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Critical Gagliardo–Nirenberg, Trudinger, Brezis–Gallouet–Wainger inequalities on graded groups and ground states

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In this paper, we investigate critical Gagliardo–Nirenberg, Trudinger-type and Brezis–Gallouet–Wainger inequalities associated with the positive Rockland operators on graded Lie groups, which include the cases of $\mathbb{R}^n$, Heisenberg, and general stratified Lie groups. As an application, using the critical Gagliardo–Nirenberg inequality, the existence of least energy solutions of nonlinear Schrödinger type equations is obtained. We also express the best constant in the critical Gagliardo–Nirenberg and Trudinger inequalities in the variational form as well as in terms of the ground state solutions of the corresponding nonlinear subelliptic equations. The obtained results are already new in the setting of general stratified Lie groups (homogeneous Carnot groups). Among new technical methods, we also extend Folland's analysis of Hölder spaces from stratified Lie groups to general homogeneous Lie groups.

Keywords: Trudinger inequality; Gagliardo–Nirenberg inequality; Sobolev inequality; Rockland operator; graded Lie group; stratified Lie group; sub-Laplacian.

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1. Introduction

Consider the following Trudinger–Moser inequality

$$\int_{\mathbb{R}^n} (\exp(\alpha|f(x)|^{\frac{p}{p-1}}) - 1)dx \leq C, \quad 1 < p < \infty, \tag{1.1}$$

for $f \in L^p_{n/p}(\mathbb{R}^n) = (1 - \triangle)^{-n/2p} L^p(\mathbb{R}^n)$ with $\|f\|_{L^p_{n/p}} \leq 1$ and for some positive constants C and α. This inequality has been generalized in many directions. In bounded domains of $\mathbb{R}^n$ with $p = n \geq 2$, we refer to [34, 3, 13, 21, 32, 48] for the finding of the best exponents in (1.1), and to [4] for the singular version of this inequality. In unbounded domains, we refer to [3, 2, 36, 37, 45, 35, 1, 37] for Sobolev spaces of fractional order and higher order.

In [28], the authors developed a rearrangement-free argument without using symmetrization to establish the Trudinger–Moser inequalities in the unbounded space $\mathbb{R}^n$ including Adams type inequalities on the higher order derivatives (and fractional order derivatives). Rearrangement fails on the Heisenberg group or higher order Sobolev spaces. In [28] the authors avoid such a rearrangement which is only available in the first order in the Euclidean spaces. We also refer to [51] for a rearrangement free argument on the Heisenberg group, where the author obtained the Trudinger–Moser inequalities when $p = Q$ by gluing local estimates with the cut-off functions. On the Heisenberg group, we also refer to [16, 29] for an analogue of inequality (1.1) on domains of finite measure, and to [27, 30, 17, 50] on the entire Heisenberg group as well as to [28, 43, 44] on stratified (Lie) groups. We also refer to [31] for the results of concentration-compactness type on the Heisenberg group and beyond using the level set argument.

In this paper, we are interested in obtaining such inequalities on graded Lie groups. We use the strategy developed in [37, 38] on $\mathbb{R}^n$.

A connected simply connected Lie group $\mathbb{G}$ is called a graded (Lie) group if its Lie algebra admits a gradation. The graded groups form the subclass of homogeneous nilpotent Lie groups admitting homogeneous hypoelliptic left-invariant differential operators ([33, 47], see also a discussion in [20, Sec. 4.1]). These operators are called Rockland operators from the Rockland conjecture, solved by Helffer and Nourrigat [25]. So, we understand by a Rockland operator *any left-invariant homogeneous hypoelliptic differential operator on* $\mathbb{G}$.

In this paper, we are interested in obtaining the inequality (1.1) associated with positive Rockland operators on graded groups. We are also interested to obtain critical Gagliardo–Nirenberg and Brezis–Gallouet–Wainger inequalities. Consequently, we give applications of these inequalities to the nonlinear subelliptic equations. As such, this is essentially the most general framework for such inequalities in the setting of nilpotent Lie groups. Indeed, if a nilpotent Lie group has a left-invariant hypoelliptic differential operator, then the group is graded, see Sec. 2 for definitions and some details.

From now on we let $\mathcal{R}$ be a positive Rockland operator, that is, a positive left-invariant homogeneous hypoelliptic invariant differential operator on $\mathbb{G}$

of homogeneous degree ν. Its powers $\mathcal{R}^a$ for any $a > 0$ are understood through the functional calculus on the whole of $\mathbb{G}$, extensively analyzed in [20, 19]. We denote the Sobolev space by $L^p_a(\mathbb{G}) = L^p_{a,\mathcal{R}}(\mathbb{G})$, for $a > 0$, defined by the norm

$$\|u\|_{L^p_{a,\mathcal{R}}(\mathbb{G})} := \left(\int_{\mathbb{G}} (|\mathcal{R}^{a/\nu} u(x)|^p + |u(x)|^p) dx\right)^{1/p}. \tag{1.2}$$

We refer to [20, Theorem 4.4.20] for the independence of the spaces $L^p_a(\mathbb{G})$ of a particular choice of the Rockland operator $\mathcal{R}$.

Thus, in this paper we will show that for a graded group $\mathbb{G}$ of homogeneous dimension Q and for a positive Rockland operator $\mathcal{R}$ of homogeneous degree ν we have the following results:

- **(Critical Gagliardo–Nirenberg inequality)** Let $1 < p < \infty$. Then there exists a constant C_1 depending only on p and Q such that

$$\|f\|_{L^q(\mathbb{G})} \leq C_1 q^{1-1/p} \|\mathcal{R}^{\frac{Q}{\nu p}} f\|^{1-p/q}_{L^p(\mathbb{G})} \|f\|^{p/q}_{L^p(\mathbb{G})} \tag{1.3}$$

  holds for any q with $p \leq q < \infty$ and for any function f from the Sobolev space $L^p_{Q/p}(\mathbb{G})$ on graded group $\mathbb{G}$.
- **(Trudinger inequality with remainders)** Let $1 < p < \infty$. Then there exist positive α and C_2 such that

$$\int_{\mathbb{G}} \left(\exp(\alpha|f(x)|^{p'}) - \sum_{0\leq k<p-1,\ k\in\mathbb{N}} \frac{1}{k!}(\alpha|f(x)|^{p'})^k\right) dx \leq C_2 \|f\|^p_{L^p(\mathbb{G})} \tag{1.4}$$

  holds for any function $f \in L^p_{Q/p}(\mathbb{G})$ with $\|\mathcal{R}^{\frac{Q}{\nu p}} f\|_{L^p(\mathbb{G})} \leq 1$, where $1/p + 1/p' = 1$. Furthermore, we show that (1.3) and (1.4) are actually equivalent and give the relation between their best constants. In [42] using this result we obtained weighted versions of (1.4) on graded groups. In the case $p = Q$, for the best constant α in the weighted Trudinger–Moser inequalities we refer to [27, Theorem 1.6] on the Heisenberg group and to [28, Theorem G] on general stratified groups when $[\int_{\mathbb{G}} |\nabla_H u(\xi)|^Q d\xi + \tau \int_{\mathbb{G}} |u(\xi)|^Q d\xi]^{1/Q} \leq 1$ for any fixed positive real number τ, and to [30, Theorem 1.1] on the Heisenberg group when $\|\nabla_H u\|_{L^Q(\mathbb{H}_n)} \leq 1$.
- **(Brezis–Gallouet–Wainger inequality)** Let $a, p, q \in \mathbb{R}$ with $1 < p, q < \infty$ and $a > Q/q$. Then there exists a constant C_1 depending only on p and Q such that

$$\|f\|_{L^\infty} \leq C_3(1 + \log(1 + \|\mathcal{R}^{a/\nu} f\|_{L^q(\mathbb{G})}))^{1/p'} \tag{1.5}$$

  holds for any function $f \in L^p_{Q/p}(\mathbb{G}) \cap L^q_a(\mathbb{G})$.
- **(Existence of ground state solutions)** Let $1 < p < q < \infty$. Then the Schrödinger type equation (5.1) has a least energy solution $\phi \in L^p_{Q/p}(\mathbb{G})$.

  Furthermore, we have $d = \mathfrak{L}(\phi)$, for the variational problem (5.4)–(5.7).

  The nonlinear equation (5.1) mentioned above appears naturally in the analysis of the best constants for the above inequalities:

- **(Best constants in critical Gagliardo–Nirenberg inequality)** Let $1 < p < q < \infty$. Let ϕ be a least energy solution of (5.1) and let $C_{GN,\mathcal{R}}$ be the smallest positive constant of C_1 in (1.3). Then we have

$$\begin{aligned} C_{GN,\mathcal{R}} &= q^{-q+q/p}\frac{q}{p}\left(\frac{q-p}{p}\right)^{\frac{p-q}{p}}\|\phi\|_{L^p(\mathbb{G})}^{p-q} \\ &= q^{-q+q/p}\frac{q}{p}\left(\frac{q-p}{p}\right)^{\frac{p-q}{p}}\left(\frac{p^2}{q-p}d\right)^{\frac{p-q}{p}}. \end{aligned} \tag{1.6}$$

Since (1.3) and (1.4) are equivalent with a relation between constants, this also gives the best constant in Trudinger inequalities.

We note that the above results are already new if $\mathbb{G}$ *is a stratified group and* $\mathcal{R}$ *is the* (*positive*) *sub-Laplacian on* $\mathbb{G}$ (*so that also* $\nu = 2$).

The paper is organized as follows. In Sec. 2, we briefly recall main concepts of graded groups and fix the notation. The critical Gagliardo–Nirenberg inequality and Trudinger-type inequality (1.1) are obtained on graded groups in Sec. 3, where the constant C is given more explicitly. In Sec. 4, we prove the Brezis–Gallouet–Wainger inequalities on graded groups. Finally, applications are given to the nonlinear Schrödinger type equations in Sec. 5.

2. Preliminaries

Following Folland and Stein [23, Chap. 1] and the recent exposition in [20, Chap. 3] we recall that $\mathbb{G}$ is a graded (Lie) group if its Lie algebra $\mathfrak{g}$ admits a gradation

$$\mathfrak{g} = \bigoplus_{\ell=1}^{\infty}\mathfrak{g}_\ell,$$

where the $\mathfrak{g}_\ell$, $\ell = 1, 2, \ldots$, are vector subspaces of $\mathfrak{g}$, all but finitely many equal to $\{0\}$, and satisfying

$$[\mathfrak{g}_\ell, \mathfrak{g}_{\ell'}] \subset \mathfrak{g}_{\ell+\ell'} \quad \forall\, \ell, \ell' \in \mathbb{N}.$$

It is called stratified if $\mathfrak{g}_1$ generates the whole of $\mathfrak{g}$ through these commutators.

