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On the chromatic number of some generalized Kneser graphs

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Abstract

We determine the chromatic number of the Kneser graph $q\Gamma_{7,\{3,4\}}$ of flags of vectorial type $\{3, 4\}$ of a rank 7 vector space over the finite field $\text{GF}(q)$ for large q and describe the colorings that attain the bound. This result relies heavily, not only on the independence number, but also on the structure of all *large* independent sets. Furthermore, our proof is more general in the following sense: it provides the chromatic number of the Kneser graphs $q\Gamma_{2d+1,\{d,d+1\}}$ of flags of vectorial type $\{d, d+1\}$ of a rank $2d+1$ vector space over $\text{GF}(q)$ for large q as long as the *large* independent sets of the graphs are only the ones that are known.

KEYWORDS

chromatic number, q -analog of generalized Kneser graph

1 | INTRODUCTION

The introduction is split into two parts. In the first part we introduce most of the required notation and in the second part we state our main results and give an outline of the strategy of our proof.

1.1 | Notation

Let V be a vector space of some dimension $n \in \mathbb{N}$. A *flag* of V is a set f of nontrivial proper subspaces of V such that $U \leq W$ or $W \leq U$ for all $U, W \in f$. The *vectorial type* of a flag f is the set $\{\text{rank}(U) \mid U \in f\}$, which is a subset of $\{1, 2, \dots, n-1\}$. Furthermore, two flags f_1 and f_2 are said to be in *general position*, if $U_1 \cap U_2 = \{0\}$ or $U_1 + U_2 = V$ for all $U_1 \in f_1$ and $U_2 \in f_2$.

Given $n \geq 3$, a subset $J \subseteq \{1, \dots, n-1\}$ and a finite field $\text{GF}(q)$, the q -Kneser graph $qK_{n,J}$ is the graph whose vertex set is the set of all flags of vectorial type J of the vector space $\text{GF}(q)^n$ and in which two vertices are adjacent if they are in general position. Note that we only consider finite graphs in this work.

Now, let Γ be a graph with vertex set X . An *independent set* of Γ is a set of pairwise nonadjacent vertices of the graph. The *independence number* $\alpha(\Gamma)$ of Γ is the cardinality of its largest independent sets. A *coloring* of Γ is a map $g : X \rightarrow C$ such that $g^{-1}(c)$ is an independent set for all $c \in C$. The smallest cardinality of a set C such that there exists a coloring $g : X \rightarrow C$ is called the *chromatic number* of Γ and is denoted by $\chi(\Gamma)$. Clearly $\chi(\Gamma)$ is the smallest integer χ such that X is the union of χ maximal independent sets.

Finally, let V be a vector space of rank $2d+1 \geq 3$ over $\text{GF}(q)$. For every rank 1 subspace P we denote by $F(P)$ the set of all flags of V of vectorial type $\{d, d+1\}$ whose rank d subspace contains P and call this set a *point-pencil*. Dually for every rank $2d$ subspace H of V we denote by $F(H)$ the set of all flags of vectorial type $\{d, d+1\}$ whose rank $d+1$ subspace is contained in H and call this set a *dual point-pencil*.

Notice that point-pencils and dual point-pencils are independent sets of cardinality $\approx q^{d^2+d-1}$ but they are not maximal independent sets. In fact, for $d=2$ every maximal independent set containing a point-pencil or a dual point-pencil has cardinality $\alpha(q\Gamma_{2d+1,\{d,d+1\}})$, see [5]. However, for $d \geq 3$ this is no longer true. There are different maximal independent sets containing a point-pencil or a dual point-pencil and they do not all have the same size. Nevertheless, the structure of these examples can still be described quite precisely (as we explain in Section 2).

1.2 | Results and strategy of proof

We determine the chromatic number of the Kneser graph $qK_{7,\{3,4\}}$ for large values of q . To give an outline of the strategy of our approach we consider a graph Γ with vertex set X .

Clearly, for any coloring $g : X \rightarrow C$ the set C satisfies the bound $|C| \geq \frac{|X|}{\alpha(\Gamma)}$. Now, the key tool in our proof is, that there is some value $\alpha' < \alpha(\Gamma)$ such that structural information on any independent set of Γ of size larger than α' is known. In that situation, if $g : X \rightarrow C$ is a coloring of Γ such that $\alpha' \cdot |C| \ll |X|$, then at least $\frac{|X| - \alpha' \cdot |C|}{\alpha(\Gamma) - \alpha'}$ color classes of g have cardinality larger than α' and satisfy the given structural conditions. We use this structural information to provide a lower bound on $|C|$ that coincides with the cardinality of a known coloring of Γ and thus in fact determine $\chi(\Gamma)$.

Note that a similar approach was successfully applied for many Kneser graphs $q\Gamma_{n,J}$ with $|J|=1$ in [2, 3] as well as in one special case for $|J|=2$ in [5]. However, for many Kneser graphs $q\Gamma_{n,J}$, with $|J| \geq 2$ the independence number is not known and only in very few cases both the independence number as well as structural information on independent sets of maximal size is given. One reason for the lack of this structural information is, that in some cases the independence number was determined by algebraic arguments and these do not automatically give the structure of the largest independent sets, see for example the recent work [4].

Now, for the case covered in this work, that is in the Kneser graph $q\Gamma_{7,\{3,4\}}$, the required structural information as well as the bound α' is provided in [8]. Furthermore, we remark that the proof provided here does not only cover that case, but also every graph $q\Gamma_{2d+1,\{d,d+1\}}$, for which an analogous result to that given for $d=3$ in [8] holds. We state this requirement more precisely in the following conjecture.

Conjecture 1.1. *For every integer $d \geq 2$ there is an integer $\rho(d)$ such that every maximal independent set of the Kneser graph $q\Gamma_{2d+1,\{d,d+1\}}$ contains a point-pencil, a dual point-pencil, or has at most $\rho(d) \cdot q^{d^2+d-2}$ elements.*

Note that it has been shown that this conjecture holds for $d = 2$, which was implicitly proven in [1], as well as for $d = 3$, as is shown in [8].

Using this conjecture we may now state our main result as follows.

Theorem 1.2. *If Conjecture 1.1 holds for some integer $d \geq 3$, then*

$$\chi(q\Gamma_{2d+1,\{d,d+1\}}) = \frac{q^{d+2} - 1}{q - 1} - q$$

provided $q > 3 \cdot 112^{2^{d+1}-1} \cdot 2^{-d-1}$ and $q \geq \frac{3}{2}\alpha^2 + \frac{21}{2}\alpha + 17$ where $\alpha = \max\{5, \rho(d)\}$. Moreover, if $\mathcal{F}$ is a family of this many maximal independent sets that cover the vertex set, then—up to duality—there exists a rank $d + 2$ subspace U of the underlying vector space and an injective map μ from $\mathcal{F}$ to the set of rank 1 subspaces of U such that $F(\mu(C)) \subseteq C$ for all $C \in \mathcal{F}$.

Since Conjecture 1.1 holds for $d = 3$, we find the following corollary.

Theorem 1.3. *For $q > 3 \cdot 7^{15} \cdot 2^{56}$ we have $\chi(q\Gamma_{7,\{3,4\}}) = q^4 + q^3 + q^2 + 1$.*

Finally, we note that, regardless of whether or not Conjecture 1.1 holds for some integer d , in Section 2 we will see that $q\Gamma_{2d+1,\{d,d+1\}}$ can be covered with $\frac{q^{d+2}-1}{q-1} - q$ maximal independent sets in different ways. In fact, we thus have $\chi(q\Gamma_{2d+1,\{d,d+1\}}) \leq \frac{q^{d+2}-1}{q-1} - q$ for all integers d and, if this holds with equality, then there are different optimal colorings of this graph. We provide examples of these colorings as well as some structural information on maximal independent sets in Section 2.

2 | INDEPENDENT SETS AND COLORINGS

To gain geometric intuition we switch to projective language, that is we pass from vector spaces of rank $2d + 1$ over the finite field $\text{GF}(q)$ to the projective space $\text{PG}(2d, q)$ of (projective) dimension $2d$. Note that throughout this work we use *dimension* whenever we refer to projective dimension and rank whenever we refer to the rank of a vector space.

In this setting the Kneser graph $q\Gamma_{2d+1,\{d,d+1\}}$ is isomorphic to the graph $\Gamma_d(q)$, which we define as follows. The vertices of $\Gamma_d(q)$ are the pairs (π, τ) of subspaces π and τ of $\text{PG}(2d, q)$ of respective dimensions $d - 1$ and d with $\pi \subseteq \tau$. Two vertices (π, τ) and (π', τ') of $\Gamma_d(q)$ are adjacent if and only if $\pi \cap \tau' = \pi' \cap \tau = \emptyset$, where $\emptyset$ is the empty subspace of $\text{PG}(2d, q)$. The vertex set of $\Gamma_d(q)$ will be denoted by $X(\Gamma_d(q))$. The vertices (π, τ) of this graph are called flags, too, and their (projective) type is $\{d - 1, d\}$. Note that in the following we will always refer to this (projective) type, when we say type.

Example 2.1 (Independent sets of $\Gamma_d(q)$).

1. For a point P and a set $\mathcal{U}$ of d -dimensional subspaces through P , such that for all $\tau, \tau' \in \mathcal{U}$ we have $\dim(\tau \cap \tau') \geq 1$, we define

$$F(P, \mathcal{U}) := \{(\pi, \tau) \in X(\Gamma_d(q)) \mid P \in \pi \text{ or } \tau \in \mathcal{U}\}.$$

We call $\{(\pi, \tau) \in F(P, \mathcal{U}) \mid P \in \pi\}$ the *generic part* and $\{(\pi, \tau) \in F(P, \mathcal{U}) \mid P \notin \pi\}$ the *special part* of $F(P, \mathcal{U})$. We also say that $F(P, \mathcal{U})$ is *based on the point* P and call P the *base point* of $F(P, \mathcal{U})$.

If there exists a line ℓ on P such that $\mathcal{U}$ consists of all d -dimensional subspaces τ with $\ell \subseteq \tau$, then we denote $F(P, \mathcal{U})$ also by $F(P, \ell)$ and say that the special part of this set is *based on the line* ℓ .

If there exists a hyperplane H on P such that $\mathcal{U}$ consists of all d -dimensional subspaces τ with $P \in \tau \subseteq H$, then we denote $F(P, \mathcal{U})$ also by $F(P, H)$ and say that the special part of this set is *based on the hyperplane* H .

2. Dually, for a hyperplane H and a set $\mathcal{E}$ of subspaces of dimension $d - 1$ of H with pairwise nonempty intersection, we define

$$F(H, \mathcal{E}) := \{(\pi, \tau) \in X(\Gamma_d(q)) \mid \tau \subseteq H \text{ or } \pi \in \mathcal{E}\}.$$

We call $\{(\pi, \tau) \in F(H, \mathcal{E}) \mid \tau \subseteq H\}$ the *generic part* and $\{(\pi, \tau) \in F(H, \mathcal{E}) \mid \tau \not\subseteq H\}$ the *special part* of $F(H, \mathcal{E})$. We also say that $F(H, \mathcal{E})$ is *based on the hyperplane* H .

