We start from the composition $\circ_B$ in Eq. (2) and use the 3-permutations to allow for a repositioning of the variables x, y, z and t, resulting in as many as 216 different 4-point compositions.

Definition 3.1. Let $p \in \{0, \ldots, 215\}$ with $p = (kji)_6 = 36k + 6j + i$ and $i, j, k \in \{0, \ldots, 5\}$. For any ternary relations S and T on X, we define the composition $S\square^0_p T$ as follows:

$$S\square^0_p T \;=\; \{\sigma_k(x, y, z) \in X^3 \mid (\exists t \in X)(\sigma_i(x, y, t) \in S \wedge \sigma_j(x, t, z) \in T)\}\,.$$

It is indeed easy to see that the compositions $\square^0_p$, $p \in \{0, \ldots, 215\}$, are all different.

However, not all 4-point compositions of ternary relations can be obtained in this way. Indeed, one can also envisage compositions that match the following description. A 3-tuple belongs to such a composition if there exist two 3-tuples of which one shares two components with the given tuple, while the other one contains the third component and a second occurrence of the corresponding element, again complemented with one common degree of freedom t.

Definition 3.2. Let $p \in \{0, \ldots, 215\}$ with $p = (kji)_6 = 36k+6j+i$ and $i, j, k \in \{0, \ldots, 5\}$. For any ternary relations S and T on X, we define the compositions $S\square^1_p T$ and $S\square^2_p T$ as follows:

$$S\square^1_p T \;=\; \{\sigma_k(x, y, z) \in X^3 \mid (\exists t \in X)(\sigma_i(x, y, t) \in S \wedge \sigma_j(z, t, z) \in T)\}\,;$$
$$S\square^2_p T \;=\; \{\sigma_k(x, y, z) \in X^3 \mid (\exists t \in X)(\sigma_i(y, y, t) \in S \wedge \sigma_j(x, t, z) \in T)\}\,.$$

As the following proposition shows, we have only 54 different compositions of type $\square^1$ and as many of type $\square^2$.

Proposition 3.3. *Let $i, j, k \in \{0, \ldots, 5\}$. For any ternary relations S and T on X, it holds that*

$$S\square^1_{(kji)_6}T = S\square^1_{(\pi_1(k)j\pi_1(i))_6}T = S\square^1_{(k\pi_2(j)i)_6}T = S\square^1_{(\pi_1(k)\pi_2(j)\pi_1(i))_6}T$$

and

$$S\square^2_{(kji)_6}T = S\square^2_{(\pi_2(k)\pi_2(j)i)_6}T = S\square^2_{(kj\pi_1(i))_6}T = S\square^2_{(\pi_2(k)\pi_2(j)\pi_1(i))_6}T,$$

with π_1 the permutation of $\{0, \ldots, 5\}$ given in Table 1:

Table 1 The permutation π_1.

u	0	1	2	3	4	5
$\pi_1(u)$	2	3	0	1	5	4

and π_2 the permutation of $\{0, \ldots, 5\}$ given in Table 2:

Table 2 The permutation π_2.

u	0	1	2	3	4	5
$\pi_2(u)$	5	4	3	2	1	0

Note that if $u \in \{0,3,4\}$, then $\pi_1(u), \pi_2(u) \in \{1,2,5\}$. This implies that we can select the unique 54 compositions of type $\Box^1$ and the 54 unique compositions of type $\Box^2$ by restricting i, j to belong to $\{0,3,4\}$ (the same could be achieved by restricting i, j to belong to $\{1,2,5\}$). This corresponds to the compositions $\Box_p^r$ ($r \in \{1,2\}$) with $p \bmod 36$ belonging to $\{0,3,4,18,21,22,24,27,28\}$.

The following examples illustrate the above definition of 4-point compositions of ternary relations.

Example 3.4. Consider $p = 119$ and two ternary relations S and T on X. In senary notation, p can be written as $119 = 3 * 36 + 1 * 6 + 5 = (315)_6$. Hence,

$$\begin{aligned} T\Box_{119}^0 S &= \{\sigma_3(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_5(x,y,t) \in T \wedge \sigma_1(x,t,z) \in S)\} \\ &= \{(y,z,x) \in X^3 \mid (\exists t \in X)((t,y,x) \in T \wedge (x,z,t) \in S)\} \\ &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,z) \in T \wedge (z,y,t) \in S)\}\,, \end{aligned}$$

$$\begin{aligned} T\Box_{119}^1 S &= \{\sigma_3(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_5(x,y,t) \in T \wedge \sigma_1(z,t,z) \in S)\} \\ &= \{(y,z,x) \in X^3 \mid (\exists t \in X)((t,y,x) \in T \wedge (z,z,t) \in S)\} \\ &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,z) \in T \wedge (y,y,t) \in S)\}\,, \end{aligned}$$

$$\begin{aligned} T\Box_{119}^2 S &= \{\sigma_3(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_5(y,y,t) \in T \wedge \sigma_1(x,t,z) \in S)\} \\ &= \{(y,z,x) \in X^3 \mid (\exists t \in X)((t,y,y) \in T \wedge (x,z,t) \in S)\} \\ &= \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,x,x) \in T \wedge (z,y,t) \in S)\}\,. \end{aligned}$$

Conversely, the following example shows how to find $r \in \{0,1,2\}$ and $p = (kji)_6 \in \{0,\ldots,215\}$ for a given composition.

Example 3.5. Consider two ternary relations S and T on X and the binary operation $*$ defined as follows:

$$S * T = \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,x) \in T \wedge (y,z,t) \in S)\}\,.$$

The operation $*$ corresponds to a composition of the type $\Box_p^0$, with $p \in \{0,\ldots,215\}$. We express this composition as in Definition 3.1 by identifying the proper 3-permutations:

$$\begin{aligned} S * T &= \{(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_4(z,x,t) \in T \wedge \sigma_4(z,t,y) \in S)\} \\ &= \{(y,z,x) \in X^3 \mid (\exists t \in X)(\sigma_4(x,y,t) \in T \wedge \sigma_4(x,t,z) \in S)\} \end{aligned}$$

$$= \{\sigma_3(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_4(x,y,t) \in T \wedge \sigma_4(x,t,z) \in S)\}\,.$$

Hence, $i = 4$, $j = 4$ and $k = 3$. Thus, $p = (344)_6 = 136$ and $* = \square^0_{136}$.

Example 3.6. Consider two ternary relations S and T on X and the binary operation $\star$ defined as follows:

$$S \star T = \{(x,y,z) \in X^3 \mid (\exists t \in X)((t,z,y) \in T \wedge (t,x,x) \in S)\}\,.$$

The operation $\star$ corresponds to a composition of the type $\square^1_p$, with $p \in \{0,\ldots,215\}$. We express this composition as in Definition 3.2 by identifying the proper 3-permutations:

$$\begin{aligned} S \star T &= \{(z,y,x) \in X^3 \mid (\exists t \in X)((t,x,y) \in T \wedge (t,z,z) \in S)\} \\ &= \{\sigma_5(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_4(x,y,t) \in T \wedge \sigma_3(z,t,z) \in S)\}\,. \end{aligned}$$

Hence, $i = 4$, $j = 3$ and $k = 5$. Thus, $p = (534)_6 = 202$ and $\star = \square^1_{202}$. According to Proposition 3.3, $*$ can also be written as $\square^1_{161}$, $\square^1_{167}$ and $\square^1_{196}$.

The following proposition expresses all the 4-point compositions in terms of the compositions $\square^r_0$, $r \in \{0,1,2\}$, which we will refer to as the basic 4-point compositions.

Proposition 3.7. *Let $p = (kji)_6 \in \{0,\ldots,215\}$ and $r \in \{0,1,2\}$. For any ternary relations S and T on X, the 4-point composition $S\square^r_p T$ can be written in terms of the composition $\square^r_0$ as follows:*

$$S\square^r_p T = (S^{\sigma_i^{-1}} \square^r_0 T^{\sigma_j^{-1}})^{\sigma_k}\,.$$

Proof. We give the proof for $r = 0$.

$$\begin{aligned} S\square^0_p T &= \{\sigma_k(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_i(x,y,t) \in S \wedge \sigma_j(x,t,z) \in T)\} \\ &= \{\sigma_k(x,y,z) \in X^3 \mid (\exists t \in X)((x,y,t) \in S^{\sigma_i^{-1}} \wedge (x,t,z) \in T^{\sigma_j^{-1}})\} \\ &= \{\sigma_k(x,y,z) \in X^3 \mid (x,y,z) \in S^{\sigma_i^{-1}} \square^0_0 T^{\sigma_j^{-1}}\} \\ &= (S^{\sigma_i^{-1}} \square^0_0 T^{\sigma_j^{-1}})^{\sigma_k}\,. \end{aligned}$$

□

Next, we study some basic properties of the 4-point compositions of ternary relations. The following proposition identifies the right and left neutral elements of the basic 4-point compositions. To that end, we introduce the following definition and lemma.

