A decomposition theorem for number-conserving multi-state cellular automata on triangular grids

Barbara Wolnik[a,c], Anna Nenca[1b], Bernard De Baets[c]

[a]Institute of Mathematics, Faculty of Mathematics, Physics and Informatics, University of Gdańsk, 80-308 Gdańsk, Poland

[b]Institute of Informatics, Faculty of Mathematics, Physics and Informatics, University of Gdańsk, 80-308 Gdańsk, Poland

[c]KERMIT, Department of Data Analysis and Mathematical Modelling, Faculty of Bioscience Engineering, Ghent University, Coupure links 653, B-9000 Gent, Belgium

Abstract

This paper concerns two-dimensional cellular automata on a triangular grid that preserve the sum of the states of all the cells. To study such cellular automata, we adapt the idea of the split-and-perturb decomposition of a number-conserving local rule, developed first for square grids, to the setting of triangular grids. As a result, we obtain a new mathematical tool that allows, for example, to enumerate all so-called k-ary (*i.e.*, binary, ternary, quaternary, quinary, etc.) number-conserving cellular automata on a triangular grid, regardless of the value of k.

Keywords: multi-state cellular automata, number conservation, triangular grids

1. Introduction

The main advantage that makes cellular automata (CAs) attractive mathematical study objects is the fact that they are simple models of computation capable of simulating complex phenomena. Moreover, the action of each CA is completely determined by its local transformation rule, which reflects the assumption that most of the laws in physics, economy, chemistry, biology, sociology and so on, must result from interactions that are strictly local. Therefore, it is not surprising that CAs are of much interest to scientists in a variety of research fields (for several recent examples, see e.g. [2, 6, 7, 11, 14, 18, 23, 31, 32, 34]).

In order to accommodate for conservation laws — one of the most important concepts in physics — usually number-conserving CAs (NCCAs) are used, for which the sum of the states of all the cells is preserved at every update. Unfortunately, although there is an extensive literature on one-dimensional NCCAs (see, for example, [4, 5, 12, 13, 16, 21, 22, 26]), the two-dimensional ones have not been studied in a satisfactory manner and most of the results are not applicable globally. For example, thanks to existing necessary and sufficient conditions it is possible (at least theoretically) to check whether a given CA is number conserving or not [9], or it is possible to design a number-conserving CA [28], but usually enumeration of all number-conserving CAs with a given state set is not feasible. Furthermore, although there are three possible regular tessellations of a plane, in the case of two dimensions, the existing results predominantly concern square grids (as in [9, 17, 28, 29]), while the results for triangular or hexagonal grids are rather scarce.

In this paper, we study number conservation of two-dimensional CAs defined on a regular triangular grid (triangular CAs). The triangular tiling of a plane was first considered by Bays [3] at the end of the last century. Triangular CAs regularly appear in different contexts (see, for instance, [1, 25, 19, 33]). Here, we focus on the simplest triangular CAs: automata that update the states

[1]Corresponding author: anna.nenca@ug.edu.pl

of the cells on the basis of the states of the adjacent cells only. Even for this case, there are no satisfactory results for the most classical CAs: the so-called k-ary CAs, *i.e.*, CAs with the state set $Q = \{0, 1, 2, \ldots, k-1\}$, where k is some positive integer greater than 1. In particular, till now, there are no tools to count all such CAs. In order to study these triangular NCCAs, we will adapt the new approach presented in [30] for NCCAs defined on a regular square grid, *i.e.*, the split-and-perturb decomposition of a number-conserving local rule. We will show that adjusting this line of reasoning to the setting of triangular grids will lead to the full solution of the problem of finding all k-ary triangular NCCAs, regardless of the value of k.

It is worth emphasizing that in general enumerating of all NCCAs in a given setting is a real challenge. For example, even in the simplest case of one-dimensional CAs with radius one little is known. Until recently, full lists were available only for binary, ternary and quaternary such NCCAs, *i.e.*, for the state sets $\{0, 1\}$, $\{0, 1, 2\}$ and $\{0, 1, 2, 3\}$ (see, for example, [22]). Recently, a complete list of 1 876 088 314 quinary NCCAs has been found (this was done using a computer and it was made possible thanks to the split-and-perturb decomposition also used in the present paper) [30]. There are no results for the state sets $\{0, 1, 2, \ldots, k-1\}$ for $k > 5$.

For the square grid with the von Neumann neighborhood in two dimensions, even if we assume rotation symmetry (a very natural assumption when modelling various phenomena, especially physical ones), the situation does not look any better. The complete list for the state set $\{0, 1, 2, 3, 4\}$ appeared in 2015 (see [17]), while for the state set $\{0, 1, 2, 3, 4, 5\}$ this was only done in 2021, as a result of very technical and tedious considerations (see [10]). That approach has been combined with the split-and-perturb decomposition, which allowed to find the complete list of rotation-symmetric NCCAs for the state set $\{0, 1, 2, 3, 4, 5, 6\}$, unfortunately, only using a computer (the list is available as a data set in [24]). So far, a similar attempt for the state set $\{0, 1, 2, 3, 4, 5, 6, 7\}$ is still beyond the computational capacity of computers. For the square grid with the Moore neighborhood, there is even no list of all NCCAs in the binary case.

From the above review, it should be clear that the enumeration of all k-ary triangular NCCAs, regardless of the value of k, presented further on in this paper, is something quite unexpected. In essence, this enumeration was only made possible thanks to additional properties of the split-and-perturb decomposition adjusted to the two-dimensional triangular grid, which unfortunately do not hold when considering other grids or other dimensions, making this paper indeed quite a unicum.

The remainder of this paper is organized as follows. In Section 2, we introduce the terminology on CAs. In Section 3, we present the idea of the split-and-perturb decomposition, while in Section 4 we use this idea to enumerate all k-ary triangular NCCAs. We conclude the paper in Section 5.

2. Preliminaries

In this section, we introduce CAs on a triangular grid and recall some results that will be of use in the following sections. To define a CA, one needs to specify a space of cells, a neighborhood, a state set and a local rule. However, to develop our idea of decomposition, we additionally need to generalize local rules, in the sense that they can yield values outside the considered state set. The structure of this section follows that of the corresponding section in [30].

2.1. The cellular space and the neighborhood of a cell

As cellular space we consider a triangular tessellation of a plane into equilateral triangles. We assume that the cells (the triangles) are labeled by elements of $\mathbb{Z}^2$, but we do not specify in which way; for example, this can be done as in Figure 1. Note that $\mathbb{Z}$ and $\mathbb{R}$ denote the set of integers and the set of real numbers, respectively. There are two types of cells in such a triangular grid, namely $\triangle$- and $\triangledown$-cells, depending on the orientation of the triangle.

A cell $\mathbf{i} \in \mathbb{Z}^2$ is *adjacent to* a cell $\mathbf{j} \in \mathbb{Z}^2$ if they have one common edge and we denote this fact by $\mathbf{i} \sim \mathbf{j}$. In this paper, we consider the von Neumann neighborhood, thus, for each cell $\mathbf{i} \in \mathbb{Z}^2$, its neighborhood $P(\mathbf{i})$ consists of exactly four cells: the cell $\mathbf{i}$ itself and its three adjacent cells, *i.e.*,

$$P(\mathbf{i}) = \{\mathbf{i}\} \cup \{\mathbf{j} \in \mathbb{Z}^2 \mid \mathbf{j} \sim \mathbf{i}\}.$$

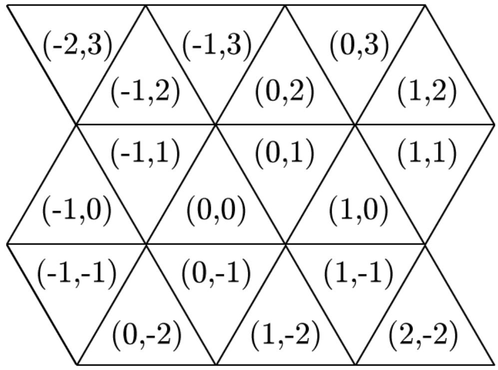

Figure 1: One way of labeling cells by elements of $\mathbb{Z}^2$. The cell $(0,0)$ is a $\triangle$-cell, while the cell $(0,1)$ is a $\bigtriangledown$- cell.

For example, if we label the cells as in Figure 1, the neighborhood of $(0,0)$ consists of four cells: $(0,0)$, $(-1,1)$, $(0,1)$ and $(0,-1)$, while the neighborhood of $(0,1)$ equals $\{(0,1),(0,2),(1,0),(0,0)\}$.

If $\mathbf{i} \neq \mathbf{j}$, then the relation between $P(\mathbf{i})$ and $P(\mathbf{j})$ is very simple and is described in the following lemma.

Lemma 2.1. *Let $\mathbf{i},\mathbf{j} \in \mathbb{Z}^2$ and $\mathbf{i} \neq \mathbf{j}$. Then either $P(\mathbf{i}) \cap P(\mathbf{j}) = \emptyset$ or $|P(\mathbf{i}) \cap P(\mathbf{j})| = 1$ or $|P(\mathbf{i}) \cap P(\mathbf{j})| = 2$. Moreover, in the latter case, it follows that $P(\mathbf{i}) \cap P(\mathbf{j}) = \{\mathbf{i},\mathbf{j}\}$.*

2.2. Configurations

As state set we consider an arbitrary set $Q \subseteq \mathbb{R}$ containing zero and at least one more number (there is nothing interesting in the case $Q = \{0\}$), with Q_* denoting the set of all non-zero states, *i.e.*, $Q_* = Q \setminus \{0\}$. The assumption that $0 \in Q$ is only for our convenience (see the discussion at the end of Section 3).

A *configuration* is any mapping from the grid $\mathbb{Z}^2$ to Q. The set of all configurations is denoted by X, *i.e.*, $X = Q^{\mathbb{Z}^2}$. However, to develop the decomposition idea, we also need to consider *configurations in a wider sense*: mappings from the grid $\mathbb{Z}^2$ to $\mathbb{R}$ and the set of all such mappings is denoted by $\widetilde{X}$, *i.e.*, $\widetilde{X} = \mathbb{R}^{\mathbb{Z}^2}$. This allows to exploit the field structure of $\mathbb{R}$ and rely on tools from linear algebra; note that we could have used the set of rational numbers instead. For the sake of simplicity, we write configuration also for elements from $\widetilde{X}$, unless confusion is possible (note that $X \subseteq \widetilde{X}$). The value of a cell $\mathbf{i} \in \mathbb{Z}^2$ in a configuration $\mathbf{x} \in \widetilde{X}$ is denoted by $x_{\mathbf{i}}$. We denote the set of *finite configurations* by X_F, *i.e.*, configurations that have a finite number of non-zero states:

$$X_F = \Big\{ \mathbf{x} \in X \mid \{\mathbf{i} \in \mathbb{Z}^2 \mid x_{\mathbf{i}} \neq 0\} \text{ is finite} \Big\}.$$

Analogously, we define $\widetilde{X}_F = \Big\{ \mathbf{x} \in \widetilde{X} \mid \{\mathbf{i} \in \mathbb{Z}^2 \mid x_{\mathbf{i}} \neq 0\} \text{ is finite} \Big\}$.