Notation 2.2. To formulate the cardinality of the sets in the previous examples in a compact form, we use the *Gaussian binomial coefficient*

$$\begin{bmatrix} a \\ b \end{bmatrix}_q := \prod_{i=1}^b \frac{q^{a-b+i} - 1}{q^i - 1}$$

for integers $q \geq 2$ and $a \geq b \geq 0$. Whenever q is clear from the context we omit the index q and set $\theta_d := \begin{bmatrix} d+1 \\ 1 \end{bmatrix}$.

Lemma 2.3. *With the notation of Example 2.1 we have:*

- (a) $F(P, \mathcal{U})$ is an independent set of $\Gamma_d(q)$. Its generic part has cardinality $\begin{bmatrix} 2d \\ d+1 \end{bmatrix} \theta_d$ and its special part has cardinality $|\mathcal{U}|q^d$.
- (b) If the special part of $F(P, \mathcal{U})$ is based on a line or a hyperplane, then $|\mathcal{U}| = \begin{bmatrix} 2d-1 \\ d-1 \end{bmatrix}$. Furthermore, if $F(P, \mathcal{U})$ is maximal but its special part is based neither on a line or a hyperplane, then

$$|\mathcal{U}| < (1 + q^{-1})\theta_{d-2}\theta_{d-1}^{d-1}. \quad (1)$$

Proof.

- (a) It is immediate that $F(P, \mathcal{U})$ is an independent set of $\Gamma_d(q)$. To verify the cardinality of its generic part we notice that the number of $(d-1)$ -dimensional subspaces on P is $\begin{bmatrix} 2d \\ d+1 \end{bmatrix}$ and that each of these subspaces lies in θ_d subspaces of dimension d . For the cardinality of the special part we notice that each subspace $\tau \in \mathcal{U}$ contains exactly q^d subspaces π of dimension $d-1$ with $P \notin \pi$.
- (b) If the special part is based on a line, then its cardinality is the number of d -subspaces on a line, which is $\begin{bmatrix} 2d-1 \\ d-1 \end{bmatrix}$. Furthermore, if it is based on a hyperplane H on P , then its cardinality is the number of d -subspaces on P in H , which is the same number. Now suppose that $F(P, \mathcal{U})$ is maximal but its special part is based neither on a line nor a hyperplane. Consider a d -subspace τ on P that does not lie in $\mathcal{U}$ and let π be a $(d-1)$ -subspace of τ with $P \notin \pi$. Then $(\pi, \tau) \notin F(P, \mathcal{U})$. The maximality of $F(P, \mathcal{U})$ implies that (π, τ) is in general position to some $(\pi', \tau') \in F(P, \mathcal{U})$. Then $\pi' \cap \tau = \emptyset$ and hence $P \notin \pi'$. Thus (π', τ') lies in the special part of $F(P, \mathcal{U})$ and thus $\tau' \in \mathcal{U}$. As (π, τ) and (π', τ') are in general position, this implies that $\tau \cap \tau' = P$. Hence $\mathcal{U}$ is a maximal set of d -subspaces on P such that any two subspaces of $\mathcal{U}$ intersect in at least a line. Finally, a result of Blokhuis, Brouwer, and Szőnyi [3, Section 3] applied to $\mathcal{U}$ in the quotient space $\text{PG}(2d, q)/P$ implies (1). $\square$

By definition, the chromatic number of a graph is the smallest cardinality of a partition of its vertex set into independent sets, but of course it is also the smallest cardinality of a cover of its vertex set by independent sets. We now provide examples of such coverings.

Example 2.4 (Coverings of $X(\Gamma_d(q))$ by independent sets). Let $U \leq \text{PG}(2d, q)$ be a subspace of dimension $d+1$, consider a set W of q points of U and let L be the set of lines of U that meet W . Furthermore, suppose that there exists an injective map $\nu : L \rightarrow U \setminus W$ such that $\nu(l) \in l$ for all $l \in L$. Then

$$\{F(\nu(l), l) \mid l \in L\} \cup \{F(P, \emptyset) \mid P \in U \setminus (\nu(L) \cup W)\}$$

is a set of independent sets of $\Gamma_d(q)$ whose union contains all vertices of $\Gamma_d(q)$.

Remark 2.5.

- (a) There are different possibilities for (W, ν) satisfying the required condition in Example 2.4 and we provide an example.

Let $P_0, \dots, P_q$ be the points of a line $\ell \leq U$ and set $W = \{P_1, \dots, P_q\}$. In each plane π of U on ℓ there are q lines through P_0 distinct from ℓ and thus there is a bijective map h_π from W to the set of these lines. Now we may define ν by $\nu(\ell) = P_0$ and $\nu(l) := l \cap h_{\langle \ell, l \rangle}(l \cap \ell)$ for $l \in L \setminus \{\ell\}$. The pair (W, ν) has the properties required in Example 2.4 and moreover satisfies $U = \nu(L) \cup W$.

It is also possible to construct pairs (W, ν) satisfying $U \neq \nu(L) \cup W$, for example for odd $q \geq 5$ when W consists of q points of a conic in a plane of U , but we omit the details.

- (b) We can find different coverings with independent sets by dualizing the coverings described in Example 2.4 and part (a) of this remark.
- (c) Since there are $\begin{bmatrix} d+2 \\ 1 \end{bmatrix}_q - q$ independent sets in the given coverings, we find

$$\chi(\Gamma_d(q)) \leq \begin{bmatrix} d+2 \\ 1 \end{bmatrix}_q - q.$$

This was already noticed for $d = 2$ in [5].

3 | PRELIMINARIES

This section contains a result on point sets in Proposition 3.2 as well as some technical bounds in Lemmas 3.3 and 3.4, that we will need later on.

Lemma 3.1. *Consider a projective space $\mathcal{P}$ of order q , a set M of points of $\mathcal{P}$ and points $P_1, \dots, P_{s+1}$ of $\mathcal{P}$, $s \geq 0$, such that $\langle P_1, \dots, P_{s+1} \rangle$ is a subspace of dimension s with no point in M . Let μ be an upper bound on the number of lines on P_{s+1} that meet M . Let $0 < c \in \mathbb{R}$ and let $\mathcal{V}$ be a set of s -dimensional subspaces such that $P_1, \dots, P_s \in V$ and $|V \cap M| \geq cq^s$ for all $V \in \mathcal{V}$.*

Then for every real number γ with $0 < \gamma < 1$, there exist at least $\frac{1-\gamma}{q}|\mathcal{V}|$ subspaces W of dimension $s+1$ satisfying $P_1, \dots, P_{s+1} \in W$ and $|W \cap M| > \frac{\gamma}{\mu}c^2q^{2s}|\mathcal{V}|$.

Proof. We may assume that $\mathcal{V} \neq \emptyset$. For $V \in \mathcal{V}$ we have $V \cap M \neq \emptyset$ and hence $P_{s+1} \notin V$. Put $x := \gamma \frac{|\mathcal{V}|cq^s}{\mu}$,

$$\begin{aligned} \mathfrak{W} &:= \{\langle V, P_{s+1} \rangle : V \in \mathcal{V}\}, \\ \mathcal{W} &:= \{W \in \mathfrak{W} : |\{V \in \mathcal{V} : V \subseteq W\}| > x\}, \end{aligned}$$

and $\overline{\mathcal{W}} := \mathfrak{W} \setminus \mathcal{W}$. Note that the elements of $\mathfrak{W}$ are subspaces of dimension $s+1$. Now, for $W \in \mathfrak{W}$ we have $W = \langle V, P_{s+1} \rangle$ for some $V \in \mathcal{V}$ and P_{s+1} lies on at least $|V \cap M| \geq cq^s$ lines of W which meet M . Furthermore, if W and W' are distinct elements of $\mathfrak{W}$ and l is a line on P_{s+1} with $l \subseteq W, W'$, then $l \subseteq W \cap W' = \langle P_1, \dots, P_{s+1} \rangle$ and thus $l \cap M = \emptyset$. Since μ is an upper bound on the number of lines on P_{s+1} which meet M , this proves that $|\mathfrak{W}| \leq \frac{\mu}{c}q^{-s}$. Since every element of $\overline{\mathcal{W}}$ contains at most x elements of $\mathcal{V}$, it follows that

$$|\{V \in \mathcal{V} : \langle V, P_{s+1} \rangle \in \overline{\mathcal{W}}\}| \leq \frac{\mu}{c}q^{-s} \cdot x = \gamma|\mathcal{V}|$$

and hence $\langle V, P_{s+1} \rangle \in \mathcal{W}$ for least $(1-\gamma)|\mathcal{V}|$ elements of $\mathcal{V}$. Since every subspace $W \in \mathcal{W}$ contains at most q subspaces $V \in \mathcal{V}$, we find $|\mathcal{W}| \geq (1-\gamma)|\mathcal{V}|/q$. Finally, since distinct elements V and V' of $\mathcal{V}$ satisfy $(V \cap V') \cap M = \langle P_1, \dots, P_s \rangle \cap M = \emptyset$, we see that every $W \in \mathcal{W}$ satisfies

$$|W \cap M| > x \cdot cq^s = \frac{\gamma}{\mu}c^2q^{2s}|\mathcal{V}|.$$

□

Proposition 3.2. Suppose that M is a set of points in $\text{PG}(2d, q)$ and there are $d + 1$ points $P_1, P_2, \dots, P_{d+1}$ that span a d -dimensional subspace τ with $\tau \cap M = \emptyset$. Furthermore, let n_0 and d_0 be positive real numbers such that the following hold:

1. Each of the points $P_1, P_2, \dots, P_{d+1}$ lies on at most $n_0 q^d$ lines that meet M .
2. $|M| \geq d_0 q^{d+1}$.

Then there exists a $(d + 1)$ -dimensional subspace U on τ with

$$|U \cap M| > (2q)^{d+1} \left(\frac{d_0}{4n_0} \right)^{2^{d+1}-1}. \quad (2)$$

Proof. We prove the following more general result. For each $s \in \{0, \dots, d + 1\}$, there exists a set $\mathcal{V}_s$ of s -dimensional subspaces satisfying $|\mathcal{V}_s| \geq (\frac{1}{2})^s d_0 q^{d+1-s}$ such that each $V \in \mathcal{V}_s$ satisfies

$$\{P_i | 1 \leq i \leq s\} \subseteq V \quad \text{and} \quad |V \cap M| \geq (2q)^s \left(\frac{d_0}{4n_0} \right)^{2^s-1}. \quad (3)$$

We use induction on s . For $s = 0$ we take $\mathcal{V}_0 = M$. For the induction step $s \rightarrow s + 1$, we assume the existence of $\mathcal{V}_s$ with the desired properties. For $V \in \mathcal{V}_s$ we know from the induction hypothesis that (3) holds and, since $\tau \cap M = \emptyset$ by hypothesis of the present lemma, this also implies $V \not\subseteq \tau$, so that $P_{s+1} \notin V$. Now the previous lemma, applied with $c = 2^s \left(\frac{d_0}{4n_0} \right)^{2^s-1}$, $\mathcal{V} = \mathcal{V}_s$ and $\mu = n_0 q^d$ and $\gamma = \frac{1}{2}$, proves the existence of a set $\mathcal{V}_{s+1} = \mathcal{W}$ with the desired properties.

