Fuzzy structures induced by fuzzy betweenness relations

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Abstract

In the setting of complete residuated lattices, we explore the relationships between the recently introduced fuzzy betweenness relations and three important mathematical notions: fuzzy interval operators, fuzzy partial orders and fuzzy Peano–Pasch spaces. After recalling the concept of a fuzzy betweenness relation w.r.t. a fuzzy equivalence relation, we prove that the resulting category is isomorphic to that of geometric fuzzy interval spaces w.r.t. the same fuzzy equivalence relation. Next, we construct a fuzzy partial order via a fuzzy betweenness relation w.r.t. a fuzzy equivalence relation and analyze their relationships in depth. Finally, taking a field as underlying set, we introduce the concept of a fuzzy betweenness field. Furthermore, in the setting of completely distributive lattices, we provide an interesting example showing that a vector space over a fuzzy betweenness field can yield a fuzzy Peano–Pasch space.

Keywords: Fuzzy betweenness relation, Fuzzy interval operator, Fuzzy E-partial order, Fuzzy betweenness field, Fuzzy Peano-Pasch space

1. Introduction

1.1. Origin

The origin of betweenness relations can be traced back to the letter by Gauss to Bolyai in 1832, pointing out the absence of betweenness postulates in the Euclidean treatment [14]. Elimination of this defect did not happen until 50 years later when Pasch [26] initiated his investigations. A broad range of follow-up studies focused on connections with other mathematical notions, such as algebraic structures [23, 38, 43], metrics [6], order and topology [19, 25, 54], antimatroids [10], convex structures [17, 39] and the like. In a more general contemporary approach, Pitcher and Smiley [34] systematically investigated a wide variety of settings in which

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betweenness relations arise. This development was not only motivated by the need for an abstract theory of betweenness relations, but also by the need to unify betweenness relations on different mathematical structures. Among the many different definitions of a betweenness relation, we adopt the definition in the sense of Hedliková ([17], [39, Further Topics 4.22]), which has been proven to characterize geometric interval operators and to have close connections with other geometric properties in the framework of (abstract) convex spaces. For more research on betweenness relations, we refer to [2, 8, 11, 33, 36].

More than half a century ago, Zadeh introduced the seminal concept of a fuzzy set [51]. Since then, theoretical and applied research on fuzzy sets has been growing steadily, one of the central notions being that of a binary fuzzy relation. Fuzzy order relations, in particular, have contributed to the highly intensive development of new areas of fuzzy mathematics, including fuzzy topology, fuzzy convex structures, fuzzy rough sets, fuzzy formal concept analysis, to name but a few. However, as far as we know, research on ternary fuzzy relations such as fuzzy betweenness relations is extremely limited [20, 37, 53]. Similar to the classical crisp case, when a ternary fuzzy relation fulfills certain axioms, we will call it a fuzzy betweenness relation. Jacas and Recasens [20] introduced a notion of fuzzy betweenness relation valued in the real unit interval equipped with a strict Archimedean t-norm T. In particular, it was shown that there is a one-to-one correspondence between these fuzzy betweenness relations and separating T-equivalences. Shi and Shi [37] investigated lattice-valued betweenness relations motivated by the theory of fuzzy convex structures. Zhang et al. [53, 55] studied fuzzy betweenness relations valued in a bounded lattice equipped with a t-norm and discussed the connection with fuzzy order relations and metrics.

In recent years, the study of fuzzy convex structures has witnessed an increasing interest, and this from points of view, such as topology [27, 48, 50, 57], convergence theory [32, 49, 56], category theory [21, 22, 30], interval theory [28, 31, 47], geometry [9, 44, 45, 46], and so on. From the perspective of fuzzy order relations, Li and Shi [24] studied some properties of L-fuzzifying convex structures induced by L-orders, where L is a completely distributive lattice. It was shown that an L-fuzzifying convexity induced by an L-order is L-fuzzifying JHC (a geometric property of fuzzy convex structures) with arity of at most 2, and that an L-fuzzifying convexity is an L-fuzzifying antimatroid and that its segment operator is a geometric L-interval operator whenever the L-order satisfies strong antisymmetry. In addition, considering an integral commutative quantale $\mathcal{Q}$ as underlying lattice, Wang and Shi [41] presented essential connections among $\mathcal{Q}$-fuzzifying convex structures, $\mathcal{Q}$-fuzzifying interval operators and $\mathcal{Q}$-preorders. These studies also reflect that there exist important relationships among fuzzy order relations, fuzzy interval operators and fuzzy convex structures.

1.2. Aims and outline

Motivated by the above-mentioned works (related to fuzzy betweenness relations, fuzzy order relations, fuzzy interval operators, fuzzy convex structures, etc.), in this paper we explore the relationships among fuzzy betweenness relations and fuzzy interval operators, fuzzy partial orders, and fuzzy Peano–Pasch spaces (a special class of fuzzy interval spaces). Zhang et al. [53] gave several definitions of fuzzy betweenness relations, the most general of which is based on a fuzzy equivalence relation E, and is called a *fuzzy E-betweenness relation*. Considering the corresponding fuzzy interval operators, we aim to propose a suitable notion of fuzzy interval operator that allows to establish connections with fuzzy E-betweenness relations. In fact, the answer will turn out to be a geometric fuzzy interval operator w.r.t. a fuzzy equivalence relation E, which will be called an *E-geometric fuzzy interval operator*, see Definition 3.8. Since fuzzy interval operators and fuzzy orders are also inextricably linked, we will proceed with a discussion of the relationships between fuzzy E-betweenness relations and fuzzy E-partial orders. Explicitly, we give necessary and sufficient conditions for a ternary fuzzy relation to be a fuzzy E-betweenness relation in terms of fuzzy E-partial orders, see Theorem 4.5. Further, based on the mentioned geometric background, we propose the notion of a fuzzy betweenness field and show that the fuzzy interval operator induced by a vector space over a fuzzy betweenness field satisfies geometric properties of the Peano type and the Pasch type, see Theorem 5.12.

This paper is organized as follows. In Section 2, we recall basic notions and notations concerning residuated lattices, fuzzy sets and fuzzy relations. In Section 3, using complete residuated lattices for expressing degrees of membership and relationship, we introduce the notions of fuzzy E-betweenness relation and E-geometric fuzzy interval space and show that they are categorically isomorphic. In Section 4, we first present a representation of fuzzy E-betweenness relations in terms of E-partial orders. Then we construct a fuzzy E-partial order using a ternary fuzzy relation and show that a special ternary fuzzy relation is a fuzzy E-betweenness relation if and only if its fuzzy E-partial order is E-antisymmetric and satisfies two additional conditions. Section 5 introduces the notion of a fuzzy betweenness field w.r.t. the crisp equality. We give an example showing that a vector space over a fuzzy betweenness field can also induce a fuzzy interval operator. In particular, considering a completely distributive lattice as underlying lattice, the fuzzy interval operator induced by a vector space over a fuzzy betweenness field satisfies the Peano property and the Pasch property. Some conclusions and future work are summarized in Section 6.

2. Preliminaries

In this section, we present the terminology and basic notions used throughout this paper. We refer to [16, 42] for more information concerning residuated lattices, to [1] for general category theory, and to [4, 15] for fuzzy sets and fuzzy relations.

2.1. Residuated lattices

As a logical algebra, a residuated lattice is an algebraic structure that is simultaneously a lattice. Ward and Dilworth [42] introduced the noncommutative version of a residuated lattice, but we will restrict to the commutative case in this paper.

Definition 2.1. A residuated lattice is an algebra $(L; \wedge, \vee, *, \rightarrow)$ of type $(2, 2, 2, 2)$ that satisfies:

(R1) $(L, \wedge, \vee)$ is a bounded lattice with top element 1 and bottom element 0;

(R2) $(L, *, 1)$ is a commutative monoid with identity 1 and the operation $*$ is isotone w.r.t. the lattice order in both arguments;

(R3) $a * b \leq c$ if and only if $b \leq a \to c$ for all $a, b, c \in L$.

If the underlying lattice L is complete, then it is called a complete residuated lattice.

In a complete residuated lattice L, the condition (R3) is equivalent to

(R3′) $a * (\bigvee_i b_i) = \bigvee_i (a * b_i)$, for all $a \in L$ and $(b_i)_{i \in I}$ in L.

Some basic properties of complete residuated lattices are gathered in the following proposition.

Proposition 2.2. *Let L be a complete residuated lattice. For any $a, b, c, d \in L$ and any two families $(a_i)_{i \in I}$ and $(b_i)_{i \in I}$ of L, the following statements hold:*

(1) $a \to b = 1$ *if and only if* $a \leq b$;

(2) $1 \to a = a$;

(3) $(a * b) \to c = a \to (b \to c) = b \to (a \to c)$;

(4) $(a \to b) * (c \to d) \leq (a * c) \to (b * d)$;

(5) $(a \to b) \leq (c \to a) \to (c \to b)$;

(6) $a \to \bigwedge_{i \in I} b_i = \bigwedge_{i \in I} (a \to b_i)$;

(7) $(\bigvee_{i \in I} a_i) \to b = \bigwedge_{i \in I} (a_i \to b)$.

A complete lattice L is called *completely distributive* [13] if for any nonempty family $(a_{jk})_{j \in J, k \in K(j)}$ in L, the identity

$$\bigwedge_{j \in J} \bigvee_{k \in K(j)} a_{jk} = \bigvee_{f \in M} \bigwedge_{j \in J} a_{j, f(j)}$$

holds, where M is the set of all choice functions $f : J \to \bigcup_{j \in J} K(j)$ with $f(j) \in K(j)$ for all $j \in J$. In a completely distributive complete lattice L, we can define

$$a \to c = \bigvee \{ b \in L \mid a \wedge b \leqslant c \}$$

for each $a, c \in L$. It then holds that $b \leq a \to c \Leftrightarrow a \wedge b \leq c$, *i.e.*, $(L; \wedge, \vee, \wedge, \to)$ is a complete residuated lattice. In other words, any completely distributive complete lattice is a particular complete residuated lattice with $* = \wedge$.

In order to give an elegant characterization of the completely distributive property of complete lattices, we recall the following definition.