Definition 3.8. We define the special ternary relations E, E_ℓ, E_m and E_r on X as follows:

(i) $E = \{(x,x,x) \in X^3 \mid x \in X\}$;

(ii) $E_\ell = \{(x,x,y) \in X^3 \mid x,y \in X\}$;

(iii) $E_m = \{(x,y,x) \in X^3 \mid x,y \in X\}$;

(iv) $E_r = \{(x, y, y) \in X^3 \mid x, y \in X\}$.

Lemma 3.9. *For any ternary relation T on X, it holds that*
(i) $T\square_0^0 E_r = T$;
(ii) $E_r\square_0^0 T = T$.

Proof. We prove (i). Let $(x, y, z) \in T$, then with $(x, z, z) \in E_r$, it follows that $(x, y, z) \in T\square_0^0 E_r$. Conversely, let $(x, y, z) \in T\square_0^0 E_r$, then there exists $t \in X$ such that $(x, y, t) \in T$ and $(x, t, z) \in E_r$. The fact that $(x, t, z) \in E_r$ implies that $t = z$. Hence $(x, y, z) \in T$, and thus $T\square_0^0 E_r \subseteq T$. □

Proposition 3.10. *Let $i, j, k \in \{0, \ldots, 5\}$. For any ternary relation T on X, it holds that*
(i) $T\square^0_{(kjk)_6} E_{\zeta(j)} = T$;
(ii) $E_{\zeta(i)}\square^0_{(kki)_6} T = T$,
with $\zeta : \{0, \ldots, 5\} \to \{\ell, m, r\}$ the mapping given in Table 3:

Table 3 The mapping ζ.

u	0	1	2	3	4	5
$\zeta(u)$	r	r	m	ℓ	m	ℓ

Proof. We prove (i). Due to Lemma 3.9, it holds for any $k \in \{0, \ldots, 5\}$ that

$$T^{\sigma_k^{-1}}\square_0^0 E_r = T^{\sigma_k^{-1}} .$$

Moreover, one can verify that $E_r = (E_{\zeta(j)})^{\sigma_j^{-1}}$ for any $j \in \{0, \ldots, 5\}$. Hence,

$$T^{\sigma_k^{-1}}\square_0^0 E_{\zeta(j)}{}^{\sigma_k^{-1}} = T^{\sigma_k^{-1}} ,$$

and thus

$$\left(T^{\sigma_k^{-1}}\square_0^0 E_{\zeta(j)}{}^{\sigma_j^{-1}}\right)^{\sigma_k} = T .$$

Therefore, Proposition 3.7 ensures that

$$T\square^0_{(kjk)_6} E_{\zeta(j)} = T .$$

□

Note that $E_{\zeta(k)}$ is simultaneously the left and right neutral element of $\square^0_{(kkk)_6}$, and can hence be called the neutral element of $\square^0_{(kkk)_6}$.

Proposition 3.11. *Let $p \in \{0, \ldots, 215\}$. For any ternary relation T on X, it holds that*
(i) $T\square_p^1 E = T$;
(ii) $E\square_p^2 T = T$.

Proof. The proof is similar to that of Proposition 3.10. □

Note that the compositions $\square_p^1$ and $\square_p^2$ do not have a neutral element.

In the following definition, we introduce the notion of mixed-commutativity of compositions of ternary relations.

Definition 3.12. Let $*$ and $\star$ be two binary operations on a set X. We say that the couple $(*, \star)$ is mixed-commutative if $x * y = y \star x$, for any $x, y \in X$.

Obviously, if a couple $(*, \star)$ is mixed-commutative, then also the converse couple $(\star, *)$ is mixed-commutative. Four-point compositions of ternary relations are not commutative in general, but to any 4-point composition corresponds another one so that they make up a mixed-commutative couple.

Proposition 3.13. *Let $p = (kji)_6 \in \{0, \dots, 215\}$. The couples $(\square_p^0, \square_{p'}^0)$ and $(\square_p^1, \square_{p'}^2)$ are mixed-commutative, with $p' = (\pi_3(k)\pi_3(i)\pi_3(j))_6$ and π_3 the permutation of $\{0, \dots, 5\}$ given in Table 4:*

Table 4 The permutation π_3.

u	0	1	2	3	4	5
$\pi_3(u)$	1	0	4	5	2	3

Proof. We give the proof for $\left(\square_p^0, \square_{p'}^0\right)$. Let S and T be two ternary relations on X, then it holds that

$$\begin{aligned} S\square_p^0 T &= \{\sigma_k(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_i(x,y,t) \in S \wedge \sigma_j(x,t,z) \in T)\} \\ &= \{\sigma_k(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_j(x,t,z) \in T \wedge \sigma_i(x,y,t) \in S)\} \\ &= \{\sigma_k(\sigma_1(x,z,y)) \in X^3 \mid (\exists t \in X)(\sigma_j(\sigma_1(x,z,t)) \in T \wedge \sigma_i(\sigma_1(x,t,y)) \in S)\} \\ &= \{\sigma_k(\sigma_1(x,y,z)) \in X^3 \mid (\exists t \in X)(\sigma_j(\sigma_1(x,y,t)) \in T \wedge \sigma_i(\sigma_1(x,t,z)) \in S)\}\,. \end{aligned}$$

Since $\sigma_u(\sigma_1(x,y,z)) = \sigma_{\pi_3(u)}(x,y,z)$, it follows that

$$S\square_p^0 T = \{\sigma_{\pi_3(k)}(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_{\pi_3(j)}(x,y,t) \in T \wedge \sigma_{\pi_3(i)}(x,t,z) \in S)\} = T\square_{p'}^0 S\,.$$

Hence, the couple $\left(\square_p^0, \square_{p'}^0\right)$ is mixed-commutative. □

In view of Proposition 3.13, indeed no 4-point composition is commutative.

3.2.2 Five-point compositions of ternary relations

In this subsection, we study the 5-point compositions of ternary relations. In view of the notation used for 4-point compositions, we use the mnemonic notation $\pentagon$ for 5-point compositions. A 4-point composition can be modified to obtain a 5-point composition by introducing an additional degree of freedom, substituting one of the occurrences of the element appearing twice. More precisely, the 4-point composition $\square^0_p$ in Definition 3.1 can be modified in two ways to obtain a 5-point composition. The first one is obtained by replacing the element x in the first 3-tuple (x, y, t) by a new degree of freedom s as follows:

$$S \,{\scriptstyle\pentagon}'_p\, T = \{\sigma_k(x, y, z) \in X^3 \mid (\exists (s, t) \in X^2)(\sigma_i(s, y, t) \in S \wedge \sigma_j(x, t, z) \in T)\}\,, \tag{4}$$

while the second one is obtained by replacing the element x in the second 3-tuple (x, t, z) as follows:

$$S \,{\scriptstyle\pentagon}''_p\, T = \{\sigma_k(x, y, z) \in X^3 \mid (\exists (s, t) \in X^2)(\sigma_i(x, y, t) \in S \wedge \sigma_j(s, t, z) \in T)\}\,. \tag{5}$$

This reasoning could give the impression that there are twice as many 5-point compositions as there are 4-point compositions. However, there are only 108 different compositions of type ${\scriptstyle\pentagon}'_p$ and 108 different compositions of type ${\scriptstyle\pentagon}''_p$. Indeed, each 5-point composition is obtained in two different ways. For $p = (kji)_6 \in \{0, \dots, 215\}$, it holds that $S \,{\scriptstyle\pentagon}'_p\, T = S \,{\scriptstyle\pentagon}'_q\, T$, with $q = (\pi_1(k) j \pi_1(i))_6$ and π_1 the permutation of $\{0, \dots, 5\}$ given in Table 1. Similarly, for $p = (kji)_6 \in \{0, \dots, 215\}$, it holds that $S \,{\scriptstyle\pentagon}''_p\, T = S \,{\scriptstyle\pentagon}''_q\, T$, with $q = (\pi_2(k)\pi_2(j) i)_6$ and π_2 the permutation of $\{0, \dots, 5\}$ given in Table 2.

In the following definition, we identify the 216 different 5-point compositions of ternary relations by using an elegant enumeration system.

