Classification of minimal blocking sets in $\mathrm{PG}(2,9)$

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Abstract. A full classification (up to equivalence) of all minimal blocking sets in $\mathrm{PG}(2,9)$ was obtained by computer. The resulting numbers of minimal blocking sets are tabulated according to size of the set and order of the automorphism group. For the minimal blocking sets with the larger automorphism groups explicit (geometric) descriptions are given. Some of these results can also be generalised to Desarguesian projective planes of higher order. We also give a complete list of all blocking semiovals in $\mathrm{PG}(2,9)$ (up to equivalence).

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1. Introduction

A *blocking set* in a finite plane is a set B of points such that every line of the plane is incident with at least one point of B. A blocking set B is called *trivial* when it contains an entire line. It is called *minimal* if no proper subset of B is a blocking set. Given a set B, we define the *weight* $w_B(\ell)$ of a line ℓ w.r.t. B (usually denoted $w(\ell)$ if B is clear from context) to be the number of points of B incident with ℓ.

Two blocking sets are called $\mathrm{P\Gamma L}$-*equivalent* if there exists a collineation of $\mathrm{PG}(2,q)$, i.e., an element of $\mathrm{P\Gamma L}(3,q)$, mapping one to the other. Two blocking sets are called PGL-*equivalent* or *projectively* equivalent, if there exists a projectivity, i.e., an element of $\mathrm{PGL}(3,q)$, mapping one to the other. The subgroup of $\mathrm{PGL}(3,q)$ that stabilises a blocking set B is the *projective automorphism group* of B, denoted by G_B (or G if B is clear from context). The subgroup of $\mathrm{P\Gamma L}(3,q)$ that stabilises B will be called the *collineation group* (or: *full* automorphism group), denoted by Γ_B or Γ.

We shall write $\mathbf{F}_q$ for the finite field of order $q = p^h$, p prime. We will usually, but not exclusively, work over $\mathbf{F}_9$ where the field elements will be written as $a + bi$ where $a, b \in \mathbf{F}_3$ and $i^2 = -1$. Note that $(i-1)^2 = i$ and therefore $i - 1$ is a primitive element of $\mathbf{F}_9$. The map $x \mapsto x^q$ is the

Frobenius automorphism (of the field extension $\mathbf{F}_{q^2}/\mathbf{F}_q$) and we shall often write $\bar{x}$ instead of x^q. The *norm* (resp. *trace*) of a field element x is given by $N(x) = x\bar{x}$ (resp. $T(x) = x + \bar{x}$).

The main result of this paper is that we now have a complete list of all minimal blocking sets in $\mathrm{PG}(2, 9)$, up to equivalence (i.e., 15 429 238 in total). This classification was obtained by computer.

The algorithm used is briefly described in Section 2 below. Section 3 provides some overview tables listing our results according to size and automorphism groups. Sections 4 and beyond give some explicit geometric descriptions of the more 'interesting' cases, in particular those with a full automorphism group of size at least 42.

2. Generation algorithm

The classification of all minimal blocking sets in $\mathrm{PG}(2, 9)$ (up to equivalence) was produced by a computer program[1] (written in C++) using a generation algorithm that is fairly straightforward:

Call a set B a *pre-blocking set* if every point of B lies on at least one line ℓ such that $w_B(\ell) = 1$. Note that a subset of a pre-blocking set is again a pre-blocking set and that every *minimal* blocking set is also a pre-blocking set. To generate all minimal blocking sets it is therefore sufficient to construct all pre-blocking sets point by point and stop whenever a pre-blocking set is reached that is a true blocking set, i.e., when every line of the plane has weight at least 1.

The main difficulty in this generation process is to ensure that only a single (pre-)blocking set is generated for each equivalence class. For this we have used the well-established technique of *canonical augmentation* [17].

Although we use essentially the same algorithm as in [8], the new implementation for the particular case of $\mathrm{PG}(2, 9)$ turned out to be significantly faster. The earlier implementation represented points and lines by their coordinate triples and used a matrix representation for projectivities, while the most recent version represents the plane as a perfect difference set and uses a permutation group representation, employing Nauty [18] for isomorphism checks. (It is however not clear whether this change in representation is the main reason for the increase in speed.)

Using the latest version, the full classification of the minimal blocking sets of $\mathrm{PG}(2, 9)$ takes three days of computer time. Preliminary tests have shown that using the same program for $\mathrm{PG}(2, 11)$ is not feasible for two reasons: the current program would probably take more than 50 years of CPU-time, and maybe more importantly, the total number of non-isomorphic blocking sets in $\mathrm{PG}(2, 11)$ will be too large to store all of them for further processing. The number of blocking sets of $\mathrm{PG}(2, 11)$ that have a non-trivial

[1] Source code can be found at https://caagt.ugent.be/bsets/

automorphism group will probably be small enough to be practical, but generating these without also generating those with a trivial automorphism group will require further research.

3. Tables

In Table 1 we present a full classification of the non-trivial minimal blocking sets in $\mathrm{PG}(2, 9)$, up to equivalence. For each of these sets B we have determined the full and projective automorphism groups. Each column in the table corresponds to a different set size $|B|$. The second row (labelled $\#_{\mathrm{PGL}}$) denotes the number of PGL-inequivalent minimal blocking sets of that size. The third row (labelled $\#_{\mathrm{P\Gamma L}}$) denotes the number of PΓL-inequivalent minimal blocking sets of that size. For each set size $|B|$ we specify a list of possible full and projective automorphism group orders (denoted by $|\Gamma|$ and $|G|$, respectively) and for each of these, the number (#) of minimal blocking sets with a full and projective automorphism group of that order. (For example, there are 5 inequivalent (with respect to $\mathrm{P\Gamma L}(3, 9)$) blocking sets of size 17 with a projective automorphism group of order 4 and a full automorphism group of order 8.)

Table 2 provides references to geometric descriptions for the minimal blocking sets that are treated in this text, including all those with a full automorphism group of size at least 42. Each line of the table refers to a description of (at least) one blocking set of given size and orders of the automorphism groups. Note that for smaller automorphism group sizes this table is not complete: for example there are 2 inequivalent blocking sets of size 24 with $|\Gamma| = 12$ and $|G| = 6$, but only one is described by Theorem 5.5 with $|S| = 2$.

4. The icosidodecahedron

It is well known [12, Lemma 13.9] that a blocking set can be constructed by taking the dual of the secants of a complete arc. There are only four projectively inequivalent complete arcs in $\mathrm{PG}(2, 9)$ ([12, p386], [27, p100]), of which two provide a minimal blocking set with this construction. One is the unique blocking semioval of size 21 which will be discussed in Sections 8 and 9 (Theorem 9.1). The other is presented in [22] where the minimal blocking sets of size 15 in $\mathrm{PG}(2, 9)$ are classified. We now determine the stabilizer group of this minimal blocking set.

It is well known [3] that $\mathrm{PG}(2, q)$ over a finite field of order $q = \pm 1 \bmod 10$ admits a group of projectivities isomorphic to the alternating group $\mathrm{Alt}(5)$. This group can be generated by the following elements of order 2, 3 and 5, respectively:

$$\begin{aligned} g_2 &: (x\ y\ z) \mapsto (-x\ y\ z) \\ g_3 &: (x\ y\ z) \mapsto (y\ z\ x) \\ g_5 &: (x\ y\ z) \mapsto \frac{1}{2}(x\ y\ z) \begin{pmatrix} 1 & -\phi^{-1} & \phi \\ \phi^{-1} & -\phi & -1 \\ \phi & 1 & -\phi^{-1} \end{pmatrix}. \end{aligned}$$

where $\phi = \frac{1+\sqrt{5}}{2}$. This group acts very much like the icosahedral group on the surface of the real 3-dimensional sphere where antipodal points are identified. For that reason we shall borrow some of the terminology of Archimedean solids below.

