



Blow-up solutions of damped Klein–Gordon equation on the Heisenberg group

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Abstract

We prove the blow-up of solutions of the semilinear damped Klein–Gordon equation in a finite time for arbitrary positive initial energy on the Heisenberg group. This work complements the paper Ruzhansky and Tokmagambetov (J Differ Equ 265(10):5212–5236, 2018), where the global in time well-posedness was proved for the small energy solutions.

Keywords Blow-up · Sub-Laplacian · Heisenberg group · Damped Klein–Gordon equation

Mathematics Subject Classification 35L71 · 35R03 · 35B44

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1 Introduction

1.1 Setting of the problem

This note studies the blow-up of solutions of the Cauchy problem for the semilinear damped Klein–Gordon equation for the sub-Laplacian \mathcal{L} on the Heisenberg group \mathbb{H}^n :

$$\begin{cases} u_{tt}(t) - \mathcal{L}u(t) + bu_t(t) + mu(t) = f(u), & t > 0, \\ u(x, 0) = u_0(x), & u_0 \in H^1_{\mathcal{L}}(\mathbb{H}^n), \\ u_t(x, 0) = u_1(x), & u_1 \in L^2(\mathbb{H}^n), \end{cases} \tag{1.1}$$

with the damping term determined by $b > 0$ and the mass $m > 0$. A total energy of problem (1.1) is defined as

$$E(t) = \frac{1}{2} \|u_t\|^2_{L^2(\mathbb{H}^n)} + \frac{m}{2} \|u\|^2_{L^2(\mathbb{H}^n)} + \frac{1}{2} \|\nabla_H u\|^2_{L^2(\mathbb{H}^n)} - \int_{\mathbb{H}^n} F(u) \, dx,$$

where we assume the following:

$$\begin{aligned} g &: [0, \infty] \rightarrow \mathbb{R}, \\ F(z) &= g(|z|) \quad \text{for } z \in \mathbb{C}, \\ f(z) &= \frac{g'(|z|)z}{|z|}. \end{aligned} \tag{1.2}$$

Then we have

$$\begin{aligned} \frac{\partial}{\partial \varepsilon} F(z + \varepsilon \xi)|_{\varepsilon=0} &= \frac{\partial}{\partial \varepsilon} g(|z + \varepsilon \xi|)|_{\varepsilon=0} = g'(|z + \varepsilon \xi|) \frac{\partial}{\partial \varepsilon} (|z + \varepsilon \xi|)|_{\varepsilon=0} \\ &= \frac{g'(|z|)}{|z|} \frac{1}{2} (\bar{z}\xi + z\bar{\xi}) = \operatorname{Re}(f(z)\bar{\xi}), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial x_j} F(u(x)) &= g'(|u(x)|) \frac{\partial |u(x)|}{\partial x_j} \\ &= \frac{g'(|u(x)|)}{2|u(x)|} \left(u(x) \frac{\partial \bar{u}(x)}{\partial x_j} + \frac{\partial u(x)}{\partial x_j} \bar{u}(x) \right) = \operatorname{Re} \left(f(u(x)) \frac{\partial \bar{u}(x)}{\partial x_j} \right). \end{aligned}$$

1.2 Literature overview

The study of the damped wave equation on the Heisenberg group was started in Bahouri–Gerrard–Xu [1] to prove the dispersive and Strichartz inequalities based on the analysis in Besov-type spaces. Later, Greiner–Holcman–Kannai [7] explicitly computed the wave kernel for the class of second-order subelliptic operators, where

their class contains degenerate elliptic and hypoelliptic operators such as the sub-Laplacian and the Grushin operator. Also, Müller–Stein [16] established L^p -estimates for the wave equation on the Heisenberg group. Recently, Müller–Seeger [15] obtained the sharp version of L^p -estimates on the H -type groups.

