Discrete Weierstrass transform: generalisations

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Abstract

The classical Weierstrass transform is an isometric operator mapping elements of the weighted L_2-space $\mathcal{L}_2(\mathbb{R}, \exp(-x^2/2))$ to the Fock space. We defined an analogue version of this transform in discrete Hermitian Clifford analysis in one dimension, where functions are defined on a discrete line rather than the continuous space. This transform is based on the classical definition, in combination with a discrete version of the Gaussian function and discrete counterparts of the classical Hermite polynomials. The aim of this paper is to extend the definition to higher dimensions, where we must take into account the anticommutativity of the basic Clifford elements and use the generalised discrete Hermite polynomials. Furthermore, we investigate, back in dimension 1, the asymptotic behaviour if the mesh width approaches 0.

Keywords: Discrete Clifford analysis, Weierstrass transform, Generalised Hermite polynomials, Monogenic polynomials

1 Introduction

The classical Weierstrass transform is an isometric operator mapping elements of the weighted L_2-space $\mathcal{L}_2(\mathbb{R}, \exp(-x^2/2))$ to the Fock space [1]. The result $\mathcal{W}[f]$ is a "smoothed" version of f, obtained by averaging the values of f, weighted with the Gaussian function at the corresponding points. It has numerous applications in physics and applied mathematics. It is for example closely related to the heat equation, see e.g. [2] and [3]: if the function f describes the initial temperature at each point of an infinitely long rod with constant thermal conductivity 1, the temperature of the rod at $t = 1$ time units later is given by $\mathcal{W}[f]$. The Weierstrass transform is also often used in image processing, where it is called the Gaussian blur, see for example [4], [5] and [6]. Another use case of the Weierstrass transform is in statistics where it is used for data smoothening, see e.g. [7]. Given a function that respresents noisy data, one wants to build a function that approximates the important patterns of the data, omitting the noise. Furthermore, the Weierstrass transform is used in signal processing, for example in [8]. The use of the Gaussian kernel also leads to the name "Gauss transform" or "Gauss-Weierstrass transform". The classical Weierstrass transform is closely related to the Segal-Bargman transform, which takes a function in the Hilbert space $\mathcal{L}_2(\mathbb{R}, \exp(-x^2/2))$ and sending it to the Fock space. We refer to [9] and [10] for this topic in Clifford analysis.

In a previous paper [11], we defined the Weierstrass transform in a discrete Clifford setting in one dimension. This was done based on the classical definition, applied on Hermite polynomials, which form the basis of the space of squared integrable functions. We translated this classical definition to the discrete Hermite polynomials and created the discrete Weierstrass space, for which the discrete Hermite polynomials form the basis. This way, we could extend the new discrete Weierstrass transform to more general discrete functions: those in the closure of the space spanned by the discrete Hermite polynomials.

In this paper, we ask ourselves two questions.

1. How do we adapt the definition from [11] when we work on a discrete line with width $h \neq 1$? What happens when h tends to 0 and what is the link to the continuous setting? It will appear that the

definitions are easily extended based on previous basis definitions from [12] and that taking the limit for h to 0 brings us to the continuous case.

2. How can we extend this definition to higher dimensions, i.e. $m \geq 2$. Two main differences will be:
   (a) The basis Clifford elements $e_1, \ldots, e_m$ are not commutative. In particular, it holds that $e_j e_k = -e_k e_j$ if $j \neq k$. To handle this issue, we will use the discrete rotation invariant operators R_j, introduced in [13]:
   $$R_j = \boldsymbol{e}_j^+ R_j^+ + \boldsymbol{e}_j^- R_j^-,$$
   with $R_j^{\pm}$ scalar operators.
   (b) The basis functions of the discrete Weierstrass space $\mathcal{W}$ in one dimension, the discrete Hermite polynomials, must be generalised to higher dimensions. Therefore, we use the generalised discrete Hermite polynomials, as described in [12]. The discrete operators $H_{n,m,r}P_r$ are the composition of a discrete spherical monogenic operator P_r, i.e. $\partial P_r = 0$ and $\mathbb{E}P_r = rP_r$, of degree r with the (discrete) Hermite polynomial $H_{n,m,r}$ of degree n. Note the dependency of $H_{n,m,r}$ on the degree r of the monogenic P_r.

   By means of recurrence relations for n on the one side, and an alternative equivalent definition on the other side, we will be able to give explicit expressions for the Weierstrass transform in dimension $m \geq 2$.

We start by giving a general introduction in discrete Clifford analysis in the preliminaries. Definitions and properties are given for general mesh width h and in general dimension m.

2 Preliminaries

Starting from the $m-$dimensional Euclidean space $\mathbb{R}^m$, with orthonormal basis $e_1, \ldots, e_m$, construct the Clifford algebra $\mathbb{R}_{m,0}$. Consider its complexification $\mathbb{C}_m = \mathbb{C} \otimes \mathbb{R}_{m,0}$ of dimension 2^m. For every basis element e_j $(j = 1 \ldots m)$, it holds that $e_j^2 = 1$.

Now consider the discrete grid $\mathbb{Z}_h^m = \{\underline{x} = (n_1 h, n_2 h, \ldots, n_m h) \mid \underline{n} \in \mathbb{Z}^m\}$ and split every basis vector e_j in a forward and backward basis element $\boldsymbol{e}_j^+$, resp. $\boldsymbol{e}_j^-$, such that $\boldsymbol{e}_j^+ + \boldsymbol{e}_j^- = e_j$. They satisfy the following commutator relations:

$$\boldsymbol{e}_j^{\pm}\boldsymbol{e}_k^{\pm} + \boldsymbol{e}_k^{\pm}\boldsymbol{e}_j^{\pm} = 0 \tag{1}$$
$$\boldsymbol{e}_j^{+}\boldsymbol{e}_k^{-} + \boldsymbol{e}_k^{-}\boldsymbol{e}_j^{+} = \delta_{jk}. \tag{2}$$

A main type of involution is the **Hermitian conjugation** $\dagger$, defined on the basis elements as $(\boldsymbol{e}_j^+)^\dagger := \boldsymbol{e}_j^-$ and $(\boldsymbol{e}_j^-)^\dagger := \boldsymbol{e}_j^+$. It reverses the order of multiplication: $(ab)^\dagger := b^\dagger a^\dagger$, with $a, b \in \mathbb{C}_m$.

To construct a discrete Dirac operator, define the forward and backward differences for $j = 1, \ldots, m$ by

$$\Delta_j^+ f(x) := \frac{f(x + he_j) - f(x)}{h},$$
$$\Delta_j^- f(x) := \frac{f(x) - f(x - he_j)}{h},$$

with $x \in \mathbb{Z}_h^m$.

Denote $\partial_j = \boldsymbol{e}_j^+ \Delta_j^+ + \boldsymbol{e}_j^- \Delta_j^-$. The **discrete Dirac operator** is then defined as

$$\partial = \sum_{j=1}^{m} \partial_j = \sum_{j=1}^{m} \boldsymbol{e}_j^+ \Delta_j^+ + \boldsymbol{e}_j^- \Delta_j^-.$$

A discrete function f satisfying $\partial f = 0$, is called left monogenic.

Likewise, denote $\xi_j = \boldsymbol{e}_j^+ X_j^- + \boldsymbol{e}_j^- X_j^+$. The discrete vector variable operator

$$\xi = \sum_{j=1}^{m} \xi_j = \sum_{j=1}^{m} \boldsymbol{e}_j^+ X_j^- + \boldsymbol{e}_j^- X_j^+,$$

with $X_j^{\pm}$ scalar operators, is defined in such a way that it satisfies the skew-Weyl relations

$$\partial_j \xi_i - \xi_i \partial_j = \delta_{ij}, \qquad j = 1, \ldots, m.$$

The following interaction relations regarding ξ_j and ∂_j complement and are in accordance with these skew Weyl relations:

$$\{\xi_j, \xi_i\} = \{\partial_j, \partial_i\} = \{\partial_j, \xi_i\} = 0 \qquad j \neq i$$

We emphasise that the operators ξ_j and ∂_j are vector operators, since they contain the basis elements $\boldsymbol{e}_j^{\pm}$. Hence also the operators ξ and ∂ are vector operators.

Let

$$\mathbb{E} := \sum_{j=1}^{m} \xi_j \partial_j$$

be the **discrete Euler operator**. The natural powers of the operators ξ_j, acting on the ground state 1, i.e. $\xi_j^k[1], k \in \mathbb{N}$, fullfill the relation $\mathbb{E}\xi_j^k[1] = k\xi_j^k[1]$ and are thus called **discrete homogeneous polynomials**. They constitute a basis for all discrete polynomials. Explicitly, they are given by

$$\begin{aligned} \xi_{j,h}^{2k+1}[1](x) &= x_j \prod_{s=1}^{k} \left(x_j^2 - s^2 h^2\right) \left(\boldsymbol{e}_j^+ + \boldsymbol{e}_j^-\right), \\ \xi_{j,h}^{2k}[1](x) &= \left(x_j^2 + k\, h\, x_j \left(\boldsymbol{e}_j^+ \boldsymbol{e}_j^- - \boldsymbol{e}_j^- \boldsymbol{e}_j^+\right)\right) \prod_{s=1}^{k-1} \left(x_j^2 - s^2 h^2\right). \end{aligned} \tag{3}$$

We will omit the sub index h in $\xi_{j,h}$ for ease of notation. The natural powers of the Hermitian conjugation $\dagger$ of ξ are analogous: they only differ by a different sign of the bivector part.

$$\begin{aligned} \left(\xi_j^\dagger\right)^{2k+1}[1] &= \xi_j^{2k+1}[1] = x_j \prod_{s=1}^{k} \left(x_j^2 - s^2 h^2\right) \left(\boldsymbol{e}_j^+ + \boldsymbol{e}_j^-\right), \\ \left(\xi_j^\dagger\right)^{2k}[1] &= \left(x_j^2 - k\, h\, x_j \left(\boldsymbol{e}_j^+ \boldsymbol{e}_j^- - \boldsymbol{e}_j^- \boldsymbol{e}_j^+\right)\right) \prod_{s=1}^{k-1} \left(x_j^2 - s^2 h^2\right) \end{aligned} \tag{4}$$

In (3) and (4), the discrete vector variable ξ acts from the left on the base state [1]. As the Clifford algebra we are working in is not commutative, the action from the right is, in general, not equal to the action on the left. In [12], it is proven that

$$\left(\xi^k[1]\right) \partial^\dagger = \partial \left(\xi^k[1]\right) \tag{5}$$

$$\left(\xi^k[1]\right) \xi^\dagger = \xi^{k+1}[1]. \tag{6}$$

To overcome the lack of community of ξ and ∂, let us introduce the rotational invariant operators $R_j = \boldsymbol{e}_j^+ R_j^+ + \boldsymbol{e}_j^- R_j^-$, $j = 1, \ldots m$, which were defined in [13].

