A note on the representation theorem for nullnorms on bounded lattices

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Abstract

Recently, we presented a complete representation for nullnorms on bounded lattices, a class of ordered semigroups, in terms of two order-preserving maps, a triangular conorm, a triangular norm and a conditionally associative function, which together satisfy seven conditions. In this paper, we illustrate that each of these seven conditions is independent of the other ones by rewriting the seventh condition. We also discuss in which cases the sixth or the seventh condition becomes dependent.

Keywords: Lattice; Nullnorm; Representation; Ordered semigroup

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1. Introduction

In 2015, Karaçal et al. [3] straightforwardly generalized the notion of a nullnorm from the real unit interval [1] to the more general setting of a

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bounded lattice as follows. Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and $a \in L$. A binary operation $N\colon L^2 \to L$ is called a *nullnorm* on L if, for any $x, y, z, w \in L$, the following conditions are fulfilled:

(i) $N(x, y) = N(y, x)$ (commutativity);

(ii) If $x \leq y$ and $z \leq w$, then $N(x, z) \leq N(y, w)$ (increasingness);

(iii) $N(N(x, y), z) = N(x, N(y, z))$ (associativity);

(iv) $N(0, x) = x$ for any $x \in [0, a]$, and $N(1, x) = x$ for any $x \in [a, 1]$.

It is easy to see that $N(a, x) = a$ for all $x \in L$. Therefore, a is called the annihilator of N. Note that the increasingness and associativity indicate that a nullnorm is an ordered semigroup [2]. Since the introduction of nullnorms, their structure has received wide interest. The present authors [6] introduced three classes of nullnorms on a bounded lattice and represented their members in terms of so-called beam and dual beam operations. In the meantime, Sun and Liu [4] introduced a broader class of nullnorms that only take values that are comparable with the annihilator and presented an elegant representation of these nullnorms in terms of two order-preserving maps, a t-conorm and a t-norm. Recently, we presented a complete representation theorem for nullnorms on bounded lattices [5].

Before stating this theorem, let us recall several notations. We denote by $\mathcal{N}(a)$ the class of all nullnorms on L with annihilator a and by $\mathcal{F}(a)$ (resp. $\mathcal{G}(a)$) the class of all maps $f : L \to [0, a]$ (resp. $g : L \to [a, 1]$) satisfying the following conditions:

(i) f (resp. g) is order-preserving;

(ii) $f(x) = x$, for all $x \in [0, a]$ (resp. $g(x) = x$, for all $x \in [a, 1]$).

For other notations and concepts, we refer to [5]. Our representation theorem reads as follows.

Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice, $a \in L\backslash\{0,1\}$ and N be a binary operation on L. Then $N \in \mathcal{N}(a)$ if and only if there exist $A_0, B_0, C \subseteq I_a^2$, $f \in \mathcal{F}(a)$, $g \in \mathcal{G}(a)$, a t-conorm S on $[0,a]$, a t-norm T on $[a,1]$, and a commutative and increasing function $\mathcal{C} : C \to I_a$ such that

$$N(x,y) = \begin{cases} S(f(x), f(y)) & , \text{ if } \ (x,y) \in A \\ T(g(x), g(y)) & , \text{ if } \ (x,y) \in B \\ \mathcal{C}(x,y) & , \text{ if } \ (x,y) \in C, \end{cases}$$

where

$A = ([0,a] \times L) \cup (L \times [0,a]) \cup A_0$ is a symmetric lower set;

$B = ([a,1] \times L) \cup (L \times [a,1]) \cup B_0$ is a symmetric upper set;

$C = L^2 \setminus (A \cup B) = I_a^2 \setminus (A_0 \cup B_0)$ is symmetric.