Definition 2.3 ([12])**.** Let L be a complete lattice. An element $a \in L$ is said to be wedge below an element $b \in L$, denoted $a \lhd b$, if for any subset $D \subseteq L$, the relation $b \le \bigvee D$ always implies the existence of $d \in D$ such that $a \le d$.

Theorem 2.4 ([12, 13])**.** *A complete lattice L is completely distributive if and only if $a = \bigvee\{b \in L \mid b \lhd a\}$ for all $a \in L$.*

Proposition 2.5. *Let L be a completely distributive complete lattice. Then for any $a, b \in L$ and any family $(b_j)_{j\in J}$ of L, the following statements hold:*

(1) *$a \lhd b$ implies $a \le b$;*

(2) *$a \neq 0$ implies $0 \lhd a$;*

(3) *$a \lhd \bigvee_{j\in J} b_j$ if and only if $(\exists j \in J)(a \lhd b_j)$.*

Throughout this paper, L always denotes a complete residuated lattice, unless otherwise specified.

2.2. Fuzzy sets

A *fuzzy subset* of a set X over L, or simply a *fuzzy subset of* X, is a map φ from X to L, *i.e.*, $\varphi : X \longrightarrow L$. The value $\varphi(x)$ is interpreted as the membership degree of x in the fuzzy subset φ. Crisp subsets of X are considered as fuzzy subsets of X taking membership values in the set $\{0, 1\} \subseteq L$. The collection of all fuzzy subsets of X over L is denoted by L^X. The operators on L can be extended to L^X in a pointwise manner. Doing so, L^X is also a complete residuated lattice.

Let φ, ϕ be two fuzzy subsets of a set X. The map $\mathrm{sub}_X : L^X \times L^X \longrightarrow L$ defined by

$$\mathrm{sub}_X(\varphi, \phi) = \bigwedge_{x\in X} \varphi(x) \to \phi(x)\,,$$

is called the *fuzzy inclusion order* on L^X [4]. The value $\mathrm{sub}_X(\varphi, \phi)$ expresses the degree to which φ is contained in ϕ.

For each map $f : X \longrightarrow Y$ and $\varphi \in L^X$, $f^{\to}(\varphi)$ denotes the fuzzy subset of Y obtained by applying Zadeh's extension principle, *i.e.*, $f^{\to}(\varphi)(y) = \bigvee\{\varphi(x) \mid f(x) = y\}$ for each $y \in Y$. For each map $g : X \longrightarrow Y$ and $\phi \in L^Y$, $g^{\leftarrow}(\phi)$ denotes the fuzzy subset of X given by $g^{\leftarrow}(\phi)(x) = \phi(g(x))$ for all $x \in X$.

2.3. Fuzzy relations

Let X be a nonempty set. A binary fuzzy relation R on X is a map from $X \times X$ to L, *i.e.*, a fuzzy subset of $X \times X$. Similarly, a ternary fuzzy relation T on X is a map from $X \times X \times X$ to L, *i.e.*, a fuzzy subset of $X \times X \times X$. A crisp binary (resp. ternary) relation is a binary (resp. ternary) fuzzy relation that takes values only in the set $\{0, 1\}$, and if R (resp. T) is a crisp binary (resp. ternary) relation on X, then the expressions $R(x, y) = 1$ (resp. $(T(x, y, z) = 1)$ and $(x, y) \in R$ (resp. $(x, y, z) \in T$) have the same meaning. By fuzzy relations we usually mean binary fuzzy relations. We are interested in the following properties of fuzzy relations.

A fuzzy relation R on a set X is said to be:

- *reflexive* if $R(x,x) = 1$ for all $x \in X$;
- *symmetric* if $R(x,y) = R(y,x)$ for all $x, y \in X$;
- *antisymmetric* if $R(x,y) * R(y,x) \neq 0$ implies $x = y$ for all $x, y \in X$;
- *transitive* if $R(x,y) * R(y,z) \leq R(x,z)$ for all $x, y, z \in X$.

A reflexive and transitive fuzzy relation (resp. crisp relation) R on a set X is called a *fuzzy quasi-order* (resp. *quasi-order*), and in this case we call (X, R) a *fuzzy quasi-ordered set* (resp. *quasi-ordered set*). In some sources, quasi-orders and fuzzy quasi-orders are called preorders and fuzzy preorders, respectively, but here we use the original name introduced by Birkhoff [5]. Note that there exist alternative generalizations of the antisymmetry of a crisp binary relation. Here we adopt the one that was introduced in [7, 52].

We say that a map $f : (X, R_X) \longrightarrow (Y, R_Y)$ between two fuzzy quasi-ordered sets is order-preserving if

$$R_X(x_1, x_2) \leq R_Y(f(x_1), f(x_2))$$

for all $x_1, x_2 \in X$.

A symmetric fuzzy quasi-order on X is called a *fuzzy equivalence relation*. Using fuzzy equivalence relations, we can deal with more general notions of reflexivity and weak antisymmetry of fuzzy relations. Namely, if E is a fuzzy equivalence relation on a set X, then a fuzzy relation R on X is said to be:

- E-*reflexive* if $E(x,y) \leq R(x,y)$ for all $x \in X$;
- E-*antisymmetric* if $R(x,y) * R(y,x) \leq E(x,y)$ for all $x, y \in X$.

A fuzzy relation R on X is called a *fuzzy E-partial order* if it is E-reflexive, E-antisymmetric and transitive [7, 18]. In this case, the triplet (X, E, R) is called a *fuzzy E-partially ordered set*.

We say that a map $f : (X, E_X, R_X) \longrightarrow (Y, E_Y, R_Y)$ is order-preserving if

$$R_X(x_1, x_2) \leq R_Y(f(x_1), f(x_2))$$

and

$$E_X(x_1, x_2) \leq E_Y(f(x_1), f(x_2))$$

for all $x_1, x_2 \in X$.

3. Fuzzy betweenness relations and fuzzy interval operators

By endowing a fuzzy betweenness relation with a fuzzy equivalence relation E, we start this section by introducing the notion of a fuzzy E-betweenness relation. Although different axiomatic definitions of betweenness relations have been proposed in the literatures [2, 25], our fuzzification in this work is based on the one presented in [17, 39] since there is a bijection between geometric interval operators and betweenness relations, as mentioned in the introduction.

Definition 3.1. Let E be a fuzzy equivalence relation on a set X. A fuzzy E-betweenness relation on X is a ternary fuzzy relation B such that

(FEB1) $B(x, y, z) = B(z, y, x)$ for all $x, y, z \in X$;

(FEB2) $E(y, z) \leq B(x, y, z)$ for all $x, y, z \in X$;

(FEB3) $B(x, y, z) * B(x, z, y) \leq E(y, z)$ for all $x, y, z \in X$;

(FEB4) $B(o, x, y) * B(o, y, z) \leq B(o, x, z)$ for all $o, x, y, z \in X$;

(FEB5) $B(o, x, y) * B(o, y, z) \leq B(x, y, z)$ for all $o, x, y, z \in X$.

For a fuzzy E-betweenness relation B on a set X, the triplet (X, E, B) is called a fuzzy E-betweenness set.

In the particular case that E is the crisp equality, fuzzy E-betweenness relations are simply called fuzzy betweenness relations, and can be described by the following definition.

Definition 3.2. A fuzzy betweenness relation on X is a ternary fuzzy relation B such that

(FB1) $B(x, y, z) = B(z, y, x)$ for all $x, y, z \in X$;

(FB2) $B(x, y, y) = 1$ for all $x, y \in X$;

(FB3) $B(x, y, z) * B(x, z, y) \neq 0$ implies $y = z$, for all $x, y, z \in X$;

(FB4) $B(o, x, y) * B(o, y, z) \leq B(o, x, z)$ for all $o, x, y, z \in X$;

(FB5) $B(o, x, y) * B(o, y, z) \leq B(x, y, z)$ for all $o, x, y, z \in X$.

For a fuzzy betweenness relation B on a set X, the pair (X, B) is called a fuzzy betweenness set.

Intuitively speaking, the value $B(x, y, z)$ is interpreted as the degree to which the element y is between the elements x and z.

Remark 3.3. Zhang et al. [53] introduced related concepts of fuzzy (E-)betweenness relations with a focus on the representability in terms of a family of fuzzy orders. It is easy to see that Definition 3.1 (resp. Definition 3.2) contains the additional axiom (FEB5) (resp. (FB5)). The main reason is that we will focus on the connection with fuzzy interval operators which play an important role in the theory of fuzzy convex structures. Since there is no confusion possible here, we will still use the term fuzzy (E-)betweenness relation in our work.

A map $f : (X, E_X, B_X) \longrightarrow (Y, E_Y, B_Y)$ is said to be *betweenness-preserving* if

$$B_X(x, y, z) \leq B_Y(f(x), f(y), f(z))$$

and

$$E_X(x, y) \leq E_Y(f(x), f(y))$$

for all $x, y, z \in X$.

We write **FEBet** (resp. **FBet**) for the category of fuzzy E-betweenness sets (resp. fuzzy betweenness sets) as objects and betweenness-preserving maps as morphisms.

When $L = \{0, 1\}$, Definition 3.2 reduces the following definition of a betweenness relation.

Definition 3.4. A betweenness relation on X is a ternary relation B such that

(B1) $(x, y, z) \in B \Leftrightarrow (z, y, x) \in B$ for all $x, y, z \in X$;

(B2) $(x, y, y) \in B$ for all $x, y \in X$;

(B3) $(x, y, z) \in B$ and $(x, z, y) \in B$ imply $y = z$ for all $x, y, z \in X$;

(B4) $(o, x, y) \in B$ and $(o, y, z) \in B$ imply $(o, x, z) \in B$ for all $o, x, y, z \in X$;

(B5) $(o, x, y) \in B$ and $(o, y, z) \in B$ imply $(x, y, z) \in B$ for all $o, x, y, z \in X$.

For a betweenness relation B on a set X, the pair (X, B) is called a betweenness set.