We fix a basis $\{X_1, \ldots, X_n\}$ of a Lie algebra $\mathfrak{g}$ adapted to the gradation. By the exponential mapping $\exp_{\mathbb{G}} : \mathfrak{g} \to \mathbb{G}$ we get points in $\mathbb{G}$:

$$x = \exp_{\mathbb{G}}(x_1X_1 + \cdots + x_nX_n).$$

A family of linear mappings of the form

$$D_r = \mathrm{Exp}(A\ln r) = \sum_{k=0}^{\infty}\frac{1}{k!}(\ln(r)A)^k$$

is a family of dilations of $\mathfrak{g}$. Here A is a diagonalizable linear operator on $\mathfrak{g}$ with positive eigenvalues. Every D_r is a morphism of the Lie algebra $\mathfrak{g}$, i.e. D_r is a linear mapping from $\mathfrak{g}$ to itself with the property

$$\forall\, X, Y \in \mathfrak{g}, \quad r > 0, \;\; [D_r X, D_r Y] = D_r[X, Y],$$

as usual $[X, Y] := XY - YX$ is the Lie bracket. One can extend these dilations through the exponential mapping to the group $\mathbb{G}$ by

$$D_r(x) = rx := (r^{\nu_1} x_1, \ldots, r^{\nu_n} x_n), \quad x = (x_1, \ldots, x_n) \in \mathbb{G}, \quad r > 0, \tag{2.1}$$

where $\nu_1, \ldots, \nu_n$ are weights of the dilations. The sum of these weights

$$Q := \operatorname{Tr} A = \nu_1 + \cdots + \nu_n$$

is called the homogeneous dimension of $\mathbb{G}$. We also recall that the standard Lebesgue measure dx on $\mathbb{R}^n$ is the Haar measure for $\mathbb{G}$ (see, e.g. [20, Proposition 1.6.6]). A homogeneous quasi-norm on $\mathbb{G}$ is a continuous non-negative function

$$\mathbb{G} \ni x \mapsto |x| \in [0, \infty)$$

with the properties:

- $|x^{-1}| = |x|$ for any $x \in \mathbb{G}$,
- $|\lambda x| = \lambda |x|$ for any $x \in \mathbb{G}$ and $\lambda > 0$,
- $|x| = 0$ if and only if $x = 0$.

We will use the following polar decomposition for our analysis: there is a (unique) positive Borel measure σ on the unit sphere

$$\mathfrak{S} := \{x \in \mathbb{G} : |x| = 1\}, \tag{2.2}$$

such that for any function $f \in L^1(\mathbb{G})$ we have

$$\int_{\mathbb{G}} f(x)dx = \int_0^\infty \int_{\mathfrak{S}} f(ry) r^{Q-1} d\sigma(y) dr. \tag{2.3}$$

Let $\widehat{\mathbb{G}}$ be a unitary dual of $\mathbb{G}$ and let $\mathcal{H}_\pi^\infty$ be the space of smooth vectors for a representation $\pi \in \widehat{\mathbb{G}}$. A Rockland operator $\mathcal{R}$ on $\mathbb{G}$ is a left-invariant differential operator which is homogeneous of positive degree and satisfies the condition:

(**Rockland condition**) For each representation $\pi \in \widehat{\mathbb{G}}$, except for the trivial representation, the operator $\pi(\mathcal{R})$ is injective on $\mathcal{H}_\pi^\infty$, i.e.

$$\forall\, v \in \mathcal{H}_\pi^\infty, \quad \pi(\mathcal{R})v = 0 \Rightarrow v = 0.$$

Here $\pi(\mathcal{R}) := d\pi(\mathcal{R})$ is the infinitesimal representation of the Rockland operator $\mathcal{R}$ as of an element of the universal enveloping algebra of $\mathbb{G}$.

Different characterizations of such operators have been obtained by Rockland [39] and Beals [6]. For an extensive presentation about Rockland operators and for the theory of Sobolev spaces on graded groups we refer to [19] and [20, Chap. 4], and for the Besov spaces on graded groups we refer to [12].

Since we will not be using the representation theoretic interpretation of these operators in this paper, we define *Rockland operators as left-invariant homogeneous* *hypoelliptic differential operators on* $\mathbb{G}$. This is equivalent to the Rockland condition as it was shown by Helffer and Nourrigat in [25].

The homogeneous and inhomogeneous Sobolev spaces $\dot{L}^p_a(\mathbb{G})$ and $L^p_a(\mathbb{G})$ based on the positive left-invariant hypoelliptic differential operator $\mathcal{R}$ have been extensively analyzed in [19] and [20, Sec. 4.4] to which we refer for the details of their properties. They generalize the Sobolev spaces based on the sub-Laplacian on stratified groups analyzed by Folland in [22]. We refer to the above papers for (non-critical) Sobolev inequalities in the setting of graded groups, and to [41] to the determination of the best constants in non-critical Sobolev and Gagliardo–Nirenberg inequalities on graded groups.

3. Critical Gagliardo–Nirenberg and Trudinger inequalities

We recall the following Gagliardo–Nirenberg inequality (see [41, Theorem 3.2]): Let $a \geq 0$, $1 < p < \frac{Q}{a}$ and $p \leq q \leq \frac{pQ}{Q-ap}$. Then we have for all functions u from the homogeneous Sobolev space $\dot{L}^p_a(\mathbb{G})$ on the graded group $\mathbb{G}$:

$$\int_{\mathbb{G}} |u(x)|^q dx \leq C \left(\int_{\mathbb{G}} |\mathcal{R}^{\frac{a}{\nu}} u(x)|^p dx \right)^{\frac{Q(q-p)}{ap^2}} \left(\int_{\mathbb{G}} |u(x)|^p dx \right)^{\frac{apq-Q(q-p)}{ap^2}}. \tag{3.1}$$

In this section, we show this inequality for $a = Q/p$, which can be viewed as a critical Gagliardo–Nirenberg inequality. Then, we prove Trudinger-type inequality on graded groups, and we show the equivalence of these two inequalities. We note that another version of the Gagliardo–Nirenberg inequalities was also given in [5].

In order to prove the critical Gagliardo–Nirenberg inequality, we need to recall the following results.

Theorem 3.1 ([20, Theorem 4.4.28, Part 6]). *Let* $\mathbb{G}$ *be a graded Lie group of homogeneous dimension* Q. *Let* $1 < p < \infty$ *and* $a, b, c \in \mathbb{R}$ *with* $a < c < b$. *Then we have*

$$\|f\|_{\dot{L}^p_c(\mathbb{G})} \leq C \|f\|^{1-\theta}_{\dot{L}^p_a(\mathbb{G})} \|f\|^{\theta}_{\dot{L}^p_b(\mathbb{G})} \tag{3.2}$$

for any function $f \in \dot{L}^p_b(\mathbb{G})$, *where* $\theta := (c-a)/(b-a)$ *and the constant* C *depends only on* a, b *and* c.

We will also need the following statement where we refer to Definition 4.1 for the precise definition of the operators of type ν.

Corollary 3.2 ([20, Corollary 3.2.32]). *Let* $\mathbb{G}$ *be a homogeneous Lie group and let* ν *be a complex number such that* $0 \leq \mathrm{Re}\nu < Q$. *Then, all operators of type* ν *on* $\mathbb{G}$ *are* $(-\nu)$*-homogeneous and extend to a bounded operator from* $L^p(\mathbb{G})$ *to* $L^q(\mathbb{G})$ *provided that* $1/p - 1/q = \mathrm{Re}\nu/Q$ *for* $1 < p \leq q < \infty$.

Now let us state the critical Gagliardo–Nirenberg inequality.

Theorem 3.3. *Let* $\mathbb{G}$ *be a graded Lie group of homogeneous dimension* Q *and let* $\mathcal{R}$ *be a positive Rockland operator of homogeneous degree* ν. *Let* $1 < p < \infty$. *Then* *there exists a constant C_1 depending only on p and Q such that*

$$\|f\|_{L^q(\mathbb{G})} \leq C_1 q^{1-1/p} \|\mathcal{R}^{\frac{Q}{\nu p}} f\|_{L^p(\mathbb{G})}^{1-p/q} \|f\|_{L^p(\mathbb{G})}^{p/q} \tag{3.3}$$

holds for every q with $p \leq q < \infty$ and for every function $f \in L^p_{Q/p}(\mathbb{G})$.

Remark 3.4. We note that when $\mathbb{G} = (\mathbb{R}^n, +)$ and $\mathcal{R} = -\triangle$ is the Laplacian, the inequality (3.3) was obtained in [36] for $p = 2$, in [26] for $p = n$ and in [37] for general p as in Theorem 3.3.

Proof of Theorem 3.3. We can assume that $f \not\equiv 0$. Let us consider the Riesz potential I_λ with $0 < \lambda < Q$ (see e.g. [20, Sec. 4.3.4] on graded groups and [40] on general homogeneous groups), given by

$$(I_\lambda f)(x) := \int_{\mathbb{G}} |xy^{-1}|^{\lambda - Q} f(y) dy = (K_\lambda * f)(x), \tag{3.4}$$

where $K_\lambda(x) = |x|^{\lambda - Q}$. Now we decompose, for some $s > 0$ to be chosen later,

$$(I_\lambda f)(x) = (I_\lambda^{(1)}(s) f)(x) + (I_\lambda^{(2)}(s) f)(x) := (K_{\lambda,s}^{(1)} * f)(x) + (K_{\lambda,s}^{(2)} * f)(x),$$

where $K_{\lambda,s}^{(1)}$ and $K_{\lambda,s}^{(2)}$ are defined by

$$K_\lambda(x) =: \begin{cases} K_{\lambda,s}^{(1)}, & |x| < s; \\ K_{\lambda,s}^{(2)}, & |x| \geq s. \end{cases}$$

Let $1/\widetilde{q} = 1/\widetilde{p} - \lambda/Q$ and $1 \leq \widetilde{p} < \widetilde{q} < \infty$. Using Young's inequality (see e.g. [20, Proposition 1.5.2]), and introducing polar coordinates $(r, y) = (|x|, \frac{x}{|x|}) \in (0, \infty) \times \mathfrak{S}$ on $\mathbb{G}$, where $\mathfrak{S}$ is the sphere as in (2.2), and by (2.3), we have

$$\|I_\lambda^{(1)}(s) f\|_{L^{\widetilde{p}}(\mathbb{G})} \leq \|K_{\lambda,s}^{(1)}\|_{L^1(\mathbb{G})} \|f\|_{L^{\widetilde{p}}(\mathbb{G})} = \frac{|\mathfrak{S}|}{\lambda} s^\lambda \|f\|_{L^{\widetilde{p}}(\mathbb{G})}.$$

Similarly, we get

$$\|I_\lambda^{(2)}(s) f\|_{L^\infty(\mathbb{G})} \leq \|K_{\lambda,s}^{(2)}\|_{L^{\widetilde{p}'}(\mathbb{G})} \|f\|_{L^{\widetilde{p}}(\mathbb{G})} = \left(\frac{|\mathfrak{S}| \widetilde{q}}{Q \widetilde{p}'} \right)^{1/\widetilde{p}'} s^{\lambda - Q/\widetilde{p}} \|f\|_{L^{\widetilde{p}}(\mathbb{G})},$$

where $1/\widetilde{p} + 1/\widetilde{p}' = 1$ and $|\mathfrak{S}|$ is the $Q - 1$ dimensional surface measure of the unit sphere $\mathfrak{S}$.

Then, as in [37, Sec. 2, Formula (2.1)], one can observe that

$$\begin{aligned} &\sup_{z>0} z |\{x \in \mathbb{G} : |(I_\lambda f)(x)| > z\}|^{1/\widetilde{q}} \\ &\quad \leq 2 \left(\frac{|\mathfrak{S}| \widetilde{q}}{Q} \right)^{1 - 1/\widetilde{p} + 1/\widetilde{q}} \frac{(\widetilde{p} - 1)^{\frac{(\widetilde{p}-1)(\widetilde{q}-\widetilde{p})}{\widetilde{p}\widetilde{q}}}}{\widetilde{p}^{1 - 1/\widetilde{p} - (2\widetilde{p} - 1)/\widetilde{q}} (\widetilde{q} - \widetilde{p})^{\widetilde{p}/\widetilde{q}}} \|f\|_{L^{\widetilde{p}}(\mathbb{G})}, \end{aligned} \tag{3.5}$$

where $1/\widetilde{q} = 1/\widetilde{p} - \lambda/Q$, $1 \leq \widetilde{p} < \widetilde{q} < \infty$, $0 < \lambda < Q$.

