Nonsingular hypercubes and nonintersecting hyperboloids

Kris Coolsaet
Department of Applied Mathematics, Computer Science and Statistics,
Ghent University,
Krijgslaan 281–S9, B–9000 Gent, Belgium
Kris.Coolsaet@UGent.be

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Abstract: Some ten years ago we managed to take a first step in the classification of nonsingular $2\times2\times2\times2$ hypercubes over a finite field by resolving the special case where the hypercubes can be written as a product of two $2\times2\times2$ hypercubes, i.e., nonsingular $2\times2\times2\times2$ hypercubes of 12-rank two.

We have now been able to extend this classification to hypercubes of 12-rank three, based on the connection between nonsingular hypercubes and bundles of nonintersecting quadrics in 3 dimensions. A bit surprisingly, the number of inequivalent nonsingular hypercubes of 12-rank three is only of the same order of magnitude as in the case of 12-rank two.

We also made some headway into the remaining case of 12-rank four. In particular, we prove that there are essentially only 2 nonsingular $2\times2\times2\times2$ hypercubes that correspond to a hyperbolic fibration of 3-dimensional projective space.

Keywords: nonsingular hypercube · classification · bundle of quadrics · hyperbolic fibration

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1 Introduction

In 1965 D. Knuth [5] introduced the notion of *nonsingular hypercube*, a 'simple' case of which, namely the nonsingular hypercubes of dimension 3, he used to describe semifields. In [2] we started investigating the next case, i.e., the hypercubes of dimension 4. These should not only be of interest as a generalization of the notion of semifield (they can be used to define a linear *ternary* product $\langle x, y, z\rangle$ which has value zero if and only if x, y or z is zero) but also because they can serve as 'factories', 'collections' or 'bundles' of semifields (fixing either x, y, z in the ternary product yields a *binary* product that defines a semifield).

Although the case $n = 2$ only yields proper fields as semifields, we start with this simple case in the hope that the techniques developed here may eventually prove useful for larger values of n. Our goal is to classify nonsingular $2\times2\times2\times2$ hypercubes up to equivalence, a task which turns out to more difficult than expected. In [2] we only succeeded in classifying a small well-defined subset of $2\times2\times2\times2$ hypercubes, those of 12-rank 2 (for definitions, see Section 2). In this paper we extend this classification to 12-rank 3 (Theorem 1), leaving only the case of 12-rank 4 still to be solved. In that most difficult case we already present a classification theorem on hypercubes of 'fibration type' (Theorems 2 and 3).

After a standstill of almost 10 years, the main idea that allowed us to make progress is a little lemma (Lemma 1) with a simple proof about bundles of quadrics in projective 3-space, which ultimately forces two conditions on a hypercube to be nonsingular, showing that nonsingular hypercubes are fewer in number than expected.

In Section 2 we start with the necessary definitions and preliminary properties used further in the text. Section 3 introduces the connection with bundles of quadrics and establishes the basic lemma, which is then used in Section 4 to obtain Proposition 2 which provides the basis for the classification of nonsingular hypercubes of 12-rank 3 in Section 5 and for further investigation of some hypercubes of 12-rank 4 in Section 6.

We end with some final remarks on hypercubes of 12-rank 4 and a small note correcting the formulation of a theorem in [2].

2 Definitions and preliminaries

We follow [2] where you can find more details, including proofs of some of the properties mentioned below.

Let K denote a field. A $2 \times \cdots \times 2$ hypercube M of dimension m is an array of 2^m elements $M_{11\cdots1}, \ldots, M_{22\cdots2}$ of K, indexed by m-tuples of subscripts taken from the set $\{1, 2\}$. The set of all such hypercubes will be denoted by $K^{2\times\cdots\times2}$. In this paper we shall mostly deal with hypercubes of dimension 4. Matrices are hypercubes of dimension 2.

Let M be a hypercube of dimension m, $m > 1$. Let $i \in \{1, 2\}$. We denote by $M^{[i]}$ the hypercube of dimension $m-1$ with coefficients $M_{i1\cdots 1}, \ldots, M_{i2\cdots 2}$. $M^{[1]}$ and $M^{[2]}$ are called *subcubes* of M. Additional subcubes of M can be obtained by fixing the index value at a position other than the first.

In the cases $m = 3, 4$ we shall arrange the elements of a hypercube in a block matrix, as follows :

$$\left(\begin{array}{cc|cc} M_{1111} & M_{1112} & M_{1211} & M_{1212} \\ M_{1121} & M_{1122} & M_{1221} & M_{1222} \\ \hline M_{2111} & M_{2112} & M_{2211} & M_{2212} \\ M_{2121} & M_{2122} & M_{2221} & M_{2222} \end{array}\right), \qquad \left(\begin{array}{cc|cc} S_{111} & S_{112} & S_{211} & S_{212} \\ S_{121} & S_{122} & S_{221} & S_{222} \end{array}\right)$$

with M and S hypercubes of dimension 4 and 3, i.e.,

$$M = \left(\begin{array}{c|c} M^{[1][1]} & M^{[1][2]} \\ \hline M^{[2][1]} & M^{[2][2]} \end{array}\right), \qquad S = \left(S^{[1]} \mid S^{[2]}\right).$$

The set $K^{2\times\cdots\times 2}$ of hypercubes of dimension m can be made into a vector space by using addition and multiplication on the coefficients:

$$(k_1 M + k_2 N)_{i_1 i_2 \cdots i_m} \stackrel{\text{def}}{=} k_1 M_{i_1 i_2 \cdots i_m} + k_2 N_{i_1 i_2 \cdots i_m}, \text{ for all } i_1, \ldots, i_m \in \{1, 2\}.$$

where $k_1, k_2 \in K$, $M, N \in K^{2\times\cdots\times 2}$.

Consider an m-tuple $Z = (Z^{(1)}, \ldots, Z^{(m)})$ of nonsingular matrices $Z^{(1)}, \ldots, Z^{(m)} \in \mathrm{GL}(2, K)$. We can use Z to transform a hypercube M of dimension m into another hypercube M^Z of the same dimension, with the following coefficients:

$$(M^Z)_{i_1 \cdots i_m} \stackrel{\text{def}}{=} \sum_{j_1, \ldots, j_m = 1}^{2} Z^{(1)}_{i_1 j_1} \cdots Z^{(m)}_{i_m j_m} M_{j_1 \cdots j_m}.$$

Here, M^Z is said to be *equivalent* to M and Z will be called an *equivalence*.

This defines an action of the group $\mathrm{GL}(2, K)^m$ on $K^{2\times\cdots\times 2}$. This action is not faithful. Its kernel consists of all equivalences of the form $(k^{(1)} I, \ldots, k^{(m)} I)$, where I denotes the 2×2 identity matrix and $k^{(1)}, \ldots, k^{(m)} \in K$ such that $k^{(1)} \cdots k^{(m)} = 1$.

Let $d \in \{1, \ldots, m\}$. An equivalence $Z \in \mathrm{GL}(2, K)^m$ such that $Z^{(i)} = I$ for all $i \neq d$, will be called a *fundamental* equivalence *in direction* d. Clearly each equivalence can be written as a product of m fundamental equivalences, one for each direction $d = 1, \ldots, m$.

For the case $m = 4$ the fundamental equivalences can be expressed as left and write block matrix multiplications, in the following way. Let $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in \mathrm{GL}(2, K)$. The following table lists the images of $M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$, with $A, B, C, D \in K^{2\times 2}$, for fundamental equivalences in the four different directions :

$$
\begin{aligned}
(X,I,I,I) &: M \mapsto \left(\begin{array}{c|c} x_1A+x_2C & x_1B+x_2D \\ \hline x_3A+x_4C & x_3B+x_4D \end{array}\right) = \left(\begin{array}{c|c} x_1I & x_2I \\ \hline x_3I & x_4I \end{array}\right)\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right), \\
(I,X,I,I) &: M \mapsto \left(\begin{array}{c|c} x_1A+x_2B & x_3A+x_4B \\ \hline x_1C+x_2D & x_3C+x_4D \end{array}\right) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\left(\begin{array}{c|c} x_1I & x_3I \\ \hline x_2I & x_4I \end{array}\right), \\
(I,I,X,I) &: M \mapsto \left(\begin{array}{c|c} XA & XB \\ \hline XC & XD \end{array}\right) = \left(\begin{array}{c|c} X & 0 \\ \hline 0 & X \end{array}\right)\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right), \\
(I,I,I,X) &: M \mapsto \left(\begin{array}{c|c} AX^T & BX^T \\ \hline CX^T & DX^T \end{array}\right) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\left(\begin{array}{c|c} X^T & 0 \\ \hline 0 & X^T \end{array}\right).
\end{aligned} \tag{1}
$$

We define the *determinant* of $Z \in \mathrm{GL}(2,K)^m$ as $\det Z \overset{\mathrm{def}}{=} \det Z^{(1)} \cdots \det Z^{(m)}$.

A function $f : K^{2\times\cdots K^{2\times 2}} \to K$ is called an *invariant of exponent* e if and only if $f(M^Z) = (\det Z)^e f(M)$, for all $M \in K^{2\times\cdots K^{2\times 2}}$ and for all $Z \in \mathrm{GL}(2,K)^m$.