Proposition 3.5 ([53])**.** *Let E be a fuzzy equivalence relation on a set X and B be a fuzzy E-betweenness relation on X. Then*

(1) $B(x, x, y) = B(x, y, y) = 1$ *for all* $x, y \in X$*;*

(2) $B(x, y, x) = E(y, x)$ *for all* $x, y \in X$*.*

Interval operators provide a useful tool to describe geometric properties of convex structures [39]. With the development of fuzzy convex structures, interval operators have been generalized to the fuzzy setting in recent years (see, e.g., [29, 41, 47]). In what follows, we will generalize geometric fuzzy interval operators to E-geometric fuzzy interval operators and then study their relationships with fuzzy E-betweenness relations. Let us first recall the definition of a geometric fuzzy interval operator.

Definition 3.6 ([47])**.** A map $I : X \times X \longrightarrow L^X$ is called a fuzzy interval operator on X if

(FI1) $I(x, y)(x) = I(x, y)(y) = 1$ for all $x, y \in X$;

(FI2) $I(x, y) = I(y, x)$ for all $x, y \in X$.

For a fuzzy interval operator I on X, the pair (X, I) is called a fuzzy interval space.

A map $f : (X, I_X) \longrightarrow (Y, I_Y)$ between fuzzy interval spaces is said to be *interval-preserving* if

$$f^{\rightarrow}(I_X(x, y)) \leq I_Y(f(x), f(y))$$

for all $x, y \in X$.

We write **FIS** for the category of fuzzy interval spaces as objects and interval-preserving maps as morphisms.

Considering the geometric properties of fuzzy interval operators, Wang and Shi [41] introduced the concept of geometric fuzzy interval operators.

Definition 3.7 ([41])**.** A fuzzy interval operator I on a set X is called geometric if

(GFI1) $\mathrm{sub}_X(I(x, x), I(x, y)) = 1$ and $I(x, x)(y) \neq 1$ when $x \neq y$ for all $x, y \in X$;

(GFI2) $I(x, y)(z) = \mathrm{sub}_X(I(x, z), I(x, y))$ for all $x, y, z \in X$;

(GFI3) $I(x, y)(z) \leq I(x, z)(o) \rightarrow I(o, y)(z)$ for all $x, y, z, o \in X$.

For a geometric fuzzy interval operator I on X, the pair (X, I) is called a geometric fuzzy interval space.

The full subcategory of **FIS** composed of geometric fuzzy interval spaces is denoted **GFIS**.

By equipping a fuzzy equivalence E on geometric fuzzy interval operators, we present the following definition.

Definition 3.8. Let E be a fuzzy equivalence relation on a set X. A fuzzy interval operator I on X is said to be E-geometric if

(EGFI) $\mathrm{sub}_X(I(x, x), I(x, y)) = 1$ and $I(x, x)(y) = E(x, y)$ for all $x, y \in X$;

(GFI2) $I(x, y)(z) = \mathrm{sub}_X(I(x, z), I(x, y))$ for all $x, y, z \in X$;

(GFI3) $I(x, y)(z) \leq I(x, z)(o) \rightarrow I(o, y)(z)$ for all $x, y, z, o \in X$.

For an E-geometric fuzzy interval operator I on X, the pair (X, E, I) is called an E-geometric fuzzy interval space.

Remark 3.9. Definition 3.8 is introduced based on a fuzzy equivalence relation E. If E is the crisp equality, then this definition reduces to Definition 3.7 in the sense of Wang and Shi [41].

A map $f : (X, E_X, I_X) \longrightarrow (Y, E_Y, I_Y)$ is said to be *interval-preserving* if

$$f^{\rightarrow}(I_X(x, y)) \leq I_Y(f(x), f(y))$$

and

$$E_X(x, y) \leq E_Y(f(x), f(y))$$

for all $x, y \in X$.

We write **EGFIS** for the category of E-geometric fuzzy interval spaces as objects and interval-preserving maps as morphisms.

In the following, we will study the relationships between **EFBet** (resp. **FBet**) and **EGFIS** (resp. **GFIS**).

Proposition 3.10. *Let (X, E, I) be an E-geometric fuzzy interval space and define a ternary fuzzy relation $B_I : X \times X \times X \longrightarrow L$ by*

$$B_I(x, y, z) = I(x, z)(y).$$

Then B_I is a fuzzy E-betweenness relation on X.

Proof. We verify that B_I satisfies (FEB1)–(FEB5).

(FEB1) Straightforward.

(FEB2) Let $x, y, z \in X$. Then

$$\begin{aligned} B_I(x, y, z) = I(x, z)(y) &= I(z, x)(y) && \text{(by (FI2))} \\ &\geq I(z, z)(y) && \text{(by (EGFI))} \\ &= E(z, y). && \text{(by (EGFI))} \end{aligned}$$

(FEB3) Let $x, y, z \in X$. Then

$$\begin{aligned} B_I(x, y, z) * B_I(x, z, y) &= I(x, z)(y) * I(x, y)(z) \\ &\leq I(y, y)(z) && \text{(by (GFI3))} \\ &= E(z, y). && \text{(by (EGFI))} \end{aligned}$$

(FEB4) Let $o, x, y, z \in X$. By (GFI2), we have

$$I(o, z)(y) \leq \mathrm{sub}_X(I(o, y), I(o, z)) \leq I(o, y)(x) \to I(o, z)(x),$$

which implies

$$I(o, y)(x) * I(o, z)(y) \leq I(o, z)(x).$$

Hence

$$B_I(o, x, y) * B_I(o, y, z) = I(o, y)(x) * I(o, z)(y) \leq I(o, z)(x) = B_I(o, x, z).$$

(FEB5) Let $o, x, y, z \in X$. By (GFI3), we know that

$$I(o, z)(y) * I(o, y)(x) \leq I(x, z)(y).$$

Thus

$$B_I(o, x, y) * B_I(o, y, z) = I(o, y)(x) * I(o, z)(y) \leq I(x, z)(y) = B_I(x, y, z).$$

The proof is completed. □

When E is the crisp equality, we obtain the following result as a corollary.

Corollary 3.11. *Let (X, I) be a geometric fuzzy interval space. Then the ternary fuzzy relation B_I is a fuzzy betweenness relation on X.*

Proposition 3.12. *If $f : (X, E_X, I_X) \longrightarrow (Y, E_Y, I_Y)$ is interval-preserving, then $f : (X, E_X, B_{I_X}) \longrightarrow (Y, E_Y, B_{I_Y})$ is betweenness-preserving.*

Proof. Since $f : (X, E_X, I_X) \longrightarrow (Y, E_Y, I_Y)$ is interval-preserving, we have

$$f^{\rightarrow}(I_X(x, z)) \leq I_Y(f(x), f(z))$$

for all $x, z \in X$. Then for any $y \in X$, it follows that

$$\begin{aligned} B_{I_Y}(f(x), f(y), f(z)) &= I_Y(f(x), f(z))(f(y)) \\ &\geq f^{\rightarrow}(I_X(x, z))(f(y)) \\ &= \bigvee\{I_X(x, z)(w) \mid f(w) = f(y)\} \\ &\geq I_X(x, z)(y) \\ &= B_{I_X}(x, y, z), \end{aligned}$$

as desired. □

By Propositions 3.10 and 3.12, we obtain a functor $\mathbb{F}$: **EGFIS** $\longrightarrow$ **EFBet** as follows:

$$\mathbb{F} : \begin{cases} \textbf{EGFIS} & \longrightarrow & \textbf{EFBet} \\ (X, E, I) & \longmapsto & (X, E, B_I), \\ f & \longmapsto & f. \end{cases}$$

Conversely, we will induce an E-geometric fuzzy interval operator from a fuzzy E-betweenness relation.

Proposition 3.13. *Let (X, E, B) be a fuzzy E-betweenness set and define the map $I_B : X \times X \longrightarrow L^X$ by*

$$I_B(x, y)(o) = B(x, o, y).$$

Then I_B is an E-geometric fuzzy interval operator.

Proof. We verify that I_B satisfies (FI1), (FI2) and (EGFI)–(GFI3).

(FI1) It follows from Proposition 3.5(1) that

$$I_B(x, y)(x) = B(x, x, y) = 1 \text{ and } I_B(x, y)(y) = B(x, y, y) = 1$$

for all $x, y \in X$.

(FI2) It is clear by (FEB1).

(EGFI) Let $x, y \in X$. Then

$$\mathrm{sub}_X(I_B(x, x), I_B(x, y)) = \bigwedge_{z \in X} B(x, z, x) \rightarrow B(x, z, y)$$

$$
\begin{aligned}
&= \bigwedge_{z\in X} E(z,x) \to B(x,z,y) && \text{(by Proposition 3.5(2))}\\
&= 1, && \text{(by (FEB1) \& (FEB2))}
\end{aligned}
$$

and it follows from Proposition 3.5(2) that

$$I_B(x,x)(y) = B(x,y,x) = E(y,x) = E(x,y).$$

(GFI2) The key is to prove that $I_B(x,y)(z) \leq \mathrm{sub}_X(I_B(x,z), I_B(x,y))$ for all $x,y,z \in X$. In fact,

$$
\begin{aligned}
\mathrm{sub}_X(I_B(x,z), I_B(x,y)) &= \bigwedge_{w\in X} I_B(x,z)(w) \to I_B(x,y)(w)\\
&= \bigwedge_{w\in X} B(x,w,z) \to B(x,w,y)\\
&\geq \bigwedge_{w\in X} B(x,z,y) && \text{(by (FEB4))}\\
&= I_B(x,y)(z).
\end{aligned}
$$

(GFI3) Let $o,x,y,z \in X$. By (FEB5), we have

$$I_B(x,z)(o) * I_B(x,y)(z) = B(x,o,z) * B(x,z,y) \leq B(o,z,y) = I_B(o,y)(z),$$

which implies $I_B(x,y)(z) \leq I_B(x,z)(o) \to I_B(o,y)(z)$. □

When E is the crisp equality, we obtain the following result.