If $(1/p_1, 1/q_1) = (1, 1-\lambda/Q)$ with $0 < \lambda < Q$, then (3.5) implies that

$$\sup_{z>0} z|\{x \in \mathbb{G} : |(I_\lambda f)(x)| > z\}|^{1/q_1} \leq 2\left(\frac{|\mathfrak{S}|q_1}{Q(q_1-1)}\right)^{1/q_1} \|f\|_{L^1(\mathbb{G})}. \tag{3.6}$$

If $(\frac{1}{p_2}, \frac{1}{q_2}) = (\frac{1}{p} - \frac{(Q-\lambda p)^2}{pQ(pQ+Q-\lambda p)}, \frac{Q-\lambda p}{Qp+Q-\lambda p})$ with $0 < \lambda < Q$ and $1 < p < Q/\lambda$, then (3.5) gives that

$$\sup_{z>0} z|\{x \in \mathbb{G} : |(I_\lambda f)(x)| > z\}|^{1/q_2}$$

$$\leq 2\left(\frac{|\mathfrak{S}|q_2}{Q}\right)^{1-1/p_2+1/q_2} \frac{(p_2-1)^{\frac{(p_2-1)(q_2-p_2)}{p_2q_2}}}{p_2^{1-1/p_2-(2p_2-1)/q_2}(q_2-p_2)^{p_2/q_2}} \|f\|_{L^{p_2}(\mathbb{G})}. \tag{3.7}$$

We fix $p \in (1, \infty)$ and define $\lambda = Q(1/p - 1/q)$ for all q with $p < q < \infty$. Then $0 < \lambda < Q$, $1 < p < Q/\lambda$, $(1/p_1, 1/q_1) = (1, 1-1/p+1/q)$ and $(1/p_2, 1/q_2) = (1/p - 1/q + 1/(q+1), 1/(q+1))$, so that (3.6) and (3.7) show that $I_{Q(1/p-1/q)}$ is of weak types (p_1, q_1) and (p_2, q_2), respectively. Setting $\theta = \theta(q) = (1-1/p)/(1-1/p+1/q-1/(q+1))$, we get $0 < \theta < 1$, $1/p = (1-\theta)/p_1 + \theta/p_2$ and $1/q = (1-\theta)/q_1 + \theta/q_2$.

As in [37, Sec. 2], we can now use the Marcinkiewicz interpolation theorem to obtain

$$\|I_\lambda f\|_{L^q(\mathbb{G})} \leq 4\left(q + \frac{pq}{q-p}\right)^{1/q} (M_1(q))^{1-\theta}(M_2(q))^{\theta}\|f\|_{L^p(\mathbb{G})}, \tag{3.8}$$

where $\lambda = Q(1/p - 1/q)$, and

$$M_1(q) = \left(\frac{|\mathfrak{S}|q_1}{Q(q_1-1)}\right)^{1/q_1},$$

$$M_2(q) = \left(\frac{|\mathfrak{S}|q_2}{Q}\right)^{1-1/p_2+1/q_2} \frac{(p_2-1)^{\frac{(p_2-1)(q_2-p_2)}{p_2q_2}}}{p_2^{1-1/p_2-(2p_2-1)/q_2}(q_2-p_2)^{p_2/q_2}}.$$

Considering the growth property of the right-hand side of (3.8) with respect to q, one gets

$$\lim_{q\to\infty} \theta(q) = 1, \quad \lim_{q\to\infty} M_1(q) = \left(\frac{|\mathfrak{S}|p}{Q}\right)^{1-1/p}$$

and

$$\lim_{q\to\infty} q^{1/p-1}M_2(q) = \left(\frac{|\mathfrak{S}|(p-1)}{Qp}\right)^{1-1/p}.$$

So, there is a constant C depending on p and Q for any q with $p < q < \infty$ such that

$$4\left(q + \frac{pq}{q-p}\right)^{1/q} (M_1(q))^{1-\theta}(M_2(q))^{\theta} \leq Cq^{1-1/p}$$

holds. Using this and (3.8), we deduce

$$\|I_\lambda f\|_{L^q(\mathbb{G})} \leq Cq^{1-1/p}\|f\|_{L^p(\mathbb{G})}, \tag{3.9}$$

for any q with $p < q < \infty$. Since $\mathcal{R}^{-\lambda/\nu}$ is the Riesz potential (see e.g. [20, Sec. 4.3.4]), then the inequality (3.9) with Corollary 3.2 implies that

$$\|f\|_{L^q(\mathbb{G})} \leq Cq^{1-1/p}\|\mathcal{R}^{(Q/\nu)(1/p-1/q)}f\|_{L^p(\mathbb{G})}, \tag{3.10}$$

for $0 < \nu < Q$ and for any q with $p < q < \infty$, where C depends only on p and Q.

By Theorem 3.1 with $a = 0$, $b = Q/p$ and $c = Q(1/p - 1/q)$, hence $\theta = 1 - p/q$, we have

$$\|\mathcal{R}^{(Q/\nu)(1/p-1/q)}f\|_{L^p(\mathbb{G})} \leq C\|\mathcal{R}^{\frac{Q}{\nu p}}f\|^{1-p/q}_{L^p(\mathbb{G})}\|f\|^{p/q}_{L^p(\mathbb{G})}, \tag{3.11}$$

which implies (3.3) in view of (3.10). □

Now we state the Trudinger-type inequality with the remainder estimate on graded groups.

Theorem 3.5. *Let $\mathbb{G}$ be a graded Lie group of homogeneous dimension Q and let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν. Let $1 < p < \infty$. Then there exist positive α and C_2 such that*

$$\int_{\mathbb{G}} \left(\exp(\alpha|f(x)|^{p'}) - \sum_{0\leq k<p-1,\ k\in\mathbb{N}} \frac{1}{k!}(\alpha|f(x)|^{p'})^k \right) dx \leq C_2\|f\|^p_{L^p(\mathbb{G})} \tag{3.12}$$

holds for all functions $f \in L^p_{Q/p}(\mathbb{G})$ with $\|\mathcal{R}^{\frac{Q}{\nu p}}f\|_{L^p(\mathbb{G})} \leq 1$, where $1/p + 1/p' = 1$.

Remark 3.6. The constant C_2 can be expressed in terms of the constant $C_1 = C_1(p, Q)$ in (3.3) as follows

$$C_2 = C_2(\alpha) = \sum_{k\geq p-1,\ k\in\mathbb{N}} \frac{k^k}{k!}(p'C_1^{p'}\alpha)^k.$$

Then, we have (3.12) for all $\alpha \in (0, (ep'C_1^{p'})^{-1})$ and $C_2(\alpha)$.

Remark 3.7. In the case $\mathbb{G} = (\mathbb{R}^n, +)$ and $\mathcal{R} = -\triangle$ is the Laplacian, the inequality (3.12) was obtained in [37]. In this abelian case, we can also refer to [35] for $p = 2$, [36] for $p = n = 2$, and to [1] for $p = n \geq 2$.

Proof of Theorem 3.5. A direct calculation gives

$$\int_{\mathbb{G}} \left(\exp(\alpha|f(x)|^{p'}) - \sum_{0\leq k<p-1,\ k\in\mathbb{N}} \frac{1}{k!}(\alpha|f(x)|^{p'})^k \right) dx$$
$$= \sum_{k\geq p-1,\ k\in\mathbb{N}} \frac{\alpha^k}{k!} \int_{\mathbb{G}} |f(x)|^{p'k}dx.$$

Since $k \geq p-1$, we have $p'k \geq p$, then using Theorem 3.3 for the above integrals in the last line, we calculate

$$\begin{aligned}
&\int_{\mathbb{G}} (\exp(\alpha|f(x)|^{p'}) - \sum_{0\leq k<p-1, \quad k\in\mathbb{N}} \frac{1}{k!}(\alpha|f(x)|^{p'})^k)dx \\
&\leq \sum_{k\geq p-1, \quad k\in\mathbb{N}} \frac{\alpha^k}{k!} C_1^{p'k}(p'k)^{p'k(1-1/p)} \int_{\mathbb{G}} |f(x)|^p dx \\
&= \sum_{k\geq p-1, \quad k\in\mathbb{N}} \frac{k^k}{k!}(p'C_1^{p'}\alpha)^k \|f\|^p_{L^p(\mathbb{G})},
\end{aligned}$$

which implies (3.12). □

Now we show that the obtained critical Gagliardo–Nirenberg inequality (3.3) and Trudinger-type inequality (3.12) are actually equivalent on general graded groups. Note that it is already known in $\mathbb{R}^n$, see [38].

Theorem 3.8. *The inequalities* (3.3) *and* (3.12) *are equivalent. Furthermore, we have*

$$\frac{1}{\widetilde{\alpha} p' e} = A^{p'} = B^{p'}, \tag{3.13}$$

where

$$\begin{aligned}
&\widetilde{\alpha} = \sup\{\alpha > 0; \exists C_2 = C_2(\alpha) : (3.12) \text{ holds} \\
&\qquad \forall f \in L^p_{Q/p}(\mathbb{G}) \text{ with } \|\mathcal{R}^{\frac{Q}{\nu p}} f\|_{L^p(\mathbb{G})} \leq 1\}, \\
&A = \inf\{C_1 > 0; \exists r = r(C_1) \text{ with } r \geq p : (3.3) \text{ holds} \\
&\qquad \forall f \in L^p_{Q/p}(\mathbb{G}), \forall q \text{ with } r \leq q < \infty\}, \\
&B = \limsup_{q\to\infty} \frac{\|f\|_{L^q(\mathbb{G})}}{q^{1-1/p}\|\mathcal{R}^{\frac{Q}{\nu p}} f\|^{1-p/q}_{L^p(\mathbb{G})}\|f\|^{p/q}_{L^p(\mathbb{G})}}.
\end{aligned} \tag{3.14}$$

Proof. We have already shown that (3.3) implies (3.12) in the proof of Theorem 3.5. By Remark 3.6, (3.14) and taking into account that $A \geq B$, we note that $\alpha < (ep'C_1^{p'})^{-1}$ implies $\alpha < (ep'B^{p'})^{-1}$, that is, (3.3) implies (3.12) with $\widetilde{\alpha} \geq (ep'B^{p'})^{-1}$.