The group has an orbit O_6 of six points (corresponding to the vertices of the *icosahedron* on the real 3-dimensional sphere) having the following coordinates:

$$(\pm\phi, 0, 1), \quad (0, 1, \pm\phi), \quad (1, \pm\phi, 0),$$

an orbit O_{10} of ten points (the *dodecahedron*), with coordinates

$$(\pm 1, \pm 1, \pm 1), \quad (\pm\phi^2, 1, 0), \quad (0, \pm\phi^2, 1), \quad (1, 0, \pm\phi^2)$$

and an orbit O_{15} of 15 points (the *icosidodecahedron*), with coordinates

$$\begin{aligned} &(1,0,0), \quad (0,1,0), \quad (0,0,1), \\ &(1, \pm\phi^2, \pm\phi), \quad (\pm\phi^2, \pm\phi, 1), \quad (\pm\phi, 1, \pm\phi^2). \end{aligned}$$

If q is a square, $\mathrm{Alt}(5)$ can be extended to $\mathrm{Sym}(5)$ by adding the collineation $f : (x, y, z) \mapsto (\bar{y}, \bar{x}, \bar{z})$. We have $\bar{\phi} = -\phi^{-1}$ and it is easily seen that f leaves all three orbits O_6, O_{10} and O_{15} invariant.

We may specialise the above to the case $q = 9$, with $\phi = -1 - i$ to obtain the following

Theorem 4.1 ([22]). *In* $\mathrm{PG}(2, 9)$ *the set* O_6 *is a complete arc and the set* O_{15} *is a minimal blocking set.*

It now easily follows from the above that the projective (resp. full) automorphism group of O_{15} is $\mathrm{Alt}(5)$ (resp. $\mathrm{Sym}(5)$), a maximal subgroup of $\mathrm{PGL}(3, 9)$ (resp. $\mathrm{P\Gamma L}(3, 9)$).

5. Blocking sets derived from a unital

Let $\mathcal{U}$ denote a unital in $\mathrm{PG}(2, q^2)$. Recall that lines intersect $\mathcal{U}$ in either 1 or $q + 1$ points and are then called *tangents* or *secants* to $\mathcal{U}$, respectively. Through a point of $\mathcal{U}$ there are one tangent and q^2 secants. Through a point not on $\mathcal{U}$ there are $q+1$ tangents and $q^2 - q$ secants. This makes $\mathcal{U}$ a minimal blocking set of size $q^3 + 1$. The $q + 1$ tangents through a point $P \notin \mathcal{U}$ define $q + 1$ points on $\mathcal{U}$; these points are called the *feet* of P and together they form the *pedal set* of P, denoted by $\tau(P)$.

There are several known constructions of blocking sets obtained from a Hermitian unital by adding a set of points off the unital and removing (parts of) their pedal sets; see for example [14, Section 4] and [19, Construction 3.1 and 3.5]. We generalize some of these results to obtain a larger family of minimal blocking sets, and also calculate the automorphism groups of the resulting sets. Moreover, we apply this idea to the union of conics unital, a Buekenhout-Metz unital [6, 20] whose properties were explored by Szőnyi [29].

5.1. Blocking sets derived from the Hermitian curve

In what follows we consider a Hermitian curve $\mathcal{H}$ in $\mathrm{PG}(2,q^2)$. Note that for a point P not on $\mathcal{H}$ we have $\tau(P) = \mathcal{H} \cap \ell_P$, with ℓ_P the polar line of P.

The following is a reformulation of Construction 3.1 from [19]:

Construction 5.1. *[19, Construction 3.1] Let $\mathcal{H}$ be a Hermitian curve in $\mathrm{PG}(2,q^2)$, $q > 2$, and $Q \in \mathcal{H}$. Let ℓ_Q denote the tangent line to $\mathcal{H}$ through Q. Let S denote a set of points of $\ell_Q \setminus \{Q\}$ of size $|S| = n \leq q$. Then $B = \mathcal{H} \setminus \{\tau(P) \mid P \in S\} \cup \{Q\} \cup S$ is a minimal blocking set of size $q^3 - nq + n + 1$.*

We will expand on this in two ways: firstly, we will show that certain sets S of size bigger than q can still yield a minimal blocking set with the above construction; secondly, we will determine the sizes of the automorphism groups of B for certain choices of S. For both cases, we need the following

Lemma 5.2. *Let $\mathcal{H}$ be a Hermitian curve in $\mathrm{PG}(2,q^2)$, $q > 2$, and $Q \in \mathcal{H}$. Let ℓ_Q denote the tangent line to $\mathcal{H}$ through Q. Let S denote a set of q points such that $S \cup \{Q\}$ is a Baer subline of ℓ_Q. Then, any secant of $\mathcal{H}$ not through Q has at most two points in common with $\{\tau(P) | P \in S\}$.*

Proof. Without loss of generality, we can choose $\mathcal{H} : x\bar{y} + y\bar{x} + z\bar{z} = 0$, and choose Q to have coordinates $(0, 1, 0)$ implying $\ell_Q : x = 0$. We may also take a point P_a of S to have coordinates $(0, a, 1)$ with $a \in \mathbf{F}_q$ and corresponding polar line ℓ_a with equation $z = -ax$.

A point $P(x, y, z)$ on $\tau(P_a) = \ell_a \cap \mathcal{H}$ must satisfy $x\bar{y} + y\bar{x} + a^2 x\bar{x} = 0$. For $x = 0$ we obtain the point Q, otherwise we may set $x = 1$, $z = -a$ and $P(1, y, -a)$ must satisfy $\bar{y} + y + a^2 = 0$, i.e., $\mathrm{Tr}\, y = -a^2$. Hence $\tau(P_a) = \{Q\} \cup \{(1, y, -a) | \mathrm{Tr}\, y = -a^2\}$.

Let $a_i \in \mathbf{F}_q$, $i = 1, 2, 3$ and consider three points $(1, y_i, -a_i) \in \tau(P_{a_i})$. If these points are collinear, then

$$\begin{vmatrix} 1 & y_1 & -a_1 \\ 1 & y_2 & -a_2 \\ 1 & y_3 & -a_3 \end{vmatrix} = 0.$$

Because $a_1, a_2, a_3 \in \mathbf{F}_q$, we have

$$0 = \mathrm{Tr} \begin{vmatrix} 1 & y_1 & -a_1 \\ 1 & y_2 & -a_2 \\ 1 & y_3 & -a_3 \end{vmatrix} = \begin{vmatrix} 1 & \mathrm{Tr}\, y_1 & -a_1 \\ 1 & \mathrm{Tr}\, y_2 & -a_2 \\ 1 & \mathrm{Tr}\, y_3 & -a_3 \end{vmatrix} = \begin{vmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{vmatrix}.$$

This is a Vandermonde determinant which is zero if and only if not all of a_1, a_2, a_3 are different. $\square$

Theorem 5.3. *For q odd, any set B of Construction* 5.1 *where S is contained in $\frac{q-1}{2}$ Baer sublines of ℓ_Q through Q, is a minimal blocking set. Adding one more point of ℓ_Q to S still yields a minimal blocking set.*

For q even, the set B is a minimal blocking set if S is contained in $\frac{q}{2}$ Baer sublines of ℓ_Q through Q.

Proof. Assume q odd and let S be contained in $\frac{q-1}{2}$ Baer sublines of ℓ_Q through Q. Due to Lemma 5.2, any secant ℓ of $\mathcal{H}$, not through Q, has at most two points in common with the pedal sets of the points of a Baer subline of ℓ_Q through Q. Since S is contained in $\frac{q-1}{2}$ such Baer sublines, ℓ has at most $q-1$ points in common with $\{\tau(P)|P \in S\}$ and is hence blocked by at least two points of B. Thus, adding one more point to S still yields a blocking set.

For q even and S contained in $\frac{q}{2}$ Baer sublines of ℓ_Q through Q, a similar argument shows that ℓ has at most q points in common with $\{\tau(P)|P \in S\}$ and is hence blocked by at least one point of B.