The operators interact with $X_j^{\pm}$ and $\Delta_j^{\pm}$ in the following way:

$$\begin{aligned} &R_j^{\pm}[1] = e_k^{\pm} \\ &\quad R_j^+ X_j^+ = X_j^- R_j^-, \quad R_j^- X_j^- = X_j^+ R_j^+, \\ &\quad R_j^+ \Delta_j^- = \Delta_j^+ R_j^-, \quad R_j^- \Delta_j^+ = \Delta_j^- R_j^+ \end{aligned}$$

It follows that, on co-ordinate level, they satisfy the following (anti-)commuting relations:

$$\begin{aligned} R_j \xi_j - \xi_j R_j &= 0 = R_j \partial_j - \partial_j R_j \\ R_j \xi_k + \xi_k R_j &= 0 = R_j \partial_k + \partial_k R_j \qquad j \neq k. \end{aligned}$$

These operators R_j will be implemented in the definition of the action of operators on Clifford elements.

The **discrete delta functions** are the building blocks of discrete function theory.

$$\delta_{nh}(x) = \begin{cases} \frac{1}{h^m} & \text{if } x = nh \\ 0 & \text{else} \end{cases}$$

In one dimension, discrete function f can be decomposed into discrete delta functions by

$$f(jh) = \sum_{n\in\mathbb{Z}} f(nh)\, h\, \delta_{nh}(jh).$$

The same function f can also be expressed as an infinite series of powers of the basis vector variables, its **Taylor series**:

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \xi_h^k[1](x)[\partial_h^k f(u)]_{u=0}.$$

In particular, the Taylor series of the delta functions are

$$\delta_0(x) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!\ell!h^{2\ell+1}} \xi_h^{2\ell}[1](x) + \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+1}}{(\ell+1)!\ell!h^{2\ell+2}} \xi_h^{2\ell+1}[1](x)(\boldsymbol{e}^+ - \boldsymbol{e}^-).$$

A **discrete distribution** is a linear functional defined on the set of discrete functions, with values in the Clifford algebra. As in the classical setting, a regular distribution F is one that is associated with a so called density function f, such that $\langle F, V\rangle = \int_{\mathbb{R}} V(x)f(x)dx$. The translation to the discrete setting is immediate:

$$\langle F, V\rangle = \sum_{x\in\mathbb{Z}^h} V(x)f(x)h.$$

Let F denote a (not necessary regular) discrete distribution, V a discrete polynomial and a a Clifford number. Then

$$\langle \partial_i F, V\rangle = -\left\langle F, V\partial_i^\dagger \right\rangle \tag{7}$$

$$\langle \xi_i F, V\rangle = \left\langle F, V\xi_i^\dagger \right\rangle \tag{8}$$

$$\langle Fa, V\rangle = \langle F, V\rangle\, a \tag{9}$$

$$\langle F, aV\rangle = a\, \langle F, V\rangle \tag{10}$$

The building blocks of discrete distributions are the discrete delta distributions $\boldsymbol{\delta}_p$, associated with the discrete delta functions δ_p

$$\left\langle \boldsymbol{\delta}_p, f \right\rangle = f(p), \qquad p \in \mathbb{Z}.$$

There is a natural correspondence between the discrete delta function and the delta distribution, given by

$$\left\langle \boldsymbol{\delta}_{jh}, f \right\rangle = \sum_{n\in\mathbb{Z}} f(nh)\, \delta_{jh}(nh)h = f(jh).$$

For the derivatives of the discrete delta distribution, we then have

$$\left\langle \partial_h^k \boldsymbol{\delta}_{jh}, f \right\rangle = (-1)^k \left\langle \boldsymbol{\delta}_{jh}, \partial_h^k f \right\rangle = \partial_h^k f(jh).$$

In particular, if $f = \xi^\ell[1](x)$, then

$$\left\langle \partial_h^k \boldsymbol{\delta}_{jh}, \xi^\ell[1] \right\rangle = \begin{cases} (-1)^k \frac{\ell!}{(\ell-k)!} \xi^{\ell-k}[1](jh) & k \leq \ell \\ 0 \quad k > l. \end{cases}$$

The **dual Taylor series** are for distributions what the Taylor series are for functions. Every discrete distrubtuion F can be written in terms of derivatives of the delta distribution.

$$F = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \partial_h^{k_1} \partial_h^{k_2} \ldots \partial_h^{k_m} \boldsymbol{\delta}_0 \left\langle F, \xi_1^{k_1} \xi_2^{k_2} \ldots \xi_m^{k_m}[1] \right\rangle . \tag{11}$$

The **discrete Gauss distribution** is another important distribution in our theory. It is uniquely defined via its action on the discrete homogeneous polynomials:

$$\left\langle G, \xi_1^{k_1} \xi_2^{k_2} \ldots \xi_m^{k_m}[1] \right\rangle = \begin{cases} (2\pi)^{\frac{m}{2}} \prod_{i=1}^{m} (k_i - 1)!! & \text{if all } k_i \text{ even} \\ 0 & \text{else,} \end{cases} \tag{12}$$

with its Taylor formula, immediately derived from (11):

$$\begin{aligned} G &= \sum_{k_1,\ldots,k_m=0}^{\infty} \frac{(-1)^{|k|}}{k_1! k_2! \ldots k_m!} \partial_1^{k_1} \partial_2^{k_2} \ldots \partial_m^{k_m} \boldsymbol{\delta}_0 \left\langle G, \xi_1^{k_1} \xi_2^{k_2} \ldots \xi_m^{k_m}[1] \right\rangle \\ &= \sum_{k=0}^{\infty} (2\pi)^{\frac{m}{2}} \frac{1}{2^{k_1+\ldots+k_m} k_1! \ldots k_m!} \partial_1^{k_1} \partial_2^{k_2} \ldots \partial_m^{k_m} \boldsymbol{\delta}_0 \\ &= (2\pi)^{\frac{m}{2}} \exp\left(\frac{\partial_h^2}{2} \right) \boldsymbol{\delta}_0 . \end{aligned} \tag{13}$$

The discrete **Hermite polynomials** are defined using the Gauss distribution. They are polynomials in ξ, defined by the recurrence relation $H_{k+1}G = (-1)^{k+1} \partial H_k G$. Using the relation $\partial G = -\xi G$, they satisfy Rodriguez' formulae

$$H_{2k}G = (-1)^k \partial^{2k} G \text{ and } H_{2k+1}G = (-1)^{k+1} \partial^{2k+1} G. \tag{14}$$

In dimension $m = 1$, one can use the radial Hermite polynomials. However, from dimension $m \geq 2$ on, one needs the *generalised* Hermite polynomials, which are the composition of a spherical monogenic operator P_r of degree r with the Hermite polynomial $H_{n,m,r}$ of degree r. They satisfy the same corresponding Rodriguez' formulae:

$$\begin{aligned} H_{2k,m,r} P_r G &= (-1)^k \partial^{2k} P_r G \\ H_{2k+1,m,r} P_r G &= (-1)^{k+1} \partial^{2k+1} P_r G. \end{aligned} \tag{15}$$

An explicit form for the Hermite polynomials is

$$H_{2k,m,r} = \sum_{j=0}^{k} a_{2j}^{2k} \xi^{2j} \qquad H_{2k+1,m,r} = \sum_{j=0}^{k} a_{2j+1}^{2k+1} \xi^{2j+1} \tag{16}$$

with

$$a_{2j}^{2k} = (-1)^j 2^{k-j} \binom{k}{j} \frac{\Gamma(k + \frac{m}{2} + r)}{\Gamma(j + \frac{m}{2} + r)} \tag{17}$$

$$a_{2j+1}^{2k+1} = (-1)^j 2^{k-j} \binom{k}{j} \frac{\Gamma(k + \frac{m}{2} + r + 1)}{\Gamma(j + \frac{m}{2} + r + 1)} . \tag{18}$$

As the Hermite polynomials of degree n are the $n-$th derivative of the Gaussian polynomial, one can of course also reverse this relationship as given in the next lemma. The appearing coefficients are the same as the Hermite coefficients, up to sign.

Lemma 2.1. *The action of natural powers of ξ on the discrete Gauss distribution is as follows:*

$$\xi^{2l} G = \sum_{j=0}^{\ell} b_{2j}^{2\ell} \partial^{2j} G$$
$$\xi^{2l+1} G = -\sum_{j=0}^{\ell} b_{2j+1}^{2\ell+1} \partial^{2j+1} G$$

with

$$b_{2j}^{2\ell} = 2^{\ell-j} \binom{\ell}{j} \frac{\Gamma(\ell+\frac{m}{2})}{\Gamma(j+\frac{m}{2})}$$
$$b_{2j+1}^{2\ell+1} = 2^{\ell-j} \binom{\ell}{j} \frac{\Gamma(\ell+\frac{m}{2}+1)}{\Gamma(j+\frac{m}{2}+1)}$$

In [11], we defined the **discrete Weierstrass transform** in one dimension and with mesh width $h=1$, based on the transform of the discrete Hermite polynomials.

$$\langle H_n(\xi_h) G, e^{-z^2/2+\xi_h z}[1] \rangle = (-1)^{\lfloor \frac{n}{2} \rfloor} \sqrt{2\pi}\, z^n.$$

Using this outcome, we could define the Weierstrass transform on more general discrete functions: those in the so-called **Weierstrass space** $\mathcal{W}$: the closure of the span of the discrete Hermite polynomials.

Definition 2.2 (Discrete Weierstrass transform)**.** *For a discrete function $f \in \mathcal{W}$, $f = \sum_{k\in\mathbb{N}}^{\infty} H_k a_k$, its Weierstrass transform is defined as*

$$\mathcal{W}[f](z) = \frac{1}{\sqrt{2\pi}} \langle f(\xi) G, e^{-z^2/2+\xi z}[1] \rangle = \sum_{n\in\mathbb{N}} \mathcal{W}[H_n] a_n = \sum_{n\in\mathbb{N}} (-1)^{\frac{\lfloor n \rfloor}{2}} a_n z^n.$$

The result is an element of the continuous (complex) Fock space.

We now aim to generalise this definition to higher dimensions and for mesh width $h \neq 1$.

3 Discrete Weierstrass transform on mesh with width $h \neq 1$

In [11], we worked on a discrete grid with mesh width $h=1$. We ask ourselves how the value h influences the definition and results of the Weierstrass transform. What will happen if h tends to 0?

As seen in the preliminaries, the mesh width h does not appear explicitly in the definition of G. It does however appear in its density function.