Corollary 3.14. *Let (X,B) be a fuzzy betweenness set. Then I_B is a geometric fuzzy interval operator.*

Proposition 3.15. *If $f : (X,E_X,B_X) \longrightarrow (Y,E_Y,B_Y)$ is betweenness-preserving, then $f : (X,E_X,I_{B_X}) \longrightarrow (Y,E_Y,I_{B_Y})$ is interval-preserving.*

Proof. Let $x,y \in X$. Then for any $w \in Y$, we have

$$
\begin{aligned}
f^{\rightarrow}(I_{B_X}(x,y))(w) &= \bigvee\{I_{B_X}(x,y)(z) \mid f(z) = w\}\\
&= \bigvee\{B_X(x,z,y) \mid f(z) = w\}\\
&\leq \bigvee\{B_Y(f(x),f(z),f(y)) \mid f(z) = w\}\\
&= B_Y(f(x),w,f(y))\\
&= I_{B_Y}(f(x),f(y))(w).
\end{aligned}
$$

By the arbitrariness of w, we obtain that f is interval-preserving. □

By Propositions 3.13 and 3.15, we obtain a functor $\mathbb{G}:\mathbf{EFBet}\longrightarrow\mathbf{EFGIS}$ as follows:

$$\mathbb{G}:\begin{cases}\mathbf{EFBet} & \longrightarrow & \mathbf{EFGIS}\\ (X,E,B) & \longmapsto & (X,E,I_B),\\ f & \longmapsto & f.\end{cases}$$

Theorem 3.16. **EFBet** *and* **EFGIS** *are isomorphic.*

Proof. Since $\mathbb{F}$ and $\mathbb{G}$ are both concrete functors, it remains to show that $\mathbb{F}\circ\mathbb{G}=\mathbb{I}_{\mathbf{EFBet}}$ and $\mathbb{G}\circ\mathbb{F}=\mathbb{I}_{\mathbf{EFGIS}}$. To that end, it suffices to verify that (1) $B_{I_B}=B$ and (2) $I_{B_I}=I$.

For (1), let $x,y,z\in X$. Then

$$B_{I_B}(x,y,z)=I_B(x,z)(y)=B(x,y,z),$$

For (2), let $x,y,z\in X$. Then

$$I_{B_I}(x,y)(z)=B_I(x,z,y)=I(x,y)(z).$$

This completes the proof. □

Considering the relationships between geometric fuzzy interval spaces and fuzzy betweenness sets, we obtain the following result as a corollary.

Corollary 3.17. **FBet** *and* **FGIS** *are isomorphic.*

4. Fuzzy *E*-betweenness relations and fuzzy *E*-partial orders

Wang and Shi [41] established the connections between fuzzy base-point orders and geometric fuzzy interval operators. On the other hand, Zhang et al. [53] extensively discussed the connections between fuzzy betweenness relations and fuzzy orders. In this section, we will explore these connections by equipping these fuzzy structures with a fuzzy equivalence relation. To this end, the following definition is necessary.

Definition 4.1 ([3, 53])**.** A ternary fuzzy relation T on a set X is said to be middle compatible with a fuzzy equivalence relation E on X if

$$E(x,y)*T(o,y,z)\leq T(o,x,z)$$

for all $o,x,y,z\in X$.

Now, let us construct a fuzzy E-betweenness relation from a fuzzy E-partially ordered set and show its relationship with the middle compatibility of E.

Theorem 4.2. *Let (X,E,R) be a fuzzy E-partially ordered set and define a ternary fuzzy relation $B_R:X\times X\times X\longrightarrow L$ by*

$$B_R(x,y,z)=(R(x,y)\vee R(z,y))*(R(y,z)\vee R(y,x)).$$

Then B_R is a fuzzy E-betweenness relation if and only if it is middle compatible with E.

Proof. Necessity. The proof can be found in [53, Proposition 4].

Sufficiency. We verify that B_R satisfies (FEB1)–(FEB5) as follows:

(FEB1) Straightforward.

(FEB2) Let $x, y, z \in X$. Since B_R is middle compatible with E and $B_R(x, z, z) = 1$, it follows that $E(y, z) = E(y, z) * B_R(x, z, z) \leq B_R(x, y, z)$.

(FEB3) Let $x, y, z \in X$. By the transitivity and the E-antisymmetry of R, we then have

$$\begin{aligned}
& B_R(x,y,z) * B_R(x,z,y) \\
= & (R(x,y) \vee R(z,y)) * (R(y,z) \vee R(y,x)) * (R(x,z) \vee R(y,z)) * (R(z,y) \vee R(z,x)) \\
= & [R(x,y) * R(y,z) * R(x,z) * R(z,y)] \vee [R(x,y) * R(y,z) * R(x,z) * R(z,x)] \vee \\
& [R(x,y) * R(y,z) * R(y,z) * R(z,y)] \vee [R(x,y) * R(y,z) * R(y,z) * R(z,x)] \vee \\
& [R(z,y) * R(y,z) * R(x,z) * R(z,y)] \vee [R(z,y) * R(y,z) * R(x,z) * R(z,x)] \vee \\
& [R(z,y) * R(y,z) * R(y,z) * R(z,y)] \vee [R(z,y) * R(y,z) * R(y,z) * R(z,x)] \vee \\
& [R(x,y) * R(y,x) * R(x,z) * R(z,y)] \vee [R(x,y) * R(y,x) * R(x,z) * R(z,x)] \vee \\
& [R(x,y) * R(y,x) * R(y,z) * R(z,y)] \vee [R(x,y) * R(y,x) * R(y,z) * R(z,x)] \vee \\
& [R(z,y) * R(y,x) * R(x,z) * R(z,y)] \vee [R(z,y) * R(y,x) * R(x,z) * R(z,x)] \vee \\
& [R(z,y) * R(y,x) * R(y,z) * R(z,y)] \vee [R(z,y) * R(y,x) * R(y,z) * R(z,x)] \\
\leq & [R(y,z) * R(z,y)] \vee [R(y,z) * 1 * R(z,y)] \vee [1 * 1 * R(y,z) * R(z,y)] \vee \\
& [1 * R(y,z) * R(z,y)] \vee [1 * R(y,z) * 1 * R(z,y)] \vee [R(z,y) * R(y,z) * 1 * 1] \vee \\
& [1 * 1 * R(y,z) * R(z,y)] \vee [R(z,y) * R(y,z) * 1 * 1] \vee [1 * R(y,z) * R(z,y)] \vee [R(y,z) * R(z,y)] \vee \\
& [1 * 1 * R(y,z) * R(z,y)] \vee [1 * R(y,z) * R(z,y)] \vee [R(z,y) * R(y,z) * 1] \vee [R(z,y) * R(y,z) * 1] \vee \\
& [1 * 1 * R(y,z) * R(z,y)] \vee [R(z,y) * 1 * R(y,z) * 1] \\
= & R(y,z) * R(z,y) \\
\leq & E(y,z).
\end{aligned}$$

(FEB4) Let $o, x, y, z \in X$. By the transitivity of R, we then have

$$\begin{aligned}
& B_R(o,x,y) * B_R(o,y,z) \\
= & (R(o,x) \vee R(y,x)) * (R(x,y) \vee R(x,o)) * (R(o,y) \vee R(z,y)) * (R(y,z) \vee R(y,o)) \\
= & [R(o,x) * R(x,y) * R(o,y) * R(y,z)] \vee [R(o,x) * R(x,y) * R(o,y) * R(y,o)] \vee \\
& [R(o,x) * R(x,y) * R(z,y) * R(y,z)] \vee [R(o,x) * R(x,y) * R(z,y) * R(y,o)] \vee \\
& [R(y,x) * R(x,y) * R(o,y) * R(y,z)] \vee [R(y,x) * R(x,y) * R(o,y) * R(y,o)] \vee \\
& [R(y,x) * R(x,y) * R(z,y) * R(y,z)] \vee [R(y,x) * R(x,y) * R(z,y) * R(y,o)] \vee \\
& [R(o,x) * R(x,o) * R(o,y) * R(y,z)] \vee [R(o,x) * R(x,o) * R(o,y) * R(y,o)] \vee \\
& [R(o,x) * R(x,o) * R(z,y) * R(y,z)] \vee [R(o,x) * R(x,o) * R(z,y) * R(y,o)] \vee \\
& [R(y,x) * R(x,o) * R(o,y) * R(y,z)] \vee [R(y,x) * R(x,o) * R(o,y) * R(y,o)] \vee \\
& [R(y,x) * R(x,o) * R(z,y) * R(y,z)] \vee [R(y,x) * R(x,o) * R(o,y) * R(y,o)] \\
\leq & (R(o,x) * R(x,z)) \vee (R(o,x) * R(x,o)) \vee (R(z,x) * R(x,z)) \vee (R(z,x) * R(x,o))
\end{aligned}$$

$$
\begin{aligned}
&= (R(o,x) \vee R(z,x)) * (R(x,z) \vee R(x,o)) \\
&= B_R(o,x,z).
\end{aligned}
$$

(FEB5) Let $o, x, y, z \in X$. By the transitivity of R, we then have

$$
\begin{aligned}
& B_R(o,x,y) * B_R(o,y,z) \\
&= (R(o,x) \vee R(y,x)) * (R(x,y) \vee R(x,o)) * (R(o,y) \vee R(z,y)) * (R(y,z) \vee R(y,o)) \\
&= [R(o,x) * R(x,y) * R(o,y) * R(y,z)] \vee [R(o,x) * R(x,y) * R(o,y) * R(y,o)] \vee \\
&\quad [R(o,x) * R(x,y) * R(z,y) * R(y,z)] \vee [R(o,x) * R(x,y) * R(z,y) * R(y,o)] \vee \\
&\quad [R(y,x) * R(x,y) * R(o,y) * R(y,z)] \vee [R(y,x) * R(x,y) * R(o,y) * R(y,o)] \vee \\
&\quad [R(y,x) * R(x,y) * R(z,y) * R(y,z)] \vee [R(y,x) * R(x,y) * R(z,y) * R(y,o)] \vee \\
&\quad [R(o,x) * R(x,o) * R(o,y) * R(y,z)] \vee [R(o,x) * R(x,o) * R(o,y) * R(y,o)] \vee \\
&\quad [R(o,x) * R(x,o) * R(z,y) * R(y,z)] \vee [R(o,x) * R(x,o) * R(z,y) * R(y,o)] \vee \\
&\quad [R(y,x) * R(x,o) * R(o,y) * R(y,z)] \vee [R(y,x) * R(x,o) * R(o,y) * R(y,o)] \vee \\
&\quad [R(y,x) * R(x,o) * R(z,y) * R(y,z)] \vee [R(y,x) * R(x,o) * R(o,y) * R(y,o)] \\
&\leq (R(x,y) * R(y,z)) \vee (R(x,y) * R(y,x)) \vee (R(z,y) * R(y,z)) \vee (R(z,y) * R(y,x)) \\
&= (R(x,y) \vee R(z,y)) * (R(y,z) \vee R(y,x)) \\
&= B_R(x,y,z).
\end{aligned}
$$