Definition 3.14. Let $p \in \{0, \dots, 215\}$ with $p = (kji)_6$. For any ternary relations S and T on X, we define the composition $S \,{\scriptstyle\pentagon}_p\, T$ as follows:

$$S \,{\scriptstyle\pentagon}_p\, T = \{\sigma_k(x, y, z) \in X^3 \mid (\exists (s, t) \in X^2)(\sigma_i(\hat{x}, y, t) \in S \wedge \sigma_j(\check{x}, t, z) \in T)\}\,, \tag{6}$$

with

$$\hat{x} = \begin{cases} x & \text{, if } k \in \{0, 3, 4\} \\ s & \text{, if } k \in \{1, 2, 5\} \end{cases}\,, \quad \check{x} = \begin{cases} s & \text{, if } k \in \{0, 3, 4\} \\ x & \text{, if } k \in \{1, 2, 5\} \end{cases}\,.$$

The same 5-point compositions would have been obtained when starting from the 4-point compositions $\square^1_p$ and $\square^2_p$. Indeed, by substituting one of the occurrences of the element z by s in the 54 compositions of the type $\square^1_p$, we obtain the 108 five-point compositions ${\scriptstyle\pentagon}_p$, with $k \in \{0, 3, 4\}$. Similarly, starting from the compositions of the type $\square^2_p$, we obtain the other 108 five-point compositions ${\scriptstyle\pentagon}_p$, with $k \in \{1, 2, 5\}$. Note that the 5-point compositions ${\scriptstyle\pentagon}_0$, ${\scriptstyle\pentagon}_{12}$, ${\scriptstyle\pentagon}_{18}$, ${\scriptstyle\pentagon}_{84}$, ${\scriptstyle\pentagon}_{86}$ and ${\scriptstyle\pentagon}_{87}$ were already introduced in (Bakri, Zedam, and De Baets 2021).

The following examples illustrate Definition 3.14.

Example 3.15. Consider $p = 175$ and two ternary relations S and T on X. In senary notation, it holds that $p = (451)_6$. Since $k = 4$, we have

$$\begin{aligned} S \,{\scriptstyle\pentagon}_{175}\, T &= \{\sigma_4(x, y, z) \in X^3 \mid (\exists (s, t) \in X^2)(\sigma_1(\hat{x}, y, t) \in S \wedge \sigma_5(\check{x}, t, z) \in T)\} \\ &= \{\sigma_4(x, y, z) \in X^3 \mid (\exists (s, t) \in X^2)(\sigma_1(x, y, t) \in S \wedge \sigma_5(s, t, z) \in T)\} \\ &= \{(z, x, y) \in X^3 \mid (\exists (s, t) \in X^2)((x, t, y) \in S \wedge (z, t, s) \in T)\} \end{aligned}$$

$$= \{(x,y,z) \in X^3 \mid (\exists (s,t) \in X^2)((y,t,z) \in S \wedge (x,t,s) \in T)\}\,.$$

Example 3.16. Consider two ternary relations S and T on X and the binary operation $*$ defined as follows:

$$S * T = \{(x,y,z) \in X^3 \mid (\exists (s,t) \in X^2)((s,t,z) \in S \wedge (y,t,x) \in T)\}\,.$$

The operation $*$ corresponds to a composition of the type $\circ_p$, with $p \in \{0,\dots,215\}$. We express this composition as in Definition 3.14 by identifying the proper 3-permutations. We distinguish two possibilities:

$$S * T = \{\sigma_1(x,y,z) \in X^3 \mid (\exists (s,t) \in X^2)(\sigma_1(s,y,t) \in S \wedge \sigma_5(x,t,z) \in T)\} \qquad (7)$$

or

$$S * T = \{\sigma_3(x,y,z) \in X^3 \mid (\exists (s,t) \in X^2)(\sigma_1(s,y,t) \in S \wedge \sigma_0(x,t,z) \in T)\}\,. \qquad (8)$$

According to our enumeration scheme, it holds that $k \in \{1,2,5\}$. Therefore, Eq. (7) is the proper one, *i.e.*, $i = 1$, $j = 5$ and $k = 1$. Hence $* = \circ_{67}$.

The following proposition expresses all the 5-point compositions in terms of the compositions $\circ_0$ or $\circ_{210}$, which we will refer to as the basic 5-point compositions.

Proposition 3.17. *Let $p = (kji)_6 \in \{0,\dots,215\}$. For any ternary relations S and T on X, the 5-point composition $S \circ_p T$ can be written in terms of the compositions $\circ_0$ or $\circ_{210}$ as follows:*

$$S \circ_p T = \begin{cases} \left(S^{\sigma_i^{-1}} \circ_0 T^{\sigma_j^{-1}}\right)^{\sigma_k} & \text{, if } k \in \{0,3,4\}\,, \\ \left(S^{\sigma_i^{-1}} \circ_{210} T^{\sigma_j^{-1}}\right)^{\sigma_k} & \text{, if } k \in \{1,2,5\}\,. \end{cases}$$

The following proposition shows some inclusion relationships between the 4-point and 5-point compositions.

Proposition 3.18. *Let $p = (kji)_6 \in \{0,\dots,215\}$. For any ternary relations S and T on X, it holds that*

(i) $S \square_p^0 T \subseteq S \circ_p T$;

(ii) $S \square_p^1 T \subseteq S \circ_p T$, *if* $k \in \{0,3,4\}$;

(iii) $S \square_p^2 T \subseteq S \circ_p T$, *if* $k \in \{1,2,5\}$.

Proof. We give the proof of (i). If $k \in \{0,3,4\}$, then

$$\begin{aligned} S \square_p^0 T &= \{\sigma_k(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_i(x,y,t) \in S \wedge \sigma_j(x,t,z) \in T)\} \\ &\subseteq \{\sigma_k(x,y,z) \in X^3 \mid (\exists (s,t) \in X^2)(\sigma_i(x,y,t) \in S \wedge \sigma_j(s,t,z) \in T)\} \\ &= S \circ_p T\,. \end{aligned}$$

Similarly, if $k \in \{1,2,5\}$, then

$$\begin{aligned} S\square_p^0 T &= \{\sigma_k(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_i(x,y,t) \in S \wedge \sigma_j(x,t,z) \in T)\} \\ &\subseteq \{\sigma_k(x,y,z) \in X^3 \mid (\exists (s,t) \in X^2)(\sigma_i(s,y,t) \in S \wedge \sigma_j(x,t,z) \in T)\} \\ &= S\pentagon_p T\,. \end{aligned}$$

□

Next, we study some basic properties of the 5-point compositions of ternary relations, some of them being similar to those of the 4-point compositions, while others being different.

The following proposition identifies the right and left neutral elements of the 5-point compositions of ternary relations.

Proposition 3.19. *Let $i,j,k \in \{0,\dots,5\}$. For any ternary relation T on X, it holds that*
(i) $T\pentagon_{(kjk)_6} E_{\zeta(j)} = T$*, if* $k \in \{0,3,4\}$;
(ii) $E_{\zeta(i)}\pentagon_{(kki)_6} T = T$*, if* $k \in \{1,2,5\}$,
with ζ as in Proposition 3.10.

It is worth noting that in contrast to the 4-point case, no 5-point composition has a neutral element.

The following proposition discusses the mixed-commutativity of the 5-point compositions. This result is analogous to that for the 4-point compositions.

Proposition 3.20. *Let $p = (kji)_6 \in \{0,\dots,215\}$. The couple $(\pentagon_p, \pentagon_{p'})$ is mixed-commutative, with $p' = (\pi_3(k)\pi_3(i)\pi_3(j))_6$ and π_3 as in Proposition 3.13.*

4 Associativity and mixed-associativity

The most desirable property of a relational composition undeniably is its associativity. In this section, we study the associativity and mixed-associativity of the 4-point and 5-point compositions of ternary relations introduced in this paper.

4.1 Associativity of compositions of ternary relations

The following proposition identifies the associative 4-point compositions.