Now let us show (3.12) $\Rightarrow$ (3.3) with $\widetilde{\alpha} \leq (ep'A^{p'})^{-1}$. Since $\|\mathcal{R}^{\frac{Q}{\nu p}} f\|_{L^p(\mathbb{G})} \leq 1$, replacing f by $f/\|\mathcal{R}^{\frac{Q}{\nu p}} f\|_{L^p(\mathbb{G})}$ in (3.12) we obtain

$$\begin{aligned}
&\int_{\mathbb{G}} \left(\exp\left(\frac{\alpha|f(x)|^{p'}}{\|\mathcal{R}^{\frac{Q}{\nu p}} f\|^{p'}_{L^p(\mathbb{G})}}\right) - \sum_{0\leq k<p-1,\, k\in\mathbb{N}} \frac{1}{k!}\left(\frac{\alpha|f(x)|^{p'}}{\|\mathcal{R}^{\frac{Q}{\nu p}} f\|^{p'}_{L^p(\mathbb{G})}}\right)^k\right) dx \\
&\quad \leq C_2 \frac{\|f\|^p_{L^p(\mathbb{G})}}{\|\mathcal{R}^{\frac{Q}{\nu p}} f\|^p_{L^p(\mathbb{G})}}.
\end{aligned} \tag{3.15}$$

Then, we see that for any ε with $0<\varepsilon<\widetilde{\alpha}$ there is C_ε such that

$$\int_{\mathbb{G}}\left(\exp\left(\frac{(\widetilde{\alpha}-\varepsilon)|f(x)|^{p'}}{\|\mathcal{R}^{\frac{Q}{\nu p}}f\|_{L^p(\mathbb{G})}^{p'}}\right)-\sum_{0\le k<p-1,\ k\in\mathbb{N}}\frac{1}{k!}\left(\frac{(\widetilde{\alpha}-\varepsilon)|f(x)|^{p'}}{\|\mathcal{R}^{\frac{Q}{\nu p}}f\|_{L^p(\mathbb{G})}^{p'}}\right)^k\right)dx$$

$$\le C_\varepsilon\frac{\|f\|_{L^p(\mathbb{G})}^p}{\|\mathcal{R}^{\frac{Q}{\nu p}}f\|_{L^p(\mathbb{G})}^p} \tag{3.16}$$

holds for all $f\in L^p_{Q/p}(\mathbb{G})$. It follows that

$$\int_{\mathbb{G}}\sum_{k\ge p-1,\ k\in\mathbb{N}}\frac{1}{k!}\left(\frac{(\widetilde{\alpha}-\varepsilon)|f(x)|^{p'}}{\|\mathcal{R}^{\frac{Q}{\nu p}}f\|_{L^p(\mathbb{G})}^{p'}}\right)^k dx\le C_\varepsilon\frac{\|f\|_{L^p(\mathbb{G})}^p}{\|\mathcal{R}^{\frac{Q}{\nu p}}f\|_{L^p(\mathbb{G})}^p},$$

that is,

$$\|f\|_{L^{p'k}(\mathbb{G})}\le (C_\varepsilon k!)^{1/p'k}(\widetilde{\alpha}-\varepsilon)^{-1/p'}\|\mathcal{R}^{\frac{Q}{\nu p}}f\|_{L^p(\mathbb{G})}^{1-(p-1)/k}\|f\|_{L^p(\mathbb{G})}^{(p-1)/k} \tag{3.17}$$

for all $k\in\mathbb{N}$ with $k\ge p-1$. Let $q>p$ and $p'k\le q<p'(k+1)$. Then, by interpolating this between $L^{p'k}(\mathbb{G})$ and $L^{p'(k+1)}(\mathbb{G})$ and taking into account $(k+1)!\le\Gamma(2+q/p')$, we get

$$\|f\|_{L^q(\mathbb{G})}\le (C_\varepsilon\Gamma(2+q/p'))^{1/p'k}(\widetilde{\alpha}-\varepsilon)^{-1/p'}\|\mathcal{R}^{\frac{Q}{\nu p}}f\|_{L^p(\mathbb{G})}^{1-p/q}\|f\|_{L^p(\mathbb{G})}^{p/q}, \tag{3.18}$$

where Γ is the gamma function. Applying here the Stirling's formula and $p'k\ge q-p'$, we obtain that there is r such that

$$\|f\|_{L^q(\mathbb{G})}\le ((p'e(\widetilde{\alpha}-\varepsilon))^{-1/p'}+\delta)q^{1-1/p}\|\mathcal{R}^{\frac{Q}{\nu p}}f\|_{L^p(\mathbb{G})}^{1-p/q}\|f\|_{L^p(\mathbb{G})}^{p/q} \tag{3.19}$$

holds for all $f\in L^p_{Q/p}(\mathbb{G})$ and q with $r\le q<\infty$. Thus, $A\le (p'e(\widetilde{\alpha}-\varepsilon))^{-1/p'}+\delta$, then by arbitrariness of ε and δ we obtain $\widetilde{\alpha}\le(ep'A^{p'})^{-1}$.

This completes the proof of Theorem 3.8. □

4. Brezis–Gallouet–Wainger inequalities

In this section, we investigate Brezis–Gallouet–Wainger inequalities, which concern the limiting case of the Sobolev estimates (see [7, 8, 10]). As part of the proof we extend the analysis of Folland [22] related to Hölder spaces from the setting of stratified to general homogeneous groups. For the background analysis on homogeneous groups we refer to Folland and Stein's fundamental book [23] as well as to a more recent treatment in [20].

We recall some definitions from [22], see also [20, Chap. 3].

Definition 4.1. Let $\mathbb{G}$ be a nilpotent Lie group and let λ be a complex number.

- A measurable function f on $\mathbb{G}$ is called homogeneous of degree λ if $f\circ D_r=r^\lambda f$ for all positive $r>0$, where D_r is the family of dilations on $\mathbb{G}$.

- A distribution $\tau \in \mathcal{D}'$ is called homogeneous of degree λ if $\langle \tau, \phi \circ D_r \rangle = r^{-Q-\lambda}\langle \tau, \phi\rangle$ for all $\phi \in \mathcal{D}$ and all positive $r > 0$.
- A distribution which is smooth away from the origin and homogeneous of degree $\lambda - Q$ is called a kernel of type λ on $\mathbb{G}$.

We also need the following results:

Proposition 4.2 ([22, Proposition 1.4]). *Let $\mathbb{G}$ be a nilpotent Lie group and let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. Then, there is a positive constant C such that*

$$|xy| \leq C(|x| + |y|), \quad \forall\, x, y \in \mathbb{G}.$$

Proposition 4.3 ([22, Proposition 1.15]). *Let $\mathbb{G}$ be a nilpotent Lie group and let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. For any $f \in C^2(\mathbb{G}\backslash\{0\})$ homogeneous of degree $\lambda \in \mathbb{R}$, there are constants $C, \varepsilon > 0$ such that*

$$|f(xy) - f(x)| \leq C|y||x|^{\lambda-1} \quad \textit{whenever } |y| \leq \frac{1}{2}|x|, \tag{4.1}$$

$$|f(xy) + f(xy^{-1}) - 2f(x)| \leq C|y|^2|x|^{\lambda-2} \quad \textit{whenever } |y| \leq \varepsilon|x|. \tag{4.2}$$

Let us now state the first main result of this section.

Theorem 4.4. *Let $\mathbb{G}$ be a graded Lie group of homogeneous dimension Q and let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν. Let $a, p, q \in \mathbb{R}$ with $1 < p, q < \infty$ and $a > Q/q$. Then there exists $C_3 > 0$ such that we have*

$$\|f\|_{L^\infty(\mathbb{G})} \leq C_3(1 + \log(1 + \|\mathcal{R}^{a/\nu} f\|_{L^q(\mathbb{G})}))^{1/p'} \tag{4.3}$$

for all functions $f \in L^p_{Q/p}(\mathbb{G}) \cap L^q_a(\mathbb{G})$ with $\|f\|_{L^p_{Q/p}(\mathbb{G})} \leq 1$.

Remark 4.5. In the case $\mathbb{G} = (\mathbb{R}^n, +)$ and $\mathcal{R} = -\triangle$ is the Laplacian, the inequality (4.3) was obtained in [10] by employing Fourier transform methods, and in [18] for $n/p, m \in \mathbb{Z}$, and in [37] for the general case without using the Fourier transform.

We recall the Lipschitz spaces and obtain an estimate on nilpotent group $\mathbb{G}$, which will be used in the proof of Theorem 4.4. So, let $C_b(\mathbb{G})$ be the space of bounded continuous functions on $\mathbb{G}$. Then we define

$$\Gamma_\alpha := \left\{ f \in C_b(\mathbb{G}) : |f|_\alpha := \sup_{x,y} |f(xy) - f(x)|/|y|^\alpha < \infty \right\}$$

for $0 < \alpha < 1$, and when $\alpha = 1$, we define

$$\Gamma_1 := \left\{ f \in C_b(\mathbb{G}) : |f|_1 := \sup_{x,y} |f(xy) + f(xy^{-1}) - 2f(x)|/|y| < \infty \right\}.$$

Note that the Lipschitz space Γ_α with $0 < \alpha \leq 1$ is a Banach space with norm $\|f\|_{\Gamma_\alpha(\mathbb{G})} = \|f\|_{L^\infty(\mathbb{G})} + |f|_\alpha$.

Lemma 4.6. *Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension Q. Let $0 < \lambda < Q$, $0 < \alpha \leq 1$ and $1 < p \leq \infty$ with $\alpha = \lambda - Q/p$. Let K be a kernel of type*

λ. Then we have

$$|Tf|_\alpha \leq C\|f\|_{L^p(\mathbb{G})} \tag{4.4}$$

for the mapping $T : f \mapsto f * K$.

Proof. First, let us consider the case $0 < \alpha < 1$. We know that

$$Tf(xy) - Tf(x) = \int_{\mathbb{G}} f(xz^{-1})(K(zy) - K(z))dz. \tag{4.5}$$

As in the case of stratified groups (see [22, Theorem 5.14]), we write it as follows

$$\begin{aligned} Tf(xy) - Tf(x) = &\int_{|z|>2|y|} f(xz^{-1})(K(zy) - K(z))dz \\ &+ \int_{|z|\leq 2|y|} f(xz^{-1})(K(zy) - K(z))dz =: I_1 + I_2. \end{aligned} \tag{4.6}$$

Then, we estimate I_1 using (4.1) and Hölder's inequality

$$\begin{aligned} |I_1| &\leq C\|f\|_{L^p(\mathbb{G})} \left(\int_{|z|>2|y|} |y|^{p'} |z|^{(\lambda-Q-1)p'} dz \right)^{1/p'} \\ &\leq C\|f\|_{L^p(\mathbb{G})} |y| (2|y|)^{\lambda-Q-1+Q/p'} \leq C\|f\|_{L^p(\mathbb{G})} |y|^\alpha. \end{aligned} \tag{4.7}$$

Now for I_2, by Proposition 4.2 and $|z| \leq 2|y|$ there exists a constant $M_1 \geq 2$ such that $|zy| \leq M_1|y|$, so by Hölder's inequality we have

$$\begin{aligned} |I_2| &\leq \|f\|_{L^p(\mathbb{G})} \left(2\int_{|z|\leq M_1|y|} |K(z)|^{p'} dz \right)^{1/p'} \\ &\leq C\|f\|_{L^p(\mathbb{G})} \left(\int_{|z|\leq M_1|y|} |z|^{(\lambda-Q)p'} dz \right)^{1/p'} \leq C\|f\|_{L^p(\mathbb{G})} |y|^\alpha. \end{aligned} \tag{4.8}$$

Combining (4.7) and (4.8), we obtain (4.4) for $0 < \alpha < 1$.

Now, in the case $\alpha = 1$, we prove it in the same way as above but using (4.2). So, we write for some positive number M_2:

$$\begin{aligned} &Tf(xy) + Tf(xy^{-1}) - 2Tf(x) \\ &\quad = \int_{\mathbb{G}} f(xz^{-1})(K(zy) + K(zy^{-1}) - 2K(z))dz \\ &\quad = \int_{|z|>M_2|y|} f(xz^{-1})(K(zy) + K(zy^{-1}) - 2K(z))dz \\ &\qquad + \int_{|z|\leq M_2|y|} f(xz^{-1})(K(zy) + K(zy^{-1}) - 2K(z))dz \\ &\quad =: I_3 + I_4. \end{aligned} \tag{4.9}$$

For I_3, using (4.2) and Hölder's inequality one calculates

$$\begin{aligned} |I_3| &\leq C\|f\|_{L^p(\mathbb{G})}\left(\int_{|z|>M_2|y|}|y|^{2p'}|z|^{(\lambda-Q-2)p'}dz\right)^{1/p'} \\ &\leq C\|f\|_{L^p(\mathbb{G})}|y|^2(M_2|y|)^{\lambda-Q-2+Q/p'} \leq C\|f\|_{L^p(\mathbb{G})}|y|, \end{aligned} \tag{4.10}$$

since $\alpha=1$. Now for I_4, by Proposition 4.2 and $|z|\leq M_2|y|$ there exists a constant $M_3\geq M_2$ such that $|zy|\leq M_3|y|$ and $|zy^{-1}|\leq M_3|y|$, so by Hölder's inequality we have

$$\begin{aligned} |I_4| &\leq \|f\|_{L^p(\mathbb{G})}\left(4\int_{|z|\leq M_3|y|}|K(z)|^{p'}dz\right)^{1/p'} \\ &\leq C\|f\|_{L^p(\mathbb{G})}\left(\int_{|z|\leq M_3|y|}|z|^{(\lambda-Q)p'}dz\right)^{1/p'} \leq C\|f\|_{L^p(\mathbb{G})}|y|. \end{aligned} \tag{4.11}$$

From (4.10) and (4.11), we obtain (4.4) for $\alpha=1$. $\square$

Now we are ready to prove Theorem 4.4.