In addition, for all $(x,y,z) \in L^3$, the following seven conditions hold:

(i-1) $(x,y) \in A_0$ implies $T(g(x), g(y)) = a$;

(i-2) $(x,y) \in B_0$ implies $S(f(x), f(y)) = a$;

(i-3) $(x,y) \in C$ implies

$$S(f(x), f(y)) = f(\mathcal{C}(x,y)) \text{ and } T(g(x), g(y)) = g(\mathcal{C}(x,y));$$

(ii-1) $y \leq z$, $(x,y) \in A$ and $(x,z) \in C$ imply $S(f(x), f(y)) \leq \mathcal{C}(x,z)$;

(ii-2) $y \leq z$, $(x,y) \in C$ and $(x,z) \in B$ imply $\mathcal{C}(x,y) \leq T(g(x), g(z))$;

(iii-1) If $(x,y) \in C$ and $(y,z) \in C$, then $(\mathcal{C}(x,y), z) \in C$ if and only if $(x, \mathcal{C}(y,z)) \in C$;

(iii-2) $\mathcal{C}$ is conditionally associative, *i.e.*, if $(x, y) \in C$, $(y, z) \in C$ and $(\mathcal{C}(x, y), z) \in C$, then $\mathcal{C}(\mathcal{C}(x, y), z) = \mathcal{C}(x, \mathcal{C}(y, z))$.

It seems natural to wonder whether each and every one of the above seven conditions is independent of the other ones. As a supplement to [5], the aim of this paper is to address this independence problem. Observing that condition (iii-2) implicitly implies condition (iii-1), thus (iii-1) being dependent, we rewrite condition (iii-2) as follows:

(iii-2)$'$ if $(x, y) \in C$, $(y, z) \in C$, $(\mathcal{C}(x, y), z) \in C$ and $(x, \mathcal{C}(y, z)) \in C$, then $\mathcal{C}(\mathcal{C}(x, y), z) = \mathcal{C}(x, \mathcal{C}(y, z))$.

Since condition (iii-2)$'$ does not seem to imply condition (iii-1), condition (iii-1) may be independent due to this slight change. Note that condition (iii-2)$'$ and condition (iii-2) are equivalent if condition (iii-1) is fulfilled. As a result, condition (iii-2) can be replaced by condition (iii-2)$'$ in the representation theorem. In the next section, we study the independence of the seven conditions (i-1)-(iii-1) and (iii-2)$'$.

2. Independence of the seven conditions in the representation theorem

First, we illustrate that in general each of the seven conditions (i-1)-(iii-1) and (iii-2)$'$ is independent of the other ones. Interestingly, the independence of the seven conditions except condition (iii-1) can be verified on the same lattice shown in Figure 1, *i.e.*, one of the two non-distributive five-element lattices.

The following example shows that condition (i-1) is independent.

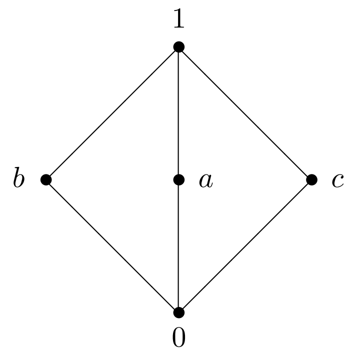

Figure 1: Hasse diagram of the lattice L in Examples 2.1–2.4.

Example 2.1. Let $L = \{0, a, b, c, 1\}$ be the lattice shown in Figure 1. Consider $A_0 = \{(b, c), (c, b)\}$, $B_0 = \emptyset$, $C = \{(b, b), (c, c)\}$, $f \in \mathcal{F}(a)$ defined by $f(x) = x \wedge a$, $g \in \mathcal{G}(a)$ defined by $g(x) = x \vee a$, the t-conorm $S = \vee|_{[0,a]^2}$ on $[0, a]$, the t-norm $T = \wedge|_{[a,1]^2}$ on $[a, 1]$, and the commutative and increasing function $\mathcal{C} : C \to I_a$ defined by

$$\mathcal{C}(x, y) = \begin{cases} b & , \text{if } (x, y) = (b, b) \\ c & , \text{if } (x, y) = (c, c). \end{cases}$$

Note that condition (i-1) fails since

$$T(g(b), g(c)) = 1 \neq a.$$

However, the other six conditions hold.