2.3. Local and global functions and rules

Definition 2.2. *A* local function *is any function $f : Q^4 \to \mathbb{R}$ fulfilling two basic assumptions:*

(L1) $f(0,0,0,0) = 0$,

(L2) *for any $q, p_1, p_2, p_3 \in Q$, it holds that $f(q,p_1,p_2,p_3) = f(q,p_2,p_3,p_1)$.*

Note that both assumptions are required if we want to deal with triangular CAs. The latter assumption is called *rotation symmetry* and allows to define a homogeneous CA despite the fact that in a triangular grid there are both $\triangle$- and $\bigtriangledown$- cells (compare Eq. (1) below).

If additionally $f(Q^4) \subseteq Q$, which means that the local function f only takes values in Q, then we call f a *local rule*. Thus, as for configurations, we generalize the concept of local rule in the sense that a local function may take values outside the state set Q.

Each local function f induces a *global function* $A_f : X \to \widetilde{X}$ defined for $\mathbf{x} \in X$ and $\mathbf{i} \in \mathbb{Z}^2$ by

$$A_f(\mathbf{x})_{\mathbf{i}} = f(x_{\mathbf{i}}, x_{\mathbf{j}_1(\mathbf{i})}, x_{\mathbf{j}_2(\mathbf{i})}, x_{\mathbf{j}_3(\mathbf{i})}), \tag{1}$$

where (here and subsequently) $\mathbf{j}_1(\mathbf{i})$, $\mathbf{j}_2(\mathbf{i})$, $\mathbf{j}_3(\mathbf{i})$ are the cells adjacent to $\mathbf{i}$, numbered clockwisely. As mentioned above, thanks to assumption (L2), the function A_f is correctly defined, moreover, if $\mathbf{x} \in X_F$, then $A_f(\mathbf{x}) \in \widetilde{X}_F$, according to (L1).

In this paper, we will deal with a special kind of CAs, the so-called *number-conserving* CAs (NCCAs), whose global rule, roughly speaking, preserves the sum of the states of all the cells constant throughout its evolution. Of course, if the grid is infinite, then there is a problem with the consideration of such sum. There are different ways to overcome this difficulty, which result in different definitions of number conservation. Without loss of generality, we can use any of them, since Durand *et al.* [9] proved that all classical definitions of number-conserving CAs found in literature (finite, infinite, periodic) are equivalent.

Definition 2.3. *A local function f is called number–conserving if for each finite configuration $\mathbf{x} \in X_F$ it holds that*

$$\sum_{\mathbf{i}\in\mathbb{Z}^2} A_f(\mathbf{x})_{\mathbf{i}} = \sum_{\mathbf{i}\in\mathbb{Z}^2} x_{\mathbf{i}}. \tag{2}$$

For a local function f, the state $q \in Q$ is called *quiescent* if $f(q, q, q, q) = q$. Note that according to Definition 2.2, the state 0 is quiescent for each local function. The next lemma shows that if we focus on number-conserving local functions only, then being quiescent is not something special.

Lemma 2.4. *If a local function f is number-conserving, then each state $q \in Q$ is quiescent. Moreover, for each $q \in Q$ it holds that $f(0, q, 0, 0) = f(0, 0, q, 0) = f(0, 0, 0, q) = \frac{1}{3}\big(q - f(q, 0, 0, 0)\big)$.*

Proof. Let us fix $q \in Q$ and denote $\alpha = f(q, q, q, q)$, $\beta = f(q, q, 0, 0)$ and $\gamma = f(q, 0, 0, 0)$. Then considering the configurations presented in Figure 2, we obtain from Eq. (2) that it holds that $2\beta + 4\gamma = 2q$ and $\alpha + 3\beta + 6\gamma = 4q$. This yields, in particular, that $\alpha = q$.

The second part follows from Eq. (2) written for the configuration $\mathbf{x}$ having only one non-zero state (equal to q): $f(q, 0, 0, 0) + f(0, q, 0, 0) + f(0, 0, q, 0) + f(0, 0, 0, q) = q$, since $f(0, q, 0, 0) = f(0, 0, q, 0) = f(0, 0, 0, q)$ according to Definition 2.2. □

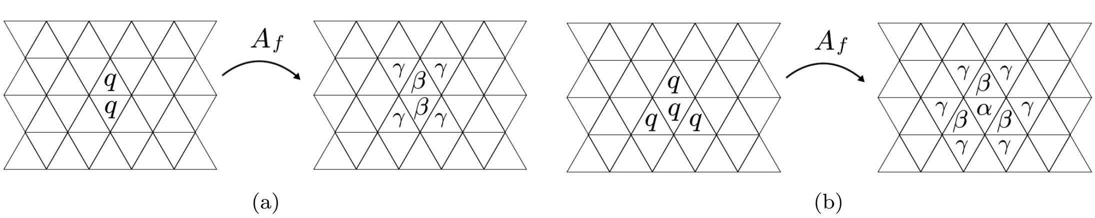

Figure 2: Examples of the action of the global function A_f on sample configurations assuming that $\alpha = f(q, q, q, q)$, $\beta = f(q, q, 0, 0)$ and $\gamma = f(q, 0, 0, 0)$ (blank cells have state 0). If f is number-conserving, then we get the following equality: (a) $2\beta + 4\gamma = 2q$ (b) $\alpha + 3\beta + 6\gamma = 4q$.

In this paper, a key role will be played by *monomers*, *i.e.*, those elements of Q^4 of which at most one coordinate is non-zero. For example, $(q, 0, 0, 0)$ and $(0, q, 0, 0)$, where $q \in Q$, are monomers.

2.4. The number of all local rules

We end this section with a description of the set of all local rules in the case of a finite state set Q. Recall that we assume that $0 \in Q$ to be able to formulate (L1).

Of course, the set of all functions from Q^4 to Q has cardinality $|Q|^{|Q|^4}$, but we want to count only the ones that satisfy both (L1) and (L2). Thus, we divide Q^4 into orbits

$$\Big\{(q, p_1, p_2, p_3), (q, p_2, p_3, p_1), (q, p_3, p_1, p_2)\Big\} \tag{3}$$

and, since the values for a local rule have to be the same for any two elements belonging to the same orbit and can be chosen independently for any two elements belonging to different orbits, the number of all local rules equals $|Q|^{l-1}$, where l is the number of orbits (the term "-1" is added because we do not have freedom in choosing a value for $(0,0,0,0)$, since we require (L1)). The orbits in (3) arise as a result of the action of the cyclic group G of order 3 generated by the permutation $\pi : Q^4 \to Q^4$ defined for any $q, p_1, p_2, p_3 \in Q$ by

$$\pi(q, p_1, p_2, p_3) = (q, p_2, p_3, p_1).$$

Thus, the number of orbits can be easily counted using Burnside's lemma (see, for example, [15]). This lemma states that the number of distinct orbits is equal to the average size of the fixed sets of individual elements of G, so

$$l = \frac{1}{3}\left(|Q|^4 + |Q|^2 + |Q|^2\right).$$

A summary of our considerations is formulated in the following theorem.

Theorem 2.5. *Consider a finite state set $Q \subset \mathbb{R}$ containing 0. The number of all local rules equals*

$$\mathrm{L}(|Q|) = |Q|^{\frac{1}{3}(|Q|^2-1)(|Q|^2+3)}.$$

Note that the number $\mathrm{L}(|Q|)$ does not depend on the structure of the set Q, but only on its cardinality. For example, it is the same for $Q = \{0, 1, 2, 3\}$ and for $Q = \{-100, 0, 1, \sqrt{3}\}$.

When dealing with two-dimensional CAs, many researchers prefer more restrictive conditions on local rules. For example, one can consider *permutation-symmetric* local rules only, *i.e.*, local rules that are 'symmetric' with respect to all permutations of cells surrounding the central cell. In our notations this means that a local rule f satisfies (L1) but that assumption (L2) is replaced by

(L2') for any permutation $\pi : \{1, 2, 3\} \to \{1, 2, 3\}$ and for any $q, p_1, p_2, p_3 \in Q$ it holds that

$$f(q, p_{\pi(1)}, p_{\pi(2)}, p_{\pi(3)}) = f(q, p_1, p_2, p_3).$$

Obviously, the number of all permutation-symmetric local rules is smaller than the number of all local rules, but it is greater than the square root of $\mathrm{L}(|Q|)$.

Theorem 2.6. *Consider a finite state set $Q \subset \mathbb{R}$ containing 0. The number of all permutation-symmetric local rules equals*

$$\mathrm{LP}(|Q|) = |Q|^{\frac{1}{6}(|Q|-1)(|Q|^3+4|Q|^2+6|Q|+6)}.$$

The proof is similar to the proof of Theorem 2.5 and can be found, for example, in [20].

The main aim of this paper is to enumerate all number-conserving local rules. As we will see, number conservation of a local rule implies permutation symmetry. In other words, if a local rule (which is assumed to be rotation-symmetric only) is number-conserving, then it is automatically permutation-symmetric (see Corollary 3.11). Unfortunately, the number $\mathrm{LP}(|Q|)$ become huge as $|Q|$ increases, so it is not possible to use an exhaustive search through the set of all permutation-symmetric local rules to find which of them are number-conserving, even if we know some necessary and sufficient conditions. Moreover, the cardinality of the set of all number-conserving local rules can depend on the structure of the state set Q (see [10]). For example, as we will see, there are exactly two number-conserving local rules with the state set $Q = \{0, 1, 2, 3\}$, but only one with the state set $Q = \{-100, 0, 1, \sqrt{3}\}$. In the next section, we present how we overcame these difficulties.

3. The split-and-perturb decomposition

In this section, we show that any number-conserving local rule f can be decomposed into the sum $f = h + g$, where h is a *split function* and g is a *perturbation*, and that this decomposition is unique. A split function is a special local function that acts as follows: each state of a cell splits according to its recipe irrespective of the states of the neighbors of this cell. A perturbation is a local function that transforms any initial configuration into a configuration having as sum of its states the value zero.

3.1. Split functions

Definition 3.1. *A local function $h : Q^4 \to \mathbb{R}$ is called a* split function *if it satisfies the following two conditions:*

(S1) *for any $q \in Q$, the values $h(q,0,0,0)$, $h(0,q,0,0)$, $h(0,0,q,0)$, $h(0,0,0,q)$ belong to Q and*

$$h(q,0,0,0) + h(0,q,0,0) + h(0,0,q,0) + h(0,0,0,q) = q \, ;$$

(S2) *for any $q, p_1, p_2, p_3 \in Q$, it holds that*

$$h(q,p_1,p_2,p_3) = h(q,0,0,0) + h(0,p_1,0,0) + h(0,0,p_2,0) + h(0,0,0,p_3).$$

The set of all split functions is denoted by $\mathcal{S}$.

Note that each split function h is unambiguously defined by its values on monomers $(q,0,0,0)$. Indeed, for $q \in Q$, let us denote $c_q = h(q,0,0,0)$. If the values $(c_q)_{q\in Q}$ are known, then any value of h can be calculated.