It is easy to see that the resulting blocking set B is minimal for all of the above cases. Through Q there are $|S|$ lines of weight one, any other point of $B \cap \mathcal{H}$ is the only point blocking the tangent to $\mathcal{H}$ through that point. A point of S is the only point blocking the tangents to $\mathcal{H}$ through its feet. We conclude that B is a minimal blocking set. □

In what follows, we determine the sizes of the automorphism groups of B for certain choices of S. Although the structure of the automorphism group of $\mathcal{H}$ and its subgroup G_Q that stabilises a point $Q \in \mathcal{H}$ is well known [13, 21], we need to establish an explicit relation between the subgroups of G_Q that fix further points on the tangent ℓ through Q and the action of G_Q on the points of ℓ:

Lemma 5.4. *Let $\mathcal{H}$ be a Hermitian curve in* $\mathrm{PG}(2, q^2)$*. Let $Q \in \mathcal{H}$ and let ℓ denote the tangent line to $\mathcal{H}$ through Q. Let G_Q denote the group of projectivities that stabilises $\mathcal{H}$ and fixes Q. Let $G_{Q,\ell}$ denote the subgroup of G_Q that fixes every point of ℓ. Then $G_{Q,\ell}$ is isomorphic to the additive group of $\mathbf{F}_q$ and the quotient group $G_Q/G_{Q,\ell}$ is isomorphic to the affine group* $\mathrm{AGL}(1, q^2)$ *of the affine line $\ell \setminus Q$.*

In particular, let $S = \{P_1, P_2, \ldots\}$ denote a subset of $\ell \setminus Q$ and let H denote the subgroup of $\mathrm{AGL}(1, q^2)$ *that leaves S invariant. Then the projective automorphism group that stabilises $\mathcal{H} \cup S$ is isomorphic to $H : q$.*

Proof. Without loss of generality we can choose $\mathcal{H}$ to be the Hermitian curve with equation $x\bar{y} + y\bar{x} + z\bar{z} = 0$ and choose $Q \in \mathcal{H}$ with coordinates $(0, 1, 0)$. Then ℓ is the line with equation $x = 0$.

Any projectivity ψ that fixes Q and ℓ has a matrix that can be written in the form

$$\begin{pmatrix} a & b & c \\ 0 & d & 0 \\ 0 & e & 1 \end{pmatrix} \text{ with } a, d \neq 0.$$

Applying ψ to the equation of $\mathcal{H}$ yields

$$\begin{aligned} ax(\bar{b}\bar{x} + \bar{d}\bar{y} + \bar{e}\bar{z}) + \bar{a}\bar{x}(bx + dy + ez) + (cx + z)(\bar{c}\bar{x} + \bar{z}) \\ = (a\bar{b} + \bar{a}b + c\bar{c})x\bar{x} + a\bar{d}x\bar{y} + \bar{a}d\bar{x}y + (a\bar{e} + c)x\bar{z} + (\bar{a}e + \bar{c})\bar{x}z + z\bar{z} = 0 \end{aligned}$$

Hence for $\mathcal{H}$ to be left invariant we must have

$$a\bar{d} = 1, \quad a\bar{b} + \bar{a}b + c\bar{c} = a\bar{e} + c = 0.$$

Choosing d, e determines $a = 1/\bar{d}$, $c = -\bar{e}/\bar{d}$ uniquely and requires b to be such that $\mathrm{Tr}(\bar{b}/\bar{d}) = -e\bar{e}/d\bar{d}$, or equivalently $\mathrm{Tr}(b\bar{d}) = -e\bar{e}$, which leaves q possibilities for b.

The action of ψ on the line $x = 0$ then amounts to $(y, z) \mapsto (dy + ez, z)$ which corresponds to the affine group $\mathrm{AGL}(1, q^2)$. Fixing every point on that line requires that $d = 1$, $e = 0$ and then $a = 1$, $c = 0$ and $\mathrm{Tr}\, b = 0$. The resulting matrices are of the form

$$\begin{pmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ with } \mathrm{Tr}\, b = 0$$

and form a group $G_{Q,\ell}$ that is isomorphic to the additive group of $\mathbf{F}_q$. Because $G_{Q,\ell}$ is the pointwise stabiliser of a set that is fixed by G_Q it is a normal subgroup of G_Q. □

Theorem 5.5. *Let B denote the minimal blocking set of Construction* 5.1 *and let G_B denote its projective automorphism group.*

- *If $n = 1$ then $|G_B| = q(q^2 - 1)$.*
- *If $n = 2$ then $|G_B| = 2q$.*
- *If $S \cup \{Q\}$ can be extended to a Baer subline of ℓ_Q by adding one point, then $|G_B| = q(q - 1)$.*
- *If $S \cup \{Q\}$ is a Baer subline of ℓ_Q, then $|G_B| = q^2(q - 1)$.*

In each of the above cases the blocking set is unique up to equivalence and the collineation group of B has size twice that of G_B.

Proof. When $n < q$, it is easy to see that the points of S are the only points of B that lie on q lines of weight one. Indeed, through Q there are only n lines of weight one, through a point of $B \cap \mathcal{H}$ there is one line of weight one (this is the tangent to $\mathcal{H}$) and all other lines through this point are secant lines to $\mathcal{H}$ and have weight at least two in B. Hence, any element of G_B must fix the set S, and of course also Q, being the only other point on the line defined by S.

Two distinct Hermitian curves can intersect in at most $(q + 1)^2$ points [16], so because $|B \setminus S| = q^3 - nq + 1 > (q + 1)^2$ for all $q > 2$ and $n < q$, $\mathcal{H}$ can be identified as the only Hermitian curve that contains all points of $B \setminus S$. Therefore, G_B is the subgroup of the automorphism group of $\mathcal{H}$ that fixes both S and Q. The subgroup of $\mathrm{AGL}(1, q^2)$ that fixes a point, a pair of points, resp. a set of points that can be extended to a Baer subline by adding one point, has size $q^2 - 1$, 2, resp. $q - 1$. By Lemma 5.4, the statements about the size of G_B follow.

Finally, let $n = q$ with $S \cup Q$ a Baer subline of ℓ_Q. By Lemma 5.2, any secant of $\mathcal{H}$ not through Q has weight at least $q-1$. So, in B, the line ℓ_Q can be uniquely identified as the only line of weight $q+1$ such that each of its points is incident with q lines of weight one. The points of $B \setminus \ell_Q$ lie on a unique Hermitian curve, $\mathcal{H}$, as $|B \setminus \ell_Q| = q^3 - q^2 > (q+1)^2$. It follows that G_B is the subgroup of the automorphism group of $\mathcal{H}$ that fixes $Q = \mathcal{H} \cap \ell_Q$ and $S = \mathcal{B} \setminus \mathcal{H}$. By Lemma 5.4, $|G_B| = q^2(q-1)$.

Note that in each of these cases the set of points S can be chosen to be a subset of the canonical Baer subline of ℓ_Q. Then the Frobenius automorphism fixes S proving $|\Gamma_B| = 2|G_B|$. □

Applied to $\mathrm{PG}(2,9)$, Theorem 5.5 explains the existence of three minimal blocking sets, namely of size 26, 24 and 22 and respective automorphism group sizes $(2\times)24, (2\times)6$ and $(2\times)18$.

In addition, for $|S| = 3$ and $S \cup \{Q\}$ not a Baer subline of ℓ_Q, the resulting minimal blocking set is of size 22 and has automorphism groups of size $(2\times)3$. If S is chosen such that $S \cup \{Q\}$ is a Baer subline of ℓ_Q, Theorem 5.3 shows that we can expand S by any point of ℓ_Q (and delete its pedal set except for Q) to again obtain a minimal blocking set. This yields a projectively unique blocking set of size 20, with automorphism group sizes $(2\times)3$.

There are some further choices for S that also give rise to minimal blocking sets computationally, though this is not guaranteed by Construction 5.1. Choosing S with $|S| = 4$ in a different manner than in Theorem 5.3 does not necessarily yield a blocking set, but when it does, it yields one with projective automorphism group of size 6 and collineation group of size 12, and it is projectively unique. There exist sets S of size 5 such that the set B of Construction 5.1 still yields a blocking set in $\mathrm{PG}(2,9)$. These cases are listed in Table 2. One of them has a fairly large automorphism group and is also given by Theorem 7.1. In $\mathrm{PG}(2,9)$ there exist no sets S of size 6 that yield a blocking set.

In the above we have added points on a fixed tangent to $\mathcal{H}$ (and deleted their pedal sets) to obtain minimal blocking sets. The theorems below show that doing the same with points on multiple tangents to $\mathcal{H}$ can again yield minimal blocking sets, some of which have fairly large automorphism groups.