Proposition 4.1. *The 4-point composition $\square_p^0$ is associative if and only if $p = (iii)_6 = 43i$, with $i \in \{0,\dots,5\}$.*

Proof. Let $p = (kji)_6$. For any three ternary relations R,S,T on X, we have

$$(R\square_p^0 S)\square_p^0 T = \{\sigma_k(x,y,z) \in X^3 \mid (\exists t \in X)(\sigma_i(x,y,t) \in R\square_p^0 S \wedge \sigma_j(x,t,z) \in T)\}\,.$$

Since the set of 3-permutations is a group, there exists $m \in \{0, \ldots, 5\}$ such that $\sigma_i = \sigma_k \sigma_m$. Hence, we can write

$$
\begin{aligned}
& (R \square_p^0 S) \square_p^0 T \\
= \ & \{\sigma_k(x, y, z) \in X^3 \mid (\exists t \in X)(\sigma_k(\sigma_m(x, y, t)) \in R \square_p^0 S \wedge \sigma_j(x, t, z) \in T)\} \\
= \ & \{\sigma_k(x, y, z) \in X^3 \mid (\exists (s, t) \in X^2)(\sigma_i(\sigma_m(x, y, s)) \in R \wedge \sigma_j(\sigma_m(x, s, t)) \in S \wedge \sigma_j(x, t, z) \in T)\}.
\end{aligned}
$$

Using the same argument, there exists $n \in \{0, \ldots, 5\}$ such that $\sigma_j = \sigma_k \sigma_n$, and we can write

$$
\begin{aligned}
& R \square_p^0 (S \square_p^0 T) \\
= \ & \{\sigma_k(x, y, z) \in X^3 \mid (\exists u \in X)(\sigma_i(x, y, u) \in R \wedge \sigma_j(x, u, z) \in S \square_p^0 T)\} \\
= \ & \{\sigma_k(x, y, z) \in X^3 \mid (\exists u \in X)(\sigma_i(x, y, u) \in R \wedge \sigma_k(\sigma_n(x, u, z)) \in S \square_p^0 T)\} \\
= \ & \{\sigma_k(x, y, z) \in X^3 \mid (\exists (u, v) \in X^2)(\sigma_i(x, y, u) \in R \wedge \sigma_i(\sigma_n(x, u, v)) \in S \wedge \sigma_j(\sigma_n(x, v, z)) \in T)\}.
\end{aligned}
$$

It clearly holds that $(R \square_p^0 S) \square_p^0 T = R \square_p^0 (S \square_p^0 T)$ if and only if $m = n = 0$, *i.e.*, if and only if $i = j = k$. Therefore, $\square_p^0$ is associative if and only if $i = j = k$. □

Remark 1. For any $p \in \{0, \ldots, 215\}$, the compositions $\square_p^1$ and $\square_p^2$ are not associative. Indeed, consider the three ternary relations R, S, T on the set $X = \{a, b, c, d, e\}$ given as:

$$
\begin{aligned}
R &= \{(a, b, c), (a, c, b), (b, a, c), (b, c, a), (c, a, b), (c, b, a)\}; \\
S &= \{(d, c, d), (d, d, c), (c, d, d)\}; \\
T &= \{(e, d, e), (e, e, d), (d, e, e)\}.
\end{aligned}
$$

One can verify that for any $p \in \{0, \ldots, 215\}$, there exists $i \in \{0, \ldots 5\}$ such that

$$
\sigma_i(a, b, e) \in (R \square_p^1 S) \square_p^1 T ,
$$

while $(R \square_p^1 S) \square_p^1 T = \emptyset$. The same holds for the compositions $\square_p^2$ by considering the relations

$$
\begin{aligned}
R &= \{(a, a, b), (a, b, a), (b, a, a)\}; \\
S &= \{(b, b, c), (b, c, b), (c, c, b)\}; \\
T &= \{(c, d, e), (c, e, d), (d, c, e), (d, e, c), (e, c, d), (e, d, c)\}.
\end{aligned}
$$

In Table 5, we present the six associative 4-point compositions denoted by the new notation $\diamond_i$, $i \in \{1, \ldots, 6\}$.

Proposition 3.13 leads to the following corollary. It expresses that the associative 4-point compositions make up three couples of compositions that are mixed-commutative.

Corollary 4.2. *The couples* $(\diamond_1, \diamond_2)$, $(\diamond_3, \diamond_5)$ *and* $(\diamond_4, \diamond_6)$ *are mixed-commutative.*

Table 5 The associative 4-point compositions. The 3-tuple (x, y, z) belongs to $S \diamond_i T$ if there exists an element $t \in X$ such that the other listed 3-tuples belong to S and T, respectively.

$\diamond_i$	$S \diamond_i T$	S	T
$\diamond_1 = \Box^0_0$	(x, y, z)	(x, y, t)	(x, t, z)
$\diamond_2 = \Box^0_{43}$	(x, y, z)	(x, t, z)	(x, y, t)
$\diamond_3 = \Box^0_{86}$	(x, y, z)	(x, y, t)	(t, y, z)
$\diamond_4 = \Box^0_{129}$	(x, y, z)	(x, t, z)	(t, y, z)
$\diamond_5 = \Box^0_{172}$	(x, y, z)	(t, y, z)	(x, y, t)
$\diamond_6 = \Box^0_{215}$	(x, y, z)	(t, y, z)	(x, t, z)

The following result is a particular case of Proposition 3.10. It identifies the neutral element of the associative 4-point compositions.

Corollary 4.3.

(i) *E_ℓ is the neutral element of $\diamond_1$ and $\diamond_2$;*

(ii) *E_m is the neutral element of $\diamond_3$ and $\diamond_5$;*

(iii) *E_r is the neutral element of $\diamond_4$ and $\diamond_6$.*

Although there are more 4-point compositions than 5-point compositions, it turns out that there are more associative 5-point compositions than associative 4-point compositions. The following proposition identifies the associative 5-point compositions.

Proposition 4.4. *Let $p = (kji)_6 \in \{0, \ldots, 215\}$. The composition $\circ_p$ is associative if and only if one of the following conditions holds:*

(i) *$k = j = i$;*

(ii) *$k \in \{0, 3, 4\}$, $i = k$ and $j = \pi_4(k)$;*

(iii) *$k \in \{1, 2, 5\}$, $i = \pi_4(k)$ and $j = k$,*

with π_4 is the permutation of $\{0, \ldots, 5\}$ given in Table 6:

Table 6 The permutation π_4.

u	0	1	2	3	4	5
$\pi_4(u)$	2	4	3	1	5	0

Proof. Let R, S, T be three ternary relations on X. We consider the case that $k \in \{0, 3, 4\}$, the other case being similar. For m as in Proposition 4.1, we have

$$\begin{aligned}
(R \circ_p S) \circ_p T &= \{\sigma_k(x, y, z) \in X^3 \mid (\exists (s, t) \in X^2)(\sigma_i(x, y, t) \in R \circ_p S \wedge \sigma_j(s, t, z) \in T)\} \\
&= \{\sigma_k(x, y, z) \in X^3 \mid (\exists (s, t) \in X^2)(\sigma_k(\sigma_m(x, y, t)) \in R \circ_p S \wedge \sigma_j(s, t, z) \in T)\} \\
&= \{\sigma_k(x, y, z) \in X^3 \mid (\exists (s, t, s', t') \in X^4)(\sigma_i(\sigma_m(x, y, t')) \in R \\
&\quad \wedge \sigma_j(\sigma_m(s', t', t)) \in S \wedge \sigma_j(s, t, z) \in T)\}\,.
\end{aligned}$$

For n as in Proposition 4.1, we have

$$\begin{aligned}R\circ_p(S\circ_p T) &= \{\sigma_k(x,y,z)\in X^3 \mid (\exists(u,v)\in X^2)(\sigma_i(x,y,u)\in R\wedge\sigma_j(v,u,z)\in S\circ_p T)\}\\ &= \{\sigma_k(x,y,z)\in X^3 \mid (\exists(u,v)\in X^2)(\sigma_i(x,y,u)\in R\wedge\sigma_k(\sigma_n(v,u,z))\in S\circ_p T)\}\\ &= \{\sigma_k(x,y,z)\in X^3 \mid (\exists(u,u',v,v')\in X^4)(\sigma_i(x,y,u)\in R\\ &\quad \wedge\sigma_i(\sigma_n(v,u,u'))\in S\wedge\sigma_j(\sigma_n(v',u',z))\in T)\}\,.\end{aligned}$$

By identification, $(R\circ_p S)\circ_p T = R\circ_p(S\circ_p T)$ if and only if $(m,n)=(0,0)$ or $(m,n)=(0,2)$, *i.e.*, if and only if

$$k=j=i \qquad \text{or} \qquad i=k \text{ and } j=\pi_4(k).$$

□

In Table 7, we present the associative 5-point compositions denoted by the new notation $\circ_i$, $i\in\{1,\ldots,12\}$.

Table 7 The associative 5-point compositions. The 3-tuple (x,y,z) belongs to $S\circ_i T$ if there exists two elements $s,t\in X$ such that the other listed 3-tuples belong to S and T, respectively.