Proof of Theorem 4.4. Let us first prove (4.3) when $a-Q/q=:\alpha\in(0,1)$. Since $\mathcal{R}^{-a/\nu}$ is the Riesz potential (see e.g. [20, Sec. 4.3.4]), by Lemma 4.6 we have

$$|f(xy)-f(x)|\leq C\|\mathcal{R}^{a/\nu}f\|_{L^q(\mathbb{G})}|y|^{\alpha}. \tag{4.12}$$

Let $z\in\mathbb{G}$ with $|z|\leq 1$ and $0<\beta<e^{-p}$. Then, (4.12) implies that

$$|f(x)-f(x\beta z)|\leq C\beta^{\alpha}\|\mathcal{R}^{a/\nu}f\|_{L^q(\mathbb{G})}. \tag{4.13}$$

Applying Hölder's inequality and Theorem 3.3, one gets

$$\begin{aligned} \int_{|z|\leq 1}|f(x\beta z)|dz &\leq \left(\frac{|\mathfrak{S}|}{Q}\right)^{1/r'}\left(\int_{|z|\leq 1}|f(x\beta z)|^r dz\right)^{1/r} \\ &\leq \left(\frac{|\mathfrak{S}|}{Q}\right)^{1/r'}\beta^{-Q/r}\|f\|_{L^r(\mathbb{G})} \\ &\leq C\beta^{-Q/r}r^{1/p'}\|f\|_{L^p_{Q/p}(\mathbb{G})} \leq C\beta^{-Q/r}r^{1/p'} \end{aligned} \tag{4.14}$$

for any r with $p\leq r<\infty$. We take $r=\log(1/\beta)$ to obtain

$$\int_{|z|\leq 1}|f(x\beta z)|dz\leq C\left(\log\left(\frac{1}{\beta}\right)\right)^{1/p'} \tag{4.15}$$

for any β with $0<\beta<e^{-p}$, where the constant C depends only on p and Q.

Then, by a direct calculation, one has

$$\begin{aligned}|f(x)| &= |f(x)| \cdot \frac{Q}{|\mathfrak{S}|} \int_{|z|\le 1} dz \\ &\le \frac{Q}{|\mathfrak{S}|} \int_{|z|\le 1} (|f(x) - f(x\beta z)| + |f(x\beta z)|)dz \\ &\le C\beta^\alpha \|\mathcal{R}^{a/\nu} f\|_{L^q(\mathbb{G})} + C\left(\log\left(\frac{1}{\beta}\right)\right)^{1/p'},\end{aligned}$$

which implies (4.3), after setting $\beta = 1/(e^p + \|\mathcal{R}^{a/\nu} f\|_{L^q(\mathbb{G})}^{1/\alpha})$.

Now it remains to consider the case $a - Q/q \ge 1$. Let $a_0 \in \mathbb{R}$ with $Q/q+1 > a_0 > Q/q$. As in [12, Sec. 2] (see also [11, Sec. 3]), we consider the operator $\chi_L(\mathcal{R})$ for every $L > 0$, defined by functional calculus, where χ_L is the characteristic function of $[0, L]$. Then, taking into account these and using (4.3) for $a \ge Q/q+1 > a_0 > Q/q$ as already proved, we have

$$\begin{aligned}&\|(1 - \chi_L(\mathcal{R}))f\|_{L^\infty(\mathbb{G})} \\ &\quad \le C(1 + \log(1 + \|\mathcal{R}^{a_0/\nu}((1 - \chi_L(\mathcal{R}))f)\|_{L^q(\mathbb{G})}))^{1/p'} \\ &\quad \le C(1 + \log(1 + \|\mathcal{R}^{a/\nu}((1 - \chi_L(\mathcal{R}))f)\|_{L^q(\mathbb{G})}))^{1/p'}. \qquad (4.16)\end{aligned}$$

For $\chi_L(\mathcal{R})f$, we use the Nikolskii inequality on graded groups [12, Theorem 2.1]:

$$\|\chi_L(\mathcal{R})f\|_{L^\infty(\mathbb{G})} \lesssim \|\chi_L(\mathcal{R})f\|_{L^p} \lesssim \|f\|_{L^p(\mathbb{G})} \le 1. \qquad (4.17)$$

Finally, we obtain

$$\begin{aligned}\|f\|_{L^\infty(\mathbb{G})} &\le \|(1 - \chi_L(\mathcal{R}))f\|_{L^\infty(\mathbb{G})} + \|\chi_L(\mathcal{R})f\|_{L^\infty(\mathbb{G})} \\ &\le C(1 + \log(1 + \|\mathcal{R}^{a/\nu} f\|_{L^q(\mathbb{G})}))^{1/p'},\end{aligned}$$

completing the proof. □

We can also obtain the following estimate using Theorem 3.3:

Theorem 4.7. *Let $\mathbb{G}$ be a graded Lie group of homogeneous dimension Q and let $1 < p < \infty$. Then there exists a constant C_4 depending only on p and Q such that*

$$\int_\Omega |f(x)|dx \le C_4 \|f\|_{L^p_{Q/p}(\mathbb{G})} |\Omega|(1 + |\log|\Omega||)^{1/p'} \qquad (4.18)$$

holds for any function $f \in L^p_{Q/p}(\mathbb{G})$ and for any Lebesgue measurable set Ω with $|\Omega| < \infty$, where $|\Omega|$ denotes the Lebesgue measure of Ω.

Remark 4.8. When $\mathbb{G} = (\mathbb{R}^n, +)$ and $\mathcal{R} = -\triangle$ is the Laplacian, the inequality (4.18) was obtained in [10, Lemma 2] and in [37, Theorem 3]. In [10], using this estimate and Morrey's technique the authors proved the Brezis–Wainger inequality: for any function $f \in L^p_{n/p+1}(\mathbb{R}^n)$ and for each $x, y \in \mathbb{R}^n$, the inequality

$$|f(x) - f(y)| \leq C\|f\|_{L^p_{n/p+1}(\mathbb{R}^n)}|x-y|(1+|\log|x-y||)^{1/p'}, \quad 1 < p < \infty,$$

holds true, where the constant C depends only on n and p.

Proof of Theorem 4.7. Using Hölder's inequality, we get

$$\int_\Omega |f(x)|dx \leq |\Omega|^{1/q'}\|f\|_{L^q(\mathbb{G})}. \tag{4.19}$$

Then, by Theorem 3.3, we can deduce that

$$\int_\Omega |f(x)|dx \leq C|\Omega|^{1/q'}q^{1/p'}\|f\|_{L^p_{Q/p}(\mathbb{G})}, \tag{4.20}$$

for $p \leq q < \infty$ with the constant C depending only p and Q. Plugging $q = p$ into (4.19) for $|\Omega| > e^{-p}$, hence $|\Omega|^{1/q'} = |\Omega|^{1-1/p} < e|\Omega|$, one gets

$$\int_\Omega |f(x)|dx \leq e|\Omega|\|f\|_{L^p(\mathbb{G})}. \tag{4.21}$$

Now plugging $q = \log(1/|\Omega|)$ into (4.20) for $0 < |\Omega| \leq e^{-p}$, we have

$$\int_\Omega |f(x)|dx \leq C|\Omega||\log|\Omega||^{1/p'}\|f\|_{L^p_{Q/p}(\mathbb{G})}. \tag{4.22}$$

Combining (4.21) and (4.22) we obtain (4.18). □

5. Best Constants and Nonlinear Schrödinger Type Equations

In this section using the critical case of the Gagliardo–Nirenberg inequality (3.3) we show the existence of least energy solutions for nonlinear Schrödinger type equations, and we obtain a sharp expression for the smallest positive constant C_1 in (3.3). For non-critical case on nilpotent Lie group, when $a \neq Q/p$ in the inequality (3.1), similar results were obtained in [14] on the Heisenberg group and in [41] on graded groups.

Let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν, on a graded group $\mathbb{G}$ of homogeneous dimension Q. Let $1 < p < q < \infty$. We consider the nonlinear equation with the power nonlinearity

$$\mathcal{R}^{\frac{Q}{\nu p}}(|\mathcal{R}^{\frac{Q}{\nu p}}u|^{p-2}\mathcal{R}^{\frac{Q}{\nu p}}u) + |u|^{p-2}u = |u|^{q-2}u, \quad u \in L^p_{Q/p}(\mathbb{G}). \tag{5.1}$$

Such an equation appears naturally in the analysis of best constants of the established critical inequalities. We prove the existence of least energy solutions to (5.1) and their relation to the best constants in the Gagliardo–Nirenberg inequalities and hence, in view of Theorem 3.8, also to the best constants in the Trudinger inequalities.

For example, for $p = 2$, if $\mathbb{G}$ is a stratified Lie group of homogeneous dimension Q and $\mathcal{R} = \mathcal{L}$ is the positive sub-Laplacian, Eq. (5.1) becomes

$$\mathcal{L}^{\frac{Q}{2}} u + u = |u|^{q-2}u, \quad u \in L^2_{Q/2}(\mathbb{G}), \tag{5.2}$$

and if $p \neq 2$, Eq. (5.1) can be regarded as the p-sub-Laplacian version of (5.2). In the Euclidean case, when $Q = 2$, $q = 4$ and $\mathfrak{L} = -\Delta$, one can note that if $u(x)$ is a solution of (5.2) then the function $w(x) = u(x)e^{it/2}$ solves the following nonlinear Schrödinger equation

$$2iw_t + \Delta w + |w|^2 w = 0, \quad t \in \mathbb{R}^+, \quad x \in \mathbb{R}^2.$$

Such equations arise in modeling the propagation of a thin electromagnetic beam through a medium with an index of refraction dependent on the field intensity (see for example [15] and [49, Sec. V]).

Now, let us give some notations and definitions for this section.

Definition 5.1. A function $u \in L^p_{Q/p}(\mathbb{G})$ is said to be a solution of (5.1) if and only if for any function $\psi \in L^p_{Q/p}(\mathbb{G})$ the equality

$$\begin{aligned}\int_{\mathbb{G}} (|\mathcal{R}^{\frac{Q}{\nu p}} u(x)|^{p-2} \mathcal{R}^{\frac{Q}{\nu p}} u(x) \overline{\mathcal{R}^{\frac{Q}{\nu p}} \psi(x)} dx + \int_{\mathbb{G}} (|u(x)|^{p-2} u(x) \overline{\psi(x)} \\ - |u(x)|^{q-2} u(x) \overline{\psi(x)}) dx = 0\end{aligned} \tag{5.3}$$

holds true.