Theorem 5.6. *Let $\mathcal{H}$ be a Hermitian curve in $\mathrm{PG}(2,q^2)$. Consider two points P_1, P_2 not on $\mathcal{H}$ with corresponding polar lines ℓ_1, ℓ_2 such that $P_1 \in \ell_2$ and $P_2 \in \ell_1$, (i.e., P_1, P_2 are two vertices and ℓ_1, ℓ_2 are two sides of a polar triangle). Then removing the points of ℓ_1, ℓ_2 from $\mathcal{H}$ and adding P_1, P_2 yields a minimal blocking set B_2 of size $q^3 - 2q + 1$. When $q > 2$ the projective automorphism group of B has size $2(q+1)^2$ and the full automorphism group has size $4(q+1)^2$.*

Proof. Any secant of $\mathcal{H}$ different from ℓ_1 and ℓ_2, has its weight reduced by at most 2 by the removal of these two lines. Also, the tangents through the points of $\ell_1 \cap \mathcal{H}$ and $\ell_2 \cap \mathcal{H}$ are blocked by the points P_1 and P_2. Moreover, because we have chosen $P_1 \in \ell_2$ and $P_2 \in \ell_1$, also ℓ_1 and ℓ_2 are blocked by B. Hence, B is a blocking set. Note that all tangents to $\mathcal{H}$ have weight 1 w.r.t. B, and each point of B lies on at least one such tangent. Hence B is minimal.

Let $q > 2$. We may identify P_1 and P_2 as those points of B that lie on $q+2$ lines of weight 1. The remaining $q^3 - 2q - 1$ points of B uniquely determine the Hermitian curve $\mathcal{H}$, because $q^3 - 2q - 1 > (q+1)^2$. And then also ℓ_1, ℓ_2 are determined as the polar lines of P_1, P_2. Therefore, the automorphism group of B is the subgroup of the automorphism group of $\mathcal{H}$ that fixes $\{P_1, P_2\}$ (and therefore also the line P_1P_2 and the pole $P_3 = \ell_1 \cap \ell_2$ of that line).

Without loss of generality we may take $\mathcal{H}$ with equation $x\bar{x}+y\bar{y}+z\bar{z} = 0$. Still without loss of generality we may choose P_1, P_2 to have coordinates $(0,0,1)$ and $(0,1,0)$, and then P_3 has coordinates $(1,0,0)$. Let ψ denote a projectivity that fixes the three points P_1, P_2, P_3. Then ψ is of the form $(x,y,z) \mapsto (ax, by, z)$ with $a, b \neq 0$. To stabilise $\mathcal{H}$, ψ must also satisfy $a\bar{a} = b\bar{b} = 1$, i.e., $N(a) = N(b) = 1$. This yields $q+1$ possibilities for a and $q+1$ possibilities for b. Because only the set $\{P_1, P_2\}$ is combinatorially determined and not each point separately, we must also consider the possibility of interchanging P_1 and P_2. The projectivity $(x,y,z) \mapsto (x,z,y)$ does just that and also fixes $\mathcal{H}$.

Hence the projective automorphism group of B has size $2(q+1)^2$, and it is easily seen that the Frobenius automorphism extends this to a collineation group of twice this size. □

For $q = 3$ this yields a blocking set of size 22 and automorphism groups of size $(2\times)32$. For $q = 2$ this blocking set is a line.

Theorem 5.9 below yields another minimal blocking set with an automorphism group of reasonable size. For the case $q = 3$ of the proof of that theorem, we need two lemmas.

Lemma 5.7. *Let $\mathcal{H}$ be a Hermitian curve in $\mathrm{PG}(2,q^2)$ with q odd. Consider a polar triangle w.r.t. $\mathcal{H}$ with vertices P_1, P_2, P_3 and sides ℓ_1, ℓ_2, ℓ_3. Then a line of $\mathrm{PG}(2,q^2)$ can intersect $\ell_1 \cup \ell_2 \cup \ell_3$ in at most two points of $\mathcal{H}$.*

Proof. We may choose coordinates $P_1(1,0,0)$, $P_2(0,1,0)$, $P_3(0,0,1)$ and take $\mathcal{H}$ to have equation $x\bar{x} + y\bar{y} + z\bar{z} = 0$, without loss of generality. Consider a line that intersects $\ell_1 \cup \ell_2 \cup \ell_3$ in three different points. The coordinates of these points can then be taken as $(1,a,0)$, $(0,1,b)$, $(c,0,1)$. Collinearity of these points is equivalent to $abc = -1$. If all three points belong to $\mathcal{H}$, we have $a\bar{a} = b\bar{b} = c\bar{c} = -1$, and then $-1 = (a\bar{a})(b\bar{b})(c\bar{c}) = (abc)(\overline{abc}) = (-1)(\overline{-1}) = 1$, a contradiction when q is odd. □

Lemma 5.8. *Let $\mathcal{H}$, $\mathcal{H}'$ be two distinct Hermitian curves in $\mathrm{PG}(2,9)$. If $\mathcal{H}$, $\mathcal{H}'$ share a polar triangle P_1, P_2, P_3, then $|\mathcal{H} \cap \mathcal{H}'| = 4$.*

Proof. We may choose coordinates $P_1(1,0,0)$, $P_2(0,1,0)$, $P_3(0,0,1)$ without loss of generality. The equation of a Hermitian curve that has $P_1P_2P_3$ as a polar triangle, is of the form $ax\bar{x} + by\bar{y} + cz\bar{z} = 0$ with $a, b, c \in \mathbf{F}_3 \setminus \{0\}$, in other words, of the form $\pm x\bar{x} \pm y\bar{y} \pm z\bar{z} = 0$. Again without loss of generality we may choose $\mathcal{H}, \mathcal{H}'$ to have equations

$$\mathcal{H} : x\bar{x} + y\bar{y} + z\bar{z} = 0, \quad \mathcal{H}' : x\bar{x} + y\bar{y} - z\bar{z} = 0.$$

It is now easily computed that $\mathcal{H} \cap \mathcal{H}'$ consists of the points with coordinates $(1, \alpha, 0)$ where $\alpha \in \mathbf{F}_9$ such that $N(\alpha) = -1$. There are exactly 4 values α that satisfy these properties. $\square$

Theorem 5.9. *Let $\mathcal{H}$ be a Hermitian curve in $\mathrm{PG}(2, q^2)$ with $q > 2$. Consider a polar triangle w.r.t. $\mathcal{H}$ with vertices P_1, P_2, P_3 and sides ℓ_1, ℓ_2, ℓ_3. Then $B = (\mathcal{H} \backslash (\ell_1 \cup \ell_2 \cup \ell_3)) \cup \{P_1, P_2, P_3\}$ is a minimal blocking set of size $q^3 - 3q + 1$, projective automorphism group of size $6(q+1)^2$ and full automorphism group twice this size.*

Proof. The proof that this is a minimal blocking set runs along the same lines as the proof of Theorem 5.6, except that the weight of a secant to $\mathcal{H}$ can now be as low as $q - 2$ w.r.t. B.

For any $q > 2$, the tangents to $\mathcal{H}$ are the only lines of weight 1. This is easy to see when $q > 3$, for $q = 3$ this is due to Lemma 5.7. As a consequence, the points P_1, P_2, P_3 can be identified as those points of B that lie on $q + 1$ lines of weight 1. When $q > 3$, the $q^3 - 3q - 2$ remaining points of B lie on a unique Hermitian curve $\mathcal{H}$ since $q^3 - 3q - 2 > (q+1)^2$. When $q = 3$, Lemma 5.8 is required to ensure that the points of B lie on a unique Hermitian curve. To obtain the group we use the same argument as in the proof of Theorem 5.6. However, because we can now freely permute P_1, P_2, P_3 we obtain a factor 6 instead of 2. $\square$

For $q = 3$ this yields a blocking set of size 19 and automorphism groups of size $(2\times)96$.

Theorem 5.10. *Let $\mathcal{H}$ be a Hermitian curve in $\mathrm{PG}(2, q^2)$ with $q > 2$. Consider a polar triangle w.r.t. $\mathcal{H}$ with vertices P_1, P_2, P_3 and sides ℓ_1, ℓ_2, ℓ_3. Let $P \in (\ell_1 \triangle \ell_2 \triangle \ell_3) \setminus \mathcal{H}$ with polar line ℓ. The set $B = (\mathcal{H} \setminus \{\ell_1 \cup \ell_2 \cup \ell_3 \cup \ell\}) \cup \{P_1, P_2, P_3, P\}$ is a minimal blocking set of size $q^3 - 4q + 1$.*

Proof. Without loss of generality, $P \in \ell_1$ and hence $P_1 \in \ell$. We have to prove that removing ℓ (except P_1) from and adding P to the minimal blocking set of Theorem 5.9 (which we will refer to as B'), again yields a minimal blocking set.