To calculate its density function, we will need an expression for the even derivatives of $\boldsymbol{\delta}_0$:

$$\begin{aligned}\partial^{2k} \boldsymbol{\delta}_0 &= \sum_{i=0}^{2k} \frac{(-1)^i}{h^{2k}} \binom{2k}{i} \boldsymbol{\delta}_{-(k-i)h} \\ &= \sum_{i=-k}^{k} \frac{(-1)^{k+i}}{h^{2k}} \binom{2k}{i+k} \boldsymbol{\delta}_{ih} \\ &= \sum_{i=-k}^{k} \frac{(-1)^{k+i}}{h^{2k}} \binom{2k}{i+k} \boldsymbol{\delta}_{ih}.\end{aligned}$$

Then

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \partial_h^k \boldsymbol{\delta}_0 \left\langle G, \xi_h^k[1] \right\rangle$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{(-1)^{2k}\sqrt{2\pi}(2k)!}{(2k)!2^k k!} \partial_h^{2k} \boldsymbol{\delta}_0 \\
&= \sum_{k=0}^{\infty} \frac{\sqrt{2\pi}}{2^k k!} \left[\sum_{i=-k}^{k} \frac{(-1)^{k+i}}{h^{2k}} \binom{2k}{i+k} \boldsymbol{\delta}_{ih} \right] \\
&= \sqrt{2\pi} \sum_{n\in\mathbb{Z}} \left[\sum_{\ell=|n|}^{\infty} \frac{(-1)^{n+l}}{2^\ell \ell! h^{2\ell}} \binom{2\ell}{n+\ell} \right] \boldsymbol{\delta}_{nh} \\
&= \sqrt{2\pi} \sum_{n\in\mathbb{Z}} \mathcal{I}_n(\frac{1}{h^2}) \exp\left(-\frac{1}{h^2}\right) \boldsymbol{\delta}_{nh}, \quad \text{ with } n = \frac{x}{h}.
\end{aligned}
$$

Here, $\mathcal{I}_n(z)$ is the modified Bessel function of the first kind. Therefore, in a point $x = nh \in \mathbb{Z}_h$, the value of G is

$$
\frac{\sqrt{2\pi}}{h} \exp\left(-\frac{1}{h^2}\right) \mathcal{I}_{\frac{x}{h}}(\frac{1}{h^2}) \tag{19}
$$

and thus the density function (in general dimension m) reads as

$$
g = \frac{(\sqrt{2\pi})^m}{h^m} \exp\left(\frac{m}{h^2}\right) \prod_{j=1}^{m} \mathcal{I}_{\frac{x_j}{h}} \left(\frac{1}{h^2}\right) \text{ with } \frac{x_j}{h} \in \mathbb{Z}. \tag{20}
$$

We again proceed in dimension $m = 1$. To analytically investigate the asymptotic behaviour of the Gauss distribution as $h \to 0$, we use formula 9.7.7 from [14]:

$$
\mathcal{I}_\nu(\nu z) \approx \frac{1}{\sqrt{2\pi\nu}} \frac{\exp(\nu\eta)}{(1+z^2)^{1/4}} \left(1 + \sum_{k=1}^{\infty} \frac{U_k(p)}{\nu^k}\right), \qquad \eta = \sqrt{1+z^2} + \ln\left(\frac{z}{1+\sqrt{1+z^2}}\right), \tag{21}
$$

which describes the uniform asymptotic expansion for large orders ν and is valid for z in the sector $\left|\arg(z)\right| < \frac{\pi}{2}$. The terms $U_k(p)$ are polynomials in $p = (1+z^2)^{-\frac{1}{2}}$ of degree $3k$, recursively given by

$$
\begin{aligned}
U_0(p) &= 1, \\
U_{k+1}(p) &= \frac{1}{2} p^2 (1-p^2) U_k'(p) + \frac{1}{8} \int_0^p (1-5t^2) U_k(t) dt
\end{aligned}
$$

As the density function of g is even in its argument, we may apply it for our case in both positive and negative $x-$axis. As seen in figure 1, cases for $h = 1$, $h = \frac{1}{10}$ and $h = \frac{1}{100}$ approach the continuous Gaussian. This is clearly confirmed by implementing formula (21) into the definition of g with $\nu = \frac{x}{h}$ and $z = \frac{1}{xh}$. Taking the limit for $h \to 0$ gives $\exp\left(-\frac{x^2}{2}\right)$, the continuous Gaussian distribution.

Another way to visualise the effect in G of h approaching 0 is given in figure 2, where we let h approach 0 in (20) Be aware that $x = nh$, hence the absolute value of x increases with the same factor as h decreases, resulting in a rescaling of the plots and x-axis. As $h \to 0$, all points of the grid collapse to the origin, hence g tends to 1.

The discrete Weierstrass transform was defined based on the transform of the Hermite polynomials. Because there is no explicit appearance of the mesh width h in formulas (16), the outcome of the Weierstrass transform will not change, as h tends to 0.

$$
\begin{aligned}
\langle H_n(\xi_h) G, e^{-z^2/2+\xi_h z}[1] \rangle &= \langle (-1)^{\lceil \frac{n}{2} \rceil} \partial^n G, e^{-z^2/2+\xi_h z}[1] \rangle \\
&= (-1)^{\lceil \frac{n}{2} \rceil} e^{-z^2/2} \langle \partial^n G, \sum_{i=0}^{\infty} \frac{\xi_h^i z^i [1]}{i!} \rangle
\end{aligned}
$$

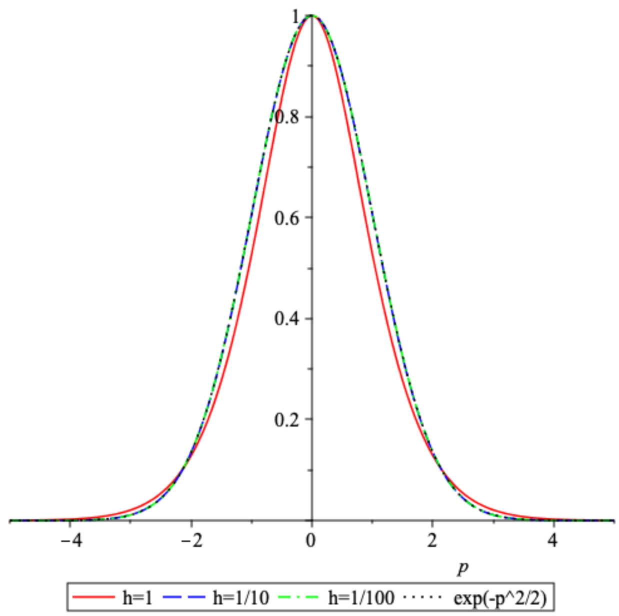

Figure 1: Asymptotic behaviour of discrete Gauss distribution for $h = 1, h = \frac{1}{10}, h = \frac{1}{100}$, compared to the continuous Gauss distribution.

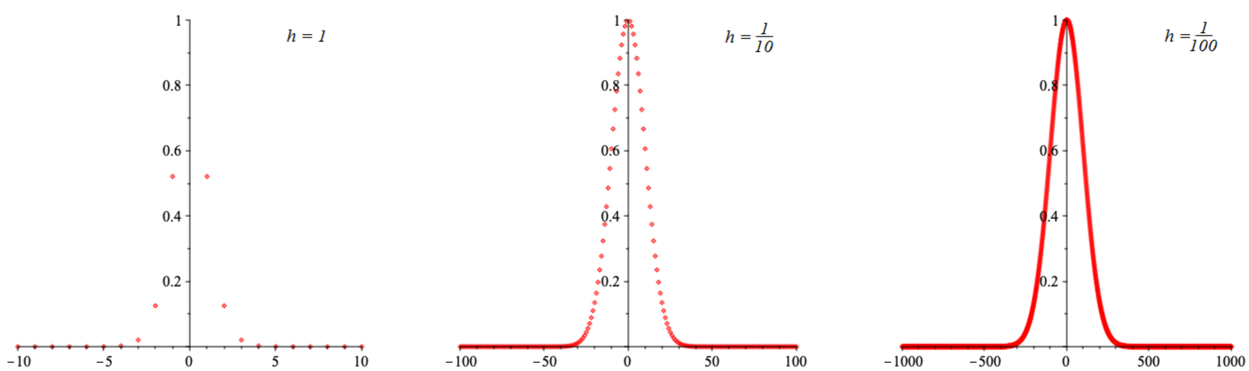

Figure 2: Density function of discrete Gauss distribution for $h = 1, h = \frac{1}{10}, h = \frac{1}{100}$ respectively.

$$
\begin{aligned}
&= (-1)^{\lfloor \frac{n}{2} \rfloor} e^{-z^2/2} \sum_{i=0}^{\infty} \frac{z^i}{i!} \langle G, \xi_h^i[1](\partial^\dagger)^n \rangle \\
&= (-1)^{\lfloor \frac{n}{2} \rfloor} e^{-z^2/2} \sum_{i=0}^{\infty} \frac{z^i}{i!} \langle G, \partial^n \xi_h^i[1] \rangle \\
&= (-1)^{\lfloor \frac{n}{2} \rfloor} e^{-z^2/2} \sum_{i=n}^{\infty} \frac{z^i}{i!} \frac{i!}{(i-n)!} \langle G, \xi_h^{i-n}[1] \rangle \\
&= (-1)^{\lfloor \frac{n}{2} \rfloor} e^{-z^2/2} \sum_{j=0}^{\infty} \frac{z^{j+n}}{j!} \langle G, \xi_h^j[1] \rangle \\
&= (-1)^{\lfloor \frac{n}{2} \rfloor} e^{-z^2/2} z^n \sum_{j=0}^{\infty} \frac{z^{2j}}{(2j)!} \sqrt{2\pi} \frac{(2j)!}{2^j j!} \\
&= (-1)^{\lfloor \frac{n}{2} \rfloor} \sqrt{2\pi}\, z^n.
\end{aligned}
$$

Also in [11], we found that the discrete delta function is an element of the discrete Weierstrass space, defined as the the completion of the right Clifford module of Hermite polynomials in ξ in the norm defined by the inner product

$$
(f, g) = \big(f(\xi)[1], g(\xi)[1]\big) := \left\langle f(\xi)G, [1](g(\xi))^\dagger \right\rangle, \tag{22}
$$

with $f(\xi)$ and $g(\xi)$ the Taylor series expansions of the discrete functions f and g.

4 Weierstrass transform in dimension $m = 2$

As a direct generalisation of the discrete Weierstrass transform of discrete Hermite polynomials, studied in [11], we find

Definition 4.1. *The definition of the Weierstrass transform of the $n-$th degree generalised Hermite polynomial in dimension m is*

$$
\mathcal{W}[H_{n,m,r}P_r](\underline{z}) = \sqrt{2\pi}^{-m} \left\langle H_{n,m,r}P_r G, \exp\left(\frac{-|\underline{z}|^2}{2} + \xi R \underline{z} \right)[1] \right\rangle,
$$

where $\underline{z} = \sum_{j=1}^{m} z_j e_j$ is a continuous complex Clifford variable.

In order to fix ideas and limit notations, let us start in dimension $m = 2$. It is furthermore clear that, amongst other, all discrete polynomials are in the span of the generalised Hermite polynomials.