We define the functionals $\mathfrak{L} : L^p_{Q/p}(\mathbb{G}) \to \mathbb{R}$ and $\mathfrak{I} : L^p_{Q/p}(\mathbb{G}) \to \mathbb{R}$ acting on $L^p_{Q/p}(\mathbb{G}) \cap L^q(\mathbb{G})$ as follows:

$$\mathfrak{L}(u) := \frac{1}{p} \int_{\mathbb{G}} |\mathcal{R}^{\frac{Q}{\nu p}} u(x)|^p dx + \frac{1}{p} \int_{\mathbb{G}} |u(x)|^p dx - \frac{1}{q} \int_{\mathbb{G}} |u(x)|^q dx \tag{5.4}$$

and

$$\mathfrak{I}(u) := \int_{\mathbb{G}} (|\mathcal{R}^{\frac{Q}{\nu p}} u(x)|^p + |u(x)|^p - |u(x)|^q) dx. \tag{5.5}$$

We use the Nehari set

$$\mathcal{N} := \{u \in L^p_{Q/p}(\mathbb{G}) \setminus \{0\} : \mathfrak{I}(u) = 0\} \tag{5.6}$$

and denote

$$d := \inf\{\mathfrak{L}(u) : u \in \mathcal{N}\}. \tag{5.7}$$

Definition 5.2. Let

$$\Gamma = \{\phi \in L^p_{Q/p}(\mathbb{G}) : \mathfrak{L}'(\phi) = 0 \text{ and } \phi \neq 0\}$$

and

$$\mathcal{G} = \{u \in \Gamma : \mathfrak{L}(u) \leq \mathfrak{L}(v) \text{ for any } v \in \Gamma\}$$

be the set of the solutions and the set of least energy solutions of (5.1), respectively.

Theorem 5.3. *Let* $\mathbb{G}$ *be a graded Lie group of homogeneous dimension* Q, *let* $1 < p < q < \infty$. *Then the Schrödinger type equation* (5.1) *has a least energy solution* $\phi \in L^p_{Q/p}(\mathbb{G})$.

Furthermore, we have $d = \mathfrak{L}(\phi)$.

In the sequel, we assume $1 < p < q < \infty$.

Let us state and prove the following lemmas, which will be used in the proof of Theorem 5.3.

Lemma 5.4. *For any function* $u \in L^p_{Q/p}(\mathbb{G})\backslash\{0\}$, *there is a unique* $\mu_u > 0$ *such that* $\mu_u u \in \mathcal{N}$. *Moreover, we have* $0 < \mu_u < 1$ *when* $\mathfrak{I}(u) < 0$.

Proof. For any $u \in L^p_{Q/p}(\mathbb{G}) \setminus \{0\}$ and

$$\mu_u = \|u\|^{\frac{p}{q-p}}_{L^p_{Q/p}(\mathbb{G})}\|u\|^{-\frac{q}{q-p}}_{L^q(\mathbb{G})}, \tag{5.8}$$

we note that $\mu_u u \in \mathcal{N}$. It is easy to see that this μ_u is unique. Then, by (5.8) one gets $0 < \mu_u < 1$ provided that $\|u\|^p_{L^p_{Q/p}(\mathbb{G})} < \|u\|^q_{L^q(\mathbb{G})}$. □

Lemma 5.5. *For all functions* $u \in \mathcal{N}$, *we have* $\inf_u \|u\|_{L^p_{Q/p}(\mathbb{G})} > 0$.

Proof. Using (3.3) we calculate

$$\begin{aligned}
\|u\|^p_{L^p_{Q/p}(\mathbb{G})} &= \|u\|^q_{L^q(\mathbb{G})} \\
&\leq Cq^{q-q/p}\|\mathcal{R}^{\frac{Q}{\nu p}}u\|^{q-p}_{L^p(\mathbb{G})}\|u\|^p_{L^p(\mathbb{G})} \\
&\leq Cq^{q-q/p}\|u\|^{q-p}_{L^p_{Q/p}(\mathbb{G})}\|u\|^p_{L^p_{Q/p}(\mathbb{G})} \\
&\leq C\|u\|^q_{L^p_{Q/p}(\mathbb{G})}
\end{aligned}$$

for all $u \in \mathcal{N}$. From this we obtain $\|u\|^{q-p}_{L^p_{Q/p}(\mathbb{G})} \geq C^{-1}$, which implies $\|u\|_{L^p_{Q/p}(\mathbb{G})} \geq \kappa$ for any function $u \in \mathcal{N}$ after setting $\kappa = C^{-\frac{1}{q-p}} > 0$. □

Let us show the following Rellich–Kondrachev type lemma. On the Heisenberg group a similar result was obtained by Garofalo and Lanconelli [24].

Lemma 5.6. *Let* $1 < p < q < \infty$. *Then, we have the compact embedding* $L^p_{Q/p}(D) \hookrightarrow L^q(D)$ *for any smooth bounded domain* $D \subset \mathbb{G}$, *where*

$$L^p_{Q/p}(D) = \{f \in L^p_{Q/p}(\mathbb{G}) : \operatorname{supp} f \subset D\}.$$

Proof. Because of the density argument, it is enough to prove $\mathring{L}^p_{Q/p}(D) \hookrightarrow L^q(D)$, where $\mathring{L}^p_{Q/p}(D)$ denotes the closure of $C_0^\infty(D)$ with respect to the norm (1.2) with $a = Q/p$.

Let $\phi \in C_0^\infty(\mathbb{G})$, with $0 \leq \phi \leq 1$, supp $\phi \subset \overline{B(0,1)}$ and $\int_{\mathbb{G}} \phi(x)dx = 1$. We define

$$K_\varepsilon f := \phi_\varepsilon * f(x), \tag{5.9}$$

where $f \in L^1_{\mathrm{loc}}(\mathbb{G})$ and $\phi_\varepsilon(x) := \varepsilon^{-Q}\phi(\varepsilon^{-1}x)$ for every $\varepsilon > 0$.

We will also use the following lemma, which is an analogue of [24, Lemma 3.1] for graded groups.

Lemma 5.7. *Let $D \subset \mathbb{G}$ be a bounded open set and $1 < q < \infty$. Then, $Z \subset L^q(D)$ is relatively compact in $L^q(D)$, if and only if*

(1) *Z is bounded*;
(2) *$\|K_\varepsilon f - f\|_q \to 0$ as $\varepsilon \to 0$, uniformly in $f \in Z$.*

Proof. Let us briefly sketch the proof of the lemma. To show the necessity, we extend functions in $L^q(D)$ with zero outside D. We can take $f_1, \ldots, f_n$ from Z and $r > 0$, so that the balls in $L^q(D)$ centered at f_k with radius r cover Z. For a given f, let us take f_k such that $\|f_k - f\|_q < r$. Then we have

$$\|f - K_\varepsilon f\|_q \leq \|f - f_k\|_q + \|f_k - K_\varepsilon f_k\|_q + \|K_\varepsilon f_k - K_\varepsilon f\|_q.$$

Taking into account $\|K_\varepsilon f\|_q \leq \|f\|_q$ and $K_\varepsilon f \to f$ in $L^q(D)$ as $\varepsilon \to 0$ also letting $r \to 0$, we get (2).

We now show sufficiency. Let f_n be a bounded sequence in Z. By the Banach–Alouglu theorem and the reflexivity of L^q, $1 < q < \infty$, we know that there exists a subsequence, still denoted by f_n weakly convergent in L^q, that is, there exists $f \in L^q$ such that

$$\int (f_n - f)h \to 0, \quad \forall h \in L^{q'}. \tag{5.10}$$

Let us now show that it actually converges strongly. For this, we write

$$\|f_n - f\|_q \leq \|f_n - K_\varepsilon f_n\|_q + \|K_\varepsilon f_n - K_\varepsilon f\|_q + \|K_\varepsilon f - f\|_q. \tag{5.11}$$

By the assumption (2), we note that the first and third summands vanish when $\varepsilon \to 0$. For the second summand, (5.10) implies that for all x and for all ε we have $K_\varepsilon(f_n - f)(x) \to 0$ as $n \to \infty$. Since we also have

$$\||K_\varepsilon(f_n - f)|^q\|_{L^1} = \|K_\varepsilon(f_n - f)\|^q_{L^q} \leq \|\phi_\varepsilon\|^q_{L^1}\|f_n - f\|^q_{L^q} < \infty,$$

by the Lebesgue dominated convergence theorem we observe that

$$\int |K_\varepsilon(f_n - f)(x)|^q dx \to 0$$

as $n \to \infty$.

Thus, from (5.11) we can conclude that Z is relatively compact in $L^q(D)$. $\square$

We now come back to the proof of Lemma 5.6. Setting $f \equiv 0$ in $\mathbb{G}\backslash D$ for $f \in \mathring{L}^p_{Q/p}(D)$, we get a function in $L^p_{Q/p}(\mathbb{G})$. Let us now use Lemma 5.7. Let Z be a bounded set in $\mathring{L}^p_{Q/p}(D)$. Then, using $|B(x,r)| = r^Q|B(0,1)|$ for the Haar measure of any open quasi-ball (see e.g. [20, p. 140]) and the critical Gagliardo–Nirenberg inequality (3.3), we note that Z is bounded in $L^q(D)$.

To complete the proof, it remains to verify the second part of Lemma 5.7. For $f \in Z$ by denoting

$$\psi_\varepsilon := K_\varepsilon f - f$$

and using (3.3), we obtain

$$\|\psi_\varepsilon\|_{L^q(D)} \leq C\|\psi_\varepsilon\|^{1-q/p}_{\dot{L}^p_{Q/p}(D)}\|\psi_\varepsilon\|^{q/p}_{\dot{L}^p(D)}. \tag{5.12}$$

Therefore, it is enough to show that

$$\|K_\varepsilon f - f\|_{\dot{L}^p_{Q/p}(D)} \to 0,$$

that is,

$$\|\mathcal{R}^{\frac{Q}{\nu p}}K_\varepsilon f - \mathcal{R}^{\frac{Q}{\nu p}}f\|_{L^p(D)} \to 0.$$

Indeed, it holds since by [20, Part (i) of Lemma 3.1.58] we have

$$\mathcal{R}^{\frac{Q}{\nu p}}K_\varepsilon f - \mathcal{R}^{\frac{Q}{\nu p}}f = \phi_\varepsilon * \mathcal{R}^{\frac{Q}{\nu p}}f - \mathcal{R}^{\frac{Q}{\nu p}}f \to 0$$

as $\varepsilon \to 0$.

Therefore, by Lemma 5.7 we can conclude that Z is relatively compact in $L^q(D)$.

We also note the following property of least energy solutions.

Lemma 5.8. *If $v \in \mathcal{N}$ and $\mathfrak{L}(v) = d$ then v must be a least energy solution of the Schrödinger type equation* (5.1).

Proof. By the Lagrange multiplier rule there is $\theta \in \mathbb{R}$ such that for any $\psi \in L^p_{Q/p}(\mathbb{G})$ we have

$$\langle \mathfrak{L}'(v), \psi\rangle_{\mathbb{G}} = \theta\langle \mathfrak{I}'(v), \psi\rangle_{\mathbb{G}}, \tag{5.13}$$

due to the assumption on v, where $\langle\cdot,\cdot\rangle_{\mathbb{G}}$ is a dual product between $L^p_{Q/p}(\mathbb{G})$ and its dual space.

Taking into account $q > p$, one gets

$$\langle \mathfrak{I}'(v), v\rangle_{\mathbb{G}} = p\|v\|^p_{L^p_{Q/p}(\mathbb{G})} - q\int_{\mathbb{G}}|v|^q dx = (p-q)\int_{\mathbb{G}}|v|^q dx < 0. \tag{5.14}$$

On the other hand, we have

$$\langle \mathfrak{L}'(v), v\rangle_{\mathbb{G}} = \mathfrak{I}(v) = 0. \tag{5.15}$$

Combining (5.14) and (5.15), we obtain $\theta = 0$ from (5.13). It implies that $\mathfrak{L}'(v) = 0$. By Definition 5.2, we obtain that v is a least energy solution of the nonlinear equation (5.1). □

Now we are ready to prove Theorem 5.3.