The following maps are invariants of exponent 2:

$$
\det{}_2 : K^{2\times2\times2\times2} \to K \quad : \quad \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \mapsto \begin{vmatrix} A & B \\ C & D \end{vmatrix}, \tag{2}
$$

$$
\det{}_2' : K^{2\times2\times2\times2} \to K \quad : \quad \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \mapsto \begin{vmatrix} A & C \\ B & D \end{vmatrix}, \tag{3}
$$

$$
\det{}_2'' : K^{2\times2\times2\times2} \to K \quad : \quad \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \mapsto \begin{vmatrix} A_{11} & A_{12} & A_{21} & A_{22} \\ B_{11} & B_{12} & B_{21} & B_{22} \\ C_{11} & C_{12} & C_{21} & C_{22} \\ D_{11} & D_{12} & D_{21} & D_{22} \end{vmatrix}, \tag{4}
$$

where in the first two cases the block matrix should now be interpreted as an element of $K^{4\times4}$. It turns out that $\det_2'' = \det - \det_2'$.

Let $A, B \in K^{2\times2}$. Consider the following bilinear dot product, i.e., the polarization of the determinant:

$$A \cdot B \overset{\mathrm{def}}{=} \det(A+B) - \det A - \det B.$$

We have $A\cdot B = B\cdot A = \operatorname{tr} A\bar{B} = \operatorname{tr} B\bar{A} = \operatorname{tr} \bar{A}B = \operatorname{tr} \bar{B}A$ where $\bar{A}$ denotes the adjoint of A. Note that $A\cdot A = 2\det A$ and that $AX\cdot BX = XA\cdot XB = (\det X)(A\cdot B)$ for any matrix $X \in K^{2\times2}$.

With these notations, the map

$$\det{}_1 : K^{2\times2\times2\times2} \to K : \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \mapsto A\cdot D - B\cdot C$$

is an invariant of exponent 1.

Finally, the following function is an invariant of exponent 3:

$$\det_3 : K^{2\times2\times2\times2} \to K : \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \mapsto$$

$$\begin{vmatrix} \det A & A\cdot B & \det B \\ A\cdot C & A\cdot D + B\cdot C & B\cdot C \\ \det C & C\cdot D & \det D \end{vmatrix} = \begin{vmatrix} \det A & \det B & \det(A+B) \\ \det C & \det D & \det(C+D) \\ \det(A+C) & \det(B+D) & \det(A+B+C+D) \end{vmatrix}.$$

The notation $\det_e$ used here for the invariants differs slightly from that used in [2]. The index e is now chosen to reflect the exponent of the invariant.

Central to this paper is the notion of a *nonsingular* hypercube, which we define by induction on the dimension m. If $m = 1$ then M is nonsingular if and only if M is not the zero vector. If $m > 1$ then M is nonsingular if and only if all linear combinations $k_1 M^{[1]} + k_2 M^{[2]}$ are nonsingular, with $k_1, k_2 \in K$, $(k_1, k_2) \neq (0, 0)$. Note that when $m = 2$ the hypercube M is nonsingular if and only if it is nonsingular as a matrix. Nonsingularity is preserved by the action of $\mathrm{GL}(2, K)^m$.

A $2\times2\times2$ hypercube $(A \mid B)$ is nonsingular if and only if $\det A \neq 0$ and the quadratic equation

$$(\det A)x^2 - (A\cdot B)x + (\det B) = 0,$$

has no solutions for x in K, or equivalently if and only if A is nonsingular and the eigenvalues of BA^{-1} do not belong to K.

A $2\times2\times2\times2$ hypercube $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ will be called *trivially singular* if any of the $2\times2\times2$ hypercubes $(A \mid B)$, $(A \mid C)$, $(B \mid D)$, $(C \mid D)$ is singular. This is a technical definition mainly used to simplify the statements of some of the lemmas and theorems below. See also the note at the end of this paper.

In [2] we introduced the notion of *ij-rank* of a hypercube $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ for $\{i, j\} \subseteq \{1, 2, 3, 4\}$, $i \neq j$. We can express all ij-ranks as ranks of the 4×4 matrices that occur in (2–4) :

$$\begin{aligned} \text{12-rank} &= \text{34-rank} = \operatorname{rank}\begin{pmatrix} A_{11} & A_{12} & A_{21} & A_{22} \\ B_{11} & B_{12} & B_{21} & B_{22} \\ C_{11} & C_{12} & C_{21} & C_{22} \\ D_{11} & D_{12} & D_{21} & D_{22} \end{pmatrix}, \\ \text{13-rank} &= \text{24-rank} = \operatorname{rank}\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \\ \text{14-rank} &= \text{23-rank} = \operatorname{rank}\begin{pmatrix} A & C \\ B & D \end{pmatrix}. \end{aligned} \tag{5}$$

The 12-rank of $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ is precisely the dimension of the subspace of $K^{2\times 2}$ generated by A, B, C and D.

If $M, M' \in K^{2\times 2\times 2\times 2}$ are equivalent, then they have the same ij-rank. The ij-rank of a nonsingular hypercube is at least 2 and at most 4. In [2] the hypercubes with 12-rank 2 were classified up to equivalence. In this paper we will consider hypercubes of higher 12-rank.

3 Hypercubes and bundles of quadrics

Let $M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \in K^{2\times 2\times 2\times 2}$. Define the following quadratic form Q_M on K^4:

$$\begin{array}{rcl} Q_M(x,y,z,t) & = & \det(xA + yB + zC + tD) \\ & = & (\det A)x^2 + (\det B)y^2 + (\det C)z^2 + (\det D)t^2 + \\ & & \quad (A\cdot B)xy + (A\cdot C)xz + (A\cdot D)xt + (B\cdot C)yz + (B\cdot D)yt + (C\cdot D)zt. \end{array}$$

Then

$$\begin{array}{rcl} Q_M(x,y,z,t) & = & (A_{11}x + B_{11}y + C_{11}z + D_{11}t)(A_{22}x + B_{22}y + C_{22}z + D_{22}t) \\ & & \quad - (A_{12}x + B_{12}y + C_{12}z + D_{12}t)(A_{21}x + B_{21}y + C_{21}z + D_{21}t) \end{array}$$

and hence $Q_M(x,y,z,t) = H(x',y',z',t')$ where H is the standard hyperbolic quadratic form H with $H(x,y,z,t) = xt - yz$ and

$$(x'\ y'\ z'\ t') = (x\ y\ z\ t)\begin{pmatrix} A_{11} & A_{12} & A_{21} & A_{22} \\ B_{11} & B_{12} & B_{21} & B_{22} \\ C_{11} & C_{12} & C_{21} & C_{22} \\ D_{11} & D_{12} & D_{21} & D_{22} \end{pmatrix}.$$

It follows that Q_M is equivalent (as a quadratic form) to H when $\det_2'' M \neq 0$, i.e., when M has 12-rank equal to 4.

Proposition 1 $M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ *is nonsingular if and only if*

$$\left\{ \begin{array}{rcl} Q_M(x,y,z,t) & = & 0, \\ xt - yz & = & 0, \end{array} \right.$$

has no solutions for $x,y,z,t \in K$ except $x = y = z = t = 0$.

Proof : By definition M is nonsingular if and only if all $2\times 2\times 2$ hypercubes $(kA+lC|kB+lD)$ are nonsingular with $k,l \in K$, $(k,l) \neq (0,0)$. And this in turn is by definition equivalent

Class	Description	Nr. of points when $K = \mathrm{GF}(q)$
0	Plane (doubly counted)	$q^2 + q + 1$
1^+	Pair of distinct planes	$2q^2 + q + 1$
1^-	Line (pair of conjugate planes)	$q + 1$
2	Cone	$q^2 + q + 1$
3^+	Hyperboloid	$q^2 + 2q + 1$
3^-	Ellipsoid	$q^2 + 1$

Table 1: The six classes of quadrics in $\mathrm{PG}(3, K)$.

to stating that the matrix $P = k'(kA + lC) + l'(kB + lD)$ is nonsingular for $k', l' \in K$, $(k', l') \neq (0, 0)$. We may write $P = xA + yB + zC + tD$ with

$$x = kk', \quad y = kl', \quad z = k'l, \quad t = ll'. \tag{6}$$

We have $xt = yz$. Conversely with every x, y, z, t such that $xt = yz$ we may associate (k, k', l, l') such that (6) holds. Indeed, when $x \neq 0$, take $(k, k', l, l') = (1, x, z/x, y)$. When $x = 0$ then either $y = 0$ or $z = 0$. In the first case take $(k, k', l, l') = (0, z, t, 1)$ and in the second case $(k, k', l, l') = (1, 0, y, t)$. Also note that $x = y = z = t = 0$ if and only if $(k, l) = (0, 0)$ or $(k', l') = (0, 0)$.

Finally, P is nonsingular if and only if $\det P \neq 0$ which by definition is equivalent to $Q_M(x, y, z, t) \neq 0$. ∎

Recall that a quadratic form in 4 variables defines a quadric (quadratic algebraic variety) on the 3-dimensional projective space $\mathrm{PG}(3, K)$. Table 3 lists all classes of such quadrics, together with the number of points they contain when K is a finite field of order q [4].

Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ denote distinct quadrics with corresponding quadratic form Q_1 and Q_2. The *bundle* $\mathcal{B}$ generated by $\mathcal{Q}_1$ and $\mathcal{Q}_2$ consists of the quadrics whose quadratic form is a linear combination of Q_1 and Q_2. The intersection of two distinct quadrics in $\mathcal{B}$ is independent of the choice of quadrics in $\mathcal{B}$ and every point of $\mathrm{PG}(3, K)$ not in that intersection lies in exactly one element of the bundle. When K is finite of order q then each bundle contains exactly $q + 1$ quadrics.