$\circ_i$	$S\circ_i T$	S	T
$\circ_1=\circ_0$	(x,y,z)	(x,y,t)	(s,t,z)
$\circ_2=\circ_{12}$	(x,y,z)	(x,y,t)	(t,s,z)
$\circ_3=\circ_{43}$	(x,y,z)	(s,t,z)	(x,y,t)
$\circ_4=\circ_{46}$	(x,y,z)	(t,s,z)	(x,y,t)
$\circ_5=\circ_{86}$	(x,y,z)	(x,s,t)	(t,y,z)
$\circ_6=\circ_{87}$	(x,y,z)	(x,t,s)	(t,y,z)
$\circ_7=\circ_{117}$	(x,y,z)	(x,t,z)	(s,y,t)
$\circ_8=\circ_{129}$	(x,y,z)	(x,t,z)	(t,y,s)
$\circ_9=\circ_{172}$	(x,y,z)	(t,y,z)	(x,s,t)
$\circ_{10}=\circ_{178}$	(x,y,z)	(t,y,z)	(x,t,s)
$\circ_{11}=\circ_{210}$	(x,y,z)	(s,y,t)	(x,t,z)
$\circ_{12}=\circ_{215}$	(x,y,z)	(t,y,s)	(x,t,z)

Proposition 3.20 leads to the following corollary. It expresses that the associative 5-point compositions make up six couples of compositions that are mixed-commutative.

Corollary 4.5. *The couples $(\circ_1,\circ_3)$, $(\circ_2,\circ_4)$, $(\circ_5,\circ_9)$, $(\circ_6,\circ_{10})$, $(\circ_7,\circ_{11})$ and $(\circ_8,\circ_{12})$ are mixed-commutative.*

4.2 Mixed-associativity of compositions of ternary relations

As a sequel to the study of the associativity property of the compositions of ternary relations, we next investigate the mixed-associativity property for the 4-point compositions. First, we present a proposition expressing a useful identity for further purposes.

Proposition 4.6. *Let $\alpha, \beta, \gamma, \delta \in \{0, \dots, 5\}$. For any ternary relations R, S, T on X, the following equalities hold:*

(i) $(R \square^0_{(\alpha\beta\gamma)_6} S) \square^0_{(\alpha\delta\alpha)_6} T = R \square^0_{(\alpha\alpha\gamma)_6} (S \square^0_{(\alpha\delta\beta)_6} T)$;

(ii) $(R \square^0_{(\alpha\beta\gamma)_6} S) \square^1_{(\alpha\delta\alpha)_6} T = R \square^0_{(\alpha\alpha\gamma)_6} (S \square^1_{(\alpha\delta\beta)_6} T)$;

(iii) $(R \square^2_{(\alpha\beta\gamma)_6} S) \square^0_{(\alpha\delta\alpha)_6} T = R \square^2_{(\alpha\alpha\gamma)_6} (S \square^0_{(\alpha\delta\beta)_6} T)$;

(iv) $(R \square^2_{(\alpha\beta\gamma)_6} S) \square^1_{(\alpha\delta\alpha)_6} T = R \square^2_{(\alpha\alpha\gamma)_6} (S \square^1_{(\alpha\delta\beta)_6} T)$.

Proof. We prove case (i).

$$
\begin{aligned}
(R \square^0_{(\alpha\beta\gamma)_6} S) \square^0_{(\alpha\delta\alpha)_6} T &= \left(\left(R \square^0_{(\alpha\beta\gamma)_6} S \right)^{\sigma_\alpha^{-1}} \square^0_0 T^{\sigma_\delta^{-1}} \right)^{\sigma_\alpha} \\
&= \left(\left(\left(R^{\sigma_\gamma^{-1}} \square^0_0 S^{\sigma_\beta^{-1}} \right)^{\sigma_\alpha} \right)^{\sigma_\alpha^{-1}} \square^0_0 T^{\sigma_\delta^{-1}} \right)^{\sigma_\alpha} \\
&= \left(\left(R^{\sigma_\gamma^{-1}} \square^0_0 S^{\sigma_\beta^{-1}} \right) \square^0_0 T^{\sigma_\delta^{-1}} \right)^{\sigma_\alpha} \\
&= \left(R^{\sigma_\gamma^{-1}} \square^0_0 \left(S^{\sigma_\beta^{-1}} \square^0_0 T^{\sigma_\delta^{-1}} \right) \right)^{\sigma_\alpha} \\
&= \left(R^{\sigma_\gamma^{-1}} \square^0_0 \left(\left(S^{\sigma_\beta^{-1}} \square^0_0 T^{\sigma_\delta^{-1}} \right)^{\sigma_\alpha} \right)^{\sigma_\alpha^{-1}} \right)^{\sigma_\alpha} \\
&= R \square^0_{(\alpha\alpha\gamma)_6} \left(\left(S^{\sigma_\beta^{-1}} \square^0_0 T^{\sigma_\delta^{-1}} \right)^{\sigma_\alpha} \right) \\
&= R \square^0_{(\alpha\alpha\gamma)_6} \left(S \square^0_{(\alpha\delta\beta)_6} T \right) .
\end{aligned}
$$

□

One can verify that for any $r \in \{0, 1, 2\}$ and $p, q \in \{0, \dots, 215\}$, the equalities

$$(R \square^1_p S) \square^r_q T = R \square^1_p (S \square^r_q T) \text{ and } (R \square^r_p S) \square^2_q T = R \square^r_p (S \square^2_q T)$$

do not hold in general.

Next, we recall the property of mixed-associativity of binary operations, also called linear distributivity (Koslowski 2003). Note that remarkable instances of this property have been identified for the relational compositions (in the crisp as well as in the fuzzy case) in the groundbreaking studies of Bandler and Kohout (Bandler and Kohout 1980a,b; De Baets and Kerre 1993a,b).

Definition 4.7. Let $*$ and $\star$ be two binary operations on a set X. We say that the couple $(*, \star)$ is mixed-associative if $(x * y) \star z = x * (y \star z)$, for any $x, y, z \in X$.

Obviously, if $*$ is an associative operation, then $(*, *)$ is a trivial mixed-associative couple.

The following proposition presents the couples of 4-point compositions that satisfy the mixed-associativity property. This result follows from Proposition 4.6.

Corollary 4.8. *Let $\alpha, \beta, \gamma \in \{0, \dots, 5\}$. The following couples of 4-point compositions are mixed-associative:*

(i) $(\square^0_{(\alpha\alpha\beta)_6}, \square^0_{(\alpha\gamma\alpha)_6})$;

(ii) $(\square^0_{(\alpha\alpha\beta)_6}, \square^1_{(\alpha\gamma\alpha)_6})$;

(iii) $(\square^2_{(\alpha\alpha\beta)_6}, \square^0_{(\alpha\gamma\alpha)_6})$;

(iv) $(\square^2_{(\alpha\alpha\beta)_6}, \square^1_{(\alpha\gamma\alpha)_6})$.

Remark 2. Note that $(\square^0_p, \square^0_{p'})$ and $(\square^0_{p'}, \square^0_p)$ are simultaneously mixed-associative if and only if $p = p'$ and $\square^0_p$ is an associative 4-point composition.

Next, we study the mixed-associativity property for the 5-point compositions. To that end, we first present some more general compositional identities. The following lemma is a matter of simple verification.

Lemma 4.9. *The permutations π_3 in Proposition 3.13 and π_4 in Proposition 4.4 commute,* i.e., *for any $u \in \{0, \dots, 5\}$, it holds that*

$$\pi_3(\pi_4(u)) = \pi_4(\pi_3(u)) .$$

Proposition 4.10. *Let $\alpha, \beta, \gamma, \delta \in \{0, \dots, 5\}$. For any ternary relations R, S, T on X, the following equalities hold:*

(i) $(R \pentagon_{(\alpha\beta\gamma)_6} S) \pentagon_{(\alpha\delta\alpha)_6} T = R \pentagon_{(\alpha\alpha\gamma)_6} (S \pentagon_{(\alpha\delta\beta)_6} T)$;

(ii) $(R \pentagon_{(\alpha\pi_4(\beta)\gamma)_6} S) \pentagon_{(\alpha\delta\alpha)_6} T = R \pentagon_{(\alpha\pi_4(\alpha)\gamma)_6} (S \pentagon_{(\alpha\delta\beta)_6} T)$, *if $\alpha, \beta \in \{0, 3, 4\}$;*

(iii) $(R \pentagon_{(\pi_4(\alpha)\pi_4(\beta)\gamma)_6} S) \pentagon_{(\alpha\delta\alpha)_6} T = R \pentagon_{(\alpha\pi_4(\alpha)\pi_4(\gamma))_6} (S \pentagon_{(\alpha\delta\beta)_6} T)$, *if $\alpha, \beta \in \{0, 3, 4\}$;*