4.1 Recurrence relation for the degree n of the Hermite polynomial

Our first goal is to express $\mathcal{W}[H_{n,2,r}P_r](\underline{z})$ in terms of $\mathcal{W}[H_{n-1,2,r}P_r](\underline{z})$. This is mainly based on the recurrence relation of the generalised Hermite polynomials (15), however some technical lemmas will finish the trick.

$$
\begin{aligned}
\mathcal{W}[H_{n,2,r}P_r](\underline{z}) &= \frac{1}{2\pi}\left\langle H_{n,2,r}P_rG, \exp\left(\frac{-|\underline{z}|^2}{2} + \xi R\underline{z}\right)[1]\right\rangle \\
&= \frac{1}{2\pi}\exp\left(\frac{-|\underline{z}|^2}{2}\right)\left\langle (-1)^n \partial H_{n-1,2,r}P_rG, \sum_{l=0}^{\infty}\frac{(\xi R\underline{z})^l}{l!}[1]\right\rangle \\
&= \frac{(-1)^{n+1}}{2\pi}\exp\left(\frac{-|\underline{z}|^2}{2}\right)\sum_{l=0}^{\infty}\frac{1}{\ell!}\left\langle H_{n-1,2,r}P_rG, (\xi_1R_1z_1+\xi_2R_2z_2)^l[1](\partial_1+\partial_2)^\dagger\right\rangle \\
&= \frac{(-1)^{n+1}}{2\pi}\exp\left(\frac{-|\underline{z}|^2}{2}\right)\sum_{l=0}^{\infty}\frac{1}{l!}\left\langle H_{n-1,2,r}P_rG, (\xi_1R_1z_1+\xi_2R_2z_2)^l[1]\left(\partial_1^\dagger+\partial_2^\dagger\right)\right\rangle \\
&= \frac{(-1)^{n+1}}{2\pi}\exp\left(\frac{-|\underline{z}|^2}{2}\right)\sum_{l=0}^{\infty}\sum_{j=0}^{l}\binom{l}{j}\frac{1}{l!}\left[\left\langle H_{n-1,2,r}P_rG, \xi_1^jR_1^jz_1^j\xi_2^{l-j}R_2^{l-j}z_2^{l-j}[1]\partial_1^\dagger\right\rangle\right. \\
&\quad \left. + \left\langle H_{n-1,2,r}P_rG, \xi_1^jR_1^jz_1^j\xi_2^{l-j}R_2^{l-j}z_2^{l-j}[1]\partial_2^\dagger\right\rangle\right].
\end{aligned} \tag{23}
$$

Lemma 4.2. *$\forall j,k,l \in \mathbb{N}$: If j and k have equal parity, then $\xi_i^jR_i^k[1]\left(\partial_i^\dagger\right)^l = e_i^k\partial_i^l\xi_i^j[1]$.*

Proof. If j and k are even, then $R_i^k = 1$, hence apply calculation rule (5) to find

$$
\xi_i^jR_i^k[1]\left(\partial_i^\dagger\right)^l = \xi_i^j[1]\left(\partial_i^\dagger\right)^l = \partial_i^l\xi_i^j[1] = e_i^k\partial_i^l\xi_i^j[1].
$$

If j and k are odd, then $\xi_i^j[1] = \left(\xi_i^\dagger\right)^j[1]$ and $R_i^k[1] = R_i[1] = e_i[1]$. There are two options for the dirac operator ∂_i, keeping in mind that $\partial_i^2 = \left(\partial_i^\dagger\right)^2$ is scalar. If l is even, say $2l$, then scalar

$$
\xi_i^jR_i[1]\left(\partial_i^\dagger\right)^{2l} = \xi_i^je_i[1]\left(\partial_i^\dagger\right)^{2l} = \left(\partial_i^\dagger\right)^{2l}\left(\xi_i^\dagger\right)^je_i[1] = e_i\partial_i^{2l}\xi_i^j[1].
$$

If l is odd, say $2l+1$, then

$$
\xi_i^jR[1]\left(\partial_i^\dagger\right)^{2l+1} = \partial_i^{2l}\left(\xi_i^je[1]\partial_i^\dagger\right) = \partial_i^{2l}\left(\partial_i^\dagger\left(\xi_i^\dagger\right)^je_i\right)[1] = \partial_i^{2l}\left(e_i\partial_i\xi^j[1]\right) = e_i\partial_i^{2l+1}\xi_i^j[1].
$$

□

Remark 4.3. *It should be noted that interaction of different indices i in ξ_i, ∂_i, R_i does not affect the order or involutions $\dagger$ of operators in the previous lemma. They can only introduce a change of sign.*

We continue from (23)

$$
\begin{aligned}
\mathcal{W}[H_{n,2,r}P_r](\underline{z}) &= \frac{(-1)^{n+1}}{2\pi}\exp\left(\frac{-|\underline{z}|^2}{2}\right)\sum_{l=0}^{\infty}\sum_{j=0}^{l}\binom{l}{j}\frac{1}{l!}\left[\left\langle H_{n-1,2,r}P_rG, e_1^j\underbrace{\partial_1\xi_1^j}_{j\xi_1^{j-1}}z_1^j\xi_2^{l-j}R_2^{l-j}z_2^{l-j}[1]\right\rangle\right. \\
&\quad \left. + \left\langle H_{n-1,2,r}P_rG, \xi_1^jR_1^jz_1^je_2^{l-j}\underbrace{\partial_2\xi_2^{l-j}}_{(l-j)\xi_2^{l-j-1}}z_2^{l-j}[1]\right\rangle\right] \\
&= \frac{(-1)^{n+1}}{2\pi}\exp\left(\frac{-|\underline{z}|^2}{2}\right)\sum_{l=0}^{\infty}\frac{1}{l!}\left[\sum_{j=1}^{l}\binom{l}{j}j\left\langle H_{n-1,2,r}P_rG, e_1^j\xi_1^{j-1}z_1^j\xi_2^{l-j}R_2^{l-j}z_2^{l-j}[1]\right\rangle\right. \\
&\quad \left. + \sum_{j=0}^{l-1}\binom{l}{j}(l-j)\left\langle H_{n-1,2,r}P_rG, \xi_1^jR_1^jz_1^je_2^{l-j}\xi_2^{l-j-1}z_2^{l-j}[1]\right\rangle\right].
\end{aligned} \tag{24}
$$

Lemma 4.4. *For any index $i = 1 \dots m$ and any power $j \in \mathbb{N}$,*

$$e_i^j \xi_i^{j-1}[1] = e_i \xi_i^{j-1} R_i^{j-1}[1].$$

Proof. The aim is to bring the factor e_i^{j-1} in the left hand side through the basic discrete polynomial ξ_i^{j-1}, in order to re-write it as the operator R_i^{j-1}, acting on [1]. However, $e_i \xi_i = \xi_i^\dagger e_i$. If j is even, $j-1$ is odd, which means $\left(\xi_i^\dagger\right)^{j-1}[1] = \xi_i^{j-1}[1]$. If j is odd, $j-1$ is even and $e_i^{j-1} = 1$. In both cases, the lemma is proven. □

Let us proceed with (24)

$$\begin{aligned}
\mathcal{W}[H_{n,2,r}P_r](\underline{z}) &= \frac{(-1)^{n+1}}{2\pi} \exp\left(\frac{-|\underline{z}|^2}{2}\right) \sum_{l=0}^{\infty} \frac{1}{l!} \left[\sum_{j=1}^{l} \binom{l}{j} j e_1 z_1 \left\langle H_{n-1,2,r}P_r G, \xi_1^{j-1} R_1^{j-1} z_1^{j-1} \xi_2^{l-j} R_2^{l-j} z_2^{l-j}[1] \right\rangle \right.\\
&\left. + \sum_{j=0}^{l-1} \binom{l}{j} (l-j) e_2 z_2 \left\langle H_{n-1,2,r}P_r G, \xi_1^{j} R_1^{j} z_1^{j} \xi_2^{l-j-1} R_2^{l-j-1} z_2^{l-j-1}[1] \right\rangle \right]\\
&= \frac{(-1)^{n+1}}{2\pi} \exp\left(\frac{-|\underline{z}|^2}{2}\right) \sum_{l=0}^{\infty} \sum_{j=0}^{l-1} \frac{1}{l!} \left(\binom{l}{j+1} (j+1) z_1 e_1 + \binom{l}{j} (l-j) z_2 e_2 \right)\\
&\times \left\langle H_{n-1,2,r}P_r G, \xi_1^{j} R_1^{j} z_1^{j} \xi_2^{l-j-1} R_2^{l-j-1} z_2^{l-j-1}[1] \right\rangle\\
&= \frac{(-1)^{n+1}}{2\pi} \exp\left(\frac{-|\underline{z}|^2}{2}\right) \sum_{l=1}^{\infty} \sum_{j=0}^{l-1} \frac{1}{(l-1)!} \binom{l-1}{j} (z_1 e_1 + z_2 e_2) \left\langle H_{n-1,2,r}P_r G, \xi_1^{j} R_1^{j} z_1^{j} \xi_2^{l-j-1} R_2^{l-j-1} z_2^{l-j-1}[1] \right\rangle\\
&= \frac{(-1)^{n+1}}{2\pi} \exp\left(\frac{-|\underline{z}|^2}{2}\right) \sum_{l=1}^{\infty} \frac{1}{(l-1)!} (z_1 e_1 + z_2 e_2) \left\langle H_{n-1,2,r}P_r G, (\xi_1 R_1 z_1 + \xi_2 R_2 z_2)^{l-1}[1] \right\rangle\\
&= (-1)^{n+1} (z_1 e_1 + z_2 e_2)\, \mathcal{W}[H_{n-1,2,r}P_r](\underline{z}).
\end{aligned} \tag{25}$$

We have proven

Theorem 4.5. *For the discrete Weierstrass transform of the discrete generalised Hermite polynomials in two dimensions, it holds that*

$$\mathcal{W}[H_{n,2,r}P_r](\underline{z}) = (-1)^{n+1} (z_1 e_1 + z_2 e_2) \mathcal{W}[H_{n-1,2,r}P_r](\underline{z}).$$

Let us check this theorem by looking at some examples for low values of n.

Example 4.6. *For $r = 0$, the results of the general definition 4.1 for the discrete Weierstrass transform must coincide with the former definition of [11], i.e. the transform of the $n-$th Hermite polynomial should be the $n-$th power of*
underlinez. Let us check this for $n = 0$ and $n = 1$. As $P_0 = 1$, we will omit this notation.

$$\begin{aligned}
\mathcal{W}[H_{0,2,0}](\underline{z}) &= \frac{1}{2\pi} \exp\left(\frac{-|\underline{z}|^2}{2}\right) \left\langle H_{0,2,0}G, \exp(\xi R \underline{z})[1] \right\rangle\\
&= \frac{1}{2\pi} \exp\left(\frac{-|\underline{z}|^2}{2}\right) \sum_{l=0}^{\infty} \frac{1}{l!} \left\langle G, (\xi_1 R_1 z_1 + \xi_2 R_2 z_2)^l[1] \right\rangle\\
&= \frac{1}{2\pi} \exp\left(\frac{-|\underline{z}|^2}{2}\right) \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{j=0}^{l} \binom{l}{j} \left\langle G, \xi_1^j R_1^j z_1^j \xi_2^{l-j} R_2^{l-j} z_2^{l-j}[1] \right\rangle\\
&= \frac{1}{2\pi} \exp\left(\frac{-|\underline{z}|^2}{2}\right) \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{j=0}^{l} \binom{l}{j} z_1^j z_2^{l-j} \left\langle G, e_1^j \xi_1^j e_2^{l-j} \xi_2^{l-j}[1] \right\rangle.
\end{aligned} \tag{26}$$

The Gaussian distribution vanishes when acting on odd powers of $\xi[1]$, see (12). Hence, the only remaining indices are those where j and l are both even.

$$\begin{aligned}
\mathcal{W}[H_{0,2,0}](\underline{z}) &= \frac{1}{2\pi}\exp\left(\frac{-|\underline{z}|^2}{2}\right)\sum_{l=0}^{\infty}\frac{1}{(2l)!}\sum_{j=0}^{l}\binom{2l}{2j}z_1^{2j}z_2^{2l-2j}\left\langle G, e_1^{2j}\xi_1^{2j}e_2^{2l-2j}\xi_2^{2l-2j}[1]\right\rangle \\
&= \exp\left(\frac{-|\underline{z}|^2}{2}\right)\sum_{l=0}^{\infty}\frac{1}{(2l)!}\sum_{j=0}^{l}\binom{2l}{2j}z_1^{2j}z_2^{2l-2j}\frac{(2j)!}{2^j j!}\frac{(2l-2j)!}{2^{l-j}(l-j)!} \\
&= \exp\left(\frac{-|\underline{z}|^2}{2}\right)\sum_{l=0}^{\infty}z_1^{2j}z_2^{2l-2j}\frac{1}{j!(l-j)!2^l} = 1.
\end{aligned} \tag{27}$$