The following simple lemma turns out to be a crucial tool in the classification of nonsingular hypercubes over a finite field.

Lemma 1 *Let K be the finite field of order q. Let $\mathcal{B}$ be the bundle generated by two quadrics in* $\mathrm{PG}(3, K)$ *that have empty intersection. Then*

- *every element of $\mathcal{B}$ is an ellipsoid (quadric of class 3^-), or*
- *$\mathcal{B}$ contains at least on line (quadric of class 1^-).*

If $\mathcal{B}$ contains (at least) 2 lines, then it contains (exactly) 2 lines and $q-1$ hyperboloids (quadrics of class 3^+).

Proof : Because the quadrics have empty intersection, the $q+1$ elements of $\mathcal{B}$ partition the q^3+q^2+q+1 points of $\mathrm{PG}(3, K)$. Hence on average, an element of $\mathcal{B}$ contains q^2+1 points. So either all elements have size q^2+1 (and then must be ellipsoids) or at least one element has a size smaller than q^2+1. And by Table 3 this element must necessarily be of class 1^-.

Now, assume that the bundle contains at least two lines. Then the remaining $q-1$ quadrics contain on average q^2+2q+1 points. These cannot be of class 1^+ (as these quadrics intersect each line in at least one point) and therefore must be of class 3^-, the only remaining class of quadric that contains a sufficient number of points.

Note that this Lemma is also a consequence of the classification given in [1], except that to be able to use that result it is necessary to prove that the quartic base curve of the bundle is not absolutely irreducible, which is not obvious. ■

In the case where the bundle contains 2 lines, the corresponding partition of $\mathrm{PG}(3, K)$ is called a *hyperbolic fibration*. In Section 6 we shall investigate this case in more detail.

Corollary 1 *Let M be a nonsingular hypercube, then there exists at least one $k \in K$ such that $Q_M(x, y, z, t) + kH(x, y, z, t)$ is of the form*

$$Q_M(x, y, z, t) + kH(x, y, z, t) = (\alpha x + \beta y + \gamma z + \delta t)(\bar{\alpha} x + \bar{\beta} y + \bar{\gamma} z + \bar{\delta} t)$$

with $\alpha, \beta, \gamma, \delta \in L$.

Proof : Let $\mathcal{H}, \mathcal{Q}$ denote the quadrics corresponding to H, Q respectively. By Proposition 1 $\mathcal{H}$ and $\mathcal{Q}$ have empty intersection. Because $\mathcal{H}$ is a hyperboloid, Lemma 1 shows that the bundle generated by $\mathcal{H}, \mathcal{Q}$ contains at least one element $\mathcal{L}$ that has class 1^-. On the one hand $\mathcal{L}$ corresponds to a quadratic form $Q+kH$ with $k \in K$ because it belongs to the bundle and is different from $\mathcal{H}$. On the other hand $\mathcal{L}$ is the intersection of two conjugate planes and therefore corresponds to a quadratic form of the form $(\alpha x+\beta y+\gamma z+\delta t)(\bar{\alpha} x+\bar{\beta} y+\bar{\gamma} z+\bar{\delta} t)$. Both quadratic forms must be equal up to a scalar factor $f \in K, f \neq 0$. If $f \neq 1$ we can always find $\phi \in L$ such that $N(\phi) = \phi\bar{\phi} = f$ and then multiplying $\alpha, \beta, \gamma, \delta$ by ϕ yields the stated result. ■

4 Hypercubes in reduced form

Henceforth we shall assume that K is a *finite* field of order q and denote the quadratic extension field of K by L.

In the lemmas below we choose a fixed matrix $\lambda \in K^{2\times 2}$ with (conjugate) eigenvalues in $L \setminus K$. The subset $K[\lambda]$ of $K^{2\times 2}$ of all linear combinations of the identity matrix I and λ behaves like a field isomorphic to L. We shall therefore identify $K[\lambda]$ with L and the set of all scalar multiples of I with K.

The eigenvalues of λ (as a matrix) are then λ and $\bar{\lambda}$ (as elements of L), where $\bar{\cdot}$ denotes conjugation in $L\!:\!K$ but at the same time corresponds to the adjoint as defined over $K^{2\times 2}$. Similarly the determinant (resp. trace) of an element $\alpha \in K[\lambda]$ corresponds to its norm $\alpha\bar{\alpha}$ (resp. trace $\alpha + \bar{\alpha}$) in $L\!:\!K$. We write T for the trace of λ and N for its norm.

We have $\bar{\lambda} = T - \lambda = N/\lambda$ and $\lambda^2 = T\lambda - N$ (i.e., λ satisfies its characteristic equation). The discriminant of this equation is $\Delta = T^2 - 4N = (\lambda - \bar{\lambda})^2 \neq 0$. Note that $T \neq 0$ when the characteristic is even (otherwise $\lambda = -\bar{\lambda} = \bar{\lambda} \in K$).

To help in recognizing whether commutativity of multiplication applies, we shall (loosely) adopt the convention that upper case letters correspond to general elements of $K^{2\times 2}$, greek letters to elements of $L = K[\lambda]$ and lower case letters to elements of K, with already the exceptions $T, N, \Delta \in K$ and $\epsilon \in K^{2\times 2} \setminus L$ (in the lemma below).

Not only will it be useful to identify $K[\lambda]$ with L but also to use a *split quaternion* representation for $K^{2\times 2}$ as provided by the following

Lemma 2 *Let $\lambda \in K^{2\times 2}$ be as above. Then there exists $\epsilon \in K^{2\times 2}$, $\epsilon \notin K[\lambda]$ such that*

$$\epsilon^2 = 1, \quad \epsilon = -\bar{\epsilon}, \quad \epsilon\beta = \bar{\beta}\epsilon, \quad \beta \cdot \epsilon = 0, \text{ for every } \beta \in K[\lambda]. \tag{7}$$

Every element $D \in K^{2\times 2}$ can be written in a unique way as $D = \alpha + \beta\epsilon$ with $\alpha, \beta \in K[\lambda]$, and then

$$\bar{D} = \bar{\alpha} - \beta\epsilon, \quad \operatorname{tr} D = \alpha + \bar{\alpha}, \quad \det D = \alpha\bar{\alpha} - \beta\bar{\beta}, \quad D \cdot \lambda = \alpha \cdot \lambda = \lambda\bar{\alpha} + \alpha\bar{\lambda}. \tag{8}$$

Proof : We shall construct $\epsilon \in K^{2\times 2}$ such that

$$\epsilon^2 = 1, \quad \epsilon = -\bar{\epsilon}, \quad 1 \cdot \epsilon = 0, \quad \lambda \cdot \epsilon = 0, \quad \epsilon\lambda = \bar{\lambda}\epsilon \tag{9}$$

and then use these properties to prove the remainder of the lemma. Note that (9) is invariant under a similarity transformation $\lambda \mapsto G\lambda G^{-1}, \epsilon \mapsto G\epsilon G^{-1}$ with $G \in \mathrm{GL}(2, K)$. It is therefore sufficient to prove (9) when λ is in Frobenius canonical form, i.e.,

$$\lambda = \begin{pmatrix} 0 & 1 \\ -N & T \end{pmatrix}.$$

It is now easily calculated that

$$\epsilon = \begin{pmatrix} 1 & 0 \\ T & -1 \end{pmatrix}$$

satisfies the first four properties of (9). Also

$$\epsilon\lambda + \lambda\epsilon = \begin{pmatrix} 0 & 1 \\ N & 0 \end{pmatrix} + \begin{pmatrix} T & -1 \\ T^2 - N & -T \end{pmatrix} = \begin{pmatrix} T & 0 \\ T^2 & -T \end{pmatrix} = T\epsilon$$

and then $\epsilon\lambda = (T - \lambda)\epsilon = \bar{\lambda}\epsilon$. This proves te last part of (9).

Consider $\beta = a + b\lambda \in K[\lambda]$. From (9) it follows that $\epsilon\beta = a\epsilon + b\bar{\lambda}\epsilon = \bar{\beta}\epsilon$. Also, $\beta \cdot \epsilon = a(1 \cdot \epsilon) + b(\lambda \cdot \epsilon) = 0$. This proves (7).

Now, consider an element $\gamma\epsilon$ with $\gamma \in K[\lambda]$, $\gamma \neq 0$. We have $(\gamma\epsilon)\lambda = \bar{\lambda}(\gamma\epsilon)$. Hence no nonzero element of $K[\lambda]\epsilon$ commutes with λ. On the other hand, every nonzero element of $K[\lambda]$ does commute with λ. It follows that $K[\lambda]$ and $K[\lambda]\epsilon$ have trivial intersection and together generate $K^{2\times 2}$ as a vector space. Hence every element $D \in K^{2\times 2}$ can be written in a unique way as $D = \alpha + \beta\epsilon$ with $\alpha, \beta \in K[\lambda]$.

Finally $\bar{D} = \bar{\alpha} + \overline{\beta\epsilon} = \bar{\alpha} + \bar{\epsilon}\bar{\beta} = \bar{\alpha} - \epsilon\bar{\beta} = \bar{\alpha} - \beta\epsilon$, and then $\operatorname{tr} D = D + \bar{D} = \alpha + \bar{\alpha}$. Also $\det D = D\bar{D} = (\alpha+\beta\epsilon)(\bar{\alpha}-\beta\epsilon) = \alpha\bar{\alpha}+\beta\epsilon\bar{\alpha}-\alpha\beta\epsilon-\beta\epsilon\beta\epsilon = \alpha\bar{\alpha}+\alpha\beta\epsilon-\alpha\beta\epsilon-\beta\bar{\beta}\epsilon^2 = \alpha\bar{\alpha}-\beta\bar{\beta}$.
■

For $D = \alpha + \beta\epsilon$ we call α the *projection* of D onto L and $\beta\bar{\beta}$ the *distance* of D from L. Note that this distance is 0 if and only if $D \in L$.