For $s = d + 1$ we find $|\mathcal{V}_{d+1}| > 0$, so each element U of $\mathcal{V}_{d+1}$ satisfies the claim of the lemma. $\square$

As mentioned earlier, we end this section with two technical lemmas that will be needed several times in Section 4.

Lemma 3.3. Let $n, k, c \geq 1$ and $q \geq 2$ be integers.

- (a) If $n > k > 0$ and $q \geq 4$ then

$$(q + 1)q^{k(n-k)-1} \leq \begin{bmatrix} n \\ k \end{bmatrix}_q \leq (q + 2)q^{k(n-k)-1}.$$

- (b) If $q > c^2 + c$ then

$$(q^2 + q + 2)^c \leq (q + c + 1)q^{2c-1}.$$

- (c) If $q > c^2 + c$ we have $\begin{bmatrix} c + 1 \\ 1 \end{bmatrix}^c \leq (q + c + 1)q^{c^2-1}$.

Proof.

- (a) The lower bound follows from $0 < k < n$ and for the upper bound we refer to [7, Lemma 34].
- (b) This can be checked easily for $c = 1$ and 2 , so we assume that $c \geq 3$. By expansion we see that $(q^2 + q + 2)^c = \sum_{i=0}^{2c} a_i q^i$, where

$$a_{2c-i} = \sum_{j=0}^{\lfloor i/2 \rfloor} \binom{c}{j} \binom{c-j}{i-2j} 2^j,$$

since a term q^{2c-i} occurs in the expansion, if for some j with $2j \leq i$ we first choose the number 2 from j factors $(q^2 + q + 2)$, second we choose the number q from $i - 2j$ of the remaining $c - j$ factors $(q^2 + q + 2)$ and finally we choose the number q^2 from the remaining factors $(q^2 + q + 2)$.

Now, we claim $a_{2c-i} \leq c^i$ for all i . Using $c \geq 3$, this can be verified for $i \leq 5$ by straightforward calculation. Thus, suppose that $i \geq 6$. Then

$$a_{2c-i} = \sum_{j=0}^{\lfloor i/2 \rfloor} \frac{c! \cdot 2^j}{(c+j-i)!(i-2j)!j!} \leq c^i \sum_{j=0}^{\lfloor i/2 \rfloor} \frac{2^j}{\underbrace{c^j(i-2j)!j!}_{=:b_{ij}}}. \quad (4)$$

We next show $b_{ij} \leq \frac{2}{i+2}$ for admissible i, j , that is, for i, j with $2j \leq i \leq 2c$ and $i \geq 6$. Using $i \geq 6$, this follows from the direct calculation if $j \leq 3$. Otherwise $j \geq 4$ and $i \geq 8$, so $j! \geq 2^j$ and hence $b_{ij} \leq c^{-j} \leq \frac{2}{i+2}$, since $i \leq 2c$. Thus we have established the bound for b_{ij} and using it in (4) we find $a_{2c-i} \leq c^i$ for $i \geq 6$. Hence $a_{2c-i} \leq c^i$ for all $i \in \{0, \dots, 2c\}$.

It follows that

$$\sum_{i=0}^{2c-2} a_i q^i \leq \sum_{i=2}^{2c} c^i q^{2c-i} = q^{2c-2} c^2 \sum_{i=0}^{2c-2} \frac{c^i}{q^i} \leq \frac{q^{2c-2} c^2}{1 - c/q} < q^{2c-1},$$

where we have used $q > c^2 + c$ in the last step. Since $a_{2c} = 1$ and $a_{2c-1} = c$, this proves the claim in (b).

- (c) Since $\begin{bmatrix} c+1 \\ 1 \end{bmatrix}_q \leq (q^2 + q + 2)q^{c-2}$ this is a corollary to the previous claim.

□

Lemma 3.4. *Let U be a subspace of dimension $d + 1$ of $\text{PG}(2d, q)$, let Y be a point of U and let T denote the set of all $(d - 1)$ -subspaces π with $\pi \cap U = Y$. Then the following holds:*

- (a) $|T| = q^{d^2-1}$.
- (b) If ℓ is a line with $\ell \cap U = Y$, then ℓ lies on q^{d^2-d-2} subspaces of T .
- (c) If ℓ is a line such that $\ell \cap U$ is a point different from Y , then q^{d^2-d-1} subspaces of T meet ℓ in a point.

- (d) If H is a hyperplane of $PG(2d, q)$ with $Y \in H$, then H contains no subspace of T if H contains U and it contains q^{d^2-d} subspaces of T if H does not contain U .

Proof. See Theorem 3.3 in Hirschfeld [6]. $\square$

4 | THE CHROMATIC NUMBER OF THE GRAPH $\Gamma_d(q)$

In this section we prove the main result of this paper. In particular, we show that, given a *large* value of q and an integer $d \geq 3$ for which Conjecture 1.1 holds, the chromatic number of $\Gamma_d(q)$ is $\left[\begin{smallmatrix} d+2 \\ 1 \end{smallmatrix} \right]_q - q$. From Remark 2.5 we have $\chi(\Gamma_d(q)) \leq \left[\begin{smallmatrix} d+2 \\ 1 \end{smallmatrix} \right]_q - q$.

In fact we will prove the following more general result, which does not depend on Conjecture 1.1.

Theorem 4.1. *Let $d \geq 3$ and $\alpha \geq 5$ be integers and let $\mathcal{F} = \left\{ F_i \mid i = 1, \dots, \left[\begin{smallmatrix} d+2 \\ 1 \end{smallmatrix} \right]_q - q \right\}$ be a multiset (so we allow $F_i = F_j$ for $i \neq j$) of $\left[\begin{smallmatrix} d+2 \\ 1 \end{smallmatrix} \right]_q - q$ independent sets of flags of type $\{d-1, d\}$ in $PG(2d, q)$, whose union consists of all flags of this type in $PG(2d, q)$. We put $S := \left\{ 1 \leq i \leq \left[\begin{smallmatrix} d+2 \\ 1 \end{smallmatrix} \right]_q - q : |F_i| > e_1 \right\}$ with*

$$e_1 := \alpha q^{d^2+d-2},$$

and suppose the following:

- (I) $q > 3 \cdot 112^{2d+1-1} \cdot 2^{-d-1}$ and $q \geq \frac{3}{2}\alpha^2 + \frac{21}{2}\alpha + 17$.
- (II) For $i \in S$ the set F_i is one of the independent sets defined in Example 2.1, which implies $g_0 \leq |F_i| \leq e_0$, where

$$g_0 := \left[\begin{smallmatrix} 2d \\ d+1 \end{smallmatrix} \right]_q \cdot \left[\begin{smallmatrix} d+1 \\ 1 \end{smallmatrix} \right]_q \quad \text{and} \quad e_0 := g_0 + \left[\begin{smallmatrix} 2d-1 \\ d-1 \end{smallmatrix} \right]_q q^d.$$

- (III) For distinct $i, j \in S$ the independent sets F_i and F_j have distinct generic parts.
- (IV) For at least $\frac{1}{2}|S|$ indices $i \in S$ the generic part of F_i is based on a point.

Then, each $F \in \mathcal{F}$ is a set described in Example 2.1.1 (i.e., it is based on a point P_F) and the points P_F , $F \in \mathcal{F}$, are $\left[\begin{smallmatrix} d+2 \\ 1 \end{smallmatrix} \right]_q - q$ mutually distinct points of a subspace of dimension $d+1$.

The proof of this theorem is carried out in Lemma 4.3 through Theorem 4.13. In all these lemmas and results, we suppose that $\mathcal{F}$ is as in the theorem, we assume that $d \geq 3$, $\alpha \geq 5$ and that (I)–(IV) are satisfied. Using (II) we define

$$I := \{i \in S \mid i \in S, \text{ the generic part of } F_i \text{ is based on a point } P_i\}.$$

Remark 4.2. As $q > \alpha$, then $e_1 < g_0 < e_0$. Hence, every $F \in \mathcal{F}$ satisfies $|F| \leq e_0$. More precisely, Hypothesis (II) shows that $g_0 \leq |F_i| \leq e_0$ for $i \in S$, and $|F| \leq e_1$ for the remaining sets F of $\mathcal{F}$.

4.1 | Construction of the subspace U

In this section we will construct a subspace U of dimension $d + 1$ that contains the base point P_i of F_i for many indices $i \in I$, similar to Example 2.4.

Lemma 4.3. *For every subset $\mathcal{G}$ of $\mathcal{F}$ we have*

$$\left| \bigcup_{F \in \mathcal{G}} F \right| \geq |\mathcal{G}|e_0 - \left(q^2 + \frac{9}{2}q + 10 \right) q^{d^2+2d-3}.$$

Proof. Since $|F| \leq e_0$ for every $F \in \mathcal{F}$, it is sufficient to prove the statement in the case when $\mathcal{G} = \mathcal{F}$. Then $|\mathcal{G}| = \theta_{d+1} - q$ and $|\bigcup_{F \in \mathcal{G}} F|$ is equal to the number $\left[\begin{smallmatrix} 2d+1 \\ d+1 \end{smallmatrix} \right] \theta_d$ of all flags of type $\{d-1, d\}$ in $\text{PG}(d, q)$. Using this, a direct calculation proves the statement when $d = 3$ or 4 . For $d \geq 5$ we use $|\mathcal{G}| \leq q\theta_d$ and find

$$\begin{aligned} |\mathcal{G}|e_0 - \left| \bigcup_{F \in \mathcal{G}} F \right| &\leq q\theta_d e_0 - \theta_d \left[\begin{smallmatrix} 2d+1 \\ d+1 \end{smallmatrix} \right] \\ &\leq \theta_d \left(\left[\begin{smallmatrix} 2d \\ d+1 \end{smallmatrix} \right] q\theta_d + \left[\begin{smallmatrix} 2d-1 \\ d \end{smallmatrix} \right] q^{d+1} - \left[\begin{smallmatrix} 2d+1 \\ d+1 \end{smallmatrix} \right] \right) \\ &= \theta_d \left[\begin{smallmatrix} 2d-1 \\ d \end{smallmatrix} \right] \left(\frac{q^{2d}-1}{q^{d+1}-1} q\theta_d + q^{d+1} - \frac{(q^{2d+1}-1)(q^d+1)}{q^{d+1}-1} \right) \\ &\leq \theta_d \left[\begin{smallmatrix} 2d-1 \\ d \end{smallmatrix} \right] (q^2 + q + 2) q^{2d-3} \quad (\text{use } d \geq 5 \text{ and } q \text{ and} \\ &\hspace{15em} \text{Theorem 4.1(I)}) \\ &\leq \theta_d (q+2)(q^2 + q + 2) q^{d(d+1)-4}. \end{aligned}$$