Similarly, we can illustrate that condition (i-2) is independent by exchanging A_0 and B_0 in Example 2.1.

The following example shows that condition (i-3) is independent.

Example 2.2. Let $L = \{0, a, b, c, 1\}$ be the lattice shown in Figure 1. Consider $A_0 = \{(b, c), (c, b), (c, c)\}$, $B_0 = \emptyset$, $C = \{(b, b)\}$, $f \in \mathcal{F}(a)$ defined by

$f(x) = x \wedge a$, $g \in \mathcal{G}(a)$ defined by

$$g(x) = \begin{cases} a & \text{, if } \ x = c \\ x \vee a & \text{, otherwise,} \end{cases}$$

the t-conorm $S = \vee|_{[0,a]^2}$ on $[0, a]$, the t-norm $T = \wedge|_{[a,1]^2}$ on $[a, 1]$, and the commutative and increasing function $\mathcal{C} : C \to I_a$ defined by $\mathcal{C}(b, b) = c$. Note that condition (i-3) fails since

$$T(g(b), g(b)) = 1 \neq a = g(\mathcal{C}(b, b)).$$

However, the other six conditions hold.

The following example shows that condition (ii-1) is independent.

Example 2.3. Let $L = \{0, a, b, c, 1\}$ be the lattice shown in Figure 1. Consider $A_0 = B_0 = \emptyset$, $C = I_a^2$, $f \in \mathcal{F}(a)$ defined by

$$f(x) = \begin{cases} a & \text{, if } \ x \in \{b, c\} \\ x \wedge a & \text{, otherwise,} \end{cases}$$

$g \in \mathcal{G}(a)$ defined by $g(x) = x \vee a$, the t-conorm $S = \vee|_{[0,a]^2}$ on $[0, a]$, the t-norm $T = \wedge|_{[a,1]^2}$ on $[a, 1]$, and the commutative and increasing function $\mathcal{C} : C \to I_a$ defined by $\mathcal{C}(x, y) \equiv b$. Note that condition (ii-1) fails since

$$S(f(b), f(0)) = a \not\leq b = \mathcal{C}(b, b).$$

However, the other six conditions hold.

Similarly, one can easily illustrate that condition (ii-2) is independent by modifying f and g in Example 2.3.

The following example shows that condition (iii-2)′ is independent.

Example 2.4. Let $L = \{0, a, b, c, 1\}$ be the lattice shown in Figure 1. Consider $A_0 = B_0 = \emptyset$, $C = I_a^2$, $f \in \mathcal{F}(a)$ defined by $f(x) = x \wedge a$, $g \in \mathcal{G}(a)$ defined by $g(x) = x \vee a$, the t-conorm $S = \vee|_{[0,a]^2}$ on $[0, a]$, the t-norm $T = \wedge|_{[a,1]^2}$ on $[a, 1]$, and the commutative and increasing function $\mathcal{C} : C \to I_a$ defined by

$$\mathcal{C}(x, y) = \begin{cases} c & \text{, if } (x, y) = (b, b) \\ b & \text{, if } (x, y) \in \{(c, c), (b, c), (c, b)\}. \end{cases}$$

Note that condition (iii-2)$'$ fails since

$$\mathcal{C}(\mathcal{C}(b, c), c) = \mathcal{C}(b, c) = b \neq c = \mathcal{C}(b, b) = \mathcal{C}(b, \mathcal{C}(c, c)).$$

However, the other six conditions hold.

The following example shows that condition (iii-1) is independent. Different from the above four examples, here I_a consists of four elements since such an example does not exist when I_a consists of no more than three elements (see Proposition 2.10).