The following lemma shows that we can restrict the investigation of nonsingular hypercubes to those of a special form:

Lemma 3 *Let λ be as above. Then every hypercube M of $K^{2\times 2\times 2\times 2}$ that is not trivially singular is equivalent to a hypercube of the form* $\left(\begin{array}{c|c} I & \lambda \\ \hline C & D \end{array}\right)$ *where $C \in K^{2\times 2}$ has the same eigenvalues as λ.*

Moreover, when $\det''_2 M = 0$, i.e., when M has 12-rank less than 4, then C can be taken to be equal to λ.

Proof : The case $\det''_2 M = 0$ is essentially Theorem 6 of [2]. Proof of the first statement runs along the lines of the proof of that theorem and amounts to 'reducing' M in subsequent steps:

Let $M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$. Note that $A \neq 0$ because M is not trivially singular. Multiply A, B, C, D to the left with A^{-1} to reduce A to I. This is a fundamental equivalence in direction 3. Henceforth we may therefore assume that $A = I$.

Note that M remains not trivially singular after this transformation. Hence both (new versions of) B, C have conjugate eigenvalues in $L \setminus K$. Let $a + b\lambda$ be an eigenvalue of B, with $a, b \in K$, $b \neq 0$. Subtract the first (block matrix) column a times from the second and divide the second column by b . This fundamental equivalence in direction 2 reduces B to a matrix with the same eigenvalues as λ.

These transformations leave C invariant. Using row operations instead of column operations (fundamental equivalence in direction 1) we may then similarly reduce C to a matrix with the same eigenvalues as λ (without affecting B).

Finally, there exists G such that $GBG^{-1} = \lambda$. Multiplying every block matrix element to the left by G and to the right by G^{-1} then transforms B to λ (directions 3 and 4.) This leaves I invariant and also the eigenvalues of C. ∎

Note that λ and C have the same conjugate eigenvalues if and only if $\operatorname{tr} C = \operatorname{tr} \lambda = T$ and $\det C = \det \lambda = N$.

A hypercube of the form $M = \left(\begin{array}{c|c} I & \lambda \\ \hline C & D \end{array}\right)$ where λ, C have the same eigenvalues will be said to be *in reduced form w.r.t.* λ. The following lemmas show that this reduced form is not necessarily unique up to equivalence, even for fixed λ.

Lemma 4 *Consider* $M = \left(\begin{array}{c|c} I & \lambda \\ \hline C & D \end{array}\right)$ *in reduced form. Let* $\sigma \in K[\lambda]$, $\sigma \neq 0$. *Then* $M' = \left(\begin{array}{c|c} I & \lambda \\ \hline \sigma C \sigma^{-1} & \sigma D \sigma^{-1} \end{array}\right)$ *is also in reduced form, and equivalent to* M.

Proof : Note that $\sigma C \sigma^{-1}$ has the same eigenvalues as C and hence as λ and therefore M' is in reduced form with respect to λ. M' can be obtained by left multiplication of every block matrix element by σ and right multiplication by σ^{-1} (equivalences in directions 3 and 4), hence M and M' are equivalent. ∎

Writing $C = \alpha + \beta\epsilon$ in the split quaternion representation of Lemma 2, we find

$$\sigma C \sigma^{-1} = \sigma\alpha\sigma^{-1} + \sigma\beta\epsilon\sigma^{-1} = \alpha + \beta\frac{\sigma}{\bar{\sigma}}\epsilon.$$

Therefore $\sigma C \sigma^{-1}$ and C have the same projection onto L and the same distance to L. In fact, every element in L of norm 1 can be written as $\sigma/\bar{\sigma}$ so β can freely be chosen as long as it has norm $\alpha\alpha - N$.

We are now in a position to formulate a strong criterion for nonsingular hypercubes that are in reduced form :

Proposition 2 *Let λ, T, N, Δ be as before. Let* $M = \left(\begin{array}{c|c} I & \lambda \\ \hline C & D \end{array}\right)$ *be in reduced form.*

Then M is nonsingular if and only if at least one of the following cases hold:

1. (Straight case) $\lambda \cdot D = C \cdot D$ and

$$(\lambda \cdot D)^2 - T(\operatorname{tr} D + k)(\lambda \cdot D) + N(\operatorname{tr} D + k)^2 + \Delta \det D = 0, \tag{10}$$

and

$$-\Delta X^2 + [2\lambda \cdot D - T(\operatorname{tr} D + k)]\, X + [T\lambda \cdot D - 2N(\operatorname{tr} D + k)] = 0 \tag{11}$$

has no solutions for $X \in K$, with $k = \lambda \cdot C - 2N$.

2. (Skew case) $\lambda \cdot D + C \cdot D = T(\operatorname{tr} D + k^*)$ and

$$(\lambda \cdot D)^2 - T(\operatorname{tr} D + k^*)(\lambda \cdot D) + N(\operatorname{tr} D + k^*)^2 + \Delta \det D = 0, \tag{12}$$

and

$$\Delta X^2 + [2\lambda \cdot D - T(\operatorname{tr} D + k^* - \Delta)]\, X + [T\lambda \cdot D - 2N(\operatorname{tr} D + k^*)] = 0 \tag{13}$$

has no solutions for $X \in K$, with $k^* = \lambda \cdot C - T^2 + 2N$.

Proof : From the definition we easily compute the quadratic form Q_M for this particular case :

$$Q_M(x, y, z, t) = x^2 + N\, y^2 + N\, z^2 + N_D\, t^2 + T\, xy + T\, xz + T_D xt + a\, yz + b\, yt + b'\, zt,$$

using the abbreviations $a = \lambda \cdot C$, $b = \lambda \cdot D$, $b' = C \cdot D$, $T_D = \operatorname{tr} D$ and $N_D = \det D$.

By Corollary 1 there must exist $k \in K$ such that $Q_M + kH$ is of class 1^-, i.e., such that

$$Q_M(x, y, z, t) + k(xt - yz) = (a_1 x + a_2 y + a_3 z + a_4 t)(b_1 x + b_2 y + b_3 z + b_4 t)$$

for certain $a_i, b_i \in L$. Without loss of generality we may assume $a_1 = b_1 = 1$, and then

$$\begin{array}{rclrcl} a_2 + b_2 & = & T, & a_3 + b_3 & = & T, \\ a_2 b_2 & = & N, & a_3 b_3 & = & N, \\ a_2 b_3 + a_3 b_2 & = & a - k, & & & \\ a_4 + b_4 & = & T_D + k, & a_4 b_4 & = & N_D, \\ a_2 b_4 + a_4 b_2 & = & b, & a_3 b_4 + a_4 b_3 & = & b'. \end{array} \tag{14}$$

From the expression for $a_4 + b_4$ and $a_2 b_4 + a_4 b_2$ we compute

$$a_4 = \frac{a_2(T_D + k) - b}{a_2 - b_2}, \quad b_4 = \frac{b_2(T_D + k) - b}{b_2 - a_2}. \tag{15}$$

and then

$$\begin{aligned} N_D = a_4 b_4 &= -\frac{(a_2(T_D+k)-b)(b_2(T_D+k)-b)}{(a_2-b_2)^2} \\ &= -\frac{b^2-(a_2+b_2)(T_D+k)+a_2b_2(T_D+k)^2}{(a_2+b_2)^2-4a_2b_2} \\ &= -\frac{b^2-T(T_D+k)b+N(T_D+k)^2}{T^2-4N} \end{aligned}$$

which will prove (10) and (12) once the correct values of k (and k^*) are established below.

From the first two lines of (14) we derive that $\{a_2, b_2\}$ and $\{a_3, b_3\}$ are the pairs of roots of the equation $x^2 - Tx + N = 0$, i.e., $\{\lambda, \bar{\lambda}\}$. Switching the roles of a_i and b_i if necessary, we may assume that $a_2 = \lambda$, $b_2 = \bar{\lambda}$. From (15) we obtain

$$a_4 = \frac{\lambda(T_D+k)-b}{\lambda-\bar{\lambda}}, \quad b_4 = \frac{\bar{\lambda}(T_D+k)-b}{\bar{\lambda}-\lambda}.$$

It follows that $b_i = \bar{a}_i$ for $i = 1, 2, 3, 4$, as required by Corollary 1.