This completes the proof. □

Remark 4.3. For any partially ordered set $(X, \leq)$, Bankston [2], Pérez-Fernández and De Baets [33] constructed a betweenness relation on X via

$$B_{\leq} = \{(x,y,z) \in X^3 \mid x = y \text{ or } y = z \text{ or } x \leq y \leq z \text{ or } z \leq y \leq x\},$$

which is called an *order-betweenness relation*. Subsequently, Zhang et al. [53] generalized this construction to the fuzzy case. Concisely, for a given fuzzy E-partially ordered set (X, R), a fuzzy E-betweenness relation can be constructed in the following way.

$$B^R(x,y,z) = E(x,y) \vee E(y,z) \vee (R(x,y) * R(y,z)) \vee (R(z,y) * R(y,x)).$$

Actually, in the classical case, $B_{\leq}$ has another equivalent form. That is,

$$B_{\leq} = \{(x,y,z) \in X^3 \mid x \leq y \text{ or } z \leq y\} \cap \{(x,y,z) \in X^3 \mid\ y \leq x \text{ or } y \leq z\}.$$

So in Theorem 4.2, we considered the fuzzy counterpart of this formula,

$$B_R(x,y,z) = (R(x,y) \vee R(z,y)) * (R(y,z) \vee R(y,x)),$$

which is much simpler compared with B^R.

Next, we will consider how to construct a fuzzy E-partial order from a fuzzy E-betweenness relation. Let B be a ternary fuzzy relation on a set X. For every $o \in X$, define a binary fuzzy relation R_o by

$$R_o(x, y) = \bigwedge_{z \in X} B(o, z, x) \to B(o, z, y).$$

Proposition 4.4. *Let E be a fuzzy equivalence relation on a set X. Suppose that B is a ternary fuzzy relation on X satisfying* (FEB2) *and* (FEB4). *Then R_o is E-reflexive and transitive.*

Proof. *E-reflexivity*: Let $x, y, z \in X$. Then

$$\begin{aligned} R_o(x, y) &= \bigwedge_{z \in X} B(o, z, x) \to B(o, z, y) \\ &\geq \bigwedge_{z \in X} B(o, x, y) && \text{(by (FEB4))} \\ &\geq E(x, y). && \text{(by (FEB2))} \end{aligned}$$

Transitivity: Let $x, y \in X$. Then

$$\begin{aligned} & R_o(x, y) \to R_o(x, z) \\ &= \left(\bigwedge_{v \in X} B(o, v, x) \to B(o, v, y) \right) \to \left(\bigwedge_{u \in X} B(o, u, x) \to B(o, u, z) \right) \\ &\geq \bigwedge_{v \in X} \left((B(o, v, x) \to B(o, v, y)) \to (B(o, v, x) \to B(o, v, z)) \right) \\ &\geq \bigwedge_{v \in X} B(o, v, y) \to B(o, v, z) && \text{(by Proposition 2.2(5))} \\ &= R_o(y, z), \end{aligned}$$

which implies that $R_o(x, y) * R_o(y, z) \leq R_o(x, z)$. □

Theorem 4.5. *Let E be a fuzzy equivalence relation on a set X and B be a ternary fuzzy relation on X satisfying* (FEB1). *Then the following statements are equivalent:*

(1) *B is a fuzzy E-betweenness relation;*

(2) *For each $o \in X$, R_o is a fuzzy E-partial order and the following statements hold:*

  (i) *$R_o(o, x) = 1$ for all $x \in X$.*

  (ii) *$B(u, x, v) \leq R_u(y, x) \to R_v(x, y)$ for all $u, v, x, y \in X$.*

  (iii) *$B(x, y, y) = 1$ for all $x, y \in X$.*

Proof. (1) $\Longrightarrow$ (2): We first prove that R_o is a fuzzy E-partial order. By Proposition 4.4, it remains to show that R_o is E-antisymmetric. Let $x, y \in X$. Then

$$\begin{aligned}
& R_o(x,y) * R_o(y,x) \\
&= \left(\bigwedge_{u\in X} B(o,u,x) \to B(o,u,y)\right) * \left(\bigwedge_{u\in X} B(o,u,y) \to B(o,u,x)\right) \\
&\le (B(o,x,x) \to B(o,x,y)) * (B(o,y,y) \to B(o,y,x)) \\
&= B(o,x,y) * B(o,y,x) && \text{(by Proposition 3.5(1))} \\
&\le B(x,y,x) && \text{(by (FEB5))} \\
&= E(x,y). && \text{(by Proposition 3.5(2))}
\end{aligned}$$

For (i), let $x \in X$. Then

$$\begin{aligned}
R_o(o,x) &= \bigwedge_{z\in X} B(o,z,o) \to B(o,z,x) \\
&= \bigwedge_{z\in X} E(z,o) \to B(o,z,x) && \text{(by Proposition 3.5(2))} \\
&= \bigwedge_{z\in X} E(z,o) \to B(x,z,o) && \text{(by (FEB1))} \\
&= 1. && \text{(by (FEB2))}
\end{aligned}$$

For (ii), let $u, v, x, y \in X$. Then

$$\begin{aligned}
& R_u(y,x) \to R_v(x,y) \\
&= \left(\bigwedge_{z\in X} B(u,z,y) \to B(u,z,x)\right) \to \left(\bigwedge_{z\in X} B(v,z,x) \to B(v,z,y)\right) \\
&\ge (B(u,y,y) \to B(u,y,x)) \to \left(\bigwedge_{z\in X} B(v,z,x) \to B(v,z,y)\right) \\
&= B(u,y,x) \to \left(\bigwedge_{z\in X} B(v,z,x) \to B(v,z,y)\right) && \text{(by Proposition 3.5(1))} \\
&\ge B(u,y,x) \to \bigwedge_{z\in X} B(v,x,y) && \text{(by (FEB4))} \\
&= B(u,y,x) \to B(y,x,v) && \text{(by (FEB1))} \\
&\ge B(u,x,v). && \text{(by (FEB5))}
\end{aligned}$$

For (iii), it follows from Proposition 3.5.

(2) $\Longrightarrow$ (1): It suffices to prove that B satisfies (FEB2)–(FEB5).

(FEB2) Let $x, y, z \in X$. Since R_z is a fuzzy E-partial order, we have $E(y,z) \leq R_z(y,z)$. By the definition of R_z, we immediately obtain that

$$\begin{aligned}
E(y,z) &\leq \bigwedge_{u \in X} B(z,u,y) \to B(z,u,z) \\
&\leq B(z,y,y) \to B(z,y,z) \\
&= 1 \to B(z,y,z) && \text{(by (iii))} \\
&= B(z,y,z).
\end{aligned}$$

Thus,

$$\begin{aligned}
E(y,z) \to B(x,y,z) &\geq B(z,y,z) \to B(x,y,z) \\
&= B(z,y,z) \to B(z,y,x) && \text{(by (FEB1))} \\
&\geq \bigwedge_{w \in X} B(z,w,z) \to B(z,w,x) \\
&= R_z(z,x) \\
&= 1, && \text{(by (i))}
\end{aligned}$$

357 which is equivalent to $E(y,z) \leq B(x,y,z)$.

(FEB3) Let $x, y, z \in X$. Then

$$\begin{aligned}
& B(x,y,z) * B(x,z,y) \\
&= B(z,y,x) * B(y,z,x) && \text{(by (FEB1))} \\
&\leq (R_z(z,y) \to R_x(y,z)) * (R_y(y,z) \to R_x(z,y)) && \text{(by (ii))} \\
&\leq (R_z(z,y) * R_y(y,z)) \to (R_x(y,z) * R_x(z,y)) && \text{(by Proposition 2.2(4))} \\
&= 1 \to (R_x(y,z) * R_x(z,y)) && \text{(by (i))} \\
&\leq E(y,z). && \text{(by E-antisymmetry of R_x)}
\end{aligned}$$

(FEB4) Let $o, x, y, z \in X$. Then

$$\begin{aligned}
B(o,y,z) &= B(z,y,o) && \text{(by (FEB1))} \\
&\leq R_z(z,y) \to R_o(y,z) && \text{(by (ii))} \\
&= R_o(y,z) && \text{(by (i))} \\
&= \bigwedge_{u \in X} B(o,u,y) \to B(o,u,z) && \text{(by the defintion of R_o)} \\
&\leq B(o,x,y) \to B(o,x,z),
\end{aligned}$$

358 which implies $B(o,x,y) * B(o,y,z) \leq B(o,x,z)$.

(FEB5) Let $o, x, y, z \in X$. Then

$$B(o,y,z)$$

$$
\begin{aligned}
&\leq R_o(x,y) \to R_z(y,x) && \text{(by (ii))}\\
&= \left(\bigwedge_{u\in X} B(o,u,x) \to B(o,u,y)\right) \to \left(\bigwedge_{v\in X} B(z,v,y) \to B(z,v,x)\right) &&\\
&\leq \left(\bigwedge_{u\in X} B(o,x,y)\right) \to (B(z,y,y) \to B(z,y,x)) && \text{(by (FEB4))}\\
&= B(o,x,y) \to B(z,y,x) && \text{(by Proposition 3.5(1))}\\
&= B(o,x,y) \to B(x,y,z), && \text{(by (FEB1))}
\end{aligned}
$$

which implies $B(o,x,y) * B(o,y,z) \leq B(x,y,z)$. □

Remark 4.6. In [24] and [41], the authors discussed the properties of fuzzy partial orders from the point of view of geometric fuzzy interval operators. Here, we equipped fuzzy partial orders and geometric fuzzy interval operators with a fuzzy equivalence relation E.