Example 4.7. *The next example uses the same calculations as seen in the general proof and again the fact that the Gaussian vanishes when acting on odd powers of* $\xi[1]$.

$$\begin{aligned}
\mathcal{W}[H_{1,2,0}](\underline{z}) &= \frac{1}{2\pi}\exp\left(\frac{-|\underline{z}|^2}{2}\right)\left\langle H_{1,2,0}G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle \\
&= \frac{1}{2\pi}\exp\left(\frac{-|\underline{z}|^2}{2}\right)\sum_{l=0}^{\infty}\frac{1}{l!}\left\langle -\left(\partial_1+\partial_2\right)G, \left(\xi_1R_1z_1+\xi_2R_2z_2\right)^l[1]\right\rangle \\
&= \frac{1}{2\pi}\exp\left(\frac{-|\underline{z}|^2}{2}\right)\sum_{l=0}^{\infty}\frac{1}{l!}\sum_{j=0}^{l}\binom{l}{j}\left[\left\langle G, \xi_1^jR_1^jz_1^j\xi_2^{l-j}R_2^{l-j}z_2^{l-j}[1]\partial_1^\dagger\right\rangle + \left\langle G, \xi_1^jR_1^jz_1^j\xi_2^{l-j}R_2^{l-j}z_2^{l-j}[1]\partial_2^\dagger\right\rangle\right] \\
&= \frac{1}{2\pi}\exp\left(\frac{-|\underline{z}|^2}{2}\right)\sum_{l=0}^{\infty}\frac{1}{l!}\left[\sum_{j=1}^{l}\binom{l}{j}z_1^jz_2^{l-j}j\left\langle G, e_1^j\xi_1^{j-1}\xi_2^{l-j}e_2^{l-j}[1]\right\rangle\right. \\
&\left.+\sum_{j=0}^{l-1}\binom{l}{j}z_1^jz_2^{l-j}(l-j)\left\langle G, \xi_1^je_1^j\xi_2^{l-j-1}e_2^{l-j}[1]\right\rangle\right] \\
&= \frac{1}{2\pi}\exp\left(\frac{-|\underline{z}|^2}{2}\right)\sum_{l=0}^{\infty}\frac{1}{(2l+1)!}\left[\sum_{j=0}^{l}\binom{2l+1}{2j+1}z_1^{2j+1}z_2^{2l-2j}e_1(2j+1)\left\langle G, \xi_1^{2j}\xi_2^{2l-2j}[1]\right\rangle\right. \\
&\qquad\left.+\sum_{j=0}^{l}\binom{2l+1}{2j}z_1^{2j}z_2^{2l+1-2j}e_2(2l+1-2j)\left\langle G, \xi_1^{2j}\xi_2^{2l-2j}[1]\right\rangle\right] \\
&= \exp\left(\frac{-|\underline{z}|^2}{2}\right)\sum_{l=0}^{\infty}\frac{1}{(2l+1)!}\left[\sum_{j=0}^{l}\binom{2l+1}{2j+1}z_1^{2j+1}z_2^{2l-2j}e_1(2j+1)\frac{(2j)!}{2^jj!}\frac{(2l-2j)!}{2^{l-j}(l-j)!}\right. \\
&\qquad\left.+\sum_{j=0}^{l}\binom{2l+1}{2j}z_1^{2j}z_2^{2l+1-2j}e_2(2l+1-2j)\frac{(2j)!}{2^jj!}\frac{(2l-2j)!}{2^{l-j}(l-j)!}\right] \\
&= \exp\left(\frac{-|\underline{z}|^2}{2}\right)\sum_{l=0}^{\infty}\sum_{j=0}^{l}\frac{1}{2^lj!(l-j)!}\left[z_1^{2j+1}z_2^{2l-2j}e_1+z_1^{2j}z_2^{2l+1-2j}e_2\right] \\
&= \exp\left(\frac{-|\underline{z}|^2}{2}\right)\exp\left(\frac{|\underline{z}|^2}{2}\right)\left(e_1z_1+e_2z_2\right) = e_1z_1+e_2z_2.
\end{aligned} \tag{28}$$

The end results were calculated with a computational program.

Based on the above calculations in examples 4.6 and 4.7, together with the results of theorem 4.5, we have an explicit expression for the $n-th$ degree discrete Hermite polynomial in dimension 2 with $r=0$.

$$\begin{aligned}\mathcal{W}[H_{2k,2,0}](\underline{z}) &= (-1)^k (z_1 e_1 + z_2 e_2)^{2k},\\ \mathcal{W}[H_{2k+1,2,0}](\underline{z}) &= (-1)^{k+1} (z_1 e_1 + z_2 e_2)^{2k+1}.\end{aligned} \tag{29}$$

Having found a recurrence relation for the degree n of the Hermite polynomial, we seek for an analogous formula, expressing the Weierstrass transform of $H_{n,2,r}P_r$ in function of $H_{n,2,r-1P_{r-1}}$.

4.2 Recurrence relation for the degree r of the monogenic

In what follows, results will be proven for the basis monogenic polynomials, and hence we will use the notation P_r for $\underbrace{(\xi_2 - \xi_1)(\xi_2 + \xi_1)...(\xi_2 \pm \xi_1)}_{r \text{ times}}$. Let us furthermore introduce the notations $\tilde{P}_r$ and $\tilde{\partial}$.

Notation 1. *We denote*

$$\begin{aligned}P_r &= \underbrace{(\xi_2 - \xi_1)(\xi_2 + \xi_1)...(\xi_2 \pm \xi_1)}_{r \text{ times}},\\ \tilde{P}_r &\coloneqq \underbrace{(\xi_2 + \xi_1)(\xi_2 - \xi_1)...(\xi_2 \mp \xi_1)}_{r \text{ times}}\end{aligned}$$

and

$$\begin{aligned}\partial &= \partial_2 + \partial_1,\\ \tilde{\partial} &\coloneqq \partial_2 - \partial_1.\end{aligned} \tag{30}$$

In combination with the discrete Gauss distribution, we know that $\partial P_r G = -\xi P_r G$, which we now can write as

$$\partial P_r G = -\xi P_r G = -\tilde{P}_{r+1} G.$$

We now try to exploit this relationship to find the recurrence relation we are looking for. Having in mind that $H_{0,m,r'} = 1, \forall m, \forall r'$, we calculate $\mathcal{W}[\tilde{P}_r](\underline{z}) = \mathcal{W}[H_{0,2,r}\tilde{P}_r](\underline{z})$.

$$\begin{aligned}
\mathcal{W}[H_{0,2,r}\tilde{P}_r](\underline{z}) &= \frac{1}{2\pi} \exp\left(-\frac{|\underline{z}|^2}{2}\right) \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{j=0}^{l} \binom{l}{j} \left\langle -(\partial_1 + \partial_2) P_{r-1} G, \xi_1^j R_1^j z_1^j \xi_2^{l-j} R_2^{l-j} z_2^{l-j}[1] \right\rangle \\
&= \frac{1}{2\pi} \exp\left(-\frac{|\underline{z}|^2}{2}\right) \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{j=0}^{l} \binom{l}{j} \left\langle P_{r-1} G, \xi_1^j R_1^j z_1^j \xi_2^{l-j} R_2^{l-j} z_2^{l-j}[1](\partial_1^\dagger + \partial_2^\dagger) \right\rangle \\
&= \frac{1}{2\pi} \exp\left(-\frac{|\underline{z}|^2}{2}\right) \sum_{l=0}^{\infty} \frac{1}{l!} \left[\sum_{j=1}^{l} \binom{l}{j} j \left\langle P_{r-1} G, e_1^j \xi_1^{j-1} z_1^j \xi_2^{l-j} R_2^{l-j} z_2^{l-j}[1] \right\rangle \right. \\
&\quad \left. + \sum_{j=0}^{l-1} \binom{l}{j} (l-j) \left\langle P_{r-1} G, \xi_1^j R_1^j z_1^j e_2^{l-j} \xi_2^{l-j-1} z_2^{l-j}[1] \right\rangle \right] \\
&= \frac{1}{2\pi} \exp\left(-\frac{|\underline{z}|^2}{2}\right) \sum_{l=1}^{\infty} \frac{1}{l!} \left[\sum_{j=0}^{l-1} \binom{l}{j+1} (j+1) z_1 e_1 \left\langle P_{r-1} G, \xi_1^j R_1^j z_1^j \xi_2^{l-j-1} R_2^{l-j-1} z_2^{l-j-1}[1] \right\rangle \right. \\
&\quad \left. + \sum_{j=0}^{l-1} \binom{l}{j} (l-j) z_2 e_2 \left\langle P_{r-1} G, \xi_1^j R_1^j z_1^j \xi_2^{l-j-1} R_2^{l-j-1} z_2^{l-j-1}[1] \right\rangle \right] \\
&= \frac{1}{2\pi} \exp\left(-\frac{|\underline{z}|^2}{2}\right) \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{j=0}^{l} \binom{l}{j} (z_1 e_1 + z_2 e_2) \left\langle P_{r-1} G, (\xi_1 R_1 z_1 + \xi_2 R_2 z_2)^l [1] \right\rangle \\
&= (z_1 e_1 + z_2 e_2)\, \mathcal{W}[H_{0,2,r-1} P_{r-1}](\underline{z}),
\end{aligned} \tag{31}$$

H_n \ $P_r, \tilde{P}_r$	0	1	2	3
0	1	$(z_1e_1 - z_2e_2)$ $(z_1e_1 + z_2e_2)$	$(z_1e_1 + z_2e_2)(z_1e_1 - z_2e_2)$ $(z_1e_1 - z_2e_2)(z_1e_1 + z_2e_2)$	$(z_1e_1 - z_2e_2)(z_1e_1 + z_2e_2)(z_1e_1 - z_2e_2)$ $(z_1e_1 + z_2e_2)(z_1e_1 - z_2e_2)(z_1e_1 + z_2e_2)$
1	$(z_1e_1 + z_2e_2)$	$(z_1e_1 + z_2e_2)(z_1e_1 - z_2e_2)$	$(z_1e_1 + z_2e_2)(z_1e_1 - z_2e_2)(z_1e_1 + z_2e_2)$	$(z_1e_1 + z_2e_2)(z_1e_1 - z_2e_2)(z_1e_1 + z_2e_2)(z_1e_1 - z_2e_2)$
2	$-(z_1e_1 + z_2e_2)^2$	$-(z_1e_1 + z_2e_2)^2(z_1e_1 - z_2e_2)$	$-(z_1e_1 + z_2e_2)^2(z_1e_1 - z_2e_2)(z_1e_1 + z_2e_2)$	$-(z_1e_1 + z_2e_2)^2(z_1e_1 - z_2e_2)(z_1e_1 + z_2e_2)(z_1e_1 - z_2e_2)$
3	$-(z_1e_1 + z_2e_2)^3$	$-(z_1e_1 + z_2e_2)^3(z_1e_1 - z_2e_2)$	$-(z_1e_1 + z_2e_2)^3(z_1e_1 - z_2e_2)(z_1e_1 + z_2e_2)$	$-(z_1e_1 + z_2e_2)^3(z_1e_1 - z_2e_2)(z_1e_1 + z_2e_2)(z_1e_1 - z_2e_2)$
4	$(z_1e_1 + z_2e_2)^4$	$(z_1e_1 + z_2e_2)^4(z_1e_1 - z_2e_2)$	$(z_1e_1 + z_2e_2)^4(z_1e_1 - z_2e_2)(z_1e_1 + z_2e_2)$	$(z_1e_1 + z_2e_2)^4(z_1e_1 - z_2e_2)(z_1e_1 + z_2e_2)(z_1e_1 - z_2e_2)$
5	$(z_1e_1 + z_2e_2)^5$	$(z_1e_1 + z_2e_2)^5(z_1e_1 - z_2e_2)$	$(z_1e_1 + z_2e_2)^5(z_1e_1 - z_2e_2)(z_1e_1 + z_2e_2)$	$(z_1e_1 + z_2e_2)^5(z_1e_1 - z_2e_2)(z_1e_1 + z_2e_2)(z_1e_1 - z_2e_2)$