From Hypothesis (I) of Theorem 4.1 we deduce $\theta_d \leq (q + \frac{3}{2})q^{d-1}$. Using Hypothesis (I) again we see that $(q + \frac{3}{2})(q+2)(q^2 + q + 2) \leq q^2(q^2 + \frac{9}{2}q + 10)$ and the statement follows. $\square$

Lemma 4.4. *Let U be a $(d+1)$ -dimensional subspace. Denote by c_1 the number of indices $i \in I$ with $P_i \notin U$ and by c_3 the number of independent sets $F \in \mathcal{F}$ with $|F| \leq e_1$. Then there is some $x \in \{c_1, |I| - c_1\}$ with $x + 2c_3 \leq 2(q + 4 + \alpha)q^{d-1}$.*

Proof. From Hypothesis (IV) in Theorem 4.1 we know that $|I| \geq \frac{1}{2}(|\mathcal{F}| - c_3)$ and we define $J := \{i \in I : P_i \notin U\}$. Then, for all $j \in J$ and all $i \in I \setminus J$ the generic parts of the sets F_i and F_j share the flags (π, τ) with $P_i = \pi \cap U$ and $P_j \in \pi$ and Lemma 3.4(b) implies that there are $\theta_d q^{d^2-d-2}$ such flags. For given $j \in J$ it is obvious that distinct i in $I \setminus J$ yield

distinct $\theta_d q^{d^2-d-2}$ such flags. Hence, for all $j \in J$ the set F_j contains at least $|I \setminus J| \theta_d q^{d^2-d-2}$ flags that are contained in F_i for some $i \in I \setminus J$. Using Remark 4.2 it follows that

$$\left| \bigcup_{i \in I} F_i \right| \leq |I|e_0 - \underbrace{|J||I \setminus J|}_{=c_1(|I|-c_1)} \theta_d q^{d^2-d-2}$$

and Lemma 4.3 applied to the set $\mathcal{G} := \{F_i | i \in I\} \cup \{F \in \mathcal{F} : |F| \leq e_1\}$ shows

$$c_3(e_0 - e_1) + c_1(|I| - c_1)\theta_d q^{d^2-d-2} \leq A := \left(q^2 + \frac{9}{2}q + 10\right)q^{d^2+2d-3}. \quad (5)$$

In particular, we already have $c_3(e_0 - e_1) \leq A$ and we set $B := (q + 4 + \alpha)q^{d-1}$. From the definition of e_0 and e_1 we see that $e_0 - e_1 \geq ((q + 1)^2 - \alpha q)q^{d^2+d-3}$. Hypothesis (I) of Theorem 4.1 implies that $B(e_0 - e_1) > A$ and hence we have $c_3 < B$. We now show that one of the numbers in $\{c_1, |I| - c_1\}$ is at most $2(B - c_3)$. Suppose that this is wrong. Then

$$\begin{aligned} (c_1 - 2(B - c_3))(|I| - c_1 - 2(B - c_3)) &\geq 0 \\ \Leftrightarrow c_1(|I| - c_1) &\geq 2(B - c_3)(|I| - 2B + 2c_3). \end{aligned} \quad (6)$$

Since $|I| \geq \frac{1}{2}(|\mathcal{F}| - c_3)$, it follows from (5) and (6) that $f(c_3) \leq A$ where f is the polynomial in x given by

$$f := x(e_0 - e_1) + 2(B - x)\left(\frac{1}{2}(|\mathcal{F}| - x) - 2B + 2x\right)\theta_d q^{d^2-d-2}.$$

Since f has degree two with a negative coefficient in x^2 and since $0 \leq c_3 < B$, we have $\min\{f(0), f(B)\} \leq f(c_3)$ so $f(0) \leq A$ or $f(B) \leq A$. But $f(B) = B(e_0 - e_1)$ and we have already seen that this is larger than A . Hence $f(0) \leq A$, that is,

$$2B\left(\frac{1}{2}|\mathcal{F}| - 2B\right)\theta_d q^{d^2-d-2} \leq A.$$

Using $|\mathcal{F}| = \theta_{d+1} - q \geq (q + 1)q^d$ and $\theta_d \geq (q + 1)q^{d-1}$, it follows that

$$(q + 1)B((q + 1)q^d - 4B) \leq \left(q^2 + \frac{9}{2}q + 10\right)q^{2d}.$$

Using the definition of B , this gives

$$(q + 1)(q + 4 + \alpha)(q^2 - 3q - 16 - 4\alpha) \leq \left(q^2 + \frac{9}{2}q + 10\right)q^2. \quad (7)$$

From the hypothesis of Theorem 4.1 we have $5 \leq \alpha \leq q$. This implies that (7) must also be satisfied when α is replaced by 5 or by q , but this contradicts the lower bound in Hypothesis (I) of Theorem 4.1 for q . $\square$

Lemma 4.5. *There exists a $(d+1)$ -dimensional subspace U such that*

$$|\{i \in \Pi P_i \notin U\}| + 2 \cdot |\{F \in \mathcal{F} : |F| < g_0\}| \leq 2(q + 4 + \alpha)q^{d-1}.$$

Proof. Let c_3 be the number of $F \in \mathcal{F}$ with $|F| < g_0$ and thus $|F| \leq e_1$, see Remark 4.2. Let β be the number of $F \in \mathcal{F}$ with $|F| \geq g_0$ whose generic parts are based on points. Let $G_1, \dots, G_\beta$ denote these independent sets and let $R_1, \dots, R_\beta$ be the respective base points of their generic parts. As $|\mathcal{F}| = \theta_{d+1} - q$, Hypothesis (IV) of Theorem 4.1 shows that $\beta \geq \frac{1}{2}(\theta_{d+1} - q - c_3)$. For all $i \in \{1, \dots, \beta\}$ define

$$g_i := \left| G_i \cap \bigcup_{j=1}^{i-1} G_j \right|.$$

From Remark 4.2, we have $|\bigcup_{j=1}^i G_j| \leq ie_0 - \sum_{j=1}^i g_i$ for all $i \leq \beta$. We may assume that the sequence $g_1, \dots, g_\beta$ is monotone increasing. We define $j := \lceil \frac{1}{4}q^{d+1} \rceil + \theta_d + \theta_{d-1} - d$ and claim that $g_j < 5q^{d^2-2}\theta_d$.

Assume this is not true. Then $\sum_{i=j}^\beta g_i \geq (\beta - j + 1)5q^{d^2-2}\theta_d$ and Lemma 4.3 applied to the set $\{F \in \mathcal{F} : |F| \leq e_1\} \cup \{G_i : j \leq i \leq \beta\}$ implies

$$(\beta - j + 1)5q^{d^2-2}\theta_d + c_3(e_0 - e_1) \leq \left(q^2 + \frac{9}{2}q + 10 \right) q^{d^2+2d-3}.$$

On the left-hand side we use the definition of j and use $\beta \geq \frac{1}{2}(\theta_{d+1} - q - c_3)$. In the resulting expression the coefficient of c_3 is $e_0 - e_1 - \frac{5}{2}q^{d^2-2}\theta_d$ and in view of Lemma 3.3 and Hypothesis (I) of Theorem 4.1 this is positive. Hence, we may omit the term with c_3 and find

$$5q^{d^2-2}\theta_d \left(\left\lfloor \frac{1}{4}q^{d+1} \right\rfloor - \frac{1}{2}(\theta_d + q) - \theta_{d-1} + d \right) \leq \left(q^2 + \frac{9}{2}q + 10 \right) q^{d^2+2d-3}.$$

However, this contradicts Hypothesis (I) of Theorem 4.1.

Hence $g_j < 5q^{d^2-2}\theta_d$. Now, let $Q_1, \dots, Q_{d+1} \in \{R_{j-\theta_{d-1}}, \dots, R_j\}$ be such that $\tau := \langle Q_1, \dots, Q_{d+1} \rangle$ is a d -dimensional subspace and set

$$\mathcal{R} := \{R_i : i \in \{1, \dots, j - \theta_{d-1} - 1\} \text{ and } R_i \notin \tau\}.$$

Then $|\mathcal{R}| \geq j - \theta_{d-1} - 1 - (|\tau| - d - 1) = \left\lfloor \frac{1}{4}q^{d+1} \right\rfloor$.

In the next step we show for all $i \in \{1, \dots, d+1\}$ that the point Q_i lies on fewer than $7q^d$ lines that meet $\mathcal{R}$. Assume that this is false and let $i \in \{1, \dots, d+1\}$ be such that Q_i lies on at least $7q^d$ lines that meet $\mathcal{R}$. Each of these lines lies in $\left\lfloor \frac{2d-1}{d-2} \right\rfloor$ subspaces of dimension $d-1$ and two of these lines occur together in $\left\lfloor \frac{2d-2}{d-3} \right\rfloor$ such subspaces. Hence there exist at least

$$7q^d \left(\binom{2d-1}{d-2} - 7q^d \binom{2d-2}{d-3} \right) = 7q^d \left(\frac{q^{2d-1} - 1}{q^{d-2} - 1} - 7q^d \right) \binom{2d-2}{d-3} \\ \geq 7q^{d^2-3}(q-7) =: z,$$

$(d-1)$ -dimensional subspaces that contain one of the $7q^d$ lines. This shows that there exist $z\theta_d$ flags (E, S) of type $\{d-1, d\}$ with $Q_i \in E$ and such that E contains a point of $\mathcal{R}$. Since $Q_i = R_k$ for some k with $j - \theta_{d-1} \leq k \leq j$, this implies that $z\theta_d \leq g_k \leq g_j < 5q^{d^2-2}\theta_d$, which contradicts Hypothesis (I) of Theorem 4.1. Consequently, for all $i \in \{1, \dots, d+1\}$ the point Q_i lies in fact on fewer than $7q^d$ lines that meet $\mathcal{R}$.