The existence and nonexistence solutions of evolution equations on the Heisenberg group were considered by Georgiev and Palmieri in [8], where they proved the global existence and nonexistence results of the Cauchy problem for the semilinear damped wave equation on the Heisenberg group with the power nonlinear term. The linear damped wave equation in this setting was studied by Palmieri [17]. The proof of blow-up solutions is based on the test function method. The first author and Yessirkegenov [23] established the existence and nonexistence of global solutions for semilinear heat equations and inequalities on sub-Riemannian manifolds. In [24], by using the comparison principle, they obtained blow-up type results and global in t -boundedness of solutions of nonlinear equations for the heat p -sub-Laplacian on the stratified Lie groups. The global existence and nonexistence of solutions for the nonlinear porous medium equation were studied by the authors in [20] on the stratified Lie groups.

This work is motivated by the paper [22] of the first author and Tokmagambetov, where the global existence of solutions for small data of problem (1.1) was shown on the Heisenberg group and on general graded Lie groups. In the sense of the potential wells theory, we can understand this result in the sense that when the initial energy is less than the mountain pass level $E(0) < d$ and the Nehari functional is positive $I(u_0) > 0$, there exists a global solution of problem (1.1). A natural question arises when the solution of problem (1.1) blows up in a finite time for $E(0) > 0$ and $I(u_0) < 0$.

The main aim of this paper is to obtain the blow-up solutions of problem (1.1) in a finite time for arbitrary positive initial energy. Our proof is based on an adopted concavity method, which was introduced by Levine [10] to establish the blow-up solutions of the abstract wave equation of the form $Pu_{tt} = -Au + F(u)$ (including the Klein–Gordon equation) for the negative initial energy. It was also used for parabolic type equations (see [9, 11–14]). Also, we would like to mention the potential wells method was introduced by Sattinger [25], see also [19]. Modifying the concavity method, Wang [26] proved the nonexistence of global solutions to nonlinear damped Klein–Gordon equation for arbitrary positive initial energy under sufficient conditions. Later, Yang–Xu [28] extended this result by introducing a new auxiliary function and the adopted concavity method. Xu–Ding [27] and Pang–Yang [18] studied the blow-up solutions of the dissipative Klein–Gordon equation for an arbitrary positive initial energy.

1.3 Preliminaries on the Heisenberg group

Let us give a brief introduction to the Heisenberg group. Let \mathbb{H}^n be the Heisenberg group, that is, the set \mathbb{R}^{2n+1} equipped with the group law

$$\xi \circ \tilde{\xi} := \left(x + \tilde{x}, y + \tilde{y}, t + \tilde{t} + 2 \sum_{i=1}^n (\tilde{x}_i y_i - x_i \tilde{y}_i) \right),$$

where $\xi := (x, y, t) \in \mathbb{H}^n$, $x := (x_1, \dots, x_n)$, $y := (y_1, \dots, y_n)$, and $\xi^{-1} = -\xi$ is the inverse element of ξ with respect to the group law. The dilation operation of the Heisenberg group with respect to the group law has the following form (see e.g. [4, 21]):

$$\delta_\lambda(\xi) := (\lambda x, \lambda y, \lambda^2 t) \text{ for } \lambda > 0.$$

The Lie algebra \mathfrak{h} of the left-invariant vector fields on the Heisenberg group \mathbb{H}^n is spanned by

$$\begin{aligned} X_i &:= \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t} \text{ for } 1 \leq i \leq n, \\ Y_i &:= \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t} \text{ for } 1 \leq i \leq n, \end{aligned}$$

and with their (non-zero) commutator

$$[X_i, Y_i] = -4 \frac{\partial}{\partial t}.$$

The horizontal gradient of \mathbb{H}^n is given by

$$\nabla_H := (X_1, \dots, X_n, Y_1, \dots, Y_n),$$

so the sub-Laplacian on \mathbb{H}^n is given by

$$\mathcal{L} := \sum_{i=1}^n (X_i^2 + Y_i^2).$$

Definition 1.1 (*Weak solution*) A function

$$\begin{aligned} u &\in C([0, T_1]; H_{\mathcal{L}}^1(\mathbb{H}^n)) \cap C^1([0, T_1]; L^2(\mathbb{H}^n)), \\ u_t &\in L^2([0, T_1]; H_{\mathcal{L}}^1(\mathbb{H}^n)), \\ u_{tt} &\in L^2([0, T_1]; H_{\mathcal{L}}^{-1}(\mathbb{H}^n)), \end{aligned}$$