For B to be a blocking set, we have to make sure each line through a point of $\ell \cap \mathcal{H}$ is still blocked. The line ℓ is blocked by P_1, each of the tangents to $\mathcal{H}$ through the $q + 1$ points of $\ell \cap \mathcal{H}$ is blocked by the newly added point P. Every other line through a point of $\ell \cap \mathcal{H}$ has weight at least 2 w.r.t. B', and hence has weight at least 1 w.r.t. B.

Note that any point of $\{P_1, P_2, P_3, P\}$ is incident with at least $q + 1$ lines of weight 1 and are hence essential. Any other point of B is a point of $\mathcal{H}$ and is the only one blocking the tangent line to $\mathcal{H}$ through this point. This proves that B is minimal. $\square$

For $q = 3$ this yields a minimal blocking set of size 16 with automorphism groups of size (2×)8.

5.2. Blocking sets derived from the union of conics unital

Consider the projective plane $\mathrm{PG}(2, q^2)$ with q odd. Let $\delta \in \mathbf{F}_{q^2}$. Define $\mathcal{B}_\delta$ to be the set of (affine) points with coordinates $(x, y, 1)$ such that $y - \delta x^2 \in \mathbf{F}_q$, together with the point $P_\infty(0, 1, 0)$. Note that $\mathcal{B}_\delta$ can also be seen as the union of q parabolas $C_{\delta,f} : y = \delta x^2 + f$, with $f \in \mathbf{F}_q$, intersecting in the common point P_∞.

Theorem 5.11. [12, §12.3] *Let $\delta \in \mathbf{F}_{q^2}$. If δ is not a square, then $\mathcal{B}_\delta$ is a minimal blocking set. Each point of $\mathcal{B}_\delta$ lies on exactly one line of weight 1. All other lines have weight $q + 1$.*

This blocking set is an example of a *Buekenhout-Metz unital* (of which there exists only one up to equivalence in PG(2, 9)) and although it provides the same weight distribution as a Hermitian curve, it is not equivalent to it. Note that the choice of the non-square δ in the construction is not important, all sets constructed in this way are projectively equivalent.

The set $\mathcal{B}_\delta$ has q^3+1 points. There are q^3+1 lines of weight 1 w.r.t. $\mathcal{B}_\delta$. These are precisely the tangents of the constituent parabolas with the line at infinity ℓ_∞ $(z = 0)$ their common tangent in P_∞.

The projective automorphism group of $\mathcal{B}_\delta$ consists of the transformations of the form

$$(x, y, z) \mapsto (ax + bz, a^2 y + 2\delta abx + (\delta b^2 + f)z, z)$$

with $a^2 \in \mathbf{F}_q$, $a \neq 0$, $b \in \mathbf{F}_{q^2}$ and $f \in \mathbf{F}_q$. This group has size $2(q-1)q^3$. The semilinear map

$$(x, y, z) \mapsto (\frac{\bar{\delta}\bar{x}}{\sqrt{\delta\bar{\delta}}}, \bar{y}, \bar{z})$$

extends this group to the full collineation group of $\mathcal{B}_\delta$ of size $4(q-1)q^3$.

An important difference with the Hermitian curve is the structure of the pedal sets. For a study on this topic, see [1]; we can restrict ourselves to the following lemma, which is a reformulation of [1, Corollary 1].

Lemma 5.12. *Let P be a point of $\mathrm{PG}(2, q^2)$ that does not belong to $\mathcal{B}_\delta$. If $P \notin \ell_\infty$ then $\tau(P)$ is contained in an arc (a parabola through P), if $P \in \ell_\infty$ then $\tau(P)$ lies on a line ℓ_P through P_∞.*

Theorem 5.13. *Let $\mathcal{B}_\delta$ denote the union of conics unital as described above and consider a point $P \notin \mathcal{B}_\delta$.*

1. *If $P \notin \ell_\infty$, then $B_1 = (\mathcal{B}_\delta \setminus \tau(P)) \cup \{P\}$ is a minimal blocking set of $\mathrm{PG}(2, q^2)$ of size $q^3 - q + 1$.*
2. *If $P \in \ell_\infty$, then $B_2 = (\mathcal{B}_\delta \setminus \tau(P)) \cup \{P, Q_1\}$ for any $Q_1 \in \tau(P)$ is a minimal blocking set of $\mathrm{PG}(2, q^2)$ of size $q^3 - q + 2$.*

Proof. Let $P \notin \ell_\infty$. To prove that B_1 is a blocking set, we have to show that any line through a deleted point $Q_i \in \tau(P)$ is blocked by a point of B_1. The tangent line to B_δ through Q_i is now blocked by P, any other line through Q_i is a secant of weight $q+1$ w.r.t. B_δ, and has its weight reduced by at most 2 due to the removal of $\tau(P)$ as these points lie on an arc (Lemma 5.12) and hence no three are collinear. We conclude that B_1 is a blocking set.

Every point of B_1 is essential as it lies on at least one line of weight 1: P lies on the $q+1$ lines PQ_i of weight 1, any other point of B_1 lies on the unique tangent to B_δ through that point, which has weight one w.r.t. B_1. So, this blocking set is minimal.

Now, let $P \in \ell_\infty$. Let $Q_i, i \in \{2, \ldots, q+1\}$ denote the points of $B_\delta \setminus B_2$. The tangent line to B_δ through Q_i is again blocked by P and is now a tangent line to B_2. The line ℓ_P (using the notation of Lemma 5.12) is blocked by Q_1, any other line through $Q_i, i \in \{2, \ldots, q+1\}$ has weight q.

Again, any point of $B_2 \cap B_\delta$ is essential in B_2, P lies on the q lines PQ_i, $i \in \{2, \ldots, q+1\}$, of weight one and Q_1 lies on the line ℓ_P of weight one. We conclude that B_2 is a minimal blocking set. □

In $\mathrm{PG}(2, 9)$, B_1 is a minimal blocking set of size 25 with projective automorphism group of size 2 and collineation group of size 4. B_2 is a minimal blocking set of size 26, but its automorphism group depends on the choice of Q_1. When $Q_1 = P_\infty$, the projective automorphism group and collineation group have size 12 and 24, respectively, in the other case, 4 and 8, respectively.

It is possible to add and delete more points to and from B_1 and B_2, but this can get technical quite fast and reduces the already small sizes of automorphism groups, hence yielding minimal blocking sets with little or no symmetry. We limit ourselves to the following theorem, the proof of which immediately follows from Lemma 5.12.

Theorem 5.14. *Let $\mathcal{B}_\delta$ denote the union of conics unital as described above. Let S denote a set of points on $\ell_\infty \setminus \{P_\infty\}$ of size $|S| = n \leq q$. Then $B = \mathcal{B}_\delta \setminus \{\tau(P) \mid P \in S\} \cup \{P_\infty\} \cup S$ is a minimal blocking set of size $q^3 - nq + n + 1$.*

6. A blocking set of index three

The *index* of a blocking set is the minimum number of lines that can be used to cover the blocking set. The following is a well-known blocking set of index three, a class of blocking sets about which quite a lot is known (see, for example, [7]). It is also a blocking set of Rédei type. More specifically it contains two Rédei lines, all of which have been characterized [5, 25, 26]. The following theorem establishes the automorphism group of this blocking set.

Theorem 6.1. *Let π be a Baer subplane of $\mathrm{PG}(2, q^2)$ with $q > 2$. Let ℓ_1, ℓ_2, ℓ_3 be three secant lines of π intersecting in a common point P (cf. Figure 1). Then $B = (\ell_1 \setminus \pi) \cup (\ell_2 \setminus \pi) \cup (\ell_3 \cap \pi)$ is a minimal blocking set of $\mathrm{PG}(2, q^2)$ of size $2q^2 - q + 1$.*

The full automorphism group of B has size $4q^2(q-1)$ and is generated by the Frobenius automorphism and projectivities $(x, y, z) \mapsto (x, z, y)$ and $(x, y, z) \mapsto (ax + by + cz, y, z)$ where $a, b, c \in \mathbf{F}_q$, $a \neq 0$.