(iv) $(R \pentagon_{(\alpha\beta\gamma)_6} S) \pentagon_{(\alpha\delta\pi_4(\alpha))_6} T = R \pentagon_{(\alpha\alpha\gamma)_6} (S \pentagon_{(\alpha\delta\pi_4(\beta))_6} T)$, *if $\alpha, \beta \in \{1, 2, 5\}$;*

(v) $(R \pentagon_{(\alpha\beta\gamma)_6} S) \pentagon_{(\pi_4(\alpha)\delta\pi_4(\alpha))_6} T = R \pentagon_{(\alpha\alpha\gamma)_6} (S \pentagon_{(\pi_4(\alpha)\delta\pi_4(\beta))_6} T)$, *if $\alpha, \beta \in \{1, 2, 5\}$.*

Proof. The proof of (i) is similar to that of Proposition 4.6. It follows by expressing $(R \pentagon_{(\alpha\beta\gamma)_6} S) \pentagon_{(\alpha\delta\alpha)_6} T$ in terms of $\pentagon_0$ in the case that $\alpha \in \{0, 3, 4\}$, and in terms of $\pentagon_{210}$ in the case that $\alpha \in \{1, 2, 5\}$. Next, we prove items (ii) and (iv), items (iii) and (v) being similar. Let $\alpha, \beta \in \{0, 3, 4\}$. Using $\alpha \in \{0, 3, 4\}$, it follows that

$$\begin{aligned}
&(R \pentagon_{(\alpha\pi_4(\beta)\gamma)_6} S) \pentagon_{(\alpha\delta\alpha)_6} T \\
&= \{\sigma_\alpha(x, y, z) \in X^3 \mid (\exists (s, t) \in X^2)(\sigma_\alpha(x, y, t) \in R \pentagon_{(\alpha\pi_4(\beta)\gamma)_6} S \wedge \sigma_\delta(s, t, z) \in T)\} \\
&= \{\sigma_\alpha(x, y, z) \in X^3 \mid (\exists (s, t, s', t') \in X^4)(\sigma_\gamma(x, y, t') \in R \wedge \sigma_{\pi_4(\beta)}(s', t', t) \in S \wedge \sigma_\delta(s, t, z) \in T)\} .
\end{aligned}$$

Moreover, since $\beta \in \{0, 3, 4\}$, it holds that $\sigma_{\pi_4(\beta)} = \sigma_\beta \sigma_2$, and the proof continues

$$\begin{aligned}
&\{\sigma_\alpha(x, y, z) \in X^3 \mid (\exists (s, t, s', t') \in X^4)(\sigma_\gamma(x, y, t') \in R \wedge \sigma_{\pi_4(\beta)}(s', t', t) \in S \wedge \sigma_\delta(s, t, z) \in T)\} \\
&= \{\sigma_\alpha(x, y, z) \in X^3 \mid (\exists (s, t, s', t') \in X^4)(\sigma_\gamma(x, y, t') \in R \wedge \sigma_\beta(t', s', t) \in S \wedge \sigma_\delta(s, t, z) \in T)\} \\
&= \{\sigma_\alpha(x, y, z) \in X^3 \mid (\exists (s, t, s', t') \in X^4)(\sigma_\gamma(x, y, t') \in R \wedge \sigma_\alpha(t', s', z) \in S \pentagon_{(\alpha\delta\beta)_6} T)\} \\
&= \{\sigma_\alpha(x, y, z) \in X^3 \mid (\exists (s, t, s', t') \in X^4)(\sigma_\gamma(x, y, t') \in R \wedge \sigma_{\pi_4(\alpha)}(s', t', z) \in S \pentagon_{(\alpha\delta\beta)_6} T)\}
\end{aligned}$$

$$= R\pentagon_{(\alpha\pi_4(\alpha)\gamma)_6}(S\pentagon_{(\alpha\delta\beta)_6}T)\,.$$

This concludes the proof of (ii). Next, starting from (ii), we obtain using Proposition 3.20 that

$$T\pentagon_{(\pi_3(\alpha)\pi_3(\alpha)\pi_3(\delta))_6}(S\pentagon_{(\pi_3(\alpha)\pi_3(\gamma)\pi_3(\pi_4(\beta)))_6}R) = (T\pentagon_{(\pi_3(\alpha)\pi_3(\beta)\pi_3(\delta))_6}S)\pentagon_{(\pi_3(\alpha)\pi_3(\gamma)\pi_3(\pi_4(\alpha)))_6}R\,.$$

Using Lemma 4.9, we can rewrite this as:

$$T\pentagon_{(\pi_3(\alpha)\pi_3(\alpha)\pi_3(\delta))_6}(S\pentagon_{(\pi_3(\alpha)\pi_3(\gamma)\pi_4(\pi_3(\beta)))_6}R) = (T\pentagon_{(\pi_3(\alpha)\pi_3(\beta)\pi_3(\delta))_6}S)\pentagon_{(\pi_3(\alpha)\pi_3(\gamma)\pi_4(\pi_3(\alpha)))_6}R\,.$$

Now, a simple renaming $(\alpha',\beta',\delta',\gamma') := (\pi_3(\alpha),\pi_3(\beta),\pi_3(\delta),\pi_3(\gamma))$ and $(R',S',T') := (T,S,R)$ yields

$$R'\pentagon_{(\alpha'\alpha'\gamma')_6}(S'\pentagon_{(\alpha'\delta'\pi_4(\beta'))_6}T') = (R'\pentagon_{(\alpha'\beta'\gamma')_6}S')\pentagon_{(\alpha'\delta'\pi_4(\alpha'))_6}T'\,.$$

Realizing that now $\alpha',\beta' \in \{1,2,5\}$, the proof of (iv) is complete. □

By taking $\alpha = \beta$ in Proposition 4.10, we obtain the following proposition about the mixed-associativity of 5-point compositions.

Corollary 4.11. *Let $\alpha,\beta,\gamma \in \{0,\dots,5\}$. The following couples of 5-point compositions are mixed-associative:*

(i) $\left(\pentagon_{(\alpha\alpha\beta)_6},\pentagon_{(\alpha\gamma\alpha)_6}\right)$;
(ii) $\left(\pentagon_{(\alpha\pi_4(\alpha)\beta)_6},\pentagon_{(\alpha\gamma\alpha)_6}\right)$, *if* $\alpha \in \{0,3,4\}$;
(iii) $\left(\pentagon_{(\pi_4(\alpha)\pi_4(\alpha)\beta)_6},\pentagon_{(\alpha\gamma\alpha)_6}\right)$, *if* $\alpha \in \{0,3,4\}$;
(iv) $\left(\pentagon_{(\alpha\alpha\beta)_6},\pentagon_{(\alpha\gamma\pi_4(\alpha))_6}\right)$, *if* $\alpha \in \{1,2,5\}$;
(v) $\left(\pentagon_{(\alpha\alpha\beta)_6},\pentagon_{(\pi_4(\alpha)\gamma\pi_4(\alpha))_6}\right)$, *if* $\alpha \in \{1,2,5\}$.

Remark 3. Although there exists no mixed-associative couple consisting of different 4-point compositions for which also the converse couple is mixed-associative, such couples can be found in the case of 5-point compositions:

(i) $(\pentagon_{(\alpha\alpha\alpha)_6},\pentagon_{(\alpha\pi_4(\alpha)\alpha)_6})$, with $\alpha \in \{0,3,4\}$;
(ii) $(\pentagon_{(\alpha\alpha\alpha)_6},\pentagon_{(\alpha\alpha\pi_4(\alpha))_6})$, with $\alpha \in \{1,2,5\}$.

Interestingly, according to Proposition 4.4, these couples consist of two different associative compositions.

The following proposition allows to identify mixed-associative couples consisting of a 4-point and a 5-point composition.

Proposition 4.12. *Let $\alpha,\beta,\gamma,\delta \in \{0,\dots,5\}$. For any ternary relations R,S,T on X, it holds that*

(i) $(R\square^0_{(\alpha\beta\gamma)_6}S)\pentagon_{(\alpha\delta\alpha)_6}T = R\square^0_{(\alpha\alpha\gamma)_6}(S\pentagon_{(\alpha\delta\beta)_6}T)$, *if* $\alpha \in \{0,3,4\}$;
(ii) $(R\pentagon_{(\alpha\beta\gamma)_6}S)\square^0_{(\alpha\delta\alpha)_6}T = R\pentagon_{(\alpha\alpha\gamma)_6}(S\square^0_{(\alpha\delta\beta)_6}T)$, *if* $\alpha \in \{1,2,5\}$;
(iii) $(R\pentagon_{(\alpha\beta\gamma)_6}S)\square^1_{(\alpha\delta\alpha)_6}T = R\pentagon_{(\alpha\alpha\gamma)_6}(S\square^1_{(\alpha\delta\beta)_6}T)$, *if* $\alpha \in \{1,2,5\}$;

(iv) $(R\square^2_{(\alpha\beta\gamma)_6} S)\pentagon_{(\alpha\delta\alpha)_6} T = R\square^2_{(\alpha\alpha\gamma)_6}(S\pentagon_{(\alpha\delta\beta)_6} T)$, *if* $\alpha \in \{0,3,4\}$.