satisfying

$$\begin{aligned} \operatorname{Re} \langle u_{tt}, v \rangle + \operatorname{Re} \int_{\mathbb{H}^n} \nabla_H u \cdot \nabla_H v \, dx + m \operatorname{Re} \int_{\mathbb{H}^n} uv \, dx \\ + b \operatorname{Re} \int_{\mathbb{H}^n} u_t v \, dx = \operatorname{Re} \int_{\mathbb{H}^n} f(u) v \, dx, \end{aligned} \tag{1.3}$$

for all $v \in H_{\mathcal{L}}^1(\mathbb{H}^n)$ and a.e. $t \in [0, T_1]$ with $u(0) = u_0(x)$ and $u_t(0) = u_1(x)$ represents a weak solution of problem (1.1).

Note that T_1 denotes the lifespan of the solution $u(x, t)$ and $\langle \cdot, \cdot \rangle$ is the duality between $H_{\mathcal{L}}^{-1}(\mathbb{H}^n)$ and $H_{\mathcal{L}}^1(\mathbb{H}^n)$. Here $H_{\mathcal{L}}^1(\mathbb{H}^n)$ denotes the sub-Laplacian Sobolev space, analysed by Folland [5], see also [4, 6].

The conservation of energy law follows from

$$\begin{aligned} \frac{\partial E(t)}{\partial t} &= \frac{\partial}{\partial t} \left[\frac{1}{2} \|u_t\|_{L^2(\mathbb{H}^n)}^2 + \frac{m}{2} \|u\|_{L^2(\mathbb{H}^n)}^2 + \frac{1}{2} \|\nabla_H u\|_{L^2(\mathbb{H}^n)}^2 - \int_{\mathbb{H}^n} F(u) \, dx \right] \\ &= \operatorname{Re} \int_{\mathbb{H}^n} \bar{u}_t [u_{tt} + mu - \mathcal{L}u - f(u)] \, dx = -b \int_{\mathbb{H}^n} |u_t|^2 \, dx, \end{aligned}$$

this gives

$$E(t) + b \int_0^t \|u_s(s)\|_{L^2(\mathbb{H}^n)}^2 \, ds = E(0), \tag{1.4}$$

where

$$E(0) := \frac{1}{2} \|u_1\|_{L^2(\mathbb{H}^n)}^2 + \frac{m}{2} \|u_0\|_{L^2(\mathbb{H}^n)}^2 + \frac{1}{2} \|\nabla_H u_0\|_{L^2(\mathbb{H}^n)}^2 - \int_{\mathbb{H}^n} F(u_0) \, dx.$$

Also, let us define the Nehari functional

$$I(u) = m \|u\|_{L^2(\mathbb{H}^n)}^2 + \|\nabla_H u\|_{L^2(\mathbb{H}^n)}^2 - \operatorname{Re} \int_{\mathbb{H}^n} f(u) \bar{u} \, dx.$$

We assume that the nonlinear term $f(u)$ satisfies the condition

$$f(0) = 0, \quad \text{and} \quad \alpha F(u) \leq \operatorname{Re}[f(u) \bar{u}],$$

where $\alpha > 2$. In particular, this includes the case

$$f(u) = |u|^{p-1}u \quad \text{for } p > 1.$$

2 Main result

We now present the main result of this paper.

Theorem 2.1 *Let $b > 0$, $m > 0$ and $\mu = \max\{b, m, \alpha\}$. Assume that nonlinearity $f(u)$ satisfies*

$$\alpha F(u) \leq \operatorname{Re}[f(u) \bar{u}] \quad \text{for } \alpha > 2, \tag{2.1}$$

where $F(u)$ is as in (1.2). Assume that the Cauchy data $u_0 \in H_{\mathcal{L}}^1(\mathbb{H}^n)$ and $u_1 \in L^2(\mathbb{H}^n)$ satisfy

$$I(u_0) = m \|u_0\|_{L^2(\mathbb{H}^n)}^2 + \|\nabla_H u_0\|_{L^2(\mathbb{H}^n)}^2 - \operatorname{Re} \int_{\mathbb{H}^n} \bar{u}_0 f(u_0) \, dx < 0, \tag{2.2}$$

and

$$\operatorname{Re}(u_0, u_1)_{L^2(\mathbb{H}^n)} \geq \frac{\alpha(\mu + 1)}{m(\alpha - 2)} E(0). \tag{2.3}$$