Finally, we apply Proposition 3.2 with $d_0 = |\mathcal{R}|/q^{d+1} \geq \frac{1}{4}$, $n_0 = 7$, and $M := \mathcal{R}$ to find a $(d+1)$ -dimensional subspace U satisfying (2). Using the lower bounds for q of Hypothesis (I) of Theorem 4.1 we conclude that $|U \cap \mathcal{R}| \geq 3q^d > 2(q+4+\alpha)q^{d-1}$. The statement of the lemma follows now from Lemma 4.4. $\square$

4.2 | Notation and strategy

Notation 4.6. From now on we let U be the $(d+1)$ -dimensional subspace provided by Lemma 4.5 and define the following sets:

- $C_0 := \{F_i \in \mathcal{F} \mid i \in I, P_i \in U\}$, $c_0 := |C_0|$,
- $C_1 := \{F_i \in \mathcal{F} \mid i \in I, P_i \notin U\}$, $c_1 := |C_1|$,
- $C_2 := \{F_i \in \mathcal{F} \mid i \notin I, |F_i| \geq g_0\}$, $c_2 := |C_2|$,
- $C_3 := \{F_i \in \mathcal{F} \mid i \notin I, |F_i| < g_0\}$, $c_3 := |C_3|$,
- $W := \{P \in U \mid P \neq P_i \ \forall i \in I\}$,
- $M := \{(\pi, \tau) \in \bigcup_{F \in \mathcal{F}} F \mid \pi \cap U \text{ is a point and } \pi \cap U \in W\}$.

To establish Theorem 4.1, we want to show that $\mathcal{F} = C_0$, that is $C_1 = C_2 = C_3 = \emptyset$. Our strategy is the following. From Notation 4.6 we see that W and M are large when C_0 is small and we quantify this in Lemma 4.7(d) and (f). As every flag lies in a member of $\mathcal{F}$, then all flags of M must lie in a member of $\mathcal{F}$. In Section 4.3 we will control the contribution of the flags of C_1 , C_2 , or C_3 to M . This information will then be used later to show that W and M are quite small, so C_0 is quite large, which in the end forces $\mathcal{F}$ to be equal to C_0 .

4.3 | The contribution of $C_1 \cup C_2 \cup C_3$ to M

In this subsection we will control the contribution of the members of $C_1 \cup C_2 \cup C_3$ to M . The members of C_2 are sets F of flags that are based on a hyperplane and our upper bound for $|F \cap M|$ will depend on the intersection of this hyperplane with W . This is why we investigate in Lemmas 4.10 and 4.11 a bound for W depending on the intersection sizes of W with hyperplanes.

We start with a lemma that gives several basic observations and boundaries.

Lemma 4.7. *Using Notation 4.6, the following hold:*

- (a) *For all $F \in C_3$ we have $|F| \leq e_1 < g_0$.*
- (b) *$C_0 \cup C_1 \cup C_2 \cup C_3$ is a partition of $\mathcal{F}$.*
- (c) *$c_1 + 2c_3 \leq 2(q + 4 + \alpha)q^{d-1}$.*
- (d) *$|W| = \theta_{d+1} - c_0 \geq q$.*
- (e) *For all $P \in W$ there are exactly $q^{d^2-1}\theta_d$ flags (π, τ) with $\pi \cap U = P$.*
- (f) *$|M| = |W|q^{d^2-1}\theta_d$.*
- (g) *$|I| = c_0 + c_1 \geq \frac{1}{2}(\theta_{d+1} - q - c_3)$.*

Proof. The first claim is implied by the choice of $\mathcal{F}$ with the properties given in Theorem 4.1, especially Hypothesis (II). Claim (b) is obvious from the choice of C_0, C_1, C_2 , and C_3 . The choice of U implies (c). From Hypothesis (III) in Theorem 4.1, we know that the base points P_i of the sets F_i with $i \in I$ are pairwise distinct. Therefore we have $|W| = |U \setminus C_0| = |U| - |C_0| = \theta_{d+1} - c_0$. Since $c_0 \leq |\mathcal{F}| \leq \theta_{d+1} - q$, this gives (d). Furthermore, from Lemma 3.4(a) we know that each point $P \in W$ lies on q^{d^2-1} subspaces of dimension $(d-1)$ that meet U only in P and each such subspace lies in θ_d subspaces of dimension d . Hence, for every point P in W exactly $q^{d^2-1}\theta_d$ flags (π, τ) of M satisfy $\pi \cap U = P$, which proves (e) and (f). To see (g) we first note that our definitions imply $|I| = c_0 + c_1$ and that exactly $|\mathcal{F}| - c_3 = \theta_{d+1} - q - c_3$ elements of $\mathcal{F}$ have at least g_0 elements. Then we recall from Hypotheses (II) and (IV) in Theorem 4.1, that every element of $\mathcal{F}$ with at least g_0 elements is based on a point or a hyperplane and that at least half of these are based on a point. $\square$

Notation 4.8. Recall from Lemma 2.3 that the special parts of all independent sets given by Example 2.1—in particular of all independent sets $F \in \mathcal{F}$ with $|F| \geq g_0$ —have cardinality at most Δ , where

$$\Delta := \left\lfloor \frac{2d-1}{d} \right\rfloor q^d \leq (q+2)q^{d^2-1}. \quad (8)$$

Lemma 4.9.

- (a) *If $F \in C_0$, then the generic part of F does not contain a flag of M . In particular $|F \cap M| \leq \Delta$.*
- (b) *If $F \in C_1$, then $|F \cap M| \leq |W|q^{d^2-d-2}\theta_d + \Delta$.*
- (c) *If $F \in C_2$, then F is based on a hyperplane H and we have*

$$|F \cap M| \leq \begin{cases} \Delta & \text{if } U \leq H, \\ \Delta + |H \cap W|q^{d^2-d}\theta_{d-1} & \text{otherwise.} \end{cases}$$

Proof.

- (a) For all flags (π, τ) of the generic part of F we have $\dim(\pi \cap U) \geq 1$ or $\pi \cap U$ is the base point of F . Since M only contains flags (π, τ) such that π meets U in a point that

is not a base point of the generic part of some $F \in C_0$, this implies that these flags do not belong to M . Therefore $|F \cap M| \leq \Delta$ follows from Notation 4.8.

- (b) As $F \in C_1$ it is based on a point P with $P \notin U$. If $Y \in W$, then according to Lemma 3.4(b) the point P lies on exactly q^{d^2-d-2} subspaces π of dimension $d-1$ satisfying $\pi \cap U = Y$. Each of these lies on θ_d subspaces of dimension d . Hence, the generic part of F contains exactly $|W|q^{d^2-d-2}\theta_d$ flags of M . Furthermore, the special part of F contains at most Δ flags and thus at most this many flags of M .
- (c) Since F is not based on a point and has cardinality at least g_0 , Hypothesis (II) of Theorem 4.1 shows that F is based on a hyperplane H . The generic part of F consists of all flags (π, τ) of type $\{d-1, d\}$ with $\tau \leq H$ and thus also $\pi \leq H$. If $Y \in H \cap W$, then according to Lemma 3.4(d) the number of $(d-1)$ -subspaces π of H with $\pi \cap U = Y$ is zero for $U \leq H$ and it is q^{d^2-d} for $U \not\leq H$. Since for every $(d-1)$ -subspace of H the number of d -subspaces of H containing it is θ_{d-1} , it follows that the generic part of F contains no flag of F for $U \leq H$ and exactly $|H \cap W|q^{d^2-d}\theta_{d-1}$ flags of M for $U \not\leq H$. Finally, since the special part of F contains at most Δ flags, this implies the claim. $\square$

Lemma 4.10. *Suppose that z is an integer such that there is at most one hyperplane of U which contains more than z points of W . Then M has size at most*

$$(c_0 + c_1 + c_2)\Delta + c_1|W|q^{d^2-d-2}\theta_d + c_2zq^{d^2-d}\theta_{d-1} + c_3e_1 + q^{d^2-1}\theta_{d-1}\theta_d.$$

Proof. Since every flag of M is covered by some $F \in \mathcal{F}$, we have, using Lemma 4.7(f)

$$|W|q^{d^2-1}\theta_d \leq \sum_{i=0}^3 |\bigcup_{F \in C_i} F \cap M|.$$

Now, if there exists a hyperplane of U with more than z points in W , then let z' denote the number of its points in W and otherwise put $z' := z$. Since every hyperplane of U lies in q^{d-1} hyperplanes of $\text{PG}(2d, q)$ which do not contain U , Lemma 4.9(c) shows

$$\begin{aligned} \left| \bigcup_{F \in C_2} F \cap M \right| &\leq (c_2 - q^{d-1})(\Delta + zq^{d^2-d}\theta_{d-1}) + q^{d-1}(\Delta + z'q^{d^2-d}\theta_{d-1}) \\ &= c_2(\Delta + zq^{d^2-d}\theta_{d-1}) + \underbrace{(z' - z)q^{d^2-1}\theta_{d-1}}_{\leq \theta_d}. \end{aligned}$$

Finally, since $|F| \leq e_1$ for $F \in C_3$ via Lemma 4.7(a), the assertion follows from Lemma 4.9(a) and (b). $\square$

Lemma 4.11. *Let τ_1 and τ_2 be distinct hyperplanes of U and set $W' := (\tau_1 \cup \tau_2) \cap W$. Then*

$$|W'|\theta_{d-1} \leq (c_1 + c_2)q^{d-3}(2q + 7) + q^{2d-3}((\alpha + 3)q + \alpha^2 + 4\alpha).$$

Proof. We have $|W'| \leq |(\tau_1 \cup \tau_2) \cap U| = q^d + \theta_d$. We set

$$M' := \{(\pi, \tau) \in M : \pi \cap U \in W'\},$$

that is, M' consists of all flags (π, τ) of type $\{d-1, d\}$ such that $\pi \cap U$ is a point that lies in W' . Lemma 4.7(e) shows that $|M'| = |W'|q^{d^2-1}\theta_d$. Each flag of M' lies in at least one of the independent sets of $\mathcal{F} = C_0 \cup C_1 \cup C_2 \cup C_3$. Hence

$$|W'|q^{d^2}\theta_{d-1} \leq |W'|q^{d^2-1}\theta_d = |M'| \leq d_0 + d_1 + d_2 + d_3, \quad (9)$$

where for all $i \in \{1, \dots, 4\}$ we let d_i denote the number of elements of M' that lie in some member of C_i . Now, we determine upper bounds on these numbers d_0, d_1, d_2 , and d_3 in four steps.

First, we consider an independent set $F \in C_0$. Then $|F| \geq g_0$ and F is based on a point $P \in U$. We know from Lemma 4.9 that only the special part T of F may contribute to M' . Therefore, we study T and the three possible structures that T may have. Note that we frequently make use of Lemma 3.4 without explicit mention.