For M to be nonsingular it is not sufficient that $Q - kH$ is of class 1^- but it is also necessary that the line it represents does not intersect the hyperboloid $\mathcal{H}$ over K. If we write the equation of $Q_M + kH$ as a product $l(x, y, z, t)\bar{l}(x, y, z, t)$ of two conjugate linear forms, with $l = x + a_2 y + a_3 z + a_4 t$, then $l(x, y, z, t) = \bar{l}(x, y, z, t) = 0$ if and only if $m_1(x, y, z, t) = m_2(x, y, z, t) = 0$ where $m_1 = l + \bar{l}$, $m_2 = \bar{\lambda} l + \lambda \bar{l}$. Note that m_1, m_2 have coefficients in K, and are easily computed from (14), for m_2 using the fact that $\lambda = a_2$. We find

$$m_1(x, y, z, t) = 2x + Ty + Tz + (T_D + k)t, \quad m_2(x, y, z, t) = Tx + 2Ny + (a - k)z + bt.$$

The system $m_1(x, y, z, t) = m_2(x, y, z, t) = 0$ reduces to

$$\left\{ \begin{array}{rcl} \Delta y + (T^2 - 2a + 2k)z + (T(T_D + k) - 2b)t &=& 0, \\ \Delta x + T(a - k - 2N)z + (Tb - 2N(T_D + k))t &=& 0. \end{array} \right.$$

Substituting the values for x, y into $\Delta(yz - xt) = 0$ then yields

$$(Tb - 2N(T_D + k))t^2 + (2b + T(a - 2k - 2N - T_D))tz - (T^2 - 2a + 2k)z^2 = 0. \tag{16}$$

This equation must have no nontrivial solutions for $t, z \in K$.

For the values of a_3, b_3 we need to distinguish between two different cases :

1. (Straight case) $a_3 = a_2 = \lambda$, $b_3 = b_2 = \bar{\lambda}$. From the third line of (14) we obtain $a - k = 2\lambda\bar{\lambda} = 2N$, and hence $k = a - 2N$. The last line of (14) then implies $b = b'$.

For this case, equation (16) can be rewritten as

$$(Tb - 2N(T_D + k))t^2 + (2b - T(T_D + k))tz - \Delta z^2 = 0.$$

This equation should not have nontrivial solutions for $t, z \in K$. Note that $\Delta \neq 0$ and therefore setting $X = z/t$ yields the stated condition on (11).

2. (Skew case) $a_3 = b_2 = \bar{\lambda}$, $b_3 = a_2 = \lambda$. We proceed in a similar way as before. From (14) we obtain $a - k = \lambda^2 + \bar{\lambda}^2 = T^2 - 2N$ and $b + b' = T(T_D + k)$.

For this case, equation (16) can be rewritten as

$$(Tb - 2N(T_D + k))t^2 + (2b - T(T_D + k) + T\Delta)tz + \Delta z^2 = 0$$

Renaming k to k^* yields the stated condition on (13). ∎

There are various equivalent ways to reformulate (10–13) :

1. (Straight case) We may reformulate (10–11) as

$$\rho\bar{\rho} = -\Delta \det D, \quad -\Delta X^2 + (\rho + \bar{\rho})X + \lambda \cdot \rho = 0 \text{ has no solutions } X \in K, \tag{17}$$

with

$$\rho = \lambda \cdot D - (\operatorname{tr} D + k)\lambda, \qquad k = \lambda \cdot C - 2N.$$

2. (Skew case) We may reformulate (12–13) as

$$\rho^*\overline{\rho^*} = -\Delta \det D,\ \Delta X^2 + (\rho^* + \overline{\rho^*} + T\Delta)X + \lambda \cdot \rho^* = 0 \text{ has no solutions } X \in K, \tag{18}$$

with

$$\rho^* = \lambda \cdot D - (\operatorname{tr} D + k^*)\lambda, \qquad k^* = \lambda \cdot C + 2N - T^2.$$

Note that (16) can also be rewritten as

$$-(\lambda \cdot D)(C \cdot D) + N(\operatorname{tr} D + k^*)^2 + \Delta \det D = 0, \tag{19}$$

which shows that the roles of λ and C in Proposition 2 can be interchanged.

5 The case of 12-rank equal to 3

Applying Proposition 2 to the special case of reduced forms with $\lambda = C$ will yield the classification of nonsingular hypercubes of 12-rank 3. For this we need to be able to compute the invariants of hypercubes in reduced form :

Lemma 5 *Let λ, T, N, Δ be as before. Let* $M = \left(\begin{array}{c|c} I & \lambda \\ \hline C & D \end{array}\right)$ *be in reduced form.*

Then

$$\det_1 M = \operatorname{tr} D - \lambda \cdot C, \quad \det_2 M = \det(D - C\lambda), \quad \det_2' M = \det(D - \lambda C),$$

$$\det_2'' M = \det(D - C\lambda) - \det(D - \lambda C) = D \cdot (\lambda C - C\lambda),$$

$$\det_3 M = \begin{vmatrix} 1 & T & N \\ T & \operatorname{tr} D + \lambda \cdot C & \lambda \cdot D \\ N & C \cdot D & \det D \end{vmatrix},$$

Proof : To compute $\det_2 M$ we use the following identity over $K^{4\times4}$ which multiplies M by a matrix of determinant 1.

$$\det_2 M = \begin{vmatrix} I & \lambda \\ C & D \end{vmatrix} \begin{vmatrix} I & -\lambda \\ 0 & I \end{vmatrix} = \begin{vmatrix} I & 0 \\ C & D - C\lambda \end{vmatrix} = \det(D - C\lambda)$$

For $\det_2'' M$ note that $\det(D - \lambda C) = \det D + (\det \lambda)(\det C) - D \cdot \lambda C$ by definition of the dot product. The other expressions are straight forward applications of the definitions. ∎

We can now prove the main theorem of this section :

Theorem 1 *Let K be the finite field of order q. Let L denote the quadratic extension of K. Let $\lambda \in L \setminus K$, $T = \lambda + \bar{\lambda}$, $N = \lambda\bar{\lambda}$, $\Delta = T^2 - 4N = (\lambda - \bar{\lambda})^2$.*

Then every nonsingular hypercube $M \in K^{2\times2\times2\times2}$ with 12-rank equal to 3 is equivalent to a hypercube of the form $\left(\begin{array}{c|c} I & \lambda \\ \hline \lambda & \lambda^2 + a + \beta\epsilon \end{array}\right)$ *where $a \in K$, $\beta \in K[\lambda]$ are such that $a \neq 0$, $\beta\bar{\beta} = a\Delta$ and $X^2 + TX + (a + N) = 0$ has no solutions for X in K.*

Conversely, every hypercube of the given form is nonsingular if and only if a, β satisfy the given conditions.

Different values of $a \in K$ yield inequivalent hypercubes. For a given $a \in K$ several $\beta \in L$ can be found that satisfy $\beta\bar{\beta} = a\Delta$ but all of those yield equivalent hypercubes.

Up to equivalence there are $\lfloor \frac{q-2}{2} \rfloor$ nonsingular hypercubes of 12-rank 3.

Proof : By Lemma 3 we may without loss of generality assume that M is of the form $M = \left(\begin{array}{c|c} I & \lambda \\ \hline \lambda & D \end{array}\right)$. We apply Proposition 2 to the case $\lambda = C$. Note that $\lambda \cdot C = \lambda \cdot \lambda = 2N$. We write $D = \alpha + \beta\epsilon$ with $\alpha, \beta \in L$.

We consider two cases:

1. (Straight case) Proposition 2 applies with $k = 0$. Note that $\lambda \cdot D = C \cdot D$ already trivially holds. We compute ρ of (17) :

$$\rho = \lambda \cdot D - (\operatorname{tr} D + k)\lambda = \lambda\bar{\alpha} + \alpha\bar{\lambda} - (\alpha + \bar{\alpha})\lambda = \alpha(\lambda - \bar{\lambda}).$$

The first condition of (17) now translates to $-(\lambda - \bar{\lambda})^2\alpha\bar{\alpha} = -\Delta(\alpha\bar{\alpha} - \beta\bar{\beta})$ and therefore $\beta\bar{\beta} = 0$, which implies $D \in \lambda$. But then M has 12-rank equal to 2 and not 3.

2. (Skew case) Proposition 2 applies with $k^* = 4N - T^2 = -\Delta$ and $2\lambda \cdot D = T(\operatorname{tr} D - \Delta)$. This can be rewritten as

$$T\Delta = T \operatorname{tr} D - 2\lambda \cdot D = (\lambda + \bar{\lambda})(\alpha + \bar{\alpha}) - 2\lambda\bar{\alpha} - 2\alpha\bar{\lambda} = (\lambda - \bar{\lambda})(\alpha - \bar{\alpha}),$$

and then $\alpha - \bar{\alpha} = T(\lambda - \bar{\lambda}) = \lambda^2 - \bar{\lambda}^2$. It follows that $\alpha - \lambda^2 \in K$. Writing $\alpha = \lambda^2 + a$ we find

$$\operatorname{tr} D = \operatorname{tr} \alpha = 2a + T^2 - 2N, \quad \lambda \cdot D = \lambda \cdot \alpha = T(a + N), \quad \operatorname{tr} D - \Delta = 2(a + N),$$

and ρ^* of (18) becomes

$$\rho^* = T(a + N) - 2(a + N)\lambda = (a + N)(T - 2\lambda) = (a + N)(\bar{\lambda} - \lambda).$$

We have $\rho^* + \bar{\rho^*} = 0$ and $\rho^*\overline{\rho^*} = -(a+N)^2\Delta$ and therefore $\det D = (a+N)^2$. The quadratic equation of (18) becomes

$$\Delta X^2 + T\Delta X + (a + N)\Delta = 0,$$

which is equivalent to $X^2 + TX + (a + N) = 0$.

Note that $\alpha\bar{\alpha} = a^2 + a(T^2 - 2N) + N^2$ and then $\beta\bar{\beta} = \alpha\bar{\alpha} - \det D = (a^2 + a(T^2 - 2N) + N^2) - (a + N)^2 = a\Delta$. The hypercube has 12-rank smaller than 3 if and only if $\beta = 0$, i.e., if and only if $a = 0$.