Lemma 3.2. *Let $h : Q^4 \to \mathbb{R}$ be a split function. For any $q, p_1, p_2, p_3 \in Q$, it holds that*

$$h(q,p_1,p_2,p_3) = c_q + \frac{1}{3}(p_1 - c_{p_1}) + \frac{1}{3}(p_2 - c_{p_2}) + \frac{1}{3}(p_3 - c_{p_3}) \, . \tag{4}$$

Proof. From (S1) and the rotation-symmetry property (L2), we have for any $q \in Q$ that

$$h(0,q,0,0) = h(0,0,q,0) = h(0,0,0,q) = \frac{1}{3}(q - c_q). \tag{5}$$

Thus, according to (S2), we get (4). □

The following lemma shows that each split function is number-conserving.

Lemma 3.3. *Let h be a split function. Then for any finite configuration $\mathbf{x} \in X_F$, it holds that*

$$\sum_{\mathbf{i}\in\mathbb{Z}^2} A_h(\mathbf{x})_{\mathbf{i}} = \sum_{\mathbf{i}\in\mathbb{Z}^2} x_{\mathbf{i}}.$$

Proof. First of all, recall that $c_0 = 0$ (see (L1)), which guarantees that in each of the sums considered below there are only finitely many non-zero terms and therefore all operations on these sums are well-defined. Let us note that in the multiset

$$\left\{\mathbf{j}_1(\mathbf{i}),\ \mathbf{j}_2(\mathbf{i}),\ \mathbf{j}_3(\mathbf{i}) \mid \mathbf{i} \in \mathbb{Z}^2\right\}$$

each element has multiplicity 3, since each cell $\mathbf{j} \in \mathbb{Z}^2$ is adjacent to three different cells $\mathbf{i} \in \mathbb{Z}^2$. Thus, we have both

$$\sum_{\mathbf{i}\in\mathbb{Z}^2} (x_{\mathbf{j}_1(\mathbf{i})} + x_{\mathbf{j}_2(\mathbf{i})} + x_{\mathbf{j}_3(\mathbf{i})}) = 3 \sum_{\mathbf{i}\in\mathbb{Z}^2} x_{\mathbf{i}}$$

and

$$\sum_{\mathbf{i}\in\mathbb{Z}^2} (c_{x_{\mathbf{j}_1(\mathbf{i})}} + c_{x_{\mathbf{j}_2(\mathbf{i})}} + c_{x_{\mathbf{j}_3(\mathbf{i})}}) = 3 \sum_{\mathbf{i}\in\mathbb{Z}^2} c_{x_{\mathbf{i}}}.$$

Hence, we have

$$\begin{aligned}
\sum_{\mathbf{i}\in\mathbb{Z}^2} A_h(\mathbf{x})_{\mathbf{i}} &= \sum_{\mathbf{i}\in\mathbb{Z}^2} h(x_{\mathbf{i}}, x_{\mathbf{j}_1(\mathbf{i})}, x_{\mathbf{j}_2(\mathbf{i})}, x_{\mathbf{j}_3(\mathbf{i})}) \\
&= \sum_{\mathbf{i}\in\mathbb{Z}^2} \Big(c_{x_{\mathbf{i}}} + \frac{1}{3}(x_{\mathbf{j}_1(\mathbf{i})} - c_{x_{\mathbf{j}_1(\mathbf{i})}}) + \frac{1}{3}(x_{\mathbf{j}_2(\mathbf{i})} - c_{x_{\mathbf{j}_2(\mathbf{i})}}) + \frac{1}{3}(x_{\mathbf{j}_3(\mathbf{i})} - c_{x_{\mathbf{j}_3(\mathbf{i})}}) \Big) \\
&= \sum_{\mathbf{i}\in\mathbb{Z}^2} c_{x_{\mathbf{i}}} + \frac{1}{3}\sum_{\mathbf{i}\in\mathbb{Z}^2}(x_{\mathbf{j}_1(\mathbf{i})} + x_{\mathbf{j}_2(\mathbf{i})} + x_{\mathbf{j}_3(\mathbf{i})}) - \frac{1}{3}\sum_{\mathbf{i}\in\mathbb{Z}^2}(c_{x_{\mathbf{j}_1(\mathbf{i})}} + c_{x_{\mathbf{j}_2(\mathbf{i})}} + c_{x_{\mathbf{j}_3(\mathbf{i})}}) \\
&= \sum_{\mathbf{i}\in\mathbb{Z}^2} c_{x_{\mathbf{i}}} + \sum_{\mathbf{i}\in\mathbb{Z}^2} x_{\mathbf{i}} - \sum_{\mathbf{i}\in\mathbb{Z}^2} c_{x_{\mathbf{i}}} = \sum_{\mathbf{i}\in\mathbb{Z}^2} x_{\mathbf{i}} \; .
\end{aligned}$$

□

Note that for any state set Q there exists at least one split function, which is also a number-conserving local rule. Indeed, if we put $c_q = q$ for each $q \in Q$, then we obtain a split function h for which A_h acts as the identity on X. We will denote this split function by h_{Id}.

3.2. Perturbations

Definition 3.4. *A local function $g : Q^4 \to \mathbb{R}$ is called a* perturbation *if it satisfies the following two conditions:*

(P1) *g takes value* 0 *on every monomer,* i.e.*, $g(q,0,0,0) = g(0,q,0,0) = g(0,0,q,0) = g(0,0,0,q) =$ 0, for each $q \in Q$;*

(P2) *for every $\mathbf{x} \in X_F$, it holds that $\sum_{\mathbf{i}\in\mathbb{Z}^2} A_g(\mathbf{x})_{\mathbf{i}} = 0$.*

The set of all perturbations is denoted by $\mathcal{P}$.

It turns out that from a mathematical point of view, the set $\mathcal{P}$ has a very ordered structure.

Lemma 3.5. *The set of all perturbations $\mathcal{P}$ forms a linear space.*

Proof. Let $g = \alpha g_1 + \beta g_2$, where $g_1, g_2 \in \mathcal{P}$ and α, β be arbitrary real numbers. Obviously, g satisfies (P1). Moreover, for any $\mathbf{x} \in X_F$ we have

$$\begin{aligned}
\sum_{\mathbf{i}\in\mathbb{Z}^2} A_g(\mathbf{x})_{\mathbf{i}} &= \sum_{\mathbf{i}\in\mathbb{Z}^2} g\left(x_{\mathbf{i}}, x_{\mathbf{j}_1(\mathbf{i})}, x_{\mathbf{j}_2(\mathbf{i})}, x_{\mathbf{j}_3(\mathbf{i})}\right) \\
&= \sum_{\mathbf{i}\in\mathbb{Z}^2} \Big(\alpha g_1\left(x_{\mathbf{i}}, x_{\mathbf{j}_1(\mathbf{i})}, x_{\mathbf{j}_2(\mathbf{i})}, x_{\mathbf{j}_3(\mathbf{i})}\right) + \beta g_2\left(x_{\mathbf{i}}, x_{\mathbf{j}_1(\mathbf{i})}, x_{\mathbf{j}_2(\mathbf{i})}, x_{\mathbf{j}_3(\mathbf{i})}\right) \Big) \\
&= \alpha \sum_{\mathbf{i}\in\mathbb{Z}^2} A_{g_1}(\mathbf{x})_{\mathbf{i}} + \beta \sum_{\mathbf{i}\in\mathbb{Z}^2} A_{g_2}(\mathbf{x})_{\mathbf{i}} = 0 \; ,
\end{aligned}$$

so, g fulfills also (P2). □

Lemma 3.6. *If a local function $g : Q^4 \to \mathbb{R}$ is a perturbation, then for any $q, p \in Q$ it holds that*

$$g(q,p,0,0) = -g(p,q,0,0).$$

Proof. Let us consider the configuration $\mathbf{x} \in X_F$ shown in Figure 3(a). As g is a perturbation, according to (P1), $A_g(\mathbf{x})$ has at most two non-zero states: $g(q,p,0,0)$ and $g(p,q,0,0)$. Thus, from (P2), we get

$$g(q,p,0,0) + g(p,q,0,0) = 0,$$

which concludes the proof. □

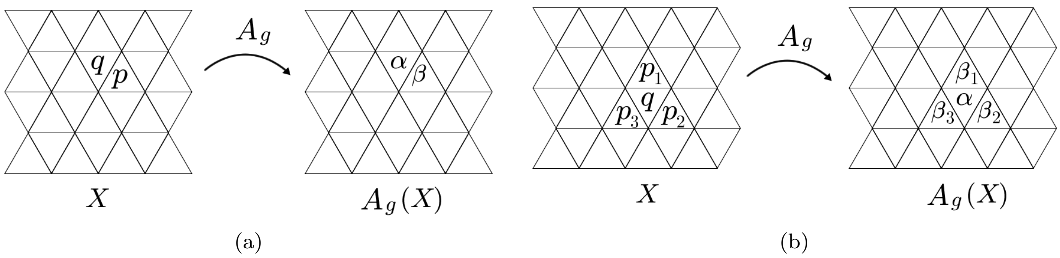

Figure 3: The action of the global function A_g when g is a perturbation satisfying (a) $\alpha = g(q,p,0,0)$ and $\beta = g(p,q,0,0)$ (b) $\alpha = g(q,p_1,p_2,p_3)$, $\beta_1 = g(p_1,q,0,0)$, $\beta_2 = g(p_2,q,0,0)$ and $\beta_3 = g(p_3,q,0,0)$ (blank cells have state 0).

In contrast to the case of a square grid, in the case of a triangular grid, every perturbation is completely determined by a function of two variables, which will be called a *perturbation root.*

Definition 3.7. *A* perturbation root *is a function* $\widetilde{g} : Q^2 \to \mathbb{R}$ *satisfying the following two conditions*

(R1) $\widetilde{g}(q,0) = 0$*, for each* $q \in Q$*;*

(R2) $\widetilde{g}(q,p) = -\widetilde{g}(p,q)$*, for each* $q,p \in Q$*.*

It is obvious that the set of all perturbation roots is a linear space and that the zero vector in this space is the function $\widetilde{g}_0$ such that $\widetilde{g}_0(q,p) = 0$, for each $q,p \in Q$.

Theorem 3.8. *A function* $g : Q^4 \to \mathbb{R}$ *is a perturbation if and only if there exists a perturbation root* $\widetilde{g} : Q^2 \to \mathbb{R}$ *such that for any* $q,p_1,p_2,p_3 \in Q$*, it holds that*

$$g(q,p_1,p_2,p_3) = \widetilde{g}(q,p_1) + \widetilde{g}(q,p_2) + \widetilde{g}(q,p_3) \,. \tag{6}$$

Proof. Let us assume that $g : Q^4 \to \mathbb{R}$ is a perturbation and define $\widetilde{g}(q,p) = g(q,p,0,0)$. Using the configuration $\mathbf{x} \in X_F$ presented in Figure 3(b), we get, according to (P1) and (P2), that

$$g(q,p_1,p_2,p_3) + g(p_1,q,0,0) + g(p_2,q,0,0) + g(p_3,q,0,0) = 0.$$

Thus, from Lemma 3.6, we have

$$g(q,p_1,p_2,p_3) = g(q,p_1,0,0) + g(q,p_2,0,0) + g(q,p_3,0,0) \,,$$

which yields Eq. (6), since we put $\widetilde{g}(q,p) = g(q,p,0,0)$.

Moreover, from Lemma 3.6 and (P1), we get that $\widetilde{g}$ satisfies conditions (R1) and (R2).