Proof of Theorem 5.3. We choose $(v_k)_k \subset \mathcal{N}$ as a minimizing sequence. By Ekeland variational principle we obtain a sequence $(u_k)_k \subset \mathcal{N}$ satisfying $\mathfrak{L}(u_k) \to d$ and $\mathfrak{L}'(u_k) \to 0$. Applying the Sobolev inequality and Lemma 5.5 we see that there exist two positive constants A_1 and A_2 with the properties

$$A_1 \leq \|u_k\|_{L^p_{Q/p}(\mathbb{G})} \leq A_2.$$

From this and the equality

$$\|u_k\|^p_{L^p_{Q/p}(\mathbb{G})} = \int_{\mathbb{G}} |u_k(x)|^q dx,$$

we obtain the existence of a positive constant A_3 so that

$$\limsup_{k\to\infty} \int_{\mathbb{G}} |u_k(x)|^q dx \geq A_3 > 0. \tag{5.16}$$

Applying Lemma 5.6 and the concentration compactness argument of [46, Lemma 3.1], we have that $u_k \to 0$ in $L^q(\mathbb{G})$ with $1 < p < q < \infty$ if the following

$$\lim_{k\to\infty} \sup_{\eta\in\mathbb{G}} \int_{B(\eta,r)} |u_k(x)|^q dx = 0 \tag{5.17}$$

holds true for some $r > 0$, where $B(\eta, r)$ is a quasi-ball on $\mathbb{G}$ centered at η with radius r. By (5.16) there is a constant $A_4 > 0$ and $r > 1$ such that

$$\liminf_{k\to\infty} \sup_{\eta\in\mathbb{G}} \int_{B(\eta,r)} |u_k(x)|^q dx \geq A_4 > 0. \tag{5.18}$$

We hence may assume that there are $\tilde{x}^k \in \mathbb{G}$ with

$$\liminf_{k\to\infty} \int_{B(\tilde{x}^k,r)} |u_k(x)|^q dx \geq \frac{A_4}{2} > 0. \tag{5.19}$$

Taking into account the bi-invariance of the Haar measure and the left invariance of the operator $\mathcal{R}$ one has for all $\tilde{x}^k \in \mathbb{G}$

$$\mathfrak{L}(u_k(\tilde{x}x)) = \mathfrak{L}(u_k(x)) \quad \text{and} \quad \mathfrak{I}(u_k(\tilde{x}x)) = \mathfrak{I}(u_k(x)).$$

Let us denote $\omega_k(x) := u_k(\tilde{x}x)$. Then we have $\mathfrak{L}(\omega_k) = \mathfrak{L}(u_k)$ and $\mathfrak{I}(\omega_k) = \mathfrak{I}(u_k)$. Moreover, it gives the bounded sequence $(\omega_k)_k$ from $L^p_{Q/p}(\mathbb{G})$ with

$$\liminf_{k\to\infty} \int_{B(0,r)} |\omega_k(x)|^q dx \geq \frac{A_4}{2} > 0. \tag{5.20}$$

There exists a subsequence, denoted by ω_k that weakly converges to ϕ in $L^p_{Q/p}(\mathbb{G})$. Then from Lemma 5.6 we see that ω_k strongly converges to ϕ in $L^q_{\text{loc}}(\mathbb{G})$. By this and (5.20), we have $\phi \neq 0$.

Now let us show that ω_k converges strongly to ϕ in $L^p_{Q/p}(\mathbb{G})$. First we show that $\mathfrak{I}(\phi) = 0$. We proceed by contradiction. Suppose that $\mathfrak{I}(\phi) < 0$. Lemma 5.4

gives that there is a positive number $\mu_\phi < 1$ with $\mu_\phi \phi \in \mathcal{N}$ for $\mathfrak{I}(\phi) < 0$. Since $\mathfrak{I}(\omega_k) = 0$, applying the Fatou lemma we calculate

$$\begin{aligned} d + o(1) = \mathfrak{L}(\omega_k) &= \left(\frac{1}{p} - \frac{1}{q}\right) \int_{\mathbb{G}} |\omega_k(x)|^q dx \\ &\geq \left(\frac{1}{p} - \frac{1}{q}\right) \int_{\mathbb{G}} |\phi(x)|^q dx + o(1), \end{aligned}$$

that is,

$$\begin{aligned} d + o(1) &\geq \left(\frac{1}{p} - \frac{1}{q}\right) \mu_\phi^{-q} \int_{\mathbb{G}} |\mu_\phi \phi(x)|^q dx + o(1) \\ &= \mu_\phi^{-q} \mathfrak{L}(\mu_\phi \phi) + o(1), \end{aligned} \tag{5.21}$$

which implies $d > \mathfrak{L}(\mu_\phi \phi)$. Since $\mu_\phi \phi \in \mathcal{N}$, we arrive at a contradiction.

Suppose now that $\mathfrak{I}(\phi) > 0$. We need the following lemma:

Lemma 5.9 ([9, Lemma 3]). *Let $\ell : \mathbb{C} \to \mathbb{R}$ be convex. Then*

$$|\ell(a+b) - \ell(a)| \leq \varepsilon[\ell(ma) - m\ell(a)] + |\ell(C_\varepsilon b)| + |\ell(-C_\varepsilon b)|$$

for any $a, b \in \mathbb{C}$, $0 < \varepsilon < \frac{1}{m} < 1$ and $\frac{1}{C_\varepsilon} = \varepsilon(m-1)$.

As in [14, Proof of Theorem 1.2], using this lemma for $\psi_k = \omega_k - \phi$ we have

$$0 = \mathfrak{I}(\omega_k) = \mathfrak{I}(\phi) + \mathfrak{I}(\psi_k) + o(1),$$

where $\mathfrak{I}(\phi) > 0$. Then, one has

$$\limsup_{k\to\infty} \mathfrak{I}(\psi_k) < 0. \tag{5.22}$$

Applying Lemma 5.4, there exists a sequence $\mu_k := \mu_{\psi_k}$ with $\mu_k \psi_k \in \mathcal{N}$ and $\limsup_{k\to\infty} \mu_k \in (0,1)$. Indeed, assume that $\limsup_{k\to\infty} \mu_k = 1$. Then we have a subsequence $(\mu_{k_j})_{j\in\mathbb{N}}$ with the property $\lim_{j\to\infty} \mu_{k_j} = 1$. Since $\mu_{k_j} \psi_{k_j} \in \mathcal{N}$ we get that $\mathfrak{I}(\psi_{k_j}) = \mathfrak{I}(\mu_{k_j} \psi_{k_j}) + o(1) = o(1)$, which is a contradiction because of (5.22). So, we have that $\limsup_{k\to\infty} \mu_k \in (0,1)$. A direct calculation gives that

$$\begin{aligned} d + o(1) = \mathfrak{L}(\omega_k) &= \left(\frac{1}{p} - \frac{1}{q}\right) \int_{\mathbb{G}} |\omega_k(x)|^q dx \\ &\geq \left(\frac{1}{p} - \frac{1}{q}\right) \int_{\mathbb{G}} |\psi_k(x)|^q dx, \end{aligned}$$

that is,

$$\begin{aligned} d + o(1) &\geq \left(\frac{1}{p} - \frac{1}{q}\right) \mu_k^{-q} \int_{\mathbb{G}} |\mu_k \psi_k(x)|^q dx + o(1) \\ &= \mu_k^{-q} \mathfrak{L}(\mu_k \psi_k) + o(1). \end{aligned} \tag{5.23}$$

Therefore, the fact $\limsup_{k\to\infty} \mu_k \in (0,1)$ gives that $d > \mathfrak{L}(\mu_k \psi_k)$. It contradicts $\mu_k \phi_k \in \mathcal{N}$.

Thus, we must have $\mathfrak{I}(\phi) = 0$. Now we prove that $\psi_k = \omega_k - \phi \to 0$ in the space $L^p_{Q/p}(\mathbb{G})$. Indeed, suppose that $\|\psi_k\|_{L^p_{Q/p}(\mathbb{G})}$ does not vanish as $k \to \infty$. Then we consider the following cases. The first one is when $\int_{\mathbb{G}} |\psi_k(x)|^q dx$ does not converge to 0 as $k \to \infty$. Then the following identity

$$0 = \mathfrak{I}(\omega_k) = \mathfrak{I}(\phi) + \mathfrak{I}(\psi_k) + o(1) = \mathfrak{I}(\psi_k) + o(1)$$

with the Brezis–Lieb lemma implies the contradiction

$$d + o(1) = \mathfrak{L}(\omega_k) = \mathfrak{L}(\phi) + \mathfrak{L}(\psi_k) + o(1) \geq d + d + o(1).$$

In the case, when $\int_{\mathbb{G}} |\psi_k(x)|^q dx$ converges to 0 as $k \to \infty$, again we obtain a contradiction:

$$\begin{aligned} d + o(1) = \mathfrak{L}(\omega_k) &= \mathfrak{L}(\phi) + \frac{1}{p}\|\psi_k\|^p_{L^p_{Q/p}(\mathbb{G})} + o(1) \\ &\geq d + \frac{1}{p}\|\psi_k\|^p_{L^p_{Q/p}(\mathbb{G})} + o(1) > d, \end{aligned}$$

when $\|\psi_k\|_{L^p_{Q/p}(\mathbb{G})} \nrightarrow 0$ as $k \to \infty$. Thus, we conclude that ω_k converges strongly to ϕ in $L^p_{Q/p}(\mathbb{G})$ and $d = \mathfrak{L}(\phi)$, where ϕ is a least energy solution of (5.1) by Lemma 5.8. $\square$

Theorem 5.10. *Let $1 < p < q < \infty$. Let ϕ be a least energy solution of* (5.1) *and let $C_{GN,\mathcal{R}}$ be the smallest positive constant C_1 in* (3.3)*. Then we have*

$$\begin{aligned} C_{GN,\mathcal{R}} &= q^{-q+q/p}\frac{q}{p}\left(\frac{q-p}{p}\right)^{\frac{p-q}{p}} \|\phi\|^{p-q}_{L^p(\mathbb{G})} \\ &= q^{-q+q/p}\frac{q}{p}\left(\frac{q-p}{p}\right)^{\frac{p-q}{p}} \left(\frac{p^2}{q-p}d\right)^{\frac{p-q}{p}}, \end{aligned} \tag{5.24}$$

where d is defined in (5.7)*.*

We will use the following lemmas to prove Theorem 5.10:

Lemma 5.11. *Let ϕ be a least energy solution of* (5.1)*. Then we have*

$$\int_{\mathbb{G}} |\mathcal{R}^{\frac{Q}{\nu p}}\phi(x)|^p dx = \frac{q-p}{p}\int_{\mathbb{G}} |\phi(x)|^p dx \tag{5.25}$$

and

$$\int_{\mathbb{G}} |\phi(x)|^q dx = \frac{q}{p}\int_{\mathbb{G}} |\phi(x)|^p dx. \tag{5.26}$$

Moreover, we have

$$\int_{\mathbb{G}} |\phi(x)|^p dx = \frac{p^2}{q-p}d. \tag{5.27}$$

Proof. Since ϕ is a least energy solution of (5.1), we have

$$\int_{\mathbb{G}} |\mathcal{R}^{\frac{Q}{\nu p}}\phi(x)|^p dx + \int_{\mathbb{G}} |\phi(x)|^p dx = \int_{\mathbb{G}} |\phi(x)|^q dx. \tag{5.28}$$

On the other hand, a direct calculation gives for $\lambda > 0$ and $\widetilde{\phi}_\lambda(x) := \lambda^{\frac{Q}{p}}\phi(\lambda x)$ that

$$\begin{aligned}\mathfrak{L}(\widetilde{\phi}_\lambda(x)) &= \frac{\lambda^Q}{p}\int_{\mathbb{G}} |\mathcal{R}^{\frac{Q}{\nu p}}(\phi(\lambda x))|^p dx \\ &\quad + \frac{\lambda^Q}{p}\int_{\mathbb{G}} |\phi(\lambda x)|^p dx - \frac{\lambda^{\frac{Qq}{p}}}{q}\int_{\mathbb{G}} |\phi(\lambda x)|^q dx \\ &= \frac{\lambda^Q}{p}\int_{\mathbb{G}} |\mathcal{R}^{\frac{Q}{\nu p}}\phi(x)|^p dx + \frac{1}{p}\int_{\mathbb{G}} |\phi(x)|^p dx - \frac{\lambda^{\frac{Qq}{p}-Q}}{q}\int_{\mathbb{G}} |\phi(x)|^q dx,\end{aligned}$$

which implies that

$$\begin{aligned}0 &= \frac{\partial}{\partial\lambda}\mathfrak{L}(\widetilde{\phi}_\lambda)|_{\lambda=1} \\ &= \frac{Q}{p}\int_{\mathbb{G}} |\mathcal{R}^{\frac{Q}{\nu p}}\phi(x)|^p dx - \frac{Q(q-p)}{pq}\int_{\mathbb{G}} |\phi(x)|^q dx.\end{aligned} \tag{5.29}$$

Thus, from (5.28) and (5.29) we obtain (5.25) and (5.26), respectively.