Proof. We give the proof of (i).

$$
\begin{aligned}
& (R\square^0_{(\alpha\beta\gamma)_6} S)\pentagon_{(\alpha\delta\alpha)_6} T \\
= \ & \{\sigma_\alpha(x,y,z) \in X^3 \mid (\exists (s,t) \in X^2)(\sigma_\alpha(x,y,t) \in R\square^0_{(\alpha\beta\gamma)_6} S \wedge \sigma_\delta(s,t,z) \in T)\} \\
= \ & \{\sigma_\alpha(x,y,z) \in X^3 \mid (\exists (s,t,u) \in X^3)(\sigma_\gamma(x,y,u) \in R \wedge \sigma_\beta(x,u,t) \in S \wedge \sigma_\delta(s,t,z) \in T)\} \\
= \ & \{\sigma_\alpha(x,y,z) \in X^3 \mid (\exists u \in X)(\sigma_\gamma(x,y,u) \in R \wedge \sigma_\alpha(x,u,z) \in S\pentagon_{(\alpha\delta\beta)_6} T)\} \\
= \ & R\square^0_{(\alpha\alpha\gamma)_6}(S\pentagon_{(\alpha\delta\beta)_6} T)\,.
\end{aligned}
$$

□

The following proposition presents the couples of 4-point and 5-point compositions that satisfy the mixed-associativity property. This result follows from Proposition 4.12.

Corollary 4.13. *Let* $\alpha,\beta,\gamma \in \{0,\ldots,5\}$. *The following couples of* 4*-point and* 5*-point compositions are mixed-associative:*

(i) $\left(\square^0_{(\alpha\alpha\beta)_6}, \pentagon_{(\alpha\gamma\alpha)_6}\right)$, *if* $\alpha \in \{0,3,4\}$;

(ii) $\left(\pentagon_{(\alpha\alpha\beta)_6}, \square^0_{(\alpha\gamma\alpha)_6}\right)$, *if* $\alpha \in \{1,2,5\}$;

(iii) $\left(\pentagon_{(\alpha\alpha\beta)_6}, \square^1_{(\alpha\gamma\alpha)_6}\right)$, *if* $\alpha \in \{1,2,5\}$;

(iv) $\left(\square^2_{(\alpha\alpha\beta)_6}, \pentagon_{(\alpha\gamma\alpha)_6}\right)$, *if* $\alpha \in \{0,3,4\}$.

5 Links between compositions of binary relations and compositions of ternary relations

In this section, we explore the links between the 3-point compositions of binary relations and the 4- and 5-point compositions of ternary relations. The notions of binary projections of a ternary relation and of cylindrical extensions of a binary relation (Zadeh 1975) play a central role in establishing these links.

5.1 From three-point compositions of binary relations to four-point compositions of ternary relations

In this subsection, we show that any cylindrical extension of a 3-point composition of binary relations contains a 4-point composition of their cylindrical extensions. First, we recall the definition of the cylindrical extensions of a binary relation.

Definition 5.1. (Zedam, Barkat and De Baets 2018) Let R be a binary relation on X.

(i) The left cylindrical extension of R is the ternary relation $C_\ell(R)$ on X defined as:

$$C_\ell(R) = \{(x,y,z) \in X^3 \mid (y,z) \in R\};$$

(ii) The middle cylindrical extension of R is the ternary relation $C_m(R)$ on X defined as:

$$C_m(R) = \{(x, y, z) \in X^3 \mid (x, z) \in R\};$$

(iii) The right cylindrical extension of R is the ternary relation $C_r(R)$ on X defined as:

$$C_r(R) = \{(x, y, z) \in X^3 \mid (x, y) \in R\}.$$

For further use, we introduce the following notation and lemma. Consider the mapping $\Gamma : \{0, 1\} \times \{\ell, m, r\} \to \{0, \ldots, 5\}$ given in Table 8:

Table 8 The mapping Γ.

i \ λ	ℓ	m	r
0	0	2	3
1	1	4	5

Lemma 5.2. *Let $i \in \{0, 1\}$ and $\lambda \in \{\ell, m, r\}$. For any binary relation R on X, it holds that*

$$\sigma_{\Gamma(\lambda,i)}(x, y, z) \in C_\lambda(R) \iff \rho_i(y, z) \in R\,.$$

Proof. We give the proof for $\lambda = \ell$. From Table 8, it follows that

$$\begin{aligned} \sigma_{\Gamma(\ell,i)}(x, y, z) \in C_\ell(R) &\iff \sigma_i(x, y, z) \in C_\ell(R) \\ &\iff \rho_i(y, z) \in R\,. \end{aligned}$$

□

Proposition 5.3. *Let $q = (kji)_2 \in \{0, \ldots, 7\}$, $r \in \{0, 1, 2\}$ and $\alpha, \beta, \gamma \in \{\ell, m, r\}$. For any binary relations R and P on X, the following inclusion holds:*

$$C_\alpha(R) \square_p^r C_\beta(P) \subseteq C_\gamma(R \circ_q P)\,,$$

with $p = (\Gamma(\gamma, k)\Gamma(\beta, j)\Gamma(\alpha, i))_6$.

Proof. We give the proof for $r = 0$.

$$\begin{aligned} & C_\alpha(R) \square_p^0 C_\beta(P) \\ = & \left\{\sigma_{\Gamma(\gamma,k)}(x, y, z) \mid (\exists t \in X)(\sigma_{\Gamma(\alpha,i)}(x, y, t) \in C_\alpha(R) \wedge \sigma_{\Gamma(\beta,j)}(x, t, z) \in C_\beta(P))\right\} \\ \subseteq & \left\{\sigma_{\Gamma(\gamma,k)}(x, y, z) \mid (\exists t \in X)(\rho_i(y, t) \in R \wedge \rho_j(t, z) \in P)\right\} \end{aligned}$$

$$= \left\{\sigma_{\Gamma(\gamma,k)}(x,y,z) \mid \rho_k(y,z) \in R\circ_q P\right\}$$
$$= \left\{\sigma_{\Gamma(\gamma,k)}(x,y,z) \mid \sigma_{\Gamma(\gamma,k)}(x,y,z) \in C_\gamma\left(R\circ_q P\right)\right\}$$
$$= C_\gamma\left(R\circ_q P\right) .$$

□

The following example shows that the inclusions in Proposition 5.3 are proper inclusions, *i.e.*, the cylindrical extension of the 3-point composition of two binary relations is in general not equal to the corresponding 4-point composition of the cylindrical extensions of these binary relations.

Example 5.4. Let R and S be the binary relations on the set $X = \{x_1, x_2, x_3, x_4\}$ given by $R = \{(x_1, x_2), (x_4, x_1)\}$ and $S = \{(x_1, x_3), (x_3, x_2)\}$. On the one hand, it holds that $R\circ_0 S = \{(x_4, x_3)\}$, and thus $C_\ell(R\circ_0 S) = \{(x_1, x_4, x_3), (x_2, x_4, x_3), (x_3, x_4, x_3), (x_4, x_4, x_3)\}$. On the other hand, we have $C_\ell(R)\square_0^0 C_\ell(S) = \{(x_1, x_4, x_3)\}$. It is clear that $C_\ell(R)\square_0^0 C_\ell(S) \subsetneq C_\ell(R\circ_0 S)$.

5.2 From four-point compositions of ternary relations to three-point compositions of binary relations

In this subsection, we show that any projection of a 4-point composition of two ternary relations is included in a corresponding 3-point composition of their binary projections. First, we recall the definition of the binary projections of a ternary relation.

Definition 5.5. (Zedam, Barkat and De Baets 2018) Let T be a ternary relation on X.