Now, let us assume that a function $g : Q^4 \to \mathbb{R}$ is given by Eq. (6), where the function $\widetilde{g} : Q^2 \to \mathbb{R}$ is a perturbation root. As condition (R1) implies that $\widetilde{g}(0,0) = 0$, we have $g(0,0,0,0) = 0$ and, according to Eq. (6), g is rotation-symmetric, *i.e.*, g satisfies both (L1) and (L2), thus it is a local function. Moreover, condition (R1) implies (P1), so, it is sufficient to show that g satisfies (P2).

Let $\mathbf{x} \in X_F$. Then

$$\begin{aligned}
\sum_{\mathbf{i}\in\mathbb{Z}^2} A_g(\mathbf{x})_{\mathbf{i}} &= \sum_{\mathbf{i}\in\mathbb{Z}^2} g\left(x_{\mathbf{i}}, x_{\mathbf{j}_1(\mathbf{i})}, x_{\mathbf{j}_2(\mathbf{i})}, x_{\mathbf{j}_3(\mathbf{i})}\right) \\
&= \sum_{\mathbf{i}\in\mathbb{Z}^2} \left(\widetilde{g}(x_{\mathbf{i}}, x_{\mathbf{j}_1(\mathbf{i})}) + \widetilde{g}(x_{\mathbf{i}}, x_{\mathbf{j}_2(\mathbf{i})}) + \widetilde{g}(x_{\mathbf{i}}, x_{\mathbf{j}_3(\mathbf{i})})\right) \\
&= \sum_{\mathbf{i}\in\mathbb{Z}^2} \sum_{\mathbf{j}\in\mathbb{Z}^2:\, \mathbf{j}\sim\mathbf{i}} \widetilde{g}(x_{\mathbf{i}}, x_{\mathbf{j}}) \,.
\end{aligned}$$

Note that every pair of adjacent cells $(\mathbf{i}, \mathbf{j})$ appears twice in this double sum: once in $\widetilde{g}(x_\mathbf{i}, x_\mathbf{j})$ and once in $\widetilde{g}(x_\mathbf{j}, x_\mathbf{i})$. But since $\widetilde{g}(x_\mathbf{i}, x_\mathbf{j}) = -\widetilde{g}(x_\mathbf{j}, x_\mathbf{i})$, these terms cancel out. Finally, we get $\sum_{\mathbf{i}\in\mathbb{Z}^2} A_g(\mathbf{x})_\mathbf{i} = 0$, which concludes the proof. □

Since the correspondence between perturbations and perturbation roots is one-to-one, for a given perturbation g, its perturbation root will be denoted by $\widetilde{g}$.

3.3. The decomposition theorem

Now, we are ready to show that each number-conserving local rule can be decomposed into the sum of a split function and a perturbation.

Theorem 3.9. *A local rule $f : Q^4 \to Q$ is number-conserving if and only if there exist a split function $h : Q^4 \to \mathbb{R}$ and a perturbation $g : Q^4 \to \mathbb{R}$ such that $f = h + g$. Moreover, for a given local rule f, the functions h and g are uniquely determined.*

Proof. If a local rule f is the sum of some split function h and a perturbation g, then it is number-conserving. Indeed, let $\mathbf{x} \in X_F$ be any finite configuration, then

$$\sum_{\mathbf{i}\in\mathbb{Z}^2} A_f(\mathbf{x})_\mathbf{i} = \sum_{\mathbf{i}\in\mathbb{Z}^2} (A_h(\mathbf{x})_\mathbf{i} + A_g(\mathbf{x})_\mathbf{i}) = \sum_{\mathbf{i}\in\mathbb{Z}^2} A_h(\mathbf{x})_\mathbf{i} + \sum_{\mathbf{i}\in\mathbb{Z}^2} A_g(\mathbf{x})_\mathbf{i} = \sum_{\mathbf{i}\in\mathbb{Z}^2} x_\mathbf{i}\,,$$

as each split function preserves the sum of states, while $\sum_{\mathbf{i}\in\mathbb{Z}^2} A_g(\mathbf{x})_\mathbf{i} = 0$ for any perturbation g.

Now, let us assume that a local rule f is number-conserving. Thus, in particular, $f(q, 0, 0, 0) + f(0, q, 0, 0) + f(0, 0, q, 0) + f(0, 0, 0, q) = q$, for each state $q \in Q$, according to Lemma 2.4. Let us define the local function h as follows:

(i) $h = f$ on monomers, *i.e.*, for any $q \in Q$, we set $h(q, 0, 0, 0) = f(q, 0, 0, 0)$ and

$$h(0, q, 0, 0) = h(0, 0, q, 0) = h(0, 0, 0, q) = f(0, q, 0, 0) = \frac{1}{3}\big(q - f(q, 0, 0, 0)\big),$$

(ii) for any $q, p_1, p_2, p_3 \in Q$, we define

$$h(q, p_1, p_2, p_3) = h(q, 0, 0, 0) + h(0, p_1, 0, 0) + h(0, 0, p_2, 0) + h(0, 0, 0, p_3).$$

It is easy to see that h is a split function. Let $g = f - h$. As both f and h are number-conserving, for any $\mathbf{x} \in X_F$, we have that $\sum_{\mathbf{i}\in\mathbb{Z}^2} A_g(\mathbf{x})_\mathbf{i} = 0$. Moreover, given the definition of h, we see that g satisfies (P1). Thus, g is a perturbation.

The decomposition of a number-conserving local rule into a split function and a perturbation is unique. Indeed, if for two split functions h_1 and h_2 and two perturbations g_1 and g_2, we have $h_1 + g_1 = h_2 + g_2$, then $h_1 = h_2$ on monomers, and therefore on the entire Q^4, according to (S2). Consequently, it must hold that also $g_1 = g_2$. □

One of the advantages of the above theorem is the following. For the enumeration of all number-conserving local rules, we can consider each split function h separately and find all perturbations g that are *compatible with* h in the sense that $h + g$ takes values in Q.

The next corollary summarizes Theorems 3.8 and 3.9.

Corollary 3.10. *A local rule $f : Q^4 \to Q$ is number-conserving if and only if there exist a split function h and a perturbation root $\widetilde{g}$ such that for any $q, p_1, p_2, p_3 \in Q$ we have*

$$f(q, p_1, p_2, p_3) = h(q, p_1, p_2, p_3) + \widetilde{g}(q, p_1) + \widetilde{g}(q, p_2) + \widetilde{g}(q, p_3)\,. \tag{7}$$

Moreover, for a given local rule f, the functions h and $\widetilde{g}$ are uniquely determined.

As a straightforward consequence of Eqs. (4) and (7), we get the following fact.

Corollary 3.11. *Every number-conserving local rule f is permutation-symmetric,* i.e., *for any permutation $\pi : \{1, 2, 3\} \to \{1, 2, 3\}$ and for any $q, p_1, p_2, p_3 \in Q$, it holds that*

$$f(q, p_{\pi(1)}, p_{\pi(2)}, p_{\pi(3)}) = f(q, p_1, p_2, p_3) \ .$$

Although it seems that the development presented in this section requires the assumption that 0 is one of the considered states, in fact, we do not need this restriction. To deal with number conservation on an infinite grid, it is sufficient to assume that in the considered state set Q, there exists at least one quiescent element q_0 (see, for instance, [17]). Then having a CA with some state set $Q \subseteq \mathbb{R}$ and a quiescent state q_0, we can consider an equivalent CA with the state set $Q' = \{q - q_0 \mid q \in Q\}$. Since subtracting a constant from all states does not affect the number conservation of the considered CA, all results obtained for Q' can be easily translated in terms of Q. Thanks to that, without loss of generality, we may assume that the considered state set contains 0.

4. The case of k-ary CAs

In this section, we present an application of the theory presented to find all triangular NCCAs with the state set $\{0, 1, \dots, k-1\}$, where $k > 1$ is some natural number. Such CAs are a generalization of binary and ternary CAs and are often called k-ary CAs (see, for example [27]).

4.1. Enumeration of split functions

As mentioned in Lemma 3.2, a split function h is given by the values $(c_q)_{q\in Q}$, where $c_q = h(q, 0, 0, 0)$. For the state set $\{0, 1, \dots, k-1\}$, this means that to define a split function we should fill Table 1(a), but it cannot be an unrestricted filling. Note that condition (S1) and Eq. (5) imply that for any $q \in \{0, 1, \dots, k-1\}$, it holds that $0 \leqslant c_q \leqslant q$ and that $c_q - q$ is divisible by 3. Thus, we have the following fact.

Lemma 4.1. *Let $h : \{0, 1, \dots, k-1\}^4 \to \mathbb{R}$ be a split function. Then for any $q \in \{0, 1, \dots, k-1\}$, we have*

$$c_q \in \left\{q, q-3, \dots, q - 3\left\lfloor \frac{q}{3} \right\rfloor + 3, q - 3\left\lfloor \frac{q}{3} \right\rfloor \right\} , \tag{8}$$

where $\lfloor x \rfloor$ denotes the integer part of x.
Furthermore, if we chose the numbers $(c_q)_{q\in\{0,1,\dots,k-1\}}$ satisfying condition (8), then the function h defined by Eq. (4) is a split function.

q	0	1	2	...	$k-1$
c_q					

(a)

q	0	1	2	3	4	5	6
c_q	0	1	2	$3 \vee 0$	$4 \vee 1$	$5 \vee 2$	$6 \vee 3 \vee 0$

(b)

Table 1: (a) The table to be completed in order to define a split function for the state set $\{0, 1, \dots, k-1\}$ (b) all possibilities for c_q, $q = 0, \dots, 6$, in the case $k = 7$.

For example, if we consider $k = 7$, *i.e.*, the state set $\{0, 1, 2, 3, 4, 5, 6\}$, then all possible fillings of Table 1(a) are shown in Table 1(b). Thus, it can be filled in only 24 ways. We can generalize this remark in the following theorem.

Theorem 4.2. *Consider the state set $\{0, 1, \dots, k-1\}$ for some natural $k > 1$. If $k \le 3$, then there is exactly one split function: h_{Id}. If $k > 3$, then there are exactly*

$$\mathrm{S}(k) = \left(\left\lfloor \frac{k-1}{3} \right\rfloor !\right)^3 \cdot \left(\left\lfloor \frac{k-1}{3} \right\rfloor + 1\right)^{k-3\left\lfloor \frac{k-1}{3} \right\rfloor} \tag{9}$$

split functions.