Example 2.5. Let $L = \{0, a, b, c, d, s, u, v, w, 1\}$ be the lattice shown in Figure 2. Consider
$A_0 = \{(s, s), (u, u), (v, v), (s, u), (u, s), (s, v), (v, s), (s, w), (w, s), (u, v), (v, u)\}$,
$B_0 = \emptyset$, $C = \{(w, w), (u, w), (w, u), (v, w), (w, v)\}$, $f \in \mathcal{F}(a)$ defined by $f(x) = x \wedge a$, $g \in \mathcal{G}(a)$ defined by

$$g(x) = \begin{cases} a & \text{, if } x = s \\ b & \text{, if } x = u \\ c & \text{, if } x = v \\ d & \text{, if } x = w \\ x \vee a & \text{, otherwise,} \end{cases}$$

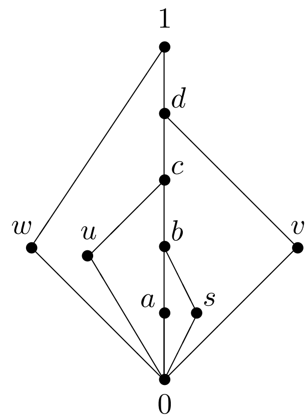

Figure 2: Hasse diagram of the lattice L in Example 2.5.

T	a	b	c	d	1
a	a	a	a	a	a
b	a	a	a	a	b
c	a	a	a	b	c
d	a	a	b	c	d
1	a	b	c	d	1

Table 1: The t-norm in Example 2.5.

the t-conorm $S = \vee|_{[0,a]^2}$ on $[0, a]$, the t-norm T on $[a, 1]$ shown in Table 1, and the commutative and increasing function $\mathcal{C} : C \to I_a$ defined by

$$\mathcal{C}(x,y) = \begin{cases} v & , \text{ if } (x,y) = (w,w) \\ s & , \text{ if } (x,y) \in \{(u,w),(w,u)\} \\ u & , \text{ if } (x,y) \in \{(v,w),(w,v)\}. \end{cases}$$

Note that $(v, w) \in C$, $(w, w) \in C$ and $(\mathcal{C}(v, w), w) = (u, w) \in C$, while $(v, \mathcal{C}(w, w)) = (v, v) \notin C$. Thus, condition (iii-1) fails. However, the other six conditions hold.

Although none of the seven conditions (i-1)-(iii-1) and (iii-2)$'$ is dependent in general, this may not be the case under some additional assumptions. Next, we study when condition (iii-1) or condition (iii-2)$'$ is dependent.

In the following, we always assume that $(L, \leq, \wedge, \vee, 0, 1)$ is a bounded lattice with $a \in L \setminus \{0, 1\}$, $A_0, B_0, C \subseteq I_a^2$ such that $A = ([0, a] \times L) \cup (L \times [0, a]) \cup A_0$ is a symmetric lower set, $B = ([a, 1] \times L) \cup (L \times [a, 1]) \cup B_0$ is a symmetric upper set, $C = I_a^2 \setminus (A_0 \cup B_0) \neq \emptyset$, $f \in \mathcal{F}(a)$, $g \in \mathcal{G}(a)$, S is a t-conorm on $[0, a]$, T is a t-norm on $[a, 1]$, and $\mathcal{C} : C \to I_a$ is a commutative and increasing function.

The values of f and g on I_a are influenced by C.

Proposition 2.6. *For any $(x, y) \in C$, the following results hold:*

(1) *if condition* (ii-1) *is satisfied, then $f(x) \vee f(y) \leq \mathcal{C}(x, y) \wedge a < a$. If condition* (i-3) *is also satisfied, $(y, z) \in C$ and $(\mathcal{C}(x, y), z) \in C$, then $f(\mathcal{C}(y, z)) \leq \mathcal{C}(\mathcal{C}(x, y), z) \wedge a < a$;*

(2) *if condition* (ii-2) *is satisfied, then $a < \mathcal{C}(x, y) \vee a \leq g(x) \wedge g(y)$. If condition* (i-3) *is also satisfied, $(y, z) \in C$ and $(\mathcal{C}(x, y), z) \in C$, then $a < \mathcal{C}(\mathcal{C}(x, y), z) \vee a \leq g(\mathcal{C}(y, z))$.*