We count the number of values of $a \in K$, $a \neq 0$ such that $X^2 + TX + (a + N)$ has no solutions. For this we count the number of different values of $X^2 + TX$ when $X \in K$. Clearly $X^2 + TX = X'^2 + TX'$ if and only if $X = X'$ or $X' = -T - X$. So the number of possible values for $X^2 + TX$ with $X \in K$ is equal to the number of sets $\{X, -T - X\}$ with $X \in K$. Such a set is a singleton only if $2X = -T$. Because $T \neq 0$ in even characteristic, there is one such singleton value if and only if q is odd. The number of possible values for $X^2 + TX$ in K is therefore equal to $q/2$ when q is even, and $(q + 1)/2$ when q is odd. The values of a in which we are interested correspond to the values of $-a - N$ that are *not* of the form $X^2 + TX$. There are $q/2$ such values of a when q is even, and $(q-1)/2$ when q is odd. From this total we must subtract the single case $a = 0$.

That different choices of β for the same value of a yield equivalent hypercubes is an immediate consequence of Lemma 4. It remains to be proved that different values of a yield hypercubes that are not equivalent.

For this we shall use the fact that hypercubes can only be equivalent if the values of all invariants of exponent 0 are the same. From Lemma 5 we obtain,

$$\det_1 M = 2a + \Delta, \quad \det_2 M = \det_2' M = a(a-\Delta), \quad \det_2'' M = 0, \quad \det_3 M = a^2(2a - \Delta).$$

and then

$$(\det_1 M)^2 + 12 \det_2 M = (4a - \Delta)^2, \quad \det_3 M - \det_1 M \det_2 M = \Delta^2 a.$$

The following are therefore invariants of exponent 0:

$$\delta_0(a) = \frac{(4a-\Delta)^2}{(2a+\Delta)^2}, \quad \delta_0'(a) = \frac{\Delta^2 a}{(2a+\Delta)^3}.$$

We shall prove that a is necessarily equal to b when both $\delta_0(a) = \delta_0(b)$ and $\delta_0'(a) = \delta_0'(b)$. In even characteristic $\delta_0'(a) = a/\Delta$, proving this statement almost directly. Otherwise, let $\delta_0(a) = \delta_0(b)$. Then

$$\frac{4a-\Delta}{2a+\Delta} = \pm \frac{4b-\Delta}{2b+\Delta}. \tag{20}$$

(The special case where $2a + \Delta = 0$ or $2b + \Delta = 0$ shall be treated later.)

The 'plus' case of (20) easily leads to $a = b$. The 'minus' case yields

$$(4a-\Delta)(2b+\Delta) + (4b-\Delta)(2a+\Delta) = 0,$$

and then $16ab + 2\Delta a + 2\Delta b - 2\Delta^2 = 0$, i.e., $(8a + \Delta)b = \Delta(\Delta - a)$. If $8a + \Delta = 0$ we find $\Delta - a = 0$, implying $\Delta = a = 0$, a contradiction. Hence we may write $b = \frac{\Delta(\Delta-a)}{8a+\Delta}$.

We compute

$$\delta_0'(b) = \frac{\Delta^2 \frac{\Delta(\Delta-a)}{8a+\Delta}}{\left(2\frac{\Delta(\Delta-a)}{8a+\Delta} + \Delta\right)^3} = \frac{\frac{\Delta-a}{8a+\Delta}}{\left(\frac{3\Delta+6a}{8a+\Delta}\right)^3} = \frac{(\Delta-a)(8a+\Delta)^2}{27(\Delta+2a)^3}.$$

Then $\delta_0'(b) = \delta_0'(a)$ if and only if $(\Delta - a)(8a+\Delta)^2 = 27\Delta^2 a$, i.e., $\Delta^3 - 12a\Delta^2 + 48a^2\Delta - 64 = 0$. This evaluates to $(\Delta - 4a)^3 = 0$, implying $\Delta = 4a$. But then $b = a$.

Finally, let $2a + \Delta = 0$, i.e., $\det_1 M = 0$. The fact that $\det_1 M$ is zero is an invariant, and therefore we also must have $2b + \Delta = 0$, and then $a = b$. ∎

6 Nonsingular hypercubes of fibration type

A nonsingular hypercube M will be called *of fibration type* if the associated bundle of quadrics defines a hyperbolic fibration (cf. Lemma 1). This notion is invariant under hypercube equivalence. The following lemma provides a simple example.

Lemma 6 *Let* $M = \left(\begin{array}{c|c} I & B \\ \hline C & BC \end{array}\right)$ *with* $B, C \in K^{2\times 2}$*. If* B, C *have eigenvalues in* $L \setminus K$ *then* M *is nonsingular and of fibration type.*

Proof : Following the proof of Proposition 1, we find that M is nonsingular if and only if $P = k'(kI + lC) + l'(kB + lBC)$ is nonsingular for every $k, k', l, l' \in K$ except when $k = l = 0$ or $k' = l' = 0$. Now $P = (k'I + l'B)(kI + lC)$ and both factors are nonsingular matrices because the eigenvalues of B and C are not in K. This proves that M is nonsingular.

Now consider the associated quadric. We have

$$\begin{aligned} Q_M(x,y,z,t) &= x^2 + N_B y^2 + N_C z^2 + N_B N_C t^2 \\ &\qquad + T_B xy + T_C xz + N_B T_C yt + N_C T_B zt + (B \cdot C) yz + (\operatorname{tr} BC) xt, \end{aligned}$$

with $T_B = \operatorname{tr} B$, $T_C = \operatorname{tr} C$, $N_B = \det B$, $N_C = \det C$, and using the identities $B \cdot BC = \det B (I \cdot C) = N_B T_C$ and $C \cdot BC = (I \cdot B) C = N_C T_B$.

Let $\lambda, \bar\lambda$ denote the eigenvalues of B and $\mu, \bar\mu$ the eigenvalues of C. Consider the following quadratic form of class 1^- :

$$\begin{aligned} Q_{\lambda\mu}(x,y,z,t) &= (x + \lambda y + \mu z + \lambda\mu t)(x + \bar\lambda y + \bar\mu z + \bar\lambda\bar\mu t) \\ &= x^2 + N_B y^2 + N_C z^2 + N_B N_C t^2 \\ &\qquad + T_B xy + T_C xz + N_B T_C yt + N_C T_B zt + (\lambda\bar\mu + \mu\bar\lambda) yz + (\lambda\mu + \bar\lambda\bar\mu) xt. \end{aligned}$$

Note that $Q_{\lambda\mu}$ has the same coefficients as Q_M except for those of yz and xt. Moreover, in both cases the sum of the coefficients of yz and xt is the same :

$$B \cdot C + \operatorname{tr} BC = \operatorname{tr} B\bar{C} + \operatorname{tr} BC = (\operatorname{tr} B)(\operatorname{tr} C) = (\lambda + \bar\lambda)(\mu + \bar\mu) = (\lambda\bar\mu + \mu\bar\lambda) + (\lambda\mu + \bar\lambda\bar\mu)$$

This proves that $Q_{\lambda\mu}$ belongs to the bundle $\mathcal{B}$ generated by Q_M and $xt - yz$.

Interchanging μ and $\bar\mu$ in the above yields another quadratic form $Q_{\lambda\bar\mu}$ of class 1^- that belongs to $\mathcal{B}$. That both 'lines' we have constructed are different, can be seen from the difference

$$Q_{\lambda\mu}(x,y,z,t) - Q_{\lambda\bar\mu}(x,y,z,t) = (\lambda - \bar\lambda)(\mu - \bar\mu)(xt - yz)$$

which is zero only if $\lambda \in K$ or $\mu \in K$. The bundle contains two different lines and therefore represents a hyperbolic fibration. ∎

Much in the same way it can be proved that $\left(\begin{array}{c|c} I & B \\ \hline C & CB \end{array}\right)$ is a hyperbolic fibration when nonsingular. It turns out that these are essentially the only two examples.

Lemma 7 *Let* $M = \left(\begin{array}{c|c} I & \lambda \\ \hline C & D \end{array}\right)$ *be in reduced form such that both the straight and skew cases of Proposition 2 apply. Then* $D = \lambda C$ *or* $D = C\lambda$.

Proof : Because both cases apply, we have $\lambda \cdot D = C \cdot D$ and $2\lambda \cdot D = T(\operatorname{tr} D + k^*)$. Note that $k - k^* = \Delta$. Subtracting (12) from (10) yields $T(k-k^*)(\lambda \cdot D) = N(k-k^*)(2\operatorname{tr} D + k + k^*)$, i.e., $T\Delta(\lambda \cdot D) = N\Delta(2\operatorname{tr} D + k + k^*)$. Dividing by Δ, multiplying by T and subtracting $4N\lambda \cdot D$ yields

$$\begin{aligned}(T^2 - 4N)(\lambda \cdot D) &= TN(2\operatorname{tr} D + k + k^*) - 2N(2\lambda \cdot D) \\ &= TN(2\operatorname{tr} D + k + k^*) - 2TN(\operatorname{tr} D + k^*) \\ &= TN(k - k^*) = TN\Delta\end{aligned}$$

and therefore $\lambda \cdot D = TN$. It follows that $T(\operatorname{tr} D + k^*) = 2TN$.

First assume that $T \neq 0$. We find $\operatorname{tr} D + k^* = 2N$ and $\rho^* = N(T - 2\lambda) = N(\bar{\lambda} - \lambda)$. Then $\rho^*\overline{\rho^*} = -N^2\Delta$ and (18) implies $\det D = N^2$. We have $\rho^* + \overline{\rho^*} = 0$ and $\lambda \cdot \rho^* = N(T^2 - 4N) = \Delta N$. The quadratic equation of (18) translates to $\Delta X^2 + T\Delta X + N\Delta = 0$, which is essentially the characteristic equation of λ and has no roots in K.