By Theorem 4.5, we know that from any fuzzy E-betweenness relation B we can generate a fuzzy E-partial order R_x for any $x \in X$. Furthermore, we have the following result.

Proposition 4.7. *If $f : (X, E_X, B_X) \longrightarrow (Y, E_Y, B_Y)$ is betweenness-preserving, then $f : (X, E_X, R^X_x) \longrightarrow (Y, E_Y, R^Y_{f(x)})$ is order-preserving for all $x \in X$, where R^X_x and $R^Y_{f(x)}$ denote the fuzzy E-partial orders generated from B_X and B_Y, respectively.*

Proof. Let $x \in X$. Since $f : (X, B_X) \longrightarrow (Y, B_Y)$ is betweenness-preserving, it follows that

$$B_X(x, x_1, x_2) \leq B_Y(f(x), f(x_1), f(x_2))$$

for each $x_1, x_2 \in X$. Then we have

$$
\begin{aligned}
R^Y_{f(x)}(f(x_1), f(x_2)) &= \bigwedge_{z\in Y} B_Y(f(x), z, f(x_1)) \to B_Y(f(x), z, f(x_2)) &&\\
&\geq \bigwedge_{z\in Y} B_Y(f(x), f(x_1), f(x_2)) && \text{(by (FEB4))}\\
&\geq B_X(x, x_1, x_2) &&\\
&= B_X(x, x_1, x_1) \to B_X(x, x_1, x_2) &&\\
&\geq \bigwedge_{u\in X} B_X(x, u, x_1) \to B_X(x, u, x_2) &&\\
&= R^X_x(x_1, x_2). &&
\end{aligned}
$$

This completes the proof. □

5. Fuzzy betweenness fields and fuzzy Peano–Pasch spaces

In this section, we will focus on fuzzy betweenness relations, *i.e.*, fuzzy E-betweenness relations when E is the crisp equality. Specifically, we will consider the relationships between fuzzy betweenness relations and other geometric features of fuzzy interval operators. For convenience, we first recall the notion of a field.

A *field* [40] is an algebraic structure $(F, +_F, \cdot_F, -_F, ^{-1}, 0_F, 1_F)$ of type $(2, 2, 2, 1, 0, 0)$, such that 0_F^{-1} is not defined, $(F, +_F, -_F, 0_F)$ and $(F \setminus \{0_F\}, \cdot_F, ^{-1}, 1_F)$ are Abelian groups, and $\cdot_F$ is distributive over $+_F$.

From now on, we write the quintuple $(F, +, \cdot, 0_F, 1_F)$ for $(F, +_F, \cdot_F, -_F, ^{-1}, 0_F, 1_F)$ if no confusion can arise. For convenience, we write $\frac{a}{b}$ or a/b for $a \cdot_F b^{-1}$, and write $a - b$ for $a -_F b$, for all $a, b \in F$.

Considering a ternary relation on a field, the concept of a ternary field is proposed in the following way.

Definition 5.1 ([39])**.** A ternary field $(F, +, \cdot, 0_F, 1_F, T)$ consists of a field $(F, +, \cdot, 0_F, 1_F)$, together with a ternary relation T on F such that

(BF1) $(r, s, t) \in T$ implies $(r + a, s + a, t + a) \in T$ for all $r, s, t, a \in F$;

(BF2) $(r, s, t) \in T$ implies $(r \cdot a, s \cdot a, t \cdot a) \in T$ for all $r, s, t, a \in F$.

In a natural way, the notion of *a betweenness field* can be obtained by instantiating the ternary relation in Definition 5.1 with a betweenness relation according to Definition 3.4.

Definition 5.2. A betweenness field $(F, +, \cdot, 0_F, 1_F, B)$ consists of a field $(F, +, \cdot, 0_F, 1_F)$, together with a betweenness relation B on F such that (BF1) and (BF2) hold.

Now let us introduce the concept of *a fuzzy betweenness field* as the fuzzy counterpart of a betweenness field.

Definition 5.3. A fuzzy betweenness field $(F, +, \cdot, 0_F, 1_F, B)$ consists of a field $(F, +, \cdot, 0_F, 1_F)$, together with a fuzzy betweenness relation B on F such that

(FBF0) $B(0_F, 1_F, 0_F) = 0$;

(FBF1) $B(r, s, t) \leq B(r + a, s + a, t + a)$ for all $r, s, t, a \in F$;

(FBF2) $B(r, s, t) \leq B(r \cdot a, s \cdot a, t \cdot a)$ for all $r, s, t, a \in F$.

Remark 5.4. Comparing to Definitions 5.2 and 5.3, (FBF0) appears as an additional condition. However, in the classical case this condition trivially holds. Indeed, if $L = \{0, 1\}$, then $B(0_F, 1_F, 0_F) = 0$ means "$(0_F, 1_F, 0_F) \notin B$". Suppose the opposite, *i.e.*, $(0_F, 1_F, 0_F) \in B$. By (B2) in Definition 3.4, we get $(0_F, 0_F, 1_F) \in B$, and so we obtain from (B3) that $0_F = 1_F$, a contradiction. Condition (FBF0) will play an important role in the properties of fuzzy betweenness fields.

In the following lemma, we present some elementary properties of fuzzy betweenness fields.

Lemma 5.5. *Let* $(F, +, \cdot, 0_F, 1_F, B)$ *be a fuzzy betweenness field. Then the following statements hold:*

(1) $B(0_F, r, 1_F) = B(0_F, 1_F - r, 1_F)$ *for all* $r \in F$;

(2) $B(0_F, r, 1_F) * B(0_F, s, 1_F) \leq B(0_F, r \cdot s, 1_F)$ *for all* $r, s \in F$;

(3) $B(0_F, r, 0_F) = 0$ *for all* $r \in F \backslash \{0_F\}$;

(4) $B(r, s, t) = B(r \cdot a, s \cdot a, t \cdot a)$ *for all* $r, s, t \in F$ *and* $a \in F \backslash \{0_F\}$.

Proof. The proofs of (3) and (4) are trivial. We verify (1) and (2).

(1) Let $r \in F$. Then

$$B(0_F, r, 1_F) \leq B(0_F, -r, -1_F) \leq B(1_F, 1_F - r, 0_F) \leq B(0_F, 1_F - r, 1_F).$$

Analogously, we have $B(0_F, 1_F - r, 1_F) \leq B(0_F, r, 1_F)$. Hence $B(0_F, r, 1_F) = B(0_F, 1_F - r, 1_F)$.

(2) This is valid since

$$\begin{aligned} B(0_F, r, 1_F) * B(0_F, s, 1_F) &\leq B(0_F, r \cdot s, s) * B(0_F, s, 1_F) && \text{(by (FBF2))} \\ &\leq B(0_F, r \cdot s, 1_F) && \text{(by (FB4))} \end{aligned}$$

□

In the classical setting, a vector space over a betweenness field can induce an interval operator in a natural way. Here, we will consider its fuzzy counterpart. That is to say, we will show that a vector space over a fuzzy betweenness field induces a fuzzy interval operator.

Let V be a vector space over a fuzzy ternary field $(F, +, \cdot, 0_F, 1_F, B)$ and define the map $I_V : V \times V \longrightarrow L^V$ by

$$I_V(x, y)(z) = \bigvee_{\substack{z = t \cdot x + (1_F - t) \cdot y \\ t \in F}} B(0_F, t, 1_F).$$

Proposition 5.6. *If* V *is a vector space over a fuzzy betweenness field* $(F, +, \cdot, 0_F, 1_F, B)$, *then* I_V *is a fuzzy interval operator on* V.

Proof. We verify that I_V satisfies (FI1) and (FI2).

(FI1) Let $x, y \in V$. Then

$$I_V(x, y)(x) = \bigvee_{\substack{x = t \cdot x + (1_F - t) \cdot y \\ t \in F}} B(0_F, t, 1_F) \geq B(0_F, 1_F, 1_F) = 1.$$

Similarly, we have $I_V(x, y)(y) = 1$. Hence $I_V(x, y)(x) = I_V(x, y)(y) = 1$.

(FI2) Let $x, y, z \in V$. Then

$$\begin{aligned} I_V(x,y)(z) &= \bigvee_{\substack{z=t\cdot x+(1_F-t)\cdot y \\ t\in F}} B(0_F, t, 1_F) \\ &= \bigvee_{\substack{z=t\cdot x+(1_F-t)\cdot y \\ t\in F}} B(0_F, 1_F - t, 1_F) \qquad \text{(by Lemma 5.5(1))} \\ &= \bigvee_{\substack{z=(1_F-s)\cdot x+s\cdot y \\ s\in F}} B(0_F, s, 1_F) \\ &= I_V(y,x)(z). \end{aligned}$$

Hence, $I_V(x, y) = I_V(y, x)$ by the arbitrariness of z. □

In order to study partially ordered fields and related geometric facts, Prenowitz [35] proposed a postulate (denoted by (B3) in [35]) to describe the transitivity (called *overlap transitivity*) of a ternary relation T on X defined by:

(OT) $(a, r, s) \in T$ and $(r, s, t) \in T$ imply $(a, r, t) \in T$ and $(a, s, t) \in T$ for all $r, s, t, a \in X$.

Example 5.7. Let $(F, +, \cdot, 0_F, 1_F)$ be a field and define $T_F \subseteq F^3$ as

$$T_F = \{(r, s, t) \in F^3 \mid (\exists a \in F)(s = a \cdot r + (1_F - a) \cdot t)\}.$$

Then T_F is a ternary relation on F satisfying (OT).

Next, we generalize this postulate to the fuzzy case for the study of the Peano property and the Pasch property of fuzzy interval spaces generated by a vector space over a fuzzy betweenness field.

Definition 5.8. Let T be a ternary fuzzy relation on a set X. We say that T is overlap-transitive provided that

(FOT) $T(a, r, s) * T(r, s, t) \leq T(a, r, t) * T(a, s, t)$ for all $r, s, t, a \in X$.

It is easy to see that (FOT) reduces to (OT) when we replace L by $\{0, 1\}$.