Proof. We only prove (2) (the proof of (1) is similar). For any $(x, y) \in C$, it follows from condition (ii-2) that

$$\mathcal{C}(x, y) \leq T(g(x), g(1)) \wedge T(g(1), g(y)) = g(x) \wedge g(y).$$

Note that $\mathcal{C}(x, y) \in I_a$ and $a \leq g(x) \wedge g(y)$. Hence, we have

$$a < \mathcal{C}(x, y) \vee a \leq g(x) \wedge g(y).$$

It follows from conditions (ii-2) and (i-3) that

$$\mathcal{C}(\mathcal{C}(x, y), z) \leq T(g(g(y)), g(z)) = T(g(y), g(z)) = g(\mathcal{C}(y, z)).$$

Therefore, $a < \mathcal{C}(\mathcal{C}(x,y),z) \vee a \leq g(\mathcal{C}(y,z))$. $\square$

The following three propositions discuss when condition (iii-1) is dependent. Due to the symmetry of C and the commutativity of $\mathcal{C}$, condition (iii-1) can be equivalently stated as follows:

If $(x,y) \in C$, $(y,z) \in C$ and $(\mathcal{C}(x,y),z) \in C$, then $(x, \mathcal{C}(y,z)) \in C$.

Proposition 2.7. *Suppose that conditions* (i-1)-(i-3) *are satisfied and* $f(x) < a < g(x)$ *for all* $x \in I_a$. *Then condition* (iii-1) *holds.*

Proof. Suppose that $(x,y) \in C$, $(y,z) \in C$ and $(\mathcal{C}(x,y),z) \in C$, but $(x,\mathcal{C}(y,z)) \notin C$. Then $(x,\mathcal{C}(y,z)) \in A_0 \cup B_0$. We consider the case that $(x,\mathcal{C}(y,z)) \in A_0$ (the case $(x,\mathcal{C}(y,z)) \in B_0$ is similar). It follows from conditions (i-1) and (i-3) that

$$a = T(g(x), g(\mathcal{C}(y,z))) = T(g(x), T(g(y), g(z))).$$

On the other hand, since $(\mathcal{C}(x,y),z) \in C$, it follows from condition (i-3) that

$$g(\mathcal{C}(\mathcal{C}(x,y),z)) = T(g(\mathcal{C}(x,y)), g(z)) = T(T(g(x),g(y)), g(z)).$$

So, the associativity of T implies that $g(\mathcal{C}(\mathcal{C}(x,y),z)) = a$, a contradiction. $\square$

Consider the set $P_C = \{x \in I_a \mid (\exists y \in I_a)((x,y) \in C)\}$.

Proposition 2.8. *Suppose that conditions* (i-1)-(ii-2) *are satisfied and* $P_C = I_a$. *Then condition* (iii-1) *holds.*

Proof. It follows from Propositions 2.6 and 2.7. $\square$

The following lemma will be used in the subsequent proposition.

Lemma 2.9. *Suppose that conditions* (i-1)-(ii-2) *are satisfied,* $(x,y) \in C$, $(y,z) \in C$, $(\mathcal{C}(x,y),z) \in C$ *and* $\mathcal{C}(x,y) \in \{x,y\}$. *Then* $(x,\mathcal{C}(y,z)) \in C$.

Proof. Suppose that $(x, \mathcal{C}(y,z)) \notin C$. Then $(x, \mathcal{C}(y,z)) \in A_0 \cup B_0$. We consider the case that $(x, \mathcal{C}(y,z)) \in A_0$ (the case $(x, \mathcal{C}(y,z)) \in B_0$ is similar). It follows from condition (ii-2) that

$$\mathcal{C}(x,y) \leq T(g(\mathcal{C}(x,y)), g(1)) = g(\mathcal{C}(x,y)).$$

Hence, it follows from conditions (ii-2), (i-3) and (i-1) that

$$\begin{aligned}\mathcal{C}(\mathcal{C}(x,y),z) &\leq T(g(g(\mathcal{C}(x,y))), g(z)) \\ &= T(g(\mathcal{C}(x,y)), g(z)) \\ &= T(g(x), g(\mathcal{C}(y,z))) = a,\end{aligned}$$

a contradiction. □

Proposition 2.10. *Suppose that conditions* (i-1)-(ii-2) *are satisfied and* I_a *consists of no more than three elements. Then condition* (iii-1) *holds.*