In order to show (5.27), using (5.4) and (5.26), we calculate

$$\begin{aligned}d = \mathfrak{L}(\phi) &= \frac{1}{p}\|\phi\|^p_{L^p_{Q/p}(\mathbb{G})} - \frac{1}{q}\|\phi\|^q_{L^q(\mathbb{G})} \\ &= \left(\frac{1}{p} - \frac{1}{q}\right)\|\phi\|^q_{L^q(\mathbb{G})} = \frac{q-p}{p^2}\int_{\mathbb{G}} |\phi(x)|^p dx.\end{aligned}$$

Then, it follows that

$$\int_{\mathbb{G}} |\phi(x)|^p dx = \frac{p^2}{q-p}d,$$

which is (5.27). $\square$

Lemma 5.12. *Let $T_{\rho,p,q}$ be defined as*

$$T_{\rho,p,q} := \inf\left\{\|u\|^p_{L^p_{Q/p}(\mathbb{G})} : u \in L^p_{Q/p}(\mathbb{G}) \text{ and } \int_{\mathbb{G}} |u(x)|^q dx = \rho\right\}, \tag{5.30}$$

for $\rho > 0$. If ϕ is a minimizer obtained in Theorem 5.3, then ϕ is a minimizer of $T_{\rho_0,p,q}$ such that $\rho_0 = \int_{\mathbb{G}} |\phi(x)|^q dx$.

Proof. From the definition of $T_{\rho_0,p,q}$, one has $\|\phi\|^p_{L^p_{Q/p}(\mathbb{G})} \geq T_{\rho_0,p,q}$. We claim that $T_{\rho_0,p,q} \geq \|\phi\|^p_{L^p_{Q/p}(\mathbb{G})}$. Indeed, using Lemma 5.4 for any $u \in L^p_{Q/p}(\mathbb{G})$ satisfying $\int_{\mathbb{G}} |u(x)|^q dx = \int_{\mathbb{G}} |\phi(x)|^q dx$ there is a unique

$$\lambda_0 = \|u\|^{\frac{p}{q-p}}_{L^p_{Q/p}(\mathbb{G})}\left(\int_{\mathbb{G}} |u|^q dx\right)^{-\frac{1}{q-p}}$$

with $\mathfrak{I}(\lambda_0 u) = 0$. Since we know that $\lambda_0 u \neq 0$ and ϕ achieves the minimum d, then a direct calculation gives

$$\begin{aligned}\left(\frac{1}{p}-\frac{1}{q}\right)\|\phi\|^p_{L^p_{Q/p}(\mathbb{G})} &= \mathfrak{L}(\phi) \leq \mathfrak{L}(\lambda_0 u) = \left(\frac{1}{p}-\frac{1}{q}\right)\lambda_0^p\|u\|^p_{L^p_{Q/p}(\mathbb{G})}\\ &= \left(\frac{1}{p}-\frac{1}{q}\right)\|u\|^{\frac{p^2}{q-p}}_{L^p_{Q/p}(\mathbb{G})}\left(\int_{\mathbb{G}}|u(x)|^q dx\right)^{-\frac{p}{q-p}}\|u\|^p_{L^p_{Q/p}(\mathbb{G})}.\end{aligned}$$

Then, from $\int_{\mathbb{G}}|u(x)|^q dx = \int_{\mathbb{G}}|\phi(x)|^q dx$ and $\int_{\mathbb{G}}|\phi(x)|^q dx = \|\phi\|^p_{L^p_{Q/p}(\mathbb{G})}$, we obtain $\|\phi\|^p_{L^p_{Q/p}(\mathbb{G})} \leq \|u\|^p_{L^p_{Q/p}(\mathbb{G})}$. From the arbitrariness of the function u one gets $T_{\rho_0,p,q} \geq \|\phi\|^p_{L^p_{Q/p}(\mathbb{G})}$. Thus, by $T_{\rho_0,p,q} = \|\phi\|^p_{L^p_{Q/p}(\mathbb{G})}$, we conclude that ϕ is a minimizer of $T_{\rho_0,p,q}$. □

Now let us prove Theorem 5.10.

Proof of Theorem 5.10. For $u \neq 0$ we define

$$J(u) := q^{q-q/p}\left(\int_{\mathbb{G}}|\mathcal{R}^{\frac{Q}{\nu p}}u(x)|^p dx\right)^{\frac{q-p}{p}}\left(\int_{\mathbb{G}}|u(x)|^p dx\right)\left(\int_{\mathbb{G}}|u(x)|^q dx\right)^{-1}. \tag{5.31}$$

We estimate the sharp expression $C_{GN,\mathcal{R}}$ by studying the following minimization problem

$$C^{-1}_{GN,\mathcal{R}} = \inf\{J(u) : u \in L^p_{Q/p}(\mathbb{G}) \setminus \{0\}\}.$$

Taking into account $\phi(x) \neq 0$ and $\phi \in L^p_{Q/p}(\mathbb{G})$, and using Lemma 5.11 we get

$$J(\phi) = q^{q-q/p-1}p\left(\frac{q-p}{p}\right)^{\frac{q-p}{p}}\left(\int_{\mathbb{G}}|\phi(x)|^p dx\right)^{\frac{q-p}{p}},$$

which gives that

$$C^{-1}_{GN,\mathcal{R}} \leq q^{q-q/p-1}p\left(\frac{q-p}{p}\right)^{\frac{q-p}{p}}\left(\int_{\mathbb{G}}|\phi(x)|^p dx\right)^{\frac{q-p}{p}}. \tag{5.32}$$

Now we obtain a lower estimate for $C^{-1}_{GN,\mathcal{R}}$. We denote $\omega(x) := \lambda u(\mu x)$ for $\lambda, \mu > 0$ and for all $u \in L^p_{Q/p}(\mathbb{G}) \setminus \{0\}$. Then, we calculate

$$\begin{aligned}\int_{\mathbb{G}}|\mathcal{R}^{\frac{Q}{\nu p}}\omega(x)|^p dx &= \lambda^p\int_{\mathbb{G}}|\mathcal{R}^{\frac{Q}{\nu p}}u(x)|^p dx,\\ \int_{\mathbb{G}}|\omega(x)|^p dx &= \lambda^p\mu^{-Q}\int_{\mathbb{G}}|u(x)|^p dx\end{aligned} \tag{5.33}$$

and

$$\int_{\mathbb{G}}|\omega(x)|^q dx = \lambda^q\mu^{-Q}\int_{\mathbb{G}}|u(x)|^q dx.$$

Choosing λ and μ such that

$$\lambda^p \mu^{-Q} \int_{\mathbb{G}} |u(x)|^p dx = \int_{\mathbb{G}} |\phi(x)|^p dx \tag{5.34}$$

and

$$\lambda^q \mu^{-Q} \int_{\mathbb{G}} |u(x)|^q dx = \int_{\mathbb{G}} |\phi(x)|^q dx, \tag{5.35}$$

and using (5.26), we obtain

$$\begin{aligned}\lambda^q \mu^{-Q} \int_{\mathbb{G}} |u(x)|^q dx &= \int_{\mathbb{G}} |\phi(x)|^q dx = \frac{q}{p} \int_{\mathbb{G}} |\phi(x)|^p dx \\ &= \frac{q}{p} \lambda^p \mu^{-Q} \int_{\mathbb{G}} |u(x)|^p dx,\end{aligned}$$

which implies that

$$\lambda^p = \left(\frac{q}{p}\right)^{\frac{p}{q-p}} \left(\int_{\mathbb{G}} |u(x)|^p dx\right)^{\frac{p}{q-p}} \left(\int_{\mathbb{G}} |u(x)|^q dx\right)^{-\frac{p}{q-p}}. \tag{5.36}$$

From (5.34) and (5.35) one has

$$\int_{\mathbb{G}} |\omega(x)|^p dx = \int_{\mathbb{G}} |\phi(x)|^p dx \quad \text{and} \quad \int_{\mathbb{G}} |\omega(x)|^q dx = \int_{\mathbb{G}} |\phi(x)|^q dx.$$

Since ϕ is a minimizer of $T_{\rho_0,p,q}$ such that $\rho_0 = \int_{\mathbb{G}} |\phi(x)|^q dx$, then using Lemma 5.12, we have

$$\int_{\mathbb{G}} |\mathcal{R}^{\frac{Q}{\nu p}} \omega(x)|^p dx \geq \int_{\mathbb{G}} |\mathcal{R}^{\frac{Q}{\nu p}} \phi(x)|^p dx.$$

Then by (5.33) and (5.25), one calculates

$$\begin{aligned}\lambda^p \int_{\mathbb{G}} |\mathcal{R}^{\frac{Q}{\nu p}} u(x)|^p dx &= \int_{\mathbb{G}} |\mathcal{R}^{\frac{Q}{\nu p}} \omega(x)|^p dx \\ &\geq \int_{\mathbb{G}} |\mathcal{R}^{\frac{Q}{\nu p}} \phi(x)|^p dx = \frac{q-p}{p} \int_{\mathbb{G}} |\phi(x)|^p dx.\end{aligned}$$

Plugging here (5.36), we obtain that

$$\begin{aligned}&\left(\frac{q}{p}\right)^{\frac{p}{q-p}} \left(\int_{\mathbb{G}} |u(x)|^p dx\right)^{\frac{p}{q-p}} \times \left(\int_{\mathbb{G}} |u(x)|^q dx\right)^{-\frac{p}{q-p}} \int_{\mathbb{G}} |\mathcal{R}^{\frac{Q}{\nu p}} u(x)|^p dx \\ &\quad \geq \frac{q-p}{p} \int_{\mathbb{G}} |\phi(x)|^p dx.\end{aligned}$$

Now, by the definition of $J(u)$ in (5.31), we arrive at

$$J(u) \geq q^{q-q/p-1} p \left(\frac{q-p}{p}\right)^{\frac{q-p}{p}} \left(\int_{\mathbb{G}} |\phi(x)|^p dx\right)^{\frac{q-p}{p}}.$$

Using again the arbitrariness of the function u, we get

$$C_{GN,\mathcal{R}}^{-1} \geq q^{q-q/p-1} p \left(\frac{q-p}{p}\right)^{\frac{q-p}{p}} \left(\int_{\mathbb{G}} |\phi(x)|^p dx\right)^{\frac{q-p}{p}}. \tag{5.37}$$

Combining (5.32) and (5.37), one obtains the first equality in (5.29). Then, this first equality with (5.27) implies the second equality in (5.29). □

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