Example 5.9. Let $(F, +, \cdot, 0_F, 1_F)$ be a field and μ be a fuzzy set on F. Define $T_F^\mu : F^3 \longrightarrow L$ by

$$T_F^\mu(r, s, t) = \begin{cases} \mu(s), & (\exists a \in F)(s = a \cdot r + (1_F - a) \cdot t), \\ 0, & \text{otherwise.} \end{cases}$$

Then T_F^μ is a ternary fuzzy relation on F that satisfies (FOT).

Interestingly, the interval spaces induced by vector spaces over fuzzy betweenness fields have nice geometric properties. Naturally, we shall discuss the geometric properties of fuzzy interval spaces induced by vector spaces over fuzzy betweenness fields, including the Peano property and the Pasch property. In order to get these ideal results, in the rest of this section, we need to assume that L is a completely distributive lattice.

Definition 5.10 ([44, 45]). A fuzzy interval space (X, I) is called

(1) a fuzzy Peano space if

$$I(b,c)(y) \wedge I(a,y)(z) \leq \bigvee_{x\in X} I(a,b)(x) \wedge I(c,x)(z)$$

for all $a, b, c, y, z \in X$;

(2) a fuzzy Pasch space if

$$I(b,e)(a) \wedge I(d,e)(c) \leq \bigvee_{x\in X} I(a,d)(x) \wedge I(b,c)(x)$$

for all $a, b, c, d, e \in X$.

When L is a completely distributive lattice, we obtain the following properties of a fuzzy betweenness field.

Lemma 5.11. *Let $(F, +, \cdot, 0_F, 1_F, B)$ be a fuzzy betweenness field such that B is overlap-transitive. Then*

(1) $B(0_F, r, 1_F) \leq B(0_F, r, 1_F + r) \wedge B(0_F, 1_F, 1_F + r)$ *for all* $r \in F$;

(2) $B(0_F, r, 1_F) \wedge B(0_F, s, 1_F) \leq B(0_F, r, r + s)$ *for all* $r, s \in F$.

Proof. (1) By Lemma 5.5(1) and (BF1), we have

$$B(0_F, r, 1_F) = B(0_F, 1_F - r, 1_F) \leq B(r, 1_F, 1_F + r).$$

Then it follows from (FOT) that

$$B(0_F, r, 1_F) = B(0_F, r, 1_F) \wedge B(r, 1_F, 1_F + r) \leq B(0_F, r, 1_F + r) \wedge B(0_F, 1_F, 1_F + r).$$

(2) By (FB1) and (FBF1), we get

$$B(0_F, s, 1_F) \leq B(1_F + r, r + s, r). \qquad \text{(i)}$$

By (1) and (FBF1), we get

$$B(0_F, r, 1_F) \leq B(0_F, r, 1_F + r) \wedge B(0_F, 1_F, 1_F + r)$$

$$
\begin{aligned}
&\leq B(0_F, r, 1_F + r) \\
&= B(1_F + r, r, 0_F). \qquad \text{(ii)}
\end{aligned}
$$

Combining (i) with (ii), we have

$$
\begin{aligned}
B(0_F, r, 1_F) \wedge B(0_F, s, 1_F) &\leq B(1_F + r, r + s, r) \wedge B(1_F + r, r, 0_F) \\
&\leq B(r + s, r, 0_F) && \text{(by (FB5))} \\
&= B(0_F, r, r + s). && \text{(by (FB1))}
\end{aligned}
$$

This completes the proof. □

Theorem 5.12. *Let V be a vector space over a fuzzy betweenness field $(F, +, \cdot, 0_F, 1_F, B)$ such that B is overlap-transitive. Then*

(1) *(V, I_V) is a fuzzy Peano space;*

(2) *(V, I_V) is a fuzzy Pasch space.*

Proof. By Lemma 5.5(1), we first show an alternative expression for $I_V(x, y)$:

$$
I_V(x, y)(z) = \bigvee_{\substack{t+s=1_F \\ z=t\cdot x+s\cdot y}} B(0_F, t, 1_F) \wedge B(0_F, s, 1_F).
$$

For (1), by Proposition 5.6, it remains to prove that

$$
I_V(b, c)(y) \wedge I_V(a, y)(z) \leq \bigvee_{x \in V} I_V(a, b)(x) \wedge I_V(c, x)(z) \tag{5.i}
$$

for all $a, b, c, y, z \in V$. For this purpose, let $\alpha \in L\backslash\{0\}$ such that

$$
\alpha \lhd I_V(b,c)(y) \wedge I_V(a,y)(z) = \bigvee_{\substack{y=t_1\cdot b+s_1\cdot c \\ z=t_2\cdot a+s_2\cdot y \\ t_1+s_1=t_2+s_2=1_F}} B(0_F,t_1,1_F) \wedge B(0_F,s_1,1_F) \wedge B(0_F,t_2,1_F) \wedge B(0_F,s_2,1_F).
$$

Then there exist $t_i, s_i \in F$ $(i = 1, 2)$ such that $t_1 + s_1 = t_2 + s_2 = 1_F$, $y = t_1 \cdot b + s_1 \cdot c$, $z = t_2 \cdot a + s_2 \cdot y$ and $\alpha \leq B(0_F, t_1, 1_F) \wedge B(0_F, s_1, 1_F) \wedge B(0_F, t_2, 1_F) \wedge B(0_F, s_2, 1_F)$. It is easy to obtain that $z = t_2 \cdot a + s_2 \cdot t_1 \cdot b + s_2 \cdot s_1 \cdot c$. Denote $t = 1_F - s_2 \cdot s_1$. Then $t = t_2 + s_2 \cdot t_1$. Thus, we have

$$
\begin{aligned}
\alpha &\leq B(0_F, t_1, 1_F) \wedge B(0_F, s_1, 1_F) \wedge B(0_F, t_2, 1_F) \wedge B(0_F, s_2, 1_F) \\
&\leq B(0_F, t_2, 1_F) \wedge B(0_F, t_1, 1_F) \wedge B(0_F, s_2, 1_F) \\
&\leq B(0_F, t_2, 1_F) \wedge B(0_F, t_1 \cdot s_2, 1_F) && \text{(by Lemma 5.5(2))} \\
&\leq B(0_F, t_2, t_2 + t_1 \cdot s_2) && \text{(by Lemma 5.11(2))} \\
&= B(0_F, t_2, t).
\end{aligned}
$$

Now we consider two cases depending on the value of t.

Case 1. Suppose $t = 0_F$. If $t_2 \neq 0_F$, then it follows from Lemma 5.5(3) that

$$B(0_F, t_2, t) = B(0_F, t_2, 0_F) = 0 \ngeq \alpha,$$

a contradiction, and hence $t_2 = 0_F$. It follows from $t = t_2 = 0_F$ and $t = 1_F - s_2 \cdot s_1$ that $s_2 = s_1 = 1_F$ and $t_1 = 0_F$. Thus $z = c$. Whence,

$$\begin{aligned}\bigvee_{x\in V} I_V(a,b)(x) \wedge I_V(c,x)(z) &= \bigvee_{x\in V} I_V(a,b)(x) \wedge I_V(c,x)(c)\\ &\geq I_V(a,b)(a) \wedge I_V(c,a)(a)\\ &= 1. && \text{(by Proposition 5.6)}\end{aligned}$$

This shows that (5.i) holds.

Case 2. Suppose $t \neq 0_F$. Let $l = t_2/t$, $k = (s_2 \cdot t_1)/t$ and $d = l \cdot a + k \cdot b$. Then

$$z = (1_F - t) \cdot c + t \cdot d.$$

By Lemma 5.5 (4), we have

$$B(0_F, l, 1_F) = B(0_F, t_2/t, 1_F) = B(0_F, t_2, t) \geq \alpha,$$

and

$$B(0_F, 1_F - t, 1_F) = B(0_F, s_2 \cdot s_1, 1_F) \geq B(0_F, s_2, 1_F) \wedge B(0_F, s_1, 1_F) \geq \alpha.$$

Hence

$$\begin{aligned}&\bigvee_{x\in V} I_V(a,b)(x) \wedge I_V(c,x)(z)\\ &= \bigvee_{x\in V}\left(\bigvee_{\substack{l_1+k_1=1_F\\ x=l_1\cdot a+k_1\cdot b}} B(0_F,l_1,1_F)\wedge B(0_F,k_1,1_F)\right)\wedge\left(\bigvee_{\substack{l_2+k_2=1_F\\ z=l_2\cdot c+k_2\cdot x}} B(0_F,l_2,1_F)\wedge B(0_F,k_2,1_F)\right)\\ &\geq \bigvee_{\substack{l_1+k_1=l_2+k_2=1_F\\ d=l_1\cdot a+k_1\cdot b\\ z=l_2\cdot c+k_2\cdot d}} B(0_F,l_1,1_F)\wedge B(0_F,k_1,1_F)\wedge B(0_F,l_2,1_F)\wedge B(0_F,k_2,1_F)\\ &\geq B(0_F,l,1_F)\wedge B(0_F,k,1_F)\wedge B(0_F,1_F-t,1_F)\wedge B(0_F,t,1_F)\\ &= B(0_F,l,1_F)\wedge B(0_F,1_F-t,1-F) && \text{(by Lemma 5.5(1))}\\ &\geq \alpha.\end{aligned}$$

By the arbitrariness of α, we obtain that (5.i) holds. This shows that (V, I_V) is a fuzzy Peano space.

For (2), it likewise remains to prove that

$$I_V(b,e)(a) \wedge I_V(d,e)(c) \leq \bigvee_{x\in V} I_V(a,d)(x) \wedge I_V(b,c)(x) \tag{5.ii}$$

for all $a, b, c, d, e \in V$. To this end, let $\alpha \in L\backslash\{0\}$ such that

$$\alpha \lhd I_V(b,e)(a) \wedge I_V(d,e)(c) = \bigvee_{\substack{t_1+s_1=t_2+s_2=1_F \\ a=t_1\cdot b+s_1\cdot e \\ c=t_2\cdot d+s_2\cdot e}} B(0_F,t_1,1_F) \wedge B(0_F,s_1,1_F) \wedge B(0_F,t_2,1_F) \wedge B(0_F,s_2,1_F).$$

Then there exist $t_i, s_i \in F$ $(i = 1, 2)$ such that $t_1 + s_1 = t_2 + s_2 = 1_F$, $a = t_1 \cdot b + s_1 \cdot e$, $c = t_2 \cdot d + s_2 \cdot e$ and $\alpha \leq B(0_F, t_1, 1_F) \wedge B(0_F, s_1, 1_F) \wedge B(0_F, t_2, 1_F) \wedge B(0_F, s_2, 1_F)$. We consider three cases depending on the values of s_1 and s_2.