- First, assume that there is a line l with $P \in l$ such that T consists of all flags (π, τ) of type $\{d-1, d\}$ with $l \leq \tau$ and $P \notin \pi$. Recall from Example 2.1 that every flag $(\pi, \tau) \in T$ is determined by π , since $\tau = \langle P, \pi \rangle$. Then we have

$$|T \cap M'| = \begin{cases} |l \cap W'|q^{d^2-1} & \text{if } l \leq U, \\ |W'|q^{d^2-d-1} & \text{if } l \cap U = P, \end{cases}$$

and, using $|W'| \leq q^d + \theta_d$ as well as the fact that $|l \cap W'|$ is at most q for $P \in \tau_1 \cup \tau_2$ and at most 2 otherwise, we have

$$|T \cap M'| \leq \begin{cases} q^{d^2} & \text{if } P \in \tau_1 \cup \tau_2, \\ (q^d + \theta_d)q^{d^2-d-1} & \text{otherwise.} \end{cases}$$

- Second, assume that there is a hyperplane H with $P \in H$ such that T consists of all flags (π, τ) of type $\{d-1, d\}$ with $P \in \tau \leq H$ and $P \notin \pi$. As before, every flag $(\pi, \tau) \in T$ is determined by π , since $\tau = \langle P, \pi \rangle$. If $U \leq H$, then Lemma 3.4(d) implies $T \cap M' = \emptyset$ and otherwise it implies

$$\begin{aligned} |T \cap M'| &= |H \cap W'|q^{d^2-d} \\ &\leq \begin{cases} |W' \cap \tau_i|q^{d^2-d} & \text{if } H \cap U = \tau_i \text{ for some } i \in \{1, 2\}, \\ (q^{d-1} + \theta_{d-1})q^{d^2-d} & \text{otherwise.} \end{cases} \end{aligned}$$

Notice that $H \cap U = \tau_i$ for some $i \in \{1, 2\}$ implies $P \in \tau_i$ and thus $|W' \cap \tau_i| \leq \theta_d - 1 = q\theta_{d-1}$. Therefore, we have

$$|T \cap M'| \leq \begin{cases} \theta_{d-1}q^{d^2-d+1} & \text{if } P \in \tau_1 \cup \tau_2, \\ (q^{d-1} + \theta_{d-1})q^{d^2-d} & \text{otherwise.} \end{cases}$$

- Finally, suppose that the special part T is not based on a line or a hyperplane. Then Lemma 2.3 shows

$$|T \cap M'| \leq |T| \leq q^d(1 + q^{-1})\theta_{d-2}\theta_{d-1}^{d-1} \leq \theta_1(q + d)\theta_{d-2}q^{d^2-d-1}.$$

Using Hypothesis (I) of Theorem 4.1 we may summarize these three upper bounds into

$$|T \cap M'| \leq \begin{cases} \theta_{d-1}q^{d^2-d+1} & \text{for } P \in \tau_1 \cup \tau_2, \\ (q^d + \theta_d)q^{d^2-d-1} & \text{otherwise.} \end{cases}$$

Note that the bound given for $P \in \tau_1 \cup \tau_2$ is a weaker bound than the bound for $P \notin \tau_1 \cup \tau_2$. Now, since different sets $F \in C_0$ are based on different points P (see Hypothesis [III] in Theorem 4.1) and since $\tau_1 \cup \tau_2$ contains $q^d + \theta_d$ points, we find

$$d_0 \leq c_0(q^d + \theta_d)q^{d^2-d-1} + (q^d + \theta_d)\theta_{d-1}q^{d^2-d+1} \leq 12q^{d^2+d},$$

where the last step uses the trivial bounds $c_0 \leq \theta_{d+1} \leq 2q^{d+1}$, $\theta_d \leq 2q^d$, and $\theta_{d-1} \leq 2q^{d-1}$.

Second, for $F \in C_1$ we see that F contains at most $|W'|q^{d^2-d-2}\theta_d + \Delta$ flags of M' analogously to Lemma 4.9(b), which already proves $d_1 \leq c_1(|W'|q^{d^2-d-2}\theta_d + \Delta)$. Now, we use $\Delta \leq (q + 2)q^{d^2-1}$ given in Inequality (8) as well as $|W'| \leq q^d + \theta_d$ and have

$$d_1 \leq c_1q^{d^2-d-2}((q^d + \theta_d)\theta_d + q^{d+2} + 2q^{d+1}).$$

For $d = 3$ simple calculations show that this is smaller than $c_1q^{d^2+d-3}(2q + 7)$ and for $d \geq 4$ we receive the same upper bound via

$$d_1 \leq c_1q^{d^2-1}(2q^{d-1} + 6q^{d-2} + 4q^{d-3} + q + 2) \leq c_1q^{d^2+d-3}(2q + 7).$$

Third, we consider $F \in C_2$. Then $|F| \geq g_0$ and F is based on a hyperplane H . If $U \subseteq H$ and (π, τ) is a flag of F , then $\dim(\pi \cap U) \geq 1$ and thus $F \cap M' = \emptyset$. Therefore, we only need to study the case $U \not\subseteq H$, which implies $\dim(U \cap H) = d$. Then, analogously to the proof of Lemma 4.9(c), we see that the number of flags of M' in the generic part of F is $|H \cap W'|q^{d^2-d}\theta_{d-1}$ and we have

$$|H \cap W'|q^{d^2-d}\theta_{d-1} \leq \begin{cases} |W' \cap \tau_i|q^{d^2-d}\theta_{d-1} & \text{if } H \cap U = \tau_i \text{ for } i \in \{1, 2\}, \\ (q^{d-1} + \theta_{d-1})q^{d^2-d}\theta_{d-1} & \text{otherwise.} \end{cases}$$

Since there are exactly q^{d-1} hyperplanes that meet U in τ_1 and as many that meet U in τ_2 , it follows that the number of flags of M' that lie in the generic part of at least one independent set of C_2 is at most

$$c_2(q^{d-1} + \theta_{d-1})q^{d^2-d}\theta_{d-1} + q^{d^2-1}(|W' \cap \tau_1| + |W' \cap \tau_2|)\theta_{d-1}.$$

The special part of each independent set of C_2 has Δ flags and thus at most this many flags of M' . Using

$$|W' \cap \tau_1| + |W' \cap \tau_2| \leq |\tau_1| + |\tau_2| = 2\theta_d \leq 2(\theta_1 + 1)q^{d-1}$$

it follows that

$$d_2 \leq c_2\Delta + c_2(q^{d-1} + \theta_{d-1})q^{d^2-d}\theta_{d-1} + 2q^{d^2+d-2}(\theta_1 + 1)\theta_{d-1}.$$

We now show that this bound implies

$$d_2 \leq c_2q^{d^2+d-3}(2q + 7) + 2q^{d^2+2d-4}(q^2 + 4q + 4). \quad (10)$$

For $d = 3$, this can easily be verified for all $q \geq 3$. For $d \geq 4$, we use $\Delta \leq (q + 2)q^{d^2-1}$ given in Inequality (8) as well as the upper bound given in Lemma 3.3(a) to find

$$d_2 \leq c_2q^{d^2-1}(2q^{d-1} + 6q^{d-2} + 4q^{d-3} + q + 2) + 2q^{d^2+2d-4}(\theta_1 + 1)^2$$

and Theorem 4.1(I) implies Equation (10).

Finally, we note that for $F \in C_3$ we trivially have $|F \cap M'| \leq |F| \leq e_1$ and, using $c_3 \leq (q + 4 + \alpha)q^{d-1}$ from Lemma 4.7(c) as well as $e_1 = \alpha q^{d^2+d-2}$, this shows

$$d_3 \leq c_3e_1 \leq (\alpha q + \alpha^2 + 4\alpha)q^{d^2+2d-3}.$$

We have now proved upper bounds of d_0 , d_1 , d_2 , and d_3 . Substituting these upper bounds in Equation (9) and dividing by q^{d^2-d-1} yields

$$\begin{aligned} |W'|q^{d+1}\theta_{d-1} &\leq (c_1 + c_2)q^{2d-2}(2q + 7) + (\alpha q + \alpha^2 + 4\alpha)q^{3d-2} \\ &\quad + q^{2d}(2q^{d-1} + 8q^{d-2} + 8q^{d-3} + 12q). \end{aligned}$$

and, using the lower bound on q given in Hypothesis (I) of Theorem 4.1, this implies the claim. $\square$

4.4 | The proof of Theorem 4.1

In this part we use the information of the last subsection to first bound the size of W . Then we use this bound to show that C_0 must be equal to $\mathcal{F}$.

Lemma 4.12. *We have $|W| \leq (\alpha + 3)q^{d-1}$.*

Proof. Let τ_1 and τ_2 be hyperplanes of U such that $|\tau_1 \cap W| \geq |\tau_2 \cap W| \geq |\tau \cap W|$ for every hyperplane τ of U other than τ_1 and set $z := |\tau_2 \cap W|$. Then, Lemmas 4.10 and 4.7(f) show

$$\begin{aligned} |W|(q^{d+1} - c_1)q^{d^2-d-2}\theta_d &\leq (c_0 + c_1 + c_2)\Delta + c_2zq^{d^2-d}\theta_{d-1} + c_3e_1 \\ &\quad + q^{d^2-1}\theta_{d-1}\theta_d, \end{aligned}$$

and, if we set $\delta := c_1 + c_2 + c_3$ and use $c_0 + c_1 + c_2 + c_3 = |\mathcal{F}| = \theta_{d+1} - q$ as well as $|W| = \theta_{d+1} - c_0 = \delta + q$ from Lemma 4.7(d), then this is equivalent to

$$0 \leq (\theta_{d+1} - q)\Delta + q^{d^2-1}\theta_{d-1}\theta_d + c_3(e_1 - \Delta) \\ + c_2q^{d^2-d}z\theta_{d-1} + (\delta + q)(c_1 - q^{d+1})q^{d^2-d-2}\theta_d.$$

Before we proceed, we simplify this inequality:

- in the first term, since Δ is positive, we may replace $(\theta_{d+1} - q)$ by its upper bound $(q + 2)q^d$ given in Lemma 3.3(a);
- in the second term we use $q^{d^2-1}\theta_{d-1}\theta_d \leq (q + 5)q^{d^2+2d-3}$, which follows from Lemma 3.3(a) and the lower bound on q from Hypothesis (I) of Theorem 4.1;
- in the third term, since the coefficient $e_1 - \Delta$ of c_3 is positive (this is implied by our assumption $\alpha \geq 5$), we may replace c_3 by its upper bound $(q + 4 + \alpha)q^{d-1}$ given in Lemma 4.7(c);
- and in the final term, since $c_1 - q^{d+1}$ is negative (consider the upper bound $c_1 \leq 2(q + 4 + \alpha)q^{d-1}$ given in Lemma 4.7(c)), we may replace $(\delta + q)q^{d^2-d-2}\theta_d$ by its lower bound $\delta(q + 1)q^{d^2-3}$ implied by Lemma 3.3(a).