Proof. Since the Frobenius automorphism stabilises π pointwise, it also stabilises ℓ_1 and ℓ_2 and hence B. It remains to establish the *projective* automorphism group G of B. Note that any element of G must stabilise the set $\{\ell_1, \ell_2\}$ as these are the only lines of weight $q^2 - q + 1$ w.r.t. B (and $q > 2$). Similarly, any element of G must stabilise the line ℓ_3 because it is the only line of weight $q+1$. (All other lines have weight 3 or lower since B lies entirely on the three lines ℓ_i.) Without loss of generality we may choose π to be the Baer subplane of all points with coordinates in $\mathbf{F}_q$ and the lines ℓ_i to have equations $\ell_1 : y = 0$, $\ell_2 : z = 0$, $\ell_3 : y = z$. If a projectivity g stabilises the three lines ℓ_1, ℓ_2, ℓ_3, then $g : (x, y, z) \mapsto (ax + by + cz, y, z)$ with $a \neq 0$. If moreover g fixes π then $a, b, c \in \mathbf{F}_q$. Note that only the set $\{\ell_1, \ell_2\}$ has to be stabilised and not each line individually; the projectivity $(x, y, z) \mapsto (x, z, y)$ interchanges ℓ_1 and ℓ_2 and fixes B. □

In PG(2, 9), the above yields a minimal blocking set of size 16 with projective and full automorphism group size of 36 and 72, respectively.

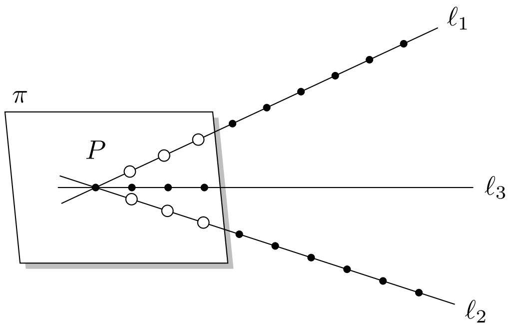

FIGURE 1. Blocking set B of Theorem 6.1 for the case $q^2 = 9$.

7. Full automorphism groups of order 48

One can see from Table 1 that PG(2, 9) has four minimal blocking sets with a full automorphism group of size 48 and that they all have a projective automorphism group of size 24. These blocking sets have sizes 17, 18, 22 and 26. The one of size 26 is given by Theorem 5.5 with $|S| = 1$. The set of size 22 is described in [10], albeit without description of the automorphism group. The theorem below gives a description of the sets of size 17 and 18.

Theorem 7.1. *Let O_{12} denote the set of twelve points of* PG(2, 9) *with coordinates of the form*

$$(0, \pm 1, 1) \quad (0, \pm i, 1) \quad (\pm 1, \pm 1, 1) \quad (\pm 1, \pm i, 1)$$

Let e_1, e_2 *denote the points with coordinates* $(1,0,0)$ *and* $(0,1,0)$ *respectively. Let* O_3 *denote the set* $\{(0,0,1),(1,0,1),(-1,0,1)\}$ *and let* O_4 *denote the set of points with coordinates of the form* $(1,\pm i \pm 1, 0)$*. Define* $B_{17} = O_{12} \cup \{e_1, e_2\} \cup O_3$ *and* $B_{18} = O_{12} \cup \{e_1, e_2\} \cup O_4$*.*

Then B_{17} *and* B_{18} *are minimal blocking sets of* $\mathrm{PG}(2,9)$ *of size 17 and 18 respectively. The projective automorphism group of both sets is isomorphic to* $4D_6$ *(of size 24) and the full automorphism group is isomorphic to* D_8D_6 *(of size 48).*

Proof. Consider the group G generated by the following projectivities

$$(x,y,z) \mapsto (x+z,y,z), \quad (x,y,z) \mapsto (-x,y,z), \quad (x,y,z) \mapsto (x,iy,z).$$

These generators act on the lines with equations $Ax + By + Cz = 0$ in the following manner:

$$(A,B,C) \mapsto (A,B,C-A), \quad (A,B,C) \mapsto (-A,B,C), \quad (A,B,C) \mapsto (A,-iB,C).$$

It is easily seen that the third generator generates a cyclic group of order 4, and the first two generate a group isomorphic to D_6. It follows that G is isomorphic to $4D_6$.

To prove that B_{17} and B_{18} are minimal blocking sets, we note that all of O_{12}, $\{e_1\}$, $\{e_2\}$, O_3 and O_4 are point orbits under the group G. The group G has 10 orbits on lines of $\mathrm{PG}(2,9)$, whose interactions with these point orbits are summarized in the following table.

A	B	C	$\lvert O \rvert$	$w_{O_{12}}$	w_{e_1}	w_{e_2}	w_{O_3}	w_{O_4}	$w_{B_{17}}$	$w_{B_{18}}$
1	± 1 $\pm i$	$\pm i$ $\pm i \pm 1$	24	1					1	1
1	$\pm i \pm 1$	$\pm i$ $\pm i \pm 1$	24	2				1	2	3
1	± 1 $\pm i$	0 ± 1	12	2			1		3	2
1	$\pm i \pm 1$	0 ± 1	12				1	1	1	1
1	0	$\pm i$ $\pm i \pm 1$	6			1			1	1
1	0	0 ± 1	3	4		1	1		6	5
0	1	± 1 $\pm i$	4	3	1				4	4
0	1	$\pm i \pm 1$	4		1				1	1
0	1	0	1		1		3		4	1
0	0	1	1		1	1		4	2	6

Column 4 gives the size of these orbits and columns 5–9 list the weight $w_S(\ell)$ of a line ℓ in that orbit with respect to any of the sets S from the statement of this theorem. (A blank entry denotes a zero weight.) The last two columns list the weights with respect to B_{17} and B_{18} and are obtained by summing the appropriate columns to the left.

Although a bit tedious, the entries in this table can easily be computed by hand, as for each line orbit only a single line must be investigated. For example, for the third row, we may take the line $x = y$ which clearly intersects O_{12} (only) in $(1, 1, 1)$ and $(-1, -1, 1)$ and O_3 in $(0, 0, 1)$.

From this table it is now easily read that B_{17} and B_{18} block all lines and that each of their points lies on at least one line of weight 1.

It is easily proven that the projective automorphism group of B_{17} and B_{18} cannot be larger than G. Furthermore, note that the Frobenius automorphism leaves B_{17} and B_{18} invariant, and extends the cyclic group generated by $(x, y, z) \mapsto (x, iy, z)$ to a group isomorphic to the dihedral group of 8 elements. □

Note that the blocking set B_{18} of Theorem 7.1 can also be produced from a Hermitian curve by applying Construction 5.1 (with $|S| = 5$); for example, choose $\mathcal{H} : i(\bar{x}z - x\bar{z}) + y\bar{y} - z\bar{z} = 0$, $Q = e_1$ and $S = \{e_2\} \cup O_4$.

8. Blocking semiovals

A semioval is a set S of points in a projective plane such that there is a unique tangent through each point of S. A blocking semioval is a blocking set that is also a semioval. It follows immediately that this must be a minimal blocking set. A full classification (up to equivalence) of the blocking semiovals in PG(2, 7) and PG(2, 8) is given in [24] and [2], respectively.

Let x_i denote the number of lines of the plane with weight i. Then $X(S) = (x_1, x_2, \ldots, x_{q-1})$ is called the weight distribution of the blocking semioval S. In [28, Theorem 6.3], Suetake classifies all blocking semiovals in PG(2, 9) with an 8-secant. This classification distinguishes between sets of different size, construction, and weight distribution. We extend this classification by also including information about the automorphism groups of these sets:

Let S be a blocking semioval in PG(2, 9) with $x_8 \neq 0$. For $|S| = 22$, there are two distinct possibilities, one with $|\Gamma| = 8$, $|G| = 4$, the other with $|\Gamma| = 4$, $|G| = 2$. For $|S| = 23$, there are two distinct possibilities for a blocking semioval constructed by [28, Theorem 4.2], one with $|\Gamma| = 16$, $|G| = 8$, the other with $|\Gamma| = 8$, $|G| = 4$. If S is a blocking semioval constructed by Dover [9], then $X(S) = (23, 21, 32, 12, 0, 1, 1, 1)$, $|\Gamma| = 4$ and $|G| = 2$.

The full classification of the blocking semiovals in PG(2, 9) is given in Table 3. References to descriptions for some of these blocking semiovals can be found in Table 2.

The minimum size of a blocking semioval was unknown for PG(2, 9) until one of size 21 was constructed in [11]. We can now confirm that this is the only blocking semioval of size 21 in PG(2, 9) as the authors of [11] suspected. This blocking set is constructed by taking the union of three (specific) Kestenband arcs, cf. Theorem 9.1 below.