Proof. As shown in Table 1(b), the values of c_0, c_1 and c_2 are unambiguously determined, while for each of c_3, c_4 and c_5 there are two possibilities, for each of c_6, c_7 and c_8 there are three possibilities, and so on, and finally, at the end of Table 1(a), there are $\lfloor \frac{k-1}{3} \rfloor$ possibilities for $c_{3\lfloor \frac{k-1}{3} \rfloor-3}$, $c_{3\lfloor \frac{k-1}{3} \rfloor-2}$, $c_{3\lfloor \frac{k-1}{3} \rfloor-1}$, and $\lfloor \frac{k-1}{3} \rfloor + 1$ possibilities for $c_{3\lfloor \frac{k-1}{3} \rfloor}, \dots, c_{k-1}$. Thus

$$\mathrm{S}(k) = 1^3 \cdot 2^3 \cdot 3^3 \cdot \ldots \cdot \left\lfloor \frac{k-1}{3} \right\rfloor^3 \cdot \left(\left\lfloor \frac{k-1}{3} \right\rfloor + 1 \right)^{k-3\lfloor \frac{k-1}{3} \rfloor},$$

which gives Eq. (9). □

The number of split functions for several values of k is given in Table 2, while a list of all split functions for the state set $\{0, 1, \dots, k-1\}$ can be obtained by Algorithm 1.

k	$\mathrm{L}(k)$	$\mathrm{LP}(k)$	$\mathrm{S}(k)$
2	2^7	2^7	1
3	3^{32}	3^{29}	1
4	4^{95}	4^{79}	2
5	5^{224}	5^{174}	4
6	6^{455}	6^{335}	8
7	7^{832}	7^{587}	24
8	8^{1407}	8^{959}	72
9	9^{2240}	9^{1484}	216
10	10^{3399}	10^{2199}	864

Table 2: The numbers of local rules ($\mathrm{L}(k)$), permutation-symmetric local rules ($\mathrm{LP}(k)$) and split functions ($\mathrm{S}(k)$), for the state set $\{0, 1, \dots, k-1\}$.

Algorithm 1: Generating split functions

Input: k – natural number, $k > 1$
Output: list of vectors $(c_0, c_1, \dots, c_{k-1})$

procedure GenSplitFun(k)
  SplitList ← `empty list`
  for $c_0 \in \{0\}$ **do**
    for $c_1 \in \{0, 1\}$ **do**
      ⋱
      for $c_{k-1} \in \{0, 1, \dots, k-1\}$ **do**
        if *each* c_q *satisfies condition:* $q - c_q$ *is divisible by 3* **then**
          add $(c_0, c_1, \dots, c_{k-1})$ to the *SplitList*
  return *SplitList*

4.2. The space of perturbation roots

According to Definition 3.7, it is obvious that if Q contains only one non-zero element (for example, for the binary case $k = 2$), then there is only one perturbation root: the zero function $\widetilde{g}_0$. However, for any two non-zero elements $p, q \in Q_*$, one can define the simplest non-trivial perturbation root $\widetilde{g}$ having exactly two non-zero values: $\widetilde{g}(p, q) = 1$ and $\widetilde{g}(q, p) = -1$. This simple observation allows us to completely characterize the space of perturbation roots $\mathcal{R}$.

Theorem 4.3. *Consider the state set* $\{0, 1, \dots, k-1\}$ *for some natural* $k > 1$*. If* $k = 2$*, then* $\mathcal{R} = \{\widetilde{g}_0\}$*. If* $k > 2$*, then the linear space* $\mathcal{R}$ *has dimension* $\binom{k-1}{2}$ *and as its basis one can take the* *set* $\left\{\widetilde{g}_{q,p} \mid q,p \in \{1,2,\ldots,k-1\} \wedge q>p\right\}$, *where*

$$\widetilde{g}_{q,p}(a,b) = \begin{cases} 1, & \text{if } (a,b)=(q,p) \\ -1, & \text{if } (a,b)=(p,q) \\ 0, & \text{otherwise .} \end{cases}$$

As a consequence, we get the following corollary.

Corollary 4.4. *Each perturbation root* $\widetilde{g}$ *can be written as*

$$\widetilde{g} = \sum_{q,p\in\{1,2,\ldots,k-1\}:q>p} \alpha_{q,p}\, \widetilde{g}_{q,p} \ ,$$

for some real coefficients $\alpha_{q,p}$.

4.3. Enumeration of number-conserving local rules

From Corollaries 3.10 and 4.4, it follows that

$$\begin{aligned} f(q_0,p_1,p_2,p_3) &= h(q_0,p_1,p_2,p_3) \\ &+ \sum_{q,p\in\{1,2,\ldots,k-1\}:q>p} \alpha_{q,p}\left(\widetilde{g}_{q,p}(q_0,p_1)+\widetilde{g}_{q,p}(q_0,p_2)+\widetilde{g}_{q,p}(q_0,p_3)\right) , \end{aligned} \tag{10}$$

for any $q_0,p_1,p_2,p_3 \in \{0,1,2,\ldots,k-1\}$. Moreover, we know that each split function h is determined by the values c_q described in Lemma 4.1. Now, for a given split function h, we are interested in finding all possibilities for $\alpha_{q,p}$, for which the right–hand side of Eq. (10) gives a number belonging $\{0,1,2,\ldots,k-1\}$.

Theorem 4.5. *Let a split function* $h : \{0,1,\ldots,k-1\}^4 \to \mathbb{R}$ *be given and let* $c_q = h(q,0,0,0)$. *The local function* f *defined by Eq. (10) takes values in* $\{0,1,2,\ldots,k-1\}$ *if and only if the coefficients* $\alpha_{q,p}$ *are integers satisfying the following bounds:*

(I) *if* $q+p \leq k-1$, *then*

$$-\frac{1}{3}p+\frac{1}{3}(c_p-c_q) \leq \alpha_{q,p} \leq \frac{1}{3}q+\frac{1}{3}(c_p-c_q) \ , \tag{11}$$

(II) *if* $q+p \geq k-1$, *then*

$$-\frac{1}{3}(k-1-q)+\frac{1}{3}(c_p-c_q) \leq \alpha_{q,p} \leq \frac{1}{3}(k-1-p)+\frac{1}{3}(c_p-c_q). \tag{12}$$

Proof. First, we prove that conditions (11) and (12) are necessary. So, let us assume that the local function f defined by Eq. (10) is a local rule and let $q,p \in \{1,2,\ldots,k-1\}$ be such that $q>p$. Since

$$f(q,p,p,p) = c_q + (p-c_p) + 3\alpha_{q,p}$$

and

$$f(p,q,q,q) = c_p + (q-c_q) - 3\alpha_{q,p} \ ,$$

both $0 \leq c_q+(p-c_p)+3\alpha_{q,p} \leq k-1$ and $0 \leq c_p+(q-c_q)-3\alpha_{q,p} \leq k-1$. Thus

$$\max\left(\frac{1}{3}(c_p-c_q-p), \frac{1}{3}(c_p-c_q+q-(k-1))\right) \leq \alpha_{q,p} \leq \min\left(\frac{1}{3}(c_p-c_q+k-1-p), \frac{1}{3}(c_p-c_q+q)\right).$$

As

$$\max\left(\frac{1}{3}(c_p-c_q-p), \frac{1}{3}(c_p-c_q+q-(k-1))\right) = \begin{cases} \frac{1}{3}(c_p-c_q-p), & \text{if } q+p \leq k-1 \\ \frac{1}{3}(c_p-c_q+q-(k-1)), & \text{if } q+p \geq k-1 \ , \end{cases}$$

and

$$\min\left(\frac{1}{3}(c_p - c_q + k - 1 - p), \frac{1}{3}(c_p - c_q + q)\right) = \begin{cases} \frac{1}{3}(c_p - c_q + q), & \text{if } q + p \le k-1 \\ \frac{1}{3}(c_p - c_q + k - 1 - p), & \text{if } q + p \ge k-1\ , \end{cases}$$

we get (11) and (12). Moreover, since $\alpha_{q,p}$ equals $f(q,p,0,0) - h(q,p,0,0)$, it has to be an integer.

Now, we show that if given integers $\alpha_{q,p}$ fulfill (11) and (12), then f takes values in $\{0,1,2,\ldots,k-1\}$. Note that since $\alpha_{q,p}$ are integers, f takes integer values, so it is sufficient to argue that, for any $q_0, p_1, p_2, p_3 \in \{0,1,2,\ldots,k-1\}$, the value of the right–hand side of Eq. (10) lies between 0 and $k-1$. Let $\widetilde{g} = \sum_{q,p\in\{1,2,\ldots,k-1\}:\, q>p} \alpha_{q,p}\widetilde{g}_{q,p}$. Note that for any $q_0, p_0 \in \{0,1,2,\ldots,k-1\}$, it holds that

$$\widetilde{g}(q_0,p_0) = \begin{cases} \alpha_{q_0,p_0}, & \text{if } 0 < p_0 < q_0 \\ -\alpha_{p_0,q_0}, & \text{if } 0 < q_0 < p_0 \\ 0, & \text{otherwise}\ , \end{cases}$$

so,

$$\frac{1}{3}c_{q_0} + \frac{1}{3}(p_0 - c_{p_0}) + \widetilde{g}(q_0,p_0) = \begin{cases} \frac{1}{3}(p_0 - c_{p_0}), & \text{if } q_0 = 0 \\ \frac{1}{3}c_{q_0}, & \text{if } p_0 = 0 \\ \frac{1}{3}p_0, & \text{if } p_0 = q_0 \\ \frac{1}{3}c_{q_0} + \frac{1}{3}(p_0 - c_{p_0}) + \alpha_{q_0,p_0}, & \text{if } 0 < p_0 < q_0 \\ \frac{1}{3}c_{q_0} + \frac{1}{3}(p_0 - c_{p_0}) - \alpha_{p_0,q_0}, & \text{if } 0 < q_0 < p_0. \end{cases}$$

Thus, according to Eqs. (8), (11) and (12), in all cases we have

$$0 \le \frac{1}{3}c_{q_0} + \frac{1}{3}(p_0 - c_{p_0}) + \widetilde{g}(q_0,p_0) \le \frac{1}{3}(k-1)\ . \tag{13}$$

Therefore, since for $q_0, p_1, p_2, p_3 \in \{0,1,2,\ldots,k-1\}$, it holds that

$$\begin{aligned} f(q_0,p_1,p_2,p_3) &= c_{q_0} + \frac{1}{3}(p_1 - c_{p_1}) + \frac{1}{3}(p_2 - c_{p_2}) + \frac{1}{3}(p_3 - c_{p_3}) + \widetilde{g}(q_0,p_1) + \widetilde{g}(q_0,p_2) + \widetilde{g}(q_0,p_3) \\ &= \left(\frac{1}{3}c_{q_0} + \frac{1}{3}(p_1 - c_{p_1}) + \widetilde{g}(q_0,p_1)\right) + \left(\frac{1}{3}c_{q_0} + \frac{1}{3}(p_2 - c_{p_2}) + \widetilde{g}(q_0,p_2)\right) \\ &\quad + \left(\frac{1}{3}c_{q_0} + \frac{1}{3}(p_3 - c_{p_3}) + \widetilde{g}(q_0,p_3)\right)\ , \end{aligned}$$

we get from (13) that $0 \le f(q_0,p_1,p_2,p_3) \le k-1$, which concludes the proof. □

With the help of the above theorem, we can enumerate all triangular NCCAs with the state set $\{0,1,\ldots,k-1\}$ with the following scenario. For every split function h, given by a sequence (c_q), we calculate the bounds (I) and (II) to find all perturbations, given as sequences $(\alpha_{q,p})$, that are compatible with h. For example, if $k=5$, then there are only four split functions h_1, h_2, h_3 and h_4 and the possible values of $\alpha_{q,p}$ for every h_i are presented in Table 3. We can see that the number of perturbations that are compatible with a given split function h_i equals 4, independently of the choice of h_i.