If on the other hand $T = 0$ then $N\Delta(2\operatorname{tr} D + k + k^*) = 0$ and therefore $0 = 2\operatorname{tr} D + k + k^* = 2(\operatorname{tr} D + \lambda \cdot C)$. When $T = 0$ the characteristic of K cannot be even and we can divide by 2. Hence $\operatorname{tr} D = -\lambda \cdot C$ and $\operatorname{tr} D + k^* = -\lambda \cdot C + \lambda \cdot C + 2N = 2N$. The rest of the argument for $T \neq 0$ now applies. Summarising, we find

$$\lambda \cdot D = C \cdot D = TN, \quad \det D = N^2, \quad \operatorname{tr} D = T^2 - \lambda \cdot C. \tag{21}$$

Now $\operatorname{tr} D + k = T^2 - 2N$. Then $\rho = TN - (T^2 - 2N)\lambda$ and $\rho\bar{\rho} = T^2N^2 - T^2N(T^2 - 2N) + N(T^2 - 2N)^2 = -N^2\Delta$, as expected. Also $\rho + \bar{\rho} = -T\Delta$ and $\rho \cdot \lambda = -N\Delta$. The quadratic equation of (17) becomes $-\Delta X^2 - T\Delta X - N\Delta = 0$, which again has no solutions in K.

Finally, write $C = \alpha + \beta\epsilon$ and $D = \gamma + \delta\epsilon$. Note that $\alpha + \bar{\alpha} = T$ and $\alpha\bar{\alpha} - \beta\bar{\beta} = N$. Also $C \cdot D = \alpha\bar{\gamma} + \gamma\bar{\alpha} - \beta\bar{\delta} - \delta\bar{\beta}$. Now, (21) translates to

$$\bar{\lambda}\gamma + \lambda\bar{\gamma} = TN, \quad \gamma + \bar{\gamma} = T^2 - \lambda\bar{\alpha} - \alpha\bar{\lambda}, \quad \gamma\bar{\gamma} - \delta\bar{\delta} = N^2, \quad \alpha\bar{\gamma} + \gamma\bar{\alpha} - \beta\bar{\delta} - \delta\bar{\beta} = TN.$$

From the first two equalities we derive

$$\begin{aligned}\gamma(\lambda - \bar{\lambda}) &= (T^2 - \lambda\bar{\alpha} - \alpha\bar{\lambda})\lambda - TN \\ &= T^2\lambda - \lambda^2\bar{\alpha} - N\alpha - TN \\ &= T^2\lambda + \lambda^2\alpha - T\lambda^2 - N\alpha - TN \\ &= T\lambda(T - \lambda) + \alpha(\lambda^2 - N) - TN \\ &= TN + \alpha\lambda(\lambda - \bar{\lambda}) - TN = \alpha\lambda(\lambda - \bar{\lambda})\end{aligned}$$

and therefore $\gamma = \alpha\lambda$. Substituting this in earlier equations yields

$$\alpha\bar{\alpha} - \beta\bar{\beta} = N, \quad N\alpha\bar{\alpha} - \delta\bar{\delta} = N^2, \quad \beta\bar{\delta} + \delta\bar{\beta} = T\beta\bar{\beta}$$

and then $\delta\bar{\delta} = N\beta\bar{\beta}$.

If $\beta \neq 0$ then it follows that δ/β has trace T and norm N and must therefore be λ or $\bar{\lambda}$. This corresponds to $\delta = \lambda\beta$ (resp. $\delta = \bar{\lambda}\beta$) and then $D = \lambda C$ (resp. $D = C\lambda$).

If $\beta = 0$ then $\delta = 0$ and $\gamma = \lambda\alpha$ is equivalent to $D = \lambda C$. Note that in this case $C = \lambda$ or $C = \bar{\lambda}$. ∎

Theorem 2 *Let K be a finite field. A nonsingular hypercube $M \in K^{2\times2\times2\times2}$ is of fibration type if and only if the matrix*

$$M_3 = \begin{pmatrix} \det A & A \cdot B & \det B \\ A \cdot C & A \cdot D + B \cdot C & B \cdot C \\ \det C & C \cdot D & \det D \end{pmatrix}$$

has rank 1.

Proof : Without loss of generality we may assume that M is in reduced form w.r.t. λ. Note that the rank of M_3 is an invariant. We easily compute

$$M_3 = \begin{pmatrix} 1 & T & N \\ T & \operatorname{tr} D + \lambda \cdot C & \lambda \cdot D \\ N & C \cdot D & \det D \end{pmatrix}. \tag{22}$$

If M is of fibration type, then from (21) in the proof of Lemma 7 we obtain

$$M_3 = \begin{pmatrix} 1 & T & N \\ T & T^2 & TN \\ N & TN & N^2 \end{pmatrix} = \begin{pmatrix} 1 \\ T \\ N \end{pmatrix} \begin{pmatrix} 1 & T & N \end{pmatrix},$$

having rank 1. Conversely, if M_3 in (22) has rank 1, then (21) must hold. We may then repeat the last part of the proof of Lemma 7 to prove that $D = \lambda C$ or $D = C\lambda$, and hence that M is of fibration type by Lemma 6. ∎

Theorem 3 *Let K the finite field of order q. If M is a nonsingular hypercube of fibration type, then M is equivalent to a hypercube of the form $\left(\begin{array}{c|c} I & B \\ \hline C & BC \end{array}\right)$ or of the form $\left(\begin{array}{c|c} I & B \\ \hline C & CB \end{array}\right)$.*

Up to equivalence the number of nonsingular hypercubes of fibration type is equal to q when q is odd, and to $q - 1$ when q is even.

Proof : That M is equivalent to the stated form is already a consequence of Lemma 7. It remains to be counted how many inequivalent hypercubes exist of these forms.

Assume without loss of generality that M is in reduced form $M = \left(\begin{array}{c|c} I & \lambda \\ \hline C & D \end{array}\right)$. Write $C = \alpha + \beta\epsilon$ with $\alpha, \beta \in K$ and first consider the case $D = \lambda C$. Lemma 4 shows that there is most one inequivalent hypercube for each possible value of α. Also note that α must have trace T because C is similar to λ.

Now, replacing the last row of the block matrix M with T times the first row minus the last row yields the matrix

$$M' = \left(\begin{array}{c|c} I & \lambda \\ \hline T - C & T\lambda - \lambda C \end{array}\right) = \left(\begin{array}{c|c} I & \lambda \\ \hline \bar{C} & \lambda\bar{C} \end{array}\right)$$

which is equivalent with M (through an equivalence in direction 1). M' is of the same form as M but corresponds to the value $\bar{\alpha}$ instead of α. As a consequence there is at most one inequivalent hypercube of this type for each conjugate pair $\{\alpha, \bar{\alpha}\}$ in L. The number of conjugate pairs in L with a given trace is equal to $\lfloor \frac{q+1}{2} \rfloor$ (with $T \neq 0$ in even characteristic). To prove that no further equivalence can arise, we shall establish an invariant that will turn out to be different for elements of L that are not conjugate.

From Lemma 5 we compute

$$\det\nolimits_1 M = \operatorname{tr} \lambda C - \lambda \cdot C = \lambda C + \bar{\lambda}\bar{C} - \lambda\bar{C} - \bar{\lambda}C = (\lambda - \bar{\lambda})(\alpha - \bar{\alpha})$$

We also have

$$\begin{aligned} (C\lambda) \cdot (\lambda C) &= (\alpha\lambda + \beta\bar{\lambda}\epsilon) \cdot (\alpha\lambda + \beta\lambda\epsilon) \\ &= 2N\alpha\bar{\alpha} - \beta\bar{\beta}(\lambda^2 + \bar{\lambda}^2) \\ &= 2N(N + \beta\bar{\beta}) - \beta\bar{\beta}(\Delta + 2N) = 2N^2 - \Delta\beta\bar{\beta} \end{aligned}$$

and then

$$\det\nolimits_2' M = 0, \quad \det\nolimits_2 M = \det(C\lambda - \lambda C) = 2N^2 - (C\lambda) \cdot (\lambda C) = \Delta\beta\bar{\beta} = \Delta(\alpha\bar{\alpha} - N)$$

From this we obtain the following invariant of exponent 0.

$$\delta_0''(\alpha) = 4 + \frac{(\det_1 M)^2}{\det_2 M} = 4 + \frac{(\alpha - \bar{\alpha})^2}{\alpha\bar{\alpha} - N} = \frac{(\alpha + \bar{\alpha})^2 - 4N}{\alpha\bar{\alpha} - N} = \frac{T^2 - 4N}{\alpha\bar{\alpha} - N} = \frac{\Delta}{\alpha\bar{\alpha} - N}$$

with values in $K \cup \{\infty\}$. Two different values of α can give rise to two equivalent hypercubes if and only if they have the same value of δ_0'', in other words, if they have the same norm and the same trace, i.e., if they are equal or conjugate.

A similar reasoning can be followed for the case where $D = C\lambda$, yielding the same number of equivalence classes (the roles of $\det_2$ and $\det_2'$ are now interchanged). It remains to be established whether (and when) both cases overlap.