Case 1. Suppose $s_1 = 0_F$. Then $a = b$. Hence

$$\bigvee_{x \in V} I_V(a,d)(x) \wedge I_V(b,c)(x) \geq I_V(a,d)(b) \wedge I_V(b,c)(b) = I_V(b,d)(b) \wedge I_V(b,c)(b) = 1_F.$$

Case 2. Suppose $s_2 = 0_F$. Then $c = d$. The rest of this proof is analogous to Case 1.

Case 3. Suppose $s_1 \neq 0_F$ and $s_2 \neq 0_F$. Then we have

$$e = \frac{a - t_1 \cdot b}{s_1} = \frac{c - t_2 \cdot d}{s_2}.$$

Hence $s_2 \cdot a + s_1 \cdot t_2 \cdot d = s_1 \cdot c + t_1 \cdot s_2 \cdot b$. Now we claim that

$$s_2 + s_1 \cdot t_2 = s_1 + t_1 \cdot s_2 \neq 0_F.$$

Firstly, since $s_2 + s_1 \cdot t_2 = 1_F - t_2 \cdot (1_F - s_1) = 1_F - t_2 \cdot t_1$ and $s_1 + t_1 \cdot s_2 = 1_F - t_1 \cdot (1_F - s_2) = 1_F - t_1 \cdot t_2$, we have $s_2 + s_1 \cdot t_2 = s_1 + t_1 \cdot s_2$.

Secondly, if $s_2 + s_1 \cdot t_2 = 0_F$, then we have

$$\begin{aligned}
\alpha &\leq B(0_F, t_1, 1_F) \wedge B(0_F, s_1, 1_F) \wedge B(0_F, t_2, 1_F) \wedge B(0_F, s_2, 1_F) \\
&\leq B(0_F, s_2, 1_F) \wedge B(0_F, s_1, 1_F) \wedge B(0_F, t_2, 1_F) \\
&\leq B(0_F, s_2, 1_F) \wedge B(0_F, s_1 \cdot t_2, 1_F) && \text{(by Lemma 5.5(2))} \\
&\leq B(0_F, s_2, s_2 + s_1 \cdot t_2) && \text{(by Lemma 5.11(2))} \\
&= B(0_F, s_2, 0_F).
\end{aligned}$$

Since $s_2 \neq 0_F$, it follows from Lemma 5.5(3) that $B(0_F, s_2, 0_F) = 0 \not\geq \alpha$, a contradiction. Hence $s_2 + s_1 \cdot t_2 \neq 0_F$. This shows that $s_1 + t_1 \cdot s_2 = s_2 + s_1 \cdot t_2 \neq 0_F$. Now let

$$h = \frac{s_2 \cdot a + s_1 \cdot t_2 \cdot d}{s_2 + s_1 \cdot t_2} \Big(= \frac{s_1 \cdot c + t_1 \cdot s_2 \cdot b}{s_1 + t_1 \cdot s_2} \Big).$$

By Lemma 5.5(4), we have

$$B\left(0_F, \frac{s_2}{s_2 + s_1 \cdot t_2}, 1_F\right) = B(0_F, s_2, s_2 + s_1 \cdot t_2) \geq \alpha$$

and

$$
\begin{aligned}
B\left(0_F, \frac{s_1}{s_1 + t_1 \cdot s_2}, 1_F\right) &= B(0_F, s_1, s_1 + t_1 \cdot s_2) \\
&\geq B(0_F, s_1, 1_F) \wedge B(0_F, t_1 \cdot s_2, 1_F) && \text{(by Lemma 5.11(2))} \\
&\geq B(0_F, s_1, 1_F) \wedge B(0_F, t_1, 1_F) \wedge B(0_F, s_2, 1_F) && \text{(by Lemma 5.5(2))} \\
&\geq \alpha.
\end{aligned}
$$

Since

$$
\frac{s_2}{s_2 + s_1 \cdot t_2} + \frac{s_1 \cdot t_2}{s_2 + s_1 \cdot t_2} = 1 = \frac{s_1}{s_1 + t_1 \cdot s_2} + \frac{t_1 \cdot s_2}{s_1 + t_1 \cdot s_2},
$$

it follows that

$$
\begin{aligned}
&\bigvee_{x \in V} I_V(a, d)(x) \wedge I_V(b, c)(x) \\
&= \bigvee_{x \in V} \left(\bigvee_{\substack{l_1 + k_1 = 1_F \\ x = l_1 \cdot a + k_1 \cdot d}} B(0_F, l_1, 1_F) \wedge B(0_F, k_1, 1_F) \right) \wedge \left(\bigvee_{\substack{l_2 + k_2 = 1_F \\ x = l_2 \cdot b + k_2 \cdot c}} B(0_F, l_2, 1_F) \wedge B(0_F, k_2, 1_F) \right) \\
&\geq \left(\bigvee_{\substack{l_1 + k_1 = 1_F \\ h = l_1 \cdot a + k_1 \cdot d}} B(0_F, l_1, 1_F) \wedge B(0_F, k_1, 1_F) \right) \wedge \left(\bigvee_{\substack{l_2 + k_2 = 1_F \\ h = l_2 \cdot b + k_2 \cdot c}} B(0_F, l_2, 1_F) \wedge B(0_F, k_2, 1_F) \right) \\
&\geq B\left(0_F, \frac{s_2}{s_2 + s_1 \cdot t_2}, 1_F\right) \wedge B\left(0_F, \frac{s_1 \cdot t_2}{s_2 + s_1 \cdot t_2}, 1_F\right) \wedge B\left(0_F, \frac{s_1}{s_1 + t_1 \cdot s_2}, 1_F\right) \wedge \\
&\quad B\left(0_F, \frac{t_1 \cdot s_2}{s_1 + t_1 \cdot s_2}, 1_F\right) \\
&= B\left(0_F, \frac{s_2}{s_2 + s_1 \cdot t_2}, 1_F\right) \wedge B\left(0_F, \frac{s_1}{s_1 + t_1 \cdot s_2}, 1_F\right) \qquad \text{(by Lemma 5.5(1))} \\
&\geq \alpha.
\end{aligned}
$$

By the arbitrariness of α, we get that (5.ii) holds. This shows that (V, I_V) is a fuzzy Pasch space. □

The Peano property and the Pasch property are important geometric properties in the theory of interval opeators [39]. Wu et al. [44, 45] generalized these properties to the fuzzy case and studied them from the viewpoint of fuzzy convex structures. Here, we provide a typical example to show the existence of fuzzy Peano spaces and fuzzy Pasch spaces in Theorem 5.12. As an application of Theorem 5.12, we will show that the fuzzy interval operator constructed in Proposition 5.6 is a geometric fuzzy interval operator. To this end, we first recall a lemma from [45].

Lemma 5.13 ([45]). *Let (X, I) be a fuzzy interval space. Then*

(1) *The fuzzy Peano property implies* (GFI2)*;*

(2) *The fuzzy Pasch property implies* (GFI3)*.*

Theorem 5.14. *Let V be a vector space over a fuzzy betweenness field $(F,+,\cdot,0_F,1_F,B)$ such that B is overlap-transitive. Then (V,I_V) is a geometric fuzzy interval space.*

Proof. By Theorem 5.12 and Lemma 5.13, it suffices to verify that I_V satisfies (GFI1).

Let $a,b\in V$. Then

$$\begin{aligned}
&\mathrm{sub}_V(I_V(a,a),I_V(a,b))\\
&=(I_V(a,a)(a)\to I_V(a,b)(a))\wedge(I_V(a,a)(b)\to I_V(a,b)(b))\wedge\\
&\quad\left(\bigwedge_{x\in V\setminus\{a,b\}}(I_V(a,a)(x)\to I_V(a,b)(x))\right)\\
&=1\wedge 1\wedge\left(\bigwedge_{x\in V\setminus\{a,b\}}\left(\bigvee\emptyset\to I_V(a,b)(x)\right)\right)\\
&=1.
\end{aligned}$$

If $a\neq b$, then

$$I_V(a,a)(b)=\bigvee_{\substack{b=t\cdot a+(1_F-t)\cdot a\\ t\in F}}B(0_F,t,1_F)=\bigvee\emptyset=0\neq 1,$$

as desired. □

6. Conclusions and future research

In this paper, we presented essential connections between fuzzy betweenness relations and three kinds of induced fuzzy structures. The main results include (i) fuzzy betweenness relations w.r.t. a fuzzy equivalence relation are categorically isomorphic to geometric fuzzy interval operators w.r.t. the same fuzzy equivalence relation; (ii) the interrelationship between fuzzy E-partial orders and fuzzy E-betweenness relations was established; (iii) the concept of a fuzzy betweenness field w.r.t. the crisp equality was introduced and a fuzzy Peano-Pasch space was constructed from a vector space over a fuzzy betweenness field. This collection of results illustrates that fuzzy betweenness relations play an important role in fuzzy set theory.

We conclude the manuscript with some problems and topics for further exploration.

(1) In Section 3, although the notion of an E-geometric fuzzy interval space w.r.t. a fuzzy equivalence relation was introduced, the notion of a fuzzy interval operator w.r.t. a fuzzy equivalence relation remains unexplored. Precisely, we wonder what is the relationship between the category of the new fuzzy interval spaces w.r.t. a fuzzy equivalence relation and that of fuzzy E-betweenness sets. Also, we will consider fuzzy betweenness fields w.r.t. a fuzzy equivalence relation.

(2) Theorem 5.12 in Section 5 is obtained in the framework of completely distributive lattices. It would be interesting to know whether this result also holds true in a more general lattice-valued background.

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