This fact holds for every $k > 1$. Indeed, if we change the values c_q and c_p by a multiple of three, the bounds in (11) and (12) will undergo the same change. Thus the number of integers $\alpha_{q,p}$ that fulfill (11) and (12) is independent of the choice of c_q and c_p. As a consequence, we get the following remark.

Corollary 4.6. *Let $k > 1$ be fixed. The number of perturbations that are compatible with a given split function h is independent of h. Thus it can be counted for the split function h_{Id}.*

Lemma 4.7. *The number of perturbations that are compatible with the split function h_{Id} equals*

$$\mathrm{M}(k) = \prod_{q,p\in\{1,2,\ldots,k-1\}:\, q>p} m(k)_{q,p}\ ,$$

	h_1	h_2	h_3	h_4
	$c_0 = 0$ $c_1 = 1$ $c_2 = 2$ $c_3 = 3$ $c_4 = 4$	$c_0 = 0$ $c_1 = 1$ $c_2 = 2$ $c_3 = 3$ $c_4 = 1$	$c_0 = 0$ $c_1 = 1$ $c_2 = 2$ $c_3 = 0$ $c_4 = 4$	$c_0 = 0$ $c_1 = 1$ $c_2 = 2$ $c_3 = 0$ $c_4 = 1$
$\alpha_{2,1}$	0	0	0	0
$\alpha_{3,1}$	$-1 \vee 0$	$-1 \vee 0$	$0 \vee 1$	$0 \vee 1$
$\alpha_{3,2}$	0	0	1	1
$\alpha_{4,1}$	$-1 \vee 0$	$0 \vee 1$	$-1 \vee 0$	$0 \vee 1$
$\alpha_{4,2}$	0	1	0	1
$\alpha_{4,3}$	0	1	-1	0

Table 3: The possible values of $\alpha_{q,p}$ for each of the split functions h_1, h_2, h_3, h_4 in the case of triangular NCCAs with the state set $\{0, 1, 2, 3, 4\}$.

where

$$m(k)_{q,p} = \begin{cases} \left\lfloor \frac{1}{3}q \right\rfloor + \left\lfloor \frac{1}{3}p \right\rfloor + 1, & \text{if } q + p \leq k - 1 \\ \left\lfloor \frac{1}{3}(k-1-p) \right\rfloor + \left\lfloor \frac{1}{3}(k-1-q) \right\rfloor + 1, & \text{if } q + p \geq k - 1. \end{cases}$$

Proof. As we consider the split function h_{Id}, for which $c_q = q$ for every $q \in Q$, the bounds (11) and (12) simplify to the following:

(I') if $q + p \leq k - 1$, then

$$-\frac{1}{3}p \leq \alpha_{q,p} \leq \frac{1}{3}q \,,$$

(II') if $q + p \geq k - 1$, then

$$-\frac{1}{3}(k-1-q) \leq \alpha_{q,p} \leq \frac{1}{3}(k-1-p).$$

Thus, if $q + p \leq k - 1$, then $\alpha_{q,p}$ can take one of the values $-\left\lfloor \frac{1}{3}p \right\rfloor, -\left\lfloor \frac{1}{3}p \right\rfloor + 1, \ldots, \left\lfloor \frac{1}{3}q \right\rfloor$, which gives $m(k)_{q,p} = \left\lfloor \frac{1}{3}q \right\rfloor + \left\lfloor \frac{1}{3}p \right\rfloor + 1$ possibilities, while for $q+p \geq k-1$, $\alpha_{q,p}$ can take one of the values $-\left\lfloor \frac{1}{3}(k-1-q) \right\rfloor, -\left\lfloor \frac{1}{3}(k-1-q) \right\rfloor + 1, \ldots, \left\lfloor \frac{1}{3}(k-1-p) \right\rfloor$, which gives $m(k)_{q,p} = \left\lfloor \frac{1}{3}(k-1-p) \right\rfloor + \left\lfloor \frac{1}{3}(k-1-q) \right\rfloor + 1$ possibilities. □

(a)

q \ p	1	2	3	4
1	×	×	×	×
2	1	×	×	×
3	2	1	×	×
4	2	1	1	×

(b)

q \ p	1	2	3	4	5
1	×	×	×	×	×
2	1	×	×	×	×
3	2	2	×	×	×
4	2	2	1	×	×
5	2	2	1	1	×

(c)

q \ p	1	2	3	4	5	6
1	×	×	×	×	×	×
2	1	×	×	×	×	×
3	2	2	×	×	×	×
4	2	2	2	×	×	×
5	2	2	2	1	×	×
6	2	2	2	1	1	×

Table 4: The numbers $m(k)_{q,p}$ for (a) $k = 5$ (b) $k = 6$ (c) $k = 7$.

Having the numbers $m(k)_{q,p}$, we can easily calculate the number of all perturbations that are compatible with the split function h_{Id}, for example, according to Table 4: $\mathrm{M}(7) = 1^4 \cdot 2^{11} =$

2048 (analogously, $M(2) = 1^0 = 1$, $M(3) = 1^1 = 1$, $M(4) = 1^3 = 1$, $M(5) = 1^4 \cdot 2^2 = 4$, $M(6) = 1^4 \cdot 2^6 = 64$). Moreover, from Corollary 4.6 it follows that the number of all k-ary triangular NCCAs equals $N(k)$, where $N(k) = S(k) \cdot M(k)$. Thus, using Eq. (9), one obtains that $N(7) = S(7)M(7) = 24 \cdot 2048 = 49152$. Although for any given k, it is not too difficult to calculate $M(k)$ (and thus $N(k)$) using Lemma 4.7, we decided to present the explicit formula for $M(k)$.

Theorem 4.8. *Let $k \geq 7$ and denote $k_* = \left\lfloor \frac{k-1}{3} \right\rfloor$. Then*

$$M(k) = 1^{e_1} \cdot 2^{e_2} \cdot \ldots (k_* - 1)^{e_{k_*-1}} \cdot k_*^{e_{k_*}} \cdot (k_* + 1)^{e_{k_*+1}} , \tag{14}$$

where the exponents e_i satisfy

$$e_1 + e_2 + \ldots + e_{k_*-1} + e_{k_*} + e_{k_*2} = \binom{k-1}{2} \tag{15}$$

and are defined as follows:

$$e_1 = 4, \text{ while for } i = 2, 3, \ldots, k_* - 1,\ e_i = \begin{cases} 9i - 3, & \text{if } i \text{ is even} \\ 9i - 6, & \text{if } i \text{ is odd} \end{cases}$$

$$e_{k_*} = \begin{cases} 14n - 3, & \text{if } k - 1 = 6n \\ 17n - 3, & \text{if } k - 1 = 6n + 1 \\ 18n - 3, & \text{if } k - 1 = 6n + 2 \\ 14n + 2, & \text{if } k - 1 = 6n + 3 \\ 17n + 3, & \text{if } k - 1 = 6n + 4 \\ 18n + 3, & \text{if } k - 1 = 6n + 5 \end{cases} \quad \text{and} \quad e_{k_*+1} = \begin{cases} n - 1, & \text{if } k - 1 = 6n \\ 4n - 1, & \text{if } k - 1 = 6n + 1 \\ 9n, & \text{if } k - 1 = 6n + 2 \\ n, & \text{if } k - 1 = 6n + 3 \\ 4n + 2, & \text{if } k - 1 = 6n + 4 \\ 9n + 6, & \text{if } k - 1 = 6n + 5 \end{cases}$$

Proof. Note that Eq. (15) is an obvious consequence of the definition of $M(k)$ as a product of $\binom{k-1}{2}$ factors. For $k \in \{7, 8, 9, 10, 11, 12\}$, formula (14) can be easily verified using tables similar to the ones presented in Table 4. Thus, it is sufficient to prove that if for some $k \geq 7$ the number $M(k)$ satisfies Eq. (14), then also $M(k + 6)$ satisfies Eq. (14).

We start with obtaining some recursion formula for $M(k)$. Let $k \geq 7$. Note that if $q + p \leq k - 2$, then $m(k)_{q,p} = m(k-1)_{q,p}$. Moreover, if $q + p > k$, then $m(k)_{q,p} = m(k-1)_{q',p'}$, where $q' = q - 1$ and $p' = p - 1$. Thus,

$$\begin{aligned} M(k) &= \prod_{q,p \in \{1,2,\ldots,k-1\}:\, q>p} m(k)_{q,p} \\ &= \prod_{q+p \leq k-2} m(k)_{q,p} \cdot \prod_{q+p=k-1} m(k)_{q,p} \cdot \prod_{q+p=k} m(k)_{q,p} \cdot \prod_{q+p \geq k+1} m(k)_{q,p} \\ &= \prod_{q+p \leq k-2} m(k-1)_{q,p} \cdot \prod_{q+p=k-1} m(k)_{q,p} \cdot \prod_{q+p=k} m(k)_{q,p} \cdot \prod_{q'+p' \geq k-1} m(k-1)_{q',p'} \\ &= M(k-1) \cdot \prod_{q+p=k-1} m(k)_{q,p} \cdot \prod_{q+p=k} m(k)_{q,p} =: M(k-1) \cdot A(k) \cdot B(k) =: M(k-1) \cdot C(k) . \end{aligned}$$

So, we are interested in finding the numbers $m(k)_{q,p}$ with $q+p = k-1$ and $q+p = k$ to calculate the factors $A(k)$ and $B(k)$ (and hence $C(k)$). To simplify the description of our argumentation, we will say that a finite sequence $(a_d)_{d=1}^{D}$ is *periodic with period T in its range* if for each $d \in \{1, 2, \ldots, D\}$ such that $d + T$ also belongs to $\{1, 2, \ldots, D\}$, it holds that $a_{d+T} = a_d$.

If we consider $m(k)_{q,p}$, with $q, p \in \{1, 2, \ldots, k-1\}$, $q > p$ and $q + p = k - 1$, then we can arrange these numbers in a sequence $\left(m(k)_{k-1-d,d}\right)_{d=1}^{\lfloor \frac{1}{2}(k-2) \rfloor}$. If $1 \leq d \leq \lfloor \frac{1}{2}(k-2) \rfloor - 3$, then

$$m(k)_{k-1-(d+3),d+3} = \left\lfloor \frac{1}{3}(k-d-4) \right\rfloor + \left\lfloor \frac{1}{3}(d+3) \right\rfloor + 1 = \left\lfloor \frac{1}{3}(k-1-d) \right\rfloor + \left\lfloor \frac{1}{3} d \right\rfloor + 1 = m(k)_{k-1-d,d} ,$$

which proves that the sequence $\left(m(k)_{k-1-d,d}\right)_{d=1}^{\lfloor\frac{1}{2}(k-2)\rfloor}$ is periodic with period 3 in its range. Its length and its first three elements depend on the remainder of dividing $k-1$ by 6 in the way shown in Table 5.