9. Blocking sets from three Kestenband arcs

In $\mathrm{PG}(2, q^2)$, let σ denote a Singer cycle. Take a point $P_0 \in \mathrm{PG}(2, q^2)$ and put $P_i = \sigma^i(P_0)$, which allows us to identify the points P_i of $\mathrm{PG}(2, q^2)$ with the integers i modulo $q^4 + q^2 + 1$. Select a line ℓ_0 (the points on this line form a perfect difference set) and put $\ell_j = \ell_0 - j$. It now follows that $i \in \ell_j \iff i + j \in \ell_0$.

Because σ has order $q^4 + q^2 + 1$ it follows that $\langle \sigma^{q^2+q+1} \rangle$ and $\langle \sigma^{q^2-q+1} \rangle$ generate subgroups of $\langle \sigma \rangle$ of size $q^2 - q + 1$ and $q^2 + q + 1$, respectively. The orbits of the points of the plane under the action of these subgroups are complete arcs (called *Kestenband* arcs) and Baer subplanes, respectively. Let us denote these orbits by $K_r = \{r + i(q^2 + q + 1) | i = 0, \ldots, q^2 - q\}$ for $r = 0, \ldots, q^2 + q$ and $B_s = \{s + i(q^2 - q + 1) | i = 0, \ldots, q^2 + q\}$ for $s = 0, \ldots, q^2 - q$. Let $\mathcal{H}$ denote the Hermitian curve associated with the Frobenius automorphism $\phi : i \mapsto q^3 i$. Note that a point i lies on $\mathcal{H}$ if and only if $\phi(i) \in \ell_i \iff q^3 i \in \ell_i \iff (q^3 + 1)i \in \ell_0$.

(We refer the reader to [11, 12] for further information about blocking semiovals, Kestenband arcs and perfect difference sets.)

From this point on, let us look at $\mathrm{PG}(2, 9)$ with perfect difference set $\ell_0 = \{0, 1, 3, 9, 27, 81, 61, 49, 56, 77\}$. Then $i \in \mathcal{H} \iff 28i \in \ell_0 \iff i \equiv 0, 2, 5$ or $6 \mod 13$; in other words, $\mathcal{H}$ is the union of the four Kestenband arcs K_0, K_2, K_5 and K_6.

The unique blocking semioval of size 21 from [11] is constructed as the union of three Kestenband arcs. However, that is not the only minimal blocking set that can be constructed in this way. Note that $|K_i \cap \ell_j| = |K_i \cap \ell_{j+13}| = \ldots = |K_i \cap \ell_{j+13k}| = w_{K_i}(\ell_j)$ and hence, when studying the intersection of an arc K_i with the lines of the plane, we can limit ourselves to the lines $\ell_0 \ldots \ell_{12}$. Using the fact that $w_{K_i}(\ell_j) = w_{K_{i+1}}(\ell_{j-1})$ it becomes fairly easy to set up a table that shows the sizes of the intersections of the arcs and the lines of the plane. For example, for $\mathcal{H} = K_0 \cup K_2 \cup K_5 \cup K_6$, we have

	ℓ_0	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7	ℓ_8	ℓ_9	ℓ_{10}	ℓ_{11}	ℓ_{12}
K_0	1	2		2	1					2	1		1
K_2		2	1					2	1		1	1	2
K_5					2	1		1	1	2		2	1
K_6				2	1		1	1	2		2	1	
$\mathcal{H}$	1	4	1	4	4	1	1	4	4	4	4	4	4

The bottom row of the table contains the weights of the lines $\ell_0, \ldots, \ell_{12}$ (and hence implicitly of all lines). Because there is no entry 0, $\mathcal{H}$ is a blocking set. Also the minimality of this blocking set can be established from the table: the point P_0 lies on a line ℓ_0 of weight 1, the point P_2 lies on a line ℓ_2 of weight 1, etc. Applying the Singer cycle, we see that every point of K_0, K_2, K_5 and K_6 is essential. (We shall follow a similar reasoning on different tables in the proof of the theorem below, without explicit proof.)

Theorem 9.1. *Up to equivalence, there are three minimal blocking sets in* $\mathrm{PG}(2,9)$ *that consist of the union of three Kestenband arcs:*

1. $B_1 = K_0 \cup K_2 \cup K_4$ *with* $|G| = 7$ *and* $|\Gamma| = 14$*;*
2. $B_2 = K_0 \cup K_2 \cup K_8$ *with* $|G| = 21$ *and* $|\Gamma| = 42$ *(this is the blocking semioval from [11]);*
3. $B_3 = K_0 \cup K_2 \cup K_7$ *with* $|G| = 168$ *and* $|\Gamma| = 336$*.*

Proof. The tables below prove that these are minimal blocking sets.

	ℓ_0	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7	ℓ_8	ℓ_9	ℓ_{10}	ℓ_{11}	ℓ_{12}
K_0	1	2		2	1					2	1		1
K_2		2	1					2	1		1	1	2
K_4	1					2	1		1	1	2		2
B_1	2	4	1	2	1	2	1	2	2	3	4	1	5

	ℓ_0	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7	ℓ_8	ℓ_9	ℓ_{10}	ℓ_{11}	ℓ_{12}
K_0	1	2		2	1					2	1		1
K_2		2	1					2	1		1	1	2
K_8		2	1		1	1	2		2	1			
B_2	1	6	2	2	2	1	2	2	3	3	2	1	3

	ℓ_0	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7	ℓ_8	ℓ_9	ℓ_{10}	ℓ_{11}	ℓ_{12}
K_0	1	2		2	1					2	1		1
K_2		2	1					2	1		1	1	2
K_7			2	1		1	1	2		2	1		
B_3	1	4	3	3	1	1	1	4	1	4	3	1	3

The sizes of the automorphism groups were calculated by computer. □

Now, let us look at some additional properties of $B = K_{10} \cup K_{12} \cup K_4$, which is equivalent to B_3, the example with the largest automorphism group. First, note that $\delta = P_i P_j P_k$ forms a polar triangle w.r.t. $\mathcal{H}$ if and only if $i \in \ell_{27j}$, $j \in \ell_{27k}$ and $k \in \ell_{27i}$, in other words, if and only if $i + 27j$, $j + 27k$ and $k + 27i$ are all in ℓ_0.

Is it possible for the polar triangle δ to lie entirely in B? And, more specifically, with $i \in K_4$, $j \in K_{10}$ and $k \in K_{12}$? Put $i = 4 + 13a$, $j = 10 + 13b$ and $k = 12 + 13c$ with $a, b, c \in \{0, \ldots, 6\}$ and then this is the case if and only if $1 + 13(a - b)$, $61 + 13(b - c)$ and $29 + 13(c - a)$ are all in ℓ_0. This can easily be checked to be equivalent to $(a = b \wedge c = a + 4) \vee (b = c \wedge a = b + 2)$. We see that there are two ways to partition the points of B into 7 polar triangles:

a	b	c	δ_1	a	b	c	δ_2
0	0	4	4,10,64	2	0	0	30,10,12
1	1	5	17,23,77	3	1	1	43,23,25
2	2	6	30,36,90	4	2	2	56,36,38
3	3	0	43,49,12	5	3	3	69,49,51
4	4	1	56,62,25	6	4	4	82,62,64
5	5	2	69,75,38	0	5	5	4,75,77
6	6	3	82,88,51	1	6	6	17,88,90

Notice that any point of B lies on two polar triangles: one of type δ_1, one of type δ_2. Let us define an incidence geometry $\mathscr{G} = (\mathscr{P}, \mathscr{L}, I)$ with $\mathscr{P}$ the polar triangles of type δ_1, $\mathscr{L}$ the polar triangles of type δ_2, and an element of $\mathscr{P}$ is incident with an element of $\mathscr{L}$ if the polar triangles have a point of $\mathrm{PG}(2,9)$ in common. For example, $\mathscr{P} \ni (4,10,64)I(4,75,77) \in \mathscr{L}$ because they both contain P_4. It is now easy to show that $\mathscr{G}$ is in fact a Fano plane.

There is another way of looking at this blocking set. For this, let us look at the following construction of a generalized hexagon by Tits [30] (notation and formulation taken from [23, Construction 8]). Recall that a generalized n-gon of order (s,t) is a rank 2 point-line geometry whose incidence graph has diameter n and girth $2n$, where each vertex corresponding to a point has valency $t+1$ and each vertex corresponding to a line has valency $s+1$.