This yields

$$0 \leq (q + 2)q^d\Delta + (q + 5)q^{d^2+2d-3} + (q + 4 + \alpha)q^{d-1}(e_1 - \Delta) \\ + c_2q^{d^2-d}z\theta_{d-1} + \delta(q + 1)q^{d^2-3}(c_1 - q^{d+1}). \quad (11)$$

Next we want to take care of the variable z in the fourth term of this inequality. For that purpose we note that the preceding lemma is applicable to the set $W' := (\tau_1 \cup \tau_2) \cap W$ and that W' satisfies

$$|W'| \geq |\tau_1 \cap W| + |\tau_2 \cap W| - \theta_{d-1} \geq 2z - \theta_{d-1},$$

that is, we have

$$2z\theta_{d-1} \leq |W'|\theta_{d-1} + \theta_{d-1}^2 \leq |W'|\theta_{d-1} + (q + 2)^2q^{2d-4} \\ \leq |W'|\theta_{d-1} + 2q^{2d-2}.$$

We use the bound given by Lemma 4.11 (where, for convenience, we replace the 7 by an 8) as well as $c_1 + c_2 \leq \delta$ to first replace the first term on the right-hand side and then simplify the result and have

$$z\theta_{d-1} \leq \delta q^{d-3}(q + 4) + \frac{q^{2d-3}}{2}((\alpha + 5)q + \alpha^2 + 4\alpha). \quad (12)$$

Now, we reconsider Inequality (11):

- using Hypothesis (I) of Theorem 4.1 we see that the coefficient $(q^2 + q - 4 - \alpha)q^{d-1}$ of Δ therein is positive and thus we may replace Δ by its upper bound $(q + 2)q^{d^2-1}$ given in Inequality (8);

- the coefficients of c_1 and c_2 therein are nonnegative and so we may substitute c_1 and c_2 by their respective upper bounds $2(q + 4 + \alpha)q^{d-1}$ and δ , the first of which is given in Lemma 4.7(c) and the second is trivial;
- we use the upper bound found in Inequality (12);
- we substitute $e_1 = \alpha q^{d^2+d-2}$ and, finally, we divide by q^{d^2-3} .

This yields

$$0 \leq \delta^2(q + 4) + \delta q^{d-1} \left(-q^3 + \frac{\alpha + 7}{2} q^2 + \left(\frac{\alpha^2}{2} + 4\alpha + 10 \right) q + (2\alpha + 8) \right) + q^{2d}((\alpha + 1)q + \alpha^2 + 4\alpha + 5) + q^{d+1}(q^3 + 3q^2 - (\alpha + 2)q - (2\alpha + 8)),$$

and, using Hypothesis (I) of Theorem 4.1 and $\alpha \geq 5$, this implies

$$0 \leq \delta^2(q + 4) + \delta q^{d+1} \left(\frac{\alpha}{2} + 4 - q \right) + q^{2d}((\alpha + 1)q + \alpha^2 + 5\alpha) + q^{d+3}(q + 3).$$

Let $f(\delta)$ denote the right-hand side of this equation and set $\delta_1 := (\alpha + 3)q^{d-1} - q$ as well as $\delta_2 := q^{d+1} - \left(\frac{\alpha}{2} + 8 \right) q^d$. We want to show that δ does not lie in the interval $[\delta_1, \delta_2]$. To see this it suffices to show that $f(\delta_1) < 0$ and $f(\delta_2) < 0$ hold; the reason for that is, that f is a quadratic polynomial in δ with a positive leading coefficient. Simple calculations show

$$\begin{aligned} f(\delta_1) = & -2q^{2d+1} + \left(\frac{3}{2}\alpha^2 + \frac{21}{2}\alpha + 12 \right) q^{2d} + (\alpha^2 + 6\alpha + 9) q^{2d-1} \\ & + (4\alpha^2 + 24\alpha + 36) q^{2d-2} + q^{d+4} + 4q^{d+3} - \left(\frac{\alpha}{2} + 4 \right) q^{d+2} \\ & - (2\alpha + 6) q^{d+1} - (8\alpha + 24) q^d + q^3 + 4q^2 \end{aligned}$$

as well as

$$f(\delta_2) = -(\alpha + 31)q^{2d+1} + (2\alpha^2 + 37\alpha + 256)q^{2d} + q^{d+4} + 3q^{d+3},$$

and in view of $q \geq \frac{3}{2}\alpha^2 + \frac{21}{2}\alpha + 17$ from Hypothesis (I) of Theorem 4.1 both of these are negative. Hence, $\delta \notin [\delta_1, \delta_2]$. Finally, we have

$$\begin{aligned} \delta = \theta_{d+1} - q - c_0 & \leq \frac{1}{2}(\theta_{d+1} - q) + c_1 + \frac{1}{2}c_3 \\ & \leq \frac{1}{2}(q^{d+1} + (q + 2)q^{d-1} - q) + c_1 + \frac{1}{2}c_3 \\ & \leq \frac{1}{2}(q^{d+1} + (q + 2)q^{d-1} - q) + 2(q + 4 + \alpha)q^{d-1} \\ & < q^{d+1} - \left(\frac{\alpha}{2} + 8 \right) q^d = \delta_2. \end{aligned}$$

In the last step we also used $\alpha \geq 5$. From $\delta < \delta_2$ and $\delta \notin [\delta_1, \delta_2]$ we find $\delta < \delta_1$, as claimed. $\square$

Theorem 4.13. *We have $\mathcal{F} = C_0$.*

Proof. Put $\delta := c_1 + c_2 + c_3$ and note that $c_0 + c_1 + c_2 + c_3 = \theta_{d+1} - q$ implies $\delta = \theta_{d+1} - q - c_0$. The strategy of the proof is to show that $\delta = 0$ by using Lemma 4.7(f) and by finding a good upper bound for $|M|$. From Lemma 4.7(d) we have $|W| = q + \delta$ and from Lemma 4.12 we have $|W| \leq (\alpha + 3)q^{d-1}$. Therefore, for all $F \in C_1 \cup C_2$ we have

$$\begin{aligned} |F \cap M| &\leq (\alpha + 3)q^{d^2-1}\theta_{d-1} + \Delta \leq (\alpha + 3)(q + 2)q^{d^2+d-3} + \Delta \\ &\leq (\alpha + 3)(q + 2)q^{d^2+d-3} + (q + 2)q^{d^2-1} \leq (\alpha + 4)q^{d^2+d-2}. \end{aligned}$$

Since $e_1 = \alpha q^{d^2+d-2}$, we know from Lemma 4.7(a) that $|F \cap M| \leq (\alpha + 4)q^{d^2+d-2}$ for all $F \in C_1 \cup C_2 \cup C_3$. Therefore, the total contribution from the independent sets in $C_1 \cup C_2 \cup C_3$ to M is at most $\delta(\alpha + 4)q^{d^2+d-2}$.

By Lemma 4.9(a), the generic parts of the independent sets in C_0 are disjoint from M . To determine the contribution of the sets of C_0 to M , it remains to consider the special parts T of independent sets $F \in C_0$ and we denote by

- ω_1 the number of those with T based on a line that is contained in U ,
- ω_2 the number of those with T based on a line that is not contained in U ,
- ω_3 the number of those with T based on a hyperplane of $\text{PG}(2d, q)$,
- ω_4 the number of the remaining ones, which, according to Lemma 2.3, are those with $|T| \leq q^{d-1}\theta_1\theta_{d-2}\theta_{d-1}^{d-1}$.

Furthermore, we let Ω_1 be the set of lines ℓ of U such that $F(P, \ell) \in C_0$ for some point P of ℓ , we let Ω_3 be the set of all point-hyperplane pairs (P, H) with $F(P, H) \in C_0$ such that U is not contained in H , and we let Ω_4 be the set of indices $i \in I$ such that F_i is an element of C_0 and its special part T has size at most $q^{d-1}\theta_1\theta_{d-2}\theta_{d-1}^{d-1}$. Finally, we let $\hat{\Omega}_4$ be the set of all flags $f \in M$ such that f is an element of the special part of some independent set F_x with $x \in \Omega_4$.

Then we have $\omega_1 + \omega_2 + \omega_3 + \omega_4 = c_0$, $|\Omega_1| \leq \omega_1$, $|\Omega_3| = \omega_3$, and $|\Omega_4| = \omega_4$. In view of the definition of Ω_3 we notice that hyperplane-based special parts T only contribute to M when the underlying hyperplane of $\text{PG}(2d, q)$ does not contain U .

Altogether, using Lemmas 3.4 and 4.7(f), it follows that

$$\begin{aligned} |W|q^{d^2-1}\theta_d = |M| &\leq \delta(\alpha + 4)q^{d^2+d-2} + \sum_{l \in \Omega_1} |l \cap W|q^{d^2-1} \\ &\quad + \omega_2|W|q^{d^2-d-1} + \sum_{(P,H) \in \Omega_3} |W|q^{d^2-d} + |\hat{\Omega}_4|. \end{aligned} \quad (13)$$

We proceed by replacing the first sum in the right-hand side of the previous inequality by an upper bound. We have

$$\begin{aligned}
0 &\leq \sum_{l \in \Omega_1} (|l \cap W| - 1)(|l \cap W| - 2) \\
&= \sum_{l \in \Omega_1} |l \cap W|(|l \cap W| - 1) - 2 \sum_{l \in \Omega_1} |l \cap W| + 2|\Omega_1| \\
&\leq |W|(|W| - 1) - 2 \sum_{l \in \Omega_1} |l \cap W| + 2|\Omega_1|,
\end{aligned}$$

where the last inequality holds, since any pair of distinct points of W is contained in at most one line of Ω_1 . Since $|\Omega_1| \leq \omega_1$ and $\omega_1 + \omega_2 + \omega_3 + \omega_4 = c_0 = \theta_{d+1} - |W|$ (by part (d) of Lemma 4.7), it follows that

$$\sum_{l \in \Omega_1} |l \cap W| \leq \frac{1}{2}|W|(|W| - 3) + \theta_{d+1} - \omega_2 - \omega_3 - \omega_4.$$

Using this as well as $|\Omega_3| = \omega_3$ in (13) and dividing by q^{d^2-d} we find

$$\begin{aligned}
|W|q^{d-1} \left(\theta_d - \frac{|W| - 3}{2} \right) &\leq \delta(\alpha + 4)q^{2d-2} + \theta_{d+1}q^{d-1} + \omega_2 \frac{|W| - q^d}{q} \\
&\quad + \sum_{(P,H) \in \Omega_3} (|W| - q^{d-1}) + \frac{|\hat{\Omega}_4| - \omega_4 q^{d^2-1}}{q^{d^2-d}}.
\end{aligned}$$

Since $|W| \leq (\alpha + 3)q^{d-1}$ by Lemma 4.12, the coefficient of ω_2 in the above inequality is not positive and therefore the term with ω_2 can be omitted. Doing so and substituting then $|W| = \delta + q$ we find

$$\begin{aligned}
(\delta + q)q^{d-1} \underbrace{\left(\theta_d - \frac{\delta + q - 3}{2} \right)}_{\geq \theta_1 q^{d-1} - \delta} &\leq \delta(\alpha + 4)q^{2d-2} + \theta_{d+1}q^{d-1} \\
&\quad + \sum_{(P,H) \in \Omega_3} (|W| - q^{d-1}) + \frac{|\hat{\Omega}_4| - \omega_4 q^{d^2-1}}{q^{d^2-d}}.
\end{aligned} \tag{14}$$