$k-1$	$\lfloor\frac{1}{2}(k-2)\rfloor$	$m(k)_{k-2,1}$	$m(k)_{k-3,2}$	$m(k)_{k-4,3}$	$\mathrm{A}(k)$
$6n$	$3n-1$	k_*	k_*	k_*+1	$k_*^{2n}(k_*+1)^{n-1}$
$6n+1$	$3n$	k_*+1	k_*	k_*+1	$k_*^{n}(k_*+1)^{2n}$
$6n+2$	$3n$	k_*+1	k_*+1	k_*+1	$(k_*+1)^{3n}$
$6n+3$	$3n+1$	k_*	k_*	k_*+1	$k_*^{2n+1}(k_*+1)^{n}$
$6n+4$	$3n+2$	k_*+1	k_*	k_*+1	$k_*^{n}(k_*+1)^{2n+1}$
$6n+5$	$3n+2$	k_*+1	k_*+1	k_*+1	$(k_*+1)^{3n+2}$

Table 5: The length, the first three elements of the sequence $\left(m(k)_{k-1-d,d}\right)_{d=1}^{\lfloor\frac{1}{2}(k-2)\rfloor}$ and the factor $\mathrm{A}(k)$ depending on the remainder of dividing $k-1$ by 6.

Indeed, if $k-1=6n$ for some natural number n, then the length of the sequence $\left(m(k)_{k-1-d,d}\right)_{d=1}^{\lfloor\frac{1}{2}(k-2)\rfloor}$ is $3n-1$ and its first three elements equal

$$m(k)_{k-2,1} = \left\lfloor\frac{1}{3}(6n-1)\right\rfloor + \left\lfloor\frac{1}{3}\cdot 1\right\rfloor + 1 = 2n-1+1 = \frac{k-1}{3} = k_*$$

$$m(k)_{k-3,2} = \left\lfloor\frac{1}{3}(6n-2)\right\rfloor + \left\lfloor\frac{1}{3}\cdot 2\right\rfloor + 1 = 2n-1+1 = \frac{k-1}{3} = k_*$$

$$m(k)_{k-4,3} = \left\lfloor\frac{1}{3}(6n-3)\right\rfloor + \left\lfloor\frac{1}{3}\cdot 3\right\rfloor + 1 = 2n-1+1+1 = 2n+1 = k_*+1.$$

We omit the calculations in the other cases as they are rather tedious, yet not difficult.

Analogously, if we consider $m(k)_{q,p}$, with $q,p\in\{1,2,\ldots,k-1\}$, $q>p$ and $q+p=k$, then we can arrange these numbers in a sequence $\left(m(k)_{k-d,d}\right)_{d=1}^{\lfloor\frac{k-1}{2}\rfloor}$, which also is periodic with period 3 in its range. Its length and its first three elements depend on the remainder of dividing $k-1$ by 6 in the way shown in Table 6, which can be easily verified.

$k-1$	$\lfloor\frac{k-1}{2}\rfloor$	$m(k)_{k-1,1}$	$m(k)_{k-2,2}$	$m(k)_{k-3,3}$	$\mathrm{B}(k)$	$\mathrm{C}(k)=\mathrm{A}(k)\cdot\mathrm{B}(k)$
$6n$	$3n$	k_*	k_*	k_*	k_*^{3n}	$k_*^{5n}(k_*+1)^{n-1}$
$6n+1$	$3n$	k_*+1	k_*	k_*	$k_*^{2n}(k_*+1)^{n}$	$k_*^{3n}(k_*+1)^{3n}$
$6n+2$	$3n+1$	k_*+1	k_*+1	k_*	$k_*^{n}(k_*+1)^{2n+1}$	$k_*^{n}(k_*+1)^{5n+1}$
$6n+3$	$3n+1$	k_*	k_*	k_*	k_*^{3n+1}	$k_*^{5n+2}(k_*+1)^{n}$
$6n+4$	$3n+2$	k_*+1	k_*	k_*	$k_*^{2n+1}(k_*+1)^{n+1}$	$k_*^{3n+1}(k_*+1)^{3n+2}$
$6n+5$	$3n+2$	k_*+1	k_*+1	k_*	$k_*^{n}(k_*+1)^{2n+2}$	$k_*^{n}(k_*+1)^{5n+4}$

Table 6: The length, the first three elements of the sequence $\left(m(k)_{k-d,d}\right)_{d=1}^{\lfloor\frac{k-1}{2}\rfloor}$ and the factors $\mathrm{B}(k)$ and $\mathrm{C}(k)$ depending on the remainder of dividing $k-1$ by 6.

Let $k-1=6n$ for some natural number n (since $k\geq 7$, it holds $n\geq 1$). According to the recursion formula above, we find

$$\begin{aligned}\mathrm{M}(k+6) &= \mathrm{M}(k)\cdot\mathrm{C}(k+1)\cdot\mathrm{C}(k+2)\cdot\mathrm{C}(k+3)\cdot\mathrm{C}(k+4)\cdot\mathrm{C}(k+5)\cdot\mathrm{C}(k+6)\\ &= \mathrm{M}(k)\cdot\mathrm{C}(6n+2)\cdot\mathrm{C}(6n+3)\cdot\mathrm{C}(6n+4)\cdot\mathrm{C}(6n+5)\cdot\mathrm{C}(6n+6)\cdot\mathrm{C}(6n+7).\end{aligned}$$

Using the formulas for $\mathrm{C}(k)$ listed in Table 6, we get

$$\begin{aligned}\mathrm{M}(k+6) &= \mathrm{M}(k)\cdot(2n)^{3n}(2n+1)^{3n}\cdot(2n)^{n}(2n+1)^{5n+1}\cdot(2n+1)^{5n+2}(2n+2)^{n}\\ &\quad\cdot(2n+1)^{3n+1}(2n+2)^{3n+2}\cdot(2n+1)^{n}(2n+2)^{5n+4}\cdot(2n+2)^{5n+5}(n)^{n}\\ &= = \mathrm{M}(k)\cdot(2n)^{4n}(2n+1)^{17n+4}(2n+2)^{14n+11}(2n+3)^{n}\ ,\end{aligned}$$

which coincides with Eq. (14), since $(k+6)-1 = 6(n+1)$. The calculations in the cases $k-1 = 6n+1$, $k-1 = 6n+2$, $k-1 = 6n+3$, $k-1 = 6n+4$ and $k-1 = 6n+5$ are just as elementary. □

The numbers $\mathrm{M}(k)$ for several values of k are shown in Table 7. As a consequence, we get the following final result.

Corollary 4.9. *Let $k > 1$ be an integer and denote $k_* = \left\lfloor \frac{k-1}{3} \right\rfloor$. The number of all k-ary triangular NCCAs equals $\mathrm{N}(k)$, where*

$$\mathrm{N}(2) = 1,\ \mathrm{N}(3) = 1,\ \mathrm{N}(4) = 2,\ \mathrm{N}(5) = 16,\ \mathrm{N}(6) = 512,$$

while for $k \geq 7$

$$\mathrm{N}(k) = 1^{E_1}\cdot 2^{E_2}\cdot\ldots(k_*-1)^{E_{k_*-1}}\cdot k_*^{E_{k_*}}\cdot(k_*+1)^{E_{k_*+1}}\ , \tag{16}$$

where the exponents E_i are defined as follows:

$$E_1 = 7 \text{ and for } i = 2,3,\ldots,k_*-1,\ E_i = \begin{cases} 9i, & \text{if } i \text{ is even} \\ 9i-3, & \text{if } i \text{ is odd} \end{cases}$$

$$E_{k_*} = \begin{cases} 14n, & \text{if } k-1 = 6n \\ 17n, & \text{if } k-1 = 6n+1 \\ 18n, & \text{if } k-1 = 6n+2 \\ 14n+5, & \text{if } k-1 = 6n+3 \\ 17n+6, & \text{if } k-1 = 6n+4 \\ 18n+6, & \text{if } k-1 = 6n+5 \end{cases} \quad \text{and} \quad E_{k_*+1} = \begin{cases} n, & \text{if } k-1 = 6n \\ 4n+1, & \text{if } k-1 = 6n+1 \\ 9n+3, & \text{if } k-1 = 6n+2 \\ n+1, & \text{if } k-1 = 6n+3 \\ 4n+4, & \text{if } k-1 = 6n+4 \\ 9n+9, & \text{if } k-1 = 6n+5 \end{cases}$$

The numbers $\mathrm{N}(k)$ for several values of k are shown in Table 7.

For a given $k > 1$, the set of all k-ary triangular NCCAs can be enumerated using Algorithm 2.

k	$\mathrm{M}(k)$	$\mathrm{N}(k) = \mathrm{S}(k) \cdot \mathrm{M}(k)$
2	1^0	$1^2 = 1$
3	1^1	$1^4 = 1$
4	1^3	$1^6 \cdot 2^1 = 2$
5	$1^4 \cdot 2^2$	$1^7 \cdot 2^4 = 16$
6	$1^4 \cdot 2^6$	$1^7 \cdot 2^9 = 512$
7	$1^4 \cdot 2^{11}$	$1^7 \cdot 2^{14} \cdot 3^1 = 49\,152$
8	$1^4 \cdot 2^{14} \cdot 3^3$	$1^7 \cdot 2^{17} \cdot 3^5 = 31\,850\,496$
9	$1^4 \cdot 2^{15} \cdot 3^9$	$1^7 \cdot 2^{18} \cdot 3^{12} = 139\,314\,069\,504$
10	$1^4 \cdot 2^{15} \cdot 3^{16} \cdot 4^1$	$1^7 \cdot 2^{18} \cdot 3^{19} \cdot 4^2 \approx 4.9 \cdot 10^{15}$
11	$1^4 \cdot 2^{15} \cdot 3^{20} \cdot 4^6$	$1^7 \cdot 2^{18} \cdot 3^{23} \cdot 4^8 \approx 1.6 \cdot 10^{21}$
12	$1^4 \cdot 2^{15} \cdot 3^{21} \cdot 4^{15}$	$1^7 \cdot 2^{18} \cdot 3^{24} \cdot 4^{18} \approx 5.1 \cdot 10^{27}$
13	$1^4 \cdot 2^{15} \cdot 3^{21} \cdot 4^{25} \cdot 5^1$	$1^7 \cdot 2^{18} \cdot 3^{24} \cdot 4^{28} \cdot 5^2 \approx 1.3 \cdot 10^{35}$
14	$1^4 \cdot 2^{15} \cdot 3^{21} \cdot 4^{31} \cdot 5^7$	$1^7 \cdot 2^{18} \cdot 3^{24} \cdot 4^{34} \cdot 5^9 \approx 4.3 \cdot 10^{43}$
15	$1^4 \cdot 2^{15} \cdot 3^{21} \cdot 4^{33} \cdot 5^{18}$	$1^7 \cdot 2^{18} \cdot 3^{24} \cdot 4^{36} \cdot 5^{21} \approx 1.7 \cdot 10^{53}$
16	$1^4 \cdot 2^{15} \cdot 3^{21} \cdot 4^{33} \cdot 5^{30} \cdot 6^2$	$1^7 \cdot 2^{18} \cdot 3^{24} \cdot 4^{36} \cdot 5^{33} \cdot 6^3 \approx 8.8 \cdot 10^{63}$

Table 7: The numbers $\mathrm{M}(k)$ and $\mathrm{N}(k)$ for $k \leq 16$.