Theorem 9.2. [30] *Let $\mathscr{U}$ be a Hermitian curve in* $\mathrm{PG}(2,9)$. *Let $\mathscr{P}$ be the set of points off $\mathscr{U}$ and let $\mathscr{L}$ be the set of polar triangles with respect to $\mathscr{U}$. Then $(\mathscr{P}, \mathscr{L}, I_{nat})$ is a generalized hexagon of order* $(2,2)$ *isomorphic to the dual of* $\mathbf{H}(2)$.

It is now easy to see that the following holds:

Theorem 9.3. *Let $\mathscr{U}$ be a Hermitian curve in* $\mathrm{PG}(2,9)$. *Let $\mathscr{P}$ be the set of 21 points of $B = K_{10} \cup K_{12} \cup K_4$ and let $\mathscr{L}$ be the set of 14 polar triangles that lie entirely in B. Then $(\mathscr{P}, \mathscr{L}, I_{nat})$ is a generalized hexagon of order* $(2,1)$.

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$\|B\|$	13			15			16			17			18		
#PGL	1			2			3			132			30726		
#PΓL	1			2			3			91			15855		
	$\|\Gamma\|$	$\|G\|$	#	$\|\Gamma\|$	$\|G\|$	#	$\|\Gamma\|$	$\|G\|$	#	$\|\Gamma\|$	$\|G\|$	#	$\|\Gamma\|$	$\|G\|$	#
	11232	5616	1	120	60	1	6	3	1	1	1	14	1	1	14263
				192	96	1	16	8	1	2	1	20	2	1	795
							72	36	1	2	2	24	2	2	570
										4	2	17	3	3	15
										4	4	2	4	2	144
										6	6	1	4	4	11
										8	4	5	6	3	11
										12	6	2	6	6	11
										16	8	3	8	4	15
										24	12	1	8	8	1
										32	16	1	12	6	9
										48	24	1	16	8	4
													18	9	1
													24	12	2
													32	16	1
													48	24	1
													144	72	1

$\|B\|$	19			20			21			22		
#PGL	524394			4544050			12508783			10899207		
#PΓL	263904			2276093			6259366			5453644		
	$\|\Gamma\|$	$\|G\|$	#	$\|\Gamma\|$	$\|G\|$	#	$\|\Gamma\|$	$\|G\|$	#	$\|\Gamma\|$	$\|G\|$	#
	1	1	259106	1	1	2263708	1	1	6247527	1	1	5442318
	2	1	3255	2	1	7850	2	1	9820	2	1	7794
	2	2	1320	2	2	4218	2	2	1704	2	2	3019
	3	3	32	3	3	7	3	3	164	3	3	196
	4	2	110	4	2	261	4	2	93	4	2	219
	4	4	30	4	4	22	4	4	19	4	4	26
	6	3	21	6	3	4	6	3	20	6	3	34
	6	6	2	6	6	1	6	6	2	6	6	3
	8	4	21	8	4	11	8	4	8	8	4	21
	12	6	2	10	10	1	8	8	1	12	6	5
	16	8	2	12	6	4	12	6	1	16	8	4
	18	9	1	16	8	4	14	7	1	18	18	1
	36	18	1	24	12	2	16	8	3	32	16	1
	192	96	1				32	16	1	36	18	1
							42	21	1	48	24	1
							336	168	1	64	32	1

$\|B\|$	23			24			25			26			28		
#PGL	2252493			65702			195			6			2		
#PΓL	1127161			33011			100			5			2		
	$\|\Gamma\|$	$\|G\|$	#	$\|\Gamma\|$	$\|G\|$	#	$\|\Gamma\|$	$\|G\|$	#	$\|\Gamma\|$	$\|G\|$	#	$\|\Gamma\|$	$\|G\|$	#
	1	1	1125123	1	1	32551	1	1	86	4	4	1	216	108	1
	2	1	1800	2	1	299	2	1	2	8	4	2	12096	6048	1
	2	2	202	2	2	88	2	2	2	24	12	1			
	4	2	23	3	3	47	3	3	7	48	24	1			
	4	4	7	4	2	9	4	2	2						
	8	4	5	4	4	1	6	3	1						
	16	8	1	6	3	8									
				6	6	4									
				12	6	2									
				24	12	1									
				768	384	1									

TABLE 1. Number of (non-trivial) minimal blocking sets of PG(2, 9) according to size of the set and order of automorphism group, up to equivalence. Recall that there are no minimal blocking sets of size 14 [4] or 27 [15].

$\|B\|$	$\|\Gamma\|$	$\|G\|$	
10	933120	466560	Line.
13	11232	5616	Baer subplane.
15	120	60	Icosidodecahedron. Secants to a complete 6-arc. Theorem 4.1.
15	192	96	Projective triangle.
16	16	8	Theorem 5.10.
16	72	36	Theorem 6.1.
17	32	16	[12, Lemma 13.2(ii)]
17	48	24	Theorem 7.1.
18	12	6	Construction 5.1 with $\|S\| = 5$.
18	48	24	Theorem 7.1. Construction 5.1 with $\|S\| = 5$.
18	144	72	Conic-tangent [8, Example 1].
19	192	96	Theorem 5.9.
20	6	3	Construction 5.1 with $\|S\| = 4$ and $S \cup \{Q\}$ contains a Baer subline through Q.
20	12	6	Construction 5.1 with $\|S\| = 4$.
21	14	7	Theorem 9.1(1).
21^s	42	21	Theorem 9.1(2). Secants to a complete 7-arc. [11].
21	336	168	Subhexagon. Theorem 9.3. Theorem 9.1(3).
22^s	4	2	Section 8 [28, Theorem 5.2].
22	6	3	Construction 5.1 with $\|S\| = 3$ and $S \cup Q$ not a Baer subline.
22^s	8	4	Section 8 [28, Theorem 5.2].
22	36	18	Construction 5.1 with $\|S\| = 3$ and $S \cup Q$ a Baer subline, see also Theorem 5.5.
22^s	48	24	[10].
22	64	32	Theorem 5.6.
23^s	4	2	Section 8 [9].
23^s	8	4	Section 8 [28, Theorem 4.2].
23^s	16	8	Section 8 [28, Theorem 4.2].
24	12	6	Construction 5.1 with $\|S\| = 2$, see also Theorem 5.5.
24^s	768	384	Triangle without vertices [8, Example 3].
25	4	2	Theorem 5.13, B_1.
26	8	4	Theorem 5.13, B_2 with $Q_1 \neq P_\infty$.
26	24	12	Theorem 5.13, B_2 with $Q_1 = P_\infty$.
26	48	24	Construction 5.1 with $\|S\| = 1$, see also Theorem 5.5.
28^s	216	108	Union of conics unital
28^s	12096	6048	Hermitian curve.

TABLE 2. Descriptions of selected minimal blocking sets of PG$(2,9)$, including all those with a full automorphism group of size at least 42, and some related examples. The superscript ^{s} indicates that the blocking set is a semioval (see Section 8).

$\vert B\vert$	21			22			23			24			25			26			28		
$\#_{\mathrm{PGL}}$	1			104			3645			5440			54			3			2		
$\#_{\mathrm{P\Gamma L}}$	1			60			1832			2751			28			2			2		
	$\vert\Gamma\vert$	$\vert G\vert$	#	$\vert\Gamma\vert$	$\vert G\vert$	#	$\vert\Gamma\vert$	$\vert G\vert$	#	$\vert\Gamma\vert$	$\vert G\vert$	#	$\vert\Gamma\vert$	$\vert G\vert$	#	$\vert\Gamma\vert$	$\vert G\vert$	#	$\vert\Gamma\vert$	$\vert G\vert$	#
	42	21	1	1	1	41	1	1	1812	1	1	2655	1	1	21	4	4	1	216	108	1
				2	1	9	2	1	16	2	1	55	2	1	1	8	4	1	12096	6048	1
				3	3	3	2	2	1	2	2	14	3	3	5						
				4	2	3	4	2	1	3	3	19	6	3	1						
				6	3	2	8	4	1	4	2	4									
				8	4	1	16	8	1	6	3	1									
				48	24	1				6	6	1									
										24	12	1									
										768	384	1									

TABLE 3. All blocking semiovals in $\mathrm{PG}(2,9)$, arranged by size of the set $|B|$, size of the collineation group $|\Gamma|$ and size of the projective automorphism group $|G|$.