Then the solution of equation (1.1) blows up in finite time T^* such that

$$0 < T^* \leq \frac{2(\mu + 1)(bT_0 + 1)}{(\alpha - 2)(\mu + 1 - m)} \frac{\|u_0\|_{L^2(\mathbb{H}^n)}^2}{\operatorname{Re}(u_0, u_1)},$$

where the blow-up time $T^* \in (0, T_0)$ with $T_0 < +\infty$.

Remark 2.2 (i) Note that the relationship between times T^* , T_0 and T_1 has the following form $T^* \in (0, T_0) \subset (0, T_1)$ where T^* is the blow-up time $T_0 < +\infty$ and $T_1 = +\infty$.

(ii) The local existence for the Klein–Gordon equation was shown in [2, 3]. The global in time well-posedness of problem (1.1) was proved by the first author and Tokmagambetov [22] for the small energy solutions and the nonlinearity $f(u)$ satisfying

$$|f(u) - f(v)| \leq C(|u|^{p-1} + |v|^{p-1})|u - v|,$$

with $1 < p \leq 1 + 1/n$.

(iii) This result generalises the Klein–Gordon equation in the Euclidean case (see [18]). Also it can be extended to the stratified Lie groups with the same method.

Proof of Theorem 2.1 First, recall the Nehari functional

$$I(u) = m\|u\|_{L^2(\mathbb{H}^n)}^2 + \|\nabla_H u\|_{L^2(\mathbb{H}^n)}^2 - \operatorname{Re} \int_{\mathbb{H}^n} \bar{u} f(u) \, dx.$$

Then the proof includes two steps.

Step I. In this step, we claim that

$$I(u(t)) < 0, \quad \text{and} \quad A(t) > \frac{2\alpha(\mu + 1)}{m(\alpha - 2)} E(0),$$

for $0 \leq t < T_1$ where $\mu = \max\{b, m, \alpha\}$ and

$$A(t) = 2\operatorname{Re}(u, u_t) + b\|u\|_{L^2(\mathbb{H}^n)}^2.$$

By using (1.3) along with $v = \bar{u}$, we get

$$\begin{aligned} A'(t) &= 2\|u_t\|_{L^2(\mathbb{H}^n)}^2 + 2\operatorname{Re}\langle u_{tt}, u \rangle + 2b\operatorname{Re} \int_{\mathbb{H}^n} \bar{u} u_t \, dx \\ &= 2\|u_t\|_{L^2(\mathbb{H}^n)}^2 - 2I(u), \quad 0 \leq t < T_1. \end{aligned} \tag{2.4}$$

In the last line we have used that

$$\begin{aligned} \operatorname{Re} \langle u_{tt}, u \rangle &= \operatorname{Re} \int_{\mathbb{H}^n} f(u) \bar{u} \, dx - \int_{\mathbb{H}^n} |\nabla_H u|^2 \, dx - m \int_{\mathbb{H}^n} |u|^2 \, dx - b \operatorname{Re} \int_{\mathbb{H}^n} \bar{u} u_t \, dx \\ &= -I(u) - b \operatorname{Re} \int_{\mathbb{H}^n} \bar{u} u_t \, dx. \end{aligned}$$

Now let us suppose by contradiction that

$$I(u(t)) < 0 \quad \text{for all } 0 \leq t < t_0,$$

and

$$I(u(t_0)) = 0.$$

Hereafter $0 < t_0 < T_1$. It is easy to see that $A'(t) > 0$ over $[0, t_0)$ and

$$A(t) > A(0) \geq 2\operatorname{Re} \langle u_0, u_1 \rangle \geq \frac{2\alpha(\mu + 1)}{m(\alpha - 2)} E(0). \tag{2.5}$$