Algorithm 2: Generating all k-ary triangular NCCAs

Input: k – natural number, $k > 1$; list of split functions *SplitList*(k);
Output: list of all k-ary triangular NCCAs

procedure GenAllNCCAs(k)
    ALLNCCAList $\leftarrow$ `empty list`
    foreach $(c_0, c_1, \ldots, c_{k-1})$ *from SplitList*(k) **do**
        for $q, p_1, p_2, p_3 \in \{0, 1, \ldots, k-1\}$ **do**
            $h(q, p_1, p_2, p_3) = c_q + \frac{1}{3}(p_1 - c_{p_1}) + \frac{1}{3}(p_2 - c_{p_2}) + \frac{1}{3}(p_3 - c_{p_3})$
        `/* set value of every` $\alpha_{q,p}$ `for` $q, p \in \{1, 2, \ldots, k-1\} : q > p$ `*/`
        foreach *value of* $\alpha_{2,1}$ *satysfying Eq. (11) or Eq. (12) of Theorem 4.5* **do**
            foreach *value of* $\alpha_{3,1}$ *satysfying Eq. (11) or Eq. (12) of Theorem 4.5* **do**
                $\ddots$
                    foreach *value of* $\alpha_{k-1,k-2}$ *satysfying Eq. (11) or Eq. (12) of Theorem 4.5* **do**
                        foreach $q, p_1, p_2, p_3 \in \{0, 1, \ldots, k-1\}$ **do**
                            calculate $f(q, p_1, p_2, p_3)$ from Eq. (10)
                        add f to list
    return *ALLNCCAList*

5. Conclusions and future work

In this paper, we have investigated two-dimensional cellular automata defined on a regular triangular grid that update their states on the basis of adjacent cells only. We have provided a very useful representation for such number-conserving cellular automata: the split-and-perturb decomposition theorem (Theorem 3.9). Based on this theorem, we have been able to enumerate all triangular number-conserving cellular automata with the classical state set $\{0, 1, 2, \ldots, k-1\}$, where k is some positive integer. In this way, we have completely solved the problem of finding all triangular k-ary (binary, ternary, quaternary, quinary, etc.) cellular automata that preserve the sum of states of all the cells, even for large values of k. As far as we know, up to now, there was no tool that allows to do this, since previous approaches mainly performed an exhaustive search through the entire space of all cellular automata.

This work can be extended in at least two ways. Firstly, certain additional properties of cellular automata can be studied. For example, one can attempt to identify which of the triangular k-ary number-conserving cellular automata are reversible, are computationally universal or are intrinsically universal (for the latter, see [8]). Secondly, it might be possible to generalize the results in this paper to the case of number-conserving cellular automata defined on regular graphs. We are going to investigate the latter topic in the near future.

References

References

[1] Alonso-Sanz, R. [2007] "A structurally dynamic cellular automaton with memory in the triangular tessellation," *Complex Systems* **17**, 1–15.

[2] Bakhtiari, M. & Salehi, M. S. [2018] "Reconstruction of deformed microstructure using cellular automata method," *Computational Materials Science* **149**, 1–13, doi: 10.1016/j.commatsci.2018.02.053.

[3] Bays, C. [1994] "Cellular automata in the triangular tessellation," *Complex Systems* **8**, 127–150.

[4] Boccara, N. & Fukś, H. [1998] "Cellular automaton rules conserving the number of active sites," *Journal of Physics A: Mathematical and General* **31**, 6007–6018, doi:10.1088/0305-4470/31/28/014.

[5] Boccara, N. & Fukś, H. [2002] "Number-conserving cellular automaton rules," *Fundamenta Informaticae* **52**, 1–13.

[6] Bouaine, A. & Rachik, M. [2018] "Modeling the impact of immigration and climatic conditions on the epidemic spreading based on cellular automata approach," *Ecological Informatics* **46**, 36–44, doi:10.1016/j.ecoinf.2018.05.004.

[7] Bowness, R., Chaplain, M. A., Powathil, G. G. & Gillespie, S. H. [2018] "Modelling the effects of bacterial cell state and spatial location on tuberculosis treatment: Insights from a hybrid multiscale cellular automaton model," *Journal of Theoretical Biology* **446**, 87–100, doi:10.1016/j.jtbi.2018.03.006.

[8] Delorme, M., Mazoyer, J., Ollinger, N. & Theyssier, G. [2011] "Bulking I: An abstract theory of bulking," *Theoretical Computer Science* **412**, 3866–3880, doi:10.1016/j.tcs.2011.02.023.

[9] Durand, B., Formenti, E. & Róka, Z. [2003] "Number-conserving cellular automata I: decidability," *Theoretical Computer Science* **299**, 523–535, doi:10.1016/S0304-3975(02)00534-0.

[10] Dzedzej, A., Wolnik, B., Nenca, A., Baetens, J. M. & De Baets, B. [2021] "Two-dimensional rotation-symmetric number-conserving cellular automata," *Information Sciences* **577**, 599–621.

[11] Fu, Z., Jia, Q., Chen, J., Ma, J., Han, K. & Luo, L. [2018] "A fine discrete field cellular automaton for pedestrian dynamics integrating pedestrian heterogeneity, anisotropy, and time-dependent characteristics," *Transportation Research Part C: Emerging Technologies* **91**, 37–61, doi:10.1016/j.trc.2018.03.022.

[12] Fukś, H. & Sullivan, K. [2007] "Enumeration of number-conserving cellular automata rules with two inputs," *Journal of Cellular Automata* **2**, 141–148.

[13] Goles, E., Moreira, A. & Rapaport, I. [2011] "Communication complexity in number-conserving and monotone cellular automata," *Theoretical Computer Science* **412**, 3616–3628, doi:10.1016/j.tcs.2011.03.008.

[14] Gounaridis, D., Chorianopoulos, I., Symeonakis, E. & Koukoulas, S. [2019] "A random forest-cellular automata modelling approach to explore future land use/cover change in attica (greece), under different socio-economic realities and scales," *Science of The Total Environment* **646**, 320–335, doi:10.1016/j.scitotenv.2018.07.302.

[15] Harary, F. [1969] *Graph Theory* (Addison-Wesley).

[16] Hattori, T. & Takesue, S. [1991] "Additive conserved quantities in discrete-time lattice dynamical systems," *Physica D: Nonlinear Phenomena* **49**, 295–322, doi:https://doi.org/10.1016/0167-2789(91)90150-8, URL http://www.sciencedirect.com/science/article/pii/0167278991901508.

[17] Imai, K., Ishizaka, H. & Poupet, V. [2015] *5-State Rotation-Symmetric Number-Conserving Cellular Automata are not Strongly Universal* (Springer International Publishing, Cham, Switzerland), ISBN 978-3-319-18812-6, pp. 31–43, doi:10.1007/978-3-319-18812-6_3.

[18] Iwan, S., Kijewska, K., Johansen, B. G., Eidhammer, O., Ma³ecki, K., Konicki, W. & Thompson, R. G. [2018] "Analysis of the environmental impacts of unloading bays based on cellular automata simulation," *Transportation Research Part D: Transport and Environment* **61**, 104–117, doi:10.1016/j.trd.2017.03.020.

[19] Ji, J., Lu, L., Jin, Z., Wei, S. & Ni, L. [2018] "A cellular automata model for high-density crowd evacuation using triangle grids," *Physica A: Statistical Mechanics and its Applications* **509**, 1034–1045, doi:https://doi.org/10.1016/j.physa.2018.06.055, URL http://www.sciencedirect.com/science/article/pii/S0378437118307799.

[20] Kornyak, V. V. [2007] "Symmetric cellular automata," *Programming and Computer Software* **33**, 87–93, doi:10.1134/S0361768807020065, URL https://doi.org/10.1134/S0361768807020065.

[21] Moreira, A. [2003] "Universality and decidability of number-conserving cellular automata," *Theoretical Computer Science* **292**, 711–721, doi:10.1016/S0304-3975(02)00065-8.

[22] Moreira, A., Boccara, N. & Goles, E. [2004] "On conservative and monotone one-dimensional cellular automata and their particle representation," *Theoretical Computer Science* **325**, 285–316, doi:10.1016/j.tcs.2004.06.010.

[23] Nagatani, T. & Tainaka, K. [2018] "Cellular automaton for migration in ecosystem: Application of traffic model to a predator–prey system," *Physica A: Statistical Mechanics and its Applications* **490**, 803–807, doi:10.1016/j.physa.2017.08.151.

[24] Nenca, A., Dzedzej, A. & Wolnik, B. [2022] "The complete list of two-dimensional rotation-symmetric number-conserving septenary cellular automata" [Data set]. Gdańsk University of Technology, doi:10.34808/jr67-r637.

[25] Ortigoza, G. M. [2015] "Unstructured triangular cellular automata for modeling geographic spread," *Appl. Math. Comput.* **258**, 520–536, doi:10.1016/j.amc.2015.01.116, URL https://doi.org/10.1016/j.amc.2015.01.116.

[26] Pivato, M. [2002] "Conservation laws in cellular automata," *Nonlinearity* **15**, 1781.

[27] Ruivo, E. L., de Oliveira, P. P., Lobos, F. & Goles, E. [2018] "Shift-equivalence of k-ary, one-dimensional cellular automata rules," *Communications in Nonlinear Science and Numerical Simulation* **63**, 280 – 291, doi:https://doi.org/10.1016/j.cnsns.2018.03.017, URL http://www.sciencedirect.com/science/article/pii/S100757041830100X.

[28] Tanimoto, N. & Imai, K. [2009] "A characterization of von Neumann neighbor number-conserving cellular automata," *Journal of Cellular Automata* **4**, 39–54.

[29] Wolnik, B., Dzedzej, A., Baetens, J. M. & De Baets, B. [2017] "Number-conserving cellular automata with a von Neumann neighborhood of range one," *Journal of Physics A: Mathematical and Theoretical* **50**, 435101.

[30] Wolnik, B., Nenca, A., Baetens, J. M. & De Baets, B. [2020] "A split-and-perturb decomposition of number-conserving cellular automata," *Physica D: Nonlinear Phenomena* **413**, 132645.

[31] Wu, Y., Abdel-Aty, M., Ding, Y., Jia, B., Shi, Q. & Yan, X. [2018] "Comparison of proposed countermeasures for dilemma zone at signalized intersections based on cellular automata simulations," *Accident Analysis & Prevention* **116**, 69–78, doi:10.1016/j.aap.2017.09.009, simulation of Traffic Safety in the Era of Advances in Technologies.

[32] Yang, J., Yu, H., Yang, H., Li, F., Wang, Z. & Zeng, X. [2018] "Prediction of microstructure in selective laser melted Ti-6Al-4V alloy by cellular automaton," *Journal of Alloys and Compounds* **748**, 281–290, doi:10.1016/j.jallcom.2018.03.116.

[33] Zawidzki, M. [2011] "Application of semitotalistic 2D cellular automata on a triangulated 3D surface," *International Journal of Design & Nature and Ecodynamics* **6**, 34–51, doi: 10.2495/DNE-V6-N1-34-51.

[34] Zhang, R., Tian, Q., Jiang, L., Crooks, A., Qi, S. & Yang, R. [2018] "Projecting cropping patterns around poyang lake and prioritizing areas for policy intervention to promote rice: A cellular automata model," *Land Use Policy* **74**, 248–260, doi:10.1016/j.landusepol.2017.09.040.