A hypercube that conforms to both cases must have $\det_2 M = \det_2' M = 0$ (and then also $\det_2'' M = 0$ making M of 12-rank 2). From the above it follows that $\alpha\bar{\alpha} = N$ resulting in $C \in K[\lambda]$ and then $C = \lambda$ or $C = \bar{\lambda}$. This is therefore the only 'overlapping' case, yielding a count of q equivalence classes when q is odd, and $q-1$ when q is even. ∎

7 Some additional remarks

As mentioned in the introduction, an important part of the classification of nonsingular 2×2×2×2 hypercubes is still missing, namely for those of 12-rank equal to 4. For the moment it is even not clear what order of magnitude to expect for the number of equivalence classes of hypercubes of that type.

Looking at Lemma 3 we have $O(q^2)$ choices for C with the same eigenvalues as λ, but Lemma 4 essentially reduces this to $O(q)$. Proposition 2 then imposes two restrictions on D which would otherwise have 4 degrees of freedom. In conclusion we would therefore expect $O(q^3)$ equivalence classes. But as already transpired in [2], there maybe some unexpected equivalences that further reduce this number. For example, let M be in reduced form, and let $k, l \in K$. We apply an equivalence in direction 2 to M :

$$\left(\begin{array}{c|c} I & \lambda \\ \hline C & D \end{array}\right)\left(\begin{array}{c|c} k & -lN \\ \hline l & k+Tl \end{array}\right) = \left(\begin{array}{c|c} k+l\lambda & lN+(k+Tl)\lambda \\ \hline kC+lD & ? \end{array}\right) = \left(\begin{array}{c|c} k+l\lambda & (k+l\lambda)\lambda \\ \hline kC+lD & ? \end{array}\right)$$

Multiplying every block in this matrix to the left with $(k+l\lambda)^{-1}$ (an equivalence in direction 3) then yields a hypercube of the form

$$\left(\begin{array}{c|c} I & \lambda \\ \hline (k+l\lambda)^{-1}(kC+lD) & ? \end{array}\right)$$

which is equivalent to M and can now further be put into reduced form using the techniques of the proof of Lemma 3. It is to be expected that for several values of k, l the resulting C-value (bottom left block) will have a different projection onto L than the original C, reducing the number of equivalence classes. But because both C and D are involved in this operation, it is difficult to get a handle on this.

In Proposition 2 we distinguish between a straight and a skew case. The following lemma proves that in investigating hypercubes of 12-rank 4 in reduced form, we need concentrate on one case only.

Lemma 8 *Let* $M = \left(\begin{array}{c|c} I & \lambda \\ \hline C & D \end{array}\right)$ *be in reduced form. Let* $M' = \left(\begin{array}{c|c} I & \lambda \\ \hline \overline{C} & T\lambda - D \end{array}\right)$. *Then* M' *is in reduced form,* M *and* M' *are equivalent, and* M *conforms to the straight case (resp. skew case) of Proposition 2 if and only if* M' *conforms to the skew case (resp. straight case).*

Proof : Because $C + \overline{C} = T$ we see that M' is obtained from M by subtracting T times the first (block matrix) row from the second and then changing signs of the second row. Hence M and M' are equivalent. Applying the same transformation to M' yields M and therefore we need to prove the lemma only in one direction.

Assume that M conforms to the straight case of Proposition 2, and write $k = \lambda \cdot C - 2N$. Let $k^{*\prime}$ denote the k^*-parameter for M' corresponding to the skew case of Proposition 2. We find $k^{*\prime} = \lambda \cdot \overline{C} - T^2 + 2N = \lambda \cdot (T - C) - T^2 + 2N = T^2 - \lambda \cdot C - T^2 + 2N$ and hence $k^{*\prime} = -k$.

Write $D' = T\lambda - D$. We have $\operatorname{tr} D' = \operatorname{tr}(T\lambda - D) = T^2 - \operatorname{tr} D$ and then $\operatorname{tr} D' + k^{*\prime} = T^2 - (\operatorname{tr} D + k)$. Also $\det D' = \det(T\lambda - D) = T^2N - T\lambda \cdot D + \det D$.

Now $\lambda \cdot D' = \lambda \cdot (T\lambda - D) = 2TN - \lambda \cdot D$ and $\overline{C} \cdot D' = \overline{C} \cdot (T\lambda - D) = T\overline{C} \cdot \lambda - \overline{C} \cdot D = T\lambda \cdot \overline{C} - (T - C) \cdot D = T(\lambda \cdot \overline{C} - \operatorname{tr} D) + C \cdot D$ and hence

$$\lambda \cdot D' + C \cdot D' = T(\lambda \cdot \overline{C} + 2N - \operatorname{tr} D) - \lambda \cdot D + C \cdot D = T(\operatorname{tr} D' + k^{*\prime}) - \lambda \cdot D + C \cdot D$$

which when $\lambda \cdot D = C \cdot D$ is equivalent to the requirement on $\lambda \cdot D'$, $C \cdot D'$ for M' to belong to the skew case of Proposition 2.

Write $\rho^{*\prime}$ for the value of ρ^* in (18) for M'. We find

$$\rho^{*\prime} = \lambda \cdot D' - (\operatorname{tr} D' + k^{*\prime})\lambda = 2TN - \lambda \cdot D - (T^2 - \operatorname{tr} D - k)\lambda = 2TN - T^2\lambda - \rho,$$

with ρ as in (17). This yields

$$\begin{aligned} \rho^{*\prime} + \overline{\rho^{*\prime}} + T\Delta &= 4TN - T^3 - \rho - \bar{\rho} + T(T^2 - 4N) = -(\rho + \bar{\rho}) \\ \lambda \cdot \rho^* &= 2T^2N - 2T^2N - \lambda \cdot \rho = -\lambda \cdot \rho \end{aligned}$$

which shows the equivalence of the quadratic equations in (17), for M, and in (18), for M'. Finally

$$\begin{aligned} \rho^{*\prime}\overline{\rho^{*\prime}} &= (2TN - T^2\lambda)(2TN - T^2\bar{\lambda}) - \rho \cdot (2TN - T^2\lambda) + \rho\bar{\rho} \\ &= (4T^2N^2 - 2T^4N + T^4N) - 2TN(\rho + \bar{\rho}) + T^2\rho \cdot \lambda + \rho\bar{\rho} \\ &= -T^2N\Delta - 2TN(\rho + \bar{\rho}) + T^2\rho \cdot \lambda - \Delta \det D \end{aligned}$$

Note that $\rho + \bar{\rho} = 2\lambda \cdot D - T(\operatorname{tr} D + k)$ and $\lambda \cdot \rho = T\lambda \cdot D - 2N(\operatorname{tr} D + k)$, being the coefficients of (11), and hence $2N(\rho + \bar{\rho}) - T\rho \cdot \lambda = (4N - T^2)\lambda \cdot D = -\Delta\lambda \cdot D$. Therefore

$$\rho^{*\prime}\overline{\rho^{*\prime}} = -T^2N\Delta + T\Delta\lambda \cdot D - \Delta D = \Delta \det D'$$

showing that also the first equalities in (17) and (18) become equivalent. ∎

In Section 3 we have already established a connection between a hypercube $M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ of 12-rank 4 and a hyperbolic quadric in $\mathrm{PG}(3, K)$, namely the quadric with equation

$Q_M(x,y,z,t) = 0$. There is another connection which is in some sense dual to the first: regarding elements of $K^{2\times 2}$ as elements of a 4-dimensional K-vector space and hence as coordinates of points of $\mathrm{PG}(3,K)$, i.e., $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \leftrightarrow (A_1, A_2, A_3, A_4)$, the set S_M of points with coordinates $k'(kA+lC)+l'(kB+lD)$ forms a grid that coincides with the generators of a hyperbolic quadric. Again M is nonsingular if and only if S_M does not intersect the standard hyperboloid $\mathcal{H}$. This was already remarked in [3, Theorem 6.5]. It is however dangerous to extend this representation to hypercubes of lower 12-rank. When M has 12-rank 2 then S_M indeed corresponds to a line external to $\mathcal{H}$ (but with every point counted $2(q+1)$ times). However, when M has 12-rank 3, S_M is not a cone (like the quadric associated with Q_M), but forms a subset of a plane.

In [3] D. Glynn considers the existence of nonsingular hypercubes that are not factorizable, i.e., that are not products of (nonsingular) hypercubes of lower dimension. The hypercubes of 12-rank equal to 2 are factorizable (using the 'cross product' of [2]). Also the hypercubes of fibration type are factorizable. For a nonsingular hypercube of 12-rank 3 to be factorizable, we must (at least) have $\det_2 M = \det_2' M = 0$ and then $a = 0$ or $a = \Delta$ in the notation of Theorem 1. The case $a = 0$ makes the 12-rank of M equal to 2. It can be proved that also the case $a = \Delta$ yields an M that is not factorizable, but in any case, as soon as q is not too small, Theorem 1 will produce many nonsingular hypercubes that are not factorizable.

Note on a theorem of [2]

In [2, Theorem 7] is is stated that a hypercube M with 12-rank 2 is nonsingular if and only if the quadratic equation $X^2 - \det_1 MX + \det_2 M$ has two distinct solutions $X \in K$. This is not entirely correct, as the condition that M is not trivially singular should be added.

A counterexample is given by $M = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right)$ which has $\det_1 M = 4$ and $\det_2 M = \det_2' M = \det_2'' M = 0$. Hence $X^2 - \det_1 MX + \det_2 M = X(X-4) = 0$ has two different solutions (when $q \neq 4$) but M is trivially singular.

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