Table 1: Weierstrass transform of generalized Hermite polynomials in two dimensions.

where we used the same methods as in (25). Analogously, it is easily checked that, with the new notation, $\tilde{\partial}\tilde{P}_r = 0$. It then holds that

$$\tilde{\partial}\tilde{P}_r G = -(\xi_2 - \xi_1)\tilde{P}_r G = -P_{r+1}G,$$

so we can calculate

$$\begin{aligned}
\mathcal{W}[H_{0,2,r}P_r](\underline{z}) &= \frac{1}{2\pi}\exp\left(-\frac{|\underline{z}|^2}{2}\right)\sum_{l=0}^{\infty}\frac{1}{l!}\sum_{j=0}^{l}\binom{l}{j}\left\langle -(\partial_2-\partial_1)\tilde{P}_{r-1}G, \xi_1^j R_1^j z_1^j \xi_2^{l-j} R_2^{l-j} z_2^{l-j}[1]\right\rangle \\
&= \frac{1}{2\pi}\exp\left(-\frac{|\underline{z}|^2}{2}\right)\sum_{l=0}^{\infty}\frac{1}{l!}\sum_{j=0}^{l}\binom{l}{j}\left\langle \tilde{P}_{r-1}G, \xi_1^j R_1^j z_1^j \xi_2^{l-j} R_2^{l-j} z_2^{l-j}[1](\partial_1^\dagger-\partial_2^\dagger)\right\rangle \\
&= \frac{1}{2\pi}\exp\left(-\frac{|\underline{z}|^2}{2}\right)\sum_{l=0}^{\infty}\frac{1}{l!}\left[\sum_{j=1}^{l}\binom{l}{j} j\left\langle \tilde{P}_{r-1}G, e_1^j\xi_1^{j-1} z_1^j \xi_2^{l-j} R_2^{l-j} z_2^{l-j}[1]\right\rangle\right. \\
&\qquad \left. -\sum_{j=0}^{l-1}\binom{l}{j}(l-j)\left\langle \tilde{P}_{r-1}G, \xi_1^j R_1^j z_1^j e_2^{l-j}\xi_2^{l-j-1} z_2^{l-j}[1]\right\rangle\right] \\
&= \frac{1}{2\pi}\exp\left(-\frac{|\underline{z}|^2}{2}\right)\sum_{l=1}^{\infty}\frac{1}{l!}\left[\sum_{j=1}^{l}\binom{l}{j} j z_1 e_1\left\langle \tilde{P}_{r-1}G, \xi_1^j R_1^j z_1^j \xi_2^{l-j-1} R_2^{l-j-1} z_2^{l-j-1}[1]\right\rangle\right. \\
&\qquad \left. -\sum_{j=0}^{l-1}\binom{l}{j}(l-j) z_2 e_2\left\langle \tilde{P}_{r-1}G, \xi_1^j R_1^j z_1^j \xi_2^{l-j-1} R_2^{l-j-1} z_2^{l-j-1}[1]\right\rangle\right] \\
&= \frac{1}{2\pi}\exp\left(-\frac{|\underline{z}|^2}{2}\right)\sum_{l=1}^{\infty}\frac{1}{l!}\sum_{j=0}^{l-1}\binom{l}{j}(z_1e_1 - z_2e_2)\left\langle \tilde{P}_{r-1}G, \xi_1^j R_1^j z_1^j \xi_2^{l-j-1} R_2^{l-j-1} z_2^{l-j-1}[1]\right\rangle \\
&= (z_1e_1 - z_2e_2)\,\mathcal{W}[H_{0,2,r-1}\tilde{P}_{r-1}](\underline{z}).
\end{aligned} \tag{32}$$

We conclude the results of the previous calculations in the next theorem:

Theorem 4.8. *For the discrete Weierstrass transform in dimension 2, it holds that*

$$\mathcal{W}[H_{0,2,r}\tilde{P}_r](\underline{z}) = (z_1e_1 + z_2e_2)\ \mathcal{W}[H_{0,2,r-1}P_{r-1}(\underline{z})]$$
$$\mathcal{W}[H_{0,2,r}P_r](\underline{z}) = (z_1e_1 - z_2e_2)\ \mathcal{W}[H_{0,2,r-1}\tilde{P}_{r-1}](\underline{z}).$$

Combining theorems the recurrence relations in 4.5, 4.8 and the trivial example in (29), we can calculate the Weierstrass transform of every generalised Hermite polynomial in two dimensions, see table 1. For example: $W[H_{3,2,3}P_r](\underline{z}) = -(z_1e_1 + z_2e_2)^3(z_1e_1 - z_2e_2)(z_1e_1 + z_2e_2)(z_1e_1 - z_2e_2)$.

The previous method in section 4.2 might be perceived as quite artificial. It is, in addition, not directly extendable to higher dimensions, because it relies on the specific representation of the basic monogenics P_r if $m = 2$. It gives us, however, a good idea of the results when $m > 2$. Another approach is needed when we enter higher dimensions.

5 Weierstrass transform in dim $m > 2$

This section will follow the same structure as the previous one. The first subsection will be straightforward, however the second subsection will outline an alternative expression for the Weierstrass transform which implicates the recurrence relation found above.

5.1 Recurrence relation for the degree n of the Hermite polynomial

Definition 5.1. *The definition of the $n-th$ degree generalised Hermite polynomial in dimension m is*

$$\mathcal{W}[H_{n,m,r}P_r](\underline{z}) = \sqrt{2\pi}^{-m} \left\langle H_{n,m,r}P_rG, \exp\left(\frac{-|\underline{z}|^2}{2} + \xi R\underline{z}\right)[1] \right\rangle.$$

All notations, calculations and results from subsection 4.1 apply for $m > 2$. However, due to the overload in notations, we limit ourselves to the results.

As a direct generalisation of theorem 4.5, we find

Theorem 5.2. *The discrete Weierstrass transform of the discrete generalised Hermite polynomials in m dimensions is recursively given by*

$$\mathcal{W}[H_{n,m,r}P_r](\underline{z}) = \left(\sum_{j=1}^{m} z_j e_j\right)(-1)^{n+1}\,\mathcal{W}[H_{n-1,m,r}P_r](\underline{z}). \tag{33}$$

To start the recursive definition, it can be easily calculated that

$$\begin{aligned} \mathcal{W}[H_{0,m,0}](\underline{z}) &= 1, \\ \mathcal{W}[H_{1,m,0}](\underline{z}) &= \sum_{j=1}^{m} z_j e_j. \end{aligned} \tag{34}$$

5.2 Recurrence relation for the degree r of the monogenic

To generalise the recurrence relation of the Weierstrass transform of $H_{n,2,r}$ in function of $H_{n,2,r-1}$ in section 4, we will first enlist some examples to fix ideas. Let $n = 0$ and let us calculate the discrete Weierstrass transform of the basic spherical monogenic polynomials P_r. Therefore, let us rephrase some important notions.

Theorem 5.3 (Cauchy-Kovalevskaya extension for discrete monogenic functions, [12])**.** *Let f be a discrete function in the variables $x_2, \ldots, x_m$, defined on the grid $\mathbb{Z}^{m-1}$ and taking values in the algebra over $\{\mathbf{e}_2^+, \mathbf{e}_2^-, \ldots, \mathbf{e}_m^+, \mathbf{e}_m^-\}$. Then there exists a unique discrete monogenic function F in the variables $x_1, \ldots, x_m$, defined on the grid $\mathbb{Z}^m$ and taking values in the algebra over $\{\mathbf{e}_1^+, \mathbf{e}_1^-, \ldots, \mathbf{e}_m^+, \mathbf{e}_m^-\}$, such that the restriction of F to the hyperplane $x_1 = 0$ equals f. This function F is given by*

$$CK[f](x_1, \ldots, x_m) = \sum_{k=0}^{\infty} \frac{\xi_1^k[1](x_1)}{k!} f_k(x_2, \ldots, x_m),$$

where $f_0 = f$ and $f_{k+1} = (-1)^{k+1} \sum_{j=2}^{m} \partial_j f_k$.

Theorem 5.4. *The set*

$$\{CK[\xi^{\underline{\alpha}}] \mid \underline{\alpha} = (\alpha_2, \dots, \alpha_m), \alpha_2 + \dots + \alpha_m = r\}$$

constitutes a basis for the set of discrete spherical monogenics of degree r in dimension m.

Let us introduce some notations.

Notation 2. *We denote*

- $\eta_i = \xi_i - \xi_1$,
- $\hat{\eta_i} = \xi_i + \xi_1$,
- $y_i = z_i - z_1$,
- $\hat{y_i} = z_i + z_1$,
- $(\eta_{l_1}, \dots \eta_{l_k})^{E_j}$ *means that every even occurrence (i.e. second, fourth, sixth,...) of η_j is replaced by $\hat{\eta_j}$ and vice versa. The composition of $E_{r_1}, \dots E_{r_k}$ is denoted in short by $E_{r_1,\dots r_k}$.*
- *Analogously, $(\eta_{l_1}, \dots \eta_{l_k})^{O_j}$ means that every odd occurrence (i.e. first, third, fifth,...) of η_j is replaced by $\hat{\eta_j}$ and vice versa. The composition of $O_{r_1}, \dots O_{r_k}$ is denoted in short by $O_{r_1,\dots r_k}$.*
- *For $\underline{\alpha} = (\alpha_2, \dots, \alpha_m) \in N^{m-1}$, $\xi^{\underline{\alpha}} = \xi_2^{\alpha_2} \dots \xi_m^{\alpha_m}$. The degree of the operator $\xi^{\underline{\alpha}}$ is $k = \alpha_2 + \dots + \alpha_m$.*
- *With every $\underline{\alpha}$, we associate the $k-$tupple $(l_1, \dots l_k)$, with every $l_j \in \{2, \dots m\}$, $l_i \leq l_j$ if $i \leq j$ and the number of times that j appears in $(l_1, \dots, l_k)$ is α_j.*

From [15], we know that

$$CK[\xi^{\underline{\alpha}}] = \frac{\alpha_2! \dots \alpha_m!}{k!} \sum_{\pi(l_1,\dots l_k)} \operatorname{sgn}(\pi)(\eta_{\pi(l_1)} \dots \eta_{\pi(l_k)})^{E_{2,\dots,m}},$$

where the sum runs over all distinguishable permutations π of $(l_1, \dots l_k)$. $\operatorname{sgn}(\pi)$ is $+1$ or -1, according to the sign of the permutation π.