Since $u(t)$ and $u_t(t)$ are both continuous in t that gives

$$A(t_0) \geq \frac{2\alpha(\mu + 1)}{m(\alpha - 2)} E(0). \tag{2.6}$$

Next we need to show a contradiction to (2.6). Using (1.4) and (2.1), we have

$$\begin{aligned} E(0) &= E(t) + b \int_0^t \|u_s\|_{L^2(\mathbb{H}^n)}^2 \, ds \\ &= \frac{1}{2} \|u_t\|_{L^2(\mathbb{H}^n)}^2 + \frac{m}{2} \|u\|_{L^2(\mathbb{H}^n)}^2 + \frac{1}{2} \|\nabla_H u\|_{L^2(\mathbb{H}^n)}^2 \\ &\quad - \int_{\mathbb{H}^n} F(u) \, dx + b \int_0^t \|u_s\|_{L^2(\mathbb{H}^n)}^2 \, ds \\ &\geq \frac{1}{2} \|u_t\|_{L^2(\mathbb{H}^n)}^2 + \frac{m}{2} \|u\|_{L^2(\mathbb{H}^n)}^2 + \frac{1}{2} \|\nabla_H u\|_{L^2(\mathbb{H}^n)}^2 \\ &\quad - \frac{1}{\alpha} \operatorname{Re} \int_{\mathbb{H}^n} \bar{u} f(u) \, dx + b \int_0^t \|u_s\|_{L^2(\mathbb{H}^n)}^2 \, ds \\ &= \frac{1}{2} \|u_t\|_{L^2(\mathbb{H}^n)}^2 + \frac{1}{\alpha} I(u) + \left(\frac{m}{2} - \frac{m}{\alpha}\right) \|u\|_{L^2(\mathbb{H}^n)}^2 \\ &\quad + \left(\frac{\alpha - 2}{2\alpha}\right) \|\nabla_H u\|_{L^2(\mathbb{H}^n)}^2 + b \int_0^t \|u_s\|_{L^2(\mathbb{H}^n)}^2 \, ds. \end{aligned}$$

If we use $I(u(t_0)) = 0$ and $\frac{m(\alpha-2)}{\alpha(\mu+1)} < 1$, then

$$\begin{aligned}
 E(0) &\geq \frac{1}{2} \|u_t(t_0)\|_{L^2(\mathbb{H}^n)}^2 + \frac{m(\alpha-2)}{2\alpha} \|u(t_0)\|_{L^2(\mathbb{H}^n)}^2 \\
 &\geq \frac{m(\alpha-2)}{2\alpha(\mu+1)} \left(\|u_t(t_0)\|_{L^2(\mathbb{H}^n)}^2 + (\mu+1) \|u(t_0)\|_{L^2(\mathbb{H}^n)}^2 \right) \\
 &> \frac{m(\alpha-2)}{2\alpha(\mu+1)} \left(2\operatorname{Re}(u(t_0), u_t(t_0)) + \mu \|u(t_0)\|_{L^2(\mathbb{H}^n)}^2 \right) \\
 &\geq \frac{m(\alpha-2)}{2\alpha(\mu+1)} A(t_0).
 \end{aligned}
 \tag{2.7}$$

Note that for the strict inequality above we use that the assumption (2.2) implies that $\|u_0\|_{L^2(\mathbb{H}^n)} \neq 0$. We have also used the fact $a^2 + b^2 - 2ab \geq 0$, where $a = \|u_t(t_0)\|_{L^2(\mathbb{H}^n)}$ and $b = \|u(t_0)\|_{L^2(\mathbb{H}^n)}$. It gives the contradiction to (2.6). This proves our claim.

Step II. Define the functional

$$M(t) = \|u\|_{L^2(\mathbb{H}^n)}^2 + b \int_0^t \|u(s)\|_{L^2(\mathbb{H}^n)}^2 ds + b(T_0 - t) \|u_0\|_{L^2(\mathbb{H}^n)}^2,$$

for $0 \leq t \leq T_0$. Then

$$\begin{aligned}
 M'(t) &= 2\operatorname{Re}(u, u_t) + b\|u(t)\|_{L^2(\mathbb{H}^n)}^2 - b\|u_0\|_{L^2(\mathbb{H}^n)}^2 \\
 &= 2\operatorname{Re}(u, u_t) + 2b \int_0^t \operatorname{Re}(u(s), u_s(s)) ds,
 \end{aligned}$$

since

$$\int_0^t \frac{d}{ds} \|u(s)\|_{L^2(\mathbb{H}^n)}^2 ds = \|u(t)\|_{L^2(\mathbb{H}^n)}^2 - \|u(0)\|_{L^2(\mathbb{H}^n)}^2.$$