We next show that

$$L := \sum_{(P,H) \in \Omega_3} (|W| - q^{d-1}) \leq \delta q^{d+1}. \tag{15}$$

This is trivial, if $|W| \leq q^{d-1}$. If $|W| > q^{d-1}$, then we use $|\Omega_3| \leq |C_0| = \theta_{d+1} - |W|$ to see that (where $y := \frac{1}{2}(\theta_d + q^{d-1})$)

$$L \leq (\theta_{d+1} - |W|)(|W| - q^{d-1}) \tag{16}$$

$$= (|W| - q)q^{d+1} - (|W| - y)^2 + y^2 - \theta_{d+1}q^{d-1} + q^{d+2}. \tag{17}$$

From Lemma 3.3(a) we find $y \leq \frac{1}{2}q^{d-1}(q+3)$ and, using Hypothesis (I) of Theorem 4.1, this implies that $y^2 - \theta_{d+1}q^{d-1} + q^{d+2} < 0$. Therefore (16) shows that $L \leq (|W| - q)q^{d+1}$. Since $|W| = q + \delta$, this establishes $|L| \leq \delta q^{d+1}$ in any case. Using this and $\theta_{d+1} \leq (q^2 + q + 2)q^{d-1}$ (see Lemma 3.3(a)) in Inequality (14) we find

$$(\delta + q)q^{d-1}(\theta_1 q^{d-1} - \delta) \leq \delta(\alpha + 4)q^{2d-2} + (q^2 + q + 2)q^{2d-2} + \delta q^{d+1} + \frac{|\hat{\Omega}_4| - \omega_4 q^{d^2-1}}{q^{d^2-d}},$$

which is equivalent to

$$0 \leq \delta^2 q^{d-1} - \delta q^d (q^{d-1} - (\alpha + 3)q^{d-2} - q - 1) + 2q^{2d-2} + \frac{|\hat{\Omega}_4| - \omega_4 q^{d^2-1}}{q^{d^2-d}}, \quad (18)$$

Finally, we study the cardinality of $\hat{\Omega}_4$. For all $x \in \Omega_4$ we know from Lemma 2.3 that $F_x = F(P_x, \mathcal{U}_x)$ for a set $\mathcal{U}_x$ of subspaces of dimension d with

$$|\mathcal{U}_x| \leq (1 + q^{-1})\theta_{d-2}\theta_{d-1}^{d-1}.$$

Furthermore, for all $x \in \Omega_4$ every flag (π, τ) of the special part of F_x that lies in $\hat{\Omega}_4$ satisfies $\dim(\tau \cap U) = 1$ and $\pi \cap U$ is a point of $\tau \cap W$. Motivated by this, we define

$$\forall x \in \Omega_4 : \zeta_x := \max\{|\tau \cap W| : \tau \in \mathcal{U}_x, \dim(\tau \cap U) = 1\}.$$

We put $\zeta := \max\{\zeta_x : x \in \Omega_4\}$ if $\Omega_4 \neq \emptyset$, and otherwise we put $\zeta := q$. There are two remarks to be made.

- For all $x \in \Omega_4$ and all $\tau \in \mathcal{U}_x$ with $\dim(\tau \cap U) = 1$ we have $W \not\subseteq P_x \in \tau \cap U$, which implies $\zeta_x \leq q$. Thus we also have $\zeta \leq q$.
- The definition of ζ implies that there is a line $l \leq U$ with $|l \cap W| = \zeta$. For all $x \in \Omega_4$ with $P_x \in l$ and all $\tau \in \mathcal{U}_x$ with $\dim(\tau \cap U) = 1$ we have $|\tau \cap W| \leq \zeta$. For all $x \in \Omega_4$ with $P_x \notin l$ and all $\tau \in \mathcal{U}_x$ with $\dim(\tau \cap U) = 1$ we have $|\tau \cap l| \leq 1$, which implies $|\tau \cap W| \leq \min\{\zeta, |W| - (\zeta - 1)\}$.

This implies

$$\begin{aligned} |\hat{\Omega}_4| &\leq (q\zeta + \max\{0, \omega_4 - q\} \cdot \min\{\zeta, |W| + 1 - \zeta\})q^{d-1}(1 + q^{-1})\theta_{d-2}\theta_{d-1}^{d-1} \\ &\leq (q^2 + \omega_4 \cdot \min\{\zeta, |W| + 1 - \zeta\})(q+1)(q+2)(q+d)q^{d^2-5} \\ &\leq q^{d^2+1} + \omega_4 \cdot \min\{\zeta, |W| + 1 - \zeta\}(q+1)(q+2)(q+d)q^{d^2-5}. \end{aligned}$$

The second step uses parts (a) and (c) of Lemma 3.3. Substituting this in Equation (18) and dividing by q^{d-5} yields

$$0 \leq \delta^2 q^4 - \delta q^5 (q^{d-1} - (\alpha + 3)q^{d-2} - q - 1) + 2q^{d+3} + q^6 + \omega_4 (\min\{\zeta, |W| + 1 - \zeta\} (q + 1)(q + 2)(q + d) - q^4). \quad (19)$$

Since $\min\{\zeta, |W| + 1 - \zeta\} \leq \zeta \leq q$ and $\omega_4 \leq |C_0| \leq |\mathcal{F}| = \theta_{d+1} - q \leq (q + 2)q^d$ we find

$$0 \leq \delta^2 q^4 - \delta q^5 (q^{d-1} - (\alpha + 3)q^{d-2} - q - 1) + 2q^{d+3} + q^6 + (q + 2)(q(q + 1)(q + 2)(q + d) - q^4)q^d.$$

If we denote the right-hand side of this inequality by $g(\delta)$, then g is polynomial in δ of degree 2 with a positive leading coefficient. For $\delta_1 := d + 4$ and $\delta_2 := (\alpha + 3)q^{d-1}$ we have

$$\begin{aligned} g(\delta_1) &= -q^{d+1}(q^3 - (\alpha d + 8d + 4\alpha + 22)q^2 - (8d + 4)q - 4d) \\ &\quad + (d + 5)q^6 + (d + 4)q^5 + (d^2 + 8d + 16)q^4, \\ g(\delta_2) &= -q^{2d+2}((\alpha + 3)q - 2\alpha^2 - 12\alpha - 18) \\ &\quad + (\alpha + 3)q^{d+5} + (d + \alpha + 6)q^{d+4} \\ &\quad + (5d + 10)q^{d+3} + (8d + 4)q^{d+2} + 4dq^{d+1} + q^6. \end{aligned}$$

Using Hypothesis (I) of Theorem 4.1, it follows that $g(\delta_1) < 0$ and $g(\delta_2) < 0$ and hence $g(x) < 0$ for all real numbers x with $\delta_1 \leq x \leq \delta_2$. Since $g(\delta) \geq 0$ and $\delta = |W| - q \leq (\alpha + 3)q^{d-1}$, it follows that $\delta < \delta_1$, that is $\delta \leq d + 3$.

Put $\beta := \min\{\zeta, |W| + 1 - \zeta\}$. Then $|W| + 1 \geq 2\beta$. Since $|W| = q + \delta$ and $\delta \leq d + 3$, it follows that $2\beta \leq q + d + 4$. Using Hypothesis (I) of Theorem 4.1 it follows that $\beta \leq q - d - 3$. This implies that the coefficient of ω_4 in (19) is negative. Hence

$$0 \leq -\delta q^4 (q^d - \delta - (\alpha + 3)q^{d-1} - q^2 - q) + 2q^{d+3} + q^6$$

and since $\delta = |W| - q \leq (\alpha + 3)q^{d-1}$ (Lemma 4.13) we find $\delta < 1$ from Hypothesis (I) of Theorem 4.1. Hence $\delta = 0$, that is, $|C_0| = \theta_{d+1} - q - \delta = \theta_{d+1} - q = |\mathcal{F}|$ and thus $C_0 = \mathcal{F}$. $\square$

Note that Theorem 4.13 concludes the proof of Theorem 4.1.

4.5 | Proof of Theorem 1.2

In this subsection we use

Theorem 4.1 to prove Theorem 1.2

Proof of Theorem 1.2. Suppose that d is such that Conjecture 1.1 holds with $\alpha = \max\{5, \rho(d)\}$ and suppose that $q > 3 \cdot 112^{2^{d+1}-1} \cdot 2^{-d-1}$ as well as

$q \geq \frac{3}{2}\alpha^2 + \frac{21}{2}\alpha + 17$. Consider a coloring of the Kneser graph Γ_d , with $t \leq \theta_{d+1} - q$ color classes $C_1, \dots, C_t$. Define $C_i := \emptyset$ for $t < i \leq \theta_{d+1} - q$. Each set C_i is an independent set of flags of type $\{d-1, d\}$ in $\text{PG}(2d, q)$. If $|C_i| > e_1 = \alpha q^{d^2+d-2}$, then let $\bar{C}_i$ be a maximal independent set containing C_i ; it follows from Conjecture 1.1 that $\bar{C}_i$ is one of the sets defined in Example 2.1 and thus we have $g_0 \leq |\bar{C}_i| \leq e_0$. For each i we now define a set F_i and for all i with $|C_i| \leq e_1$ we simply set $F_i := C_i$. Now, consider an index i with $|C_i| > e_1$. If there exists an index $j < i$ with $|C_j| > e_1$ and such that $\bar{C}_i$ and $\bar{C}_j$ have the same generic part, then let F_i be the special part of $\bar{C}_i$ (this implies $|F_i| \leq q^d \binom{2d-1}{d-1} < e_1$) and otherwise set $F_i := \bar{C}_i$. Let S be the set of indices i with $|F_i| > e_1$. Consider the multiset $\mathcal{F} = \{F_i | 1 \leq i \leq \theta_{d+1} - q\}$. Then each F_i is an independent set and the union of the F_i is the set of all flags of type $\{d-1, d\}$ in $\text{PG}(2d, q)$. We consider two cases.

Case 1: For at least $\frac{1}{2}|S|$ indices $i \in S$ the generic part of F_i is based on a point. Then $\mathcal{F}$ satisfies all hypotheses of Theorem 4.1. The conclusion of this theorem implies $S = \{1, 2, \dots, \theta_{d+1} - q\}$, we know that the generic part of every set F_i is based on a point and the base points are $\theta_{d+1} - q$ distinct points of a subspace of dimension $d+1$. This implies $t = \theta_{d+1} - q$ as well as $|C_i| > e_1$ and $F_i = \bar{C}_i$ for all i . Notice that $F_i = \bar{C}_i$ might not be uniquely determined by C_i , however its base point is. This follows from the fact that two maximal independent sets based on distinct points (are easily seen to) have less than e_1 elements in common and hence C_i cannot be contained in both. This proves Theorem 1.2 in this case.

Case 2: For less than $\frac{1}{2}|S|$ indices $i \in S$ the generic part of F_i is based on a point. Then for more than $\frac{1}{2}|S|$ indices i the generic part is based on a hyperplane and we can apply the first case in the dual space. This proves Theorem 1.2. $\square$

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