Example 5.5. *For $m = 2$, we find*

$$CK[\xi_2^r] = \underbrace{(\xi_2 - \xi_1)(\xi_2 + \xi_1)...(\xi_2 \pm \xi_1)}_{r \text{ times}} = P_r.$$

This P_r is unique, up to a scalar multiplication. Hence the dimension of the discrete monogenic polynomials of degree r in two dimensions is one.

With these notations, we can rewrite, for example, the result of example 4.7:

$$\mathcal{W}[H_{1,2,0}](\underline{z}) = \mathcal{W}[\hat{\eta_2}] = \hat{y_2}.$$

In the next section, we will give some examples of the Weierstrass transform of $CK[\xi^{\underline{\alpha}}]$, for low values of $\|\underline{\alpha}\| = r$. This can be interpreted as the Weierstrass transform $\mathcal{W}[H_{0,m,r}CK[\xi^{\underline{\alpha}}]]$, as any Hermite polynomial of degree 0 equals 1.

5.3 Examples

Example 5.6. $\|\alpha\| = 1$
$CK[\xi^{\underline{\alpha}}] = \eta_j$, when $\underline{\ell} = (j)$. Hence $P_1 = \xi_j - \xi_1$. Having in mind that $\xi G = -\partial G$, this leads us to the calculation of example 4.7.

Example 5.7. $\|\alpha\| = 2$
Two combinations are possible $(i, j \neq 1, i \neq j)$:

1. $\ell = (i, j)$,
2. $\ell = (j, j)$.

In the first case, $CK[\xi^{\underline{\alpha}}] = \frac{1}{2}\left(\eta_i\eta_j - \eta_j\eta_i\right) = \xi_i\xi_j - \xi_1\xi_j + \xi_1\xi_i$*. This will result in the Weierstrass transform* $\frac{1}{2}\left(y_iy_j - y_jy_i\right)$*. For example, take* $m = 3$ *and* $\ell = (2,3)$*.*

$$\mathcal{W}[\xi_2\xi_3 - \xi_1\xi_3 + \xi_1\xi_2] = \frac{1}{\sqrt{2\pi}^{\frac{3}{2}}}\exp\left(\frac{-|\underline{z}|^2}{2}\right)\left[\left\langle \xi_2\xi_3 G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle - \left\langle \xi_1\xi_3 G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle + \left\langle \xi_1\xi_2 G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle\right]$$

The three terms in the RHS are completely similar, so we will only work out the first term.

$$\begin{aligned}
\left\langle \xi_2\xi_3 G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle &= \sum_{\ell=0}^{\infty}\frac{1}{\ell!}\sum_{j_1+j_2+j_3=\ell}\binom{\ell}{j_1j_2j_3}\left\langle \xi_2\xi_3 G, \xi_1^{j_1}R_1^{j_1}z_1^{j_1}\xi_2^{j_2}R_2^{j_2}z_2^{j_2}\xi_3^{j_3}R_3^{j_3}z_3^{j_3}[1]\right\rangle \\
&= \sum_{\ell=0}^{\infty}\frac{1}{\ell!}\sum_{j_1+j_2+j_3=\ell}\binom{\ell}{j_1j_2j_3}\left\langle \partial_2\partial_3 G, \xi_1^{j_1}R_1^{j_1}z_1^{j_1}\xi_2^{j_2}R_2^{j_2}z_2^{j_2}\xi_3^{j_3}R_3^{j_3}z_3^{j_3}[1]\right\rangle \\
&= \sum_{\ell=0}^{\infty}\frac{1}{\ell!}\sum_{j_1+j_2+j_3=\ell}\binom{\ell}{j_1j_2j_3}\left\langle G, \xi_1^{j_1}R_1^{j_1}z_1^{j_1}\xi_2^{j_2}R_2^{j_2}z_2^{j_2}\xi_3^{j_3}R_3^{j_3}z_3^{j_3}[1]\partial_2^{\dagger}\partial_3^{\dagger}\right\rangle \\
&= \sum_{\ell=0}^{\infty}\frac{1}{\ell!}\sum_{j_1+j_2+j_3=\ell}\binom{\ell}{j_1j_2j_3}\left\langle G, \xi_1^{j_1}R_1^{j_1}z_1^{j_1}\partial_2\xi_2^{j_2}R_2^{j_2}z_2^{j_2}\partial_3\xi_3^{j_3}R_3^{j_3}z_3^{j_3}[1]\right\rangle \\
&= \sum_{\ell=0}^{\infty}\frac{1}{\ell!}\sum_{j_1+j_2+j_3=\ell}\binom{\ell}{j_1j_2j_3}j_2j_3\left\langle G, \xi_1^{j_1}R_1^{j_1}z_1^{j_1}\xi_2^{j_2-1}R_2^{j_2}z_2^{j_2}\xi_3^{j_3-1}R_3^{j_3}z_3^{j_3}[1]\right\rangle \\
&= \sum_{\ell=0}^{\infty}\frac{1}{(2\ell)!}\sum_{j_1+j_2+j_3=\frac{\ell-2}{2}}\frac{(2\ell)!}{(2j_1)!(2j_2+1)!(2j_3+1)!}(2j_2+1)(2j_3+1) \\
&\qquad \left\langle G, \xi_1^{2j_1}R_1^{2j_1}z_1^{2j_1}\xi_2^{2j_2}R_2^{2j_2+1}z_2^{2j_2+1}\xi_3^{2j_3}R_3^{2j_3+1}z_3^{2j_3+1}[1]\right\rangle \\
&= \sum_{\ell=0}^{\infty}\frac{1}{(2\ell)!}\sum_{j_1+j_2+j_3=\frac{\ell-2}{2}}\frac{(2\ell)!}{(2j_1)!(2j_2)!(2j_3)!}z_1^{2j_1}z_2^{2j_2+1}z_3^{2j_3+1}e_2e_3\left\langle G, \xi_1^{2j_1}\xi_2^{2j_2}\xi_3^{2j_3}[1]\right\rangle \\
&= \sum_{\ell=0}^{\infty}\sum_{j_1+j_2+j_3=\frac{\ell-2}{2}}\frac{1}{(2j_1)!(2j_2)!(2j_3)!}z_1^{2j_1}z_2^{2j_2+1}z_3^{2j_3+1}e_2e_3\frac{(2j_1)!}{2^{j_1}j_1!}\frac{(2j_2)!}{2^{j_2}j_2!}\frac{(2j_3)!}{2^{j_3}j_3!} \\
&= \sum_{\ell=0}^{\infty}\sum_{j_1+j_2+j_3=\frac{\ell-2}{2}}\left(\frac{z_1^2}{2}\right)^{j}_{1}\frac{1}{j_1!}\left(\frac{z_2^2}{2}\right)^{j}_{2}\frac{1}{j_2!}\left(\frac{z_3^2}{2}\right)^{j}_{3}\frac{1}{j_3!}z_2e_2z_3e_3 \\
&= \exp\left(\frac{|\underline{z}|^2}{2}\right)z_2e_2z_3e_3.
\end{aligned}$$

During this calculation, we used this lemma:

Lemma 5.8. *Let* $i \in \{1,\dots,m\}$ *and let* $j \in \mathbb{N}$*. Then it holds that* $\left\langle G, \xi_i^j R_i[1]\right\rangle = e_i\left\langle G, \xi_i^j[1]\right\rangle$*.*

Proof. First, note that $\xi_i^j R_i[1] = \xi_i^j e_i[1]$. To switch the order of ξ_i and e_i, remark that $\xi_i e_i = e_i\xi_i^{\dagger}$. However, if j is odd, $\xi_i^{\dagger j}[1] = \xi_i^j[1]$, so this case is covered. If j is even, one can write $\{\xi_i^{\dagger}\}^j[1]$ as $\xi_i^j[1] - j\xi_i^{j-1}[1](e_i^- - e_i^+)$. But as the Gaussian distribution G vanishes on odd powers of ξ, we can also cover this case. □

The result of the three terms together will then be

$$\begin{aligned}
\mathcal{W}[\frac{1}{2}\left(\eta_2\eta_3 - \eta_3\eta_2\right)] &= z_2e_2z_3e_3 - z_1e_1z_3e_3 + z_1e_1z_2e_2 \\
&= \frac{1}{2}\left(y_2y_3 - y_3y_2\right).
\end{aligned}$$

In the second case, $CK[\xi^{\underline{\alpha}}] = \eta_j \hat{\eta}_j = \xi_j^2 - 2\xi_1\xi_j - \xi_1^2$, which will give $y_j\hat{y}_j$ as a result. Let us verify this for $m = 3$ and $\ell = (2,2)$.

$$\mathcal{W}[\xi_2^2 - 2\xi_1\xi_2 - \xi_1^2] = \frac{1}{\sqrt{2\pi}^{\frac{3}{2}}} \exp\left(\frac{-|\underline{z}|^2}{2}\right)\left[\left\langle \xi_2^2 G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle - 2\left\langle \xi_1\xi_2 G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle - \left\langle \xi_1^2 G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle\right]$$

Using the result of the previous calculation, we only need to know $\left\langle \xi_2^2 G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle$.