(i) The left projection of T is the binary relation $P_\ell(T)$ on X defined as:

$$P_\ell(T) = \{(x,y) \in X^2 \mid (\exists z \in X)((z,x,y) \in T)\};$$

(ii) The middle projection of T is the binary relation $P_m(T)$ on X defined as:

$$P_m(T) = \{(x,y) \in X^2 \mid (\exists z \in X)((x,z,y) \in T)\};$$

(iii) The right projection of T is the binary relation $P_r(T)$ on X defined as:

$$P_r(T) = \{(x,y) \in X^2 \mid (\exists z \in X)((x,y,z) \in T)\}.$$

Consider the mapping $\Pi : \{0,\dots,5\} \times \{\ell, m, r\} \to \{0,1\}$ defined as:

$$\Pi(k,\lambda) = \begin{cases} 1 & \text{, if } \Gamma(1,\lambda) = k \\ 0 & \text{, otherwise.} \end{cases} \tag{9}$$

Explicitly, $\Pi(k,\lambda)$ is given in Table 9. The following lemma is straightforward.

Table 9 The mapping Π.

k \ λ	ℓ	m	r
0	0	0	0
1	1	0	0
2	0	0	0
3	0	0	0
4	0	1	0
5	0	0	1

Lemma 5.6. *Let T be a ternary relation on X. For any $k \in \{0, \ldots, 5\}$, it holds that*

$$P_{\omega(k)}(T^{\sigma_k}) = \left(P_{\ell}(T)\right)^{\rho_{\Pi(k,\omega(k))}},$$

with $\omega : \{0, \ldots, 5\} \to \{\ell, m, r\}$ the mapping given in Table 10:

Table 10 The mapping ω.

k	0	1	2	3	4	5
$\omega(k)$	ℓ	ℓ	m	r	m	r

Proposition 5.7. *Let $p = (kji)_6 \in \{0, \ldots, 215\}$ and $r \in \{0, 1, 2\}$. For any ternary relations S and T on X, the following inclusion holds:*

$$P_{\omega(k)}(S \square_p^r T) \subseteq P_{\omega(i)}(S) \circ_q P_{\omega(j)}(T), \tag{10}$$

with $q = (\Pi(k, \omega(k))\Pi(j, \omega(j))\Pi(i, \omega(i)))_2$.

Proof. We give the proof for the case $r = 0$. From Proposition 3.7 and Lemma 5.6, it follows that

$$\begin{aligned} P_{\omega(k)}(S \square_p^0 T) &= P_{\omega(k)}\left((S^{\sigma_i^{-1}} \square_0^0 T^{\sigma_j^{-1}})^{\sigma_k}\right) \\ &= \left(P_{\ell}(S^{\sigma_i^{-1}} \square_0^0 T^{\sigma_j^{-1}})\right)^{\rho_{\Pi(k,\omega(k))}} \\ &\subseteq \left(P_{\ell}(S^{\sigma_i^{-1}}) \circ P_{\ell}(T^{\sigma_j^{-1}})\right)^{\rho_{\Pi(k,\omega(k))}} . \end{aligned}$$

Lemma 5.6 also implies that $P_{\ell}(T^{\sigma_i^{-1}}) = (P_{\omega(i)}(T))^{\rho_{\Pi(i,\omega(i))}}$, and hence,

$$P_{\omega(k)}(S \square_p^0 T) \subseteq \left(P_{\omega(i)}(S)^{\rho_{\Pi(i,\omega(i))}} \circ P_{\omega(j)}(T)^{\rho_{\Pi(j,\omega(j))}}\right)^{\rho_{\Pi(k,\omega(k))}} = P_{\omega(i)}(S) \circ_q P_{\omega(j)}(T).$$

□

The following example shows that the inclusions in Proposition 5.7 are proper inclusions.

Example 5.8. Let S and T be the ternary relations on the set $X = \{x_1, x_2, x_3, x_4\}$ given by

$$\begin{aligned} S &= \{(x_1, x_1, x_2), (x_1, x_2, x_3)\}, \\ T &= \{(x_1, x_2, x_4), (x_2, x_4, x_1), (x_3, x_2, x_2)\}. \end{aligned}$$

We have $S \Box_0^0 T = \{(x_1, x_1, x_4)\}$ and $P_\ell(S \Box_0^0 T) = \{(x_1, x_4)\}$. Since $P_\ell(S) \circ P_\ell(T) = \{(x_1, x_2), (x_1, x_4)\}$, it holds that $P_\ell(S \Box_0^0 T) \neq P_\ell(S) \circ P_\ell(T)$.

5.3 From three-point compositions of binary relations to five-point compositions of ternary relations and vice versa

The following result shows that any cylindrical extension of a 3-point composition of binary relations is equal to a corresponding 5-point composition of their cylindrical extensions.

Proposition 5.9. *Let $q = (kji)_2 \in \{0, \dots, 7\}$ and $\alpha, \beta, \gamma \in \{\ell, m, r\}$. For any binary relations R and P on X, the following equality holds:*

$$C_\gamma(R \circ_q P) = C_\alpha(R) \star_p C_\beta(P),$$

with p as in Proposition 5.3.

Proof. For $p = (\Gamma(\gamma, k)\Gamma(\beta, j)\Gamma(\alpha, i))_6$, it holds that

$$\begin{aligned} C_\alpha(R) \star_p C_\beta(P) &= \left\{\sigma_{\Gamma(\gamma,k)}(x, y, z) \mid (\exists t \in X)(\sigma_{\Gamma(\alpha,i)}(x, y, t) \in C_\alpha(R) \wedge \sigma_{\Gamma(\beta,j)}(x, t, z) \in C_\beta(P))\right\} \\ &= \left\{\sigma_{\Gamma(\gamma,k)}(x, y, z) \mid (\exists t \in X)(\rho_i(y, t) \in R \wedge \rho_j(t, z) \in P)\right\} \\ &= \left\{\sigma_{\Gamma(\gamma,k)}(x, y, z) \mid \rho_k(y, z) \in R \circ_q P\right\} \\ &= \left\{\sigma_{\Gamma(\gamma,k)}(x, y, z) \mid \sigma_{\Gamma(\gamma,k)}(x, y, z) \in C_\gamma\left(R \circ_q P\right)\right\} \\ &= C_\gamma\left(R \circ_q P\right). \end{aligned}$$

□

Conversely, we have the following result.

Proposition 5.10. *Let $p = (kji)_6 \in \{0, \dots, 215\}$. For any ternary relations S and T on X, the following equality holds:*

$$P_{\omega(k)}(S \star_p T) = P_{\omega(i)}(S) \circ_q P_{\omega(j)}(T), \tag{11}$$

with q as in Proposition 5.7.

Proof. We set $\eta = 0$ if $k \in \{0, 3, 4\}$, and $\eta = 210$ if $k \in \{1, 2, 5\}$. From Proposition 3.17 and Lemma 5.6, it follows that

$$\begin{aligned} P_{\omega(k)}(S \pentagon_p T) &= P_{\omega(k)}\left((S^{\sigma_i^{-1}} \pentagon_\eta T^{\sigma_j^{-1}})^{\sigma_k} \right) \\ &= \left(P_\ell(S^{\sigma_i^{-1}} \pentagon_\eta T^{\sigma_j^{-1}}) \right)^{\rho_{\Pi(k,\omega(k))}} \\ &= \left(P_\ell(S^{\sigma_i^{-1}}) \circ P_\ell(T^{\sigma_j^{-1}}) \right)^{\rho_{\Pi(k,\omega(k))}} . \end{aligned}$$

From Lemma 5.6, it follows that

$$P_{\omega(k)}(S \pentagon_p T) = \left(P_{\omega(i)}\,(S)^{\rho_{\Pi(i,\omega(i))}} \circ P_{\omega(j)}\,(T)^{\rho_{\Pi(j,\omega(j))}} \right)^{\rho_{\Pi(k,\omega(k))}} = P_{\omega(i)}(S) \circ_q P_{\omega(j)}(T) .$$

□

6 Conclusion

In this work, we have expounded a general approach to the study of the composition of ternary relations, resulting in a first type of 324 4-point compositions and a second type of 216 5-point compositions. For each type, we have provided a convenient enumeration scheme based on senary numbers that allows to generate all compositions. Also, we have presented a way of expressing all compositions in terms of a limited number of representative ones. We have identified all associative 4-point and 5-point-compositions, as well as couples of compositions satisfying the mixed-associativity property. Although there exist plenty such couples, only in the case of 5-point compositions, there exist mixed-associative couples consisting of different compositions for which also the converse couples are mixed-associative. Furthermore, we have provided some links between the compositions of binary relations and the two types of compositions of ternary relations. We anticipate that these types of compositions will be useful in some of the many fields of application mentioned in the introduction. They will most likely prove to be essential in the study of the transitivity of ternary relations.

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Conflict of interest The authors declare that they have no conflict of interest.

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