We observe the following estimates:

$$\begin{aligned}
 |\operatorname{Re}(u, u_t)|^2 &\leq \|u_t\|_{L^2(\mathbb{H}^n)}^2 \|u\|_{L^2(\mathbb{H}^n)}^2, \\
 \left(\int_0^t |\operatorname{Re}(u(s), u_s(s))| ds \right)^2 &\leq \left(\int_0^t \|u(s)\|_{L^2(\mathbb{H}^n)}^2 ds \right) \left(\int_0^t \|u_s(s)\|_{L^2(\mathbb{H}^n)}^2 ds \right),
 \end{aligned}$$

and

$$\begin{aligned}
 2\operatorname{Re}(u, u_t) \int_0^t \operatorname{Re}(u, u_s) ds &\leq 2\|u\|_{L^2(\mathbb{H}^n)} \|u_t\|_{L^2(\mathbb{H}^n)} \\
 &\quad \times \left(\int_0^t \|u\|_{L^2(\mathbb{H}^n)}^2 ds \right)^{1/2} \left(\int_0^t \|u_s\|_{L^2(\mathbb{H}^n)}^2 ds \right)^{1/2}
 \end{aligned}$$

$$\leq \|u\|_{L^2(\mathbb{H}^n)}^2 \int_0^t \|u_s\|_{L^2(\mathbb{H}^n)}^2 \, ds + \|u_t\|_{L^2(\mathbb{H}^n)}^2 \int_0^t \|u\|_{L^2(\mathbb{H}^n)}^2 \, ds.$$

Using the above inequalities, we calculate

$$\begin{aligned} (M'(t))^2 &= 4 \left(|\operatorname{Re}(u, u_t)|^2 + 2b \operatorname{Re}(u, u_t) \int_0^t \operatorname{Re}(u, u_s) \, ds + b^2 \left(\int_0^t \operatorname{Re}(u, u_s) \, ds \right)^2 \right) \\ &\leq 4 \left(\|u\|_{L^2(\mathbb{H}^n)}^2 + b \int_0^t \|u\|_{L^2(\mathbb{H}^n)}^2 \, ds \right) \left(\|u_t\|_{L^2(\mathbb{H}^n)}^2 + b \int_0^t \|u_s\|_{L^2(\mathbb{H}^n)}^2 \, ds \right), \end{aligned}$$

for all $0 \leq t \leq T_0$. The second derivate with respect to time of $M(t)$ is

$$M''(t) = 2\|u_t\|_{L^2(\mathbb{H}^n)}^2 - 2I(u),$$

for all $0 \leq t \leq T_0$, where we have used the equality from (2.4). Then we construct the differential inequality as follows

$$\begin{aligned} M''(t)M(t) - \frac{\omega + 3}{4} (M'(t))^2 &\geq M(t) \left(M''(t) - (\omega + 3) \left(\|u_t\|^2 + b \int_0^t \|u_s\|_{L^2(\mathbb{H}^n)}^2 \, ds \right) \right) \\ &= M(t) \left(-(\omega + 1)\|u_t\|_{L^2(\mathbb{H}^n)}^2 - (\omega + 3)b \int_0^t \|u_s\|_{L^2(\mathbb{H}^n)}^2 \, ds - 2I(u) \right), \end{aligned}$$

where we assume that $\omega > 1$. We shall now show that the following term is nonnegative:

$$\begin{aligned} \eta(t) &= -(\omega + 1)\|u_t\|_{L^2(\mathbb{H}^n)}^2 - (\omega + 3)b \int_0^t \|u_s(s)\|_{L^2(\mathbb{H}^n)}^2 \, ds - 2I(u) \\ &\geq (\alpha - \omega - 1)\|u_t\|_{L^2(\mathbb{H}^n)}^2 + b(2\alpha - \omega - 3) \int_0^t \|u_s(s)\|_{L^2(\mathbb{H}^n)}^2 \, ds \\ &\quad + m(\alpha - 2)\|u\|_{L^2(\mathbb{H}^n)}^2 + (\alpha - 2)\|\nabla_H u\|_{L^2(\mathbb{H}^n)}^2 - 2\alpha E(0) \\ &= (\alpha - \omega - 1)[\|u_t\|_{L^2(\mathbb{H}^n)}^2 + (b + 1)\|u\|_{L^2(\mathbb{H}^n)}^2] + (\alpha - 2)\|\nabla_H u\|_{L^2(\mathbb{H}^n)}^2 \\ &\quad - 2\alpha E(0) + b(2\alpha - \omega - 3) \int_0^t \|u_s(s)\|_{L^2(\mathbb{H}^n)}^2 \, ds \\ &\quad + (m(\alpha - 2) - (b + 1)(\alpha - \omega - 1))\|u\|_{L^2(\mathbb{H}^n)}^2 \\ &\geq (\alpha - \omega - 1)[2\operatorname{Re}(u, u_t) + b\|u\|_{L^2(\mathbb{H}^n)}^2] + (\alpha - 2)\|\nabla_H u\|_{L^2(\mathbb{H}^n)}^2 - 2\alpha E(0) \\ &\quad + b(2\alpha - \omega - 3) \int_0^t \|u_s(s)\|_{L^2(\mathbb{H}^n)}^2 \, ds \\ &\quad + (m(\alpha - 2) - (b + 1)(\alpha - \omega - 1))\|u\|_{L^2(\mathbb{H}^n)}^2. \end{aligned}$$

In the second line we have used (2.7). By selecting $\omega = \alpha - 1 - \frac{m(\alpha-2)}{\mu+1}$ which satisfies $\omega > 1$ since $\mu + 1 > m$ and using the argument from Step I, we obtain

$$\begin{aligned} \eta(t) &> \frac{m(\alpha - 2)}{\mu + 1} (2\text{Re}(u, u_t) + b\|u\|_{L^2(\mathbb{H}^n)}^2) - 2\alpha E(0) \\ &> \frac{m(\alpha - 2)}{\mu + 1} (2\text{Re}(u_0, u_1) + b\|u_0\|_{L^2(\mathbb{H}^n)}^2) - 2\alpha E(0) \\ &> \left(\frac{m(\alpha - 2)}{\mu + 1}\right) 2\text{Re}(u_0, u_1) - 2\alpha E(0) \\ &\geq 0. \end{aligned}$$

Note that we have used the fact $A'(t) > 0$, expression (2.5) with $A(t) = 2\text{Re}(u, u_t) + b\|u\|_{L^2(\mathbb{H}^n)}^2$, and condition (2.3) in the last line, respectively. Then we obtain the inequality

$$M''(t)M(t) - \frac{\omega + 3}{4} (M'(t))^2 > 0.$$

Then

$$\frac{d}{dt} \left[\frac{M'(t)}{M^{\frac{\omega+3}{4}}(t)} \right] > 0 \implies \begin{cases} M'(t) \geq \left[\frac{M'(0)}{M^{\frac{\omega+3}{4}}(0)} \right] M^{\frac{\omega+3}{4}}(t), \\ M(0) = (bT_0 + 1)\|u_0\|_{L^2(\mathbb{H}^n)}^2. \end{cases}$$

Let us denote $\sigma = \frac{\omega-1}{4}$. Then we have

$$-\frac{1}{\sigma} \left[M^{-\sigma}(t) - M^{-\sigma}(0) \right] \geq \frac{M'(0)}{M^{\sigma+1}(0)} t,$$

that gives

$$M(t) \geq \left(\frac{1}{M^\sigma(0)} - \frac{\sigma M'(0)}{M^{\sigma+1}(0)} t \right)^{-\frac{1}{\sigma}}.$$

Then the blow-up time T^* satisfies

$$0 < T^* \leq \frac{M(0)}{\sigma M'(0)},$$

where $M'(0) = 2\text{Re}(u_0, u_1)$. □

Author Contributions MR and BS wrote the main manuscript. Both authors reviewed the manuscript.

Declarations

Conflict of interest The authors declare no competing interests.

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