$$\begin{aligned}
\left\langle \xi_2^2 G, \exp\left(\xi R\underline{z}\right)[1]\right\rangle &= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{j_1+j_2+j_3=\ell} \binom{\ell}{j_1 j_2 j_3} \left\langle \xi_2^2 G, \xi_1^{j_1} R_1^{j_1} z_1^{j_1} \xi_2^{j_2} R_2^{j_2} z_2^{j_2} \xi_3^{j_3} R_3^{j_3} z_3^{j_3}[1]\right\rangle \\
&= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{j_1+j_2+j_3=\ell} \binom{\ell}{j_1 j_2 j_3} \left\langle (\partial_2^2+1) G, \xi_1^{j_1} R_1^{j_1} z_1^{j_1} \xi_2^{j_2} R_2^{j_2} z_2^{j_2} \xi_3^{j_3} R_3^{j_3} z_3^{j_3}[1]\right\rangle \\
&= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{j_1+j_2+j_3=\ell} \binom{\ell}{j_1 j_2 j_3} \left\langle G, \xi_1^{j_1} R_1^{j_1} z_1^{j_1} \xi_2^{j_2} R_2^{j_2} z_2^{j_2} \xi_3^{j_3} R_3^{j_3} z_3^{j_3}[1](\partial_2^{\dagger^2}+1)\right\rangle \\
&= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{j_1+j_2+j_3=\ell} \binom{\ell}{j_1 j_2 j_3} \left\langle G, \xi_1^{j_1} R_1^{j_1} z_1^{j_1} (\partial_2^2+1) \xi_2^{j_2} R_2^{j_2} z_2^{j_2} \xi_3^{j_3} R_3^{j_3} z_3^{j_3}[1]\right\rangle \\
&= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{j_1+j_2+j_3=\ell} \binom{\ell}{j_1 j_2 j_3} \Big(j_2(j_2-1) \left\langle G, \xi_1^{j_1} R_1^{j_1} z_1^{j_1} \xi_2^{j_2-2} R_2^{j_2} z_2^{j_2} \xi_3^{j_3} R_3^{j_3} z_3^{j_3}[1]\right\rangle \\
&\quad + \left\langle G, \xi_1^{j_1} R_1^{j_1} z_1^{j_1} \xi_2^{j_2} R_2^{j_2} z_2^{j_2} \xi_3^{j_3} R_3^{j_3} z_3^{j_3}[1]\right\rangle \Big) \\
&= \sum_{\ell=0}^{\infty} \frac{1}{(2\ell)!} \sum_{j_1+j_2+j_3=\ell} \frac{(2\ell)!}{(2j_1)!(2j_2)!(2j_3)!} \Big(2j_2(2j_2-1) \left\langle G, \xi_1^{2j_1} R_1^{2j_1} z_1^{2j_1} \xi_2^{2j_2-2} R_2^{2j_2} z_2^{2j_2} \xi_3^{2j_3} R_3^{2j_3} z_3^{2j_3}[1]\right\rangle \\
&\quad + \left\langle G, \xi_1^{2j_1} R_1^{2j_1} z_1^{2j_1} \xi_2^{2j_2} R_2^{2j_2} z_2^{2j_2} \xi_3^{2j_3} R_3^{2j_3} z_3^{2j_3}[1]\right\rangle \Big) \\
&= \sum_{\ell=0}^{\infty} \frac{1}{(2\ell)!} \sum_{j_1+j_2+j_3=\ell} \frac{(2\ell)!}{(2j_1)!(2j_2-2)!(2j_3)!} \Big(\left\langle G, \xi_1^{2j_1} R_1^{2j_1} z_1^{2j_1} \xi_2^{2j_2-2} R_2^{2j_2} z_2^{2j_2} \xi_3^{2j_3} R_3^{2j_3} z_3^{2j_3}[1]\right\rangle \\
&\quad + \left\langle G, \xi_1^{2j_1} R_1^{2j_1} z_1^{2j_1} \xi_2^{2j_2} R_2^{2j_2} z_2^{2j_2} \xi_3^{2j_3} R_3^{2j_3} z_3^{2j_3}[1]\right\rangle \Big) \\
&= \sum_{\ell=0}^{\infty} \sum_{j_1+j_2+j_3=\ell} \frac{1}{(2j_1)!(2j_2-2)!(2j_3)!} \Big(z_2^2 \left\langle G, \xi_1^{2j_1} z_1^{2j_1} \xi_2^{2j_2-2} z_2^{2j_2} \xi_3^{2j_3} z_3^{2j_3}[1]\right\rangle \\
&\quad + \left\langle G, \xi_1^{2j_1} z_1^{2j_1} \xi_2^{2j_2} z_2^{2j_2} \xi_3^{2j_3} z_3^{2j_3}[1]\right\rangle \Big) \\
&= z_2^2 + 1 = (z_2 e_2)^2 + 1.
\end{aligned}$$

We used lemma 2.1 and lemma 4.2. As a result, $\mathcal{W}[\eta_2\hat{\eta}_2] = \mathcal{W}[\xi_2^2 - 2\xi_1\xi_2 - \xi_1^2] = z_2^2 + 1 - 2z_1e_1z_2e_2 - z_1^2 - 1 = z_2^2 - 2z_1e_1z_2e_3 - z_1^2 = y_2\hat{y}_2$.

Example 5.9. $\|\alpha\| = 3$
Three combinations are possible

1. $\ell = (i, j, k)$
2. $\ell = (i, j, j)$ *or* $\ell = (i, i, j)$
3. $\ell = (j, j, j)$

For the first case, $CK[\xi^{\underline{\alpha}}] = \frac{1}{3!}\left(\eta_i\eta_j\eta_k - \eta_j\eta_i\eta_k + \eta_j\eta_k\eta_i - \eta_k\eta_j\eta_i - \eta_i\eta_k\eta_j + \eta_k\eta_i\eta_j\right)$*. Using the same reasoning as in the previous examples, this is simply transformed into* $\frac{1}{3!}\left(y_iy_jy_k - y_jy_iy_k + y_jy_ky_i - y_ky_jy_i - y_iy_ky_j + y_ky_iy_j\right)$*. The second case splits up in two sub cases, however both are identically calculated.* $\mathcal{W}\left[CK[\xi^{\underline{\alpha}}]\right] = \frac{1}{3!}\mathcal{W}\left[\eta_i\hat{\eta}_i\eta_j \ - \eta_i\eta_j\hat{\eta}_i + \eta_j\eta_i\right.$ $y_i\hat{y}_iy_j \ - y_iy_j\hat{y}_i + y_jy_i\hat{y}_i$*. Therefor, we need a combination of previous calculations. Finally,* $CK[\xi_j^3] = \eta_j\hat{\eta}_j\eta_j$ *becomes* $z_j\hat{z}_jz_j$*.*

The structure of the Weierstrass transform of the basic monogenic polynomials $CK[\xi^{\underline{\alpha}}]$ is clear: every factor η_j or $\hat{\eta}_j$ translates into y_j or $\hat{y}_j$, in the same order. Looking at lemma 2.1, every power of ξ_j, acting on G, corresponds to a polynomial of the same degree in ∂, acting on G. This polynomial will result in the same polynomial in z_je_j. Another way to obtain this result, is by bringing the powers of ξ, acting on G in the left hand side, to an action from the right on [1] in the right hand side.

Now that we know what to expect, the only thing that remains is to prove the result. Therefore, we have another look at the situation in the classical setting.

It is now clear how to calculate the discrete Weierstrass transform of discrete monogenic polynomials. There is no particular need for a recurrence relation on the degree of that monogenic. Although we now know what to expect, these results are not proven yet. If we want to do so, we have to look for another approach, which is discussed in the next section.

6 Alternative approach

In the classical setting, the Weierstrass transform has a an alternative definition. However more informal, it leads to certain advantages such as the idea of the inverse of the Weierstrass transform [16].

Definition 6.1. *The continuous Weierstrass transform can be written as*

$$W[f] = \exp\left(\frac{1}{2}\partial_x^2\right) f(x) = \sum_{j=0}^{\infty} \frac{1}{j!2^j}\partial_x^{2j} f(x).$$

This definition plays with convergence of the series. There are functions that are Weierstrass transformable, but for which this series does not converge. For details, also see [16]. Nonetheless, it inspires us to look for an alternative approach for the discrete Weierstrass transform. Therefore, let us calculate $\exp\left(-\frac{\partial^2}{2}\right)\xi^n P_r(\xi)[1]$, with P_r a monogenic homogeneous polynomial of degree r (as is $CK[\xi^{\underline{\alpha}}]$).

For the even case, $n = 2\ell$

$$\begin{aligned}
\exp\left(-\frac{\partial^2}{2}\right)\xi^{2\ell}P_r(\xi)[1] &= \sum_{j=0}^{\ell} \frac{(-1)^j}{j!\,2^j}\partial^{2j}\xi^{2\ell}P_r[1] \\
&= \sum_{j=0}^{\ell} \frac{(-1)^j}{j!\,2^j}\frac{4^j\,\Gamma(\ell + r + \frac{m}{2})\,\Gamma(\ell+1)}{\Gamma(\ell + r + \frac{m}{2} - j)\,\Gamma(l + 1 - j)}\xi^{2\ell-2j}P_r[1] \\
&= \sum_{i=0}^{\ell} \frac{(-1)^{l-i}}{(l-i)!\,2^{l-i}}\frac{\Gamma(\ell + r + \frac{m}{2})\,\Gamma(\ell+1)}{\Gamma(i + r + \frac{m}{2})\,\Gamma(i+1)}\xi^{2i}P_r[1] \\
&= (-1)^\ell \sum_{i=0}^{\ell} (-1)^i\,2^{\ell-i}\binom{\ell}{i}\frac{\Gamma(\ell + \frac{m}{2} + r)}{\Gamma(i + \frac{m}{2} + r)}\xi^{2i}P_r[1] \\
&= (-1)^\ell\, H_{2\ell,m,r}(\xi)P_r[1].
\end{aligned}$$

For the odd case, $n = 2\ell + 1$

$$\begin{aligned}
\exp\left(-\frac{\partial^2}{2}\right)\xi^{2\ell+1}P_r(\xi)[1] &= \sum_{j=0}^{\ell}\frac{(-1)^j}{j!\,2^j}\partial^{2j}\xi^{2\ell+1}P_r[1] \\
&= \sum_{j=0}^{\ell}\frac{(-1)^j}{j!\,2^j}\frac{4^j\,\Gamma(\ell+r+\frac{m}{2}+1)\,\Gamma(\ell+1)}{\Gamma(\ell+r+\frac{m}{2}-j+1)\,\Gamma(l+1-j)}\xi^{2\ell+1-2j}P_r[1] \\
&= \sum_{i=0}^{\ell}\frac{(-1)^{l-i}}{(l-i)!\,2^{l-i}}\frac{\Gamma(\ell+r+\frac{m}{2}+1)\,\Gamma(\ell+1)}{\Gamma(i+r+\frac{m}{2}+1)\,\Gamma(i+1)}\xi^{2i+1}P_r[1] \\
&= (-1)^{\ell}\sum_{i=0}^{\ell}(-1)^i\,2^{\ell-i}\binom{\ell}{i}\frac{\Gamma(\ell+\frac{m}{2}+r+1)}{\Gamma(i+\frac{m}{2}+r+1)}\xi^{2i+1}P_r[1] \\
&= (-1)^{\ell}\,H_{2\ell+1,m,r}(\xi)P_r[1].
\end{aligned}$$

Together, we see that

$$\exp\left(-\frac{\partial^2}{2}\right)\xi^n P_r(\xi)[1] = (-1)^{\lfloor\frac{n}{2}\rfloor}H_{n,m,r}(\xi)P_r(\xi)[1]$$

or thus

$$\exp\left(\frac{\partial^2}{2}\right)H_{n,m,r}(\xi)P_r(\xi)[1] = (-1)^{\lfloor\frac{n}{2}\rfloor}\xi^n P_r(\xi)[1]$$

After replacing the discrete variable ξ into the continuous variable $\underline{z} = \sum_{j=1}^{m} z_j e_j$, we see that

1. For $r = 0$, this is exactly the result obtained by calculation with the original defintion.
2. For $n = 0$, this is exactly the result obtained by direct calculation of the examples in the previous paragraph.

As the generalised Hermite polynomials form a basis for the of the space of functions which are Weierstrass transformable and as the result obtained by the original definition 4.1 is identical, we can state that this alternative approach makes sense and is valid to work with.

This is a short, but clear way to deduce the structure of the discrete Weierstrass transform of the (generalized) Hermite polynomials.

Conjecture 6.2. *The discrete Weierstrass transform of the generalised Hermite polynomials is given by*

$$\mathcal{W}\left[H_{n,m,r}P_r\right](\underline{z}) = (-1)^{\lfloor\frac{n}{2}\rfloor}\underline{z}^n P_r(\underline{z}) \tag{35}$$

where $\underline{z} = \sum_{j=1}^{m} z_j e_j$.

7 Conclusion

We established two generalisations of the Weierstrass transform in discrete Clifford analysis, based on the initial definition from [11]. First, we investigated the consistency of the definition with the classical setting, by considering a mesh with width $h \neq 1$ and letting h tend to zero. The discrete Gauss distribution approaches the continuous Gauss distribution as the mesh width approaches zero. Furthermore, we proved the structure of the discrete Weierstrass transform in dimension $m > 1$. This was done based on the recurrence relation of the discrete generalised Hermite polynomials and on the alternative definition in the classical setting.

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