Proof. Suppose that $(x,y) \in C$, $(y,z) \in C$ and $(\mathcal{C}(x,y), z) \in C$. We need to prove that $(x, \mathcal{C}(y,z)) \in C$. We consider four cases:

(1) The case $x = z$.

In this case, it follows from the symmetry of C and the commutativity of $\mathcal{C}$ that $(x, \mathcal{C}(y,z)) \in C$.

(2) The case $x \neq z \neq y \neq x$.

In this case, we have $P_C = I_a$. Hence, it follows from Proposition 2.8 that $(x, \mathcal{C}(y,z)) \in C$.

(3) The case $y = x \neq z$.

That is to say, $(x,x) \in C$, $(x,z) \in C$ and $(\mathcal{C}(x,x), z) \in C$. We need to prove that $(x, \mathcal{C}(x,z)) \in C$.

If $\mathcal{C}(x,z) \in \{x,z\}$, then it trivially holds that $(x, \mathcal{C}(x,z)) \in C$.

If $\mathcal{C}(x,z) \notin \{x,z\}$, then it follows from Proposition 2.6 that

$$f(\mathcal{C}(x,z)) < a < g(\mathcal{C}(x,z)).$$

Since $(x,z) \in C$, it also follows from Proposition 2.6 that

$$f(x), f(z) < a < g(x), g(z).$$

Noting that I_a consists of no more than three elements, it follows from Proposition 2.7 that $(x, \mathcal{C}(x,z)) \in C$.

(4) The case $x \neq z = y$.

That is to say, $(x,z) \in C$, $(z,z) \in C$ and $(\mathcal{C}(x,z), z) \in C$. We need to prove that $(x, \mathcal{C}(z,z)) \in C$.

If $\mathcal{C}(x,z) \notin \{x,z\}$, then we have $P_C = I_a$. Hence, it follows from Proposition 2.8 that $(x, \mathcal{C}(z,z)) \in C$.

If $\mathcal{C}(x,z) \in \{x,z\}$, then it follows from Lemma 2.9 that $(x, \mathcal{C}(z,z)) \in C$.

We conclude that condition (iii-1) holds. □

The following proposition shows when condition (iii-2)′ is dependent.

Proposition 2.11. *Suppose that condition* (i-3) *is satisfied and* $f|_{I_a} : I_a \to [0,a]$ *or* $g|_{I_a} : I_a \to [a,1]$ *is injective. Then condition* (iii-2)′ *holds.*

Proof. We only consider the case that $f|_{I_a}$ is injective (the case when $g|_{I_a}$ is injective is similar). For $(x,y) \in C$, $(y,z) \in C$, $(\mathcal{C}(x,y), z) \in C$, and $(x, \mathcal{C}(y,z)) \in C$, it follows from condition (i-3) that

$$\begin{aligned} f(\mathcal{C}(\mathcal{C}(x,y),z)) &= S(S(f(x), f(y)), f(z)) \\ &= S(f(x), S(f(y), f(z))) = f(\mathcal{C}(x, \mathcal{C}(y,z))), \end{aligned}$$

together with the injectivity of $f|_{I_a}$ implying that $\mathcal{C}(\mathcal{C}(x, y), z) = \mathcal{C}(x, \mathcal{C}(y, z))$.
□

3. Conclusion

As a supplement to the complete representation theorem for nullnorms on bounded lattices [5], we have illustrated that each of the seven conditions in this theorem is independent of the other ones by rewriting the seventh condition (see Examples 2.1–2.5). We have also discussed when the sixth or the seventh condition becomes dependent (see Propositions 2.7, 2.8, 2.10 and 2.11).

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